STUDIA UBB. PHILOSOPHIA, Vol. 63 (2018), 2, pp. 19-30
(RECOMMENDED CITATION)
DOI:10.24193/subbphil.2018.2.02

MUSIC AND MATHEMATICS: SOME REMARKS ON THE IDEAS OF SPACE AND FORM

CRISTINA CRISTE[*]

ABSTRACT. Mathematics has been used to describe, analyse and create music for millennia. The use of specific mathematical apparatus is manifest in the organization of basic musical notions into forms or objects that inhabit a sonorous landscape. The collecting of musical units into a set constitutes a starting point for analysing a musical object under different mathematical structures, and although there is no unified approach, musical analysis and composition use geometry and discrete mathematics to describe different musical relations (particularly group theory) in the process of modelling the basic elements of music, such as scales, intervals, chords or rhythms.

Keywords: music, mathematics, sound, space, form

1. Introduction

In the last centuries, the description of basic music-theoretical terms underwent an increasing level of abstraction and generalization under the influence of profound changes in physics, but especially mathematics, an endeavour assumed by musicians, composers, and mathematicians.

This late involvement of mathematics is not synonymous with the Pythagorean tradition, still found in some musical interpretations, that music is number, or *sonorous number*. The new relation between mathematics and music[1] is rather found in the models offered by the first. In other words, music is not mathematics, but it is mathematical, such that the organization of musical sound can be described by a strict mathematical structure.

[*] Babeş-Bolyai University Cluj-Napoca; Center for Applied Philosophy, Faculty of History and Philosophy, Babeş-Bolyai University.

[1] Predominantly characterizing the 20th century, but found in previous centuries as well. For a more comprehensive approach in this regard, see J. Fauvel & R. Flood & R. Wilson (eds.), *Music and Mathematics. From Pythagoras to Fractals*; Arbonés & Milrud, *L'harmonie est numérique. Musique et mathématiques.*

Still, there is no unified mathematical perspective with regard to music, particularly a geometrical one, although there are theories more encompassing than others, ‘encompassing’ here referring to the depth and power of the mathematical tool and/or the breadth of the musical implications[2].

Such an approach acknowledges the fact that every object, including strings, vibrate at certain frequencies and that the waves dissipate, radiate away energy as heat and/or sound, without transmitting matter, through a medium such as air. If energy is not continuously supplied to the vibrating object, it will get to its equilibrium position.

2. Mathematical music

The movement of the wave is organized and described as a characteristic shape, further associated to a particular sound source. But the simple harmonic motion, characterizing the vibrating motion of sound is fundamental not only to music, but to the quantum mechanical motion of the atom as well.

In mathematical parlance, such a motion is called sine motion, while its every physical approximation, including the sound, is considered sinusoidal. And being based on the circle, “sine motion is a timeless description of motion having no beginning or end”[3].

Music becomes possible in the process of distinguishing discrete events in this idealised motion, one further connected with the physically abstract notion of a *continuous flow of sound*. And such a discrete event is called *tone*, “probably the fundamental unit of musical experience"[4], with music also described as a set of tonal structures.

The abstraction of the tonal space, allowing the spatial representations of tonal distance and tonal relationships, was first introduced as a conceptual diagram named *Tonnetz* (German for *tone-network*) by Euler in 1739, was further developed to explore the properties of pitch structures, and has even been used to visualize non-tonal triadic relationships (by Neo-Riemannian[5] theorists).

When a tone, characterized by 3 sonic qualities - pitch, musical loudness, and timbre - is combined with the temporal qualities of onset and duration, it becomes a *note*. The combination of notes under the temporal dimension determines

---

[2] For example: D. Lewin, *Generalized Musical Intervals and Transformations*; G. Mazzola, *The Topos of Music*; D. Tymoczko, *A Geometry of Music.*

[3] G. Loy, *Musimathics. The Mathematical Foundations of Music*, Vol. 1. MIT Press, 2006, p. 7.

[4] *Ibidem*, p.12.

[5] The notion alludes to Hugo Riemann, the music theorist and composer, and not Bernhard Riemann, the mathematician.

three contexts of analysis: (i) *the musical score*; (ii) *the melody*, as notes performed in sequence; and (iii) *harmony*, when the focus is the simultaneous performance of the notes.

Pitch is an essential organizing principle, generally defined, not unambiguously, as the "auditory attribute of sound according to which sounds can be ordered on a scale from low to high (ANSI 1999)"[6]. The complexity of the analysis lies in the meaning of that "auditory attribute". Acoustically, the pitch can also be represented by the logarithm of its fundamental frequency.

A sound has a pitch if its wave shape – represented as a Fourier series - is highly redundant through time. Otherwise, it is said, we hear noise. The existence of pitch already seems to involve a structure. So there is sound and there are sound structures.

Now, of course, there is always the question: when does sound become music? According to John Cage's "4'33''", any sound can constitute music. But what the notion of pitch already implies is that sound might represent structures different from what is generally called music. That being said, shaping the nature of their connection requires further distinctions.

The difference in pitch between two tones determines the interval, which is perceived regardless of the degrees of variations in sounds' frequency, duration, timbre and amplitude. "Alterations of pitch in melodies – Helmoholtz wrote - take place by intervals and not by continuous transitions"[7].

The pitches create therefore an acoustic landscape determined by range, density and a subjective sense of distance characterized as height or width, constituting the *chroma*. The 'subjective' aspect is determined by the fact that a specific interval is experienced as *higher*, *lower* or *wider* than another.

Using the language of set theory[8] to categorize musical objects and describe their relationships, a class of pitches is considered an unordered subset of *n* objects, an ordered subset is a permutation, and the basic operations are transposition and inversion. The approach is based on the idea, which started with Riemann in geometry and then Cantor in set theory, that the same mathematical object (the manifold, a set) could not only carry different structures – geometries, - but there could also be different notions of equivalence between objects and structures.

---

[6] Loy, *op.cit*., p. 13.

[7] Helmholtz, *On the Sensations of Tone. As a Physiological Basis for the Theory of Music*, Dover Publications, New York, 1954 p. 250.

[8] Not set theory per se. *Musical set theory* is a rather improper term, since such an approach is more related to group theory and combinatorics than to mathematical set theory. And even though one is able to use the vocabulary of set theory to talk about finite sets, there are many differences between methods and terminology in them. Musicians, for example, use the terms *transposition* and *inversion* where mathematicians would use *translation* and *reflection*.

The mathematical transposition translates the restatement of a melody at higher and lower pitch levels in a way that preserves intervals. Inversion represents another way to create variation into a musical piece while preserving the intervallic sound of a melody, but not the exact intervals.

An example of the way in which finite sets play an important role in music analysis, in addition to group theory, is Messiaen' *Chronochromie* (1960). Messiaen uses the term "interversion" to denote a particular ordering of permuted elements, and the term "permutation" to refer to this technique, added to the two-item list of *mathematical impossibilities* mentioned in his *Traité de rythme, de couleur, et d'ornithologie:* the modes of limited transposition" (*modesà transpositions limitées*) and "non-retrogradable rhythms" (*rythmes nonrétrogradables*)[9]. The permutation is applied to a sequence of musical objects (pitches, durations, etc.) to obtain another sequence, but one has to choose among the great number of symmetries that a significant number of objects could determine. In the end, Messiaen creates a collection of rhythms obtained by the iterated application of a given symmetrical permutation to a collection of 32 durations.

A further determination within this range of available pitches, the *chroma*, involves the selection of a small subset of intervals that would become the pitch classes of a *scale*.

A musical scale can thus be defined as "an ordered set of pitches, together with a formula for specifying their frequencies", with each individual pitch called *degree*, and all degrees forming an "ordered set of names and positions for the scale pitches"[10]. Those pitch classes are replicated across the space of each octave, determining the gamut of pitches.

The diatonic scale, probably the most recognizable, known and influential of the scales, expresses but one of the many possible orderings of intervals. The chromatic scale extends the diatonic scale by breaking up the whole steps into half steps. Or, as Gareth Loy pointed out, the unique order of whole and half steps characterizing the extended diatonic scale – the chromatic scale -, "provides a crucial asymmetry that our hearing exploits in order to orient ourselves to the music we're hearing. If the interval pattern were not asymmetrical, it would be impossible for us to orient ourselves in the scale"[11]. In the words of Pierre Curie, "it's the dissymmetry that creates the phenomenon".

And like a mathematician facing the choice of a geometry or other, in conformity with particular interests and circumstances, one is free to choose any

[9] O. Messiaen, *Traité de rythme, de couleur, et d'ornithologie*, *Vol. I*. Trans. by Melody Baggech, Norman, Oklahoma, 1998.

[10] Loy, *op. cit.*, p. 16, *passim*.

[11] *Ibidem*, pp. 17-18.

musical ordering that serves her needs, although diatonic ordering has had a considerable influence on the music around the cultures of the world.

We can visualise the pitch space of the 12-tone equal-tempered scale – which constitutes the dominant tuning system in the West – as a circle (the circle of fifths), or even as a spiral if we continue to add sharps and flats. The point to be emphasized is that the key structure can be characterized in terms of circularity.

But the orientation in the pitch space is not as straightforward as it seems, an aspect noticed in an early research by Roger Shepard[12] – hence the *Shepard illusion*, and it involves the difference in experimenting the chroma, on one hand, and the tone height, on the other. In playing a sequence of major fifths intervals on a piano, starting from the lowest tone, the sequence seems to ascend if one focuses on the tone height, but it appears to decrease by semitones if the attention falls on the chroma. Hence the auditory counterpart to the impossible staircases of Escher. Or the Necker cube.

According to Callender, Quinn and Tymoczko, it is possible though to measure musical distance in other ways than the intervallic conception, when we consider chords to be close or even identical, since each contains the same total collection of intervals[13].

It is helpful to point out the fact that if we describe some notes to be close because the ratio of their fundamental frequencies can be expressed using small whole numbers, we have an *acoustic* conception of musical distance that, together with the intervallic one, represents a *perceptual* model of musical distance.

On the other hand, a*conceptual model* in the experience of hearing music is one that measures music melodically, using distance in log-frequency space, an approach based on the idea of *voice leading*, developed in recent years by Callender et al[14].

That being said, when we hear a melody consisting of several pitches, what we actually perceive are the intervals between the individual notes, with the relationship between these intervals making a melody, more or less appealing.

In other words, if we consider the interval as the distance between two notes, we hear the distance. But since the interval could also be described as the numerical proportion between the frequency of the two different notes, it seems that we are prone to perceive somehow an exponential relation between pitch

[12] R. N. Shepard, "Circularity in judgments of relative pitch", in *Journal of the Acoustical Society of America*, 36/ 1964, pp. 2345-2353.

[13] C. Callender & I. Quinn & D. Tymoczko, "Generalized Voice-Leading Spaces", in *Science*, Vol. 320, Issue 5873/ 18, Apr. 2008

[14] *Ibidem* for the distinction among different conceptions of musical distance.

and frequency. "Frequency *f* goes up exponentially as pitch *p* goes up linearly: to double pitch, we must quadruple frequency"[15].

The unison expresses identity, the octave expresses equivalence, but the singularity of the other intervals (minor, major or perfect) is determined by the subjective perception of *distance* (in terms of 'higher', 'lower', 'wider'), despite significant variation in other sound properties, such as duration, amplitude, frequency, or timbre.

With regard to the frequency in the production of articulations in the emission of sounds, we get *rhythm*, the understructure of a melody, the distribution of pitch and intensity in sequences, or the organization of various dynamics of sounds and silence in time. As fundamental property of every perceived, aware, determined pattern of sound, its *presence* acknowledged at every level of encounter with a musical element (and not only), it represents a fundamental condition for a mathematical analysis of music. Due to its forms, it involves different types of temporality and, simultaneously, various ways of pacing the acoustic space.

As J. Arbonés & P. Milrud pointed out, "it is possible to compare this situation with the general mathematical problem of tessellating the plane, that is, when one has to tile an entire plane with regular geometrical forms. In our case, what we desire to cover completely is the sonorous plane"[16].

We perceive therefore a movement through different forms determined as ratios (of frequency), and together they are shaping, at another level, the form of the melody, through rhythm. Forms creating and/or determining other forms.

The question is, is it possible to perceive the transformation of different forms or musical objects and their perception? The question involves several aspects:

- firstly, it is important to address the concept of continuity;
- secondly, it involves the composite notion of movement and change;
- and thirdly, it points to the context in which ideas of continuity and motion are valid events; in other words, it points to the musical space organized as score.

On one hand, there is a continuity when considering the passage from sound to music. On the other hand, there is another idea of continuity in the movement among the elements of a specific piece of music, irrespective of the temporal and spatial aspects; or between different sound patterns. David Lewin's *transformational theory*[17], in applying group theory to music, is particularly appropriate in the analysis of the musical motion.

---

[15] Loy, *op.cit*., p. 15

[16] J. Arbonés & P. Milrud, *L'harmonie est numérique. Musique et mathématiques*. RBA France, 2013, p. 48.

[17] See David Lewin, *Generalized Musical Intervals and Transformations,* Oxford University Press, [1987] 2007.

In this second type of continuity, music becomes – it is becoming, it becomes in the specific movements and rhythm of its forms, even if such a form is silence – silence in all its musical distinctions (the types of rest; the hold or pause; fermata, or caesura). That is because as part of a composition, silence is determined in time, just like a unit fragment on the real line that simultaneously represents a set of infinite cardinality (of the continuum).

In mathematics, the analysis of the definition of continuity – from the ϵ - δ definition, through that involving open intervals, open discs and then in terms of the set of open subsets - led to the institution of metric and topological spaces. This development proved that continuity, viewed invariably, depends not on distance, but on something more intrinsic.

Each level of abstraction in this development describes space as a collection of points with an added structure satisfying certain axioms, and with topology as the most abstract of these structures or geometries.

There are recent studies that "interpret a scale as defining a metric according to which adjacent scale tones are one "unit" apart; scalar transposition is just translation relative to this metric"[18].

How did topology come into music? An example, analysed by Richard Cohn, Jack Douthett and Peter Steinbach, and Thomas Fiore is Beethoven's *Ninth Symphony*, specifically the measures 143-176 of the second movement, which contain an extraordinary sequence of 19 chords. There are certain conversions by some functions of these chords with finally obtaining a graph that makes a torus, so the chord progression from Beethoven is a path on it. For a complete torus we would have needed 24 chords. Another example of music on a topological object is Bach's *Musical Offering*, which contains a passage that is music on a Möbius strip.

Now returning to the abstract structures of the musical space, it is possible to describe some rigid and continuous motions named symmetries. They represent the transformations that leave a pattern in space unchanged, and they are modelled as functions acting on the entire space, meaning that every transformation must be applicable to every object.

In geometry, the study of an object, of a specific space involves the description of its possible transformations. Geometers study a space by describing its possible transformations. A musical space can be approached as a mathematical one, since an object, including a musical one, is defined by its symmetries. So we analyse those properties that remain invariant under its possible transformations;

[18] C. Callender & I. Quinn, & D. Tymoczko, *op.cit*., p. 2.

and a pattern in space by asking what transformations leave the pattern unchanged, what symmetries the pattern has.

Now symmetry, as Saunders Mac Lane notes, "is neither geometric nor algebraic; or perhaps both"[19]. Its essence, as Thyssen and Ceulemans pointed out, is "[T]hinking *difference* and *identity* together"[20]. It doesn't have to depend on numbers, and it basically shows that algebra and geometry have in common some underlying, more abstract, form: the abstract structure of a mathematical group, which is particularly important when applied to music because it allows us to see/conceive things that we otherwise would not be able to see. This property of groups to make visible the hidden connections and ultimately to reveal hidden connections among phenomena is probably not far from Paul Klee' principle, "Not to render the visible, but to render visible".

Group theory describes the way that sets and pitches relate and how they can be transformed from one to the other, it shows how different musical voices move against and through each other and, ultimately, how sound, including the silences, creates or is organized into a musical space.

The basic musical transformations – which constitute the elements of a group and can be used to analyse both tonal and atonal music – are: translations, rotations, reflections and glide reflections, although in practice, composers usually use only reflection on two lines and rotation through 180º. The transpositions and inversions (of intervals, chords, melodies, or voices in counterpoint) mentioned above also form a group in the mathematical sense of the word.

All these transformations form a group by the act of group composition, of interacting with each other.

Another way would be to take a set of functions whose inputs are major and minor chords and whose outputs are major and minor chords.

With regard to the dimensions of the space in which to study these transformations, they are described in terms of the basic theoretical musical concepts mentioned above. What varies are the dimensions chosen in a specific analysis. Most often, there are two dimensions - time and pitch, or loudness and pitch, although some focus on sounds defined by their continuously changing time, pitch and loudness, *i.e.* as points of a 3-dimensional manifold.

But there are some limits in the analogy. Helmholtz noted a difference between the geometric manifolds, where it is possible to compare distances between points everywhere ("*i.e*. we can freely move a rigid measurement body

[19] Saunders Mac Lane, *Mathematics. Form and Function*, Springer-Verlag, 1986, p. 19.

[20] P. Thyssen & A. Ceulemans, *Shattered Symmetry. Group Theory from the Eightfold Way to the Periodic Table*, Oxford University Press, 2017, p. 33.

everywhere over the manifold"), and the physiological manifolds, the 2-dimensional acoustical manifold for example, where there is no comparability of distances between points everywhere, where "[T]wo sounds of the same pitch and different volumes are not comparable to two sounds of different pitch but the same volume"[21].

In recent years, some researches[22] proposed an approach that would subsume a great variety of previous geometric models. They started by expressing different musical terms as symmetries of an *n*-dimensional space, where each dimension representing a voice in the score and then creating specific mathematical spaces, singular quotient spaces, named *or bifolds*, by identifying (or "gluing together") points in $\mathbb{R}^n$.

There are many musical examples[23] expressing these transformations that do not change the distance between two points in the plane, isometries that do not alter the scale of distances (the metric). We take a musical motif – and many tunes express an asymmetric one - as consisting of a set of notes described by a pitch over time, that is, a subset of $\mathbb{R}^2$, the 2-dimensional space.

Rachmaninov's remake of Paganini's *Capricio no. 24* for violin solo, *Rhapsody on a Theme by Paganini*, Op.43, creates a chromatic *inversion* by changing minor to major[24] (variation 18), although he follows Paganini's score bar by bar. He also imposes a slight change in rhythm and, at one point, there is an octave jump.

Hindemith's *Ludus tonalis* contains a rotational symmetry, through 180º: with the exception of the last chord, the final movement is the same as the first, but rotated through 180º.

Brahms's *Intermezzo in E minor, Op. 116, no.5* is an example that Callender et al. use to show how a symmetry, though hard to spot in the musical notation, but rather easy to hear, becomes obvious in the geometrical representation[25]. The researchers identified two pairs of voice leadings, represented as vectors, to measure the similarity of musical progressions. And since "the two pairs of vectors are related by reflection, the figure also illustrates the Möbius strip's nonorientability"[26].

---

[21] See K. Mainzer, *Symmetry and Complexity. The Spirit of Beauty of Nonlinear Science*, World Scientific Publishing, 2005, p. 80, *passim*.

[22] See C. Callender & I. Quinn & D. Tymoczko, *op.cit*.; R. Wells Hall, "Geometrical Music Theory", in *Science*, Vol. 320, Issue 5874/ 18, Apr. 2008 pp. 328-329.

[23] See, for example, Wilfrid Hodges, "The Geometry of Music", in J. Fauvel & R. Flood & R. Wilson (eds.), *Music and Mathematics. From Pythagoras to Fractals*, Oxford University Press, 2003, pp. 91-111.

[24] The A minor of Paganini theme becomes D♭ major.

[25] See the *Supporting Online Material* for Callender et al., op.cit.

[26] *Ibidem*, p. 15.

3. Conclusion

Beyond the organizing ideas of rhythm, musical tone, pitch, loudness, timbre, musical note, interval, then melody and harmony, there is a continuous flow of sound permitting the configuration and the description of determinate, discrete patterns known as music. That being said, it should also be emphasized the fact that although the substratum of sound is present in music, it is present in the diversity of languages as well.

Still further beyond this underlying stream of continuously changing fluctuations there is the simple harmonic motion of vibrating objects, abstractly and mathematically represented by the ideal concept of the timeless sine motion. In other words, change and timeless form, constant ratios of continuous movement and perceived rest permeates the various manifestations of music and the act of its creation.

The employment of mathematics in analysing music unveils structures that are inherent in all its expressions. As a result, it is possible to underlie musical aspects otherwise less discernible, but also to secure a specific basis for the act itself of creation. What is more, the use of the same mathematical theories and models to pinpoint basic facts and principles in music and different sciences could constitute a starting point to various philosophical interrogations. It certainly refines Nietzsche's idea of music as a continuous endeavour to get to a specific intuition of things, to think the world in the most universal form of being.

The organization of vibrations as music into discrete patterns already delineates a specific, sonorous space, to be mathematically characterized by certain dimensions. The choice of basic musical objects delineates the space.

It is possible to acknowledge this space without the comfortable aid of time, particularly when we approach music through its notation. In its notational form and like any other scriptural act, including the mathematical one, it offers clues and indications to connections not 'visible' as such.

We meet then another way to perceive its articulations. But the question remains: is music grasped among its articulations, *articulation of forces, not of thing*s[27], or through them? The understanding of the passage from sound to music, to specific tonal structures, involves not only a thoughtful aesthetic interrogation, but also an epistemological examination regarding the cognition or perception of continuity.

[27] To reprise Deleuze's formula in his discussion of Cézanne. See Gilles Deleuze, *Francis Bacon. Logique de la sensation*, Éditions du Seuil, 2002

Within this space with a determinate number of dimensions, to be chosen, a musical object is reduced, in its description, to a determinate set of points. But a point is also an idealized concept in mathematics, it doesn't have extension: the physical dot and the mathematical point are not identical. Considering Dedekind's definition of the real number[28] as a cut, 'gap' in the rational numbers, a "point" is a whole universe to someone who knows that it is actually a cut in the infinite set of rational numbers"[29].

In this complex process of abstraction, both music and mathematics curl around in their expressions such as to signal various registers of understanding. But their connection is just as salient in the moment when both come "alive", become "part of our experience", to use Keith Devlin's words: "the symbols on a page are just a representation of the mathematics. When read by a competent performer (…) the mathematics lives and breathes in the mind of the reader like some abstract symphony"[30].

So maybe the idea of music itself as a spatial structure of sonic forms developing in time is a construction by which one seeks to grasp movements at different levels of description, a process of discovery that must spiral though models that prove continuously incomplete.

BIBLIOGRAPHY

Arbonés, Javier & Milrud, Pablo, *L'harmonie est numérique. Musique et mathématiques*, RBA France, 2013

Callender, Clifton, & Quinn, Ian, & Tymoczko, Dmitri, "Generalized Voice-Leading Spaces", in *Science*, Vol. 320, Issue 5873/ 18, Apr, 2008, AAAS, pp. 346-348, DOI: 10.1126/science.1153021

Callender, Clifton, & Quinn, Ian, & Tymoczko, Dmitri, "Online Supporting Material for Generalized Voice-Leading Spaces", http://science.sciencemag.org/content/suppl/2008/04/17/320.5874.346.DC1.

Cohn, Richard L., "Neo-Riemannian Operations, Parsimonious Trichords, and Their *Tonnetz* Representations", in *Journal of Music Theory* 41/ 1/ 1997, Duke University Press Journals, pp. 1-66.

[28] He builds his abstract definition of number on the notions of set and mapping (*Abbildungen*) and defines the real numbers as cuts, 'gaps' in the rational numbers.

[29] J. Stillwell, *Roads to Infinity. The Mathematics of Truth and Proof,* A K Peters, 2010, p. 24.

[30] Keith Devlin, *The Language of Mathematics. Making the Invisible Visible*, W.H. Freeman & Co., 2000, p. 5, *passim*.

Deleuze, Gilles, *Francis Bacon. Logique de la sensation*, Éditions du Seuil, 2002.
Devlin, Keith, *The Language of Mathematics. Making the Invisible Visible*, W. H. Freeman & Co., 2000.
Douthett, Jack, & Steinbach, Peter, "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition", in *Journal of Music Theory* 42/ 2 /1998, Duke University Press Journals, pp. 241-263.
Fauvel, John, & Flood, Raymond & Wilson, Robin (eds.), *Music and Mathematics. From Pythagoras to Fractals*, Oxford University Press, 2003.
Fiore, Thomas M., *Music and Mathematics. Lecture Notes*, http://www.personal.umd.umich.edu/~tmfiore/1/music.html
Helmholtz, Hermann L. F., *On the Sensations of Tone. As a Physiological Basis for the Theory of Music*, Dover Publications, New York, 1954.
Lewin, David, *Generalized Musical Intervals and Transformations,* Oxford University Press, 2007.
Loy, Gareth, *Musimathics. The Mathematical Foundations of Music,* Vol. 1. MIT Press, 2006.
Mac Lane, Saunders, *Mathematics. Form and Function*, Springer-Verlag, 1986.
Mainzer, Klaus, *Symmetry and Complexity. The Spirit of Beauty of Nonlinear Science*, World Scientific Publishing, 2005.
Mazzola, Guerino, *The Topos of Music. Geometric Logic of Concepts, Theory, and Performance, Birkhäuser, 2003.*
Messiaen, Olivier, *Traité de rythme, de couleur, et d'ornithologie*, *Vol. I*., Trans. by Melody Baggech, A document submitted to the Graduate Faculty of the School of Music in partial fulfilment of the requirements for the degree of Doctor of Musical Arts, Norman, Oklahoma, 1998
Shepard, R. N. "Circularity in judgments of relative pitch", in *Journal of the Acoustical Society of America*, 36/ 1964, AIP, 2345-2353.
Stillwell, John, *Roads to Infinity. The Mathematics of Truth and Proof*, A K Peters, 2010.
Thyssen, Pieter,& Ceulemans, Arnout, *Shattered Symmetry. Group Theory from the Eightfold Way to the Periodic Table*, Oxford University Press, 2017.
Tymoczko, Dmitri, *A Geometry of Music. Harmony and Counterpoint in the Extended Common Practice*, Oxford University Press, 2011.
Wells Hall Rachel, "Geometrical Music Theory", in *Science,* Vol. 320, Issue 5874/ 18 Apr 2008, AAAS, pp. 328-329, DOI: 10.1126/science.1155463.