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INVOLUNTARY UNEMPLOYMENT IN A NEOCLASSICAL GROWTH MODEL WITH PUBLIC DEBT AND HUMAN CAPITAL

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Abstract: Even more than eight decades since the publication of Keynes' "General Theory of Employment, Interest, and Money" modern macroeconomists disagree on the notion of "underemployment equilibrium" with so-called "involuntary unemployment". While the majority of macro theorists trace involuntary unemployment back to frictions and rigidities in the adaptation of wages and output prices to market imbalances, a minority position holds that even under perfectly flexible output prices and wage rates involuntary unemployment might occur. Morishima in "Walras' Economics" and more recently Magnani presume that contrary to the majority view aggregate investment is not perfectly flexible but governed by "animal spirits" of investors. The aim of the present paper is to integrate the Morishima-Magnani approach into a Diamond-type overlapping generations' (OLG) model with internal public debt subsequently extended by human capital accumulation. It turns out that in spite of perfectly flexible real wage and interest rate involuntary unemployment occurs in intertemporal general equilibrium when aggregate investor sentiments are too pessimistic regarding the rentability of investment in real capital. In the model extended by human capital a higher public debt to output ratio decreases unambiguously involuntary unemployment, if initially the endogenous output growth rate is higher than the real interest rate.

JEL classification: D91, E13, E24

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1. Introduction

At first sight, involuntary unemployment in a neoclassical growth model seams to be a contradiction in terms, at least from the viewpoint of Diamond's (1965) seminal work on "National debt in a neoclassical growth model". Similarly as in

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Solow's (1956) neoclassical growth model, Diamond (1965) assumed full employment of the work force for the overlapping generations' (OLG) economy with production and capital accumulation. Thus, in this economy with exogenous growth unemployment is purely voluntary.

As is well-known, involuntary unemployment is usually associated with Keynesian macroeconomics (Keynes (1936); Hicks (1937)). Involuntary unemployment is traced back to lacking aggregate demand (aggregate demand failures). But on the reasons why aggregate demand remains below full employment output in a perfectly functioning market economy there is no consensus among mainstream economists to this date. The majority of mainstream macro-economists adhered in the past to the macroeconomic disequilibrium or quantity rationing approach of Clower (1965), Barro and Grossman (1971), Malinvaud (1977) and in an overlapping generations context Rankin (1986, 1987) which presumes that rigid nominal price and wage levels are too high (in comparison to Walrasian price and wage levels) such that aggregate demand falls short from full employment supply. Due to the rationing rule that the short side of the market determines the quantity traded both on output and labor market demand determines the quantity produced and labor input into production. More recently, mainstream macroeconomists follow the New Keynesian approach in which prices and wages adapt sluggishly to market imbalances due to imperfect competition and other market failures (see Taylor (1979, 1980); Mankiw (1985), Akerlof and Yellen (1985); Blanchard and Kiyotaki (1987); Ball and Romer (1990); for a survey see Dixon (2000)).

In contrast to the majority, a minority of macroeconomists follow the lead of Morishima (1977) and more recently Magnani (2015) who attribute aggregate demand failures to not perfectly flexible aggregate investment determined by pessimistic "animal spirits" of investors independently from aggregate savings of households. In contradistinction to the older fixed-price and the newer imperfectly flexible price approach of the majority view, Morishima (1977) and Magnani (2015) assume perfectly flexible and perfectly competitive output prices, wages and interest rates. The reason why in spite of these strong assumptions full employment does not occur is that with an independent macro-founded investment function the system of general equilibrium equations is overdeterminate. Overdeterminacy disappears only if at least one market clearing condition is cancelled, and it is the labor market clearing condition which is lost.

Magnani (2015) - without noting precursor Morishima (1977) - proposes to integrate these static macroeconomic reasoning into Solow's (1956) neoclassical growth model without public debt. Since we utltimately want to study the effects of public debt on capital accumulation, output growth and involuntary unemployment in the long run, we stick to Diamond's (1965) OLG model with non-neutral internal public debt. However, output growth in Diamond's (1965) OLG model is exogenous which precludes the steady-state (long run) investigation of public debt variations on output growth and involuntary unemployment. Thus, we shall introduce in a second step human capital accumulation à la Glomm and Ravikumar (1992) and Lin (2000) in order to be able to endogenize the output growth rate. The first step is - as already suggested by Magnani (2015) - to introduce involuntary unemployment into Diamond's exogenous growth model to parallel Magnani's (2015) integration of an endogenous unemployment rate into Solow's neoclassical growth model.

The purpose of this article is two-fold: First, we intend to show how the structure of the intertemporal equilibrium dynamics derived from household's and firm's optimization problems, from government's budget constraint and the intertemporal market clearing conditions changes when aggregate investment is determined by an independent macro investment function and the unemployment rate is endogenized in a log-linear utility and Cobb-Douglas production function version of Diamond's (1965) OLG model with internal public debt. Our contribution to the literature consists here in reiterating Magnani's (2015) modification of Solow's neoclassical growth model within an OLG framework. In line with Magnani (2015) we will show analytically how a higher saving rate or more pessimistic animal spirits of investors affect the unemployment rate along the intertemporal equilibrium path. Our second purpose is to investigate the effects of a higher public debt to output ratio on the output growth rate, on the capital output ratio and on the unemployment rate in a steady state of the Diamond OLG model extended by human capital accumulation which are financed by public human capital investment expenditures. Our contribution to literature consists here in introducing involuntary unemployment into Lin's (2000) and Farmer's (2014) OLG model with human capital accumulation and public debt and exploring analytically and numerically the steady state effects of a higher public debt to output ratio on output growth, the capital output ratio and the unemployment rate. In particular, we will demonstrate on which factors it depends whether a higher public debt to output ratio decreases/increases the growth rate and increases/decreases the unemployment rate.

The remainder of the article is organized as follows: Section 2 outlines the basic model: Diamond's log-linear utility, Cobb-Douglas production function example (as in Farmer (2006)) of an OLG economy with internal public debt and involuntary unemployment. Section 3 extends the basic model by introducing human capital accumulation in line with Glomm and Ravikumar (1992) and Lin (2000) and explores how a higher public debt to output ratio affects the output growth rate, the capital output ratio and the unemployment rate. Section 4 summarizes the main results and concludes.

2. The basic model: The log-linear Cobb-Douglas OLG model with internal public debt and involuntary unemployment

Consider as in F (2006) an infinite-horizon economy composed of perfectly competitive firms, finitely lived households, and a non-optimizing government. A new generation, called generation t, enters the economy in each period $t = 0,1,2,...$. Generation t is composed of a continuum of $L_t > 0$ units of identical agents. It is assumed that the growth rate of the population is $g^L > -1$ which implies that $L_{t+1} = G^L L_t$, $G^L = 1 + g^L$.

The Household Sector

Households, each consisting of one agent, are non-altruistic: The old do not care for the young and the young do not care for the old. They live a maximum of two periods, youth (adult) and old age. In youth, each agent is endowed with one unit of labor, which is supplied inelastically to firms. In exchange for the labor supply, each agent of generation t obtains the real wage rate w_t, which denotes the units of the produced good per unit of labor. However, in contrast to the original Diamond (1965) OLG model, not the whole labor supply is employed but only $(1-u_t)L_t$, where $0<u_t<1$ denotes the unemployment rate. The government collects taxes on wages quoted as a fixed proportion of wage income, $\tau_t w_t(1-u_t)$, $0<\tau_t<1$. Young agents split the net wage income $(1-\tau_t)(1-u_t)w_t$ each period between current consumption c_t^1 and savings s_t. Savings are invested into real capital in period t per capita, I_t^D/L_t, demanded by households in youth, and into real government bonds per capita, B_{t+1}^D/L_t, also demanded by housholds in youth. For simplicity we assume a depreciation rate of one with respect to real capital. Thus, in old age, the household supplies inelastically K_{t+1}^S/L_t to firms, whereby $K_{t+1}^S/L_t = I_t^D/L_t$, and B_{t+1}^S/L_t to young households in period $t+1$, whereby $B_{t+1}^S/L_t = B_{t+1}^D/L_t$. Thus, the per-capita savings are invested as follows: $s_t = K_{t+1}^S/L_t + B_{t+1}^S/L_t$. In old age, households consume their gross return on assets: $c_{t+1}^2 = q_{t+1}K_{t+1}^S/L_t + (1+i_{t+1})B_{t+1}^S/L_t$, where c_{t+1}^2 is consumption in old age, q_{t+1} denotes the gross rental rate on real capital, and i_{t+1} denotes the real interest rate on government bonds in period $t+1$. For simplicity, there are no taxes on rental and interest income.

The intertemporal preferences of the typical two-period lived household are represented by a log-linear intertemporal utility function slightly generalized in comparison to Diamond's (1965, p. 1134) leading example. As usual, this simple specification aims at closed form solutions for the intertemporal equilibrium dynamics (see e.g. de la Croix and Michel (2002, pp. 181-184)).

The typical younger household maximizes the following intertemporal utility function subject to the budget contraints of the active period (i) and the retirement period (ii):

$$\text{Max} \rightarrow \varepsilon \ln c_t^1 + \beta \ln c_{t+1}^2$$

subject to:

$$(i)\ c_t^1 + I_t^D/L_t + B_{t+1}^D/L_t = (1-\tau_t)(1-u_t)w_t,$$

$$(ii)\ c_{t+1}^2 = q_{t+1}K_{t+1}^S/L_t + (1+i_{t+1})B_{t+1}^S/L_t,\ K_{t+1}^S = I_t^D,\ B_{t+1}^S = B_{t+1}^D.$$

$0<\varepsilon\le 1$ denotes the utility elasticity of consumption in youth, while $0<\beta<1$ depicts the subjective future utility discount factor. As is well-known, the log-linear intertemporal utility function ensures the existence of a unique, interior solution of the above optimization problem. Hence, we are entitled to solve the old-age budget constraint for B_{t+1}^S/L_t and insert the result into the young-age budget constraint (i), and we obtain:

$$c_t^1 + c_{t+1}^2/(1+i_{t+1}) + [1 - q_{t+1}/(1+i_{t+1})]K_{t+1}^S/L_t = (1-\tau_t)(1-u_t)w_t. \quad (1)$$

Obviously, a strictly positive and finite solution of maximizing the intertemporal utility function subject to constraint (1) requires that the following no-arbitrage condition holds:

$$q_{t+1} = 1 + i_{t+1} \quad (2)$$

The no-arbitrage condition (2) implies that K_{t+1}^S/L_t is optimally indeterminate, and the first-order conditions for a maximum solution read as follows:

$$c_t^1 + c_{t+1}^2/(1+i_{t+1}) = (1-\tau_t)(1-u_t)w_t, \quad (3)$$

$$(\beta/\varepsilon)c_t^1 = c_{t+1}^2/(1+i_{t+1}). \quad (4)$$

Solving equations (3) and (4) for c_t^1 and c_{t+1}^2 yields the following optimal consumption in youth and old age:

$$c_t^1 = \varepsilon/(\varepsilon+\beta)(1-\tau_t)(1-u_t)w_t, \quad (5)$$

$$c_{t+1}^2 = \beta/(\varepsilon+\beta)(1+i_{t+1})(1-\tau_t)(1-u_t)w_t. \quad (6)$$

Since $s_t = K_{t+1}^S/L_t + B_{t+1}^S/L_t$, we find for the utility maximizing savings:

$$s_t = \beta/(\varepsilon+\beta)(1-\tau_t)(1-u_t)w_t. \quad (7)$$

The Production Sector

All firms are endowed with an identical (linear-homogeneous) Cobb-Douglas production function: $Y_t = (a_t N_t)^{1-\alpha}(K_t)^\alpha$, $0 < \alpha < 1$. Here, Y_t denotes aggregate output or gross domestic product (GDP), a_t features the efficiency level of employed laborers N_t, while K_t denotes the input of capital services, all in period t, and $1-\alpha$ (α) depicts the production elasticity (production share) of labor (capital services). The profit of the production sector (in terms of the single output) is $Y_t - w_t N_t - q_t K_t$. In addition, it is assumed that labor efficiency grows at the exogenously given rate $g^a \geq 0$ such that $a_{t+1} = G^a a_t$ with $G^a = 1 + g^a$ and $a_0 = \overline{a} > 0$. Following normal practice in growth theory, $G^n \equiv G^L G^a$ is called the natural growth factor.

Maximization of $Y_t - w_t N_t - q_t K_t$ subject to the Cobb-Douglas production implies the following first-order conditions:

$$(1-\alpha)a_t[K_t/(a_tN_t)]^{\alpha} = w_t\,, \tag{8}$$
$$\alpha[K_t/(a_tN_t)]^{(\alpha-1)} = q_t\,. \tag{9}$$

However, since the number of employed workers is $N_t = L_t(1-u_t)$, we can rewrite the profit maximization conditions (8) and (9) as follows:

$$(1-\alpha)a_t[K_t/(a_tL_t(1-u_t))]^{\alpha} = w_t\,, \tag{10}$$
$$\alpha[K_t/(a_tL_t(1-u_t))]^{(\alpha-1)} = q_t\,. \tag{11}$$

Finally, the GDP function can be rewritten as follows:

$$Y_t = (a_tL_t(1-u_t))^{1-\alpha}(K_t)^{\alpha}\,. \tag{12}$$

The Public Sector

The government does not optimize but is subject to the following constraint period by period:

$$B_{t+1} = (1+i_t)B_t + \Gamma_t - \tau_t(1-u_t)w_tL_t\,, \tag{13}$$

where B_t denotes the aggregate stock of real public debt at the beginning of period t and Γ_t indicates total government expenditures during period t.

Magnani's* (*2015*) *macro-founded investment function

Magnani (2015, pp. 13-14) rightly claims that in Solow's (1956) neoclassical growth model investment in real capital is not micro- but macro-founded since aggregate investment is determined by aggregate savings. This is also true in Diamond's (1965) OLG model of neoclassical growth in which perfectly flexible aggregate investment is determined by aggregate savings of households in youth age. In contrast to these neoclassical growth models, Morishima (1977) in "Walras' Economics" and more recently Magnani (2015, p. 14) assume that "investments are determined by an independent investment function." This function is specified in discrete time as follows:

$$I_t^D = \varphi L_o(G^L)^t a_o(G^a)^t(1+i_t)^{-\theta}\,,\ \varphi > 0, \theta \geq 0\,. \tag{14}$$

For ease of exposition and for the sake of closed-form solutions, we assume in line with Magnani's (2015, p. 14) base model the simpler version of equation (14) with $\theta = 0$ which implies that aggregate investment is exogenous as in simple Keynesian good market models. The positive parameter φ reflects "Keynesian investors' animal spirits". (ibid)

Market clearing conditions

In addition to the restrictions imposed by household and firm optimization and the above government budget constraint, markets for labor and capital services as well as for the assets have to clear in all periods (the market for the output of production is cleared by means of Walras' Law[1]).

$$L_t(1-u_t) = N_t, \forall t \tag{15}$$

$$K_t^S = K_t, \forall t \tag{16}$$

$$B_t = B_t^D = B_t^S, \forall t \tag{17}$$

Unemployment rate period t

Before deriving the intertemporal equilibrium dynamics it is apt to determine the unemployment rate in period t, $t = 0,1,,2,...$. To this end, we use the output market clearing identity

$$Y_t = L_t c_t^1 + L_{t-1} c_t^2 + I_t^D + \Gamma_t. \tag{18}$$

Inserting into (18) equations (12), (5), (ii) for period t, (14) and the market clearing conditions (16) and (17), we obtain the following equation:

$$\begin{aligned}(A_t)^{1-\alpha}(1-u_t)^{1-\alpha}(K_t)^{\alpha} = L_t\left(\frac{\varepsilon}{\varepsilon+\beta}\right)(1-\tau_t)(1-u_t)w_t + L_{t-1}[q_t K_t + (1+i_t)B_t] \\ + \varphi A_t + \Gamma_t,\end{aligned} \tag{19}$$

with $A_t \equiv a_t L_t$ as potential effective labor. Dividing equation (19) on both sides through A_t, the following equation turns out:

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[1] The proof of Walras' law proceeds as follows: Denote by $P_t > 0$ the nominal price (level) of GDP. Then, the current period budget constraint of households in youth age can be rewritten as follows: $P_t L_t c_t^1 + P_t I_t^D + P_t B_{t+1}^D = (1-\tau_t)P_t w_t (1-u_t)L_t$ (F.1). Moreover, the budget constraint of households in old age reads as follows: $P_t L_{t-1} c_t^2 = P_t q_t K_t^S + P_t(1+i_t)B_t^S$ (F.2). In addition, maximum profits are zero, which implies: $P_t Y_t = P_t w_t N_t + P_t q_t K_t$ (F.3). Finally, government's budget constraint is rewritten as follows: $P_t B_{t+1} = P_t(1+i_t)B_t + P_t\Gamma_t$ $-P_t\tau_t w_t L_t(1-u_t)$ (F.4). Adding up the left and right hand side of equations (F.1) and (F.2) yields: $P_t L_t c_t^1 + P_t I_t^D + P_t L_{t-1} c_t^2 = (1-\tau_t)P_t w_t(1-u_t)L_t - P_t B_{t+1}^D + P_t q_t K_t^S + P_t(1+i_t)B_t^S$ (F.5). Considering (14), (15) and (16) in (F.5) we get: $P_t L_t c_t^1 + P_t L_{t-1} c_t^2 + P_t I_t^D = P_t w_t N_t$ $+P_t q_t K_t - P_t \tau_t w_t N_t - P_t B_{t+1} + P_t(1+i_t)B_t$ (F.6). Inserting (F.3) into (F.6) and taking account of (F.4) in (F.6) yields: $P_t L_t c_t^1 + P_t L_{t-1} c_t^2 + P_t I_t^D = P_t Y_t - P_t \Gamma_t$, which represents production output market clearing. Since this equation is always true, P_t is indeterminate and can be fixed as $P_t = 1$. Q.E.D.

$$(1-u_t)^{1-\alpha}(k_t)^{\alpha}=\left(\frac{\varepsilon}{\varepsilon+\beta}\right)(1-\tau_t)(1-u_t)\frac{w_t}{a_t}+(G^L)^{-1}[q_t k_t+(1+i_t)b_t]+\varphi+\Gamma_t/A_t, \quad (20)$$

with $k_t \equiv K_t/(a_t L_t)$ denoting capital per potential unit of effective labor (= potential capital intensity) and $b_t \equiv B_t / A_t$ as public debt per potential unit of effective labor.

On account of the linear-homogeneity of the production function we can rewrite the profit-maximizing conditions (10) and (11) as follows:

$$w_t / a_t = (1-\alpha)(k_t)^{\alpha}(1-u_t)^{-\alpha}, \quad (21)$$

$$q_t = \alpha(k_t)^{\alpha-1}(1-u_t)^{1-\alpha}. \quad (22)$$

Considering the no-arbitrage condition (2), defining $\gamma_t \equiv \Gamma_t / A_t$ and inserting (21) and (22) into (20) we obtain the following equation:

$$\begin{aligned}(1-u_t)^{1-\alpha}(k_t)^{\alpha}&=\left(\frac{\varepsilon}{\varepsilon+\beta}\right)(1-\tau_t)(1-\alpha)(k_t)^{\alpha}(1-u_t)^{1-\alpha}\\&+(G^L)^{-1}[\alpha(k_t)^{\alpha-1}(1-u_t)^{1-\alpha}k_t+\alpha(k_t)^{\alpha-1}(1-u_t)^{1-\alpha}b_t]+\varphi+\gamma_t.\end{aligned} \quad (23)$$

Next we have to specify how the government determines its intertemporal policy profile. In accordance with the established literature, we assume that the government sticks to a policy of time-stationary ("constant") tax rates and expenditure rates. This means that $\tau_t = \tau_{t+1} = \tau, \quad \gamma_t = \gamma_{t+1} = \gamma$ hold where $0 < \tau < 1$ respectively $\gamma > 0$ are exogenously fixed by the government. Under this proviso (23) can be rewritten as follows:

$$(1-u_t)^{1-\alpha}=\frac{\varphi+\gamma}{(k_t)^{\alpha}[1-\left(\frac{\varepsilon}{\varepsilon+\beta}\right)(1-\tau)(1-\alpha)-(G^L)^{-1}\alpha(1+b_t/k_t)]} \quad (24)$$

Equation (24) shows the intertemporal equilibrium unemployment rate in period t as determined by structural and policy parameters and historically given potential capital and public debt intensity.

Intertemporal equilibrium dynamics

In order to derive the intertemporal equilibrium dynamics we use $K_{t+1}^S = I_t^D$, market clearing conditions (16) and (14), and after dividing the resulting equation on both sides through A_t we obtain:

$$K_{t+1}/A_{t+1}(A_{t+1}/A_t) \equiv k_{t+1}G^n = I_t^D / A_t = \varphi. \quad (25)$$

Next we divide government's budget constraint (13) through A_t, and we get the following equation:

$$b_{t+1}G^n = (1+i_t)b_t + \gamma - \tau(1-u_t)w_t / a_t . \tag{26}$$

Considering again the no-arbitrage condition (2) and the first-order conditions for maximum profit (18) and (19) equation (26) can be rewritten as follows:

$$b_{t+1}G^n = \alpha(k_t)^{\alpha-1}(1-u_t)^{1-\alpha} b_t + \gamma - \tau(1-\alpha)(k_t)^{\alpha}(1-u_t)^{1-\alpha} \tag{27}$$

Introducing the new variable $\eta_t \equiv b_t / k_t$ (see Michaelis (1989)), which can be termed the debt to output ratio, equation (27) is changed to the following equation:

$$\eta_{t+1}k_{t+1}G^n = (k_t)^{\alpha}(1-u_t)^{1-\alpha}[\alpha\eta_t + \gamma - \tau(1-\alpha)]. \tag{28}$$

Respecting equations (24) and (25) we obtain eventually the following two-dimensional system of the intertemporal equilibrium dynamics:

$$k_{t+1}G^n = \varphi , \tag{29}$$

$$\eta_{t+1} = \frac{\varphi+\gamma}{\varphi[1-\left(\frac{\varepsilon}{\varepsilon+\beta}\right)(1-\tau)(1-\alpha)-(G^L)^{-1}\alpha(1+\eta_t)]}[\alpha\eta_t + \gamma - \tau(1-\alpha)]. \tag{30}$$

The intertemporal equilibrium unemployment rate reads as follows:

$$u_t = 1 - \left\{ \frac{\varphi+\gamma}{(k_t)^{\alpha}[1-\left(\frac{\varepsilon}{\varepsilon+\beta}\right)(1-\tau)(1-\alpha)-(G^L)^{-1}\alpha(1+\eta_t)]} \right\}^{1/(1-\alpha)} \tag{31}$$

Existence and dynamic stability of non-trivial steady states

As usual, the steady states of the intertemporal equilibrium dynamics (29) and (30) are defined as $\lim_{t->\infty} k_t = k$ and $\lim_{t->\infty} \eta_t = \eta$. It is obvious that a trivial steady state does not exist. Thus, we are entitled to focus on non-trivial steady state solutions.

Proposition 1. Suppose that $\tau(1-\alpha) > \gamma$. Then, exactly two non-trivial steady state solutions (η_1, k) and (η_2, k) exist. Thus, the solutions read as follows:

$$\begin{aligned}(\eta_1,k) &= \{\text{-E1} + [\text{E1}^2 + 4\,\text{E0}\,\text{E2}]^2)/(2\,\text{E0}), \varphi/\text{G}^\text{n}\},\\ (\eta_2,k) &= \{\text{-E1} - [\text{E1}^2 + 4\,\text{E0}\,\text{E2}]^2)/(2\,\text{E0}), \varphi/\text{G}^\text{n}\},\end{aligned} \tag{32}$$

with $\mathrm{E0} \equiv \alpha(\mathrm{G^L})^{-1}\varphi$, $\mathrm{E1} \equiv (\varphi+\gamma)\alpha + \mathrm{E0} - \varphi[1-(\varepsilon/(\varepsilon+\beta))(1-\tau)(1-\alpha)]$,
$\mathrm{E2} \equiv (\varphi+\gamma)[\tau(1-\alpha)-\gamma]$ and $\eta_1 > 0, \eta_2 < 0$ and $k > 0$.

Proof. Immediate.

Remark 1. While in the first steady state solution the debt to output ratio is larger than zero, and the government is a debtor of the private sector, in the second steady state the debt to output ratio is less than zero and the government is a lender to the private sector.

The next step is to consider the local dynamic stability of the steady state solutions (η_1, k) and (η_2, k), respectively.

Proposition 2. The non-trivial steady state solution (η_1, k) in (32) of the equilibrium dynamics (29) and (30) is asymptotically unstable, while (η_2, k) in (32) is asymptotically stable.

Proof. Differentiating (30) with respect to η_t yields:

$$\frac{d\eta_{t+1}}{d\eta_t} = \frac{\alpha[(\gamma+\varphi)/\varphi + (G^L)^{-1}\eta)}{[1-\left(\frac{\varepsilon}{\varepsilon+\beta}\right)(1-\tau)(1-\alpha)-(G^L)^{-1}\alpha(1+\eta)]} = \frac{\alpha[(\gamma+\varphi)/\varphi + (G^L)^{-1}\eta)\varphi\eta}{(\varphi+\gamma)[\alpha\eta+\gamma-(1-\alpha)\tau]} \quad (33)$$

The total derivative in (33) evaluated at $\eta_1 > 0$ is certainly larger than zero since the denominator is larger than zero and it is also larger than unity since $\eta_1 > 0$ and $\alpha[(\gamma+\varphi)/\varphi + (G^L)^{-1}\eta_1)\eta_1 > (\gamma+\varphi)/\varphi(\alpha\eta_1+\gamma-(1-\alpha)\tau) \Leftrightarrow$ $\alpha(G^L)^{-1}(\eta_1)^2 > \gamma-(1-\alpha)\tau$ since $\gamma-(1-\alpha)\tau < 0$ on account of the assumption in Proposition 1.

The total derivative in (33) evaluated at $\eta_2 < 0$ is certainly larger than zero since both the numerator and the denominator are smaller than zero and it is also smaller than unity since $\eta_2 < 0$ and $\alpha[(\gamma+\varphi)/\varphi + (G^L)^{-1}\eta_2)\eta_2$ $/(\gamma+\varphi)/\varphi(\alpha\eta_2+\gamma-(1-\alpha)\tau) < 1 \Leftrightarrow \alpha[(\gamma+\varphi)/\varphi + (G^L)^{-1}\eta_2)\eta_2 > (\gamma+\varphi)/\varphi(\alpha\eta_2+\gamma$ $-(1-\alpha)\tau) \Leftrightarrow \alpha(G^L)^{-1}(\eta_2)^2 > \gamma-(1-\alpha)\tau$ since $\gamma-(1-\alpha)\tau < 0$ on account of the assumption in Proposition 1. Q.E.D.

In view of economic reality the second steady state solution (η_2, k) although dynamically stable can be disregarded. Thus, we focus on the unstable first steady state solution (η_1, k). In order to overcome the problem of dynamic instability we focus on the following specific intertemporal equilibrium path:

$$k_{t+1} = k,\ t = 0,1,2,...,\ k_0 = \bar{k} > 0\,, \quad (34)$$

$$\eta_t = \eta_1,\ t = 0,1,2...\,, \quad (35)$$

$$b_t = \eta_1 k_t, t = 0,1,2.... \quad (36)$$

Usually, the economy commences with historically given capital intensity $k_0 = \bar{k} > 0$ and debt intensity $b_0 = \bar{b} > 0$. However, equation (36) then transpires that the dynamic system (43)-(35) becomes overdeterminate. In order to restore determinacy the initital debt intensity b_0 must be accidentally such that $\eta_0 = b_0 / \bar{k} = \eta_1$. Where this is (realistically) not the case, one of the policy parameters in η_1 has to become endogenous. The natural candidate is either the tax rate τ or the expenditure rate γ. To avoid overdeterminacy or a break down of the equilibrium dynamics within a finite number of periods, the government is no longer free to choose any feasible tax or expenditure rate if $\eta_0 = \bar{b} / \bar{k} = \eta_1$ but must rather choose the following tax rate (given γ):

$$\tau_0 = \frac{[(\varepsilon/(\varepsilon+\beta)(1-\alpha)+\alpha(G^L)^{-1}(1+\eta_0)-1]\varphi\eta_0+(\gamma+\varphi)(\gamma+\alpha\eta_0)}{(1-\alpha)(\gamma+\varphi)+[\varepsilon/(\varepsilon+\beta)](1-\alpha)\varphi\eta_0}. \tag{37}$$

By sticking to τ_0 in (37), the government ensures that a historically given $\bar{b} > 0$ (and $\bar{k} > 0$) can be maintained indefinitely. On the contrary, if the government fails to fix the tax rate exactly at the value of τ_0 in (37), the economy breaks down in finite time or it converges to the dynamically stable second steady state where the government is a lender to the private sector.

Comparative Steady State Analysis*: *How More Optimistic Investor's Animal Spirits or a Lower Saving Rate Affect the Unemployment Rate

We are now prepared to investigate how more optimistic investor's animal spirits or a lower saving rate of households affect the equilibrium unemployment rate.

Suppose first that in the initial period $t = 0$ investor's animal spirits become more optimistic, i.e. $d\varphi > 0$ while the OLG economy commences with $\eta_0 = \bar{k} / \bar{b}$. To obtain the requested answer we use in addition to equation (37) the equation (31) rewritten as follows:

$$u_0 = 1 - \left\{ \frac{\varphi+\gamma}{(k_0)^\alpha [1 - \left(\frac{\varepsilon}{\varepsilon+\beta}\right)(1-\tau_0)(1-\alpha) - (G^L)^{-1}\alpha(1+\eta_0)]} \right\}^{1/(1-\alpha)} \tag{38}$$

Totally differentiating u_0 with respect to φ yields after simplification:

$$\frac{du_0}{d\varphi} = -\frac{[\beta+\varepsilon(1+\eta_0)]\left\{\frac{G^L(k_0)^\alpha - \alpha[\beta(\gamma+\varphi(1+\eta_0)}{(\beta+\varepsilon\gamma)G^L - \alpha(1+\eta_0)(\beta+\varepsilon(1-G^L)}\right\}^{1/(1-\alpha)}}{(1-\alpha)[\beta(\gamma+\varphi)+\varepsilon(\gamma+\varphi(1+\eta_0))]} \tag{39}$$

Since the term in curled brackets is certainly larger than zero we are assured that $du_0 / d\varphi < 0$, or in other words that more optimistic animal spirits of investors reduce the equilibrium unemployment rate. In view of the right hand side of equation (38) this unique analytical result is the consequence of more aggregate demand due to better investor's expectations, i.e. $d\varphi > 0$ and a dampening effect on the tax rate which ensures that the historically given ratio of the debt to capital intensity can be maintained indefinitely. A lower tax rate raises disposable income of households in youth age and hence aggregate demand. Thus, we are able to confirm the unemployment reducing effect of more optimistic investor's expectations of Magnani (2015, p. 14) in our more complicated OLG model with internal public debt.

It remains to be seen whether Magnani's (2015) second main result in regard to the unemployment increasing effect of a higher saving rate (= a larger old-age utility weight $d\beta > 0$) can also be confirmed within our more elaborated OLG model. Differentiation of u_0 in (38) with respect β reads as follows:

$$\frac{du_0}{d\beta} = \frac{\{\varepsilon G^L (k_0)^{-\alpha} [\gamma G^L (1-\alpha-\gamma-(\varphi+\alpha\eta_0)) + (1+\eta_0)\varphi(G^L(1-\alpha)-\alpha\eta_0)]\}\Theta}{(1-\alpha)[(\beta+\varepsilon\gamma)G^L - \alpha(1+\eta_0)(\beta+\varepsilon(1-G^L))]^2} \qquad (40)$$

with $$\Theta \equiv \left\{ \frac{G^L (k_0)^{\alpha} - \alpha[\beta(\gamma+\varphi)+\varepsilon(\gamma+\varphi(1+\eta_0))]}{(\beta+\varepsilon\gamma)G^L - \alpha(1+\eta_0)(\beta+\varepsilon(1-G^L))} \right\}^{\alpha/(1-\alpha)}.$$

The numerator on the right hand side of (40) and Θ is certainly larger than zero which is also true for $(1-\alpha-\gamma-(\varphi+\alpha\eta_0))$ and $(G^L(1-\alpha)-\alpha\eta_0)$ as extensive simulation of feasible parameter sets comprising both terms shows. Thus, the term in curled brackets in the numerator of the ratio on the right hand side of (40) is strictly larger than zero and hence $du_0 / d\beta > 0$. A larger weight of old-age consumption utilty raises the equilibrium unemployment rate. Intuitively, a larger β decreases the marginal propensity to consumption and hence aggregate demand while it also decreases τ_0 which raises young-age consumption and aggregate demand. However, the tax reducing effect of a higher β is weaker than the decreasing effect on the marginal propensity to consumption. As a consequence, we are prepared to confirm also Magnani's (2015, p. 15) second main result regarding the unemployment raising effect of a higher saving rate in our more complex OLG model with internal public debt.

3. The basic model extended by human capital accumulation: The Log-linear Cobb-Douglas OLG model with endogenous growth and involuntary unemployment

In the basic OLG model of the previous section growth was exogenous. Fiscal policy was unable to affect the output (GDP) growth rate. Of particular interest is the question whether a higher public debt can raise GDP growth and decrease unemployment in the long run. In order to address this question we extend in this section the basic OLG model by introducing human capital accumulation. In order to point out most clearly the growth enhancing effects of human capital accumulation we assume in this section no population growth ($g^L = 0 \Leftrightarrow G^L = 1$) and no exogenous growth in labor efficiency, i.e. $G^a = 1$. As a consequence of the first assumption the number of households remains constant over time: $L_t = L_{t-1} = L$.

The Household Sector

The optimization problem of the two-period lived household which enters the economy in period t is essentially the same as in the basic model in the previous section with the exception of human capital h_t which the household accumulated in the period $t-1$ and which enters her labor supply and wage income.

Thus, the typical younger household maximizes the following intertemporal utility function subject to the budget contraints of the active period (i) and the retirement period (ii):

$$\text{Max} \to \varepsilon \ln c_t^1 + \beta \ln c_{t+1}^2$$

subject to:

$$(i)\ c_t^1 + I_t^D / L_t + B_{t+1}^D / L_t = w_t h_t (1-\tau_t)(1-u_t),$$
$$(ii)\ c_{t+1}^2 = q_{t+1} K_{t+1}^S / L_t + (1+i_{t+1}) B_{t+1}^S / L_t,\ K_{t+1}^S = I_t^D,\ B_{t+1}^S = B_{t+1}^D.$$

Obviously, the utility maximizing consumption and savings function read as follows:

$$c_t^1 = \varepsilon / (\varepsilon + \beta)(1-\tau_t)(1-u_t) w_t h_t, \tag{41}$$

$$c_{t+1}^2 = \beta / (\varepsilon + \beta)(1+i_{t+1})(1-\tau_t)(1-u_t) w_t h_t, \tag{42}$$

$$s_t = \beta / (\varepsilon + \beta)(1-\tau_t)(1-u_t) w_t h_t. \tag{43}$$

The Production Sector

All firms are now endowed with an identical (linear-homogeneous) Cobb-Douglas production function: $Y_t = A(h_t N_t)^{1-\alpha} (K_t)^\alpha,\ 0 < \alpha < 1, A > 0$. Maximization of $Y_t - w_t h_t N_t - q_t K_t$ subject to the Cobb-Douglas production function implies the following first-order conditions:

$$(1-\alpha) A [K_t / (h_t N_t)]^\alpha = w_t, \tag{44}$$

$$\alpha A [K_t / (h_t N_t)]^{(\alpha-1)} = q_t. \tag{45}$$

However, since the number of employed workers is $N_t = L(1-u_t)$ we can rewrite the profit maximization conditions (8) and (9) as follows:

$$(1-\alpha)A[K_t/(h_t L(1-u_t))]^{\alpha} = w_t, \quad (46)$$

$$\alpha A[K_t/(h_t L(1-u_t))]^{(\alpha-1)} = q_t. \quad (47)$$

Finally, the GDP function can be rewritten as follows:

$$Y_t = A(h_t L(1-u_t))^{1-\alpha}(K_t)^{\alpha}. \quad (48)$$

The Public Sector

Similarly as in the basic model, the government does not optimize but is subject to the following constraint period by period:

$$B_{t+1} = (1+i_t)B_t + \Delta_t + \Gamma_t - \tau_t(1-u_t)w_t h_t L, \quad (49)$$

where Γ_t denotes now human capital investment (HCI) expenditures and Δ_t denotes all non-HCI expenditures of the government during period t.

Human Capital Accumulation and GDP Growth

In line with Glomm and Ravikumar (1992, 1997), human capital in period t is determined by human capital of the generation which entered the economy in period $t-1$ and by government HCI spending in period $t-1$, Γ_{t-1}:

$$h_t = H_0(h_{t-1})^{\mu}(\Gamma_{t-1}/L)^{1-\mu}, H_0 = \overline{H} > 0,\ 0 < \mu < 1, \quad (50)$$

where H_0 represents a level parameter, μ denotes the production elasticity of human capital, and $1-\mu$ features the production elasticity of public HCI spending. Multiplying equation (50) on both sides by L, we obtain the aggregate version of (45):

$$Lh_t \equiv H_t = H_0(Lh_{t-1})^{1-\mu}(\Gamma_{t-1})^{\mu} \equiv H_0(H_{t-1})^{1-\mu}(\Gamma_{t-1})^{\mu}. \quad (51)$$

The economy grows, even in the absence of population growth and exogenous progress in labor efficiency. The growth factor (= gross growth rate) of GDP is defined as follows:

$$G_{t+1}^{Y} = \frac{Y_{t+1}}{Y_t} \quad (52)$$

Using equations (48) and (51), equation (52) can be rewritten as:

$$G_{t+1}^{Y} = \frac{H_{t+1}}{H_t} \frac{(1-u_{t+1})^{1-\alpha}}{(1-u_t)^{1-\alpha}} \frac{(k_{t+1})^{\alpha}}{(k_t)^{\alpha}},\ k_t \equiv \frac{K_t}{H_t}. \quad (53)$$

Magnani's Macro-founded Investment Function

It is easy to see that Magnani's (2015) macro-founded investment function reads in our new model context as follows:

$$I_t^D = \varphi H_t (1+i_t)^{-\theta},\ \varphi > 0, \theta \geq 0. \quad (54)$$

In order to obtain closed-form solutions we assume as above $\theta = 0$.

Since the market clearing conditions remain as in the basic model, we do not explicitely mention them again.

The Intertemporal Equilibrium Dynamics in Terms of per GDP ratios

In order to be able to calibrate the present model to empirical data we first transform main endogenous variables into per GDP ratios.

To start with, the government budget constraint (49) is rewritten as follows:

$$\begin{aligned} \frac{B_{t+1}}{Y_{t+1}} \frac{Y_{t+1}}{Y_t} &\equiv b_{t+1} G_{t+1}^{Y} = (1+i_t) \frac{B_t}{Y_t} + \frac{\Delta_t}{Y_t} + \frac{\Gamma_t}{Y_t} - \frac{\tau_t (1-u_t) w_t h_t L}{Y_t} \\ &\equiv (1+i_t) b_t + \gamma_t + \delta_t - \frac{\tau_t (1-u_t) w_t h_t L}{Y_t}, \end{aligned} \quad (55)$$

with $b_t \equiv B_t / Y_t$, $\delta_t \equiv \Delta_t / Y_t$ and $\gamma_t \equiv \Gamma_t / Y_t$.

Since the profit maximizing condition (41) can rewritten as

$$w_t h_t L (1-u_t) = (1-\alpha)(K_t)^{\alpha} (h_t L (1-u_t))^{1-\alpha} = (1-\alpha) Y_t, \quad (56)$$

government's budget constraint can be written as follows:

$$b_{t+1} G_{t+1}^{Y} = (1+i_t) b_t + \delta_t + \gamma_t - \tau_t (1-\alpha). \quad (57)$$

In line with the initial assumption in the previous section we assume that the government sticks to time-stationary wage tax rates $\tau_t = \tau_{t+1} = \tau$ and non-HCI expenditure ratios: $\delta_t = \delta_{t+1} = \delta$. Moreover, in accordance with the empirical reality in most advanced countries before the global financial crisis 2007/2008 we assume time-stationary government debt to GDP ratios: $b_{t+1} = b_t = b$. Acknowledging these assumptions the budget constraint of the government reads eventually as follows:

$$\gamma_t = b[G_{t+1}^Y - (1+i_t)] + \chi \ , \tag{58}$$

with $\chi \equiv \tau(1-\alpha) - \delta$ denoting the primary surplus ratio (excluding government HCI-expenditures) of the government.

The next dynamic variable we want to transform into a per GDP ratio is the real capital stock K_t . The capital stock to GDP ratio is known as capital output ratio denoted as v_t . The relationship to the real capital to human capital ratio k_t is as follows:

$$v_t \equiv \frac{K_t}{Y_t} = \frac{K_t}{A(H_t)^{1-\alpha}(1-u_t)^{1-\alpha}(K_t)^{\alpha}} = \frac{(K_t)^{1-\alpha}}{A(H_t)^{1-\alpha}(1-u_t)^{1-\alpha}} = \frac{(k_t)^{1-\alpha}}{A(1-u_t)^{1-\alpha}} \ . \tag{59}$$

Using the relationship between v_t and k_t in (59), we can rewrite the profit maximizing condition (42) as follows:

$$q_t = \frac{\alpha}{v_t} \ . \tag{60}$$

The growth factor of human capital reads in terms of the transformed variables as follows:

$$\begin{aligned} \frac{H_{t+1}}{H_t} &= H_0 (H_t)^{\mu-1}(\gamma_t)^{1-\mu}(Y_t)^{1-\mu} = \\ & H_0 (H_t)^{\mu-1}(\gamma_t)^{1-\mu} A^{(1-\mu)}(K_t)^{\alpha(1-\mu)}(H_t)^{(1-\alpha)(1-\mu)}(1-u_t)^{(1-\alpha)(1-\mu)} \\ &= H_0 A^{(1-\mu)/(1-\alpha)}(\gamma_t)^{1-\mu}(v_t)^{\alpha(1-\mu)/(1-\alpha)}(1-u_t)^{(1-\mu)} . \end{aligned} \tag{61}$$

The GDP growth factor in terms of the capital output ratio can be rewritten as follows:

$$\begin{aligned} G_{t+1}^Y &= \frac{H_{t+1}}{H_t}\left(\frac{v_{t+1}}{v_t}\right)^{\alpha/(1-\alpha)} \frac{(1-u_{t+1})}{(1-u_t)} \\ &= H_0 A^{(1-\mu)/(1-\alpha)}(\gamma_t)^{1-\mu}(v_{t+1})^{\alpha/(1-\alpha)}(v_t)^{-\alpha\mu/(1-\alpha)}(1-u_{t+1})(1-u_t)^{-\mu} . \end{aligned} \tag{62}$$

In order to derive the intertemporal equilibrium dynamics we again use $K^{S}_{t+1} = I^{D}_{t}$, market clearing condition (16) and (14), but dividing the resulting equation on both sides now through Y_t such that we obtain by using (54) and (59):

$$\frac{K_{t+1}}{Y_{t+1}} G^{Y}_{t+1} \equiv v_{t+1} G^{Y}_{t+1} = \frac{I_t}{Y_t} = \varphi \frac{H_t}{K_t}\frac{K_t}{Y_t} = \varphi \frac{1}{k_t} v_t = \varphi A^{\frac{-1}{(1-\alpha)}} (v_t)^{\frac{-\alpha}{(1-\alpha)}} (1-u_t)^{-1} \quad (63)$$

The intertemporal equilibrium unemployment rate can be derived analogously to the derivation in the basic model by using the output market identity (18) and inserting the optimal consumption of youth- and old-age consumers, the investment function and government expenditures as a proportion of GDP yields:

$$Y_t = \varepsilon/(\varepsilon+\beta)(1-\alpha)(1-\tau)Y_t + (\alpha/v_t)(K_t + B_t) + \delta Y_t + \gamma_t Y_t + \varphi H_t \quad (64)$$

Dividing equation (64) on both sides through Y_t, we obtain after introducing the capital output and the debt to GDP ratio the following intertemporal equilibrium unemployment rate:

$$u_t = 1 - \frac{\varphi}{[1-\varepsilon/(\varepsilon+\beta)(1-\alpha)(1-\tau) - \alpha(1+b/v_t) - \delta - \gamma_t] A^{\frac{1}{(1-\alpha)}} (v_t)^{\alpha/(1-\alpha)}} \quad (65)$$

The final steps needed to obtain the equation of motion for the capital output ratio entail inserting the GDP growth factor equation (62) into equations (58) and (63). This procedure yields:

$$\gamma_t = b[H_0 A^{\frac{(1-\mu)}{(1-\alpha)}} (\gamma_t)^{1-\mu} (v_{t+1})^{\frac{\alpha}{(1-\alpha)}} (v_t)^{\frac{-\alpha\mu}{(1-\alpha)}} (1-u_{t+1})(1-u_t)^{-\mu} - \frac{\alpha}{v_t}] + \chi, \quad (66)$$

$$H_0 A^{\frac{(2-\mu)}{(1-\alpha)}} (\gamma_t)^{1-\mu} (v_{t+1})^{\frac{1}{(1-\alpha)}} (1-u_{t+1}) = \varphi (v_t)^{\frac{\alpha(\mu-1)}{(1-\alpha)}} (1-u_t)^{\mu-1} \quad (67)$$

Solving (62) for γ_t, we obtain:

$$\gamma_t = [\varphi/(A^{\frac{(2-\mu)}{(1-\alpha)}} H_0)]^{\frac{1}{(1-\mu)}} (v_{t+1})^{\frac{-1}{(1-\alpha)(1-\mu)}} (1-u_{t+1})^{\frac{-1}{(1-\mu)}} (1-u_t)^{-1} (v_t)^{\frac{-\alpha}{(1-\alpha)}} \quad (68)$$

Inserting γ_t from (68) into equations (66) and (65), we obtain two implicit non-linear dynamic equations with v_t and u_t as dynamic variables:

$$\left[\frac{\varphi}{(H_0 A^{\frac{2-\mu}{(1-\alpha)}})}\right]^{\frac{1}{(1-\mu)}} (v_{t+1})^{\frac{-1}{(1-\alpha)(1-\mu)}} (1-u_{t+1})^{\frac{-1}{(1-\mu)}} =$$

$$\left[\frac{(\varphi b)}{A^{\frac{1}{(1-\alpha)}}}\right] (v_{t+1})^{-1} + [\chi - \frac{\alpha b}{v_t}] (v_t)^{\frac{\alpha}{(1-\alpha)}} (1-u_t), \tag{69}$$

$$\left[\frac{\varphi}{(H_0 A^{\frac{1}{(1-\alpha)}})}\right]^{\frac{1}{(1-\mu)}} (v_{t+1})^{\frac{-1}{(1-\alpha)(1-\mu)}} (1-u_{t+1})^{\frac{-1}{(1-\mu)}} =$$

$$\left[1 - \frac{\varepsilon(1-\alpha)(1-\tau)}{(\varepsilon+\beta)} - \delta - \alpha(1+\frac{b}{v_t})\right] A^{\frac{1}{(1-\alpha)}} (v_t)^{\frac{\alpha}{(1-\alpha)}} (1-u_t) - \varphi. \tag{70}$$

Equating the left hand sides of equations (69) and (70) we obtain after rearranging an explicit solution for v_{t+1} by using the short hand $w_t \equiv 1-u_t$ as follows:

$$v_{t+1} = \frac{\varphi b}{\left[1 - \frac{\varepsilon(1-\alpha)(1-\tau)}{(\varepsilon+\beta)} - \alpha - \tau(1-\alpha)\right] A^{\frac{1}{(1-\alpha)}} (v_t)^{\alpha/(1-\alpha)} w_t - \varphi} \tag{71}$$

Inserting the v_{t+1} from equation (71) into the left hand side of equation (70) we obtain after rearraging also an explicit solution for w_{t+1} as follows:

$$w_{t+1} = \frac{\left\{[1 - \frac{\varepsilon(1-\alpha)(1-\tau)}{(\varepsilon+\beta)} - \alpha - \tau(1-\alpha)] A^{\frac{1}{1-\alpha}} (v_t)^{\frac{\alpha}{(1-\alpha)}} w_t - \varphi\right\}^{\frac{1}{(1-\alpha)}}}{H_0 b^{\frac{1}{(1-\alpha)}} \varphi^{\frac{\alpha}{(1-\alpha)}} A^{\frac{1}{(1-\alpha)}} \left\{\left[1 - \frac{\varepsilon(1-\alpha)(1-\tau)}{(\varepsilon+\beta)} - \delta - \alpha\left(1+\frac{b}{v_t}\right)\right] A^{\frac{1}{(1-\alpha)}} (v_t)^{\frac{\alpha}{(1-\alpha)}} w_t - \varphi\right\}^{(1-\mu)}}. \tag{72}$$

Thus, we finally arrive at a two-dimensional system of first-order difference equations embracing the dynamic variables v_t and w_t.

Existence and dynamic stability of non-trivial steady states

As above, the steady states of the equilibrium dynamics depicted by the difference equations (71) and (72) are defined as $\lim_{t->\infty} v_t = v$ and $\lim_{t->\infty} w_t = w$. In contrast to the basic model, explicit steady state solutions are now impossible. Thus, we have to resort to an intermediate value theorem in order to prove the existence of at least one feasible steady state solution $v_{\min} < v < \infty$ and $0 < w < 1$.

To this end, let us first define for given structural and policy parameters (with the exception of φ) that maximal investor's animal spirits parameter denoted as $\varphi^{\max}$ and that minimal capital output ratio $v_{\min}$ which ensure full employment. Using the steady state version of equation (66) which can be explicitely solved for w as follows:

$$w = \frac{\varphi(b+v)}{[1-\varepsilon(1-\alpha)(1-\tau)/(\varepsilon+\beta)-\alpha-\tau(1-\alpha)]A^{1/(1-\alpha)}v^{1/(1-\alpha)}} \tag{73}$$

and setting $w=1$ in equation (68), we get immediately:

$$\varphi^{\max} = \frac{[1-\varepsilon(1-\alpha)(1-\tau)/(\varepsilon+\beta)-\alpha-\tau(1-\alpha)]A^{1/(1-\alpha)}(v_{\min})^{1/(1-\alpha)}}{b+v_{\min}}. \tag{74}$$

Using the steady state version of equation (72), setting $w=1$ and inserting for $\varphi^{\max}$ the right hand side of equation (74) we obtain the following equation in order to determine $v_{\min}$:

$$\begin{aligned}
&H_0 b^{\frac{1}{(1-\alpha)}}\{\frac{[1-\varepsilon(1-\alpha)(1-\tau)/(\varepsilon+\beta)-\alpha-\tau(1-\alpha)]A^{1/(1-\alpha)}(v_{\min})^{1/(1-\alpha)}}{b+v_{\min}}\}^{\frac{\alpha}{(1-\alpha)}} A^{\frac{1}{(1-\alpha)}} \\
&\{[1-\frac{\varepsilon(1-\alpha)(1-\tau)}{(\varepsilon+\beta)}-\delta-\alpha(1+b/v_{\min})]A^{\frac{1}{(1-\alpha)}}(v_{\min})^{\frac{\alpha}{(1-\alpha)}} - \\
&\frac{[1-\varepsilon(1-\alpha)(1-\tau)/(\varepsilon+\beta)-\alpha-\tau(1-\alpha)]A^{1/(1-\alpha)}(v_{\min})^{1/(1-\alpha)}}{b+v_{\min}}\}^{(1-\mu)} = \\
&\{[1-\frac{\varepsilon(1-\alpha)(1-\tau)}{(\varepsilon+\beta)}-\alpha-\tau(1-\alpha)]A^{\frac{1}{1-\alpha}}(v_{\min})^{\frac{\alpha}{(1-\alpha)}} \\
&-\frac{[1-\varepsilon(1-\alpha)(1-\tau)/(\varepsilon+\beta)-\alpha-\tau(1-\alpha)]A^{1/(1-\alpha)}(v_{\min})^{1/(1-\alpha)}}{b+v_{\min}}\}^{\frac{1}{(1-\alpha)}}.
\end{aligned} \tag{75}$$

Now we are prepared to state the following parameter restriction (PR):

PR: Let $0<\alpha<1, 0<\varepsilon<1, 0<\beta<1, 0<\tau<1, 0<\mu<1, b>0, H_0>0$, $A>0$ such that equation (75) can be solved for a real and strictly positive $v_{\min}$.

Moreover, for the proof of the following proposition it is useful to define the following continuous real-valued functions $LHS(v)$ and $RHS(v)$:

$$LHS(v) \equiv H_0 A^{\frac{1}{(1-\alpha)}} b^{\frac{1}{(1-\alpha)}} \varphi^{\frac{\alpha}{(1-\alpha)}} \{[1-\frac{\varepsilon(1-\alpha)(1-\tau)}{(\varepsilon+\beta)} - \delta - \alpha(1+\frac{b}{v})] A^{\frac{1}{(1-\alpha)}} v^{\frac{\alpha}{(1-\alpha)}} w - \varphi\}^{1-\mu}, \tag{76}$$

$$RHS(v) \equiv \frac{\left\{[1-\frac{\varepsilon(1-\alpha)(1-\tau)}{(\varepsilon+\beta)} - \alpha - \tau(1-\alpha)] A^{\frac{1}{(1-\alpha)}} v^{\frac{\alpha}{(1-\alpha)}} w - \varphi\right\}^{\frac{1}{(1-\alpha)}}}{w}. \tag{77}$$

Proposition 3. Suppose that PR holds with $1/(1-\alpha) > \mu$ and $\chi > 0$. Then, at least one steady state solution $v_{\min} \le v < \infty$ and $0 < w \le 1$ of the equilibrium dynamics (71) and (72) exists.

Proof.

For $\varphi = \varphi^{\max}$ we know from PR that $w = 1$ and $v = v_{\min}$. Thus, let be $\varphi < \varphi^{\max}$. Under this presumption, we want to show that $v_{\min} < v < \infty$ and $0 < w < 1$. From equation (73) and $dw/dv = -A^{-1/(1-\alpha)}(\varepsilon+\beta)\varphi v^{(\alpha-2)/(1-\alpha)} \; /[(1-\alpha)^2 \beta(1-\tau)] < 0$ follows immediately $0 < w < 1$. In order to show that $v_{\min} < v < \infty$ whereby v is the solution of $LHS(v) = RHS(v)$ we apply an intermediate value theorem. To this end, we have to show that $LHS(v_{\min}) < RHS(v_{\min})$ and $\lim_{v->\infty} LHS(v) > \lim_{v->\infty} RHS(v)$. In order to show $LHS(v_{\min}) < RHS(v_{\min})$ note first that $RHS(v_{\min}) = \{A^{1/(1-\alpha)}(1-\alpha)b\beta(1-\tau)(v_{\min})^{\alpha/(1-\alpha)} \; /[(\varepsilon+\beta)H_0(b+v_{\min})]\}^{1/(1-\alpha)}$ which does not depend on φ. In contrast, $LHS(v_{\min})$ does depend on φ and one can show for $1/(1-\alpha) > \mu$ that $\partial \mathrm{LHS}(\mathrm{v}_{\min})/\partial\varphi > 0$. By definition we know that $LHS(v_{\min})(\varphi^{\max}) = RHS(v_{\min})(\varphi^{\max})$. Since $RHS(v_{\min})$ does not change with $\varphi < \varphi^{\max}$ while $LHS(v_{\min})(\varphi) < LHS(v_{\min})(\varphi^{\max})$ because of $\partial \mathrm{LHS}(\mathrm{v}_{\min})/\partial\varphi > 0$ we are ensured that $LHS(v_{\min})(\varphi) < RHS(v_{\min})(\varphi) \; \forall \varphi < \varphi^{\max}$. In order to show $\lim_{v->\infty} LHS(v) > \lim_{v->\infty} RHS(v)$ note that $\lim_{v->\infty} LHS(v) = A^{1/(1-\alpha)} b^{1/(1-\alpha)} \varphi^{1/(1-\alpha)-\mu} H_0 \{(\beta+\varepsilon)\chi/[(1-\alpha)\beta(1-\tau)]\}^{1-\mu} > 0$ while $\lim_{v->\infty} RHS(v) = 0$. Thus, $\lim_{v->\infty} LHS(v) > \lim_{v->\infty} RHS(v)$. Since both $LHS(v)$ and $RHS(v)$ are continuous functions on the interval $(v_{\min}, \infty)$ we are ensured that $LHS(v) = RHS(v)$. Q.E.D.

The next step is to investigate the local dynamic stability of steady state solutions $\{(v,w)\}$. To this end, we do, however, not directly differentiate the first-order difference equation system (71) und (72) with respect to v_t and w_t but use the intertemporal equilibrium equations (58) (62), (63) and (65) and differentiate these equations totally with respect to all endogenous variables $v_{t+1}, v_t, w_{t+1}, w_t, G^Y_{t+1}, \gamma_t$. Then, we form the Jacobian matrix of partial differentials evaluated at a steady state solution (v,w) as follows:

$$J(v,w) \equiv \begin{bmatrix} \frac{\partial v_{t+1}}{\partial v_t}(v,w) & \frac{\partial v_{t+1}}{\partial w_t}(v,w) \\ \frac{\partial w_{t+1}}{\partial v_t}(v,w) & \frac{\partial w_{t+1}}{\partial w_t}(v,w) \end{bmatrix}, \tag{78}$$

with

$$\frac{\partial v_{t+1}}{\partial v_t}(v,w) \equiv j_{11} = -\frac{\alpha[v+b]}{(1-\alpha)b} < 0,$$

$$\frac{\partial v_{t+1}}{\partial w_t}(v,w) \equiv j_{12} = -\frac{v[v+b]}{bw} < 0,$$

$$\frac{\partial w_{t+1}}{\partial v_t}(v,w) \equiv j_{21}$$

$$= \frac{\alpha w \left\langle v^2[\gamma-(1-\alpha)(1-\mu)bG^Y] + b\{\gamma v[\alpha+\mu(1-\alpha)] - (1-\alpha)^2(1-\mu)b\}\right\rangle}{(1-\alpha)^2 b\gamma v^2},$$

$$\frac{\partial w_{t+1}}{\partial w_t}(v,w) \equiv j_{22}$$

$$= \frac{[\gamma-(1-\alpha)(1-\mu)bG^Y] + b\gamma[\mu+\alpha(1-\mu)]}{(1-\alpha)b\gamma}.$$

While obviously the first row of the Jacobian (78) exhibits negative entries, the sign of the first element in the second row is in general unkown while $j_{22} > 0$. In order to investigate the dynamic stability of steady state solutions following from $LHS(v) = RHS(v)$, we need to know the signs and the magnitudes of the eigenvalues of Jacobian (78). As is well-known the eigenvalues of the 2x2 Jacobian matrix (78) are defined as follows: $\lambda_{1,2} \equiv [trJ(v,w) \pm \sqrt{\Delta(v,w)}]/2$ with $trJ(v,w)$ denoting the trace of the Jacobian matrix (78) and $\Delta(v,w)$ being the discriminant of this matrix defined as $\Delta(v,w) \equiv trJ(v,w)^2 - 4\det J(v,w)$, respectively, where $\det J(v,w)$ denotes the determinant of the Jacobian matrix (78).

The algebraic calculation of the trace and of the determinant of the Jacobian matrix (78) yields the following results:

$$trJ(v,w) = \mu + \frac{[\gamma - (1-\mu)bG^Y]v}{b\gamma} > 0\,, \tag{79}$$

$$\det J(v,w) = -\frac{\alpha(1-\mu)(b+v)}{\gamma\ v} < 0\,. \tag{80}$$

Obviously, the determinant of the Jacobian is for at all feasible steady state solutions less than zero where the trace is certainly larger then zero if $\gamma - (1-\mu)bG^Y > 0$. Using the trace (79) and the Jacobian (80), the eigenvalues of the Jacobian (78) read as follows:

$$\lambda_{1,2} = \frac{1}{2}\left\{\mu + \frac{[\gamma-(1-\mu)bG^Y]v}{b\gamma} \pm \left[\left(\mu + \frac{[\gamma-(1-\mu)bG^Y]v}{b\gamma}\right)^2 + \frac{4\alpha(1-\mu)(b+v)}{\gamma v}\right\}^{\frac{1}{2}}. \tag{81}$$

A glance on the eigenvalue formula (81) reveals immediately that both eigenvalues are real since the term under the square root is certainly larger than zero. It is also clear from formula (81) that for $\gamma > (1-\mu)bG^Y$ the first eigenvalue is strictly larger than zero while the second eigenvalue is less than zero. To obtain information on the magnitudes of the eigenvalues we make use of Lemma A1 in the Appendix of Galor (1992, p. 1383) which says inter alia: $\lambda_1 > 1$ and $-1 < \lambda_2 < 0$ if and only if $trJ(v,w) \geq 2 \vee 1 - trJ(v,w) + \det J(v,w) < 0$ and $trJ(v,w) > -2 \wedge 1 + trJ(v,w) + \det J(v,w) > 0 \wedge \det J(v,w) < 0$. We know for sure that $\det J(v,w) < 0$ and $trJ(v,w) > -2$ for $\gamma - (1-\mu)bG^Y > 0$. It is also true that $1 - trJ(v,w) + \det J(v,w) < 0$ and $1 + trJ(v,w) + \det J(v,w) > 0$ at least for a broad set of feasible parameter combinations.

Thus, these results suggest the following proposition:

Proposition 4. Suppose the assumptions of Proposition 3 hold. Then, the algebraic calculation of the eigenvalues λ_1 and λ_2 of the Jacobian at a steady state solution $v_{\min} < v < \infty$ and $0 < w < 1$ brings forth that $\lambda_1 > 1$ and $-1 < \lambda_2 < 0$.

In other words: steady state solutions represent oscillating saddle points with v_t as slowly moving variable and w_t as jump variable. With $v_0 = \bar{v} > 0$ historically given, w_0 jumps on the saddle-path along which both variables converge in oscillations towards a steady state solution.

Being assured that steady state solutions of the equilibrium dynamics (71) and (72) exist and are saddle-point stable we are entitled to perform comparative steady analysis.

Comparative Steady State Analysis*: *How a Larger Government Debt to GDP ratio affects the Unemployment Rate

In this section we want to explore how a larger government debt to GDP ratio affects the GDP growth rate and the unemployment rate in a steady state. Lin (2000) finds in a comparable OLG model with endogenous growth and full employment that a higher government debt to GDP ratio raises the GDP growth rate if the growth rate is larger than the real interest rate in the initial steady state while a larger government debt to GDP ratio lowers the GDP growth rate if the real interest rate is higher than the GDP growth rate in the initial steady state. It will be interesting to see whether the growth rate effect of more government debt will in our model with an independent aggregate investment function also depend on the initial GDP growth rate and interest rate difference or not. Moreover, of particular interest is the public debt effect on the unemployment rate which could not be explored by Lin (2000).

In order to be ready to answer these open questions, we use now the steady state version of the intertemporal equilibrium equations (58) and (62), (63) and (65) and differentiate the resulting static equation system totally with respect to γ, G^Y, v, w and b. We obtain the following linear equation system with respect to the total differentials $d\gamma, dG^Y, dv, dw, db$:

$$d\gamma = (G^Y - q)db + bdG^Y + \alpha b / v^2 dv \,, \tag{82}$$

$$dG^Y = (1-\mu)G^Y\left(\frac{d\gamma}{\gamma} + \frac{\alpha}{(1-\alpha)}\frac{dv}{v} + \frac{dw}{w}\right), \tag{83}$$

$$\frac{dG^Y}{G^Y} + \frac{dw}{w} + \frac{dv}{(1-\alpha)v} = 0 \,, \tag{84}$$

$$\frac{dw}{w} + \left[\frac{1}{(1-\alpha)} + \frac{b}{(G^Y v^2)}\right]qdv - \frac{d\gamma}{(G^Y v)} = \frac{qdb}{G^Y v} \,. \tag{85}$$

Solving simultaneously equations (82) and (83) for $d\gamma$ and dG^Y and inserting the resulting equations into equations (84) and (85) we obtain after rearranging the following two-dimensional linear equations system:

$$\begin{aligned} &(1 - j_{11})dv - j_{12}dw = j_{13}db, \\ &- j_{21}dv + (1 - j_{22})dw = j_{23}db, \end{aligned} \tag{86}$$

with

$$j_{13} = \frac{\alpha + (G^Y - q)v}{bG^Y} \,,$$

$$j_{23} = -\frac{\{\alpha[\gamma-(1-\mu)bG^Y] + (1-\mu)\alpha^2 bG^Y + \gamma(G^Y - q)v\}w}{(1-\alpha)b\gamma G^Y v}.$$

In order to get dv/db and dw/db we apply Cramer's rule to the linear equation system (86) the result of which reads as follows:

$$\frac{dv}{db} = \frac{\begin{vmatrix} j_{13} & -j_{12} \\ j_{23} & 1-j_{22} \end{vmatrix}}{\{1-\mu - \frac{[\gamma-(1-\mu)bG^Y]v}{b\gamma} + DetJ\}} \tag{87}$$

and

$$\frac{dw}{db} = \frac{\begin{vmatrix} 1-j_{11} & j_{13} \\ -j_{21} & j_{23} \end{vmatrix}}{\{1-\mu - \frac{[\gamma-(1-\mu)bG^Y]v}{b\gamma} + DetJ\}} \tag{88}$$

Using the elements of the Jacobian matrix (78) and the additional coefficients of the linear equation system (86) to calculate the determinants in (87) and (88) we obtain the following results:

$$\frac{dv}{db} = \frac{\alpha\{[\gamma(2-\mu)-(1-\mu)bG^Y] + (G^Y - q)v[(1-\mu)G^Y v + (2-\mu)\gamma]\}}{b\gamma\, G^Y \{1-\mu - \frac{[\gamma-(1-\mu)bG^Y]v}{b\gamma} + DetJ\}}, \tag{89}$$

$$\begin{aligned}\frac{dw}{db} = &\Big\langle\{(G^Y - q)w\{\alpha(1-\mu)G^Y v^3 + \alpha(1-\alpha)(1-\mu)bv + \gamma[1+\alpha(1-\mu)]v^2\} \\ &+ \{(1-\alpha)(1-\mu)\alpha^2 b + \alpha v[\gamma-(1-\mu)bG^Y] + \alpha^2\gamma(1-\mu)\}v\}\Big\rangle \\ &/\Big\langle (\alpha-1)b\gamma G^Y v^2\{1-\mu - \frac{[\gamma-(1-\mu)bG^Y]v}{b\gamma} + DetJ\}\Big\rangle.\end{aligned} \tag{90}$$

The right hand side of the differential quotient (89) shows that a higher public debt to GDP ratio affects the capital output ratio unambiguously negatively if $\gamma > (1-\mu)bG^Y$ and dynamic inefficiency prevails, i.e. the GDP growth factor is larger than the real interest factor since $1 - TrJ + DetJ < 0$. Under dynamic efficiency, i.e. the real interest factor is larger than the GDP growth factor the response of the capital output ratio to a higher public debt to GDP ratio becomes in general ambiguous. Depending on the numerical values of structural and policy parameters there are dynamic efficient parameter combinations at which the capital output ratio decreases when the debt to GDP ratio increases and other dynamic efficient parameter combinations which induce an increasing capital output ratio with a higher debt to GDP ratio.

More interesting from a policy point of view is the response of the unemployment rate to a higher government debt to GDP ratio which is shown by the term on the right hand side of equation (90). Again, when $\gamma > (1-\mu)bG^Y$ and dynamic inefficiency prevails the response of unity minus the unemployment rate is unambiguously positive and hence a higher public debt to GDP ratio reduces the unemployment rate. Why this is so is partly explained by a look on the output market equilibrium equation (65). It shows that unity minus the unemployment rate balances the fixed investment to GDP ratio with the other aggregate demand to GDP ratios. When due to a higher debt to GDP ratio old consumers demand on account of higher wealth a larger proportion of the GDP the unemployment must fall. Moreover, higher public debt raises the real interest rate which necessitates a decline of the capital output ratio due to profit maximization which additionally increases labor demand and hence reduces unemployment.

While the negative response of the capital output ratio and the unemployment rate to a higher public debt ratio can for dynamic inefficiency demonstrated analytically (see equations (89) and (90)) it is interesting to know whether this is also possible for the HCI expenditure ratio and the GDP growth factor.

The calculation of the steady state HCI expenditure ratio differential from equations (82) and (83) brings forth the following result:

$$d\gamma = \frac{\gamma\{(1-\alpha)(G^Y - q)vw\,db + (1-\alpha)(1-\mu)bvG^Y dw + bqw[G^Y v(1-\mu) + (1-\alpha)]dv\}}{(1-\alpha)[\gamma - (1-\mu)bG^Y]vw}. \quad (91)$$

While on the right hand side of equation (91) all coefficients in front of the differentials dv, dw and db are unambiguously positive for the case of dynamic inefficiency the total reaction of the steady state HCI expenditure ratio to a change in the government debt to GDP ratio cannot be unambiguously stated in general because under dynamic inefficiency w increases while v decreases with a higher debt to GDP ratio. This ambiguity cannot even be resolved if one solves equations (82)-(86) simultaneously with respect to $dv, dw, d\gamma, dG^Y$, and calculates $d\gamma / db$ which yields:

$$\frac{d\gamma}{db} = \frac{\gamma[(G^Y - q)v - qb]}{[\gamma - b(1-\mu)G^Y]v + b(1-\mu)[\alpha + bq - \gamma]}. \quad (92)$$

A similar observation pertains to the steady state response of the GDP growth factor to a higher government debt to GDP ratio which shows the following result from the simultaneous solution of equations (82) und (83):

$$dG^Y = (1-\mu)G^Y\{(1-\alpha)vw(G^Y - q)\,db + (1-\alpha)\gamma v dw + qw[(1-\alpha)b + \gamma v]dv\}/\{(1-\alpha)[\gamma - (1-\mu)bG^Y]vw\}. \tag{93}$$

Again, the ambiguity of the sign of dG^Y/db does not vanish if one calculates dG^Y/db from the simultaneous solution of equations (82)-(86) which yields:

$$\frac{dG^Y}{db} = \frac{(1-\mu)G^Y[(G^Y - q)v + \gamma - qb]}{[\gamma - b(1-\mu)G^Y]v + b(1-\mu)[\alpha + bq - \gamma]}. \tag{94}$$

Because even in case of dynamic inefficiency both the response of the HCI ratio and the GDP growth factor is in general ambiguous we resort now to four typical numerical parameter sets in line with the assumptions of Proposition 3 in order to obtain unambiguously numerical results. The first parameter set (denoted as parameter combination A) induces a dynamic inefficient steady state while the remaining parameter sets (denoted as parameter combinations B, C and D) imply dynamic efficiency.

Parameter combination A: $\beta = 0.6, \varepsilon = 0.2, \alpha = 0.25, \tau = 0.35, \mu = 0.5,$ $A = 10, H_0 = 2$, $\delta = 0.2, \varphi = 3.6, b = 0.024$. Parameter combination B is identical to A with the exception that $\alpha = 0.3, \tau = 0.4, \varphi = 3.$ Parameter set C ist identical to parameter set B with the exception that $\tau = 0.5, \varepsilon = 0.4, \varphi = 1.4$. Finally, parameter set D is identical to parameter set C with the exception of $\alpha = 0.4, \varphi = 1.5$. Parameter set A can be characterized as exhibiting a relatively low tax rate, a high labor income share and an optimistitic animal spirits scenario. Parameter set B exhibits also optimistic animal spirits but the wage tax rate is higher and the labor income share lower than in the case before. Both parameter sets A and B comprise a relatively low youth age propensity to consume. In contrast, the youth age propensity to consume is significantly higher in parameter sets C and D: Moreover, the latter parameter sets exhibit the highest wage tax rate and relatively low animal spirits parameter. The main difference between parameter set C and D is the relatively high capital income share in the latter. The calculation of the steady state solutions for the capital output ratio v , the GDP growth factor G^Y , the HCI expenditure ratio γ , unity minus the unemployment rate w and the real interest factor under all parameter combinations are depicted in the following Table 1.

Table 1. Steady solutions under parameter combinations A-D before the policy shock.

	capital-output ratio	GDP growth factor	One minus unemployment rate	HCI expenditure ratio	Interest factor
A	0.179660	1.79527	0.918128	0.072190	1.39151
B	0.148516	1.82593	0.933794	0.0753423	2.01999
C	0.083712	1.94965	0.925716	0.110782	3.58374
D	0.109539	1.34792	0.956043	0.0447103	3.65165

For the policy shock, assume that b is increased from $b = 0.024$ towards $b = 0.03$. How the steady state solutions change after the policy shock is delineated in the following Table 2.

Table 2. Steady solutions under parameter combinations A-D after the policy shock.

	capital-output ratio	GDP growth factor	One minus unemployment rate	HCI expenditure ratio	Interest factor
A	0.168313	1.84368	0.975277	0.073251	1.48533
B	0.143287	1.81779	0.987242	0.071723	2.09369
C	0.082018	1.87478	0.991294	0.096505	3.65796
D	0.118573	1.21152	0.932070	0.035142	3.37344

The comparison of the steady state solutions before and after the policy shock depicted in Table 1 und Table 2 shows two extreme scenarios from a policy perspective: while the results of higher public debt under parameter combination A can be termed Keynesian, those under parameter combination D resemble neo-classical policy expectations. Higher public debt under dynamic inefficiency in Keynesian case A increases the HCI expenditure ratio, accelerates GDP growth and the real interest rate while it reduces the capital output ratio and the unemployment rate. On the contrary, higher public debt under dynamic efficiency in neo-classical case D reduces the HCI expenditure ratio, the GDP growth and the real interest rate, while it raises the capital output ratio and the unemployment rate. The latter is remarkable because only in case D higher public debt raises the unemployment rate. Parameter combination C is close to parameter combination D with the expception of a lower capital income share which is associated with the unemployment reducing effect of higher public debt. Parameter combinations B and C are both dynamically efficient scenarios (as combination D) with rather optimistic animal spirits but the wage tax rate and youth age marginal propensity to consumption are lower in case B than in case C. As a consequence, a higher public debt triggers in both cases qualitatively the same response of main endogenous variables: the HCI expenditure ratio, the GDP growth rate, the capital output ratio and the unemployment rate decline while the real interest rate rises.

An empirical application: Debt reduction in Euro Area and selected European countries

Before concluding, we apply the theoretical framework of a large closed economy presented in previous section on economic areas with approximately balanced current accounts. We present both a dynamically inefficient scenario, i.e. the real interest factor is smaller than the GDP growth factor, and a dynamically efficient scenario, i.e. the real interest factor is larger than the GDP growth factor. The first sample is called the “dynamic inefficiency sample” (Sample DYNIE) which consists of 19 Euro Area countries. The second sample is called the „dynamic efficiency sample“ (Sample DYNE) and consists of 12 selected European countries, i.e. Austria, Bulgaria, Croatia, Germany, Greece, Hungary, Italy, Netherlands, Portugal, Romania, Slovenia, and Spain. For our empirical application, we mainly use data from the AMECO database of the European Commission. Data on average labor income tax rates come from the OECD tax database and the annual reports on taxation trends of the European Commission. Regarding the closed economy feature of our modeling framework, note that the current account of the Euro Area has been roughly balanced over the period 2008 to 2016. The same holds true for the selected European countries of the second sample as a whole.

The numerical specification of all model parameters proceeds in several steps. First, we identify the numerical values of the fiscal policy parameters and the unemployment rate corresponding to DYNIE as follows: the HCI expenditure ratio, the tax rate on labor income and the unemployment rate in the Euro Area exhibit on average 5.26%, 37.37% and 9.20%, respectively. For the numerical values of the fiscal policy parameters and the unemployment rate corresponding to the DYNE we encounter an average HCI expenditure ratio of 5.26%, an average tax rate on labor income of 37.37% and an average unemployment rate of 9.20% in the selected European countries.

Second, we fix in the DYNIE the GDP growth factor at 1,711239. This corresponds to a yearly growth rate of 2.17% in the Euro Area. The DYNE GDP growth factor is fixed at 1,511787. This corresponds to a yearly growth rate of 1.67% in the selected European countries. We set the DYNIE real interest factor at 1,669357 and the DYNE real interest factor at 1,719198, which is equivalent to an average annual real interest rate of 2.07% in the Euro Area and of 2.19% in the selected European countries respectively. The DYNIE public debt to GDP ratio is fixed at 0,024706. This corresponds to a yearly Euro Area public debt to GDP ratio of 61.77%. The public debt to GDP ratio DYNE is fixed at 0,027157. This corresponds to a yearly average public debt to GDP ratio of the selected countries of 67.89%.

Third, there are some structural parameter values, which we take from the relevant literature. These include the total factor productivity A, the utility elasticity of consumption in youth ε, the level parameter H_0 and the human capital production elasticity μ. The total factor productivity in DYNIE is set at 17, that in DYNE is set at 14.5 (see the similar values in Auerbach and Kotlikoff, 1998). The utility elasticity of consumption in youth ε, the level parameter H_0 and the human capital production elasticity μ is set in both samples at 0.5, 1.65 and 0.45 respectively (for the numerical value of the utility elasticity of consumption in youth see Auerbach and Kotlikoff, 1987, for the level parameter as well as for the human capital production elasticity see Lin, 2000).

Forth we calibrate the remaining structural and policy parameters α, β and δ such that the steady state versions of the equations (58), (62) and (71) hold and get the following calibration results for DYNIE: $\alpha = 0.260379,\ \beta = 0.903367,\ \delta = 0.224878$ and for DYNE: $\alpha = 0.285637,\ \beta = 0.938301,\ \delta = 0.228447$. The calibrated parameter values for the capital income share, the future utility discount factor und the non-HCI government expenditure ratio correspond roughly to the parameters values found in the literature (see e.g. De la Croix and Michel (2002)).

Given all parameter values so far we are able to calculate the benchmark solutions for the capital-output-ratios from the steady state version of equation (71). The calculation yields a DYNIE capital-output-ratio of 0.155976 and a DYNE capital-output-ratio of 0.166146 which corresponds to a yearly capital-output-ratio of 3.90 in the Euro Area and to a yearly capital-output-ratio in the selected European countries of 4.15, respectively. Finally, the animal spirits parameter results from equation (65) and is equal to 5.807785 in DYNIE and equals 4.681602 in DYNE.

Knowing the benchmark solution for both samples, we are now in a position to quantify the steady state impacts of an increase in public debt from $b = 0.024$ to $b = 0.03$, i.e. an increase of 25%. Table 3 reports the results for DYNIE.

Table 3. Steady solutions before and after the policy shock in DYNIE.

Benchmark solution		Public debt increased by 25%	
Capital output ratio	0.155976	Capital output ratio	0.150248
GDP growth factor	1.711239	GDP growth factor	1.715358
Unity minus unemployment rate	0.908034	One minus unemployment rate	0.952856
HCI expenditure ratio	0.052570	HCI expenditure ratio	0.051006
Interest factor	1.669357	Interest factor	1.732998

Comparing the benchmark case to the policy case of an increased public debt we see that the overall effects on the capital output ratio, the GDP growth factor, the unemployment rate and the real interest rate are the same as under parameter combination A. Higher public debt increases the GDP growth factor and the real interest rate while it reduces the capital output ratio and the unemployment rate. Only the HCI expenditure ratio declines whereas under parameter combination A a higher public debt to GDP ratio increases the HCI expenditure ratio. The reason is that the numerical values of the GDP growth factor and the real interest factor in DYNIE are too close to each other such that the positive impact of the bracket term $\left(G^{Y} - q\right)$ in equation (82) is too small to boost the HCI expenditure ratio due to higher public debt as under parameter combination A.

Finally, we present the steady state solutions before and after the policy shock for DYNE. The results are reported in Table 4.

Table 4. Steady solutions before and after the policy shock for DYNE.

Benchmark solution		Public debt increased by 25%	
Capital output ratio	0.166146	Capital output ratio	0.164577
GDP growth factor	1.511787	GDP growth factor	1.501886
One minus unemployment rate	0.904402	One minus unemployment rate	0.922534
HCI expenditure ratio	0.047899	HCI expenditure ratio	0.046521
Interest factor	1.719198	Interest factor	1.735585

In DYNE a higher public debt to GDP ratio boosts the real interest rate and impacts negatively the capital output ratio, the GDP growth factor, the unemployment rate and the HCI expenditure ratio. Thus, we can observe the same response of endogenous variables as under parameter combination B.

4. Conclusions

As outlined in the introduction this paper seeks to integrate involuntary unemployment in a neoclassical growth model with internal public debt and human capital accumulation. In contrast to mainstream new-Keynesian macro models in which involuntary unemployment is traced back to inflexible wages, output prices and interest rates vis-à-vis market imbalances, real wages and real interest rates are perfectly flexible in our basic neo-classical growth model with internal public debt à la Diamond (1965). The real wages and the real interest rate are also perfectly flexible in the subsequent OLG model extended by human capital accumulation. In both growth models involuntary unemployment occurs since in line with Morishima (1977) and Magnani (2015) aggregate investment is inflexible due to investors' animal spirits.

Both for the functionally specified Diamond (1965) and the extended OLG model with inflexible aggregate investment we demonstrate analytically the existence and saddle-path stability of a steady state solution of the intertemporal equilibrium dynamics. While in the basic model the government has to choose a certain tax rate such that with historically given public debt and private real capital stocks a steady state solution exists and is immediately reached, in the extended model the capital output ratio converges in damped oscillations from a historically given towards its steady state level.

Being assured of the existence and dynamic stability of steady state solutions we performed comparative steady state analysis for both models. For the basic model we could show analytically that less optimistic animal spirits of investors and thriftier youth age households raise the unemployment rate as in Keynesian macro models. Hereby we confirm the main results which Magnani (2015) obtains in his Solow (1956) growth model with inflexible aggregate investment. Due to the exogenity of growth and full employment Magnani (2015) could not investigate the effects of public debt on GDP growth and unemployment in Solow's neo-classical model. What Magnani (2015) was not able to do we investigated in the OLG model extended by human capital accumulation and an exogenously fixed government debt to GDP ratio. Only for the case of dynamic inefficiency where the initial GDP growth rate is higher than the real interest rate we could demonstrate analytically that a higher public debt raises the real interest rate while both the capital output ratio and the unemployment rate decline. The effect on the real GDP growth rate as well as on the HCI expenditure ratio is in general ambiguous.

With dynamic efficiency the effects of higher public debt on main endogenous variables are in general ambiguous. However, for a parameter set with a rather high capital income share, a high wage tax rate and a high youth-age propensity to consume we obtained results in line with neo-classical policy expectations: higher public debt raises the unemployment rate and the capital output ratio while it reduces the HCI expenditure ratio, the GDP growth rate and (somewhat unexpected) the real interest rate. Two other dynamically efficient scenarios in between the extreme scenarios A and D demonstrate that higher public debt is associated with lower unemployment although GDP growth and HCI expenditures are lower. The main reason for this result is that higher public debt creates a positive wealth effect with old-age consumers which raises aggregate demand and hence reduces unemployment.

The limitations of the present research results are obvious. First, a more general interest-rate dependent aggregate investment function as indicated above could be used in order to see whether the main results we obtained carry over to the more general case. Second, stock-market foundations for the aggregate investment function in line with Farmer's (2012, 2013) investor's belief function should be provided in order to overcome the purely macro-foundation of the aggregate investment function. Both subjects are left to future research.

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