STUDIA UBB PHYSICA, Vol. 61 (LXI), 2, 2016, pp. 65-81
(RECOMMENDED CITATION)

Dedicated to Professor Dr. Sorin Dan Anghel on His 65th Anniversary

ON THE STRUCTURE AND STABILITY OF NEUTRON STARS. A GENERAL RELATIVISTIC APPROACH

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ABSTRACT. The structure of Neutron Stars (NS) is still unclear and for this reason this paper serves as an attempt to couple the Tolman-Oppenheimer-Volkoff (TOV) equations with a polytropic Equation of State (EoS). For different EoS models coupled with the relativistic TOV equations it could be calculated the critical mass and radius for a neutron star, underlining consequences of the type of EoS used on the mass-radius stable configurations of the NS. Another briefly investigated topic in this paper is to see if the positive cosmological constant bears any role in the evolution of the neutron star.

Keywords: *Neutron Stars, Computational Physics, Theoretical Physics, Astrophysics*

INTRODUCTION

After a relatively massive star(~10 – 28 $M_{\circ}$) burns out its fuel one of the possible outcomes is the star to become a celestial entity known as a Neutron Star. The remaining matter is a very dense, collapsed core, with masses between 0.9-1.9 $M_{\circ}$ and a radius just about 10 km [2]. The core is at several times nuclear density and may be composed of exotic matter. In the interior is superconducting and superfluid, with transition temperatures around a billion degrees Kelvin, the only thing stopping gravitational collapse being neutron degeneracy pressure. This structure represents a key interest in physics, because it gives one the chance to study matter in exotic states, which will probably never be available to scientists in controlled environments here, on Earth.

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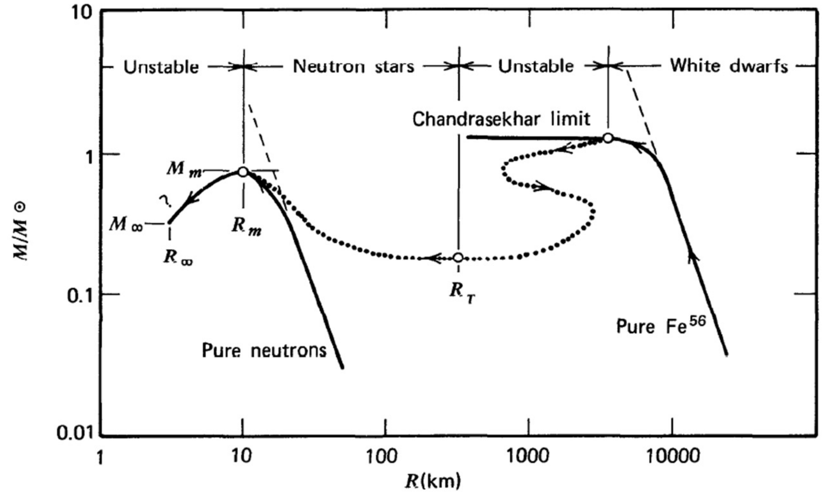

Fig. 1. Different configurations for stellar equilibrium. Note that transitions between stability and instability occur at the maxima and minima points of the curve [1].

The aim of this paper is to tackle the structure problems of this types of stars by analyzing them in a general relativistic framework, using the Tolman-Oppenheimer-Volkoff equations in order to study the internal structure and stable configurations of neutron stars (Figure 1) and the influence of the cosmological constant upon the star. In our approach we used the non-interacting Fermi gas model and, as an ingredient, the introduction of nucleon-nucleon interactions in order to obtain a realistic model.

THEORETICAL DETAILES

Neutron Stars are relativistic objects and for this reason, their structure must be analyzed in a general-relativistic framework. Starting from the Einstein Equations [1],

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \tag{1}$$

where $G^{\mu\nu}$ is the Einstein tensor, describing the curvature of space-time, and $T^{\mu\nu}$ is the stress-energy tensor, describing matter/energy sources of space-time curvature. The stress-energy tensor, in the case of an ideal fluid, takes the form:

$$T^{\mu\nu} = \left(\rho\left(1+\frac{\varepsilon}{c^2}\right)+\frac{p}{c^2}\right)u^{\mu}u^{\nu}+pg^{\mu\nu} \quad (2)$$

where ρ is the baryon rest mass density, ϵ is the specific energy density, p is the fluid pressure, and $g^{\mu\nu}$ is the 4-metric. In the case of vanishing space velocity u^i = (0,0,0) and $T^{\mu\nu}$= 0(in vacuum) we adopt the form of the interior Schwarzschild metric:

$$ds^2 = -e^{2\phi(r)}c^2dt^2+\left(1-\frac{2Gm(r)}{rc^2}\right)^{-1}dr^2+r^2d\Omega^2 \quad (3)$$

where, $e^{\Phi}(r)$ is the lapse function, $\Phi(\mathrm{r}) = \frac{1}{2}\ln(1-\frac{2GM}{rc^2})$ is the metric potential and m(r) is the gravitational mass inside the radius r. For this model, m(r)=M(r) is the total mass inside the sphere of radius R.

In this framework the Tolman-Oppenheimer-Volkoff equations are [1]:

$$\frac{dp}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2}\left\{1+\frac{p(r)}{\epsilon(r)}\right\}\left\{1+\frac{4\pi r^3p(r)}{M(r)c^2}\right\}\left\{1-\frac{2GM(r)}{c^2r}\right\}^{-1} \quad (4)$$

$$\frac{dM}{dr} = 4\pi r^2\rho\left(1+\frac{\epsilon(r)}{c^2}\right) \quad (5)$$

$$\frac{d\Phi}{dr} = \frac{m+4\pi r^3\frac{p}{c^2}}{r\left(1-\frac{2Gm}{c^2}\right)} \quad (6)$$

In (4) the first two factors in curly brackets, represent special relativity corrections of order $v/_{c^2}$ (these factors reduce to 1 in the non-relativistic limit) and the last factor is a general relativistic correction. The correction factors are all positively defined. To solve these equations is important to invoke the balance between gravitational forces and internal pressure, the pressure being a function of the Equation of State (EoS). It is necessary to find the conditions to withstand the gravitational attraction (and so the structure equations imply there is a maximum mass that a star can have). Finding the most appropriate and complete EoS will be one of the goals pursued in this paper.

White Dwarfs. Fermi EoS

For free electrons, knowing the number of states, dn, available at momentum k per unit of volume, the electron number density can be calculated [1]:

$$n = \frac{k_F^3}{3\pi^2\hbar^3} \tag{7}$$

where k_F is the Fermi energy, a quantity that varies according to the star's total mass.

Because the electrons are neutralized by protons, accompanied by neutrons, one can neglect the electron mass, m_e, with respect to nucleon mass m_N and so the total mass density of the star is:

$$\rho = nm_N\beta \tag{8}$$

where β = $A/_Z$ is the number of nucleons per electron.

From (7) and (8) yields:

$$k_F = \hbar \sqrt[3]{\left(\frac{3\pi^2\rho}{\beta m_N}\right)} \tag{9}$$

In the total energy expression the contributions of nucleon masses is proportional with ρ.

$$\epsilon = \beta n m_N + \epsilon_{e^-}(k_F) \tag{10}$$

In the relativist case ($k_F \gg m_e$) the pressure is [8]:

$$p(k_F) = \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_F}(k^2c^2 + m_e^2c^4)^{-1/2} k^4 dk = \tag{11}$$

simplifies as seen in [8]:

$$p(k_F) = \frac{\epsilon_0}{3}\int_0^{\frac{k_F}{m_e c}} u^3 du = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2\rho}{\beta m_N}\right)^{4/3} = K_r\epsilon^{4/3} \tag{12}$$

where $K_R = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2}{m_N\beta}\right)^{4/3}$.

For a star having a simple polytropic EoS $p = k\rho^\gamma \sim k\epsilon^\gamma$ it is clear now, from (12), that the relativistic electron Fermi gas has a polytropic EoS with $\gamma = 4/_3$.

In a similar way one can establish another polytropic EoS for non-interacting electron Fermi gas model in a non-relativistic case (k_F<<m_e) that yields :

$$K_{NR} = \frac{\hbar c}{15\pi^2 m_e}\left(\frac{3\pi^2}{m_N c^2\beta}\right)^{\frac{5}{3}} \tag{13}$$

Now the TOV equations(4)(5) can be used, coupled with one of the EoS, to obtain the relationship between p=p(r) and M=M(r) for White Dwarfs(WD). This paper refers to the structural study of neutron stars, but it's simpler to use (8)-(13) for WD (electron degeneracy) and shift to neutrons for the neutron star EoS.

A. Neutron Stars. First EoS model: Fermi neutron gas

Other than (4), we need a EoS for the pure neutron star therefore, our first choice is a Fermi gas model for neutrons instead of electrons.

a. Non-Relativistic Case

For the neutron star the value of K_{NR} is:

$$K_{NR} = \frac{\hbar^2}{15\pi^2 m_N}\left(\frac{3\pi^2}{\beta m_N c^2}\right)^{5/3} = 6.484\ x\ 10^{-26}\ \frac{cm^2}{ergs^{2/3}} \tag{14}$$

b. Relativistic Case

The EoS is still polytrope with $\gamma = 1$ [8].

The central pressures expected when computing this case are greater than 10^{-4}. The problem that arises for this EoS because the pressure $\overline{p}(r)$ has never computing zero value and the loop on $\overline{r}$ runs through the whole range, thus giving enormous values for the radii when compared with the expected results. In order to fix this, we need to find a EoS that works for every value of the relativity parameter $\frac{k_F}{m_N c}$. We can do this by trying to fit the energy density as two transcendental functions of pressure [8].

$$\overline{\epsilon}(p) = B_{NR}\overline{p}^{3/5} + B_R\overline{p}$$

The values of B can be calculated using Mathematica's build in fitting function:

$$B_{NR} = 2.4216,\ B_R = 2.8663 \tag{15}$$

Eq. (4) can be integrated from r = 0 to r = R, knowing that $\overline{p}(R) = 0$ and using the EoS from above. The initial values are $\overline{p}(0) > 0$; $\overline{\mathcal{M}}(0) = 0$.

Non-Relativistic			Relativistic		
p_0	R(km)	M($M_{\circ}$)	p_0	R(km)	M($M_{\circ}$)
10^{-2}	15.0	1.0370	10^{-2}	13.4	0.7166

Using general relativistic corrections, one can see a significant difference between in the star's maximum mass and radius, for a given central pressure (Figures 2 and 3).

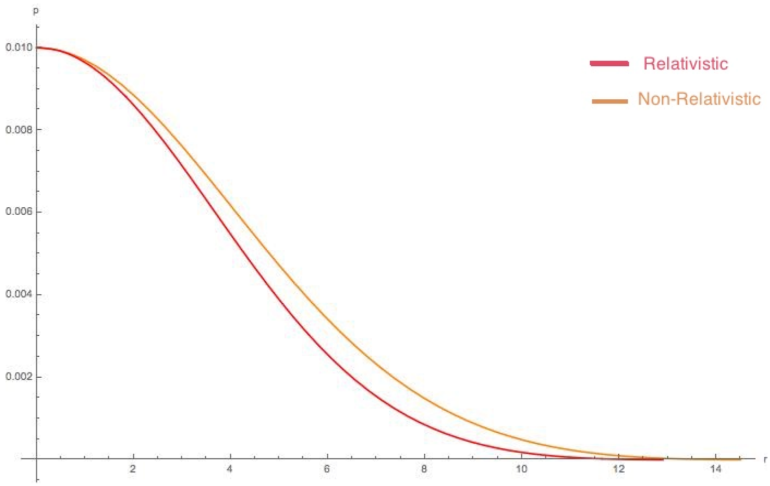

Fig. 2. $\overline{\mathrm{p}}(\mathrm{r})$ for a neutron star with central pressure of 0.01 with a non-interacting Fermi EoS fit.

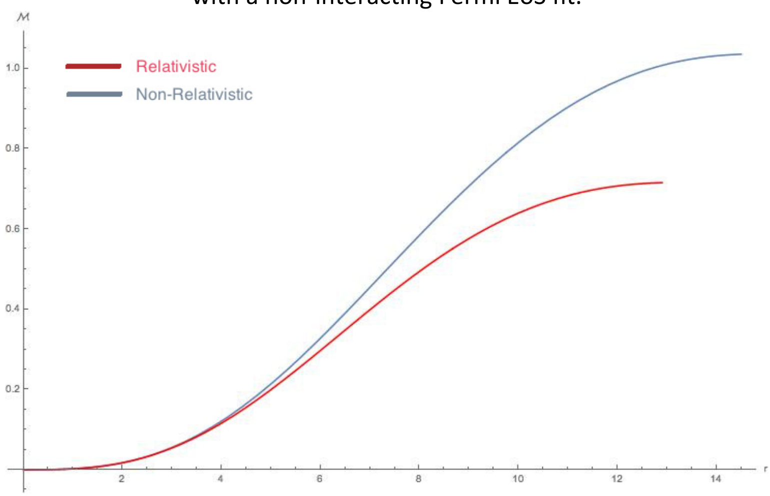

Fig. 3. $\overline{\boldsymbol{\mathcal{M}}}(\boldsymbol{r})$ for a pure neutron star with central pressure of 0.01 using non-interacting Fermi EoS fit.

It can be seen in Figure 3 that for radii bigger than r = 4 km, general relativistic (GR) corrections are by all means not negligible, the allowed masses being significantly smaller when using a GR framework.

B. Second EoS model: non-interacting Fermi gas with p^+ and e^-

The presence of protons and electrons in a neutron star is due to the weak decay:

$$n \rightarrow p + e^- + \overline{\nu_e} \tag{16}$$

If this situation is so, and knowing that neutrons have a lifetime of about 15 minutes, then the following question arises: why don't all the neutrons inside the star decay into protons and electrons? Because all the available low-energy levels for the decay proton are already filled up by other protons present and the Pauli exclusion principle kicks, in preventing the neutron beta decay [1].

$$k_{F,p} = k_{F,e} \tag{17}$$

$$\mu_n = \mu_p + \mu_e \tag{18}$$

Charge neutrality is ensured through (17) and weak interaction equilibrium through (18). Integrals for the total pressure and energy density are [8]:

$$p_i(k_{F,i}) = \int_0^{k_{F,i}} (k^2 + m_i)^{-1/2} k^4 dk \tag{19}$$

$$\epsilon_i(k_{F,i}) = \int_0^{k_{F,i}} (k^2 + m_i)^{1/2} k^2 dk \tag{20}$$

where m_i(i=1,..N) is the mass of every individual nucleon and N is the total number of nucleons.

Using Mathematica we can generate a table of values for ϵ_t, p_t over a range of $k_{F,n}$ that can be fitted to the same form used in (15).

The new coefficients obtained are:

$$B_{NR} = 2.572 \quad B_R = 2.891 \tag{21}$$

It can be observed that the coefficients are very similar to those in the case of a pure neutron star and, in fact, the results are extremely similar to that case. We continue on to a more exact EoS, including nuclear interactions.

C. Third EoS model. Prakash EoS

In our attempt to introduce nucleon-nucleon interaction we started by developing a simple model for the nuclear potential that reproduces the characteristics of nuclear matter[4].

We know from the von Weizacker mass formula [8] that for symmetric nuclear matter (N=Z) the equilibrium density n_0= 0.16 nucleons/fm^3, that when compared with m_N = 939 MeV/c^2 tells us that we can use a non-relativistic approach.

$$E_{binding} = \frac{E}{A} - m_N = -16MeV \quad (22)$$

We want our potential to respect (22) by introducing the nuclear compressibility K_0 that is not exactly determined to this day, but it is known to take values between 200-400MeV, and the symmetry energy which brings a contribution of about 30 MeV above the minimum at n_0 [8].

a. Symmetric nuclear matter

For symmetric nuclear matter we have:

$$n_n = n_p \quad (23)$$

$$n = n_n + n_p \quad (24)$$

The potential will be constructed with the aid of two functions and three parameters and the nuclear potential will be included in the energy density.

The potential in $\epsilon(n)$ will be [8]:

$$\frac{\epsilon(n)}{n} = m_N + \frac{3}{5}\frac{\hbar^2 k_F^2}{2m_N} + \frac{A}{2}u + \frac{B}{\sigma+1}u^\sigma \quad (25)$$

where $u = \frac{n}{n_0}$

In (25) the first term represents the rest mass energy and the second is the average kinetic energy/nucleon.

The kinetic energy term will be abbreviated as $\langle E_F^0 \rangle \cong 22.1\, MeV$ for $k_F^0 = k_F(n_0)$.

We can find out the values of A,B and σ:

$$\begin{cases} \langle E_F^0 \rangle + \dfrac{A}{2} + \dfrac{B}{\sigma + 1} = E_{binding} & (26) \\ \dfrac{2}{3}\langle E_F^0 \rangle + \dfrac{A}{2} + \dfrac{B\sigma}{\sigma + 1} = 0 & (27) \\ \dfrac{10}{9}\langle E_F^0 \rangle + A + B\sigma = \dfrac{K_0}{9} & (28) \end{cases}$$

Solving this system we get the values for A, B, σ in terms of K_0 as seen in Table 1.

Table 1. Values obtained by solving the system (26)(27)(28) using Wolfram Mathematica for different values of the compressibility K_0.

K_0(MeV)	A(MeV)	B(MeV)	σ
200	-366.188	313.348	1.16117
250	-193.367	140.527	1.39891
300	-149.617	96.7769	1.63665
350	-129.658	76.8176	1.8744
400	-118.232	65.3916	2.11214

The pressure is [8] (Figure 4)

$$p(n) = n^2 \frac{d}{dn}\left(\frac{\epsilon}{n}\right) = n_0 \left[\frac{2}{3}\langle E_F^0 \rangle u^{5/3} + \frac{A}{2}u^2 + \frac{B\sigma}{\sigma+1}u^{\sigma+1}\right] \quad (29)$$

where $u=n/n_0$.

We can see that the minima is located at u=1 (Figure 4) and has the same depth of -16MeV independent of the compressibility. The second derivatives of these curves correspond to the nuclear compressibility (K_0).

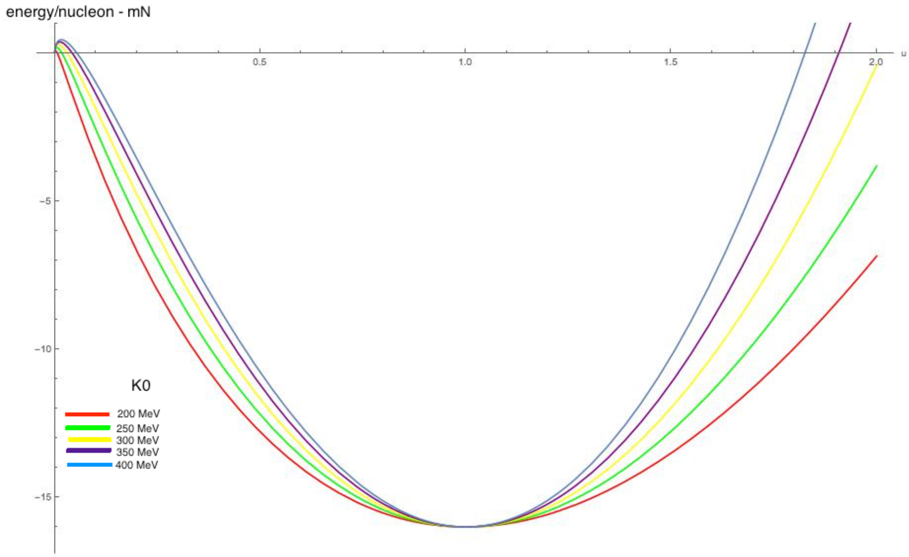

Fig. 4. The average energy/nucleon minus it's rest mass as a function of $u=\frac{n}{n_0}$ for different values of K_0.

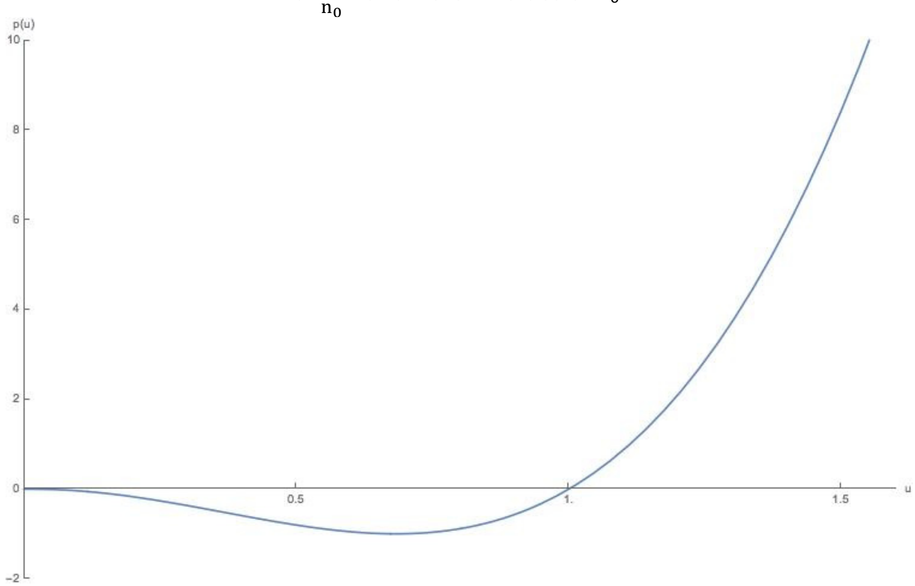

Fig. 5. Dependence of pressure on u.

Figure 5 shows that the pressure is negative for values of u between [0,1] which denotes instability for $u<1(n<n_0)$.

b. Non-symmetric Nuclear matter

We tackle the non symmetric nuclear matter by introducing a parameter, α, to represent the neutron and proton densities [4]:

$$n_n = \frac{1+\alpha}{2}n \qquad n_p = \frac{1-\alpha}{2}n \tag{30}$$

where $n = n_p + n_e$.
α is real and has a range from 0 to 1, being 1 for pure neutron matter and 0 for symmetric nuclear matter discussed above.

$$\alpha = \frac{n_n - n_p}{n} = \frac{N-Z}{A} \tag{31}$$

Following Prakash[4] we can expect the isospin symmetry breaking interaction to depend through proportionality to α^p, where p is an integer.

Taking into account the kinetic energy contributions of neutrons and protons results:

$$\epsilon_{KE}(n,\alpha) = \frac{3}{5}\frac{k_{F,n}^2}{2m_N}n_n + \frac{k_{F,p}^2}{2m_N}n_p \tag{32}$$

where $m_N \cong m_P$.

The kinetic energy parametrized by α [4] has the expression:

$$\epsilon_{KE}(n,\alpha) = n\langle E_F \rangle \frac{1}{2}\left[(1+\alpha)^{5/3} + (1-\alpha)^{5/3}\right] \tag{33}$$

with $\langle E_F \rangle = \frac{3}{5}\frac{\hbar^2}{2m_N}\left(\frac{3\pi^2 n}{2}\right)^{2/3}$ being the mean kinetic energy of symmetric nuclear matter

The excess kinetic energy has the form [8]:

$$\Delta\epsilon_{KE}(n,\alpha) = n\langle E_F \rangle \left\{\frac{1}{2}\left[(1+\alpha)^{\frac{5}{3}} - (1-\alpha)^{\frac{5}{3}}\right] - 1\right\} \tag{34}$$

Making $\alpha = 1$(pure neutron matter) yields:

$$\Delta\epsilon_{KE} = n\langle E_F\rangle\left(2^{2/3} - 1\right) \tag{35}$$

If we expand to leading order in α, we reach all our goals keeping the terms of order α^2. Assuming a quadratic approximation in α, for the potential contribution, the total energy per particle will be [8]:

$$E(n,\alpha) = E(n,0) + \alpha^2 S(n) \tag{36}$$

The isospin symmetry breaking is proportional to α^2[4], therefore we assume a form for S(u):

$$S(u) = \left(2^{\frac{2}{3}} - 1\right)\frac{3}{5}\langle E_F\rangle\left(u^{\frac{2}{3}} - F(u)\right) + S_0 F(u) \tag{37}$$

where F(u) is a arbitrary function. The following conditions must be satisfied: F(1) = 1 because S(u=1) = S_0 and F(0) = 0. If choosing to use F(u)=u form [4] and [8], then S_0 (the bulk symmetry energy parameter) is 30 MeV (Figure 6).

From Figure 6 it can be seen that, in the vicinity of u=1, the average energy per neutron is independent of the values of the compressibility(K_0).

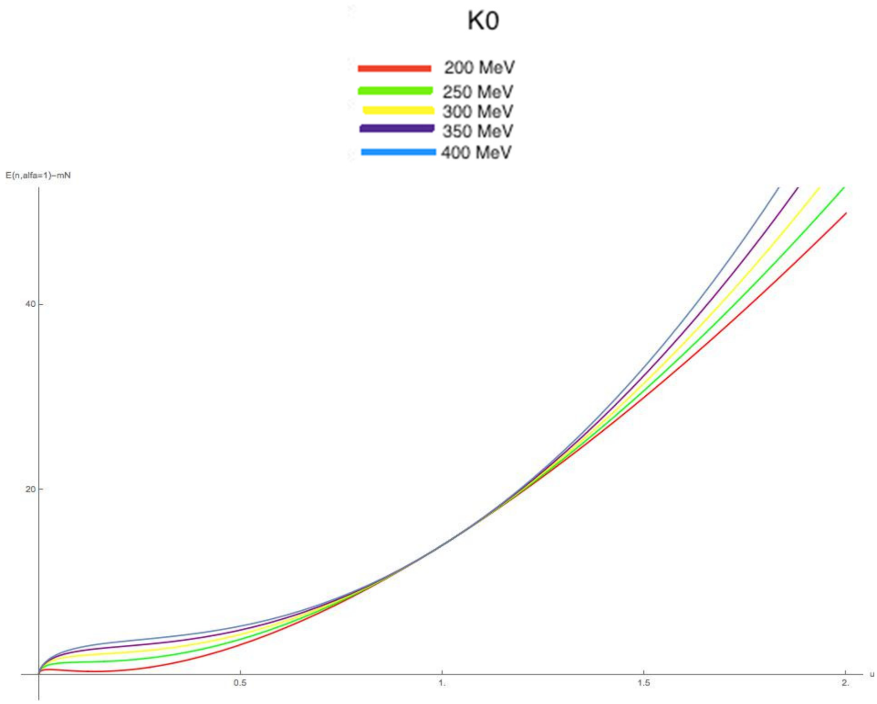

Fig. 6. Average energy per neutron minus it's rest mass as a function of u=n/n_0 for different values of compressibility(K_0).

The pressure [8],[4] in this case yields (Figures 7 and 8):

$$p(n,x) = u\frac{d}{du}\epsilon(n,\alpha) - \epsilon(n,\alpha) \tag{38}$$

$$p(n,x) = p(n,0) + n_0\alpha^2\left[\frac{2^{\frac{2}{3}}-1}{5}\langle E_F^0\rangle\left(2u^{\frac{5}{3}} - 3u^2\right) + S_0u^2\right] \tag{39}$$

where $u = \frac{n}{n_0}$.

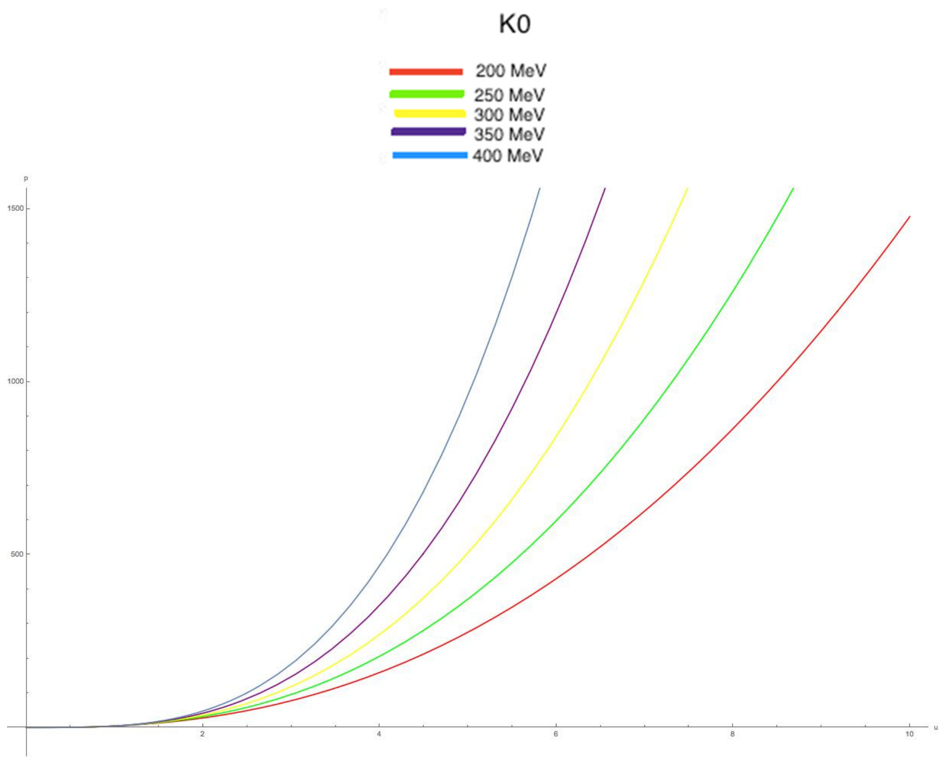

Fig. 7. Pressure as a function of $u=n/n_0$ for different values of K_0 ranging from [200-400]MeV

Figure 7 shows that the pressure increase smoothly from u=0, we have a monotonic, non-negative pressure. This suggests that we can try another polytropic fit.

By fitting the data in Mathematica, we have obtained:

$$K_0 = 3.54842 \times 10^{-4}$$
$$\gamma = 2.1$$

Using this polytropic equation now we can repeat the procedures shown above to see the maximum mass and radius of a neutron star using Fermi gas with nucleon-nucleon interactions.

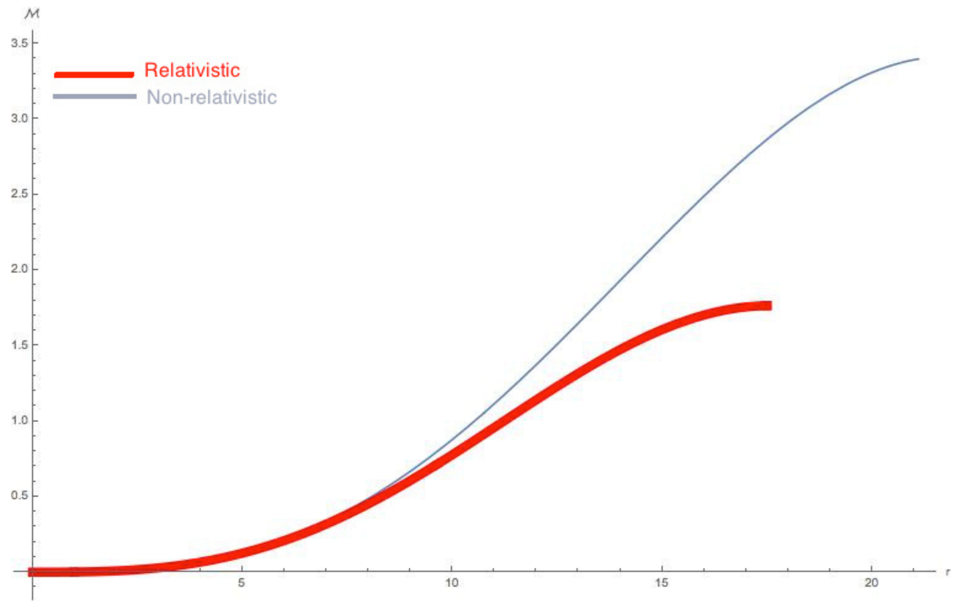

Fig. 8. Mass dependency on r for both relativistic and non-relativistic TOV equation using Prakash EoS

In Figure 8, we have obtained similar curves for r < 7 km, but for radii over 7 km, general relativistic effects can not to be neglected, similarly with the Fermi EoS (Figure 3), the actual allowed mass is smaller than it's Newtonian mass.

Figure 9 shows, for general relativistic case, a much steeper decrease in pressure slope for r > 4km.

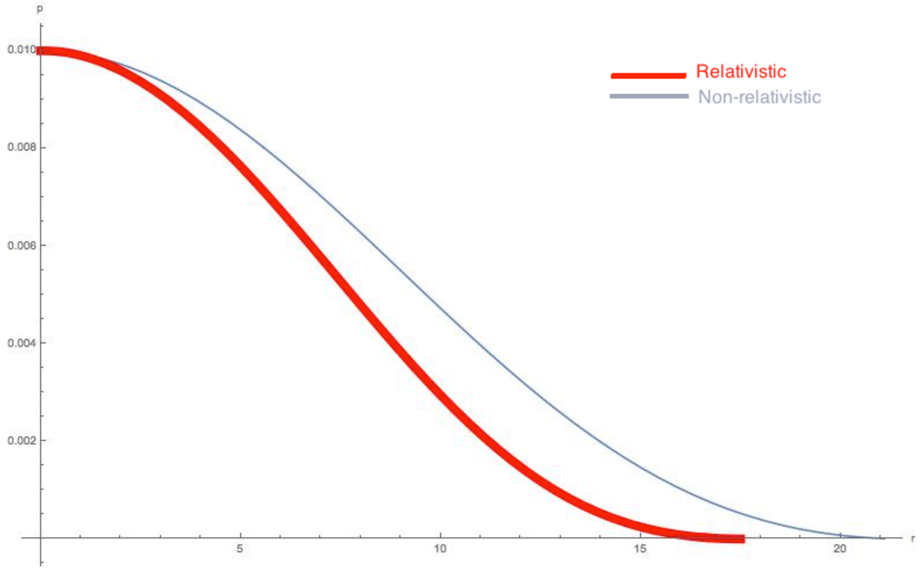

Fig. 9. Pressure dependency on r using Prakash EoS.

RESULTS AND DISCUSSION

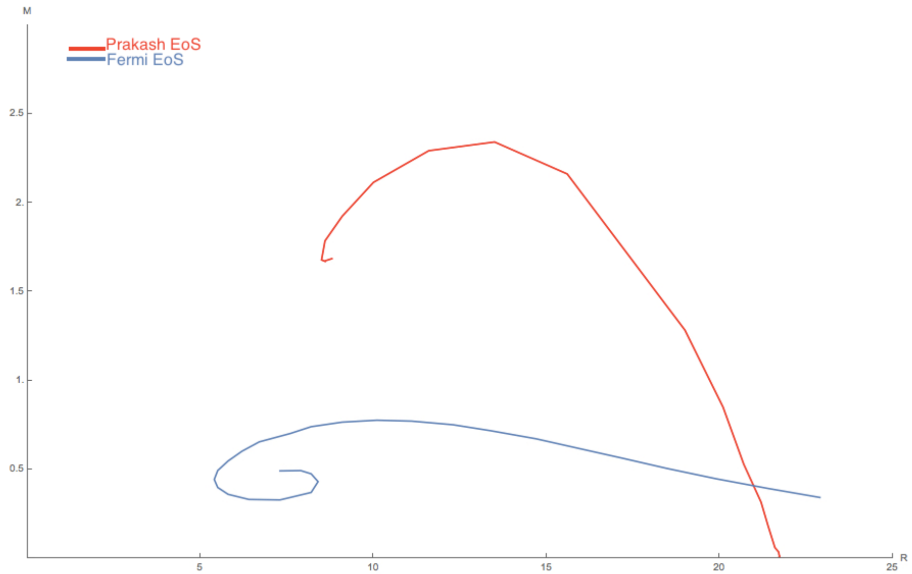

Fig. 10. The mass M(in M_o) and radius R(in km) for pure neutron stars using a Fermi EoS(blue) and using Prakash EoS.

From Figure 10 one can see that stars that have low-mass and large radius are solutions of the TOV equations for small central pressures. The maxima of this graph occurs at R=11(Fermi EoS) and R=13.6(Prakash EoS) and stars that are positioned to the right of the maximum are stable, while those on the left suffer from gravitational collapse. Thus we can conclude that changes in energy density and pressure are caused by changes in density (given that the thermal component in stars that are cold is negligible). Also it can be seen that including nucleon-nucleon interactions into the EoS has a strong effect on the stability of the star, increasing the maximum mass that a stable neutron star can have.

Including the cosmological constant

For a non-zero cosmological constant the modified Einstein Equations solution get a modified version of equation (4) [8]

$$\frac{dp}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2}\left(1+\frac{p(r)}{\epsilon(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{M(r)c^2}-\frac{\Lambda r^3}{2GM(r)}\right)\left(1-\frac{2GM(r)}{c^2 r}\right)^{-1} \quad (40)$$

where $\Lambda \approx 1{,}76\ x\ 10^{-52}$ (m^{-2}) [9].

Using (40) instead of (4) in our simulations we concluded that there is no noticeable difference in the variation of the pressure and energy density as a function of r(Figures 9 and 10).

CONCLUSIONS

The structure of neutron stars is to the day a very active topic in theoretical and computational physics. Our goal was to construct an Equation of State as simple and efficient as possible, starting with some idealistic approximations and to compute and analyze the consequences. This model is still upgradable, for instance the assumption that inside the neutron star is a QCP(quark-gluon plasma) can be used and then a polytropic EoS of this type won’t fit anymore.

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