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On the Adequacy and Substantiality of the Structuralist Thesis

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ABSTRACT. The idea that positions in structures have no mathematically significant non-fundamental features is a constitutive trait of non-eliminative structuralism; it underpins the restricted structuralist thesis that all fundamental properties are structural. So, a seemingly straightforward strategy to uphold the eligibility of non-eliminative structuralism is to prove a formal rendition of the thesis. However, the soundness of the strategy depends on two key aspects: the thesis has to be substantial, and materially adequate. The substantiality of the thesis is predicated on the non-synonymy of fundamental and structural properties. The adequacy is predicated on the synonymy between the formal definition of fundamental properties and the intuitive content of the notion. Two remarkable abstractionists accounts claim to have proven a formal, non-trivial, consistent version of the thesis. The first one, developed Linnebo and Pettigrew, arguably fails to satisfactorily accomplish this goal. However, the more formally sophisticated second one, developed by Schiemer and Wigglesworth, succeeds. This will be focus of the paper. I am going to argue that, precisely because it proves a non-trivial formal version of the thesis, their account of fundamental properties fails to be adequate. More precisely, I will show that the formal specifications of the fundamental properties needed to ensure the substantiality and soundness of the proof undergenerate and overgenerate structural properties. In the end, it seems that there is a trade-off between substantiality and adequacy. The arguments will inform some pessimistic conclusions about the overall strategy of establishing the eligibility of non-eliminative structuralism by means of such a proof of the structuralist thesis.

Keywords: *non-eliminative structuralism, variable domain Kripke models, abstraction principles, structural relations, fundamental relations.*

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1. The structuralist thesis

The idea that positions or places in structures have no non-structural properties is a fundamental and constitutive tenet of structuralism in the philosophy of mathematics. It was formulated over and over by prominent structuralists[1]. Benacerraf 's assertion[2], for example, that 'the "elements" of the structure have no properties other than those relating them to other "elements" of the same structure' is an illustrative formulation of this core trait of structuralism, which has come to be known as the *structuralist thesis*. Following the nomenclature, I will call this first pass of the thesis, *the unrestricted structuralist thesis*:

The unrestricted structuralist thesis: positions in pure/abstract structures have only structural properties/relations.

Of course, the thesis needs unpacking, specifically, by an operational characterization[3] of structural relations. The characterization employed by Linnebo and Pettigrew[4] and Schiemer and Wigglesworth[5] in their versions of a particular type of structuralism, namely *non-eliminative structuralism*[6], falls under what Korbmacher and Schiemer[7] call the *invariance account* of structural properties, namely that structural relations are those relations that remain invariant under isomorphism. Such an explication invites a view of pure structures as the result of a process of abstraction from 'concrete' isomorphic systems of objects; in accordance with the etymology of 'isomorphic' and along a non-eliminative structuralist line, we say that such systems exemplify or instantiate the same 'form' or structure. So, according to these versions of *nes*, a pure structure is the sediment of isomorphic systems obtained through abstraction.

Burgess[8] has convincingly argued that the unrestricted version of the structuralist thesis is incoherent: consider the (second-order) property of having only structural properties; according to the unrestricted structuralist thesis, positions in pure structures enjoy such a property, yet the property is not shared by

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1 See for example (Benacerraf, 1983, p. 291), (Resnik, 1981, p. 530), (Parsons, 2004, p. 57).
2 (Benacerraf, 1983, p. 291)
3 Given such characterization, the status of positions could also be clarified.
4 (Linnebo & Pettigrew, 2014)
5 (Schiemer & Wigglesworth, 2019)
6 Henceforth abbreviated by *nes*. A formal description of *nes* is provided in the next section.
7 (Korbmacher & Schiemer, 2018)
8 (Burgess, 1999)

the corresponding objects in 'concrete' isomorphic systems, as can be easily observed by inspecting, for example, the properties of the set-theoretic objects, {Ø, {Ø}}, {{Ø}}, playing the role of 2 in von Neumann's, respectively Zermelo's, reconstruction of the natural numbers. Consequently, having only structural properties is a peculiar, non-structural property of positions in pure structures. One easy way out of Burgess's criticism is to circumscribe the range of relevant properties of positions to first-order properties. But this manoeuvre, as Linnebo and Pettigrew and Pettigrew[9] have argued, cannot account for mundane, first-order, mathematically extraneous properties of positions such as 'being John's favourite number', 'being the favourite example of a mathematical structure'. And such properties are unavoidable for any candidate for the reference of number theory discourse. Accordingly, the pure properties of positions have to be further restricted to first-order, 'intrinsic'[10] properties. Obviously, this move just pushes the problem under the rug of the meaning of 'intrinsic', so not much progress has been accomplished. Instead of providing a rigorous, intuition-sound definition of 'intrinsic', and then a satisfying characterization of the class of relevant properties of positions by tackling other possible shortcomings, I will follow Linnebo and Pettigrew, and assume that such a characterization has been provided under the label of *fundamental* properties of positions for the sake of articulating *the structuralist thesis*. Distilling the discussion in a slogan, the non-eliminative structuralist adheres to:

The structuralist thesis: positions in pure structures have no mathematically relevant non-fundamental properties; moreover, all fundamental properties are structural properties.

Their way[11] of establishing the thesis is by proving a formal rendition of it – called *Purity*. Of course, a lot of formal work needs to be laid down in order to express and prove such a thesis. The task of the next sections is precisely that. But the guiding principle of the effort is that the specification of the class of fundamental properties should be independent of the invariance account of structural properties: "It is not an option simply to stipulate that 'fundamental' is to mean structural, as this would trivialize Purity: any object is such that all of its structural properties are structural'[12]. Thus, the non-synonymy of structural and fundamental relations is what gives substantiality to the structuralist thesis. So, it is no surprise that Linnebo and Pettigrew

[9] (Pettigrew, 2018)
[10] As Pettigrew (2018) qualifies them.
[11] As well as Schiemer and Wigglesworth's.
[12] (Linnebo & Pettigrew, 2014, p. 279)

list as a capital merit of their proposal that "it provides a principled and precise definition of 'fundamental' that makes Purity a substantial and philosophically interesting claim; and, moreover, one that is true"[13].

As mentioned, the task of the next sections is to set up the formal medium in which the thesis is couched and proved. This is done in steps and with a certain proviso. The first step is to formally characterize the type of structuralism that accommodates the thesis, followed by a specification of Linnebo and Pettigrew abstractionist version of it, and an assessment of the structuralist thesis in this framework. Afterwards, I will focus on a 'new and improved' abstractionist version, that of Schiemer and Wigglesworth, and asses the significance of the formal rendition of the thesis in it. The assessment will inform some pessimistic conclusions about the general strategy of providing decisive support for non-eliminative structuralism by proving a formal rendition of the structuralist thesis.

2. Non-eliminative structuralism

Following Linnebo and Pettigrew, I will present a formal characterization of the kernel of *nes* restricted to relational systems – conceived[14] as set-theoretic entities of the form $S = \langle D, R_1, R_2, \ldots R_n \rangle$, where, as the convention dictates, D is a set, and $R_1, R_2, \ldots R_n$ are relations on D. Accordingly, from now on, unqualified talk about systems and pure structures should be understood as set-theoretic talk about relational systems and relational pure structures. The technical concept underlying the precis characterization of *nes* is that of isomorphism of (relational) systems.

Definition 2.1:

Two relational systems, S and S', are isomorphic, in symbols,
$S \cong S'$, if $\exists f, f: D \rightarrow D'$, such that
a) f is bijective;
b) f is an embedding: for each R_i of arity n in S, the following holds:
$$\forall x_1, \ldots x_n \in D\, [R_i(x_1, \ldots, x_n) \equiv R_i'(f(x_1), \ldots, f(x_n))]$$

Non-eliminative structuralism can now be formally characterized by the adherence to the following theses:

[13] (ibidem)
[14] As the standard practice in model theory dictates.

Instantiation: Let *S* be a system, and [*S*] its corresponding pure structure. Then, [*S*] ≅ *S*, that is, every pure structure [S] is isomorphic with its instantiated system.

A moment's reflection shows that *Instantiation* is essential for giving the face-value reading of singular terms (of a non-algebraic theory), for any discourse about a particular object in the domain of a system S involving structural properties could, in virtue of *Instantiation*, be rendered as a discourse about the corresponding position in the pure structure [*S*]; thus, *instantiation* is a rigour demanded by the semantic constraint directed at singular terms purportedly denoting simple objects, like numbers, vertices, etc.

Purity: If Φ is a *fundamental* property of a position *a* in a pure structure [S], that is Φ(*a*), then, for any *S'* such that *f:* [*S*] ≅ *S'*, Φ is a property of *f*(*a*), that is Φ(*f*(*a*)).

Obviously, *Purity* is the formal counterpart of the quintessential restricted structuralist thesis: "purity is our consistent reformulation of the structuralists' claim that positions in pure structures have no non-structural properties".[15]

As it is formulated, *Purity* invites an intensional conception of properties, on pain of insurmountable difficulties concerning the structural character of an extensionally construed property. I will briefly discuss some of these difficulties in the context of Schiemer and Wigglesworth's proposal to overcome them by an articulation of an intensional view of properties.

Uniqueness: [*S*] uniquely satisfies *Instantiation* and *Purity*. Specifically, uniqueness demands that for every *S* ≅ *S'*, and pure structures [*S*] ≅*S*, [*S'*] ≅ *S'*, [*S*] = [*S'*].

Uniqueness is demanded by the face-value reading of singular terms purportedly denoting complex objects i.e., unique structures (purportedly described by non-algebraic theories), so, again, *Uniqueness* is an implementation of the self-imposed semantic constraint of *nes* directed, this time, at structure-denoting singular terms, such as ℕ or ℝ.

[15] (Linnebo & Pettigrew, 2014, p. 272)

3. LP-structuralism: Structural abstraction via Frege abstraction

In 'Two types of abstraction for structuralism'[16], Linnebo and Pettigrew seek to provide a defensible non-eliminative structuralist account of pure structures by appeal to (neo)-Fregean abstraction principles. Their proposal, from now referred to as *LP-structuralism*, resides, roughly, in indicating how pure structures can be soundly detached from systems *via* abstraction principles. Beginning with Frege, abstraction principles were used to legitimate the introduction of new, more abstract, concepts and objects out of already accepted 'old' ones. To this end, abstraction principles provide identity conditions of abstracta in terms of equivalence relations of the old type of objects. For example, Frege's well-known abstraction principle for the directions of lines,

(DL): for every l, l', d(l) = d(l') iff l || l',

establishes the legitimacy of the concept of *direction* by giving the necessary and sufficient conditions of identity of the new objects falling under it – directions d(l), d(l') – in terms of the equivalence relation of parallelism || of good old lines l, l'.

Similarly, Linnebo and Pettigrew develop the abstraction principle that provides the identity conditions for pure structures:[17]

Frege Abstraction for Pure Structures:

Given systems *S* and *S'*, [*S*] = [*S'*] iff *S* ≌ *S'*.

As a *nes* candidate, pure structures obtained by abstraction principles should satisfy *Instantiation* so they should contain positions[18] corresponding to 'concrete' elements in isomorphic systems, and relations between those positions matching the relations between the corresponding 'concrete' elements in systems. Accordingly, the next obvious step is to provide abstraction principles for positions.

[16] (Linnebo & Pettigrew, 2014)

[17] Linnebo and Pettigrew propose that pure structures are *sui-generis* entities in order to avoid the Burali-Forti paradox.

[18] playing the role of simple mathematical objects and referents of singular terms in non-algebraic theories.

Frege Abstraction for Positions in Pure Structures:

Given systems S and S', and elements x of S and x' of S':
$[x]_S = [x']_{S'}$ iff $\exists f$ ($f: S \cong S'$ and $f(x) = x'$)

Collecting all such positions leads to the pure domain of a pure structure:

Pure Domains in Frege Abstraction:

For all x in S and their matching positions [x] in the pure structure [S] of S,
$[D]_S = \{[x]_S: x \in D\}$

Now that the positions and domain of a pure structure have been defined, Linnebo and Pettigrew proceed by specifying how to abstract relations on positions that isomorphically match relations on corresponding elements of systems.

Pure Relations on Pure Domains.

Suppose S is a system and Φ is an n-ary relation on the domain D of S.

Then: $[\Phi]_S(x_1, x_2, \ldots x_n)$ iff there are elements $u_1, u_2, \ldots u_n$ of D such that, for each i, $[u_i]_S = x_i$, and $\Phi(u_1, u_2, \ldots u_n)$.

As I have indicated in the first section, the restricted structuralist thesis presupposes the non-trivial identification[19] of a class of pure relations, called *fundamental* relations, that are provably structural. The success of such a non-trivial identification will substantiate the purity thesis that the only mathematically relevant relations that pure positions have are structural. The specific candidate for the role of fundamental relations that Linnebo and Pettigrew propose is:

Fundamental Relations among Positions

Suppose Φ is a relation on the positions of [S]. Then Φ is fundamental if there is a relation Ψ on the domain of S such that [Ψ] = Φ.

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[19] Meaning that the identification should be given in independent terms than those used for specifying what a structural relation is; in our case, this amounts to an identification that is independent of the invariance under isomorphism account.

Clearly, LP-structuralism is an insightful *nes* proposal that articulates a stepwise comprehensive mechanism for obtaining the philosophical stone of *nes*, pure structures. If it succeeds, then significant philosophical progress has been achieved. The goal of the next section consists precisely in the assessment of the proof and substantiality of *Purity* in Linnebo and Pettigrew's formal framework.

4. The structuralist thesis in LP-structuralism

Linnebo and Pettigrew contend that, modulo rigid systems, LP-structuralism proves the structuralist thesis under the formal guise of *Purity*[20]:

If S is rigid and $x_1, x_2, \ldots x_k$ are elements of D,
$[\Psi]_S([x_1], [x_2], \ldots [x_k])$ iff $\Psi(x_1, x_2, \ldots x_k)$

However, Schiemer and Wigglesworth dispute the claim of Linnebo and Pettigrew, rightly pointing out that LP-structuralism is at odds with *Purity*, and set out to give a corrected *Purity*-proof version of it – call it *SW-structuralism*. I'll discuss their criticism of LP-structuralism in relation to *Purity* next, and I'll outline their solution in the next section.

Schiemer and Wigglesworth's critique is two-folded. First, they contend that the abstraction principle for pure positions makes *Purity* irreconcilable with Linnebo and Pettigrew's formal proposal for fundamental relations and properties. Then, they argue that any attempt to reconcile the definition of fundamental relations with *Purity* has to rely on an intensional approach to relations. Let us tackle the issues in the order just presented.

W.l.o.g. consider a fundamental pure property $[\Phi]_S$ of a pure position $[x]_S$ in a correspondingly pure structure [S]. On the one hand, by being fundamental, *Purity* requires that $[\Phi]_S$ be shared by all elements in isomorphic systems S', i.e. $[\Phi]_S(f([x]_S))$, $f: [S] \cong S'$. On the other hand, according to the gloss accompanying *Pure Relations* – '$[\Phi]_S$ is the property that holds of an object iff that object is a pure position in the pure structure [S]'[21] – $[\Phi]_S$ is attributable only to $[x]_S$, so, in any isomorphic system S', $[\Phi]_S$ cannot hold of $f([x]_S)$ i.e. it is **not** the case that $[\Phi]_S(f([x]_S))$, contradicting, thus, *Purity*. Given this incompatibility, Schiemer and Wigglesworth propose to alter the definition of fundamental relations in a manner

[20] This is their *Proposition 5.2*

[21] (Linnebo & Pettigrew, 2014, p. 275)

consistent with Purity and the requirement of non-triviality. In order for this local definitional patch to work, an explicit general account of relations should be first articulated. To this end, they argue for an intensional understanding LP-structuralism by showing that, in the presence of the abstraction principle for pure relations, an extensional treatment of relations overgenerates fundamental relations. To make their argument transparent, let us recall that in an extensional account, relations are bound to particular systems and completely determined by their relata. In more precise terms, relations are identified with the set of ordered tuples of elements from a given system, acting as relata. For obvious reasons, this set is known as the extension of the relation; hence, a relation just is a 'local' or system-relative extension of ordered tuples. As a consequence, any relations that consist in the same set of tuples of relata are identical.

Now, let Φ be an *arbitrary* pure relation of pure positions $a_1, a_2, \ldots a_n$ in a pure structure [S] of a system S, $\Phi(a_1, a_2, \ldots a_n)$. By *Instantiation*, there is an isomorphism $f\colon [S] \cong S$ such that $\Phi(a_1, a_2, \ldots a_n) \equiv \psi(f(a_1), \ldots, f(a_n))$; by abstraction principle for pure relations, $[\psi]_S(a_1, \ldots, a_n)$; by fundamental relations, $[\psi]_S$ is a fundamental relation; by extensionality $\Phi = [\psi]_S$. So, any arbitrary pure property Φ can be transformed in LP-structuralism into a fundamental property. It is worth noting the contribution of the extensional treatment of relations to the argument in order to sharply understand the mechanism in Schiemer and Wigglesworth's proposal that effectively blocks the conclusion that every pure relation is fundamental.

5. SW-structuralism: structural abstraction via Kripke models

In light of the previous section's discussion, Schiemer and Wigglesworth articulate[22] a formal framework for structural abstraction capable of entertaining an intensional construal of relations, in which to define fundamental relations of positions in pure structures, not only respecting the non-triviality condition, but enabling a proof of the structuralist thesis. The formal framework they consider adequate for this purpose is that of variable domain Kripke models. As they emphasize, one of the perks of using such a versatile framework is that it permits not only to provide 'an intensional account of mathematical properties' but, importantly, 'to formally capture a dynamic version of abstraction'.[23]

[22] In this section I am going to follow closely Schiemer and Wigglesworth's exposition.

[23] (Schiemer & Wigglesworth, 2019, p. 1204)

A variable domain Kripke model is a quadruple $\mathcal{M} = \langle D, W, \sim_{acc}, v\rangle$ equipped with the usual interpretation (D is a non-empty set acting as the universal domain, W a non-empty set of worlds, $\sim_{acc}$ an accessibility relation on W, v an interpretation of relations) plus the extra charge for v of assigning a set $D_w \subseteq D$, the local domain of quantification, to each world $w \in W$. Embedding LP-structuralism in such a framework is done by considering that the worlds $w \in W$ are relational systems, $w = \langle D_w, R_{1w}, \ldots, R_{nw}\rangle$, D is the set of all objects in all D_w, $w \in W$, and the accessibility relation $\sim_{acc}$ is the isomorphism $\cong$ relation between relational systems. Now, in this setting, intensional relations R^n are interpreted as functions $f^n : W \rightarrow \mathbb{P}(D^n)$, where $\mathbb{P}(D^n)$ is the powerset of all n-tuples from D. Accordingly, a nonempty n-ary relation R_w of a world–system w is essentially the local extension of R^n in w (henceforth $R^n{}_w$), defined by the value $R^n{}_w \in \mathbb{P}(D^n{}_w)$. Identity conditions for intensional relations[24] easily follow: $R_1 = R_2$ iff $R_{1w} = R_{2w}$ for all $w \in W$, i.e., two intensional relations are $\mathcal{M}$-identical iff they have the same local extensions. The accessibility relation $\sim_{acc}$ is defined unsurprisingly:

Definition 4.2.1. ($\sim_{acc}$):

$$\text{Given } w = \langle D_w, R_{1w}, \ldots, R_{nw}\rangle,\ v = \langle D_v, R_{1v}, \ldots, R_{nv}\rangle,$$
$$w \sim_{acc} v \text{ iff } \exists f\colon D_w \rightarrow D_v, \text{ such that}$$

a) f is bijective;

b) f is an embedding: for each k-ary relation R, the following holds:

$$\forall x_1, \ldots x_k \in D_w\ [R_w(x_1, \ldots x_k) \equiv R_v(f(x_1), \ldots, f(x_k))]$$

At this point in the presentation, it is instructive to mention another critique to LP-structuralism that Schiemer and Wigglesworth address[25], envisaging the nature of the abstraction process as articulated in the abstraction principles and operator. What particularly troubles them is that Linnebo and Pettigrew left unspecified how exactly the abstraction operators, represented by the square bracket notation [], work: they are supposed to act as functions []: $S \rightarrow [S]$, and []: $x \in S \rightarrow [x]_S \in [S]$, but their codomain is unspecified. The need of such a clarification is fundamental to any structural abstractionist project, let alone one that intends to rigorously recasts the LP-abstraction principles in a Kripke-models mould. For this reason, Schiemer and Wigglesworth turn to a predicative and dynamic understanding of abstraction.

24 For readability purposes, I will drop the specification of the arity of the relations from now on.
25 (Schiemer & Wigglesworth, 2019, p. 1208)

As conveyed by LP-structuralism, the structural abstraction process distils from each particular system *S* its corresponding pure structure [S]. Now, suppose that a collection of such systems is given. What dynamic and predicative abstraction does, in Schiemer and Wigglesworth's account, is to extend this initial collection of systems *S* by appending all the corresponding pure structures [S] obtained through structural abstraction. As they put it, 'mathematical abstraction, understood as a predicative and dynamic process, simply allows one to consider larger and larger domains of mathematical entities, independently of the question of their objective existence. In the case of LP-structuralism, the relevant abstraction principles introduce pure structures into the domain of consideration by giving their identity conditions, as well as the identity conditions for the pure positions that belong to those structures'.[26] Dynamic abstraction is implemented in SW-structuralism through the operation of Kripke model extension. As the name of the operation suggests, model extension refers to the embedding *via* the inclusion or identity function of a Kripke model $\mathcal{M} = \langle D, W, \sim_{acc}, v\rangle$ into another, larger one, $\mathcal{M}' = \langle D', W', \sim_{acc}', v'\rangle$; obviously, the embedding implies that $D \subseteq D'$, $W \subseteq W'$, $\sim_{acc} \subseteq \sim_{acc}'$, $v \subseteq v'$. Accordingly, the extended model is specified in three steps: 1) by defining *W'* and *D'*, 2) by defining $\sim_{acc}'$, and 3) by defining[27] *v'*. I proceed the exposition in order. For the first step, this means supplementing *W* with the members of the set of pure structures W_S, and D with the members of the set of pure positions D_P, thus obtaining $W' = W \cup W_S$, and $D' = D \cup D_P$. W_S and D_P are given by the well-defined operators echoing '*Frege Abstraction for Pure Structures*' and '*Frege Abstraction for Positions in Pure Structures*':

Definition 4.2.2. Pure structure abstraction operator §:

Given $\mathcal{M} = \langle D, W, \sim_{acc}, v\rangle$, call a pure structure operator a function §: $W \rightarrow W_S$, $W \cap W_S = \emptyset$, such that

§(w_1) = §(w_2) iff $w_1 \sim_{acc} w_2$, for all $w_1, w_2 \in W$.

Collecting the pure structures in a set gives $W_S = \{§(w) \backslash w \in W\}$

Definition 4.2.3. Pure positions abstraction operator σ:

Given $\mathcal{M} = \langle D, W, \sim_{acc}, v\rangle$, and relational systems $w_1, w_2 \in W$, call a pure position operator, a function $\sigma: D \rightarrow D_P$, such that for all $a \in D_{w1}$, $b \in D_{w2}$, $\sigma(a) = \sigma(b)$ iff there is an isomorphism f between w_1 and w_2 ($w_1 \sim_{acc} w_2$) and $f(a) = b$,

[26] (Schiemer & Wigglesworth, 2019, p. 1215)

[27] It will become clear that Schiemer and Wigglesworth define the valuation function *v'* only partially, restricting the specification of extensions to the members of the class of pure relations obtained by abstraction.

The set of pure positions is, then, easily defined as $D_P = \{\sigma(a) \backslash a \in D\}$.

The accessibility relation $\sim_{acc}'$ can now be defined by extending $\sim_{acc}$ to include the unaccounted interactions between world-systems and pure structures in the new setting $W' = W \cup W_S$:

Definition 4.2.4. ($\sim_{acc}'$):

$$\text{For all } w_1, w_2 \in W', w_1 \sim_{acc}' w_2 \text{ iff } w_1 \sim_{acc} w_2 \vee w_1 = §(w_2) \vee w_2 = §(w_1)$$

Note that the pure structures $§(w_1)$, $§(w_2)$ are either identical or not $\sim_{acc}$ related.

Before presenting the third and final step in the construction of the extended model $\mathcal{M}'$ and, based on it, the account of fundamental and structural relations, let's take stock, as Schiemer and Wigglesworth do, of two significant advantages of adopting Kripke models doubled by an intensional understanding of properties and relations, to convey a sound structural abstraction form of *nes*.

First, they claim that the intensional construal developed in the Kripke models infrastructure effectively blocks the argument that brought havoc to LP-structuralism, that all pure relations are fundamental. What blocks the argument according to their account is the fine-grained identification of relations: to be LP-fundamental, an arbitrary pure relation Φ on positions has to be co-extensional with the induced abstract relation $[\psi]_S$ in every system *S* in order to be identical with it. Note, however, that all Schiemer and Wigglesworth have managed to show is that extra-work is required in SW-structuralism for proving that all pure relations are LP-fundamental, not that this verdict is 'effectively blocked' by their account. I will succinctly return to this issue in the next section, after clarifying their take on fundamental relations.

The second, significant benefit of using Kripke models consists in the elegant explanation of the nature of pure structures, positions, and pure relations. To circumvent the Burali-Forti paradox, LP-structuralism simply asserts that they are *sui-generis* entities, keeping their status ambiguous. On SW-account, they can have the same status as the systems from which were abstracted without the threat of the Burali-Forti paradox because 'the pure structures are not members of the set of worlds in the initial Kripke model, but are introduced through a dynamic abstraction process as captured by extending the initial model'.[28].

[28] (Schiemer & Wigglesworth, 2019, p. 1217).

5.1. Pure, fundamental, and structural relations in SW-structuralism …

As advertised above, I am going to present Schiemer and Wigglesworth's definition of the valuation function *v'* and sketch their account of fundamental relations. Briefly, their strategy is to assign extensions at pure structures under *v'* only to those pure relations that are the extensions of relations that already have local extensions fixed by *v*, i.e., relations already interpreted by *v*. Of course, for this strategy to succeed one has to define what an extended relation is.

Definition 4.2.1.1. Extended relations:

Given $\mathcal{M} = \langle D, W, \sim_{acc}, v\rangle$, $D', W', \sim_{acc}'$, say that an n-ary relation R extends an n-ary relation Q iff:
a) $R_w = Q_w$ for all $w \in W$
b) For all $u \in W_S$ and all $d_1, \ldots, d_n \in D_u$, $\langle d_1, \ldots, d_n\rangle \in R_u$ iff there is a $w \in W$ with $b_1, \ldots, b_n \in D_w$, such that
i) $d_i = \sigma(b_i)$, $i \in n$
ii) $\langle b_1, \ldots, b_n\rangle \in Q_w$

In other words, a relation R in $\mathcal{M}'$ is an extension of a relation Q in $\mathcal{M}$ iff R is $\mathcal{M}$-identical to Q and is LP-abstracted from Q. The definition has a couple of remarkable features worth stating. First, it enables a rigorous characterization of the pure abstracted relations in SW-structuralism by considering them to be extended relations. Second, it leaves room for pure relations that are not the product of abstraction, allowing these unattended relations to act as the deposit of the unavoidable extraneous properties of positions discussed in section 1. Third, it does justice to the intuition that the relevant pure relations of a structure have to be connected with relations in 'concrete' systems. However attractive are these features, and this is highly significant, the definition cannot capture the class of fundamental relations, on pain of falsifying the structuralist thesis. This is due to condition b) of the definition permitting the generation of an extended relation by abstraction from an idiosyncratic,[29] arbitrary, relation of a system. It is both illustrative and highly relevant to see why using Schiemer and Wigglesworth's example[30]. Consider the Kripke model $\mathcal{M} = \langle D = \{N_Z \cup N_N\}, W = \{z, n\}, \sim, v\rangle$, where the system *z* consists of the set of finite Zermelo ordinals, N_Z, and *n* of the set of

[29] In the sense that it is specific to the system in question by not being preserved under isomorphism.
[30] (Schiemer & Wigglesworth, 2019, p. 1219)

finite von Neumann ordinals, N_N, both equipped with their 'usual ordering'.[31] Both systems exhibit the same pure structure ℕ, so §(z) = §(n) = ℕ. The 'set-theoretic property P of having exactly two members'[32] has the local extensions $P_z = \emptyset$, and $P_N = \{\{\emptyset, \{\emptyset\}\}\}$. According to the previous definition, one can extend P to P* by ensuring that P* is co-extensional with P relative to *z* and *n*, (condition a)), and that $P^*_{\mathbb{N}} = \{2\}$, where $2 = \sigma(\{\emptyset, \{\emptyset\}\}) = \sigma(\{\{\emptyset\}\})$ (condition b)). Under the naïve hypothesis that fundamental relations are just extended world-bound relations abstracted by means of the previous definition, P* would count as fundamental. But this would falsify the structuralist thesis, for the property of having exactly two members is not structural, as witnessed by the third Zermelo ordinal.

Consequently, if the class of fundamental relations would be completely defined by the class of extended relations, then not all fundamental relations will turn out to be structural. So, some extra conditions should be added to the definition of fundamental properties, to the effect that, in conjunction with the desirable and intuitively sound condition of being an extended relation, they will ensure the provability of the structuralist thesis. The missing, satisfactory condition that Schiemer and Wigglesworth propose to complete the definition of fundamental relations is definability.

Definition 4.2.1.2. Definable relation:

Given a language $\mathcal{L}$ and an $\mathcal{L}$-systems $w = \langle D_w, R_{1w}, \ldots, R_{kw} \rangle$, we say that an n-ary relation R_i is definable iff there is an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and for all w there are elements $b_1, \ldots, b_m \in D_w$ such that for all $d_1, \ldots, d_n \in D_w$:

$$\langle d1, \ldots, dn \rangle \in R_{iw} \Leftrightarrow w \models \varphi(d_1, \ldots, d_n, b_1, \ldots, b_m)$$

Their justification for choosing the definability condition is that as the troublesome definition of extended relations indicates, the fundamental properties should not be abstracted from arbitrary relations in 'concrete' systems, but 'from relations dealing with (or about) the internal structure of the systems in question',[33] and 'the special class of relations admissible for this kind of abstraction'[34] is that of definable relations, taken to reflect the inner structure of systems.

31 ibidem
32 ibidem
33 (Schiemer & Wigglesworth, 2019, p. 1220)
34 Ibidem.

As advertised, fundamental relations can now be fully characterized by the simultaneous satisfaction of the above two conditions.

Definition 4.2.1.3. Fundamental relations:

An n-ary relation R on the positions of a pure structure §(w) is fundamental iff there is an n-ary relation Q on the elements of an $\mathcal{L}$-systems w and a formula φ in $\mathcal{L}$ such that

i) Q is defined by φ, and
ii) R is an extension of Q.

Again, there are some remarkable consequences worth stating of this definition. The first thing to note is that fundamental relations become language-dependent.[35] This is not a peculiar structuralist position, as the practice of model theory indicates:

‘Model theorists are forever talking about symbols, names and labels. A group theorist will happily write the same abelian group multiplicatively or additively, whichever is more convenient for the matter in hand. Not so the model theorist: for him or her the group with '•' is one structure and the group with '+' is a different structure. Change the name and you change the structure’.[36]

Secondly, not only are fundamental relations pure (in Linnebo and Pettigrew’s sense), but are induced by abstraction (according to condition ii). Thirdly, fundamental relations are defined by the same formulas that define the relations from which they are abstracted, given that definability is preserved under extension of relations. Lastly, all intuitively fundamental properties concerning some familiar mathematical systems, mentioned and discussed by Linnebo and Pettigrew 2014, are captured formally by this definition.

The only thing that’s missing for proving the structuralist thesis is a definition of structural properties. The definition goes as expected:

Definition 4.2.1.4. Structural properties:

R is a structural property of position a in the domain of §(w) iff for all systems $w \in W$ and for all isomorphisms f: §(w) → w, the following holds:

$$a \in R_{\S(w)} \Rightarrow f(a) \in R_w$$

[35] This language relativity is not unprecedented in the structuralist literature, as Schiemer and Wigglesworth (2014) acknowledge by citing (Resnik, 1997).

[36] (Hodges, 1997, p. 1)

The structural thesis is proven in section 7 of Schiemer and Wigglesworth's paper under the label *proposition 1*:

'Suppose a is a position in structure §(w). If R is a fundamental property of a in §(w), then R is a structural property of a in §(w)'.

6. Adequacy *vs* substantiality of the structuralist thesis in SW-structuralism

6.1. The role of the intensional construal of relations

On a careful examination, one can observe that what effectively blocks the argument that arbitrary pure relations ($R_{§(w)}$) turn out to be fundamental in SW-structuralism is not the appeal to intensional relations, but to definability, as Schiemer and Wigglesworth certainly recognize: 'Properties such as being John's favourite number fail to be fundamental since there are no definable properties from which they can be abstracted'.[37] More precisely, the appeal to signatures in specifying what counts as fundamental relations is what blocks the argument, but this move is available to Linnebo and Pettigrew 's version of structuralism.

6.2. Why definable?

It is instructive to ponder why the definition of extended relations is formulated in such a manner that it doesn't prohibit right from the start the abstraction from, or extension of, arbitrary, idiosyncratic relations (recall that it is sufficient for a relation Q to occur in a system *w* in order to be extendable). The definition can be easily adjusted so that the admissible relations for abstraction or extension are structural. Here is one way of rectifying it:

Given $\mathcal{M} = \langle D, W, \sim_{acc}, v\rangle$, D', W', $\sim_{acc}'$, *say that an n-ary relation R extends an n-ary relation Q iff:*
a) $R_w = Q_w$ *for all* $w \in W$
b) For all $u \in W_S$ *and all* $d_1, \ldots, d_n \in D_u$, $\langle d_1, \ldots, d_n\rangle \in R_u$ *iff there is a* $w \in W$ *with* $b_1, \ldots, b_n \in D_w$, *such that*
i) $d_i = \sigma(b_i)$, $i \in n$
ii) $\langle b_1, \ldots, b_n\rangle \in Q_w$

[37] (Schiemer & Wigglesworth, 2019, p. 1222)

c) *For all* $w' \in W$*, such that* $w \sim_{acc} w'$*, there are* $c_1, \ldots, c_n \in D_{w'}$ *such that* $\langle c_1, \ldots, c_n \rangle \in T_{w'}$ *iff* $\langle b_1, \ldots, b_n \rangle \in Q_w$ *where T is an n-ary relation.*

Amended in this manner, the definition of extended relations becomes synonymous to that of structural relations. And now we can clearly see why such an altered definition is untenable in *nes*: it trivializes the structuralist thesis by making all fundamental relations, understood as extended relations, definitionally synonymous with structural relations.

At this point, it becomes apparent that the wrinkle of definability as an essential condition for the characterization of fundamental relations has everything to do with the substantiality of the structuralist thesis. Stated in terms of extended and definable relations, the definition of fundamental relations is non-synonymous to that of structural relations, substantiating, thus, the claim of the structuralist thesis. Consequently, a proof of the structuralist thesis becomes a significant and revealing result.

6.3. Adequacy of the definition of fundamental relations

However clever and ingenious, Schiemer and Wigglesworth's rendition of the structuralist thesis, I argue, misses its mark. Just to be clear, I will argue that their definition of fundamental relations fails to capture what their explicit formal purpose was: essential features of the underlying structure instantiated in systems sharing the same signature. More precisely, it fails in two aspects: it overgenerates and undegenerates fundamental relations. To unpack my claim, consider some of the examples of relations and properties that they include in the 'intuitively fundamental' target set of the formal fundamental relations:

'Being the additive identity in a complete ordered field is one such property (of the zero position). Being an annihilating element for multiplication in such a field is another. The list could be extended for other types of structures: being an even number or being the second successor of the zero position are fundamental properties of certain places in the natural number structure. Being a node with a certain degree, that is, having a certain number of edges incident to it, is a fundamental property of nodes in a graph structure.'[38]

All these cases seem to be easily and quite naturally captured by their definition of fundamental relations:

[38] (Schiemer & Wigglesworth, 2019, p. 1220)

"Consider, for instance, properties of the positions in the natural number structure discussed above. Each of these can be induced by abstraction from a concrete property of elements in a natural number system that is definable in terms of the primitive non-logical vocabulary of the language of Peano arithmetic. The property of being an even number, for example, is clearly fundamental in this sense, since it can be abstracted from a property of numbers in a given concrete natural number system that is definable by a first-order formula '$\exists y(y+y = x)$'".[39]

But, as they claim, and I wholeheartedly agree, if the properties listed above are 'intuitively fundamental', then so it should be the property of being a non-standard number or an infinite successor of the zero position in a non-standard model of arithmetic. The non-standard numbers are at least as constitutive[40] for non-standard structures, as zero is for a field. Accordingly, their properties should count as intuitively fundamental for the non-standard structure of arithmetic. But it is an elementary result[41] regarding non-standard models of arithmetic that their characteristic properties are not definable in first-order Peano Arithmetic (PA). So, consider the Kripke model
$\mathcal{M} = \langle D, W = \{n^*, n^{*\prime}\}, \sim, v\rangle$, where n* and n*' are isomorphic non-standard models of arithmetic. Their pure structure, §(n*) = §(n*') = $\mathbb{N}^*$, should contain non-standard positions having the pure properties corresponding to those mentioned above as intuitive. But since these properties are not definable, they are not fundamental according to Schiemer & Wigglesworth's definition, although, as argued, they are intuitively fundamental by their own lights. By Dedekind's categoricity theorem, second order Peano Arithmetic has only standard models, so changing the logic in this context doesn't help, as it excludes non-standard numbers. Other types of logics, or the appeal to open-ended versions of PA or schematic theories could be excluded on appropriate grounds. And, in the end, it isn't even a matter of switching to the 'right logic', afterall, the study of such models in not only lucrative for better understanding and illuminating the standard model, but also worth pursuing for the mathematics of it. So, the definition undergenerates fundamental relations.

To see that it also overgenerates it is enough to consider properties representable in PA. With a sensible coding scheme of Gödel numbering, there will be a PA-formula coding the abovementioned 'property P of having exactly two members', or of being a formula of PA, or a term, or a sentence etc. All these recursive properties are representable in PA. Now, consider Schiemer and Wigglesworth's example of the Kripke model $\mathcal{M} = \langle D, W = \{z, n\}, \sim, v\rangle$, consisting of the set of finite

[39] (Schiemer & Wigglesworth, 2019, p. 1222)

[40] Although proving this takes a bit of effort: 'is not quite trivial to show that there must *be* some nonstandard numbers in any nonstandard model $\mathcal{M}$.' (Boolos, 2007, p. 303)

[41] For details see the 'overspill lemma' or 'principle' in (Boolos, 2007, p. 309) or (van Dalen, 2004, p. 122)

Zermelo and von Neumann ordinals equipped with their 'usual ordering' and expand the signature to that of PA, importing the usual interpretation of the symbols. In this model, the arithmetized version of the property P turns out to be fundamental, although, intuitively it shouldn't be. I am aware of the difference between the property as expressed and understood in metalanguage and its artihmetized counterpart, but that is beside the point, for in virtue of being representable there is a canonical way of reconstructing the metalinguistic meaning. The same holds for other syntactic properties, and all the other non-arithmetic recursive properties. But these arithmetized versions of non-arithmetical properties obviously go over and beyond what an intuitively fundamental property of the natural numbers is supposed to be. The definition overgenerates.

7. Concluding remarks

The substantiality of the structuralist thesis is predicated on the non-synonymy of fundamental and structural relations. The adequacy is predicated on the synonymy between the formal definition of fundamental properties and the intuitive content of the notion. If *structural* is taken to be invariance under isomorphism, as both Linnebo and Pettigrew, and Schiemer and Wigglesworth explicitly consider, then both abstractionist proposals fall short of upholding the eligibility of non-eliminative structuralism by proving the structuralist thesis. Linnebo and Pettigrew fail not only to convincingly circumscribe the class of fundamental properties, but also to prove the structuralist thesis. In Schiemer and Wigglesworth's reconstruction, the structuralist thesis is successfully proven, but it amounts to the fundamental but elementary result that definable relations are preserved under isomorphism. Now, how substantial is this result for *nes* is a matter of debate, given that the notion of isomorphism is intimately connected with the signature of a system. Isomorphisms are essentially defined with respect to a signature and a language. That much is an elementary observation.

'The isomorphism concept is intricately linked with that of formal language, which is a way of making precise exactly which mathematical structure one is considering. Whether a given one-to-one correspondence is an isomorphism depends crucially, after all, on which structural features are deemed salient'. (Hamkins, 2020, p. 30)

Nevertheless, the main point of my contention is that, even though the substantiality concerns raised above are surpassed, the class of fundamental relations, as sharply and ingeniously defined by Schiemer and Wigglesworth still won't cut it by their own standards: it unequivocally undergenerates and arguably overgenerates.

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