STUDIA UBB. PHILOSOPHIA, Vol. 70 (2025), Sp. Iss., pp. 27-46
(RECOMMENDED CITATION)
DOI:10.24193/subbphil.2025.sp.iss.02

Brouwer–Hilbert on the Limits of Mathematical Knowledge

Silviu-Constantin FEDEROVICI[*]

ABSTRACT. Brouwer famously challenged the limits of mathematical knowledge by arguing that classical formalism obscures intuitive evidence. Hilbert, by contrast, considered that intuitive insights could safely be ignored as long as formal systems remained consistent and complete. Such a disagreement created a paradigmatic tension between intuitionism and formalism in how the foundations of mathematics should be regarded. This paper evaluates Hilbert's eventual pragmatic dominance and explores, via a shared Kantian heritage, how intuitionistic insights might coexist with formal approaches. Focusing on axioms, the analysis reveals how neglecting certain epistemic values while admitting alternative forms of evidence shapes our understanding of mathematical limits.

Keywords: *philosophy of mathematics, Brouwer-Hilbert controversy, epistemic limits, Kantian heritage*

I. Introduction: Mathematics Between Knowledge and Ignorance

Among the sciences and other systematic forms of reasoning, mathematics has long stood as a model of knowledge, providing an epistemological pillar for our inquiry into empirical phenomena. Unlike other domains marked by radical conceptual shifts, mathematics has traditionally projected the image of a complete and self-contained body of knowledge, seemingly immune to internal gaps or inconsistencies. As Kant noted, the results of this discipline provide the most powerful instruments for scientific evidence through the precision of its synthetic *a priori* judgments: "Here is a great and proved field of knowledge, which is already of admirable compass and for the future promises unbounded extension, which carries with it

* Alexandru Ioan Cuza University of Iași, Doctoral School of Philosophy and Social-Political Sciences. Email: silviu.federovici@student.uaic.ro

thoroughly apodictic certainty, i.e., absolute necessity”[1]. Mathematics not only consolidates our reasoning with remarkable rigour but also enables the systematic construction of new results upon established foundations, without the apparent risk of encountering essential breakdowns. Indeed, its internal coherence, logical stability, and resistance to counterfactual variation[2] have long underpinned its distinctive epistemic status.

Few thinkers have shaped the modern conception of mathematics as decisively as Hilbert, whose efforts to establish the important results of this domain upon universal foundations and resolve all major open problems were intended to shield it from the prospect of *ignorabimus*[3]. His contributions extended beyond the systematic consolidation of prior developments, as Hilbert founded a formalist school of thought, alongside prominent mathematicians such as Bernays, Ackermann, and von Neumann, who advanced the axiomatic method and developed proof theory as a rigorous framework for analysing mathematical reasoning well into the contemporary era. These achievements remain landmarks in the foundations of mathematics. Despite the unrestricted ambition of Hilbert’s early 20th-century programme to formalise mathematics as a complete system, its limitations became increasingly evident, particularly after the groundbreaking discovery of Gödel’s incompleteness theorems. Even before these results, the historical episode of the *Grundlagenkrise* had already revealed cracks in this foundational optimism, most notably through the challenges posed by Brouwer’s intuitionism. His critique questioned the ideal of completeness, thus anticipating the limits of the formalist perspective that Gödel would later prove.

At the same time, the privileged position Hilbert assigned to the axiomatic method as the sole reliable path toward a definitive basis of mathematics has faced various challenges over time, though it ultimately proved to be the most influential strategy. One of the most radical critiques came from Brouwer’s intuitionism, which viewed axioms not as true foundations, but as linguistic artefacts that illegitimately

---

[1] I. Kant, *Prolegomena to Any Future Metaphysics That Will Be Able to Present Itself as a Science*, P. G. Lucas (ed.), Manchester, Manchester University Press, 1953, p. 36.

[2] For instance, mathematical judgments usually cannot be meaningfully evaluated through counterfactual hypotheses. There is no epistemic gain in supposing that 1 equals 2, since such an assumption merely generates a contradiction within the established system rather than illuminating any consistent alternative structure. This is why mathematics is associated with a stronger form of necessity, as counterfactual statements have a far more limited application within its framework compared to other fields.

[3] See D. Hilbert, “From Mathematical Problems”, in W. Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics,* Vol. II, Oxford, Clarendon Press, 1996, pp. 1096–1105; and D. Hilbert, “On the Infinite”, in P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics: Selected Readings*, 2nd ed., Cambridge, Cambridge University Press, 1983, p. 200.

diverted attention from the true source of mathematical reasoning, namely, temporal intuition, to the features and rules of formal manipulation. Whereas Hilbert regarded axioms as the bedrock of mathematical foundations, Brouwer argued that such linguistic expressions lacked epistemic substance. They were not only redundant in relation to the insights and evidence provided by intuition, but also potential sources of weakness, susceptible to generating antinomies and unfounded results. Why, then, did Brouwer's objections fail to dismantle Hilbert's image of mathematics? How might mathematics be threatened by all these fissures, and why is it advantageous, even necessary, to overlook them? Lastly, how might the acceptance or rejection of axiomatic systems reflect the distinction between knowledge and ignorance in the foundational debates of mathematics, and what does the formalism–intuitionism polemic reveal about the nature of these two epistemological states? These will be the guiding questions addressed in the sections that follow.

The first part of this essay examines the foundational tensions between intuitionism and formalism, with particular emphasis on the role of axiomatic systems. This dispute is emblematic for the epistemology of mathematics, insofar as the problem of axioms reveals not only the intrinsic limits of formalisation but also the possibility of absolute boundaries of mathematical knowledge itself. I will argue that the Hilbertian approach largely overlooks Brouwer's objections, illustrating this claim through a simple intuitionistic counterexample to the unrestricted use of transfinite axioms, together with the formalist response devised to address this particular challenge. The second part evaluates the competing arguments of formalism and intuitionism by means of a method that, despite its apparent simplicity, carries considerable philosophical and mathematical significance: a comparative table designed to illustrate the pragmatic value of these two foundational positions. Inspired by formal epistemology, this approach is designed to quantify the epistemic trade-offs inherent to each standpoint, offering a novel explanation for the prevailing status of the Hilbertian position. The final chapter revisits the guiding questions and the formalist–intuitionist opposition in light of a philosophically based analysis, which departs from the pragmatic criteria previously considered. Drawing on Țurlea's observation that intuitionism and formalism share a Kantian root, I trace their deeper interpretative divergence and explain how this split has gradually favoured a Hilbertian position. Ultimately, I argue that the image of mathematical knowledge should be re-situated within a broader epistemological framework, one that acknowledges the reductive assumptions underlying its formal structures. These omissions, far from negligible, reveal vulnerabilities that may threaten the very foundation of mathematical knowledge.

Regarding the literature, the Brouwer-Hilbert dispute has been extensively studied, covering the historical controversy between their schools of thought as well as the broader contemporary tension between intuitionism (or constructivism) and formalism[4], particularly concerning the status of axioms on both sides[5]. Our concern here, however, is to investigate a possible philosophical link between these two approaches, which were initially separated by epistemological considerations and divergences of mathematical practice. Even if various forms of synthesis have been attempted from a mathematical point of view, for instance by integrating constructive structures into classical results[6], the philosophical question of how intuitionism and formalism might be bridged remains unclear. Recent studies have explored such possibilities by focusing on the refined Kantian underpinnings of mathematical intuitionism and the interpretative shifts that led to the success of the Hilbertian vision, while the emergence of *Homotopy Type Theory* has re-contextualised the Brouwerian legacy as a robust framework for constructive mathematics[7]. Therefore, this inquiry aims to outline a possible route of co-existence, beginning from the knowledge–ignorance opposition and the Kantian influence shared by both thinkers.

II. Axiomatic Tensions and the Epistemic Divide Between Intuitionism and Formalism

The foundational dispute between formalism and intuitionism[8] from the beginning of last century revealed among other aspects how contrasting conceptions of knowledge and ignorance can unsettle the apparent solidity of mathematics. For

[4] Important references on the tensions between intuitionism (or constructivism) and formalism (or classical mathematics) include P. Mancosu (ed.), *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s*, New York, Oxford University Press, 1998; Michael Dummett, *Elements of Intuitionism*, Oxford, Oxford University Press, 1977; and Arend Heyting, *Intuitionism: An Introduction*, Amsterdam, North-Holland Publishing Company, 1956.

[5] Discussions on the status of axioms in intuitionism and their epistemic implications in mathematics can be found in A. S. Troelstra and D. van Dalen, *Constructivism in Mathematics: An Introduction,* Vol. I, Amsterdam, Elsevier Science Publishers, 1988; for a more technical analysis, see A. S. Troelstra, "Axioms for Intuitionistic Mathematics Incompatible with Classical Logic", in R. E. Butts and J. Hintikka (eds.), *Logic, Foundations of Mathematics, and Computability Theory*, Dordrecht, D. Reidel Publishing Company, 1977, pp. 59–86.

[6] See, for example, the texts in S. Shapiro (ed.), *Intensional Mathematics. Studies in Logic and the Foundations of Mathematics,* Vol. 113, Amsterdam, Elsevier Science Publishers B.V., 1985.

[7] For a detailed reappraisal of these foundational tensions, see Carl J. Posy, *Mathematical Intuitionism*, Cambridge, Cambridge University Press, 2020; Paolo Mancosu, *The Adventure of Reason: Interplay between Philosophy of Mathematics and Mathematical Logic, 1900-1940*, Oxford, Oxford University Press, 2010; and The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, Princeton, Institute for Advanced Study, 2013.

[8] For a general historical context of the *Grundlagenkrise* at the turn of the 19th–20th centuries, see, e.g., I. Grattan-Guinness, *The Search for Mathematical Roots: 1870–1940: Logics, Set Theories, and the Foundations of Mathematics from Cantor through Russell to Gödel*, Princeton, Princeton University Press, 2000.

instance, what formalists present as stable epistemic ground, established through the axiomatic method, is regarded by intuitionism as a linguistic surface that conceals deeper indeterminacies and ignores the real source of mathematics, namely the mental faculty of intuition[9]. In Brouwer's view, these states of indeterminacy are inherent to mathematics, contrary to the image of completeness, and they appear mostly when we operate with the concept of infinity. This clearly suggests that the Hilbertian side tends to ignore, in the form of unrecognised ambiguity of mathematical reasoning, what resides at the very heart of what is classically considered rigour. From the formalist perspective, such cases of indeterminacy reflect a deficiency or limitation of our previous formal systems in fully articulating and systematising knowledge, a shortcoming that the axiomatic method seeks to overcome. Thus, there is no need to proclaim a crisis in mathematics or the necessity of reconstructing it entirely. Ultimately, the opposition rests not merely on methodological differences regarding the norms admitted for doing mathematics, but more generally on seemingly incompatible epistemological commitments regarding the nature of mathematical knowledge. For Hilbert, mathematics is fundamentally tied to the possibility of formalising its content in a consistent and complete manner, with axioms, rules of inference, and formulas systematically structured, and with the conviction that the semantics can be entirely captured within this linguistic framework. Intuitionism, for its part, offers non-classical modes of construction based on intuitive insights that challenge the assumption of completeness and expose the blind spots of formal abstractions.

Among the key points of contention between intuitionism and classical mathematics are the unrestricted use of certain logical laws, most notably the principle of the excluded middle, particularly when applied to transfinite sets, the interpretation and mathematical treatment of infinity, and differing conceptions of intuition. Yet perhaps the most profound divergence concerns the status of the axiomatic method, upheld by Hilbert and sharply contested by Brouwer[10]. The intuitionist critique, particularly during the *Grundlagenkrise*, elicited markedly different reactions within the scholarly community, depending on how mathematicians assessed both the severity of the problems confronting the classical conception and the

---

9 Brouwer consistently argued that temporal intuition should serve as the primary source of mathematical knowledge. Across his career, he attempted to establish a constructive method based on the essential properties of intuitive evidence. See, for example, L. E. J. Brouwer, "On the Foundations of Mathematics", in A. Heyting (ed.), *Collected Works,* Vol. I, Amsterdam, North-Holland, 1975, p. 53; and L. E. J. Brouwer, "Intuitionism and Formalism", in *Collected Works,* Vol. I, p. 127.

10 L. E. J. Brouwer, "On the Foundations of Mathematics", pp. 92–95, and "Formalism and Intuitionism", pp. 123–138, in A. Heyting (ed.), *Collected Works,* Vol. I, Amsterdam, North-Holland, 1975.

possible strategies for resolving them. Some, like Weyl[11], recognised the importance of exposing the epistemic fissures in the classical image of the continuum. In this view, previous atomistic descriptions of the continuum no longer met the rigour of constructive reasoning, as mainly established by Brouwer. Others maintained that the foundational crisis could be simply addressed by refining classical tools, especially through the adjustment of axiomatic systems, to prevent the emergence of antinomies and preclude various forms of indeterminacy. While Brouwer's initial doubts evolved into an ambitious project to reconstruct mathematics on intuitionistic grounds, his alternative vision gradually lost momentum, though its critical potential continues to be influential today. Ongoing debates about constructive procedures, the study of impredicative definitions, and scepticism toward certain logical principles continue to reflect its enduring legacy.

One of Brouwer's most radical claims in his early writings[12] was that the use of axioms, i.e. foundational statements assumed without proof, should be entirely avoided in mathematical constructions, as they merely formalise ideas already known through intuition without providing additional evidence. A clear example comes from arithmetic, where he rejected axiomatic foundations[13], in favour of constructions directly derived from the primordial intuition of time (ger. *Ur-Intuition*)[14], thereby grounding the generation of natural numbers on a philosophical framework[15]. For Brouwer, formalisation, especially that built upon Hilbertian ideals of completeness and consistency, did not represent either the starting point of mathematical construction or the authentic medium of reasoning, but rather an

---

[11] H. Weyl, "On the New Foundational Crisis of Mathematics", in P. Mancosu (ed.), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, New York, Oxford University Press, 1998, pp. 86–118.

[12] See L. E. J. Brouwer, "On the Foundations of Mathematics", pp. 77–81, and "Intuitionism and Formalism", p. 125, in A. Heyting (ed.), *Collected Works*, Vol. I, Amsterdam, North-Holland, 1975. Later, he recognised the utility of axioms in Heyting's formalisation of intuitionistic mathematics, although in his own writings he continued to avoid them.

[13] As classically formalised in arithmetic by G. Peano, or in the logical approach of Bertrand Russell, *The Principles of Mathematics*, London, Bradford & Dickens, 1942, p. 128.

[14] The foundational stages of mathematics, grounded in temporal intuition, are articulated by Brouwer through what he calls the two acts of intuitionism, as presented in his work "Historical Background, Principles and Methods of Intuitionism", in *South African Journal of Science*, 49 / 1952, South African Association for the Advancement of Science, pp. 140–142.

[15] More specifically, the philosophical method used here could be seen as a form of genetic constructivism, meaning that the origin of mathematics must be established in correspondence with certain foundational mental phenomena, such as the perception of change in time. Even if some commentators have interpreted this as a form of psychologism, however, I endorse the explanation from M. van Atten, *On Brouwer*, Belmont, Wadsworth Philosophers Series, 2004, pp. 72–76, that Brouwer had in mind transcendental phenomena, and not empirical ones.

arbitrary linguistic rendering of our intuitions[16]. From an epistemological perspective, intuitionism challenges the formalist conviction that axioms define the absolute limits of mathematical knowledge within which the reasoning operates. Such a manner of establishing boundaries in mathematics according to criteria that carry no intrinsic meaning beyond their syntactic function may itself be regarded as a form of ignorance, since it imposes an arbitrary condition that requires one to overlook intuitive insights[17].

Hilbert's decision to renounce any kind of meaning to the mathematical objects derived from an external source marked an essential step in safeguarding mathematics from potential sources of error. Accordingly, he regarded the complete elimination of such external meanings as the best means of overcoming epistemic vulnerabilities, since they did not belong to the content of mathematics as a pure formal discipline. This act followed from Hilbert's conviction that axiomatisation constitutes the most reliable path to secure the foundations of mathematics. As a natural consequence, his objective was to establish the whole of mathematics on a universal basis through the adequate choice of axioms. These axioms would generate a set of sentences that would be consistent and complete, relying on the mechanical manipulations prescribed by the rules of inference rather than on intuitive guidance. Although Hilbert acknowledged the heuristic role of intuition, he confined it to the restricted status of intellectual recognition of symbolic tokens, relevant at a pre-mathematical stage but epistemically insecure and undesirable later on. As Kreisel observed: "Hilbert's programme begins where the semantic leaves off"[18], thus representing a clear shift that dissolves all variations of meaning into purely formal language governed by syntactical rules. Moreover, driven by the ambition that every major mathematical problem could ultimately be solved, i.e. we have either a proof or a disproof for every well-formed formula A, Hilbert's approach reflected a deeply positivist stance. In *The Knowledge of Nature*, he famously declared: "For the mathematician there is no *ignorabimus*... We must know. We

---

[16] L. E. J. Brouwer, "Intuitionism and Formalism", in *Collected Works,* Vol. I, p. 128: "(...) neither the ordinary language nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary".

[17] In formalism, such formulas serve purely syntactical functions, with no semantic content. See David Hilbert, "On the Infinite", in *Philosophy of Mathematics: Selected Readings*, p. 197: "The symbols of the logical calculus originally were introduced only in order to communicate. Still it is consistent with our finitary viewpoint to deny any meaning to logical symbols, just as we denied meaning to mathematical symbols, and to declare that the formulas of the logical calculus are ideal statements which mean nothing in themselves".

[18] G. Kreisel, "Foundations of Intuitionistic Logic", in E. Nagel, P. Suppes and A. Tarski (eds.), *Logic, Methodology and Philosophy of Science*, Stanford, Stanford University Press, 1962, p. 201.

shall know"[19]. As a reason for the rejection of the possibility of inherent epistemic limits regarding the completeness of mathematics, Hilbert considered the axiomatic method as part of a broader scientific optimism of his time, inspired by breakthrough discoveries such as the theory of relativity and radioactivity. Within this historical atmosphere, mathematics was envisioned as the ultimate foundation of the natural sciences[20], and axioms were intended to preserve this apodictic character of mathematical status at any cost. This conviction had practical consequences in the development of mathematics: principles such as the Axiom of Infinity or the Axiom of Choice[21], though lacking constructive or intuitive justification, provided powerful tools that decisively influenced the axiomatisation of arithmetic, set theory and analysis. Ultimately, the authority of axioms in mathematics rested not on their semantic clarity, but on their syntactic efficiency and fruitfulness.

In his 1912 inaugural lecture *Intuitionism and Formalism*, Brouwer contested the prevailing formalist approach to the foundations of mathematics. More specifically, he argued that axiomatic systems fail to resolve the emergence of paradoxes, such as those stemming from the axiom of comprehension (or inclusion) in ZFC set theory, as well as various instances of vicious reasoning, like the axiom of induction in number theory[22]. For Brouwer, mathematical truth derives directly from intuitive constructions, not from the mere absence of contradiction within a linguistic framework. Making consistency within formal reasoning the sole criterion for mathematical validity, as Hilbert did, illegitimately subordinates mathematics to its linguistic representation. Moreover, Brouwer regarded completeness as a property of linguistic expressions rather than of mathematics itself. In Hilbert's vision, to achieve completeness within a formal system, every mathematical well-formed formula must be decidable: given any formula A, one must be able either to construct a proof of A or to derive a contradiction from its proof. In other words, *tertium non datur* must apply to every possible mathematical statement in our set of formulas.

[19] D. Hilbert, "Logic and the Knowledge of Nature", in W. Ewald (ed.), *From Kant to Hilbert. A Source Book in the Foundations of Mathematics,* Vol. II, Oxford, Clarendon Press, 1996, p. 1165.

[20] This reflects Kant's claim that mathematics defines the very possibility of genuine science. See I. Kant, *Metaphysical Foundations of Natural Science*, M. Friedman (ed.), Cambridge, Cambridge University Press, 2004, p. 6: "In any special doctrine of Nature there is only as much genuine science as there is mathematics".

[21] See E. Zermelo, "Untersuchungen über die Grundlagen der Mengenlehre I", in *Mathematische Annalen*, 65 / 1908, B. G. Teubner, pp. 261–281. For a discussion on their non-constructivity, see M. Dummett, *Elements of Intuitionism*, Oxford, Clarendon Press, 1977, pp. 52–55.

[22] For example, Brouwer pointed to paradoxes such as the Burali-Forti paradox, concerning the well-ordering of sets, and the axiom of induction, which becomes impredicative in the formalist account. For further references regarding the axioms admitted by Hilbert, see L. E. J. Brouwer, "Intuitionism and Formalism", in *Collected Works,* Vol. I, p. 133.

As an objection to this ideal, Brouwer offered several counterexamples, including the unresolved question of whether the digit sequence "0123456789" appears in the infinite decimal expansion of π, to show his point by highlighting the limits of classical logic and completeness[23]. This proposition is currently neither provable nor disprovable, since there is no constructive method to verify or refute the presence of this sequence in the decimal expansion of π. Even if the sequence were eventually located at some stage *k* of a constructive enumeration of the digits[24], one could simply replace it with another sequence not yet encountered by stage *k*, thereby preserving the indeterminacy. Such examples illustrate that questions about infinite collections inevitably give rise to indeterminacy, as long as we want to talk meaningfully from an intuitive viewpoint about these kinds of sets. Unlike Hilbert, who formalised transfinite sets as complete mathematical objects[25], Brouwer maintained that infinite sets, such as the decimal expansion of π, cannot be meaningfully captured without acknowledging this inherent context of indeterminacy. This is not merely a practical limitation arising from our inability to examine every element of infinite sets, but a principled one: there exists no rule that fully determines the generation of all elements of such a set in a constructive manner.

To illustrate this contrast, let us briefly examine Hilbert's axiomatic approach to transfinite sets through the operator τ, introduced to reconcile infinite totalities with finitary mathematics. Hilbert acknowledged the need for a distinct axiomatic approach to the transfinite sets, yet insisted that such reasoning must be reducible to finite methods: "the free use and the full mastery of the transfinite is to be achieved on the territory of the finite"[26]. Consequently, he proposed a transfinite axiom, formulated as A(τA) → A(a), which allows the inference that if a predicate A applies to some specific object τA, then it applies to all objects *a*[27]. In other words, τ represents an arbitrary object satisfying property A and serves as a generic

[23] L. E. J. Brouwer, "The Unreliability of the Logical Principles", in A. Heyting (ed.), *Collected Works*, Vol. I, Amsterdam, North-Holland, 1975, p. 110.

[24] Meanwhile, this sequence was indeed found, but, as we have seen, it can be replaced with one that does not appear in the decimal expansion of π (see D. E. Hesseling, *Gnomes in the Fog: The Reception of Brouwer's Intuitionism in the 1920s*, Basel, Springer, 2003, p. 71).

[25] D. Hilbert, "On the Infinite", in *Philosophy of Mathematics: Selected Readings*, pp. 198–199.

[26] D. Hilbert, "The Logical Foundations of Mathematics", in W. Ewald (ed.), *From Kant to Hilbert. A Source Book in the Foundations of Mathematics,* Vol. II, Oxford, Clarendon Press, 1996, p. 1140.

[27] To clarify this axiom, Hilbert uses the example of the predicate "bribable": if τA designates an ideally just person for whom it has been proven that they are bribable, then, according to the axiom A(τA) → A(a), it follows that all people are bribable. From an intuitionistic perspective, however, this inference appears meaningless, since such an ideal instance says nothing about the actual bribability of other individuals. The example reveals the gap between formal generalisation and intuitive meaning, highlighting how the transfinite operator abstracts away from constructive content (*Ibid.*, p. 1141).

placeholder, allowing quantified statements to be reduced to finitary terms and supporting the formalisation of transfinite reasoning within finite logic. This enables inferences from τ-objects to general domains, aiming to preserve consistency and completeness, even when dealing with infinite sets. However, from an intuitionistic position, this apparently elegant technique fails to resolve the epistemic ambiguity of the infinite. For instance, if A denotes the property "every possible finite digit sequence appears in the decimal expansion of π", then τA would designate a hypothetical decimal expansion satisfying A. By the axiom $A(\tau A) \rightarrow A(a)$, one could infer that this property holds for all decimal expansions, seemingly addressing the earlier counterexample to the law of the excluded middle. Yet from Brouwer's perspective, this inference does not address how or when such a sequence as "0123456789" actually appears. It merely postulates existence implicitly, without constructive proof. Thus, the τ-operator shifts the problem into formal language, bypassing intuitive justification. Infinite entities, in the intuitionist view, lack meaning unless constructively supported. While such axioms give the appearance of completeness, they remain detached from constructive grounding, relying on the law of the excluded middle without restriction, a principle whose absolute validity Brouwer confines to finite reasoning. His critique of formal axiomatic notions, such as the τ-operator, thus exposes ambiguities in formalist foundations and underscores the need to reconsider the limits imposed by intuition.

III. Pragmatic Success vs Epistemic Limits

To understand why formalist practices continue to shape the prevailing image of mathematics, while intuitionistic perspectives are often marginalised or regarded as historical curiosity, I will adopt a pragmatic method of comparison between these two foundational positions. Drawing on approaches from formal epistemology[28], this method evaluates the main strengths of each perspective, especially regarding the acceptance and use of axiomatic systems, via a structured comparative table, which may be further extended. The purpose is not to claim strict objectivity, but rather to highlight which framework currently offers greater epistemic utility in the foundations of mathematics. Therefore, in the current

[28] For example, D. Lewis employed a similar approach by pragmatically arguing that possible worlds should be regarded as equally real as our actual world, since this assumption better serves formal understanding. See D. Lewis, *On the Plurality of Worlds*, Oxford, Basil Blackwell, 1986, pp. 3–5 (Ch. 1, "A Philosopher's Paradise"). In a comparable manner, Hilbert admitted the unrestricted notion of the transfinite to preserve and extend the developments initiated by Cantor in set theory.

context, mathematical utility serves as the principal criterion for assessing the use of axioms. Utility is understood in terms of practical advantages, such as the ease of integrating existing results, generating new theorems, the effectiveness of proof techniques, and sustaining productive mathematical development.

Before presenting Table 1, two clarifications are in order. First, the scoring system, based on an arbitrary scale from 0 to 15 points for illustrative purposes, is not intended as a rigorous evaluation of the arguments themselves. The goal is instead to provide a broader perspective on the conflict between intuitionism and formalism and to explore some of its immediate consequences in mathematics. Within this framework, pragmatic considerations must be the factor explaining the enduring dominance of the formalist image of mathematics, which will serve as the primary focus of analysis. Second, the scores should be regarded as flexible, approximate estimates, reflecting relative epistemic weight rather than absolute values. For example, if intuitionism were able to provide a compelling alternative to classical theorems, its score would increase substantially. In the current context, however, certain structural advantages of formalism, such as the preservation and consolidation of important classical results through axiomatic systems, constitute fairly objective benefits. By contrast, the richer semantics offered by intuitionism, while philosophically significant and valuable in constructive analyses, does not exert the same impact on mainstream mathematical practice. In moving beyond a purely descriptive account, the following table proposes a decision-theoretic lens through which to identify the specific utility thresholds that favoured the formalist image of mathematics. This heuristic reveals how the prioritisation of different epistemic values, such as constructive clarity versus axiomatic efficiency, fundamentally shapes the resulting conception of mathematical knowledge.

Table 1. Arguments Accounting

No.	Argument / Criterion	Intuitionism	Pts.	Formalism	Pts.
1	Consolidation and preservation of previous results	Theorems and propositions are partially reconstructed and generally weakened due to constructive constraints	5	Results are easily reproducible and reinforced within axiomatic systems	13
2	Epistemic foundation of constructions	Grounds mathematical activity in meaningful concepts (e.g. *Ur-Intuition*) that provide direct epistemic justification	7	Axioms are accepted for their efficiency and clarity, without additional semantic justification	4

No.	Argument / Criterion	Intuitionism	Pts.	Formalism	Pts.
3	Completeness and theoretical adaptability	Cannot reconstruct many classical theories (e.g. Cantor's transfinite set theory and certain axioms from real analysis); the system is conservative and restricts the uncritical acceptance of new mathematical objects	4	Covers a large part of classical mathematics and can easily integrate extensions, new theories, and additional axioms (e.g. ZFC, type theory)	11
4	Recognition of epistemic limits in mathematics	Acknowledges irresolvable problems and treats notions such as infinity, the existential quantifier, and the application of logical laws with appropriate restrictions.	8	Tends to conceal such limits, promoting unlimited confidence in the power of axiomatic systems	6
Sum			24		34

As the table indicates, one of formalism's major strengths lies in its ability to preserve and extend prior mathematical achievements without necessitating radical reconstruction. Thus, the formalist approach emphasises the continuity between established results and the axioms from which they are derived. Key examples include postulates like Zermelo's Axiom of Choice and set-theoretical results such as Cantor's construction of transfinite sets. As long as intuitionism cannot provide alternatives with comparable rigour and simplicity without simultaneously discarding results that are mathematically valid yet lack intuitive justification, it struggles to assert an objectively superior position in foundational debates. The mere fact that certain objects, such as higher cardinalities, cannot be meaningfully described does not, from the standpoint of mathematical utility, justify dismissing them wholesale as erroneous. In this regard, formalism possesses a clear pragmatic *raison d'être*, ensuring both continuity and productivity in mathematical research. From Brouwer's perspective, however, this pragmatic advantage conceals a deeper epistemic flaw: the detachment of mathematical knowledge from the intuitive meaning that endows it with valid significance. The demand for ubiquitous intuitive meaning, moreover, may reflect a philosophical commitment rather than mathematical necessity. For intuitionism, meaning must accompany every formal manipulation; semantic grounding in intuitive capacities is not optional but essential for legitimate mathematical construction. Brouwer's critique thus exposes not only the limits of formal reasoning but also the inherent difficulty of reconstructing the edifice of classical mathematics on purely intuitionistic foundations.

Our analysis reveals a dual tension: while formalism ensures stability and extensibility, intuitionism uncovers the hidden vulnerabilities underlying formal precision. Each framework thus embodies both strengths and weaknesses. Formalism ensures continuity and wide applicability, but at the cost of detaching mathematics from its intuitive origins. Intuitionism, by contrast, preserves epistemic authenticity grounded in our intuitive mental capacities, yet struggles to reconstruct and extend certain classical results. This interplay highlights the intrinsic limits of formal foundations, where aspects such as intuitive insights, emphasised by Brouwer and dismissed by Hilbert, are systematically overlooked. Paradoxically, this very omission has become a decisive advantage: by privileging clarity, generality, and technical effectiveness, formalism has enabled the expansion and eventual dominance of mathematics.

IV. Kantian Roots as a Basis for Revisiting the Brouwer–Hilbert Controversy

As we have seen, intuitionism challenges the traditional image of mathematics as a complete and determinate body of knowledge, exposing fissures within formal reasoning. Although it offers valuable insights into the limits of mathematical knowledge, intuitionism has not established a sufficiently robust alternative to the dominant formalist paradigm. Our analysis so far has examined the epistemic and methodological divergences between these two perspectives, aiming to explain, from a pragmatic standpoint, how formalism achieved success with axiomatic method, despite its detachment from intuitive meaning. In this final part of the paper, we turn to a shared historical root: the distinct interpretation of Kant's philosophy of mathematics. Both Brouwer and Hilbert drew on Kantian ideas, yet they interpreted them in radically different ways, ultimately developing opposing visions of mathematical knowledge. These divergent readings reveal their contrasting approaches to epistemic limits and the role of ignorance, as each thinker emphasised particular elements of Kant's perspective while neglecting others. Understanding this interpretative shift clarifies how these parallel approaches shaped the trajectories that formalism and intuitionism ultimately followed. Adopting a Kantian root also allows us to see formalism and intuitionism not simply as radically opposed, but as distinct elaborations of shared philosophical foundation. This lens explains why their debate was so sharp, each side selecting one dimension of Kant's thought in contrast to the other, while also showing that both schools could legitimately claim philosophical grounding in his legacy. Overall, revisiting their Kantian roots provides a deeper, more integrated understanding of how these seemingly incompatible positions emerged from a common philosophical background.

According to Țurlea, "The Kantian philosophy of mathematics inspired divergent and even rival foundational programmes: Fregean logicism, Hilbertian formalism, and Brouwerian intuitionism"[29]. Hilbert, for instance, developed his conception of geometry and mathematics more broadly, by explicitly invoking Kant's dictum that "all human knowledge begins with intuitions, proceeds through concepts, and ends with ideas"[30]. Brouwer, in contrast, sought a more radical reading of Kant, grounding mathematics entirely in temporal intuition, which he regarded as its authentic source. Neither thinker derived their positions systematically from Kant, yet both were influenced by his ideas. For Hilbert, Kant's legacy provided justification for the formalisation and systematic organisation of mathematics; for Brouwer, it supported a return to the mind's intuitive, pre-conceptual activity. A schematic reading of Kant's sequence, from intuition to concepts and finally to ideas, elucidates how each thinker reinterpreted these stages to demarcate the limits of mathematical knowledge.

intuitions → concepts → ideas

Interpreting Kant's sequence in two different ways clarifies how this shared philosophical root influenced Hilbert's and Brouwer's view of the origin and epistemic status of mathematics, particularly regarding the adoption of axioms. First, Hilbert interpreted these stages hierarchically, as a progressive ascent in which each level contributes increasing clarity. In this framework, intuition serves only a preliminary role, limited to the recognition of symbol strings, before being superseded by formal concepts and, ultimately, the pure ideas of reason, such as mathematical infinity. Accordingly, Kant's epistemic sequence provided Hilbert with a rationale for grounding mathematics primarily in formal concepts rather than intuition. He therefore situated the limits of mathematical knowledge after the initial stage, holding that mathematics should be built from pure concepts stripped of intuitive content. As a consequence, axioms are not intended to capture any intuition; instead, they function to ensure internal coherence and universality, allowing a systematic exploration of mathematical ideas free from uncertainty.

---

[29] M. Țurlea, *Filosofia matematicii*, București, Editura Universității din București, 2002, p. 195.

[30] *Ibid.*, p. 209. Note that this is a paraphrase of Kant's original formulation, which refers to "sensibility" (or "sensation") rather than "intuition" as the initial stage of knowledge (I. Kant, *Critique of Pure Reason*, P. Guyer and A. W. Wood (eds.), Cambridge, Cambridge University Press, 1998, A298/B355, p. 387).

Brouwer regarded Hilbert's interpretations as epistemically flawed. In his dissertation[31], he argued that mathematics is founded entirely in mental acts, with intuition serving not as a preliminary step but as the primary and genuine source of truth during the construction of mathematical reasoning. As he wrote: "the only possible foundation of mathematics must be sought in this construction, under the obligation carefully to observe which constructions intuition allows and which not"[32]. For Brouwer, intuition cannot be treated as a source to be later discarded, since the ultimate meaning of mathematics depends entirely on its presence. Concepts and ideas are valid only insofar as they carry intuitive content; they serve merely as tools to encode, communicate, and recall previous constructions, with their significance deriving primarily from the unfolding of intuition itself. Hence, in Brouwer's intuitionism, this schema must be understood as a derivative structure, in which intuition is primary, while concepts and linguistic ideas are essentially auxiliary. The epistemic boundary, in this case, lies between intuition and conceptualisation, whereas in Hilbert's framework, mathematics begins only after the intuitive step, once formal language has been established. These constitute two opposed directions of development along the Kantian sequence. Finally, Brouwer emphasised this limit to highlight that formal language alone can be misleading, suggesting clear scepticism about its ability to generate valid mathematical knowledge in comparison with direct intuitive construction, an approach which is, in some respects, more faithful to Kant's original intentions[33]. Significantly, we must distinguish Kant's formalist stance on general logic from his requirements for mathematics. Drawing on MacFarlane's analysis of logical hylomorphism, we observe that Kant characterises general logic as formal precisely because it must abstract from all semantic content to function as a constitutive norm for thought[34]. Since such logic remains epistemically blind to objects, valid mathematical knowledge conversely requires a content-based (transcendental) logic rooted in pure intuition, anticipating Brouwer's rejection of empty formalism.

---

31 L. E. J. Brouwer, "On the Foundations of Mathematics", in *Collected Works,* Vol. I, p. 52.

32 *Ibid.*, pp. 94–95.

33 Kant was indeed, with respect to pure linguistic constructs, an anti-formalist, as we can see in his critiques of the metaphysicians who created philosophical systems in forced correspondence with the results of science, for instance, those from astronomical calculations, calling them "subtle fictions which have no truth to them outside the field of mathematics" (See I. Kant, "Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality", in D. Walford and R. Meerbote (eds.), *Theoretical Philosophy, 1755–1770*, Cambridge, Cambridge University Press, 2000, p. 168).

34 For a detailed analysis of how Kant's conception of generality implies the complete formality of logic, see J. G. MacFarlane, *What does it mean to say that Logic is Formal?*, PhD Thesis, Pittsburgh, University of Pittsburgh, 2000, pp. 79–81.

At least two Kantian-based factors can explain why Hilbert's formalist interpretation prevailed. First, formalism, and other foundational schools such as logicism, considered intuition as an unstable and equivocal notion to serve as a reliable foundation for mathematical knowledge, privileging instead the clarity and universality of logical principles and formal language. This orientation fostered a shared epistemic framework that unified the mathematical community, gradually marginalising intuitionism as a deviation from the classical norms. The transparency of formal reasoning and the unrestricted application of logical laws provided Hilbert's approach with a pragmatic and institutional advantage, supporting its consolidation and success. Moreover, the areas of mathematics which were detached from intuitive meaning[35] and against intuitionism's criteria of validation developed consistently and could not be reconstructed satisfactorily. The choice to renounce these areas was a matter of preference rather than a real mathematical necessity. Second, Brouwer positioned himself more as a philosopher seeking to actualise Kant's legacy by grounding mathematics entirely in intuitive acts. His interpretation was guided by a philosophical demand, in which the mental faculties establish a direct epistemic relationship with mathematical constructions. In contrast, Hilbert sought to consolidate his existing mathematical edifice through eventual philosophical justification, representing an opposite approach to establishing the foundations of this domain. Consequently, while Hilbert drew on Kant to justify axiomatic clarity, Brouwer rejected this manner of reading the German philosopher, insisting that the concept of intuition, although modified and actualised, must remain the foundational basis of mathematics. Their divergent interpretation had implications in various areas of mathematics, such as the problem related to non-Euclidean geometries: for intuitionism, it exposed the limits of axiomatic systems and underscored the need to ground mathematics in intuition, as a more universal faculty from which we can take various perspectives on the structure of space, whereas Hilbert treated it as a challenge to refine and complete the system of axioms, a strategy that ultimately proved to be more influential. Ultimately, Hilbert's vision benefited from the universality and malleable character of axioms, while Brouwer's intuitionism faced challenges by relying on the philosophical notion of intuition, which is debatable and imposes significant constraints.

Yet formalism's dominance has not extinguished intuitionistic inquiry. Even within its internal coherence and impressive capacity for systematic development, mathematics conceived purely formally retains zones of epistemic opacity. These gaps, though not immediately destabilising, allow the system to operate without confronting foundational ambiguities that Brouwer insisted could not be overcome

[35] For example, set-theoretic arithmetic based on higher cardinalities demonstrates how certain mathematical constructions, though formally consistent, extend beyond the bounds of intuitive evidence.

by formalism. In turn, the polished image of mathematics as seamless and complete thus relies on bracketing questions of intuitive meaning, questions that remain essential for a deeper understanding of its foundations. At the core, the tension between formalism and intuitionism centres on the epistemic status of intuition as a limit of knowledge: should it be regarded as constitutive of the entire edifice of mathematical truths, or merely as a preliminary guide to a system of formal entities whose further external significance is suspended? Ultimately, this divergence directly affects how indeterminacy is treated, because if we confer authority to our intuition, then we must conclude that these results of incompleteness are inherent to mental construction. In this light, intuitionism functions as a critical counterpoint, highlighting the reductive assumptions embedded within formal structures and providing a framework for reassessing classical mathematics from an intuitive perspective. Although it cannot replace formalist practice, it continues to challenge its basic assumptions. As Bourbaki once remarked, intuitionism may eventually become a "historical curiosity"[36], but only after classical mathematics has addressed the foundational uncertainties it reveals, underscoring that the polished image of mathematical knowledge rests on selective omission and epistemic compromises.

By tracing the Kantian sequence, we can see how both Brouwer and Hilbert developed their positions through different ways of setting limits on the foundations of mathematics. A possible way to balance their seemingly opposing interpretations is to keep these boundaries as open and flexible as possible: to cultivate intuition in relation to formal structures without restricting the latter, especially when they prove consistent and mathematically fruitful. In this way, formal results may be seen not as opposed to intuition but as potential paths still awaiting fulfilment from an intuitive standpoint. Recognising the limitations and blind spots of formalism allows us to appreciate the epistemic value of intuitionistic critique, not as an alternative system to replace classical methods, but as a lens to expose the assumptions (or their absence) underlying them. By situating mathematical knowledge within a broader epistemological framework, informed by a Kantian understanding of intuition, concepts, and ideas, we can acknowledge both the power of formal structures and the irreducible role of intuition in shaping mathematical understanding. This perspective shows that the apparent dichotomy between formalism and intuitionism is not absolute; rather, it reflects complementary insights into the ways humans construct, justify, and interpret mathematical truth. Ultimately, embracing this dual awareness fosters a more reflective and philosophically grounded conception of mathematics, one that preserves rigour while remaining attentive to its foundational ambiguities.

[36] N. Bourbaki, *Éléments d'histoire des mathématiques*, Paris, Hermann, 1960, p. 56: "L'école intuitionniste, dont le souvenir n'est sans doute destiné à subsister qu'à titre de curiosité historique...".

V. Conclusion

Returning to our preliminary question, we now ask what truly distinguishes knowledge from ignorance in the foundations of mathematics? In certain domains, clear norms apply: empirical validation in natural sciences, moral action in ethics, or effective organisation in politics. On the other hand, in mathematics the validation criteria are non-experiential and diverge sharply from these examples. For Hilbert, knowledge is equated with formal provability, based on sets of axioms and rules of syntactic derivations. Intuitionism grounds proof in constructive acts of the mind rather than formal manipulations. Each approach thus advances a distinct epistemic ideal: one that values the universality of formal language, the other that emphasises the evidential force of intuitive construction. The opposition becomes especially acute in the case of axioms, which formalism treats as defining the boundaries of mathematical reasoning, while intuitionism sees them as potential sources of error. Yet the history of mathematics demonstrates the indispensability of axioms, though they are no longer preserved in Hilbert's initial form. Rather than undermining mathematics as a linguistic discipline, intuitionism broadens its epistemic roots by acknowledging ambiguity and treating indeterminacy as an intrinsic and meaningful component of the domain. Such prudence may ultimately offer a wiser and more sustainable stance than Hilbert's unreserved optimism. The debate over foundations between intuitionism and formalism does not expose a weakness of mathematics *per se*, but rather indicates a deeper truth: absolute clarity and certainty are inseparable from the risk of deliberate ignorance. A Kantian-inspired synthesis of intuitionism and formalism encourages us to view mathematics not simply as a self-sufficient, hierarchically ordered edifice, but as grounded in intuitive construction, conceptual meaning, and epistemic limitation. Recognising these limits does not diminish the status of mathematics, but completes it within a broader epistemological context. As Martin-Löf has noted, the Hilbert–Brouwer controversy has reached a form of resolution through developments like the double-negation interpretation and the Curry-Howard correspondence[37]. Furthermore, as Posy suggests, this Kantian-inspired perspective finds a contemporary revival in the necessity of a humanly graspable mathematics. For instance, by acknowledging the temporal and flowing character of intuition, characteristics rooted in the Kantian tradition, against the splittable nature of the classical set-theoretic continuum, we can reveal the transcendental limits of our finite minds as a necessary epistemological constraint

[37] P. Martin-Löf, "The Hilbert-Brouwer Controversy Resolved?", in M. Schirn (ed.), *The Philosophy of Mathematics Today*, Oxford, Clarendon Press, 1998, pp. 243–256.

on the reach of formal language[38]. Today, mathematical knowledge appears as a layered structure, balancing formal precision with constructive reasoning. The law of the excluded middle is no longer an unquestioned principle, but a contextual tool within epistemic boundaries. Ultimately, knowledge and ignorance in mathematics are not opposites, but intertwined in a dynamic and evolving process.

BIBLIOGRAPHY

1. Bourbaki, Nicolas, *Éléments d'histoire des mathématiques*, Paris, Hermann, 1960.
2. Brouwer, L. E. J., "Historical Background, Principles and Methods of Intuitionism", in *South African Journal of Science*, 49 / 1952, South African Association for the Advancement of Science, pp. 140–142.
3. Brouwer, L. E. J., "Intuitionism and Formalism", in A. Heyting (ed.), *Collected Works*, Vol. I, Amsterdam, North-Holland, 1975, pp. 123-138.
4. Brouwer, L. E. J., "On the Foundations of Mathematics", in A. Heyting (ed.), *Collected Works*, Vol. I, Amsterdam, North-Holland, 1975, pp. 11-101.
5. Brouwer, L. E. J., "The Unreliability of the Logical Principles", in A. Heyting (ed.), *Collected Works*, Vol. I, Amsterdam, North-Holland, 1975, pp. 107-111.
6. Dummett, Michael, *Elements of Intuitionism*, Oxford, Oxford University Press, 1977.
7. Grattan-Guinness, Ivor, *The Search for Mathematical Roots: 1870–1940: Logics, Set Theories, and the Foundations of Mathematics from Cantor through Russell to Gödel*, Princeton, Princeton University Press, 2000.
8. Hesseling, Dennis E., *Gnomes in the Fog: The Reception of Brouwer's Intuitionism in the 1920s*, Basel, Springer, 2003.
9. Heyting, Arend, *Intuitionism: An Introduction*, Amsterdam, North-Holland Publishing Company, 1956.
10. Hilbert, David, "From Mathematical Problems", in W. Ewald (ed.), *From Kant to Hilbert. A Source Book in the Foundations of Mathematics,* Vol. II, Oxford, Clarendon Press, 1996, pp. 1096–1105.
11. Hilbert, David, "Logic and the Knowledge of Nature", in W. Ewald (ed.), *From Kant to Hilbert. A Source Book in the Foundations of Mathematics,* Vol. II, Oxford, Clarendon Press, 1996, pp. 1157–1165.
12. Hilbert, David, "On the Infinite", in P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics: Selected Readings*, 2nd ed., Cambridge, Cambridge University Press, 1983, pp. 198–199.
13. Hilbert, David, "The Logical Foundations of Mathematics", in W. Ewald (ed.), *From Kant to Hilbert. A Source Book in the Foundations of Mathematics,* Vol. II, Oxford, Clarendon Press, 1996, pp. 1134–1148.

[38] Carl J. Posy, *Mathematical Intuitionism*, Cambridge, Cambridge University Press, 2020, p. 11.

14. Kant, Immanuel, *Critique of Pure Reason*, P. Guyer and A. W. Wood (eds.), Cambridge, Cambridge University Press, 1998.
15. Kant, Immanuel, "Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality", in D. Walford and R. Meerbote (eds.), *Theoretical Philosophy, 1755–1770*, Cambridge, Cambridge University Press, 2000.
16. Kant, Immanuel, *Metaphysical Foundations of Natural Science*, M. Friedman (ed.), Cambridge, Cambridge University Press, 2004.
17. Kant, Immanuel, *Prolegomena to Any Future Metaphysics That Will Be Able to Present Itself as a Science*, P. G. Lucas (ed.), Manchester, Manchester University Press, 1953.
18. Kreisel, Georg, "Foundations of Intuitionistic Logic", in E. Nagel, P. Suppes and A. Tarski (eds.), *Logic, Methodology and Philosophy of Science*, Stanford, Stanford University Press, 1962, pp. 198–210.
19. Lewis, David, *On the Plurality of Worlds*, Oxford, Basil Blackwell, 1986.
20. MacFarlane, John G., *What does it mean to say that Logic is Formal?*, PhD Thesis, Pittsburgh, University of Pittsburgh, 2000.
21. Mancosu, Paolo, *The Adventure of Reason: Interplay between Philosophy of Mathematics and Mathematical Logic, 1900-1940*, Oxford, Oxford University Press, 2010.
22. Martin-Löf, Per, "The Hilbert-Brouwer Controversy Resolved?", in M. Schirn (ed.), *The Philosophy of Mathematics Today*, Oxford, Clarendon Press, 1998, pp. 243–256.
23. Posy, Carl J., *Mathematical Intuitionism*, Cambridge, Cambridge University Press, 2020.
24. Russell, Bertrand, *The Principles of Mathematics*, London, Bradford & Dickens, 1942.
25. Shapiro, Stewart (ed.), *Intensional Mathematics. Studies in Logic and the Foundations of Mathematics, Vol. 113*, Amsterdam, Elsevier Science Publishers B.V., 1985.
26. The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, Princeton, Institute for Advanced Study, 2013.
27. Troelstra, A. Sjerp, "Axioms for Intuitionistic Mathematics Incompatible with Classical Logic", in R. E. Butts and J. Hintikka (eds.), *Logic, Foundations of Mathematics, and Computability Theory*, Dordrecht, D. Reidel Publishing Company, 1977, pp. 59-86.
28. Troelstra, A. Sjerp, and van Dalen, Dirk, *Constructivism in Mathematics: An Introduction,* Vol. I, Amsterdam, Elsevier Science Publishers B.V., 1988.
29. Țurlea, Marin, *Filosofia Matematicii*, București, Editura Universității din București, 2002.
30. van Atten, Mark, *On Brouwer*, Belmont, Wadsworth Philosophers Series, 2004.
31. Weyl, Hermann, "On the New Foundational Crisis of Mathematics", in P. Mancosu (ed.), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, New York, Oxford University Press, 1998, pp. 86–118.
32. Zermelo, Ernst, "Untersuchungen über die Grundlagen der Mengenlehre I", in *Mathematische Annalen*, 65 / 1908, B. G. Teubner, pp. 261–281.