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On the Completeness Interpretation of Representation Theorems

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ABSTRACT. Representation theorems, similar to their counterparts, categoricity theorems, establish an isomorphism between certain algebraic systems. However, in contrast to categoricity theorems, they have received considerably less attention in the philosophy of mathematics. The paper attempts to rectify this shortcoming by excavating the philosophical potential of representation theorems through an analysis of one of their most popular interpretations in the mathematical literature, the completeness interpretation. The meaning of this notion of completeness and the mechanism through which representation theorems are supposed to achieve it are still unclear. The paper addresses both issues. First, it proposes a definition of completeness that best suits the informal notion used in the mathematical interpretation of the theorems. Second, it formally details the mechanism responsible for achieving it. In the process, I'll issue some remarks on the significance and relevance of the formal reconstruction of the completeness interpretation for non-eliminative structuralism. For exegetical as well as evidential reasons, I'll focus on Cayley's representation theorem and use it as a case study.

Keywords: Cayleys representation theorem, semantic completeness, categoricity, axiomatic completeness, representation theorems.

Technical preliminaries

In this section, I'll briefly introduce [2] some standard definitions, and state without proof some elementary results in model theory that will be used in the analysis of representation theorems. I assume familiarity with these notions and results, so I'll skip accompanying clarificatory commentaries or examples, but I'll emphasize where I deviate from standard terminology in order to accommodate the discussion with the structuralist jargon.

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[2] To this end I'll follow the presentation in (Marker 2002)

Definition 1. A first order language [3] $\mathcal{L}$ consists of:

(i) A set $\mathcal{C}$ of constant symbols

(ii) A set $\mathcal{F}$ of function symbols associated with a positive integer n_F specifying the arity of each $F \in \mathcal{F}$.

(iii) A set $\mathcal{R}$ of relation symbols associated with a positive integer n_R specifying the arity of each $R \in \mathcal{R}$

I assume familiarity with the inductive definitions of $\mathcal{L}$-terms, formulas, and sentences.

Definition 2. An $\mathcal{L} - \text{system } \mathcal{M}$ $(\text{or } \mathcal{L} - \text{model } \mathcal{M})$ consists of:

(i) A nonempty set *M* called the *domain* of $\mathcal{M}$

(ii) An element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$

(iii) A set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$

(iv) A function $F^{\mathcal{M}}: M^{n_F} \to M$ for each $F \in \mathcal{F}$

As in the case of syntax, I assume familiarity with the tarskian truth definition of $\mathcal{L}$-formulas $\varphi(\bar{x})$ and $\mathcal{L}$-sentences φ in a $\mathcal{L}$-system or model $\mathcal{M}$.

In what follows, I will drop the prefix where the signature or language $\mathcal{L}$ is implicit, so that talk about systems $\mathcal{M}, \mathcal{N}$ is always understood as talk about the implicit $\mathcal{L}$-systems or models. Also, I customarily abbreviate tuples of elements or variables, so $\bar{a} = (a_1, .., a_{n_R})$, or $\bar{a} = (a_1, .., a_{n_F})$ and $\bar{x} = (x_1, .., x_n)$. Definition 2 is the first place where I deviate from standard terminology: $\mathcal{M}$ is usually referred to by 'structure' (or 'model'), but this term is highly qualified in the structuralist literature, so I reserve the structuralist meaning to the term and detail its relations with systems and models in due time.

Now, as Hodges observed[4], definition 2 doesn't tell us how these ingredients are assembled, that's the role of axioms. I will generally denote the set of axioms by Λ and add indicative subscripts for specific types or sets of them.

Definition 3. Two systems, $\mathcal{M} = \langle M, c^{\mathcal{M}}, R^{\mathcal{M}}, F^{\mathcal{M}} \rangle$ and N=$\langle N, c^{\mathcal{N}}, R^{\mathcal{N}}, F^{\mathcal{N}}, \rangle$ are isomorphic, in symbols, $\mathcal{M} \cong \mathcal{N}$, if there exists a bijective function $f: M \to N$, such that

(i) for each $c \in \mathcal{C}, f\left(c^{\mathcal{M}}\right) = c^{\mathcal{N}}$

(ii) for each $R \in \mathcal{R}$ and each $\vec{a} \in M^{n_R}, \vec{a} \in R^{\mathcal{M}}$ iff $f\left(\vec{a}\right) \in R^{\mathcal{N}}$

(iii) for each $F \in \mathcal{F}$ and each $\vec{a} \in M^{n_F}, f\left(F^{\mathcal{M}}\left(\vec{a}\right)\right) = F^{\mathcal{N}}\left(f\left(\vec{a}\right)\right)$

[3] some authors use the more appropriate term *signature* - see (Hodges 1997)

[4] 'And second, *exactly* what is a structure? Our definition said nothing about the way in which the ingredients [(i)]-[(iv)] are packed into a single entity. True again. But this was a deliberate oversight - the packing arrangements will never matter to us. [...] The important thing is to know what the symbols and the ingredients are, and this can be indicated in any reasonable way' (Hodges 1997) p. 4

The logical counterpart of isomorphism, elementary equivalence, is defined as follows:

Definition 4. Two systems $\mathcal{M}, \mathcal{N}$ are elementary equivalent, in symbols $\mathcal{M} \equiv \mathcal{N}$, if for all sentences $\varphi \in \mathcal{L}$

$$\mathcal{M} \vDash \varphi \text{ iff } \mathcal{N} \vDash \varphi$$

Definition 5. A theory T is consistent if it has a model, that is, there exists a system $\mathcal{M}$ such that $\mathcal{M} \vDash T$.

Definition 6. A theory T is categorical iff for any two T-models $\mathcal{M}, \mathcal{N}$

$$\mathcal{M} \cong \mathcal{N}$$

Definition 7. A theory T is semantically complete iff for all sentences $\varphi \in \mathcal{L}$

$$T \vDash \varphi \text{ or } T \vDash / \varphi$$

If, as standard, we denote by $Th(\mathcal{M})$, the full theory of $\mathcal{M}$, the set of sentences φ such that $\mathcal{M} \vDash \varphi$, then we can reformulate the previous definition by imposing on the systems $\mathcal{M}, \mathcal{N}$ the condition $Th(\mathcal{M}) = Th(\mathcal{N})$.

The next elementary theorem establishes that $Th(\mathcal{M})$ is invariant under isomorphism.

Theorem 1. Isomorphic systems are elementary equivalent.

Actually, something stronger follows from isomorphism, namely that all open formulas are invariant under isomorphism.

Theorem 2. If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \vDash \varphi(\bar{x})$ iff $\mathcal{N} \vDash \varphi(\bar{x})$.

The next theorem shows that elementary equivalence implies semantic completeness.

Theorem 3. Let T be a theory consistent theory such that for any $\mathcal{M}, \mathcal{N}$ models of T, $\mathcal{M} \equiv \mathcal{N}$. Then T is semantically complete.

The completeness interpretation of the representation theorems

Representation theorems. An informal description

In an effort to describe the representation theorems in a general manner, and to accurately capture the meaning of particular statements of them, some qualifications are in order. First, representation theorems concern mathematical types such as the type of groups, or rings, or graphs, etc., customarily specified by dedicated clusters of axioms. Second, representation theorems target particular classes of systems that can be aptly described as the *intended systems*. Third, a distinction[5] between two kinds of mathematical systems should be employed. On the one hand, systems can be introduced and presented in an austere way, described solely in terms of the relevant signature $\mathcal{L}$ and axioms[6] Λ that they satisfy, with no additional information, e.g. about the identity of the elements in their respective domains. Such systems are called abstract. On the other hand, systems are introduced as exemplifying particular mathematical types, so that the identity, as well as the behavior of their elements are well understood, e.g. when the permutations on a set together with composition of functions are presented as forming the symmetric group. Such systems are called concrete. I will emphasize the aspects of the mathematical practice presented in the following sections that corroborate and substantiate these qualifications.

A representation theorem is understood to be for a mathematical type and relative to a class of intended concrete systems. In this setting, the theorem states that every abstract system is isomorphic to an intended concrete system. Star examples of representation theorems include Cayley's representation theorem for groups (every group is isomorphic to a group of permutations), Stone's representation theorem for boolean algebras (every boolean algebra is isomorphic to a field of sets), and Kuratowski's representation theorem[7]. The pressing issue of the significance of the representation theorems is addressed in the following subsection; it also motivates the focus on Cayley's theorem.

Exegetical evidence for the completeness interpretation

The first author to systematize the interpretations of representation theorems found in the mathematical literature is George Weaver. In his 1998 article[8] *Structuralism and Representation Theorems*, he distinguishes between three interpretations of the representation theorems, the completeness, the minimal generalization and the typical example interpretation, noting that some theorems have been given more than one. The focus of this paper, the completeness interpretation, is attested in (Birkhoff and Mac Lane 1953) *vis-a-vis* Cayley's representation theorem: after proving the theorem, the authors state that it 'can be interpreted as demonstrating the completeness of our postulates on the multiplication

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[5] The distinction made a career in contemporary philosophy of mathematics, especially in structuralism, so is employed both in philosophical and mathematical literature - see the next sections for details.

[6] of a specific mathematical type, of course.

[7] see (Weaver 1998), p. 262 for details

[8] (Weaver 1998)

of transforms'. In the second edition[9], they develop the interpretation, changing assertively its tonality in the process, and take Cayley's theorem to show 'that the axioms for the operation of multiplication in a group imply all formal properties valid for the operation "composition of permutations"'[10]. Notably, Weaver's gloss of this quotation shows the murky state of the interpretation: 'Thus, each 'formal' proposition/property true on all groups of transformations (where multiplication is interpreted as the composition of permutations) is true on all abstract groups; and, hence, follows from the associative, inverse and identity laws. Like Veblen's claim that categoricity implies semantic completeness, the above presupposes that isomorphic systems are indistinguishable in the class of propositions along with the modern definition of logical consequence'[11]. The notion of completeness alluded in both Birkhoff & Mac Lane and Weaver's quotations has to be distinguished from semantic completeness[12]. That much is clear from the elementary observation that the theory of groups can accommodate both abelian and non-abelian groups. Now, it is true that representation theorems establish an isomorphism, but not to the extent of categoricity[13]. But precisely what is this notion of completeness and how exactly follows from the isomorphism involved in the representation theorems remains highly unclear. In the next section, I'll address both issues, thus providing a formal explication of the interpretation. In his article, Weaver goes on to provide textual evidence for two more authors[14] endorsing the completeness interpretation for other representation theorems: Sikorski, in addressing the significance of Stone's representation theorem for Boolean algebra, and Stoll, in considering Kuratowski's representation theorem for partially ordered sets. These last additions suggest a lack of predilection for a particular representation theorem in conveying the completeness interpretation. However, as I'm going to show in the remainder of this subsection, meanwhile, a notable exegetical evidence surfaced, in which the proof of Cayley's theorem is purposely aligned to the goal of proving the completeness of the axioms of group theory. This explicit case-study in the completeness interpretation is the reason for focusing on Cayley's theorem.

9 (Birkhoff and Mac Lane 1967)
10 (Birkhoff and Mac Lane 1967) pp. 97-98.
11 (Weaver 1998) p. 263
12 as defined in the technical preamble. Weaver certainly is aware of the difference between the two notions, as can be seen by his assessment that 'like Veblen's claim that categoricity implies semantic completeness' the interpretation presupposes that isomorphism implies propositional indistinguishability.
13 there is also indirect evidence for this, representation theorems do not imply semantic completeness
14 see (Weaver 1998), pp. 263-264 for details.

Completeness of axiomatization via Cayley's theorem

In his dedicated video[15] on it, Richard Borcherds emphatically presents Cayley's theorem as proving the completeness of the axioms of group theory. In fact, in opposition to the other advocates of the interpretation, Borcherds doesn't fill with a scarce completeness moral the after-proof of Cayley's theorem, he states the issue of the completeness of the axioms of group theory followed by the role of Cayley's theorem as providing a proof of it right from the beginning of the exposition. Only then he proceeds to prove a purposely altered version of Cayley's theorem as a way of securing the sought completeness. As in his courses[16], he begins by defining the notion of a 'concrete group' as 'the set of symmetries of something'[17], for example the symmetries of a rectangle, followed by the standard definition of groups, under the label 'abstract group'[18]. He then remarks that the former definition easily 'translates' into the latter, and posits the problem of the converse: 'It is clear that if we take the composition of symmetries as our binary relation, then the concrete notion of a group can be translated to the abstract notion. It is a *subtle and important point* (my emphasis) that the converse is true[19]'. And, again, in his 2005 lectures: 'The first, "concrete" definition can be thought of in the second, "abstract" way by making the operation composition of symmetries. Is the reverse true? Cayley proved that the answer is yes' [20]. In the video, Borcherds explains the 'subtle and important point' in terms of securing the completeness of the axioms of group theory: the question of the converse, namely, whether an abstract group 'is [...] the symmetries of some object', is equivalated ('in other words', as he puts it) to that of finding '*all* (my emphasis) the axioms for a group that we need'[21].

It is instructive to sweep through Borcherds's dedicated-to-the-goal-of-completeness proof of Cayley's theorem, and the first thing to note is the essential use of group actions.

Definition 8. A *left action* of a group G on a set S, $G \curvearrowright S$, is a map $\cdot : G \times S \rightarrow S$ such that:

1. $g \cdot (h \cdot s) = (gh) \cdot s$, for all $g, h \in G, s \in S$
2. $e \cdot s = s$ for all $s \in S$

[15] (Borcherds 2020)
[16] (Borcherds 2017, 2005)
[17] (Borcherds 2005, 2017)
[18] (Borcherds 2017)
[19] (Borcherds 2017) p. 6.
[20] (Borcherds 2005)
[21] (Borcherds 2020), 1:05 - 1:14

Now, a group G acting on a set S induces a homomorphism[22] $\sigma: G \to Sym(S)$ defined by $\sigma(g) = \pi_g$, where $\pi_g: S \to S$ is the permutation associated with each $g \in G$ by $\pi_g(s) = g \cdot s$. The proof of these claims is easy and can be found in (Dummit and Foote 2004) p. 42. The converse direction, from a homomorphism σ to the induced left action, $G \overset{\sigma}{\curvearrowright} S$, is, again, an easy exercise. The core of the proof consists in finding 'some object S such that G comprises the symmetries of S'[23]. Actually, the 'finding' is implemented as a construction of such an object from the action of G, considered as a group, to G, considered as a set S (=G). A routine check shows that $G \curvearrowright S(= G)$ given by $g \cdot s = gs$, where gs is the group operation, is well-defined, and it induces the homomorphism $\sigma: G \to Sym(G)$ defined by $\pi_g: G \to G, \pi_g(s) = g \cdot s = gs$. However, as Borcherds explains in the video, S (=G) has to have some structure, in order for *all* its symmetries to be exhausted by G. This additional structure is provided by the right action of G that the symmetries must preserve.

A *right action* is defined correspondingly to a left action and, obviously, generates corresponding results. A symmetry $f: S \to S$ preserves the right-action iff $f(s \cdot g) = f(s) \cdot g$ for all $g \in G, s \in S$. Associativity ensures that every left action of a group acting on itself preserves the right action[24]. So, G acting on S (=G) by the left action is subsumed under the symmetries of the object S (=G) with the additional structure given by the right action of G on itself. All that remains to show is that the converse also holds, i.e., every symmetry $f: S \to S$ that preserves the right action corresponds to an element of G considered as a group. The point of the proof is to look for the element $g \in S(= G)$ to which such a symmetry f maps $e \in S(= G)$, i.e. $f(e) = g$. Then, for all $s \in S(= G), f(s) = f(e \cdot s) = f(e) \cdot s = g \cdot s$. The first equality is justified by the identity $s = es$, the second equality by the condition that f preserves the right action, and the third by the supposition that $f(e) = g$. So any symmetry f is just multiplication on the left by an element $g \in G$. Thus, 'G is exactly the symmetries of G preserving the right action of G' [25]. In both lectures, Borcherds ends the proof with a recipe for constructing a[26] Cayley graph of the 'structured object' G plus the right action component, which has all its symmetries given by the left action of G: 'We can picture G as a graph, where the elements are vertices, and the edges between elements are labeled by their right actions. Then the left action

[22] This homomorphism σ is also known as the permutation representation of the action.
[23] (Borcherds 2017) p. 7
[24] This can be easily seen by setting $f(s) = g \cdot s$
[25] (Borcherds 2005) p. 8.
[26] There are multiple definitions of what constitutes a Cayley graph - see (Pegg, Rowland, and Weisstein n.d.)

of G gives the symmetries of the graph'[27]. In the video, he presents two color-coded examples of such graphs. Immediately afterwards[28], he summarizes the moral of the recipe for constructing 'a Cayley graph from any group [such that] the group is then the group of symmetries of the Cayley graph' by writing that the 'axioms of group capture [the] concept of [the] "symmetries of an object"' and adds that 'anything we can prove about all symmetries of an object can be deduced from the axioms; we haven't missed out any vital axiom'[29]. A similar procedure is hinted[30] in Wilfrid Hodges's *A shorter model theory*. *Exercise 4.1.1* asks to '[s]how that for every abstract group G there is a structure with domain G whose automorphism group is isomorphic to G'. The hint on how to proceed aligns with Borcherds' recipe: 'Let X be a set of generators of G, and for each $x \in X$ introduce a function $f_x: g \to g \cdot x$ on the set G. Consider the structure consisting of the set G and the functions f_x. *This structure is essentially the* ***Cayley graph*** *of the group G*' (emphasis in original).

This cluster of expositions, by means of explicit definitions for abstract and concrete groups, testifies for and solidifies the corresponding general distinction, mentioned in the first subsection, between abstract and concrete systems. Explicit in Borcherds's presentation of Cayley's theorem, and, *mutatis mutandi*, in any discussion on the significance of representation theorems, the distinction is, in a sense, inherent to their formulation. In fact, in his 2005 lectures, the first theorem[31] proven by Borcherds, theorem 1.1., concerns exactly the equivalency of the two systems in the case of group theory. The left to right direction is, essentially, Cayley' theorem. This way of framing Cayley's theorem is by no means peculiar or idiosyncratic. (Cameron 2008), in the second edition of his *Introduction to algebra*, notes that '[b]efore the rise of the axiomatic method', 'a group was either a permutation group (whose elements are permutations of a set, and whose operation is composition of permutations), or a matrix group (whose elements are matrices, and whose operation is matrix multiplication). In modern terminology, we could say that the early group theorists studied subgroups of the symmetric group Sym(Ω) or of the general linear group GL(n, F)[32].' He then goes on to prove a variant of Borcherds' theorem 1.1.: 'In order that this body of knowledge should not be lost, it is necessary to ensure that the new groups (axiomatically defined) are really the same as the old ones. We already showed [...] that the symmetric group and the general linear group

[27] (Borcherds 2017) p. 8.
[28] (Borcherds 2020) 23:00
[29] (Borcherds 2020)23:00-23:38
[30] (Hodges 1997), p. 100.
[31] A set G is an abstract group iff it is a concrete group. (Borcherds 2005) p. 7
[32] (Cameron 2008, p. 132).

are groups in the axiomatic sense, and hence their subgroups are too. The point of Cayley's Theorem is to show the converse of this for permutation groups: that is, every group 'is' a permutation group'[33].

Obviously, what Cameron's describes as the object of study of the early group theorists, subgroups of the symmetric group Sym(Ω) or of the general linear group GL(n, F), are 'concrete' groups in Borcherds's sense, and the 'axiomatically defined' groups are essentially Borcherds's 'abstract' groups. This last correspondence points to the general significance of the axiomatic approach in shaping abstract systems. In the next section, I'm going to recover this significance in model-theoretic terms giving a distinct role to the axioms of a mathematical type.

Embroidering on Cameron's comment, in order that this body of evidence not be lost in irrelevance relative to the first subsection, let's remark the ample support it provides for the other two qualifications succinctly described there. First, Cayley's theorem is fundamentally and categorically about the type of groups. This is evident from the very formulations presented above, also, from some descriptions of the theorem as 'Cayley's representation theorem for groups' [34]. Second, again, the very formulations of the theorem highlight particular classes of systems as intentional: in Borcherds's framing, those that fall under the notion of 'symmetries of an object', and in Cameron's case, the traditional permutations of a set.

Completeness of axiomatization

All these considerations are formally distilled in the following definitions and results. The connection to the informal discussions and arguments presented in the previous sections are easy to spot as I am following them closely.

Definition 9. $\varphi\left(\vec{x}\right) \in \mathcal{L}_{\mathbf{T}}$ expresses a $\mathbf{T}$-property or a structural or formal property P of $\mathbf{T}$-concrete systems S_C if for all S_C, $S_C \vDash \varphi\left(\vec{x}\right)$. Alternatively, denoting the appropriate closure of $\varphi\left(\vec{x}\right)$ by $cl\varphi\left(\vec{x}\right)$, $\varphi\left(\vec{x}\right) \in \mathcal{L}_{\mathbf{T}}$ expresses a $\mathbf{T}$-property or a structural or formal property P if $cl\varphi\left(\vec{x}\right) \in \bigcap_{S_I} Th(S_I)$.

Definition 10. An axiomatization $\Lambda_{\mathbf{T}}$ is complete relative to the $\mathbf{T}$-intentional concrete systems S_I if for all $\varphi\left(\vec{x}\right) \in \mathcal{L}_{\mathbf{T}}$ expressing $\mathbf{T}$-properties of S_I, $\Lambda_{\mathbf{T}} \vDash \varphi\left(\vec{x}\right)$.

33 (ibidem)
34 see, for example, (Givant 2017)

The previous definition is quite trivial, as can be seen by considering abelian permutation groups, S_{APG}, to be the intentional concrete systems, and their axiomatization, $\Lambda_{\mathbf{G}}$, to be the standard group axiomatization.

Then, $\Lambda_{\mathbf{G}}$ is not complete relative to **G**-systems S_{APG} for obvious reasons: $\varphi(x_1, x_2)$, expressing the commutative property, is a **G**-property of S_{APG}, although $\Lambda_{\mathbf{G}} \nvDash \varphi(x_1, x_2)$. That is, the axioms of group theory are not complete *vis-a-vis* abelian groups.

Definition 11. Call *abstract* a standard **T**-model-theoretic system

$$\mathcal{M}_{\mathbf{T}} = \langle M, c_1^{\mathcal{M}}, c_2^{\mathcal{M}}, \ldots, c_k^{\mathcal{M}}, R_1^{\mathcal{M}}, R_2^{\mathcal{M}}, \ldots, R_n^{\mathcal{M}}, f_1^{\mathcal{M}}, f_2^{\mathcal{M}}, \ldots, f_m^{\mathcal{M}} \rangle$$

satisfying only the axioms $\Lambda_{\mathbf{T}}$.

With these utensils at hand, we can recast the formulation of a representation theorem as:

Theorem 4. Every $\mathcal{M}_{\mathbf{T}}$ system is isomorphic to a **T**-intentional concrete system S_I.

As a consequence, *via* theorem 2 every formal property P of intentional systems S_I is a property of their abstract model-theoretic counterparts $\mathcal{M}_{\mathbf{T}}$.

From this consequence, the justification of the completeness claim follows easily.

Theorem 5. Let $\Lambda_{\mathbf{T}}$ be the axiomatization of the mathematical type **T** such that every abstract system $\mathcal{M}_{\mathbf{T}}$ system is isomorphic to an intentional system S_I. Then $\Lambda_{\mathbf{T}}$ is complete.

Proof. Given that for all $\mathcal{M}_{\mathbf{T}}$ there is a **T**-intentional system S_I such that $\mathcal{M}_{\mathbf{T}} \cong S_I$, and that open formulas are invariant under isomorphism, it follows that $\mathcal{M}_{\mathbf{T}} \vDash \varphi(\bar{x})$, for all $\varphi(\bar{x})$ such that $cl\varphi\left(\vec{x}\right) \in \bigcap_{S_I} Th(S_I)$ so, according to the definition of completeness of axiomatization, $\Lambda_{\mathbf{T}} \vDash \varphi(\bar{x})$ □

Conclusions

There are three relevant consequences that follow almost immediately from the proposed formalization and result.

1. Categoricity theorems can be construed as limit cases of representation theorems. If one intends to axiomatically capture just one system S_I, then *theorem 4* implies categoricity: there is just one isomorphism class of systems, that of S_I. For example, in this setting, Dedekind'categoricity theorem can be restated as: Every PA2-system is isomorphic to the standard system.
2. Semantic completeness implies completeness of axiomatization. This can be easily proved using the alternative of *definition 9* and noting that by semantic completeness $Th(SI) = Th(Sj)$ for all concrete systems S_I, S_j.

3. The distinction between concrete and abstract systems and the way in which abstract systems are construed offer support to non-eliminative forms of structuralism, especially to the structuralist thesis that positions in pure structures have no mathematically relevant non-structural properties. As emphasized and argued the distinction is at the heart of a representation theorem. Furthermore, the essential trait of abstract systems is that of having no other discernable properties over and beyond what the axiomatization prescribes. Their elements are akin positions in pure structures and their properties and relations are confined to $\Lambda_{\mathbf{T}}$. Of course, this observation amounts to nothing more than a confirmation from mathematical practice of the non-eliminativist stance in the philosophy of mathematics. It is by no means an indication of what the pure structures are – for technical reasons, such as the Burali-Forti paradox, pure structures, whatever they may be, should have a different regime than systems[35].

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[35] A standard maneuver for setting apart pure structures from systems, in the non-eliminativist literature, is to endow the former a *sui-generis* status