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Telefon: 0264 405300

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On the existence of solutions to psi-Hilfer fractional neutral integro-differential equations with delay

Vinitha Ravi , Nirmalkumar Rajendran  and Shyamsunder 


Abstract. This paper investigates a class of neutral-type fractional differential equations with finite delays, formulated through the generalized Ψ -Hilfer fractional derivative. This operator, being a broad framework that unifies various fractional derivatives, is highly effective in modeling dynamical processes with memory and hereditary characteristics. The primary objective is to establish sufficient conditions for the existence and uniqueness of solutions to such equations. The analysis employs fixed point theory—specifically Banach’s contraction principle and Krasnoselskii’s fixed point theorem—within an appropriately weighted function space. These tools ensure that the solutions are not only well-defined but also uniquely determined. Furthermore, two stability notions, namely Ulam–Hyers stability and its generalized form, are studied to verify that solutions remain close to the expected behavior under small perturbations in initial conditions or parameters. To demonstrate the applicability of the theoretical framework, an illustrative example with explicit functions and parameters is provided. The results strengthen the theoretical foundations of fractional calculus and open directions for further research on more generalized and complex delayed fractional systems.

Mathematics Subject Classification (2010): 26A33, 34K37, 34A12, 34K40, 34K20.

Keywords: Ψ -Hilfer fractional derivative, fractional integro-differential equations, existence and uniqueness, Ulam–Hyers Stability.

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Abbreviations

- Fractional Differential Equation (\mathcal{FDE})
- Fractional Integral Equation (\mathcal{FIE})
- Fractional Calculus (\mathcal{FC})
- Fractional Integral (\mathcal{FI})
- Fractional Derivatives (\mathcal{FD})
- Riemann-Liouville (\mathcal{RL})
- Ψ -Hilfer Fractional Derivative ($\Psi - \mathcal{HFD}$)
- Ulam-Hyers Stability (\mathcal{UHS})
- Ulam-Hyers Rassias Stability (\mathcal{UHRs})
- Fractional Functional Integro-Differential Equation (\mathcal{FIDE})

1. Introduction

\mathcal{FC} has emerged as a powerful framework in modern mathematics and applied sciences, owing to its capability of describing processes influenced by memory and hereditary characteristics. Such systems frequently appear in diverse fields, including control systems, viscoelastic materials, biomedical engineering, signal transmission, and fluid mechanics. Classical integer-order models, fractional formulations provide greater flexibility and accuracy in capturing the underlying dynamics. Consequently, both theoretical investigations and practical applications of \mathcal{FD} and \mathcal{FI} have received considerable attention in recent decades [3, 5, 11].

To address different modeling needs, several operators have been introduced in the literature, such as \mathcal{RL} , Caputo, and Hadamard derivatives. A more generalized form is the Ψ -Hilfer operator, which integrates these classical definitions into a unified structure. Each operator offers unique interpretations, enhancing the scope of \mathcal{FC} [3, 10, 11, 12].

Parallel to these developments, significant research has focused on the solvability of \mathcal{FDE} and \mathcal{FIE} . Both linear and nonlinear models have been investigated using advanced mathematical tools, including fixed point theory, semigroup methods, and techniques from functional analysis [1, 2]. Moreover, stability analysis—particularly \mathcal{UHS} and its extended form, \mathcal{UHRs} —plays a vital role in assessing the reliability of fractional models when subjected to perturbations. These approaches have recently been combined with Ψ -Hilfer derivatives to establish stability results for nonlinear equations [7, 8, 9].

Although a wide range of studies have considered fractional systems without delays or impulsive effects, fewer works address models involving both neutral terms and delay components within the Ψ -Hilfer framework. Foundational contributions can be found in [13, 14], yet further research is necessary to explore existence and stability under such conditions.

Motivated by this gap, the present paper investigates a class of \mathcal{FIDE} with delay, governed by the Ψ -Hilfer fractional derivative. The general model under study is expressed as:

$$\begin{cases} \mathcal{H}\mathcal{D}_{\mathbf{a}_1^+}^{\varsigma_1, \varsigma_2; \Psi} \left[\mathbf{f}(\iota) - \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1^+}^{\xi_i} \mathcal{Z}_i(\iota, \mathbf{f}_i) \right] & = \mathcal{G}(\iota, \mathbf{f}_\iota), \quad \iota \in \mathcal{J} = [\mathbf{a}_1, \mathbf{b}_1], \\ \mathbf{f}(\iota) & = \Phi(\iota), \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1], \end{cases} \quad (1.1)$$

Here, $\mathcal{H}\mathcal{D}_{\mathbf{a}_1^+}^{\varsigma_1, \varsigma_2; \Psi}$ denotes the Ψ - \mathcal{HFD} of order $\varsigma_1 \in (0, 1)$ and type $\varsigma_2 \in [0, 1]$, and $\mathcal{I}_{\mathbf{a}_1^+}^{\xi_i; \Psi}$ represents the Ψ - \mathcal{RL} fractional integral. The functions \mathcal{Z}_i and \mathcal{G} are assumed to be continuous with respect to their arguments, and the initial condition Φ provides the required history for the system.

The remainder of this paper is organized as follows. Section 2 introduces the required background, notations, and fundamental tools from Ψ -Hilfer calculus and fixed point theory. Section 3 establishes existence, uniqueness, and \mathcal{UHS} . Section 4 provides a detailed illustrative example that supports the theoretical results.

2. Preliminaries

In the following section, to introduce the essential preliminaries, including definitions, notations, and results from Ψ -Hilfer fractional calculus and nonlinear analysis, required for the later developments. These definitions are adapted to the weighted function space framework appropriate for our study.

Let $\mathcal{J} = [\mathbf{a}_1, \mathbf{b}_1] \subset \mathcal{R}$ be a compact interval, and let $\Psi \in \mathcal{C}^1(\mathcal{J}, \mathcal{R})$ be an increasing function with $\Psi'(\tau) \neq 0 \quad \forall \tau \in \mathcal{J}$. This function Ψ defines the generalized fractional operator's kernel.

We introduce the following weighted spaces

$$\mathcal{C}_{\Psi}^{\eta}(\mathcal{J}) = \{ \mathbf{f} : \mathcal{J} \rightarrow \mathcal{R} \mid (\Psi(\iota) - \Psi(\mathbf{a}_1))^{\eta} \mathbf{f}(\iota) \in \mathcal{C}(\mathcal{J}) \}$$

with norm:

$$\| \mathbf{f} \|_{\mathcal{C}_{\Psi}^{\eta}} = \sup_{\iota \in \mathcal{J}} |(\Psi(\iota) - \Psi(\mathbf{a}_1))^{\eta} \mathbf{f}(\iota)|.$$

$$\mathcal{C}_{\Psi, \delta}^{\eta}(\mathcal{J}) = \{ \mathbf{f} : [\mathbf{a}_1 - \delta, \mathbf{b}_1] \rightarrow \mathcal{R} \mid (\Psi(\iota) - \Psi(\mathbf{a}_1))^{\eta} \mathbf{f}(\iota) \in \mathcal{C}([\mathbf{a}_1 - \delta, \mathbf{b}_1]) \}$$

with norm:

$$\| \mathbf{f} \|_{\mathcal{C}_{\Psi}^{\eta}} = \sup_{\iota \in [\mathbf{a}_1 - \delta, \mathbf{b}_1]} |(\Psi(\iota) - \Psi(\mathbf{a}_1))^{\eta} \mathbf{f}(\iota)|.$$

For a fixed delay parameter $\delta > 0$, we denote $\mathcal{C}_{\delta} = \mathcal{C}([-\delta, 0], \mathcal{R})$ with norm:

$$\| \phi \|_{\mathcal{C}_{\delta}} = \sup_{\tau \in [-\delta, 0]} |\phi(\tau)|.$$

For $\mathbf{f} \in \mathcal{C}([\mathbf{a}_1 - \delta, \mathbf{b}_1], \mathcal{R})$, the delay norm is given by

$$\| \mathbf{f}_{\tau} \|_{\mathcal{C}_{\delta}} = \sup_{\rho \in [-\delta, 0]} |\mathbf{f}(\tau + \rho)|.$$

Let $\mathcal{C}_{\Psi}^{\eta} = \mathcal{C}([\mathbf{a}_1 - \delta, \mathbf{b}_1], \mathcal{R})$ denote the Weighted space of real-valued continuous function defined on the extended interval $[\mathbf{a}_1 - \delta, \mathbf{b}_1]$ with the norm:

$$\|f\|_{\mathcal{C}_{\Psi}^{\eta}} = \sup_{\tau \in [\mathbf{a}_1 - \delta, \mathbf{b}_1]} |f(\tau)|.$$

Definition 2.1. [13]: Let $\varsigma_1 > 0$. The Ψ - $\mathcal{RL} - \mathcal{FI}$ of a function f on \mathcal{J} is given by

$$\mathcal{I}_{\mathbf{a}_1+}^{\varsigma_1; \Psi} f(\iota) = \frac{1}{\Gamma(\varsigma_1)} \int_{\mathbf{a}_1}^{\iota} \Psi'(\mathfrak{s}) (\Psi(\iota) - \Psi(\mathfrak{s}))^{\varsigma_1-1} f(\mathfrak{s}) d\mathfrak{s}.$$

Definition 2.2. [13] For $0 < \varsigma_1 < 1$ and $0 \leq \varsigma_2 \leq 1$, the Ψ - \mathcal{HFD} of a function f is expressed as

$$\mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1; \Psi} f(\iota) = \left(\frac{1}{\Psi'(\iota)} \frac{d}{d\iota} \right)^{\mathbf{n}} (\mathcal{I}_{\mathbf{a}_1+}^{(\mathbf{n}-\varsigma_1); \Psi}) f(\iota), \quad \mathbf{n} = [\varsigma_1] + 1$$

Definition 2.3. [13] For $\mathbf{n} - 1 < \varsigma_1 < \mathbf{n} (\mathbf{n} \in \mathcal{N})$ and $f, \Psi \in \mathcal{C}^{\mathbf{n}}(\mathcal{J}, \mathcal{R})$, then Ψ - Caputo fractional derivative of a function f of order ς_1 is defined by

$${}^{\mathcal{C}}\mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1; \Psi} f(\iota) = \mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1; \Psi} \left(f(\iota) - \sum_{i=0}^{\mathbf{n}-1} \frac{f_{\Psi}^{[i]}(\mathbf{a}_1)}{k!} (\Psi(\iota) - \Psi(\mathfrak{s}))^i \right)$$

Where $\mathbf{n} = [\varsigma_1] + 1$ for $\varsigma_1 \neq \mathcal{N}$, and $f_{\Psi}^{[i]}(\iota) = \left(\frac{1}{\Psi'(\iota)} \frac{d}{d\iota} \right)^i f(\iota)$.

In particular, if $\mathbf{n} = \varsigma_1$, we have ${}^{\mathcal{C}}\mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1; \Psi} f(\iota) = f_{\Psi}^{[\mathbf{n}]}(\iota)$.

Definition 2.4. ([13]) For $\mathbf{n} - 1 < \varsigma_1 < \mathbf{n} (\mathbf{n} \in \mathcal{N})$ and $f, \Psi \in \mathcal{C}^{\mathbf{n}}(\mathbf{a}_1, \mathbf{b}_1)$ such that Ψ is an increasing with $\Psi'(\iota) \neq 0$ for all $\iota \in [\mathbf{a}_1, \mathbf{b}_1]$. Then the left-sided Ψ -Hilfer fractional derivative of f of order ς_1 and type $0 \leq \varsigma_2 \leq 1$ is defined by

$${}^{\mathcal{H}}\mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1, \varsigma_2; \Psi} f(\iota) = \mathcal{I}_{\mathbf{a}_1+}^{\varsigma_2(\mathbf{n}-\varsigma_1); \Psi} \left(\frac{1}{\Psi'(\iota)} \frac{d}{d\iota} \right)^{\mathbf{n}} (\mathcal{I}_{\mathbf{a}_1+}^{(1-\varsigma_2)(\mathbf{n}-\varsigma_1); \Psi}) f(\iota)$$

One has

$${}^{\mathcal{H}}\mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1, \varsigma_2; \Psi} f(\iota) = \mathcal{I}_{\mathbf{a}_1+}^{\varsigma_2(\mathbf{n}-\varsigma_1); \Psi} \mathcal{D}_{\mathbf{a}_1+}^{\eta; \Psi} f(\iota)$$

where

$$\mathcal{D}_{\mathbf{a}_1+}^{\eta; \Psi} f(\iota) = \left(\frac{1}{\Psi'(\iota)} \frac{d}{d\iota} \right)^{\mathbf{n}} (\mathcal{I}_{\mathbf{a}_1+}^{(\mathbf{n}-\eta); \Psi}) f(\iota), \quad \eta = \varsigma_1 + \varsigma_2(\mathbf{n} - \varsigma_1)$$

Lemma 2.5. [13] If $\varsigma_1, \varsigma_2 > 0$, $f \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ then

$$(\mathcal{I}_{\mathbf{a}_1+}^{\varsigma_1; \Psi}) (\mathcal{I}_{\mathbf{a}_1+}^{\varsigma_2; \Psi}) f(\iota) = \mathcal{I}_{\mathbf{a}_1+}^{(\varsigma_1+\varsigma_2); \Psi} f(\iota)$$

Theorem 2.6. [13] If $\varsigma_1, \sigma > 0$, and $0 \leq \varsigma_2 \leq 1$. Then

$$\mathcal{I}_{\mathbf{a}_1+}^{\varsigma_1; \Psi} [\Psi(\iota) - \Psi(\mathbf{a}_1)]^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\varsigma_1 + \sigma)} (\Psi(\iota) - \Psi(\mathbf{a}_1))^{\varsigma_1 + \sigma - 1}$$

and

$${}^{\mathcal{H}}\mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1, \varsigma_2; \Psi} [\Psi(\iota) - \Psi(\mathbf{a}_1)]^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma - \varsigma_1)} (\Psi(\iota) - \Psi(\mathbf{a}_1))^{\sigma - \varsigma_1 - 1}.$$

Theorem 2.7. [13] Let $f \in \mathcal{C}_{\eta[\mathbf{a}_1, \mathbf{b}_1]}$, $0 < \eta < \varsigma_1 < 1$. Then we have

$$\mathcal{I}_{\mathbf{a}_1+}^{\varsigma_1; \Psi} f(\mathbf{a}_1) = \lim_{\iota \rightarrow \mathbf{a}_1+} \mathcal{I}_{\mathbf{a}_1+}^{\varsigma_1; \Psi} f(\iota) = 0.$$

Theorem 2.8. (Banach fixed point theorem) Let (\mathcal{X}, d) be a nonempty complete metric space, and let $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a contraction mapping. Then, there exists a unique point $\mathfrak{x} \in \mathcal{X}$ such that $\mathcal{T}(\mathfrak{x}) = \mathfrak{x}$; that is, \mathfrak{x} is a fixed point of \mathcal{T} .

Theorem 2.9. Let \mathcal{X} be a Banach space, and let $\mathcal{S} \subset \mathcal{X}$ be a nonempty, closed, bounded, and convex subset. Consider two mappings $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{S} \rightarrow \mathcal{X}$ such that for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{S}$, the point $\mathcal{T}_1(\mathfrak{x}) + \mathcal{T}_2(\mathfrak{y})$ belongs to \mathcal{S} . If \mathcal{T}_1 is a contraction and \mathcal{T}_2 is completely continuous, then the equation $\mathfrak{x} = \mathcal{T}_1\mathfrak{x} + \mathcal{T}_2\mathfrak{y}$ admits at least one solution $\mathfrak{x} \in \mathcal{S}$.

3. Main results

For convenience define the parameter:

$$\begin{aligned} \Lambda &= \Lambda_1 + \Lambda_2 \\ &= \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \|\omega\| + \sum_{i=1}^n \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^{\xi_i}}{\Gamma(\xi_i + 1)} \|\nu_i\| \end{aligned} \tag{3.1}$$

The assumptions listed below will be used in establishing the main results.

- (A₁) The function $\mathcal{G}, \mathcal{Z}_i : \mathcal{J} \times \mathcal{C}_\delta \rightarrow \mathcal{R}$ are continuous for each $i = 1, 2, \dots, n$.
- (A₂) There exist measurable function $\omega(\iota), \nu_i(\iota)$ such that for all $\mathfrak{f}, \bar{\mathfrak{f}} \in \mathcal{C}_\delta$ and $\iota \in \mathcal{J}$,

$$|\mathcal{G}(\iota, \mathfrak{f}) - \mathcal{G}(\iota, \bar{\mathfrak{f}})| \leq \omega(\iota) \|\mathfrak{f} - \bar{\mathfrak{f}}\|_{\mathcal{C}_\delta},$$

$$|\mathcal{Z}_i(\iota, \mathfrak{f}) - \mathcal{Z}_i(\iota, \bar{\mathfrak{f}})| \leq \nu_i(\iota) \|\mathfrak{f} - \bar{\mathfrak{f}}\|_{\mathcal{C}_\delta}$$

- (A₃) There exist constants $\Phi, \Phi_i \geq 0$ for all $(\iota, \mathfrak{f}) \in \mathcal{J} \times \mathcal{C}_\delta$,

$$|\mathcal{G}(\iota, \mathfrak{f})| \leq \Phi \|\mathfrak{f}\|_{\mathcal{C}_\delta}, |\mathcal{Z}_i(\iota, \mathfrak{f})| \leq \Phi_i.$$

Definition 3.1. A function $f \in \mathcal{C}_{\Psi, \delta}^\eta$ is said to be a solution of (1) if it satisfies

$$\mathcal{H} \mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1, \varsigma_2; \Psi} \left[f(\iota) - \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i} \mathcal{Z}_i(\iota, f_i) \right] = \mathcal{G}(\iota, f_\iota), \quad \iota \in \mathcal{J} = [a, b],$$

and the initial condition $f(\iota) = \Phi(\iota), \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1]$.

Lemma 3.2. Let $0 < \varsigma_1, \sigma < 1, 0 \leq \varsigma_2 \leq 1$ and $\Psi'(\iota) \neq 0$ on $\mathcal{J} = [a, b]$. If $\mathfrak{h}, \mathfrak{g} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ with $\mathfrak{h}(\mathbf{a}_1) = 0$ and $\lambda(\iota) = 0$, then the linear problem

$$\begin{aligned} \mathcal{H} \mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1, \varsigma_2; \Psi} [f(\iota) - \mathfrak{h}(\iota)] &= \mathfrak{g}(\iota), \quad \iota \in \mathcal{J}, \\ f(\iota) &= \lambda(\iota), \quad \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1], \end{aligned}$$

has a unique solution given by

$$f(\iota) = \begin{cases} \mathfrak{h}(\iota) + \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi} \mathfrak{g}(\iota), & \iota \in \mathcal{J}, \\ \lambda(\iota), & \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1] \end{cases}$$

Theorem 3.3. *If assumption $(A_1) - (A_2)$ hold and*

$$\Lambda < 1 \tag{3.2}$$

then equation (1.1) admits a unique solution on $[\mathbf{a}_1 - \delta, \mathbf{b}_1]$.

Proof. Let $\mathcal{T} : \mathcal{C}_{\Psi, \delta}^\eta \rightarrow \mathcal{C}_{\Psi, \delta}^\eta$ as

$$(\mathcal{T}\mathbf{f})(\iota) = \begin{cases} \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi} [\mathcal{G}(\iota, \mathbf{f}_\iota)] + \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i; \Psi} [\mathcal{Z}_i(\iota, \mathbf{f}_\iota)], & \iota \in \mathcal{J}, \\ \lambda(\iota), & \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1]. \end{cases}$$

Now we need to show that \mathcal{T} is a contraction under the Λ -condition.

Using assumption (A_2) , for $\iota \in \mathcal{J}$,

$$\begin{aligned} |\mathcal{T}_{\mathbf{f}}(\iota) - \mathcal{T}_{\bar{\mathbf{f}}}(\iota)| &\leq \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi} |\mathcal{G}(\iota, \mathbf{f}_\iota) - \mathcal{G}(\iota, \bar{\mathbf{f}}_\iota)| + \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i; \Psi} |\mathcal{Z}_i(\iota, \mathbf{f}_\iota) - \mathcal{Z}_i(\iota, \bar{\mathbf{f}}_\iota)| \\ &\leq \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \|\omega\| \|\mathbf{f}_\iota - \bar{\mathbf{f}}_\iota\|_{\mathcal{C}_\delta} \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi}(\mathbf{1})(\iota) \\ &\quad + \sum_{i=1}^n \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^{\xi_i}}{\Gamma(\xi_i + 1)} \|\nu_i\| \|\mathbf{f}_\iota - \bar{\mathbf{f}}_\iota\|_{\mathcal{C}_\delta} \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi}(\mathbf{1})(\iota) \\ &\leq \left[\frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \|\omega\| + \sum_{i=1}^n \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^{\xi_i}}{\Gamma(\xi_i + 1)} \|\nu_i\| \right] \|\mathbf{f}_\iota - \bar{\mathbf{f}}_\iota\|_{\mathcal{C}_{\Psi, \delta}^\eta} \\ &\leq (\Lambda_1 + \Lambda_2) \|\mathbf{f}_\iota - \bar{\mathbf{f}}_\iota\|_{\mathcal{C}_{\Psi, \delta}^\eta}. \end{aligned}$$

Hence,

$$\|\mathcal{T}_{\mathbf{f}} - \mathcal{T}_{\bar{\mathbf{f}}}\|_{\mathcal{C}_{\Psi, \delta}^\eta} \leq \Lambda \|\mathbf{f}_\iota - \bar{\mathbf{f}}_\iota\|_{\mathcal{C}_{\Psi, \delta}^\eta}.$$

Since $\Lambda < 1$, \mathcal{T} is a contraction and Theorem 2.8 guaranties a unique fixed point, which corresponds to the unique solution of (1.1) □

Theorem 3.4. *If assumptions $(A_1) - (A_3)$ are satisfied and*

$$\frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \Phi < 1. \tag{3.3}$$

then (1) has at least one solution on $[\mathbf{a}_1 - \delta, \mathbf{b}_1]$

Proof. Suppose $\mathcal{C}_{\Psi, \delta}^\eta = \mathcal{C}([\mathbf{a}_1 - \delta, \mathbf{b}_1], \mathcal{R})$ be the weighted space of continuous real-valued functions with the supremum norm, and define the closed ball:

$$\mathcal{B}_\varrho = \{\mathbf{f} \in \mathcal{C}_{\Psi, \delta}^\eta : \|\mathbf{f}\|_{\mathcal{C}_{\Psi, \delta}^\eta} \leq \varrho\},$$

where

$$\varrho \geq \frac{\sum_{i=1}^n \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^{\xi_i}}{\Gamma(\xi_i + 1)} \Phi_i}{\mathbf{1} - \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \Phi}$$

Define the operator $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ where

$$\mathcal{T}_1(\mathbf{f})(\iota) = \begin{cases} \mathcal{I}_{\mathbf{a}_1+}^{\eta;\Psi}[\mathcal{G}(\iota, \mathbf{f}_\iota)], & \iota \in \mathcal{J}, \\ 0, & \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1]. \end{cases}$$

$$\mathcal{T}_2(\mathbf{f})(\iota) = \begin{cases} \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i;\Psi}[\mathcal{Z}_i(\iota, \mathbf{f}_\iota)], & \iota \in \mathcal{J}, \\ \lambda, & \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1]. \end{cases}$$

Step 1: $\mathcal{T}_1(\mathbf{f}) + \mathcal{T}_2(\mathbf{f}) \in \mathcal{B}_\varrho$.

For $\iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1]$,

$$|\mathcal{T}_1(\mathbf{f})(\iota) + \mathcal{T}_2(\mathbf{f})(\iota)| \leq |\lambda(\iota)| \leq \|\lambda\|_{\mathcal{C}_\delta} \leq \varrho.$$

For $\iota \in \mathcal{J}$, using assumption (A_3) , we estimate

$$\begin{aligned} |\mathcal{T}_1(\mathbf{f})(\iota) + \mathcal{T}_2(\mathbf{f})(\iota)| &\leq \mathcal{I}_{\mathbf{a}_1+}^{\eta;\Psi}|\mathcal{G}(\iota, \mathbf{f}_\iota)| + \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i;\Psi}|\mathcal{Z}_i(\iota, \mathbf{f}_\iota)| \\ &\leq \Phi \|\mathbf{f}\|_{\mathcal{C}_{\Psi,\delta}^\eta} \frac{(\Psi(\iota) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} + \sum_{i=1}^n \frac{(\Psi(\iota) - \Psi(\mathbf{a}_1))^{\xi_i}}{\Gamma(\xi_i + 1)} \Phi_i \\ &\leq \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \Phi \varrho + \sum_{i=1}^n \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^{\xi_i}}{\Gamma(\xi_i + 1)} \Phi_i \\ &\leq \varrho. \end{aligned}$$

So, $\mathcal{T}_1(\mathbf{f}) + \mathcal{T}_2(\mathbf{f}) \in \mathcal{B}_\varrho$.

Step 2: Since \mathcal{T} is a contraction as shown in Theorem 3.3, it follows that \mathcal{T}_1 also satisfies the contraction property.

Step 3: \mathcal{T}_2 is completely continuous on \mathcal{B}_ϱ .

Due to the continuity of $\mathcal{Z}_i(\cdot, \mathbf{f}(\cdot))$ and $\lambda(\cdot)$, it follows that \mathcal{T}_2 is continuous.

Moreover

$$\|\mathcal{T}_2\mathbf{f}\|_{\mathcal{C}_\delta} = \sup_{\iota \in [-\delta, 0]} |\mathcal{T}_2\mathbf{f}(\iota)| = \sup_{\iota \in [-\delta, 0]} |\lambda(\iota)| = \|\lambda\|_{\mathcal{C}_\delta} \leq \|\lambda\|_{\mathcal{C}_{\Psi,\delta}^\eta} \leq \varrho, \quad \mathbf{f} \in \mathcal{B}_\varrho.$$

and

$$\|\mathcal{T}_2\mathbf{f}\|_{\mathcal{C}} = \sup_{\iota \in \mathcal{J}} |\mathcal{Z}_2\mathbf{f}(\iota)| \leq \sum_{i=1}^n \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^{\xi_i}}{\Gamma(\xi_i + 1)} \Phi_i = \mathcal{L}, \quad \mathbf{f} \in \mathcal{B}_\varrho.$$

Thus, combining these estimates

$$\|\mathcal{T}_2\mathbf{f}\|_{\mathcal{C}_{\Psi,\delta}^\eta} \leq \varrho + \mathcal{L}$$

This confirms that \mathcal{T}_2 is uniformly bounded on the set \mathcal{B}_ϱ .

It remains to establish the equicontinuity of \mathcal{T}_2 on \mathcal{B}_ϱ .

The operator derivative can be estimate as follows,

$$\begin{aligned} |(\mathcal{T}_2)^{(1)}(\iota)| &\leq \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i-1; \Psi} |\mathcal{Z}_i(\iota, \mathfrak{f}_\iota)| \\ &\leq \sum_{i=1}^n \Phi_i \mathcal{I}_{\mathbf{a}_1+}^{\xi_i-1; \Psi} (1)(\iota) \\ &\leq \sum_{i=1}^n \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^{\xi_i-1}}{\Gamma(\xi_i)} \Phi_i = l \end{aligned}$$

Now, for any $\iota_1, \iota_2 \in \mathcal{J}$,

$$|(\mathcal{T}_2\mathfrak{f})(\iota_2) - (\mathcal{T}_2\mathfrak{f})(\iota_1)| = \int_{\iota_1}^{\iota_2} |(\mathcal{T}_2)(s)| ds \leq l(\iota_2 - \iota_1).$$

As $\iota_2 - \iota_1 \rightarrow 0$, this expression tends to zero uniformly, proving equicontinuity. By the Arzela- Ascoli theorem, since \mathcal{T}_2 is uniformly bounded and equicontinuous on \mathcal{B}_ϱ , it is relatively compact in $C_{\Psi, \delta}^\eta$. Together with continuity we conclude that \mathcal{T}_2 is compact on \mathcal{B}_ϱ . \square

Definition 3.5. *Problem (1.1) is UHS if there exists $\mathfrak{c} \in \mathcal{R}^+$ such that, for every $\epsilon > 0$ and $\bar{\mathfrak{f}}$ satisfying*

$$\begin{cases} \left| \left[\mathcal{H} \mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1, \varsigma_2; \Psi} \left[\bar{\mathfrak{f}}(\iota) - \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i; \Psi} \mathcal{Z}_i(\iota, \bar{\mathfrak{f}}_i) \right] - \mathcal{G}(\iota, \bar{\mathfrak{f}}_\iota) \right] \right| \leq \epsilon, & \iota \in \mathcal{J} = [\mathbf{a}_1, \mathbf{b}_1], \\ |\bar{\mathfrak{f}}(\iota) - \lambda(\iota)| \leq \epsilon, & \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1], \end{cases} \quad (3.4)$$

there exists a unique solution $\mathfrak{f} \in C_\delta$ with

$$\|\bar{\mathfrak{f}} - \mathfrak{f}\|_{C_\delta} \leq \mathfrak{c}\epsilon.$$

Definition 3.6. *Problem (1.1) is generalized UHS if there exists $\sigma \in \mathcal{C}(\mathcal{R}^+, \mathcal{R}^+)$, with $\sigma(0) = 0$, such that, for every $\epsilon > 0$, $\bar{\mathfrak{f}} \in C_{\Psi, \delta}^\eta$ satisfying the inequality in (3.4), there exist a unique solution $\mathfrak{f} \in C_\delta$ of (1.1) such that,*

$$\|\bar{\mathfrak{f}} - \mathfrak{f}\|_{C_\delta} \leq \sigma(\epsilon).$$

Remark 3.7. $\bar{\mathfrak{f}} \in C_\delta$ is a solution of the inequality (3.4) if and only if there exist $\mathfrak{g} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ such that

- $|\mathfrak{g}(\iota)| \leq \epsilon, \quad \forall \iota \in \mathcal{J}$ and
- $\mathcal{H} \mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1, \varsigma_2; \Psi} \left[\bar{\mathfrak{f}}(\iota) - \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i; \Psi} \mathcal{Z}_i(\iota, \bar{\mathfrak{f}}_i) \right] = \mathcal{G}(\iota, \bar{\mathfrak{f}}_\iota) + \mathfrak{g}(\iota), \quad \forall \iota \in \mathcal{J}.$

Theorem 3.8. *If the condition (A_2) and (3.2) are satisfied. then the solution of (1.1) is UHS and generalized UHS.*

Proof. Suppose $\epsilon \in \mathcal{R}^+$, let $\bar{\mathfrak{f}} \in C_{\Psi, \delta}^\eta$ be any solution of the inequality. By Remark 3.7, there exists $\mathfrak{g} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ with $|\mathfrak{g}(\iota)| \leq \epsilon, \quad \iota \in \mathcal{J}$, such that the perturbed problem is

given by

$$\begin{cases} \mathcal{H}\mathcal{D}_{\mathbf{a}_1+}^{\varsigma_1, \varsigma_2; \Psi} \left[\bar{f}(\iota) - \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i; \Psi} \mathcal{Z}_i(\iota, \bar{f}_i) \right] = \mathcal{G}(\iota, \bar{f}_\iota) + \mathbf{g}(\iota), & \forall \iota \in \mathcal{J}. \\ \bar{f}(\iota) = \lambda(\iota), & \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1]. \end{cases} \quad (3.5)$$

By Lemma 3.2 , the solution $\bar{f}(\iota)$ to the perturbed problem is given by

$$\bar{f}(\iota) = \begin{cases} \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi} [\mathcal{G}(\iota, \bar{f}_\iota) + \mathbf{g}(\iota)] + \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i; \Psi} [\mathcal{Z}_i(\iota, \bar{f}_i)], & \iota \in \mathcal{J}, \\ \lambda(\iota), & \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1]. \end{cases}$$

Let $f \in \mathcal{C}_\delta$ be the solution of (1.1) which satisfies:

$$f(\iota) = \begin{cases} \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi} [\mathcal{G}(\iota, f_\iota)] + \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i; \Psi} [\mathcal{Z}_i(\iota, f_i)], & \iota \in \mathcal{J}, \\ \lambda(\iota), & \iota \in [\mathbf{a}_1 - \delta, \mathbf{a}_1]. \end{cases}$$

Now, estimate the difference $|\bar{f} - f|$ for $\iota \in \mathcal{J}$:

$$|\bar{f}(\iota) - f(\iota)| \leq \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi} |\mathcal{G}(\iota, \bar{f}_\iota) - \mathcal{G}(\iota, f_\iota)| + \mathcal{I}_{\mathbf{a}_1+}^{\eta; \Psi} |\mathbf{g}(\iota)| + \sum_{i=1}^n \mathcal{I}_{\mathbf{a}_1+}^{\xi_i; \Psi} |\mathcal{Z}_i(\iota, \bar{f}_i) - \mathcal{Z}_i(\iota, f_i)|$$

Applying assumption (A_2) and estimate $|\mathbf{g}(\iota)| \leq \epsilon$, we have

$$\begin{aligned} |\bar{f}(\iota) - f(\iota)| &\leq \left(\frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \|\omega\| + \sum_{i=1}^n \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^{\xi_i}}{\Gamma(\xi_i + 1)} \|\nu_i\| \right) \|\bar{f}_\iota - f_\iota\|_{\mathcal{C}_\delta} \\ &+ \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \epsilon. \end{aligned}$$

That is,

$$\|\bar{f} - f\|_{\mathcal{C}_{\Psi, \delta}^\eta} \leq \Lambda \|\bar{f} - f\|_{\mathcal{C}_{\Psi, \delta}^\eta} + \mathbf{p}\epsilon$$

where

$$\mathbf{p} = \frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)}$$

Therefore

$$\begin{aligned} (\mathbf{1} - \Lambda) \|\bar{f}_\iota - f_\iota\|_{\mathcal{C}_{\Psi, \delta}^\eta} &\leq \mathbf{p}\epsilon \\ \|\bar{f}_\iota - f_\iota\|_{\mathcal{C}_{\Psi, \delta}^\eta} &\leq \frac{\mathbf{p}}{(\mathbf{1} - \Lambda)} \epsilon. \end{aligned}$$

Let $\mathbf{c} = \frac{\mathbf{p}}{(\mathbf{1} - \Lambda)} > 0$. Then,

$$\|\bar{f}_\iota - f_\iota\|_{\mathcal{C}_{\Psi, \delta}^\eta} \leq \mathbf{c}\epsilon$$

Which proves the problem (1.1) is \mathcal{UHS} .

To show generalized \mathcal{UHS} , defined $\sigma(\epsilon) = \frac{\mathbf{p}}{(\mathbf{1} - \Lambda)} \epsilon \in \mathcal{C}(\mathcal{R}^+, \mathcal{R}^+)$ with $\sigma(0) = 0$.

Thus

$$\|\bar{f}_\iota - f_\iota\|_{\mathcal{C}_{\Psi, \delta}^\eta} \leq \sigma(\epsilon)$$

and the solution of (1.1) is generalized \mathcal{UHS} . □

4. An example

To demonstrate the applicability of the theoretical framework, consider the specific form of equation (1.1) on the interval $\mathcal{J} = [1, 2]$.

Let the function $\psi(\iota) = \iota^2$, with parameters:

$$a = 1, b = 2, \varsigma_1 = \frac{1}{2}, \varsigma_2 = \frac{1}{2}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{3}{2}, \delta > 0.$$

Define the nonlinear terms as

$$\mathcal{Z}_i(\iota, \mathbf{r}) = \frac{\iota^2}{(10 + \iota)} \tan(\mathbf{r}), \quad \mathcal{G}(\iota, \mathbf{r}) = \frac{1}{[5(1 + \iota^2)]} \cos(\mathbf{r}) + \frac{\iota}{(20 + \iota^2)} \quad (4.1)$$

Using the expressions, the constant are evaluated as

$$\Lambda_1 = 0.3261 \text{ and } \Lambda_2 = 0.1452, \quad \Lambda = \Lambda_1 + \Lambda_2 = 0.4713 < 1.$$

Hence, Theorem 3.3 is satisfied, guaranteeing uniqueness of the solution.

Moreover, with

$$\frac{(\Psi(\mathbf{b}_1) - \Psi(\mathbf{a}_1))^\eta}{\Gamma(\eta + 1)} \Phi = 0.1355 < 1$$

The hypotheses of both Theorem 3.3 and Theorem 3.4 are satisfied. Therefore, we conclude that the problem (1.1) with $\mathcal{G}(\iota, \mathbf{r})$ and $\mathcal{Z}_i(\iota, \mathbf{r})$ defined in (4.1), admits a unique solution on the interval $[1, 2]$. Moreover, existence and uniqueness are guaranteed.

5. Conclusion

This paper examined a class of fractional neutral integro-differential equations with delay involving the Ψ - \mathcal{HFD} . By applying Banach's contraction principle and Krasnoselskii's fixed point theorem, conditions were established to guarantee the existence and uniqueness of solutions. In addition, Ulam–Hyers stability and its generalized form were demonstrated, confirming that the solutions remain stable under small perturbations. An illustrative example verified the theoretical results and highlighted the role of fractional orders, delay effects, and nonlinear terms in ensuring solvability. The study reinforces the utility of the Ψ -Hilfer operator as a versatile tool for analyzing systems with memory and hereditary properties.

References

- [1] Abbas, S., Benchohra, M., Graef, J.R., Henderson, J., *Implicit Fractional Differential and Integral Equations: Existence and Stability*, Walter de Gruyter GmbH and Co KG, **26**(2018).
- [2] Gu, H., Trujillo, J. J., *Existence of mild solution for evolution equation with Hilfer fractional derivative*, Appl. Math. Comput., **257**(2015), 344–354.
- [3] Herrmann, R., *Fractional Calculus: An Introduction for Physicists*, World Scientific Publishing Company, Singapore, 2011.
- [4] Huang, J., Li, Y., *Hyers–Ulam stability of delay differential equations of first order*, Mathematische Nachrichten **289**(1)(2016), 60–66.

- [5] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, **204**(2006).
- [6] Manikandan, S., Kanagarajan, K., Elsayed, E. M., Vivek, D., *Nonlocal initial value problems for Ambartsumian equation with Hilfer generalized proportional fractional derivative*, Appl. Sci. Uni. J., **7**(2023), no.1, 1-15.
- [7] Medved, M., Brestovanska, E., *Differential equations with tempered ψ -Caputo fractional derivative*, Math. Model Anal. **26**(2021), 631-650.
- [8] Mohammed, S., Abdo., Abdulkafi, M., Saeed., Sathish, K., Panchal., *Existence and Ulam-Hyers Mittag-Leffler Stability of Ψ -Hilfer Fractional Functional Integro-differential Equation*, Math. AP., 2020, 1-14.
- [9] Ntouyas. S. K., Vivek. D., *Existence and uniqueness results for sequential ψ -Hilfer fractional differential equations with multi-point boundary conditions*, Acta. Math. Univ. Comenianae. **2**(2021), 171-185.
- [10] Oldham, K., Spanier, J., *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, Elsevier, Academic Press, New York, **111**(1974).
- [11] Podlubny, I., *Fractional Differential Equations*, Mathematics in Science and Engineering, Academic Press, San Diego, **198**(1999).
- [12] Samko, S.G., Kilbas, A. A., Marichev, O. I., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, New York, 1993.
- [13] Sousa, J. V. C., Oliveira, E.C., *On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the Ψ -Hilfer operator*, Fixed Point Theory Appl., **20(96)**(2018). <https://doi.org/10.1007/s11784-018-0587-5>.
- [14] Wang, J., Fečkan, M., Tian, Y., *Stability analysis for a general class of noninstantaneous impulsive differential equations*, Mediterr. J. Math., **14(2)**(2017), pp. 46.
- [15] Yong, Z., Jinrong, W., Lu, Z., *Basic Theory of Fractional Differential Equations*, World Scientific Publishing Company, Singapore, 2014.

Vinitha Ravi 


Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore–641020, India

Department of Mathematics, KG College of Arts and Science, Coimbatore–641035, India
e-mail: vinithasrmv2023@gmail.com

Nirmalkumar Rajendran 


Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore–641020, India

e-mail: nirmalkumarsrmvcas@gmail.com

Shyamsunder 

Department of Mathematics, SRM University Delhi-NCR, Sonapat–131029, Haryana, India
e-mail: skumawatmath@gmail.com

Characterizations of pseudolinear and semistrictly quasilinear functions

Vsevolod I. Ivanov 

Abstract. In this paper, we obtain several new complete characterizations of pseudolinear functions. Two of the results are of first-order and one is derivative free. The results are derived in terms of the Clarke-Rockafellar subdifferential. Additionally, we prove a characterization of the semistrictly quasilinear functions. It is similar to the derivative free characterization of the pseudolinear functions. We also find the conditions such that a semistrictly quasilinear function becomes pseudolinear.

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1. Introduction

The concepts of pseudolinearity and semistrict quasilinearity provide in a very natural way generalizations of the linearity. These classes include many useful functions. Among them are, for instance, the linear fractional functions. Several interesting properties of pseudolinear functions appeared in [6, 14, 15, 17, 21, 22]. More generalized convex functions have been studied for example in the books [5, 9, 18, 19, 20]. Quasilinear functions have been studied in [19] under the name quasimonotone functions.

In this paper, we obtain more characterizations of pseudolinear functions. In Theorem 2.4, we obtain necessary conditions for pseudolinearity, but they are not sufficient. The respective necessary and sufficient conditions are obtained in Theorem 2.5. Theorem 2.2 is well known and we prove it for reader's convenience. In fact, it is a generalization to non-differentiable functions of a theorem due to Chew and Choo

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[6]. It is interesting which is the largest class of functions such that the conditions of Theorem 2.4 hold. We prove that they become necessary and sufficient when the function is semistrictly quasilinear, which implies that the mentioned class is the exact one. Every pseudolinear function, which is subdifferentiable with respect to the Clarke-Rockafellar subdifferential, is semistrictly quasilinear. We also find the conditions that a semistrictly quasilinear function must satisfy to become pseudolinear.

In the sequel, we suppose that \mathbb{E} is a Banach space. We denote by \mathbb{E}^* its dual and the duality pairing between the vectors $a \in \mathbb{E}^*$ and $b \in \mathbb{E}$ by $\langle a, b \rangle$, by \mathbb{R} the set of reals, by $\overline{\mathbb{R}}$ the union $\mathbb{R} \cup \{+\infty\}$, by $B(x, r)$ the closed ball of a center x with a radius r . Let $f : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a proper extended real-valued function, whose domain is the set

$$\text{dom}(f) := \{x \in \mathbb{E} \mid f(x) < +\infty\}.$$

Definition 1.1. Let $f : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a proper extended real-valued function and $x \in \text{dom}(f)$. The Clarke-Rockafellar generalized derivative of f at x in direction v is defined by

$$f^\uparrow(x, v) = \sup_{\varepsilon > 0} \limsup_{(y, \alpha) \downarrow_f x; t \downarrow 0} \inf_{u \in B(v, \varepsilon)} [f(y + tu) - \alpha]/t,$$

where $(y, \alpha) \downarrow_f x$ means that $y \rightarrow x$, $\alpha \rightarrow f(x)$, $\alpha \geq f(y)$ (see [23]), and $y \rightarrow x$ implies that the norm $\|y - x\|$ approaches 0. If f happens to be lower semicontinuous at x the definition can be expressed in the slightly simpler form

$$f^\uparrow(x, v) = \sup_{\varepsilon > 0} \limsup_{y \downarrow_f x; t \downarrow 0} \inf_{u \in B(v, \varepsilon)} [f(y + tu) - f(y)]/t,$$

where $y \downarrow_f x$ means that $y \rightarrow x$, $f(y) \rightarrow f(x)$. When f is finite and locally Lipschitz, this derivative coincides with the Clarke generalized derivative [7], which is defined by

$$f^0(x, v) = \limsup_{y \rightarrow x; t \downarrow 0} [f(y + tv) - f(y)]/t.$$

The Clarke-Rockafellar subdifferential of f at x is defined as follows:

$$\partial^\uparrow f(x) = \{x^* \in \mathbb{E}^* \mid \langle x^*, v \rangle \leq f^\uparrow(x, v), \quad \forall v \in \mathbb{E}\}$$

with the convention that $\partial^\uparrow f(x) = \emptyset$ if $x \notin \text{dom}(f)$. The Clarke's subdifferential (or Clarke's generalized gradient) of f at x is defined as follows:

$$\partial f(x) = \{x^* \in \mathbb{E}^* \mid \langle x^*, v \rangle \leq f^0(x, v), \quad \forall v \in \mathbb{E}\}.$$

Definition 1.2. A real-valued function $f : \mathbb{E} \rightarrow \mathbb{R}$ is called pseudoconvex (in terms of the Clarke directional derivative) iff the following implication is satisfied

$$f(y) < f(x) \quad \Rightarrow \quad \langle x^*, y - x \rangle < 0, \quad \forall x^* \in \partial f(x). \tag{1.1}$$

A proper extended real function $f : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is called pseudoconvex (in terms of the Clarke-Rockafellar subdifferential) iff the following implication is satisfied

$$f(y) < f(x) \quad \Rightarrow \quad \langle x^*, y - x \rangle < 0, \quad \forall x^* \in \partial^\uparrow f(x). \tag{1.2}$$

Recall that a real function f is said to be quasiconvex on a convex set X if,

$$f[x + t(y - x)] \leq \max\{f(x), f(y)\}, \quad \forall x \in X \quad \forall y \in X, \quad \forall t \in [0, 1].$$

The following result is due to Daniilidis, Hadjisavvas [8, Proposition 2.2].

Lemma 1.3. *Let $f : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a lower semicontinuous pseudoconvex function with a convex domain. Then f is quasiconvex.*

The following result is a particular case of Lemma 3 in the paper by Ivanov [10]:

Lemma 1.4. *Let f be a lower semicontinuous proper extended pseudoconvex real function, defined on some convex set S , included in an open set in a Banach space \mathbb{E} . Then the following implication holds*

$$x \in S, y \in S, f(y) \leq f(x) \quad \Rightarrow \quad \langle x^*, y - x \rangle \leq 0, \quad \forall x^* \in \partial^\dagger f(x).$$

The next results were also derived by Ivanov [10]:

Lemma 1.5. *Let $f : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a lower semicontinuous and radially continuous proper extended real-valued function with a convex domain. Then, f is pseudoconvex if and only if there exists a positive function $p : \mathcal{E} \times \mathcal{E} \times \mathbb{E}^* \rightarrow (0, +\infty)$ with*

$$p(x, y, x^*) \langle x^*, y - x \rangle + p(y, x, y^*) \langle y^*, x - y \rangle \leq 0, \quad (1.3)$$

$$\forall (x, y) \in \text{dom}(f) \times \text{dom}(f), \forall (x^*, y^*) \in \partial f(x) \times \partial f(y).$$

2. Characterizations of pseudolinear functions

In this section, we apply the characterizations of pseudoconvex functions to obtain characterizations of pseudolinear ones.

Recall that a function f is said to be pseudoconcave iff $-f$ is pseudoconvex. A function f is said to be pseudolinear iff f is both pseudoconvex and pseudoconcave. In the characterizations of pseudoconvex functions, we suppose that f is a proper extended real-valued function, which implies that $f(x) > -\infty$ for every $x \in \mathbb{E}$. We want to apply these results to functions such that both f and $-f$ are proper. Therefore f should be a finite function.

Remark 2.1. We repeat once again that the Clarke generalized gradient is denoted by ∂f and the Clarke-Rockafellar subdifferential by ∂^\dagger .

The next theorem is well known. Its proof appears in terms of various derivatives and subdifferential in [1, 6, 17, 22]. It was proved initially in [6]. Our proof is more general, but similar. We prove it, because the construction in it is used in the proof of another result.

Theorem 2.2. *Let f be a lower semicontinuous proper extended function with a convex domain S . Then f is pseudolinear on S with respect to the Clarke-Rockafellar subdifferential if and only if there exists a positive function $p : S \times S \times \mathbb{E}^* \rightarrow (0, +\infty)$ with*

$$f(y) - f(x) = p(x, y, x^*) \langle x^*, y - x \rangle, \quad \forall x \in S \quad \forall y \in S, \quad \forall x^* \in \partial^\dagger f(x) \quad (2.1)$$

such that $\partial^\dagger f(x) \neq \emptyset$.

Proof. Let f be pseudolinear. We prove that there exists a function p satisfying (2.1). Consider the function, defined as follows:

$$p(x, y, x^*) = \begin{cases} \frac{f(y)-f(x)}{\langle x^*, y-x \rangle}, & \text{if } \langle x^*, y-x \rangle \neq 0 \\ 1, & \text{if } \langle x^*, y-x \rangle = 0. \end{cases} \quad (2.2)$$

We prove that it is positive. Let $\langle x^*, y-x \rangle > 0$. It follows from pseudoconvexity that $f(y) \geq f(x)$. Suppose that it is possible that $f(y) = f(x)$. Then by Lemma 1.4 we obtain that $\langle x^*, y-x \rangle \leq 0$, which is a contradiction. Let $\langle x^*, y-x \rangle < 0$. Since $\partial^\uparrow(sf)(x) = s\partial^\uparrow f(x)$ for every $s \in \mathbb{R}$, then $-x^* \in \partial^\uparrow(-f)(x)$. We conclude from here that $f(y) \leq f(x)$. By Lemma 1.4 we obtain that the case $f(y) = f(x)$ is impossible. Therefore $p > 0$.

We prove that the function p satisfies (2.1). It is enough to show that $\langle x^*, y-x \rangle = 0$ implies that $f(y) = f(x)$. Indeed, assume the contrary. If $f(y) < f(x)$, then by pseudoconvexity we obtain that $\langle x^*, y-x \rangle < 0$, a contradiction. If $f(y) > f(x)$, then by pseudoconcavity we again get a contradiction.

Let $x \in S, y \in S, x^* \in \partial^\uparrow f(x)$ and equation (2.1) is satisfied. Obviously (2.1) implies that f is pseudolinear. □

Theorem 2.3. *Let f be a locally Lipschitz real-valued function, defined on some convex set S included in an open set Γ in a Banach space \mathbb{E} . Then, f is pseudolinear on S if and only if there exists a positive function $p : S \times S \times \mathbb{E}^* \rightarrow (0, +\infty)$ with*

$$p(x, y, x^*) \langle x^*, y-x \rangle + p(y, x, y^*) \langle y^*, x-y \rangle = 0, \quad \forall (x, y) \in S \times S, \forall (x^*, y^*) \in \partial f(x) \times \partial f(y). \quad (2.3)$$

Proof. Let f be pseudolinear. We prove that inequality (2.3) holds. Choose arbitrary $x \in S, y \in S$. It follows from Theorem 2.2 that there exists a function $p : S \times S \times \mathbb{E}^* \rightarrow (0, +\infty)$ with

$$f(y) - f(x) = p(x, y, x^*) \langle x^*, y-x \rangle, \quad \forall x^* \in \partial f(x) \quad (2.4)$$

and

$$f(x) - f(y) = p(y, x, y^*) \langle y^*, x-y \rangle, \quad \forall y^* \in \partial f(y). \quad (2.5)$$

If we add (2.4) and (2.5), then we obtain (2.3).

The converse claim follows from Lemma 1.5, applied both to f and $-f$. □

Equation (2.3) is a type of monotonicity of the subdifferential of a pseudolinear function. More properties of such maps are studied in [4]. More types monotone maps have been defined in [12, 13].

Theorem 2.4. *Let S be a convex set, included in some open set Γ in a Banach space \mathbb{E} . Suppose that f is a locally Lipschitz function, which is pseudolinear on S with respect to the Clarke's derivative. Then for all $x \in S, y \in S$, and $\lambda \in [0, 1]$ there exists a number $b > 0$, which depends on x, y, λ such that the following conditions are satisfied:*

$$f[x + \lambda(y-x)] = \lambda b f(y) + (1 - \lambda b)f(x), \quad (2.6)$$

$$0 < b \leq 1/\lambda, \quad \forall \lambda \in (0, 1]. \quad (2.7)$$

Proof. Choose arbitrary points $x \in S$, $y \in S$ and a number $\lambda \in (0, 1)$. Denote $z(\lambda) = x + \lambda(y - x)$. We have $\partial f(z(\lambda)) \neq \emptyset$. Take arbitrary $\xi \in \partial f(z(\lambda))$. It follows from Theorem 2.2 that there exists a positive function $q : S \times S \times \mathbb{E}^* \rightarrow (0, +\infty)$ such that

$$q(z(\lambda), x, \xi)[f(x) - f(z(\lambda))] = \langle \xi, x - z(\lambda) \rangle = \lambda \langle \xi, x - y \rangle \tag{2.8}$$

and

$$q(z(\lambda), y, \xi)[f(y) - f(z(\lambda))] = \langle \xi, y - z(\lambda) \rangle = (1 - \lambda) \langle \xi, y - x \rangle \tag{2.9}$$

where $q = 1/p$. Let us multiply (2.8) by $(1 - \lambda)$, (2.9) by λ , and add the obtained inequalities. Then we obtain that (2.6) holds where

$$b = q(z(\lambda), y, \xi) / [\lambda q(z(\lambda), y, \xi) + (1 - \lambda) q(z(\lambda), x, \xi)]. \tag{2.10}$$

It follows from (2.10) that $0 < \lambda b < 1$ if $0 < \lambda < 1$ and $x \neq y$.

Equality (2.6) is satisfied with $b = 1$ if $\lambda = 1$. It also holds if $\lambda = 0$.

We prove that b does not depend on ξ . Equation (2.6) implies that if $f(y) \neq f(x)$, then

$$b = \frac{f[x + \lambda(y - x)] - f(x)}{\lambda[f(y) - f(x)]}.$$

Suppose that $f(y) = f(x)$. Since the function f is both pseudoconvex and pseudoconcave, then by Lemma 1.3, f is both quasiconvex and quasiconcave. Therefore,

$$f(x) = \min\{f(x), f(y)\} \leq f[x + \lambda(y - x)] \leq \max\{f(x), f(y)\} = f(x) \quad \text{for all } \lambda \in [0, 1].$$

We conclude from here that $f[x + \lambda(y - x)] = f(x)$ for all $\lambda \in [0, 1]$.

It is seen that really b does not depend on ξ . □

The next result gives us a derivative-free complete characterization of pseudolinear functions.

Proposition 2.5. *Let S be a convex set in a Banach space \mathbb{E} . Suppose that f is a continuously differentiable function, defined on some open convex set, which contains S . Then the following claims are equivalent:*

- (a) f is pseudolinear on S ;
- (b) there is a function $b : S \times S \times [0, 1] \rightarrow (0, +\infty)$ such that for all $x \in S$, $y \in S$ there exists the limit

$$q(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda), \tag{2.11}$$

$q(x, y)$ is strictly positive, and for each $\lambda \in [0, 1]$ equation (2.6) and inequality (2.7) are satisfied.

Proof. We prove implication (a) \Rightarrow (b). Let f be pseudolinear on S . It follows from Theorem 2.4, taking into account that for continuously differentiable functions the Clarke directional derivative coincides with the usual Frechet derivative, that there exists a function $b(x, y, \lambda)$ which satisfy (2.6) and (2.7). We prove that there exists the limit $\lim_{\lambda \downarrow 0} b(x, y, \lambda)$, and it is strictly positive. Take arbitrary points $x, y \in S$. Let us consider two cases:

- 1) $f(x) \neq f(y)$. The function

$$b(x, y, \lambda) = \frac{f[x + \lambda(y - x)] - f(x)}{\lambda[f(y) - f(x)]}$$

satisfies (2.6) and (2.7). The limit

$$q(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda) = \frac{\nabla f(x)(y - x)}{f(y) - f(x)}.$$

If $f(y) < f(x)$, then, by pseudoconvexity $\nabla f(x)(y - x) < 0$ and $q(x, y) > 0$. If $f(y) > f(x)$, then by pseudoconcavity $\nabla f(x)(y - x) > 0$ and again $q(x, y) > 0$.

2) $f(x) = f(y)$. In this case we can choose $b(x, y, \lambda) = 1$. It is obvious that $q(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda) = 1 > 0$.

We prove implication (b) \Rightarrow (a). If $\lambda > 0$, then (2.6) can be written as

$$\frac{f[x + \lambda(y - x)] - f(x)}{\lambda} = b(x, y, \lambda)[f(y) - f(x)].$$

Taking the limit as $\lambda \downarrow 0$, we obtain that

$$\nabla f(x)(y - x) = q(x, y)[f(y) - f(x)].$$

It follows from here that f is both pseudoconvex and pseudoconcave. □

Example 2.6. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x) = x_2/x_1$, where

$$S = \{x = (x_1, x_2) \mid x_1 > 0\}.$$

The function f is pseudolinear over S . This function satisfies (2.6) and (2.7). We have

$$(f(x + \lambda(y - x)) - f(x))/\lambda = (x_1y_2 - x_2y_1)/(x_1(x_1 + \lambda(y_1 - x_1)))$$

and

$$b(x, y, \lambda) = (f(x + \lambda(y - x)) - f(x))/(\lambda(f(y) - f(x))) = y_1/(x_1 + \lambda(y_1 - x_1)).$$

Therefore $0 < \lambda b \leq 1$ for all $x \in S, y \in S, \lambda \in (0, 1]$.

3. On semistrictly quasilinear and pseudolinear functions

It is interesting which is the class of functions such that the necessary conditions from Theorem 2.4 become both necessary and sufficient. In this section, we show that this property can be generalized and it becomes necessary and sufficient when the function is semistrictly quasilinear.

Definition 3.1. A function f , defined on a convex set S is called semistrictly quasiconvex iff for all $x, y \in S, \lambda \in (0, 1)$ the following implication holds:

$$f(y) < f(x) \quad \Rightarrow \quad f[x + \lambda(y - x)] < f(x).$$

If the function $-f$ is semistrictly quasiconvex, then f is called semistrictly quasiconcave.

Definition 3.2. A function f , defined on a convex set, is called semistrictly quasilinear iff it is both semistrictly quasiconvex and semistrictly quasiconcave.

Proposition 3.3 ([11]). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a radially lower semicontinuous semistrictly quasiconvex function. Then f is quasiconvex.

Remark 3.4. It is well known that every differentiable pseudoconvex function, defined on a convex set, is semistrictly quasiconvex on this set; see, for example, [2, Theorem 3.5.11]. A generalization of this claim to non-differentiable functions in terms of Clarke-Rockafellar derivative was obtained in Ref. [10].

Proposition 3.5. *Let S be a convex set, included in some open convex set Γ in a Banach space \mathbb{E} . Suppose that f is a continuous function, which is pseudolinear on S . Then f is both semistrictly quasiconvex and semistrictly quasiconcave.*

Proof. The claim follows directly from the known theorem that every pseudoconvex function is semistrictly quasiconvex. \square

Lemma 3.6. *A function f defined on a Banach space \mathbb{E} is both semistrictly quasiconvex and semistrictly quasiconcave if and only if the following implication holds*

$$x \in \text{dom } f, y \in \text{dom } f, f(y) < f(x), \lambda \in (0, 1) \Rightarrow f(y) < f[x + \lambda(y - x)] < f(x).$$

Proof. The proof follows immediately from the definitions of semistrict quasiconvexity and semistrict quasiconcavity. \square

It is interesting which is the widest class of functions, which satisfy the conditions (2.6) and $0 < \lambda b(x, y, \lambda) < 1$.

Theorem 3.7. *Let f be a continuous function defined on some convex set in a Banach space \mathbb{E} . Then f is both semistrictly quasiconvex and semistrictly quasiconcave if and only if for all $x \in S, y \in S$ and $\lambda \in (0, 1)$ there exists a number $b > 0$, which depend on x, y, λ such that $0 < \lambda b(x, y, \lambda) < 1$ and Condition (2.6) is satisfied.*

Proof. Let f be both semistrictly quasiconvex and semistrictly quasiconcave. Consider the function $b(x, y, \lambda)$ defined by

$$b = \frac{f[x + \lambda(y - x)] - f(x)}{\lambda[f(y) - f(x)]}.$$

Let $f(y) < f(x)$ and $0 < \lambda < 1$. We prove that

$$0 < f[x + \lambda(y - x)] - f(x) / [f(y) - f(x)] < 1. \tag{3.1}$$

It follows from the definition of semistrict quasiconvexity that $f[x + \lambda(y - x)] < f(x)$. Therefore

$$f[x + \lambda(y - x)] - f(x) / [f(y) - f(x)] > 0.$$

It follows from Lemma 3.6 that

$$f(y) < f[x + \lambda(y - x)] < f(x).$$

Hence (3.1) is satisfied

The case $f(y) > f(x)$ is similar. It follows from semistrict quasiconvexity that $f[x + \lambda(y - x)] < f(y)$. Therefore $f[x + \lambda(y - x)] - f(x) < f(y) - f(x)$, which implies that (3.1) is also satisfied. Therefore, Condition (2.6) holds and $0 < \lambda b < 1$.

Let $x \in S, y \in S, f(x) = f(y)$. By Proposition 3.3 f is both quasiconvex and quasiconcave. Therefore,

$$f[x + \lambda(y - x)] \leq \max\{f(x), f(y)\} = f(x) \quad \text{for all } \lambda \in [0, 1],$$

$$f[x + \lambda(y - x)] \geq \min\{f(x), f(y)\} = f(x) \quad \text{for all } \lambda \in [0, 1].$$

We conclude from here that $f[x + \lambda(y - x)] = f(x)$ for all $\lambda \in [0, 1]$. Hence, (2.6) is satisfied with $b = 1$ for every $\lambda \in (0, 1)$ and $0 < \lambda b < 1$.

Conversely, suppose that $x \in S$, $y \in S$, $f(y) < f(x)$, $0 < \lambda < 1$, and $0 < \lambda b(x, y, \lambda) < 1$. We prove that f is both semistrictly quasiconvex and semistrictly quasiconcave. Since $0 < \lambda b(x, y, \lambda) < 1$, then we have

$$\lambda b f(y) + (1 - \lambda b) f(x) < \lambda b f(x) + (1 - \lambda b) f(x) = f(x),$$

and

$$\lambda b f(y) + (1 - \lambda b) f(x) > \lambda b f(y) + (1 - \lambda b) f(y) = f(y).$$

It follows from (2.6) that $f(y) < f[x + \lambda(y - x)] < f(x)$. By Lemma 3.6 f is both semistrictly quasiconvex and semistrictly quasiconcave. \square

The following claim is well known [15].

Proposition 3.8. *Let S be an open convex set in a finite-dimensional space E , f be a Fréchet differentiable function, defined on S . Then f is pseudolinear on S if and only if the following sets are equal for all $x \in S$: $\{y \in S : \nabla f(x)(y - x) = 0\}$ and $\{y \in S : f(y) = f(x)\}$.*

The next theorem is similar, but different from Proposition 3.8.

Theorem 3.9. *Let S be a convex set in a Banach space E , f be a Fréchet differentiable semistrictly quasilinear function, defined on some open set Γ , containing S . Then f is pseudolinear on S if and only if the following implication holds:*

$$x \in S, y \in S, \nabla f(x)(y - x) = 0 \quad \Rightarrow \quad f(y) = f(x). \quad (3.2)$$

Proof. Let implication (3.2) be satisfied. We prove that f is pseudolinear. Take arbitrary $x \in S$, $y \in S$ such that $f(y) < f(x)$. By semistrict quasiconvexity we have $f[x + \lambda(y - x)] < f(x)$ for every $\lambda \in (0, 1)$. Therefore $\nabla f(x)(y - x) \leq 0$. It follows from (3.2) that the case $\nabla f(x)(y - x) = 0$ is impossible, because $f(y) < f(x)$. Hence f is pseudoconvex. Using similar arguments we can prove that f is pseudoconcave. Both pseudoconvexity and pseudoconcavity imply that f is pseudolinear.

Suppose that f is pseudolinear. We prove that implication (3.2) holds. Let $x \in S$, $y \in S$, $\nabla f(x)(y - x) = 0$, but $f(x) \neq f(y)$. If $f(y) < f(x)$, by pseudoconvexity we have $\nabla f(x)(y - x) < 0$, which is a contradiction. If $f(y) > f(x)$, by pseudoconcavity we have $\nabla f(x)(y - x) > 0$, which is also a contradiction. Therefore (3.2) holds. \square

We prove a generalization of this result.

Theorem 3.10. *Let S be a convex subset of an open set Γ in a Banach space \mathbb{E} . Suppose that $f : \Gamma \rightarrow \mathbb{R}$ is a continuous function, which is both semistrictly quasiconvex and semistrictly quasiconcave on S and $\partial^\dagger f(x) \neq 0$, $\partial^\dagger(-f)(x) \neq 0$ for all $x \in S$. Then f is pseudolinear with respect to the Clarke-Rockafellar subdifferential if and only if both implications hold:*

$$x \in S, y \in S, \xi \in \partial^\dagger f(x), \langle \xi, y - x \rangle = 0 \quad \Rightarrow \quad f(y) \geq f(x). \quad (3.3)$$

and

$$x \in S, y \in S, \eta \in \partial^\dagger(-f)(x), \langle \eta, y - x \rangle = 0 \Rightarrow f(y) \leq f(x). \quad (3.4)$$

Proof. Let f be pseudolinear. We prove implication (3.3). Take arbitrary $x \in S, y \in S, \xi \in \partial^\dagger f(x)$ such that $\langle \xi, y - x \rangle = 0$. If $f(y) < f(x)$, then by pseudoconvexity we have $\langle \xi, y - x \rangle < 0$, which contradicts $\langle \xi, y - x \rangle = 0$. The proof of implication (3.4) is similar. Take arbitrary $x \in S, y \in S, \eta \in \partial^\dagger(-f)(x)$ such that $\langle \eta, y - x \rangle = 0$. If $f(y) > f(x)$, then by pseudoconcavity we have $\langle \eta, y - x \rangle > 0$, which is also a contradiction.

Conversely, suppose that implications (3.3) and (3.4) are fulfilled. We prove that f is pseudoconvex. Let x and y be arbitrary points from S . We prove that

$$\langle \xi, y - x \rangle > 0, \xi \in \partial^\dagger f(x) \text{ implies } f(y) \geq f(x).$$

Indeed, it follows from $\langle \xi, y - x \rangle > 0$ that $f^\dagger(x, y - x) > 0$. By the definition of the Clarke-Rockafellar derivative, there exist $\varepsilon > 0$ and sequences $\{x_i\}_{i=1}^\infty, x_i \in \Gamma, \{t_i\}_{i=1}^\infty, t_i > 0$ such that $x_i \rightarrow x, t_i \downarrow 0$ and

$$\inf_{u \in B(y-x, \varepsilon)} [f(x_i + t_i u) - f(x_i)]/t_i > 0, \quad \forall i.$$

Taking the number i sufficiently large we ensure that $x_i \in B(x, \varepsilon)$. Therefore, we have $y - x_i \in B(y - x, \varepsilon)$ and $f[x_i + t_i(y - x_i)] > f(x_i)$. Using that f is lower semicontinuous and semistrictly quasiconvex we conclude from Proposition 3.3 that it is quasiconvex on S . Therefore,

$$f(x_i) < f[x_i + t_i(y - x_i)] \leq f(y).$$

Hence, $f(x) \leq \liminf_{i \rightarrow \infty} f(x_i) \leq f(y)$. It follows from the converse implication that

$$x \in S, y \in S, f(y) < f(x) \text{ imply } \langle \xi, y - x \rangle \leq 0, \quad \forall \xi \in \partial^\dagger f(x).$$

Therefore, according to implication (3.3), we obtain that f is pseudoconvex.

We prove that $-f$ is pseudoconvex. Let x and y be arbitrary points from S . We prove that

$$\langle \eta, y - x \rangle > 0, \eta \in \partial^\dagger(-f)(x) \text{ imply } f(y) \leq f(x).$$

Indeed, it follows from $\langle \eta, y - x \rangle > 0$ that $(-f)^\dagger(x, y - x) > 0$. By the definition of the Clarke-Rockafellar derivative, there exist $\varepsilon > 0$ and sequences $\{x_i\}_{i=1}^\infty, x_i \in \Gamma, \{t_i\}_{i=1}^\infty, t_i > 0$ such that $x_i \rightarrow x, t_i \downarrow 0$ and

$$\inf_{u \in B(y-x, \varepsilon)} [-f(x_i + t_i u) + f(x_i)]/t_i > 0, \quad \forall i.$$

Taking the number i sufficiently large we ensure that $x_i \in B(x, \varepsilon)$. Therefore, we have $y - x_i \in B(y - x, \varepsilon)$ and $f[x_i + t_i(y - x_i)] < f(x_i)$. Using that f is upper semicontinuous and semistrictly quasiconcave, we conclude from Proposition 3.3 that it is quasiconcave. Therefore,

$$f(x_i) > f[x_i + t_i(y - x_i)] \geq f(y).$$

Hence, $f(x) \geq \limsup_{i \rightarrow \infty} f(x_i) \geq f(y)$. It follows from the converse implication that

$$x \in S, y \in S, f(y) > f(x) \text{ imply } \langle \eta, y - x \rangle \leq 0, \quad \forall \eta \in \partial^\dagger(-f)(x).$$

Therefore, by implication (3.4), we obtain that $-f$ is pseudoconvex, which implies that the function f is pseudolinear. □

Corollary 3.11. *Let S be an open convex set in a Banach space \mathbb{E} . Suppose that $f : S \rightarrow \mathbb{R}$ is a locally Lipschitz semistrictly quasilinear on S function. Then, f is pseudolinear with respect to the Clarke generalized directional derivative if and only if the following implication holds:*


$$x \in S, y \in S, \xi \in \partial f(x), \langle \xi, y - x \rangle = 0 \quad \Rightarrow \quad f(y) = f(x).$$

Remark 3.12. Theorem 3.9 is not a consequence of Corollary 3.11, because the Clarke subdifferential $\partial f(x)$ does not coincides with the gradient $\nabla f(x)$ when the function is Fréchet differentiable, but it is not necessarily continuously differentiable.

References

- [1] Ansari, Q. H., Schaible, S., Yao, J. C., *η -pseudolinearity*, Riviste di Matematiche per le Scienze Economiche e Sociali, **22**(1999), 31–39.
- [2] Bazaraa, M. S., Shetty, C. M., *Nonlinear Programming - Theory and Algorithms*, John Wiley & Sons, New York, 1979.
- [3] Bector, C. R., Singh, C., *B- vex functions*, J. Optim. Theory Appl., **71**(1991), no. 2, 237–253.
- [4] Bianchi, M., Hadjisavvas, N., Schaible, S., *On pseudomonotone maps T for which $-T$ is also pseudomonotone*, J. Convex Anal. **10**(2003), no. 1, 149–168.
- [5] Cambini, A., Martein, L., *Generalized Convexity and Optimization*, Lecture Notes in Econom. and Math. Systems, Springer, Berlin, **616**(2009).
- [6] Chew, K. L., Choo, E. U., *Pseudolinearity and efficiency*, Math. Program. **28**(1984), 226–239.
- [7] Clarke, F. H., *Optimization and nonsmooth analysis*, A Wiley-Interscience Publication, John Wiley & Sons, New York, 1983.
- [8] Daniilidis, A., Hadjisavvas, N., *On the subdifferentials of quasiconvex and pseudoconvex functions and cyclic monotonicity*, J. Math. Anal. Appl., **237**(1999), 30–42.
- [9] Giorgi, G., Guerraggio, A., Thierfelder, J., *Mathematics of optimization*, Elsevier, Amsterdam, 2004.
- [10] Ivanov, V. I., *Characterizations of pseudoconvex functions and semistrictly quasiconvex ones*, J. Glob. Optim., **57**(2013), 677–693.
- [11] Karamardian, S., *Strictly quasi-convex (concave) functions and duality in mathematical programming*, J. Math. Anal. Appl., **20**(1967), 344–358.
- [12] Karamardian, S., *Complementarity over cones with monotone and pseudomonotone maps*, J. Optim. Theory Appl., **18**(1976), 445–454.
- [13] Karamardian, S., Schaible, S., *Seven kinds of monotone maps*, J. Optim. Theory Appl. **66**(1990), no. 1, 37–46.
- [14] Komlósi, S., *First and second order characterizations of pseudolinear functions*, European J. Oper. Res. **67**(1993), 278–286.
- [15] Kortanek, K. O., Evans, J. P., *Pseudoconcave programming and Lagrange regularity*, Oper. Res **15**(1967), 882–892.
- [16] Lalitha, C. S., Mehta, M., *A note of pseudolinearity in terms of bifunctions*, Asia Pacific J. Oper. Res. **24**(2007), 83–91.

- [17] Lu, Q. H., Zhu, D. L., *Some characterizations of locally Lipschitz pseudolinear functions*, Math. Appl., **18**(2005), 272–278.
- [18] Mangasarian, O. L., *Nonlinear programming*, Mc Graw-Hill, New York, 1969.
- [19] Martos, B., *Nonlinear programming theory and methods*, Akademiai Kiado, Budapest, 1975.
- [20] Mishra, S. K., Upadhyay, B. B., *Pseudolinear functions and optimization*, Chapman & Hall Book, Boca Raton, 2015.
- [21] Rapcsák, T., *On pseudolinear functions*, European J. Oper. Res. **50**(1991), 353–360.
- [22] Rezaei, M., Gazor, H., *Nonsmooth pseudolinearity*, Journal of Mathematical Extensions, **6**(2012), no. 4, 63–77.
- [23] Rockafellar, R. T., *Generalized directional derivatives and subgradients of nonconvex functions*, Canad. J. Math., **32**(1980), 257–280.
- [24] Schaible, S., *Bibliography in fractional programming*, Zeitschrift für Operations Research, **26**(1982), no. 1, 211–241.
- [25] Schaible, S., Ibaraki, T., *Fractional programming*, European J. Oper. Res, **12**(1983), no. 4, 325–338.
- [26] Stancu-Minasian, I. M., *Fractional Programming. Theory, Methods, Applications*, Mathematics and Its Applications, Springer, Berlin, **409**(1997).

Vsevolod I. Ivanov 

Department of Mathematics, Technical University of Varna,
9010 Varna, Bulgaria,

e-mail: vsevolodivanov@yahoo.com

Notes on certain complex-type special functions in which the Gaussian function and its integral play essential roles

Hüseyin Irmak 

Abstract. The primary aim of this scientific note is first to review the essential background on several special functions in which the Gaussian function in certain complex domains and its integral play fundamental roles, and subsequently to establish (or organize) a number of relevant results together with some of their potential implications.

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1. Introduction and motivation

In mathematics and several related scientific fields, we frequently encounter various types of academic investigations involving special functions, both in theoretical and applied contexts. Different forms of these functions – with complex (or real) variables (or parameters) – play an important role in a wide range of scientific studies in the written literature. In particular, they have undeniable significance in mathematical transformations and operator theory as is known.

This special research is closely related to various complex-type special functions. For example, one of them is the Faddeeva function of a complex variable (z) (or real variable (x)). It is a complex-valued function that is closely connected to several

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other well-known functions, such as the Fresnel integral, Dawson's integral, and the Voigt function. In the literature, as we know this special function is also referred to as the Kramp function, typically expressed in terms of the familiar complex variable (z).

In addition, the Faddeeva function plays a significant role in various physical problems, especially those concerning electromagnetic responses in complex media. At the same time, it appears in problems involving small-amplitude waves propagating through Maxwellian plasmas. In this connection, it is particularly notable in the expression for plasma permittivity, from which dispersion relations are also derived. Consequently, it is sometimes called the plasma dispersion function, although this name is often reserved for a certain rescaled version of the function.

Furthermore, the infrared permittivity of amorphous oxides exhibits phonon-induced resonances that can be too complicated to model with simple harmonic oscillators. The Brendel-Bormann oscillator model, which represents an infinite superposition of oscillators with slightly different frequencies following a Gaussian distribution, provides a more accurate description; its integrated response can be expressed in terms of the Faddeeva function.

Additionally, the Faddeeva function plays an essential role in analyzing electromagnetic waves used in AM radio. In particular, ground waves – vertically polarized waves that propagate over a lossy ground with finite resistivity and permittivity – can be characterized using this function. It also provides valuable information about changes in neutron cross sections of materials as temperature varies.

The primary aim of this special study is to examine the relevant function – which occupies a significant place in science and technology – in the complex plane, to present essential information about it, and to focus on certain key propositions accompanied by various illustrative examples.

At the same time, especially, this special investigation also seeks to explore specific aspects of the Gaussian function, which holds a prominent position in science and technology and plays a fundamental role among the special functions mentioned above. Subsequently, this study addresses the Faddeeva function (with complex variable z) within selected domains of the complex plane, with particular emphasis on fundamental propositions and several illustrative examples. Especially, we point out that the work [1] is only one of the fundamental references for the above material.

We can now move on to the next sections to introduce the necessary definitions and establish various theoretical concepts along with some related implications.

2. Necessary definitions and related information

In the mathematical and related scientific literature, the classical forms of the Gaussian function (with the variable ρ) and the Gaussian integral frequently appear in the expressions given by

$$\mathcal{G}(\rho) = \exp(-\rho^2) \quad (2.1)$$

and

$$\int_{-\infty}^{\infty} \exp(-\rho^2) d\rho, \tag{2.2}$$

respectively.

The relevant parameter (or variable) ρ , used in the mathematical expressions given in (2.1) and (2.2), may also appear as either a real or a complex variable in various mathematical documents. Since these situations play a big role in our specific results, we believe it is useful to highlight some pertinent information here. The first of these concerns the following expressions, which contain specific information directly related to the exponential function given in (2.1) and the generalized integral given in (2.2).

Remark 2.1. Let the notation \mathbb{R} denote the set of real numbers, and let the variable ρ be of the real variable x , serving as the independent variable of the function in (2.1). In this case, for the real-valued function

$$\mathcal{G}(x) = \exp(-x^2) \quad (x \in \mathbb{R})$$

and its associated improper integral being of the form in (2.2), each of the following assertions is supplied.

- (a) It is a decreasing function for all $x > 0$, and an increasing function for all $x < 0$.
- (b) For all $x \in \mathbb{R}$, $0 < \mathcal{G}(x) \leq 1$. Hence, it is a bounded function on \mathbb{R} .
- (c) It is an even function; namely, $\int_{-\infty}^0 \mathcal{G}(x) dx = \int_0^{\infty} \mathcal{G}(x) dx$.
- (d) It is a continuous and integrable function on \mathbb{R} .
- (e) $\int_{-\infty}^{\infty} \mathcal{G}(x) dx = 2 \int_{-\infty}^0 \mathcal{G}(x) dx = 2 \int_0^{\infty} \mathcal{G}(x) dx = \sqrt{\pi}$.

The second special information is some specific results directly related to the complex variable form of the Gaussian function given in (2.2).

Remark 2.2. Let the notation \mathbb{C} denote the set of complex numbers, and also let the mentioned variable ρ be the familiar complex variable z , which is the independent variable of the function given as in (2.1). In this case, for the complex-type function

$$\mathcal{G}(z) = \exp(-z^2) \quad (z = x + iy; i^2 = -1; x, y \in \mathbb{R}),$$

each of the following assertions is satisfied.

- (a) It is a continuous, differentiating and integrable function on \mathbb{C} . Hence, it is an entire function.
- (b) $|\mathcal{G}(z)| = \exp[-\Re e(z^2)] = \exp(y^2 - x^2)$.
- (c) $\Re e\{\mathcal{G}(z)\} = \exp[-\Re e(z^2)] \text{Cos}[\Re e(-z^2)] = e^{y^2 - x^2} \text{Cos}(2xy)$.
- (d) $\Im m\{\mathcal{G}(z)\} = \exp[-\Re e(z^2)] \text{Sin}[\Im m(-z^2)] = -e^{y^2 - x^2} \text{Sin}(2xy)$.
- (e) $\text{Arg}\{\mathcal{G}(z)\} = \Im m(-z^2) = -2xy \pmod{2\pi}$.

It is well known in the theory of complex functions that curves lying in certain regions of the complex plane, whether continuous or piecewise continuous, play a fundamental role in the evaluation of complex integrals. Accordingly, drawing on classical analysis and taking into account the detailed statements presented in Remarks 2.1 and 2.2, the following observations will significantly facilitate the analysis of the principal complex error functions along the relevant curves.

Remark 2.3. Let z be any parameter in the complex plane \mathbb{C} , and also let Ω be a complex curve defined by

$$\Omega : \Omega(\sigma) = z\sigma \quad (0 \leq \sigma \leq 1),$$

which connects the origin of the plane \mathbb{C} to a given complex point $z \in \mathbb{C}$, and also let the real-valued function

$$\kappa(\rho) = \exp(-\rho^2) \quad (\rho \in \mathbb{R}).$$

Then, when considering the curve Ω , each of the following statements is provided.

- (a) $|\exp(-(z\sigma)^2)| = \exp(-\Re[(z\sigma)^2])$.
- (b) The complex-type function $\kappa(\rho z)$ is continuous and its modulus is also bounded on the curve Ω .
- (c) $\int_0^z \kappa(\rho) d\rho = \int_{\Omega} \kappa(\rho) d\rho = z \int_0^1 \kappa(z\sigma) d\sigma$.
- (d) $\int_0^1 |\kappa(z\sigma)| d\sigma \leq \int_0^1 \exp(-\sigma^2 |z|^2) d\sigma$.
- (e) $|z| \int_0^1 \exp(-\sigma^2 |z|^2) d\sigma = \int_0^{|z|} \exp(-\tau^2) d\sigma$.
- (f) $|\int_0^z \kappa(\rho) d\rho| = |\int_{\Omega} \kappa(\rho) d\rho| = |z| \int_0^1 |\kappa(z\sigma)| d\sigma \leq \max\left\{|z|, \frac{\sqrt{\pi}}{2}\right\}$.
- (g) $\int_0^z \kappa(\rho) d\rho = \int_0^{\infty} \kappa(\rho) d\rho - \int_z^{\infty} \kappa(\rho) d\rho$.
- (h) $\int_z^{\infty} \kappa(\rho) d\rho = \int_0^{\infty} \kappa(\rho) d\rho - \int_0^z \kappa(\rho) d\rho$.
- (i) $\int_{|z|}^{\infty} \kappa(\rho) d\rho = \int_0^{\infty} \kappa(\rho) d\rho - \int_0^{|z|} \kappa(\rho) d\rho$.
- (j) $\int_0^{|z|} \kappa(\rho) d\rho = \int_0^{\infty} \kappa(\rho) d\rho - \int_{|z|}^{\infty} \kappa(\rho) d\rho$.
- (k) $\int_{|z|}^{\infty} \kappa(\rho) d\rho \leq \int_0^{|z|} \kappa(\rho) d\rho \leq \int_0^{\infty} \kappa(\rho) d\rho$.
- (l) $\left| \int_0^z \kappa(\rho) d\rho \right| \leq \int_0^{|z|} \kappa(\rho) d\rho$.
- (m) $\left| \int_z^{\infty} \kappa(\rho) d\rho \right| \leq \int_{|z|}^{\infty} \kappa(\rho) d\rho$.

It is well established that the essential expressions introduced in Remarks 2.1, 2.2, and 2.3 encode significant information commonly encountered in both the complex and Cartesian planes, as well as in a wide range of computations and definitions in function and integral theory. For further details, the comprehensive references cited in [1]-[8] may be consulted. In line with the objectives of the present work, we now

turn our attention to a collection of fundamental definitions and their associated calculations.

In the mathematical literature, the function of the complex variable z , commonly referred to as the Faddeeva (or Kramp) function and denoted by $\mathcal{W}(z)$, represents a scaled version of the complex complementary error function. As detailed in [9], it is defined by

$$\mathcal{W}(z) = \exp(-z^2) \operatorname{erfc}(-iz) \quad (i^2 = -1; z \in \mathbb{C}). \tag{2.3}$$

Here, the function $\operatorname{erfc}(z)$ denotes the complementary error function of the complex variable z , which is defined as

$$\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-\rho^2) d\rho \quad (z \in \mathbb{C}). \tag{2.4}$$

Taking into account the fundamental results of classical analysis and the computational result stated in Remark 2.1(e), the following identity can be readily obtained:

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \left(\int_0^z \exp(-\rho^2) d\rho + \int_z^\infty \exp(-\rho^2) d\rho \right) \\ = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-\rho^2) d\rho \\ = 1. \end{aligned} \tag{2.5}$$

Here, the complex-valued integral appearing above, namely,

$$\frac{2}{\sqrt{\pi}} \int_0^z \exp(-\rho^2) d\rho \quad (z \in \mathbb{C}), \tag{2.6}$$

is known as the fundamental definition of the error function with complex variable z , and it is commonly denoted by $\operatorname{erf}(z)$.

In particular, in view of the special information given in (2.4)-(2.6), it is straightforward to obtain

$$\operatorname{erf}(z) + \operatorname{erfc}(z) = 1 \quad (z \in \mathbb{C}), \tag{2.7}$$

that is, this identity is only one of the fundamental relationships between the error function and the complementary error function. For more relations between the main error functions and some of related applications, see [1, 4, 8, 9, 13, 15, 16].

We further emphasize that the principal error functions (of the complex variable z), namely the complex functions:

$$\operatorname{erf}(z) \quad \text{and} \quad \operatorname{erfc}(z) \quad (z \in \mathbb{C}) \tag{2.8}$$

are only two of the most important functions within the family of all error functions.

At the same time, as indicated in the first part of this special study, these complex functions also possess several broad applications (or special relationships) with many special functions arising in various fields of science and technology. As a representative example, the rescaled function of the complex variable z ,

$$\mathcal{Z}(z) = i\sqrt{\pi} \mathcal{W}(z) \quad (z \in \mathbb{C}) \tag{2.9}$$

is also defined in terms of the Faddeeva function defined by (2.3). When the complex parameter z is expressed in the familiar form $z = x + iy$, the Faddeeva function $\mathcal{W}(z)$ can be also written as

$$\mathcal{W}(z) \equiv \mathcal{W}(x + iy) = u(x, y) + iv(x, y) = u + iv, \quad (2.10)$$

where the real-valued functions $u(x, y)$ and $v(x, y)$ are known, respectively, as the real Voigt function and the imaginary Voigt function with real parameters x and y . These special functions represent the real and imaginary parts of the (complex) Faddeeva function. Moreover, for the rescaled function being of the form:

$$\mathcal{Z}(z) \equiv \mathcal{Z}(x + iy),$$

it is clear that

$$\Re(\mathcal{Z}(z)) = -\sqrt{\pi} v, \quad (2.11)$$

$$\Im(\mathcal{Z}(z)) = \sqrt{\pi} u, \quad (2.12)$$

$$|\mathcal{Z}(z)| = \sqrt{\pi} |\mathcal{W}(z)| = \sqrt{\pi(u^2 + v^2)} \quad (2.13)$$

and

$$\arg(\mathcal{Z}(z)) = -\arctan\left(\frac{v}{u}\right) + 2\ell\pi \quad (\ell \in \mathbb{Z}),$$

which immediately yields that

$$\text{Arg}(\mathcal{Z}(z)) = -\arctan\left(\frac{v}{u}\right) \pmod{2\pi}. \quad (2.14)$$

Thus, the complex function $\mathcal{Z}(z)$ is fully characterized by the above expressions once the relationship between u and v , determined by the complex functions defined in (2.3), is specified.

For discussions of the relationships and applications between complex error functions and other special functions, the reader is referred to the earlier studies listed in [9]-[16]. Moreover, because the present study focuses on complex functions of a single variable and their applications, the works cited in [17]-[19] constitute essential references for researchers in this area.

The next section is devoted to presenting several propositions involving complex error functions defined in (2.4) and (2.6), as well as their associated complex functions.

3. Various propositions, relations and some of possible implications

In this main section, several special results concerning the Faddeeva function of the complex variable z are presented, including comprehensive statements, properties, and their potential implications and applications. Selected results are subsequently proved.

Proposition 3.1. *Let z be any member of the set \mathbb{C} . Then, each of the following propositions dealing with the Faddeeva function of the complex variable z holds:*

$$\mathcal{W}(-z) = \overline{\mathcal{W}(\bar{z})}, \tag{3.1}$$

$$\mathcal{W}(z) + \mathcal{W}(-z) = 2 \exp(-z^2), \tag{3.2}$$

$$\mathcal{W}(z) = \exp(-z^2) [1 - \operatorname{erf}(-iz)], \tag{3.3}$$

$$\mathcal{W}(iz) = \exp(z^2) [1 - \operatorname{erf}(z)] \tag{3.4}$$

and

$$\sqrt{\pi} \mathcal{W}'(z) + 2\sqrt{\pi} z \mathcal{W}(z) - 2i = 0. \tag{3.5}$$

Proof. For the relevant proofs, it suffices to make appropriate use of the main results presented in (2.3)-(2.8). Detailed proofs of Propositions (3.2) and (3.4) are provided, whereas the proofs of Propositions (3.1), (3.3), and (3.5) are left to the interested reader.

For the proof of the statement given in (3.2), the complex function defined by (2.3) immediately yields the assertion:

$$\mathcal{W}(-z) = \exp(-z^2) \operatorname{erfc}(iz) \quad (z \in \mathbb{C}), \tag{3.6}$$

thus, from (2.3) and (3.6), the desired result follows immediately.

For the proof of the statement given in (3.4), using the definition in (2.3), we obtain

$$\begin{aligned} \mathcal{W}(iz) &= \exp\left[-(-iz)^2\right] \operatorname{erfc}[-i(iz)] \\ &= \exp(z^2) \operatorname{erfc}(z) \quad (z \in \mathbb{C}), \end{aligned} \tag{3.7}$$

and by using the main relationship given in (2.7) together with (3.7), the proof can be readily completed based on the principal complex functions described in (2.8). \square

As additional results of potential interest regarding the complex function in (2.3), and given that the complex functions in (2.8) are analytic, several propositions naturally arise concerning series expansions with real or complex parameters, which are particularly suitable for approximate calculations. We now proceed to establish some of these results.

Proposition 3.2. *Let z belong to the set \mathbb{C} . Then, each of the following statements is true.*

$$\exp(-z^2) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^{2j}, \tag{3.8}$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1}, \tag{3.9}$$

$$\operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1}, \tag{3.10}$$

$$\operatorname{erf}(-iz) = -\frac{2i}{\sqrt{\pi}} \int_0^z \exp(\xi^2) d\xi \tag{3.11}$$

$$= -\frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1}, \tag{3.12}$$

$$\operatorname{erfc}(-iz) = 1 + \frac{2i}{\sqrt{\pi}} \int_0^z \exp(\xi^2) d\xi \tag{3.13}$$

$$= 1 + \frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1} \tag{3.14}$$

and

$$\mathcal{W}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^{2j} + \frac{2i}{\sqrt{\pi}} z \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^k \frac{(-1)^k}{(k-\ell)! [2(k-\ell)+1]} z^{2k} \right). \tag{3.15}$$

Proof. In light of the detailed information provided in the special studies [1], [4], [9], [15], and [16], since the main complex error function $\operatorname{erf}(z)$, defined as in (2.6), is an entire function, it has no singularities except at infinity, and its Taylor-Maclaurin expansion always converges as we know. The integral mentioned in (2.6) cannot be evaluated in closed form in terms of elementary complex functions, but by expanding the integrand of the analytic function, in the complex form, namely

$$\mathcal{K}(z) := \exp(z) \quad (z \in \mathbb{C}), \tag{3.16}$$

into its series expansion and then integrating term by term, one can easily obtain its Maclaurin series. However, for convenience, we also want to share two pieces of necessary-special information (just below).

The first is the series expansion for the necessary proof of the exponential function, *which* plays an important role for each of the propositions from (3.8) to (3.15), *which* is

$$\exp(z) = \sum_{j=0}^{\infty} \frac{1}{j!} z^j \quad (z \in \mathbb{C}). \tag{3.17}$$

The second is the revised new form with the help of the definition in (2.6), *which* relates to the error function $\operatorname{erf}(z)$. For it, by changing of the variable in the form $\rho = -i\xi$ for the integral presented by (2.6), the assertion given by

$$\begin{aligned} \operatorname{erf}(-iz) &= \frac{2}{\sqrt{\pi}} \int_0^{-iz} \exp(\rho^2) d\rho \\ &= -\frac{2i}{\sqrt{\pi}} \int_0^z \exp(\xi^2) d\xi \end{aligned} \tag{3.18}$$

is then obtained, *which* also implies that the power series being of the equivalent forms given by

$$\begin{aligned} \operatorname{erf}(-iz) &= -\frac{2i}{\sqrt{\pi}} \int_0^z \left(\sum_{j=0}^{\infty} \frac{1}{j!} \xi^{2j} \right) d\xi \\ &= -\frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \left(\frac{1}{j!} \int_0^z \xi^{2j} d\xi \right) \end{aligned}$$

$$= -\frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1} \quad (z \in \mathbb{C}). \tag{3.19}$$

□

Specifically, with regard to the relevant research and the remaining pending proofs, we would like to offer the following guidance:

- By setting $z := -z^2$ in the mentioned function given in (3.16) and then considering the series expansion in (3.17), the elementary result constituted as in (3.8) is first obtained.

- Through the instrumentality of the assertions given in (3.16) and (3.17), the essential equalities given in (3.9) and (3.10) are easily seen, respectively.

- By combining the assertions presented in (2.7) and (3.17), the essential results given in (3.12) and (3.14) are also obtained.

- For the proof of the final proposition, namely the assertion in (3.15), it suffices to consider the well-known Cauchy product of the associated power series of the complex functions erf and $erfc$.

- Furthermore, since the expressions given in (3.9) and (3.11), as well as those in (3.10) and (3.12), are directly related, the detailed series expansions corresponding to (3.9) and (3.11) have been omitted.

In terms of various possible implications and suggestions, the mentioned propositions, namely Proposition 3.1 and Proposition 3.2, which contain comprehensive information, will aid both us and related researchers. Now, we want to focus on some specific information in this direction, highlighting some specific examples. We also leave the determination of other implications to the dedicated efforts of interested researchers.

For some of those special implications, let us consider the information given in (2.9) and (3.2) (or (3.3)). With the help of the (complex) Faddeeva function defined in (2.3), we can easily arrive at the following special results:

$$\mathcal{W}(0) = 1, \tag{3.20}$$

$$\begin{aligned} \mathcal{W}(i) &= e \cdot erfc(1) \\ &= e [1 - erf(1)] \\ &= e \left(1 - \frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)j!} \right), \end{aligned} \tag{3.21}$$

$$\mathcal{Z}(0) = i\sqrt{\pi}, \tag{3.22}$$

$$\mathcal{Z}(i) = ie\sqrt{\pi} erfc(1) \tag{3.23}$$

$$= ie\sqrt{\pi} \left(1 - \frac{2i}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)j!} \right), \tag{3.24}$$

and

$$\mathcal{W}(z) = \frac{1}{i\sqrt{\pi}} \mathcal{Z}(z) \implies \mathcal{W}(-z) = \frac{1}{i\sqrt{\pi}} \mathcal{Z}(-z) \quad (z \in \mathbb{C}), \quad (3.25)$$

which immediately yields the logical implications given by

$$\begin{aligned} 2 \exp(-z^2) &= \mathcal{W}(z) + \mathcal{W}(-z) \\ &= \frac{1}{i\sqrt{\pi}} [\mathcal{Z}(z) + \mathcal{Z}(-z)] \\ \implies \mathcal{Z}(z) + \mathcal{Z}(-z) &= 2i\sqrt{\pi} \exp(-z^2) \quad (z \in \mathbb{C}). \end{aligned} \quad (3.26)$$

For an additional implication, let us focus on the complex-type differential equation related to the Faddeeva function and its first derivative, which is given in (3.5). For this, consider the following initial value problem:

$$\begin{cases} \sqrt{\pi} \mathcal{W}'(z) - 2z\sqrt{\pi} \mathcal{W}(z) - 2i = 0 \\ \mathcal{W}(0) = 1 \end{cases} \quad (3.27)$$

can easily be obtained by the help of the information given by (3.5) and (3.6), where $z \in \mathbb{C}$.

In fact, new higher-order initial value problems can also be created by taking various order derivatives of the differential equation mentioned above. We leave such new determinations to interested researchers.

4. Conclusion and recommendations

In the first section, the extensive role of complex error functions in science and technology, as well as some motivating factors, was emphasized.

The second section presented special definitions related to certain mathematical forms (integrals and functions), the complex error functions, and specific associated functions, organized as several detailed remarks. An illustrative example covering applications between (2.9) and (2.14) was also discussed.

In the third section, several essential statements concerning the aforementioned complex-valued functions are formulated, and some of them are proven. Possible implications of these results are presented in (3.1)-(3.15) as examples for interested researchers. Furthermore, the variety of complex-valued transformations – including numerous elementary computations that can be derived from (3.20)-(3.24) as well as the special computations highlighted in (2.11)-(2.14) – can, of course, be further expanded.

By using the Faddeeva function and related error functions derived from the main error functions indicated in (2.8), numerous propositions and their corresponding implications can be formulated. For instance, considering the complex imaginary error function

$$erfi(z) = -i \operatorname{erf}(iz) \quad (i^2 = -1; z \in \mathbb{C}), \quad (4.1)$$

many analogous results can be explored. For these, it suffices to refer to the detailed results in (2.3)-(4.2).

Additionally, in light of the specific information highlighted in (3.25)-(3.27), the special function given in (2.7) can be applied to the differential equation in (3.6), which takes the differential forms:

$$\mathcal{Z}'(z) + 2z\mathcal{Z}(z) + 2 = 0 \quad \text{or} \quad \begin{cases} \mathcal{Z}'(z) + 2z\mathcal{Z}(z) + 2 = 0 \\ \mathcal{Z}(0) = i\sqrt{\pi} \end{cases}, \quad (4.2)$$

and can be readily solved, where $z \in \mathbb{C}$. Of course, this equation can be readily solved. Moreover, the special-form differential equations in both (3.5) and (4.2) may be extended to other classes of equations, including higher-order differential equations.

In specially, building on the extensive information in Remarks 2.1-2.3, one can also determine lower and/or upper bounds for the modulus of complex-type functions $w(z)$, $\operatorname{erfc}(z)$, $\operatorname{erf}(z)$, $w(z)$, $w(iz)$, $\operatorname{erf}(iz)$ and $\operatorname{erfi}(z)$, as defined in (2.3), (2.4), (2.6), (3.6), (2.7), (3.18), and (4.2), respectively. For each of the studies mentioned here that involve complex functions, it is important to emphasize the role of constructing and analyzing the complex curve Ω in different ways, as highlighted in Remark 2.3. Furthermore, approximate evaluations of function series – arising as consequences of reasonable constraints on the relevant z (or $|z|$) parameters – can introduce additional perspectives on the limitations associated with the specified moduli.

As final words, for further details and applications of complex error functions derived from the main functions indicated in (2.8), interested researchers may consult the studies in [2], [3], [6], [8], [10], [13], and [19]-[24].

References

- [1] Abramowitz, M., Stegun, I. A., *Handbook of Mathematical Functions with formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series 55, Tenth Printing, National Bureau of Standards, Washington, DC, 1972.
- [2] Fitzpatrick, R., *Plasma Dispersion Function*, Plasma Physics lecture notes, University of Texas, Austin, 2011.
- [3] Zaghoul, M. R., Ali, A. N., *Algorithm 916: Computing the Faddeyeva and Voigt Functions*, ACM Trans. Math. Soft., **38**(2011), no. 1, pp. 15.
- [4] Faddeeva, V. N., Terent'ev, N. N., *Tables of values of the function $W(z) = e^{-z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt\right)$ for complex argument*, Gosud. Izdat. Teh.-Teor. Lit., Moscow, 1954, English transl., Pergamon Press, New York, 1961.
- [5] Chiarella, C., Reiche, A., *On the evaluation of integrals related to the error function*, Math. Comput., **22**(1968), 137-143.
- [6] Olver, F. W. J., Lozier, D. W., Boisvert, R. F., Clark, C. W., *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, USA, 2010.
- [7] Andrews, L. C., *Special functions of mathematics for engineers*, SPIE Press, 1998.
- [8] Alzer, H., *Error function inequalities*, Adv. Comput. Math., **33**(2010), no. 3, 349-379.

- [9] Irmak, H., *A Note on Error Functions and Some Related Special Functions*, IJESA, **10**(2026), no. 1, 9-14.
- [10] Salzer, H. E., *Formulas for calculating the error function of a complex variable*, MTAC, **5**(1951), no. 13, 67-70.
- [11] Lebedev, N. N., *Special Functions and their Applications*, (Translated by Richard A. Silverman), Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965.
- [12] Schreier, F., *The Voigt and complex error function: A comparison of computational methods*, J. Quant. Spectrosc. Radiat. Transfer, **48**(1992), no. 5-6, 743-762.
- [13] Salem, F. A. S., Irmak, H., *A special note on the error functions defined in certain domains of the complex plane and some of their implication*, Transactions on Applied and Theoretical Mechanics, **18**(2023), 94-101.
- [14] Korn, G. A., Korn, T. M., *Mathematical handbook for scientists and engineers: Definitions, theorems and formulas for reference and review*, Dover Publications, 2000.
- [15] Irmak, H., *Various results for series expansions of the error functions with the complex variable and some of their implications*, Turkish J. Math., **44**(2020), no. 5, 1640-1648.
- [16] Mohammed, N. H., Cho, N. C., Adegani, E. A. A., Bulboacă, T., *Geometric properties of normalized imaginary error function*, Stud. Univ. Babeş-Bolyai Math., **67**(2022), no. 2, 455-462.
- [17] Dettman, J., *Applied Complex Variables*, Dover Publications, INC, New York, 1970.
- [18] Rudin, W., *Principles of Mathematical Analysis*, International Series in Pure & Applied Mathematics, McGraw-Hill Education, USA, 1964.
- [19] Abrarov, S. M., Quine, B. M., Jagpal, R. K., *A sampling-based approximation of the complex error function and its implementation without poles*, Appl. Numer. Math., **129**(2018), 181-191.
- [20] Poppe, G. P. M., Wijers, C. M. J., *More efficient computation of the complex error function*, ACM Trans. Math. Softw., **16**(1990), no. 1, 38-46.
- [21] Fettis, H. E., Caslin, J. C., Cramer, K. R., *Complex Zeros of the Error Function and of the Complementary Error Function*, Math. Comput., **27**(1973), no. 122, 401-407.
- [22] Brendel, R., Bormann, D., *An infrared dielectric function model for amorphous solids*, J. Appl. Phys., **71**(1992), no. 1, pp. 1.
- [23] Irmak, H., Agarwal, P., Agarwal, R. P., *The complex error functions and various extensive results together with implications pertaining to certain special functions*, Turkish J. Math., **46**(2022), no. 2, 662-667.
- [24] Behnad, A., *A Novel Extension to Craig's Q-Function Formula and Its Application in Dual-Branch EGC Performance Analysis*, IEEE Trans. Commun., **68**(2020), no. 7, 4117-4125.

Hüseyin Irmak 

Department of Computer Engineering




Faculty of Engineering & Architecture

Nişantaşı University

TR-34481742, Sarıyer, İstanbul, Turkey

e-mail: huseyin.irmak@nisantasi.edu.tr or hirmak70@gmail.com

Analysis of certain determinants for a defined subclass of analytic functions using Poisson distribution series in petal-shaped domain

Sangarambadi Padmanabhan Vijayalakshmi , Martin Mehdha ,
Saurabh Porwal , Raman Ezhilarasi  and Thirumalai Vinjimur
Sudharsan 

Abstract. The current study focuses on obtaining the sharp coefficient estimates and Fekete-Szegő inequality for the class $\Psi_{\vartheta}(m, \lambda)$ and uses the Poisson distribution series to obtain the sharp estimates of coefficient inequalities, Fekete-Szegő inequality, second order Toeplitz determinants and upper bounds of third order Toeplitz determinants and second order Hankel determinants for a certain analytic function $\mathbb{U}(z) = z + \delta_2 z^2 + \delta_3 z^3 + \dots, \mathbb{U}(z) \neq 0, z \in \Delta$ belonging to the class $\mathbb{P}\Psi_{\vartheta}(m, \lambda, \Upsilon) = \{\mathbb{U} \in \mathcal{H} : I^k \mathbb{U} \in \Psi_{\vartheta}(m, \lambda)\}, m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, \lambda, \vartheta \in \mathbb{N} = \{1, 2, \dots\}, \Upsilon = \Upsilon_i(k) = \frac{k^{i-1}}{(i-1)!} e^{-k}$, defined on the open unit disc ($z \in \Delta := \{z : |z| < 1\}$). This research could motivate others to delve deeper into the coefficient functional problem related to the Poisson distribution series of analytic functions across different categories of univalent functions.

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1. Introduction

Consider the class \mathcal{H} comprising analytic functions of the form

$$\mathbb{U}(z) = z + \delta_2 z^2 + \delta_3 z^3 + \dots, \quad (z \in \Delta := \{z : |z| < 1\}), \quad (1.1)$$

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and $\mathcal{S} = \{U(z) \in \mathcal{H} : U(z) \text{ is univalent in } \Delta\}$. Subordination of U to V in Δ is written as $U \prec V$, where $U, V \in \mathcal{H}$, provided there exists a Schwarz function w such that $w(0) = 0$ and $|w(z)| \leq |z|$ such that $U(z) = V(w(z))$, $z \in \Delta$.

Let \mathcal{P} denote the class of all functions $p(z)$ given by

$$p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i, \quad (z \in \Delta), \operatorname{Re}\{p(z)\} > 0 \text{ and } p(0) = 1. \tag{1.2}$$

Let $h_j(z) = \sum_{n=0}^{\infty} \delta_{n,j} z^n$ ($j = 1, 2$) be analytic in Δ . The Hadamard (or convolution) product of h_1 and h_2 is defined by $(h_1 * h_2)(z) = \sum_{n=0}^{\infty} \delta_{n,1} \delta_{n,2} z^n$, $z \in \Delta$. Following [10], define

$$(U(z))^\vartheta = \left(z + \sum_{k=2}^{\infty} \delta_k z^k \right)^\vartheta, \tag{1.3}$$

for $U(z) \neq 0$ for all $z \neq 0$ and $\vartheta \in \mathbb{N} := \{1, 2, \dots\}$.

We observe that the power function given by formula (1.3) has an analytic branch in Δ if and only if $U(z) \neq 0$ for all $z \neq 0$ and $\vartheta \in \mathbb{N}$. Therefore,

$$(U(z))^\vartheta = (z + \delta_2 z^2 + \delta_3 z^3 + \delta_4 z^4 + \dots)^\vartheta.$$

Applying binomial formula, we observe that

$$(U(z))^\vartheta = z^\vartheta [1 + \vartheta (\delta_2 z + \delta_3 z^2 + \delta_4 z^3 + \dots) + \frac{\vartheta(\vartheta-1)}{2!} (\delta_2 z + \delta_3 z^2 + \delta_4 z^3 + \dots)^2 + \dots].$$

From [20], we have

$$(U(z))^\vartheta = z^\vartheta + \sum_{k=2}^{\infty} A_k z^{\vartheta+k-1}, \tag{1.4}$$

where

$$\begin{aligned} A_2 &= \vartheta \delta_2; \quad A_3 = \delta_3 \vartheta + \frac{\vartheta(\vartheta-1)}{2} \delta_2^2; \quad A_4 = \delta_4 \vartheta + \vartheta(\vartheta-1) \delta_2 \delta_3 + \frac{\vartheta(\vartheta-1)(\vartheta-2)}{6} \delta_2^3; \\ A_5 &= \delta_5 \vartheta + \frac{\vartheta(\vartheta-1)}{2} [\delta_3^2 + 2\delta_2 \delta_4] + \frac{\vartheta(\vartheta-1)(\vartheta-2)}{2} \delta_2^2 \delta_3 + \frac{\vartheta(\vartheta-1)(\vartheta-2)(\vartheta-3)}{24} \delta_2^4, \dots \end{aligned} \tag{1.5}$$

If $U(z) \neq 0$ for all $z \neq 0$, define

$$\Lambda(z) = z \left(\frac{U(z)}{z} \right)^\vartheta = z \left[\frac{(z^\vartheta + \sum_{k=2}^{\infty} A_k z^{\vartheta+k-1})}{z^\vartheta} \right] = z + \sum_{k=2}^{\infty} A_k z^k. \tag{1.6}$$

Since $\left(\frac{U(z)}{z} \right)^\vartheta \neq 0$ for all $z \in \Delta$, $\left(\frac{U(z)}{z} \right)^\vartheta$ is analytic in Δ and so $\Lambda(z)$ is analytic in Δ . Indeed, $\Lambda \in \mathcal{H}$.

Bernardi [5] studied the general operator

$$I_2 U(z) = \frac{1+\gamma}{z^\gamma} \int_0^z U(t) t^{\gamma-1} dt, \quad \gamma \in \mathbb{N}, \tag{1.7}$$

and showed that $I_2(\mathcal{S}^*) \subset \mathcal{S}^*$. The important subfamily of \mathcal{S} is the family \mathcal{S}^* called the starlike function.

By [10] and applying the operator to the function $\Lambda(z)$ defined in (1.6), we define the

integral operator J^m iteratively. To ensure a well-defined sequence, we first establish the base cases J^0 and J^1 :

$$J^0(\Lambda(z)) = \Lambda(z) \tag{1.8}$$

The first-order integral operator J^1 is defined as:

$$J^1(\Lambda(z)) = \frac{1 + \lambda}{z^\lambda} \int_0^z \Lambda(t)t^{\lambda-1} dt, \quad (\lambda \in \mathbb{N}). \tag{1.9}$$

Substituting the power series expansion $\Lambda(z) = z + \sum_{k=2}^\infty A_k z^k$ into the expression above, we obtain:

$$J^1(\Lambda(z)) = z + \sum_{k=2}^\infty \left(\frac{1 + \lambda}{k + \lambda} \right) A_k z^k. \tag{1.10}$$

For $m \in \{2, 3, \dots\}$, the higher-order operators are defined via the following recursive relation:

$$J^m(\Lambda(z)) = J(J^{m-1}(\Lambda(z))). \tag{1.11}$$

Consequently, by the principle of mathematical induction, the general form of the operator for any $m \in \mathbb{N}_0$ where $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ is given by:

$$J^m(\Lambda(z)) = z + \sum_{k=2}^\infty \left(\frac{1 + \lambda}{k + \lambda} \right)^m A_k z^k. \tag{1.12}$$

For $k \geq 1$ and $n \geq 1$, Pommerenke [21] introduced the k^{th} Hankel determinant defined as

$$H_k(n) = \begin{vmatrix} \delta_n & \delta_{n+1} & \cdots & \delta_{n+k-1} \\ \delta_{n+1} & \cdots & \cdots & \delta_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{n+k-1} & \delta_{n+k} & \cdots & \delta_{n+2k-2} \end{vmatrix}, \quad (\delta_1 = 1).$$

In particular, the second Hankel determinant takes the form $\mathbb{H}_2(2) = \begin{vmatrix} \delta_2 & \delta_3 \\ \delta_3 & \delta_4 \end{vmatrix}$. It follows that Fekete-Szegő inequality is $\mathbb{H}_2(1)$. Toeplitz determinants and the Hankel determinants are closely connected.

As described in [25], the symmetric Toeplitz determinant $\mathcal{T}_k(n)$ takes the form

$$\mathcal{T}_k(n) = \begin{vmatrix} \delta_n & \delta_{n+1} & \cdots & \delta_{n+k-1} \\ \delta_{n+1} & \delta_n & \cdots & \delta_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n+k-1} & \delta_{n+k} & \cdots & \delta_n \end{vmatrix}.$$

In particular,

$$\mathcal{T}_2(2) = \begin{vmatrix} \delta_2 & \delta_3 \\ \delta_3 & \delta_2 \end{vmatrix}, \quad \mathcal{T}_3(1) = \begin{vmatrix} 1 & \delta_2 & \delta_3 \\ \delta_2 & 1 & \delta_2 \\ \delta_3 & \delta_2 & 1 \end{vmatrix}.$$

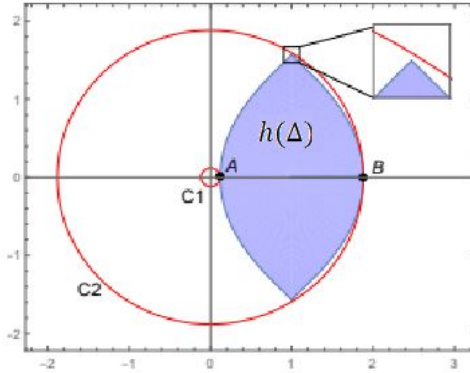


FIGURE 1. Boundary curve: $h(\Delta)$ lies in the annular region bounded between the circles $C1$ and $C2$.

Toeplitz matrices has applications in pure and applied mathematics [28]. Arora and Kumar [2] considered the petal-shaped region

$$\Delta_p = \{w \in \mathbb{C} : |\sinh(w - 1)| < 1\}, \tag{1.13}$$

which admits the following functional characterization:

$$h(z) = 1 + \sinh^{-1}z. \tag{1.14}$$

Its boundary curve is given in Figure 1.

Although $h(z)$ is multivalued, it becomes holomorphic in Δ upon choosing branch cuts along $(-\infty, -i) \cup (i, \infty)$ on the imaginary axis, and hence is analytic in Δ . Geometrically, the function $h(z)$ maps the unit disc Δ onto the petal-shaped domain Δ_p . This domain has attracted considerable attention from researchers [1, 2, 4, 12, 17].

2. Preliminary results

In [10], the authors studied the class $\Psi_\lambda^n(m, \delta)$ for $m \in \mathbb{N}_0 = \{0, 1, 2, 3 \dots\}$, $n \in \mathbb{N}$, $\lambda > 0$, $z \in \Delta$ and $0 \leq \delta < 1$. Motivated by this study, we introduce and investigate a subclass of \mathcal{H} defined on a petal-shaped domain, as described below.

Definition 2.1. Let $\mathbb{U}(z) \neq 0$ for $z \in \Delta$ and let $\vartheta \in \mathbb{N}$. Define $\Lambda(z)$ by (1.6). Then the function \mathbb{U} is said to belong to the class $\Psi_\vartheta(m, \lambda)$, where $m \in \mathbb{N}_0$ and $\lambda, \vartheta \in \mathbb{N}$, if $\Lambda(z)$ satisfies the condition

$$\frac{(J^m \Lambda(z))}{z} \prec 1 + \sinh^{-1}z, \tag{2.1}$$

Lemma 2.2 ([6], p.41, Carathéodory’s Lemma). If $p \in \mathcal{P}$, then

$$|c_k| \leq 2 \quad (k \geq 1), \tag{2.2}$$

and the estimate is sharp.

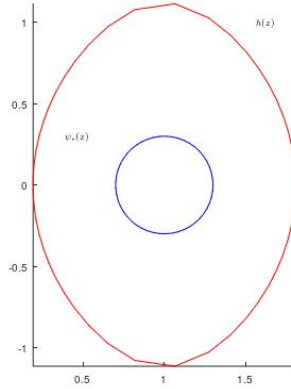


FIGURE 2. The images of $\Psi_*(\Delta)$ (blue color) and $h(\Delta)$ (red color).

Lemma 2.3. [13] *Suppose $p \in \mathcal{P}$. Then, for every complex number μ , one finds*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}. \tag{2.3}$$

Lemma 2.4. [13] *Let $p \in \mathcal{P}$ of the form (1.2). Then there exist some $\xi, \zeta \in \mathbb{C}$ with $|\xi| \leq 1, |\zeta| \leq 1$, such that*

$$2c_2 = c_1^2 + (4 - |c_1^2|)\xi, \tag{2.4}$$

$$4c_3 = c_1^3 + 2c_1\xi(4 - c_1^2) - (4 - c_1^2)c_1\xi^2 + 2(4 - c_1^2)(1 - |\xi|^2)\zeta. \tag{2.5}$$

Remark 2.5. We shall illustrate with an example that the aforementioned class $\Psi_\vartheta(m, \lambda)$ is non-empty.

With $F_1(z) = z + az^2$, we obtain $\Lambda(z) = z(1 + az)^\vartheta$, consequently

$$\Psi_*(z) = \frac{(J^m(\Lambda(z)))}{z} = 1 + t_2A_2z + t_3A_3z^2 + \dots,$$

where $\vartheta \in \mathbb{N}, m \in \mathbb{N}_0, \lambda \in \mathbb{N}$.

For the values of $\vartheta = 1$, we get $\Psi_*(z) = 1 + t_2az$, when $m = \lambda = 1, a = 1/2$ and $z = 0.9e^{i\theta}$, we get $t_2 = \frac{2}{3}$ and hence

$$\Psi_*(0.9e^{i\theta}) = 1 + \frac{1}{3}(0.9e^{i\theta}).$$

Thus from the figure (Figure 2) it is clear that $\Psi_*(\Delta) \subset h(\Delta)$ and hence the class $\Psi_\vartheta(m, \lambda)$ is non-empty.

Remark 2.6. The function $g(z)$, given explicitly by $g(z) = \frac{z}{1-z}, z \in \Delta$ is in the class $\Psi_\vartheta(m, \lambda)$, since

$$\Psi_1(m, \lambda) = \frac{(J^m(\Lambda(z)))}{z} = 1 + t_2A_2z + t_3A_3z^2 + \dots,$$

where $A_2 = \vartheta$; $A_3 = \vartheta + \frac{\vartheta(\vartheta-1)}{2}$; $A_4 = \vartheta + \vartheta(\vartheta-1) + \frac{\vartheta(\vartheta-1)(\vartheta-2)}{6}$; ... and $\Lambda(z) = z\left(\frac{\vartheta(z)}{z}\right)^\vartheta$ and $\vartheta \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$. Proceeding as like previous remark, we have $\Psi_1(\Delta) \subset h(\Delta)$.

Extensive research has been devoted to obtaining the upper bounds of the second and third order Hankel determinants [3, 7, 8, 9, 10, 11, 14, 15, 18, 19, 26, 27]. However, determining the bounds for a subclass of analytic functions utilizing Poisson distribution series in a petal-shaped domain presents numerous applications in science and engineering.

Hence, in the present paper we derive coefficient estimates and Fekete-Szegő inequality for the class $\Psi_\vartheta(m, \lambda)$ and by using the Poisson distribution series, we determine the precise bounds for coefficient inequalities, Fekete-Szegő inequality, second order Toeplitz determinants, upper bounds of third order Toeplitz determinants, second order Hankel determinants for the class $\mathbb{P}\Psi_\vartheta(m, \lambda, \Upsilon)$, $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$ and $\vartheta \in \mathbb{N}$ associated with the petal-shaped domain.

3. Coefficient bounds and the Fekete–Szegő inequality for the class $\Psi_\vartheta(m, \lambda)$

Theorem 3.1. *If the function \mathbb{U} given by (1.1) is in the class $\Psi_\vartheta(m, \lambda)$, for $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$, $\vartheta \in \mathbb{N}$ then*

$$|\delta_2| \leq \frac{1}{\vartheta t_2}, |\delta_3| \leq 2\sigma_2 \max \left\{ 1, \left| \frac{2\sigma_1}{\sigma_2} - 1 \right| \right\},$$

where $t_i = \left(\frac{\lambda+1}{\lambda+i}\right)^m$, $\sigma_1 = \frac{1}{4t_3\vartheta} + \frac{(\vartheta-1)}{8t_2^2\vartheta^2}$, $\sigma_2 = \frac{1}{2t_3\vartheta}$. The inequalities are sharp.

Proof. Let $\mathbb{U}(z) \in \Psi_\vartheta(m, \lambda)$, according to the subordination relationship, there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, satisfying

$$\frac{J^m(\Lambda(z))}{z} = 1 + \sinh^{-1}(w(z)) \quad (z \in \Delta). \tag{3.1}$$

Consider the function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots, \quad p \in \mathbb{P}. \tag{3.2}$$

$$w(z) = \frac{p(z) - 1}{1 + p(z)} = \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right)z^3 + \dots. \tag{3.3}$$

After simple computation, we get

$$1 + \sinh^{-1}(w(z)) = 1 + \frac{1}{2}c_1z + \left(-\frac{c_1^2}{4} + \frac{c_2}{2}\right)z^2 + \left(\frac{5c_1^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right)z^3 + \dots. \tag{3.4}$$

Using series expansion, we obtain

$$\frac{J^m(\Lambda(z))}{z} = 1 + \sum_{i=2}^{\infty} t_i A_i z^{i-1} = 1 + t_2 A_2 z + t_3 A_3 z^2 + t_4 A_4 z^3 + \dots, \tag{3.5}$$

where $t_i = \left(\frac{1+\lambda}{i+\lambda}\right)^m$.

Equating the coefficients between (3.4) and (3.5), we determine that

$$A_2 = \frac{c_1}{2t_2} \Rightarrow \delta_2 = \frac{c_1}{2t_2\vartheta} = c_1\Lambda_1, \tag{3.6}$$

$$\delta_3 = -c_1^2\sigma_1 + c_2\sigma_2, \tag{3.7}$$

$$\delta_4 = \gamma_1c_1^3 - \gamma_2c_1c_2 + \gamma_3c_3, \tag{3.8}$$

where,

$$\Lambda_1 = \frac{1}{2t_2\vartheta}, \sigma_1 = \frac{1}{4t_3\vartheta} + \frac{(\vartheta-1)}{8t_2^2\vartheta^2}, \sigma_2 = \frac{1}{2t_3\vartheta},$$

$$\gamma_1 = \frac{5}{48t_4\vartheta} + \frac{\vartheta-1}{2t_2\vartheta} \left[\frac{1}{4t_3\vartheta} + \frac{\vartheta-1}{8t_2^2\vartheta^2} - \frac{(\vartheta-2)}{24t_2^2\vartheta^2} \right], \gamma_2 = \frac{1}{2t_4\vartheta} + \frac{(\vartheta-1)}{4\vartheta^2t_2t_3}, \gamma_3 = \frac{1}{2\vartheta t_4}.$$

Applying Lemma 2.2 in (3.6), we get

$$|\delta_2| \leq 2\Lambda_1 \leq \frac{1}{\vartheta t_2}.$$

Sharpness is achieved by the function (3.2) with $p(z) = 1 + 2z$.

From (3.7), we have

$$|\delta_3| = | -c_1^2\sigma_1 + c_2\sigma_2 | = \sigma_2 \left| c_2 - c_1^2 \frac{\sigma_1}{\sigma_2} \right|.$$

By implementing Lemma 2.3, we obtain

$$|\delta_3| \leq 2\sigma_2 \max \left\{ 1, \left| \frac{2\sigma_1}{\sigma_2} - 1 \right| \right\}.$$

Equality is attained for the function defined by (3.2) with $p(z) = 1 + 2z^2$, when $\max \left\{ 1, \left| \frac{2\sigma_1}{\sigma_2} - 1 \right| \right\} = 1$ and the inequality is sharp for the function (3.2) with $p(z) = 1 + 2z + 2z^2$, when $\max \left\{ 1, \left| \frac{2\sigma_1}{\sigma_2} - 1 \right| \right\} = \left| \frac{2\sigma_1}{\sigma_2} - 1 \right|$. □

Corollary 3.2. *If $\vartheta = 1$, then $|\delta_2| \leq \frac{1}{t_2}$, $|\delta_3| \leq \frac{2}{t_3}$.*

Theorem 3.3. *Suppose the function \mathbb{U} defined in (1.1) lies in the class $\Psi_\vartheta(m, \lambda)$, for $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$, $\vartheta \in \mathbb{N}$, Consequently, for any $\nu \in \mathbb{C}$*

$$|\delta_3 - \nu\delta_2^2| \leq 2\sigma_2 \max \left\{ 1, \left| 2 \left(\frac{\sigma_1 + \nu\Lambda_1^2}{\sigma_2} \right) - 1 \right| \right\}.$$

The inequality is sharp.

Proof. From (3.6) and (3.7), we get

$$|\delta_3 - \nu\delta_2^2| = |c_2\sigma_2 - c_1^2\sigma_1 - \nu(c_1^2\Lambda_1^2)| = \sigma_2 \left| c_2 - c_1^2 \left(\frac{\sigma_1 + \nu\Lambda_1^2}{\sigma_2} \right) \right|.$$

Applying (2.3), we get

$$|\delta_3 - \nu\delta_2^2| \leq 2\sigma_2 \max \left\{ 1, \left| 2 \left(\frac{\sigma_1 + \nu\Lambda_1^2}{\sigma_2} \right) - 1 \right| \right\}.$$

Equality is attained for the function given by (3.2) with $p(z) = 1 + 2z^2$, when $\max \left\{ 1, \left| 2 \left(\frac{\sigma_1 + \nu \Lambda_1^2}{\sigma_2} \right) - 1 \right| \right\} = 1$ and when $\max \left\{ 1, \left| 2 \left(\frac{\sigma_1 + \nu \Lambda_1^2}{\sigma_2} \right) - 1 \right| \right\} = \left| 2 \left(\frac{\sigma_1 + \nu \Lambda_1^2}{\sigma_2} \right) - 1 \right|$, sharpness is attained for the function given by (3.2) with $p(z) = 1 + 2z + 2z^2$. \square

Corollary 3.4. *If $\vartheta = 1$, then $|\delta_3 - \nu \delta_2^2| \leq \frac{1}{t_3}$.*

4. Functions generated from the Poisson distribution

A random variable \mathcal{X} is said to follow a Poisson distribution with parameter $k > 0$ if it assumes the values $0, 1, 2, 3, \dots$ with respective probabilities

$$P(\mathcal{X} = i) = \frac{k^i e^{-k}}{i!}, \quad i = 0, 1, 2, 3, \dots$$

Thus $P(\mathcal{X} = \tau) = k^\tau \frac{e^{-k}}{\tau!}$, $\tau = 0, 1, 2, \dots$.

Porwal [22] investigated a power series whose coefficients correspond to the probabilities of a Poisson distribution.

$$I(k, z) = z + \sum_{i=2}^{\infty} \frac{k^{i-1}}{(i-1)!} e^{-k} z^i, \quad z \in \Delta, \tag{4.1}$$

where $k > 0$ also by applying ratio test one can find the radius of convergence of the above series is infinity and $J^m(\Lambda(z))$ is defined in (1.12).

Following recent research [16, 24, 23], consider the linear operator $I^k(z) : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$I^k(\Lambda(z)) = I(k, z) * (\Lambda(z)) = z + \sum_{i=2}^{\infty} \frac{k^{i-1}}{(i-1)!} e^{-k} A_i z^i \tag{4.2}$$

$$= z + \sum_{i=2}^{\infty} \Upsilon_i(k) A_i z^i, \tag{4.3}$$

where A_i 's are defined in (1.5) and

$$\Upsilon = \Upsilon_i(k) = \frac{k^{i-1}}{(i-1)!} e^{-k} \tag{4.4}$$

and $*$ denotes the convolution of two series. In particular, $\Upsilon_2 = k e^{-k}$ and $\Upsilon_3 = \frac{k^2}{2} e^{-k}$. Implementing (1.12), we have

$$J^m(I^k \Lambda(z)) = z + \sum_{i=2}^{\infty} \Upsilon_i t_i A_i z^i, \tag{4.5}$$

where $t_i = \left(\frac{1+\lambda}{\lambda+i} \right)^m$.

Definition 4.1. Let $\mathbb{U}(z) \neq 0$ for $z \in \Delta$ and let $\vartheta \in \mathbb{N}$. If the function $\Lambda(z)$ is defined by (1.6) then, the function $\mathbb{U}(z)$ is said to belong to the class $\mathbb{P}\Psi_{\vartheta}(m, \lambda, \Upsilon)$, where $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$ and $\vartheta \in \mathbb{N}$, if the function $\Lambda(z)$ satisfies the following condition:

$$\frac{(J^m(I^k \Lambda(z)))}{z} \prec 1 + \sinh^{-1} z, \tag{4.6}$$

or equivalently, $\mathbb{P}\Psi_{\vartheta}(m, \lambda, \Upsilon) = \{\mathbb{U}(z) \in \mathcal{H} : I^k \mathbb{U}(z) \in \Psi_{\vartheta}(m, \lambda, \Upsilon)\}$, where $J^m(I^k \Lambda(z))$ is given in (4.5) and $\Upsilon_i, i = 2, 3, \dots$ is defined in (4.4).

By employing the corresponding coefficient estimates together with the Fekete–Szegő inequality for functions belonging to the class $\Psi_{\vartheta}(m, \lambda)$, we derive the coefficient estimates and the associated Fekete–Szegő inequality for functions in the class $\mathbb{P}\Psi_{\vartheta}(m, \lambda, \Upsilon)$. Applying Theorems (3.1) and (3.3) yields the following theorems for the function $I^k \Lambda(z)$.

5. Fekete–Szegő inequality and related estimates for the class

$\mathbb{P}\Psi_{\vartheta}(m, \lambda, \Upsilon)$

Theorem 5.1. Let \mathbb{U} be the function defined in (1.1). If \mathbb{U} is in the class $\mathbb{P}\Psi_{\vartheta}(m, \lambda, \Upsilon)$, for $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$, $\vartheta \in \mathbb{N}$ and $\Upsilon = \Upsilon_i(k) = \frac{k^{i-1}}{(i-1)!} e^{-k}$, then

$$|\delta_2| \leq \frac{1}{\vartheta t_2 \Upsilon_2}, \quad |\delta_3| \leq 2\beta_2 \max \left\{ 1, \left| \frac{2\beta_1}{\beta_2} - 1 \right| \right\},$$

where $t_i = \left(\frac{\lambda+1}{\lambda+i}\right)^m$, $\beta_1 = \frac{1}{4t_3 \Upsilon_3 \vartheta} + \frac{(\vartheta-1)}{8t_2^2 \Upsilon_2^2 \vartheta^2}$, $\beta_2 = \frac{1}{2t_3 \Upsilon_3 \vartheta}$. The inequalities are sharp.

Proof. Since $\mathbb{U}(z) \in \mathbb{P}\Psi_{\vartheta}(m, \lambda, \Upsilon)$, the subordination condition ensures the existence of a Schwarz function $w(z)$ for which $w(0) = 0$ and $|w(z)| < 1$, satisfying

$$\frac{J^m(I^k \Lambda(z))}{z} = 1 + \sinh^{-1}(w(z)) \quad (z \in \Delta). \tag{5.1}$$

Consider the function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \dots, \tag{5.2}$$

where $p \in \mathbb{P}$. Consequently,

$$w(z) = \frac{p(z) - 1}{1 + p(z)} = \frac{c_1}{2} z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^3}{8}\right) z^3 + \dots \tag{5.3}$$

After simple computation, we get

$$1 + \sinh^{-1}(w(z)) = 1 + \frac{1}{2} c_1 z + \left(-\frac{c_1^2}{4} + \frac{c_2}{2}\right) z^2 + \left(\frac{5c_1^3}{48} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right) z^3 + \dots \tag{5.4}$$

Using series expansion, we obtain

$$\frac{J^m(I^k \Lambda(z))}{z} = 1 + \sum_{i=2}^{\infty} t_i \Upsilon_i(k) A_i z^{i-1} = 1 + t_2 \Upsilon_2 A_2 z + t_3 \Upsilon_3 A_3 z^2 + \dots, \tag{5.5}$$

where $t_i = \left(\frac{1+\lambda}{i+\lambda}\right)^m$.

Equating the coefficients between (5.4) and (5.5), we determine that $A_2 = \frac{c_1}{2t_2\Upsilon_2}$. Hence,

$$\delta_2 = \frac{c_1}{2t_2\vartheta\Upsilon_2} = c_1\alpha_1, \tag{5.6}$$

$$\delta_3 = -c_1^2\beta_1 + c_2\beta_2, \tag{5.7}$$

$$\delta_4 = \eta_1c_1^3 - \eta_2c_1c_2 + \eta_3c_3, \tag{5.8}$$

where,

$$\begin{aligned} \alpha_1 &= \frac{1}{2t_2\vartheta\Upsilon_2}, \beta_1 = \frac{1}{4t_3\Upsilon_3\vartheta} + \frac{(\vartheta-1)}{8t_2^2\Upsilon_2^2\vartheta^2}, \beta_2 = \frac{1}{2t_3\Upsilon_3\vartheta}, \\ \eta_1 &= \frac{5}{48t_4\Upsilon_4\vartheta} + \frac{\vartheta-1}{2t_2\vartheta\Upsilon_2} \left[\frac{1}{4t_3\Upsilon_3\vartheta} + \frac{\vartheta-1}{8t_2^2\Upsilon_2^2\vartheta^2} - \frac{(\vartheta-2)}{24t_2^2\vartheta^2\Upsilon_2^2} \right], \\ \eta_2 &= \frac{1}{2t_4\Upsilon_4\vartheta} + \frac{(\vartheta-1)}{4\vartheta^2t_2t_3\Upsilon_2\Upsilon_3}, \eta_3 = \frac{1}{2\vartheta t_4\Upsilon_4}. \end{aligned}$$

Applying (2.2) in (5.6), we get

$$|\delta_2| \leq 2\alpha_1 \leq \frac{1}{\vartheta t_2\Upsilon_2}.$$

The bound is attained by the function defined by (5.2) with $p(z) = 1 + 2z$. From (5.7), we have

$$|\delta_3| = | -c_1^2\beta_1 + c_2\beta_2 | = \beta_2|c_2 - c_1^2\frac{\beta_1}{\beta_2}|.$$

By implementing (2.3), we obtain

$$|\delta_3| \leq 2\beta_2 \max \left\{ 1, \left| \frac{2\beta_1}{\beta_2} - 1 \right| \right\}.$$

The inequality is sharp for the function given by (5.2) with $p(z) = 1 + 2z^2$, when $\max \left\{ 1, \left| \frac{2\beta_1}{\beta_2} - 1 \right| \right\} = 1$ and it is sharp for the function given by (5.2) with $p(z) = 1 + 2z + 2z^2$, when $\max \left\{ 1, \left| \frac{2\beta_1}{\beta_2} - 1 \right| \right\} = \left| \frac{2\beta_1}{\beta_2} - 1 \right|$. □

Corollary 5.2. *If $\vartheta = 1$, then*

$$|\delta_2| \leq \frac{1}{t_2\Upsilon_2}, \quad |\delta_3| \leq \frac{1}{t_3\Upsilon_3}.$$

In view of Theorem (3.3), let us prove the following theorem.

Theorem 5.3. *Suppose that the function \mathbb{U} given by (1.1) belongs to the class $\mathbb{P}\Psi_\vartheta(m, \lambda, \Upsilon)$, for $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$, $\vartheta \in \mathbb{N}$ and $\Upsilon = \Upsilon_i(k) = \frac{k^{i-1}}{(i-1)!}e^{-k}$, then for any $\nu \in \mathbb{C}$,*

$$|\delta_3 - \nu\delta_2^2| \leq 2\beta_2 \max \left\{ 1, \left| 2 \left(\frac{\beta_1 + \nu\alpha_1^2}{\beta_2} \right) - 1 \right| \right\}.$$

The inequality is sharp.

Proof. From (5.6) and (5.7), we get

$$|\delta_3 - \nu\delta_2^2| = |c_2\beta_2 - c_1^2[\alpha_1^2\nu + \beta_1]| = \beta_2 \left| c_2 - c_1^2 \left(\frac{\alpha_1^2\nu + \beta_1}{\beta_2} \right) \right|.$$

Applying (2.3), we get

$$|\delta_3 - \nu\delta_2^2| \leq 2 \beta_2 \max \left\{ 1, \left| 2 \left(\frac{\alpha_1^2\nu + \beta_1}{\beta_2} \right) - 1 \right| \right\}.$$

The estimate is attained by the function defined in (5.2) with $p(z) = 1 + 2z^2$, whenever $\max \left\{ 1, \left| 2 \left(\frac{\beta_1 + \nu\alpha_1^2}{\beta_2} \right) - 1 \right| \right\} = 1$ and the bound is attained for the function defined in (5.2) with $p(z) = 1 + 2z + 2z^2$, whenever $\max \left\{ 1, \left| 2 \left(\frac{\beta_1 + \nu\alpha_1^2}{\beta_2} \right) - 1 \right| \right\} = \left| 2 \left(\frac{\beta_1 + \nu\alpha_1^2}{\beta_2} \right) - 1 \right|$. □

6. Toeplitz determinant $\mathcal{T}_2(2)$ and $\mathcal{T}_3(1)$ for the class $\mathbb{P}\Psi_\vartheta(m, \lambda, \Upsilon)$

Theorem 6.1. Consider the function \mathbb{U} , as defined in (1.1), to be a member of the class $\mathbb{P}\Psi_\vartheta(m, \lambda, \Upsilon)$, for $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$, $\vartheta \in \mathbb{N}$ and $\Upsilon = \Upsilon_i(k) = \frac{k^{i-1}}{(i-1)!}e^{-k}$. Then,

$$|\mathcal{T}_2(2)| \leq 16\beta_1^2 + 36\beta_2^2 + 48\beta_1\beta_2 + 4\alpha_1^2,$$

where $\alpha_1, \beta_1, \beta_2$ are presented in theorem (5.1), and it is the best estimate.

Proof. The bound of $\mathcal{T}_2(2)$ is denoted by $|\mathcal{T}_2(2)| = |\delta_3^2 - \delta_2^2|$.

Applying (5.6) and (5.7)

$$|\mathcal{T}_2(2)| = |c_1^4\beta_1^2 - 2c_1^2c_2\beta_1\beta_2 - c_1^2\alpha_1^2 + c_2^2\beta_2^2|. \tag{6.1}$$

Using the values of c_2 from Lemma 2.4, yields

$$|\mathcal{T}_2(2)| = \left| c_1^4\beta_1^2 - c_1^2\beta_1\beta_2 [c_1^2 + (4 - c_1^2)\xi] - c_1^2\alpha_1^2 + \frac{1}{4} [c_1^2 + (4 - c_1^2)\xi]^2 \beta_2^2 \right|. \tag{6.2}$$

Denoting $|c_1| = c$ and $|\xi| = \rho$, then $c \in [0, 2]$ and $\rho \in [0, 1]$, and applying triangle inequality, we get

$$|\mathcal{T}_2(2)| \leq h_1c^4 + h_2\rho c^2|4 - c_1^2| + \alpha_1^2c^2 + \frac{\beta_2^2}{4}\rho^2|4 - c_1^2|^2,$$

where $h_1 = \frac{\beta_2^2}{4} + \beta_1^2 + \beta_1\beta_2$, $h_2 = \beta_2\beta_1 + \frac{\beta_2^2}{2}$. Again, applying triangle inequality for this terms $|4 - c_1^2| \leq 4 + |c_1^2| = 4 + c^2 \geq 0$ similarly, $|(4 - c_1^2)|^2 \leq (4 + c^2)^2$, we get

$$|\mathcal{T}_2(2)| \leq h_1c^4 + h_2\rho c^2(4 + c^2) + c^2\alpha_1^2 + \frac{\beta_2^2}{4}\rho^2(4 + c^2)^2 = A(c, \rho).$$

Next, we maximize the function $A(c, \rho)$ for $(c, \rho) \in [0, 2] \times [0, 1]$. Partially differentiating $A(c, \rho)$ with respect to ρ , we obtain $\frac{\partial A}{\partial \rho} = h_2c^2(4 + c^2) + \frac{\beta_2^2}{2}\rho(4 + c^2)^2 \geq 0$, for $0 \leq c \leq 2$ and $0 \leq \rho \leq 1$. Hence $A(c, \rho)$ is increasing in ρ and attains its maximum at $\rho = 1$. Therefore, $A(c, \rho) \leq A(c, 1) = B(c)$, where $B(c) =$

$|h_1|c^4 + |h_2|c^2(4 + c^2) + \alpha_1^2c^2 + \frac{\beta_2^2}{4}(4 + c^2)^2$. We note that $B'(c) \geq 0$ implies $B(c)$ increases with c and thus attains its maximum value at $c = 2$. Therefore,

$$\begin{aligned} |\mathcal{T}_2(2)| &\leq 16|h_1| + 32|h_2| + 4\alpha_1^2 + 16\beta_2^2 \\ &\leq 16\beta_1^2 + 36\beta_2^2 + 48\beta_1\beta_2 + 4\alpha_1^2. \end{aligned}$$

The obtained bound is sharp. Equality holds for the extremal Carathéodory function $p(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}$, $\theta \in \mathbb{R}$, for which $c_1 = 2e^{i\theta}$, $c_2 = 2e^{2i\theta}$. Substituting these values into the expression (6.1), $\mathcal{T}_2(2)$ yields equality. Hence, the result is sharp. □

Corollary 6.2. *If $\vartheta = 1$, then $|\mathcal{T}_2(2)| \leq 16B_1^2 + 36B_2^2 + 48B_1B_2 + 4A_1^2$, where $A_1 = \frac{1}{2t_2\Upsilon_2}$, $B_1 = \frac{1}{4t_3\Upsilon_3}$, $B_2 = \frac{1}{2t_3\Upsilon_3}$.*

Theorem 6.3. *Suppose the function \mathbb{U} , defined by (1.1), belongs to the class $\mathbb{P}\Psi_\vartheta(m, \lambda, \Upsilon)$, for $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$, $\vartheta \in \mathbb{N}$ and $\Upsilon = \Upsilon_i(k) = \frac{k^{i-1}}{(i-1)!}e^{-k}$, then*

$$|\mathcal{T}_3(1)| \leq 1 + 8\alpha_1^2 + 16(\beta_1^2 + 2\alpha_1^2\beta_1) + 36\beta_2^2 + 48(\alpha_1^2\beta_2 + \beta_1\beta_2).$$

Proof. The bounds of $\mathcal{T}_3(1)$ is denoted by $|\mathcal{T}_3(1)| = |1 - 2\delta_2^2 + 2\delta_2^2\delta_3 - \delta_3^2|$. Proceeding on the similar lines of theorem (6.1), the upper bound is obtained as

$$|\mathcal{T}_3(1)| \leq 1 + 8\alpha_1^2 + 16(\beta_1^2 + 2\alpha_1^2\beta_1) + 36\beta_2^2 + 48(\alpha_1^2\beta_2 + \beta_1\beta_2).$$

□

Corollary 6.4. *If $\vartheta = 1$, then*

$$|\mathcal{T}_3(1)| \leq 1 + 8A_1^2 + 16(B_1^2 + 2A_1^2B_1) + 36B_2^2 + 48(A_1^2B_2 + B_1B_2),$$

where A_1, B_1, B_2 are defined in corollary (6.2).

7. Second Hankel determinant associated with the class $\mathbb{P}\Psi_\vartheta(m, \lambda, \Upsilon)$

Theorem 7.1. *If the function \mathbb{U} of the form (1.1) is in the class $\mathbb{P}\Psi_\vartheta(m, \lambda, \Upsilon)$, for $m \in \mathbb{N}_0$, $\vartheta \in \mathbb{N}$ and $\Upsilon = \Upsilon_i(k) = \frac{k^{i-1}}{(i-1)!}e^{-k}$, then*

$$\mathbb{H}_2(2) = |\delta_2\delta_4 - \delta_3^2| \leq 16(\eta_1\alpha_1 - \beta_1^2) + 24(2\beta_1\beta_2 - \eta_2\alpha_1) + 44\eta_3\alpha_1 + 36\beta_2^2.$$

Proof. Using (5.6) to (5.8), we have

$$|\delta_2\delta_4 - \delta_3^2| = |[\eta_1\alpha_1 - \beta_1^2]c_1^4 + [2\beta_1\beta_2 - \eta_2\alpha_1]c_1^2c_2 + \eta_3\alpha_1c_1c_3 - c_2^2\beta_2^2|. \tag{7.1}$$

Using c_2 and c_3 values from Lemma 2.4, yields

$$\begin{aligned} |\delta_2\delta_4 - \delta_3^2| &= \left| [\eta_1\alpha_1 - \beta_1^2]c_1^4 + [2\beta_1\beta_2 - \eta_2\alpha_1]c_1^2\frac{1}{2}[c_1^2 + \xi(4 - c_1^2)] + \eta_3\alpha_1c_1\frac{1}{4}[c_1^3 \right. \\ &\quad \left. + 2\xi c_1(4 - c_1^2) - \xi^2c_1(4 - c_1^2) + 2(4 - c_1^2)(1 - |\xi|^2)\zeta] - \frac{\beta_2^2}{4}[c_1^2 + \xi(4 - c_1^2)]^2 \right|. \end{aligned}$$

Denoting $|c_1| = c$ and $|\zeta| = \rho$, then $c \in [0, 2]$ and $\rho \in [0, 1]$ also using the fact that $|\zeta| \leq 1$ and implementing triangle inequality, we arrive

$$|\delta_2\delta_4 - \delta_3^2| \leq |q_1|c^4 + q_2\rho c^2|4 - c_1^2| + 2q_3c|4 - c_1^2| + q_3c\rho^2|4 - c_1^2|(c + 2) + q_4\rho^2|4 - c_1^2|^2$$

where $q_1 = \frac{1}{4}[4\eta_1\alpha_1 - 4\beta_1^2 - 2\eta_2\alpha_1 + 4\beta_1\beta_2 + \eta_3\alpha_1 + \beta_2^2]$, $q_2 = \frac{1}{2}[\eta_3\alpha_1 + \beta_2^2 - \eta_2\alpha_1 + 2\beta_1\beta_2]$, $q_3 = \frac{1}{4}[\eta_3\alpha_1]$, $q_4 = \frac{1}{4}\beta_2^2$.

For $c \in [0, 2]$, applying triangle inequality again, we have

$$|4 - c_1^2| \leq 4 + |c_1^2| = (4 + c^2) \geq 0, \text{ and } |4 - c_1^2|^2 \leq (4 + c^2)^2, \text{ thus}$$

$$\begin{aligned} |\delta_2\delta_4 - \delta_3^2| &\leq |q_1|c^4 + |q_2|\rho c^2(4 + c^2) + 2|q_3|c(4 + c^2) + |q_3|c\rho^2(4 + c^2)(c + 2) + |q_4|\rho^2(4 + c^2)^2 \\ &= V(c, \rho), \end{aligned}$$

Proceeding as in Theorem 6.1, we get $V'(\rho) > 0$. Thus $V(c, \rho)$ attains the maximum value at $\rho = 1$. Hence

$$V(c, \rho) \leq V(c, 1) = U(c) = |q_1|c^4 + |q_2|c^2(4 + c^2) + 2|q_3|c(4 + c^2) + |q_3|c(4 + c^2)(c + 2) + |q_4|(4 + c^2)^2.$$

$U'(c) \geq 0$ implies $U(c)$ is an increasing function and it attains the maximum value at $c = 2$. Therefore,

$$\begin{aligned} |\delta_2\delta_4 - \delta_3^2| &\leq 16|q_1| + 32|q_2| + 96|q_3| + 64|q_4| \\ &\leq 16(\eta_1\alpha_1 - \beta_1^2) + 24(2\beta_1\beta_2 - \eta_2\alpha_1) + 44\eta_3\alpha_1 + 36\beta_2^2. \end{aligned}$$

□

Corollary 7.2. *If $\vartheta = 1$, then $|\mathbb{H}_2(2)| \leq 16(D_1A_1 - B_1^2) + 48B_1B_2 + 20D_2A_1 + 36B_2^2$, where A_1, B_1, B_2 are defined in Corollary 6.2 and $D_1 = \frac{5}{48t_4\Upsilon_4}, D_2 = \frac{1}{2t_4\Upsilon_4}$.*

8. Conclusion

Toeplitz and Hankel matrices are open to a broad variety of diverse algorithms and offer some of the most appealing computational features. For a given analytic function $\mathbb{U}(z)$ belonging to the class $\Psi_\vartheta(m, \lambda)$, for $\mathbb{U}(z) \neq 0$ for all $z \in \Delta, m \in \mathbb{N}_0, \lambda \in \mathbb{N}, \vartheta \in \mathbb{N}$ defined on the open unit disc Δ , we have obtained sharp estimates of coefficient inequalities and the Fekete-Szegő inequality. Also, using Poisson distribution series, we have obtained sharp coefficient inequalities, the Fekete-Szegő inequality, sharp second-order Toeplitz determinants, upper bounds of third-order Toeplitz determinants, and second-order Hankel determinants in a petal-shaped domain. We anticipate that these findings will have a significant impact on many different types of science, technology, engineering, and mathematics-related sectors.


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References

- [1] Abbas, M., Alhefthi, R. K., Breaz, D., Arif, M., *On coefficient inequalities for functions of symmetric starlike related to a petal-shaped domain*, *Axioms* **14**(2025), no. 165, 1–25.
- [2] Arora, K., Kumar, S. S., *Starlike functions associated with a petal-shaped domain*, *Bull. Korean Math. Soc.*, **59**(2022), no. 4, 993–1010.
- [3] Wahid, N. H. A., Tumiran, A., Shaba, T. G., *Hankel and Toeplitz determinants of logarithmic coefficients of inverse functions for the subclass of starlike functions with respect to symmetric conjugate points*, *Eur. J. Pure Appl. Math.*, **17**(2024), no. 3, 1818–1830.
- [4] Barukab, O. M., Arif, M., Abbas, M., Khan, S. A., *Sharp bounds of the coefficient results for the family of bounded turning functions associated with a petal-shaped domain*, *J. Funct. Spaces*, Art. ID 5535629, 2021, pp. 9.
- [5] Bernardi, S. D., *Convex and starlike univalent functions*, *Trans. Amer. Math. Soc.*, **135**(1969), 429–446.
- [6] Duren, P. L., *Univalent Functions*, Grundlehren Math. Wiss., Springer-Verlag, New York, vol. 259, 1983.
- [7] Ehrenborg, R., *The Hankel determinant of exponential polynomials*, *Amer. Math. Monthly*, **107**(2000), no. 6, 557–560.
- [8] Janteng, A., Halim, S. A., Darus, M., *Hankel determinant for starlike and convex functions*, *Int. J. Math. Anal.*, **1**(2007), no. 13, 619–625.
- [9] Janteng, A., Halim, S. A., Darus, M., *Hankel determinant for functions starlike and convex with respect to symmetric points*, *J. Qual. Meas. Anal.* **2**(2006), no. 1, 37–43.
- [10] Juma, A. R., Al-Fayadh, A., Vijayalakshmi, S. P., Sudharsan, T. V., *Upper bound on the third Hankel determinant for the class of univalent functions using an integral operator*, *Afr. Mat.*, **33**(2022), no. 56, 33–56.
- [11] Kanas, S., Adegani, E. A., Zireh, A., *A unified approach to second Hankel determinant of bi-subordinate functions*, *Mediterr. J. Math.*, **14**(2017), no. 233, 1–12.
- [12] Shi, L., Srivastava, H. M., Khan, M. G., Khan, N., Ahmad, B., Khan, B., Mashwani, W. K., *Certain subclasses of analytic multivalent functions associated with petal-shaped domain*, *Axioms*, **10**(2021), no. 4, 1–14.
- [13] Libera, R. J., Zlotkiewicz, E. J., *Coefficient bounds for the inverse of a function with derivative in \mathcal{P}* , *Proc. Amer. Math. Soc.*, **87**(1983), no. 2, 251–257.
- [14] Mandal, S., Roy, P. P., Ahamed, M. B., *Hankel and Toeplitz determinants of logarithmic coefficients of inverse functions for certain classes of univalent functions*, *Iran. J. Sci. Technol. Trans. A Sci.*, **49**(2025), 243–252.
- [15] Motamednezhad, A., Bulboacă, T., Adegani, E. A., Dibagar, N., *Second Hankel determinant for a subclass of analytic bi-univalent functions defined by subordination*, *Turk. J. Math.*, **42**(2018), 2798–2808.

- [16] Murugusundaramoorthy, G., *Subclasses of starlike and convex functions involving Poisson distribution series*, Afr. Mat., **28**(2017), 1357–1366.
- [17] Nithiyandham, E. K., Srutha Keerthi, B., *Fekete–Szegő inequality for Sakaguchi type of functions in petal-shaped domain*, Aust. J. Math. Anal. Appl., **19**(2022), no. 2, pp. 8.
- [18] Noonan, J. W., Thomas, D. K., *On the second Hankel determinant of areally mean p -valent functions*, Trans. Amer. Math. Soc., **223**(1976), 337–346.
- [19] Noor, K. I., *Hankel determinant problem for the class of functions with bounded boundary rotation*, Rev. Roumaine Math. Pures Appl., **28**(1983), no. 8, 731–739.
- [20] Opoola, T. O., *On a new subclass of univalent functions*, Mathematica (Cluj) **36** (1994), no. 2, 195–200.
- [21] Pommerenke, C., *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc., **41**(1966), 111–122.
- [22] Porwal, S., *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal., Art. ID 984135, 2014, 3 pp.
- [23] Porwal, S., Kumar, M., *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat., **27**(2016), 1021–1027.
- [24] Tang, H., Murugusundaramoorthy, G., Li, S., Ma, L., *Fekete–Szegő and Hankel inequalities for certain class of analytic functions related to the sine function*, AIMS Math., **7**(2022), no. 4, 6365–6380.
- [25] Thomas, D. K., and Abdul Halim, S., *Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions*, Bull. Malays. Math. Sci. Soc., **40**(2017), no. 4, 1781–1790.
- [26] Vamshee K. D., Ramreddy, T., *Hankel determinant for p -valent starlike and convex functions of order α* , Novi Sad J. Math., **42**(2012), no. 2, 89–102.
- [27] Vijayalakshmi, S. P., Sudharsan, T. V., Breaz, D., Subramanian, K. G., *Bounds on the third order Hankel determinant for certain subclasses of analytic functions*, An. Științ. Univ. Ovidius Constanța Ser. Mat., **25**(2017), no. 3, 199–214.
- [28] Ye, K., Lim, L., *Every matrix is a product of Toeplitz matrices*, Found. Comput. Math., **16**(2016), 577–598.

Sangarambadi Padmanabhan Vijayalakshmi 

Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College,
Chennai, India

e-mail: vijishreekanth@gmail.com

Martin Mehdha 


Research Scholar, University of Madras and Department of Mathematics,
DRBCCC Hindu College, Pattabiram,
Chennai, India


e-mail: mehdhamartin@gmail.com

Saurabh Porwal 

Department of Mathematics, Ram Sahai Government Degree College,
Bairi-Shivrajpur, Kanpur,
Uttar Pradesh, India

e-mail: saurabhjcb@rediffmail.com

Raman Ezhilarasi 
Department of Mathematics, S.I.V.E.T College,
Gowrivakkam, Chennai, India
e-mail: ezhilarasi2906@gmail.com

Thirumalai Vinjimur Sudharsan 
Department of Mathematics, S.I.V.E.T College,
Gowrivakkam, Chennai, India
e-mail: tvsudharsan@gmail.com

Coefficient bounds and Fekete-Szegő inequality for a unified subclass of m -fold symmetric bi-univalent functions

Navyodh Singh , Gagandeep Singh  and Navjeet Singh 

Abstract. In this paper, we introduce a new and unified subclass of m -fold symmetric bi-univalent functions by subordinating to generalized Janowski function, in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$. Bounds for the initial coefficients and Fekete-Szegő inequality for the functions in this class are studied. Particular cases of the results derived here, are also discussed.

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1. Introduction

Let \mathcal{A} denote the class of analytic functions f having Taylor-Maclaurin series of the form


$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

defined in the unit disc $\mathbb{E} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Further, the class of functions $f \in \mathcal{A}$ and univalent in \mathbb{E} , is denoted by \mathcal{S} .

By \mathcal{U} , we denote the class of Schwarz functions of the form $u(z) = \sum_{k=1}^{\infty} c_k z^k$, which are analytic in the unit disc \mathbb{E} and satisfy the conditions $u(0) = 0$ and $|u(z)| < 1$.

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The fundamental classes of starlike functions and convex functions are denoted by \mathcal{S}^* and \mathcal{K} respectively, and given by

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in \mathbb{E} \right\},$$

and

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{(zf'(z))'}{f'(z)} \right) > 0, z \in \mathbb{E} \right\}.$$

It is obvious that the classes \mathcal{S}^* and \mathcal{K} are related as $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$. This relation was established by Alexander [1] and is known as Alexander relation.

A function $f \in \mathcal{A}$ is said to be in the class \mathcal{C} of close-to-convex functions if there exists a convex function $h \in \mathcal{K}$ such that $\operatorname{Re} \left(\frac{f'(z)}{h'(z)} \right) > 0$ or equivalently there exists a starlike function $g \in \mathcal{S}^*$ such that $\operatorname{Re} \left(\frac{zf'(z)}{g(z)} \right) > 0$. This class was introduced by Kaplan [20].

Further, Noor [30] introduced the class \mathcal{C}^* of quasi-convex functions. A function $f \in \mathcal{A}$ is said to be quasi-convex if there exists a convex function $h \in \mathcal{K}$ such that

$$\operatorname{Re} \left(\frac{(zf'(z))'}{h'(z)} \right) > 0, z \in \mathbb{E}.$$

Every quasi-convex function is convex. Obviously, $f \in \mathcal{C}^*$ if and only if $zf' \in \mathcal{C}$.

Sakaguchi [33] established the class \mathcal{S}_s^* of functions $f \in \mathcal{A}$ which satisfy the following condition:

$$\operatorname{Re} \left(\frac{2zf'(z)}{f(z) - f(-z)} \right) > 0.$$

The functions in the class \mathcal{S}_s^* are called starlike functions with respect to symmetric points. Das and Singh [11] proved that $\frac{f(z) - f(-z)}{2}$ is a starlike function in \mathbb{E} . So it is obvious that the class \mathcal{S}_s^* is contained in the class \mathcal{C} of close-to-convex functions.

Later on, Das and Singh [11] introduced the class \mathcal{K}_s of the functions $f \in \mathcal{A}$ which satisfy the following condition:

$$\operatorname{Re} \left(\frac{2(zf'(z))'}{(f(z) - f(-z))'} \right) > 0.$$

The functions in the class \mathcal{K}_s are called convex functions with respect to symmetric points. It is easy to verify that $f \in \mathcal{K}_s$ if and only if $zf' \in \mathcal{S}_s^*$. Further, some more subclasses of Sakaguchi-type functions were investigated by several authors including [21, 31, 36, 37].

For $0 \leq \alpha < 1$ and $s, t \in \mathbb{C}$ (Complex plane) with $s \neq t$, Frasin [13] introduced and studied the class $\mathcal{S}(\alpha, s, t)$ which consists of the functions $f \in \mathcal{A}$ and satisfy the condition

$$\operatorname{Re} \left(\frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right) > \alpha.$$

Analogously, the class $\mathcal{T}(\alpha, s, t)$ consists of the functions $f \in \mathcal{A}$ which satisfy the condition

$$\operatorname{Re} \left(\frac{(s-t)(zf'(z))'}{(f(sz) - f(tz))'} \right) > \alpha.$$

Obviously $f \in \mathcal{T}(\alpha, s, t)$ if and only if $zf' \in \mathcal{S}(\alpha, s, t)$. Also the following observations are obvious:

- (i) $\mathcal{S}(0, 1, -1) \equiv \mathcal{S}_s^*$.
- (ii) $\mathcal{T}(0, 1, -1) \equiv \mathcal{K}_s$.
- (iii) $\mathcal{S}(0, 1, 0) \equiv \mathcal{S}^*$.
- (iv) $\mathcal{T}(0, 1, 0) \equiv \mathcal{K}$.

Let f and g be two analytic functions in \mathbb{E} . Then f is said to be subordinate to g , if there exists a Schwarz function $u \in \mathcal{U}$ such that $f(z) = g(u(z))$. If f is subordinated to g , then it is written as $f \prec g$. Further, if g is univalent in \mathbb{E} , then $f \prec g$ implies $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Univalent functions are one-one and so are invertible. Also the inverse functions need not be defined on the entire unit disc \mathbb{E} . The Koebe one-quarter theorem [12] ensures that the image of \mathbb{E} under the functions $f \in \mathcal{S}$ contains a disc of radius $\frac{1}{4}$. It is obvious that every function $f \in \mathcal{S}$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z(z \in \mathbb{E}),$$

and

$$f(f^{-1}(w)) = w \left(|w| < r_0(f) : r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{E} if both f and f^{-1} are univalent in \mathbb{E} . The class of functions $f \in \mathcal{A}$ which are bi-univalent in \mathbb{E} , is denoted by Σ . Some examples of functions in the class Σ are $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2} \log \left(\frac{1+z}{1-z} \right)$ with the corresponding inverse functions $\frac{e^w - 1}{e^w}$, $\frac{w}{1+w}$, $\frac{e^{2w} - 1}{e^{2w} + 1}$. But, the well known Koebe function $f(z) = \frac{z}{(1-z)^2}$ is not a member of Σ .

Estimating the Taylor-Maclaurin coefficients a_n , is an important problem in geometric function theory as it provides information about the geometric properties of the functions in \mathcal{A} . Lewin [22] was the first, who investigated the class Σ and proved that $|a_2| < 1.51$. Further, Brannan and Clunie [7] conjectured that $|a_2| < \sqrt{2}$.

Subsequently, non-sharp bounds for $|a_2|$ and $|a_3|$ for various sub-classes of Σ were studied by several authors in [10, 14, 17, 19, 24, 25, 26, 28, 29, 37, 41, 42, 43, 44, 46, 47] and more recently by Singh and Singh [39, 40].

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, Polatoglu et al. [32] introduced the class $\mathcal{P}(A, B; \alpha)$, the subclass of \mathcal{A} which consists of functions of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \frac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}$. Also for $\alpha = 0$, the class $\mathcal{P}(A, B; \alpha)$ agrees with $\mathcal{P}(A, B)$, which is a subclass of \mathcal{A} introduced by Janowski [18].

For $m \in \mathbb{N}$, a domain \mathbb{D} is said to be m -fold symmetric if a rotation of \mathbb{D} about the origin through an angle $\frac{2\pi}{m}$, carries \mathbb{D} on itself. It follows that a function f analytic in \mathbb{E} is said to be m -fold symmetric if

$$f(e^{\frac{2\pi}{m}} z) = e^{\frac{2\pi}{m}} f(z).$$

We denote the class of m -fold symmetric univalent functions by \mathcal{S}_m . A function $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}.$$

Srivastava et al. [45] defined m -fold symmetric bi-univalent functions analogues to the concept of m -fold symmetric univalent functions. They gave some important results such as each function $f \in \Sigma$ generates an m -fold symmetric bi-univalent function for every $m \in \mathbb{N}$. Furthermore, for the normalized form of f , they obtained the series expansion of f^{-1} as follows:

$$g(w) = f^{-1}(w) = w - a_{m+1} w^{m+1} + ((m + 1)a_{m+1}^2 - a_{2m+1}) w^{2m+1} + \dots,$$

where $f^{-1} = g$. We denote the class of m -fold symmetric bi-univalent functions by Σ_m . For $m = 1$, the class Σ_m coincides with Σ . Various examples of m -fold symmetric

bi-univalent functions are $\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}}$, $[-\log(1 - z^m)]^{\frac{1}{m}}$, $\frac{1}{2} \log\left(\frac{1 + z^m}{1 - z^m}\right)^{\frac{1}{m}}$, with

the corresponding inverse functions $\left(\frac{w^m}{1 + w^m}\right)^{\frac{1}{m}}$, $\left(\frac{e^{wm} - 1}{e^{wm} + 1}\right)^{\frac{1}{m}}$, $\left(\frac{e^{2wm} - 1}{e^{2wm} + 1}\right)^{\frac{1}{m}}$. Some

important subclasses of m -fold symmetric bi-univalent functions were studied by a few researchers including Al-Khafaji [3], Altinkya and Yalcin [5], Bulut [9], Hussain et al. [15], Ibrahim et al. [16], Sakar and Tasar [34] and Senthil and Keerthi [35].

Getting inspired from the research work mentioned above, now we define a new and generalized subclass $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ of Σ_m . This class is a generalization of many earlier established classes. By giving particular values to the parameters introduced in this class, several known results established by various authors, can be easily obtained. Each new class and its associated results add to the theoretical understanding of bi-univalent functions and their geometric properties. Here we

used the concept of subordination, which provides a powerful and flexible method to establish various interesting results of the subclasses of univalent and bi-univalent functions, in the unit disc \mathbb{E} . So the class proposed here is very significant and plays an important role in motivating the other researchers to study some more generalized subclasses of m -fold symmetric bi-univalent functions. In this paper, we establish the non-sharp bounds for the Taylor-Maclaurin coefficients such as $|a_{m+1}|$, $|a_{2m+1}|$ and Fekete-Szegő inequality for the defined class.

Definition 1.1. A function $f \in \Sigma_m$ is said to be in the class $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ if the following conditions are satisfied:

$$(1 - \alpha) \frac{(s - t)z[f'(z)]^\lambda}{f(sz) - f(tz)} + \alpha \frac{(s - t)[(zf'(z))']^\lambda}{(f(sz) - f(tz))'} \prec \left(\frac{1 + [B + (A - B)(1 - \eta)]z^m}{1 + Bz^m} \right)^\beta,$$

and

$$(1 - \alpha) \frac{(s - t)w[g'(w)]^\lambda}{g(sw) - g(tw)} + \alpha \frac{(s - t)[(wg'(w))']^\lambda}{(g(sw) - g(tw))'} \prec \left(\frac{1 + [B + (A - B)(1 - \eta)]w^m}{1 + Bw^m} \right)^\beta,$$

where $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$, $0 \leq \alpha \leq 1$, $\lambda \geq 0$, $0 < \beta \leq 1$, $0 \leq \eta < 1$, $z \in \mathbb{E}$, $w \in \mathbb{E}$ and $g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + ((m + 1)a_{m+1}^2 - a_{2m+1})w^{2m+1} + \dots$

The following particular cases are obvious:

- (i) For $m = 1$, the class $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ reduces to $\mathcal{S}_{\Sigma}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$.
- (ii) For $m = 1, \eta = 0$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ agrees with $\mathcal{S}_{\Sigma}^{\lambda, \alpha, \beta}(A, B; s, t)$, the class studied by Singh et al. [44].
- (iii) On putting $m = 1, \eta = 0, \lambda = 1, s = 1, t = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ coincides with $\mathcal{M}_{\Sigma}^*(\beta, \alpha; A, B)$, the class studied by Singh [38].
- (iv) By substituting $m = 1, \alpha = 0, \eta = 0, A = 1, B = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ reduces to $\mathcal{S}_{\Sigma}^{\lambda}(s, t, \beta)$, the class studied by Mazi and Opoola [24].
- (v) For $0 \leq \gamma < 1$, on putting $m = 1, \alpha = 0, \eta = 0, \beta = 1, A = 1 - 2\gamma, B = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ agrees with $\mathcal{S}_{\Sigma}^{\lambda}(s, t, \gamma)$, the class studied by Mazi and Opoola [24].
- (vi) On Substituting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1, \alpha = 0$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ reduces to $\mathcal{S}_{\Sigma_m}^{\beta}$, the class studied by Altinkya and Yalcin [5].
- (vii) By putting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1 - 2\gamma, B = -1, \alpha = 0$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ reduces to $\mathcal{S}_{\Sigma_m}^{\gamma}$, the class studied by Altinkya and Yalcin [5].
- (viii) For $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1, m = 1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ coincides with $\mathcal{M}_{\Sigma}(\beta, \alpha)$, the class studied by Li and Wang [23].
- (ix) For $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1, m = 1, \alpha = 1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ reduces to $\mathcal{M}_{\Sigma}(\beta, 1)$, the class studied by Li and Wang [23].
- (x) On Putting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1, m = 1, \alpha = 0$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ agrees with $\mathcal{S}_{\Sigma}^*(\beta)$, the class studied by Brannan and Taha [8].
- (xi) For $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1 - 2\gamma, B = -1, m = 1, \alpha = 0, \beta = 1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ reduces to $\mathcal{S}_{\Sigma}^*(\gamma)$, the class of bi-starlike functions of order γ studied by Brannan and Taha [8].

(xii) For $s = 1, t = 0, \eta = 0, A = 1 - 2\gamma, B = -1, m = 1, \alpha = 1, \beta = 1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ reduces to $\mathcal{B}_{\Sigma}(\gamma, 1)$, the class of bi-convex functions of order γ studied by Li and Wang [23].

(xiii) By substituting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1, B = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ agrees with $\mathcal{M}_{\Sigma_m}(\beta, \alpha, 1)$, the class established by Motamednezhad et al. [27].

(xiv) On putting $\lambda = 1, s = 1, t = 0, \eta = 0, A = 1 - 2\gamma, B = -1$, $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$ agrees with $\mathcal{M}_{\Sigma_m}(\gamma, \alpha, 1)$, the class studied by Motamednezhad et al. [27].

In the sequel, we lay down once for all that $s, t \in \mathbb{C}$ with $s \neq t, |t| \leq 1, 0 \leq \alpha \leq 1, \lambda \geq 0, 0 < \beta \leq 1, 0 \leq \eta < 1, -1 \leq B < A \leq 1, z \in \mathbb{E}, w \in \mathbb{E}$ and $g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + ((m + 1)a_{m+1}^2 - a_{2m+1})w^{2m+1} + \dots$

For deriving our main results, we need the following lemma:

Lemma 1.2. [6] If $p(z) = \frac{1 + [B + (A - B)(1 - \eta)]u(z)}{1 + Bu(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k, u(z) \in \mathcal{U}$, then

$$|p_n| \leq (A - B)(1 - \eta), n \geq 1.$$

2. Coefficient estimates for the function class $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$

Theorem 2.1. If $f \in \mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$, then

$$|a_{m+1}| \leq \frac{\beta \sqrt{2(A - B)(1 - \eta)}}{\sqrt{\Delta}}, \tag{2.1}$$

and

$$|a_{2m+1}| \leq \frac{\beta(A - B)(1 - \eta)}{(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right]} + \frac{(A - B)^2(1 - \eta)^2 \beta^2(m + 1)}{2(1 + \alpha m)^2 \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right) \right]^2}, \tag{2.2}$$

where

$$\begin{aligned} \Delta = & \beta \left[2(1 + 2\alpha m + \alpha m^2) \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right) \left(\frac{s^{m+1} - t^{m+1}}{s - t} - \lambda(m + 1) \right) \right. \\ & - (1 + 2\alpha m)(m + 1) \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) + \lambda(m + 1)((1 + 2\alpha m)(2m + 1) \\ & \left. + (\lambda - 1)(m + 1)(1 + 2\alpha m + \alpha m^2)) \right] - (\beta - 1)(1 + \alpha m)^2 \\ & \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right) \right]^2. \end{aligned} \tag{2.3}$$

Proof. Applying the principle of subordination in Definition 1.1, it yields

$$\begin{aligned}
 & (1 - \alpha) \frac{(s - t)z[f'(z)]^\lambda}{f(sz) - f(tz)} + \alpha \frac{(s - t)[(zf'(z))']^\lambda}{(f(sz) - f(tz))'} \\
 &= \left(\frac{1 + [B + (A - B)(1 - \eta)](u(z))^m}{1 + B(u(z))^m} \right)^\beta \\
 &= [p(z)]^\beta, u \in \mathcal{U},
 \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
 & (1 - \alpha) \frac{(s - t)w[g'(w)]^\lambda}{g(sw) - g(tw)} + \alpha \frac{(s - t)[(wg'(w))']^\lambda}{(g(sw) - g(tw))'} \\
 &= \left(\frac{1 + [B + (A - B)(1 - \eta)](v(w))^m}{1 + B(v(w))^m} \right)^\beta \\
 &= [q(w)]^\beta, v \in \mathcal{U},
 \end{aligned} \tag{2.5}$$

where $p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \dots$ and $q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \dots$

Expanding (2.4), it takes the form

$$\begin{aligned}
 & 1 + (1 + \alpha m) \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right) \right] a_{m+1} z^m \\
 &+ \left[(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right] a_{2m+1} \right. \\
 &+ (1 + 2\alpha m + \alpha m^2) \left[\left(\frac{s^{m+1} - t^{m+1}}{s - t} \right)^2 - \lambda(m + 1) \left[\frac{s^{m+1} - t^{m+1}}{s - t} - \frac{(\lambda - 1)(m + 1)}{2} \right] \right] a_{m+1}^2 \Big] z^{2m} + \dots \\
 &= 1 + \beta p_m z^m + \left[\beta p_{2m} + \frac{\beta(\beta - 1)}{2} p_m^2 \right] z^{2m} + \dots
 \end{aligned} \tag{2.6}$$

Similarly on expanding (2.5), we obtain

$$\begin{aligned}
 & 1 - (1 + \alpha m) \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right) \right] a_{m+1} w^m \\
 &+ \left[- (1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right] a_{2m+1} + (1 + m)(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right] a_{m+1}^2 + (1 + 2\alpha m + \alpha m^2) \left[\left(\frac{s^{m+1} - t^{m+1}}{s - t} \right)^2 - \lambda(m + 1) \left[\frac{s^{m+1} - t^{m+1}}{s - t} - \frac{(\lambda - 1)(m + 1)}{2} \right] \right] a_{m+1}^2 \right] w^{2m} + \dots \\
 &= 1 + \beta q_m w^m + \left[\beta q_{2m} + \frac{\beta(\beta - 1)}{2} q_m^2 \right] w^{2m} + \dots
 \end{aligned} \tag{2.7}$$

Equating the coefficients of z^m and z^{2m} in (2.6), we obtain

$$(1 + \alpha m) \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right) \right] a_{m+1} = \beta p_m, \tag{2.8}$$

$$\begin{aligned}
& (1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right] a_{2m+1} + (1 + 2\alpha m + \alpha m^2) \\
& \left[\left(\frac{s^{m+1} - t^{m+1}}{s - t} \right)^2 - \lambda(m + 1) \left[\frac{s^{m+1} - t^{m+1}}{s - t} - \frac{(\lambda - 1)(m + 1)}{2} \right] \right] a_{m+1}^2 \quad (2.9) \\
& = \beta p_{2m} + \frac{\beta(\beta - 1)}{2} p_m^2.
\end{aligned}$$

Equating the coefficients of w^m and w^{2m} in (2.7), it gives

$$-(1 + \alpha m) \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right) \right] a_{m+1} = \beta q_m, \quad (2.10)$$

$$\begin{aligned}
& -(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right] a_{2m+1} + (1 + m)(1 + 2\alpha m) \\
& \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right] a_{m+1}^2 + (1 + 2\alpha m + \alpha m^2) \\
& \left[\left(\frac{s^{m+1} - t^{m+1}}{s - t} \right)^2 - \lambda(m + 1) \left[\frac{s^{m+1} - t^{m+1}}{s - t} - \frac{(\lambda - 1)(m + 1)}{2} \right] \right] a_{m+1}^2 \\
& = \beta q_{2m} + \frac{\beta(\beta - 1)}{2} q_m^2. \quad (2.11)
\end{aligned}$$

(2.8) and (2.10) together give,

$$p_m = -q_m. \quad (2.12)$$

Further, squaring and adding (2.8) and (2.10), it yields

$$2(1 + \alpha m)^2 \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right) \right]^2 a_{m+1}^2 = \beta^2 (p_m^2 + q_m^2). \quad (2.13)$$

Adding (2.9) and (2.11), we obtain

$$\begin{aligned}
& \left[(1 + m)(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right] + 2(1 + 2\alpha m + \alpha m^2) \right. \\
& \left. \left[\left(\frac{s^{m+1} - t^{m+1}}{s - t} \right)^2 - \lambda(m + 1) \left(\frac{s^{m+1} - t^{m+1}}{s - t} - \frac{(\lambda - 1)(m + 1)}{2} \right) \right] \right] a_{m+1}^2 \\
& = \beta(p_{2m} + q_{2m}) + \frac{\beta(\beta - 1)}{2} (p_m^2 + q_m^2). \quad (2.14)
\end{aligned}$$

Using (2.13) in (2.14), we get

$$\begin{aligned}
& \left[(1 + m)(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s - t} \right) \right] + 2(1 + 2\alpha m + \alpha m^2) \right. \\
& \left. \left[\left(\frac{s^{m+1} - t^{m+1}}{s - t} \right)^2 - \lambda(m + 1) \left(\frac{s^{m+1} - t^{m+1}}{s - t} - \frac{(\lambda - 1)(m + 1)}{2} \right) \right] \right] a_{m+1}^2 \\
& = \beta(p_{2m} + q_{2m}) + \frac{(\beta - 1)(1 + \alpha m)^2 (\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s - t} \right))^2}{\beta} a_{m+1}^2. \quad (2.15)
\end{aligned}$$

Simplifying (2.15), it gives

$$a_{m+1}^2 = \frac{\beta^2(p_{2m} + q_{2m})}{\Delta}, \tag{2.16}$$

where Δ is defined in (2.3).

Taking modulus and applying triangle inequality in (2.16), it takes the form

$$|a_{m+1}|^2 \leq \frac{\beta^2(|p_{2m}| + |q_{2m}|)}{\Delta}. \tag{2.17}$$

Applying Lemma 1.2 in (2.17), we can easily obtain (2.1).

Now subtracting (2.11) from (2.9), we get

$$\begin{aligned} & 2(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right] a_{2m+1} \\ & - (1 + m)(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right] a_{m+1}^2 \\ & = \beta(p_{2m} - q_{2m}) + \frac{\beta(\beta - 1)}{2} (p_m^2 - q_m^2). \end{aligned} \tag{2.18}$$

Using (2.12) and (2.13) in (2.18), it gives

$$\begin{aligned} & 2(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right] a_{2m+1} \\ & = \beta(p_{2m} - q_{2m}) + \frac{(1 + m)(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right] \beta^2 p_m^2}{(1 + \alpha m)^2 \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s-t} \right) \right]^2}. \end{aligned} \tag{2.19}$$

Taking modulus and applying triangle inequality in (2.19), it yields

$$\begin{aligned} |a_{2m+1}| \leq & \frac{\beta(|p_{2m}| + |q_{2m}|)}{2(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right]} \\ & + \frac{(m + 1)\beta^2 |p_m|^2}{2(1 + \alpha m)^2 \left[\lambda(m + 1) - \left(\frac{s^{m+1} - t^{m+1}}{s-t} \right) \right]^2}. \end{aligned} \tag{2.20}$$

Using Lemma 1.2 in (2.20), the inequality (2.2) can be easily obtained. □

On putting $m = 1$, Theorem 2.1 gives the following result:

Remark 2.2. If $f \in \mathcal{S}_\Sigma^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$, then

$$|a_2| \leq \frac{\beta \sqrt{2(A-B)(1-\eta)}}{\sqrt{\beta[(2\lambda - 4\lambda(s+t-\lambda) + 2st) + 2\alpha((s^2 + 4st + t^2) - 6\lambda(s+t-\lambda))] - (\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}},$$

and

$$|a_3| \leq \frac{\beta(A-B)(1-\eta)}{(1+2\alpha)(3\lambda-s^2-st-t^2)} + \frac{(A-B)^2(1-\eta)^2\beta^2}{(1+\alpha)^2(2\lambda-s-t)^2}.$$

By substituting $m = 1, \eta = 0$ in Theorem 2.1, it gives the following result due to Singh et al. [44]:

Remark 2.3. If $f \in \mathcal{S}_{\Sigma}^{\lambda, \alpha, \beta}(A, B; s, t)$, then

$$|a_2| \leq \frac{\beta \sqrt{2(A-B)}}{\sqrt{\beta[(2\lambda - 4\lambda(s+t-\lambda) + 2st) + 2\alpha((s^2 + 4st + t^2) - 6\lambda(s+t-\lambda))] - (\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}},$$

and

$$|a_3| \leq \frac{\beta(A-B)}{(1+2\alpha)(3\lambda-s^2-st-t^2)} + \frac{(A-B)^2\beta^2}{(1+\alpha)^2(2\lambda-s-t)^2}.$$

For $m = 1, \eta = 0, \lambda = 1, s = 1, t = -1$, Theorem 2.1 gives the following result due to Singh [38]:

Remark 2.4. If $f \in \mathcal{M}_{\Sigma}^s(\beta, \alpha; A, B)$, then

$$|a_2| \leq \frac{\beta \sqrt{A-B}}{\sqrt{2((1+\alpha)^2 - \beta\alpha^2)}},$$

and

$$|a_3| \leq \frac{\beta^2(A-B)^2}{4(1+\alpha)^2} + \frac{\beta(A-B)}{2(1+2\alpha)}.$$

By substituting $m = 1, \alpha = 0, \eta = 0, A = 1, B = -1$ in Theorem 2.1, it gives the following result due to Mazi and Opoola [24]:

Remark 2.5. If $f \in \mathcal{S}_{\Sigma}^{\lambda}(s, t, \beta)$, then

$$|a_2| \leq \frac{2\beta}{\sqrt{(6\lambda - 4\lambda(s+t-\lambda+1) + 2st)\beta - (\beta-1)(2\lambda-s-t)^2}},$$

and

$$|a_3| \leq \frac{2\beta}{(3\lambda-s^2-t^2-st)} + \frac{4\beta^2}{(2\lambda-s-t)^2}.$$

By putting $m = 1, \alpha = 0, \eta = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ in Theorem 2.1, the following result due to Mazi and Opoola [24], is obvious:

Remark 2.6. If $f \in \mathcal{S}_{\Sigma}^{\lambda}(s, t, \gamma)$, then

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{3\lambda - 2\lambda(s+t-\lambda+1) + st}},$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{(3\lambda-s^2-t^2-st)} + \frac{4(1-\gamma)^2}{(2\lambda-s-t)^2}.$$

For $\lambda = 1, s = 1, t = 0, \alpha = 0, \eta = 0, A = 1, B = -1$, Theorem 2.1 yields the following result due to Altinkya and Yalcin [5]:

Remark 2.7. If $f \in \mathcal{S}_{\Sigma_m}^\beta$, then

$$|a_{m+1}| \leq \frac{2\beta}{m\sqrt{\beta+1}},$$

and

$$|a_{2m+1}| \leq \frac{\beta}{m} + \frac{2(m+1)\beta^2}{m^2}.$$

On putting $\lambda = 1, s = 1, t = 0, \alpha = 0, \eta = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ in Theorem 2.1, the following result due to Altinkya and Yalcin [5], can be easily obtained:

Remark 2.8. If $f \in \mathcal{S}_{\Sigma_m}^\gamma$, then

$$|a_{m+1}| \leq \frac{\sqrt{2(1-\gamma)}}{m},$$

and

$$|a_{2m+1}| \leq \frac{1-\gamma}{m} + \frac{2(m+1)(1-\gamma)^2}{m^2}.$$

For $\lambda = 1, s = 1, t = 0, \alpha = 0, \eta = 0, m = 1, A = 1, B = -1$, Theorem 2.1 yields the following result due to Brannan and Taha [8]:

Remark 2.9. If $f \in \mathcal{S}_{\Sigma}^*(\beta)$, then

$$|a_2| \leq \frac{2\beta}{\sqrt{\beta+1}},$$

and

$$|a_3| \leq 4\beta^2 + \beta.$$

By substituting $\lambda = 1, s = 1, t = 0, \alpha = 0, \eta = 0, m = 1, \beta = 1, A = 1 - 2\gamma, B = -1$, Theorem 2.1 agrees with the following result due to Brannan and Taha [8]:

Remark 2.10. If $f \in \mathcal{S}_{\Sigma}^*(\gamma)$, then

$$|a_2| \leq \sqrt{2(1-\gamma)},$$

and

$$|a_3| \leq 4(1-\gamma)^2 + (1-\gamma).$$

3. Fekete-Szegő inequality for the function class $\mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$

Theorem 3.1. *If $f \in \mathcal{S}_{\Sigma_m}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$, then*

$$|a_{2m+1} - \mu a_{m+1}^2| \leq \begin{cases} \frac{\beta(1-\eta)(A-B)}{(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]}, & \text{if } 0 \leq |l(\mu)| < \frac{1}{2(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]} \\ \frac{2\beta(1-\eta)(A-B)|l(\mu)|}{1}, & \text{if } |l(\mu)| \geq \frac{1}{2(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]} \end{cases} \tag{3.1}$$

where

$$l(\mu) = \frac{\beta(\frac{m+1}{2} - \mu)}{\Delta}, \tag{3.2}$$

and Δ is defined in (2.3).

Proof. Using (2.13) in (2.17), we have

$$a_{2m+1} = \frac{\beta(p_{2m} - q_{2m})}{2(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]} + \left(\frac{m+1}{2}\right) a_{m+1}^2. \tag{3.3}$$

Making use of (2.16) and (3.3), it yields

$$a_{2m+1} - \mu a_{m+1}^2 = \frac{\beta(p_{2m} - q_{2m})}{2(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]} + \left(\frac{m+1}{2} - \mu\right) \left[\frac{\beta^2(p_{2m} + q_{2m})}{\Delta}\right],$$

where Δ is defined in (2.3).

Further, (3.4) can be expressed as

$$a_{2m+1} - \mu a_{m+1}^2 = \beta \left[\left(l(\mu) + \frac{1}{2(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]} \right) p_{2m} + \left(l(\mu) - \frac{1}{2(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]} \right) q_{2m} \right], \tag{3.4}$$

where $l(\mu)$ is defined in (3.2).

Taking modulus and applying triangle inequality in (3.4), we obtain

$$|a_{2m+1} - \mu a_{m+1}^2| \leq \beta \left[\left| l(\mu) + \frac{1}{2(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]} \right| |p_{2m}| + \left| l(\mu) - \frac{1}{2(1+2\alpha m)[\lambda(2m+1) - (\frac{s^{2m+1}-t^{2m+1}}{s-t})]} \right| |q_{2m}| \right]. \tag{3.5}$$

Using Lemma 1.2, (3.5) yields

$$|a_{2m+1} - \mu a_{m+1}^2| \leq \beta(1 - \eta)(A - B) \left[\left| l(\mu) + \frac{1}{2(1+2\alpha m) \left[\lambda(2m+1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right]} \right| + \left| l(\mu) - \frac{1}{2(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right]} \right| \right]. \tag{3.6}$$

For $0 \leq |l(\mu)| < \frac{1}{2(1+2\alpha m) \left[\lambda(2m+1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right]}$,

$$|a_{2m+1} - \mu a_{m+1}^2| \leq \frac{\beta(1 - \eta)(A - B)}{(1 + 2\alpha m) \left[\lambda(2m + 1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right]}. \tag{3.7}$$

For $|l(\mu)| \geq \frac{1}{2(1+2\alpha m) \left[\lambda(2m+1) - \left(\frac{s^{2m+1} - t^{2m+1}}{s-t} \right) \right]}$,

$$|a_{2m+1} - \mu a_{m+1}^2| \leq 2\beta(1 - \eta)(A - B)|l(\mu)|. \tag{3.8}$$

The proof of Theorem 3.1 is obvious from (3.7) and (3.8). □

For $m = 1$, Theorem 3.1 yields the following result:

Remark 3.2. If $f \in \mathcal{S}_{\Sigma}^{\lambda, \alpha, \beta, \eta}(A, B; s, t)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\beta(1-\eta)(A-B)}{(1+2\alpha) \left[3\lambda - \left(\frac{s^2+st+t^2}{s-t} \right) \right]}, & \text{if } 0 \leq |l(\mu)| < \frac{1}{2(1+2\alpha) \left[3\lambda - \left(\frac{s^2+st+t^2}{s-t} \right) \right]}, \\ 2\beta(1 - \eta)(A - B)|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{2(1+2\alpha) \left[3\lambda - \left(\frac{s^2+st+t^2}{s-t} \right) \right]}, \end{cases}$$

where

$$l(\mu) = \frac{\beta(1 - \mu)}{\Delta_1},$$

and Δ_1 is given by

$$\Delta_1 = \beta \left[2(1 + 3\alpha)(s + t)(s + t - 2\lambda) - 2(1 + 2\alpha)(s^2 + st + t^2) + 2\lambda(3(1 + 2\alpha) + 2(\lambda - 1)(1 + 3\alpha)) \right] - (\beta - 1)(1 + \alpha)^2 [2\lambda - (s + t)]^2. \tag{3.9}$$

On putting $m = 1, \eta = 0$ in Theorem 3.1, the following result can be easily obtained:

Remark 3.3. If $f \in \mathcal{S}_{\Sigma}^{\lambda, \alpha, \beta}(A, B; s, t)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\beta(A-B)}{(1+2\alpha) \left[3\lambda - \left(\frac{s^2+st+t^2}{s-t} \right) \right]}, & \text{if } 0 \leq |l(\mu)| < \frac{1}{2(1+2\alpha) \left[3\lambda - \left(\frac{s^2+st+t^2}{s-t} \right) \right]}, \\ 2\beta(A - B)|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{2(1+2\alpha) \left[3\lambda - \left(\frac{s^2+st+t^2}{s-t} \right) \right]}, \end{cases}$$

where

$$l(\mu) = \frac{\beta(1 - \mu)}{\Delta_1},$$

and Δ_1 is defined in (3.9).

By substituting $m = 1, \eta = 0, \lambda = 1, s = 1, t = -1$ in Theorem 3.1, the following result is obvious:

Remark 3.4. If $f \in \mathcal{M}_{\Sigma}^s(\beta, \alpha; A, B)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\beta(A-B)}{2(1+2\alpha)}, & \text{if } 0 \leq |l(\mu)| < \frac{1}{4(1+2\alpha)}, \\ 2\beta(A-B)|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{4(1+2\alpha)}, \end{cases}$$

where

$$l(\mu) = \frac{\beta(1-\mu)}{4\beta(1+2\alpha) - 4(\beta-1)(1+\alpha)^2}.$$

For $\eta = 0, \lambda = 1, s = 1, t = 0, A = 1, B = -1, \alpha = 0$, Theorem 3.1 agrees with the following result:

Remark 3.5. If $f \in \mathcal{S}_{\Sigma_m}^{\beta}$, then

$$|a_{2m+1} - \mu a_{m+1}^2| \leq \begin{cases} \frac{2\beta}{2m}, & \text{if } 0 \leq |l(\mu)| < \frac{1}{4m}, \\ 4\beta|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{4m}, \end{cases}$$

where

$$l(\mu) = \frac{\beta(\frac{m+1}{2} - \mu)}{\beta[2 + 2m + 2m^2] - m^2(\beta - 1)}.$$

For $m = 1, \eta = 0, \lambda = 1, s = 1, t = 0, A = 1, B = -1, \alpha = 0$, Theorem 3.1 coincides with the following result:

Remark 3.6. If $f \in \mathcal{S}_{\Sigma}^*(\beta)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \beta, & \text{if } 0 \leq |l(\mu)| < \frac{1}{4}, \\ 4\beta|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{4}, \end{cases}$$

where

$$l(\mu) = \frac{\beta(1-\mu)}{\beta+1}.$$

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References

- [1] Alexander, J. W., *Functions which map the interior of the unit circle upon simple regions*, Ann. Math., **17**(1915-16), 12-22.
- [2] Al-Hawary, T., Amourah, A., Frasin, B. A., *Fekete-Szegő inequality for bi-univalent functions by means of Horadam polynomials*, Bol. Soc. Mat. Mex., **79**(2021), 1-12.
- [3] Al-Khafaji, A. K., *Coefficient estimates for a new subclass of m -fold symmetric bi-univalent functions*, Iran. J. Math. Sci Infom., **18**(2023), no. 1, 33-39.
- [4] Alnajjar, O., Amourah, A., Alsoboh, A., Malkawi, A., Abri, F. A., Sasa, T., *On a new subclass of bi-univalent functions associated with Gregory numbers*, Stat., Opt. Infor. Comput., **15**(6)2026, 5313–5323, DOI: <https://doi.org/10.19139/soic-2310-5070-3241>.
- [5] Altinkya, S. and Yalcin, S., *Coefficient bounds for certain subclasses of m -fold symmetric bi-univalent functions*, J. Math., Article Id. 241683, 2015, pp. 5.
- [6] Aouf, M. K., *On a class of p -valent starlike functions of order α* , Int. J. Math. Math. Sci., **10**(1987), no. 4, 733-744.
- [7] Brannan, D., and Clunie, J. G., *Aspects of contemporary complex analysis*, Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham, July 1-20, 1979, Academic Press, London, 1980.
- [8] Brannan, D. A. and Taha, T., S., *On some classes of bi-univalent functions*, Stud. Univ. Babeş-Bolyai Math., **31**(1986), no. 2, 70-77.
- [9] Bulut, S., *Coefficient estimates for general subclasses of m -fold symmetric analytic bi-univalent functions*, Turkish J. Math., **40**(2016), no. 6, 1386-1397.
- [10] Crisan, O., *Coefficient estimates for certain subclasses of bi-univalent functions*, Gen. Math. Notes, **16**(2013), 93-102.
- [11] Das, R. N. and Singh, P., *On subclasses of schlicht mappings*, Int. J. Pure Appl. Math., **8**(1977), 864-872.
- [12] Duren, P. L., *Univalent functions*, Springer-Verlag, New York, 1983.
- [13] Frasin, B. A., *Coefficient inequalities for certain classes of Sakaguchi type functions*, Int. J. Nonlinear Sci., **10**(2010), no. 2, 206-211.
- [14] Frasin, B. A. and Aouf, M. K., *New subclasses of bi-univalent functions*, Appl. Math. Lett., **24**(2011), 1569-1573.
- [15] Hussain, S., Khan, S., Zaighum, M. A. and Darus, M., *On certain classes of bi-univalent functions related to m -fold symmetry*, J. Nonlinear Sci. Appl., **11**(2018), 490-499.
- [16] Ibrahim, I. O., Shaba, T. G. and Patil, A. B., *On some subclasses of m -fold symmetric bi-univalent functions associated with the Sakaguchi-type functions*, Earthline J. Math. Sci., **8**(2022), no. 1, 1-15.
- [17] El-Ityan, M., Al-Hawary, T., Frasin B. A. and Aldawush, I., *A new subclass of bi-univalent functions defined by subordination to Laguerre polynomials and (p, q) -derivative operator*, Symmetry, **17**(2025), no. 7, 982.
- [18] Janowski, W., *Some extremal problems for certain families of analytic functions*, Ann. Pol. Math., **28**(1973), 297-326.
- [19] Joshi S. and Pawar, H., *On some subclasses of bi-univalent functions associated with pseudo-starlike functions*, J. Egyptian Math. Soc., **24**(2016), 522-525.
- [20] Kaplan, W., *Close-to-convex schlicht functions*, Mich. Math. J., **1**(1952), 169-185.

- [21] Keerthi, B. S., Shanthi, V. G. and Stephen, B. A., *Certain coefficient inequalities for Sakaguchi type functions and applications to fractional derivative*, Bull. Math. Anal. Appl., **4**(2012), no. 2, 29-39.
- [22] Lewin, M., *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., **18**(1967), 63-68.
- [23] Li, X. F. and Wang, A. P., *Two new subclasses of bi-univalent functions*, Int. Math. Forum, **7**(2012), no. 30, 1495-1504.
- [24] Mazi, E. P. and Opoola, T. O., *On some subclasses of bi-univalent functions associating pseudo-starlike functions with Sakaguchi type functions*, Gen. Math., **25**(2017), 85-95.
- [25] Magesh, N., Rosy, T. and Varma, S., *Coefficient estimate problem for a new subclass of bi-univalent functions*, J. Complex Anal., Article ID 474231, 2013, pp. 3.
- [26] Magesh, N. and Bulut, S., *Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions*, Afr. Mat. **29**(2018), 203-209.
- [27] Motamednezhad, A., Salehian, S. and Magesh, N., *Coefficient estimates for subclass of m -fold symmetric bi-univalent functions*, Kragujevac J. Math., **46**(2022), no. 3, 395-406.
- [28] Murugusundaramoorthy, G., Magesh, N., and Prameela, V., *Coefficient bounds for certain classes of bi-univalent functions*, Abstr. Appl. Anal., Article ID. 573017, 2013, pp. 3.
- [29] Muthaiyan, E., and Wanas, A. K., *Coefficient estimates for two new subclasses of bi-univalent functions involving Laguerre polynomial*, Earthline J. Math. Sci., **15**(2024), 187-199.
- [30] Noor, K. I., *On quasi-convex functions and related topics*, Int. J. Math. Math. Sci., **10**(1987), 241-258.
- [31] Owa, S., Sekine, T. and Yamakawa, R., *On Sakaguchi type functions*, Appl. Math. Comput., **187**(2007), 356-361.
- [32] Polatoglu, Y., Bolkal, M., Sen, A. and Yavuz, E., *A study on the generalization of Janowski function in the unit disc*, Acta Math. Acad. Paedagog. Nyhazi.,(N.S.), **22**(2006), 27-31.
- [33] Sakaguchi, K., *On a certain univalent mapping*, J. Math. Soc. Japan, **11**(1959), 72-75.
- [34] Sakar, F. M. and Tasar, N., *Coefficient bounds for certain subclasses of m -fold symmetric bi-univalent functions*, New Trends Math. Sci., **7**(2019), no. 1, 62-70.
- [35] Senthil, B. and Keerthi, B. S., *Certain subclasses of m -fold symmetric Sakaguchi-type bi-univalent functions*, Int. J. Pure Appl. Math., **109**(2016), no. 10, 29-37.
- [36] Shilpa, N. and Latha, S., *Coefficient inequalities for certain classes of Janowski-Sakaguchi type functions*, Int. J. Pure Appl. Math., **81**(2012), no. 5, 663-669.
- [37] Singh, G., *Coefficient inequality for a subclass of generalized Sakaguchi type functions*, Asia-Pac. J. Math., **1**(2014), no. 1, 1-9.
- [38] Singh, G., *Coefficient estimates for bi-univalent functions with respect to symmetric points*, J. Nonlinear Func. Anal., **1**(2013), 1-9.
- [39] Singh, G. and Singh, G., *New subclasses of bi-univalent functions defined with q -derivative operator*, Adv. Stud. Euro Tbilisi Math. J., **18**(2025), no. 2, 295-305.
- [40] Singh, G., and Singh, G., *Certain subclasses of alpha-convex bi-univalent functions defined with q -derivative operator*, Arch. Math., **61**(2025), no. 2, 63-72.

- [41] Singh, G., Singh, G. and Singh, G., *Certain subclasses of Sakaguchi-type bi-univalent functions*, *Ganita*, **69**(2019), no. 2, 45-55.
- [42] Singh, G., Singh, G. and Singh, G., *A generalized subclass of alpha-convex bi-univalent functions of complex order*, *Jnanabha*, **50**(2020), no. 1, 65-71.
- [43] Singh, G., Singh, G. and Singh, G., *Certain subclasses of univalent and bi-univalent functions related to shell-like curves connected with Fibonacci numbers*, *Gen. Math.*, **28**(2020), no. 1, 125-140.
- [44] Singh, G., Singh, G. and Singh, G., *Initial coefficients for generalized subclasses of bi-univalent functions defined with subordination*, *Arch. Math.*, **58**(2022), no. 2, 105-113.
- [45] Srivastava, H. M., Sivasubramanian, S. and Sivakumar, R., *Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions*, *Tbilisi Math. J.*, **7**(2014), no. 2, 1-10.
- [46] Vijayalakshmi, S. P. and Sudharsan, T. V., *Coefficient estimates for bi-univalent Sakaguchi-type functions*, *Asia-Pac. J. Math.*, **4**(2017), no. 1, 32-37.
- [47] Yee, L. C. and Darus, M., *Coefficient results on certain subclasses of Sakaguchi-type bi-univalent functions*, *Eur. J. Pure Appl. Math.*, **18**(2025), no. 2, 1-12.

Navyodh Singh 

Department of Mathematics, Khalsa College,
Amritsar-143001, Punjab, India
e-mail: navyodh81@yahoo.co.in

Gagandeep Singh 

Department of Mathematics, Khalsa College,
Amritsar-143001, Punjab, India
e-mail: kamboj.gagandeep@yahoo.in

Navjeet Singh 

Department of Mathematics, Khalsa College,
Amritsar-143001, Punjab, India
e-mail: navjeet8386@yahoo.com

Properties of some univalent functions associated with balloon-shaped domain

Trailokya Panigrahi  and Shiba Prasad Dhal 

Abstract. This paper introduces a novel subclass of analytic functions defined by the product of a modified sigmoid function and the lemniscate Bernoulli function. We initiate the study by deriving initial coefficient bounds for functions within this subclass, followed by an investigation into several key analytic properties. Specifically, we establish the Fekete–Szegő inequality and analyze Hankel determinants of various orders. Furthermore, the study provides estimates for the logarithmic coefficients and establishes bounds for both the inverse coefficients and the logarithmic inverse coefficients of functions in the subclass. This comprehensive analysis offers a significant contribution to the theory of analytic function subclasses associated with the products of special functions.

Mathematics Subject Classification (2010): 30C45, 30C50, 30C55.

Keywords: Analytic functions, univalent functions, Fekete–Szegő inequality, Hankel determinants, logarithmic coefficients, logarithmic inverse coefficients.

1. Introduction

Let \mathcal{A} denote the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. If $f \in \mathcal{A}$, then f has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let \mathcal{S} denote the class of all univalent (i.e., one-to-one) functions in \mathcal{A} . For given analytic functions f and g in \mathbb{D} , we say f is subordinate to g in \mathbb{D} and write $f \prec g$ if there exists an analytic function w with the property $w(0) = 0$ and $|w(z)| < 1$ such

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that $f(z) = g(w(z)), (z \in \mathbb{D})$. Moreover, if the function g is univalent in \mathbb{D} , then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. A function $f \in \mathcal{A}$ is called starlike (convex respectively) if $f(\mathbb{D})$ is starlike with respect to the origin (convex respectively). Let \mathcal{S}^* and \mathcal{C} denote the class of starlike and convex functions in \mathcal{S} respectively. It is well known that a function $f \in \mathcal{A}$ is in \mathcal{S}^* if and only if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, \quad \text{for } z \in \mathbb{D}.$$

Similarly, a function $f \in \mathcal{A}$ is in \mathcal{C} if and only if

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad \text{for } z \in \mathbb{D}.$$

From the above it is easy to see that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. Given $\alpha \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is said to be close-to-convex with argument α and with respect to g if

$$\operatorname{Re} \left(e^{i\alpha} \frac{zf'(z)}{g(z)} \right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{1.2}$$

Using the notion of subordination between two analytic functions, Ma and Minda [19] introduced the more general class $\mathcal{S}^*(\phi)$ by

$$\mathcal{S}^*(\phi) := \left\{ h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec \phi(z) \right\},$$

where ϕ is a regular function with a positive real part in \mathbb{D} , with $\phi(0) = 1$ and $\phi'(0) > 0$. In addition, the function ϕ maps \mathbb{D} onto a star-shaped region with respect to $\phi(0) = 1$ and is symmetric with respect to the real axis. Varying the function ϕ we can obtain several familiar subclasses of the class $\mathcal{S}^*(\phi)$, and for details see [16] and references within.

The classic Fekete–Szegő problem [11] involves finding the exact limits of the functional $|a_3 - \mu a_2^2|$ for a compact-function family or $f \in \mathcal{A}$ with any $\mu \in \mathbb{C}$; for further details, one may refer to [27].

Pommerenke provided the following *Hankel determinant* in [21],[22], denoted by $\mathcal{H}_{q,n}(f)$, which contains the coefficients of a function $f \in \mathcal{S}$:

$$\mathcal{H}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

with $q, n \in \mathbb{N} := \{1, 2, \dots\}$. Therefore, by altering the parameters q and n we obtain the following Hankel determinants:

$$\mathcal{H}_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \quad \mathcal{H}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

That denote the first and the second order Hankel determinants. There are a few references in the literature to the Hankel determinant for functions in the general family \mathcal{S} . The best-known sharp inequality for the function $f \in \mathcal{S}$ is $\mathcal{H}_{2,n}(f) \leq \kappa\sqrt{n}$,

where κ is a constant, and it is due to Hayman ([12] Theorem 1). Additionally, for the class \mathcal{S} , it was found in [20] that

$$|\mathcal{H}_{2,2}(f)| \leq \kappa, \quad \text{where } 1 \leq \kappa \leq \frac{11}{3} \simeq 3.66 \dots$$

The precise bounds of Hankel determinants for a given family of functions have piqued the interest of several mathematicians. For the three well-known subfamilies of the set \mathcal{S} that are \mathcal{C} , \mathcal{S}^* , and \mathcal{R} (convex, starlike, and functions of a bounded turning, respectively), Janteng et al. ([13],[14]) computed the sharp bounds $|\mathcal{H}_{2,2}(f)|$. These bounds are provided by

$$|\mathcal{H}_{2,2}(f)| \leq \begin{cases} \frac{1}{8}, & \text{for } f \in \mathcal{C}, \\ 1, & \text{for } f \in \mathcal{S}^*, \\ \frac{4}{9}, & \text{for } f \in \mathcal{R}. \end{cases}$$

Moreover, the sharp bounds of this determinant for a few subclasses \mathcal{S}^* and \mathcal{K} were found in [17] and subsequently studied in [9]. This problem was solved for various families of bi-univalent functions in [2, 4, 15]. The logarithmic coefficients of $f \in \mathcal{S}$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \tag{1.3}$$

where γ_n are known as the logarithmic coefficients. The logarithmic coefficients γ_n play a central role in the theory of univalent functions. Very few exact upper bounds for γ_n seem to have been established. The significance of this problem in the context of the Bieberbach conjecture was pointed out by Milin in his conjecture. Milin conjectured that for $f \in \mathcal{S}$ and $n \geq 2$,

$$\sum_{m=1}^n \sum_{k=1}^m \left(k |\gamma_k|^2 - \frac{1}{k} \right) \leq 0, \tag{1.4}$$

which led de Branges, by proving this conjecture, to the proof of the Bieberbach conjecture [6]. More attention has been given to the results in an average sense [7, 8] than the exact upper bounds for $|\gamma_n|$. For the Koebe function $k(z) = \frac{z}{(1-z)^2}$, the logarithmic coefficients are $\gamma_n = \frac{1}{n}$. Since the Koebe function $k(z)$ plays the role of extremal function for most of the extremal problems in the class \mathcal{S} , it is expected that $|\gamma_n| \leq \frac{1}{n}$ holds for functions in \mathcal{S} . But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function f in the class \mathcal{S} with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ ([7, Theorem 8.4]). Now equating the coefficients of (1.3) we obtain

$$\begin{aligned} \gamma_1 &= \frac{a_2}{2}, & \gamma_2 &= \frac{1}{2} \left(a_3 - \frac{a_2^2}{2} \right), & \gamma_3 &= \frac{1}{2} \left(a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right), \\ \gamma_4 &= \frac{1}{2} \left(a_5 - a_2 a_4 - \frac{1}{2} a_3^2 + a_2^2 a_3 - \frac{1}{4} a_2^4 \right). \end{aligned} \tag{1.5}$$

The famous Koebe 1/4-theorem ensures that, for each univalent function f defined in \mathbb{D} , its inverse f^{-1} exists at least on a disc of radius 1/4 with Taylor’s series of the form representation

$$f^{-1}(\omega) = \omega + h_2\omega^2 + h_3\omega^3 + h_4\omega^4 + \dots. \tag{1.6}$$

Using the representation $f(f^{-1}(\omega)) = \omega$, we obtain

$$\begin{aligned} h_2 &= -a_2, & h_3 &= -a_3 + 2a_2^2, & h_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ h_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.7}$$

In 1959, Sakaguchi [26] introduced the class of starlike functions with respect to symmetric points as:

$$S_S^* := \left\{ f \in \mathcal{S} : \Re \left(\frac{2zf'(z)}{f(z) - f(-z)} \right) > 0; z \in \mathbb{D} \right\}.$$

These functions are also known as Sakaguchi functions which are close-to-convex as well as univalent. In 2004, making use of subordination between two analytic functions Ravichandran [24] introduced a unified class $S_s^*(\phi)$ as follows:

$$S_S^*(\phi) = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z); z \in \mathbb{D} \right\},$$

where $\phi(z) = 1 + \sum_{n=1}^{\infty} D_n z^n$ is univalent starlike function with respect to 1 which maps \mathbb{D} onto a symmetric region with respect to real axis in the right half plane. A good amount of literature is available for finding the upper bounds of coefficient functional for several subclasses of the class S_s^* . For details see [16] and reference within. Motivated by aforementioned works, we introduce the following subclass.

Definition 1.1. A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class S_*^r if the below condition holds true:

$$S_*^r = \left\{ f \in \mathcal{S} : \frac{(1+r)zf'(z)}{f(z) - rf(-z)} \prec \frac{2\sqrt{1+z}}{1+e^{-z}}; z \in \mathbb{D} \right\}. \tag{1.8}$$

Provided the denominator $f(z) - rf(-z) \neq 0$.

- Starlike Functions (S_*^0): By setting $r = 0$, the denominator simplifies to $f(z)$ and the scaling factor $(1+r)$ becomes unity. This reduces the condition to the cornerstone of geometric function theory:

$$\frac{zf'(z)}{f(z)} \prec \frac{2\sqrt{1+z}}{1+e^{-z}}, \quad z \in \mathbb{D}.$$

In this state, the function maps the unit disk onto a domain that is starlike with respect to the origin.

- Sakaguchi-Type Functions (S_*^1): By setting $r = 1$, the expression transforms into the functional defined by Sakaguchi:

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{2\sqrt{1+z}}{1+e^{-z}}, \quad z \in \mathbb{D}.$$

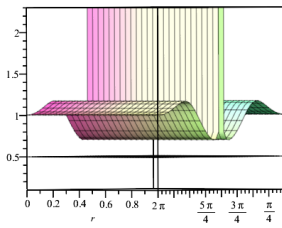
Functions satisfying this condition are known as starlike with respect to symmetric points.

Next, we need to demonstrate that the chosen class is well-defined and non-empty. Additionally, the class is characterized by the product of a sigmoid function and a Bernoulli-shaped domain. It is symmetric with respect to the origin and satisfies the Ma-Minda condition.

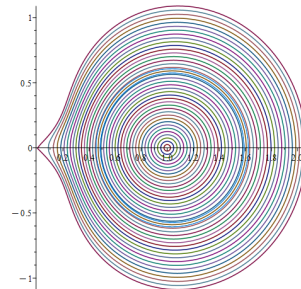
Remark 1.2. (i) Note that $p(z) := \frac{2\sqrt{1+z}}{1+e^{-z}}$ is chosen correctly, because $p(0) = 1$ and $p'(z) = \frac{2e^{-z}z+3e^{-z}+1}{\sqrt{1+z}(1+e^{-z})^2}$, hence $p'(0) = 1 \neq 0$. Also, we can see in the Figure 1a that

$$\begin{aligned}
 P(z) &= \operatorname{Re} \frac{zp'(z)}{p(z) - p(0)} = \operatorname{Re} \frac{zp'(z)}{p(z) - 1} \\
 &= \operatorname{Re} \left(-\frac{(2e^{-z}z + 3e^{-z} + 1)z}{(-2\sqrt{1+z} + 1 + e^{-z})\sqrt{1+z}(1 + e^{-z})} \right) > 0, \quad z \in \mathbb{D},
 \end{aligned}$$

and using this fact together with $p'(0) = 1 \neq 0$ it follows that $p(z) = \frac{2\sqrt{1+z}}{1+e^{-z}}$ is also a starlike (univalent) function in \mathbb{D} and because $p(\bar{z}) = \overline{p(z)}$, $z \in \mathbb{D}$, the domain $p(\mathbb{D})$ is symmetric with respect to the real axis see Figure 1b.



(A) The image of $P(re^{it})$, $r \in [0, 1]$, $t \in [0, 2\pi]$



(B) The image of $p(\mathbb{D})$ (Balloon-Shaped Domain)

FIGURE 1. Figures for Remark 1.2 (i).

(ii) The class is well chosen if we could prove that it isn't empty. First, to show that the class is non-empty for $r = 0$, take $f(z) = z + 0.3z^2 + 0.2z^3 \in \mathcal{A}$ such that

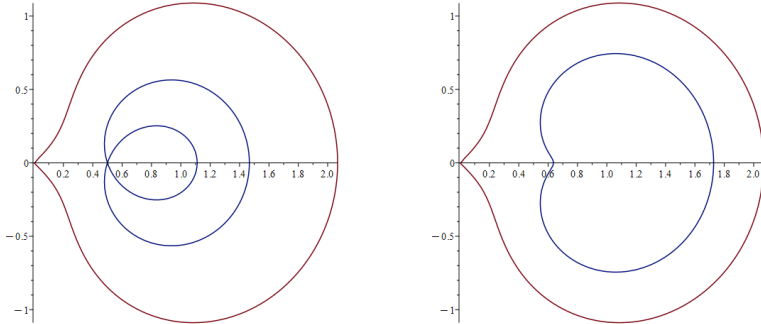
$$\phi(z) = \frac{zf'(z)}{f(z)} = \frac{0.6z^2 + 0.6z + 1.0}{0.2z^2 + 0.3z + 1.0}, z \in \mathbb{D}.$$

It can be seen from Figure 2a that the class is non-empty.

(iii) Now for $r = 1$ the class is non empty, if we take $f(z) = z + 0.3z^2 + 0.1z^3 \in \mathcal{A}$ such that

$$\psi(z) = \frac{2zf'(z)}{f(z) - f(-z)} = \frac{0.6z^2 + 1.2z + 2.0}{0.2z^2 + 2.0}, z \in \mathbb{D}.$$

It can be seen from Figure 2b. That the class is non-empty.



(A) The images of $\phi(e^{it})$ (blue) and $p(e^{it})$ (red), $t \in [0, 2\pi)$, (for $r=0$)
 (B) The images of $\psi(e^{it})$ (blue) and $p(e^{it})$ (red), $t \in [0, 2\pi)$, (for $r=1$)

FIGURE 2. Figures for Remark 1.2 (ii) and (iii).

The corresponding extremal function for the defined class is given by

$$f(z) = z + \frac{(1+r)z^2}{1+3r} + \frac{(9-5r)z^3}{16(1+3r)} + \frac{(1+r)(31-27r)z^4}{48(5r+3)(1+3r)} + \frac{(7r^2-70r+31)z^5}{192(5r+3)(1+3r)} + \dots$$

Now for $r = 0$ the corresponding extremal function is

$$f(z) = z + z^2 + \frac{9}{16}z^3 + \frac{31}{144}z^4 + \frac{31}{576}z^5 + \dots$$

Now for $r = 1$ the corresponding extremal function is

$$f(z) = z + \frac{1}{2}z^2 + \frac{1}{16}z^3 + \frac{1}{192}z^4 - \frac{1}{192}z^5 + \dots$$

In this study we adopt a novel methodology centered on establishing a direct correspondence between the coefficients of functions within a specific class and those of their associated Carathéodory functions. In various instances, this framework allows for the intuitive prediction of precise functional estimates, streamlining the subsequent computational process — a characteristic particularly evident within the class under investigation. By integrating the foundational lemmas with refined analytical techniques and more rigorous calculus, we have established enhanced bounds, the majority of which are sharp. The following lemmas serve as the essential theoretical groundwork required to substantiate these findings.

2. Preliminaries

Let us define by \mathcal{P} the well-known Carathéodory class i.e., the family of holomorphic functions d in \mathbb{D} that satisfies the condition $\text{Re } d(z) > 0, z \in \mathbb{D}$, and of the form

$$d(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, z \in \mathbb{D}. \tag{2.1}$$

We need the following lemmas in order to prove our results.

Lemma 2.1. *Let $d \in \mathcal{P}$ be of the form (2.1).*

(i) *Then, for $n \geq 1$*

$$|d_n| \leq 2. \tag{2.2}$$

The inequality holds for all $n \geq 1$ if and only if $d(z) = \frac{1 + \lambda z}{1 - \lambda z}, |\lambda| = 1$.

(ii) *Also, if $\mu \geq 0$ then*

$$|d_{n+k} - \mu d_n d_k| \leq 2 \max \{1; |2\mu - 1|\} = \begin{cases} 2, & \text{if } 0 \leq \mu \leq 1, \\ 2|2\mu - 1|, & \text{otherwise.} \end{cases} \tag{2.3}$$

If $0 < \mu < 1$, the inequality is sharp for the function $d(z) = \frac{1 + z^{n+k}}{1 - z^{n+k}}$. In the other cases, the inequality is sharp for the function $d(z) = \frac{1 + z}{1 - z}$.

Note that the inequality (2.2) is the well-known result of the Carathéodory Lemma [5] (see also [23, Corollary 2.3, p. 41] and [7, Carathéodory Lemma, p. 41]). Inequality (2.3) represents Lemma 2.3 of [25], that for $\mu = 1$ was proved in a more general form in [18, Lemma 1, p. 546]. We emphasize that the inequality (2.3) remains valid for all $\mu \in \mathbb{C}$ as it was proved in [10, Theorem 1]

Lemma 2.2. [3, Lemma 2.2.] *If $d \in \mathcal{P}$ has the form (2.1), then*

$$|\alpha d_1^3 - \beta d_1 d_2 + \gamma d_3| \leq 2(|\alpha| + |\beta - 2\alpha| + |\alpha - \beta + \gamma|). \tag{2.4}$$

Lemma 2.3. [1, Lemma 3, p. 66] *Let $d \in \mathcal{P}$ and has the expansion of the form (2.1). If $B \in [0, 1]$ with $B(2B - 1) \leq D < B$, then*

$$|d_3 - 2Bd_1 d_2 + Dd_1^3| \leq 2. \tag{2.5}$$

3. Coefficient estimates and Fekete-Szegő inequality

In this section, we aim to examine the upper bounds of the first four initial coefficients, along with the Fekete–Szegő functional $|a_3 - \mu a_2^2|$, for the considered class S_*^r . The focus is placed on deriving appropriate upper estimates for these coefficients within the framework of the class S_*^r .

Theorem 3.1. *Let the function $f \in \mathcal{A}$ be of the form (1.1) belongs to the class S_*^r . Then*

$$|a_2| \leq \frac{1+r}{1+3r}, \quad |a_3| \leq \frac{9-5r}{16(1+3r)}, \quad |a_4| \leq \frac{|(57r-5)(1+r)| + (1+r)(519r+197)}{192(1+3r)(5r+3)},$$

$$|a_5| \leq \frac{|227r^2 + 10r - 13| + 6349r^2 + 3926r + 253}{768(1+3r)(5r+3)} + \frac{1}{4}. \tag{3.1}$$

Proof. If the function $f \in \mathcal{A}$ of the form (1.1) belongs to the class S_*^r , then by Definition 1.1 there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1, (z \in \mathbb{D})$ such that

$$\frac{(1+r)zf'(z)}{f(z) - rf(-z)} = \frac{2\sqrt{1+w(z)}}{1+e^{-w(z)}}. \tag{3.2}$$

Writing the analytic function w in terms of $d \in \mathcal{P}$, that is

$$d(z) = \frac{1+w(z)}{1-w(z)} = 1 + d_1z + d_2z^2 + \dots, \quad z \in \mathbb{D}.$$

To present the transition from the Schwarz function $w(z)$ to the coefficients of the Carathéodory function $d(z)$ we get

$$w(z) = \frac{1}{2}d_1z + \left(\frac{1}{2}d_2 - \frac{1}{4}d_1^2\right)z^2 + \left(\frac{1}{8}d_1^3 - \frac{1}{2}d_1d_2 + \frac{1}{2}d_3\right)z^3$$

$$+ \left(\frac{1}{2}d_4 - \frac{1}{2}d_1d_3 - \frac{1}{4}d_2^2 - \frac{1}{16}d_1^4 + \frac{3}{8}d_1^2d_2\right)z^4 + \dots, \quad z \in \mathbb{D}.$$

Now substituting $w(z)$ in the right-hand side of (3.2) we get

$$\frac{2\sqrt{1+w(z)}}{1+e^{-w(z)}} = 1 + \frac{1}{2}d_1z + \left(\frac{d_2}{2} - \frac{7d_1^2}{32}\right)z^2 + \left(\frac{1}{2}d_3 - \frac{7}{16}d_1d_2 + \frac{17}{192}d_1^3\right)z^3$$

$$+ \left(\frac{1}{2}d_4 - \frac{7}{16}d_1d_3 - \frac{7}{32}d_2^2 + \frac{17}{64}d_2d_1^2 - \frac{203}{6144}d_1^4\right)z^4 + \left(-\frac{7}{16}d_1d_4 - \frac{7}{16}d_3d_2 + \frac{17}{64}d_3d_1^2\right)z^5$$

$$+ \frac{17}{64}d_1d_2^2 - \frac{203}{1536}d_2d_1^3 + \frac{1}{2}d_5 + \frac{733}{61440}d_1^5z^5 + \dots \tag{3.3}$$

Expanding the left-hand side of (3.2) in a Maclaurin series with respect to z , we obtain

$$\frac{(1+r)zf'(z)}{f(z) - rf(-z)} = 1 + \left(\frac{a_2(1+3r)}{1+r}\right)z + \left(\frac{(2r^2+4r+2)a_3}{(1+r)^2} + \frac{(3r^2-2r-1)a_2^2}{(1+r)^2}\right)z^2$$

$$+ \left(\frac{(5r^3+13r^2+11r+3)a_4}{(1+r)^3} + \frac{(-r^3-5r^2-7r-3)a_2a_3}{(1+r)^3} + \frac{(3r^3-5r^2+r+1)a_2^3}{(1+r)^3}\right)z^3$$

$$+ \left(\frac{(4r^4+16r^3+24r^2+16r+4)a_5}{(1+r)^4} + \frac{(8r^4+12r^3-4r^2-12r-4)a_2a_4}{(1+r)^4}\right)z^4$$

$$- \frac{(2r^4+8r^3+12r^2+8r+2)a_3^2}{(1+r)^4} - \frac{4(r-1)}{(1+r)}a_2^2a_3 + \frac{(3r+1)(r-1)^3}{(1+r)^4}a_2^4z^4 + \dots \tag{3.4}$$

Comparing the coefficients of (3.3) and (3.4) we get

$$a_2 = \frac{d_1(1+r)}{2(1+3r)}. \tag{3.5}$$

$$a_3 = \frac{d_2}{4} - \frac{(29r-1)d_1^2}{64(1+3r)} = \frac{1}{4} \left(d_2 - \frac{(29r-1)}{16(1+3r)}d_1^2 \right). \tag{3.6}$$

$$a_4 = \frac{(57r-5)(1+r)d_1^3}{384(1+3r)(5r+3)} - \frac{3(1+19r)(1+r)d_2d_1}{2(1+3r)(5r+3)} + \frac{(1+r)d_3}{2(5r+3)}. \tag{3.7}$$

$$a_5 = -\frac{(227r^2+10r-13)d_1^4}{3072(1+3r)(5r+3)} + \frac{(119r^2+42r-1)d_1^2d_2}{128(1+3r)(5r+3)} - \frac{(137r^2+82r+5)d_3d_1}{64(1+3r)(5r+3)} - \frac{(1080r^2+1008r+216)d_2^2}{3072(1+3r)(5r+3)} - \frac{(-5760r^2-5376r-1152)d_4}{3072(1+3r)(5r+3)}. \tag{3.8}$$

Now taking modulus in (3.5) and applying (2.2) of Lemma 2.1 we get bound for $|a_2|$. After that taking modulus on both sides of (3.6) and applying (2.3) of Lemma 2.1 we get the desired estimation for $|a_3|$. Now taking modulus in (3.7) and applying (2.4) of Lemma 2.2 we get our desired estimation for $|a_4|$. Now rearranging the terms of a_5 we get

$$a_5 = - \left[\left(\frac{(227r^2+10-13)d_1^3}{3072(1+3r)(5r+3)} - \frac{(119r^2+42r-1)d_1d_2}{128(1+3r)(5r+3)} + \frac{(137r^2+82r+5)d_3}{64(1+3r)(5r+3)} \right) d_1 - \frac{1}{8} \left(d_4 - \frac{3d_2^2}{16} \right) \right]. \tag{3.9}$$

Now taking modulus on both sides of (3.9) and applying triangle inequality we get

$$|a_5| \leq \left| \frac{(227r^2+10r-13)d_1^3}{3072(1+3r)(5r+3)} - \frac{(119r^2+42r-1)d_1d_2}{128(1+3r)(5r+3)} + \frac{(137r^2+82r+5)d_3}{64(1+3r)(5r+3)} \right| |d_1| + \frac{1}{8} \left| d_4 - \frac{3d_2^2}{16} \right|. \tag{3.10}$$

Using (2.4) for $\alpha = \frac{(227r^2+10r-13)}{3072(1+3r)(5r+3)}$, $\beta = \frac{(119r^2+42r-1)}{128(1+3r)(5r+3)}$, $\gamma = \frac{(137r^2+82r+5)}{64(1+3r)(5r+3)}$ in the first part of (3.10) and applying (2.3) of Lemma 2.1 for $\mu = \frac{3}{16}$ in the second part of (3.10) we get

$$|a_5| \leq \frac{|227r^2+10r-13|+6349r^2+3926r+253}{768(1+3r)(5r+3)} + \frac{1}{4}.$$

Here we get the desired estimation for $|a_5|$. □

The next theorem gives the bound of Fekete- Szegő functional for the class S_*^r .

Theorem 3.2. *If $f \in S_*^r$ has the form (1.1), then for any complex number μ we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{2} \max \left\{ 1; \left| \frac{2\mu(1+r)^2}{(1+3r)^2} + \frac{5r-9}{8(1+3r)} \right| \right\}.$$

Proof. If $f \in S_*^r$, making use of (3.5) and (3.6) we obtain

$$a_3 - \mu a_2^2 = \frac{1}{4} \left[d_2 - \left(\frac{29r - 1}{16(1 + 3r)} + \frac{(1 + r)^2 \mu}{(1 + 3r)^2} \right) d_1^2 \right] \tag{3.11}$$

Taking the modulus on both sides of (3.11) and applying (2.3) of Lemma 2.1 we get the desired estimation for this. \square

Letting $\mu = 1$ in Theorem 3.2 we obtain the following result.

Corollary 3.3. *If $f \in S_*^r$ has the form (1.1), then*

$$|a_3 - a_2^2| = |\mathcal{H}_{2,1}(f)| \leq \frac{1}{2} \max \left\{ 1; \left| \frac{31r^2 + 10r + 7}{8(1 + 3r)^2} \right| \right\}.$$

4. Hankel determinant bounds for the class S_*^r

In this section we investigate the upper bound of Hankel determinant of order two for the functions that belong to the class S_*^r .

Theorem 4.1. *If the function $f \in \mathcal{A}$ given by (1.1) belongs to the class S_*^r , then*

$$|\mathcal{H}_{2,2}(f)| \leq \frac{44649r^3 + 50245r^2 + 16715r + 1927}{1536(1 + 3r)^2(5r + 3)}. \tag{4.1}$$

Proof. From (3.8), (3.9) and (3.10) it follows that

$$a_2 a_4 - a_3^2 = - \left(\frac{(11703r^3 + 4955r^2 - 1259r + 89) d_1^3}{12288(5r + 3)(1 + 3r)^2} - \frac{(359r^3 + 235r^2 - 11r - 7) d_2 d_1}{128(5r + 3)(1 + 3r)^2} - \frac{d_3(1 + r)^2}{4(1 + 3r)(5r + 3)} \right) d_1 - \frac{d_2^2}{16}. \tag{4.2}$$

Taking modulus and applying triangle inequality in (4.2), we get

$$|a_2 a_4 - a_3^2| \leq \left| \frac{(11703r^3 + 4955r^2 - 1259r + 89) d_1^3}{12288(5r + 3)(1 + 3r)^2} - \frac{(359r^3 + 235r^2 - 11r - 7) d_1 d_2}{128(5r + 3)(1 + 3r)^2} \right| d_1 + \frac{d_2^2}{16}. \tag{4.3}$$

Making use of Lemmas 2.1, 2.2 in the right-hand side of (4.3) we get

$$|a_2 a_4 - a_3^2| \leq \frac{44649r^3 + 50245r^2 + 16715r + 1927}{1536(1 + 3r)^2(5r + 3)}.$$

Hence we get the desired estimation. \square

Theorem 4.2. *If the function $f \in \mathcal{A}$ given by (1.1) belongs to the class S_*^r , then*

$$|a_4 - a_2 a_3| \leq \frac{(1 + r) [303r^2 + 144r - 7] + (561r^2 + 432r + 103)}{96(1 + 3r)^2(5r + 3)}. \tag{4.4}$$

Proof. If $f \in S_*^r$ has the form (1.1), using the relations (3.8)–(3.10) we get

$$a_4 - a_2a_3 = \frac{(1+r)(303r^2 + 144r - 7)d_1^3}{192(5r+3)(1+3r)^2} - \frac{d_1d_2(29r+7)(1+r)}{16(5r+3)(1+3r)} + \frac{d_3(1+r)}{10r+6}. \tag{4.5}$$

Considering the modulus on both sides of (4.5) and applying (2.4) of Lemma 2.2 we get

$$\begin{aligned} |a_4 - a_2a_3| &\leq \frac{(1+r)|303r^2 + 144r - 7|}{96(5r+3)(1+3r)^2} + \frac{(1+r)(561r^2 + 432r + 103)}{96(1+3r)^2(5r+3)} \\ &= \frac{(1+r)[|303r^2 + 144r - 7| + (561r^2 + 432r + 103)]}{96(1+3r)^2(5r+3)}. \end{aligned}$$

Here we obtained the required estimation. □

5. Logarithmic coefficient estimates for the class S_*^r

In this section, we determine upper bounds estimates for the first four logarithmic coefficients of the functions f that belong to the class S_*^r .

Theorem 5.1. *If $f \in S_*^r$ given by (1.1), then*

$$\begin{aligned} |\gamma_1| &\leq \frac{(1+r)}{2(1+3r)}, & |\gamma_2| &\leq \frac{1}{4} \max \left\{ 1; \left| \frac{23r^2 - 6r - 1}{8(1+3r)^2} \right| \right\}, \\ |\gamma_3| &\leq \frac{1+r}{2(5r+3)}, & |\gamma_4| &\leq \frac{71r+33}{32(5r+3)}. \end{aligned} \tag{5.1}$$

Proof. Substituting the values of a_2, a_3, a_4 and a_5 from (3.5)–(3.8) in the relation (1.3) gives

$$\gamma_1 = \frac{(1+r)}{4(1+3r)}d_1. \tag{5.2}$$

$$\gamma_2 = \frac{d_2}{8} - \frac{(95r^2 + 42r + 7)d_1^2}{128(1+3r)^2}. \tag{5.3}$$

$$\gamma_3 = \frac{(1+r)(949r^3 + 839r^2 + 211r + 17)d_1^3}{384(1+3r)^3(5r+3)} - \frac{d_1d_2(29r+7)(1+r)}{32(5r+3)(1+3r)} + \frac{d_3(1+r)}{20r+12}. \tag{5.4}$$

$$\begin{aligned} \gamma_4 &= -\frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203)d_1^4}{16384(1+3r)^4(5r+3)} \\ &+ \frac{d_1^2d_2(1381r^3 + 1401r^2 + 495r + 51)}{512(5r+3)(1+3r)^2} - \frac{3d_1d_3(17r+7)}{640r+384} - \frac{7d_2^2}{256} + \frac{d_4}{16}. \end{aligned} \tag{5.5}$$

Now taking modulus in (5.2) and applying (2.2) of Lemma 2.1 we get bound for $|\gamma_1|$. After that taking modulus on both sides of (5.3) and applying (2.3) of Lemma 2.1 we

get the desired estimation for $|\gamma_2|$. Now taking modulus in (5.4) and applying (2.4) of Lemma 2.2 we get

$$|\gamma_3| \leq \frac{(1+r)(949r^3 + 839r^2 + 211r + 17)}{192(1+3r)^3(5r+3)} + \frac{(1+r)(617r^3 + 583r^2 + 215r + 25)}{96(1+3r)^3(5r+3)} + \frac{(1+r)(409r^3 + 587r^2 + 223r + 29)}{192(1+3r)^3(5r+3)} = \frac{1+r}{2(5r+3)}.$$

Here we obtained the desired estimation for $|\gamma_3|$. Now rearranging the terms of γ_4 we get

$$\begin{aligned} \gamma_4 = & -d_1 \left(\frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203)}{16384(1+3r)^4(5r+3)} d_1^3 \right. \\ & \left. - \frac{(1381r^3 + 1401r^2 + 495r + 51)}{512(5r+3)(1+3r)^2} d_1 d_2 + \frac{3(17r+7)}{640r+384} d_3 \right) + \left(\frac{d_4}{16} - \frac{7d_2^2}{256} \right). \end{aligned} \tag{5.6}$$

Now taking modulus on both sides of (5.6) and applying triangle inequality we get

$$\begin{aligned} |\gamma_4| \leq & |d_1| \left| \frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203)}{16384(1+3r)^4(5r+3)} d_1^3 \right. \\ & \left. - \frac{(1381r^3 + 1401r^2 + 495r + 51)}{512(5r+3)(1+3r)^2} d_1 d_2 + \frac{3(17r+7)}{640r+384} d_3 \right| + \frac{1}{16} \left| d_4 - \frac{7}{16} d_2^2 \right|. \end{aligned} \tag{5.7}$$

After that applying (2.4) of Lemma 2.2 in first part and (2.3) of Lemma 2.1 in (5.7) we get

$$|\gamma_4| \leq \frac{3(17r+7)}{32(5r+3)} + \frac{1}{8} = \frac{71r+33}{32(5r+3)}.$$

Here we get our desired estimations. □

6. Inverse coefficient bounds for the class S_*^r

In this section, we will estimate the upper bounds of the first four inverse coefficient bounds for the function belonging to the class S_*^r .

Theorem 6.1. *If $f \in S_*^r$ is given by (1.1) and its inverse f^{-1} has the form (1.6), then*

$$|h_2| \leq \frac{1+r}{1+3r}, \tag{6.1}$$

$$|h_3| \leq \frac{1}{2} \max \left\{ 1; \frac{47r^2 + 42r + 23}{8(1+3r)^2} \right\}, \tag{6.2}$$

$$|h_4| \leq \frac{(1+r)(4905r^3 + 5607r^2 + 2583r + 633 + |741r^3 + 1083r^2 + 315r - 59|)}{96(1+3r)^3(5r+3)}, \tag{6.3}$$

$$|h_5| \leq \frac{|371247r^5 + 733389r^4 + 563094r^3 + 202138r^2 + 24299r - 3863|}{1536(1+3r)^4(5r+3)} + \frac{(2r+1)(195r^2 + 206r + 79)}{8(5r+3)(1+3r)^2} + \frac{19}{32}. \tag{6.4}$$

Proof. Substitute the values of a_2, a_3, a_4 and a_5 from (3.5)-(3.8) into (1.7) we get

$$h_2 = -\frac{d_1(1+r)}{2+6r}. \tag{6.5}$$

$$h_3 = \frac{(119r^2 + 90r + 31)d_1^2}{64(1+3r)^2} - \frac{d_2}{4}. \tag{6.6}$$

$$h_4 = -\frac{(1+r)}{(5r+3)} \left[\frac{(4119r^3 + 4641r^2 + 1881r + 335)d_1^3}{192(1+3r)^3} - \frac{d_1d_2(69r+31)}{16(1+3r)} + \frac{d_3}{2} \right]. \tag{6.7}$$

$$h_5 = \frac{(906609r^5 + 2078931r^4 + 2013354r^3 + 991142r^2 + 250165r + 28727)d_1^4}{12288(1+3r)^4(5r+3)} - \frac{d_1^2d_2(1479r^3 + 2269r^2 + 1305r + 259)}{64(5r+3)(1+3r)^2} + \frac{d_1d_3(233r^2 + 274r + 101)}{64(5r+3)(1+3r)} + \frac{27d_2^2}{128} - \frac{d_4}{8}. \tag{6.8}$$

Now taking modulus in (6.5) and applying (2.2) of Lemma 2.1 we get bound for $|h_2|$. After that taking modulus on both sides of (6.6) and applying (2.3) of Lemma 2.1 we get the desired estimation for $|h_3|$. Now taking modulus in (6.7) and applying (2.4) of Lemma 2.2 we get our desired estimation for $|h_4|$. Now rearranging the terms of h_5

we get

$$\begin{aligned}
 h_5 = & \left(\frac{(906609r^5 + 2078931r^4 + 2013354r^3 + 991142r^2 + 250165r + 28727) d_1^3}{12288 (1 + 3r)^4 (5r + 3)} \right. \\
 & - \frac{d_1 d_2 (1479r^3 + 2269r^2 + 1305r + 259)}{64 (5r + 3) (1 + 3r)^2} + \frac{d_3 (233r^2 + 274r + 101)}{64 (5r + 3) (1 + 3r)} \Big) d_1 \\
 & - \frac{1}{8} \left(d_4 - \frac{27}{16} d_2^2 \right). \tag{6.9}
 \end{aligned}$$

Now taking modulus on both sides of (6.9) and applying triangle inequality we get

$$\begin{aligned}
 |h_5| \leq & \left| \frac{(906609r^5 + 2078931r^4 + 2013354r^3 + 991142r^2 + 250165r + 28727) d_1^3}{12288 (1 + 3r)^4 (5r + 3)} \right. \\
 & - \frac{d_1 d_2 (1479r^3 + 2269r^2 + 1305r + 259)}{64 (5r + 3) (1 + 3r)^2} + \frac{d_3 (233r^2 + 274r + 101)}{64 (5r + 3) (1 + 3r)} \Big| |d_1| \\
 & + \frac{1}{8} \left| d_4 - \frac{27}{16} d_2^2 \right|. \tag{6.10}
 \end{aligned}$$

After that applying (2.4) of Lemma (2.2) in first part and (2.3) of Lemma 2.1 in (6.10) we obtain

$$\begin{aligned}
 |h_5| \leq & \frac{|371247r^5 + 733389r^4 + 563094r^3 + 202138r^2 + 24299r - 3863|}{1536 (1 + 3r)^4 (5r + 3)} \\
 & + \frac{(2r + 1) (195r^2 + 206r + 79)}{8 (5r + 3) (1 + 3r)^2} + \frac{19}{32}.
 \end{aligned}$$

Hence the Theorem 6.1 is proved. □

7. Logarithmic inverse coefficient bounds for the class S_*^r

Coefficient problems represent a fundamental aspect of geometric function theory, offering a bridge between the analytic characterization of a function and its geometric properties. In particular, the study of logarithmic coefficients γ_n has gained prominence due to their utility in sharp growth estimates and their historical role in the development of the Milin and de Branges theorems. While the standard Taylor-Maclaurin coefficients a_n and the associated Fekete-Szegő functionals have been extensively documented for the class S_*^r , the corresponding logarithmic inverse coefficients remain comparatively less explored. Let $f \in S_*^r$ and let f^{-1} be the inverse function defined in a neighborhood of the origin. The logarithmic inverse coefficients Γ_n are generated by the following series expansion:

$$\log \left(\frac{f^{-1}(w)}{w} \right) = 2 \sum_{n=1}^{\infty} \Gamma_n w^n, \quad |w| < \rho. \tag{7.1}$$

where $\rho \geq 1/4$ denotes the radius of the Bloch disk. Differentiating and using (1.7), we obtain

$$\begin{aligned} \Gamma_1 &= -\frac{1}{2}h_2, & \Gamma_2 &= -\frac{1}{2}h_3 + \frac{3}{4}h_2^2, & \Gamma_3 &= -\frac{1}{2}h_4 + 2h_2h_3 - \frac{5}{3}h_2^3, \\ \Gamma_4 &= -\frac{1}{2}h_5 + \frac{5}{2}h_2h_4 - \frac{15}{2}h_2^2h_3 + \frac{5}{4}h_2h_3^2 + \frac{35}{8}h_2^4. \end{aligned} \tag{7.2}$$

In this section, we provide a systematic investigation into this problem, establishing upper bounds for the initial four coefficients $|\Gamma_1|, |\Gamma_2|, |\Gamma_3|$ and $|\Gamma_4|$.

Theorem 7.1. *If $f \in S_*^r$ given by (1.1), then*

$$\begin{aligned} |\Gamma_1| &\leq \frac{(1+r)}{2(1+3r)}, & |\Gamma_2| &\leq \frac{1}{4} \max \left\{ 1; \left| \frac{23r^2 - 6r - 1}{8(1+3r)^2} \right| \right\}, & |\Gamma_3| &\leq \frac{1+r}{2(5r+3)}, \\ |\Gamma_4| &\leq \frac{71r+33}{32(5r+3)}. \end{aligned} \tag{7.3}$$

Proof. Substituting the values of h_2, h_3, h_4 and h_5 from (1.7) in the relation (7.2) gives

$$\Gamma_1 = \frac{d_1(1+r)}{4(1+3r)}. \tag{7.4}$$

$$\Gamma_2 = \frac{d_2}{8} - \frac{(95r^2 + 42r + 7)d_1^2}{128(1+3r)^2} = \frac{1}{8} \left(d_2 - \frac{(95r^2 + 42r + 7)}{16(1+3r)^2} d_1^2 \right). \tag{7.5}$$

$$\Gamma_3 = \frac{(1+r)(949r^3 + 839r^2 + 211r + 17)d_1^3}{384(1+3r)^3(5r+3)} - \frac{d_1d_2(29r+7)(1+r)}{32(5r+3)(1+3r)} + \frac{d_3(1+r)}{20r+12}. \tag{7.6}$$

$$\begin{aligned} \Gamma_4 &= -\frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203)d_1^4}{16384(1+3r)^4(5r+3)} \\ &+ \frac{d_2d_1^2(1381r^3 + 1401r^2 + 495r + 51)}{512(5r+3)(1+3r)^2} - \frac{3d_1d_3(17r+7)}{640r+384} + \frac{d_4}{16} - \frac{7d_2^2}{256}. \end{aligned} \tag{7.7}$$

Now taking modulus in (7.4) and applying (2.2) of Lemma 2.1 we get bound for $|\Gamma_1|$. After that taking modulus on both sides of (7.5) and applying (2.3) of Lemma 2.1 we get the desired estimation for $|\Gamma_2|$. Now taking modulus in (7.6) and applying (2.4) of Lemma (2.2) we get

$$\begin{aligned} |\Gamma_3| &\leq \frac{(1+r)(949r^3 + 839r^2 + 211r + 17)}{192(1+3r)^3(5r+3)} + \frac{(1+r)(617r^3 + 583r^2 + 215r + 25)}{96(1+3r)^3(5r+3)} \\ &+ \frac{(1+r)(409r^3 + 587r^2 + 223r + 29)}{192(1+3r)^3(5r+3)} = \frac{1+r}{2(5r+3)}. \end{aligned}$$

Here we get the desired estimation for $|\Gamma_3|$. Now rearranging the terms of Γ_4 we get

$$\Gamma_4 = -d_1 \left(\frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203) d_1^3}{16384 (1 + 3r)^4 (5r + 3)} - \frac{d_1 d_2 (1381r^3 + 1401r^2 + 495r + 51)}{512 (5r + 3) (1 + 3r)^2} + \frac{3d_3 (17r + 7)}{640r + 384} \right) + \frac{1}{16} \left(d_4 - \frac{7}{16} d_2^2 \right). \tag{7.8}$$

Now taking modulus on both sides of (7.8) and applying triangle inequality we get

$$\Gamma_4 \leq |d_1| \left| \frac{(74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203) d_1^3}{16384 (1 + 3r)^4 (5r + 3)} - \frac{d_1 d_2 (1381r^3 + 1401r^2 + 495r + 51)}{512 (5r + 3) (1 + 3r)^2} + \frac{3d_3 (17r + 7)}{640r + 384} \right| + \frac{1}{16} \left| d_4 - \frac{7}{16} d_2^2 \right|. \tag{7.9}$$

After that applying (2.4) of Lemma (2.2) in first part and (2.3) of Lemma 2.1 in (7.9) we obtain

$$\begin{aligned} |\Gamma_4| &\leq \frac{74221r^5 + 119799r^4 + 80066r^3 + 26798r^2 + 4065r + 203}{4096 (1 + 3r)^4 (5r + 3)} \\ &+ \frac{124643r^5 + 214521r^4 + 147806r^3 + 50482r^2 + 8751r + 613}{2048 (1 + 3r)^4 (5r + 3)} \\ &+ \frac{205261r^5 + 373911r^4 + 267138r^3 + 95726r^2 + 17217r + 1259}{4096 (1 + 3r)^4 (5r + 3)} + \frac{1}{8} \\ &= \frac{3(17r + 7)}{32(5r + 3)} + \frac{1}{8} = \frac{71r + 33}{32(5r + 3)}. \end{aligned}$$

Hence we get our required estimations for Theorem 7.1. □

Concluding remarks

In this paper, we successfully introduced and comprehensively investigated a novel subclass of analytic functions uniquely defined through the product of a modified sigmoid function and the lemniscate Bernoulli function. Our systematic analysis began by establishing precise initial coefficient bounds for functions belonging to this new subclass.

Furthermore, we rigorously explored several critical analytic characteristics. Key findings include the establishment of the Fekete–Szegő inequality, alongside a thorough analysis of Hankel determinants of various orders, which provides valuable insight into the coefficient variability. We also successfully derived important estimates for the logarithmic coefficients and determined significant bounds for both the inverse coefficients and their corresponding logarithmic inverse coefficients.

Collectively, the results presented here offer a significant and substantial contribution to the field of geometric function theory. They not only deepen the understanding of coefficient problems within this novel class but also enrich the broader theory of analytic function subclasses associated with the products of special functions. The methods and findings lay a strong foundation for future research, particularly in exploring other differential and integral operators in conjunction with functions defined by similar special functional products.


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References

- [1] Ali, R. M., *Coefficients of the inverse of strongly starlike functions*, Bull. Malays. Math. Sci. Soc. **26**(2003), 63-71.
- [2] Altinkaya, S., Yalçın, S., *Upper bound of second Hankel determinant for bi-Bazilevič functions*, Mediterr. J. Math., **13**(2016), 4081–4090, DOI:10.1007/S00009-016-0733-5.
- [3] Arif, M., Raza, M., Tang, H., Hussain, S., Khan, H., *Hankel determinant of order three for familiar subsets of analytic functions related with sine function*, Open Math., **17**(2019), 1615-1630, DOI:10.1515/math-2019-0132.
- [4] Çağlar, M., Deniz, E., Srivastava, H. M., *Second Hankel determinant for certain subclasses of bi-univalent functions*, Turkish J. Math., **41**(2017), 694–706, DOI:10.3906/mat-1602-25.
- [5] Carathéodory, C., *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*. Math. Ann., **64**(1)(1907), 95-115.
- [6] De Branges, L., *A proof of the Bieberbach conjecture*, Acta Math. **154**(1985), no. 1-2, 137-152, DOI 10.1007/BF02392821.
- [7] Duren, P. L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, **259**(1983).
- [8] Duren, P. L., Leung, Y. J., *Logarithmic coefficients of univalent functions*, J. Anal. Math., **36**(1980), 36–43, DOI 10.1007/BF02798766.
- [9] Ebadian, A., Bulboacă, T., Cho, N. E., Analouei Adegani, E., *Coefficient bounds and differential subordinations for analytic functions associated with starlike function*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **114**(2020), pp. 128, DOI:10.1007/s13398-020-00871-x.
- [10] Efraimidis, I., *A generalization of Livingston's coefficient inequalities for functions with positive real part*, J. Math. Anal. Appl. **435**(1)(2016), 369-379, <https://doi.org/10.1016/j.jmaa.2015.10.050>.
- [11] Fekete, M., Szegő, G., *Eine Bemerkung Über ungerade schlichte Funktionen*, J. Lond. Math. Soc., **1**(1933), 85–89.
- [12] Hayman, W. K., *On the second Hankel determinant of mean univalent functions*, Proc. Lond. Math. Soc., **3**(1968), 77–94.

- [13] Janteng, A., Halim, S. A., Darus, M., *Coefficient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math, **7**(2006), 1–5.
- [14] Janteng, A., Halim, S. A., Darus, M., *Hankel determinant for starlike and convex functions*, J. Inequal. Pure Appl. Math., **1**(2007), 619–625.
- [15] Kanas, S., Analouei Adegani, E., Zireh, A., *An unified approach to second Hankel determinant of bi-subordinate functions*, Mediterr. J. Math., **14**(2007), pp. 233, <https://doi.org/10.1007/s00009-017-1031-6>.
- [16] Khatter, K., Ravichandran, V., Sivaprasad Kumar, S., *Estimates for Initial Coefficients of Certain Starlike Functions with Respect to Symmetric Points*, 2016, 10.1007/978-81-322-3640-5-24.
- [17] Lee, S. K., Ravichandran, V., Supramaniam, S., *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl., **281**(2013), <https://doi.org/10.1186/1029-242X-2013-281>.
- [18] Livingston, A. E., *The coefficients of multivalent close-to-convex functions*, Proc. Amer. Math. Soc., **21**(1969), 545–552, DOI:10.1090/S0002-9939-1969-0243054-0.
- [19] Ma, W., Minda, C., *A unified treatment of some special classes of functions*, Proceedings of the Conference on Complex Analysis, 1994.
- [20] Obradović, M., Tuneski, N., *Hankel determinants of second and third order for the class S of univalent functions*, Math. Slovaca, **71**(2021), 649–654, DOI:10.1515/ms-2021-0010.
- [21] Pommerenke, C., *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc., **1**(1966), 111–122, <https://doi.org/10.1112/JLMS/S1-41.1.111>.
- [22] Pommerenke, C., *On the Hankel determinants of univalent functions*, Mathematika, **14**(1967), 108–112, <https://doi.org/10.1112/S002557930000807X>.
- [23] Pommerenke, C., *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, Germany, 1975.
- [24] Ravichandran, V., *Starlike and convex functions with respect to conjugate points*, Acta Math. Acad. Paedagog. Nyházi, **20**(1)(2004), 31–37.
- [25] Ravichandran, V., Verma, S., *Bound for the fifth coefficient of certain starlike functions*, C. R. Math. Acad. Sci. Paris, Ser. 1 **353**(2015), 505–510, DOI:10.1016/j.crma.2015.03.003.
- [26] Sakaguchi, K., *On a certain univalent mapping*, J. Math. Soc. Japan, **11**(1959), 72–75, DOI: 10.2969/jmsj/01110072.
- [27] Srivastava, H. M., Raza, M., Abu Jarad, E. S. A., Srivastava, G., Abujarad, M. H., *Fekete-Szegő inequality for classes of (p, q) -starlike and (p, q) -convex functions*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM., **113**(2019), 3563–3584, DOI:10.1007/s13398-019-00713-5.

Trailokya Panigrahi 

Institute of Mathematics and Applications,
Andharua, Bhubaneswar-751029, Odisha, India.
e-mail: trailokya.ima@iomaorissa.ac.in

Shiba Prasad Dhal 

Department of Mathematics, Utkal University,
Vani Vihar, Bhubaneswar-751004, Odisha, India.
e-mail: 2024rs002.ima@iomaorissa.ac.in

Fractional Langevin equations involving ψ -Caputo type operators in a Banach space: a solution sets approach

Oualid Zentar , Mohamed Ziane  and Mohammed Al Horani 

Abstract. This paper investigates certain topological properties of the set of all global solutions for a class of nonlinear ψ -Caputo fractional Langevin equations. The nonlinearity, defined on an infinite-dimensional Banach space, is assumed to satisfy Nagumo-type growth conditions. An Aronszajn-type result is established using the nonlinear alternative for condensing operators, combined with the Browder–Gupta method. An illustrative example is provided to support the theoretical findings.

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1. Introduction

The theory of fractional differential equations is widely recognized for its extensive applications across numerous scientific disciplines. As a result, it has attracted significant interest from the mathematical community [14]. Over time, various definitions of fractional derivatives and integral operators have been proposed [24], each possessing distinct characteristics and properties. A recent development in this field is the emergence of “fractional calculus with respect to functions” [3], which has proven useful in modeling real-world phenomena. This type of operator appears in several contexts, including anomalous diffusion processes such as ultra-slow diffusion [16], random walks [12], financial crises [21], the Verhulst model [7], and the Heston model

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[6]. Accordingly, much research has focused on exploring both qualitative and quantitative properties of solutions to differential problems involving fractional derivatives with respect to functions [26, 27].

The Langevin model plays a crucial role in illustrating the interaction between particles and their surrounding medium, as well as the stochastic forces or fluctuations responsible for their irregular motion. However, the model's dependence on a specific relationship between a particle's position and velocity has led to the development of the fractional Langevin model, which better captures anomalous diffusion phenomena [15]. Additionally, certain real-world processes are more accurately described by systems with coupled randomness. For instance, in epidemiology, the migration of birds from different parts of the world can introduce infectious diseases, with transmission rates increasing as migratory birds converge. Such scenarios necessitate accounting for random disturbances. While these motivation-driven models offer strong explanatory power, the complexity of the corresponding mathematical formulations can pose significant challenges, particularly in analyzing the existence of solutions. Consequently, the study of qualitative properties for nonlinear ψ -Caputo Langevin equations has gained increasing importance.

In a recent study, the authors of [27] investigated certain quantitative aspects of the problem using Banach's contraction principle (equipped with a Bielecki-type norm) and fixed-point principles for convex-power condensing operators. Their analysis was conducted under the assumption that the nonlinearity satisfies a Nagumo-type growth condition in an infinite-dimensional Banach space. Building upon this foundation, the present manuscript seeks to advance the previous work by providing a new qualitative analysis of the following nonlinear fractional Langevin equation involving the ψ -Caputo fractional derivative (FD).

$$\begin{cases} \left({}^c\mathcal{D}_{a^+}^{\sigma;\psi} + \omega {}^c\mathcal{D}_{a^+}^{\sigma-1;\psi} \right) x(t) = \mathcal{W}(t, x(t)), & t \in J := [a, b], \\ x'(a) = x(a) = 0, \end{cases} \quad (1.1)$$

where $1 < \sigma < 2$, ${}^c\mathcal{D}_{a^+}^{\sigma;\psi}$ and ${}^c\mathcal{D}_{a^+}^{\sigma-1;\psi}$ represents the ψ -Caputo FD of order σ and $\sigma - 1$, respectively, ψ, ψ' are positive functions, $\mathcal{W} : J \times \mathbb{F} \rightarrow \mathbb{F}$ is a given function specified later, $(\mathbb{F}, \|\cdot\|)$ be an arbitrary Banach space and $\omega > 0$.

It is important to note that when the conditions ensuring the existence of solutions for the system under consideration do not guarantee uniqueness, it becomes necessary to explore the topological properties of the solution set. In particular, investigating the topological structure, such as acyclicity, compactness, and the R_δ -property (see Definition 2.10), is a central aspect of the qualitative theory of integral and differential equations and inclusions, due to its broad applicability [4, 28]. Notably, the R_δ -property plays a key role in demonstrating the invariance of the reachability set under nonlinear perturbations within the framework of control systems [11].

A distinctive feature of our investigation lies in the following key contributions:

- (1) Establishing sufficient conditions under which problem (1.1) admits a unique solution.

- (2) Proving a new Aronszajn-type result that addresses an open question related to fractional Langevin equations in a general framework, specifically, when the non-linearity operates in an arbitrary Banach space. This result is obtained by combining the nonlinear alternative for condensing maps with the Browder–Gupta approach.
- (3) In contrast to the approaches in [2, 27], we utilize an a priori estimation technique to prove the compactness and R_δ property of the global solution set, thereby extending, refining, and generalizing these previous findings.

The remainder of this paper is organized as follows: Section 2 collects preliminary material necessary for the developments that follow. Section 3 addresses contributions (1) and (2). Finally, illustrative examples are presented to support and demonstrate the applicability of our main results.

2. Preliminary results

In what follows, we equip the space $C(J, \mathbb{F})$ of continuous functions $f : J \rightarrow \mathbb{F}$ with the supnorm

$$\|f\|_\infty = \sup_{t \in J} \|f(t)\|, \quad \text{for all } f \in C(J, \mathbb{F}).$$

Consider the space $L_\psi^\beta(J, \mathbb{F})$ ($1 \leq \beta < \infty$) of Bochner-integrable functions x on J with the norm

$$\|f\|_{L_\psi^\beta} = \left(\int_a^b \psi'(s) \|f(s)\|^\beta ds \right)^{\frac{1}{\beta}}. \tag{2.1}$$

If $\psi(t) = t$ the space $L_\psi^\beta(J, \mathbb{F})$ coincides with the usual $L^\beta(J, \mathbb{F})$.

Define

$$\mathbb{L}_+^1(J, \mathbb{R}) = \{\psi : \psi \in C^1(J, \mathbb{R}), 0 < \psi'(t) \text{ for } t \in J\}.$$

For $t, s \in J$, ($s < t$), and $\psi \in \mathbb{L}_+^1(J, \mathbb{R})$ we pose

$$\varrho(t, s)^\sigma = (\psi(t) - \psi(s))^\sigma \quad \text{with} \quad \varrho(t, s) = \psi(t) - \psi(s).$$

Definition 2.1. [14, 3] *Let $\sigma > 0$. The ψ -fractional integral (FI) of order σ of an integrable function z is given by*

$$\mathcal{I}_{a^+}^{\sigma, \psi} z(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \varrho(t, s)^{\sigma-1} \psi'(s) z(s) ds, \quad \psi \in \mathbb{L}_+^1(J, \mathbb{R}), \text{ and } t > a,$$

where $\Gamma(\cdot)$ is the well-known gamma function.

Lemma 2.2. [14, 3] *For $\sigma, \alpha > 0$, one has*

$$\left(\mathcal{I}_{a^+}^{\sigma, \psi} \varrho(s, a)^{\alpha-1} \right) (t) = \frac{\Gamma(\alpha) \varrho(t, a)^{\sigma+\alpha-1}}{\Gamma(\sigma + \alpha)}.$$

Definition 2.3. [3] Let $\psi \in \mathbb{L}_+^1(J, \mathbb{R})$ and $n - 1 < \sigma \leq n$ with $n \in \mathbb{N}$. The ψ -Caputo FD of a n -times continuously differentiable function z of order σ is given as

$$({}^c \mathcal{D}_{a^+}^{\sigma; \psi} z)(t) = \mathcal{I}_{a^+}^{n-\sigma; \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n z(t).$$

Definition 2.4. [5] The Kuratowski measure of noncompactness (MNC) ς of a bounded set \mathbb{V} in a Banach space \mathbb{F} is defined as:

$$\varsigma(\mathbb{U}) := \inf \left\{ \varepsilon > 0 : \mathbb{U} = \bigcup_{i=1}^n \mathbb{U}_i \text{ and } \text{diam}(\mathbb{U}_i) \leq \varepsilon \text{ for } 1 \leq i \leq n \right\}.$$

Lemma 2.5. [5, 17] Let $\mathbb{V}, \mathbb{U} \subset \mathbb{F}$ be two bounded subsets. Then $\varsigma(\cdot)$ fulfills

1. $\mathbb{V} \subset \mathbb{U}$ implies that $\varsigma(\mathbb{V}) \leq \varsigma(\mathbb{U})$;
2. $\varsigma(\mathbb{V}) = 0$ if and only if \mathbb{V} is relatively compact;
3. $\varsigma(\mathbb{V} \cup \mathbb{U}) = \max\{\varsigma(\mathbb{V}), \varsigma(\mathbb{U})\}$;
4. $\varsigma(\mathbb{V}) = \varsigma(\text{conv}(\mathbb{V})) = \varsigma(\overline{\mathbb{V}})$, where $\text{conv } \mathbb{V}$ and $\overline{\mathbb{V}}$ represent the convex hull and the closure of \mathbb{V} , respectively;
5. $\varsigma(\mathbb{V} + \mathbb{U}) \leq \varsigma(\mathbb{V}) + \varsigma(\mathbb{U})$, where $\mathbb{V} + \mathbb{U} = \{v + u : v \in \mathbb{V}, u \in \mathbb{U}\}$;
6. $\varsigma(d\mathbb{V}) = |d|\varsigma(\mathbb{V})$, for any $d \in \mathbb{R}$;
7. For any bounded \mathbb{V} , there exists a countable set $\tilde{\mathbb{V}} \subset \mathbb{V}$, such that

$$\varsigma(\mathbb{V}) \leq 2\varsigma(\tilde{\mathbb{V}}).$$

Lemma 2.6. [13] Let $\{y_n\}_{n=1}^{+\infty} \subset L^1(J, \mathbb{F})$ such that $\|y_n(t)\| \leq h(t)$ for almost all $t \in J$, $n \geq 1$ and $h \in L^1(J, \mathbb{R}_+)$. Then, $t \mapsto \varsigma(\{y_n(t)\}_{n=1}^{+\infty})$ is integrable and

$$\frac{1}{2} \varsigma \left(\left\{ \int_a^t y_n(r) dr \right\}_{n=1}^{+\infty} \right) \leq \int_a^t \varsigma(\{y_n(r)\}_{n=1}^{+\infty}) dr. \tag{2.2}$$

Lemma 2.7. [9] Let \mathbb{S} be a Banach space, and let $\mathbb{V} \subset \mathbb{S}$ be a bounded and closed subset. Suppose that $\mathfrak{G} : \mathbb{V} \rightarrow \mathbb{S}$ is a condensing map. Then the operator $I - \mathfrak{G}$ maps closed subsets of \mathbb{V} onto closed subsets of \mathbb{S} , and $I - \mathfrak{G}$ is proper.

Recall that if, for every compact subset $\mathbb{U} \subset \mathbb{S}$, the set $(I - \mathfrak{G})^{-1}(\mathbb{U})$ is compact, then the continuous map $I - \mathfrak{G}$ is said to be proper.

Next, we revisit certain concepts from geometric topology [4, 11].

Definition 2.8. A subset \mathcal{B} of a Banach space \mathbb{S} is called a retract of \mathbb{S} if there exists a continuous map $f : \mathbb{S} \rightarrow \mathcal{B}$ such that $f(z) = z$, for every $z \in \mathcal{B}$.

Definition 2.9. A set \mathcal{B} is named contractible if there exists a (continuous) homotopy $H : \mathcal{B} \times [0, 1] \rightarrow \mathcal{B}$ and $z_0 \in \mathcal{B}$ such that

- (i) $H(z, 1) = z_0$, for every $z \in \mathcal{B}$.
- (ii) $H(z, 0) = z$, for every $z \in \mathcal{B}$.

Definition 2.10. *A nonempty compact space \mathfrak{C} is called an R_δ -set if there exists a decreasing sequence of compact, eventually nonempty, contractible spaces $(\mathfrak{C}_m)_{m \in \mathbb{N}}$ such that*

$$\mathfrak{C} = \bigcap_{m=1}^{\infty} \mathfrak{C}_m.$$

Lemma 2.11. [11] *Let \mathbb{G} and \mathbb{F} be normed spaces, and let $\mathfrak{G} : \mathbb{F} \rightarrow \mathbb{G}$ be a continuous map. Then, for every $\epsilon > 0$, there exists a locally Lipschitz map $\mathfrak{G}_\epsilon : \mathbb{F} \rightarrow \mathbb{G}$ such that*

$$\|\mathfrak{G}(z) - \mathfrak{G}_\epsilon(z)\| < \epsilon, \text{ for every } z \in \mathbb{F}.$$

Theorem 2.12. [8] *Let \mathbb{F} and \mathbb{S} be Banach spaces, and $\mathfrak{G} : \mathbb{F} \rightarrow \mathbb{S}$ a proper map. Assume that for every $\epsilon > 0$, there exists a proper map $\mathfrak{G}_\epsilon : \mathbb{F} \rightarrow \mathbb{S}$ such that the following two conditions hold:*

- (a) $\|\mathfrak{G}_\epsilon(z) - \mathfrak{G}(z)\| < \epsilon$, for all $z \in \mathbb{F}$.
- (b) For all $\epsilon > 0$ and $\tilde{y} \in \mathbb{S}$ in a neighborhood of the origin such that $\|\tilde{y}\| \leq \epsilon$, the equation $\mathfrak{G}_\epsilon(z) = \tilde{y}$ admits exactly one solution z_ϵ .

Then, $\mathfrak{G}^{-1}(0)$ is an R_δ -set.

Theorem 2.13. [22] *Let \mathbb{S} be a Banach space, and $\mathbb{V} \subset \mathbb{S}$ be a closed, convex set containing the origin. Suppose that $\mathcal{Z} : \mathbb{V} \rightarrow \mathbb{V}$ is a condensing map. Then, one of the following alternatives holds:*

- 1. \mathcal{Z} has a fixed point; or
- 2. the set $\mathbb{K} = \{x \in \mathbb{V} : x = \zeta \mathcal{Z}x, \zeta \in (0, 1)\}$ is unbounded.

The preceding theorem establishes the compactness of the solution set, and its proof follows the same lines as that of [5, Theorem 1.6.12].

A slight modification of [18, Theorem 1] yields the following result:

Lemma 2.14. *Let $\sigma > 0$, $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume the following conditions hold:*

- (i) $\psi \in \mathbb{L}^1(J, \mathbb{R}_+)$, and $\mathfrak{A}, \mathfrak{B}, \mathcal{W}, f$ are non-negative continuous functions defined on J ;
- (ii) $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous, positive, and non-decreasing function.

If f fulfills the inequality:

$$f(t) \leq \mathfrak{A}(t) + \mathfrak{B}(t) \int_a^t \psi'(s) \varrho(t, s)^{\sigma-1} \mathcal{W}(s) \Theta(f(s)) ds, \quad t \in J.$$

Then,

$$f(t) \leq \left[\varpi^{-1} \left(\varpi(\mathfrak{A}_1(t)) + \mathfrak{B}_1(t) \int_a^t \mathcal{W}(s)^q \psi'(s) ds \right) \right]^{1/q}, \quad t \in J,$$

where:

$$\begin{aligned} \mathfrak{A}_1(t) &= 2^{q-1} \sup_{a \leq s \leq t} \mathfrak{A}(s)^q, \\ \mathfrak{Z}_p &= \left(\frac{\varrho(b, a)^{p(\sigma-1)+1}}{p(\sigma-1)+1} \right)^{\frac{1}{p}}, \\ \mathfrak{B}_1(t) &= \mathfrak{Z}_p 2^{q-1} \sup_{a \leq s \leq t} \mathfrak{B}(s)^q, \\ \varpi(z) &= \int_{z_0}^z \frac{dr}{[\Theta(r^{1/q})]^q}, \quad \text{for } z_0, z > 0, \end{aligned}$$

and ϖ^{-1} is the inverse of ϖ .

3. Main results

To give an Aronszajn-type result for problem (1.1), Theorem 2.13 and 2.12 are applied. Henceforth, the set of all solutions of problem (1.1) will be denoted by $\mathfrak{D}(J, \mathcal{W})$. Assume that:

- (H1) The function $\mathcal{W} \in C(J \times \mathbb{F}, \mathbb{F})$.
- (H2) There exists a real-valued function $K \in L^{\frac{1}{\psi}}(J, \mathbb{R}_+)$ and a constant $\beta \in (0, \sigma - 1)$ such that

$$\|\mathcal{W}(t, v_1) - \mathcal{W}(t, v_2)\| \leq K(t)\|v_1 - v_2\|, \quad \text{for each } v_1, v_2 \in \mathbb{F} \text{ and } t \in J.$$

- (H3) There exists $\phi \in C(J, [0, \infty))$, and nondecreasing continuous functions $\Theta : [0, \infty) \rightarrow [0, \infty)$, such that

$$\|\mathcal{W}(t, u)\| \leq \phi(t)(\Theta(\|u\|) + 1), \quad \text{for each } t \in J \text{ and } u \in \mathbb{F}.$$

- (H4) There exists a function $\Lambda \in C(J, \mathbb{R}_+)$, such that

$$\varsigma(\mathcal{W}(t, \mathbb{U}(t))) \leq \Lambda(t)\varsigma(\mathbb{U}(t)), \quad \text{for all } t \in J,$$

where \mathbb{U} is bounded in $C(J, \mathbb{F})$.

For computational convenience, we denote $\phi^* = \sup_{t \in J} \phi(t)$ and $\Lambda^* = \sup_{t \in J} \Lambda(t)$.

3.1. Uniqueness result

This subsection presents the contractibility of $\mathfrak{D}(J, \mathcal{W})$, where the classical contraction principle is applied.

Theorem 3.1. *Assume that (H1)-(H2) hold. Additionally, it is supposed that*

$$\mathfrak{U}_\beta := \frac{(\sigma - 1)\|K\|_{L^{\frac{1}{\psi}}}^{\frac{1}{\beta}}}{(\sigma - \beta)\Xi \varrho(b, a)^{\beta - \sigma}} \left(\frac{1 - \beta}{\sigma - \beta - 1} \right)^{1 - \beta} < 1, \tag{3.1}$$

where $\Xi = e^{-\omega \varrho(b, a)} \Gamma(\sigma - 1)$. Then, the set $\mathfrak{D}(J, \mathcal{W})$ is a singleton in $C(J, \mathbb{F})$. Moreover, $\mathfrak{D}(J, \mathcal{W})$ is contractible.

Proof. Invoking [27, Theorem 3.1], we define the operator $\mathcal{Z} : C(J, \mathbb{F}) \rightarrow C(J, \mathbb{F})$ by

$$(\mathcal{Z}x)(t) = \int_a^t \frac{(\sigma - 1)}{e^{\omega \varrho(t,s)}} \left(\mathcal{I}_{a^+}^{\sigma-1; \psi} \mathcal{W}(\tau, x(\tau)) \right) (s) \psi'(s) ds, \quad t \in J. \tag{3.2}$$

Clearly, the set of all solutions to (1.1) coincides with the set of fixed points of the operator \mathcal{Z} . Therefore, it is enough to show that \mathcal{Z} is a contraction.

Since $\frac{1}{\beta} + \frac{\beta - 1}{\beta} = 1$, then $\psi'(\tau) = (\psi'(\tau))^{\frac{1}{\beta}} (\psi'(\tau))^{\frac{\beta-1}{\beta}}$, using the Hölder inequality, we obtain

$$\begin{aligned} \int_a^s \psi'(\tau) \varrho(s, \tau)^{\sigma-2} K(\tau) d\tau &\leq \left(\int_a^s \psi'(\tau) \varrho(s, \tau)^{\frac{\sigma-2}{1-\beta}} d\tau \right)^{1-\beta} \left(\int_a^s \psi'(\tau) (K(\tau))^{\frac{1}{\beta}} d\tau \right)^{\beta} \\ &\leq \|K\|_{L_{\psi}^{\frac{1}{\beta}}} \left(\frac{1 - \beta}{\sigma - \beta - 1} \right)^{1-\beta} \varrho(s, a)^{\sigma-\beta-1}, \end{aligned} \tag{3.3}$$

since $e^{\omega \varrho(t,a)} \geq 1$ for each $t \in J$, we obtain

$$\frac{1}{e^{\omega \varrho(t,s)}} = \frac{e^{\omega \varrho(s,a)}}{e^{\omega \varrho(t,a)}} \leq e^{\omega \varrho(s,a)} \leq e^{\omega \varrho(b,a)}, \quad \text{for all } t \geq s > a. \tag{3.4}$$

For all $\mathbf{r}, x \in C(J, \mathbb{F})$ and each $t \in J$, by (H2), the relations (3.3) and (3.4) entails

$$\begin{aligned} &\|(\mathcal{Z}x)(t) - (\mathcal{Z}\mathbf{r})(t)\| \\ &\leq \frac{\sigma - 1}{e^{-\omega \varrho(b,a)}} \int_a^t \frac{\psi'(s)}{\Gamma(\sigma - 1)} \left(\int_a^s \psi'(\tau) \varrho(s, \tau)^{\sigma-2} K(\tau) \|x(\tau) - \mathbf{r}(\tau)\| d\tau \right) ds \\ &\leq \frac{\sigma - 1}{e^{-\omega \varrho(b,a)}} \|x - \mathbf{r}\|_{\infty} \left(\frac{1 - \beta}{\sigma - \beta - 1} \right)^{1-\beta} \frac{\|K\|_{L_{\psi}^{\frac{1}{\beta}}}}{\Gamma(\sigma - 1)} \int_a^t \psi'(s) \varrho(s, a)^{\sigma-\beta-1} ds \\ &\leq \frac{(\sigma - 1) \|K\|_{L_{\psi}^{\frac{1}{\beta}}}}{(\sigma - \beta) \varrho(t, a)^{\beta-\sigma} \Xi} \left(\frac{1 - \beta}{\sigma - \beta - 1} \right)^{1-\beta} \|x - \mathbf{r}\|_{\infty}. \end{aligned}$$

So, one has

$$\|\mathcal{Z}x - \mathcal{Z}\mathbf{r}\|_{\infty} \leq \mathcal{U}_{\beta} \|x - \mathbf{r}\|_{\infty}.$$

Therefore, \mathcal{Z} is a contraction by virtue of condition (3.1). Applying the Banach contraction principle, it follows that \mathcal{Z} has a unique fixed point, which implies that $\mathfrak{D}(J, \mathcal{W}) = \{\hat{\mathbf{u}}\}$.

Now, we introduce the homotopy $\mathbb{H} : \mathfrak{D}(J, \mathcal{W}) \times [0, 1] \rightarrow \mathfrak{D}(J, \mathcal{W})$ by

$$\mathbb{H}(x, \vartheta)(t) = \begin{cases} x(t), & t \in (a, a(1 - \vartheta) + b\vartheta], \\ \hat{\mathbf{u}}(t), & t \in (a(1 - \vartheta) + b\vartheta, b]. \end{cases}$$

In particular,

$$\mathbb{H}(x, \vartheta) = \begin{cases} x, & \text{for } \vartheta = 1, \\ \widehat{\mathbf{u}}, & \text{for } \vartheta = 0. \end{cases}$$

Observe that $H(z, \lambda)(\cdot) \in \mathfrak{D}(J, \mathcal{W})$ for each $z \in \mathfrak{D}(J, \mathcal{W})$ and $\lambda \in [0, 1]$, ensuring that H is well-defined. Our goal now is to prove that $\mathbb{H}(\cdot, \cdot)$ is contractive. To achieve this, we begin by establishing its continuity. Let $(x_n, \vartheta_n) \in \mathfrak{D}(J, \mathcal{W}) \times [0, 1]$ be such that $(x_n, \vartheta_n) \rightarrow (x, \vartheta)$, one has

$$\mathbb{H}(x_n, \vartheta_n)(t) = \begin{cases} x_n(t), & t \in (a, a(1 - \vartheta_n) + b\vartheta_n], \\ \widehat{\mathbf{u}}(t), & t \in (a(1 - \vartheta_n) + b\vartheta_n, b]. \end{cases}$$

We proceed by considering the following cases:

- (i) If $\lim_{n \rightarrow \infty} \vartheta_n = 0$, it follows that

$$\mathbb{H}(x, 0)(t) = \widehat{\mathbf{u}}(t), \quad \text{for all } t \in J.$$

Then,

$$\begin{aligned} \|\mathbb{H}(x_n, \vartheta_n)(t) - \mathbb{H}(x, \vartheta)(t)\| &\leq \|\mathbb{H}(x_n, \vartheta_n)(t) - \mathbb{H}(x, \vartheta)(t)\|_{[a(1-\vartheta_n)+b\vartheta_n, b]} \\ &\quad + \|\mathbb{H}(x_n, \vartheta_n)(t) - \mathbb{H}(x, \vartheta)(t)\|_{[a, a(1-\vartheta_n)+b\vartheta_n]} \\ &\leq \|\widehat{\mathbf{u}}(t) - \widehat{\mathbf{u}}(t)\|_{[a(1-\vartheta_n)+b\vartheta_n, b]} \\ &\quad + \|x_n(\xi) - x(t)\|_{[a, a(1-\vartheta_n)+b\vartheta_n]} \\ &\leq \|x_n(t) - x(t)\|_{[a, a(1-\vartheta_n)+b\vartheta_n]}. \end{aligned}$$

Hence

$$\|\mathbb{H}(x_n, \vartheta_n) - \mathbb{H}(x, \vartheta)\|_\infty \leq \|x_n - x\|_\infty,$$

which tends to 0 when $n \rightarrow \infty$.

- (ii) An analogous argument applies in the case $\lim_{n \rightarrow \infty} \vartheta_n = 1$.
- (iii) If $0 < \lim_{n \rightarrow \infty} \vartheta_n = \vartheta < 1$ with $\vartheta_n \neq 0$, then, we distinguish between two sub-cases:

- (1). $x_n \in \mathfrak{D}(J, \mathcal{W})$. This entails that for $a \leq t \leq a(1 - \vartheta_n) + b\vartheta_n$

$$x_n(t) = (\sigma - 1) \int_a^t \frac{\psi'(s)}{e^{\omega \varrho(t,s)}} \left(\mathcal{I}_{a^+}^{\sigma-1; \psi} \mathcal{W}(\tau, x_n(\tau)) \right) (s) ds.$$

By (H1), for $t \in J$ one has

$$x(t) = (\sigma - 1) \int_a^t \frac{\psi'(s)}{e^{\omega \varrho(t,s)}} \left(\mathcal{I}_{a^+}^{\sigma-1; \psi} \mathcal{W}(\tau, x(\tau)) \right) (s) ds.$$

- (2). If $a(1 - \vartheta_n) + b\vartheta_n < t \leq b$, it follows that

$$\mathbb{H}(x_n, \vartheta_n)(t) = \widehat{\mathbf{u}}(t) = \mathbb{H}(x, \vartheta)(t).$$

Then

$$\|\mathbb{H}(x_n, \vartheta_n) - \mathbb{H}(x, \vartheta)\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Accordingly, \mathbb{H} is continuous, yielding the contractibility of $\mathfrak{D}(J, \mathcal{W})$ to the point $\widehat{\mathbf{u}}$. □

3.2. R_δ -property of $\mathfrak{D}(J, \mathcal{W})$

This subsection establishes the topological structure of $\mathfrak{D}(J, \mathcal{W})$.

Theorem 3.2. *Assume that (H1) and (H3)-(H4) are fulfilled. Then $\mathfrak{D}(J, \mathcal{W})$ is an R_δ -set.*

Proof. Let us reintroduce the operator \mathcal{Z} as defined in (3.2). For $T > 0$, we now define:

$$\mathbb{T}_T = \{x \in C(J, \mathbb{F}) : \|x\|_\infty < T\}.$$

The proof of Theorem 3.2 will proceed through several steps.

Step 1 : The continuity of the operator \mathcal{Z} .

Let $\{x_n\}$ be a sequence in \mathbb{T}_T such that $x_n \rightarrow x$ when $n \rightarrow \infty$. Recalling (3.4), one has

$$\begin{aligned} & \|(\mathcal{Z}x_n)(t) - (\mathcal{Z}x)(t)\| \\ & \leq \frac{\sigma - 1}{\Xi} \int_a^t \psi'(s) \int_a^s \psi'(\tau) \varrho(s, \tau)^{\sigma-2} \|\mathcal{W}(\tau, x_n(\tau)) - \mathcal{W}(\tau, x(\tau))\| d\tau ds \\ & \leq \frac{\sigma - 1}{\Xi} \|\mathcal{W}(\cdot, x_n(\cdot)) - \mathcal{W}(\cdot, x(\cdot))\|_\infty \int_a^t \psi'(s) \int_a^s \psi'(\tau) \varrho(s, \tau)^{\sigma-2} d\tau ds \\ & \leq \frac{1}{\Xi} \|\mathcal{W}(\cdot, x_n(\cdot)) - \mathcal{W}(\cdot, x(\cdot))\|_\infty \int_a^t \psi'(s) \varrho(s, a)^{\sigma-1} ds \\ & \leq \frac{\varrho(t, a)^\sigma}{\sigma \Xi} \|\mathcal{W}(\cdot, x_n(\cdot)) - \mathcal{W}(\cdot, x(\cdot))\|_\infty. \end{aligned}$$

From the continuity of \mathcal{W} , it follows that:

$$\|(\mathcal{Z}x_n)(t) - (\mathcal{Z}x)(t)\| \xrightarrow{n \rightarrow \infty} 0.$$

Then,

$$\|\mathcal{Z}x_n - \mathcal{Z}x\|_\infty \xrightarrow{n \rightarrow \infty} 0.$$

Consequently, \mathcal{Z} is continuous.

Step 2 : \mathcal{Z} maps bounded sets of $C(J, \mathbb{F})$ into itself.

Let $x \in \mathbb{T}_T$. Using (3.4). Then, for each $t \in J$, one gets

$$\|(\mathcal{Z}x)(t)\| \leq \frac{\sigma - 1}{\Xi} \int_a^t \left(\int_a^s \psi'(\tau) \varrho(s, \tau)^{\sigma-2} \|\mathcal{W}(\tau, x(\tau))\| d\tau \right) \psi'(s) ds.$$

By (H3), for each $t \in J$, one has

$$\|\mathcal{W}(\tau, x(\tau))\| \leq \phi(\tau)\Theta(\|x(\tau)\|) + \phi(\tau) \leq \phi^*\Theta(\|x\|_\infty) + \phi^*. \tag{3.5}$$

So, the relation (3.5) implies

$$\begin{aligned} \|(\mathcal{Z}x)(t)\| &\leq \frac{(\sigma - 1)\phi^*(\Theta(T) + 1)}{\Xi} \int_a^t \psi'(s) \int_a^s \psi'(\tau)\varrho(s, \tau)^{\sigma-2} d\tau ds \\ &\leq \frac{\phi^*(\Theta(T) + 1)}{\Xi} \int_a^t \psi'(s)\varrho(s, a)^{\sigma-1} ds \\ &\leq \frac{\phi^*(\Theta(T) + 1)}{\sigma\Xi} \varrho(t, a)^\sigma \\ &\leq \frac{\phi^*(\Theta(T) + 1)}{\sigma\Xi} \varrho(b, a)^\sigma := \widehat{T}. \end{aligned}$$

Thus,

$$\|\mathcal{Z}x\|_\infty \leq \widehat{T}.$$

Step 3 : The family $\mathcal{Z}(\mathbb{T}_T)$ is equicontinuous in $C(J, \mathbb{F})$.

For every $a \leq t_1 < t_2 \leq b$ and $x \in \mathbb{T}_T$, one obtains

$$\|(\mathcal{Z}x)(t_2) - (\mathcal{Z}x)(t_1)\| \leq O_1 + O_2,$$

where

$$O_1 = \frac{\sigma - 1}{\Xi} \int_{t_1}^{t_2} \psi'(s) \int_a^s \psi'(\tau)\varrho(s, \tau)^{\sigma-2} \|\mathcal{W}(\tau, x(\tau))\| d\tau ds,$$

and

$$O_2 = (\sigma - 1) \int_a^{t_1} \psi'(s) \left\| \frac{1}{e^{\omega\varrho(t_2, s)}} - \frac{1}{e^{\omega\varrho(t_1, s)}} \right\| \left\| \left(\mathcal{I}_{a^+}^{\sigma-1; \psi} \mathcal{W}(\tau, x(\tau)) \right) (s) \right\| ds.$$

Using (H3) and (3.5), one gets

$$\begin{aligned} O_1 &\leq \frac{(\sigma - 1)\phi^*(\Theta(T) + 1)}{\Xi} \int_{t_1}^{t_2} \psi'(s) \int_a^s \psi'(\tau)\varrho(s, \tau)^{\sigma-2} d\tau ds \\ &\leq \frac{\phi^*(\Theta(T) + 1)}{\Xi} \int_{t_1}^{t_2} \psi'(s)\varrho(s, a)^{\sigma-1} ds \\ &\leq \frac{\phi^*(\Theta(T) + 1)}{\sigma\Xi} [\varrho(t_2, a)^\sigma - \varrho(t_1, a)^\sigma]. \end{aligned}$$

Thus,

$$O_1 \xrightarrow{t_2 \rightarrow t_1} 0. \tag{3.6}$$

On the other hand,

$$O_2 = \left(\frac{\sigma - 1}{e^{\omega\varrho(t_1)}} - \frac{\sigma - 1}{e^{\omega\varrho(t_2)}} \right) \int_a^{t_1} \frac{1}{e^{-\omega\varrho(s)}} \left\| \left(\mathcal{I}_{a^+}^{\sigma-1; \psi} \mathcal{W}(\tau, x(\tau)) \right) (s) \right\| \psi'(s) ds.$$

Thus,

$$O_2 \xrightarrow{t_2 \rightarrow t_1} 0. \tag{3.7}$$

It follows from (3.6) and (3.7) that both inequalities are independent of x and tend to zero as $t_2 \rightarrow t_1$. Accordingly, $\mathcal{Z}(\mathbb{T}_T)$ is equicontinuous.

Step 4 : \mathcal{Z} is condensing.

Let \mathbb{Y} be a bounded subset of $C(J, \mathbb{F})$, we define the MNC as follows:

$$\widehat{\varsigma}_\mu(\mathbb{Y}) = \sup_{t \in J} \frac{1}{e^{\mu t}} \varsigma(\mathbb{Y}(t)), \quad \text{for } \mu > 0. \tag{3.8}$$

Now, since $\psi'(\cdot)\psi(\cdot, a)^{\sigma-1} \in L^1(J, \mathbb{R})$, we can choose μ such that

$$2\mathfrak{q}(\mu) < 1. \tag{3.9}$$

where

$$\mathfrak{q}(\mu) := \frac{4\Lambda^*}{\Xi} \sup_{t \in J} \int_a^t \frac{\psi'(s)\varrho(s, a)^{\sigma-1}}{e^{\mu(t-s)}} ds.$$

Next, for a countable set $\{X^n\}_{n=1}^{+\infty} \subseteq \mathcal{Z}(\mathbb{Y})$ one has

$$\widehat{\varsigma}_\mu(\{X^n\}_{n=1}^{+\infty}) \leq \widehat{\varsigma}_\mu(\mathcal{Z}(\mathbb{Y})). \tag{3.10}$$

Then, there exists $\{x_n\}_{n=1}^{+\infty} \subset \mathbb{Y}$, such that

$$X^n(t) = (\mathcal{Z}x_n(t)), \quad \text{for } n \geq 1, t \in J. \tag{3.11}$$

After that, from

$$(\mathcal{Z}x_n)(t) \leq \frac{\sigma-1}{\Xi} \int_a^t \int_a^s \psi'(\tau)\varrho(s, \tau)^{\sigma-2} \mathcal{W}(\tau, x_n(\tau)) d\tau \psi'(s) ds, \tag{3.12}$$

so,

$$\begin{aligned} \widehat{\varsigma}_\mu(\{X^n(t)\}_{n=1}^{+\infty}) &= \widehat{\varsigma}_\mu(\{(\mathcal{Z}x_n)(t)\}_{n=1}^{+\infty}) \\ &\leq \widehat{\varsigma}_\mu \left(\left\{ \frac{\sigma-1}{\Xi} \int_a^t \int_a^s \psi'(\tau)\varrho(s, \tau)^{\sigma-2} \mathcal{W}(\tau, x_n(\tau)) d\tau \psi'(s) ds \right\}_{n=1}^{+\infty} \right). \end{aligned} \tag{3.13}$$

By (H4), for $a \leq \tau \leq s$, we get

$$\begin{aligned} \varsigma \left(\left\{ \frac{\psi'(\tau)}{\varrho(s, \tau)^{2-\sigma}} \mathcal{W}(\tau, x_n(\tau)) \right\}_{n=1}^{+\infty} \right) &\leq \frac{\psi'(\tau)}{\varrho(s, \tau)^{2-\sigma}} \Lambda(\tau) \varsigma(\{x_n(\tau)\}_{n=1}^{+\infty}) \\ &\leq \frac{\Lambda(\tau)\psi'(\tau)e^{\mu\tau}}{\varrho(s, \tau)^{2-\sigma}} \widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty}). \end{aligned}$$

Using Lemma 2.6, for each $t \in J$, $s \in [a, t]$ and $\tau \leq s$, one gets

$$\begin{aligned}
 & \varsigma \left(\left\{ \frac{\sigma - 1}{\Xi} \int_a^t \int_a^s \psi'(\tau) \varrho(s, \tau)^{\sigma-2} \mathcal{W}(\tau, x_n(\tau)) d\tau \psi'(s) ds \right\}_{n=1}^{+\infty} \right) \\
 & \leq \frac{4(\sigma - 1)\Lambda^*}{\Xi} \widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty}) \int_a^t \int_a^s \frac{\psi'(\tau) e^{\mu\tau}}{\varrho(s, \tau)^{2-\sigma}} d\tau \psi'(s) ds \\
 & \leq \frac{4(\sigma - 1)\Lambda^*}{\Xi} \widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty}) \int_a^t \psi'(s) e^{\mu s} \int_a^s \frac{\psi'(\tau)}{\varrho(s, \tau)^{2-\sigma}} d\tau ds \\
 & \leq \frac{4\Lambda^* \widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty})}{\Xi} \int_a^t \frac{\psi'(s) e^{\mu s}}{\varrho(s, a)^{1-\sigma}} ds.
 \end{aligned}$$

Multiplying by $\exp(-\mu t)$, one obtains

$$\begin{aligned}
 & \sup_{t \in J} \frac{1}{e^{\mu t}} \varsigma \left(\left\{ \frac{\sigma - 1}{\Xi} \int_a^t \int_a^s \psi'(\tau) \varrho(s, \tau)^{\sigma-2} \mathbf{t}(\tau, x_n(\tau)) d\tau \psi'(s) ds \right\}_{n=1}^{+\infty} \right) \\
 & \leq \frac{4\Lambda^* \widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty})}{\Xi} \sup_{t \in J} \int_a^t \frac{\psi'(s) e^{-\mu(t-s)}}{\varrho(s, a)^{1-\sigma}} ds.
 \end{aligned} \tag{3.14}$$

So, by (3.9), (3.13) and (3.14), we have

$$\widehat{\varsigma}_\mu(\{(\mathcal{Z}x_n)(t)\}_{n=1}^{+\infty}) \leq \mathfrak{q}(\mu) \widehat{\varsigma}_\mu(\{x_n(\tau)\}_{n=1}^{+\infty}). \tag{3.15}$$

Therefore, one has

$$\widehat{\varsigma}_\mu(\{x_n\}_{n=1}^{+\infty}) \leq \widehat{\varsigma}_\mu(\mathbb{Y}) \leq \widehat{\varsigma}_\mu(\mathcal{Z}(\mathbb{Y})) \leq \frac{1}{2} \widehat{\varsigma}_\mu(\{X^n\}_{n=1}^{+\infty}). \tag{3.16}$$

By (3.10) and (3.16), we can get

$$\widehat{\varsigma}_\mu(\{X^n\}_{n=1}^{+\infty}) = 0.$$

Then, $\widehat{\varsigma}_\mu(\mathbb{Y}) = 0$, which proves the compactness of the set $\overline{\mathbb{Y}}$. Accordingly, \mathcal{Z} is condensing.

Step 5. The boundedness of set \mathbb{K} (see Theorem 2.13 (2)).

Let $x \in C(J, \mathbb{F})$ and $x = \rho \mathcal{Z}x$ for some $\rho \in (0, 1)$. Then, $x(t) \leq \mathcal{Z}x(t)$. By applying Dirichlet’s formula [23] together with (3.4) one obtains

$$\begin{aligned}
 \|x(t)\| & \leq \frac{\sigma - 1}{\Xi} \int_a^t \psi'(s) \left(\int_a^s \frac{\psi'(\tau)}{\varrho(s, \tau)^{2-\sigma}} \|\mathcal{W}(\tau, x(\tau))\| d\tau \right) ds \\
 & \leq \frac{\sigma - 1}{\Xi} \int_a^t \psi'(s) \|\mathcal{W}(s, x(s))\| \int_s^t \frac{\psi'(\tau)}{\varrho(\tau, s)^{2-\sigma}} d\tau ds \\
 & = \frac{1}{\Xi} \int_a^t \frac{\psi'(s)}{\varrho(t, s)^{1-\sigma}} \|\mathcal{W}(s, x(s))\| ds.
 \end{aligned}$$

Employing (H3), one gets

$$\begin{aligned} \|x(t)\| &\leq \frac{1}{\Xi} \int_a^t \psi'(s) \varrho(t, s)^{\sigma-1} \phi(s) (\Theta(\|x(s)\|) + 1) ds \\ &\leq \frac{\phi^*}{\Xi} \int_a^t \frac{\psi'(s)}{\varrho(t, s)^{1-\sigma}} ds + \frac{1}{\Xi} \int_a^t \frac{\psi'(s)}{\varrho(t, s)^{1-\sigma}} \phi(s) \Theta(\|x(s)\|) ds \\ &\leq \mathcal{Q}_0 + \mathcal{Q}_1 \int_a^t \frac{\psi'(s)}{\varrho(t, s)^{1-\sigma}} \phi(s) \Theta(\|x(s)\|) ds, \end{aligned}$$

where $\mathcal{Q}_0 = \frac{\phi^* \varrho(b, a)^\sigma}{\sigma \Xi}$ and $\mathcal{Q}_1 = \frac{1}{\Xi}$. Using Lemma 2.14, we obtain

$$\|x(t)\| \leq \left[\varpi^{-1} \left(\varpi(\widehat{\mathcal{Q}}_0) + \widehat{\mathcal{Q}}_1 \int_a^t \phi(s)^q \psi'(s) ds \right) \right]^{1/q} := \mathbf{N}, \quad t \in J,$$

where

$$\widehat{\mathcal{Q}}_0 = 2^{q-1} \mathcal{Q}_0^q, \quad \widehat{\mathcal{Q}}_1 = 2^{q-1} \mathfrak{Z}_p \mathcal{Q}_1^q, \quad \mathfrak{Z}_p = \left[\frac{\varrho(b, a)^{p(\sigma-1)+1}}{p(\sigma-1)+1} \right]^{\frac{1}{p}},$$

$$\varpi(z) = \int_{z_0}^z \left[\Theta(z^{1/q}) \right]^{-q} \psi'(s) ds, \text{ for } z_0, z > 0 \text{ and } \varpi^{-1} \text{ is the inverse of } \varpi.$$

Hence, we obtain

$$\|x\|_\infty \leq \mathbf{N},$$

which entails the boundedness of the set \mathbb{K} . From Theorem 2.13, we conclude that $\mathfrak{D}(J, \mathcal{W})$ is non-empty and compact subset of $C(J, \mathbb{F})$.

Step 6. $\mathfrak{D}(J, \mathcal{W})$ is an R_δ -set.

Let $0 < \epsilon_n < 1$ with $\epsilon_n \xrightarrow{n \rightarrow \infty} 0$. Using (H1), thanks to Lemma 2.11, there exists a sequence $\{\mathcal{W}_n\}$ of locally Lipschitz functions such that

$$\|\mathcal{W}_n(t, y) - \mathcal{W}(t, y)\| < \epsilon_n, \quad \text{for all } y \in \mathbb{F} \text{ and } t \in J. \tag{3.17}$$

From relation (3.17) and (H3). Then

$$\|\mathcal{W}_n(t, y)\| \leq 1 + \phi(t) \Theta(\|y\|) + \phi(t), \quad n \in \mathbb{N} \setminus \{0\}.$$

One can define \mathcal{O}_n by

$$\mathcal{O}_n(x)(t) = \frac{\sigma-1}{\Gamma(\sigma-1)} \int_a^t \frac{\psi'(s)}{e^{\omega \varrho(t,s)}} \left(\int_a^s \frac{\psi'(\tau)}{\varrho(s, \tau)^{2-\sigma}} \mathcal{W}_n(\tau, x(\tau)) d\tau \right) ds, \quad t \in J. \tag{3.18}$$

Theorem 3.1, together with the local Lipschitz continuity of \mathcal{W}_n , implies that (3.18) is uniquely solvable.

Let

$$\mathbb{M}(x) = (I - \mathcal{O})(x).$$

By Step 4, the approximate operators $\mathcal{O}_n : C(J, \mathbb{F}) \rightarrow C(J, \mathbb{F})$ are condensing, which allows us to define

$$\mathbb{M}_n(x) = (I - \mathcal{O}_n)(x)$$

where each \mathbb{M}_n is a condensing perturbation of the identity and, by Lemma 2.7, proper maps.

Now, the relation (3.17) permits the uniform convergence of $\{\mathbb{M}_n\}$ to \mathbb{M} in $C(J, \mathbb{F})$.

Invoking (3.4), we get

$$\begin{aligned} \|\mathbb{M}_n(x)(t) - \mathbb{M}(x)(t)\| &\leq \frac{\sigma - 1}{\Xi} \int_a^t \psi'(s) \int_a^s \frac{\psi'(\tau)}{\varrho(s, \tau)^{2-\sigma}} \|\mathcal{W}_n(\tau, x(\tau)) \\ &\quad - \mathcal{W}(\tau, x(\tau))\| d\tau ds \\ &\leq \frac{\sigma - 1}{\Xi} \varrho(b, a)^\sigma \varepsilon_n, \quad t \in J, \end{aligned}$$

and equation $\mathbb{M}_n(x) = y$ has a unique solution for all $y \in C(J, \mathbb{R})$ as well as (3.18).

Consequently, the assertions of Theorem 2.12 are verified. Therefore, the solution set $\mathbb{M}^{-1}(0)$ is an R_δ -set. □

4. Illustrative examples

In this section we introduce some examples that illustrate our theoretical results. Consider the Banach space

$$\mathbb{F} = c_0 = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \xrightarrow{n \rightarrow \infty} 0\}$$

endowed with

$$\|x\|_{c_0} = \sup_{n \in \mathbb{N} \setminus \{0\}} |x_n|.$$

Take $\psi(t) = t$. Let us define the function $\mathcal{W} : J \times c_0 \rightarrow c_0$ by

$$\begin{aligned} \mathcal{W}(t, x) = &\left\{ \frac{13(b-a)^{\beta-\sigma}}{(1+e^t)e^{\eta t+\omega(b-a)}} + \frac{(b-a)^{\beta-\sigma} \tan^{-1}(|x_n|)}{(1+e^t)e^{\eta t+\omega(b-a)}} \right. \\ &\left. + \frac{(b-a)^{\beta-\sigma}}{(1+e^t)e^{\eta t+\omega(b-a)}} \frac{|x_n|}{1+|x_n|} \right\}_{n \geq 1} \end{aligned} \tag{4.1}$$

where η is a positive constant.

Obviously, the function \mathcal{W} is continuous and for any $u, v \in c_0$ and $t \in J$, one can verify that

$$\begin{aligned} \|\mathcal{W}(t, z_1) - \mathcal{W}(t, z_2)\| &\leq \frac{2(b-a)^{\beta-\sigma}}{(1+e^t)e^{\eta t+\omega(b-a)}} \|z_1 - z_2\| \\ &\leq \frac{2(b-a)^{\beta-\sigma}}{(1+e^a)e^{\eta t+\omega(b-a)}} \|z_1 - z_2\|. \end{aligned}$$

So, the assumption (H2) fulfills with

$$K(t) = \frac{2(b-a)^{\beta-\sigma}}{(1+e^a)e^{\eta t+\omega(b-a)}} \quad \text{for } t \in J.$$

Now, for $0 < \beta < \sigma - 1$, let $K^* = \left\| \frac{2e^{-\eta t}}{1+e^a} \right\|_{L^{\frac{1}{\beta}}}$. Choosing a suitable $0 < \beta < \sigma - 1$ and $\eta > 0$ large enough, it follows that

$$\mathcal{U}_\beta < 1.$$

Therefore, all the conditions of Theorem 3.1 are met. Then, the set of solutions of (1.1) with \mathcal{W} defined by (4.1) is a singleton.

Now, for $t \in J$ and $x = \{x_n\}_n \in c_0$, consider the forcing nonlinearities,

$$\mathcal{W}(t, x) = \left\{ \frac{3}{13+e^t} + \frac{4 \tan^{-1}(|x_n|)}{13+e^t} + \frac{1}{2^n \cdot 13 + 2^n e^t} \right\}_{n \geq 1}. \tag{4.2}$$

Clearly, \mathcal{W} satisfies (H1). To illustrate (H2), let $t \in J$ and $x = \{x_n\}_n \in \mathbb{Y} \subset c_0$. Then

$$\|\mathcal{W}(t, x)\| \leq \frac{4(1+\|x\|)}{e^t+13} \leq \phi(t)\Theta(\|x\|) + \phi(t). \tag{4.3}$$

Then, (H2) is fulfilled with

$$\Theta(y) = y, \text{ for all } y \in \mathbb{R}_+, \quad \text{and} \quad \phi(t) = \frac{4}{e^t+13} \text{ for all } t \in J.$$

Next, assumption (H3) is fulfilled. Indeed, the Hausdorff MNC χ in $(c_0, \|\cdot\|_{c_0})$ can be defined by the formula

$$\chi(\mathbb{Y}) = \limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{Y}} \|(I - P_n)x\|_\infty,$$

where $\mathbb{Y} \in \mathcal{P}(c_0)$, P_n denotes the projection of the first n vectors onto the linear span in the standard basis (see [5]).

Using (4.3) (see also [25, 26]), and the relationship between the Hausdorff χ and Kuratowski ς MNCs, given by the inequality

$$\chi(\mathbb{Y}) \leq \varsigma(\mathbb{Y}) \leq 2\chi(\mathbb{Y}),$$

one gets

$$\varsigma(\mathcal{W}(t, \mathbb{Y}(t))) \leq \Lambda(t)\varsigma(\mathbb{Y}(t)), \quad \text{for each } t \in J,$$

where

$$2\phi(t) = \Lambda(t).$$

Therefore, all assumptions of Theorem 3.2 are fulfilled, the solution set of (1.1), with \mathcal{W} defined by (4.2), is an R_δ -set.

5. Conclusion

Under Nagumo-type growth conditions, we present novel results concerning the topological structure of the solution set for nonlinear ψ -Caputo fractional Langevin equations in Banach spaces. Our approach unfolds in multiple stages: the initial stage utilizes the classical contraction mapping principle, while the second is grounded in the nonlinear alternative for condensing operators, incorporating the Browder-Gupta technique (Theorem 2.12). These findings offer meaningful contributions to the evolving study of such equations.

Further developments, including periodic solutions and approximate controllability, will be addressed in a forthcoming work.

References


- [1] Abbas, S., Benchohra, M., Graef, J. R., Henderson, J., *Implicit fractional differential and integral equations. Existence and stability*, De Gruyter, Berlin, 2018.
- [2] Abu Hammad, M., Zentar, O., Alshorm, S., Ziane, M., Zitouni, I., *Theoretical analysis of a class of φ -Caputo fractional differential equations in Banach space*, AIMS Mathematics, **9**(3)(2024), 6411-6423.
- [3] Almeida, R., *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul., **44**(2017), 460-481. DOI:10.1016/j.cnsns.2016.09.006.
- [4] Andres, J., Górniewicz, L., *Topological fixed point principles for boundary value problems*, Dordrecht: Kluwer Academic Publishers, 2003.
- [5] Akhmerov, R. R., Kamenskii, M. I., Potapov, A. S., Rodkina, A. E., Sadovskii, B. N., *Measures of noncompactness and condensing operators*, Oper. Theory: Adv. Appl., Vol. 55, Basel: Birkhäuser, 1992.
- [6] Arfaoui, H., *New numerical method for solving a new generalized American options under Ψ -Caputo time-fractional derivative Heston model*, to appear in Rocky Mountain J. Math.
- [7] Awadalla, M., Yameni, N., Yves, Y., Asbeh, K., *Psi-Caputo logistic population growth model*, J. Math., Article ID 8634280, 2021, pp. 9.
- [8] Browder, F. E., Gupta, C. P., *Topological degree and nonlinear mappings of analytic type in Banach spaces*, J. Math. Anal. Appl., **26**(1969), 390-402.
- [9] Deimling, K., *Nonlinear functional analysis*, Berlin Springer-Verlag; 1985.
- [10] Develi, F., *Existence and Ulam-Hyers stability results for nonlinear fractional Langevin equation with modified argument*, Math. Methods Appl. Sci., **45**(7)(2022), 3417-3425.
- [11] Djebali, S., Górniewicz, L., Ouahab, A., *Solution sets for differential equations and inclusions*, De Gruyter Ser. Nonlinear Anal. Appl., Vol. 18, Berlin: de Gruyter, 2013.
- [12] Fan, Q., Wu, G-C., Fu, H., *A note on function space and boundedness of the general fractional integral in continuous time random walk*, J. Nonlin. Math. Phys., **29**(1)(2022), 95-102.
- [13] Kamenskii, M., Obukhovskii, V., Zecca, P., *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, De Gruyter, Berlin, 2001.

- [14] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, Netherlands, 2004.
- [15] Kobelev, V., Romanov, E., *Fractional Langevin equation to describe anomalous diffusion*, Prog. Theor. Phys. Suppl., **139**(2000), 470-476.
- [16] Kosztołowicz, T., Dutkiewicz, A., *Subdiffusion equation with Caputo fractional derivative with respect to another function*, Phys. Rev. E., **104**(2021), 014118.
- [17] Li, Y., *The positive solutions of abstract semilinear evolution equations and their applications*, Acta Math. Sin., **39**(5)(1996), 666–672.
- [18] Medved, M., Pospisil, M., Brestovanska, E., *Nonlinear Integral Inequalities Involving Tempered φ -Hilfer Fractional Integral and Fractional Equations with Tempered φ -Caputo Fractional Derivative*, Fractal And Fractional, **7**(2023), 611. <https://doi.org/10.3390/fractalfrac7080611>
- [19] Meerschaert, M. M., Zhang, Y., Baeumer, B., *Tempered anomalous diffusion in heterogeneous systems*, Geophys. Res. Lett., **35**(17)(2008), 1-5.
- [20] Nguyen, M., *Nonlinear Langevin time-delay differential equations with generalized Caputo fractional derivatives*, Filomat, **37**(19)(2023), 6487–6495.
- [21] Norouzi, F., N'Guérékata, G., *A study of ψ -Hilfer fractional differential system with application in financial crisis*, Chaos Solitons Fractals **6**(2021), 100056.
- [22] O'Regan, D., Precup, R., *Theorems of Leray-Schauder type and applications*, Ser. Math. Anal. Appl., Vol. 3, London: Gordon and Breach Science Publishers, 2001.
- [23] Samko, S. G., Kilbas, A. A., Marichev, O. I., *Fractional integrals and derivatives*, Theory and Applications, New York: Gordon and Breach, 1993.
- [24] Yang, X. J., *General fractional derivatives: theory, methods and applications*, CRC Press, Boca Raton, 2019.
- [25] Zentar, O., Al Horani, M., Ziane, M., *A study of a coupled system involving tempered Caputo derivatives with respect to functions*, Rendiconti Del Circolo Matematico Di Palermo Series 2, **74**(2025), 1-19.
- [26] Zentar, O., Ziane, M., Al Horani, M., *Theoretical study of a φ -Hilfer fractional differential system in Banach spaces*, Can. Math. Bull., **67**(3)(2024), 742–759.
- [27] Zentar, O., Ziane, M., Al Horani, M., Zitouni, I., *Theoretical study of a class of ζ -Caputo fractional differential equations in a Banach space*, J. Appl. Anal. Comput., **14**(5)(2024), 2808-2821.
- [28] Ziane, M., *On the solution set for weighted fractional differential equations in Banach spaces*, Differ. Equ. Dyn. Syst., **28**(2)(2020), 419-430.

Oualid Zentar 

Department of Mathematics, University of Tiaret, Tiaret, Algeria
 Laboratory of Research in Artificial Intelligence and Systems (LRAIS),
 University of Tiaret, Algeria

e-mail: oualid.zentar@univ-tiaret.dz

Mohamed Ziane 

Department of Mathematics, University of Tiaret, Tiaret, Algeria
Laboratory of Research in Artificial Intelligence and Systems (LRAIS),
University of Tiaret, Algeria
e-mail: mohamed.ziane@univ-tiaret.dz

Mohammed Al Horani 

Department of Mathematics, University of Jordan,
Amman, 11942, Jordan
e-mail: horani@ju.edu.jo

Existence, attractivity and controllability results for integro-differential equations with state-dependent delay

Imene Medjadj , Abdelkrim Salim  and Mouffak Benchohra 

Abstract. The objective of our research is to investigate the existence, attractivity and controllability of solutions for integro-differential equations with state-dependent delays. We employ a fixed point theorem to establish the existence of these solutions, while also utilizing the concept of measures of noncompactness. In the last section, we give an example to show that the assumed conditions can be verified and to illustrate our results.

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1. Introduction

Many branches of sciences such as physics, fluid dynamic, biology and chemistry are described and characterized by integro-differential equations. The study of integro-differential systems has captured the interest of many researchers. Grimmer was an early pioneer in this field, making significant contributions to the understanding of these complex systems through the use of resolvent operators. His work established the existence of integro-differential systems and provided critical insights into their behavior and dynamics, as documented in several of his papers [20, 21]. Resolvent operators are mathematical tools that help analyze the properties of integro-differential systems by expressing their solutions in terms of initial conditions and input functions. These operators are particularly useful for examining the stability and controllability of integro-differential systems. The existence, uniqueness, stability and controllability

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for various integro-differential problem with finite, infinite and state-dependent delay have been studied by many researchers, see [4, 12, 15, 16, 18].

In 1930, Kuratowski [25] introduced the notion of a measure of noncompactness, which has been proven invaluable in functional analysis, particularly in metric fixed point theory and the theory of operator equations in Banach spaces. Moreover, this concept plays a crucial role in exploring the existence of solutions across various types of equations, including ordinary and partial differential equations, as well as integral and integro-differential equations. More detailed information on this approach can be found in the works of Akhmerov et al. [2], Benchohra et al. [9, 10, 13, 14, 15], Banaś [4, 5, 6, 7].

Controllability theory is essential for understanding the behavior and dynamics of abstract control systems. Its primary goal is to identify a suitable control function that can guide the system’s state to a desired final state. Exact controllability refers to the ability to steer the system precisely to the target state, while approximate controllability involves guiding the system to a neighborhood arbitrarily close to the final state. Due to the inherent uncertainty or imprecision in real-world systems, approximate controllability is often more practical and desirable. Over the years, numerous researchers have examined the approximate or exact controllability of control systems, resulting in a several published studies [8, 11, 19, 26, 27, 29, 30].

This study is dedicated to demonstrating the existence, attractivity and controllability of the solutions for an integro-differential equation. Our analysis will be conducted within the Banach space of real functions that are defined, continuous, and bounded over the real axis \mathbb{R} . Specifically, we will address the following problem:

$$z'(\delta) = Az(\delta) + \int_0^\delta B(\delta - s)z(s)ds + \Psi(\delta, z_{\rho(\delta, z_\delta)}), \tag{1.1}$$

$$\int_0^\delta \zeta(\delta, s, z_{\rho(s, z_s)}) ds), \delta \in \mathfrak{R} := [0, +\infty),$$

$$z(\delta) = \varpi(\delta), \delta \in (-\infty, 0],$$

where $(\mathfrak{S}, |\cdot|)$ is a real Banach space, $\Psi : \mathfrak{R} \times \mathfrak{Q} \times \mathfrak{S} \rightarrow \mathfrak{S}$ is given function, $\zeta : \Gamma \times \mathfrak{Q} \rightarrow \mathfrak{S}$ is a continuous function with $\Gamma = \{(\delta, s) \in \mathfrak{R} \times \mathfrak{R}, \delta \leq s\}$, A is the infinitesimal generator of a C_0 -semigroup $(\Upsilon(\delta))_{\delta \geq 0}$ on \mathfrak{S} , and $B(\delta)$ is linear closed operator on \mathfrak{S} with $D(A) \subset D(B)$, $\varpi \in \mathfrak{Q}$. $\rho : \mathfrak{R} \times \mathfrak{Q} \rightarrow \mathbb{R}$, \mathfrak{Q} is the phase space which will be defined later. For any function z defined on \mathbb{R} and any $\delta \in \mathfrak{R}$, we denote by z_δ the element of \mathfrak{Q} defined by

$$z_\delta(\chi) = z(\delta + \chi), \chi \in (-\infty, 0].$$

Next, we study the controllability of the following problem:

$$z'(\delta) = Az(\delta) + \int_0^\delta B(\delta - s)z(s)ds + \Psi(\delta, z_{\rho(\delta, z_\delta)}),$$

$$\int_0^\delta \zeta(\delta, s, z_{\rho(s, z_s)}) ds + \mathcal{C}u(\delta), \text{ a.e. } \delta \in \mathfrak{R}, \quad (1.2)$$

$$z(\delta) = \varpi(\delta), \quad \delta \in (-\infty, 0],$$

where $u \in L^2(\mathfrak{R}, H)$ is the control function, H is the Banach space of admissible control functions and \mathcal{C} is bounded linear operator from H to \mathfrak{S} .

The following is the structure of our paper. In Section 2, we begin by presenting some background and preliminary information. Section 3 delves into the study of the existence of mild solutions for the system (1.1). Building on this basis, we move to Section 4, where we establish adequate requirements for the system's attractivity. In Section 5, we delve to the study of the controllability of (1.2). Finally, we offer an example as a sample application to highlight the practical relevance of our findings.

2. Preliminaries

By \mathbb{k} we denote the Banach space of all bounded and continuous functions from \mathbb{R} into \mathfrak{S} with

$$\|z\|_{\mathbb{k}} = \sup_{\delta \in \mathbb{R}} |z(\delta)|.$$

By $\widehat{\mathbb{k}}$ we denote the Banach space of all bounded and continuous functions from \mathfrak{R} into \mathfrak{S} with

$$\|z\|_{\widehat{\mathbb{k}}} = \sup_{\delta \in \mathfrak{R}} |z(\delta)|.$$

By \widehat{B} we denote the Banach space $D(A)$ with

$$\|z\|_{\widehat{B}} = \|Az\| + \|z\|, z \in \widehat{B}$$

Now let us recall some information about partial integro-differential equations and resolvent operator.

We consider the following problem :

$$z'(\delta) = Az(\delta) + \int_0^\delta B(\delta - s)z(s)ds, \text{ for } \delta \geq 0, \quad (2.1)$$

$$z(0) = z_0 \in \mathfrak{S}.$$

Definition 2.1. [20] *A resolvent operator for problem (2.1) is a bounded linear operator valued function $R(\delta) \in \mathcal{L}(\mathfrak{S})$ for $\delta \geq 0$, satisfying the following properties:*

1. $R(0) = I$ (identity map of \mathfrak{S}) and $\|R(\delta)\| \leq Me^{\beta\delta}$, for some constants $M > 0$ and $\beta \in \mathbb{R}$.
2. For each $z \in \mathfrak{S}$, $R(\delta)z$ is strongly continuous for $\delta \geq 0$.
3. For any $z \in \mathfrak{S}$, $R(\cdot)z \in \mathcal{C}^1([0, +\infty), \mathfrak{S}) \cap \mathcal{C}([0, +\infty), \widehat{B})$ and

$$R'(\delta)z = AR(\delta)z + \int_0^\delta B(\delta - s)R(s)zds = R(\delta)Az + \int_0^\delta R(\delta - s)B(s)zds \text{ for } \delta \geq 0.$$

The existence of resolvent operator has been discussed in [20, 21] under the following assumptions.

- (R₁) A is the infinitesimal generator of strongly continuous semigroup $(\Upsilon(\delta))_{\delta \geq 0}$ on \mathfrak{S} .
- (R₂) For all $\delta \geq 0$, $B(\delta)z$ is closed linear operator from $D(A) \rightarrow \mathfrak{S}$ and $B(\delta) \in \mathcal{L}(\widehat{B}, \mathfrak{S})$. For any $z \in \widehat{B}$, the map $\delta \rightarrow B(\delta)$ is bounded, differentiable and the derivative $\delta \rightarrow B'(\delta)z$ is bounded uniformly continuous on \mathbb{R}^+ .

Theorem 2.2. *If the condition (R₁), (R₂) hold, then the problem (2.1) has a unique resolvent operator.*

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.3. [2, 5] *Let \mathfrak{S} be a Banach space and $\Omega_{\mathfrak{S}}$ the bounded subsets of \mathfrak{S} . The Kuratowski measure of noncompactness is the map $\varkappa : \Omega_{\mathfrak{S}} \rightarrow [0, \infty)$ defined by*

$$\varkappa(\widehat{\mathfrak{Z}}) = \inf\{\epsilon > 0 : \widehat{\mathfrak{Z}} \subseteq \bigcup_{i=1}^n \widehat{\mathfrak{Z}}_i \text{ and } \text{diam}(\widehat{\mathfrak{Z}}_i) \leq \epsilon\}; \text{ here } \widehat{\mathfrak{Z}} \in \Omega_{\mathfrak{S}},$$

and verifies:

- $\varkappa(\widehat{\mathfrak{Z}}) = 0 \Leftrightarrow \widehat{\mathfrak{Z}}$ is compact ($\widehat{\mathfrak{Z}}$ is relatively compact);
- $\varkappa(\widehat{\mathfrak{Z}}) = \varkappa(\overline{\widehat{\mathfrak{Z}}})$;
- $\mathfrak{Z} \subset \widehat{\mathfrak{Z}} \Rightarrow \varkappa(\mathfrak{Z}) \leq \varkappa(\widehat{\mathfrak{Z}})$;
- $\varkappa(\mathfrak{Z} + \widehat{\mathfrak{Z}}) \leq \varkappa(\mathfrak{Z}) + \varkappa(\widehat{\mathfrak{Z}})$;
- $\varkappa(c\widehat{\mathfrak{Z}}) = |c|\varkappa(\widehat{\mathfrak{Z}}); c \in \mathbb{R}$;
- $\varkappa(\text{conv}\widehat{\mathfrak{Z}}) = \varkappa(\widehat{\mathfrak{Z}})$.

Here, $\overline{\widehat{\mathfrak{Z}}}$ and $\text{conv}\widehat{\mathfrak{Z}}$ denote the closure and convex hull of bounded set $\widehat{\mathfrak{Z}}$, respectively.

Lemma 2.4. *Let \mathcal{D} be a bounded subset of \mathfrak{S} . Then, for each $\epsilon > 0$, there exists a sequence $\{U_n\}_{n=1}^\infty \subset \mathcal{D}$ such that*

$$\varkappa(\mathcal{D}) \leq 2\varkappa(\{U_n\}_{n=1}^\infty) + \epsilon.$$

Lemma 2.5. *If $\{U_n\}_{n=0}^\infty \subset L^1$ is uniformly integrable, then the function $\delta \rightarrow \varkappa(\{U_n(\delta)\}_{n=0}^\infty)$ is measurable and*

$$\varkappa\left(\left\{\int_0^\delta U_n(s)ds\right\}\right) \leq 2 \int_0^\delta \varkappa(\{U_n(s)\}_{n=0}^\infty)ds.$$

For any arbitrary nonempty bounded subset D of the space $\widehat{\mathbb{k}}$ and a function h , we define

$$\eta^\infty(h, \epsilon) = \sup\{\|h(\delta) - h(s)\|; \delta, s \in \mathfrak{R}, |\delta - s| \leq \epsilon\},$$

and

$$\eta^\infty(D, \epsilon) = \sup\{\eta^\infty(h, \epsilon); h \in D\}, \eta_0^\infty(D) = \lim_{\epsilon \rightarrow 0} \eta^\infty(D, \epsilon).$$

Now we consider

$$\begin{aligned} \bar{\varkappa}_\infty(D) &= \lim_{T \rightarrow \infty} \sup\{\varkappa(D(\delta)) : \delta \in [0, T]\}, \\ \text{diam}D(\delta) &= \sup\{\|x(\delta) - y(\delta)\|_{\mathfrak{S}} : x, y \in D\}, \end{aligned}$$

$$c(D) = \lim_{\delta \rightarrow \infty} \text{diam}D(\delta).$$

Finally, we can consider the following function \varkappa^* define by:

$$\varkappa^*(D) = \eta_0^\infty(D) + \bar{\varkappa}_\infty(D) + c(D).$$

The function \varkappa^* is a measure of noncompactness in the space $\widehat{\mathbb{K}}$, for more details see for instance [3].

Now we give the definition of a Meir-Keeler contraction.

Definition 2.6. [28] *Let (X, d) be a metric space. A mapping N on X is called a Meir-Keeler contraction if, for each $\epsilon > 0$, there exists $\lambda > 0$, such that*

$$\epsilon \leq d(x, y) < \epsilon + \lambda \Rightarrow d(Nx, Ny) < \epsilon, \text{ for all } x, y \in X.$$

Definition 2.7. [1] *Let Y be a nonempty subset of Banach space \mathfrak{S} , and μ be a measure of noncompactness on \mathfrak{S} . An operator $N : Y \rightarrow Y$ is called a Meir-Keeler condensing operator if, for each $\epsilon > 0$, there exists $\lambda > 0$, such that $\epsilon \leq \mu(\Omega) < \epsilon + \lambda \Rightarrow \mu(N\Omega) < \epsilon$, for any bounded set Ω of Y .*

Now we give the fixed point theorem with respect a Meir-Keeler condensing operator.

Theorem 2.8. [1] *Let \mathcal{D} be a nonempty, bounded, closed, convex subset of a Banach space \mathfrak{S} . and μ be an arbitrary measure of noncompactness on \mathfrak{S} . If $N : \mathcal{D} \rightarrow \mathcal{D}$ is a continuous and Meir-Keeler condensing operator, then N has at least one fixed point and the set of all fixed points of N in \mathcal{D} is compact.*

In this segment, we will utilize an axiomatic definition of the phase space Ω as introduced by Hale and Kato in [22]. We will adhere to the terminology employed in [24]. Consequently, $(\Omega, |\cdot|_\Omega)$ will constitute a seminormed linear space of functions that map $(-\infty, 0]$ into \mathfrak{S} and adhere to the specified axioms:

- (A₁) If $\mathfrak{w} : (-\infty, \zeta) \rightarrow \mathfrak{S}, \zeta > 0$, is continuous on \mathfrak{R} and $\mathfrak{w}_0 \in \Omega$, then for every $\delta \in \mathfrak{R}$:
 - (i) $\mathfrak{w}_\delta \in \Omega$;
 - (ii) $\exists \widehat{l}_1 > 0$ such that $|\mathfrak{w}(\delta)| \leq \widehat{l}_1 \|\mathfrak{w}_\delta\|_\Omega$;
 - (iii) $\exists \widehat{l}_2(\cdot), \widehat{l}_3(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ independent of \mathfrak{w} with \widehat{l}_2 continuous and bounded, and \widehat{l}_3 locally bounded where:

$$\|\mathfrak{w}_\delta\|_\Omega \leq \widehat{l}_2(\delta) \sup\{|\mathfrak{w}(\varrho)| : 0 \leq \varrho \leq \delta\} + \widehat{l}_3(\delta) \|\mathfrak{w}_0\|_\Omega.$$

(A₂) For the function \mathfrak{w} in (A₁), \mathfrak{w}_δ is a Ω -valued continuous function on \mathfrak{R} .

(A₃) The space Ω is complete.

Denote

$$\widehat{l}_2 = \sup\{\widehat{l}_2(\delta) : \delta \in \mathfrak{R}\},$$

and

$$\widehat{l}_3 = \sup\{\widehat{l}_3(\delta) : \delta \in \mathfrak{R}\}.$$

3. Existence of mild solutions

Definition 3.1. We say that a function $z \in \mathbb{k}$ is a mild solution of problem (1.1) if $z(\delta) = \varpi(\delta), \delta \in (-\infty, 0]$ and

$$z(\delta) = R(\delta)\varpi(0) + \int_0^\delta R(\delta - s)\Psi(s, z_{\rho(s, z_s)}, \int_0^s \zeta(s, \theta, z_{\rho(\theta, z_\theta)}) d\theta) ds, \text{ a.e. } \delta \in \mathfrak{R}. \tag{3.1}$$

Set

$$\Lambda(\rho^-) = \{\rho(\varrho, \varpi) : (\varrho, \varpi) \in \mathfrak{R} \times \mathfrak{Q}, \rho(\varrho, \varpi) \leq 0\}.$$

Let $\rho : \mathfrak{R} \times \mathfrak{Q} \rightarrow \mathbb{R}$ be continuous and:

(EC_ϖ) The function $\delta \rightarrow \varpi_\delta$ is continuous from $\Lambda(\rho^-)$ into \mathfrak{Q} and there exists a continuous and bounded function $\mathcal{U}^\varpi : \Lambda(\rho^-) \rightarrow (0, \infty)$ where

$$\|\varpi_\delta\| \leq \mathcal{U}^\varpi(\delta)\|\varpi\| \text{ for every } \delta \in \Lambda(\rho^-).$$

Remark 3.2. The condition (EC_ϖ), is frequently verified by functions continuous and bounded. For more details, see for instance [24].

Lemma 3.3. [23] If $z : \mathbb{R} \rightarrow \mathfrak{S}$ is a function such that $z_0 = \varpi$, then

$$\|z_\varrho\|_\Omega \leq (\widehat{l}_3 + \mathcal{U}^\varpi)\|\varpi\|_\Omega + \widehat{l}_2 \sup\{|z(\chi)|; \chi \in [0, \max\{0, \varrho\}]\}, \varrho \in \Lambda(\rho^-) \cup \mathfrak{R},$$

where $\mathcal{U}^\varpi = \sup_{\delta \in \Lambda(\rho^-)} \mathcal{U}^\varpi(\delta)$.

Let us introduce the following hypotheses:

(EC_1) $A : D(A) \subset \mathfrak{S} \rightarrow \mathfrak{S}$ is the infinitesimal generator of a uniformly continuous C_0 semigroup $(\Upsilon(\delta))_{\delta \geq 0}$.

(EC_2) The conditions $(R_1), (R_2)$ hold, and there exist constants $M \geq 1, \alpha > 0$ such that

$$\sup\{\|R(\delta)\|_{B(\mathfrak{S})} : \delta \geq 0\} \leq Me^{-\alpha\delta}.$$

(EC_3) The function $\Psi : \mathfrak{R} \times \mathfrak{Q} \times \mathfrak{S} \rightarrow \mathfrak{S}$ is Carathéodory and satisfies the following conditions:

i) There exists a function $k_1 \in L^1(\mathfrak{R})$, such that :

$$\|\Psi(\delta, x, y) - \Psi(\delta, x^*, y^*)\| \leq k_1(\delta)(\|x - x^*\| + \|y - y^*\|), \delta \in \mathfrak{R}, x, x^* \in \mathfrak{Q}, y, y^* \in \mathfrak{S}.$$

ii) For each $\delta \in \mathfrak{R}$, we have and

$$\lim_{\delta \rightarrow \infty} \sup_{\delta \in \mathfrak{R}} \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) ds = 0.$$

iii) For each bounded set $\mathcal{D}_1 \subset \mathfrak{Q}$ and $\mathcal{D}_2 \subset \mathfrak{S}$, and each $\delta \in \mathfrak{S}$ we have

$$\varkappa(\Psi(\delta, \mathcal{D}_1, \mathcal{D}_2)) \leq k_1(\delta) \left[\sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}_1(\varrho)) + \varkappa(\mathcal{D}_2) \right].$$

(EC₄) i) There exists a function $k_2 \in L^1(\mathfrak{R})$ such that:

$$\|\zeta(\delta, s, z) - \zeta(\delta, s, z^*)\| \leq k_2(s)\|z - z^*\|,$$

for each $(\delta, s) \in \Gamma$ and for all $z, z^* \in \mathfrak{Q}$.

ii) For each bounded set $\mathcal{D} \subset \mathfrak{Q}$, we have

$$\varkappa(\zeta(\delta, s, \mathcal{D})) \leq k_2(\delta) \sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}(\varrho)).$$

Remark 3.4. We denote by

$$k_i^* = \|k_i\|_{L^1(\mathfrak{R})}$$

for

$$i = 1, 2, \Psi^* = \sup_{\delta \in \mathfrak{R}} \int_0^\delta \Psi(s, 0, 0) ds, \zeta^* = \sup\{\|\zeta(\delta, s, 0)\|, (\delta, s) \in \Gamma\}.$$

Theorem 3.5. Assume that (EC₁) – (EC₄), (EC_∞) hold. If $8M\widehat{M}_2k_1^*(1+k_2^*) < 1$, then the problem (1.1) has at least one mild solution on \mathbb{k} .

Proof. We transform the problem (1.1) into a fixed point problem. Consider the operator $\Xi : \mathbb{k} \rightarrow \mathbb{k}$ defined by :

$$\Xi(z) := \begin{cases} \varpi(\delta), & \text{if } \delta \in (-\infty, 0], \\ R(\delta)\varpi(0) + \int_0^\delta R(\delta-s)\Psi(s, z_{\rho(s, z_s)}, \int_0^s \zeta(s, \theta, z_{\rho(\theta, z_\theta)})d\theta)ds, & \text{if } \delta \in \mathfrak{R}. \end{cases}$$

Let $\vartheta(\cdot) : \mathbb{R} \rightarrow \mathfrak{S}$ be the function defined by:

$$\vartheta(\delta) = \begin{cases} \varpi(\delta), & \text{if } \delta \in (-\infty, 0]; \\ R(\delta)\varpi(0), & \text{if } \delta \in \mathfrak{R}. \end{cases}$$

Then $\vartheta_0 = \varpi$. For each $h \in \mathbb{k}$ with $h(0) = 0$, we denote by \bar{h} the function

$$\bar{h}(\delta) = \begin{cases} 0; & \text{if } \delta \in (-\infty, 0]; \\ h(\delta); & \text{if } \delta \in \mathfrak{R}. \end{cases}$$

If z satisfies (3.1), we can decompose it as $z(\delta) = h(\delta) + \vartheta(\delta)$, $\delta \in \mathfrak{R}$, which implies $z_\delta = h_\delta + \vartheta_\delta$ for every $\delta \in \mathfrak{R}$ and the function $h(\cdot)$ satisfies

$$h(\delta) = \int_0^\delta R(\delta-s)\Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}, \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)})d\theta)ds.$$

Set

$$\widehat{\mathbb{k}}_0 = \{h \in \widehat{\mathbb{k}} : h(0) = 0\},$$

and let

$$\|h\|_{\widehat{\mathbb{k}}_0} = \sup\{|h(\delta)| : \delta \in \mathfrak{R}\}, h \in \widehat{\mathbb{k}}_0.$$

$\widehat{\mathbb{k}}_0$ is a Banach space with the norm $\|\cdot\|_{\widehat{\mathbb{k}}_0}$.

We define the operator $\mathcal{A} : \widehat{\mathbb{k}}_0 \rightarrow \widehat{\mathbb{k}}_0$ by:

$$\mathcal{A}(h)(\delta) := \begin{cases} 0, & \text{if } \delta \leq 0, \\ \int_0^\delta R(\delta - s) \Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}, \\ \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta) ds, & \text{if } \delta \in \mathfrak{R}, \end{cases}$$

The operator \mathcal{A} maps $\widehat{\mathbb{k}}_0$ into $\widehat{\mathbb{k}}_0$, for each $\delta \in \mathfrak{R}$ we have

$$\begin{aligned} & |\mathcal{A}(h)(\delta)| \\ & \leq \left\| \int_0^\delta R(\delta - s) [\Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}, \right. \\ & \quad \left. \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta \right. \\ & \quad \left. - \Psi(s, 0, 0) + \Psi(s, 0, 0)] ds \right\| \\ & \leq M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \|h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}\| ds \\ & \quad + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \left\| \int_0^s (\zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) \right. \\ & \quad \left. - \zeta(s, \theta, 0) + \zeta(s, \theta, 0) \right\| d\theta ds \\ & \quad + M \int_0^\delta e^{-\alpha(\delta-s)} \|\Psi(s, 0, 0)\| ds. \end{aligned}$$

Thus,

$$\begin{aligned} |\mathcal{A}(h)(\delta)| & \leq M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \|h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}\| ds \\ & \quad + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s k_2(\theta) \|h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}\| d\theta ds \\ & \quad + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s \|\zeta(s, \theta, 0)\| d\theta ds \\ & \quad + M \int_0^\delta e^{-\alpha(\delta-s)} \|\Psi(s, 0, 0)\| ds. \end{aligned}$$

Then,

$$\begin{aligned} |\mathcal{A}(h)(\delta)| & \leq M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) (\widehat{l}_2 |h(s)| + (\widehat{l}_3 + \mathcal{U}^\varpi + \widehat{l}_2 M e^{-\alpha s} \widehat{l}_1) \|\varpi\|_\Omega) ds \\ & \quad + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s k_2(\theta) (\widehat{l}_2 |h(\theta)| \\ & \quad + (\widehat{l}_3 + \mathcal{U}^\varpi + \widehat{l}_2 M e^{-\alpha \theta} \widehat{l}_1) \|\varpi\|_\Omega) d\theta ds \end{aligned}$$

$$\begin{aligned}
& + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s \|\zeta(s, \theta, 0)\| d\theta ds \\
& + M \int_0^\delta e^{-\alpha(\delta-s)} \|\Psi(s, 0, 0)\| ds.
\end{aligned}$$

Let $C_1 = (\widehat{l}_3 + \mathcal{U}^\varpi + \widehat{l}_2 M \widehat{l}_1) \|\varpi\|_\Omega$. Then, we have

$$\begin{aligned}
|\mathcal{A}(h)(\delta)| & \leq M k_1^* \widehat{l}_2 \|h\|_{\widehat{\mathbb{k}}_0} + M C_1 k_1^* + M k_1^* k_2^* \widehat{l}_2 \|h\|_{\widehat{\mathbb{k}}_0} \\
& \quad + M C_1 k_1^* k_2^* + M k_1^* \zeta^* + M \Psi^* := C^* \leq C^*.
\end{aligned}$$

Hence, $\mathcal{A}(h)(\delta) \in \widehat{\mathbb{k}}_0$.

Let $\tau > 0$ be such that $\tau \geq \frac{M C_1 k_1^* + M C_1 k_1^* k_2^* + M k_1^* \zeta^* + M \Psi^*}{1 - M k_1^* \widehat{l}_2 (1 + k_2^*)}$, and σ_τ be the closed ball in $\widehat{\mathbb{k}}_0$ centered at the origin and of radius τ . Let $z \in \sigma_\tau$ and $\delta \in \mathfrak{R}$. Then,

$$|\mathcal{A}(z)(\delta)| \leq M k_1^* \widehat{l}_2 \tau + M C_1 k_1^* + M k_1^* k_2^* \widehat{l}_2 \tau + M C_1 k_1^* k_2^* + M k_1^* \zeta^* + M \Psi^*$$

Thus,

$$\|\mathcal{A}(z)(\delta)\|_{\widehat{\mathbb{k}}_0} \leq \tau,$$

then $\mathcal{A}(\sigma_\tau) \subset \sigma_\tau$.

The proof can be given by following steps.

Step 1: \mathcal{A} is continuous.

Let $\{h^n\}_{n \in \mathbb{N}}$ be a sequence such that $h^n \rightarrow h$ in σ_τ . Firstly, we are going to study the convergence of the sequences $(h_{\rho(s, h_s^n)}^n)_{n \in \mathbb{N}}$, $s \in \mathfrak{R}$.

If $s \in \mathfrak{R}$ is such that $\rho(s, h_s) > 0$, then we have,

$$\begin{aligned}
\|h_{\rho(s, h_s^n)}^n - h_{\rho(s, h_s)}\|_\Omega & \leq \|h_{\rho(s, h_s^n)}^n - h_{\rho(s, h_s^n)}\|_\Omega + \|h_{\rho(s, h_s^n)} - h_{\rho(s, h_s)}\|_\Omega \\
& \leq \widehat{l}_2 \|h_n - h\|_{\sigma_\tau} + \|h_{\rho(s, h_s^n)} - h_{\rho(s, h_s)}\|_\Omega,
\end{aligned}$$

which proves that $h_{\rho(s, h_s^n)}^n \rightarrow h_{\rho(s, h_s)}$ in Ω as $n \rightarrow \infty$ for every $s \in \mathfrak{R}$ such that $\rho(s, h_s) > 0$. Similarly, is $\rho(s, h_s) < 0$, we get

$$\|h_{\rho(s, h_s^n)}^n - h_{\rho(s, h_s)}\|_\Omega = \|\phi_{\rho(s, h_s^n)}^n - \phi_{\rho(s, h_s)}\|_\Omega = 0,$$

which also shows that $h_{\rho(s, h_s^n)}^n \rightarrow h_{\rho(s, h_s)}$ in Ω as $n \rightarrow \infty$ for every $s \in \mathfrak{R}$ such that $\rho(s, h_s) < 0$. Using the previous arguments, we can prove that $h_{\rho(s, h_s)}^n \rightarrow \phi$ for every $s \in \mathfrak{R}$ such that $\rho(s, h_s) = 0$.

$$\begin{aligned}
& |\mathcal{A}(h^n)(\delta) - \mathcal{A}(h)(\delta)| \\
& \leq \left\| \int_0^\delta R(\delta - s) \Psi(s, h_{\rho(s, h_s^n + \vartheta_s)}^n + \vartheta_{\rho(s, h_s^n + \vartheta_s)}), \right. \\
& \quad \left. \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta^n + \vartheta_\theta)}^n + \vartheta_{\rho(\theta, h_\theta^n + \vartheta_\theta)}) d\theta \right\| ds
\end{aligned}$$

$$\begin{aligned}
& - \int_0^\delta R(\delta - s) \Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}), \\
& \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta) ds \| \\
\leq & M \int_0^\delta e^{-\alpha(\delta - s)} \| \Psi(s, h_{\rho(s, h_s^n + \vartheta_s)} + \vartheta_{\rho(s, h_s^n + \vartheta_s)}), \\
& \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta^n + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta^n + \vartheta_\theta)}) d\theta) ds \\
& - \Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}, \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta) ds \|.
\end{aligned}$$

Then by (EC_3) and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(h^n) - \mathcal{A}(h)\|_{\widehat{\mathbb{K}}_0} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus \mathcal{A} is continuous.

Step 2: $\mathcal{A}(\sigma_\tau)$ is equicontinuous. Let $B = [0, b]$ be a compact of $[0, +\infty)$, for $b > 0$.

Let $\delta_1, \delta_2 \in B$, with $\delta_2 > \delta_1$, we have

$$\begin{aligned}
& |\mathcal{A}(h)(\delta_1) - \mathcal{A}(h)(\delta_2)| \\
& \leq \left| \int_0^{\delta_1} R(\delta_1 - s) \Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}), \right. \\
& \quad \left. \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta) ds \right| \\
& \quad - \int_0^{\delta_2} R(\delta_2 - s) \Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}), \\
& \quad \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta) ds \| \\
& \leq \int_0^{\delta_1} \|R(\delta_1 - s) - R(\delta_2 - s)\| \| \Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}), \\
& \quad \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta) \\
& \quad - \Psi(s, 0, 0) + \Psi(s, 0, 0) \| ds \\
& \quad + \int_{\delta_1}^{\delta_2} \|R(\delta_2 - s)\| \| \Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}), \\
& \quad \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta) \\
& \quad - \Psi(s, 0, 0) + \Psi(s, 0, 0) \| ds \\
& \leq \int_0^{\delta_1} \|R(\delta_1 - s) - R(\delta_2 - s)\| k_1(s) (\widehat{l}_2 \mathbf{r} + C_1) ds
\end{aligned}$$

$$\begin{aligned}
& + \int_0^{\delta_1} \|R(\delta_1 - s) - R(\delta_2 - s)\| \|\Psi(s, 0, 0)\| ds \\
& + \int_0^{\delta_1} \|R(\delta_1 - s) - R(\delta_2 - s)\| k_1(s) \int_0^s k_2(\theta) (\widehat{l}_2 \mathbf{r} + C_1) d\theta ds \\
& + \int_0^{\delta_1} \|R(\delta_1 - s) - R(\delta_2 - s)\| k_1(s) \int_0^s \|\zeta(s, \theta, 0)\| d\theta ds \\
& + M \int_{\delta_1}^{\delta_2} e^{-\alpha(\delta_2 - s)} k_1(s) (\widehat{l}_2 \mathbf{r} + C_1) ds \\
& + M \int_{\delta_1}^{\delta_2} e^{-\alpha(\delta_2 - s)} \|\Psi(s, 0, 0)\| ds \\
& + M \int_{\delta_1}^{\delta_2} e^{-\alpha(\delta_2 - s)} k_1(s) \int_0^s k_2(\theta) (\widehat{l}_2 \mathbf{r} + C_1) d\theta ds \\
& + M \int_{\delta_1}^{\delta_2} e^{-\alpha(\delta_2 - s)} k_1(s) \int_0^s \|\zeta(s, \theta, 0)\| d\theta ds.
\end{aligned}$$

By (EC_1) , $|\mathcal{A}(h)(\delta_1) - \mathcal{A}(h)(\delta_2)| \rightarrow 0$ as $\delta_1 \rightarrow \delta_2$. Then $\mathcal{A}(\sigma_\tau)$ is equicontinuous.

Step 3: We show that $\mathcal{A} : \sigma_\tau \rightarrow \sigma_\tau$ is a Meir-Keeler condensing operator. For $\mathcal{D} \subset \sigma_\tau$, $\mathcal{A}(\mathcal{D})$ is bounded. Then there exists a countable set $\mathcal{D}_r = \{h_n\}_{n=1}^\infty \subset \mathcal{D}$.

Claim 1. Using the fact that $\mathcal{A}(\sigma_\tau)$ is equicontinuous and by the definition of $\eta_0^\infty(\cdot)$, we get that

$$\eta_0^\infty(\mathcal{A}(\mathcal{D})) = 0.$$

Claim 2. We have

$$\begin{aligned}
\mathcal{K}(\mathcal{A}(\mathcal{D})(\delta)) & \leq 2\mathcal{K}\left(\int_0^\delta R(\delta - s) \Psi(s, h_{\rho(s, h_s^n + \vartheta_s)}^n) + \vartheta_{\rho(s, h_s^n + \vartheta_s)}, \right. \\
& \quad \left. \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)}^n + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta ds \right)_{n=0}^\infty + \epsilon \\
& \leq 4\widehat{l}_2 M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \left(\sup_{\varrho \in (-\infty, 0]} \mathcal{K}(\mathcal{D}_0(\varrho))(s) \right. \\
& \quad \left. + \mathcal{K}\left(\int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)}^n + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta\right) ds \right) + \epsilon \\
& \leq 4\widehat{l}_2 M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \sup_{\varrho \in (-\infty, 0]} \mathcal{K}(\mathcal{D}_0(\varrho)) ds \\
& \quad + 8\widehat{l}_2 M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) \left(\int_0^s k_2(\theta) \sup_{\varrho \in (-\infty, 0]} \mathcal{K}(\mathcal{D}_0(\varrho)) d\theta \right) ds + \epsilon \\
& \leq 8\widehat{l}_2 M \int_0^\delta e^{-\alpha(\delta - s)} k_1(s) (1 + k_2^*) \sup_{\varrho \in (-\infty, 0]} \mathcal{K}(\mathcal{D}_0(\varrho)) ds + \epsilon.
\end{aligned}$$

Then since ϵ is arbitrary, we get

$$\varkappa(\mathcal{A}(\mathcal{D}))(\delta) \leq 8M\widehat{l}_2k_1^*(1+k_2^*) \sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}_0(\varrho)).$$

Then, $\bar{\varkappa}_\infty(\mathcal{A}\mathcal{D}) \leq 8\widehat{l}_2Mk_1^*(1+k_2^*)\bar{\varkappa}_\infty(\mathcal{D})$.

Claim 3. We have

$$\begin{aligned} & |h(\delta) - h^*(\delta)| \\ & \leq \left\| \int_0^\delta R(\delta - s)\Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}, \right. \\ & \quad \left. \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)})d\theta \right. \\ & \quad \left. - \int_0^\delta R(\delta - s)\Psi(s, h_{\rho^*(s, h_s^* + \vartheta_s)} + \vartheta_{\rho^*(s, h_s^* + \vartheta_s)}, \right. \\ & \quad \left. \int_0^s \zeta(s, \theta, h_{\rho^*(\theta, h_\theta^* + \vartheta_\theta)} + \vartheta_{\rho^*(\theta, h_\theta^* + \vartheta_\theta)})d\theta \right\| \\ & \leq M \int_0^\delta e^{-\alpha(\delta-s)}k_1(s) \|h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)} - h_{\rho^*(s, h_s^* + \vartheta_s)} - \vartheta_{\rho^*(s, h_s^* + \vartheta_s)}\| ds \\ & \quad + M \int_0^\delta e^{-\alpha(\delta-s)}k_1(s) \int_0^s k_2(\theta) \|h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)} \\ & \quad - h_{\rho^*(\theta, h_\theta^* + \vartheta_\theta)} - \vartheta_{\rho^*(\theta, h_\theta^* + \vartheta_\theta)}\| d\theta ds \\ & \leq M \int_0^\delta e^{-\alpha(\delta-s)}k_1(s)\widehat{l}_2|h(s) - h^*(s)| ds \\ & \quad + M \int_0^\delta e^{-\alpha(\delta-s)}k_1(s) \int_0^s k_2(\theta)\widehat{l}_2|h(\theta) - h^*(\theta)| d\theta ds. \end{aligned}$$

Then,

$$c(\mathcal{A}\mathcal{D}) \leq (M\widehat{l}_2k_1^*(1+k_2^*))c(\mathcal{D}).$$

From Claims 1,2 and 3, we deduce that

$$\varkappa^*(\mathcal{A}(\mathcal{D})) \leq 8(M\widehat{l}_2k_1^*(1+k_2^*))\varkappa^*(\mathcal{D}).$$

For a fixed $\epsilon > 0$, set

$$\lambda = \frac{1 - 8\widehat{l}_2Mk_1^*(1+k_2^*)}{8\widehat{l}_2Mk_1^*(1+k_2^*)}\epsilon.$$

We get that

$$\epsilon \leq \varkappa^*((\mathcal{D})) < \epsilon + \lambda \Rightarrow \varkappa^*(\mathcal{A}(\mathcal{D})) < \epsilon.$$

That means \mathcal{A} is a Meir-Keeler condensing operator, then using Theorem 2.8 we can say that the operator \mathcal{A} has at least fixed point, so the operator $\Xi : \mathbb{k} \rightarrow \mathbb{k}$ has a fixed point $z(\delta) = h(\delta) + \vartheta(\delta)$ and the set of all fixed points of Ξ in \mathbb{k} is compact.

□

4. Attractivity

In this section, we are going to study the attractivity of solutions.

Definition 4.1. [16] *We say that the solutions of problem (1.1) are locally attractive if there exists a closed ball $\overline{B}(\xi^*, \mathbf{r}^*)$ in the space \mathbb{k} , where $\xi^* \in \mathbb{k}$, such that for any solutions ξ and $\tilde{\xi}$ of (1.1) belonging to $\overline{B}(\xi^*, \mathbf{r}^*)$, the following convergence holds:*

$$\lim_{s \rightarrow +\infty} (\xi(s) - \tilde{\xi}(s)) = 0.$$

When the limit is uniform with respect to $\overline{B}(\xi^, \mathbf{r}^*)$, we say that the solutions of (1.1) are uniformly locally attractive.*

Theorem 4.2. *Suppose that the hypotheses $(EC_1) - (EC_4), (EC_\infty)$ hold. If $8M\widehat{l}_2k_1^*(1+k_2^*) < 1$, Then the solutions of problem (1.1) are uniformly locally attractive.*

Proof. Let h^* be a solution of (1.1), for $h \in B(h^*, \mathbf{r}^*)$, by $(EC_1) - (EC_4)$, we have

$$\begin{aligned} & \| \mathcal{A}(h)(\delta) - h^*(\delta) \| \\ &= \| \mathcal{A}(h)(\delta) - \mathcal{A}(h^*)(\delta) \| \\ &\leq M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \widehat{l}_2 |h(s) - h^*(s)| ds \\ &\quad + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s k_2(\theta) \widehat{l}_2 |h(\theta) - h^*(\theta)| d\theta ds \\ &\leq M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \widehat{l}_2 \mathbf{r}^* ds + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s k_2(\theta) \widehat{l}_2 \mathbf{r}^* d\theta ds \\ &\leq (M\widehat{l}_2k_1^*(1+k_2^*))\mathbf{r}^* \\ &\leq \mathbf{r}^*. \end{aligned}$$

Then $\mathcal{A}(B_{\mathbf{r}^*}) \subset B_{\mathbf{r}^*}$.

For each $h, \tilde{h} \in B(h^*, \mathbf{r}^*), \delta \in [0, +\infty)$, we have :

$$\begin{aligned} & \| h(\delta) - \tilde{h}(\delta) \| \\ &= \| \mathcal{A}(h)(\delta) - \mathcal{A}(\tilde{h})(\delta) \| \\ &\leq M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \widehat{l}_2 |h(s) - \tilde{h}(s)| ds \\ &\quad + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s k_2(\theta) \widehat{l}_2 |h(\theta) - \tilde{h}(\theta)| d\theta ds \\ &\leq M\widehat{l}_2\mathbf{r}^* \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) ds + M\widehat{l}_2\mathbf{r}^*k_2^* \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) ds. \end{aligned}$$

We conclude that $\|h(\delta) - \tilde{h}(\delta)\| \rightarrow 0$, as $\delta \rightarrow \infty$. □

5. Controllability

Definition 5.1. *The system (1.2) is said to be controllable if for every initial function $\varpi(\delta) \in \mathfrak{Q}$, and $\widehat{z} \in \mathfrak{S}$ and for $\gamma \in \mathfrak{R}$, there is a control $z \in L^2([0, \gamma], \mathfrak{S})$, such that the mild solution $z(\cdot)$ of the problem (1.2) satisfies the condition $z(\gamma) = \widehat{z}$.*

Let us introduce the following hypotheses:

(EC₅) (i) For some $\gamma \in \mathbb{N}$, the linear operator $\mathfrak{W} : L^2([0, \gamma], H) \rightarrow \mathfrak{S}$ is defined by

$$\mathfrak{W}z = \int_0^\gamma R(\gamma - s)\mathcal{C}u(s)ds,$$

has a pseudo-invertible operator $\widetilde{\mathfrak{W}}^{-1}$ which takes values in $L^2([0, \gamma], H)/\text{Ker}\mathfrak{W}$

(ii) There exists positive constants $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}_1$ such that :

$$\|C\| \leq \mathcal{M}_1 \text{ and } \|\widetilde{\mathfrak{W}}^{-1}\| \leq \mathcal{M}_2.$$

(iii) There exists a function $k_3 \in L^1(\mathfrak{R})$, and a positive constant k_4 such that for any bounded sets $\mathcal{D} \subset \mathfrak{S}$, and $\mathcal{D}' \subset H$, we have

$$\varkappa((\widetilde{\mathfrak{W}}^{-1}\mathcal{D})(\delta)) \leq k_3(\delta)\varkappa(\mathcal{D}),$$

and

$$\varkappa(\mathcal{C}(\mathcal{D}')) \leq k_4\varkappa(\mathcal{D}'(\delta)),$$

where $\delta \in \mathfrak{R}$.

Remark 5.2. We denote by $k_3^* = \|k_3\|_{L^1(\mathfrak{R})}$.

Theorem 5.3. *Suppose that the hypotheses (EC₁) – (EC₅), (EC _{ϖ}) hold and*

$$8M\widehat{l}_2(k_1^* + k_2^*k_3^*)(1 + \gamma\mathcal{M}_2\mathcal{M}_1M) < 1.$$

Then problem (1.2) is controllable.

Proof. We define a control

$$u_z = \widetilde{\mathfrak{W}}^{-1}(\widehat{z} - R(\gamma)\varpi(0) - \int_0^\gamma R(\gamma - s)\Psi(s, z_{\rho(s, z_s)}, \int_0^s \zeta(s, \theta, z_{\rho(\theta, z_\theta)})d\theta)ds)(\delta).$$

Now we shall show that the operator $\Xi : \mathbb{k} \rightarrow \mathbb{k}$ defined by :

$$\Xi(z) := \begin{cases} \varpi(\delta), & \text{if } \delta \in (-\infty, 0], \\ R(\delta)\varpi(0) + \int_0^\delta R(\delta - s)\Psi(s, z_{\rho(s, z_s)}, \int_0^s \zeta(s, \theta, z_{\rho(\theta, z_\theta)})d\theta)ds \\ + \int_0^\delta R(\delta - s)\mathcal{C}u_z(s)ds, & \text{if } \delta \in \mathfrak{R}, \end{cases}$$

Let $\vartheta(\cdot) : \mathbb{R} \rightarrow \mathfrak{S}$ be the function defined by:

$$\vartheta(\delta) = \begin{cases} \varpi(\delta), & \text{if } \delta \in (-\infty, 0], \\ R(\delta)\varpi(0), & \text{if } \delta \in \mathfrak{R}. \end{cases}$$

Then $\vartheta_0 = \varpi$. For each $h \in \mathbb{k}$ with $h(0) = 0$, we denote by \bar{h} the function

$$\bar{h}(\delta) = \begin{cases} 0; & \text{if } \delta \in (-\infty, 0], \\ h(\delta); & \text{if } \delta \in \mathfrak{R}. \end{cases}$$

If z satisfies (3.1), we can decompose it as $z(\delta) = h(\delta) + \vartheta(\delta)$, $\delta \in \mathfrak{R}$, which implies $z_\delta = h_\delta + \vartheta_\delta$ for every $\delta \in \mathfrak{R}$ and the function $h(\cdot)$ satisfies

$$h(\delta) = \int_0^\delta R(\delta - s)\Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}), \\ \int_0^s \zeta(s, \theta, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)})d\theta ds + \int_0^\delta R(\delta - s)\mathcal{C}u_{h+\vartheta}(s)ds, \delta \in \mathfrak{R}.$$

Set

$$\widehat{\mathbb{k}}_0 = \{h \in \widehat{\mathbb{k}} : h(0) = 0\},$$

and let

$$\|h\|_{\widehat{\mathbb{k}}_0} = \sup\{|h(\delta)| : \delta \in \mathfrak{R}\}, h \in \widehat{\mathbb{k}}_0.$$

$\widehat{\mathbb{k}}_0$ is a Banach space with the norm $\|\cdot\|_{\widehat{\mathbb{k}}_0}$.

We define the operator $\mathcal{A} : \widehat{\mathbb{k}}_0 \rightarrow \widehat{\mathbb{k}}_0$ by:

$$\mathcal{A}(h)(\delta) := \begin{cases} 0, & \text{if } \delta \leq 0, \\ \int_0^\delta R(\delta - s)\Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}), \\ \int_0^s \zeta(s, \theta, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)})d\theta ds, \\ + \int_0^\delta R(\delta - s)\mathcal{C}u_{h+\vartheta}(s)ds, & \text{if } \delta \in \mathfrak{R}, \end{cases}$$

The operator \mathcal{A} maps $\widehat{\mathbb{k}}_0$ into $\widehat{\mathbb{k}}_0$, for each $\delta \in \mathfrak{R}$ we have

$$|u_{h+\vartheta}(s)| \\ \leq |\widetilde{\mathfrak{W}}^{-1}(\widehat{z} - R(\gamma)\varpi(0) + \int_0^\gamma R(\gamma - \delta)\Psi(\delta, z_{\rho(\delta, z_\delta)}, \int_0^\delta \zeta(\delta, \theta, z_{\rho(\theta, z_\theta)})d\theta dt)(s)| \\ \leq \mathcal{M}_2[\|\widehat{z}\| + M\|\varpi\| + \int_0^\gamma \|R(\gamma - \delta)\|k_1(\delta)\|h_{\rho(\delta, h_\delta + \vartheta_\delta)} + \vartheta_{\rho(\delta, h_\delta + \vartheta_\delta)}\|dt \\ + \int_0^\gamma \|R(\gamma - \delta)\Psi(\delta, 0, 0)\|d\delta \\ + \int_0^\gamma \|R(\gamma - \delta)\|k_1(\delta) \int_0^\delta k_2(\theta)\|h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}\|d\theta d\delta \\ + \int_0^\gamma \|R(\gamma - \delta)\|k_1(\delta) \int_0^\delta \zeta(\delta, \theta, 0)\|d\theta d\delta\| (s) \\ \leq \mathcal{M}_2[\|\widehat{z}\| + M\widehat{l}_1\|\mathfrak{w}_\delta\|_\Omega + Mk_1^* \int_0^\gamma (\widehat{l}_2|h(\delta)| + (\widehat{l}_3 + \mathfrak{U}^\varpi + \widehat{l}_2Me^{-\alpha s}\widehat{l}_1)\|\varpi\|_\Omega))d\delta$$

$$\begin{aligned}
 & + M\Psi^* + Mk_1^* \int_0^\gamma k_2(\theta)(\widehat{l}_2|h(\theta)| + (\widehat{l}_3 + \mathcal{U}^\varpi + \widehat{l}_2 M\widehat{l}_1)\|\varpi\|_\Omega)d\theta d\delta + Mk_1^*\zeta^*] \\
 & \leq \mathcal{M}_2[\|\widehat{z}\| + M\widehat{l}_1\|\mathbf{w}_\delta\|_\Omega + M\Psi^* + Mk_1^*\zeta^* + nMk_1^*(\widehat{l}_2\|h\| + C_1) \\
 & \quad + nMk_1^*k_2^*(\widehat{l}_2\|h\| + C_1)] := C_2.
 \end{aligned}$$

Then,

$$\begin{aligned}
 |\mathcal{A}(h)(\delta)| \leq & \left\| \int_0^\delta R(\delta - s)\Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}, \right. \\
 & \left. \int_0^s \zeta(s, \theta, h_{\rho(\theta, h_\theta + \vartheta_\theta)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)})d\theta\right) ds \Big\| \\
 & + \left| \int_0^\delta R(\delta - s)\mathcal{C}u_{h+\vartheta}(s)ds \right|
 \end{aligned}$$

Then, we have

$$\begin{aligned}
 |\mathcal{A}(h)(\delta)| \leq & Mk_1^*\widehat{l}_2\|h\|_{\widehat{\mathbb{K}}_0} + MC_1k_1^* + Mk_1^*k_2^*\widehat{l}_2\|h\|_{\widehat{\mathbb{K}}_0} + MC_1k_1^*k_2^* \\
 & + \mathcal{M}_2\mathcal{M}_1M[\|\widehat{z}\| + M\widehat{l}_1\|\mathbf{w}_\delta\|_\Omega + M\Psi^* \\
 & + Mk_1^*\zeta^* + nMk_1^*(\widehat{l}_2\|h\| + C_1) + nMk_1^*k_2^*(\widehat{l}_2\|h\| + C_1)] \\
 & \leq C^* + \mathcal{M}_1MC_2 := C_3 \\
 & \leq C_3.
 \end{aligned}$$

Hence, $\mathcal{A}(h)(\delta) \in \widehat{\mathbb{K}}_0$.

Let $\mathbf{r} > 0$ be such that

$$\mathbf{r} \geq \frac{\mathcal{M}_2\mathcal{M}_1M[\|\widehat{z}\| + M\widehat{l}_1\|\mathbf{w}_\delta\|_\Omega + M\Psi^* + Mk_1^*\zeta^* + nMC_1k_1^*(1+k_2^*)] + MC_1k_1^* + MC_1k_1^*k_2^*}{1 - M\widehat{l}_2k_1^*(1+k_1^*k_2^*) - \gamma\mathcal{M}_2\mathcal{M}_1M^2\widehat{l}_2k_1^*(1+k_2^*)},$$

and $\sigma_{\mathbf{r}}$ be the closed ball in $\widehat{\mathbb{K}}_0$ centered at the origin and of radius z . Let $z \in \sigma_{\mathbf{r}}$ and $\delta \in \mathfrak{R}$. Then,

$$\begin{aligned}
 |\mathcal{A}(z)(\delta)| \leq & Mk_1^*\widehat{l}_2\mathbf{r} + MC_1k_1^* + Mk_1^*k_2^*\widehat{l}_2\mathbf{r} + MC_1k_1^*k_2^* \\
 & + \mathcal{M}_2\mathcal{M}_1M[\|\widehat{z}\| + M\widehat{l}_1\|\mathbf{w}_\delta\|_\Omega + nMk_1^*(\widehat{l}_2\mathbf{r} + C_1) \\
 & + M\Psi^* + Mk_1^*\zeta^* + nMk_1^*k_2^*(\widehat{l}_2\mathbf{r} + C_1)]
 \end{aligned}$$

Thus,

$$\|\mathcal{A}(z)(\delta)\|_{\widehat{\mathbb{K}}_0} \leq \mathbf{r},$$

then $\mathcal{A}(\sigma_{\mathbf{r}}) \subset \sigma_{\mathbf{r}}$.

The proof can be given by following steps.

Step 1: As in the proofs of Theorem 3.5, we can prove that $\mathcal{A}(h)(\delta)$ is continuous and $\mathcal{A}(\sigma_{\mathbf{r}})$ is equicontinuous on any compact $B = [0, b]$ of $[0, +\infty)$, for $b > 0$.

Step 2: We show that $\mathcal{A} : \sigma_\tau \rightarrow \sigma_\tau$ is a Meir-Keeler condensing operator. For $\mathcal{D} \subset \sigma_\tau$, $\mathcal{A}(\mathcal{D})$ is bounded. Then there exists a countable set $\mathcal{D}_0 = \{h_\gamma\}_{\gamma=1}^\infty \subset \mathcal{D}$.

Claim 1. Using the fact that $\mathcal{A}(\sigma_\tau)$ is equicontinuous and by the definition of $\eta_0^\infty(\cdot)$, we get that $\eta_0^\infty(\mathcal{D}) \leq \eta_0^\infty(\mathcal{A}\mathcal{D}) = 0$. Then $\eta_0^\infty(\mathcal{D}) = 0$.

Claim 2. We have

$$\begin{aligned} \varkappa(\mathcal{A}(\mathcal{D})(\delta)) &\leq 8\widehat{l}_2 M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s)(1+k_2^*) \sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}_0(\varrho)) ds + \epsilon \\ &\quad + 4Mk_4 \int_0^\delta e^{-\alpha(\delta-s)} k_3(s) \\ &\quad \times \varkappa\left(\left\{ \int_0^n R(\gamma-\delta) \Psi(s, z_{\rho(s, z_s)}, \int_0^s \zeta(\delta, s, z_{\rho(\theta, z_\theta)}) d\theta \right\}_{\gamma=0}^\infty\right) \\ &\leq 8\widehat{l}_2 M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s)(1+k_2^*) \sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}_0(\varrho)) ds + \epsilon \\ &\quad + 4Mk_4 \int_0^\delta e^{-\alpha(\delta-s)} k_3(s) nM\widehat{l}_2 k_1(s)(1+2k_2^*) \sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}_0(\varrho)) ds \\ &\leq 8\widehat{l}_2 M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s)(1+k_2^*) + k_4 k_3(s) M n k_1(s)(1+k_2^*) \\ &\quad \times \sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}_0(\varrho)) ds. \end{aligned}$$

Then,

$$\varkappa(\mathcal{A}(\mathcal{D})(\delta)) \leq 8M\widehat{l}_2 k_1^*(1+k_2^*) + k_4 k_3^* M n k_1^*(1+k_2^*) \sup_{\varrho \in (-\infty, 0]} \varkappa(\mathcal{D}_0(\varrho)) + \epsilon.$$

We have

$$\bar{\varkappa}_\infty(\mathcal{A}(\mathcal{D})) \leq [8M\widehat{l}_2 k_1^*(1+k_2^*) + k_4 k_3^* M n k_1^*(1+k_2^*)] \bar{\varkappa}_\infty(\mathcal{D}).$$

Then, $\bar{\varkappa}_\infty(\mathcal{A}(\mathcal{D})) \leq L_1 \bar{\varkappa}_\infty(\mathcal{D})$.

Claim 3. We have

$$\begin{aligned} &|h(\delta) - h^*(\delta)| \\ &\leq \left| \int_0^\delta R(\delta-s) \Psi(s, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(s, h_s + \vartheta_s)}, \right. \\ &\quad \left. \int_0^s \zeta(s, \theta, h_{\rho(s, h_s + \vartheta_s)} + \vartheta_{\rho(\theta, h_\theta + \vartheta_\theta)}) d\theta \right) ds \\ &\quad + \int_0^\delta R(\delta-s) \mathcal{C}u_{h+\vartheta}(s) ds - \int_0^\delta R(\delta-s) \Psi(s, h_{\rho(s, h_s^* + \vartheta_s^*)} + \vartheta_{\rho(s, h_s^* + \vartheta_s^*)}, \end{aligned}$$

$$\begin{aligned}
 & \int_0^s \zeta(s, \theta, h_{\rho(s, h_s^* + \vartheta_s)}^* + \vartheta_{\rho(\theta, h_\theta^* + \vartheta_\theta)}) d\theta ds | \\
 & + \int_0^\delta R(\delta - s) \mathcal{C}u_{h^* + \vartheta}(s) ds \\
 \leq & M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \widehat{l}_2 |h(s) - h^*(s)| ds \\
 & + M \int_0^\delta e^{-\alpha(\delta-s)} k_1(s) \int_0^s k_2(\theta) \widehat{l}_2 |h(\theta) - h^*(\theta)| d\theta ds \\
 & + \mathcal{M}_1 \mathcal{M}_2 M \int_0^\delta e^{-\alpha(\delta-s)} n M k_1(s) \widehat{l}_2 |h(s) - h^*(s)| ds \\
 & + \mathcal{M}_1 \mathcal{M}_2 M \int_0^\delta e^{-\alpha(\delta-s)} n M k_1(s) \int_0^s k_2(\theta) \widehat{l}_2 |h(\theta) - h^*(\theta)| d\theta ds.
 \end{aligned}$$

We have $c(\mathcal{AD}) \leq [M\widehat{l}_2 k_1^*(1 + k_2^*)(1 + \mathcal{M}_1 \mathcal{M}_2 M n)]c(\mathcal{D})$. Then, $c(\mathcal{AD}) \leq L_2 c(\mathcal{D})$.

From claim 1,2 and 3 we deduce that

$$\varkappa^*(\mathcal{A}(\mathcal{D})) \leq L_3 \varkappa^*(\mathcal{D}),$$

where $L_3 = \max(L_1, L_2)$. So for a given $\epsilon > 0$, let $\lambda = \frac{1-L_3}{L_3} \epsilon$. We get that

$$\epsilon \leq \varkappa^*(\mathcal{D}) < \epsilon + \lambda \Rightarrow \varkappa^*(\mathcal{A}(\mathcal{D})) < \epsilon.$$

That means \mathcal{A} is a Meir-Keeler condensing operator, then using the Theorem 2.8 we can say that the operator \mathcal{A} has at least fixed point, so the operator $\Xi : \mathbb{k} \rightarrow \mathbb{k}$ has a fixed point $z(\delta) = h(\delta) + \vartheta(\delta)$ and the set of all fixed points of Ξ in \mathbb{k} is compact, which mean that the problem (1.2) is controllable.

6. Example

We consider the following partial integro-differential problem

$$\begin{aligned}
 \frac{\partial}{\partial \delta} h(\delta, x) = & \frac{\partial^2}{\partial x^2} h(\delta, x) + a \frac{\partial}{\partial x} h(\delta, x) + bh(\delta, x) \\
 & + \int_0^\delta Y(\delta - s) \left[\frac{\partial^2}{\partial x^2}(s, x) + a \frac{\partial}{\partial x} h(s, x) + bh(s, x) \right] \\
 & + \frac{e^{-\alpha \delta}}{21(\delta^2 + 2)} \int_{-\infty}^{-\delta} \frac{\sin(\tau) e^{-\Delta(\delta, \tau) - \alpha(\delta - \tau)}}{(\delta + \tau)^2 + 1} d\tau \\
 & + \frac{e^{-\alpha \delta}}{21(\delta^2 + 2)} \int_0^\delta \frac{\ln(1 + e^{-\delta}) e^{-\alpha(\delta - s)}}{1 + 2\delta^2 + s^2} \int_{-\infty}^{-s} \frac{\cos(\Delta(s, \tau)) e^{-\Delta(s, \tau) - \alpha(s - \tau)}}{(s + \tau)^2 + 1} d\tau ds, \\
 & x \in [0, 1], \delta \in \mathfrak{R},
 \end{aligned} \tag{6.1}$$

$$h(\delta, 0) = h(\delta, 1), \delta \in \mathfrak{R}, \tag{6.2}$$

$$h(\delta, x) = \phi(\delta), \delta \in (-\infty, 0], x \in [0, 1], \tag{6.3}$$

where $Y : \mathfrak{R} \rightarrow \mathbb{R}$ is continuous, $\Delta : \mathfrak{R} \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous function. Let $\mathfrak{S} = L^2(0, 1)$, and A is an operator induced on \mathfrak{S} define as

$$Ah = h'' + ah' + bh, a, b \in \mathbb{R} \text{ and, } D(A) = H^2(0, 1) \cap H_0^1(0, 1).$$

From [17] we know that A is the infinitesimal generator of analytic C_0 semigroup $(Y(\delta))_{\delta \geq 0}$ on \mathfrak{S} . Since the semigroup generated by A is analytic, then it's norm continuous for $\delta > 0$. So the resolvent operator is operator-norm continuous for $\delta > 0$. We define the operator $B : \mathfrak{S} \rightarrow \mathfrak{S}$ as

$$B(\delta)h = Y(\delta)Ah, \quad \text{for } \delta \geq 0.$$

We assume that there exist $\beta > \beta_1 > 1$ such that $Y(\delta) < \frac{1}{\beta_1}e^{-\beta\delta}$, for all $\delta \geq 0$. Then, we have $M = 1, \alpha = 1 - \frac{1}{\beta_1}$.

Consider $\mathfrak{Q} = BUC(\mathbb{R}^-, \mathfrak{S})$, the phase space of bounded uniformly continuous functions endowed with the following norm

$$\|\phi\|_{\mathfrak{Q}} = \sup_{\tau \in (-\infty, 0]} \|\phi(\tau)\|_{\mathfrak{S}}, \phi \in \mathfrak{Q},$$

and $\widehat{l}_2 = \widehat{l}_3 = 1$ let $\rho(\delta, h)(x) = \Delta(\delta, h(\delta + \tau, x))$. See [24] for more details.

We have also

$$\Psi(\delta, f, g) = \frac{e^{-\alpha\delta}}{21(\delta^2 + 2)} \int_{-\infty}^{-\delta} \frac{\sin(\tau)e^{-f(\delta+\tau, x) - \alpha(\delta-\tau)}}{(\delta + \tau)^2 + 1} d\tau + \frac{e^{-\alpha\delta}}{21(\delta^2 + 2)} g(\delta, x),$$

$$\zeta(\delta, s, f) = \frac{\ln(1 + e^{-\delta})e^{-\alpha(\delta-s)}}{1 + 2\delta^2 + s^2} \int_{-\infty}^{-s} \frac{\cos(f(s + \tau, x))e^{-f(s+\tau, x) - \alpha(s-\tau)}}{(s + \tau)^2 + 1} d\tau,$$

and

$$k_1 = \frac{e^{-\alpha\delta}}{21(\delta^2 + 2)}, k_2 = \ln(2) = k_2^*, k_1^* = \frac{1}{42}.$$

The problem (1.1) in an abstract formulation of the problem (6.1)-(6.3). Since $(EC_1) - (EC_4), (EC_{\varpi})$ are satisfied, and $8M\widehat{l}_2k_1^*(1 + k_2^*) < 1$, then the problem (6.1)-(6.3) has at least one mild solution on \mathbb{k} , which is uniformly locally attractive.

References


- [1] Aghajani, A., Mursaleen, M., Haghghi, A. S., *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta Math. Sci., **35(3)**(2015), 552-566.
- [2] Akhmerov, R. R., Kamenskii, M. I., A. S., Potapov, Rodkina, A. E. and Sadovskii, B. N., *Measures of Noncompactness an Condensing Operators*, Birkhauser Verlag, Basel, 1992.
- [3] Banaś, J., Chlebowicz, A. and Wos, W., *On measures of noncompactness in the space of functions defined on the half-axis with values in a Banach space*, J. Math. Anal. Appl., **489**(2020), 124187.
- [4] Banaś, J., Dhage, B. C., *Global asymptotic stability of solutions of a functional integral equation*, Nonlinear Anal., **69**(2008), 1945-1952.

- [5] Banaś, J. and Goebel, K., *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.
- [6] Banaś, J., Rocha Martin, J., and Sadarangani, K., *On monotonic solutions of some integral equations*, *Dynam. Systems Appl.*, **14**(1)(2005), 103-110.
- [7] Banaś, J. and Sadarangani, K., *On some measures of noncompactness in the space of continuous functions*, *Nonlinear Anal.*, **68**(2008), 377-383.
- [8] Benchohra, M., Bouazzaoui, F., Karapınar, E. and Salim, A., *Controllability of second order functional random differential equations with delay*, *Mathematics*, **10**(2022), pp. 16. <https://doi.org/10.3390/math10071120>
- [9] Benchohra, M., Karapınar, E., Lazreg, J. E. and Salim, A., *Advanced Topics in Fractional Differential Equations: A Fixed Point Approach*, Springer, Cham, 2023.
- [10] Benchohra, M., Karapınar, E., Lazreg, J. E. and Salim, A., *Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives*, Springer, Cham, 2023.
- [11] Benkhettou, N., Aissani, K., Salim, A., Benchohra, M. and Tunc, C., *Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses*, *Appl. Anal. Optim.*, **6**(2022), 79-94.
- [12] Bensalem, A., Salim, A., Ahmad, B. and Benchohra, M., *Existence and controllability of integrodifferential equations with non-instantaneous impulses in Fréchet spaces*, *CUBO*, **25**(2)(2023), 231-250.
- [13] Bensalem, A., Salim, A. and Benchohra, M., *Ulam-Hyers-Rassias stability of neutral functional integrodifferential evolution equations with non-instantaneous impulses on an unbounded interval*, *Qual. Theory Dyn. Syst.*, **22**(2023), 29 pages.
- [14] Bensalem, A., Salim, A., Benchohra, M. and Fečkan, M., *Approximate controllability of neutral functional integro-differential equations with state-dependent delay and non-instantaneous impulses*, *Mathematics*, **11**(2023), 1-17.
- [15] Bensalem, A., Salim, A., Benchohra, M. and N'Guérékata, G., *Functional integro-differential equations with state-dependent delay and non-instantaneous impulses: existence and qualitative results*, *Fractal Fract.*, **6**(2022), 1-27.
- [16] Dhage, B. C., Lakshmikantham, V., *On global existence and attractivity results for nonlinear functional integral equations*, *Nonlinear Anal.*, **72**(2010), 2219-2227.
- [17] Engel, K. J., Nagel, R., *One-Parameter Semigroups for Linear Evolution Equations*; Springer: New York, NY, USA, 2000.
- [18] Diop, M. A., Ezzinbi, K., Ly, M. P., *Nonlocal problems for integrodifferential equation via resolvent operators and optimal control*, *Differ. Incl. Control Optim.*, **42**(2022), 5-25.
- [19] Diop, A., Frederico, G. S. F. and Sousa, J. V. C., *On controllability for a class of multi-term time-fractional random differential equations with state-dependent delay*, *Ann. Funct. Anal.*, **13**(2022). <https://doi.org/10.1007/s43034-022-00165-w>
- [20] Grimmer, R. C., *Resolvent operators for integral equations in a Banach space*, *Trans. Amer. Math. Soc.*, **273**(1982), 333-349.
- [21] Grimmer, R. C., Pritchard, A. J., *Analytic resolvent operators for integral equations in Banach space*, *J. Differential Equations*, **50**(2)(1983), 234-259.
- [22] Hale, J. and Kato, J., *Phase space for retarded equations with infinite delay*, *Funkcial. Ekvac.*, **21**(1978), 11-41.
- [23] Hernández, E., Sakthivel, R., Rathinasamy and Tanaka, A., *Existence results for impulsive evolution differential equations with state-dependent delay*, *Electron. J. Differential Equations*, **28**(2008), 1-11.

- [24] Hino, Y., Murakami, S. and Naito, T., *Functional Differential Equations with Unbounded Delay*, Springer-Verlag, Berlin, 1991.
- [25] Kuratowski, K., *Sur les espaces complets*, Fund. Math., **15**(1930), 302-309.
- [26] Mahmudov, N. I., *Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces*, SIAM, J. Control Optim., **42**(2003), 1604-1622.
- [27] Mahmudov, N. I., *Approximate controllability of evolution systems with nonlocal conditions*, Nonlinear Anal., **68**(2008), 536-546.
- [28] Meir, A., Keeler, E., *A theorem on contraction mappings*, J. Math. Anal. Appl., **28**(1969), 326-329.
- [29] Mokkedem, F. Z. and Fu, X., *Approximate controllability of semi-linear neutral integro-differential systems with finite delay*, Appl. Math. Comput., **242**(2014), 202-215.
- [30] Mokkedem, F. Z. and Fu, X., *Approximate controllability of a semi-linear neutral evolution system with infinite delay*, Int. J. Robust Nonlinear Control, **27**(2017), 1122-1146.

Imene Medjadj 

Department of Mathematics, Faculty of mathematics and computer science,
University of Science and Technology-Mohamed Boudiaf (USTO MB),
Mathematics and Statistics and their Applications Laboratory (LMSA),
El Mnaouar, BP 1505, Bir El Djir 31000,
Oran, Algeria
e-mail: imene.medjadj@univ-usto.dz

Abdelkrim Salim 

Faculty of Technology, Hassiba Benbouali University of Chlef,
P.O. Box 151 Chlef 02000, Algeria
e-mail: a.salim@univ-chlef.dz

Mouffak Benchohra 

Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes,
P.O. Box 89, Sidi Bel-Abbes 22000, Algeria
e-mail: benchohra@yahoo.com

Variational analysis of Kirchhoff equations in Musielak-Orlicz-Zygmund spaces

Ahmed El Ouardani , Ahmed Aberqi  and Mhamed Elmassoudi 

Abstract. We study a class of nonlocal Kirchhoff problems with nonlinearities exhibiting nonstandard growth. Using variational methods in Musielak-Orlicz-Zygmund spaces, we prove the existence of nontrivial weak solutions. The analysis uses generalized N -functions, Orlicz-Zygmund embeddings, pseudo-monotone operators, and the Palais-Smale condition, which allow handling double-phase and nonlocal Kirchhoff terms. The results extend classical variational methods to settings with borderline and logarithmic growth.

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Keywords: Kirchhoff equations, double-phase problem, Musielak - Orlicz - Zygmund spaces, weak solutions, variational approaches.

1. Introduction

The study of nonlocal Kirchhoff problems with Sobolev–Zygmund type growth is motivated both by applications in nonlinear elasticity and composite materials, and they also raise important mathematical questions related to borderline phenomena in partial differential equations.

Kirchhoff-type nonlocal operators were first used to describe the vibration of elastic strings and membranes. The classical model of Kirchhoff [17] leads to equations of the form

$$-M \left(\int_{\Omega} |\nabla u|^p dx \right) \Delta_p u = f(x, u),$$

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where the tension depends on the average deformation. Lions extended these ideas to the PDE setting and motivated the study of nonlocal operators coupled with nonlinear source terms.

At the same time, double-phase problems were introduced to model materials with different behaviors in different regions. These problems have the form

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u),$$

where the coefficient $\mu(x)$ governs the transition between the p - and q -growth phases. Important contributions in this area were given by Colombo and Mingione [9] and Baroni–Colombo–Mingione [3].

In the last decade, researchers have studied double-phase and Kirchhoff-type operators in generalized Orlicz and Musielak–Orlicz spaces frameworks. Fiscella and Pinamonti [13] proved existence results for Kirchhoff problems in Orlicz spaces using variational methods. Diening et al. [10], and Boccardo–Murat [6] analyzed critical growth phenomena in Zygmund-Orlicz type spaces, where logarithmic perturbations naturally arise.

More recently, special attention has been given to sources of Zygmund type

$$f(x, u) \sim |u|^{q-2}u \log(e + |u|),$$

which combine superlinear behavior with logarithmic modulation. Aberqi et al. [1] studied double-phase obstacle problems with logarithmic convection, while Shen and Squassina [22] obtained existence and multiplicity results for Kirchhoff and nonlocal elliptic problems with logarithmic nonlinearities using critical point theory. Significant studies in this field include those by Aberqi et al. [1, 2, 5, 11]. These works highlight the crucial role of modular coercivity, monotonicity, and compactness recovery in borderline regimes.

Motivated by these works, we study a nonlocal double-phase Kirchhoff problem with logarithmic-type sources in Musielak–Orlicz–Zygmund spaces. Our main goal is to prove the strong convergence of Palais–Smale sequences for the associated energy functional.

This property ensures the existence of weak solutions and clarifies the interaction between the Kirchhoff term, the double-phase operator, and the logarithmic source. Our approach relies on combining modular coercivity with strict monotonicity, providing new insights into the borderline behavior of such equations.

The plan of the paper is as follows: In Section 2, we present the problem setting and assumptions. In Section 3, we recall some well-known preliminary properties and results on Orlicz-Zygmund Spaces. In Section 4, we introduce the Kirchhoff framework and the critical Point theory. Finally, in Section 5, we state and prove the main result of the paper (Theorem 5.1).

2. Problem setting and assumptions

We study a class of borderline double-phase problems with nonlocal effects and logarithmic convection in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. Our aim is to establish the existence of weak solutions under general assumptions, including

unbalanced growth and gradient-dependent nonlinearities. The model problem is to find $u : \Omega \rightarrow \mathbb{R}$ satisfying

$$(\mathcal{P}) \begin{cases} -K\left(\int_{\Omega} \Phi(x, |\nabla u|) dx\right) \operatorname{div} a(x, \nabla u) = g(x)|u|^{q-2}u + f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

The ingredients of (2.1) are specified as follows.

- The generalized N -function is

$$\Phi(x, t) = \frac{1}{p} t^p (1 + \log(e + t)), \quad p > 1,$$

leading to the Orlicz-Zygmund space $W_0^1 L_{\Phi}(\Omega) \equiv W_0^{1,p} \log L(\Omega)$.

- The Leray-Lions type operator $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is monotone, coercive, and continuous in the gradient variable. A typical model is

$$a(x, \nabla u) = |\nabla u|^{p-2} \left(1 + \log(e + |\nabla u|) + \frac{|\nabla u|}{p(e + |\nabla u|)} \right) \nabla u.$$

- $K : [0, \infty) \rightarrow (0, \infty)$ is a Kirchhoff function (see Definition 4.1).
- There exists $\theta \in (1, \frac{p^*}{q}]$ with $p^* = \frac{Np}{N-p}$ and $q < p^*$. such that

$$tK(t) \leq \theta \widehat{K}(t) \text{ for any } t \in [0, \infty), \tag{2.2}$$

where

$$\widehat{K}(t) := \int_0^t K(\tau) d\tau, \quad t \geq 0.$$

- The source term has the form

$$h(x, u) = g(x)|u|^{q-2}u + f(x, u),$$

with $g \in L^\infty(\Omega)$, $\operatorname{ess\,inf}_{x \in \Omega} g(x) = g_0 > 0$ and f is a Carathéodory function.

Writing $F(x, u) = \int_0^u f(x, t) dt$, we assume

$$f(x, u) \leq |u|^{q-2} \log(e + |u|), \tag{2.3}$$

and

$$0 < \lambda F(x, u) \leq u f(x, u) \text{ for } |u| \text{ large}, \tag{2.4}$$

where the parameter λ satisfies $(p + 1)\theta < \lambda < q$.

Let $X = \mathbb{W}_0$ be the Orlicz-Zygmund space (see below).

Definition 2.1. A function $u \in X$ is called a weak solution if it satisfies

$$\int_{\Omega} K\left(\int_{\Omega} \Phi(x, |\nabla u|) dx\right) a(x, \nabla u) \cdot \nabla \varphi dx = \int_{\Omega} g(x)|u|^{q-2}u \varphi dx + \int_{\Omega} f(x, u) \varphi dx.$$

Under these assumptions, the associated energy functional exhibits the Mountain–Pass geometry and satisfies the Palais–Smale condition below a suitable critical level. Therefore, existence of nontrivial weak solutions to (2.1) can be established. Our framework extends the results of Tran–Nguyen [25] and El Ouardani et al. [12] by incorporating logarithmic double-phase growth within Orlicz–Zygmund spaces.

3. Orlicz-Zygmund spaces

3.1. Classical Orlicz spaces

To handle nonlinear PDEs with borderline or critical growth, one needs a refined functional framework beyond classical Sobolev spaces. Orlicz and Orlicz–Zygmund spaces naturally provide the tools to address coercivity, compactness, and critical embeddings, as shown in classical works [8, 14, 7, 15, 18, 20, 21, 23] and in recent studies [25, 12]. We now recall the basic properties of these spaces that will serve as the foundation for our analysis.

Definition 3.1. *A function $\mathcal{A} : [0, \infty) \rightarrow [0, \infty)$ is an N -function if it is convex, continuous, strictly increasing, $\mathcal{A}(0) = 0$, $\mathcal{A}(t) > 0$ for all $t > 0$, and*

$$\lim_{t \rightarrow 0} \frac{\mathcal{A}(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{\mathcal{A}(t)}{t} = \infty.$$

Definition 3.2. *The Orlicz space associated with \mathcal{A} is*

$$L^{\mathcal{A}}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \exists \lambda > 0, \varrho_{\mathcal{A}, \Omega} \left(\frac{u}{\lambda} \right) < \infty \right\},$$

where the modular functional is

$$\varrho_{\mathcal{A}, \Omega}(u) = \int_{\Omega} \mathcal{A}(|u(x)|) dx,$$

and the Luxemburg norm is defined by

$$\|u\|_{\mathcal{A}} := \inf \left\{ \lambda > 0 : \int_{\Omega} \mathcal{A} \left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

When \mathcal{A} and its conjugate satisfy the Δ_2 -condition, the space $L^{\mathcal{A}}(\Omega)$ is reflexive [24, 15, 4].

3.2. Orlicz-Zygmund spaces

For problems with logarithmic perturbations of polynomial growth, Orlicz-Zygmund spaces provide a natural setting. Let $p \in (1, \infty)$ and $s > 0$ and define

$$\mathcal{H}_s^p(t) := t^p \log^s(e + t), \quad t \geq 0.$$

Definition 3.3. *The Orlicz-Zygmund space associated with \mathcal{H}^p is*

$$L^{\mathcal{H}_s^p}(\Omega) := \left\{ v \in L^1(\Omega) : \int_{\Omega} |v|^p \log^s(e + |v|) dx < \infty \right\},$$

equipped with the Luxemburg norm

$$\|v\|_{L^{\mathcal{H}_s^p}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|v|^p}{\lambda^p} \log^s \left(e + \frac{|v|}{\lambda} \right) dx < 1 \right\}.$$

Remark 3.4. If $\mathcal{H}_s^p(t) = t^p$, we recover the classical $L^p(\Omega)$ space.

For convenience, the modular function is defined as

$$[v]_{L^{\mathcal{H}_s^p}(\Omega)} := \left(\int_{\Omega} |v|^p \log^s(e + |v|/||v||_p) dx \right)^{1/p}.$$

Lemma 3.5 ([12]). *For $a \geq 0, b, p, s \geq 1$, the following inequalities hold:*

$$\log^s(e + ab) \leq 2^{s-1}(\log^s(e + a) + \log^s b), \tag{3.1}$$

$$\log^s b \leq (s/(ep))^s b^p. \tag{3.2}$$

Proposition 3.6 ([12]). *For all $p \in [1, \infty), s > 0$, and $q \in (p, \infty)$, the embeddings*

$$L^q(\Omega) \subset L^{\mathcal{H}_s^p}(\Omega) \subset L^p(\Omega),$$

hold. Moreover, the modular and norm satisfy

$$\|v\|_{L^{\mathcal{H}_s^p}(\Omega)} \leq [v]_{L^{\mathcal{H}_s^p}(\Omega)} \leq [2^s + (2s/(ep))^s] \|v\|_{L^{\mathcal{H}_s^p}(\Omega)}.$$

3.3. Sobolev-Orlicz-Zygmund Spaces and Compact Embeddings

The Sobolev-Orlicz-Zygmund space is defined as

$$W^{1, \mathcal{H}_s^p}(\Omega) := \left\{ u \in L^{\mathcal{H}_s^p}(\Omega) : D^\beta u \in L^{\mathcal{H}_s^p}(\Omega), |\beta| \leq 1 \right\},$$

with norm

$$\|v\|_{W^{1, \mathcal{H}_s^p}(\Omega)} := \|v\|_{L^{\mathcal{H}_s^p}(\Omega)} + \|\nabla v\|_{L^{\mathcal{H}_s^p}(\Omega)}.$$

Denote $\mathbb{W} := W^{1, \mathcal{H}_s^p}(\Omega)$, $\mathbb{W}_0 := W_0^{1, \mathcal{H}_s^p}(\Omega)$, with $\|v\|_{\mathbb{W}_0} = \|\nabla v\|_{L^{\mathcal{H}_s^p}(\Omega)}$.

Lemma 3.7. *For all $q \in (1, p^*), p^* = \frac{np}{n-p}$, the embedding $\mathbb{W}_0 \hookrightarrow L^{\mathcal{H}_s^q}(\Omega)$ is compact.*

Proof. For every Young function A and its complementary function \bar{A} satisfying the Δ_2 -condition, we introduce the following Young function

$$\bar{A}_n(t) = \int_0^t y^{\frac{n+1}{n}} \left[\bar{A}^{-1}(y^n) \right]^{-1} dy$$

where \bar{A}_n^{-1} is the right-continuous generalized inverse function of \bar{A}_n , defined by the following formula:

$$\bar{A}_n(t) = \int_0^t \frac{\bar{A}(y)}{y^{\frac{n+1}{n}}} dy$$

and

$$\bar{A}_n^{-1}(t) = \inf \{ y \geq 0 : \bar{A}_n(y) \geq t \} \quad \text{for } t \geq 0.$$

By Theorem 3 in [8], the Sobolev-Orlicz space

$$W_0^{1, \mathcal{A}}(\Omega),$$

embeds continuously into the corresponding Orlicz-Zygmund space

$$L^{\widehat{\mathcal{A}}_n}(\Omega).$$

For any N-function B satisfying the Δ_2 -condition and $B \ll \widehat{\mathcal{A}}_n$, this embedding is compact:

$$W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow L^B(\Omega).$$

Applying this to the functions

$$\mathcal{A}(t) = t^p + t^p \log(e + t) \quad \text{and} \quad B(t) = t^q \log(e + t), \quad q \in (1, p^*),$$

verifies that

$$W_0^{1,G}(\Omega) \hookrightarrow L^{\mathcal{H}_{p^*/p}^{p^*}}(\Omega),$$

where $G(t) = t^p$. The result follows directly from the general theory. □

In Orlicz-Zygmund spaces, for functions with a large norm ($\|u\|_{W_0} > 1$), there exists a constant $C > 0$ such that

$$\Psi(u) = \int_{\Omega} \frac{1}{p} |\nabla u|^p (1 + \log(e + |\nabla u_n|)) \, dx \geq C \|u\|_{W_0}^p, \tag{3.3}$$

holds.

This inequality follows from the definition of the Luxemburg norm (see [21] for the general modular-norm relationship in Musielak-Orlicz settings).

4. Kirchhoff framework and critical point theory

4.1. Kirchhoff function and operator

Kirchhoff-type models generalize classical diffusion equations by introducing a nonlocal dependence on the gradient, encoded in a Kirchhoff function.

Definition 4.1. *We assume that the Kirchhoff function $K : [0, \infty) \rightarrow (0, \infty)$ satisfies:*

- K is continuous, strictly positive, and non-decreasing;
- there exist constants $k_0, k_1 > 0$ and $\alpha \geq 0$ such that

$$k_0 \leq K(s) \leq k_1(1 + s^\alpha), \quad \forall s \geq 0. \tag{4.1}$$

A canonical example is $K(s) = k_0 + k_1 s^\alpha$, which recovers the classical Kirchhoff-type operator

$$-K\left(\int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u.$$

In the Musielak-Orlicz setting, the associated Kirchhoff operator is defined for $u \in W_0^1 L_{\Phi}(\Omega)$ by

$$A(u) := -K\left(\int_{\Omega} \Phi(x, |\nabla u|) \, dx\right) \operatorname{div}(a(x, \nabla u)),$$

where $a(x, \xi)$ is of Leray-Lions type.

Estimates for the primitive \widehat{K}

Proposition 4.2. *Under Definition 4.1, for all $s \geq 0$:*

$$k_0 s \leq \widehat{K}(s) \leq k_1 \left(s + \frac{s^{\alpha+1}}{\alpha+1} \right), \tag{4.2}$$

$$\frac{s}{2} K\left(\frac{s}{2}\right) \leq \widehat{K}(s) \leq sK(s). \tag{4.3}$$

Corollary 4.3. *We have*

$$\lim_{s \rightarrow \infty} \widehat{K}(s) = +\infty,$$

and the polynomial growth estimate

$$k_0 s \leq \widehat{K}(s) \leq C(1 + s^{\alpha+1}), \quad \forall s \geq 0,$$

for some constant $C > 0$ depending only on k_1 and α .

Based on Definition 4.1, the following properties of the Kirchhoff function K and its primitive \widehat{K} follow immediately.

Scaling inequalities for $K(t\theta)$

Proposition 4.4. *For all $t, \theta \geq 0$, there exist constants $C_1, C_2 > 0$ depending only on k_0, k_1, α such that:*

$$k_0 \leq K(t\theta) \leq k_1(1 + (t\theta)^\alpha), \tag{4.4}$$

$$K(t\theta) \leq C_1(1 + t^\alpha)K(\theta), \tag{4.5}$$

$$K(t\theta) \geq K(\theta), \quad t \geq 1. \tag{4.6}$$

Scaling inequalities for $\widehat{K}(t\theta)$

Proposition 4.5. *Under Definition 4.1, for all $t, \theta \geq 0$ there exists a constant $C > 0$ depending only on k_0, k_1, α such that*

$$t \widehat{K}(\theta) \leq \widehat{K}(t\theta) \leq C t(1 + t^\alpha) \widehat{K}(\theta). \tag{4.7}$$

Proof. Set $\tau = t\sigma$ so that $d\tau = t d\sigma$. Then

$$\widehat{K}(t\theta) = \int_0^{t\theta} K(\tau) d\tau = t \int_0^\theta K(t\sigma) d\sigma.$$

Since K is non-decreasing and $K(\sigma) \geq 0$,

$$K(t\sigma) \geq K(\sigma) \implies \widehat{K}(t\theta) \geq t\widehat{K}(\theta).$$

Using Proposition 4.4, $K(t\sigma) \leq C(1 + t^\alpha)K(\sigma)$, hence

$$\widehat{K}(t\theta) = t \int_0^\theta K(t\sigma) d\sigma \leq C t(1 + t^\alpha) \widehat{K}(\theta).$$

□

4.2. Critical point theory

The Palais–Smale theorem is a foundational result in critical point theory. Establishing the existence of critical points of a functional is a key step in variational methods for PDEs, and the Palais–Smale condition ensures compactness of sequences that might otherwise diverge.

Definition 4.6. (see [16])

- The derivative $DJ(u)$ at u is the unique continuous linear operator from X to \mathbb{R} such that $J(u+h) - J(u) - DJ(u)h = o(\|h\|)$ as $\|h\| \rightarrow 0$.
- u is a critical point if $DJ(u) = 0$; otherwise it is a regular point.
- If u is a critical point, $c = J(u)$ is called a critical value.

Definition 4.7. A C^1 -functional $J : X \rightarrow \mathbb{R}$ satisfies the Palais–Smale condition (PS) if every sequence (u_n) in X with

$$J(u_n) \text{ bounded, } J'(u_n) \rightarrow 0$$

has a convergent subsequence. Any such sequence is called a Palais–Smale sequence.

Lemma 4.8 ([16]). Let $J \in C^1(X, \mathbb{R})$ satisfy the PS condition, and let c be a regular value. Then there exists a flow $\eta \in C(\mathbb{R} \times X, X)$ satisfying properties (1)–(6) of [16].

4.3. Mountain pass theorem

Let

$$\Gamma = \{\varphi \in C^1([0, 1], X) : \varphi(0) = 0, \varphi(1) = u_0\}, \quad c = \inf_{\varphi \in \Gamma} \max_{t \in [0, 1]} J(\varphi(t)).$$

Theorem 4.9 (Mountain pass theorem, [16]). Let $J \in C^1(X, \mathbb{R})$ satisfy the PS condition. Assume:

1. $J(0) = 0$.
2. There exist $R, \delta > 0$ such that $J(u) \geq \delta$ if $\|u\|_X = R$.
3. There exists u with $\|u\|_X > R$ such that $J(u) \leq 0$.

Then c is a critical value of J , and $c \geq \delta$.

5. Main result

Equipped with the functional framework and preliminary results presented above, we now prove the existence of weak solutions to problem (2.1) by employing the variational approach based on the Palais–Smale condition.

Theorem 5.1. Under the above assumptions, there exists at least one nontrivial weak solution $u \in W_0^1 L_\Phi(\Omega)$ of the Kirchhoff problem.

Proof. The proof of Theorem 5.1 is divided into several steps.

Solutions of problem (2.1) correspond to the critical points of the Euler–Lagrange energy functional

$$\mathcal{J}(u) = \widehat{K} \left(\int_{\Omega} \Phi(x, |\nabla u|) dx \right) - \int_{\Omega} \frac{g(x)}{q} |u|^q dx - \int_{\Omega} F(x, u) dx.$$

Step 1: Verification of the Palais–Smale condition

We do not provide full proofs of compactness, coercivity, or weak lower semicontinuity. We only verify that the essential structural properties are satisfied, which allows the use of the Palais-Smale framework in this restricted setting. The key points are that the functional J is of class C^1 on X and that the nonnegativity of Φ gives $J(0) = 0$. Moreover, according to (2.2) and for $t \in [0, 1]$, we have, $\widehat{K}(t) \geq \widehat{K}(1)t^\theta$, we obtain

$$\begin{aligned}
 J(u) &= \widehat{K}(\Phi(x, |\nabla u|)) - \int_{\Omega} \frac{g(x)}{q} |u|^q dx - \int_{\Omega} F(x, u) dx \\
 &\geq \widehat{K}(1) (\Phi(x, |\nabla u|))^\theta - \frac{1}{q} \|g\|_\infty \|u\|_q^q \\
 &\geq \frac{\widehat{K}(1)}{p^\theta} \|u\|_{\mathbb{W}_0}^{p\theta} - \frac{1}{q} \|g\|_\infty \|u\|_q^q \\
 &\geq \frac{\widehat{K}(1)}{p^\theta} \|u\|_{\mathbb{W}_0}^{p\theta} - \frac{1}{q} \|g\|_\infty [u]_{L^{\mathcal{H}_g^q}(\Omega)}^q \\
 &\geq \frac{\widehat{K}(1)}{p^\theta} \|u\|_{\mathbb{W}_0}^{p\theta} - \frac{1}{q} \|g\|_\infty C \|u\|_{\mathbb{W}_0}^q \quad (\text{from injection Lemma 3.7}) \\
 &\geq \left(\frac{\widehat{K}(1)}{p^\theta} - \left[\frac{1}{q} \|g\|_\infty C \right] \|u\|_{\mathbb{W}_0}^{q-p\theta} \right) \|u\|_{\mathbb{W}_0}^{p\theta}.
 \end{aligned} \tag{5.1}$$

Therefore, for any $u \in \mathbb{W}_0$ with $\|u\|_{\mathbb{W}_0} = R \in \left(0, \min\left\{1, \left[\frac{\widehat{K}(1)}{p^\theta \left(\frac{1}{q} \|g\|_\infty C\right)} \right]^{\frac{1}{q-p\theta}} \right\} \right)$.

We conclude that $J(u) \geq \delta$ where $\delta = \left(\frac{\widehat{K}(1)}{p^\theta} - \left[\frac{1}{q} \|g\|_\infty C \right] R^{q-p\theta} \right) R^{p\theta}$.

We choose, for $u_0 \in \mathbb{W}_0$, $\|u_0\|_{\mathbb{W}_0} > 1$ and t sufficiently large, we obtain:

$$\begin{aligned}
 J(tu_0) &= \widehat{K}(\Phi(x, |\nabla tu_0|)) - \int_{\Omega} \frac{g(x)}{q} |tu_0|^q dx - \int_{\Omega} F(x, u_0) dx \\
 &\leq C \widehat{K}(1) t^{p\theta} \|u_0\|_{\mathbb{W}_0}^{p\theta} - \frac{g_0}{q} t^q \|u_0\|_q^q. \\
 &\leq t^{p\theta} \left(C \widehat{K}(1) \|u_0\|_{\mathbb{W}_0}^{p\theta} - \frac{g_0}{q} t^{q-p\theta} \|u_0\|_q^q \right).
 \end{aligned} \tag{5.2}$$

Since $1 \leq p\theta < q$, it is easy to see that $J(tu_0) \rightarrow -\infty$ when $t \rightarrow +\infty$.

Hence, we find t sufficiently large and $\epsilon > 0$ such that

$$J(tu_0) \leq -\epsilon < 0.$$

Step 2: Coercivity and monotonicity of the operator

Before establishing compactness results for Palais–Smale sequences, it is essential to analyze the structural properties of the underlying nonlinear operator,

$$a(x, \xi) = g(|\xi|) \xi, \quad g(r) = r^{p-2} \left(1 + \log(e+r) + \frac{r}{p(e+r)} \right),$$

belongs to the class of radial, Leray–Lions type operators with logarithmic perturbations, a setting naturally encountered in borderline double-phase problems. The following proposition establishes precise coercivity and monotonicity inequalities.

Proposition 5.2. *There exist constants $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$ and $C_3 = 1$, such that for all $\xi, \eta \in \mathbb{R}^N$,*

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq C_1 \psi(|\xi - \eta|), \quad a(x, \xi) \cdot \xi \geq C_2 \phi(|\xi|) - C_3,$$

with

$$\psi(r) = r^p(1 + \log(e + r)) + r^p \log(e + r), \quad \phi(r) = r^p(1 + \log(e + r)).$$

Proof. Define

$$h(r) := g(r)r = r^{p-1} \left(1 + \log(e + r) + \frac{r}{p(e + r)} \right), \quad r \geq 0.$$

Then

$$h'(r) = (p - 1)r^{p-2} \left(1 + \log(e + r) + \frac{r}{p(e + r)} \right) + r^{p-1} \left(\frac{1}{e + r} + \frac{e}{p(e + r)^2} \right),$$

hence

$$h'(r) \geq \frac{1}{2}r^{p-2}(1 + \log(e + r)), \quad \forall r \geq 0.$$

Scalar monotonicity. For $\xi, \eta \in \mathbb{R}$, by the mean value theorem,

$$(g(|\xi|)\xi - g(|\eta|)\eta)(\xi - \eta) = (h(\xi) - h(\eta))(\xi - \eta) = h'(\theta)(\xi - \eta)^2,$$

with θ between $|\xi|$ and $|\eta|$.

Hence

$$(g(|\xi|)\xi - g(|\eta|)\eta)(\xi - \eta) \geq \frac{1}{2}|\xi - \eta|^p(1 + \log(e + |\xi - \eta|)).$$

Vectorial monotonicity. For $\xi, \eta \in \mathbb{R}^N$, the standard inequality for radial operators (see [19, 26]) implies

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq C_1 \psi(|\xi - \eta|).$$

Coercivity.

$$a(x, \xi) \cdot \xi = g(|\xi|)|\xi|^2 = h(|\xi|) \geq \frac{1}{2}\phi(|\xi|) - 1, \tag{5.3}$$

so that coercivity holds with $C_2 = \frac{1}{2}$ and $C_3 = 1$. □

Remark 5.3. The constants $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$, and $C_3 = 1$ are uniform, ensuring coercivity and strict monotonicity independently of ξ, η . For more details, see Lindqvist [19] and Ziemer [26].

The coercivity and monotonicity properties established above provide the structural backbone of our operator. They ensure that the variational setting is well-posed and allow us to invoke Minty’s lemma, a key tool to upgrade weak convergence into strong convergence.

Lemma 5.4 (Minty Lemma). *Let $A : \mathbb{W}_0 \rightarrow \mathbb{W}_0^*$ be strictly monotone and weakly continuous. If $u_n \rightharpoonup u$ and $\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ strongly in \mathbb{W}_0 and $A(u_n) \rightarrow A(u)$ in \mathbb{W}_0^* .*

For a proof, see Minty [20] or Brézis [7].

5.1. Step 3: Strong convergence of Palais–Smale sequences

Proposition 5.5. *Let $\{u_n\}$ be a bounded Palais–Smale sequence for J . Then, up to a subsequence,*

$$u_n \rightarrow u \quad \text{strongly in } \mathbb{W}_0.$$

Proof. Assume, by contradiction, that $\|u_n\|_{\mathbb{W}_0} \rightarrow \infty$.

Let $\Psi(u_n) = \int_{\Omega} \Phi(x, |\nabla u_n|) dx$, then

$$\begin{aligned} J(u_n) - \frac{1}{\lambda} \langle J'(u_n), u_n \rangle &= \left[\widehat{K}(\Psi(u_n)) - \frac{1}{\lambda} K(\Psi(u_n)) \int_{\Omega} a(x, \nabla u_n) \cdot \nabla u_n dx \right] \\ &\quad + \left[\frac{1}{\lambda} \int_{\Omega} g(x) |u_n|^q dx - \int_{\Omega} \frac{g(x)}{q} |u_n|^q dx \right] \\ &\quad + \left[\frac{1}{\lambda} \int_{\Omega} f(x, u_n) u_n dx - \int_{\Omega} F(x, u_n) dx \right]. \end{aligned}$$

□

Using $a(x, \nabla u_n) \cdot \nabla u_n \leq (p+1)\Phi(x, |\nabla u_n|)$ since $\frac{|\nabla u_n|}{e+|\nabla u_n|} \leq 1$ and (2.4) we have

$$\begin{aligned} J(u_n) - \frac{1}{\lambda} \langle J'(u_n), u_n \rangle &\geq \widehat{K}(\Psi(u_n)) - \frac{p+1}{\lambda} K(\Psi(u_n)) \Psi(u_n) + \left(\frac{q-\lambda}{\lambda q} \right) \int_{\Omega} g(x) |u_n|^q dx \\ &\geq \left(\frac{1}{\theta} - \frac{p+1}{\lambda} \right) K(\Psi(u_n)) \Psi(u_n) + \left(\frac{q-\lambda}{\lambda q} \right) \int_{\Omega} g(x) |u_n|^q dx. \end{aligned}$$

Since $(p+1)\theta < \lambda < q$ and $g \geq 0$, we have

$$J(u_n) - \frac{1}{\lambda} \langle J'(u_n), u_n \rangle \geq \left(\frac{1}{\theta} - \frac{p+1}{\lambda} \right) K(\Psi(u_n)) \Psi(u_n),$$

and by (4.1) and (3.3), we deduce

$$J(u_n) - \frac{1}{\lambda} \langle J'(u_n), u_n \rangle \geq \left(\frac{1}{\theta} - \frac{p+1}{\lambda} \right) k_0 C \|u_n\|_{\mathbb{W}_0}^p.$$

The right-hand side diverges to $+\infty$, contradicting boundedness of $J(u_n)$. Thus, $\{u_n\}$ is bounded in \mathbb{W}_0 . On the other-hand, by reflexivity, there exists $u \in \mathbb{W}_0$ such that (up to subsequence)

$$u_n \rightharpoonup u \quad \text{in } \mathbb{W}_0, \quad u_n \rightarrow u \quad \text{a.e. in } \Omega.$$

Now, Take $\varphi_n = u_n - u$ in $J'(u_n)$, then

$$\langle J'(u_n), \varphi_n \rangle = K(\Psi(u_n)) \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) dx - T_g - T_f,$$

where

$$T_g = \int_{\Omega} g(x) |u_n|^{q-2} u_n (u_n - u) dx, \quad T_f = \int_{\Omega} f(x, u_n) (u_n - u) dx.$$

By compactness $\mathbb{W}_0 \hookrightarrow L^q$, we have $T_g \rightarrow 0$. By dominated convergence (subcritical growth of f), $T_f \rightarrow 0$. Hence,

$$K(\Psi(u_n)) \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx \rightarrow 0.$$

That means that $\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0$. Since u_n is bounded and K is continuous and $K \geq k_0 > 0$. Thus,

$$I_n = \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx \rightarrow 0. \quad (5.4)$$

Using strict monotonicity (Proposition 5.2), we deduce

$$\int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \geq C \int_{\Omega} \Phi(x, |\nabla u_n - \nabla u|) \, dx,$$

$$\int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx = I_n - \int_{\Omega} a(x, \nabla u) \cdot (\nabla u_n - \nabla u) \, dx.$$

Since $u_n \rightharpoonup u$ in \mathbb{W}_0 , the term $\int_{\Omega} a(x, \nabla u) \cdot (\nabla u_n - \nabla u) \, dx \rightarrow 0$ as $n \rightarrow \infty$ by the definition of weak convergence.

Combined with (5.4), we conclude that

$$\lim_{n \rightarrow \infty} \int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx = 0.$$


Hence, the Palais–Smale sequence $\{u_n\}$ is bounded and converges strongly to a critical point u of J . This establishes the existence of a weak solution in the Musielak–Orlicz–Zygmund framework. □

References

- [1] Aberqi, A., Benslimane, O., Elmassoudi, M., Ragusa, M. A., *Nonnegative solution of a class of double phase problems with logarithmic nonlinearity*, Boun. Value Probl., 2022, 1–13. <https://doi.org/10.1186/s13661-022-01639-5>
- [2] Aberqi, A., Bennouna, J., Elmassoudi, M., *Existence and uniqueness of renormalized solution for nonlinear parabolic equations in Musielak Orlicz spaces*, Bol. Soc. Parana. Mat. **3**(40)(2022), 1–22. <https://doi.org/10.5269/bspm.45234>.
- [3] Baroni, P., Colombo, M., Mingione, G., *Nonautonomous functionals, borderline cases and related function classes*, St. Petersburg Mathematical Journal, **27**(3)(2016), 347–379. <https://doi.org/10.1090/spmj/1392>
- [4] Bennett, C., Sharpley, R., *Interpolation of Operators*, Academic Press, Boston, 1988.
- [5] Benslimane, O., Aberqi, A., Elmassoudi, M., *Double phase problem with singularity and homogenous Choquard type term*, J. Appl. Anal. Comput., **14**(4)(2024), 2109–2124. <https://doi.org/10.11948/20230303>
- [6] Boccardo, L., Murat, F., *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal., **19**(6)(1992), 581–597. [https://doi.org/10.1016/0362-546X\(92\)90023-8](https://doi.org/10.1016/0362-546X(92)90023-8).

- [7] Brézis, H., *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Math. Stud., vol. 5, North-Holland Publishing Company, Amsterdam, 1973.
- [8] Cianchi, A., *A sharp embedding theorem for Orlicz-Sobolev spaces*, Indiana Univ. Math. J., **45**(1996), 39–65.
- [9] Colombo, M., Mingione, G., *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal. **218**(2015), 219–273. <https://doi.org/10.1007/s00205-015-0859-9>
- [10] Diening, L., Harjulehto, P., Hästö, P., Růžička, M., *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2010. <https://doi.org/10.1007/978-3-642-18363-8>
- [11] El Alami, B., Aberqi, A., Elmassoudi, M., *Analysis of a singular double-phase equation with homogeneous Choquard term*, J. Math. Sci. (N.Y.), 2025, 1–10. <https://doi.org/10.1007/s10958-025-07756-7>
- [12] El Ouardani, A., Aberqi, A., Benslimane, O., Elmassoudi, M., *Investigation into double-phase elliptic problems with boundary conditions*, incorporating a logarithmic convection term, J. Pseudo-Differ. Oper. Appl., **16**(2025), 1–21. <https://doi.org/10.1007/s11868-024-00671-6>
- [13] Fiscella, A., Pinamonti, A., *Existence and Multiplicity Results for Kirchhoff-Type Problems on a Double-Phase Setting*, Mediterr. J. Math. **20**(33)(2023). <https://doi.org/10.1007/s00009-022-02245-6>
- [14] Gossez, J. P., *Some approximation properties in Orlicz-Sobolev spaces*, No. MRCTSR2126, 1980.
- [15] Iwaniec, T., Verde, A., *On the Operator $L(f) = f \log|f|$* , J. Funct. Anal., **169**(1999), 391–420. <https://doi.org/10.1006/jfan.1999.3443>
- [16] Kavian, O., *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Mathématiques et applications, vol. 13, 1993, Springer-Verlag.
- [17] Kirchhoff, G., *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.
- [18] Lang, J., Mendez, O., *Analysis on Function Spaces of Musielak-Orlicz Type*, Chapman & Hall/CRC Monographs and Research Notes in Mathematics, 2019. <https://doi.org/10.1201/9781498762618>
- [19] Lindqvist, P., *Notes on the p -Laplace Equation*, University of Jyväskylä Lecture Notes, 2006. Available at arXiv: math/0605532.
- [20] Minty, G. J., *Monotone operators in Hilbert space*, Duke Math. J., **29**(3)(1962), 341–346. <https://doi.org/10.1215/S0012-7094-62-02933-2>
- [21] Musielak, J., *Modular spaces and Orlicz spaces*, Lecture Notes in Math., 1983. <https://doi.org/10.1007/BFb0072210>
- [22] Shen, L., Squassina, M., *Existence and concentration of normalized solutions for p -Laplacian equations with logarithmic nonlinearity*, J. Differential Equations, **421**(2025), 1–49. <https://doi.org/10.1016/j.jde.2024.11.049>
- [23] Simon, J., *Régularité de la solution d'une équation non linéaire dans \mathbb{R}^N* , In: Bénilan, P., Robert, J. (eds) *Journées d'Analyse Non Linéaire. Lecture Notes in Mathematics*, no. V., Springer, Berlin, Heidelberg, **665**(1978). <https://doi.org/10.1007/BFb0061807>
- [24] Stein, E. M., *Note on the Class $LlogL$* , Studia Math. **31**(1969), 305–310.

- [25] Tran, M. P., Nguyen, T. N., *Existence of weak solutions to borderline double-phase problems with logarithmic convection terms*, J. Math. Anal. Appl., **1**(2025), 129185. <https://doi.org/10.1016/j.jmaa.2024.129185>
- [26] Ziemer, W. P. *Weakly Differentiable Functions, Graduate Texts in Mathematics*, Springer, New York, vol. 120, 1989. <https://doi.org/10.1007/978-1-4612-1015-3>

Ahmed El Ouardani 


LSATE Laboratory, ENSA, Sidi Mohamed Ben Abdellah University,
Fez, Morocco.

e-mail: elouardani.ahmed.75@gmail.com

Ahmed Aberqi 

LSATE Laboratory, ENSA, Sidi Mohamed Ben Abdellah University,
Fez, Morocco.




e-mail: ahmed.aberqi@usmba.ac.ma

Mhamed Elmassoudi 

L2MASI Laboratory, Faculty of Sciences Dhar El Mahraz,
University Sidi Mohamed Ben Abdellah, Fez, Morocco.

e-mail: mhamed.elmassoudi@usmba.ac.ma

Low-regret control of a nonlinear parabolic problem with missing data

Biéliémi Lamien , Sadou Tao  and Elisée Gouba 

Abstract. In this paper, we study the optimal control of a nonlinear parabolic problem with missing data. Using the concepts of no-regret control, low-regret control and adapted low-regret control, we give a characterization of the control for ill-posed problems. More precisely, we study the control of a nonlinear parabolic problem using a regularization approach that generates incomplete information. We obtain a singular optimality system characterizing the no-regret control for the nonlinear parabolic problem.

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1. Introduction

Let Ω be a bounded open subset of \mathbb{R}^n with a sufficiently smooth boundary Γ . Let $T > 0$ and let $Q = \Omega \times (0, T)$ denote the space–time cylinder, with lateral boundary $\Sigma = \Gamma \times (0, T)$.


Consider the following Cauchy problem in Q :

$$\begin{cases} \frac{\partial z}{\partial t} - \Delta z - z^3 = v & \text{in } Q, \\ z(0) = 0 & \text{in } \Omega. \end{cases} \quad (1.1)$$

Let $z \in L^2(Q)$ denote the state of the system and $v \in L^2(Q)$ the control. The state of the system is unknown on the boundary of the domain. Thus, problem (1.1) is a problem with missing data. Due to the presence of the term z^3 in (1.1), this problem is generally unstable. In the case where a boundary condition is prescribed, the solution

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generally blows up at a time $t_0 < T$ [3, 4, 13]. This problem was studied by Lions in [3], where he obtained a characterization of optimal pairs. Optimal control problems for nonlinear distributed systems constitute an open and active field of research. Most existing works focus on deriving necessary or sufficient first- or second-order optimality conditions. Owing to the lack of obvious regularity of the state solution, the issues of existence and characterization of optimal controls still require a detailed analysis. The assumptions imposed on control problems are generally chosen so that a more or less adapted standard method can be applied to establish the existence of solutions. In the present work, we apply the low-regret control method introduced by Lions in [6] to study the control of nonlinear distributed systems with missing data. We then characterize the no-regret control of the problem with missing data through a singular optimality system. The notion of no-regret control was introduced by Leonard J. Savage [12]. Lions subsequently employed this concept in the context of optimal control, motivated by numerous applications in economics and ecology [6]. Indeed, he was the first to use both the low-regret and no-regret control concepts to control distributed systems with missing data in various areas of applied mathematics [1, 5, 7]. For example, integrating a decision criterion into a closed and nonempty subspace of uncertainties makes it possible to improve existing results and to extend the concept of low-regret control to a framework involving multiple agents in economics [1]. Picart et al. apply the concepts of no-regret and low-regret control to address problems related to the control of nutrient absorption by roots in the rhizosphere (the area surrounding the roots), considering two types of soil: healthy soil and polluted soil [9]. Nakoulima et al. successfully characterize no-regret control for missing data problems in both stationary and evolutionary cases through a singular optimality system [10, 11]. In this article, we regularize problem (1.1) as a sequence of well-posed problems to study the control problem using the low-regret method, which is particularly suited to problems with missing data. We show that the no-regret control corresponds to the limit of the low-regret control as the disturbance parameter tends to zero. The rest of the article is organized as follows: in Section 2, we present the regularization and control problem. In Section 3, we characterize low-regret and no-regret control via singular optimality systems.

2. Regularization and control problem

2.1. Regularization of the ill-posed problem

In this section, we regularize problem (1.1) following the work of Lions [2, 3] on the regularization of problems with missing data. Thus, for all $\varepsilon > 0$ and for all $g \in L^2(\Sigma)$, we consider the following regularized problem:

$$\begin{cases} \frac{\partial z_\varepsilon}{\partial t} - \Delta z_\varepsilon - z_\varepsilon^3 + \varepsilon z_\varepsilon^5 = v & \text{in } Q, \\ z_\varepsilon(0) = 0 & \text{in } \Omega, \\ z_\varepsilon = \varepsilon g & \text{on } \Sigma. \end{cases} \quad (2.1)$$

Let G be a nonempty, closed vector subspace of $L^2(\Sigma)$, equipped with the $L^2(\Sigma)$ norm. The function g ranges over the set G and may represent pollution or a disturbance.

Remark 2.1. For any fixed $\varepsilon > 0$ and $g \in L^2(\Sigma)$, the system (2.1) has a unique solution [2].

2.2. Cost function and control problem

2.2.1. Analysis of cost function and no-regret control case. Let $z_\varepsilon := z_\varepsilon(v, g)$ denote the state of system (2.1). For all $v \in L^2(Q)$ and $g \in L^2(\Sigma)$, we associate the following cost function:

$$J_\varepsilon(v, g) = \frac{1}{2} \|z_\varepsilon(v, g) - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|v\|_{L^2(Q)}^2. \tag{2.2}$$

Here, $N > 0$ and $z_d \in L^2(Q)$ denotes the desired state.

Definition 2.2. We say that $u \in L^2(Q)$ is a no-regret control for problem (1.1) if u is a solution of the following problem:

$$\inf_{v \in L^2(Q)} \left[\sup_{g \in L^2(\Sigma)} (J_\varepsilon(v, g) - J_\varepsilon(0, g)) \right]. \tag{2.3}$$

Remark 2.3. To make the nonlinear problem amenable to analysis, we consider the standard definition of no-regret control associated with a linearized cost functional, and then study the corresponding no-regret control problem.

The concepts of no-regret control and low-regret control, introduced by Lions in [6], rely on the decomposition of the solution of (2.1) in the form $z_\varepsilon(v, g) = z_\varepsilon(v, 0) + z_\varepsilon(0, g) - z_\varepsilon(0, 0)$ where $z_\varepsilon(v, 0)$ is the solution of (2.1) with $g = 0$ and $z_\varepsilon(0, g)$ is the solution of (2.1) with $v = 0$. This decomposition is no longer valid in our case because the mapping $(v, g) \mapsto z_\varepsilon(v, g)$ from $L^2(Q) \times L^2(\Sigma)$ to $L^2(Q)$ is nonlinear. For this nonlinear case, we exploit the regularity of the solution of (2.1) and replace the cost function defined in (2.2) by its linear form with respect to g . Thus, following [8], we consider the new cost function given by:

$$J_{1\varepsilon}(v, g) = J_\varepsilon(v, 0) + \frac{\partial J_\varepsilon}{\partial g}(v, 0)(g). \tag{2.4}$$

Hence, we consider the following new optimization problem:

$$\inf_{v \in L^2(Q)} \left[\sup_{g \in L^2(\Sigma)} (J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g)) \right]. \tag{2.5}$$

The solution of (2.5) if exists, is called no-regret control for the nonlinear problem. Then, we have the following proposition.

Proposition 2.4. For all $(v, g) \in L^2(Q) \times L^2(\Sigma)$, we have the following equality:

$$\begin{aligned} J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) &= J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle \\ &\quad - \left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) \right\rangle, \end{aligned} \tag{2.6}$$

where J_ε is the cost function defined in (2.2).

Proof. The mapping $(v, g) \mapsto z_\varepsilon(v, g)$ is differentiable with respect to g in $L^2(\Sigma)$. Thus, the first-order (linear) expansion of z_ε at the point $(v, 0)$ along the direction of g is:

$$\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda}.$$

We have :

$$\begin{aligned} J_\varepsilon(v, \lambda g) &= J_\varepsilon(v, 0) + \frac{1}{2} \|z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)\|_{L^2(Q)}^2 \\ &\quad + \langle z_\varepsilon(v, 0) - z_d, z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0) \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{J_\varepsilon(v, \lambda g) - J_\varepsilon(v, 0)}{\lambda} &= \left\langle z_\varepsilon(v, 0) - z_d, \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda} \right\rangle \\ &\quad + \lim_{\lambda \rightarrow 0} \frac{\lambda}{2} \left\| \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda} \right\|_{L^2(Q)}^2 \\ &= \left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle \\ &= \frac{\partial J_\varepsilon}{\partial g}(v, 0)(g). \end{aligned} \tag{2.7}$$

By substituting (2.7) into (2.5), we obtain:

$$J_{1\varepsilon}(v, g) = J_\varepsilon(v, 0) + \left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle. \tag{2.8}$$

By substituting $v = 0$ into (2.8), we obtain:

$$J_{1\varepsilon}(0, g) = J_\varepsilon(0, 0) + \left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) \right\rangle. \tag{2.9}$$

Consequently,

$$\begin{aligned} J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) &= J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle \\ &\quad - \left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) \right\rangle. \end{aligned}$$

□

In what follows, we use the propositions below to estimate the terms

$$\left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle \text{ and } \left\langle z_\varepsilon(0, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) \right\rangle \text{ in (2.6).}$$

Proposition 2.5. *The partial derivative $\frac{\partial z_\varepsilon}{\partial g} := \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda}$ is the solution of the following problem:*

$$\begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g} \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial g} \right) - 3z_\varepsilon^2(v, 0) \frac{\partial z_\varepsilon}{\partial g} + 5\varepsilon z_\varepsilon^4(v, 0) \frac{\partial z_\varepsilon}{\partial g} = 0 & \text{in } Q, \\ \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \Big|_{t=0} = 0 & \text{in } \Omega, \\ \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \varepsilon g & \text{on } \Sigma. \end{cases} \quad (2.10)$$

Proof. Let us show that $\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g)$ is a solution of (2.10).

- For the condition in Q , we have:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g} \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial g} \right) - 3z_\varepsilon^2 \frac{\partial z_\varepsilon}{\partial g} + 5\varepsilon z_\varepsilon^4 \frac{\partial z_\varepsilon}{\partial g} &= \frac{\partial}{\partial g} \left(\frac{\partial z_\varepsilon}{\partial t} - \Delta z_\varepsilon - z_\varepsilon^3 + \varepsilon z_\varepsilon^5 \right) \\ &= 0 \text{ in } Q. \end{aligned}$$

Indeed,

$$\frac{\partial z_\varepsilon}{\partial t}(v, g) - \Delta z_\varepsilon(v, g) - z_\varepsilon^3(v, g) + \varepsilon z_\varepsilon^5(v, g) = v \text{ in } Q$$

- For the initial condition in Ω , we have :

$$\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \Big|_{t=0} = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) \Big|_{t=0} - z_\varepsilon(v, 0) \Big|_{t=0}}{\lambda} = 0 \text{ because } z_\varepsilon(v, g) \Big|_{t=0} = 0 \text{ in } \Omega.$$

- For the boundary condition on Σ , we have :

$$\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v, \lambda g) - z_\varepsilon(v, 0)}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\lambda \varepsilon g - 0}{\lambda} = \varepsilon g \text{ on } \Sigma. \quad \square$$

Proposition 2.6. *The partial derivative $\frac{\partial z_\varepsilon}{\partial g} := \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(0, \lambda g) - z_\varepsilon(0, 0)}{\lambda}$ is the solution of the following problem :*

$$\begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g} \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial g} \right) - 3z_\varepsilon^2(0, 0) \frac{\partial z_\varepsilon}{\partial g} + 5\varepsilon z_\varepsilon^4(0, 0) \frac{\partial z_\varepsilon}{\partial g} = 0 & \text{in } Q, \\ \frac{\partial z_\varepsilon}{\partial g} \Big|_{t=0} = 0 & \text{in } \Omega, \\ \frac{\partial z_\varepsilon}{\partial g}(0, 0)(g) = \varepsilon g & \text{on } \Sigma. \end{cases} \quad (2.11)$$

Remark 2.7. The proof of Proposition 2.6 is carried out in the same way as that of Proposition 2.5.

Now we have the following lemma:

Lemma 2.8. For all $v \in L^2(Q)$ and for all $g \in L^2(\Sigma)$, we have:

$$J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}, \quad (2.12)$$

with $S(v) = \zeta_\varepsilon(v) - \zeta_\varepsilon(0)$ and $\zeta_\varepsilon(v) = \zeta_\varepsilon(x, t, v)$ the solution of the following system:

$$\begin{cases} -\frac{\partial \zeta_\varepsilon}{\partial t}(v) - \Delta \zeta_\varepsilon(v) - 3z_\varepsilon^2(v, 0)\zeta_\varepsilon(v) + 5\varepsilon z_\varepsilon^4(v, 0)\zeta_\varepsilon(v) = -z_\varepsilon(v, 0) + z_d & \text{in } Q, \\ \zeta_\varepsilon(T, v) = 0 & \text{in } \Omega, \\ \zeta_\varepsilon(v) = 0 & \text{on } \Sigma. \end{cases} \quad (2.13)$$

Proof. We multiply the first equation of (2.13) by $\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g)$. By integrating by parts and using the boundary conditions, $\zeta_\varepsilon(T, v) = 0$ and $\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \Big|_{t=0} = 0$ in Ω , we obtain that:

$$\left\langle -\frac{\partial \zeta_\varepsilon}{\partial t}(v), \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle_{L^2(Q)} = \left\langle \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right), \zeta_\varepsilon(v) \right\rangle_{L^2(Q)}.$$

Using the conditions $\zeta_\varepsilon(v) = 0$ and $\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = \varepsilon g$ on Σ and by applying Green's formula, we have:

$$\begin{aligned} \left\langle -\Delta \zeta_\varepsilon(v), \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle_{L^2(Q)} &= \left\langle \frac{\partial}{\partial g}(-\Delta z_\varepsilon(v, 0))(g), \zeta_\varepsilon(v) \right\rangle_{L^2(Q)} \\ &\quad - \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

Therefore:

$$\left\langle -(z_\varepsilon(v, 0) - z_d), \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle_{L^2(Q)} = - \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}.$$

Indeed,

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right) - 3z_\varepsilon^2(v, 0) \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \\ + 5\varepsilon z_\varepsilon^4(v, 0) \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) = 0 &\text{ in } Q. \end{aligned}$$

It follows that :

$$\left\langle z_\varepsilon(v, 0) - z_d, \frac{\partial z_\varepsilon}{\partial g}(v, 0)(g) \right\rangle_{L^2(Q)} = \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}. \quad (2.14)$$

For $v = 0$, the system (2.13) becomes:

$$\begin{cases} -\frac{\partial \zeta_\varepsilon}{\partial t}(0) - \Delta \zeta_\varepsilon(0) - 3z_\varepsilon^2(0, 0)\zeta_\varepsilon(0) + 5\varepsilon z_\varepsilon^4(0, 0)\zeta_\varepsilon(0) = -z_\varepsilon(0, 0) + z_d & \text{in } Q, \\ \zeta_\varepsilon(T, 0) = 0 & \text{in } \Omega, \\ \zeta_\varepsilon(0) = 0 & \text{on } \Sigma. \end{cases} \quad (2.15)$$

By multiplying the first equation of (2.15) by $\frac{\partial z_\varepsilon}{\partial g}(0,0)(g)$ and then applying the previous techniques, we obtain:

$$\left\langle -(z_\varepsilon(0,0) - z_d), \frac{\partial z_\varepsilon}{\partial g}(0,0) \right\rangle_{L^2(Q)} = - \left\langle \frac{\partial \zeta_\varepsilon}{\partial \nu}(0), \varepsilon g \right\rangle_{L^2(\Sigma)}. \quad (2.16)$$

By substituting (2.14) and (2.16) into the equation (2.6) of the Proposition 2.4, we obtain:

$$J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)},$$

with $S(v) = \zeta_\varepsilon(v) - \zeta_\varepsilon(0)$. □

Remark 2.9. Using the previous result, we obtain:

$$\sup_{g \in L^2(\Sigma)} (J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g)) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \sup_{g \in L^2(\Sigma)} \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle, \quad (2.17)$$

$$\text{with } \sup_{g \in L^2(\Sigma)} \left(\left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} \right) = \begin{cases} +\infty & \text{if } \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} \neq 0, \\ 0 & \text{if } \frac{\partial S}{\partial \nu}(v) = 0. \end{cases}$$

The result of $\sup_{g \in L^2(\Sigma)} \left(\left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} \right)$ does not allow us to characterize the no-regret control of the following problem:

$$\inf_{v \in L^2(Q)} \left[\sup_{g \in L^2(\Sigma)} (J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g)) \right]. \quad (2.18)$$

To make sense of problem (2.18), we consider the set E_1 defined by:

$$E_1 = \left\{ v \in L^2(Q), \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} = 0, \forall g \in L^2(\Sigma) \right\}.$$

In what follows, we use the notion of low-regret control to seek a control in the set E_1 .

2.2.2. Low-regret control.

In this subsection, we focus on the low-regret problem, which depends on the disturbance g and its norm. This problem is obtained by relaxing problem (2.18). We then consider the following definition:

Definition 2.10. *Let $\gamma > 0$ be a fixed number representing the relaxation parameter. Then, problem (2.18) becomes:*

$$\inf_{v \in L^2(Q)} \left[\sup_{g \in L^2(\Sigma)} \left(J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) - \gamma \|g\|_{L^2(\Sigma)}^2 \right) \right]. \quad (2.19)$$

The solution of (2.19), if it exists, is called low-regret control.

Lemma 2.11. *We consider the function J_ε defined in (2.2). Thus, for all $v \in L^2(Q)$, the problem (2.19) becomes:*

$$\inf_{v \in L^2(Q)} \left[J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu} \right\|_{L^2(\Sigma)}^2 \right]. \tag{2.20}$$

Proof. Using the result of Lemma 2.8, we have:

$$\begin{aligned} \sup_{g \in L^2(\Sigma)} \left(J_{1\varepsilon}(v, g) - J_{1\varepsilon}(0, g) - \gamma \|g\|_{L^2(\Sigma)}^2 \right) &= J_\varepsilon(v, 0) - J_\varepsilon(0, 0) \\ &- \frac{\gamma}{2} \|g\|_{L^2(\Sigma)}^2 + \sup_{g \in L^2(\Sigma)} \left\langle \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)}. \end{aligned} \tag{2.21}$$

According to the Fenchel transformation, we have:

$$\begin{aligned} 2 \sup_{g \in L^2(\Sigma)} \left(\left\langle \frac{1}{2} \frac{\partial S}{\partial \nu}(v), \varepsilon g \right\rangle_{L^2(\Sigma)} - \frac{\gamma}{2} \|g\|_{L^2(\Sigma)}^2 \right) &= \frac{\varepsilon^2}{2\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2 \\ - \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2 &= \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2. \end{aligned}$$

Thus, the problem (2.19) becomes :

$$\inf_{v \in L^2(Q)} \mathcal{J}_\varepsilon^\gamma(v) \tag{2.22}$$

with $\mathcal{J}_\varepsilon^\gamma(v) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2$. □

Now we reformulate the problem (2.19) as follows:

$$\left\{ \begin{array}{l} \text{For all } \gamma > 0, \text{ find } u_\varepsilon^\gamma \in L^2(Q) \text{ such that :} \\ \mathcal{J}_\varepsilon^\gamma(u_\varepsilon^\gamma) = \inf_{v \in L^2(Q)} \mathcal{J}_\varepsilon^\gamma(v). \end{array} \right. \tag{2.23}$$

Problem (2.23) is a low-regret problem and its solution is called low-regret control.

Remark 2.12. In contrast to the linear case, the function J_ε^γ is not convex. So we don't necessarily have uniqueness of u_ε^γ . Moreover, we are not convinced that u_ε^γ converges in E_1 . Then, we will adapt the function J_ε^γ like J.L. Lions [6] by using a regularization method for finding the low-regret control related to the problem.

2.3. Adapted low-regret control.

Since the function J_ε^γ is not convex, we define a new function to study the control problem. We are therefore interested in finding a solution to the following minimization problem:

$$\inf_{v \in L^2(Q)} \mathcal{J}_{\varepsilon a}^\gamma(v) \tag{2.24}$$

with $\mathcal{J}_{\varepsilon a}^\gamma(v) = J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{1}{2} \|v - \tilde{u}\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2$, where $\tilde{u} \in L^2(Q)$ is the no-regret control. The control solution u_ε^γ of problem (2.24) is called the adapted low-regret control.

2.3.1. Existence of adapted low-regret control. In this section, we establish the existence of the adapted low-regret control, inspired by the work of T. Thomas et al. in [14, 15]. We consider the following proposition:

Proposition 2.13. *There exists at least one solution $u_\varepsilon^\gamma \in L^2(Q)$, called the adapted low-regret control, for the optimal control problem (2.24).*

Proof. According to the definition of $\mathcal{J}_{\varepsilon a}^\gamma$, for all $v \in L^2(Q)$, we have: $\mathcal{J}_{\varepsilon a}^\gamma(v) \geq -J_\varepsilon(0, 0) \Leftrightarrow -J_\varepsilon(0, 0) \leq J_\varepsilon(v, 0) - J_\varepsilon(0, 0) + \frac{1}{2} \|v - \tilde{u}\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v) \right\|_{L^2(\Sigma)}^2$.

Let us consider the nonempty set $E_2 = \{v \in L^2(Q), \mathcal{J}_{\varepsilon a}^\gamma(v) \geq -J_\varepsilon(0, 0)\}$. Indeed, by taking $v = 0$, we have: $\mathcal{J}_{\varepsilon a}^\gamma(0) = \frac{1}{2} \|\tilde{u}\|_{L^2(Q)}^2 \geq -J_\varepsilon(0, 0)$. Thus, there exists a real constant C , which depends on ε and γ , such that $C(\varepsilon, \gamma) = \inf_{v \in L^2(Q)} \mathcal{J}_{\varepsilon a}^\gamma(v)$. Indeed, we consider a minimizing sequence $(v_n)_{n \in \mathbb{N}} = (v_n(\varepsilon, \gamma))_{n \in \mathbb{N}}$ that converges to d_ε^γ . Then, $d_\varepsilon^\gamma = \lim_{n \rightarrow +\infty} \mathcal{J}_{\varepsilon a}^\gamma(v_n)$.

$$\begin{aligned} -J_\varepsilon(0, 0) &\leq \frac{1}{2} \|z_\varepsilon(v_n, 0) - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|v_n\|_{L^2(Q)}^2 - \frac{1}{2} \|z_\varepsilon(0, 0) - z_d\|_{L^2(Q)}^2 \\ &\quad + \frac{1}{2} \|v_n - \tilde{u}\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial S}{\partial \nu}(v_n) \right\|_{L^2(\Sigma)}^2 \\ &\leq d_\varepsilon^\gamma + 1. \end{aligned}$$

We deduce the following inequalities, which show that the sequences are bounded.

$$\begin{cases} \|v_n\|_{L^2(Q)} \leq C_1(\varepsilon, \gamma), \\ \|z_\varepsilon(v_n, 0) - z_d\|_{L^2(Q)} \leq C_2(\varepsilon, \gamma), \\ \frac{\varepsilon}{\sqrt{\gamma}} \left\| \frac{\partial S}{\partial \nu}(v_n) \right\|_{L^2(\Sigma)} \leq C_3(\varepsilon, \gamma). \end{cases}$$

Here, $C_i(\varepsilon, \gamma)$, for $i \in 1, 2, 3$, are constants independent of n . Therefore, we can extract a subsequence, still denoted by $(v_n)_{n \in \mathbb{N}}$, such that $v_n(\varepsilon, \gamma) \rightharpoonup u_\varepsilon^\gamma$ in $L^2(Q)$. \square

In order to characterize low-regret control, we consider the following propositions:

Proposition 2.14. For all $w \in L^2(Q)$, the partial derivative $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0) - z_\varepsilon(v, 0)}{\lambda}$ be the solution of the following problem:

$$\begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v} \right) - \Delta \left(\frac{\partial z_\varepsilon^\gamma}{\partial v} \right) - 3z_\varepsilon^2 \frac{\partial z_\varepsilon^\gamma}{\partial v} + 5\varepsilon z_\varepsilon^4 \frac{\partial z_\varepsilon^\gamma}{\partial v} = w & \text{in } Q, \\ \frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \Big|_{t=0} = 0 & \text{in } \Omega, \\ \frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) = 0 & \text{on } \Sigma. \end{cases} \quad (2.25)$$

Proof. Let us show that $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w)$ is a solution of (2.25). $\forall w \in L^2(Q)$ and $\forall \lambda \in \mathbb{R}$, $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0) - z_\varepsilon(v, 0)}{\lambda}$

- For the condition in Q , on the one hand, we have: $\frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta(z_\varepsilon^\gamma) - (z_\varepsilon^\gamma)^3 + \varepsilon(z_\varepsilon^\gamma)^5 =$
 $\frac{\partial}{\partial v} \left(\frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta(z_\varepsilon^\gamma) - (z_\varepsilon^\gamma)^3 + \varepsilon(z_\varepsilon^\gamma)^5 \right) (v, 0)(w) = \lim_{\lambda \rightarrow 0} \frac{v + \lambda w - v}{\lambda} = w$ in Q .

On the other hand, we have:

$$\begin{aligned} & \frac{\partial}{\partial v} \left(\frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta(z_\varepsilon^\gamma) - (z_\varepsilon^\gamma)^3 + \varepsilon(z_\varepsilon^\gamma)^5 \right) (v, 0)(w) = \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \right) \\ & - \Delta \left(\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \right) - 3(z_\varepsilon^\gamma)^2(v, 0) \frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) + 5\varepsilon(z_\varepsilon^\gamma)^4(v, 0) \frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w). \end{aligned}$$

- For the initial condition in Ω .
 $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \Big|_{t=0} = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0)|_{t=0} - z_\varepsilon(v, 0)|_{t=0}}{\lambda} = 0$ in Ω .
- For the condition on Σ .
 $\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) = \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0) - z_\varepsilon(v, 0)}{\lambda} = 0$ on Σ .

□

Proposition 2.15. For all $w \in L^2(Q)$, the partial derivative $\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) = \lim_{\lambda \rightarrow 0} \frac{\zeta_\varepsilon(v + \lambda w) - \zeta_\varepsilon(v)}{\lambda}$ is the solution of the following problem:

$$\begin{cases} -\frac{\partial}{\partial t} \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v} \right) - \Delta \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v} \right) - 3z_\varepsilon^2 \frac{\partial \zeta_\varepsilon^\gamma}{\partial v} + 5\varepsilon z_\varepsilon^4 \frac{\partial \zeta_\varepsilon^\gamma}{\partial v} = -\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) & \text{in } Q, \\ \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \Big|_{t=T} = 0 & \text{in } \Omega, \\ \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) = 0 & \text{on } \Sigma. \end{cases} \quad (2.26)$$

Proof. Let us show that $\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w)$ is a solution of problem (2.26).

- For the condition in Q , we have :

$$\begin{aligned}
 & -\frac{\partial}{\partial t} \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \right) - \Delta \left(\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \right) - 3z_\varepsilon^2 \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) + 5\varepsilon z_\varepsilon^4 \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \\
 = & \lim_{\lambda \rightarrow 0} \left(\frac{\frac{\partial \zeta_\varepsilon}{\partial t}(v + \lambda w) - \Delta \zeta_\varepsilon(v + \lambda w) - 3z_\varepsilon^2 \zeta_\varepsilon(v + \lambda w) + 5\varepsilon z_\varepsilon^4 \zeta_\varepsilon(v + \lambda w)}{\lambda} \right) \\
 & - \lim_{\lambda \rightarrow 0} \left(\frac{\frac{\partial \zeta_\varepsilon}{\partial t}(v, 0) - \Delta \zeta_\varepsilon(v, 0) - 3z_\varepsilon^2 \zeta_\varepsilon(v, 0) + 5\varepsilon z_\varepsilon^4 \zeta_\varepsilon(v, 0)}{\lambda} \right) \\
 = & \lim_{\lambda \rightarrow 0} \frac{-(z_\varepsilon(v + \lambda w, 0) - z_d) + z_\varepsilon(v, 0) - z_d}{\lambda} \text{ according to (2.13)} \\
 = & -\lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(v + \lambda w, 0) - z_\varepsilon(v, 0)}{\lambda} \\
 = & -\frac{\partial z_\varepsilon^\gamma}{\partial v}(v, 0)(w) \text{ in } Q,
 \end{aligned}$$

- For the condition in Ω , we have:

$$\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) \Big|_{t=T} = \lim_{\lambda \rightarrow 0} \frac{\zeta_\varepsilon(v + \lambda w)|_{t=T} - \zeta_\varepsilon(v)|_{t=T}}{\lambda} = 0 \text{ because } \zeta_\varepsilon(v)|_{t=T} = 0 \text{ in } \Omega.$$

- For the condition on Σ , we have:

$$\frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(v)(w) = \lim_{\lambda \rightarrow 0} \frac{\zeta_\varepsilon(v + \lambda w) - \zeta_\varepsilon(v)}{\lambda} = 0 \text{ because } \zeta_\varepsilon(v) = 0 \text{ on } \Sigma.$$

□

In the following, we provide the characterization of the adapted low-regret control.

2.3.2. Characterization of adapted low-regret control.

This characterization is given by the following proposition:

Proposition 2.16. *The adapted low-regret control u_ε^γ solution of (2.24) is characterized by $\{\zeta_\varepsilon^\gamma, z_\varepsilon^\gamma, \rho_\varepsilon^\gamma, p_\varepsilon^\gamma\}$ the unique solution of:*

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} -\frac{\partial \zeta_\varepsilon^\gamma}{\partial t} - \Delta \zeta_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \zeta_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 \zeta_\varepsilon^\gamma = -(z_\varepsilon^\gamma - z_d) & \text{in } Q, \\ \zeta_\varepsilon^\gamma(T, u_\varepsilon^\gamma) = 0 & \text{in } \Omega, \\ \zeta_\varepsilon^\gamma = 0 & \text{on } \Sigma. \end{array} \right. & (2.27) \end{array} \right.$$

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta z_\varepsilon^\gamma - (z_\varepsilon^\gamma)^3 + \varepsilon(z_\varepsilon^\gamma)^5 = u_\varepsilon^\gamma & \text{in } Q, \\ z_\varepsilon^\gamma(0) = 0 & \text{in } \Omega, \\ z_\varepsilon^\gamma = 0 & \text{on } \Sigma. \end{array} \right. & (2.28) \end{array} \right.$$

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \frac{\partial \rho_\varepsilon^\gamma}{\partial t} - \Delta \rho_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \rho_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 \rho_\varepsilon^\gamma = 0 & \text{in } Q, \\ \rho_\varepsilon^\gamma(0) = 0 & \text{in } \Omega, \\ \rho_\varepsilon^\gamma = -\frac{\varepsilon^2}{2\gamma} \frac{\partial S_\varepsilon^\gamma}{\partial \nu} & \text{on } \Sigma. \end{array} \right. & (2.29) \end{array} \right.$$

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{l} -\frac{\partial p_\varepsilon^\gamma}{\partial t} - \Delta p_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 p_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 p_\varepsilon^\gamma = z_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma & \text{in } Q, \\ p_\varepsilon^\gamma(T, u_\varepsilon^\gamma) = 0 & \text{in } \Omega, \\ p_\varepsilon^\gamma = 0 & \text{on } \Sigma. \end{array} \right. & (2.30) \end{array} \right.$$

$$\left\{ \begin{array}{ll} p_\varepsilon^\gamma + (N + 1) u_\varepsilon^\gamma = \tilde{u} & \text{in } L^2(Q), \text{ where } z_\varepsilon^\gamma := z_\varepsilon(u_\varepsilon^\gamma, 0). & (2.31) \end{array} \right.$$

Proof. The Euler-Lagrange condition on $J_{\varepsilon a}^\gamma$, for all $w \in L^2(Q)$, is given by:

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{\mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma + \lambda w) - \mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma)}{\lambda} &= \left\langle \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w), z_\varepsilon(u_\varepsilon^\gamma, 0) - z_d \right\rangle_{L^2(Q)} \\ + \langle Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} &+ \frac{\varepsilon^2}{2\gamma} \left\langle \frac{\partial}{\partial \nu} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma) \right\rangle_{L^2(\Sigma)} \geq 0. \end{aligned} \quad (2.32)$$

Indeed, for all $\lambda \in \mathbb{R}$ and for all $w \in L^2(Q)$, we have:

$$\begin{aligned} &\frac{\mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma + \lambda w) - \mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma)}{\lambda} = \frac{\lambda}{2} \left\| \frac{z_\varepsilon(u_\varepsilon^\gamma + \lambda w, 0) - z_\varepsilon(u_\varepsilon^\gamma, 0)}{\lambda} \right\|_{L^2(Q)}^2 \\ &+ \frac{\lambda}{2} (N + 1) \|w\|_{L^2(Q)}^2 + \left\langle \frac{z_\varepsilon(u_\varepsilon^\gamma + \lambda w, 0) - z_\varepsilon(u_\varepsilon^\gamma, 0)}{\lambda}, z_\varepsilon(u_\varepsilon^\gamma, 0) - z_d \right\rangle_{L^2(Q)} \\ &+ N \langle u_\varepsilon^\gamma, w \rangle_{L^2(Q)} + \langle u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} + \frac{\varepsilon^2 \lambda}{4\gamma} \left\| \frac{\frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma + \lambda w) - \frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma)}{\lambda} \right\|_{L^2(\Sigma)}^2 \\ &+ \frac{\varepsilon^2}{2\gamma} \left\langle \frac{\frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma + \lambda w) - \frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma)}{\lambda}, \frac{\partial S}{\partial \nu}(u_\varepsilon^\gamma) \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

By passing to the limit, we obtain:

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{\mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma + \lambda w) - \mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma)}{\lambda} &= \left\langle \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w), z_\varepsilon(u_\varepsilon^\gamma, 0) - z_d \right\rangle_{L^2(Q)} \\ &+ \langle Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} + \frac{\varepsilon^2}{2\gamma} \left\langle \frac{\partial}{\partial v} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \frac{\partial S}{\partial v}(u_\varepsilon^\gamma) \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

Indeed, we have:

$$\begin{cases} \lim_{\lambda \rightarrow 0} \frac{z_\varepsilon(u_\varepsilon^\gamma + \lambda w, 0) - z_\varepsilon(u_\varepsilon^\gamma, 0)}{\lambda} = \frac{\partial z_\varepsilon^\gamma}{\partial v}(u_\varepsilon^\gamma, 0)(w), \\ \lim_{\lambda \rightarrow 0} \frac{S(u_\varepsilon^\gamma + \lambda w) - S(u_\varepsilon^\gamma)}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\zeta_\varepsilon(u_\varepsilon^\gamma + \lambda w) - \zeta_\varepsilon(u_\varepsilon^\gamma)}{\lambda} = \frac{\partial \zeta_\varepsilon^\gamma}{\partial v}(u_\varepsilon^\gamma)(w). \end{cases}$$

In addition, the terms $\left\| \frac{\partial z_\varepsilon^\gamma}{\partial v} \right\|_{L^2(Q)}^2$ and $\left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial v} \right\|_{L^2(\Sigma)}^2$ are bounded and tend to zero when λ tends to zero.

By multiplying the first equation of (2.29) by $\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w)$ we have:

$$\begin{aligned} \left\langle \frac{\partial \rho_\varepsilon^\gamma}{\partial t} - \Delta \rho_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \rho_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle_{L^2(Q)} &= 0 \Leftrightarrow \left\langle \frac{\partial \rho_\varepsilon^\gamma}{\partial t}, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle \\ &+ \left\langle -\Delta \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle + \left\langle -3(z_\varepsilon^\gamma)^2 \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle + \left\langle 5(z_\varepsilon^\gamma)^4 \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle = 0 \end{aligned}$$

By integrating twice by parts, we obtain:

$$\left\langle \frac{\partial \rho_\varepsilon^\gamma}{\partial t}, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle_{L^2(Q)} = \left\langle -\frac{\partial}{\partial t} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \rho_\varepsilon^\gamma \right\rangle_{L^2(Q)}.$$

By applying Green's formula, we obtain :

$$\begin{aligned} \left\langle -\Delta \rho_\varepsilon^\gamma, \frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right\rangle_{L^2(Q)} &= \left\langle -\Delta \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \rho_\varepsilon^\gamma \right\rangle_{L^2(Q)} \\ &- \frac{\varepsilon^2}{2\gamma} \left\langle \frac{\partial}{\partial v} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \frac{\partial S_\varepsilon^\gamma}{\partial v} \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

It follows that :

$$\frac{\varepsilon^2}{2\gamma} \left\langle \frac{\partial}{\partial v} \left(\frac{\partial \zeta_\varepsilon}{\partial v}(u_\varepsilon^\gamma)(w) \right), \frac{\partial S_\varepsilon^\gamma}{\partial v} \right\rangle_{L^2(\Sigma)} = \left\langle \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w), -\rho_\varepsilon^\gamma \right\rangle_{L^2(Q)}.$$

The Euler-Lagrange condition becomes : $\forall w \in L^2(Q)$,

$$\left\langle \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0), z_\varepsilon(u_\varepsilon^\gamma, 0)(w) - z_d - \rho_\varepsilon^\gamma \right\rangle_{L^2(Q)} + \langle Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} \geq 0. \quad (2.33)$$

By multiplying the first equation of (2.30) by $\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w)$, we have:

$$\begin{aligned} \left\langle -\frac{\partial p_\varepsilon^\gamma}{\partial t} - \Delta p_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 p_\varepsilon^\gamma + 5\varepsilon(z_\varepsilon^\gamma)^4 p_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle_{L^2(Q)} \\ = \left\langle z_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle. \end{aligned}$$

By integrating twice by parts, we obtain:

$$\left\langle -\frac{\partial p_\varepsilon^\gamma}{\partial t}, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle_{L^2(Q)} = \left\langle \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right), p_\varepsilon^\gamma \right\rangle_{L^2(Q)}.$$

By applying Green's formula, we obtain:

$$\left\langle -\Delta p_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle_{L^2(Q)} = \left\langle -\Delta \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right), p_\varepsilon^\gamma \right\rangle_{L^2(Q)}.$$

It follows that:

$$\begin{aligned} \left\langle z_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma, \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right\rangle_{L^2(Q)} &= \left\langle \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right), p_\varepsilon^\gamma \right\rangle_{L^2(Q)} \\ &+ \left\langle -\Delta \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right), p_\varepsilon^\gamma \right\rangle_{L^2(Q)} \\ &+ \left\langle -3(z_\varepsilon^\gamma)^2 \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) + 5\varepsilon(z_\varepsilon^\gamma)^4 \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w), p_\varepsilon^\gamma \right\rangle_{L^2(Q)} \\ &= \langle p_\varepsilon^\gamma, w \rangle_{L^2(Q)}. \end{aligned}$$

Indeed, we have:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right) - \Delta \left(\frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \right) - 3(z_\varepsilon^\gamma)^2 \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) \\ + 5\varepsilon(z_\varepsilon^\gamma)^4 \frac{\partial z_\varepsilon}{\partial v}(u_\varepsilon^\gamma, 0)(w) = w \text{ in } Q. \end{aligned}$$

Thus, the condition (2.33) becomes :

$$\langle p_\varepsilon^\gamma + Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w \rangle_{L^2(Q)} \geq 0, \forall w \in L^2(Q). \quad (2.34)$$

$\forall w \in L^2(Q)$, $-w \in L^2(Q)$ because $L^2(Q)$ is a vector space. Considering $w' = -w$, (2.34) becomes:

$$\langle p_\varepsilon^\gamma + Nu_\varepsilon^\gamma + u_\varepsilon^\gamma - \tilde{u}, w' \rangle_{L^2(Q)} \leq 0, \forall w' = -w \in L^2(Q).$$

Therefore, the adapted low-regret control u_ε^γ is characterized by the following equation:

$$p_\varepsilon^\gamma + (N+1)u_\varepsilon^\gamma = \tilde{u} \text{ in } L^2(Q).$$

□

3. Singular optimality system (SOS)

3.1. Characterization of low-regret control

In this section, we present the singular optimality system characterizing the low-regret control of system (2.19). To this end, we first establish the inequalities given by the following lemma:

Lemma 3.1. *There exists some positive constants C_i for $i \in \{1, 2, 3\}$ such that:*

$$\begin{cases} \|u_\varepsilon^\gamma\|_{L^2(Q)} \leq C_1; \\ \|z_\varepsilon^\gamma\|_{L^2(Q)} \leq C_2; \\ \frac{1}{2} \frac{\varepsilon}{\sqrt{\gamma}} \left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C_3. \end{cases} \quad (3.1)$$

Proof. According to Proposition 2.13, u_ε^γ is the solution of (2.24). Thus, we have :

$$\mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma) \leq \mathcal{J}_{\varepsilon a}^\gamma(v), \quad \forall v \in L^2(Q).$$

By taking $v = 0$, we obtain:

$$\begin{aligned} \mathcal{J}_{\varepsilon a}^\gamma(u_\varepsilon^\gamma) &= \frac{1}{2} \|z_\varepsilon^\gamma - z_d\|_{L^2(Q)}^2 + \frac{N}{2} \|u_\varepsilon^\gamma\|_{L^2(Q)}^2 + \frac{1}{2} \|u_\varepsilon^\gamma - \tilde{u}\|_{L^2(Q)}^2 + \frac{\varepsilon^2}{4\gamma} \left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)}^2 \\ &\leq \frac{1}{2} \left(\|z_d\|_{L^2(Q)}^2 + \|\tilde{u}\|_{L^2(Q)}^2 \right) = C. \end{aligned}$$

We deduce the following inequalities:

$$\begin{cases} \|z_\varepsilon^\gamma - z_d\|_{L^2(Q)} \leq C_1 \\ \|u_\varepsilon^\gamma\|_{L^2(Q)} \leq C_2 \\ \frac{\varepsilon}{2\sqrt{\gamma}} \left\| \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C_3. \end{cases}$$

By the reverse triangle inequality, we obtain:

$$\begin{aligned} \left| \|z_\varepsilon^\gamma\|_{L^2(Q)} - \|z_d\|_{L^2(Q)} \right| &\leq \|z_\varepsilon^\gamma - z_d\|_{L^2(Q)} \leq C_1 \\ \implies \|z_\varepsilon^\gamma\|_{L^2(Q)} &\leq C_1 + \|z_d\|_{L^2(Q)} = C'. \end{aligned}$$

Hence the inequalities of the Lemma 3.1. □

Theorem 3.2. *The low-regret control u^γ of the problem (2.19) is characterized by $\{\zeta^\gamma, z^\gamma, \rho^\gamma, p^\gamma\}$ the unique solution of :*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} -\frac{\partial \zeta^\gamma}{\partial t} - \Delta \zeta^\gamma - 3(z^\gamma)^2 \zeta^\gamma = -(z^\gamma - z_d) \quad \text{in } Q, \\ \zeta^\gamma(T, u^\gamma) = 0 \quad \text{in } \Omega, \\ \zeta^\gamma = 0 \quad \text{on } \Sigma. \end{array} \right. \quad (3.2) \\ \left\{ \begin{array}{l} \frac{\partial z^\gamma}{\partial t} - \Delta z^\gamma - (z^\gamma)^3 = u^\gamma \quad \text{in } Q, \\ z^\gamma(0) = 0 \quad \text{in } \Omega, \\ z^\gamma = 0 \quad \text{on } \Sigma. \end{array} \right. \quad (3.3) \\ (SOS) \left\{ \begin{array}{l} \frac{\partial \rho^\gamma}{\partial t} - \Delta \rho^\gamma - 3(z^\gamma)^2 \rho^\gamma = 0 \quad \text{in } Q, \\ \rho^\gamma(0) = 0 \quad \text{in } \Omega, \\ \rho^\gamma = 0 \quad \text{on } \Sigma. \end{array} \right. \quad (3.4) \\ \left\{ \begin{array}{l} -\frac{\partial p^\gamma}{\partial t} - \Delta p^\gamma - 3(z^\gamma)^2 p^\gamma = z^\gamma - z_d - \rho^\gamma \quad \text{in } Q, \\ p^\gamma(T, u^\gamma) = 0 \quad \text{in } \Omega, \\ p^\gamma = 0 \quad \text{on } \Sigma. \end{array} \right. \quad (3.5) \\ p^\gamma + (N + 1) u^\gamma = \tilde{u} \quad \text{in } L^2(Q), \text{ where } z^\gamma := z(u^\gamma, 0). \quad (3.6) \end{array} \right.$$

Proof. Consider z_ε^γ the solution of the problem (2.28). According to Lemma 3.1, we have : $\|z_\varepsilon^\gamma\|_{L^2(Q)} \leq C$. Therefore, we can extract a subsequence $(z_\varepsilon^\gamma)_\varepsilon$ which weakly converges to z^γ in $L^2(Q)$ when $\varepsilon \rightarrow 0$. In addition, $\|u_\varepsilon^\gamma\|_{L^2(Q)} \leq C$. Thus, $u_\varepsilon^\gamma \rightharpoonup u^\gamma$ in $L^2(Q)$. Therefore, from (2.28), we have:

$$\left\| \frac{\partial z_\varepsilon^\gamma}{\partial t} - \Delta z_\varepsilon^\gamma - (z_\varepsilon^\gamma)^3 - u_\varepsilon^\gamma \right\|_{L^2(Q)} = \|\varepsilon(z_\varepsilon^\gamma)^5\|_{L^2(Q)} \leq \varepsilon C \xrightarrow{\varepsilon \rightarrow 0} 0$$

when $\varepsilon \rightarrow 0$, the problem (2.28) becomes:

$$\left\{ \begin{array}{l} \frac{\partial z^\gamma}{\partial t} - \Delta z^\gamma - (z^\gamma)^3 = u^\gamma \quad \text{in } Q, \\ z^\gamma(0) = 0 \quad \text{in } \Omega, \\ z^\gamma = 0 \quad \text{on } \Sigma. \end{array} \right.$$

We consider $\zeta_\varepsilon^\gamma \in L^2(Q)$ as the solution to the problem (2.27). Since $\zeta_\varepsilon^\gamma \in L^2(Q)$, the sequence $(\zeta_\varepsilon^\gamma)$ is bounded in $L^2(Q)$. There exists a positive constant M_1 such that $\|\zeta_\varepsilon^\gamma\|_{L^2(Q)} \leq M_1$. Thus, from (2.27) we have:

$$\left\| -\frac{\partial \zeta_\varepsilon^\gamma}{\partial t} - \Delta \zeta_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \zeta_\varepsilon^\gamma + (z_\varepsilon^\gamma - z_d) \right\|_{L^2(Q)} = \|5\varepsilon(z_\varepsilon^\gamma)^4 \zeta_\varepsilon^\gamma\|_{L^2(Q)} \leq 5\varepsilon C M_1 \xrightarrow{\varepsilon \rightarrow 0} 0$$

Therefore, it exists a subsequence $(\zeta_\varepsilon^\gamma)_\varepsilon$ such that $\zeta_\varepsilon^\gamma \rightharpoonup \zeta^\gamma$ in $L^2(Q)$.

When $\varepsilon \rightarrow 0$, the problem (2.27) becomes:

$$\begin{cases} -\frac{\partial \zeta^\gamma}{\partial t} - \Delta \zeta^\gamma - 3(z^\gamma)^2 \zeta^\gamma = -(z^\gamma - z_d) & \text{in } Q, \\ \zeta^\gamma(T, u^\gamma) = 0 & \text{in } \Omega, \\ \zeta^\gamma = 0 & \text{on } \Sigma. \end{cases}$$

We consider $\rho_\varepsilon^\gamma \in L^2(Q)$ the solution of the problem (2.29). According to the Lemma 3.1, we deduce the following limit:

$\frac{\varepsilon}{\sqrt{\gamma}} \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \rightharpoonup \lambda^\gamma$ converges weakly in $L^2(\Sigma)$. So, when $\varepsilon \rightarrow 0$, $\frac{\varepsilon^2}{\gamma} \frac{\partial \zeta_\varepsilon^\gamma}{\partial \nu} \rightarrow 0$, $\rho_\varepsilon^\gamma \in L^2(Q)$. Therefore, the sequence $(\rho_\varepsilon^\gamma)$ is bounded in $L^2(Q)$. Then, there exists a positive constant M_2 such that $\|\rho_\varepsilon^\gamma\|_{L^2(Q)} \leq M_2$. Thus, from (2.29) we have:

$$\left\| \frac{\partial \rho_\varepsilon^\gamma}{\partial t} - \Delta \rho_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 \rho_\varepsilon^\gamma \right\|_{L^2(Q)} = \|5\varepsilon(z_\varepsilon^\gamma)^4 \rho_\varepsilon^\gamma\|_{L^2(Q)} \leq 5\varepsilon C M_2 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Therefore, it exists a subsequence $(\rho_\varepsilon^\gamma)_\varepsilon$ such that $\rho_\varepsilon^\gamma \rightharpoonup \rho^\gamma$ in $L^2(Q)$. When $\varepsilon \rightarrow 0$, the problem (2.29) becomes:

$$\begin{cases} \frac{\partial \rho^\gamma}{\partial t} - \Delta \rho^\gamma - 3(z^\gamma)^2 \rho^\gamma = 0 & \text{in } Q, \\ \rho^\gamma(0) = 0 & \text{in } \Omega, \\ \rho^\gamma = 0 & \text{on } \Sigma. \end{cases}$$

We consider p_ε^γ the solution of the problem (2.30). According to the equation (2.31), (u_ε^γ) is bounded in $L^2(Q)$ and $\tilde{u} \in L^2(Q)$. Thus, there exists a positive constant M_3 such that $\|p_\varepsilon^\gamma\|_{L^2(Q)} = \|-u_\varepsilon^\gamma + \tilde{u} - N u_\varepsilon^\gamma\| \leq M_3$. Therefore, (p_ε^γ) is bounded in $L^2(Q)$ and we have:

$$\left\| -\frac{\partial p_\varepsilon^\gamma}{\partial t} - \Delta p_\varepsilon^\gamma - 3(z_\varepsilon^\gamma)^2 p_\varepsilon^\gamma - (z_\varepsilon^\gamma - z_d - \rho_\varepsilon^\gamma) \right\|_{L^2(Q)} = \|5\varepsilon(z_\varepsilon^\gamma)^4 p_\varepsilon^\gamma\|_{L^2(Q)} \leq 5\varepsilon C M_3 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Therefore, it exists a subsequence $(p_\varepsilon^\gamma)_\varepsilon$ such that $p_\varepsilon^\gamma \rightharpoonup p^\gamma$ in $L^2(Q)$.

When $\varepsilon \rightarrow 0$, the system (2.30) becomes:

$$\begin{cases} -\frac{\partial p^\gamma}{\partial t} - \Delta p^\gamma - 3(z^\gamma)^2 p^\gamma = z^\gamma - z_d - \rho^\gamma & \text{in } Q, \\ p^\gamma(T, u^\gamma) = 0 & \text{in } \Omega, \\ p^\gamma = 0 & \text{on } \Sigma. \end{cases}$$

According to the equation (2.31) and the inequalities of the Lemma 3.1, we have:

$$u_\varepsilon^\gamma \rightharpoonup u^\gamma \text{ in } L^2(Q). \tag{3.7}$$

In addition, $p_\varepsilon^\gamma \rightharpoonup p^\gamma$ in $L^2(Q)$. We can deduce that:

$$p_\varepsilon^\gamma \rightharpoonup p^\gamma = -(N + 1) u^\gamma + \tilde{u} \text{ in } L^2(Q), \text{ where } z^\gamma := z(u^\gamma, 0). \tag{3.8}$$

□

3.2. Characterization of no-regret control

We now deduce, in this section, the singular optimality system of the no-regret control for problem (2.18) by passing to the limit as $\gamma \rightarrow 0$. The limit problem is not well posed in the sense of Hadamard. This singular optimality system is presented in the following theorem:

Theorem 3.3. *The no-regret control \tilde{u} of the problem (2.18) is characterized by $\{\zeta, z, \rho, p\}$ a solution of the optimality system:*

$$\begin{cases}
 \left\{ \begin{array}{ll}
 -\frac{\partial \zeta}{\partial t} - \Delta \zeta - 3z^2 \zeta = -(z - z_d) & \text{in } Q, \\
 \zeta(T, \tilde{u}) = 0 & \text{in } \Omega, \\
 \zeta = 0 & \text{on } \Sigma.
 \end{array} \right. & (3.9) \\
 \left\{ \begin{array}{ll}
 \frac{\partial z}{\partial t} - \Delta z - z^3 = \tilde{u} & \text{in } Q, \\
 z(0) = 0 & \text{in } \Omega, \\
 z = 0 & \text{on } \Sigma.
 \end{array} \right. & (3.10) \\
 (SOS) \left\{ \begin{array}{ll}
 \frac{\partial \rho}{\partial t} - \Delta \rho - 3z^2 \rho = 0 & \text{in } Q, \\
 \rho(0) = 0 & \text{in } \Omega, \\
 \rho = 0 & \text{on } \Sigma.
 \end{array} \right. & (3.11) \\
 \left\{ \begin{array}{ll}
 -\frac{\partial p}{\partial t} - \Delta p - 3z^2 p = z - z_d - \rho & \text{in } Q, \\
 p(T, \tilde{u}) = 0 & \text{in } \Omega, \\
 p = 0 & \text{on } \Sigma.
 \end{array} \right. & (3.12) \\
 p + N\tilde{u} = 0 & \text{in } L^2(Q), \text{ where } z := z(\tilde{u}, 0). & (3.13)
 \end{cases}$$

Proof. According to the results of Theorem 3.2 and by passing to the limit as $\gamma \rightarrow 0$, we obtain:

$$\begin{cases}
 \left\{ \begin{array}{ll}
 \zeta^\gamma \rightharpoonup \zeta = 0, \\
 z^\gamma \rightharpoonup z = 0, \\
 \rho^\gamma \rightharpoonup \rho = 0, \\
 p^\gamma \rightharpoonup p = 0.
 \end{array} \right. & \text{on } \Sigma \\
 \left\{ \begin{array}{ll}
 \zeta^\gamma \rightharpoonup \zeta = 0, \\
 z^\gamma(0) \rightharpoonup z(0) = 0, \\
 \rho^\gamma \rightharpoonup \rho = 0, \\
 p^\gamma(T, u^\gamma) \rightharpoonup p(T, \tilde{u}) = 0.
 \end{array} \right. & \text{in } \Omega
 \end{cases}$$

According to equation (3.8), we deduce that:

$$u^\gamma \rightharpoonup \tilde{u} \text{ in } L^2(Q).$$

It follows that there exists a no-regret control \tilde{u} , characterized by $\{\zeta, z, \rho, p\}$ solution of (2.18). □

4. Conclusion

In this article, we analyzed an ill-posed parabolic problem with missing data by combining regularization and low-regret control. Thanks to the regularization method, we


were able to recover the missing boundary data on Σ and transform the initial problem into a family of well-posed problems. To study the control problem, we applied the low-regret control method, which is better suited to problems involving missing data. We were able to characterize the no-regret control of system (1.1) thanks to the convergence of low-regret control, then by taking the limit as the perturbation parameter tends towards zero. This work can be approached numerically.

References

- [1] Gabay, D., Lions, J.-L., *Least regrets strategic decisions*, C. R. Math. Acad. Sci. Paris, **319**(10)(1994), 1049–1056.
- [2] Lions, J.-L., *Some methods in the mathematical analysis of systems and their control*, Science Press, Beijing, 1981.
- [3] Lions, J.-L., *Méthodes Mathématiques de l'Informatique*, Gauthier-Villars, Bordas, Paris, vol. 13, 1983.
- [4] Lions, J.-L., *Optimal control of non well posed distributed systems*, Banach Center Publ., **14**(1)(1985), 299–311.
- [5] Lions, J.-L., *Contrôle de Pareto de systèmes distribués. le cas stationnaire*, C. R. Math. Acad. Sci. Paris, **302**(6)(1986), 223–227.
- [6] Lions, J.-L., *Contrôle à moindres regrets des systèmes distribués*, C. R. Math. Acad. Sci. Paris, **315**(1992), 1253–1257.
- [7] Lions, J.-L., *Duality arguments for multi agents least-regret control*, Institut de France, 1999.
- [8] Lions, J.-L., *Least regret control, virtual control and decomposition methods*, ESAIM Math. Model. Numer. Anal., M2AN, **32**(2)(2000), 409–418.
- [9] Picart, D., Ozier-Lafontaine, H. O., Abdennebi, L., *Modeling plant nutrient uptake: Mathematical analysis and optimal control*, Evol. Equ. Control Theory, **4**(2)(2015), 193–203.
- [10] Nakoulima, O., Omrane, A., Velin, J., *Perturbations à moindres regrets dans les systèmes distribués à données manquantes*, C. R. Math. Acad. Sci. Paris, **330**(9)(2000), 801–806.
- [11] Nakoulima, O., Omrane, A., Velin, J., *On the pareto control and no-regret control for distributed systems with incomplete data*, SIAM J. Control Optim., **42**(4)(2003), 1167–1184.
- [12] Savage, L. J., *The foundations of statistics, 2nd rev. ed.*, Dover Publications, New York, 1972, pp. 310.
- [13] Smoller, J., *Shock Waves and Reaction–Diffusion Equations*, Springer-Verlag, New York, 1983.
- [14] Tindano, T., Tao, S., Sawadogo, S., *Optimal control of a nonlinear elliptic problem with missing data*, Journal of Mathematics Research, **14**(6)(2022), 46–55.
- [15] Thomas, T., Mifiamba, S., Tao, S., Somdouda, S., *Optimal control of a nonlinear elliptical evolution problem with missing data*, Adv. Differ. Equ. Control, **30**(2)(2023), 135–150.

Biéliémi Lamien 

Laboratoire Lanibio, Université Joseph Ki-Zerbo,
03 BP 7021 Ouagadougou 03,
Burkina Faso
e-mail: bieliemil@gmail.com

Sadou Tao 

Laboratoire Lanibio, Université Joseph Ki-Zerbo,
03 BP 7021 Ouagadougou 03,
Burkina Faso
e-mail: sadou.tao@ujkz.bf

Elisée Gouba 

Laboratoire Lanibio, Université Joseph Ki-Zerbo,
03 BP 7021 Ouagadougou 03,
Burkina Faso
e-mail: elisee.gouba@ujkz.bf