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Completely inert subgroups of abelian groups

Andrey R. Chekhlov  and Peter V. Danchev 

Dedicated to Patrick W. Keef on the occasion of his 70th birthday

Abstract. We define and study in-depth the so-called *completely inert* and *uniformly completely inert* subgroups of Abelian groups. We curiously show that a subgroup is completely inert exactly when it is characteristically inert. Moreover, we prove that a subgroup is uniformly completely inert precisely when it is uniformly characteristically inert. These two statements somewhat strengthen recent results due to Goldsmith-Salce established for totally inert subgroups in J. Commut. Algebra (2025). Some other closely relevant things are obtained as well.

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1. Some Fundamentals

Throughout the current brief paper, all our groups are *additively* written and *Abelian*. Our notation and terminology are mainly standard and follow those from [14]. In fact, recall the standard concepts that an arbitrary subgroup F of a group G is said to be a *fully invariant* subgroup provided $\phi(F) \subseteq F$ for any endomorphism ϕ of G , and an arbitrary subgroup C of G is said to be a *characteristic* subgroup provided $\psi(C) \subseteq C$ for any automorphism ψ of G . Moreover, in [2], an arbitrary subgroup S of G is said to be a *strongly invariant* subgroup provided $f(S) \subseteq S$ for any homomorphism $f : S \rightarrow G$.

It is obvious that strongly invariant subgroups are fully invariant subgroups, and the latter are always characteristic subgroups, while both reverses fail in general.

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Generalizing these two notions, it is well known that a subgroup N of a group G is called *fully inert* provided $(\phi(N) + N)/N$ is finite for all endomorphisms ϕ of G , and is called *characteristically inert* provided $(\psi(N) + N)/N$ is finite for all automorphisms ψ of G (for the latter see [7]). Besides, in [1], a subgroup N of G is called *strongly inert* provided $(f(N) + N)/N$ is finite for all homomorphisms $f : N \rightarrow G$. If, in both cases, the cardinalities of the finite quotients $(\phi(N) + N)/N$ and $(\psi(N) + N)/N$ are bounded by some fixed positive integer, the subgroup N is termed *uniformly fully inert* and *uniformly characteristically inert*, respectively. The description of these subgroups as being commensurable with fully invariant and, respectively, with characteristic subgroups can be found in [5].

Apparently, the following implications hold:

$$\text{strong inertness} \Rightarrow \text{full inertness} \Rightarrow \text{characteristic inertness.}$$

However, one observes in what follows that the opposite implications are, generally, untrue.

Further, in [13], the new concept of a totally inert subgroup of a group was introduced as follows: A subgroup T of an arbitrary group G is called *totally inert*, provided that the intersection $T \cap \phi(T)$ has finite index in both T and $\phi(T)$ for any non-zero endomorphism ϕ of G .

Clearly, total inertness implies full inertness.

Imitating the "uniformly" property presented above, it is reasonably natural to ask what is the behavior of uniformly totally inert subgroups defined analogously as follows: A subgroup T of an arbitrary group G is termed *uniformly totally inert*, provided that the intersection $T \cap \phi(T)$ has index bounded by a fixed positive integer in both T and $\phi(T)$ for any non-zero endomorphism ϕ of G .

Nevertheless, this does *not* give anything new in the infinite case as the next arguments illustrate – indeed, we claim that these are only the rational torsion-free group \mathbb{Q} and the quasi-cyclic p -group $\mathbb{Z}(p^\infty)$. In fact, looking for infinite subgroups H of a group G different from, respectively, \mathbb{Q} and $\mathbb{Z}(p^\infty)$, which subgroups are uniformly totally inert, looking at [13, Corollary 2.3] in order to admit infinite totally inert subgroups, G must be torsion-free reduced and indecomposable. Furthermore, for such a group G , a subgroup $H \neq \{0\}$ is infinite, but it cannot be uniformly totally inert, because there is a prime p such that $H/pH \neq \{0\}$ (noticing that H is not divisible), so H/pH has cardinality at least p and, therefore, H/p^nH has cardinality at least p^n , whence H cannot be uniformly totally inert, as expected.

In order to strengthen this, we come to the following basic tool (see the initial version in [8] as well).

Definition 1.1. *A subgroup C of an arbitrary group G is called completely inert provided the intersection $C \cap \psi(C)$ has finite index in both C and $\psi(C)$ for any automorphism ψ of G .*

Evidently, total inertness forces complete inertness, but the reciprocal implication is manifestly non-true as we will illustrate in the sequel.

As above, a reasonably logical question is to ask what happens with *uniformly completely inert subgroups* defined by analogy thus: The subgroup C has an intersection $C \cap \psi(C)$ bounded by a fixed positive integer in C and $\psi(C)$ for any automorphism ψ of G .

And so, the objective of this article is to give a systematic exploration of the so-introduced concept of complete inertness by finding its crucial properties and comparing them with these of the defined above total inertness.

2. Main results

Before establishing the principal achievements, our preliminaries here are the following ones distributed subsequently into two subsections.

2.1. Completely inert subgroups

The first technicality can easily be established by analogy with [13, Proposition 2.1].

Lemma 2.1. *A subgroup commensurable with a completely inert subgroup is again completely inert.*

Proof. It is straightforwardly analogous to Proposition 2.1 from [13]. \square

We now arrive at the following quite surprising assertion.

Proposition 2.2. *A subgroup C of a group G is completely inert subgroup if, and only if, it is characteristically inert.*

Proof. Necessity is evident, so we omit the arguments. As for sufficiency, by assumption, the quotient $\psi^{-1}(C)/(C \cap \psi^{-1}(C))$ is finite for each $\psi \in \text{Aut}(G)$. We claim that the factor-group $C/(C \cap \psi(C))$ is likewise finite as being an isomorphic image of $\psi^{-1}(C)/(C \cap \psi^{-1}(C))$ under the action of ψ .

In fact, mapping

$$\bar{\psi} : \psi^{-1}(c) + (C \cap \psi^{-1}(C)) \mapsto c + (\psi(C) \cap C),$$

one concludes that it is an isomorphism between $\psi^{-1}(C)/(C \cap \psi^{-1}(C))$ and $C/(C \cap \psi(C))$. Indeed, ψ maps $C \cap \psi^{-1}(C)$ into $\psi(C) \cap C$, so that $\bar{\psi}$ is obviously a well-defined homomorphism. Furthermore, if $c = \psi(c_1)$ for some $c, c_1 \in C$, then $c_1 = \psi^{-1}(c) \in C$, and hence $\bar{\psi}$ is an injection. It is also routinely seen that $\bar{\psi}$ is a surjection. Whence, $\bar{\psi}$ is an isomorphism, as asserted. \square

We now proceed by proving a series of technicalities as our next technical statement is the following known fact.

Lemma 2.3. ([6, Lemma 2.2]) *Let H be a fully inert subgroup of the group $G = \bigoplus_{i \in I} G_i$, where the index set I is infinite, and let each π_i denote the canonical projection from G onto G_i . Then, H is commensurable with $\bigoplus_{i \in I} \pi_i(H)$, the images $\pi_i(H)$ are fully inert in G_i , and almost all $\pi_i(H)$ are fully invariant in G_i . Furthermore, there is a finite subset $S \subset I$, such that $\bigoplus_{i \in I \setminus S} \pi_i(H)$ is fully invariant in $\bigoplus_{i \in I \setminus S} G_i$.*

We shall use, hereafter, the symbol " \sim " to designate the following symmetric relation between two fixed subgroups of a given group: We call two subgroups K and L of a group A *commensurable* and indicate that as $K \sim L$ whenever the quotients $(K + L)/K$ and $(K + L)/L$ are simultaneously finite.

We are, thereby, ready to attack the following.

Proposition 2.4. *Let H be a fully inert subgroup of the group $G = \bigoplus_{i \in I} G_i$, where $G_i \cong G_j$ for every $i, j \in I$ and the index set I is infinite. Then, H is commensurable with some fully invariant subgroup of G .*

Proof. Invoking Lemma 2.3, with no loss of generality we can assume that $H = \bigoplus_{i \in I} \pi_i(H)$. For $j \in I \setminus S$, denoting $F_j = \pi_j(H)$, we derive that $\bigoplus_{j \in I \setminus S} F_j$ is fully invariant in $\bigoplus_{j \in I \setminus S} G_j$, where it is clear that all F_j are isomorphic ($j \in I \setminus S$).

After that, in each G_i , where $i \in S$, there exists a fully invariant subgroup $F_i \cong F_j$ with $j \in I \setminus S$. Thus, $F = \bigoplus_{i \in I} F_i$ is a fully invariant subgroup of G . It next suffices to show that, for each $i \in S$, the subgroup F_i is commensurable with $\pi_i(H)$. To this aim, let i be some index in S and $\varphi \in E(G)$ such that $\varphi(G_i) = G_{j_0}$, where j_0 is a fixed index from $I \setminus S$ with $\varphi(G_k) = \{0\}$ whenever $k \neq i$. So, one checks that

$$H + \varphi(H) = \left(\bigoplus_{s \in S} \pi_s(H) \right) \oplus (F_{j_0} + F'_{j_0}) \oplus \left(\bigoplus_{l \in I \setminus (S \cup \{j_0\})} F_l \right),$$

where $F'_{j_0} = \varphi \pi_i(H)$, whence $F_{j_0} + F'_{j_0} \sim F_{j_0}$.

Let us now $\psi \in E(G)$ such that $\psi(G_{j_0}) = G_i$ and $\varphi(G_k) = \{0\}$ whenever $k \neq j_0$. Therefore, one infers that

$$H + \psi(H) = \left(\bigoplus_{s \in S \setminus \{i\}} \pi_s(H) \right) \oplus (F'_i + \pi_i(H)) \oplus \left(\bigoplus_{l \in I \setminus S} F_l \right),$$

where $F'_i = \psi(F_{j_0}) = F_i$, whence $F_i + \pi_i(H) \sim \pi_i(H)$.

But since $F'_{j_0} \cong \pi_i(H)$, $F_{j_0} \cong F_i$ and these isomorphisms both induce endomorphisms of the group G , we then arrive at the relations $F_i + \pi_i(H) \sim F_i$ and $F_i + \pi_i(H) \sim \pi_i(H)$, i.e., $\pi_i(H) \sim F_i$, as asked for. \square

The next technical assertion is well-known (compare with Lemma 2.3 as listed above).

Lemma 2.5. ([3, Lemma 7]) *Let $G = \bigoplus_{i \in I} G_i$, and let $\pi_i : G \rightarrow G_i$ be the corresponding projections. If H is a fully inert subgroup of G , then H is of finite index in the subgroup $\bigoplus_{i \in I} \pi_i(H)$.*

It is worthy of noticing that, if $G := A^n$ for some natural number n and a group A , then *not* every fully inert subgroup of G is commensurable with a fully invariant subgroup (see, for example, [4]).

We finish the series of technical claims with the last one.

Lemma 2.6. *Let $G = A_1 \oplus \cdots \oplus A_n$, $A_i \cong A$ for $i = 1, \dots, n$, where A is a non-zero group, and suppose in A each fully inert subgroup is commensurable with a fully invariant subgroup. Then, each fully inert subgroup H of G is commensurable with a fully invariant subgroup of G . In particular, the subgroup H is uniformly fully inert.*

Proof. Utilizing Lemma 2.5, it can be assumed that

$$H = \pi_1(H) \oplus \cdots \oplus \pi_n(H).$$

It is now easy to see that each $\pi_i(H)$ is fully inert in A_i for $i = 1, \dots, n$. So, each $\pi_i(H)$ is commensurable with a fully invariant subgroup F_i of A_i . Likewise, it is evident that each F_i is commensurable with $\phi(F_j)$ for each isomorphism $\phi : A_j \rightarrow A_i$. Thus, it can be supposed that $F_i \cong F_j$ for all $i, j = 1, \dots, n$. Consequently, in this case, $F = F_1 \oplus \cdots \oplus F_n$ is a fully invariant subgroup of G which is commensurable with H , as required. \square

The following construction is worthwhile, although it was documented in [7, Example 4.1] that there is a group with a characteristically inert subgroup which is *not* fully inert. Thus, Proposition 2.2 yields that there is a group with completely inert subgroup which is *not* fully inert and so, manifestly, *not* totally inert.

Example 2.7. There exists a group with a completely inert subgroup which is *not* totally inert.

Proof. Let G be a torsion-free group of infinite rank such that $E(G) \cong \mathbb{Z}$, the group of integers. Therefore, each its subgroup is characteristic and hence is completely inert. But $H \leq G$ will be totally inert exactly when the factor-group H/nH is finite for all non-zero integers n , that is, if it is a narrow subgroup in terms of [13]. But, because G must have infinite rank, in it will exist subgroups that are not totally inert, as promised. (Notice that this could also be deduced by consulting with [13, Theorem 2.8].) \square

Remark 2.8. Concerning the preceding example, note that if G is a torsion-free group each subgroup of which is completely inert and $pG = G$ for at least one prime p , then the rank of G is necessarily finite. Indeed, if the rank of G is infinite, then G will have a free subgroup H of infinite rank with the property $H \approx pH$.

Likewise, there is an abundance of completely inert subgroups which are not totally inert: in fact, such are all infinite characteristic subgroups of decomposable groups A (e.g., the subgroups nA and $(p^m A)[n]$ for some naturals m, n) as well as, if a group G is torsion-free, then such subgroups are $G(t)$ and $G(\chi)$, where t is the type and χ is the characteristic of G .

In the other vein, since each totally inert subgroup of a decomposable group is finite (see [13, Proposition 2.2]), it must be that each infinite characteristic subgroup of a decomposable group will be completely inert but *not* totally inert subgroup.

We now proceed by showing more statements in the following aspect.

Lemma 2.9. *Suppose that in a group G every subgroup is completely inert. The following two points hold:*

- (1) *if $G = A \oplus B$, then $f(B)$ is finite for every homomorphism $f : B \rightarrow A$;*
- (2) *if G is torsion-free, then G is indecomposable.*

Proof. (1) It is evident since $\varphi = 1_G + f \in \text{Aut } G$ and $\varphi(B) = f(B) \oplus B \sim B$.

(2) Let $H = \langle a + b \rangle$, where $0 \neq a \in A$, $0 \neq b \in B$. If $f = 1_A \oplus (-1_B)$, then $f(H) = \langle a - b \rangle$ and $H \cap f(H) = \{0\}$, and so $H \not\sim f(H)$, as required. \square

Thus, from this lemma, it immediately follows that in any divisible group D each subgroup is completely inert if, and only if, either $D \cong \mathbb{Q}$, the group of all rational numbers, or D is torsion and the rank of each its p -component is ≤ 1 .

Lemma 2.10. *The following three items hold:*

- (1) *If G is a reduced p -group in which each subgroup is completely inert, then G is finite.*
- (2) *In a non-reduced p -group G all subgroups are completely inert if, and only if, G is a direct sum of a quasi-cyclic group and a finite group.*
- (3) *In a torsion group all subgroups are completely inert if, and only if, almost all its p -components are co-cyclic and the non co-cyclic p -components are either finite or a direct sum of a quasi-cyclic group and a finite group.*

Proof. (1) From Lemma 2.9, it follows that the Ulm-Kaplansky invariant $f_n(G)$ is finite for all $n < \omega$. So, if we assume the contrary that G is infinite, then both G/pG and $G[p]$ are also infinite, and thus there exists a homomorphism $G/pG \rightarrow G[p]$ with infinite image.

Now, if $\pi : G \rightarrow G/pG$ is the usual canonical surjection and $\varphi = f \circ \pi$, then one checks that $\varphi^2 = 0$ and thus $\psi = 1 + \varphi \in \text{Aut } G$. But, as $H \cap \psi(H) = pH$, we infer $H \cap \psi(H) \not\sim H$, proving that G is finite, as promised.

Furthermore, items (2) and (3) follows directly from Lemma 2.9 and item (1). \square

Our next assertion is a helpful consequence of Lemma 2.9 quoted above.

Corollary 2.11. *If $G = \mathbb{Q} \oplus B$ is a group, where \mathbb{Q} is the rational group, then in G each subgroup is completely inert if, and only if, B is finite.*

Proof. Necessity. An appeal to Lemma 2.9 reaches us that B is torsion, almost all p -components of B are cyclic and the non-cyclic p -components are finite. If, in a way of contradiction, we assume for a moment that B is infinite (i.e., an infinite number of its p -components are non-zero), then in \mathbb{Q} there will exist a subgroup, X say, with an infinite image of the existing homomorphism $X \rightarrow B$, that this can not be happen, contrary to our assumption.

Sufficiency. It is obvious, so the arguments are removed. \square

Given a reduced torsion-free group G , we shall denote by $R(G)$ the *maximal subring* of the field of rational numbers \mathbb{Q} contained in $E(G)$, which subring is generated by the 1 and the inverses of the prime numbers p for which $G = pG$.

Lemma 2.12. *Let G be a torsion-free group such all endomorphism are injections. Then, in G each subgroup is completely inert if, and only if, $\text{Aut } G = U(R(G))$ and, moreover, if $R(G) \neq \mathbb{Z}$, then the rank of G is finite.*

Proof. Necessity. Suppose that H is a pure subgroup of rank 1, and $\varphi \in \text{Aut } G$. Since $H \sim \varphi(H)$, we derive $H = \varphi(H)$, so $\varphi \upharpoonright H \in R(H)$ and hence $\varphi \upharpoonright H = m/n$, where m and n are mutually simple integers. However, as $(n\varphi - m)H = \{0\}$, we deduce $n\varphi = m$. Therefore, $nG = G$ whence $\varphi = m/n \in R(G)$.

Sufficiency. If $R(G) = \mathbb{Z}$, then one sees that each subgroup is totally inert. If, however, $pG = G$ for some prime p , then under assumption G possesses finite rank, and so each $H \leq G$ will too have finite rank, whence $H \sim \varphi(H)$ for every $\varphi \in R(G)$. \square

Let us now recollect the following technicality necessary for our further presentation.

Lemma 2.13. ([11, §8, Exercise 5]) *If H is a subgroup of $G = B \oplus C$, then H is the sub-direct sum of the groups $B \cap (H + C)$ and $(B + H) \cap C$.*

We now intend to prove the following statement.

Proposition 2.14. *In the splitting group $G = T \oplus R$, where $T = T(G)$, each subgroup is completely inert if, and only if, in both T and R each subgroup is completely inert and, moreover, the image of every homomorphism $f : S \rightarrow T$ is finite for every subgroup $S \leq R$.*

Proof. Necessity. Assume that $f(S)$ is infinite for some $S \leq R$. Letting $H := \{f(x) + x \mid x \in S\}$, then for $\varphi := 1_T \oplus (-1_R)$ we have

$$T' = 2f(S) = \{2f(x) \mid x \in S\} \leq H + \varphi(H) \sim H.$$

So, T' is finite, because $H \cap T = \{0\}$. However, since $2T_q(f(S)) = T_q(f(S))$ for any prime $q > 2$, $T_2(f(S))$ has to be infinite, and so it is non-reduced. But, in this case, $2(f(S))$ must be infinite, and this contradiction guarantees that $f(S)$ is finite.

Sufficiency. Assume $H \leq G$. Then, Lemma 2.13 informs us that H is a sub-direct sum of the groups T_1 and R_1 , where $T_1 := T \cap (H + R)$ and $R_1 := (T + H) \cap R$. Consequently, if $T_0 := T_1 \cap H$ and $R_0 := H \cap R_1$, then we know with help of the property of sub-direct sums that $T_1/T_0 \cong R_1/R_0$.

If, for a moment, the factor-group T_1/T_0 is non-reduced, then given the structure of the group T , we obtain that non-reduced must be the group T as well; thus, there is a homomorphism $R_1 \rightarrow T$ with infinite image, that is manifestly wrong.

If, however, T_1/T_0 is reduced, then each of its p -components is a factor-group of either a cyclic group or of a finite group, and thus there exists a homomorphism $T_1/T_0 \rightarrow T_1$ with non-zero image of each non-zero p -component of T_1/T_0 . Hence, if T_1/T_0 is infinite, then there will exist a homomorphism $R_1 \rightarrow T$ with infinite image, that is manifestly untrue.

Therefore, T_1/T_0 has to be finite, and so finite is R_1/R_0 too. Furthermore, since $\theta(H) = R_1 \sim R_0$, where $\theta : G \rightarrow R$ is the standard projection, we similarly perceive that

$$(1 - \theta)(H) = T_1 \sim T_0.$$

But, because

$$T_0 \oplus R_0 \leq H \leq T_1 \oplus R_1,$$

we then can conclude $H \sim T_1 \oplus R_1$.

Now, letting $\varphi \in \text{Aut } G$, by assumption we have $T_1 \sim \varphi(T_1)$, and thus the action of φ on R_1 can be represented as the sum of some $\psi \in \text{Aut } R$ and $\alpha \in \text{Hom}(R, T)$. Consequently, $\psi(R_1) \sim R_1$ and $\alpha(R_1)$ is finite, so that $\varphi(R_1) \sim R_1$ means $\varphi(H) \sim H$, as needed. \square

Example 2.15. The next two statements are valid:

(1) Write $G = D \oplus R$, where R is a torsion-free group, and D is a torsion group such that each its subgroup is completely inert and, for each prime p , its p -component is a non-reduced non-zero group. Then, each subgroup in G is completely inert if, and only if, $R \cong \mathbb{Z}$.

(2) If $T = \bigoplus_{p \in \Pi} T_p$ is a torsion group such that $T_p \cong \mathbb{Z}(p)$ for each $p \in \Pi$ and $|\Pi| = \aleph_0$. Then, every direct summand of $G = \prod_{p \in \Pi} T_p$ is fully invariant, and so completely inert. Besides, in G there is a subgroup which is *not* completely inert.

Proof. (1) **Necessity.** Suppose $H \leq R$ is a pure subgroup such that the rank of R/H is exactly 1. If $R/H \not\cong \mathbb{Z}$, then there exists a homomorphism $R \rightarrow D$ with infinite image, which is an absurd. So, $R/H \cong \mathbb{Z}$ and hence R is decomposable, which is again absurdly.

Notice that in our construction the group D possesses an infinite number of non-zero primary components: in fact, if the factor-group R/H is isomorphic to the subgroup of \mathbb{Q} which is generated by all rational numbers of the form $\frac{1}{p}$ with p a prime number and D has a finite number of p -primary components, then our arguments do *not* work properly.

Sufficiency. It follows at once from Proposition 2.14.

(2) Writing $G = A \oplus B$, we then arrive at $T(G) = T(A) \oplus T(B)$, and $f(T(A)) = \{0\}$ for every $f \in \text{Hom}(A, B)$; so, $f(A) = \{0\}$ as the quotient-group $A/T(A)$ is divisible.

Setting $x = (\dots, x_p, \dots)$, where $x_p \neq 0$, if, and only if, $p \in \Pi$ and $X \leq G$ is such a torsion-free rank 1 subgroup that $x \in X$.

Furthermore, putting $y \in G$, $y := (\dots, x'_p, \dots)$ with $x'_p \neq 0$ for every $p \in \Pi$ such that the coset $y + T(G)$ is independent of $x + T(G)$ over \mathbb{Q} in $G/T(G)$, one finds that there exists $\varphi \in \text{Aut } G$ with $\varphi(x) = y$. That is why, a simple check shows that the subgroup X is not completely inert in G , as stated. \square

An extremely difficult question is that of globally characterizing those reduced (p -)groups for which every their fully inert subgroup is commensurable with a fully invariant subgroup (see, e.g., [17, Problem 2.1]). The best achievements in this directions are that the class of such groups contains both the classes of totally projective groups and torsion-complete groups (see [12], [15] and [17, Theorem 2.2], respectively).

That is why, it is quite logical to ask what happens in the case of characteristically inert subgroups (compare also with [9]). Concretely, one may ask to characterize

those reduced (p -)groups whose totally (resp., completely) inert subgroups are commensurable with some fully invariant (resp., characteristic) subgroups.

In this vein, we are now prepared to attack the following helpful observation.

Proposition 2.16. *The subgroup H of a group G is commensurable with a characteristic subgroup of G if, and only if, H is uniformly completely inert in G .*

Proof. Necessity. It is almost evident since Proposition 2.22 reaches us that the subgroup is uniformly completely inert if, and only if, it is uniformly characteristically inert, and each subgroup commensurable with a characteristic subgroup is uniformly characteristically inert (see [10]).

Sufficiency. If we assume H is uniformly completely inert, then it is uniformly characteristically inert and so [10, Corollary 1.9] works to deduce that H is commensurable with a characteristic subgroup of G , as required. \square

We, thus, now come to a major assertion which summarizes a part of the above assertions and, thereby, stimulates our further writing.

Theorem 2.17. *Suppose G is a group such that each its subgroup is commensurable with a characteristic subgroup of G . Then, G can completely be characterized.*

Proof. A consultation with Proposition 2.16 is a guarantor that every subgroup of G is uniformly completely inert. Besides, each uniformly completely inert subgroup is manifestly completely inert. That is why, we can subsequently apply all statements starting from Lemma 2.9 to Example 2.15 to get the desired complete characterization after all. \square

Note that it cannot be derived any useful information when G is a reduced group such that each its (totally inert) subgroup is commensurable with a fully invariant subgroup of G . Indeed, it was noted in [13] that a fully invariant subgroup H of a group G is totally inert in G if, and only if, $f(H)$ has finite index in H for all endomorphisms $f : G \rightarrow G$.

Notice also that both Proposition 2.4 and Lemma 2.6 are common generalizations of [17, Theorem 2.3], and the same theorem appears in [13, Theorem 3.5].

The following extra comments are, hopefully, worthwhile.

Remark 2.18. The class of groups considered in Theorem 2.17 forms a much smaller class than the classes of groups in which every completely/characteristically inert subgroup is commensurable with a characteristic subgroup. In fact, the proper inclusion follows from the simple facts alluded to above that there are too many subgroups that are definitely *not* either completely nor characteristically inert. Thus, it cannot be happen that we will succeed to obtain their comprehensive descriptions at all.

It was pointed out in [17, Theorem 2.3] that, if $X = \bigoplus_{i \in I} G_i$ is a group such that each direct summand G_i is isomorphic to a fixed unbounded fully transitive p -group in which every fully inert subgroup is commensurable with a fully invariant subgroup, then X has the same property.

We now will expand this to the case of characteristic subgroups as follows.

Theorem 2.19. *Suppose that $G = \oplus_{i \in I} G_i$ is a group such that each direct summand G_i is isomorphic to a fixed unbounded transitive p -group in which every characteristically inert subgroup is commensurable with a characteristic subgroup. Then, G possesses the same property.*

Proof. It entirely relies on the same arguments as in Proposition 2.4 and Lemma 2.6, so the drop off the complete details. \square

Some additional things of the subject are these:

Lemma 2.20. *If H is an essential subgroup in G and $\phi \in \text{Aut } G$, then $\phi(H)$ is also essential in G . In particular, if H does not contain its proper essential subgroups, then H is characteristic in G .*

Proof. If we assume $A \cap \phi(H) = \{0\}$ for some $A \leq G$, then one verifies that

$$\phi^{-1}(A \cap \phi(H)) = \phi^{-1}(A) \cap H = \{0\},$$

so both $\phi^{-1}(A)$ and A are zero, as required. \square

We terminate this section with the following.

Proposition 2.21. *Let G be a torsion-free group of finite rank, and suppose $H \leq G$ is a free essential subgroup (i.e., $r(H) = r(G)$). Then, H is completely inert in G .*

Proof. Letting $\phi \in \text{Aut } G$, then Lemma 2.20 employs to get that $\phi(H)$ is essential in G , so that $H \cap \phi(H)$ is essential in both H and $\phi(H)$; in particular, one discovers that $H \cap \phi(H) \cong H$ whence $H \cap \phi(H)$ has finite index simultaneously in H and $\phi(H)$, thus substantiating our claim. \square

In particular, if G is decomposable, then with no too many efforts we can receive additional examples of completely inert *not* totally inert subgroups.

2.2. Uniformly completely inert subgroups

Our instrument here is the following.

Proposition 2.22. *A subgroup U of a group G is uniformly completely inert if, and only if, it is uniformly characteristically inert.*

Proof. The same idea as in Proposition 2.2 is workable since, by assumption, there is $n \in \mathbb{N}$ such that

$$|\psi^{-1}(C)/(C \cap \psi^{-1}(C))| \leq n$$

for each $\psi \in \text{Aut}(G)$, and so we claim that

$$|C/(C \cap \psi(C))| \leq n$$

as being an isomorphic image of $\psi^{-1}(C)/(C \cap \psi^{-1}(C))$ acting by ψ , as demonstrated above. \square

Recall now that a group G is said to have *unit sum number* $\text{usn}(G) = n \in \mathbb{N}$ if each endomorphism of G is a sum of $\leq n$ automorphisms of G .

As the next lemma illustrates, if $\text{usn}(G) = n$, then every uniformly characteristically inert subgroup of G is uniformly fully inert in G , hopefully bounded by another fixed positive integer.

Lemma 2.23. (1) If $\text{usn}(G) = n$, then every uniformly characteristically inert subgroup C of G (bounded by a fixed integer $k > 0$) is uniformly fully inert in G (bounded by an integer $\leq nk$).

(2) If C is uniformly characteristically inert in G and $H \sim C$, then H is uniformly characteristically inert in G .

Proof. (1) Letting φ be an arbitrary endomorphism of G , then there are $\alpha_1, \dots, \alpha_n \in \text{Aut } G$ with $\varphi = \alpha_1 + \dots + \alpha_n$. But then we obtain

$$(C + \varphi(C))/C \leq (\alpha_1(C) + C)/C + \dots + (\alpha_n(C) + C)/C,$$

and so

$$|(C + \varphi(C))/C| \leq |(\alpha_1(C) + C)/C| + \dots + |(\alpha_n(C) + C)/C| \leq nk,$$

as claimed.

(2) It can be verified in two different ways:

(a) It must be shown that the order of the factor-group $\alpha(H)/(\alpha(H) \cap H)$ is finite and the same for all $\alpha \in \text{Aut } G$. To that end, since $|H/(H \cap C)|, |C/(H \cap C)| \leq n$ for some n , one inspects that

$$|\alpha(H)/(\alpha(H) \cap \alpha(C))|, |\alpha(C)/(\alpha(H) \cap \alpha(C))| \leq n$$

for all $\alpha \in \text{Aut } G$. So, $\alpha(C) \sim \alpha(H)$, but as $\alpha(C) \sim C$, in view of $C \sim H$ we can get $\alpha(H) \sim H$ for any $\alpha \in \text{Aut } G$.

(b) Knowing that each uniformly characteristically inert subgroup is commensurable with some characteristic subgroup (see [10, Corollary 1.9]), we may deduce that H is commensurable with some characteristic subgroup, and consequently is uniformly characteristically inert, as asserted. \square

As a useful consequence of the previous lemma, we record.

Corollary 2.24. Let H be an infinite uniformly characteristically inert subgroup with bounded index $k \geq 1$ of the group $G = \bigoplus_{i \in I} G_i$, where $G_i \cong G_j$ for all $i, j \in I$ and the index set I is infinite. If π_i denotes the canonical projection of G onto G_i , then $H \leq \bigoplus_{i \in I} \pi_i(H)$ with bounded index $3k$.

Proof. Recall that $\text{usn}(G) = 3$ (see, e.g., [7]). Now, Lemma 2.23 tells us that H is, actually, uniformly fully inert. If, however, we assume the contrary that

$$\left| \left(\bigoplus_{i \in I} \pi_i(H) \right) / H \right| > 3k,$$

then there will exist $i_1, \dots, i_n \in I$ such that, for the sum $\psi = \pi_{i_1} + \dots + \pi_{i_n}$, we have

$$|(\psi(H) + H)/H| = |(\pi_{i_1}(H) + \dots + \pi_{i_n}(H))/H| > 3k,$$

contradicting to Lemma 2.23, as suspected. \square

Proposition 2.25. If G is a group in which each fully inert subgroup is uniformly fully inert and A is a direct summand in G with a finitely generated additional direct summand, then in A each fully inert subgroup is uniformly fully inert.

Proof. Write $G = A \oplus B$, $H \leq A$ is fully inert in A and set $F := \text{Hom}(A, B)H$. Since F is fully invariant in B , then it is not too hard to establish that the subgroup $H \oplus F$ is fully inert in G . Indeed, since F also is finitely generated, it follows that F is, actually, the image of a finite number of homomorphisms $A \rightarrow B$. Thus, the action of each endomorphism of G on $H \oplus F$ is the action of some endomorphism $\alpha \in E(A)$ on H , of some homomorphism $\beta : A \rightarrow B$ on H , of some endomorphism $\gamma \in E(B)$ on F , and of some homomorphism $\delta : B \rightarrow A$ on F . We, furthermore, have that $\gamma(F) \leq F$, $\beta(H) \leq F$ and so $\delta(F)$ is contained in the image of a finite number of some endomorphisms of A . Hence, by assumption, $H \oplus F$ is uniformly fully inert in G . However, every endomorphism of A can be viewed as an endomorphism of G and, therefore, H is uniformly fully inert in A , as pursued. \square

We now end our entire work with the following question which, hopefully, will stimulate a further investigation on the subject.

Problem 2.26. Explore those subgroups K of a group G such that the intersection $K \cap f(K)$ has finite index in both K and $f(K)$ for all homomorphisms $f : K \rightarrow G$.

It is apparent that these subgroups are always totally inert and strongly inert simultaneously.

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Generalization and refinement of fractional Hermite-Hadamard type inequalities for m -convex functions

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Abstract. Fractional Hermite-Hadamard type inequalities are recognized as significant results in the field of convex analysis. In this work, we derive several inequalities of this type for twice differentiable m -convex functions by employing various analytical methods, including the Hölder-İşcan inequality and the improved power mean integral inequality.

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“It seems to me that the notion of a convex function is just as fundamental as that of a positive or an increasing function. If I am not mistaken, this concept deserves a place in elementary treatments of real function theory.”

– J.L.W.V. Jensen


1. Introduction

The Hermite-Hadamard inequality is a classical result in convex analysis that provides a double inequality for the integral average of a convex function over an interval. Given its simplicity and elegance, the Hermite-Hadamard inequality has been widely studied and extended in various directions, including discrete, multidimensional, and fractional settings.

In recent years, the development of fractional calculus, a generalization of classical calculus involving derivatives and integrals of arbitrary (non-integer) order, has

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led to renewed interest in extending classical inequalities to the fractional context. This inequality has opened new avenues for applications in mathematical analysis, applied sciences, and engineering, where fractional models often provide more accurate descriptions of memory and hereditary properties.

Parallel to these advancements, the concept of m -convex functions, introduced as a generalization of traditional convex functions, has gained significant attention. As introduced by Toader [20], a function $\xi : [0, \infty) \rightarrow \mathbb{R}$ is said to be m -convex for $m \in [0, 1]$ if it satisfies the inequality

$$\xi(v\zeta_1 + m(1-v)\zeta_2) \leq v\xi(\zeta_1) + m(1-v)\xi(\zeta_2),$$

for all $\zeta_1, \zeta_2 \in [0, \infty)$ and $v \in [0, 1]$. This class bridges the gap between convex and star-shaped functions, making it highly useful for extending various integral inequalities.

This article focuses on establishing fractional Hermite-Hadamard type inequalities for twice differentiable m -convex functions, employing tools from fractional calculus such as Riemann–Liouville fractional integrals. These new results not only generalize classical Hermite-Hadamard inequalities but also provide refined bounds under the broader assumption of m -convexity. Such generalizations are valuable for both theoretical research and practical applications in areas where fractional models and generalized convexity play a key role, for further investigations see ([1]–[9] and [14]–[18]).

In [12], İşcan obtained the following inequality named as Hölder – İşcan integral inequality which gives better results than the classical Hölder’s integral inequality [16] is stated as:

Theorem 1.1. *Let $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If ξ_1 and ξ_2 are real functions defined on $[\varpi_1, \varpi_2]$ and if $|\xi_1|^p$ and $|\xi_2|^q$ are integrable on $[\varpi_1, \varpi_2]$, then*

$$\int_{\varpi_1}^{\varpi_2} |\xi_1(u)\xi_2(u)| du \leq \frac{1}{\varpi_2 - \varpi_1} \left[\left(\int_{\varpi_1}^{\varpi_2} (\varpi_2 - u) |\xi_1(u)|^p du \right)^{\frac{1}{p}} \left(\int_{\varpi_1}^{\varpi_2} (\varpi_2 - u) |\xi_2(u)|^q du \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_{\varpi_1}^{\varpi_2} (u - \varpi_1) |\xi_1(u)|^p du \right)^{\frac{1}{p}} \left(\int_{\varpi_1}^{\varpi_2} (u - \varpi_1) |\xi_2(u)|^q du \right)^{\frac{1}{q}} \right]. \quad (1.1)$$

Remark 1.2. Note that if we put $p = q = 2$, the above inequality gives us improved Cauchy–Schwarz integral inequality.

A different representation of Hölder–İşcan integral inequality is stated as:

Theorem 1.3. Let ξ_1, ξ_2 are real valued functions defined on $[\varpi_1, \varpi_2]$ and if $|\xi_1|$ and $|\xi_1||\xi_2|^q$ are integrable on $[\varpi_1, \varpi_2]$, then for $q \geq 1$ we have:

$$\begin{aligned} & \int_{\varpi_1}^{\varpi_2} |\xi_1(u)\xi_2(u)|du \leq \\ & \leq \frac{1}{\varpi_2 - \varpi_1} \left[\left(\int_{\varpi_1}^{\varpi_2} (\varpi_2 - u)|\xi_1(u)|du \right)^{1-\frac{1}{q}} \left(\int_{\varpi_1}^{\varpi_2} (\varpi_2 - u)|\xi_1(u)||\xi_2(u)|^q du \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\varpi_1}^{\varpi_2} (u - \varpi_1)|\xi_1(u)|du \right)^{1-\frac{1}{q}} \left(\int_{\varpi_1}^{\varpi_2} (u - \varpi_1)|\xi_1(u)||\xi_2(u)|^q du \right)^{\frac{1}{q}} \right]. \end{aligned} \quad (1.2)$$

The above inequality is known as Improved power mean integral inequality (see [13]), which is the refinement of power mean integral inequality [18].

Now, we are going to recall some necessary definitions and mathematical results related to fractional calculus which will be use further in this article.

Definition 1.4. [19] Let $\xi \in L[\varpi_1, \varpi_2]$. The Riemann–Liouville integrals $J_{\varpi_1^+}^\alpha \xi(\zeta)$ and $J_{\varpi_2^-}^\alpha \xi(\zeta)$ of order $\alpha > 0$ are defined by

$$J_{\varpi_1^+}^\alpha \xi(\zeta) = \frac{1}{\Gamma(\alpha)} \int_{\varpi_1}^{\zeta} (\zeta - v)^{\alpha-1} \xi(v)dv, \quad \zeta > \varpi_1$$

and

$$J_{\varpi_2^-}^\alpha \xi(\zeta) = \frac{1}{\Gamma(\alpha)} \int_{\zeta}^{\varpi_2} (v - \zeta)^{\alpha-1} \xi(v)dv, \quad \zeta < \varpi_2,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$ is the Gamma function.

Remark 1.5. Note that if we take $\alpha = 0$, then $J_{\varpi_1^+}^0 \xi(\zeta) = J_{\varpi_2^-}^0 \xi(\zeta) = \xi(\zeta)$ and if we take $\alpha = 1$, then the fractional integral reduces to the classical one.

Recently, we have proved the following two distinct results related to fractional Hermite Hadamard type inequality for the class of twice differentiable m -convex functions (see [6]).

Theorem 1.6. Let $\xi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on I° , the interior of I . Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$. If $|\xi''|^q$ is m -convex on I for some $m \in (0, 1]$ and $q \geq 1$ then the following inequality for

fractional integrals with $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ holds:

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \beta^{\frac{1}{p}}(p + 1, \alpha p + 1) \\ & \times \left[\left(\frac{|\xi''(\varpi_1)|^q + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{m \left| \xi''\left(\frac{\varpi_1}{m}\right) \right|^q + |\xi''(\varpi_2)|^q}{2} \right)^{\frac{1}{q}} \right], \quad (1.3) \end{aligned}$$

where β is the Euler Beta function.

Theorem 1.7. Let $\xi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° . Assume that $q \geq 1$, such that $|\xi''|^q$ is convex function on I . Suppose that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{\alpha(\varpi_2 - \varpi_1)^2}{4(\alpha + 1)(\alpha + 2)} \left[\left(\frac{2\alpha + 4}{3\alpha + 9} |\xi''(\varpi_1)|^q + \frac{\alpha + 5}{3\alpha + 9} |\xi''(\varpi_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{\alpha + 5}{3\alpha + 9} |\xi''(\varpi_1)|^q + \frac{2\alpha + 4}{3\alpha + 9} |\xi''(\varpi_2)|^q \right)^{\frac{1}{q}} \right]. \quad (1.4) \end{aligned}$$

The structure of this article unfolds as follows: The subsequent section contains some results related to generalization and refinements of fractional Hermite Hadamard type inequalities applicable to the category of twice differentiable m -convex functions. Our approach will leverage diverse techniques, encompassing Hölder-İşcan and improved power mean integral inequalities. These findings are anticipated to exhibit a broader scope compared to those presented in [6], [9], [10] and [11]. The third section will provide a concluding statement, while the final section will offer insights and future prospects for readers interested in further exploration.

2. Various estimations for the right bound of fractional Hermite Hadamard type inequalities for twice differentiable m -Convex functions

In order to prove our main results we need to recall following lemma from [10].

Lemma 2.1. Let $\xi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° , the interior of I . Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated

identity for fractional integral with $\alpha > 0$ holds:

$$\begin{aligned} & \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \\ &= \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \int_0^1 v(1 - v^\alpha) [\xi''(v\varpi_1 + (1 - v)\varpi_2) + \xi''((1 - v)\varpi_1 + v\varpi_2)] dv, \quad (2.1) \end{aligned}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$ is the Gamma function.

Now, we are going to state and prove our new results related to fractional Hermite Hadamard type inequalities for twice differentiable m -convex functions.

Theorem 2.2. Let $\xi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on I° , the interior of I . Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$. If $|\xi''|^q$ is m -convex on I for some $m \in (0, 1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ holds:

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \left\{ (\beta(p + 1, \alpha p + 1) - \beta(p + 2, \alpha p + 1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p + 2, \alpha p + 1) \right\} \\ & \quad \times \left[\left(\frac{|\xi''(\varpi_1)|^q + 2m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{2|\xi''(\varpi_1)|^q + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q}{6} \right)^{\frac{1}{q}} \right]. \quad (2.2) \end{aligned}$$

Proof. By using Lemma 2.1 and the property of absolute value, we have,

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \int_0^1 |v(1 - v^\alpha)| [|\xi''(v\varpi_1 + (1 - v)\varpi_2)| + |\xi''((1 - v)\varpi_1 + v\varpi_2)|] dv. \quad (2.3) \end{aligned}$$

Applying (1.1) to $\int_0^1 |v(1-v^\alpha)| |\xi''(v\varpi_1 + (1-v)\varpi_2)| dv$ and $\int_0^1 |v(1-v^\alpha)| |\xi''((1-v)\varpi_1 + v\varpi_2)| dv$ implies

$$\begin{aligned} & \int_0^1 |v(1-v^\alpha)| |\xi''(v\varpi_1 + (1-v)\varpi_2)| dv \\ & \leq \left[\left(\int_0^1 (1-v) |v(1-v^\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-v) |\xi''(v\varpi_1 + (1-v)\varpi_2)|^q dv \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 v |v(1-v^\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 v |\xi''(v\varpi_1 + (1-v)\varpi_2)|^q dv \right)^{\frac{1}{q}} \right] \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 |v(1-v^\alpha)| |\xi''((1-v)\varpi_1 + v\varpi_2)| dv \\ & \leq \left[\left(\int_0^1 (1-v) |v(1-v^\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-v) |\xi''((1-v)\varpi_1 + v\varpi_2)|^q dv \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 v |v(1-v^\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 v |\xi''((1-v)\varpi_1 + v\varpi_2)|^q dv \right)^{\frac{1}{q}} \right] \end{aligned}$$

As we have $|\xi''|^q$ is an m -convex function, so we can take

$$|\xi''(v\varpi_1 + (1-v)\varpi_2)|^q \leq v |\xi''(\varpi_1)|^q + m(1-v) \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q$$

and

$$|\xi''((1-v)\varpi_1 + v\varpi_2)|^q \leq (1-v) |\xi''(\varpi_1)|^q + mv \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q.$$

Utilizing the above four results (2.3) becomes

$$\begin{aligned}
 & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\
 & \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \left[\left(\int_0^1 v^p (1 - v^\alpha)^p dv - \int_0^1 v^{p+1} (1 - v^\alpha)^p dv \right)^{\frac{1}{p}} \right. \\
 & \quad \times \left\{ \left(|\xi''(\varpi_1)|^q \int_0^1 v(1 - v) dv + m \left| \xi'' \left(\frac{\varpi_2}{m} \right) \right|^q \int_0^1 (1 - v)^2 dv \right)^{\frac{1}{q}} \right. \\
 & \quad \left. \left. + \left(|\xi''(\varpi_1)|^q \int_0^1 (1 - v)^2 dv + m \left| \xi'' \left(\frac{\varpi_2}{m} \right) \right|^q \int_0^1 v(1 - v) dv \right)^{\frac{1}{q}} \right\} \right. \\
 & \quad \left. + \left(\int_0^1 v^{p+1} (1 - v^\alpha)^p dv \right)^{\frac{1}{p}} \right. \\
 & \quad \times \left\{ \left(|\xi''(\varpi_1)|^q \int_0^1 v^2 dv + m \left| \xi'' \left(\frac{\varpi_2}{m} \right) \right|^q \int_0^1 v(1 - v) dv \right)^{\frac{1}{q}} \right. \\
 & \quad \left. \left. + \left(|\xi''(\varpi_1)|^q \int_0^1 v(1 - v) dv + m \left| \xi'' \left(\frac{\varpi_2}{m} \right) \right|^q \int_0^1 v^2 dv \right)^{\frac{1}{q}} \right\} \right].
 \end{aligned}$$

After using the following facts, the result of Theorem 2.2 is accomplished.

$$\begin{aligned}
 \int_0^1 v^p (1 - v^\alpha)^p dt & \leq \int_0^1 v^p (1 - v)^{\alpha p} dv = \beta(p + 1, \alpha p + 1), \\
 \int_0^1 v^{p+1} (1 - v^\alpha)^p dt & \leq \int_0^1 v^{p+1} (1 - v)^{\alpha p} dv = \beta(p + 2, \alpha p + 1), \\
 \int_0^1 v(1 - v) dv & = \frac{1}{6}
 \end{aligned}$$

and

$$\int_0^1 v^2 dv = \int_0^1 (1 - v)^2 dv = \frac{1}{3}.$$

□

Remark 2.3. If we choose $\alpha = m = 1$ in Theorem 2.2, then we get Theorem 2.1 of [9].

Corollary 2.4. Under the assumptions of Theorem 2.2,

1. If we put $p = q = 2$, then we get the refined result obtained by using Cauchy–Schwarz integral inequality as:

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \left\{ (\beta(3, 2\alpha + 1) - \beta(4, 2\alpha + 1))^{\frac{1}{2}} + \beta^{\frac{1}{2}}(4, 2\alpha + 1) \right\} \\ & \quad \times \left[\left(\frac{|\xi''(\varpi_1)|^2 + 2m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^2}{6} \right)^{\frac{1}{2}} + \left(\frac{2|\xi''(\varpi_1)|^2 + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^2}{6} \right)^{\frac{1}{2}} \right]. \end{aligned}$$

which is the refinement of first result of Corollary 2 of [6].

2. If we choose $m = 1$, then we get following refined result related to fractional Hermite Hadamard type inequality for twice differentiable convex function:

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \left\{ (\beta(p + 1, \alpha p + 1) - \beta(p + 2, \alpha p + 1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p + 2, \alpha p + 1) \right\} \\ & \quad \times \left[\left(\frac{|\xi''(\varpi_1)|^q + 2|\xi''(\varpi_2)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{2|\xi''(\varpi_1)|^q + |\xi''(\varpi_2)|^q}{6} \right)^{\frac{1}{q}} \right] \end{aligned}$$

which is the refinement of third result of Theorem 3 of [10].

3. If we choose $\alpha = 1$, then we get following refined result related to Hermite Hadamard type inequality for twice differentiable m -convex function:

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{1}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{4} \left\{ (\beta(p + 1, p + 1) - \beta(p + 2, p + 1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p + 2, p + 1) \right\} \\ & \quad \times \left[\left(\frac{|\xi''(\varpi_1)|^q + 2m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{2|\xi''(\varpi_1)|^q + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q}{6} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

which is the refinement of third result of Corollary 2 of [6].

Theorem 2.5. Here we claim that inequality (2.2) of Theorem 2.2 is better than the inequality (1.6).

Proof. Since the function $h : [0, \infty) \rightarrow \mathbb{R}$, $h(\zeta) = \zeta^n$, $n \in (0, 1]$ is concave, we can write:

$$\frac{\varpi_1^n + \varpi_2^n}{2} = \frac{h(\varpi_1) + h(\varpi_2)}{2} \leq h\left(\frac{\varpi_1 + \varpi_2}{2}\right) = \left(\frac{\varpi_1 + \varpi_2}{2}\right)^n \quad (2.4)$$

$\forall \varpi_1, \varpi_2 \geq 0$. For the above inequality if we choose

$$\varpi_1 = \frac{|\xi''(\varpi_1)|^q + 2m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}, \quad \varpi_2 = \frac{2|\xi''(\varpi_1)|^q + m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6}$$

and $n = \frac{1}{q}$. Applying the inequality (2.4) to the inequality (2.2), we have

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \left\{ (\beta(p + 1, \alpha p + 1) - \beta(p + 2, \alpha p + 1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p + 2, \alpha p + 1) \right\} \\ & \quad \times \left[\left(\frac{|\xi''(\varpi_1)|^q + 2m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{2|\xi''(\varpi_1)|^q + m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6} \right)^{\frac{1}{q}} \right] \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{(\alpha + 1)} \left\{ (\beta(p + 1, \alpha p + 1) - \beta(p + 2, \alpha p + 1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p + 2, \alpha p + 1) \right\} \\ & \quad \times \left[\frac{1}{2} \left(\frac{|\xi''(\varpi_1)|^q + 2m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q + 2|\xi''(\varpi_1)|^q + m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{6} \right)^{\frac{1}{q}} \right] \\ & = \frac{(\varpi_2 - \varpi_1)^2}{2^{\frac{1}{q}}(\alpha + 1)} \left\{ (\beta(p + 1, \alpha p + 1) - \beta(p + 2, \alpha p + 1))^{\frac{1}{p}} + \beta^{\frac{1}{p}}(p + 2, \alpha p + 1) \right\} \\ & \quad \times \left[\frac{|\xi''(\varpi_1)|^q + m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2} \right]^{\frac{1}{q}}. \end{aligned}$$

Again applying the inequality (2.4) to the above inequality by taking

$$\varpi_1 = \beta(p + 1, \alpha p + 1) - \beta(p + 2, \alpha p + 1), \quad \varpi_2 = \beta(p + 2, \alpha p + 1)$$

and $n = \frac{1}{p}$, we have

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{\alpha + 1} \beta^{\frac{1}{p}}(p + 1, \alpha p + 1) \left(\frac{|\xi''(\varpi_1)|^q + m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2} \right)^{\frac{1}{q}} \quad (2.5) \end{aligned}$$

Similarly. we can write

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{\alpha + 1} \beta^{\frac{1}{p}} (p + 1, \alpha p + 1) \left(\frac{m |\xi''(\frac{\varpi_1}{m})|^q + |\xi''(\varpi_2)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned} \quad (2.6)$$

By adding (2.5) and (2.6) we get our required result i.e., inequality (1.6). \square

Theorem 2.6. *Let $\xi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on I° , the interior of I . Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$. If $|\xi''|^q$ is m -convex on I for some $m \in (0, 1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{\alpha(\varpi_2 - \varpi_1)^2}{12(\alpha + 1)(\alpha + 3)(\alpha + 4)} \left[\left(\frac{(\alpha + 4)(\alpha + 5)}{(\alpha + 2)} \right)^{1 - \frac{1}{q}} \right. \\ & \quad \times \left\{ \left(\frac{(\alpha + 7)}{2} |\xi''(\varpi_1)|^q + m \frac{(\alpha^2 + 9\alpha + 26)}{2(\alpha + 2)} \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{(\alpha^2 + 9\alpha + 26)}{2(\alpha + 2)} |\xi''(\varpi_1)|^q + m \frac{(\alpha + 7)}{2} \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q \right)^{\frac{1}{q}} \right\} \\ & \quad + (2(\alpha + 4))^{1 - \frac{1}{q}} \left\{ \left(\frac{3(\alpha + 3)}{2} |\xi''(\varpi_1)|^q + m \frac{(\alpha + 7)}{2} \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{(\alpha + 7)}{2} |\xi''(\varpi_1)|^q + m \frac{3(\alpha + 3)}{2} \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q \right)^{\frac{1}{q}} \right\} \Big]. \end{aligned} \quad (2.7)$$

Proof. By using Lemma 2.1 and the property of absolute value, we have,

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \int_0^1 |v(1 - v^\alpha)| [|\xi''(v\varpi_1 + (1 - v)\varpi_2)| + |\xi''((1 - v)\varpi_1 + v\varpi_2)|] dv. \end{aligned} \quad (2.8)$$

Applying (1.2) to $\int_0^1 |v(1-v^\alpha)| |\xi''(v\varpi_1 + (1-v)\varpi_2)| dv$ and $\int_0^1 |v(1-v^\alpha)| |\xi''((1-v)\varpi_1 + v\varpi_2)| dv$ implies

$$\begin{aligned} & \int_0^1 |v(1-v^\alpha)| |\xi''(v\varpi_1 + (1-v)\varpi_2)| dv \\ & \leq \left[\left(\int_0^1 v(1-v)(1-v^\alpha) dv \right)^{1-\frac{1}{q}} \left(\int_0^1 v(1-v)(1-v^\alpha) |\xi''(v\varpi_1 + (1-v)\varpi_2)|^q dv \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 v(v(1-v^\alpha)) dv \right)^{1-\frac{1}{q}} \left(\int_0^1 v(v(1-v^\alpha)) |\xi''(v\varpi_1 + (1-v)\varpi_2)|^q dv \right)^{\frac{1}{q}} \right] \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 |v(1-v^\alpha)| |\xi''((1-v)\varpi_1 + v\varpi_2)| dv \\ & \leq \left[\left(\int_0^1 v(1-v)(1-v^\alpha) dv \right)^{1-\frac{1}{q}} \left(\int_0^1 v(1-v)(1-v^\alpha) |\xi''((1-v)\varpi_1 + v\varpi_2)|^q dv \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 v(v(1-v^\alpha)) dv \right)^{1-\frac{1}{q}} \left(\int_0^1 v(v(1-v^\alpha)) |\xi''((1-v)\varpi_1 + v\varpi_2)|^q dv \right)^{\frac{1}{q}} \right] \end{aligned}$$

As we have $|\xi''|^q$ is an m -convex function, so we can take

$$|\xi''(v\varpi_1 + (1-v)\varpi_2)|^q \leq v |\xi''(\varpi_1)|^q + m(1-v) \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q$$

and

$$|\xi''((1-v)\varpi_1 + v\varpi_2)|^q \leq (1-v) |\xi''(\varpi_1)|^q + mv \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q.$$

Utilizing the above four results (2.8) becomes

$$\begin{aligned}
& \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\
& \leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)} \left[\left(\int_0^1 v(1 - v^\alpha) dv - \int_0^1 v^2(1 - v^\alpha) dv \right)^{1 - \frac{1}{q}} \right. \\
& \quad \times \left\{ \left(|\xi''(\varpi_1)|^q \int_0^1 v^2(1 - v)(1 - v^\alpha) dv + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q \int_0^1 v(1 - v)^2(1 - v^\alpha) dv \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(|\xi''(\varpi_1)|^q \int_0^1 v(1 - v)^2(1 - v^\alpha) dv + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q \int_0^1 v^2(1 - v)(1 - v^\alpha) dv \right)^{\frac{1}{q}} \right\} \\
& \quad + \left(\int_0^1 v^2(1 - v^\alpha) dv \right)^{1 - \frac{1}{q}} \\
& \quad \times \left\{ \left(|\xi''(\varpi_1)|^q \int_0^1 v^3(1 - v^\alpha) dv + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q \int_0^1 v^2(1 - v)(1 - v^\alpha) dv \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(|\xi''(\varpi_1)|^q \int_0^1 v^2(1 - v)(1 - v^\alpha) dv + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q \int_0^1 v^3(1 - v^\alpha) dv \right)^{\frac{1}{q}} \right\} \Bigg].
\end{aligned}$$

After arranging and using the following facts the result of Theorem 2.6 is accomplished.

$$\begin{aligned}
\int_0^1 v(1 - v^\alpha) dv &= \frac{\alpha}{2(\alpha + 2)}, \\
\int_0^1 v^2(1 - v^\alpha) dv &= \frac{\alpha}{3(\alpha + 3)}, \\
\int_0^1 v^2(1 - v)(1 - v^\alpha) dv &= \frac{\alpha(\alpha + 7)}{12(\alpha + 3)(\alpha + 4)}, \\
\int_0^1 v(1 - v)^2(1 - v^\alpha) dv &= \frac{\alpha(\alpha^2 + 9\alpha + 26)}{12(\alpha + 3)(\alpha + 4)}
\end{aligned}$$

and

$$\int_0^1 v^3(1-v^\alpha)dv = \frac{\alpha}{4(\alpha+4)}.$$

□

Remark 2.7. If we choose $\alpha = m = 1$ in Theorem 2.6, then we get Theorem 2.3 of [9].

Corollary 2.8. Under the assumptions of Theorem 2.6,

1. If we choose $m = 1$, then we get following refined result related to fractional Hermite Hadamard type inequality for twice differentiable convex function:

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1+}^\alpha \xi(\varpi_2) + J_{\varpi_2-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{\alpha(\varpi_2 - \varpi_1)^2}{12(\alpha+1)(\alpha+3)(\alpha+4)} \left[\left(\frac{(\alpha+4)(\alpha+5)}{(\alpha+2)} \right)^{1-\frac{1}{q}} \right. \\ & \quad \times \left\{ \left(\frac{(\alpha+7)}{2} |\xi''(\varpi_1)|^q + \frac{(\alpha^2+9\alpha+26)}{2(\alpha+2)} |\xi''(\varpi_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left. + \left(\frac{(\alpha^2+9\alpha+26)}{2(\alpha+2)} |\xi''(\varpi_1)|^q + \frac{(\alpha+7)}{2} |\xi''(\varpi_2)|^q \right)^{\frac{1}{q}} \right\} \right. \\ & \quad \left. + (2(\alpha+4))^{1-\frac{1}{q}} \left\{ \left(\frac{3(\alpha+3)}{2} |\xi''(\varpi_1)|^q + \frac{(\alpha+7)}{2} |\xi''(\varpi_2)|^q \right)^{\frac{1}{q}} \right. \right. \\ & \quad \left. \left. + \left(\frac{(\alpha+7)}{2} |\xi''(\varpi_1)|^q + \frac{3(\alpha+3)}{2} |\xi''(\varpi_2)|^q \right)^{\frac{1}{q}} \right\} \right]. \end{aligned}$$

2. If we choose $\alpha = 1$, then we get following Hermite Hadamard type inequality for twice differentiable m -convex function:

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{1}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta \right| \leq \frac{(\varpi_2 - \varpi_1)^2}{24} \\ & \times \left[\left(\frac{2|\xi''(\varpi_1)|^q + 3m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q}{5} \right)^{\frac{1}{q}} + \left(\frac{3|\xi''(\varpi_1)|^q + 2m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q}{5} \right)^{\frac{1}{q}} \right], \end{aligned}$$

which is the refinement of Corollary 3 of [6].

Theorem 2.9. *Under the same assumptions of Theorem 2.6, we have:*

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{\alpha(\varpi_2 - \varpi_1)^2}{4(\alpha + 1)(\alpha + 2)} \\ & \times \left[\left(\frac{|\xi''(\varpi_1)|^q + m \left| \xi''\left(\frac{\varpi_2}{m}\right) \right|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{m |\xi''\left(\frac{\varpi_1}{m}\right)|^q + |\xi''(\varpi_2)|^q}{2} \right)^{\frac{1}{q}} \right]. \quad (2.9) \end{aligned}$$

Proof. The proof of the above theorem will be followed in the same way as in Theorem 2.5 by applying inequality (2.4) to the result of Theorem 2.6. \square

Remark 2.10. Following well-known results would be captured as special cases of our obtained result by varying different values of m and α in Theorem 2.9:

1. If we choose $\alpha = 1$, then we get Corollary 3 of [6].
2. If we choose $\alpha = m = 1$, then we get Remark 13 of [11].

Corollary 2.11. *If we choose $m = 1$ in Theorem 2.9, then we get following refined result related to fractional Hermite Hadamard type inequality for twice differentiable convex function:*

$$\begin{aligned} & \left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varpi_2 - \varpi_1)^\alpha} [J_{\varpi_1^+}^\alpha \xi(\varpi_2) + J_{\varpi_2^-}^\alpha \xi(\varpi_1)] \right| \\ & \leq \frac{\alpha(\varpi_2 - \varpi_1)^2}{2(\alpha + 1)(\alpha + 2)} \left(\frac{|\xi''(\varpi_1)|^q + |\xi''(\varpi_2)|^q}{2} \right)^{\frac{1}{q}}, \end{aligned}$$

which is the same result as we have obtained by applying (2.4) to the result of Theorem 1.7.

3. Conclusion

The fractional Hermite-Hadamard inequality stands out as one of the most renowned inequality within the realm of inequalities, boasting numerous generalizations across different classes of convex functions in existing literature. In this article, we have presented a comprehensive study on the generalization and refinement of fractional Hermite-Hadamard type inequalities for the class of twice differentiable m -convex functions. The main results, outlined in Section 2, extend classical inequalities by incorporating the framework of fractional calculus and the broader notion of m -convexity. These results not only generalize the existing Hermite-Hadamard type inequalities but also provide sharper and more refined bounds compared to those available in the current literature, i.e., [6], [9], [10] and [11]. Finally, last section discusses several promising directions for future research.

Overall, this study contributes significantly to the growing body of work in fractional integral inequalities and convex analysis, opening up new possibilities for both theoretical exploration and practical implementation.

Now, we are going to summarize the results of Section 2 in tabular form.

3.1. Results Summary of Section 2

S. No	m	α	Results	Found in
1	1	–	FHHTI for Convex Functions	[10] This Article
2	–	1	HHTI for m -Convex Functions	[6], This Article
3	1	1	HHTI for Convex Functions	[9, 11]

TABLE 1. Result Summary of Section 2

In the above table FHHTI, HHTI and – stands for Fractional Hermite-Hadamard type inequality, Hermite Hadamard type inequality and for any value, respectively.

4. Remarks and future ideas

In this section, we present several observations and propose potential avenues for future research based on the results established in this article.

1. All inequalities discussed throughout this study can be reformulated in the reverse direction for concave functions, utilizing the well-known equivalence that a function ξ is concave if and only if $-\xi$ is convex.
2. An interesting direction for future work is the extension of fractional Hermite-Hadamard type inequalities by incorporating weights, thereby deriving Fejér-type inequalities.
3. Similar analytical frameworks may be applied to explore other generalized classes of convex functions beyond the m -convex class considered here.
4. The results presented in this work could also be adapted to the discrete setting, providing analogous inequalities in discrete calculus.
5. Another promising extension involves generalizing these results to multidimensional settings, which may broaden their applicability.
6. The theory can further be developed within the framework of time scale calculus or quantum calculus, allowing for unified treatment of discrete and continuous cases.
7. Lastly, the ideas and methods employed in this article may be extended to the context of fuzzy analysis, opening new perspectives in uncertainty modeling and fuzzy optimization.

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Hermite–Hadamard type inequalities via (h, m) -convexity

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Abstract. In this paper, we establish a novel Hermite–Hadamard inequality for (h, m) -convex functions using Riemann–Liouville fractional integral operators, right and left. Furthermore, some new Hermite–Hadamard type fractional integral inequalities are proved for differentiable functions whose first derivative is (h, m) -convex. We demonstrate that these newly established integral inequalities generalize some existing results.

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Keywords: Hermite–Hadamard inequality, fractional integral, convex function, (h, m) -convex function.

1. Introduction

The notion of convexity is a fundamental and highly productive idea in contemporary mathematics.

Definition 1.1. A function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, $a < b$ is said to be convex if the following inequality holds for all $x, y \in [a, b]$ and $t \in [0, 1]$:


$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

From now on, throughout the paper, we always suppose that $a < b$ when we consider intervals $[a, b]$.

Convex functions play an important role in many branches of mathematics, and are of particular interest to mathematicians in inequality theory. The researchers interested in the development of convex functions can get a comprehensive overview in [11].

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Consequently, investigations in this realm have led to the development of numerous novel inequalities, significantly enriching the field. One of the most researched and most important inequalities for convex functions on $[a, b]$ is the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

First established in 1893, the Hermite–Hadamard inequality for convex functions has undergone numerous refinements over the years, with various mathematicians having contributed to its advancement and generalization, see e.g. papers [3, 4, 5, 6, 10, 17] and the references therein for more information and generalizations of inequality (1.1).

Among the many generalizations of the convex functions, we recall the concepts of m -convex, h -convex and (h, m) -convex functions, which coincide with the convex functions for $m = 1$, $h(x) = x$ and $m = 1$, $h(x) = x$, respectively.

Definition 1.2. [19] *Function $f : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ is said to be m -convex with $m \in [0, 1]$, if for all $x, y \in [a, b]$ and $t \in [0, 1]$:*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Definition 1.3. [21] *Let $h : [0, 1] \rightarrow \mathbb{R}^+$. Then $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^+$ is called an h -convex function if the following inequality holds for $x, y \in [a, b]$ and $t \in [0, 1]$:*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

Definition 1.4. [13] *Let $h : [0, 1] \rightarrow \mathbb{R}^+$. We say that $f : [a, b] \subset \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an (h, m) -convex function with $m \in [0, 1]$, if for all $x, y \in [a, b]$ and $t \in [0, 1]$:*

$$f(tx + m(1-t)y) \leq h(t)f(x) + mh(1-t)f(y). \quad (1.2)$$

Fractional calculus, a generalization of ordinary calculus, has a rich history spanning over about 300 years. Its origins date back to 1695, when Gottfried Wilhelm Leibniz and Guillaume de l'Hôpital exchanged letters discussing the possibility of extending integer-order derivatives to non-integer orders. This conversation sparked the interest of mathematicians, leading to further exploration of fractional calculus concept. Today, fractional calculus has a wide range of applications in various fields, including acoustics, optics, viscoelasticity, chemical, rheology, and control theory, electrical and mechanical engineering, statistical physics, robotics and bioengineering. To address real-world problems, mathematicians have developed numerous fractional integral operators. Generalization of Hermite–Hadamard inequality (1.1) via fractional integrals has been extensively researched, see e.g. [2, 7, 12, 18] and the references therein. We recall one of the most well-known fractional integral operator called Riemann–Liouville integral operator.

Definition 1.5. [9, 14] Let $f \in L^1_{loc}([a, b])$. Then the Riemann–Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$ are defined by

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

$$I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b.$$

From now on, throughout the paper, we always assume that $\alpha > 0$.

In [15], Sarikaya et al. proved the following Hermite–Hadamard type inequality for the sum of Riemann–Liouville fractional integrals $I_{a^+}^\alpha$ and $I_{b^-}^\alpha$.

Theorem 1.6. [16] Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive, convex function on $[a, b]$ and $a \geq 0$. Then we have the following fractional integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} (I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)) \leq \frac{f(a) + f(b)}{2}.$$

Recently, Ali et al. gave the following refinements of (1.1) for the Riemann–Liouville fractional integrals.

Theorem 1.7. [1] Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $[a, b]$ such that $f' \in L^1_{loc}([a, b])$ is h -convex on $[a, b]$. The following inequalities hold:

$$(b-a)^\alpha \frac{\frac{1}{h(\frac{1}{2})} f(\frac{a+b}{2}) + \left(\frac{1}{\alpha} + H - \frac{1}{h(\frac{1}{2})}\right) f(a)}{1+H\alpha} - (b-a)^{\alpha+1} \frac{H\alpha f'(a)}{2h(\frac{1}{2})(\alpha+1)(1+H\alpha)} \quad (1.3)$$

$$\leq \Gamma(\alpha) (I_{b^-}^\alpha f(a)) \leq (b-a)^\alpha \frac{f(a)+H\alpha f(b)}{\alpha(1+H\alpha)} + (b-a)^{\alpha+1} \frac{Hf'(a)}{(\alpha+1)(1+H\alpha)}$$

and

$$(b-a)^\alpha \frac{H\alpha f(a)+f(b)}{\alpha(1+H\alpha)} - (b-a)^{\alpha+1} \frac{Hf'(b)}{(\alpha+1)(1+H\alpha)} \leq \Gamma(\alpha) (I_{a^+}^\alpha f(b)) \quad (1.4)$$

$$\leq (b-a)^\alpha \frac{\frac{1}{h(\frac{1}{2})} f(\frac{a+b}{2}) + \left(\frac{1}{\alpha} + H - \frac{1}{h(\frac{1}{2})}\right) f(b)}{1+H\alpha} + (b-a)^{\alpha+1} \frac{H\alpha f'(b)}{2h(\frac{1}{2})(\alpha+1)(1+H\alpha)},$$

where $H = \int_0^1 h(t) dt$.

The rest of the paper is organized as follows. In Section 2, we present the extension of Theorem 1.6 for (h, m) -convex functions with some consequences for different types of general convexity. After the established results, we give the generalization of Theorem 1.7 via two theorems along with the corresponding consequences. In Section 3, applications are given in order to highlight the main results.

2. Main results

First, we formulate an Hermite–Hadamard type result for the sum of Riemann–Liouville fractional integrals $I_{a^+}^\alpha f(b)$ and $I_{b^-}^\alpha f(a)$ via (h, m) -convexity.

Theorem 2.1. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an (h, m) -convex function with $m \in (0, 1]$. If $f \in L^1[a, b]$, then we have

$$\begin{aligned} 2H_m f\left(\frac{m(a+b)}{1+m}\right) &\leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \\ &\leq \Gamma(\alpha+1) \left[I_{1^-}^\alpha h(0)(f(a) + f(b)) + I_{0^+}^\alpha h(1)m \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) \right], \end{aligned} \quad (2.1)$$

where $H_m = \left[h\left(\frac{m}{1+m}\right) + mh\left(\frac{1}{1+m}\right) \right]^{-1}$.

Proof. From the definition of (h, m) -convexity, we can write

$$\begin{aligned} f\left(\frac{m(x+y)}{1+m}\right) &= f\left(\frac{m}{1+m}x + m\frac{1}{1+m}y\right) \\ &\leq h\left(\frac{m}{1+m}\right)f(x) + mh\left(\frac{1}{1+m}\right)f(y) \end{aligned}$$

for all $x, y \geq 0$. If we choose $x = ta + (1-t)b$ and $y = tb + (1-t)a$, we get

$$\begin{aligned} f\left(\frac{m(a+b)}{1+m}\right) &= f\left(\frac{m(x+y)}{1+m}\right) \\ &\leq h\left(\frac{m}{1+m}\right)f(ta + (1-t)b) + mh\left(\frac{1}{1+m}\right)f(tb + (1-t)a) \end{aligned} \quad (2.2)$$

for all $t \in [0, 1]$. Since

$$\int_0^1 t^{\alpha-1} f(ta + (1-t)b) dt = \frac{1}{(b-a)^\alpha} \int_a^b (b-x)^{\alpha-1} f(x) dx = \frac{\Gamma(\alpha)}{(b-a)^\alpha} I_{a^+}^\alpha f(b)$$

and

$$\int_0^1 t^{\alpha-1} f(tb + (1-t)a) dt = \frac{1}{(b-a)^\alpha} \int_a^b (x-a)^{\alpha-1} f(x) dx = \frac{\Gamma(\alpha)}{(b-a)^\alpha} I_{b^-}^\alpha f(a),$$

hence, if we multiply both sides of inequality (2.2) by $t^{\alpha-1}$ and integrate with respect to t on $[0, 1]$, we obtain

$$\frac{1}{\alpha} f\left(\frac{m(a+b)}{1+m}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[h\left(\frac{m}{1+m}\right) I_{a^+}^\alpha f(b) + mh\left(\frac{1}{1+m}\right) I_{b^-}^\alpha f(a) \right]. \quad (2.3)$$

Analogously, by choosing $x = tb + (1-t)a$ and $y = ta + (1-t)b$, we get

$$\frac{1}{\alpha} f\left(\frac{m(a+b)}{1+m}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[h\left(\frac{m}{1+m}\right) I_{b^-}^\alpha f(a) + mh\left(\frac{1}{1+m}\right) I_{a^+}^\alpha f(b) \right]. \quad (2.4)$$

Putting together (2.3) and (2.4) yields the first inequality in (2.1).

To see the second inequality of (2.1), first note that (1.2) implies

$$f(ta + (1-t)b) \leq h(t)f(a) + mh(1-t)f\left(\frac{b}{m}\right)$$

and

$$f(tb + (1-t)a) \leq h(t)f(b) + mh(1-t)f\left(\frac{a}{m}\right).$$

After multiplying the sides of the previous inequalities by $t^{\alpha-1}$ and integrating with respect to t on $[0, 1]$, we get

$$\int_0^1 t^{\alpha-1} f(ta + (1-t)b) dt \leq f(a) \int_0^1 t^{\alpha-1} h(t) dt + mf\left(\frac{b}{m}\right) \int_0^1 (1-t)^{\alpha-1} h(t) dt$$

and

$$\int_0^1 t^{\alpha-1} f(tb + (1-t)a) dt \leq f(b) \int_0^1 t^{\alpha-1} h(t) dt + mf\left(\frac{a}{m}\right) \int_0^1 (1-t)^{\alpha-1} h(t) dt,$$

which inequalities together give us the second inequality of (2.1). □

If in Theorem 2.1, we suppose m -convexity or h -convexity, we get the following consequence for the sum of Riemann–Liouville fractional integrals.

Corollary 2.2. *Let us consider an m -convex or h -convex function f .*

- *If $f \in L^1[a, b]$ is m -convex with $m \in (0, 1]$, then we have*

$$\begin{aligned} \frac{1+m}{m} f\left(\frac{m(a+b)}{1+m}\right) &\leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \\ &\leq \frac{\alpha}{\alpha+1} [f(a) + f(b)] + \frac{m}{\alpha+1} \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right]. \end{aligned}$$

- *If $f \in L^1[a, b]$ is h -convex, then*

$$\begin{aligned} \frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \\ &\leq \Gamma(\alpha+1) [f(a) + f(b)] [I_{0^+}^\alpha h(1) + I_{1^-}^\alpha h(0)]. \end{aligned}$$

By analyzing the proof of Theorem 2.1, one can easily see that upper bounds for the individual Riemann–Liouville fractional integrals can be given.

Remark 2.3. For (h, m) -convex function $f \in L^1[a, b]$ with $m \in (0, 1]$, we have

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} I_{a^+}^\alpha f(b) \leq \Gamma(\alpha) \left[I_{1^-}^\alpha h(0) f(a) + I_{0^+}^\alpha h(1) mf\left(\frac{b}{m}\right) \right]$$

and

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} I_{b^-}^\alpha f(a) \leq \Gamma(\alpha) \left[I_{1^-}^\alpha h(0) f(b) + I_{0^+}^\alpha h(1) mf\left(\frac{a}{m}\right) \right].$$

If in Theorem 2.1, we suppose $\alpha = 1$ and different types of general convexity, keeping in mind Remark 2.3 or using [13, Theorem 8], we get the following corollary.

Corollary 2.4. *Let us consider an (h, m) -convex, m -convex, h -convex or convex function f .*

- *If $f \in L^1[a, b]$ is (h, m) -convex function with $m \in (0, 1]$. Then we have*

$$\begin{aligned} H_m f\left(\frac{m(a+b)}{1+m}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq H \min \left\{ f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right) \right\}, \end{aligned} \tag{2.5}$$

where $H_m = \left[h \left(\frac{m}{1+m} \right) + mh \left(\frac{1}{1+m} \right) \right]^{-1}$ and $H = \int_0^1 h(t) dt$.

- If $f \in L^1[a, b]$ is m -convex with $m \in (0, 1]$, then we have

$$\begin{aligned} \frac{1+m}{2m} f \left(\frac{m(a+b)}{1+m} \right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \frac{1}{2} \min \left\{ f(a) + mf \left(\frac{b}{m} \right), f(b) + mf \left(\frac{a}{m} \right) \right\}. \end{aligned}$$

- For h -convex functions, Theorem 2.1 becomes [15, Theorem 6].
- For convex functions, Theorem 2.1 implies inequality (1.1).

We present the extension of Theorem 1.7 for Riemann–Liouville fractional integral $I_{b-}^\alpha f(a)$ via (h, m) -convexity.

Theorem 2.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $[a, b]$ such that $f' \in L_{loc}^1([a, b])$ is (h, m) -convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned} (b-a)^\alpha \frac{\left(\frac{1+m}{m}\right)H_m \left[f \left(\frac{m(a+b)}{1+m} \right) - f \left(\frac{2ma}{1+m} \right) \right] + \left(\frac{1}{\alpha} + H \right) f(a)}{1+H\alpha} - (b-a)^{\alpha+1} \frac{mH_m H \alpha f' \left(\frac{2a}{1+m} \right)}{(\alpha+1)(1+H\alpha)} \\ \leq \Gamma(\alpha) (I_{b-}^\alpha f(a)) \leq (b-a)^\alpha \frac{f(a) + H \alpha f(b)}{\alpha(1+H\alpha)} + (b-a)^{\alpha+1} \frac{mH f' \left(\frac{a}{m} \right)}{(\alpha+1)(1+H\alpha)}, \end{aligned} \quad (2.6)$$

where $H_m = \left[h \left(\frac{m}{1+m} \right) + mh \left(\frac{1}{1+m} \right) \right]^{-1}$ and $H = \int_0^1 h(t) dt$.

Proof. Writing the inequality (2.5) for f' yields

$$\begin{aligned} H_m f' \left(\frac{m(a+b)}{1+m} \right) &\leq \frac{1}{b-a} \int_a^b f'(x) dx \\ &\leq H \min \left\{ f'(a) + m f' \left(\frac{b}{m} \right), f'(b) + m f' \left(\frac{a}{m} \right) \right\}. \end{aligned} \quad (2.7)$$

We choose to write

$$\min \left\{ f'(a) + m f' \left(\frac{b}{m} \right), f'(b) + m f' \left(\frac{a}{m} \right) \right\} = f'(b) + m f' \left(\frac{a}{m} \right).$$

Then inequality (2.7) can be written as

$$H_m f' \left(\frac{m(a+b)}{1+m} \right) \leq \frac{1}{b-a} \int_a^b f'(x) dx \leq H \left[f'(b) + m f' \left(\frac{a}{m} \right) \right].$$

By replacing b by x , multiplying all the three terms by $(x-a)^\alpha > 0$ and then integrating with respect to x over $[a, b]$, we have

$$\begin{aligned} S_1 &= H_m \int_a^b (x-a)^\alpha f' \left(\frac{m(a+x)}{1+m} \right) dx \\ &\leq S_2 = \int_a^b (x-a)^{\alpha-1} \left(\int_a^x f'(t) dt \right) dx \\ &\leq S_3 = H \int_a^b (x-a)^\alpha \left[f'(x) + m f' \left(\frac{a}{m} \right) \right] dx. \end{aligned}$$

After some simplification, we get

$$\begin{aligned} S_2 &= \int_a^b (x-a)^{\alpha-1} (f(x) - f(a)) dx \\ &= \int_a^b (x-a)^{\alpha-1} f(x) dx - \frac{1}{\alpha} (b-a)^\alpha f(a), \end{aligned} \tag{2.8}$$

and integration by parts yields

$$\begin{aligned} S_3 &= H \left[\int_a^b (x-a)^\alpha f'(x) dx + m f' \left(\frac{a}{m} \right) \int_a^b (x-a)^\alpha dx \right] \\ &= H \left[(b-a)^\alpha f(b) - \alpha \int_a^b (x-a)^{\alpha-1} f(x) dx + m \frac{(b-a)^{\alpha+1}}{\alpha+1} f' \left(\frac{a}{m} \right) \right]. \end{aligned} \tag{2.9}$$

By combining (2.8) and (2.9), we get

$$\begin{aligned} \int_a^b (x-a)^{\alpha-1} f(x) dx &\leq \frac{1}{\alpha} (b-a)^\alpha f(a) + H [(b-a)^\alpha f(b) \\ &\quad - \alpha \int_a^b (x-a)^{\alpha-1} f(x) dx + m \frac{(b-a)^{\alpha+1}}{\alpha+1} f' \left(\frac{a}{m} \right)], \end{aligned}$$

from where we obtain

$$\begin{aligned} (1 + H\alpha) \int_a^b (x-a)^{\alpha-1} f(x) dx \\ \leq (b-a)^\alpha \left(\frac{1}{\alpha} f(a) + H f(b) \right) + (b-a)^{\alpha+1} \frac{m H f' \left(\frac{a}{m} \right)}{\alpha+1}, \end{aligned} \tag{2.10}$$

hence we have the right inequality in (2.6).

Now, integrating S_1 by parts and substituting $t = \frac{m(a+x)}{1+m}$ yield

$$\begin{aligned} S_1 &= H_m \left[\frac{1+m}{m} (b-a)^\alpha f \left(\frac{m(a+b)}{1+m} \right) - \frac{1+m}{m} \alpha \int_a^b (x-a)^{\alpha-1} f \left(\frac{m(a+x)}{1+m} \right) dx \right] \\ &= (b-a)^\alpha \left(\frac{1+m}{m} \right) H_m f \left(\frac{m(a+b)}{1+m} \right) - \alpha \left(\frac{1+m}{m} \right)^{\alpha+1} H_m \int_{\frac{2ma}{1+m}}^{\frac{m(a+b)}{1+m}} \left(t - \frac{2ma}{1+m} \right)^{\alpha-1} f(t) dt. \end{aligned}$$

Using inequality (2.10) for $\frac{2ma}{1+m}$ and $\frac{m(a+b)}{1+m}$ instead of a and b , we have

$$\begin{aligned} S_1 &\geq (b-a)^\alpha \left(\frac{1+m}{m}\right) H_m f\left(\frac{m(a+b)}{1+m}\right) - \alpha \left(\frac{1+m}{m}\right)^{\alpha+1} H_m \\ &\quad \times \left[(b-a)^\alpha \left(\frac{m}{1+m}\right)^\alpha \frac{Hf\left(\frac{m(a+b)}{1+m}\right) + \frac{1}{\alpha} f\left(\frac{2ma}{1+m}\right)}{1+H\alpha} + (b-a)^{\alpha+1} \left(\frac{m}{1+m}\right)^{\alpha+1} \frac{mHf'\left(\frac{2a}{1+m}\right)}{(\alpha+1)(1+H\alpha)} \right] \\ &= (b-a)^\alpha \left(\frac{1+m}{m}\right) H_m f\left(\frac{m(a+b)}{1+m}\right) - (b-a)^\alpha \left(\frac{1+m}{m}\right) H_m \frac{H\alpha f\left(\frac{m(a+b)}{1+m}\right) + f\left(\frac{2ma}{1+m}\right)}{1+H\alpha} \\ &\quad - (b-a)^{\alpha+1} \frac{mH_m H\alpha f'\left(\frac{2a}{1+m}\right)}{(\alpha+1)(1+H\alpha)}. \end{aligned}$$

Considering $S_1 \leq S_2$ along with the above inequality implies the left inequality in (2.6). \square

If in Theorem 2.5, we suppose m -convexity or h -convexity, we get the following consequence for Riemann–Liouville fractional integral $I_{b^-}^\alpha f(a)$.

Corollary 2.6. *Let us consider a function f which has an m -convex, h -convex or convex derivative.*

- If $f' \in L_{loc}^1([a, b])$ is m -convex with $m \in (0, 1]$, then we have

$$\begin{aligned} &(b-a)^\alpha \frac{\alpha \left(\frac{1+m}{m}\right)^2 \left[f\left(\frac{m(a+b)}{1+m}\right) - f\left(\frac{2ma}{1+m}\right) \right] + (\alpha+2)f(a)}{\alpha(\alpha+2)} - (b-a)^{\alpha+1} \frac{\alpha(1+m)f'\left(\frac{2a}{1+m}\right)}{2(\alpha+1)(\alpha+2)} \\ &\leq \Gamma(\alpha) (I_{b^-}^\alpha f(a)) \leq (b-a)^\alpha \frac{2f(a) + \alpha f(b)}{\alpha(\alpha+2)} + (b-a)^{\alpha+1} \frac{mf'\left(\frac{a}{m}\right)}{(\alpha+1)(\alpha+2)}. \end{aligned}$$

- If $f' \in L_{loc}^1([a, b])$ is h -convex, then inequality (2.6) coincides with (1.3).
- If $f' \in L_{loc}^1([a, b])$ is convex, then inequality (2.6) coincides with [1, Corollary 2.1].

Moreover, if in Theorem 2.5, we suppose $\alpha = 1$ and different types of general convexity, we get the following corollary.

Corollary 2.7. *Let us consider a function f which has an (h, m) -convex, m -convex, h -convex or convex derivative.*

- If $f' \in L_{loc}^1([a, b])$ is (h, m) -convex function with $m \in (0, 1]$, then we have

$$\begin{aligned} &\frac{\left(\frac{1+m}{m}\right) H_m \left[f\left(\frac{m(a+b)}{1+m}\right) - f\left(\frac{2ma}{1+m}\right) \right] + (1+H)f(a)}{1+H} - \frac{mH_m H(b-a)f'\left(\frac{2a}{1+m}\right)}{2(1+H)} \leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \frac{f(a) + Hf(b)}{1+H} + \frac{mH(b-a)f'\left(\frac{a}{m}\right)}{2(1+H)}, \end{aligned}$$

where $H_m = \left[h\left(\frac{m}{1+m}\right) + mh\left(\frac{1}{1+m}\right) \right]^{-1}$ and $H = \int_0^1 h(t) dt$.

- If $f' \in L^1_{loc}([a, b])$ is m -convex with $m \in (0, 1]$, then we have

$$\frac{\left(\frac{1+m}{m}\right)^2 \left[f\left(\frac{m(a+b)}{1+m}\right) - f\left(\frac{2ma}{1+m}\right) \right] + 3f(a)}{3} - \frac{(1+m)(b-a)f'\left(\frac{2a}{1+m}\right)}{12} \leq \frac{1}{b-a} \int_a^b f(x) dx$$

$$\leq \frac{2f(a)+f(b)}{3} + \frac{m(b-a)f'\left(\frac{a}{m}\right)}{6}.$$

- If $f' \in L^1_{loc}([a, b])$ is h -convex, then

$$\frac{2f\left(\frac{a+b}{2}\right) + (-1+H)f(a)}{1+H} - \frac{H(b-a)f'(a)}{2(1+H)} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+Hf(b)}{1+H} + \frac{H(b-a)f'(a)}{2(1+H)}.$$

- If $f' \in L^1_{loc}([a, b])$ is convex, then inequality (2.6) coincides with [8, Corollary 3].

Theorem 2.8. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $[a, b]$ such that $f' \in L^1_{loc}([a, b])$ is (h, m) -convex on $[a, b]$, then the following inequality holds:

$$(b-a)^\alpha \frac{H\alpha f(a) + f(b)}{\alpha(1+H\alpha)} - (b-a)^{\alpha+1} \frac{mHf'\left(\frac{b}{m}\right)}{(\alpha+1)(1+H\alpha)} \leq \Gamma(\alpha)(I_{a^+}^\alpha f(b))$$

$$\leq (b-a)^\alpha \frac{\left(\frac{1+m}{m}\right)H_m \left[f\left(\frac{m(a+b)}{1+m}\right) - f\left(\frac{2mb}{1+m}\right) \right] + \left(\frac{1}{\alpha}+H\right)f(b)}{1+H\alpha} + (b-a)^{\alpha+1} \frac{mH_m H\alpha f'\left(\frac{2b}{1+m}\right)}{(\alpha+1)(1+H\alpha)}, \tag{2.11}$$

where $H_m = \left[h\left(\frac{m}{1+m}\right) + mh\left(\frac{1}{1+m}\right) \right]^{-1}$ and $H = \int_0^1 h(t) dt$.

Proof. If in (2.7), we choose to write

$$\min \left\{ f'(a) + mf'\left(\frac{b}{m}\right), f'(b) + mf'\left(\frac{a}{m}\right) \right\} = f'(a) + mf'\left(\frac{b}{m}\right),$$

then inequality (2.7) can be written as

$$H_m f'\left(\frac{m(a+b)}{1+m}\right) \leq \frac{1}{b-a} \int_a^b f'(x) dx \leq H \left[f'(a) + mf'\left(\frac{b}{m}\right) \right].$$

By replacing a by x , multiplying all the three terms by $(b-x)^\alpha > 0$ and then integrating with respect to x over $[a, b]$, we have

$$T_1 = H_m \int_a^b (b-x)^\alpha f'\left(\frac{m(x+b)}{1+m}\right) dx$$

$$\leq T_2 = \int_a^b (b-x)^{\alpha-1} \left(\int_x^b f'(t) dt \right) dx$$

$$\leq T_3 = H \int_a^b (b-x)^\alpha \left[f'(x) + mf'\left(\frac{b}{m}\right) \right] dx.$$

Some simplification yields

$$T_2 = \int_a^b (b-x)^{\alpha-1} (f(b) - f(x)) dx$$

$$= \frac{1}{\alpha} (b-a)^\alpha f(b) - \int_a^b (b-x)^{\alpha-1} f(x) dx, \tag{2.12}$$

while, by integration by parts, we have

$$\begin{aligned} T_3 &= H \left[\int_a^b (b-x)^\alpha f'(x) dx + m f' \left(\frac{b}{m} \right) \int_a^b (b-x)^\alpha dx \right] \\ &= H \left[-(b-a)^\alpha f(a) + \alpha \int_a^b (b-x)^{\alpha-1} f(x) dx + m f' \left(\frac{b}{m} \right) \frac{(b-a)^{\alpha+1}}{\alpha+1} \right]. \end{aligned} \quad (2.13)$$

Combining (2.12) and (2.13) yields

$$\begin{aligned} &\frac{1}{\alpha} (b-a)^\alpha f(b) - H \left[-(b-a)^\alpha f(a) + \alpha \int_a^b (b-x)^{\alpha-1} f(x) dx + m f' \left(\frac{b}{m} \right) \frac{(b-a)^{\alpha+1}}{\alpha+1} \right] \\ &\leq \int_a^b (b-x)^{\alpha-1} f(x) dx, \end{aligned}$$

from where we obtain

$$(b-a)^\alpha \frac{H\alpha f(a) + f(b)}{\alpha} - (b-a)^{\alpha+1} \frac{mHf' \left(\frac{b}{m} \right)}{(\alpha+1)} \leq (1+H\alpha) \int_a^b (b-x)^{\alpha-1} f(x) dx, \quad (2.14)$$

hence we got the left inequality in (2.11).

After integration by parts and substituting $t = \frac{m(x+b)}{1+m}$ yield, we calculate T_1 as

$$\begin{aligned} T_1 &= H_m \left[\left(\frac{1+m}{m} \right) (b-x)^\alpha f \left(\frac{m(x+b)}{1+m} \right) \Big|_a^b + \alpha \left(\frac{1+m}{m} \right) \int_a^b (b-x)^{\alpha-1} f \left(\frac{m(x+b)}{1+m} \right) dx \right] \\ &= -(b-a)^\alpha \left(\frac{1+m}{m} \right) H_m f \left(\frac{m(a+b)}{1+m} \right) + \alpha \left(\frac{1+m}{m} \right)^{\alpha+1} H_m \int_{\frac{m(a+b)}{1+m}}^{\frac{2mb}{1+m}} \left(\frac{2mb}{1+m} - t \right)^{\alpha-1} f(t) dt. \end{aligned}$$

Using inequality (2.14) for $\frac{m(a+b)}{1+m}$ and $\frac{2mb}{1+m}$ instead of a and b , we have

$$\begin{aligned} T_1 &\geq -(b-a)^\alpha \left(\frac{1+m}{m} \right) H_m f \left(\frac{m(a+b)}{1+m} \right) + \alpha \left(\frac{1+m}{m} \right)^{\alpha+1} H_m \\ &\quad \times \left[(b-a)^\alpha \left(\frac{m}{1+m} \right)^\alpha \frac{H\alpha f \left(\frac{m(a+b)}{1+m} \right) + f \left(\frac{2mb}{1+m} \right)}{\alpha(1+H\alpha)} \right. \\ &\quad \left. - (b-a)^{\alpha+1} \left(\frac{m}{1+m} \right)^{\alpha+1} \frac{mHf' \left(\frac{2b}{1+m} \right)}{(\alpha+1)(1+H\alpha)} \right] \\ &= -(b-a)^\alpha \left(\frac{1+m}{m} \right) H_m f \left(\frac{m(a+b)}{1+m} \right) \\ &\quad + (b-a)^\alpha \left(\frac{1+m}{m} \right) H_m \frac{H\alpha f \left(\frac{m(a+b)}{1+m} \right) + f \left(\frac{2mb}{1+m} \right)}{1+H\alpha} \end{aligned}$$

$$- (b - a)^{\alpha+1} \alpha H_m \frac{m H f' \left(\frac{2b}{1+m} \right)}{(\alpha + 1)(1 + H\alpha)}.$$

Considering $T_1 \leq T_2$ along with the above inequality implies the right inequality in (2.11). \square

If in Theorem 2.8, we suppose m -convexity or h -convexity, we get the following consequence for Riemann–Liouville fractional integral $I_{a^+}^\alpha f(b)$.

Corollary 2.9. *Let us consider a function f which has an m -convex, h -convex or convex derivative.*

- If $f' \in L_{loc}^1([a, b])$ is m -convex with $m \in (0, 1]$, then we have

$$\begin{aligned} & (b - a)^\alpha \frac{\alpha f(a) + 2f(b)}{\alpha(\alpha+2)} - (b - a)^{\alpha+1} \frac{m f' \left(\frac{b}{m} \right)}{(\alpha+1)(\alpha+2)} \leq \Gamma(\alpha) (I_{a^+}^\alpha f(b)) \\ \leq & (b - a)^\alpha \frac{\alpha \left(\frac{1+m}{m} \right)^2 \left[f \left(\frac{m(a+b)}{1+m} \right) - f \left(\frac{2mb}{1+m} \right) \right] + (\alpha+2)f(b)}{\alpha(\alpha+2)} + (b - a)^{\alpha+1} \frac{\alpha(1+m)f' \left(\frac{2b}{1+m} \right)}{2(\alpha+1)(\alpha+2)}. \end{aligned}$$

- If $f' \in L_{loc}^1([a, b])$ is h -convex, then inequality (2.11) coincides with (1.4).
- If $f' \in L_{loc}^1([a, b])$ is convex, then inequality (2.11) coincides with [1, Corollary 2.3].

Moreover, if in Theorem 2.8, we suppose $\alpha = 1$ and different types of general convexity, we get the following corollary.

Corollary 2.10. *Let us consider a function f which has an (h, m) -convex, m -convex, h -convex or convex derivative.*

- If $f' \in L_{loc}^1([a, b])$ is (h, m) -convex function with $m \in (0, 1]$, then we have

$$\begin{aligned} & \frac{Hf(a)+f(b)}{1+H} + \frac{mH(b-a)f' \left(\frac{b}{m} \right)}{2(1+H)} \leq \frac{1}{b-a} \int_a^b f(x) dx \\ \leq & \frac{\left(\frac{1+m}{m} \right) H_m \left[f \left(\frac{m(a+b)}{1+m} \right) - f \left(\frac{2mb}{1+m} \right) \right] + (1+H)f(b)}{1+H} + \frac{mH_m H(b-a)f' \left(\frac{2b}{1+m} \right)}{2(1+H)}, \end{aligned}$$

where $H_m = \left[h \left(\frac{m}{1+m} \right) + mh \left(\frac{1}{1+m} \right) \right]^{-1}$ and $H = \int_0^1 h(t) dt$.

- If $f' \in L_{loc}^1([a, b])$ is m -convex, then we have

$$\begin{aligned} & \frac{f(a)+2f(b)}{3} - \frac{m(b-a)f' \left(\frac{b}{m} \right)}{6} \leq \frac{1}{b-a} \int_a^b f(x) dx \\ \leq & \frac{\left(\frac{1+m}{m} \right)^2 \left[f \left(\frac{m(a+b)}{1+m} \right) - f \left(\frac{2mb}{1+m} \right) \right] + 3f(b)}{3} + \frac{(1+m)(b-a)f' \left(\frac{2b}{1+m} \right)}{12}. \end{aligned}$$

- If $f' \in L_{loc}^1([a, b])$ is h -convex, then

$$\frac{Hf(a)+f(b)}{1+H} - \frac{H(b-a)f'(b)}{2(1+H)} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{2f \left(\frac{a+b}{2} \right) + (-1+H)f(b)}{1+H} + \frac{H(b-a)f'(b)}{2(1+H)}.$$

- If $f' \in L^1_{loc}([a, b])$ is convex, then inequality (2.11) coincides with [1, Corollary 2.4].

3. Conclusions

We present two examples which demonstrate the usability of our main result. The first example includes an m -convex function for which we calculate the bounds for the sum of the Riemann–Liouville fractional integrals. The second one includes a function with m -convex derivative.

Example 3.1. Let us consider the function $f : [0, \infty) \rightarrow \mathbb{R}$ given by

$$f(x) = \frac{1}{12}(x^4 - 5x^3 + 9x^2 - 5x),$$

which is known to be $\frac{16}{17}$ -convex (see [20]), but not convex or concave.

Applying Corollary 2.2 on the interval $[0, 3]$ yields

$$\begin{aligned} & \frac{1+m}{m} f\left(\frac{m(a+b)}{1+m}\right) = \frac{33}{16} f\left(\frac{48}{33}\right) = 0.1474 \\ & \leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] = \frac{\alpha}{3^\alpha} \left[\int_0^3 (3-t)^{\alpha-1} f(t) dt + \int_0^3 t^{\alpha-1} f(t) dt \right] \\ & \leq \frac{\alpha}{\alpha+1} [f(a) + f(b)] + \frac{m}{\alpha+1} \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right] = \frac{\alpha + 1.318}{\alpha + 1} \end{aligned}$$

and by applying Corollary 2.4 on $[0, 3]$, we have

$$\begin{aligned} & \frac{1+m}{2m} f\left(\frac{m(a+b)}{1+m}\right) = 0.0737 \\ & \leq \frac{1}{b-a} \int_a^b f(x) dx = 0.1625 \\ & \leq \frac{1}{2} \min \left\{ f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right) \right\} = \frac{1}{2} \min \{1.3181, 1\} = 0.5. \end{aligned}$$

Example 3.2. Let us consider the function $g : [0, \infty) \rightarrow \mathbb{R}$ given by

$$g(x) = \frac{1}{60}x^5 - \frac{5}{48}x^4 + \frac{1}{4}x^3 - \frac{5}{24}x^2,$$

which has a $\frac{16}{17}$ -convex derivative $g'(x) = f(x)$, where f is from Example 3.1.

Applying Corollary 2.6 for g on $[0, 3]$ yields

$$\begin{aligned} & (b-a)^\alpha \frac{\alpha \left(\frac{1+m}{m}\right)^2 \left[g\left(\frac{m(a+b)}{1+m}\right) - g\left(\frac{2ma}{1+m}\right) \right] + (\alpha+2)g(a)}{\alpha(\alpha+2)} \\ & - (b-a)^{\alpha+1} \frac{\alpha(1+m)g'\left(\frac{2a}{1+m}\right)}{2(\alpha+1)(\alpha+2)} = \frac{-0.1241}{\alpha+2} 3^\alpha \leq \Gamma(\alpha)(I_{b^-}^\alpha g(a)) = \int_0^3 t^{\alpha-1} g(t) dt \\ & \leq (b-a)^\alpha \frac{2g(a) + \alpha g(b)}{\alpha(\alpha+2)} + (b-a)^{\alpha+1} \frac{mg'\left(\frac{a}{m}\right)}{(\alpha+1)(\alpha+2)} = \frac{0.4875}{\alpha+2} 3^\alpha, \end{aligned}$$

and by applying Corollary 2.7 for g on $[0, 3]$, we have

$$\begin{aligned} & \frac{\left(\frac{1+m}{m}\right)^2 \left[g\left(\frac{m(a+b)}{1+m}\right) - g\left(\frac{2ma}{1+m}\right) \right] + 3g(a)}{3} - \frac{(1+m)(b-a)g'\left(\frac{2a}{1+m}\right)}{12} = -0.0414 \\ & \leq \frac{1}{b-a} \int_a^b g(x) dx = 0.05 \leq \frac{2g(a) + g(b)}{3} + \frac{m(b-a)g'\left(\frac{a}{m}\right)}{6} = 0.1625. \end{aligned}$$

Similarly, by applying Corollary 2.9 for g on $[0, 3]$, we obtain

$$\begin{aligned} & (b-a)^\alpha \frac{\alpha g(a) + 2g(b)}{\alpha(\alpha+2)} - (b-a)^{\alpha+1} \frac{mg'\left(\frac{b}{m}\right)}{(\alpha+1)(\alpha+2)} = \frac{0.975}{\alpha(\alpha+2)} 3^\alpha - \frac{1.3181}{(\alpha+1)(\alpha+2)} 3^{\alpha+1} \\ & \leq \Gamma(\alpha)(I_{a^+}^\alpha g(b)) = \int_0^3 (3-t)^{\alpha-1} g(t) dt \\ & \leq (b-a)^\alpha \frac{\alpha \left(\frac{1+m}{m}\right)^2 \left[g\left(\frac{m(a+b)}{1+m}\right) - g\left(\frac{2mb}{1+m}\right) \right] + (\alpha+2)g(b)}{\alpha(\alpha+2)} + (b-a)^{\alpha+1} \frac{\alpha(1+m)g'\left(\frac{2b}{1+m}\right)}{2(\alpha+1)(\alpha+2)} \\ & = \frac{-1.3544\alpha + 0.975}{\alpha(\alpha+2)} 3^\alpha + \frac{1.1448\alpha}{(\alpha+1)(\alpha+2)} 3^{\alpha+1}, \end{aligned}$$

and Corollary 2.10 yields

$$\begin{aligned} & \frac{g(a) + 2g(b)}{3} - \frac{m(b-a)g'\left(\frac{b}{m}\right)}{6} = -0.334 \\ & \leq \frac{1}{b-a} \int_a^b g(x) dx = 0.05 \\ & \leq \frac{\left(\frac{1+m}{m}\right)^2 \left[g\left(\frac{m(a+b)}{1+m}\right) - g\left(\frac{2mb}{1+m}\right) \right] + 3g(b)}{3} + \frac{(1+m)(b-a)g'\left(\frac{2b}{1+m}\right)}{12} = 0.446. \end{aligned}$$

We note that the extension of our main results in case of other fractional integrals, such as Katugampola or Raina remains open.

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Analysis of the HIV/AIDS transmission dynamics model using Caputo fractional-order derivative

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Abstract. Although national and international institutions, such as the World Health Organization (WHO) and UNAIDS, are making significant efforts to eradicate HIV by 2030, it remains a major threat to global public health. Despite its low prevalence, HIV continues to claim lives and remains a major public health issue, especially in developing countries. Thanks to the accessibility of antiretroviral drugs, the prevalence of this scourge has been gradually declining worldwide in recent years. Thus, this article investigates the effectiveness of antiretroviral therapy in controlling viral transmission through a fractional-order extension of a deterministic model. We study the boundedness of the model's solution by applying the Laplace transform to solve the fractional Gronwall inequality. To ensure the existence and uniqueness of the model's solution, we rely on the Picard-Lindelöf theorem. We also study the stability of the disease-free equilibrium point to qualitatively analyze the behavior of the model. Next, we perform a sensitivity analysis of the basic reproduction number \mathcal{R}_0 to evaluate its robustness concerning the model parameters. Finally, we simulate the approximate solutions of the fractional-order model in MATLAB for different values of the fractional order and present the results of the sensitivity analysis and numerical simulation. Our results demonstrate that the fractional model provides real added value in modeling, thanks to its ability to incorporate memory effects and finely tune transmission dynamics according to the fractional order, thereby allowing for a more realistic representation of epidemiological processes.

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1. Introduction

In the 1980s, acquired immunodeficiency syndrome (AIDS) first emerged in the United States [41]. It was in 1983 that the human immunodeficiency virus (HIV), the retrovirus responsible for this infection and the cause of AIDS, was identified at the Institut Pasteur in Paris [41]. This retrovirus weakens the immune system by affecting most cells, with a particularly destructive impact on $CD4^+$ T cells [8]. The human body becomes vulnerable to any infection when the number of $CD4^+$ T cells falls below a certain threshold, causing the cell-mediated immune system to collapse and the overall immune system to weaken [43]. AIDS is now considered a pandemic that has claimed approximately 42.3 million lives (35.7–51.1 million) between 1981 and 2023 [40]. To date, there is no cure or vaccine for AIDS, but antiretroviral therapy (ART) improves health, prolongs life, and significantly reduces the risk of HIV transmission [46]. Advances in treatment and prevention have led some to question whether the "end of AIDS" is now within reach [23]. Studies have shown, on the one hand, that the risk of HIV transmission among serodifferent heterosexual couples is reduced by 96% when the HIV-positive partner is undergoing treatment [17], and on the other hand, that individuals living with HIV who are on antiretroviral therapy have a very low probability (estimated at less than 0.04) of transmitting the virus [18]. For more information on the biology and epidemiology of HIV/AIDS, readers are directed to references [2, 7, 41], and the associated literature cited within.

Several mathematical models have proven effective in describing and analyzing the dynamics of HIV infection; see, for example, [3, 8, 9, 10, 11, 13, 25, 26, 33, 38, 45, 46, 47, 54] and related studies.

Fractional-order integral-differential operators generalize integration and differentiation to non-integer orders [21]. In recent decades, fractional-order models have consistently demonstrated superior accuracy compared to classical integer-order models in describing certain aspects of natural phenomena [21]. Regarding the transmission dynamics of infectious diseases using fractional-order mathematical modeling, several valuable studies have been conducted. Among these, we can mention [6, 8, 19, 22, 29, 30, 36, 42, 52], and the references cited therein. The authors of [8] extend the HIV-1 infection model of $CD4^+$ T cells to the concept of the Caputo-Fabrizio fractional derivative and present numerical results for different values of the fractional order to verify the efficiency and accuracy of the new fractional model. In [6], the authors construct their model by integrating the population of mildly infected individuals into the compartmental SIR model, transforming it into an SMIR model in the form of a system of fractional-order differential equations in the sense of Caputo. They performed numerical simulations and found that population dynamics varied with different values of the fractional order. They also observed that an increase in the infection rate of mild cases leads to a corresponding increase in the total population of infected individuals. Fractional-order derivative and integral operators depend not only on the current state of the system but also on all its past states due to the non-local nature of these operators. This property makes them particularly useful and powerful for evaluating the future evolution of the system. As a result, they prove to be more effective than classical deterministic operators, as they incorporate

memory effects, providing more accurate and realistic modeling of complex dynamic phenomena [30].

We here consider the deterministic SIAHR model describing the transmission dynamics of HIV/AIDS, including a compartment for individuals in remission, as studied in [12], to which we will generalize to a fractional-order model. Our objective is to extend the deterministic model by incorporating a fractional-order time derivative in the sense of Caputo. This allows us to demonstrate the power and effectiveness of fractional-order differentiation and integration operators in evaluating the future state of the model, compared to other classical deterministic operators.

The rest of the work is organized as follows: The deterministic model is provided in Section 2. In Section 3, we formulate and prove the existence and uniqueness of the solutions to the fractional-order model in the sense of Caputo relative to the deterministic model. In Section 4, we analyze the fractional-order model in the sense of Caputo. In Section 5, we address the sensitivity analysis of the basic reproduction number \mathcal{R}_0 . Section 6 is dedicated to the study of numerical results and discussions of the fractional-order model. Finally, the conclusions are presented in Section 7.

2. The deterministic model

We consider the SIAHR model describing the transmission dynamics of HIV/AIDS, including a compartment for individuals in remission, as studied in [12]. The model segments the human population into five distinct and mutually exclusive compartments: the population of susceptible individuals S , the population of infected individuals showing no clinical symptoms of the disease I , the group of individuals with the AIDS disease A , the population of individuals undergoing antiretroviral treatment H , and the group of individuals in remission R .

The distribution of individuals across the different compartments occurs as follows: in S , a susceptible person contracts the infection after coming into contact with an infected individual. An individual in compartment I can start antiretroviral treatment and join class H , or they may ignore the treatment and naturally progress to compartment A . In class A , an individual undergoing treatment can also move to class H . An individual in class H can move to compartment R if they adhere to the treatment properly, or transition to compartment A if they fail to adhere to it.

As in [12], susceptible individuals acquire infection through contact with infected individuals at a rate K defined as

$$K = \frac{\alpha(I + \eta A)}{N}, \quad (2.1)$$

where α is the effective contact rate for HIV transmission, η is a modification factor ($\eta \geq 1$) accounting for the difference in infectiousness between individuals with AIDS symptoms and asymptomatic HIV-infected individuals. This model incorporates the fact that individuals with AIDS symptoms are more infectious than asymptomatic carriers due to their higher viral load. There are other studies in the scientific literature [46, 51] that have indeed demonstrated a positive correlation between viral load and HIV transmission potential.

The model studied in [12] is presented by the following system of ordinary differential equations:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \Lambda p I(t) - (K(t) + \mu_0)S(t) \\ \frac{dI(t)}{dt} = K(t)S(t) - k_1 I(t) \\ \frac{dA(t)}{dt} = \epsilon \beta_1 I(t) - k_2 A(t) + \rho \lambda H(t) \\ \frac{dH(t)}{dt} = (1 - \epsilon) \beta_1 I(t) + \beta_2 A(t) - k_3 H(t) \\ \frac{dR(t)}{dt} = (1 - \rho) \lambda H(t) - \mu_0 R(t), \end{cases} \quad (2.2)$$

where $k_1 = -\Lambda p + \beta_1 + \mu_0$, $k_2 = \beta_2 + \mu_0 + \mu_1$ and $k_3 = \lambda + \mu_0$, with the following initial conditions:

$$S(0) \geq 0, \quad I(0) \geq 0, \quad A(0) \geq 0, \quad H(0) \geq 0 \quad \text{and} \quad R(0) \geq 0.$$

In [12], it was also shown that the model (2.2) is mathematically and epidemiologically well-posed, meaning that the proofs of existence and uniqueness of the solutions to the model (2.2) are studied there.

3. Formulation of the fractional-order model

Drawing inspiration from the results of the deterministic HIV/AIDS transmission model presented in reference [12], and motivated by the importance of the new concepts of fractional-order derivatives, we extend the model (2.2) to a fractional-order version. Applied problems require a definition of fractional-order derivatives that allows for the use of physically interpretable initial conditions, and it is the definition of the fractional-order derivative in the sense of Caputo that satisfies this requirement [48]. Thus, we will generalize or extend the model (2.2) to a fractional-order model in the sense of Caputo, accounting for memory effects and capturing long-term dynamics such as relapses in infections or delayed responses to treatment in epidemiology. In particular, interruptions or irregularities in the administration of antiretrovirals have a cumulative impact on viral load and immune responses.

3.1. Essential preliminaries

This section reviews the principal tools of generalized fractional operators, building on earlier advances. These fundamental mathematical notions underpin the analysis developed in this study [16, 20, 22, 24, 31, 32, 34, 42, 48].

Definition 3.1. Let $x \in \mathbb{R}_*^+$, the Gamma function defined by:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt. \quad (3.1)$$

(This integral is convergent for all $x > 0$).

Proposition 3.2. For all $x > 0$ and for all $n \in \mathbb{N}$, we have:

$$\Gamma(x+1) = x\Gamma(x); \quad \Gamma(n) = (n-1)!, \quad \Gamma(n+1) = n!$$

$$\text{In particular } \Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1; \quad \Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt = \sqrt{\pi}.$$

Definition 3.3. Let $\sigma \in \mathbb{R}_*^+$, $a \in \mathbb{R}$, $n \in \mathbb{N}$ such that $n - 1 < \sigma < n$ and let f be a locally integrable function defined on $[a; +\infty[$. The Caputo fractional derivative of order σ of $f(x)$ with lower limit a is defined by:

$${}^c D_t^\sigma f(x) = \frac{1}{\Gamma(n - \sigma)} \int_a^x (x - t)^{n - \sigma - 1} f^{(n)}(t) dt, \tag{3.2}$$

where $f^{(n)}$ denotes the $n - th$ order derivative of f .

Definition 3.4. The fractional integral operator of order $\sigma > 0$ of the function $f(t)$, $t > 0$ is defined by:

$${}_0 I_t^\sigma f(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t - s)^{\sigma - 1} f(s) ds, \quad t \geq 0 \tag{3.3}$$

Definition 3.5. The function $F(s)$ of the complex variable s is defined by:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

called the Laplace transform of the function f .

Lemma 3.6. Note that the Laplace transform of a derivative operator of order n is obtained as follows:

$$\mathcal{L}\{f^{(n)}(s)\} = S^n \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} S^{n-k-1} f^{(k)}(t_0). \tag{3.4}$$

Similarly, for $\sigma \in [n - 1; n]$, we obtain the Laplace transform of the Caputo fractional operator as:

$$\mathcal{L}\{{}^c D_t^\sigma f(t)\} = S^\sigma \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} S^{\sigma-k-1} f^{(k)}(t_0). \tag{3.5}$$

Definition 3.7. The exponential function, e^z , plays a compelling role in the theory of integer-order differential equations. The generalization of the exponential function to a single parameter was introduced by G.M. Mittag-Leffler and is referred to as the function

$$E_\sigma(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\sigma k + 1)}. \tag{3.6}$$

The Mittag-Leffler function with two parameters also plays a crucial role in the theory of fractional calculus. This two-parameter Mittag-Leffler function was introduced by Agarwal and is defined by a series expansion as follows:

$$E_{\sigma,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\sigma k + \beta)}, \quad \sigma, \beta > 0. \tag{3.7}$$

Remark 3.8. For $\beta = 1$, we have $E_{\sigma,1}(z) = E_\sigma(z)$ and for $\sigma = \beta = 1$, we have

$$E_{1,1}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k + 1)} = \sum_{k=0}^{+\infty} \frac{z^k}{k!} = e^z. \tag{3.8}$$

Theorem 3.9. (Banach Contraction Theorem) Let X be a Banach space and $F : X \rightarrow X$, a contraction, that is, there exists $0 < k < 1$ such that for all $x, y \in X$, we have

$$\|F(x) - F(y)\|_X \leq k\|x - y\|_X.$$

Then F has a unique fixed point $x^* \in X$, i.e., $F(x^*) = x^*$.

Definition 3.10. (Banach's Fixed Point Theorem) Let S be a closed subset of a Banach space X and let T be a mapping that assigns S to S . Suppose that:

$$\|T(x) - T(y)\|_S \leq p\|x - y\|_S, \text{ for all } x, y \in S, 0 \leq p < 1,$$

then:

- a) There exists a unique vector $x^* \in S$ satisfying $x^* = T(x^*)$;
- b) x^* can be obtained by the method of successive approximations, starting from any arbitrary initial vector in S .

Theorem 3.11. [6] The equilibrium solutions x^* of the system

$$\begin{cases} {}^c D_t^\sigma x(t) = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{3.9}$$

are locally and asymptotically stable if all the eigenvalues λ_i of the Jacobian $\frac{\partial f}{\partial x_i}$ evaluated at the equilibrium point satisfy

$$|\arg(\lambda_i)| > \frac{\sigma\pi}{2},$$

for all $\sigma \in (0, 1)$.

3.2. Formulation of the Caputo fractional-order model

A fractional-order mathematical model of HIV/AIDS transmission is developed in this section. We thus modify the system (2.2) to obtain a fractional-order system in the sense of Caputo. By generalizing the deterministic model to the fractional-order one, we use the fractional-order time derivative and integral operators in the sense of Caputo, as defined in (3.2). The fractional-order mathematical model in the sense of Caputo, obtained from the deterministic model given in (2.2), is described as follows:

$$\begin{cases} {}^c D_t^\sigma S(t) = \Lambda - \Lambda p I(t) - (K + \mu_0)S(t) \\ {}^c D_t^\sigma I(t) = K S(t) - k_1 I(t) \\ {}^c D_t^\sigma A(t) = \epsilon \beta_1 I(t) - k_2 A(t) + \rho \lambda H(t) \\ {}^c D_t^\sigma H(t) = (1 - \epsilon) \beta_1 I(t) + \beta_2 A(t) - k_3 H(t) \\ {}^c D_t^\sigma R(t) = (1 - \rho) \lambda H(t) - \mu_0 R(t), \end{cases} \tag{3.10}$$

with the following initial conditions:

$$\begin{aligned} S(0) = S_0 \geq 0; I(0) = I_0 \geq 0; A(0) = A_0 \geq 0; \\ H(0) = H_0 \geq 0 \quad \text{and} \quad R(0) = R_0 \geq 0, \end{aligned} \tag{3.11}$$

where ${}^c_0D_t^\sigma$ is the Caputo fractional derivative operator of order σ .

Then, by applying the fractional integral operator of order σ as given in (3.3) to the system (3.10) with the initial conditions (3.11), we obtain:

$$\begin{cases} S(t) = S_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} (\Lambda - \Lambda p I(\tau) - (K + \mu_0) S(\tau)) d\tau \\ I(t) = I_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} (K S(\tau) - k_1 I(\tau)) d\tau \\ A(t) = A_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} (\epsilon \beta_1 I(\tau) - k_2 A(\tau) + \rho \lambda H(\tau)) d\tau \\ H(t) = H_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} (1 - (\epsilon) \beta_1 I(\tau) + \beta_2 A(\tau) - k_3 H(\tau)) d\tau \\ R(t) = R_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} ((1-\rho) \lambda H(\tau) - \mu_0 R(\tau)) d\tau, \end{cases} \quad (3.12)$$

where $\sigma \in (0, 1)$ and $t \geq 0$. We note that as $\sigma \rightarrow 1$, the fractional-order model (3.10) converges to the classical deterministic model (2.2).

3.3. Boundedness of solutions

The size of each population class in the model (3.10) varies over time, and the total population size $N(t)$ is given by:

$$N(t) = S(t) + I(t) + A(t) + H(t) + R(t).$$

Summing the equations of the model (3.10), we obtain the equation for the total population by linearizing the Caputo operator:

$${}^c_0D_t^\sigma N(t) = {}^c_0D_t^\sigma S(t) + {}^c_0D_t^\sigma I(t) + {}^c_0D_t^\sigma A(t) + {}^c_0D_t^\sigma H(t) + {}^c_0D_t^\sigma R(t),$$

which gives us, after simplification:

$${}^c_0D_t^\sigma N(t) = \Lambda - \mu_0 N(t) - \mu_1 A(t).$$

Applying an appropriate upper bound, we obtain the inequality:

$${}^c_0D_t^\sigma N(t) + \mu_0 N(t) \leq \Lambda. \quad (3.13)$$

Applying the Laplace transform method, we solve Grönwall's inequality (3.13) subject to the initial condition $N(0) \geq 0$:

$$\mathcal{L}\{ {}^c_0D_t^\sigma N(t) \} + \mu_0 \mathcal{L}\{ N \} \leq \mathcal{L}\{ \Lambda \}.$$

It follows that:

$$\mathcal{L}\{ N(t) \} \leq \frac{\Lambda}{S(S^\sigma + \mu_0)} + \sum_{k=0}^{n-1} \frac{S^{\sigma-k-1} N^{(k)}(0)}{S^\sigma + \mu_0}. \quad (3.14)$$

Splitting (3.14) to partial fractions gives:

$$\mathcal{L}\{ N(t) \} \leq \frac{\Lambda}{\mu_0} \left(\frac{1}{S} - \frac{1}{S \left(1 + \frac{\mu_0}{S^\sigma} \right)} \right) + \sum_{k=0}^{n-1} \frac{N^{(k)}(0)}{S^{k+1} \left(1 + \frac{\mu_0}{S^\sigma} \right)}. \quad (3.15)$$

Using the Taylor series expansion, we obtain:

$$\frac{1}{1 + \frac{\mu_0}{S^\sigma}} = \sum_{n=0}^{\infty} \left(\frac{-\mu_0}{S^\sigma} \right)^n,$$

further:

$$\mathcal{L}\{N(t)\} \leq \frac{\Lambda}{\mu_0} \left(\frac{1}{S} - \sum_{n=0}^{\infty} \left(\frac{-\mu_0}{S^{n\sigma+1}} \right) \right) + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} \left(\frac{(-\mu_0)^n N^{(k)}(0)}{S^{n\sigma+k+1}} \right). \quad (3.16)$$

Applying the inverse Laplace transform to (3.16), we obtain:

$$\begin{aligned} N(t) \leq & \frac{\Lambda}{\mu_0} \mathcal{L}^{-1} \left(\frac{1}{S} \right) - \frac{\Lambda}{\mu_0} \sum_{n=0}^{\infty} (-\mu_0)^n \mathcal{L}^{-1} \left(\frac{1}{S^{n\sigma+1}} \right) + \\ & \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} (-\mu_0)^n N^{(k)}(0) \mathcal{L}^{-1} \left\{ \frac{1}{S^{n\sigma+k+1}} \right\}. \end{aligned} \quad (3.17)$$

Let us recall that:

$$\mathcal{L}\{t^n\} = \frac{n!}{S^{n+1}} = \frac{\Gamma(n+1)}{S^{n+1}}$$

and

$$\mathcal{L}^{-1} \left\{ \frac{1}{S^{n+1}} \right\} = \frac{t^n}{\Gamma(n+1)}.$$

Thus, we have:

$$N(t) \leq \frac{\Lambda}{\mu_0} - \frac{\Lambda}{\mu_0} \sum_{n=0}^{\infty} \frac{(-\mu_0 t^\sigma)^n}{\Gamma(n\sigma+1)} + \sum_{k=0}^{n-1} \sum_{n=0}^{\infty} \frac{(-\mu_0) t^{n\sigma+k} N^{(k)}(0)}{\Gamma(\sigma n+k+1)}.$$

Replacing the Mittag-Leffler function with its definition 3.7 yields:

$$N(t) \leq \frac{\Lambda}{\mu_0} (1 - E_{\sigma,1}(-\mu_0 t^\sigma)) + \sum_{k=0}^{n-1} E_{\sigma,k+1}(-\mu_0 t^\sigma) t^k N^{(k)}(0), \quad (3.18)$$

where $E_{\sigma,1}(-\mu_0 t^\sigma)$ and $E_{\sigma,k+1}(-\mu_0 t^\sigma)$ is the series of the Mittag-Leffler function that converges for every argument, from which we can say that the solutions of the model (3.10) are bounded. Thus, we have the biologically feasible region of the model (3.10) defined by:

$$\begin{aligned} \Omega = \left\{ (S(t), I(t), A(t), H(t), R(t)) \in \mathbb{R}_+^5 \mid N(t) \leq \frac{\Lambda}{\mu_0} (1 - E_{\sigma,1}(-\mu_0 t^\sigma)) + \right. \\ \left. \sum_{k=0}^{n-1} E_{\sigma,k+1}(-\mu_0 t^\sigma) t^k N^{(k)}(0) \right\}. \end{aligned} \quad (3.19)$$

3.4. Uniqueness of the solution

Let us consider that the model (3.10) is written as follows:

$$\begin{cases} {}^c_0D_t^\sigma x(t) = F(t, x), & x(0) = x_0 \\ F(t, x) = Ax + g(x) + \eta, \end{cases} \quad (3.20)$$

where $F : \mathbb{R} \times \mathbb{R}^5 \rightarrow \mathbb{R}^5$ is a vector field of dimension 5, the unknown $x = x(t) \in \mathbb{R}^5$ is the state vector defined as:

$$x(t) = (S(t), I(t), A(t), H(t), R(t))^T,$$

$$A = \begin{pmatrix} -\mu_0 & -\Lambda p & 0 & 0 & 0 \\ 0 & -k_1 & 0 & 0 & 0 \\ 0 & \epsilon\beta_1 & -k_2 & \rho\lambda & 0 \\ 0 & (1-\epsilon)\beta_1 & \beta_2 & -k_3 & 0 \\ 0 & 0 & 0 & (1-\rho)\lambda & -\mu_0 \end{pmatrix}, \quad g(x) = \begin{pmatrix} -\frac{\alpha(I(t) + \eta A(t))}{N} \\ \frac{\alpha(I(t) + \eta A(t))}{N} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\eta = (\Lambda, 0, 0, 0, 0)^T.$$

The space \mathbb{R}^5 , endowed with the standard Euclidean norm $\|\cdot\|$, is a Banach space, i.e., a complete normed vector space.

Theorem 3.12. *The system (3.20) satisfies the Lipschitz continuity condition.*

Proof. As system (3.20) satisfies the Lipschitz continuity condition, we then obtain:

$$|F(t, x) - F(t, x^*)| = |A(x(t) - x^*(t)) + g(x(t)) - g(x^*(t))|,$$

this implies

$$|F(t, x) - F(t, x^*)| \leq (\|A\| + 1)\|x(t) - x^*(t)\|.$$

It follows that

$$\|F(t, x) - F(t, x^*)\| \leq L\|x(t) - x^*(t)\|, \quad \text{where } L = \|A\| + 1 < \infty. \quad (3.21)$$

Therefore, the function F is fully continuous and bounded. \square

Concerning the Picard-Lindelöf [19, 31], we establish the following theorem:

Theorem 3.13. *Let $0 < \sigma < 1$, $I = [0, h^*] \subseteq \mathbb{R}$ and $J = |x(t) - x(0)| \leq k$, where k is a constant lying in the interval $(0, 1)$.*

Let $f : I \times J \rightarrow \mathbb{R}$ be a bounded and continuous function, that is, $\exists! M > 0$ such that $|f(t, x)| \leq M$, we know that f satisfies the Lipschitz conditions. If $kL < M$, then there exists a unique $x \in C[0, h^]$ that satisfies the problem (3.20), where*

$$h^* = \min \left\{ h, \left(\frac{k\Gamma(\sigma + 1)}{M} \right)^{\frac{1}{\sigma}} \right\}.$$

Proof. Let

$$T = \{x \in C[0, h^*] : \|x(t) - x(0)\| \leq k\}.$$

Given that $T \subseteq \mathbb{R}$ is a closed set, then T is a complete metric space. The continuous system (3.10) can be transformed into the following equivalent integral equations:

$${}^c D_t^{-\sigma} [{}^c D_t^\sigma x(t)] = {}^c D_t^{-\sigma} f(t, x),$$

this implies

$$x(t) - x(0) = {}^c I_t^\sigma f(t, x),$$

it follows that

$$x(t) - x(0) = \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} f(\tau, x(\tau)) d\tau.$$

Thus, we obtain:

$$x(t) = x(0) + \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} f(\tau, x(\tau)) d\tau. \quad (3.22)$$

The equation (3.22) is equivalent to the Volterra integral equation, which allows us to solve (3.20). Let us define an operator F^* in T

$$F^*[x] = x_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} f(\tau, x(\tau)) d\tau. \quad (3.23)$$

We need to verify that the operator given in (3.23) satisfies the hypotheses of the Banach–Picard contraction theorem. We will first show that the operator F^* is invariant over T . To this aim, we compute:

$$\begin{aligned} |F^*[x(t)] - x(0)| &= \left| \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} f(\tau, x(\tau)) d\tau \right| \\ &\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} |f(\tau, x(\tau))| d\tau \\ &\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} M d\tau \\ &\leq \frac{M}{\Gamma(\sigma + 1)} t^\sigma \\ &\leq \frac{M}{\Gamma(\sigma + 1)} (h^*)^\sigma \\ &\leq \frac{M}{\Gamma(\sigma + 1)} \frac{k\Gamma(\sigma + 1)}{M} \end{aligned}$$

$$|F^*[x(t)] - x(0)| \leq k$$

Or equivalently,

$$x(0) - k \leq F^*[x(t)] \leq x(0) + k, \text{ for all } t \in [0, h^*].$$

Therefore, the operator F^* is a mapping from T into itself.

Next, we show that T is a contraction.

$$\begin{aligned} |F^*[x](t) - F^*[x^*](t)| &= \left| \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} [f(\tau, x(\tau)) - f(\tau, x^*(\tau))] d\tau \right| \\ &\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t - \tau)^{\sigma-1} |f(\tau, x(\tau)) - f(\tau, x^*(\tau))| d\tau. \end{aligned}$$

$$\begin{aligned} &\leq \frac{L}{\Gamma(\sigma+1)} \|x - x^*\| (h^*)^\sigma. \\ &\leq \frac{L}{\Gamma(\sigma+1)} \|x - x^*\| \frac{k\Gamma(\sigma+1)}{M}. \end{aligned}$$

Therefore,

$$\|F^*[x](t) - F^*[x^*](t)\| \leq \frac{kL}{M} \|x - x^*\|.$$

Since $\frac{kL}{M} < 1$, the operator T is a contraction and consequently, by theorem 3.13, it has a unique fixed point; thus, the problem (3.10) has a unique solution. \square

4. Mathematical analysis of the fractional-order model

In this section, we study the local and global stability of the disease-free equilibrium point. Before that, we remind that the disease-free equilibrium point E_0 and the basic reproduction number \mathcal{R}_0 are determined in the model of the reference [12], which are the same as for the model (3.10). The disease-free equilibrium point E_0 is given by:

$$E_0 = \left(\frac{\Lambda}{\mu_0}, 0, 0, 0, 0 \right) \quad (4.1)$$

and the basic reproduction number \mathcal{R}_0 is given by:

$$\mathcal{R}_0 = \frac{\alpha[k_2k_3 - \beta_2\rho\lambda + \eta\epsilon\beta_1k_3 + \eta\rho\lambda(1 - \epsilon)\beta_1]}{k_1(k_2k_3 - \beta_2\rho\lambda)}. \quad (4.2)$$

4.1. Local stability of the disease-free equilibrium point

The following theorem confirms that the disease is locally and asymptotically stable at the disease-free equilibrium point:

Theorem 4.1. *The disease-free equilibrium point $E_0 = \left(\frac{\Lambda}{\mu_0}, 0, 0, 0, 0 \right)$ is locally and asymptotically stable in Ω if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. Let us determine the Jacobian matrix $J(E_0)$ of the model (3.10) at the disease-free equilibrium:

$$J(E_0) = \begin{pmatrix} -\mu_0 & -\Lambda P - \alpha & -\alpha\eta & 0 & 0 \\ \alpha & \alpha - k_1 & \alpha\eta & 0 & 0 \\ 0 & \epsilon\beta_1 & -k_2 & \rho\lambda & 0 \\ 0 & (1 - \epsilon)\beta_1 & \beta_2 & -k_3 & 0 \\ 0 & 0 & 0 & (1 - \rho)\lambda & -\mu_0 \end{pmatrix}. \quad (4.3)$$

Let $P(\Psi) = \det(J(E_0) - \Psi I_5)$ denote the characteristic polynomial of the Jacobian matrix $J(E_0)$, where Ψ represents the set of eigenvalues of $J(E_0)$ and I_5 is the identity

matrix of size 5×5

$$P(\Psi) = \begin{vmatrix} -\mu_0 - \Psi & -\Lambda p - \alpha & -\alpha\eta & 0 & 0 \\ 0 & \alpha - k_1 - \Psi & \alpha\eta & 0 & 0 \\ 0 & \epsilon\beta_1 & -k_2 - \Psi & \rho\lambda & 0 \\ 0 & (1 - \epsilon)\beta_1 & \beta_2 & -k_3 - \Psi & 0 \\ 0 & 0 & 0 & (1 - \rho)\lambda & -\mu_0 - \Psi \end{vmatrix}. \quad (4.4)$$

The disease-free equilibrium point is locally and asymptotically stable if all the eigenvalues of the Jacobian matrix $J(E_0)$ have negative real parts, or if $|\text{Arg}(\Psi_i)| > \frac{\pi\sigma}{2}$ with $i \in \{1, 2, 3, 4, 5\}$ and $0 < \sigma < 1$.

Solving $\det(J(E_0) - \Psi I_5) = 0$ and after some algebraic manipulations, we obtain:

$$(\Psi + \mu_0)(\Psi + \mu_0) \begin{vmatrix} \alpha - k_1 - \Psi & \alpha\eta & 0 \\ \epsilon\beta_1 & -k_2 - \Psi & \rho\lambda \\ (1 - \epsilon)\beta_1 & \beta_2 & -k_3 - \Psi \end{vmatrix} = 0. \quad (4.5)$$

We have a double eigenvalue, which is negative $(-\mu_0)$, in other words, $|\text{Arg}(-\mu_0)| = \pi > \frac{\pi\sigma}{2}$. Thus, the stability of E_0 depends on the remaining eigenvalues.

After algebraic computations and defining $a_1 = k_1 + k_2 + k_3 - \alpha$;

$a_2 = k_1k_2 + k_1k_3 + k_2k_3 - \alpha(k_2 + k_3) - \beta_2\rho\lambda - \alpha\eta$,

and $a_3 = k_1k_2k_3 + \alpha\beta_2\rho\lambda - k_1\beta_2\rho\lambda - \alpha k_2k_3 - \alpha\eta\epsilon\beta_1k_3 - \alpha\eta\rho\lambda(1 - \epsilon)\beta_1$, we obtain the following characteristic polynomial $Q(\Psi)$ corresponding to the remaining undetermined eigenvalues:

$$Q(\Psi) = \Psi^3 + a_1\Psi^2 + a_2\Psi + a_3. \quad (4.6)$$

To demonstrate the local stability of the disease-free equilibrium point E_0 of the model (3.10) using the Routh-Hurwitz criteria, it is necessary to show that: $a_1 > 0$, $a_1a_2 - a_3 > 0$, and $a_3 > 0$.

$a_1 > 0$ because $k_1 + k_2 + k_3 > \alpha$.

$a_3 = k_1(k_2k_3 - \beta_2\rho\lambda)(1 - \mathcal{R}_0)$; $a_3 > 0$, if $\mathcal{R}_0 < 1$.

$a_1a_2 - a_3 = k_1k_2k_3(2 + \mathcal{R}_0) + \beta_2\rho\lambda k_1(1 - \mathcal{R}_0) + k_1k_2(k_1 + k_2 - 2\alpha) + k_1k_3(k_1 + k_3 - 2\alpha) + k_2k_3(k_2 + k_3 - 3\alpha) + (k_2 + k_3)(\alpha^2 - \alpha\eta\epsilon\beta_1) + \alpha\eta\epsilon\beta_1(\alpha - k_1) - \alpha\eta\epsilon\beta_1k_2^2 > 0$.

As the Routh-Hurwitz conditions are satisfied, we conclude that the disease-free equilibrium point is locally and asymptotically stable. \square

4.2. Global stability of the disease-free equilibrium point

The following theorem confirms that the disease is globally and asymptotically stable at the disease-free equilibrium point:

Theorem 4.2. *The disease-free equilibrium point E_0 is globally and asymptotically stable in Ω when $\mathcal{R}_0 \leq 1$. If $\mathcal{R}_0 > 1$, E_0 becomes unstable, and the disease continues to spread uniformly in the population.*

Proof. The global stability analysis of the disease-free equilibrium point is based on the construction of a Lyapunov function using the matrix method proposed by

Zhisheng Shuai and P. Van den Driessche [44] and later expanded in [12]. Thus, we obtain:

$$L = \left[\left(\frac{\mathcal{R}_0 - \frac{\alpha}{k_1}}{\alpha} + \frac{1}{k_1} \right) I + \frac{k_1 k_3 (\mathcal{R}_0 - \frac{\alpha}{k_1})}{\alpha (\epsilon \beta_1 k_3 + \rho \lambda (1 - \epsilon) \beta_1)} A + \frac{k_1 \rho \lambda (\mathcal{R}_0 - \frac{\alpha}{k_1})}{\alpha (\epsilon \beta_1 k_3 + \rho \lambda (1 - \epsilon) \beta_1)} H \right] \zeta_1, \quad (4.7)$$

where ζ_1 is one of the components of the non-negative left eigenvector, normalized to unit length, associated with the spectral radius of $V^{-1}F$.

We have: L is a Lyapunov function for the system (3.3), and $L \geq 0 \forall \mathcal{R}_0 \geq \frac{\alpha}{k_1}$

The fractional derivative of L to time t in the sense of Caputo gives

$${}^c_0 D_t^\sigma L = (\mathcal{R}_0 - 1) \left(\zeta_1 I + \frac{\mu_0 k_1 \left(\mathcal{R}_0 - \frac{\alpha}{k_1} \right) (k_2 k_3 - \beta_2 \rho \lambda)}{\alpha (\epsilon \beta_1 k_3 + \rho \lambda (1 - \epsilon) \beta_1)} \zeta_1 A \right) - \left(\frac{1}{k_1} + \frac{\left(\mathcal{R}_0 - \frac{\alpha}{k_1} \right)}{\alpha} \right) \alpha (I + \eta A) \left(1 - \frac{S}{N} \right) \zeta_1. \quad (4.8)$$

Since $S \leq N$, if $\mathcal{R}_0 \leq 1$ then ${}^c_0 D_t^\sigma L < 0$. Moreover ${}^c_0 D_t^\sigma L = 0$ for $I = A = 0$ and $S = \frac{\Lambda}{\mu_0}$. Thus, the largest invariant set of the model (3.10), when ${}^c_0 D_t^\sigma L = 0$, is the disease-free equilibrium point E_0 .

We have formulated a suitable Lyapunov function for system (3.10) and shown that it strictly decreases along the system's trajectories, i.e., ${}^c_0 D_t^\sigma L \leq 0$.

As a result, by LaSalle's invariance principle [31], E_0 is globally and asymptotically stable. \square

5. Sensitivity analysis

In this section, we study the sensitivity analysis of the basic reproduction number \mathcal{R}_0 . Sensitivity analysis is a crucial technique for complex systems, as it helps determine the influence of the model parameters in (3.10) on the transmission of HIV/AIDS. Sensitivity analysis enables us to understand how changes in the model parameters affect the disease's propagation dynamics [4, 5, 35, 53]. By performing local sensitivity analysis, we can evaluate the effect of a single parameter on the basic reproduction number \mathcal{R}_0 while keeping the other parameters constant.

According to the section 4,

$$\mathcal{R}_0 = \frac{\alpha [k_2 k_3 - \beta_2 \rho \lambda + \eta \epsilon \beta_1 k_3 + \eta \rho \lambda (1 - \epsilon) \beta_1]}{k_1 (k_2 k_3 - \beta_2 \rho \lambda)}.$$

By setting $\Gamma = k_2k_3 - \beta_2\rho\lambda + \eta\epsilon\beta_1k_3 + \eta\rho\lambda(1 - \epsilon)\beta_1$ and $\Omega = k_1(k_2k_3 - \beta_2\rho\lambda)$ then:

$$\mathcal{R}_0 = \alpha \frac{\Gamma}{\Omega}.$$

Definition 5.1. [15] *The normalized sensitivity index of \mathcal{R}_0 , which depends differentially on a parameter p , is given by:*

$$\gamma_p^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial p} \cdot \frac{p}{\mathcal{R}_0}. \tag{5.1}$$

For conducting the sensitivity analysis, other authors [4, 5, 39, 46, 47, 53] have also used this fundamental formula to calculate the sensitivity index of each parameter appearing in \mathcal{R}_0 .

The expressions for the sensitivity indices of the 11 parameters of \mathcal{R}_0 are as follows:

$$\begin{aligned} \gamma_\alpha^{\mathcal{R}_0} &= +1; \gamma_p^{\mathcal{R}_0} = \frac{\Lambda p}{k_1}; \gamma_\Lambda^{\mathcal{R}_0} = \frac{\Lambda p}{k_1}; \gamma_{\beta_1}^{\mathcal{R}_0} = \frac{\beta_1\eta[\epsilon k_3 + \rho\lambda(1 - \epsilon)]}{\Gamma} - \frac{\beta_1(k_2k_3 - \beta_2\rho\lambda)}{\Omega}; \\ \gamma_{\beta_2}^{\mathcal{R}_0} &= \frac{\beta_2(k_3 - \rho\lambda)}{\Gamma} - \frac{\beta_2k_1(k_3 - \rho\lambda)}{\Omega} \\ \gamma_{\mu_0}^{\mathcal{R}_0} &= \frac{\mu_0(k_2 + k_3 + \eta\epsilon\beta_1)}{\Gamma} - \frac{\mu_0[k_2k_3 - \beta_2\rho\lambda + k_1(k_2 + k_3)]}{\Omega}; \gamma_{\mu_1}^{\mathcal{R}_0} = \frac{\mu_1k_3}{\Gamma} - \frac{\mu_1k_1k_3}{\Omega}; \\ \gamma_\lambda^{\mathcal{R}_0} &= \frac{\lambda[k_2 - \beta_2\rho + \eta\epsilon\beta_1 + \eta\rho(1 - \epsilon)\beta_1]}{\Gamma} - \frac{\lambda k_1k_2}{\Omega} \\ \gamma_\rho^{\mathcal{R}_0} &= \frac{\rho[-\beta_2\lambda + \eta\lambda(1 - \epsilon)\beta_1]}{\Gamma} + \frac{\rho k_1\beta_2\lambda}{\Omega}; \gamma_\eta^{\mathcal{R}_0} = \frac{\eta\beta_1[\epsilon k_3 + \rho\lambda(1 - \epsilon)]}{\Gamma}; \\ \gamma_\epsilon^{\mathcal{R}_0} &= \frac{\epsilon\eta\beta_1(k_3 - \rho\lambda)}{\Gamma}. \end{aligned}$$

Table 1 gives a description and the values of the parameters, and Table 2 gives sensitivity indices of the parameters.

Parameters	Meaning	Values	Reference
N(0)	Total population	100000	assumed
Λ	Recruitment rate	380	assumed
α	HIV/AIDS transmission rate	0.00011	assumed
β_1	Rate of transfer from the compartment of infected individuals without clinical signs to the AIDS compartment	0.3	[50]
β_2	rate of transfer from the AIDS class to the compartment of individuals on treatment	0.33	[14]
λ	rate of transfer of individuals on treatment to the AIDS class	0.2	assumed
η	AIDS infectivity versus HIV infection without clinical signs of the disease	1.05	[45]
p	probability of HIV-positive newborns through vertical transmission	0.0001	Fitting
ϵ	probability of infected individuals without clinical signs progressing to the AIDS phase	0.1	assumed
ρ	Probability of individuals who have failed their treatment and move up to the AIDS class	0.09	[14]
μ_0	HIV natural mortality rate	1/57	assumed
μ_1	HIV-induced mortality rate	0.7	assumed

TABLE 1. Values of the parameters used in the model (3.3) of HIV/AIDS

Parameters	Sensitivity analysis
α	+1
Λ	+0.1359
p	+0.1359
β_1	-1.0221
β_2	-0.0152
λ	-0.0249
η	+0.0511
ϵ	+0.0269
ρ	+0.0232
μ_0	-0.0655
μ_1	-0.351

TABLE 2. Values of the sensitivity indices of the parameters of \mathcal{R}_0 .

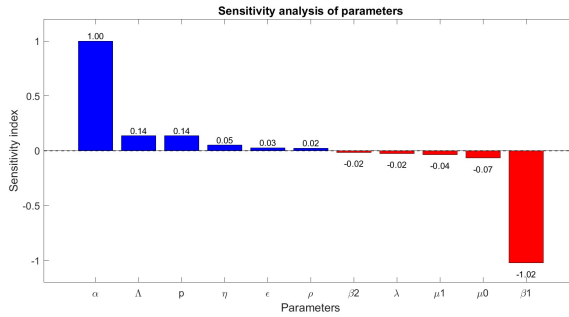


FIGURE 1. Diagram of the sensitivity analysis of the parameters of \mathcal{R}_0

The parameter with a sensitivity index of high absolute value is more influential than the one with a low sensitivity index in absolute value, and the signs of the sensitivity indices of the parameters show their positive or negative impacts in the model. From the sensitivity analysis of \mathcal{R}_0 , we observe that α and β_1 are the most influential parameters of \mathcal{R}_0 . Indeed, $\gamma\alpha^{\mathcal{R}_0} = +1$, so a 1% increase in α leads to a 1% increase in \mathcal{R}_0 ; $\gamma\beta_1^{\mathcal{R}_0} = -1.0221$ means that a 1% increase in β_1 leads to a 1.0221% decrease in \mathcal{R}_0 . The same interpretation can be applied to the remaining parameters in the expression of \mathcal{R}_0 , depending on the sign of the sensitivity index [1, 4, 5, 15, 35, 46, 47]. We can say that an increase in the parameters with positive sensitivity indices leads to a proportional increase in \mathcal{R}_0 , while an increase in the parameters with negative sensitivity indices results in an inversely proportional change in \mathcal{R}_0 . It is imperative to implement strategic initiatives aimed at reducing the transmission of HIV/AIDS and managing the progression of individuals from the asymptomatic infection stage to the stage of full-blown AIDS. In this regard, the promotion of preventive measures and the expanded access to antiretroviral therapy should be considered public health priorities.

6. Numerical results and discussions

This section focuses on the numerical simulation of the system (3.10). The goal is to investigate how the fractional order σ influences the transmission and control of HIV/AIDS, using the model represented by this system.

6.1. Discretization of the model (3.3)

In this section, we present a numerical simulation of the results of the system (3.10). The objective is to examine the impact of the fractional order σ on the transmission and control of HIV/AIDS, through the model (3.10). To achieve this, we use the iterative Euler method for fractional differential equations, where the fractional integral of each variable in system (3.3) is approximated by a discrete sum (See [27, 28, 37, 49] for more details on Euler methods for ordinary and fractional differential equations). This approach involves transforming the fractional differential equation into an integral equation and then solving it by applying a discrete method.

We introduce a uniform time discretization of the interval $[0, T]$, defined by the points $t_k = k\Delta t$ for $k = 0, \dots, n_T$, where $\Delta t = \frac{T}{n_T}$ is the step size, and $n_T \in \mathbb{N}^*$ is the number of subdivisions. This discretization is used to apply the generalized Euler method to a fractional-order differential equation.

By considering system (3.12), and setting $t = t_{n+1}$, $n \in \mathbb{N}$, we obtain:

$$\left\{ \begin{array}{l} S(t_{n+1}) = S_0 + \frac{1}{\Gamma(\sigma)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\sigma-1} \\ \quad \times \left(\Lambda - \Lambda p I(\tau) - \mu_0 S(\tau) - \frac{\alpha S(\tau)(I(\tau) + \eta A(\tau))}{N} \right) d\tau \\ I(t_{n+1}) = I_0 + \frac{1}{\Gamma(\sigma)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\sigma-1} \left(\frac{\alpha S(\tau)(I(\tau) + \eta A(\tau))}{N} - k_1 I(\tau) \right) d\tau \\ A(t_{n+1}) = A_0 + \frac{1}{\Gamma(\sigma)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\sigma-1} \left(\epsilon \beta_1 I(\tau) - k_2 A(\tau) + \rho \lambda H(\tau) \right) d\tau \\ H(t_{n+1}) = H_0 + \frac{1}{\Gamma(\sigma)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\sigma-1} \left((1 - \epsilon) \beta_1 I(\tau) + \beta_2 A(\tau) - k_2 H(\tau) \right) d\tau \\ R(t_{n+1}) = R_0 + \frac{1}{\Gamma(\sigma)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\sigma-1} \left((1 - \rho) \lambda H(\tau) - \mu_0 R(\tau) \right) d\tau. \end{array} \right. \quad (6.1)$$

After some algebraic manipulations and setting:

$$f(s, u(s))|_{[t_k, t_{k+1}]} \simeq f(t_k, u(t_k)) \quad \text{with} \quad u(t) = (S(t), I(t), A(t), H(t), R(t)), \quad (6.2)$$

we have:

$$\left\{ \begin{array}{l} S(t_{n+1}) = S_0 + \frac{\Delta t^\sigma}{\Gamma(\sigma+1)} \sum_{k=0}^n [(n-k+1)^\sigma - (n-k)^\sigma] \\ \quad \times \left(\Lambda - \Lambda p I(t_k) - \mu_0 S(t_k) - \frac{\alpha S(t_k)(I(t_k) + \eta A(t_k))}{N} \right) \\ I(t_{n+1}) = I_0 + \frac{\Delta t^\sigma}{\Gamma(\sigma+1)} \sum_{k=0}^n [(n-k+1)^\sigma - (n-k)^\sigma] \\ \quad \times \left(\frac{\alpha S(t_k)(I(t_k) + \eta A(t_k))}{N} + k_1 I(t_k) \right) \\ A(t_{n+1}) = A_0 + \frac{\Delta t^\sigma}{\Gamma(\sigma+1)} \sum_{k=0}^n [(n-k+1)^\sigma - (n-k)^\sigma] \\ \quad \times (\epsilon \beta_1 I(t_k) - k_2 A(t_k) + \rho \lambda H(t_k)) \\ H(t_{n+1}) = H_0 + \frac{\Delta t^\sigma}{\Gamma(\sigma+1)} \sum_{k=0}^n [(n-k+1)^\sigma - (n-k)^\sigma] \\ \quad \times ((1-\epsilon)\beta_1 I(t_k) + \beta_2 A(t_k) - k_3 H(t_k)) \\ R(t_{n+1}) = R_0 + \frac{\Delta t^\sigma}{\Gamma(\sigma+1)} \sum_{k=0}^n [(n-k+1)^\sigma - (n-k)^\sigma] \\ \quad \times ((1-\rho)\lambda H(t_k) - \mu_0 R(t_k)). \end{array} \right. \quad (6.3)$$

6.2. Numerical results

To study the effects of the fractional order on the model (3.3), we implemented in MATLAB, inspired by the work of [27, 28, 37, 49], the approximate solution (6.3) based on the Euler method, for different values of σ . The parameter values used for the numerical simulation of the model (3.10) are derived from the literature or assumed to be biologically plausible (see Table 1). The system (3.10) is considered subject to the initial conditions given by:

$$S(0) = 89040; I(0) = 4700; A(0) = 100; H(0) = 3580; R(0) = 2580, \quad (6.4)$$

the values of the parameters from Table 1 and a final time value of $T = 10$ (years). The following figures present the graphs of the approximate solutions:

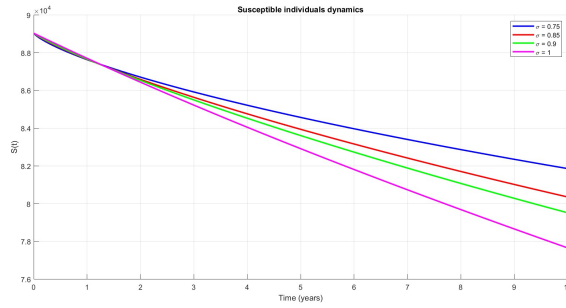


FIGURE 2. Dynamics of the susceptible individuals compartment of the model (3.10) when $\sigma = 0.75, 0.85, 0.90, 1$

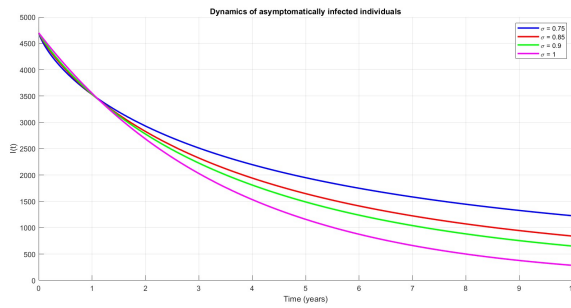


FIGURE 3. Dynamics of infected without clinical signs compartment of the model (3.10) when $\sigma = 0.75, 0.85, 0.90, 1$.

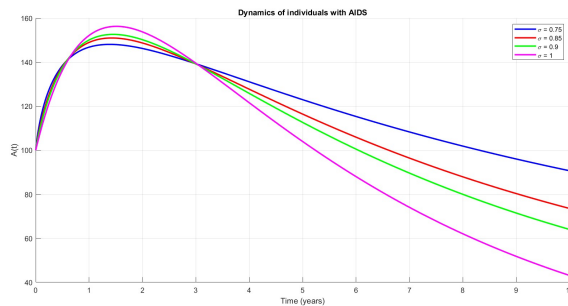


FIGURE 4. Dynamics of infected with clinical signs compartment of the model (3.10) when $\sigma = 0.75, 0.85, 0.90, 1$.

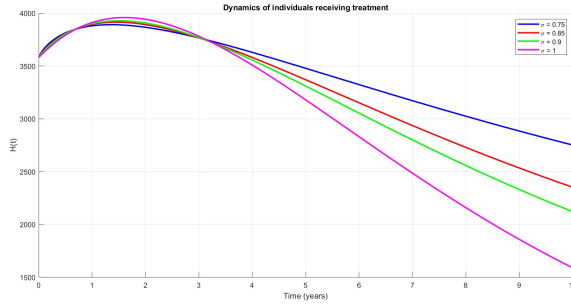


FIGURE 5. Dynamics of the treated individuals compartment of the model (3.10) when $\sigma = 0.75, 0.85, 0.90, 1$.

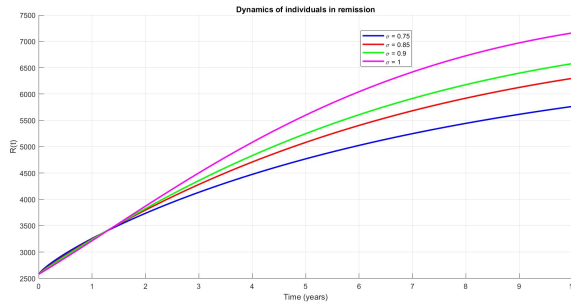


FIGURE 6. Dynamics of individuals in remission compartment of the model (3.10) when $\sigma = 0.75, 0.85, 0.90, 1$.

In figures 2 to 6, the graphs show the dynamics of susceptible individuals, individuals infected without clinical signs of the disease, individuals with AIDS, individuals infected with HIV under antiretroviral treatment, and individuals in remission for different values of the fractional order σ .

In the compartment of susceptible individuals, the decrease in the dynamics of individuals is less pronounced as the fractional order decreases. Specifically, the smallest decrease is observed for $\sigma = 0.75$, while the largest decrease is noted for $\sigma = 1$.

In the class of infected individuals, with or without clinical signs of AIDS, the dynamics of individuals' decline and the rate of decline are directly linked to the fractional order. Thus, the smallest decline is observed for $\sigma = 0.75$, and the largest decline is noted for $\sigma = 1$.

In the compartment of individuals under treatment, the dynamics of individuals decrease, with a reduction rate proportional to the fractional order, as in the previous cases.

In the class of individuals in remission, the dynamics of individuals increase, with a growth rate proportional to the fractional order. Specifically, the most significant

increase is observed for $\sigma = 1$, while the smallest increase is noted for $\sigma = 0.75$. In light of the foregoing, the results indicate that the fractional-order model, by its ability to incorporate long-term memory effects, offers a more relevant framework than the deterministic model for capturing the dynamics of HIV transmission. Specifically, variations in the fractional order produce distinct trajectories, thereby enabling a more refined representation of transmission dynamics and enhancing the effectiveness of strategies aimed at curbing the spread of the virus, as demonstrated in the figures presented above.

7. Conclusion

In this study, we first analyzed the transmission of HIV/AIDS using a fractional model in the sense of Caputo. We applied Banach's contraction principle to demonstrate the existence of a unique solution for the developed model and subsequently performed stability analyses of the disease-free equilibrium point. Next, we conducted a sensitivity analysis of the basic reproduction number \mathcal{R}_0 concerning the various parameters of the fractional model. We found that the transmission rate and the progression rate of individuals from the group of infected individuals who show no clinical signs to the AIDS compartment are the most sensitive parameters of \mathcal{R}_0 . Finally, approximate solutions of the fractional-order model were simulated for various values of the fractional order, and the corresponding results were analyzed accordingly. The study revealed that the fractional model offers a superior fit compared to the deterministic model, primarily due to its ability to capture long-term dynamics such as infection relapses and delayed responses to antiretroviral therapy. Moreover, this modeling framework allows for the generation of distinct trajectories based on variations in the fractional order, thereby providing a more nuanced representation of viral transmission. Consequently, the outcomes derived from this approach are particularly relevant for predicting the spread of HIV/AIDS within a population while accounting for the variability introduced by the fractional order.

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On a specific ratio-cosine Hardy-Hilbert-type integral inequality in the entire plane

Christophe Chesneau 

Abstract. This article focuses on a specific Hardy-Hilbert-type integral inequality that is defined in the entire plane. The main contribution is the derivation of a ratio-cosine kernel function, which sets it apart from most existing literature on the subject. As a consequence of the main theorem, a related integral inequality of independent interest is also derived. The exposition is self-contained, with full details of all proofs presented, and each step is carefully justified.

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1. Introduction


The classical Hardy-Hilbert integral inequality is a key tool in real analysis. It provides a precise upper bound for a double integral involving the product of two functions. More precisely, let $p > 1$, $q = p/(p-1)$ and $f, g : (0, +\infty) \rightarrow (0, +\infty)$ be two functions. Then the Hardy-Hilbert integral inequality states that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y) dx dy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} g^q(y) dy \right]^{1/q},$$

provided that both integrals on the right-hand side converge. The constant $\pi/\sin(\pi/p)$ is known to be the best possible and cannot be improved (see [7]). This inequality has found numerous applications in analysis, operator theory, and mathematical physics. It serves as a benchmark for a wide class of Hardy-Hilbert-type integral inequalities.

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Further details and developments can be found in the monographs of Yang [21, 20], as well as in the classical and modern contributions in [17, 15, 22, 4, 1, 3, 2].

In fact, most of the results in this field have been obtained in the quarter-plane $(0, +\infty)^2 = \{(x, y) \in \mathbb{R}^2; x > 0, y > 0\}$. In recent years, considerable effort has been devoted to extending Hardy-Hilbert-type integral inequalities beyond this domain to the entire plane \mathbb{R}^2 , i.e., the set of all points in the plane. See, for instance, the studies in [18, 8, 19, 25, 14, 16, 9, 24, 11, 27, 12, 10, 6, 26, 13, 23]. The established generalizations are quite diverse, involving various classes of kernel functions, and a broad range of parameter settings. As a well-known example, in [19], the result below holds. Let $p > 1$, $q = p/(p - 1)$ and $f, g : \mathbb{R} \rightarrow (0, +\infty)$ be two functions. Then we have

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{|1 + xy|^\theta} f(x)g(y) dx dy \\ & \leq \kappa \left[\int_{-\infty}^{+\infty} |x|^{(1-\theta/2)p-1} f^p(x) dx \right]^{1/p} \left[\int_{-\infty}^{+\infty} |y|^{(1-\theta/2)q-1} g^q(y) dy \right]^{1/q}, \end{aligned}$$

where

$$\kappa = B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) + 2B\left(1 - \theta, \frac{\theta}{2}\right)$$

and $B(a, b)$ is the standard beta function, i.e., $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$, provided that both integrals on the right-hand side converge. The constant κ is known to be the best possible and cannot be improved (see [19]).

Despite the abundance of literature on the subject, inequalities involving ratio-cosine kernel functions have received relatively little attention. This article aims to address this gap. More specifically, our objective is to present a new integral inequality based on the following kernel function:

$$k(x, y) = \frac{\{1 - \tau \cos[\alpha(\beta - x)]\}^{1/q} \{1 - \tau \cos[\alpha(\beta - y)]\}^{1/p}}{x^2 + y^2},$$

where $\alpha, \beta > 0$ and $\tau \in [-1, 1]$ are three independent and adjustable parameters. For the sake of completeness, we provide a detailed proof. A key component of our analysis is an integral formula derived from one originating in [5]. We also prove an integral inequality involving the dependence of only one function.

The remainder of the article is organized as follows: Section 2 presents a preliminary result: the aforementioned new integral formula. Section 3 contains our main results. Section 4 provides a conclusion.

2. A preliminary result

The lemma below corresponds to [5, Formula 3.723 7], with only minor changes to the notation.

Lemma 2.1. [5, Formula 3.723 7] *Let $\alpha, \beta, \eta > 0$. Then we have*

$$\int_{-\infty}^{+\infty} \frac{\cos[\alpha(\beta - x)]}{\eta^2 + x^2} dx = \frac{\pi}{\eta} \cos(\alpha\beta)e^{-\alpha\eta}.$$

Based on this result, we derive the new result below. This mainly involves modifying the integrand of the previous integral to make it non-negative.

Lemma 2.2. *Let $\alpha, \beta > 0$, $\gamma \in \mathbb{R} \setminus \{0\}$ and $\tau \in [-1, 1]$. Then we have*

$$\int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - x)]}{\gamma^2 + x^2} dx = \frac{\pi}{|\gamma|} \left[1 - \tau \cos(\alpha\beta)e^{-\alpha|\gamma|} \right].$$

Proof. Combining a linearization of the integral with the standard arctangent primitive and Lemma 2.1 yields

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - x)]}{\gamma^2 + x^2} dx &= \int_{-\infty}^{+\infty} \frac{1}{|\gamma|^2 + x^2} dx - \tau \int_{-\infty}^{+\infty} \frac{\cos[\alpha(\beta - x)]}{|\gamma|^2 + x^2} dx \\ &= \left[\frac{1}{|\gamma|} \arctan \left(\frac{x}{|\gamma|} \right) \right]_{x \rightarrow -\infty}^{x \rightarrow +\infty} - \tau \frac{\pi}{|\gamma|} \cos(\alpha\beta)e^{-\alpha|\gamma|} \\ &= \frac{\pi}{|\gamma|} - \tau \frac{\pi}{|\gamma|} \cos(\alpha\beta)e^{-\alpha|\gamma|} = \frac{\pi}{|\gamma|} \left[1 - \tau \cos(\alpha\beta)e^{-\alpha|\gamma|} \right]. \end{aligned}$$

This ends the proof of the lemma. □

The fact that the integrand is non-negative and is defined on the real line, and that the integral formula is in closed form, makes it suitable for use in Hardy-Hilbert-type integral inequalities in the entire plane. This is the focus of the subsequent section.

3. Main results

Based on Lemma 2.2, the theorem below states a new Hardy-Hilbert-type integral inequality in the entire plane. As outlined in the introduction, it is of ratio-cosine nature.

Theorem 3.1. *Let $p > 1$, $q = p/(p - 1)$, $\alpha, \beta > 0$, $\tau \in [-1, 1]$ and $f, g : \mathbb{R} \rightarrow (0, +\infty)$ be two functions. Then we have*

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\{1 - \tau \cos[\alpha(\beta - x)]\}^{1/q} \{1 - \tau \cos[\alpha(\beta - y)]\}^{1/p}}{x^2 + y^2} f(x)g(y) dx dy \\ &\leq \pi \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \tau \cos(\alpha\beta)e^{-\alpha|x|} \right] f^p(x) dx \right\}^{1/p} \\ &\quad \times \left\{ \int_{-\infty}^{+\infty} \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta)e^{-\alpha|y|} \right] g^q(y) dy \right\}^{1/q}, \end{aligned}$$

provided that both integrals on the right-hand side converge.

Proof. Note that the double integral is well-defined because, for any $x, y \in \mathbb{R}$,

$$1 - \tau \cos[\alpha(\beta - x)] \geq 1 - |\tau| \geq 0$$

and

$$1 - \tau \cos[\alpha(\beta - y)] \geq 1 - |\tau| \geq 0.$$

Decomposing the integrand using $1/p + 1/q = 1$ and applying the Hölder integral inequality, we get

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\{1 - \tau \cos[\alpha(\beta - x)]\}^{1/q} \{1 - \tau \cos[\alpha(\beta - y)]\}^{1/p}}{x^2 + y^2} f(x)g(y) dx dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ \frac{1 - \tau \cos[\alpha(\beta - y)]}{x^2 + y^2} \right\}^{1/p} f(x) \times \left\{ \frac{1 - \tau \cos[\alpha(\beta - x)]}{x^2 + y^2} \right\}^{1/q} g(y) dx dy \\ &\leq \left\{ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - y)]}{x^2 + y^2} f^p(x) dx dy \right\}^{1/p} \\ &\quad \times \left\{ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - x)]}{x^2 + y^2} g^q(y) dx dy \right\}^{1/q}. \end{aligned} \quad (3.1)$$

Let us examine these integrals. It follows from the Fubini-Tonelli integral theorem and Lemma 2.2 with $\gamma = x$ that

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - y)]}{x^2 + y^2} f^p(x) dx dy \\ &= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - y)]}{x^2 + y^2} dy \right\} f^p(x) dx \\ &= \int_{-\infty}^{+\infty} \left\{ \frac{\pi}{|x|} [1 - \tau \cos(\alpha\beta) e^{-\alpha|x|}] \right\} f^p(x) dx \\ &= \pi \int_{-\infty}^{+\infty} \frac{1}{|x|} [1 - \tau \cos(\alpha\beta) e^{-\alpha|x|}] f^p(x) dx. \end{aligned} \quad (3.2)$$

In a similar way, we find that

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - x)]}{x^2 + y^2} g^q(y) dx dy \\ &= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \frac{1 - \tau \cos[\alpha(\beta - x)]}{x^2 + y^2} dx \right\} g^q(y) dy \\ &= \int_{-\infty}^{+\infty} \left\{ \frac{\pi}{|y|} [1 - \tau \cos(\alpha\beta) e^{-\alpha|y|}] \right\} g^q(y) dy \\ &= \pi \int_{-\infty}^{+\infty} \frac{1}{|y|} [1 - \tau \cos(\alpha\beta) e^{-\alpha|y|}] g^q(y) dy. \end{aligned} \quad (3.3)$$

It follows from Equations (3.1), (3.2) and (3.3), and $1/p + 1/q = 1$ that

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\{1 - \tau \cos[\alpha(\beta - x)]\}^{1/q} \{1 - \tau \cos[\alpha(\beta - y)]\}^{1/p}}{x^2 + y^2} f(x)g(y) dx dy \\ & \leq \pi \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} [1 - \tau \cos(\alpha\beta)e^{-\alpha|x|}] f^p(x) dx \right\}^{1/p} \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} \frac{1}{|y|} [1 - \tau \cos(\alpha\beta)e^{-\alpha|y|}] g^q(y) dy \right\}^{1/q}. \end{aligned}$$

This completes the proof of the theorem. □

To the best of our knowledge, this is a new addition to the literature on the Hardy-Hilbert integral inequality in the entire plane.

The constant factor, π , was obtained in the development process with the minimum possible number of steps. However, it has not been proven rigorously that this is the best possible approach. The question of its optimality remains unanswered.

Assuming that both integrals on the right-hand side converge implies that the functions f and g must behave in a specific way near 0. More precisely, near 0, they must satisfy the equivalences $f(x) \sim |x|^\iota$ and $g(y) \sim |y|^\omega$ with $\iota, \omega > 0$, with reference to the convergence of the Riemann integral in the interval $(-1, 1)$.

Some specific examples of Theorem 3.1 are provided below.

- If we take $\tau = 0$, we get

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{x^2 + y^2} f(x)g(y) dx dy \leq \pi \left[\int_{-\infty}^{+\infty} \frac{1}{|x|} f^p(x) dx \right]^{1/p} \left[\int_{-\infty}^{+\infty} \frac{1}{|y|} g^q(y) dy \right]^{1/q}.$$

- If we take $\tau = 1$, using the following trigonometric formula: $\sin^2(a) = [1 - \cos(2a)]/2$, $a \in \mathbb{R}$, and $1/p + 1/q = 1$, we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\{\sin^2[\alpha(\beta - x)/2]\}^{1/q} \{\sin^2[\alpha(\beta - y)/2]\}^{1/p}}{x^2 + y^2} f(x)g(y) dx dy \\ & \leq \frac{\pi}{2} \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} [1 - \cos(\alpha\beta)e^{-\alpha|x|}] f^p(x) dx \right\}^{1/p} \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} \frac{1}{|y|} [1 - \cos(\alpha\beta)e^{-\alpha|y|}] g^q(y) dy \right\}^{1/q}. \end{aligned}$$

- If we take $\tau = -1$, using the following trigonometric formula: $\cos^2(a) = [1 + \cos(2a)]/2$, $a \in \mathbb{R}$, and $1/p + 1/q = 1$, we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\{\cos^2[\alpha(\beta-x)/2]\}^{1/q} \{\cos^2[\alpha(\beta-y)/2]\}^{1/p}}{x^2+y^2} f(x)g(y)dx dy \\ & \leq \frac{\pi}{2} \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 + \cos(\alpha\beta)e^{-\alpha|x|} \right] f^p(x)dx \right\}^{1/p} \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} \frac{1}{|y|} \left[1 + \cos(\alpha\beta)e^{-\alpha|y|} \right] g^q(y)dy \right\}^{1/q}. \end{aligned}$$

The proposition below is a non-trivial consequence of Theorem 3.1. It has the distinctive feature of involving only a single function.

Proposition 3.2. *Let $p > 1$, $q = p/(p-1)$, $\alpha, \beta > 0$, $\tau \in [-1, 1]$ and $f : \mathbb{R} \rightarrow (0, +\infty)$ be a function. Then we have*

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta)e^{-\alpha|y|} \right] \right\}^{-(p-1)} \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx \right\}^p dy \\ & \leq \pi^p \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \tau \cos(\alpha\beta)e^{-\alpha|x|} \right] f^p(x)dx \right\}, \end{aligned}$$

provided that the integral on the right-hand side converges.

Proof. To ease the developments, let us set

$$\begin{aligned} \mathfrak{A} &= \int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta)e^{-\alpha|y|} \right] \right\}^{-(p-1)} \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx \right\}^p dy. \end{aligned}$$

Then we can write

$$\begin{aligned} \mathfrak{A} &= \int_{-\infty}^{+\infty} \left\{ \left\{ \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta)e^{-\alpha|y|} \right] \right\}^{-1} \right. \\ & \quad \times \left. \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx \right\}^{p-1} \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)dx \right\} dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p}}{x^2+y^2} f(x)g_{\circ}(y)dx dy, \end{aligned} \quad (3.4)$$

with

$$g_{\circ}(y) = \left\{ \left\{ \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] \right\}^{-1} \times \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p} f(x) dx}{x^2+y^2} \right\}^{p-1}.$$

Applying Theorem 3.1 to f and g_{\circ} , we get

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p} f(x) g_{\circ}(y) dx dy}{x^2+y^2} \\ & \leq \pi \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|x|} \right] f^p(x) dx \right\}^{1/p} \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] g_{\circ}^q(y) dy \right\}^{1/q}. \end{aligned} \tag{3.5}$$

Let us now examine this last integral. Using $1/p + 1/q = 1$ and the expression of \mathfrak{A} , we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] g_{\circ}^q(y) dy \\ & = \int_{-\infty}^{+\infty} \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] \left\{ \left\{ \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] \right\}^{-1} \right. \\ & \quad \times \left. \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p} f(x) dx}{x^2+y^2} \right\}^{q(p-1)} dy \\ & = \int_{-\infty}^{+\infty} \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] \left\{ \left\{ \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] \right\}^{-1} \right. \\ & \quad \times \left. \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p} f(x) dx}{x^2+y^2} \right\}^p dy \\ & = \int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] \right\}^{-(p-1)} \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} \frac{\{1-\tau \cos[\alpha(\beta-x)]\}^{1/q} \{1-\tau \cos[\alpha(\beta-y)]\}^{1/p} f(x) dx}{x^2+y^2} \right\}^p dy \\ & = \mathfrak{A}. \end{aligned} \tag{3.6}$$

It follows from Equations (3.4), (3.5) and (3.6) that

$$\mathfrak{A} \leq \pi \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|x|} \right] f^p(x) dx \right\}^{1/p} \mathfrak{A}^{1/q}.$$

Using $1/p + 1/q = 1$, we obtain

$$\mathfrak{A}^{1/p} \leq \pi \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|x|} \right] f^p(x) dx \right\}^{1/p},$$

so that

$$\mathfrak{A} \leq \pi^p \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|x|} \right] f^p(x) dx.$$

This can be written as

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|y|} \right] \right\}^{-(p-1)} \\ & \times \left\{ \int_{-\infty}^{+\infty} \frac{\{1 - \tau \cos[\alpha(\beta-x)]\}^{1/q} \{1 - \tau \cos[\alpha(\beta-y)]\}^{1/p}}{x^2 + y^2} f(x) dx \right\}^p dy \\ & \leq \pi^p \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \tau \cos(\alpha\beta) e^{-\alpha|x|} \right] f^p(x) dx \right\}. \end{aligned}$$

The proof of the proposition ends. \square

This proposition can be typically used in the study of specific ratio-cosine integral operators. Some particular examples are given below.

- If we take $\tau = 0$, we get

$$\int_{-\infty}^{+\infty} |y|^{p-1} \left[\int_{-\infty}^{+\infty} \frac{1}{x^2 + y^2} f(x) dx \right]^p dy \leq \pi^p \left[\int_{-\infty}^{+\infty} \frac{1}{|x|} f^p(x) dx \right].$$

- If we take $\tau = 1$, using the following trigonometric formula: $\sin^2(a) = [1 - \cos(2a)]/2$, $a \in \mathbb{R}$, and $1/p + 1/q = 1$, we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1 - \cos(\alpha\beta) e^{-\alpha|y|} \right] \right\}^{-(p-1)} \\ & \times \left\{ \int_{-\infty}^{+\infty} \frac{\{\sin^2[\alpha(\beta-x)/2]\}^{1/q} \{\sin^2[\alpha(\beta-y)/2]\}^{1/p}}{x^2 + y^2} f(x) dx \right\}^p dy \\ & \leq \left(\frac{\pi}{2}\right)^p \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 - \cos(\alpha\beta) e^{-\alpha|x|} \right] f^p(x) dx \right\}. \end{aligned}$$

- If we take $\tau = -1$, using the following trigonometric formula: $\cos^2(a) = [1 + \cos(2a)]/2$, $a \in \mathbb{R}$, and $1/p + 1/q = 1$, we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left\{ \frac{1}{|y|} \left[1 + \cos(\alpha\beta) e^{-\alpha|y|} \right] \right\}^{-(p-1)} \\ & \times \left\{ \int_{-\infty}^{+\infty} \frac{\{\cos^2[\alpha(\beta-x)/2]\}^{1/q} \{\cos^2[\alpha(\beta-y)/2]\}^{1/p}}{x^2 + y^2} f(x) dx \right\}^p dy \\ & \leq \left(\frac{\pi}{2}\right)^p \left\{ \int_{-\infty}^{+\infty} \frac{1}{|x|} \left[1 + \cos(\alpha\beta) e^{-\alpha|x|} \right] f^p(x) dx \right\}. \end{aligned}$$

4. Conclusion

In conclusion, we have derived a new Hardy-Hilbert-type integral inequality involving a ratio-cosine kernel function in the entire plane, alongside a related inequality that depends on only one function. The presence of three independent parameters makes our results notably flexible and distinguishes them from existing inequalities in the literature. One limitation of our study is that we do not prove that π is the best possible constant factor, which remains an open problem. Proposed areas for future research include exploring multidimensional analogues, extending the method to other classes of oscillatory kernel functions, and investigating potential applications in harmonic analysis and mathematical physics.

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Sharp inverse logarithmic coefficient bounds for starlike functions associated with cosine function

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Abstract. Let \mathcal{S}_{cos}^* be the subclass of starlike functions f associated with cosine function defined by $(zf'(z)/f(z)) \prec \cos(z)$. In this paper, we obtain the sharp coefficient bounds and Hankel determinants of second order for the inverse logarithmic function for this class. We also present the best possible bounds of second order Toeplitz determinant for the functions in the same class.

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1. Introduction and preliminaries

Let f be an analytic function having Maclaurin series expansion in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$


Then f is known to be member of the class \mathcal{A} . The family of all analytic functions adhere to the conditions $f(0) = 0$ and $f'(0) = 1$ in \mathbb{D} . Denote by \mathcal{S} , a subclass of \mathcal{A} which contains all univalent (one-one) functions in \mathbb{D} .

The family of analytic functions $p(z)$ in \mathbb{D} that meet the criteria, $Re(p(z)) > 0$ is known as Caratheodory class \mathcal{P} and it's power series expansion is given as

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \quad (1.2)$$

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A domain $D \subseteq \mathbb{C}$ is said to be starlike with respect to a point $z_0 \in \mathbb{C}$ if and only if for every $z \in D$, the line segment joining z and z_0 lies entirely in D . An analytic function f is called starlike if $f(\mathbb{D})$ with respect to the origin is a starlike domain and the class of all univalent starlike functions is denoted by \mathcal{S}^* . This subclass is defined analytically as

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \right\}, \quad z \in \mathbb{D}.$$

Suppose that \mathcal{B} denote a class of analytic functions w in \mathbb{D} which satisfy $w(0) = 0$ and $|w(z)| < 1$. The functions in class \mathcal{B} are known as Schwarz functions. Now, assume two analytic functions g_1 and g_2 in \mathbb{D} , the function g_1 is subordinate to g_2 (i.e. $g_1 \prec g_2$), if there exists a function $w(z) \in \mathcal{B}$ such that $g_1(z) = g_2(w(z))$, for $z \in \mathbb{D}$. Furthermore, if g_2 is univalent and $g_1(0) = g_2(0)$, then $g_1 \prec g_2$ is equivalent to the condition $g_1(\mathbb{D}) \subset g_2(\mathbb{D})$. The class \mathcal{S}_{cos}^* defined by

$$\mathcal{S}_{cos}^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \cos(z) \right\},$$

was thoroughly investigated in [1] by Ali et al. by presenting the sharp coefficient bounds and Hankel determinants of second and third order. They also obtained the same results for logarithmic coefficients but investigating these results for inverse logarithmic function is an open problem.

The inverse f^{-1} of a univalent function f defined on some disk $|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$ possesses a series expansion given as

$$f^{-1}(z) = z + \sum_{n=2}^{\infty} A_n z^n,$$

For a function $f \in \mathcal{S}$, the logarithmic coefficients γ_n are defined as

$$F_f(z) = \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

The series expansion of inverse logarithmic coefficients Γ_n is given by

$$F_{f^{-1}}(z) = \log \left(\frac{f^{-1}(z)}{z} \right) = 2 \sum_{n=1}^{\infty} \Gamma_n z^n. \quad (1.3)$$

By using (1.1) and (1.3), we have

$$\Gamma_1 = -\frac{1}{2}a_2, \quad \Gamma_2 = \frac{3}{4}a_2^2 - \frac{1}{2}a_3, \quad (1.4)$$

$$\Gamma_3 = -\frac{5}{3}a_2^3 + 2a_2a_3 - \frac{1}{2}a_4, \quad (1.5)$$

$$\Gamma_4 = \frac{35}{8}a_2^4 - \frac{15}{2}a_2^2a_3 + \frac{5}{2}a_2a_4 + \frac{5}{4}a_3^2 - \frac{1}{2}a_5. \quad (1.6)$$

In [10], the concept of Hankel determinant was introduced by Pommerenke as a technique for investigating analytic functions and their subclasses in geometric function

theory. It is defined as

$$H_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix}$$

where $m, n \geq 1$. We see that we can obtain different Hankel determinants by changing the values of m and n . We note that

$$H_{2,1}(f) = a_1a_3 - a_2^2, \quad H_{2,2}(f) = a_2a_4 - a_3^2. \tag{1.7}$$

$H_{2,1}(f)$ demonstrates the Fekete-Szegő functional. Riaz et al. [12] investigated the bounds of Hankel determinants for starlike and convex functions associated with sigmoid functions. In [9], M. Obradović and N. Tunesk calculated the upper bounds of second and third orders Hankel determinants for the class \mathcal{S} of univalent functions. Allu et al. [3], studied the upper bound of second order Hankel determinant for logarithmic coefficients of univalent functions and In 2020, Arif et al. [4], investigated the fifth Hankel determinant for the class of bounded turning function. Similarly, many other authors studied Hankel determinant for different subclasses of analytic functions (see [1, 11]).

The symmetric Toeplitz determinant $T_m(n)(f)$ was introduced by Ali et al. [2] is defined as

$$T_m(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_n \end{vmatrix}$$

$$T_2(1)(f) = a_1^2 - a_2^2, \quad T_2(2)(f) = a_2^2 - a_3^2. \tag{1.8}$$

In [2], Ali et al. examined the bounds of $|T_2(n)(f)|, |T_3(1)(f)|$ and $|T_3(2)(f)|$ within the classes \mathcal{S}^* and \mathcal{S}^c . Cudna et al. [6] calculated sharp upper and lower estimates for $|T_2(1)(f)|$ and $|T_3(1)(f)|$ concerning the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^c(\alpha)$, where $0 \leq \alpha < 1$. Toeplitz determinant for initial values of m and n are estimated for starlike and close-to-convex functions in [2].

In this paper, we estimate the bounds of inverse logarithmic coefficients $|\Gamma_2|, |\Gamma_3|$ and $|\Gamma_4|$ for the class \mathcal{S}_{cos}^* . We also present the upper bounds for Hankel determinants $|H_{2,1}(f^{-1}/2)|$ and $|H_{2,2}(f^{-1}/2)|$ as well as symmetric Toeplitz determinant $|T_2(1)(f^{-1}/2)|$ and $|T_2(2)(f^{-1}/2)|$ for the same class.

The following lemmas will be used to obtain our main results. The first lemma is the well known Carathéodory lemma.

Lemma 1.1. [7] *Let $p \in \mathcal{P}$ be given in the form (1.2). Then*

$$|c_n| \leq 2, \quad n \geq 1.$$

The inequality holds for all $n \geq 1$ if and only if $p(z) = \frac{1 + \lambda z}{1 - \lambda z}, |\lambda| = 1$.

Lemma 1.2. [5] Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk, and $p \in \mathcal{P}$ be given by (1.2). Then,

$$\begin{aligned} c_1 &= 2\zeta_1, \\ c_2 &= 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2, \\ c_3 &= 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3, \\ c_4 &= 2\zeta_1^4 + 2(1 - |\zeta_1|^2)(3\zeta_1^2 + \overline{\zeta_1}^2\zeta_2^2 - 3|\zeta_1|^2\zeta_2 + \zeta_2)\zeta_2 \\ &\quad + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)(2\zeta_1 - 2\overline{\zeta_1}\zeta_2 - \overline{\zeta_2}\zeta_3)\zeta_3 \\ &\quad + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)(1 - |\zeta_3|^2)\zeta_4, \end{aligned}$$

for some $\zeta_i \in \overline{\mathbb{D}}$ and $i \in \{1, 2, 3, 4\}$.

The following Lemma presents the initial part of the result from [8, Remark p. 162]

Lemma 1.3. [8] Let $p \in \mathcal{P}$ be given in the form (1.2). Then

$$|c_2 - vc_1^2| \leq 2 - v|c_1|^2, \quad 0 < v \leq \frac{1}{2}.$$

2. Main results

Let $f \in \mathcal{S}_{\cos}^*$, so by using the relations (1.1) and (1.2), we get

$$a_2 = 0, \quad a_3 = -\frac{1}{16}c_1^2, \quad a_4 = \frac{1}{24}c_1^3 - \frac{1}{12}c_1c_2, \quad (2.1)$$

$$a_5 = -\frac{1}{48}c_1^4 - \frac{1}{32}c_2^2 + \frac{3}{32}c_1^2c_2 - \frac{1}{16}c_1c_3. \quad (2.2)$$

In our first theorem, we calculate the sharp bounds of inverse logarithmic coefficients for functions in the class \mathcal{S}_{\cos}^* .

Theorem 2.1. Consider $f \in \mathcal{S}_{\cos}^*$ is of the form (1.1). Then

$$|\Gamma_1| = 0, \quad |\Gamma_2| \leq \frac{1}{8}, \quad |\Gamma_3| \leq \frac{\sqrt{3}}{27}, \quad |\Gamma_4| \leq \frac{1}{16}.$$

These inequalities are sharp.

Proof. Using values of (2.1) and (2.2) in (2.3)-(2.5), we deduce that

$$\Gamma_1 = 0, \quad \Gamma_2 = \frac{1}{32}c_1^2, \quad (2.3)$$

$$\Gamma_3 = \frac{1}{24}c_1c_2 - \frac{1}{48}c_1^3, \quad (2.4)$$

$$\Gamma_4 = \frac{47}{3072}c_1^4 + \frac{1}{64}c_2^2 - \frac{3}{64}c_1^2c_2 + \frac{1}{32}c_1c_3. \quad (2.5)$$

A₁: From the second equality of (2.3), it follows that

$$|\Gamma_2| \leq \frac{1}{32}|c_1|^2.$$

Denote $c := |c_1|$, the above inequality becomes

$$|\Gamma_2| \leq \frac{1}{32}c^2 := F(c).$$

We can see that $F(c)$ attains maximum value at $c = 2$, then

$$|\Gamma_2| \leq \frac{1}{8}.$$

This inequality is sharp for the function f_1 given by

$$f_1(z) := z \exp\left(\int_0^z \frac{\cos(x) - 1}{x} dx\right) = z - \frac{1}{4}z^3 + \frac{1}{24}z^5 + \dots \quad (2.6)$$

Here, $a_2 = 0$ and $a_3 = -\frac{1}{4}$. Thus from (1.4), we have $|\Gamma_2| = \frac{1}{8}$.

A₂: Now, (2.4) can be expressed as

$$\Gamma_3 = \frac{1}{24}c_1 \left(c_2 - \frac{c_1^2}{2}\right),$$

by applying Lemma 1.3, we obtain

$$|\Gamma_3| \leq \frac{1}{12}|c_1| - \frac{1}{48}|c_1|^3.$$

Set $c := |c_1|$, for $c \in [0, 2]$, we have

$$|\Gamma_3| \leq \frac{1}{12}c - \frac{1}{48}c^3 =: L(c).$$

We can see that $L'(c)$ has zero $c_0 = \frac{2\sqrt{3}}{3}$ that lies in $(0, 2)$, satisfying $L''(c_0) < 0$, thus

$$|\Gamma_3| \leq L(c_0) = \frac{\sqrt{3}}{27}.$$

To demonstrate sharpness, consider $t_1 = \frac{2}{3}\sqrt{3}$ and $p_1(z) = \frac{1+2t_1z+z^2}{1-z^2}$ and

$$w_1(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{(\sqrt{3} + 3z)z}{z\sqrt{3} + 3}.$$

Clearly, we can see that $w_1(0) = 0$ and $|w_1(z)| < 1$. Thus, the function

$$f_2(z) = z \exp\left(\int_0^z \frac{\cos(w_1(x)) - 1}{x} dx\right) = z - \frac{1}{12}z^3 - \frac{2}{27}\sqrt{3}z^4 + \dots \quad (2.7)$$

belongs to the class \mathcal{S}_{cos}^* with $a_2 = 0$, $a_3 = -\frac{1}{12}$ and $a_4 = -\frac{2}{27}\sqrt{3}$. Therefore from (1.5), we have $|\Gamma_3| = \frac{\sqrt{3}}{27}$.

A₃: In the end, we will find the bound of $|\Gamma_4|$ by solving (2.5), we get

$$\Gamma_4 = \frac{47}{3072}c_1^4 + \frac{1}{64}c_2^2 - \frac{3}{64}c_2c_1^2 + \frac{1}{32}c_1c_3,$$

and by utilizing Lemma 1.2, we have

$$\Gamma_4 = \frac{11}{192}\zeta_1^4 + \frac{1}{16}(1 - |\zeta_1|^2)^2\zeta_2^2 - \frac{1}{8}(1 - |\zeta_1|^2)\zeta_2^2|\zeta_1|^2 + \frac{1}{8}(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_1\zeta_3.$$

Applying the triangle inequality and since $\zeta_i \in \mathbb{D}$, which implies that $|\zeta_3| \leq 1$, we obtain the following inequality

$$|\Gamma_4| \leq \frac{11}{192} |\zeta_1|^4 + \frac{1}{16} (1 - |\zeta_1|^2)^2 |\zeta_2|^2 + \frac{1}{8} (1 - |\zeta_1|^2) |\zeta_2|^2 |\zeta_1|^2 + \frac{1}{8} (1 - |\zeta_1|^2) (1 - |\zeta_2|^2) |\zeta_1|.$$

Indicating $x := |\zeta_1|$ and $y := |\zeta_2|$, where $x, y \in (0, 1)$, we obtain

$$|\Gamma_4| \leq \frac{11}{192} x^4 + \frac{1}{16} (1 - x^2)^2 y^2 + \frac{1}{8} (1 - x^2) x^2 y^2 + \frac{1}{8} (1 - x^2) (1 - y^2) x := Q(x, y). \quad (2.8)$$

Denoting $\Theta = [0, 1] \times [0, 1]$ is the closed unit square, now our aim is to find the maximum value of $Q(x, y)$ in Θ .

I. Firstly, we will evaluate the highest value of $Q(x, y)$ on the four edges of Θ .

(i) On edge $x = 0$, $Q(x, y)$ deduce to

$$q_1(y) := Q(0, y) = \frac{y^2}{16} \leq \frac{1}{16}, \quad \text{for } y \in (0, 1).$$

(ii) At edge $x = 1$, we get

$$Q(1, y) = \frac{11}{192}.$$

(iii) At the edge $y = 0$, $Q(x, y)$ becomes

$$q_2(x) := Q(x, 0) = \frac{11}{192} x^4 + \frac{x}{8} (1 - x^2), \quad x \in (0, 1).$$

It is easy to check that the zero of q_2' is $x_0 = 0.8148 \in (0, 1)$, satisfying $q_2''(x_0) < 0$. Thus

$$Q(x, 0) \leq Q(x_0, 0) = 0.0595.$$

(iv) On edge $y = 1$, we obtain

$$q_3(x) := Q(x, 1) = \frac{47}{192} x^4 + \frac{1}{16} - \frac{1}{4} x^2, \quad x \in (0, 1).$$

We can see that $q_3' = 0$ yields $x = \frac{2\sqrt{282}}{47}$ but $q_3''(\frac{2\sqrt{282}}{47}) = 1 > 0$. Therefore, q_3 has no maximum value in $(0, 1)$.

II. Now, we will maximize $Q(x, y)$ on the vertices of $\Theta := [0, 1] \times [0, 1]$.

$$Q(0, 0) = 0, \quad Q(0, 1) = \frac{1}{16}, \quad Q(1, 0) = Q(1, 1) = \frac{11}{192}.$$

We conclude that

$$\max\{Q(x, y) : (x, y) \in \Theta\} = Q(0, y) = Q(0, 1) = \frac{1}{16}.$$

Hence, (2.8) becomes

$$|\Gamma_4| \leq \frac{1}{16}.$$

This inequality is sharp for the function f_3 given by

$$f_3(z) = z \exp\left(\int_0^z \frac{\cos(x^2) - 1}{x} dx\right) = z - \frac{1}{8}z^5 + \dots$$

Here, $a_2 = a_3 = a_4 = 0$ and $a_5 = -\frac{1}{8}$. Thus, from (1.6), we have $|\Gamma_4| = \frac{1}{16}$. □

Next, we calculate the sharp upper bound on $|H_{2,1}(f^{-1}/2)|$.

Theorem 2.2. *Consider $f \in \mathcal{S}_{\cos}^*$ is given by the form (1.1). Then*

$$|H_{2,1}(f^{-1}/2)| \leq \frac{1}{64}.$$

This inequality is sharp.

Proof. From (2.3), (2.4) and (1.7), we get

$$H_{2,1}(f^{-1}/2) = -\frac{1}{1024}c_1^4$$

It implies

$$|H_{2,1}(f^{-1}/2)| \leq \frac{1}{1024}|c_1|^4.$$

Since $|c_1| \leq 2$, thus

$$|H_{2,1}(f^{-1}/2)| \leq \frac{1}{64}.$$

This inequality is sharp for the function f_1 given by (2.6). □

In next theorem, we find the sharp bound for $|H_{2,2}(f^{-1}/2)|$.

Theorem 2.3. *Consider $f \in \mathcal{S}_{\cos}^*$ is of the form (1.1). Then*

$$|H_{2,2}(f^{-1}/2)| \leq \frac{11}{1536}.$$

The inequality is sharp.

Proof. By utilizing (1.7), (2.3), (2.4) and (2.5), we obtain

$$H_{2,2}(f^{-1}/2) = \frac{13}{294912}c_1^6 - \frac{23}{18432}c_1^2c_2^2 + \frac{5}{18432}c_1^4c_2 + \frac{1}{1024}c_1^3c_3,$$

by applying Lemma (1.2), the above equality becomes

$$\begin{aligned} H_{2,2}(f^{-1}/2) &= \frac{11}{1536}\zeta_1^6 - \frac{23}{1152}(1 - |\zeta_1|^2)^2\zeta_1^2\zeta_2^2 - \frac{1}{64}(1 - |\zeta_1|^2)\zeta_1^2\zeta_2^2|\zeta_1|^2 \\ &\quad + \frac{1}{64}(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_1^3\zeta_3. \end{aligned}$$

Using triangular inequality and $|\zeta_3| \leq 1$ because $\zeta_i \in \mathbb{D}$, we have

$$\begin{aligned} |H_{2,2}(f^{-1}/2)| &\leq \frac{11}{1536}|\zeta_1|^6 + \frac{23}{1152}(1 - |\zeta_1|^2)^2|\zeta_1|^2|\zeta_2|^2 + \frac{1}{64}(1 - |\zeta_1|^2)|\zeta_1|^4|\zeta_2|^2 \\ &\quad + \frac{1}{64}(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)|\zeta_1|^3. \end{aligned}$$

Indicating $x := |\zeta_1|$ and $y := |\zeta_2|$, where $x, y \in [0, 1]$, we obtain

$$|H_{2,2}(f^{-1}/2)| \leq \frac{11}{1536}x^6 + \frac{23}{1152}(1-x^2)^2x^2y^2 + \frac{1}{64}(1-x^2)x^4y^2 + \frac{1}{64}(1-x^2)(1-y^2)x^3 = L(x, y). \quad (2.9)$$

Denoting $\Theta := [0, 1] \times [0, 1]$ be the closed unit square, we now aim to maximize $L(x, y)$ on the edges and vertices of Θ .

I. Firstly, we will evaluate the highest value of $L(x, y)$ on the four edges of Θ .

(i) On the edge $x = 0$, we get

$$L(0, y) = 0.$$

(ii) At edge $x = 1$, we get

$$L(1, y) = \frac{11}{1536}.$$

(iii) At the edge $y = 0$, $L(x, y)$ becomes

$$r_1(x) := L(x, 0) = \frac{11}{1536}x^6 - \frac{1}{64}x^5 + \frac{1}{64}x^3, \quad x \in (0, 1).$$

We note that $r_1' > 0$, it follows that r_1 is an increasing function on $(0, 1)$, so

$$L(x, 0) \leq \frac{11}{1536}.$$

(iv) On edge $y = 1$, we obtain

$$r_2(x) := L(x, 1) = \frac{53}{4608}x^6 - \frac{7}{288}x^4 + \frac{23}{1152}x^2, \quad x \in (0, 1).$$

Since, $r_2' > 0$, it follows that r_2 is an increasing function on $(0, 1)$, then

$$L(x, 1) \leq \frac{11}{1536}.$$

II. Lastly, we will maximize $L(x, y)$ on the vertices of $\Theta := [0, 1] \times [0, 1]$.

$$L(0, 0) = L(0, 1) = 0, \quad L(1, 0) = L(1, 1) = \frac{11}{1536}.$$

We conclude that

$$\max\{L(x, y) : (x, y) \in \Theta\} = L(x, 0) = L(x, 1) = L(1, y) = L(1, 0) = L(1, 1) = \frac{11}{1536}.$$

Hence, becomes

$$|H_{2,2}(f^{-1}/2)| \leq \frac{11}{1536}.$$

The equality exists for the function f_1 given by (2.6). Here, $a_2 = a_4 = 0, a_3 = -\frac{1}{4}$ and $a_5 = \frac{1}{24}$. Therefore, after some simplifications we get $|H_{2,2}(f^{-1}/2)| = \frac{11}{1536}$. \square

The sharp bound of $|T_2(1)(f^{-1}/2)| := |T_2(1)|$ is presented in this theorem.

Theorem 2.4. Consider $f \in \mathcal{S}_{cos}^*$ is given by the form (1.1). Then

$$|T_2(1)| \leq \frac{1}{64}.$$

The inequality is sharp.

Proof. By using (2.3) in (1.8), we deduce that

$$T_2(1) = -\frac{1}{1024}c_1^4,$$

applying modulus, the above expression reduces to

$$|T_2(1)| \leq \frac{1}{1024}|c_1|^4 \leq \frac{1}{64},$$

since $c_1 \in [0, 2]$.

This inequality is sharp for the function f_1 given by (2.6). □

In our last theorem, we calculate the sharp bound on $|T_2(2)(f^{-1}/2)| := |T_2(2)|$.

Theorem 2.5. Consider $f \in \mathcal{S}_{cos}^*$ is of the form (1.1). Then

$$|T_2(2)| \leq \frac{1}{64}.$$

The inequality is sharp.

Proof. By utilizing (2.3) in (1.8), we obtain

$$T_2(2) = \frac{c_1^4}{1024} - \left(\frac{c_1}{24} \left(c_2 - \frac{c_1^2}{2} \right) \right)^2,$$

by applying Lemma 1.3, the above equality becomes

$$|T_2(2)| \leq \frac{|c_1|^4}{1024} + \left(\frac{|c_1|}{12} - \frac{|c_1|^3}{48} \right)^2.$$

Denoting $c := |c_1|$, we have $c \in [0, 2]$

$$|T_2(2)| \leq \frac{c^4}{1024} + \left(\frac{c}{12} - \frac{c^3}{48} \right)^2 =: S(c), \quad c \in [0, 2].$$

We can observe that $S'(c) > 0$. Thus $S(c)$ is an increasing function and has maximum value at $c = 2$, it implies

$$|T_2(2)| \leq S(2) = \frac{1}{64}.$$

The function f_1 defined by (2.6) gives equality for this bound. □

3. Conclusion

In this article by applying the Cho et. al. [5] results, we presented the coefficient bounds of initial inverse logarithmic coefficients for the starlike functions in the class \mathcal{S}_{cos}^* . Additionally, we determined the bounds of second order Hankel and Toeplitz determinants for the same class. The bounds presented here are sharp and the extremel function for each bound is also given.

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Global existence, asymptotic behavior, and blow-up for a parabolic p -Laplacian type equation with complex interactions at the boundary

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Abstract. In this paper, we study the initial boundary value problem involving the p -Laplacian parabolic equation $u_t - \Delta_p u + \alpha|u|^{p-2}u = 0$, $(x, t) \in \Omega \times]0, +\infty[$, with logarithmic boundary condition. By using the potential wells method combined with the Nehari Manifold, we establish the existence of a weak global solution. In addition, we also obtain the decay polynomial of the weak solution. Then, by virtue of the differential inequality technique, we prove that the solutions blow up in finite time under suitable initial values.

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1. Introduction and main results

In this paper, we consider the initial-boundary value problem of the following parabolic p -Laplacian equation with a logarithmic non-linear term on the boundary as follows:


$$\begin{cases} u_t - \Delta_p u + \alpha|u|^{p-2}u = 0, & (x, t) \in \Omega \times]0, +\infty[, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \beta|u|^{q-2}u \log(|u|), & (x, t) \in \partial\Omega \times [0, +\infty[, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where Ω is an open bounded domain in \mathbb{R}^N ($N \geq 1$) with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\alpha, \beta > 0$ and p, q satisfies

(H1) : $1 < p < q < p_\partial$,

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where

$$p_{\partial} := \begin{cases} \frac{(N-1)p}{N-p}, & \text{if } p < N, \\ +\infty, & p \geq N. \end{cases}$$

Logarithmic nonlinearities have been widely studied in physics, arising in areas such as supersymmetric field theories, inflationary cosmology, nuclear physics, optics, and quantum mechanics. Parabolic p -Laplacian equations also play a significant role in various fields, such as fluid dynamics and climatology. For more applications involving the parabolic equations in fluids, climate modeling, and climatology, we can refer to [3, 2, 6, 7, 4, 5] and references therein. Furthermore, equation (1.1) models a wide range of phenomena in nature and industry, such as non-Newtonian fluids and diffusive processes that exhibit spatial heterogeneity, as found in porous media and biological systems. Similar models have been discussed in population dynamics, particularly in the work of Murray [12].

The term $\alpha|u|^{p-2}u$ represents reaction-diffusion systems commonly found in chemical reactions and heat propagation models, which are crucial for understanding phenomena like combustion and autocatalytic reactions, as shown in classical works by Vázquez and Lions [15, 11]. The term $\beta|u|^{q-2}u \log(|u|)$, accounting for logarithmic interactions at the boundary, adds a layer of complexity that models surface phenomena such as adsorption and desorption processes, which are relevant to fields like thermodynamics and chemical surface reactions, as discussed by Amann in [1].

Many works have explored similar logarithmic terms in different contexts, such as, Peng and Zhou [13] studied the following heat equation with logarithmic nonlinearity

$$u_t - \Delta u_t = |u|^{p-2}u \log(|u|),$$

under the Dirichlet boundary condition. They obtained the global existence and blow-up of solutions. Also, they discussed the upper bound of blow-up time under suitable conditions.

Nhan and Truong [14] studied the following Dirichlet nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - \Delta_p u = |u|^{p-2}u \log(|u|).$$

They obtained results regarding the existence or non-existence of global solutions. Also, He et al. [8] proved the decay and the finite time blow-up for weak solutions of the equation.

Xiao and Li [17] considered the initial boundary value problem for nonlinear higher-order heat equations of

$$u_t + (-\Delta)^m u_t + (-\Delta)^m u = f(u).$$

They established the existence of a weak solution to the static problem by using the potential well technique.

Later Lamaizi et al. [10] studied the following heat equation involving the Laplacian

$$u_t - \Delta u + u = 0,$$

under a nonlinear boundary condition

$$\frac{\partial u}{\partial \nu} = \lambda |u|^{p-1} u.$$

They prove the global existence and blow-up of solutions. Also, they discussed the upper bound of blow-up time under suitable conditions.

Motivated by the above works, in the present manuscript, using the potential well theory combined with the Nehari manifold, we discuss the global existence and finite time blow-up for the solutions to the problem (1.1) under suitable conditions on the initial data. Meanwhile, we find that the global solution decays polynomially.

The rest of the paper is organized as follows. In Sect. 2, we recall some preliminary results that will be used throughout the paper. In Sect. 3, we introduce a family of potential wells and the invariance of some sets are established. Finally, in Sect. 4 we presents our main results and its proof.

2. Preliminaries

For $p \in [1, +\infty)$, we denote the L^p -norm of a function $f \in L^p(\Omega)$ by $\|f\|_p = (\int_{\Omega} |f|^p dx)^{\frac{1}{p}}$ and the Lebesgue norm of $L^p(\partial\Omega)$ by $\|g\|_{p,\partial} = (\int_{\partial\Omega} |g|^p dx)^{\frac{1}{p}}$ for $g \in L^p(\partial\Omega)$.

The usual Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{v \in L^p(\Omega) : |\nabla v| \in L^p(\Omega)\},$$

equipped by the norm

$$\|\cdot\|_{W^{1,p}(\Omega)} = (\|\cdot\|_p^p + \|\nabla(\cdot)\|_p^p)^{\frac{1}{p}}.$$

Proposition 2.1. (See [16]) *The trace operator $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is continuous if and only if $1 \leq q \leq p_{\partial}$ if $p \neq N$ and for $1 \leq q < \infty$ if $p = N$. Where*

$$p_{\partial} = \begin{cases} \frac{(N-1)p}{N-p}, & \text{if } 1 < p < N, \\ +\infty, & p \geq N. \end{cases}$$

Lemma 2.2. (See [9]) *Suppose that a positive, twice-differentiable function ψ satisfies the inequality*

$$\ddot{\psi}(t)\psi(t) - (\mu + 1)[\dot{\psi}(t)]^2 > 0,$$

for some $\mu > 0$. Then there exists $T^ > 0$ such that $\lim_{t \rightarrow T^*} \psi(t) = +\infty$.*

Lemma 2.3. (See [14]) *For any positive number $s > 0$, we have the following inequality*

$$\log(x) \leq \frac{x^s}{se}, \quad \forall x \geq 1.$$

3. Potential wells

In this section, under the hypothesis (H1), we introduce a family of potential wells for the problem (1.1) and give a series of their properties.

For clarity and simplicity, we will adopt the following notation throughout this paper:

$$u := u(t) := u(x, t),$$

with T representing the maximal existence time of u .

We now introduce two important functionals:

$$J(u) := \frac{1}{p} \|u\|_\alpha^p - \frac{\beta}{q} \oint_{\partial\Omega} |u|^q \log(|u|) + \frac{\beta}{q^2} \|u\|_{q,\partial}^q \quad (3.1)$$

$$I(u) := \|u\|_\alpha^p - \beta \oint_{\partial\Omega} |u|^q \log(|u|) dx, \quad (3.2)$$

where $\|u\|_\alpha := (\|\nabla u\|_p^p + \alpha \|u\|_p^p)^{\frac{1}{p}}$, a norm that is equivalent to the standard Sobolev norm $\|u\|_{W^{1,p}(\Omega)}$.

Additionally, we observe that:

$$J(u) = \left(\frac{1}{p} - \frac{1}{q} \right) \|u\|_\alpha^p + \frac{1}{q} I(u) + \frac{\beta}{q^2} \|u\|_{q,\partial}^q. \quad (3.3)$$

The Nehari manifold is defined by

$$\mathcal{N} := \{u \in W^{1,p} : I(u) = 0, \|u\|_\alpha > 0\},$$

and the mountain pass level d , which represents the depth of the potential well, is given by:

$$d := \inf_{\lambda > 0} \{ \sup J(\lambda u) : u \in W^{1,p}(\Omega), \|u\|_\alpha > 0 \}. \quad (3.4)$$

Moreover, we define the potential well by

$$W := \{u \in W^{1,p}(\Omega) : I(u) > 0, J(u) < d\} \cup \{0\},$$

and the set

$$V := \{u \in W^{1,p}(\Omega) : I(u) < 0, J(u) < d\}.$$

Lemma 3.1. *For any $u \in W^{1,p}(\Omega)$, $\|u\|_{q,\partial} > 0$, and $\lambda > 0$ there hold*

1. $\lim_{\lambda \rightarrow 0^+} J(\lambda u) = 0$ and $\lim_{\lambda \rightarrow +\infty} J(\lambda u) = -\infty$;
2. *there exists a unique $\lambda_*(u) = \lambda_*$ such that $\frac{d}{d\lambda} J(\lambda_* u) = 0$;*
3. *the function $\lambda \mapsto J(\lambda u)$ is increasing on $[0, \lambda_*]$, decreasing on $[\lambda_*, +\infty)$ and takes the maximum at λ_* ;*
4. *the function $\lambda \mapsto I(\lambda u)$ satisfies $I(\lambda u) > 0$ on $(0, \lambda_*)$, $I(\lambda u) < 0$ on $(\lambda_*, +\infty)$ and $I(\lambda_* u) = 0$.*

Proof. 1. By (3.1) we get

$$J(\lambda u) = \frac{\lambda^p}{p} \|u\|_\alpha^p - \frac{\beta}{q} \lambda^q \log(\lambda) \|u\|_{q,\partial}^q - \frac{\beta}{q} \lambda^q \oint_{\partial\Omega} |u|^q \log(|u|) dx + \frac{\lambda^q}{q^2} \|u\|_{q,\partial}^q.$$

Thus, the first statement holds due to $\|u\|_{q,\partial} > 0$.

2. With a simple calculation, we get the following

$$\frac{d}{d\lambda} J(\lambda u) = \lambda^{p-1} \|u\|_\alpha^p - \beta \lambda^{q-1} \log(\lambda) \|u\|_{q,\partial}^q - \beta \lambda^{q-1} \oint_{\partial\Omega} |u|^q \log(|u|) dx,$$

making the derivative equal zero, we obtain

$$\lambda^{p-1} \|u\|_\alpha^p = \beta \lambda^{q-1} \log(\lambda) \|u\|_{q,\partial}^q + \beta \lambda^{q-1} \oint_{\partial\Omega} |u|^q \log(|u|) dx.$$

For $\lambda > 0$, define

$$K(\lambda) := \lambda^{q-p} \log(\lambda) \|u\|_{q,\partial}^q + \lambda^{q-p} \oint_{\partial\Omega} |u|^q \log(|u|) dx.$$

It is easy to see that there exists a unique $\lambda_0 > \lambda_1$ such that

$$K(\lambda_0) = 0, \quad K'(\lambda_1) = 0,$$

and the function $\lambda \mapsto K(\lambda)$ is increasing on $[\lambda_1, +\infty)$ and $K(\lambda) > 0$, for all $\lambda > \lambda_0$. Then, for any $u \in W^{1,p}(\Omega)$, $\|u\|_{q,\partial} > 0$, and $\beta, \alpha > 0$, there exists a unique $\lambda_\star > \lambda_0$ such that

$$K(\lambda_\star) = \frac{\|u\|_\alpha^p}{\beta}, \tag{3.5}$$

that is,

$$\frac{d}{d\lambda} J(\lambda_\star u) = 0.$$

Notice that

$$\lambda \frac{d}{d\lambda} J(\lambda u) = I(\lambda u) = \lambda^p (\|u\|_\alpha^p - \beta K(\lambda)).$$

Therefore, assertions (3) and (4) are satisfied. □

Remark 3.2. Based on Lemma 3.1, we observe that $d > 0$, and the depth of the potential well d is further defined by

$$d = \inf_{u \in \mathcal{N}} J(u). \tag{3.6}$$

Next, for $\delta > 0$, we define

$$I_\delta(u) := \delta \|u\|_\alpha^p - \beta \oint_{\partial\Omega} |u|^q \log(|u|) dx. \tag{3.7}$$

$$\mathcal{N}_\delta := \{u \in W^{1,p}(\Omega) : I_\delta(u) = 0, \|u\|_\alpha > 0\},$$

$$d(\delta) := \inf_{u \in \mathcal{N}_\delta} J(u) \tag{3.8}$$

$$W_\delta := \{u \in W^{1,p}(\Omega) : I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\},$$

and

$$V_\delta := \{u \in W^{1,p}(\Omega) : I_\delta(u) < 0, J(u) < d(\delta)\}.$$

Note that, from (H1), there exists $s_0 > 0$ such that $s_0 + q < p_\partial$.

We are now ready to demonstrate the following result:

Lemma 3.3. *Let $u \in W^{1,p}(\Omega)$. The following properties hold:*

1. *If $0 < \|u\|_\alpha < r(\delta)$, then $I_\delta(u) > 0$.*
2. *If $I_\delta(u) < 0$, then $\|u\|_\alpha > r(\delta)$.*
3. *If $I_\delta(u) = 0$, then either $\|u\|_\alpha = 0$ or $\|u\|_\alpha \geq r(\delta)$.*

Here, $r(\delta)$ is the unique solution to the equation

$$\phi(r) := \beta \frac{C^{q+s_0}}{s_0 e} r^{q-p+s_0} = \delta,$$

where

$$C := \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{q+s_0, \partial}}{\|u\|_{W^{1,p}(\Omega)}}.$$

Proof. 1. For any $u \in W^{1,p}(\Omega) \setminus \{0\}$, we have

$$\oint_{\partial\Omega} |u|^q \log(|u|) dx \leq \frac{1}{s_0 e} \|u\|_{q+s_0, \partial}^{q+s_0} \leq \frac{C^{q+s_0}}{s_0 e} \|u\|_\alpha^{q+s_0}.$$

Then, from $\|u\|_\alpha < r(\delta)$ and the fact that the function $r \mapsto \phi(r)$ is increasing, we get

$$\begin{aligned} \beta \oint_{\partial\Omega} |u|^q \log(|u|) dx &\leq \beta \frac{C^{q+s_0}}{s_0 e} \|u\|_\alpha^{q+s_0} \\ &= \beta \frac{C^{q+s_0}}{s_0 e} \|u\|_\alpha^{q-p+s_0} \|u\|_\alpha^p \\ &= \phi(\|u\|_\alpha) \|u\|_\alpha^p \\ &< \phi(r(\delta)) \|u\|_\alpha^p \\ &= \delta \|u\|_\alpha^p. \end{aligned}$$

Thus, $I_\delta(u) > 0$.

2. If $I_\delta(u) < 0$, then $\|u\|_\alpha^p > 0$ and

$$\delta \|u\|_\alpha^p < \beta \oint_{\partial\Omega} |u|^q \log(|u|) dx \leq \phi(\|u\|_\alpha) \|u\|_\alpha^p,$$

since $\delta = \phi(r(\delta))$ and ϕ is increasing, we get $\|u\|_\alpha^p > r(\delta)$.

3. If $I_\delta(u) = 0$ and $\|u\|_\alpha > 0$, then

$$\delta \|u\|_{W^{1,p}(\Omega)}^p = \beta \oint_{\partial\Omega} |u|^q \log(|u|) dx \leq \phi(\|u\|_\alpha) \|u\|_\alpha^p,$$

that is, $\|u\|_\alpha \geq r(\delta)$.

Note that, if $\|u\|_\alpha = 0$, there holds $I_\delta(u) = 0$.

□

Lemma 3.4. *The continuous function $\delta \mapsto d(\delta)$ satisfies the following properties:*

1. For all $\delta \in (0, \frac{q}{p}]$ and $u \in \mathcal{N}_\delta$, we have $d(\delta) \geq \theta(\delta)r^p(\delta)$,
 where $\theta(\delta) := \frac{1}{p} - \frac{\delta}{q}$, and $d(\delta) > 0$ for every $\delta \in (0, \frac{q}{p})$,
 namely $d \geq d_1 := \left(\frac{1}{p} - \frac{1}{q}\right) r^p(1)$.
2. The function $\delta \mapsto d(\delta)$ is increasing on $(0, 1]$, decreasing on $[1, +\infty]$ and reaches its maximum $d := d(1)$ at $\delta = 1$.
3. There exists $\delta_0 \geq \frac{q}{p}$ such that $d(\delta_0) = 0$ and $d(\delta) > 0$ for each $\delta \in (0, \delta_0)$.

Proof. 1. Let $u \in \mathcal{N}_\delta$, we have

$$\begin{aligned} J(u) &= \frac{1}{p} \|u\|_\alpha^p - \frac{\beta}{q} \oint_{\partial\Omega} |u|^q \log(|u|) + \frac{\beta}{q^2} \|u\|_{q,\partial}^q \\ &\geq \left[\frac{1}{p} - \frac{\delta}{q} \right] \|u\|_\alpha^p. \end{aligned}$$

Since $\|u\|_\alpha > 0$, it follows from Lemma 3.3 that $\|u\|_\alpha \geq r(\delta)$. The proof of assertion (1) is complete.

2. We prove $d(\delta_1) < d(\delta_2)$ for each $0 < \delta_1 < \delta_2 < 1$ or $1 < \delta_2 < \delta_1 < \infty$. To achieve this, it is enough to show that for any $0 < \delta_1 < \delta_2 < 1$ or $1 < \delta_2 < \delta_1 < \infty$, and for $u \in \mathcal{N}_{\delta_2}$, there exists a function $v \in \mathcal{N}_{\delta_1}$ and a positive constant $\epsilon(\delta_1, \delta_2) > 0$ such that

$$J(u) > J(v) + \epsilon(\delta_1, \delta_2).$$

Indeed, for any $u \in \mathcal{N}_{\delta_2}$, and $\delta > 0$, we can define $\lambda := \lambda(\delta)$ as in equation (3.5) such that:

$$K(\lambda) = \frac{\delta}{\beta} \|u\|_\alpha^p \quad \text{with} \quad \lambda(\delta_2) = 1,$$

which ensures $I_\delta(\lambda u) = 0$. By taking $v := \lambda(\delta_1)u$, we observe that $I_{\delta_1}(\lambda(\delta_1)u) = I_{\delta_1}(v) = 0$ and since $\|u\|_\alpha > 0$, it follows that $v \in \mathcal{N}_{\delta_1}$.

Additionally, we have:

$$\begin{aligned} \frac{d}{d\lambda} J(\lambda u) &= \frac{1}{\lambda} I(\lambda u) = \frac{1}{\lambda} [(1 - \delta) \|\lambda u\|_\alpha^p + I_\delta(\lambda u)] \\ &= (1 - \delta) \lambda^{p-1}(\delta) \|u\|_\alpha^p. \end{aligned} \tag{3.9}$$

Note that, the function $\delta \mapsto \lambda(\delta)$ is increasing on $(0, \infty)$, as shown in the proof of assertion (3).

- (a) If $0 < \delta_1 < \delta_2 < 1$, we have $\lambda(\delta_1) < \lambda(\delta_2)$. Furthermore, $u \in \mathcal{N}_{\delta_2}$ and Lemma 3.3 (3) says $\|u\|_\alpha \geq r(\delta_2)$. Integrating (3.9) over $[\delta_1, \delta_2]$, we get

$$\begin{aligned} J(u) - J(v) &= \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} \frac{d}{d\lambda} J(\lambda u) d\lambda \\ &= J(\lambda(\delta_2)u) - J(\lambda(\delta_1)u) \\ &= \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} (1 - \delta) \lambda^{p-1}(\delta) \|u\|_\alpha^p d\lambda \\ &\geq \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} (1 - \delta_2) \lambda^{p-1}(\delta_1) r^p(\delta_1) d\lambda \\ &= (\lambda(\delta_2) - \lambda(\delta_1)) (1 - \delta_2) \lambda^{p-1}(\delta_1) r^p(\delta_2) \\ &= \epsilon(\delta_1, \delta_2) > 0. \end{aligned}$$

- (b) If $1 < \delta_2 < \delta_1 < +\infty$, using the semme argument in (a) to arrive at

$$J(u) - J(v) \geq (\lambda(\delta_1) - \lambda(\delta_2)) (\delta_2 - 1) \lambda^{p-1}(\delta_2) r^p(\delta_2) = \epsilon(\delta_1, \delta_2) > 0.$$

Thus, assertion (2) follows.

3. For $u \in W^{1,p}(\Omega)$, $\|u\|_\alpha > 0$, and $\delta > 0$, from (3.5), we can define $\lambda := \lambda(\delta)$ such that

$$K(\lambda) = \frac{\delta}{\beta} \|u\|_\alpha^p,$$

so that $I_\delta(\lambda u) = 0$. The next step is to prove that the continuous function $\delta \mapsto \lambda(\delta)$ is increasing on $(0, \infty)$. Taking the derivative of $K(\lambda)$ with respect to δ , we obtain $\lambda'(\delta) K'(\lambda(\delta)) = \frac{1}{\beta} \|u\|_\alpha^p > 0$. If λ is decreasing, we have for $\delta_1 < \delta_2$, $\lambda(\delta_2) < \lambda(\delta_1)$ and $\log(\lambda(\delta_2)) < \log(\lambda(\delta_1))$, then

$$\frac{K(\lambda(\delta_2))}{\lambda^{q-p}(\delta_2)} < \frac{K(\lambda(\delta_1))}{\lambda^{q-p}(\delta_1)},$$

that is

$$\frac{\lambda^{q-p}(\delta_1)}{\lambda^{q-p}(\delta_2)} < \frac{\delta_1}{\delta_2} < 1,$$

this implies that $\delta_2 < \delta_1$, which contradict the fact that $\delta_1 < \delta_2$. According to Lemma 3.1, we have $\delta \mapsto J(\lambda u)$ is increasing on $(0, \lambda_\star]$ and decreasing on $[\lambda_\star, +\infty)$, thus there exists $\delta_0 > 0$ such that $d(\delta_0) = 0$ and from the assertion (1) of this lemma, we deduce that $\delta_0 \geq \frac{q}{p}$.

□

Remark 3.5. In vertue of Lemma 3.4, we infer that $d > d_0 := \lim_{\delta \rightarrow 0} d(\delta) \geq 0$ and for any $\mu \in (d_0, d)$, the equation $d(\delta) = \mu$ has two solutions δ_1, δ_2 such that $\delta_1 < 1 < \delta_2 < \delta_0$.

Lemma 3.6. Assume that $d_0 < J(u) < d$ for some $u \in W^{1,p}(\Omega)$. Let δ_1, δ_2 be two roots of the equation $d(\delta) = J(u)$ with $\delta_1 < 1 < \delta_2$. Then the sign of $I_\delta(u)$ is unchangeable on $\delta \in (\delta_1, \delta_2)$.

Proof. Arguing by contradiction. Assume that there exists $\delta_0 \in (\delta_1, \delta_2)$ such that $I_{\delta_0}(u) = 0$. From $J(u) > 0$, we have $\|u\|_\alpha > 0$, then $u \in \mathcal{N}_{\delta_0}$ and $J(u) \geq d(\delta_0)$. This leads to a contradiction with $J(u) = d(\delta_1) = d(\delta_2) < d(\delta_0)$. \square

Note that by the Definition 4.1, the weak solution u satisfies the following energy equality:

$$\int_0^t \|u_\tau\|_2^2 d\tau + J(u) = J(u_0), \quad \forall t \in [0, T]. \tag{3.10}$$

Next, by using the potential wells above, we show the invariance of sets W_δ and V_δ in the subcritical case $0 < J(u_0) < d$ and critical case $J(u_0) = d$.

Proposition 3.7. *Let $u_0 \in W^{1,p}(\Omega)$, such that $J(u_0) \in (0, d)$. Then there exists $\delta_1 \in (1, \delta_0)$ such that $J(u_0) = d(\delta_1)$. Furthermore, for any weak solution of problem (1) with the initial data u_0 there hold:*

for each $\delta \in [1, \delta_1)$ and $t \in [0, T)$,

1. if $I(u_0) > 0$, then $u \in W_\delta$,
2. if $I(u_0) < 0$, then $u \in V_\delta$.

Proof. Since $J(u_0) \in (0, d)$, then from Lemma 3.4, we get the existence of $\delta_1 \in (1, \delta_0)$ such that $J(u_0) = d(\delta_1)$.

1. Let $\delta \in [1, \delta_1)$ and suppose that $I(u_0) > 0$. We have

$$I_\delta(u_0) = (\delta - 1)\|u_0\|_\alpha^p + I(u_0),$$

then $I_\delta(u_0) > 0$. Further, Lemma 3.4 asserts that $J(u_0) = d(\delta_1) < d(\delta)$ since $\delta \mapsto d(\delta)$ is decreasing on $[1, +\infty)$. Thus $u_0 \in W_\delta$, for each $\delta \in [1, \delta_1)$.

Next, we prove $u \in W_\delta$ for all $\delta \in (1, \delta_1)$ and $t \in (0, T)$. By contradiction, if it is false, from the continuity of I_δ and $d(\delta)$ with respect to t and δ , there exists $t_0 \in (0, T)$ and $\delta_2 \in [1, \delta_1)$ such that $u(t_0) \in \partial W_{\delta_2}$, which means that either $I_{\delta_2}(u(t_0)) = 0, \|u(t_0)\|_{W^{1,p}(\Omega)} > 0$ or $J(u(t_0)) = d(\delta_2)$ (since 0 is an interior point of W). If $J(u(t_0)) = d(\delta_2)$, it follows from (3.10) that

$$d(\delta_2) = J(u(t_0)) \leq J(u_0) = d(\delta_1),$$

which is impossible, thus,

$$u(t_0) \in \mathcal{N}_{\delta_2} \quad \text{and} \quad J(u(t_0)) \geq d(\delta_2).$$

Again, from (3.10), we get a contradiction.

2. Let $\delta \in [1, \delta_1)$ and suppose that $I(u_0) < 0$. By contradiction, we prove that $u_0 \in V_\delta$ for all $\delta \in (1, \delta_1)$. From the continuity of I_δ and $d(\delta)$, there exists $\delta_2 \in [1, \delta_1)$ such that $u_0 \in V_\delta$ for all $\delta \in [1, \delta_2)$ and $u_0 \in \partial V_{\delta_2}$, that means that $I_\delta(u_0) < 0, J(u_0) < d(\delta), \forall \delta \in [1, \delta_2)$ and $I_{\delta_2}(u_0) = 0$, or $J(u_0) = d(\delta_2)$.

Since $J(u_0) = d(\delta_1)$, we get $J(u_0) \neq d(\delta_2)$. Thus,

$$I_{\delta_2}(u_0) = 0 \quad \text{and} \quad I_\delta(u_0) < 0, \forall \delta \in [1, \delta_2).$$

In virtue of Lemma 3.3, we get $\|u_0\|_\alpha > r(\delta)$ for all $\delta \in [1, \delta_2)$. Letting $\delta \rightarrow \delta_2$, we obtain $\|u_0\|_\alpha \geq r(\delta_2) > 0$, thus $u_0 \in \mathcal{N}_{\delta_2}$ and $J(u_0) = d(\delta_1) \geq d(\delta_2)$, which is impossible.

Now, we prove $u \in V_\delta$ for all $\delta \in [1, \delta_1)$ and $t \in (0, T)$. By contradiction, assume that there exists $t_0 \in (0, T)$ and $\delta_2 \in [1, \delta_1)$ such that $u \in V_\delta$ for all $(t, \delta) \in [0, t_0) \times [1, \delta_2)$ and $u(t_0) \in \partial V_{\delta_2}$. Then,

$$I_{\delta_2}(u(t_0)) = 0 \quad \text{or} \quad J(u(t_0)) = d(\delta_2).$$

From $u \in V_\delta$ for all $(t, \delta) \in [0, t_0) \times [1, \delta_2)$ and Lemma 3.3 we deduce that $\|u\|_\alpha > r(\delta)$ for all $(t, \delta) \in [0, t_0) \times [1, \delta_2)$. When t is near t_0 , we obtain $\|u(t_0)\|_\alpha \geq r(\delta) > 0$. This leads to $I_{\delta_2}(u(t_0)) \neq 0$. Thus $J(u(t_0)) = d(\delta_2) < d(\delta_1)$, which is impossible. Following the same argument used in the proof of $u_0 \in V_{\delta_2}$, we deduce that $u \in V_\delta$ for each $\delta \in [1, \delta_1)$ and $t \in [0, T)$. □

In view of Proposition 3.7 and Lemma 3.4, we deduce the following results:

Corollary 3.8. *Let $u_0 \in W^{1,p}(\Omega)$, $\mu \in [d_0, d)$, and δ_1, δ_2 two roots of the equation $d(\delta) = \mu$ with $\delta_1 < 1 < \delta_2$. suppose that u is a weak solution of (1.1) with $d_0 < J(u_0) \leq \mu$, then for each $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, there hold*

1. if $I(u_0) > 0$, then $u \in W_\delta$,
2. if $I(u_0) < 0$, then $u \in V_\delta$.

Proposition 3.9. *Let u be a weak solution to the problem (1.1). Suppose that $J(u_0) \leq 0$; then*

$$\|u\|_\alpha \geq \left(\frac{s_0 e q}{p \beta C^{q+s_0}} \right)^{\frac{1}{q-p+s_0}}.$$

Moreover, if $\|u_0\|_\alpha > 0$, then for any $\delta \in (0, \frac{q}{p})$, the weak solution belongs to V_δ .

Proof. Let u be any solution of problem (1.1) with the initial data that satisfies $J(u_0) \leq 0$. From the conservation of energy, we deduce that

$$J(u) \leq 0, \quad \forall t \in [0, T),$$

which implies that

$$\frac{1}{p} \|u\|_\alpha^p \leq \frac{\beta}{s_0 e q} C^{q+s_0} \|u\|_\alpha^{q+s_0}, \quad \forall t \in [0, T). \tag{3.11}$$

- (i) If $J(u_0) < 0$, we get $\|u_0\|_\alpha > 0$ and $J(u) < 0$ implies that $\|u\|_\alpha > 0$. Thus from (3.11) we get

$$\|u\|_\alpha \geq \left(\frac{s_0 e q}{p \beta C^{q+s_0}} \right)^{\frac{1}{q-p+s_0}}.$$

- (ii) If $J(u_0) = 0$, then by (3.11) we must have either $\|u_0\|_\alpha \geq \left(\frac{s_0 e q}{p \beta C^{q+s_0}} \right)^{\frac{1}{q-p+s_0}}$ or $\|u_0\|_\alpha = 0$.

- (a) If $\|u_0\|_\alpha = 0$, we infer $\|u\|_\alpha = 0$ for all $t \in [0, T)$. If it is false, there exists $t_0 \in (0, t)$ such that

$$0 < \|u(t_0)\|_\alpha < \left(\frac{s_0 e q}{p \beta C^{q+s_0}} \right)^{\frac{1}{q-p+s_0}},$$

which leads to a contradiction with (3.11).

- (b) If $\|u_0\|_\alpha \geq \left(\frac{s_0 e q}{p \beta C^{q+s_0}}\right)^{\frac{1}{q-p+s_0}}$, by a similar argument in (a) we can prove that $\|u\|_\alpha \geq \left(\frac{s_0 e q}{p \beta C^{q+s_0}}\right)^{\frac{1}{q-p+s_0}}$ for all $t \in [0, t)$.

On the other hand, for any $\delta \in (0, \frac{q}{p})$ and $t \in [0, T)$ we have

$$\theta(\delta)\|u\|_\alpha^p + \frac{1}{q}I_\delta(u) \leq J(u) \leq J(u_0). \tag{3.12}$$

1. If $J(u_0) < 0$, from (3.12), we get

$$I_\delta(u) < 0, \quad \text{and} \quad J(u) < d(\delta), \quad \forall \delta \in (0, \frac{q}{p}), t \in [0, T).$$

Thus, $u \in V_\delta$.

2. If $J(u_0) = 0$, from (ii) we have $\|u\|_\alpha \geq \left(\frac{s_0 e q}{p \beta C^{q+s_0}}\right)^{\frac{1}{q-p+s_0}}$ for all $t \in [0, T)$.

Together with (3.12) we also have

$$I_\delta(u) < 0, \quad \text{and} \quad J(u) < d(\delta), \quad \forall \delta \in (0, \frac{q}{p}), t \in [0, T).$$

Thus, $u \in V_\delta$. □

Proposition 3.10. *Suppose that $u_0 \in W^{1,p}(\Omega)$, $J(u_0) = d$ and u is a weak solution of the problem (1.1). Then for all $t \in [0, T)$ there holds*

1. if $I(u_0) > 0$, we have $I(u) > 0$,
2. if $I(u_0) < 0$, we have $I(u) < 0$.

Proof. 1. If it is false, there exists $t_0 \in (0, T)$ such that $I(u(t)) > 0$ for all $t \in (0, t_0)$ and $I(u(t_0)) = 0$. which, together with the definition of d means that $J(u(t_0)) \geq d$. Furthermore, by (3.10) we get

$$J(u(t_0)) = J(u_0) - \int_0^{t_0} \|u_s\|_2^2 ds \leq d,$$

then $J(u(t_0)) = d$ and $\int_0^{t_0} \int_\Omega |u_s|^2 dx ds = 0$, i.e., $u_t = 0$ for all $t \in [0, t_0]$. Which is a contradiction with $I(u_0) > 0$.

2. If the result is false, then there exists $t_0 \in (0, T)$ such that $I(u(t)) < 0$ for all $t \in (0, t_0)$ and $I(u(t_0)) = 0$.

Similar to the proof above, we also have a contradiction. Thus, Proposition 3.10 follows. □

4. Main results

In this section, we will present three theorems concerning the global existence, asymptotic behavior, and blow-up of weak solutions to problem (1.1).

We begin by defining the weak solution, maximal existence time, and blow-up of solutions for problem (1.1).

Definition 4.1. (*Weak solution*) A function $u(x, t)$ is called a weak solution of (1.1) on $\Omega \times [0, T[$, if $u \in L^\infty(0, T; W^{1,p}(\Omega))$ with $u_t \in L^2(0, T; L^2(\Omega))$ and satisfies (1.1) in the distribution sense, that is, $\forall v \in W^{1,p}(\Omega)$, and $t \in [0, T)$,

$$\int_{\Omega} u_t v dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + \alpha \int_{\Omega} |u|^{p-2} u v dx - \beta \oint_{\partial\Omega} |u|^{q-2} u \log(|u|) v = 0, \tag{4.1}$$

$$u(x, 0) = u_0(x) \in W^{1,p}(\Omega). \tag{4.2}$$

Definition 4.2. (*Maximal existence time*). Let $u(x, t)$ be a weak solution of (1.1). We define the maximal existence time T of $u(x, t)$ as follows:

1. if $u(x, t)$ exists for all $t \geq 0$, then $T = +\infty$,
2. if there exists $0 < t_0 < +\infty$ such that $u(x, t)$ exists for $t \in [0, t_0[$, but doesn't exist at t_0 , then $T = t_0$.

Definition 4.3. (*Finite time blow-up*). Let $u(x, t)$ be a weak solution of (1.1). We say that u blows-up in finite time, if the maximal existence time T is finite and

$$\lim_{t \rightarrow T^-} \int_0^t \|u(\cdot, s)\|_2^2 ds = +\infty.$$

4.1. Global existence and polynomial decay

Theorem 4.4. Under the Hypothesis (H1), if $u_0 \in W^{1,p}(\Omega)$ such that $J(u_0) \leq d$ and $I(u_0) \geq 0$, then the problem (1.1) has a global weak solution $u \in L^\infty(0, +\infty; W^{1,p}(\Omega))$ with $u_t \in L^2(0, +\infty; L^2(\Omega))$.

Theorem 4.5. Under the hypothesis of Theorem 4.4, if $I(u_0) > 0$ then the weak solution of problem (1.1) satisfies the following estimates :

1. if $J(u_0) < d_1$, we have $u \in W$ for all $t \geq 0$ and there exists a constant $K > 0$, such that

$$\|u(\cdot, t)\|_2 \leq \left(\|u_0\|_2^{2-p} + (p-2)Kt \right)^{\frac{1}{2-p}}, \quad \forall t \geq 0. \tag{1.1}$$

2. if $d_1 < J(u_0) \leq d$, we have $u \in \bar{W}$ for all $t \geq 0$, and for each $\eta \in (0, d)$, there exists $t_\eta > 0$ such that

$$\|u(\cdot, t)\|_2 \leq \left(\|u_0\|_2^{2-p} + (p-2)K't \right)^{\frac{1}{2-p}}, \quad \forall t \geq t_\eta. \tag{1.2}$$

Here $K, K' > 0$ are two constants that will be determined later.

Proof of Theorem 4.4. We split the proof into two cases.

Case 1 : Global existence for $(J(u_0) < d)$.

First, we can exclude some special cases as follows:

1. $0 < J(u_0) < d, I(u_0) = 0$. This is contradictory with the definition of d .
2. $J(u_0) = 0, I(u_0) = 0$. If $\|u_0\|_\alpha > 0$, it follows that $u_0 \in \mathcal{N}$ and $J(u_0) \geq d$, which is contradictive. Otherwise, $u_0 = 0$, which is a trivial case.
3. $J(u_0) = 0, I(u_0) > 0$. This contradict (3.3).
4. $J(u_0) < 0, I(u_0) \geq 0$. Similar to case (3).

It remains for us to consider the case $0 < J(u_0) < d$ and $I(u_0) > 0$.

We choose a sequence of eigenfunctions $(\phi_k(x))_{k \geq 1}$ as an orthogonal basis of the Sobolev space $W^{1,p}(\Omega)$ and construct an approximate solution $u^{(m)}(x, t)$ of problem (1.1),

$$u^{(m)}(x, t) = \sum_{k=1}^m g_{km}(t)\phi_k(x), \quad m \geq 1,$$

which satisfy

$$\begin{aligned} \int_\Omega u_t^{(m)} \phi_j dx + \int_\Omega |\nabla u^{(m)}|^{p-2} \nabla u^{(m)} \nabla \phi_j dx + \alpha \int_\Omega |u^{(m)}|^{p-2} u^{(m)} \phi_j dx \\ - \beta \oint_{\partial\Omega} |u^{(m)}|^{q-2} u^{(m)} \log(|u^{(m)}|) \phi_j dx = 0, \quad j = 1, \dots, m, \end{aligned} \tag{4.3}$$

and as $m \rightarrow +\infty$,

$$u^{(m)}(x, 0) = \sum_{k=1}^m a_{km} \phi_k(x) \rightarrow u_0(x) \text{ in } W^{1,p}(\Omega). \tag{4.4}$$

Multiplying (4.3) by $g'_{jm}(t)$, summing for j and integrating with respect to t from 0 to t , we get

$$\int_0^t \int_\Omega \|u_s^{(m)}\|_2^2 ds + J(u^{(m)}) = J(u^{(m)}(0)), \quad \forall t \geq 0. \tag{4.5}$$

By (4.4) we have $J(u^{(m)}(0)) \rightarrow J(u_0)$, which together with (4.5) implies that

$$\int_0^t \int_\Omega \|u_s^{(m)}\|_2^2 ds + J(u^{(m)}) = J(u^{(m)}(0)) < d, \quad \forall t \geq 0, \tag{4.6}$$

for sufficiently large m .

Following an argument similar to that in the proof of Proposition 3.7 together with (4.6) we can prove that $u_m(x, t) \in W$ for any $t \geq 0$ and sufficiently large m . Thus, from (3.1) and (4.6) we deduce that

$$\int_0^t \|u_s^{(m)}\|_2^2 ds dx < d, \quad \|u^{(m)}\|_\alpha^p < \frac{qpd}{q-p}.$$

On the other hand, from Lemma 2.3, we get

$$\oint_{\partial\Omega} \left| |u^{(m)}|^{q-2} u^{(m)} \log(|u^{(m)}|) \right|^{\frac{q+s_0}{q+s_0-1}} dx < \frac{C^{q+s_0}}{s_0 e} \left(\frac{qpd}{q-p} \right)^{\frac{q+s_0}{p}}.$$

Denote by \rightarrow_* the weakly star convergence. Then the estimates above imply that there exists a subsequence, still denoted by (u_m) , such that as $m \rightarrow +\infty$,

$u^{(m)} \rightharpoonup_* u$ in $L^\infty(0, \infty; W^{1,p}(\Omega))$ and a.e. in $\Omega \times [0, +\infty)$,

$u_t^{(m)} \rightharpoonup u_t$, in $L^2(0, \infty; L^2(\Omega))$,

$|u^{(m)}|^{q-2}u^{(m)} \log(|u^{(m)}|) \rightharpoonup_* |u|^{q-2}u \log(|u|)$ in $L^\infty(0, \infty; L^{\frac{q+s_0}{q+s_0-1}}(\partial\Omega))$ and a.e. in $\Omega \times [0, +\infty)$.

Hence in (4.3), for j fixed, letting $m \rightarrow +\infty$, there holds

$$\int_{\Omega} u_t \phi_j dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi_j dx + \alpha \int_{\Omega} |u|^{p-2} u \phi_j dx = \beta \oint_{\partial\Omega} |u|^{q-2} u \log(|u|) \phi_j dx, \quad \forall j \geq 1.$$

Furthermore,

$$\int_{\Omega} u_t v dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + \alpha \int_{\Omega} |u|^{p-2} u v dx = \beta \oint_{\partial\Omega} |u|^{q-2} u \log(|u|) v dx, \quad \forall v \in H_{X,0}^1(\Omega).$$

Meanwhile, from (4.4) we obtain $u(x, 0) = u_0(x)$ in $W^{1,p}(\Omega)$. It follows from Proposition 3.7 that $u \in W$ for all $t \geq 0$. Therefore, u is a global weak solution to the problem (1.1).

Case 2 : Global existence for $(J(u_0) = d)$.

Set $u_0^{(m)} = \mu_m u_0$, where $\mu_m = 1 - \frac{1}{m}$, $m \geq 2$.

We consider the initial boundary value problem

$$\begin{cases} u_t - \Delta_p u + \alpha |u|^{p-2} u = 0, & \in \Omega \times]0, +\infty[, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \beta |u|^{q-2} u \log(|u|), & (x, t) \in \partial\Omega \times [0, +\infty[, \\ u(x, 0) = u_0^{(m)}(x), & x \in \Omega \end{cases} \quad (4.7)$$

From Lemma 3.1 and the fact $I(u_0) \geq 0$, we have

$$\lambda_*(u_0) = \lambda_* \geq 1.$$

Together with $\mu_m < 1$, it follows that

$$\begin{aligned} I(u_0^{(m)}) &> 0, \\ J(u_0^{(m)}) &< J(u_0) = d. \end{aligned}$$

Thus, it follows from the results of **Case 1** that, for any m , the problem (4.7) admits a global weak solution $u^{(m)} \in L^\infty(0, \infty; W^{1,p}(\Omega))$ with $u_t^{(m)} \in L^2(0, \infty; L^2(\Omega))$ and $u^{(m)} \in W$, $\forall t \geq 0$, and satisfies

$$\begin{aligned} &\int_{\Omega} u_t^{(m)} v dx + \int_{\Omega} |\nabla u^{(m)}|^{p-2} \nabla u^{(m)} \nabla v dx + \alpha \int_{\Omega} |u^{(m)}|^{p-2} u^{(m)} v dx \\ &= \beta \oint_{\partial\Omega} |u^{(m)}|^{q-2} u^{(m)} \log(|u^{(m)}|) v dx, \end{aligned}$$

for any $v \in W^{1,p}(\Omega)$, $t > 0$, and

$$\int_0^t \int_{\Omega} \|u_s^{(m)}\|_2^2 ds + J(u^{(m)}) = J(u^{(m)}(0)) < J(u_0) = d, \quad \forall t \geq 0.$$

The proof left similar to that in the **Case 1**. The proof of Theorem 4.4 is complet. \square

Proof of Theorem 4.5. (1) If $J(u_0) < d_1$ and $I(u_0) > 0$ we see that $J(u_0) > 0$, then from Proposition 3.7, we have $u \in W_{\delta}$ for each $\delta \in [1, \delta_1)$, which implies that $I(u) > 0, \forall t \geq 0$.

Note that by (3.10), one can has $J(u) \leq J(u_0)$, together with (3.3) we deduce that

$$\|u\|_{\alpha} < \left[\left(\frac{1}{p} - \frac{1}{q} \right)^{-1} J(u_0) \right]^{\frac{1}{p}}.$$

On the other hand, we have

$$\begin{aligned} I(u) &= \|u\|_{\alpha}^p - \beta \oint_{\partial\Omega} |u|^q \log(|u|) dx \\ &\geq \|u\|_{\alpha}^p - \beta \frac{C^{q+s_0}}{s_0 e} \|u\|_{\alpha}^{q+s_0} \\ &= \left(1 - \beta \frac{C^{q+s_0}}{s_0 e} \|u\|_{\alpha}^{q-p+s_0} \right) \|u\|_{\alpha}^p \\ &\geq \left(1 - \beta \frac{C^{q+s_0}}{s_0 e} \left[\left(\frac{1}{p} - \frac{1}{q} \right)^{-1} J(u_0) \right]^{\frac{q-p+s_0}{p}} \right) \|u\|_{\alpha}^p \\ &= \sigma \|u\|_{\alpha}^p, \end{aligned}$$

where

$$\sigma = \left(1 - \beta \frac{C^{q+s_0}}{s_0 e} \left[\left(\frac{1}{p} - \frac{1}{q} \right)^{-1} J(u_0) \right]^{\frac{q-p+s_0}{p}} \right) > 0.$$

Furthermore, from Definition 4.1, we get

$$I(u) = -\frac{1}{2} \frac{d}{dt} \|u\|_2^2, \quad \forall t \geq 0.$$

Thus, we obtain

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \|u\|_2^2 &\geq \sigma \|u\|_{\alpha}^p \\ &\geq \sigma C_1^p \|u\|_2^p, \end{aligned}$$

where $C_1 > 0$, is the optimal constant of the embdding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$.

Integrating the last inequality from 0 to t , we get the decay polynomially (1.1).

(2) In this case, we see from Proposition 3.10 that

$$\int_{\Omega} u_t u dx = -I(u) < 0.$$

By Hölder’s inequality, we have

$$0 < - \int_{\Omega} u_t u dx \leq \|u_t\|_2 \|u\|_2,$$

then $\|u_t\|_2 > 0$ and $t \mapsto \int_0^t \|u_s\|_2^2 ds$ is increasing on $[0, \infty)$. According that to (3.10), we deduce that $0 < \int_0^t \|u_s\|_2^2 ds < d$ and for any $\eta \in (0, d)$, there exists $T_\eta > 0$ such that

$$\int_0^{T_\eta} \|u_s\|_2 ds = \eta,$$

i.e.,

$$d - \eta = J(u(T_\eta)).$$

Taking T_η as the initial time, similar to (1), we obtain the decay estimate (1.2). This complet the proof of Theorem 4.5. □

4.2. Blow-up in finite time

Theorem 4.6. *Under the hypothesis (H1), if $u_0 \in W^{1,p}(\Omega)$ such that $J(u_0) \leq d$ and $I(u_0) < 0$. Then, for $q > 2$ (resp. $q \leq 2$) the weak solution u of problem (1.1) blows up in finite time(resp. at ∞) in the sense of Definition 4.3.*

Proof of Theorem 4.6. We divide the proof into two cases.

Case 1 : Blow-up for $J(u_0) < d$.

Let u be a weak solution of problem (1.1) with $J(u_0) < d$ and $I(u_0) < 0$, then $\|u_0\|_\alpha > 0$. Arguing by contradiction, suppose that u is global (i.e., $T = \infty$). For $t \geq 0$, define

$$\psi(t) = \int_0^t \|u(s)\|_2^2 ds + \|u_0\|_2^2.$$

Then

$$\begin{aligned} \dot{\psi}(t) &= \|u(t)\|_2^2, \\ \ddot{\psi}(t) &= \int_{\Omega} 2uu_t dx = -2I(u), \end{aligned}$$

where $\dot{\phi}(t) := \frac{d}{dt}\phi(t)$.

It follows from (3.1), (3.10), Lemma 3.3 (2) and the embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ that

$$\begin{aligned} \ddot{\psi}(t) &= -2qJ(u) + 2 \left(\frac{q}{p} - 1 \right) \|u\|_\alpha^p + \frac{2\beta}{q} \|u\|_{q,\partial}^q \\ &\geq 2q \int_0^t \|u_s\|_2^2 ds - 2qJ(u_0) + 2 \left(\frac{q}{p} - 1 \right) C_1^2 \|u\|_2^2 r^{p-2}(1) \\ &= 2q \int_0^t \|u_s\|_2^2 ds - 2qJ(u_0) + 2 \left(\frac{q}{p} - 1 \right) C_1^2 r^{p-2}(1) \psi(t). \end{aligned}$$

Note that

$$\left(\int_0^t \int_{\Omega} uu_t dx ds \right)^2 = \frac{1}{4} \left(\int_0^t \frac{d}{ds} \|u(s)\|_2^2 ds \right)^2 = \frac{1}{4} \left([\dot{\psi}(t)]^2 - 2\|u_0\|_2^2 \dot{\psi}(t) + \|u_0\|_2^4 \right),$$

then

$$\begin{aligned} [\dot{\psi}(t)]^2 &= 4 \left(\int_0^t \int_{\Omega} uu_t dx ds \right)^2 + 2\|u_0\|_2^2 \dot{\psi}(t) - \|u_0\|_2^4 \\ &\leq 4 \left(\int_0^t \int_{\Omega} uu_t dx ds \right)^2 + 2\|u_0\|_2^2 \dot{\psi}(t). \end{aligned}$$

Thus

$$\begin{aligned} \ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 &\geq 2q \left[\left(\int_0^t \|u\|_2^2 dx ds \right) \left(\int_0^t \|u_t\|_2^2 dx ds \right) - \left(\int_0^t \int_{\Omega} uu_t dx ds \right)^2 \right] \\ &\quad - 2qJ(u_0)\psi(t) + 2 \left(\frac{q}{p} - 1 \right) C_1^2 r^{p-2}(1)\psi(t)\dot{\psi}(t) - q\|u_0\|_2^2 \dot{\psi}(t). \end{aligned}$$

Using Hölder's inequality, we obtain

$$\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 \geq -2qJ(u_0)\psi(t) + 2 \left(\frac{q}{p} - 1 \right) C_1^2 r^{p-2}(1)\psi(t)\dot{\psi}(t) - q\|u_0\|_2^2 \dot{\psi}(t). \quad (4.8)$$

1. If $J(u_0) \leq 0$, we see from Proposition 3.9 that $I(u) < 0$. Then, by (4.8) we get

$$\begin{aligned} \ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 &\geq 2 \left(\frac{q}{p} - 1 \right) C_1^2 r^{p-2}(1)\psi(t)\dot{\psi}(t) - q\|u_0\|_2^2 \dot{\psi}(t) \\ &= \left(\frac{q}{p} - 1 \right) C_1^2 r^{p-2}(1) \left[\psi(t) - \frac{qp}{(q-p)C_1^2 r^{p-2}(1)} \|u_0\|_2^2 \right] \dot{\psi}(t). \end{aligned}$$

Moreover, $\ddot{\psi}(t) > 0$, since $I(u) < 0$. Hence,

$$\psi(t) \geq t\dot{\psi}(t) + \psi(0), \quad t \geq 0,$$

i.e.,

$$\psi(t) \geq t\|u_0\|_2^2, \quad t \geq 0.$$

For $t > \frac{qp}{(q-p)C_1^2 r^{p-2}(1)}$, we have $\psi(t) - \frac{qp}{(q-p)C_1^2 r^{p-2}(1)} \|u_0\|_2^2 > 0$, that is,

$$\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 > 0, \quad \forall t > \frac{qp}{(q-p)C_1^2 r^{p-2}(1)}.$$

2. If $0 < J(u_0) < d$, by (4.8) we write

$$\begin{aligned} \ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 &\geq -2qJ(u_0)\psi(t) + 2 \left(\frac{q}{p} - 1 \right) C_1^2 r^{p-2}(1)\psi(t)\dot{\psi}(t) - q\|u_0\|_2^2 \dot{\psi}(t) \\ &= \frac{q-p}{2p} C_1^2 r^{p-2}(1) \left[\dot{\psi}(t) - \frac{4pq}{(q-p)C_1^2 r^{p-2}(1)} J(u_0) \right] \psi(t) \\ &\quad + \frac{q-p}{2p} C_1^2 r^{p-2}(1) \left[\psi(t) - \frac{2pq}{(q-p)C_1^2 r^{p-2}(1)} \|u_0\|_2^2 \right] \dot{\psi}(t). \end{aligned}$$

It follows from Proposition 3.7 that $I_{\delta_2}(u) < 0$, where $\delta_2 \in [1, \delta_1)$ and Lemma 3.3 implies that $\|u\|_{\alpha}^p > r^p(\delta_2)$. Now, By

$$I(u) = (1 - \delta_2)\|u\|_{\alpha}^p + I_{\delta_2}(u)$$

we get that

$$\ddot{\psi}(t) > 2(\delta_2 - 1)r^p(\delta_2), \quad \forall t \geq 0. \tag{4.9}$$

Integrating (4.9) over $[0, t]$, we get

$$\dot{\psi}(t) > 2(\delta_2 - 1)r^p(\delta_2)t.$$

and

$$\psi(t) > (\delta_2 - 1)r^p(\delta_2)t^2, \quad \forall t \geq 0.$$

Therefore, for sufficiently large t , we can obtain

$$\begin{aligned} \dot{\psi}(t) - \frac{4pq}{(q-p)C_1^2 r^{p-2}(1)} J(u_0) &> 0 \\ \dot{\psi}(t) - \frac{2pq}{(q-p)C_1^2 r^{p-2}(1)} \|u_0\|_2^2 &> 0. \end{aligned}$$

Thus,

$$\ddot{\psi}(t)\psi(t) - \frac{q}{2}[\dot{\psi}(t)]^2 > 0.$$

Finally, applying Lemma 2.2 for $q > 2$, we deduce that the function ψ goes to ∞ in finite time, This contradicts the assumption $T = \infty$. In other words, the weak solution blows up in finite time in the sense of Definition 4.3.

For $q \leq 2$ and $\|u_0\|_2 > 0$, we have

$$\frac{\ddot{\psi}(t)}{\dot{\psi}(t)} > \frac{q}{2} \frac{\dot{\psi}(t)}{\psi(t)}, \quad t \geq 0.$$

Solving the differential inequality above we obtain for $t \geq 0$

$$\psi(t) \geq e^{ct + \log(\psi(0))},$$

where $c := \frac{\dot{\psi}(0)}{\psi(0)^{\frac{q}{2}}}$. This shown that u blow up at ∞ .

Case 2 : Blow-up for $J(u_0) = d$.

Let u be a weak solution to problem (1.1) with $J(u_0) = d$ and $I(u_0) < 0$. By Proposition 3.10 we have $I(u) < 0$ for each $t \in [0, T)$, this combined with $-I(u) = \int_{\Omega} u_t u dx$ and using Hölder's inequality, we get $\|u_t\|_2 > 0$ and $t \mapsto \int_0^t \|u_t\|_2^2 ds$ is increasing for $t \geq 0$.


In the energy equality (3.10), choosing $t_0 > 0$ so that

$$0 < J(u(t_0)) = d - \int_0^{t_0} \|u_t\|_2^2 ds < d,$$

and taking t_0 as the initial time, we can prove that the weak solution u blows up in finite time, by using a similar argument in the **case 1**. The proof of Theorem 4.6 is complete. □


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Logarithmic Sobolev inequality in the variable exponent setting and its applications to hyperbolic differential equations with a logarithmic source term

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Abstract. We establish the generalized parametric logarithmic Sobolev inequalities in the Gagliardo-Nirenberg form for variable exponential space with log-Hölder exponential function. Employing the generalized parametric logarithmic Sobolev inequalities, we establish the existence of weak solutions to the boundary problem for the hyperbolic equation with logarithmic nonlinearity and involving variable exponents. Numerical examples and further applications will be addressed in a forthcoming paper.

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1. Introduction

In this work, we employ the logarithmic Sobolev inequality with variable exponent and investigate the solvability of the hyperbolic problem with variable exponent

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$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) + \theta |u|^{p(x)-2} u \ln |u|,$$

$$u(x, 0) = \phi_0(x),$$

$$\frac{\partial u(x, 0)}{\partial t} = \phi_1(x),$$

$$u(x, t)|_{\partial\Omega \times [0, T]} = 0,$$

$$\frac{\partial u(x, t)}{\partial t} \Big|_{\partial\Omega \times [0, T]} = 0,$$

where Ω is a bounded domain in R^l with a smooth boundary $\partial\Omega$, and ϕ_0, ϕ_1 are fixed initial functions; θ is a positive number. We define a function $p^* : \Omega \rightarrow R$ by $p^*(x) = \frac{lp(x)}{l-p(x)}$ if $p(x) < l$ and $p^*(x) = \infty$ if $p(x) \geq l$.

This type of equation models physical systems with spatially varying properties [1, 12, 27]. Materials science with non-uniform properties: The variable exponent allows the mathematical problem to model the dynamic behavior of materials where the elasticity, viscosity, or other material properties change from one location to another [24, 25]. Examples include elastoplastic materials, which can exhibit both elastic and plastic deformation. Geophysics and seismic wave propagation: The variable exponent can model different rock properties at various depths in the Earth's crust. It helps analyze how seismic waves travel through and are attenuated by these heterogeneous geological formations. In [27], S.D. Zeng, A.A. Khan, and S. Migorski used a framework of Oseen's type problems, which are fluid dynamics equations, and specifically addressed scenarios with non-smooth boundary conditions. The work establishes the existence of solutions for these inequalities and their applications in physics and engineering. The diffusive term in the equation is used in mathematical models for image denoising. In this application, the variable exponent can adapt the smoothing process to the specific local features of the image, such as preserving edges while removing noise [14]. In [14], the paper studied optimizing the algorithm for real-time applications, such as video surveillance. The optimizations included software pipeline, operation unit balancing, and other programming techniques to maximize processing efficiency.

Investigations of equations with logarithmic nonlinearity have a long history, see [11], and references therein. Variable exponents were studied in many papers, including [5, 7, 9, 15, 22, 23]. In [11], the authors analyze a hyperbolic equation with logarithmic non-linearity and a weak damping term, proving global existence of solutions using the potential well method and investigating growth and decay estimates. In [15], B.S. Wang, G.L. Hou, and B. Ge studied the existence and uniqueness of solutions for a quasilinear elliptic equation with a variable exponent and a convection term. In [19], the paper proved the existence of ground state sign-changing solutions for a class of second-order quasilinear elliptic equations. The equations are derived from models in nonlinear optics.

In the present article, we assume that a function $p : \Omega \rightarrow R$ satisfies the estimation

$$|p(x) - p(y)| \leq \frac{c}{\log(e + |x - y|^{-1})}$$

and

$$|p(x) - p_\infty| \leq \frac{c_1}{\log(e + |x|)}$$

for all $x, y \in \Omega$, with some $c > 0$, and some $c_1 > 0$, and real constant p_∞ . Let $p, r : \Omega \rightarrow [0, \infty)$ and $p, r \in C(\text{clos}(\Omega)) \cap L^\infty(\Omega)$. Let $p(x) < l, r(x) < \frac{lp(x)}{l-p(x)}$ for all $x \in \text{clos}(\Omega)$. Then, we establish that for each $\mu > 0$, there exists a positive number $c(\mu)$ such that

$$\begin{aligned} & \int_{\Omega} |u|^{p(x)} \ln \left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \right) dx \leq \\ & \leq \mu \rho_{p(\cdot)}(\nabla u) + c(\mu) \rho_{p(\cdot)}(u) + c_1 \frac{p_S}{p_m} \rho_{p(\cdot)}(u) \ln(\rho_{p(\cdot)}(u)) \end{aligned}$$

for all $u \in L^{r(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$.

Also, we prove that assume $\phi_0 \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$, $\phi_0 \in W_\delta, 0 < \delta < \ell, \phi_1 \in L^2(\Omega)$, and $E_0 < d$, then there exists a function $u \in L^\infty([0, \infty), W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\})$, $\partial_t u \in L^\infty([0, \infty), L^2(\Omega))$ such that

$$\begin{aligned} & \int_{\Omega} \frac{\partial^2 u(x, t)}{\partial t^2} \varphi(x) dx + \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx \\ & + \int_{\Omega} \frac{\partial u}{\partial t} \varphi dx = \theta \int_{\Omega} |u|^{p(x)-2} u \ln |u| \varphi dx \end{aligned}$$

for all $\varphi \in W_{1,0}^{p(\cdot)}(\Omega)$. Thus, the hyperbolic problem (4.1) has a global weak solution $u \in L^\infty([0, \infty), W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\})$, and the weak solution u grows as an exponential function as time approaches infinity (see Theorem 7).

2. Preliminaries

We assume that an exponent p belongs to the class of log-Hölder continuous functions $P^{\log}(\Omega), \Omega \subseteq R^l$, and denote $p_m = \inf_{x \in \Omega} p(x)$ and $p_S = \sup_{x \in \Omega} p(x)$. For further information, see [5, 6]. We define a modular function by

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx,$$

and the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ norm by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

In the case variable exponent $p(\cdot)$ is identical to the constant, we obtain the standard L^p spaces.

For each $a > 0$ such that $\rho\left(\frac{u}{a}\right) < \infty$, we have that $\lambda \mapsto \rho\left(\frac{u}{\lambda}\right)$ is a continuous function on $[a, \infty)$ such that $\lim_{\lambda \rightarrow \infty} \rho\left(\frac{u}{\lambda}\right) = 0$. For all $\lambda \geq 1$, we have

$$\lambda^{p_m} \rho(u) \leq \rho(\lambda u) \leq \lambda^{p_S} \rho(u).$$

For all $0 < \lambda < 1$, we obtain

$$\lambda^{p_S} \rho(u) \leq \rho(\lambda u) \leq \lambda^{p_m} \rho(u).$$

There are different modular functions, for example

$$\tilde{\rho}_{p(\cdot)}(u) = \int_{\Omega} \frac{1}{p(x)} |u(x)|^{p(x)} dx,$$

which satisfies the following inequalities

$$\tilde{\rho}_{p(\cdot)}(u) \leq \rho_{p(\cdot)}(u) \leq \tilde{\rho}_{p(\cdot)}(2u)$$

and the induced norm satisfies the relation

$$\|u\|_{L^{p(\cdot)}} = \|u\|_{\tilde{\rho}_{p(\cdot)}} \leq \|u\|_{\rho_{p(\cdot)}} \leq 2 \|u\|_{\tilde{\rho}_{p(\cdot)}}.$$

The Banach space $W_{1,0}^{p(\cdot)}(\Omega)$ is defined by

$$W_{1,0}^{p(\cdot)}(\Omega) = \left\{ u \in W_{1,0}^1(\Omega) : u, \nabla u \in L^{p(\cdot)}(\Omega) \right\}$$

endowed with the norm

$$\|u\|_{W_{1,0}^{p(\cdot)}} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for all $u \in W_{1,0}^{p(\cdot)}(\Omega)$.

Proposition 2.1. *Let $p \in P^{\log}(\Omega)$, then for all $u \in W_{1,0}^{p(\cdot)}(\Omega)$ the estimate*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c \operatorname{diam}(\Omega) \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds with a constant c depending only on $p(\cdot)$ and l .

Proposition 2.2. *Let $p, r : \Omega \rightarrow [0, \infty)$ and $p, r \in C(\operatorname{clos}(\Omega)) \cap L^\infty(\Omega)$. Let $p(x) < l, r(x) < \frac{lp(x)}{l-p(x)}$ for all $x \in \operatorname{clos}(\Omega)$. Then, there exists a continuous and compact embedding $W_1^{p(\cdot)}(\Omega) \rightarrow L^{r(\cdot)}(\Omega)$.*

The proof and additional information on the Sobolev and Poincare inequalities can be found in [5, 6, 15].

Similar to classical theory, we have that, in case $p_S < \infty$, the set $C_0^\infty(\Omega)$ of all smooth functions with compact support is dense in $L^{p(\cdot)}(\Omega)$ for every open domain Ω .

In the variable exponent setting, we reformulate the Holder theorem as follows: for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, the inequality

$$\int_{\Omega} |u(x)v(x)| dx \leq \left(1 + \frac{1}{p_m} - \frac{1}{p_S}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{q(\cdot)}(\Omega)}$$

holds with $q(x) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$.

For all $p_1, p_2 \in P^{\log}(\Omega)$ such that $p_1(x) \leq p_2(x)$ almost everywhere in Ω , the embedding $L^{p_2(\cdot)}(\Omega) \rightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.

Theorem 2.3. *Let $p \in P^{\log}(\Omega)$, $1 < p_m \leq p_S < l$. Let the function $p^* : \Omega \rightarrow R$ be defined by $p^*(x) = \frac{lp(x)}{l-p(x)}$ if $p(x) < l$ and $p^*(x) = \infty$ if $p(x) \geq l$. Then, the inequality*

$$\|u\|_{L^{p^*(\cdot)}(\Omega)} \leq c \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds for all $u \in W_{1,0}^{p(\cdot)}(\Omega)$ with a positive constant c depending only on the l and $p(\cdot)$.

For all $s(\cdot), p(\cdot) > 1$, the interpolation theory yields the inequality

$$\|u\|_{L^{s(\cdot)}(\Omega)} \leq 4 \|u\|_{L^{p(\cdot)}(\Omega)}^\theta \|u\|_{L^{r(\cdot)}(\Omega)}^{1-\theta},$$

where $\frac{1}{s(\cdot)} = \frac{\theta}{p(\cdot)} + \frac{1-\theta}{r(\cdot)}$ [5, Section 7].

Straightforwardly, from the variable exponent Sobolev-Poincare inequalities and the interpolation theory, we obtain the following interpolation theorem.

Theorem 2.4. *Let $p \in P^{\log}(\Omega)$, $1 < p_m \leq p_S < \infty$, and $1 < p(\cdot) < l$, $1 < s(\cdot) < p(\cdot)$. Then, the inequality*

$$\|u\|_{L^{s(\cdot)}(\Omega)} \leq c \|u\|_{L^{p(\cdot)}(\Omega)}^\theta \|u\|_{L^{r(\cdot)}(\Omega)}^{1-\theta}$$

holds for all $u \in L^{r(\cdot)}(\Omega) \cap W_1^{p(\cdot)}(\Omega)$ with a positive constant c depending only on the l and $p(\cdot), s(\cdot), r(\cdot)$, where $\frac{1}{s(\cdot)} = \theta \left(\frac{1}{p(\cdot)} - \frac{1}{l} \right) + \frac{1-\theta}{r(\cdot)}$, $\theta \in [0, 1)$.

Proposition 2.5. *Let $p \in P^{\log}(\Omega)$, $1 < p(\cdot) < r(\cdot) < \infty$, then the inequality*

$$\int_{\Omega} \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln \left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \right) dx \leq \frac{1}{1 - \frac{p_S}{r_m}} \ln \left(\frac{(\rho_{r(\cdot)}(u))^{\frac{p_S}{r_m}}}{\rho_{p(\cdot)}(u)} \right)$$

holds for all $L^{p(\cdot)}(\Omega) \cap L^{r(\cdot)}(\Omega)$.

Proof. We obtain that the derivative of the function of t on $(0, \infty)$

$$f(t) = t \ln \left(\int_{\Omega} |u|^{\frac{1}{t}} dx \right) \quad (\text{for } u \neq 0 \text{ a.e.})$$

in the following form

$$f'(t) = t \ln \left(\int_{\Omega} |u|^{\frac{1}{t}} dx \right) - \frac{1}{t} \left(\int_{\Omega} |u|^{\frac{1}{t}} dx \right)^{-1} \int_{\Omega} |u|^{\frac{1}{t}} \ln(|u|) dx$$

for all $t \in (0, \infty)$. Employing the convexity argument, for all $0 < \tilde{t} < t < \infty$, we have

$$f'(t) \geq \frac{f(\tilde{t}) - f(t)}{\tilde{t} - t}$$

thus, we obtain

$$\ln \left(\int_{\Omega} |u|^{p(x)} dx \right) - \int_{\Omega} p(x) \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln(|u|) dx \geq \frac{1}{\frac{p_S}{r_m} - 1} \ln \left(\frac{(\rho_{r(\cdot)}(u))^{\frac{p_S}{r_m}}}{\rho_{p(\cdot)}(u)} \right)$$

so

$$-\ln \left(\int_{\Omega} |u|^{p(x)} dx \right) + \int_{\Omega} p(x) \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln(|u|) dx \leq \frac{1}{1 - \frac{p_S}{r_m}} \ln \left(\frac{(\rho_{r(\cdot)}(u))^{\frac{p_S}{r_m}}}{\rho_{p(\cdot)}(u)} \right).$$

□

3. Logarithmic Sobolev inequalities in the variable spaces

We consider the analog of logarithmic Sobolev inequalities.

Theorem 3.1. *Let $p \in P^{\log}(\Omega)$, $1 < p(\cdot) < \infty$, then there exists a constant $c_1 > 0$ depending only on l and $p(\cdot)$ such that the inequality*

$$\int_{\Omega} \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln \left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \right) dx \leq c_1 \ln \left(c(l, p(\cdot)) \frac{\rho_{p(\cdot)}(\nabla u)}{\rho_{p(\cdot)}(u)} \right)$$

holds for all $u \in L^{p(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$.

Proof. We assume that $p \in P^{\log}(\Omega)$, $p(x) < l$, then we obtain that

$$\rho_{r(\cdot)}(u) \leq \tilde{c}(l, p(\cdot)) \rho_{p(\cdot)}(\nabla u)$$

for all $u \in W_{1,0}^{p(\cdot)}(\Omega)$ for $r(x) \leq p^*(x)$. We have

$$\rho_{r(\cdot)}(u) \leq (\tilde{c}(l, p(\cdot)) \rho_{p(\cdot)}(\nabla u))^{\theta} \rho_{p(\cdot)}(u)^{1-\theta}.$$

□

We can assume $r(x) = \frac{lp(x)}{l-1}$ and $\theta = \frac{1}{p_S}$, then we have

$$\begin{aligned} & \int_{\Omega} \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln \left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \right) dx \leq \\ & \leq \frac{r_m}{r_m - p_S} \ln \left(\frac{\left((\tilde{c}(l, p(\cdot)) \rho_{p(\cdot)}(\nabla u))^{\theta} \rho_{p(\cdot)}(u)^{1-\theta} \right)^{\frac{p_S}{r_m}}}{\rho_{p(\cdot)}(u)} \right) \end{aligned}$$

and

$$\int_{\Omega} \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln \left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \right) dx \leq \frac{r_m}{r_m - p_S} \ln \left(c(l, p(\cdot)) \frac{\rho_{p(\cdot)}(\nabla u)^{\frac{1}{r_m}}}{\rho_{p(\cdot)}(u)^{1 - \frac{p_S - 1}{r_m}}} \right),$$

$$\begin{aligned}
 & \int_{\Omega} \frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \ln \left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \right) dx \leq \\
 & \leq \frac{r_m}{r_m - p_S} \ln(c(l, p(\cdot))) \\
 & \quad + \frac{1}{r_m} \frac{r_m}{r_m - p_S} \ln(\rho_{p(\cdot)}(\nabla u)) \\
 & \quad - \left(1 - \frac{p_S - 1}{r_m}\right) \frac{r_m}{r_m - p_S} \ln(\rho_{p(\cdot)}(u)) \leq \\
 & \leq c_1 \ln \left(c(l, p(\cdot)) \frac{\rho_{p(\cdot)}(\nabla u)}{\rho_{p(\cdot)}(u)} \right)
 \end{aligned}$$

for all $u \in L^{p(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$.

Theorem 3.2. *Let $p \in P^{\log}(\Omega)$, $1 < p(\cdot) < \infty$, and $r(x) \leq p^*(x)$, then the inequality*

$$\int_{\Omega} \frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \ln \left(\frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \right) dx \leq c_1 \ln \left(c(l, p(\cdot), r(\cdot)) \frac{\rho_{p(\cdot)}(\nabla u)}{\rho_{r(\cdot)}(u)} \right)$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$

Proof. Let $p \in P^{\log}(\Omega)$, $p(x) < l$. Then, we have

$$\rho_{s(\cdot)}(u) \leq (\tilde{c}(l, p(\cdot)) \rho_{p(\cdot)}(\nabla u))^{\theta} \rho_{r(\cdot)}(u)^{1-\theta}.$$

□

We obtain

$$\int_{\Omega} \frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \ln \left(\frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \right) dx \leq \frac{1}{1 - \frac{r_S}{s_m}} \ln \left(\frac{(\rho_{s(\cdot)}(u))^{\frac{r_S}{s_m}}}{\rho_{r(\cdot)}(u)} \right),$$

so, we have

$$\begin{aligned}
 & \int_{\Omega} \frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \ln \left(\frac{|u|^{r(x)}}{\rho_{r(\cdot)}(u)} \right) dx \leq \\
 & \leq \frac{1}{1 - \frac{r_S}{s_m}} \ln \left(\frac{\left((\tilde{c}(l, p(\cdot)) \rho_{p(\cdot)}(\nabla u))^{\theta} \rho_{r(\cdot)}(u)^{1-\theta} \right)^{\frac{r_S}{s_m}}}{\rho_{r(\cdot)}(u)} \right) \leq \\
 & \leq \frac{1}{1 - \frac{r_S}{s_m}} \ln \left(c(l, p(\cdot)) \frac{\left((\rho_{p(\cdot)}(\nabla u))^{\frac{1}{r_S}} \rho_{r(\cdot)}(u)^{\frac{r_S-1}{r_S}} \right)^{\frac{r_S}{s_m}}}{\rho_{r(\cdot)}(u)} \right) \leq \\
 & \leq c_1 \ln \left(c(l, p(\cdot), r(\cdot)) \frac{\rho_{p(\cdot)}(\nabla u)}{\rho_{r(\cdot)}(u)} \right)
 \end{aligned}$$

with $\theta = \frac{1}{r_S}$ for $r(x) \leq p^*(x)$.

The results above can be presented in the parametric form.

Theorem 3.3. *Let $p \in P^{\log}(\Omega)$, $1 < p(\cdot) < \infty$, and $r(x) \leq p^*(x)$. For each $\mu > 0$, there exists a positive number $c(\mu)$ such that*

$$\begin{aligned} \int_{\Omega} |u|^{p(x)} \ln \left(\frac{|u|^{p(x)}}{\rho_{p(\cdot)}(u)} \right) dx &\leq \\ &\leq \mu \rho_{p(\cdot)}(\nabla u) + c(\mu) \rho_{p(\cdot)}(u) + c_1 \frac{p_S}{p_m} \rho_{p(\cdot)}(u) \ln(\rho_{p(\cdot)}(u)) \end{aligned}$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$. Remark: The condition $r(x) \leq p^(x)$ ensures the continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$.*

4. Existence of a weak solution to the hyperbolic equation with logarithmic nonlinearity

In this section, we consider the hyperbolic equation involving variable exponents in the following form

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) + \frac{\partial u}{\partial t} &= \theta |u|^{p(x)-2} u \ln |u|, \\ u(x, 0) &= \phi_0(x), \\ \frac{\partial u(x, 0)}{\partial t} &= \phi_1(x), \\ u(x, t)|_{\partial\Omega \times [0, T]} &= 0, \\ \frac{\partial u(x, t)}{\partial t} \Big|_{\partial\Omega \times [0, T]} &= 0, \end{aligned} \tag{4.1}$$

where Ω is a bounded domain in R^l with a smooth boundary $\partial\Omega$, and ϕ_0, ϕ_1 are fixed initial functions. Where a function $p^* : \Omega \rightarrow R$ is defined by $p^*(x) = \frac{lp(x)}{l-p(x)}$ if $p(x) < l$ and $p^*(x) = \infty$ if $p(x) \geq l$. We assume that θ is a positive number.

Definition 4.1. *The energy functional is defined by*

$$E_t(u) = \frac{1}{2} \int_{\Omega} |\partial_t u|^2 dx + \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \tag{4.2}$$

$$- \theta \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} \ln |u| dx + \theta \int_{\Omega} \frac{1}{p(x)^2} |u|^{p(x)} dx. \tag{4.3}$$

We have the following energy estimate.

Lemma 4.2. *The energy functional E_t is a non-increasing function of $t \geq 0$ such that*

$$\frac{dE_t(u)}{dt} = - \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx \leq 0 \tag{4.4}$$

for all $t \geq 0$.

Proof. Multiplying the hyperbolic equation by $\frac{\partial u}{\partial t}$ and integrating over Ω , we obtain

$$\begin{aligned} \int_{\Omega} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} dx - \int_{\Omega} \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) \frac{\partial u}{\partial t} dx \\ + \int_{\Omega} \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} dx = \theta \int_{\Omega} |u|^{p(x)-2} u \ln |u| \frac{\partial u}{\partial t} dx. \end{aligned}$$

Thus, we deduce

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right. \\ & \left. - \theta \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} \ln |u| dx + \theta \int_{\Omega} \frac{1}{p(x)^2} |u|^{p(x)} dx \right) = - \int_{\Omega} \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} dx \leq 0. \end{aligned}$$

□

The energy estimate is proven. By a density argument, this estimate remains valid for weak solutions.

Now, we are ready to prove the existence of a weak solution.

Theorem 4.3. *For each given $\phi_0 \in W_{1,0}^{p(\cdot)}(\Omega)$ and $\phi_1 \in L^2(\Omega)$, there exists a weak solution u to the boundary problem (4.1), in the following sense: $u \in C([0, T], W_{1,0}^{p(\cdot)}(\Omega))$, $\partial_t u \in C([0, T], L^2(\Omega))$ and u satisfies the identity*

$$\begin{aligned} & \int_{\Omega} \frac{\partial^2 u(x, t)}{\partial t^2} \varphi(x) dx + \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx + \\ & + \int_{\Omega} \frac{\partial u}{\partial t} \varphi dx = \theta \int_{\Omega} |u|^{p(x)-2} u \ln |u| \varphi dx \end{aligned} \tag{4.5}$$

for all $\varphi \in W_{1,0}^{p(\cdot)}(\Omega)$.

Proof. We are going to employ the Galerkin approximation method. Let $\{\psi_k, k \in N\}$ be a basis in $W_{1,0}^{p(\cdot)}(\Omega)$, orthogonal in $L^2(\Omega)$. Let Λ_j be a linear span $span\{\psi_1, \psi_2, \dots, \psi_j\}$. We define the projections of the initial conditions on the finite-dimensional subspaces of Λ_j by

$$\phi_{0,j}(x) = \sum_{k=1, \dots, j} a_k \psi_k \xrightarrow[j \rightarrow \infty]{W_{1,0}^{p(\cdot)}(\Omega)} \phi_0(x),$$

$$\phi_{1,j}(x) = \sum_{k=1, \dots, j} b_k \psi_k \xrightarrow[j \rightarrow \infty]{W_{1,0}^{p(\cdot)}(\Omega)} \phi_1(x).$$

The Galerkin approximate solution will be assumed to have the form

$$u_j(x, t) = \sum_{k=1, \dots, j} \xi_k(t) \psi_k(x),$$

with coefficients $\xi_k(t) = \langle u(x, t), \psi_k(x) \rangle$ that satisfy the approximate problems in Λ_j

$$\begin{aligned} & \int_{\Omega} \frac{\partial^2 u_j}{\partial t^2} \varphi dx + \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla \varphi dx + \\ & + \int_{\Omega} \frac{\partial u_j}{\partial t} \varphi dx = \theta \int_{\Omega} |u_j|^{p(x)-2} u_j \ln |u_j| \varphi dx, \end{aligned}$$

$$\begin{aligned}
u_j(x, 0) &= \phi_{0,j} = \sum_{k=1, \dots, j} \langle \phi_0, \psi_k \rangle \psi_k, \\
\frac{\partial u_j(x, 0)}{\partial t} &= \phi_{1,j} = \sum_{k=1, \dots, j} \langle \phi_1, \psi_k \rangle \psi_k, \\
u_j(x, t)|_{\partial\Omega \times [0, T]} &= 0, \\
\frac{\partial u_j(x, t)}{\partial t} \Big|_{\partial\Omega \times [0, T]} &= 0.
\end{aligned}$$

□

It is a system of j ordinary differential equations of the second order for the coefficients

$$\xi_k(t) = \langle u(x, t), \psi_k(x) \rangle, k = 1, \dots, j$$

on $[0, t_j)$, $0 < t_j \leq T$ for each index j . Therefore, the coefficients $\xi_k(t)$, $k = 1, \dots, j$ satisfy the integral identity

$$\begin{aligned}
&\int_{\Omega} \left(\frac{\partial^2 u_j}{\partial t^2} - \operatorname{div} \left(|\nabla u_j|^{p(x)-2} \nabla u_j \right) + \right. \\
&\quad \left. + \frac{\partial u_j}{\partial t} - \theta |u_j|^{p(x)-2} u_j \ln |u_j| \right) \psi_k(x) dx = 0, \quad k = 1, \dots, j.
\end{aligned}$$

Next, we must prove that we can take $t_j = T$ for each j . Let $\frac{\partial u_j}{\partial t}$ be a test function, then we have

$$\begin{aligned}
&\int_{\Omega} \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_j}{\partial t} dx - \int_{\Omega} \operatorname{div} \left(|\nabla u_j|^{p(x)-2} \nabla u_j \right) \frac{\partial u_j}{\partial t} dx \\
&\quad + \int_{\Omega} \frac{\partial u_j}{\partial t} \frac{\partial u_j}{\partial t} dx = \theta \int_{\Omega} |u_j|^{p(x)-2} u_j \ln |u_j| \frac{\partial u_j}{\partial t} dx,
\end{aligned}$$

integration by parts gives the following identity

$$\begin{aligned}
\frac{d}{dt} E_t(u_j) &= \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} \left(\frac{\partial u_j}{\partial t} \right)^2 dx + \int_{\Omega} \frac{1}{p(x)} |\nabla u_j|^{p(x)} dx \right. \\
&\quad \left. - \theta \int_{\Omega} \frac{1}{p(x)} |u_j|^{p(x)} \ln |u_j| dx + \theta \int_{\Omega} \frac{1}{p(x)^2} |u_j|^{p(x)} dx \right) = \\
&= - \int_{\Omega} \left(\frac{\partial u_j}{\partial t} \right)^2 dx \leq 0.
\end{aligned}$$

Integration of the last inequality with respect to time from 0 to t gives

$$E_t(u_j) \leq E_0(u_j).$$

We obtain

$$E_t(u_j) = \frac{1}{2} \int_{\Omega} \left(\frac{\partial u_j}{\partial t} \right)^2 dx + \int_{\Omega} \frac{1}{p(x)} |\nabla u_j|^{p(x)} dx - \theta \int_{\Omega} \frac{1}{p(x)} |u_j|^{p(x)} \ln |u_j| dx + \theta \int_{\Omega} \frac{1}{p(x)^2} |u_j|^{p(x)} dx.$$

By the parametric logarithmic inequality, we have

$$\begin{aligned} & \int_{\Omega} \frac{1}{p(x)} |u_j|^{p(x)} \ln |u_j| dx \leq \\ & \leq \frac{1}{p_m} \mu \int_{\Omega} |\nabla u_j|^{p(x)} dx + \frac{1}{p_m} c(\mu) \int_{\Omega} |u_j|^{p(x)} dx \\ & \quad + \frac{1}{p_m} \left(c_1 \frac{p_S}{p_m} + 1 \right) \ln \left(\int_{\Omega} |u_j|^{p(x)} dx \right) \int_{\Omega} |u_j|^{p(x)} dx. \end{aligned}$$

So,

$$\begin{aligned} E_t(u_j) & \geq \frac{1}{2} \int_{\Omega} \left(\frac{\partial u_j}{\partial t} \right)^2 dx \\ & \quad + \left(\frac{1}{p_S} - \frac{\theta}{p_m} \mu \right) \int_{\Omega} |\nabla u_j|^{p(x)} dx + \theta \left(\frac{1}{p_S^2} - \frac{1}{p_m} c(\mu) \right) \int_{\Omega} |u_j|^{p(x)} dx \\ & \quad - \frac{1}{p_m} \left(c_1 \frac{p_S}{p_m} + 1 \right) \ln \left(\int_{\Omega} |u_j|^{p(x)} dx \right) \int_{\Omega} |u_j|^{p(x)} dx. \end{aligned}$$

Assume $c = 2E_0(u_j)$, we have

$$\begin{aligned} & \int_{\Omega} \left(\frac{\partial u_j}{\partial t} \right)^2 dx + 2 \left(\frac{1}{p_S} - \frac{\theta}{p_m} \mu \right) \int_{\Omega} |\nabla u_j|^{p(x)} dx \\ & \quad + \theta \left(\frac{1}{p_S^2} - \frac{1}{p_m} c(\mu) \right) \int_{\Omega} |u_j|^{p(x)} dx \leq \\ & \leq c + 2 \frac{1}{p_m} \left(c_1 \frac{p_S}{p_m} + 1 \right) \ln \left(\int_{\Omega} |u_j|^{p(x)} dx \right) \int_{\Omega} |u_j|^{p(x)} dx. \end{aligned}$$

Now, we select μ such that both inequalities $\frac{p_m}{\theta p_S} > \mu$ and $\frac{p_m}{p_S^2} > c(\mu)$ hold together, then we obtain

$$\begin{aligned} & \int_{\Omega} \left(\frac{\partial u_j}{\partial t} \right)^2 dx + \int_{\Omega} |\nabla u_j|^{p(x)} dx + \int_{\Omega} |u_j|^{p(x)} dx \leq \\ & \leq c \left(1 + \frac{1}{p_m} \ln \left(\int_{\Omega} |u_j|^{p(x)} dx \right) \int_{\Omega} |u_j|^{p(x)} dx \right). \end{aligned}$$

We deduce

$$u_j(\cdot, t) = u_j(\cdot, 0) + \int_{[0, t]} \frac{\partial u_j(\cdot, \tau)}{\partial t} d\tau.$$

Therefore, we obtain

$$\begin{aligned} \int_{\Omega} \left(\frac{\partial u_j(t)}{\partial t} \right)^2 dx &= \int_{\Omega} \left(u_j(\cdot, 0) + \int_{[0, t]} \frac{\partial u_j(\cdot, \tau)}{\partial t} d\tau \right)^2 dx \leq \\ &\leq 2 \int_{\Omega} u_j(0)^2 dx + 2 \int_{\Omega} \left(\int_{[0, t]} \frac{\partial u_j(\cdot, \tau)}{\partial t} d\tau \right)^2 dx \leq \\ &\leq 2 \int_{\Omega} u_j(0)^2 dx + 2T \int_{[0, t]} \int_{\Omega} \left(\frac{\partial u_j(\cdot, \tau)}{\partial t} \right)^2 dx d\tau. \end{aligned}$$

Thus, we conclude

$$\int_{\Omega} \left(\frac{\partial u_j(t)}{\partial t} \right)^2 dx \leq 2 \int_{\Omega} u_j(0)^2 dx + 2Tc_2 \left(1 + \frac{1}{p_m} \ln \left(\int_{\Omega} |u_j|^{p(x)} dx \right) \int_{\Omega} |u_j|^{p(x)} dx \right).$$

We take $c_3 = \max \left(2 \int_{\Omega} u_j(0)^2 dx, 2Tc_2 \right)$, so we have

$$\begin{aligned} \int_{[0, T]} \int_{\Omega} \left(\frac{\partial u_j(t)}{\partial t} \right)^2 dx dt &\leq \\ &2c_3 \left(1 + \int_{[0, t]} \frac{1}{p_m} \ln \left(\int_{\Omega} |u_j|^{p(x)} dx \right) \int_{\Omega} |u_j|^{p(x)} dx d\tau \right). \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_{[0, T]} \int_{\Omega} \left(\frac{\partial u_j(t)}{\partial t} \right)^2 dx dt &\leq \\ &\leq 2c_2 \left(1 + \int_{[0, t]} \left(c_3 + \int_{\Omega} |u_j|^{p(x)} dx \right) \ln \left(c_3 + \left(\int_{\Omega} |u_j|^{p(x)} dx \right)^{\frac{1}{p_m}} \right) d\tau \right) \end{aligned}$$

and

$$\int_{[0, T]} \int_{\Omega} \left(\frac{\partial u_j(t)}{\partial t} \right)^2 dx dt \leq 2c_3 \exp(2c_3 t) = \text{const.}$$

We conclude

$$\int_{\Omega} \left(\frac{\partial u_j}{\partial t} \right)^2 dx + \int_{\Omega} |\nabla u_j|^{p(x)} dx + \int_{\Omega} |u_j|^{p(x)} dx \leq c$$

where a positive constant c does not depend on $t \in [0, T]$ and j . We obtain

$$\max_{t \in [0, t_j]} \int_{\Omega} \left(\frac{\partial u_j}{\partial t} \right)^2 dx + \max_{t \in [0, t_j]} \int_{\Omega} |\nabla u_j|^{p(x)} dx + \max_{t \in [0, t_j]} \int_{\Omega} |u_j|^{p(x)} dx \leq \tilde{c},$$

where a positive constant c does not depend on $t \in [0, T]$ and j . We take the limit as t_j tends to T . Thus, we have:

$$1) \text{ set } \{u_j\} \text{ is uniformly bounded in } L^2 \left([0, T], W_{1,0}^{p(\cdot)}(\Omega) \right),$$

and

2) set $\left\{ \frac{\partial u_j}{\partial t} \right\}$ is uniformly bounded in $L^\infty ([0, T], L^2(\Omega))$.

There exists a subsequence of $\{u_j, j \in N\}$, which will still be denoted by $\{u_j, j \in N\}$, such that:

1. Sequence $\{u_j\}$ converges to u *-weakly in $L^\infty ([0, T], W_{1,0}^{p(\cdot)}(\Omega))$ as j tends to infinity;
2. Sequence $\left\{ \frac{\partial u_j}{\partial t} \right\}$ converges to $\frac{\partial u}{\partial t}$ *-weakly in $L^\infty ([0, T], L^2(\Omega))$ as j tends to infinity;
3. Sequence $\{u_j\}$ converges to u weakly in $L^2 ([0, T], W_{1,0}^{p(\cdot)}(\Omega))$ as j tends to infinity;
4. Sequence $\left\{ \frac{\partial u_j}{\partial t} \right\}$ converges to $\frac{\partial u}{\partial t}$ weakly in $L^2 ([0, T], L^2(\Omega))$ as j tends to infinity.

By the Aubin–Lions–Simon lemma, we obtain that sequence $\{u_j\}$ converges to u in $L^2 ([0, T], L^2(\Omega))$ as j tends to infinity, and $\{u_j\}$ converges to u almost everywhere in $\Omega \times [0, T]$ as j tends to infinity. By continuity of $r \rightarrow r^{p(x)-1} \ln r^\theta$, we obtain that sequence $\left\{ |u_j|^{p(x)-2} u_j \ln |u_j|^\theta \right\}$ converges to $|u|^{p(x)-2} u \ln |u|^\theta$ almost everywhere in $\Omega \times [0, T]$ as j tends to infinity.

For all $\varphi \in \Lambda_j$, we obtain the identity

$$\begin{aligned} & \int_{\Omega} \frac{\partial u_j}{\partial t} \varphi dx - \int_{\Omega} \phi_{1,j} \varphi dx + \int_{[0, t]} \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla \varphi dx d\tau \\ & + \int_{[0, t]} \int_{\Omega} \frac{\partial u_j}{\partial t} \varphi dx d\tau = \theta \int_{[0, t]} \int_{\Omega} |u_j|^{p(x)-2} u_j \ln |u_j| \varphi dx d\tau. \end{aligned}$$

We pass to the limit as j goes to infinity, and deduce

$$\begin{aligned} & \int_{\Omega} \frac{\partial u}{\partial t} \varphi dx = \int_{\Omega} \phi_1 \varphi dx - \int_{[0, t]} \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx d\tau - \\ & - \int_{[0, t]} \int_{\Omega} \frac{\partial u}{\partial t} \varphi dx d\tau + \theta \int_{[0, t]} \int_{\Omega} |u|^{p(x)-2} u \ln |u| \varphi dx d\tau. \end{aligned}$$

The right-hand side of the last identity is an absolutely continuous function. We differentiate at t , thus the identity

$$\begin{aligned} & \theta \int_{\Omega} |u(x, t)|^{p(x)-2} u(x, t) \ln |u(x, t)| \varphi(x) dx = \\ & = \int_{\Omega} \frac{\partial^2 u(x, t)}{\partial t^2} \varphi(x) dx + \int_{\Omega} \frac{\partial u(x, t)}{\partial t} \varphi(x) dx \\ & + \int_{\Omega} |\nabla u(x, t)|^{p(x)-2} \nabla u(x, t) \nabla \varphi(x) dx \end{aligned}$$

holds for all $\varphi \in W_{1,0}^{p(\cdot)}(\Omega)$. Employing the initial conditions, we deduce:

- 1) sequence $\{u_j\}$ converges to u weakly in $L^2 ([0, T], W_{1,0}^{p(\cdot)}(\Omega))$ as j tends to infinity;
- 2) sequence $\left\{ \frac{\partial u_j}{\partial t} \right\}$ converges to $\frac{\partial u}{\partial t}$ weakly in $L^2 ([0, T], L^2(\Omega))$ as j tends to infinity;

3) sequence $\{u_j\}$ converges to u in $C([0, T], L^2(\Omega))$ as j tends to infinity; and,

4) sequence $\{u_j(x, 0) = \phi_{0,j}\}$ converges to $u(x, 0) = \phi_0(x)$ in $W_{1,0}^{p(\cdot)}(\Omega) \cap H_0^1(\Omega)$ as j tends to infinity.

Let $\eta \in C_0^\infty([0, T])$. Then, we have

$$\begin{aligned} & - \int_{\Omega} \frac{\partial u_j}{\partial t} \eta'(t) \varphi dx = \int_{[0, t]} \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla \varphi \eta(\tau) dx d\tau - \\ & - \int_{[0, t]} \int_{\Omega} \frac{\partial u_j}{\partial t} \varphi \eta(\tau) dx d\tau + \theta \int_{[0, t]} \int_{\Omega} |u_j|^{p(x)-2} u_j \ln |u_j| \varphi \eta(\tau) dx d\tau \end{aligned}$$

for all $\varphi \in W_{1,0}^{p(\cdot)}(\Omega)$. We pass to the limit as j tends to infinity, and deduce

$$\begin{aligned} & - \int_{\Omega} \frac{\partial u}{\partial t} \eta'(t) \varphi dx = \int_{[0, t]} \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \eta(\tau) dx d\tau - \\ & - \int_{[0, t]} \int_{\Omega} \frac{\partial u}{\partial t} \varphi \eta(\tau) dx d\tau + \theta \int_{[0, t]} \int_{\Omega} |u|^{p(x)-2} u \ln |u| \varphi \eta(\tau) dx d\tau. \end{aligned}$$

Therefore, we obtain that sequence $\left\{ \frac{\partial u_j(x, 0)}{\partial t} = \phi_{1,j} \right\}$ converges to $\frac{\partial u(x, 0)}{\partial t} = \phi_1(x)$ in $L^2(\Omega)$ as index j tends to infinity. Thus, the existence of a weak solution has been proven.

5. New potential wells technique for variable exponent spaces

For small positive numbers δ , we define the variable exponent functionals by

$$J_\delta(u) = \delta \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \theta \int_{\Omega} \frac{1}{p(x)^2} |u|^{p(x)} dx - \theta \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} \ln |u| dx,$$

and

$$I_\delta(u) = \delta \int_{\Omega} |\nabla u|^{p(x)} dx - \theta \int_{\Omega} |u|^{p(x)} \ln |u| dx.$$

For all $u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$, for $\lambda > 0$, we consider the mapping $\lambda \mapsto \Psi_\delta(\lambda) = J_\delta(\lambda u)$ defined by

$$\begin{aligned} \Psi_\delta(\lambda) &= \delta \int_{\Omega} \frac{\lambda^{p(x)}}{p(x)} |\nabla u|^{p(x)} dx + \\ &+ \theta \int_{\Omega} \frac{\lambda^{p(x)}}{p(x)^2} |u|^{p(x)} dx - \theta \int_{\Omega} \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)} \ln |\lambda u| dx. \end{aligned}$$

We estimate the logarithm term, and have

$$\begin{aligned} & \int_{\Omega} |u|^{p(x)} \ln |u| \, dx \leq \\ & \leq \int_{\{|u| \leq 1\}} |u|^{p(x)} \ln |u| \, dx + \int_{\{|u| \geq 1\}} |u|^{p(x)} \ln |u| \, dx \leq \\ & \leq c \int_{\{|u| \geq 1\}} |u|^{p(x)+\gamma} \, dx \leq c \int_{\Omega} |u|^{p(x)+\gamma} \, dx \end{aligned}$$

for all $\gamma > 0$. By application of the Young inequality, we obtain that for each $\varepsilon > 0$ there exists some constant $c(\varepsilon) > 0$ such that

$$\begin{aligned} & \int_{\Omega} |u|^{p(x)} \ln |u| \, dx \leq c \int_{\Omega} |u|^{p(x)+\gamma} \, dx \leq \\ & \leq c\tilde{\varepsilon} \rho_{p(\cdot)}(|\nabla u|) + cc(\tilde{\varepsilon}) \rho_{p(\cdot)}(u) \leq \\ & \leq \varepsilon \rho_{p(\cdot)}(|\nabla u|) + c(\varepsilon) (\rho_{p(\cdot)}(u))^\beta, \end{aligned}$$

where we select $0 < \gamma < \frac{p_m^2}{l}$ and

$$1 < \beta = p_m \frac{\left(1 - \frac{l\gamma}{p_m(p_m+\gamma)}\right) (p_m + \gamma)}{p_m^2 - l\gamma}.$$

Lemma 5.1. *Let $u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$. Then, we have $\lim_{\lambda \rightarrow +0} \Psi_\delta(\lambda) = 0$ and $\lim_{\lambda \rightarrow \infty} \Psi_\delta(\lambda) = -\infty$.*

Proof. Straightforwardly, we have

$$\begin{aligned} \Psi_\delta(\lambda) &= J_\delta(\lambda u) = \delta \int_{\Omega} \frac{1}{p(x)} |\lambda \nabla u|^{p(x)} \, dx - \\ & \quad - \theta \int_{\Omega} \frac{1}{p(x)} |\lambda u|^{p(x)} \ln |\lambda u| \, dx + \theta \int_{\Omega} \frac{1}{p(x)^2} |\lambda u|^{p(x)} \, dx \xrightarrow{\lambda \rightarrow 0} 0 \end{aligned}$$

and

$$\begin{aligned} & \Psi_\delta(\lambda) \leq \\ & \leq \int_{\Omega} \frac{1}{p(x)} \left(\left(\delta |\nabla u|^{p(x)} + \theta \frac{1}{p(x)} |u|^{p(x)} \right) \lambda^{p(x)} - \theta \lambda^{p(x)} |u|^{p(x)} \ln |\lambda u| \right) \, dx = \\ & = \int_{\Omega} \frac{1}{p(x)} \left(\left(\delta |\nabla u|^{p(x)} + \theta \frac{1}{p(x)} |u|^{p(x)} - \theta |u|^{p(x)} \ln |\lambda| \right) \lambda^{p(x)} \right) \, dx \\ & \quad - \theta \int_{\Omega} \frac{1}{p(x)} \lambda^{p(x)} |u|^{p(x)} \ln |u| \, dx \xrightarrow{\lambda \rightarrow \infty} -\infty. \end{aligned}$$

□

Lemma 5.2. *Let $u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$. Then, there exists a solution λ to the equation $\Psi'_\delta(\lambda) = 0$ such that*

$$\min \{D_1, D_1\} \leq \lambda \leq \max \{D_1, D_1\}$$

where,

$$D_1 = \exp \left(\frac{1}{\theta B(u)} \left(\delta \frac{\min(\lambda^{p_S-1}, \lambda^{p_m-1})}{\max(\lambda^{p_S-1}, \lambda^{p_m-1})} A(u) - \theta C(u) \right) \right),$$

$$D_2 = \exp \left(\frac{1}{\theta B(u)} \left(\delta \frac{\max(\lambda^{p_S-1}, \lambda^{p_m-1})}{\min(\lambda^{p_S-1}, \lambda^{p_m-1})} A(u) - \theta C(u) \right) \right),$$

and

$$A(u) = \int_{\Omega} |\nabla u|^{p(x)} dx, \quad B(u) = \int_{\Omega} |u|^{p(x)} dx, \quad C(u) = \int_{\Omega} |u|^{p(x)} \ln |u| dx.$$

Proof. We have

$$\begin{aligned} \Psi'_\delta(\lambda) &= \delta \int_{\Omega} \lambda^{p(x)-1} |\nabla u|^{p(x)} dx + \theta \int_{\Omega} \frac{\lambda^{p(x)-1}}{p(x)} |u|^{p(x)} dx \\ &\quad - \theta \int_{\Omega} \left(\lambda^{p(x)-1} |u|^{p(x)} \ln |\lambda u| + \frac{\lambda^{p(x)-1}}{p(x)} |u|^{p(x)} \right) dx \\ &= \delta \int_{\Omega} \lambda^{p(x)-1} |\nabla u|^{p(x)} dx - \theta \int_{\Omega} \lambda^{p(x)-1} |u|^{p(x)} \ln |\lambda| dx \\ &\quad - \theta \int_{\Omega} \lambda^{p(x)-1} |u|^{p(x)} \ln |u| dx. \end{aligned}$$

Thus, we have

$$\begin{aligned} \Psi'_\delta(\lambda) &\geq \delta \min(\lambda^{p_S-1}, \lambda^{p_m-1}) A(u) \\ &\quad - \theta \max(\lambda^{p_S-1}, \lambda^{p_m-1}) (\ln |\lambda| B(u) + C(u)) = f_1(\lambda) \end{aligned}$$

and

$$\begin{aligned} \Psi'_\delta(\lambda) &\leq \delta \max(\lambda^{p_S-1}, \lambda^{p_m-1}) A(u) \\ &\quad - \theta \min(\lambda^{p_S-1}, \lambda^{p_m-1}) (\ln |\lambda| B(u) + C(u)) = f_2(\lambda). \end{aligned}$$

Thus, we have that

$$\min \{D_1, D_1\} \leq \lambda \leq \max \{D_1, D_1\},$$

where, we denote

$$D_1 = \exp \left(\frac{1}{\theta B(u)} \left(\delta \frac{\min(\lambda^{p_S-1}, \lambda^{p_m-1})}{\max(\lambda^{p_S-1}, \lambda^{p_m-1})} A(u) - \theta C(u) \right) \right)$$

$$D_2 = \exp \left(\frac{1}{\theta B(u)} \left(\delta \frac{\max(\lambda^{p_S-1}, \lambda^{p_m-1})}{\min(\lambda^{p_S-1}, \lambda^{p_m-1})} A(u) - \theta C(u) \right) \right),$$

and

$$\begin{aligned} A(u) &= \int_{\Omega} |\nabla u|^{p(x)} dx, \\ B(u) &= \int_{\Omega} |u|^{p(x)} dx, \\ C(u) &= \int_{\Omega} |u|^{p(x)} \ln |u| dx. \end{aligned}$$

□

Lemma 5.3. *Let $u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$. Then, there exists a number $\lambda^* = \lambda^*(u) > 0$ such that the functional Ψ_{δ} attains its maximum at $\lambda = \lambda^*$. We have that:*

- 1) if $I_{\delta}(u) < 0$ then $\lambda^* \in (0, 1)$;
- 2) if $I_{\delta}(u) = 0$ then $\lambda^* = 1$;
- 3) if $I_{\delta}(u) > 0$ then $\lambda^* > 1$.

Proof. We have that $I_{\delta}(\lambda u) = \lambda \Psi'_{\delta}(\lambda)$, where

$$\Psi'_{\delta}(\lambda) = \delta \int_{\Omega} \lambda^{p(x)-1} |\nabla u|^{p(x)} dx - \theta \int_{\Omega} \lambda^{p(x)-1} |u|^{p(x)} \ln |\lambda u| dx$$

and

$$I_{\delta}(u) = \delta \int_{\Omega} |\nabla u|^{p(x)} dx - \theta \int_{\Omega} |u|^{p(x)} \ln |u| dx.$$

Therefore, we obtain

$$\Psi'_{\delta}(\lambda) \leq \lambda^{ps-1} I_{\delta}(u)$$

for $\lambda \in (0, 1)$, and

$$\Psi'_{\delta}(\lambda) \geq \lambda^{ps-1} I_{\delta}(u)$$

for $\lambda > 1$. The lemma is proven. □

Definition 5.4. *The Nehari manifold N_{δ} associated with the energy functional J_{δ} is defined by*

$$N_{\delta} = \left\{ u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\} : I_{\delta}(u) = \langle J'_{\delta}(u), u \rangle = 0 \right\}.$$

Straightforwardly, we obtain that the set N_{δ} is not empty, and the operator J_{δ} is coercive on the set N_{δ} . We denote

$$\begin{aligned} d_{\delta} &= \inf_{u \in N_{\delta}} J_{\delta}(u) \\ &= \inf \left\{ \sup_{\lambda \geq 0} J_{\delta}(u) : u \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\} \right\}. \end{aligned}$$

We define

$$\begin{aligned} W_{\delta} &= \left\{ u \in W_{1,0}^{p(\cdot)}(\Omega) : J_{\delta}(u) < d_{\delta}, I_{\delta}(u) > 0 \right\} \cup \{0\}, \\ V_{\delta} &= \left\{ u \in W_{1,0}^{p(\cdot)}(\Omega) : J_{\delta}(u) < d_{\delta}, I_{\delta}(u) < 0 \right\}. \end{aligned}$$

By the parametric logarithmic Sobolev inequality, we estimate

$$\begin{aligned} I_\delta(u) &= \delta \int_\Omega |\nabla u|^{p(x)} dx - \theta \int_\Omega |u|^{p(x)} \ln |u| dx \geq \\ &\geq (\delta - \theta\mu) \int_\Omega |\nabla u|^{p(x)} dx \\ &\quad - \left(\theta c(\mu) + \theta \left(c_1 \frac{p_S}{p_m} + 1 \right) \ln \left(\int_\Omega |u|^{p(x)} dx \right) \right) \int_\Omega |u|^{p(x)} dx. \end{aligned}$$

We obtain that

- 1) if $\rho_{p(\cdot)}(u) \leq \exp \left(-c(\mu) \left(c_1 \frac{p_S}{p_m} + 1 \right)^{-1} \right) = \ell$ and $\mu < \frac{\delta}{\theta}$, then $I_\delta(u) > 0$;
- 2) if $I_\delta(u) < 0$ and $\mu < \frac{\delta}{\theta}$, then $\rho_{p(\cdot)}(u) > \exp \left(-c(\mu) \left(c_1 \frac{p_S}{p_m} + 1 \right)^{-1} \right)$;
- 3) if $I_\delta(u) = 0$ and $\mu < \frac{\delta}{\theta}$, then $\rho_{p(\cdot)}(u) \geq \exp \left(-c(\mu) \left(c_1 \frac{p_S}{p_m} + 1 \right)^{-1} \right)$.

Theorem 5.5. *Let $\phi_0 \in W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\}$, $\phi_0 \in W_\delta$, $0 < \delta < \ell$ and $\phi_1 \in L^2(\Omega)$ be given. We assume $E_0 < d$, then the hyperbolic problem (1) has a global weak solution $u \in L^\infty([0, \infty), W_{1,0}^{p(\cdot)}(\Omega) \setminus \{0\})$, $\partial_t u \in L^\infty([0, \infty), L^2(\Omega))$ and the identity*

$$\begin{aligned} \int_\Omega \frac{\partial^2 u(x, t)}{\partial t^2} \varphi(x) dx + \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx \\ + \int_\Omega \frac{\partial u}{\partial t} \varphi dx = \theta \int_\Omega |u|^{p(x)-2} u \ln |u| \varphi dx \end{aligned}$$

holds for all $\varphi \in W_{1,0}^{p(\cdot)}(\Omega)$. Moreover, the weak solution u grows as an exponential function as time approaches infinity. Since $L_t(u)$ is equivalent to the energy norm, this implies exponential growth of the solution in the energy norm.

Proof. The existence can be proven similarly to the previous theorem. We have proven that

$$-E'_t(u) = \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 \geq 0.$$

We assume that $0 < E_0(u) < E_1 < \min \left(\frac{\ell^{p_m}}{p_S}, \frac{\ell^{p_S}}{p_S} \right)$, and denote

$$L_t(u) = E_1 - E_t(u) + \varepsilon \int_\Omega u \frac{\partial u}{\partial t} dx + \frac{\varepsilon}{2} \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2$$

for all $t \geq 0$. We differentiate with respect to time and obtain

$$\begin{aligned} L'_t(u) &= -E'_t(u) + \\ &+ \varepsilon \int_{\Omega} u \left(\operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) + \theta |u|^{p(x)-2} u \ln |u| \right) dx \\ &+ \varepsilon \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx + \varepsilon \int_{\Omega} u \frac{\partial u}{\partial t} dx = (1 + \varepsilon) \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx - \\ &- \varepsilon \int_{\Omega} |\nabla u|^{p(x)} dx + \varepsilon \int_{\Omega} \theta |u|^{p(x)-2} u \ln |u| dx. \end{aligned}$$

For a positive number γ , we have

$$\begin{aligned} L'_t(u) &= \varepsilon \gamma E_1 - \varepsilon \gamma E_t(u) - \varepsilon \gamma E_1 \\ &+ \left(1 + \varepsilon + \frac{1}{2} \varepsilon \gamma \right) \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx + \varepsilon \int_{\Omega} \frac{\gamma - p(x)}{p(x)} |\nabla u|^{p(x)} dx \\ &- \varepsilon \int_{\Omega} \frac{\gamma - p(x)}{p(x)} \theta |u|^{p(x)-2} u \ln |u| dx + \theta \varepsilon \gamma \int_{\Omega} \frac{1}{p(x)^2} |u|^{p(x)} dx. \end{aligned}$$

Since $\phi_0 \in W_{\delta}$, we obtain $u \in W$ and $I(u) < 0$, therefore, we have $\|u\|_{L^{p(\cdot)}} > \ell$. So, we obtain

$$\begin{aligned} L'_t(u) &\geq \varepsilon \gamma E_1 - \varepsilon \gamma E_t(u) + \\ &+ \left(1 + \varepsilon + \frac{1}{2} \varepsilon \gamma \right) \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx + \varepsilon \int_{\Omega} \frac{\gamma - p(x)}{p(x)} |\nabla u|^{p(x)} dx \\ &- \varepsilon \theta \frac{\gamma - p_m}{p_m} \left(\mu \int_{\Omega} |\nabla u|^{p(x)} dx + c(\mu) \int_{\Omega} |u|^{p(x)} dx + \right. \\ &+ \left. \left(\frac{p_S}{p_m} + 1 \right) \ln \left(\int_{\Omega} |u|^{p(x)} dx \right) \int_{\Omega} |u|^{p(x)} dx \right) \geq \\ &\geq \varepsilon \gamma E_1 - \varepsilon \gamma E_t(u) + \left(1 + \varepsilon + \frac{1}{2} \varepsilon \gamma \right) \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx \\ &+ \varepsilon \left(\frac{\gamma - p_S}{p_S} - \theta \frac{\gamma - p_m}{p_m} \mu \right) \int_{\Omega} |\nabla u|^{p(x)} dx \\ &- \varepsilon \theta \frac{\gamma - p_m}{p_m} \left(c(\mu) - \left(\frac{p_S}{p_m} + 1 \right) \ln \left(\int_{\Omega} |u|^{p(x)} dx \right) \right) \int_{\Omega} |u|^{p(x)} dx. \end{aligned}$$

For $\|u\|_{L^{p(\cdot)}} > \ell$, we choose $\gamma = 4p_S$, then we obtain

$$L'_t(u) \geq \beta \left(E_1 - E_t(u) + \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\Omega} |u|^{p(x)} dx \right),$$

where we denote

$$\beta = \min \left(E_1 - E_t(u), \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx, \int_{\Omega} |\nabla u|^{p(x)} dx, \int_{\Omega} |u|^{p(x)} dx \right) > 0.$$

for $\varepsilon > 0$.

By the Young and generalized Holder inequalities, we have

$$\begin{aligned} L_t(u) &\leq E_1 - E_t(u) + \varepsilon c_1 \left(\int_{\Omega} |u|^{p(x)} dx + E_1 - E_t(u) \right) \\ &\quad + \frac{1}{2} \varepsilon \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx \leq \\ &\leq c \left(E_1 - E_t(u) + \int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\Omega} |u|^{p(x)} dx \right), \end{aligned}$$

where $c_1 = 1 + \frac{1}{E_1 - E_0(u)}$. Thus, we obtain $L'_t(u) \geq \text{const} L_t(u)$ so $L_t(u) \geq L_0(u) \exp(ct)$. \square

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Summation process of monotone and sublinear operators in \mathfrak{B} –statistical sense

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Abstract. By employing the \mathcal{A} –summation process in the \mathfrak{B} –statistical sense, where \mathcal{A} and \mathfrak{B} are sequences of infinite matrices, we provide new results on the classical Korovkin theorem for a sequence of monotone and sublinear operators. Reported results essentially extend some theorems existing in the literature.

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1. Introduction

Approximation theory is closely related to several fields, including the theory of polynomial approximation, functional analysis, summability theory, probability theory, numerical methods for solving differential and integral equations and measure theory.

The classical Korovkin-type theorems are fundamentally focused on the approximation of real-valued functions using positive linear operators (see, e.g., [1, 25]). In fact, this theorem establishes the conditions under which a given sequence of positive linear operators converges to the identity operator in the space of continuous functions on a compact interval. Specifically, if $(U_n)_n$ is a sequence of positive linear operators that map $C([0, 1])$ into itself and the sequence $(U_n(f))_n$ converges uniformly to f on $[0, 1]$ for the three test functions $1, x$ and x^2 , then this sequence also converges to f uniformly on $[0, 1]$ for every $f \in C([0, 1])$ where $C([0, 1]) := \{f : f \text{ is continuous on } [0, 1]\}$. If the sequence of positive linear operators does not converge to the identity operator, it may be beneficial to apply certain matrix summability techniques (see, e.g., [3, 24]). A considerable number of

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researchers have examined such summability methods for various operators and sequences defined across different spaces [11, 14, 30, 31, 32, 34, 35, 38, 39, 40, 41, 42]. In the classical Korovkin theorem, many classical operators typically converge to the value of the approximated function. However, at discontinuity points, they often converge to the average of the left and right limits of the function. In such instances, Cesàro-type matrix summability methods are useful for correcting the convergence deficiencies [9]. The primary objective of using summability theory has always been to transform a non-convergent sequence into a convergent one. This motivation led to Féjér's renowned theorem, which demonstrates the effectiveness of the Cesàro method in ensuring the convergence of the Fourier series of a continuous periodic function. Matrix summability methods are also applied in the field of physics [10], in ergodic theory [27, 28, 29]. Non-matrix summability methods, such as statistical convergence (see, [15, 16, 36]) have also been studied within the approximation theory (see, e.g. [17, 18, 22]). By employing the \mathcal{A} -summation process, Atlıhan and Orhan [4] also present a Korovkin-type result.

Subsequently, the Korovkin theorem has been extended within the framework of monotone and sublinear operators acting on various function spaces, as discussed in [19, 20, 21]. For results related to approximation by sublinear operators, refer to [2]. Gal and Iancu [18] have investigated results on the Korovkin theorem using the concept of statistical convergence for a sequence of monotone and sublinear operators (see, also [22]). At present, related studies can be found in [12, 13, 43]. In this paper, by employing \mathcal{A} -summation process in \mathfrak{B} -statistical sense, we present two new versions of the Korovkin-type theorem for a sequence of monotone and sublinear operators.

2. Preliminaries

Let X be a metric measure space where X is equipped with the metric d and the measure m defined on the sigma field of Borel subsets of X .

If we consider the vector lattice $\mathcal{F}(X)$ of all real-valued functions defined on X , endowed with the pointwise ordering, then some significant vector sublattices of $\mathcal{F}(X)$ are

$$\mathcal{C}(X) = \{f \in \mathcal{F}(X) : f \text{ is bounded and continuous}\},$$

and

$$\mathcal{AC}_b(X) = \{f \in \mathcal{F}(X) : f \text{ is bounded and almost everywhere continuous}\}.$$

These spaces are equipped with the supremum norm, defined as $\|f\| = \sup_{x \in X} |f(x)|$.

As in [18], let X and Y be two metric spaces, and E and F two ordered vector subspaces (or the positive cones) of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$, respectively, that contain the unity. An operator $U : E \rightarrow F$ is called a weakly nonlinear operator (or a weakly nonlinear functional when $F = \mathbb{R}$) if it satisfies the following three conditions:

- (Sublinearity) U is subadditive and positively homogeneous, i.e.,

$$U(f + g) \leq U(f) + U(g) \text{ and } U(\alpha f) = \alpha U(f),$$

for all f, g in E and $\alpha \geq 0$.

- (Monotonicity) $f \leq g$ in E implies $U(f) \leq U(g)$.
- (Translatability) $U(f + \alpha \cdot 1) \leq U(f) + \alpha U(1)$ for all $f \in E$ and $\alpha \geq 0$.
- (Subunital property) $U(1) \leq 1$

If E and F are closed vector sublattices of the Banach lattices $\mathcal{C}(X)$ and $\mathcal{C}(Y)$, respectively, then any monotone and subadditive operator (or functional when $F = \mathbb{R}$) $U : E \rightarrow F$ satisfies the following inequality

$$|U(f) - U(g)| \leq U(|f - g|) \text{ for all } f, g.$$

Before proceeding to our main results we present below certain basic definitions, notations and results which will be needed from \mathfrak{B} -summability [8, 37].

Let $\mathfrak{B} = (\mathfrak{B}_i)$ be a sequence of infinite matrices with $\mathfrak{B}_i = (b_{nk}^{(i)})$. Then the sequence $x = (x_k)$ is \mathfrak{B} -summable to the value L if

$$\lim_n (\mathfrak{B}_i x)_n = \lim_n \sum_k b_{nk}^{(i)} x_k = L, \text{ uniformly in } i.$$

The method \mathfrak{B} is called m -multiplicative if each convergent sequence x is \mathfrak{B} -summable and there exists a $m \in \mathbb{R}$ such that $\lim_n (\mathfrak{B}_i x)_n = m \lim_n x_n$, uniformly in i [7, 8, 37].

Let us recall the following theorem from Bell's Ph.D. Thesis [7] and from [37] that characterizes m -multiplicativity and regularity.

Theorem 2.1 ([7, 37]). *The method \mathfrak{B} is m -multiplicative if and only if*

- (i) $\lim_n b_{nk}^{(i)} = 0$ for all $k \in \mathbb{N}$, uniformly in i ,
- (ii) $\lim_n \sum_k b_{nk}^{(i)} = m$, uniformly in i ,
- (iii) $\sum_k |b_{nk}^{(i)}| < \infty$ for all $n, i \in \mathbb{N}$, and there exists $N \in \mathbb{N}$ such that $\sup_{i \in \mathbb{N}, n > N} \sum_k |b_{nk}^{(i)}| < \infty$.

When $m = 1$, the method \mathfrak{B} is called regular [7, 8, 37].

A set $D := \{k_1 < k_2 < \dots < k_p < \dots\} \subset \mathbb{N}$ is said to have \mathfrak{B} -density $\delta_{\mathfrak{B}}(D)$ equal to d , if the characteristic sequence χ_D of D is \mathfrak{B} -summable to d , i.e.

$$\lim_n \sum_k b_{nk}^{(i)} \chi_D(k) = \lim_n \sum_{k \in D} b_{nk}^{(i)} = d, \text{ uniformly in } i,$$

[8, 23].

By \mathcal{R}^+ we denote the set of all regular methods \mathfrak{B} with $b_{nk}^{(i)} \geq 0$ for all n, k and i .

Let $\mathfrak{B} \in \mathcal{R}^+$. A sequence $x = (x_k)$ is called \mathfrak{B} -statistically convergent to L provided that, for every $\varepsilon > 0$, $\delta_{\mathfrak{B}}(\{k : |x_k - L| \geq \varepsilon\}) = 0$ holds [23]. In this case we write $st_{\mathfrak{B}} - \lim x_k = L$.

Now, let $\mathcal{A} := (A^{(n)}) = (a_{kj}^{(n)})$ be a sequence of infinite matrices with nonnegative real entries.

A sequence (U_j) of monotone and sublinear operators of $\mathcal{C}(X)$ into itself is called an \mathcal{A} -Summation Process in \mathfrak{B} -statistical sense on $\mathcal{C}(X)$ if $(U_j(f))$ is \mathcal{A} -summable to f in \mathfrak{B} -statistical sense for every $f \in \mathcal{C}(X)$, i.e.,

$$st_{\mathfrak{B}} - \lim_k \left\| \sum_{j=1}^{\infty} a_{k,j}^{(n)} U_j(f) - f \right\| = 0, \text{ uniformly in } n, \tag{2.1}$$

for every $f \in \mathcal{C}(X)$ where it is assumed that the series in (2.1) converges for each k, n and f .

In particular, when $A^{(n)} = A$ for some matrix A , \mathcal{A} -summability coincides with the ordinary matrix summability by A .

Consider the matrices $(a_{kj}^{(n)})$ by

$$(a_{kj}^{(n)}) = \begin{cases} \frac{1}{k}, & \text{if } 1 + n \leq j \leq k + n, \\ 0, & \text{otherwise.} \end{cases}$$

In this special case, \mathcal{A} -summability coincides with almost convergence (see [26]) (or uniform statistical convergence [5, 33]) which is a well-known example of a non-matrix method of summability.

Let $\{U_j\}$ be a sequence of monotone and sublinear operators such that for each $k, n \in \mathbb{N}$

$$\sum_{j=1}^{\infty} a_{k,j}^{(n)} |U_j(1)| < \infty. \tag{2.2}$$

For each $k, n \in \mathbb{N}$ and $f \in E \cap \mathcal{AC}_b(X)$, let

$$B_k^{(n)}(f)(x) := \sum_{j=1}^{\infty} a_{k,j}^{(n)} U_j(f)(x),$$

which is well defined by (2.2).

Remark 2.2. If each operator U_j is monotone, sublinear and translatable, then $B_k^{(n)}$ is also monotone, sublinear and translatable for each n and k , respectively.

3. Main results

In recent studies, Gal and Iancu [18] have investigated certain Korovkin-type results for sequences of monotone and sublinear operators within the framework of statistical convergence. Motivated by their work, we employ the \mathcal{A} -summation process in \mathfrak{B} -statistical sense to establish two novel versions of the Korovkin-type theorem for these operators.

Theorem 3.1. *Let $\mathfrak{B} \in \mathcal{R}^+$. Assume that X is a locally compact subset of \mathbb{R} , and E is a vector sublattice of $\mathcal{F}(X)$ that contains the four test functions: $1, x, -x, x^2$. Let $A := \{A^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries.*

(i) If (U_j) is a sequence of monotone and sublinear operators from E into itself such that

$$st_{\mathfrak{B}} - \lim_k B_k^{(n)}(f) = f \text{ a.e. and uniformly in } n, \tag{3.1}$$

for each of the four test functions: $1, x, -x, x^2$, and (2.2) holds, then (U_j) is an \mathcal{A} -summation process in \mathfrak{B} -statistical sense on $E \cap \mathcal{AC}_b(X)$, i.e., (3.1) holds for all nonnegative functions f in $E \cap \mathcal{AC}_b(X)$.

(ii) If, in addition to the hypotheses of (i), each operator U_j is translatable, then $st_{\mathfrak{B}} - \lim_k B_k^{(n)}(f) = f$ a.e. and uniformly in n for all $f \in E \cap \mathcal{AC}_b(X)$.

Proof. (i) Let $f \in E \cap \mathcal{AC}_b(X)$ such that $f \geq 0$. Suppose that s is a continuity point of f and also satisfies the condition

$$st_{\mathfrak{B}} - \lim_k B_k^{(n)}(h)(s) = h(s), \text{ uniformly in } n, \tag{3.2}$$

for each of the test functions $h \in \{1, x, -x, x^2\}$.

Then, for an arbitrarily fixed $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(s)| \leq \varepsilon \text{ for every } x \in X \text{ satisfying } |x - s| \leq \delta.$$

Moreover, if $|x - s| \geq \delta$, then

$$|f(x) - f(s)| \leq \frac{2\|f\|}{\delta^2} |x - s|^2.$$

As a result, we obtain

$$|f(x) - f(s)| \leq \varepsilon + \frac{2\|f\|}{\delta^2} |x - s|^2, \tag{3.3}$$

for all $x \in X$.

Putting $M := \max\{0, s\}$, we can write (3.3) as

$$|f(x) - f(s)| \leq \varepsilon + \frac{2\|f\|}{\delta^2} \left[x^2 + 2x(M - s) + 2M(-x) + |s|^2 \right].$$

Utilizing this inequality along with the monotonicity and sublinearity of the operators U_j , we deduce that

$$\begin{aligned} \left| B_k^{(n)}(f)(s) - f(s) \right| &\leq \left| B_k^{(n)}(f)(s) - B_k^{(n)}(f(s))(s) + f(s)B_k^{(n)}(1)(s) - f(s) \right| \\ &\leq B_k^{(n)}(|f - f(s)|)(s) + f(s) \left| B_k^{(n)}(1)(s) - 1 \right| \\ &\leq \varepsilon + \frac{2\|f\|}{\delta^2} \left[B_k^{(n)}(t^2)(s) + 2(M - s)B_k^{(n)}(t)(s) \right. \\ &\quad \left. + 2MB_k^{(n)}(-t)(s) + |s|^2 B_k^{(n)}(1)(s) \right] + f(s) \left| B_k^{(n)}(1) - 1 \right| \\ &\leq \varepsilon + C \left[\left| B_k^{(n)}(t^2)(s) + 2(M - s)B_k^{(n)}(t)(s) \right. \right. \\ &\quad \left. \left. + 2MB_k^{(n)}(-t)(s) + |s|^2 B_k^{(n)}(1)(s) \right| + \left| B_k^{(n)}(1)(s) - 1 \right| \right], \end{aligned} \tag{3.4}$$

for each $n, k \in \mathbb{N}$, where $C = \max\left\{ \frac{2\|f\|}{\delta^2}, f(s) \right\}$.

Given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. We now define the following sets:

$$D := \left\{ k \in \mathbb{N} : \sup_n \left| B_k^{(n)}(f)(s) - f(s) \right| \geq \frac{r-\varepsilon}{C} \right\},$$

$$D_1 := \left\{ k \in \mathbb{N} : \sup_n \left| B_k^{(n)}(t^2)(s) + 2(M-s)B_k^{(n)}(t)(s) \right. \right. \\ \left. \left. + 2MB_k^{(n)}(-t)(s) + |s|^2 B_k^{(n)}(1)(s) \right| \geq \frac{r-\varepsilon}{2C} \right\},$$

and

$$D_2 := \left\{ k \in \mathbb{N} : \sup_n \left| B_k^{(n)}(1)(s) - 1 \right| \geq \frac{r-\varepsilon}{2C} \right\}.$$

From (3.4), it follows that

$$D \subset D_1 \cup D_2,$$

which implies

$$\sum_{k \in D} b_{mk}^{(i)} \leq \sum_{k \in D_1} b_{mk}^{(i)} + \sum_{k \in D_2} b_{mk}^{(i)},$$

for all $m \in \mathbb{N}$.

By letting $m \rightarrow \infty$ and using (3.2), we conclude that $\delta_{\mathfrak{B}}(D) = 0$, i.e.,

$$st_{\mathfrak{B}} - \lim B_k^{(n)}(f)(s) = f(s), \text{ uniformly in } n,$$

as desired.

(ii) In addition to the hypotheses of (i), assume that each operator U_j is translatable. Then $B_k^{(n)}$ is also translatable. From (i), we obtain that

$$st_{\mathfrak{B}} - \lim B_k^{(n)}(f + \|f\|) = f + \|f\|, \text{ a.e. uniformly in } n.$$

Since $B_k^{(n)}$ is translatable, we have $B_k^{(n)}(f + \|f\|) = B_k^{(n)}(f) + \|f\| B_k^{(n)}(1)$, which yields that $st_{\mathfrak{B}} - \lim B_k^{(n)}(f) = f$, a.e. and uniformly in n . \square

Before presenting the next result, let us first introduce the concept of \mathfrak{B} -statistical convergence in the measure (see also [6]).

Definition 3.2. Let $\mathfrak{B} \in \mathcal{R}^+$ and let g_n, g be measurable functions on $[0, 1]$. We say that the sequence (g_n) converges \mathfrak{B} -statistically in the measure m to g if and only if for all $\varepsilon, \eta > 0$,

$$\delta_{\mathfrak{B}}(\{n \in \mathbb{N} : m(\{x \in [0, 1] : |g_n(x) - g(x)| \geq \varepsilon\}) \geq \eta\}) = 0.$$

We are now ready to provide the other main result.

Theorem 3.3. Let $\mathfrak{B} \in \mathcal{R}^+$, and let $\mathcal{A} := \{A^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries, and let $C([0, 1])$ represent the vector lattice of all continuous functions on $[0, 1]$. Let (U_j) be a sequence of monotone, subunital and sublinear operators from $C([0, 1])$ into itself such that the following condition holds

$$st_{\mathfrak{B}} - \lim B_k^{(n)}(f) = f \text{ in measure and uniformly in } n,$$

for each of the four test functions $h \in \{1, x, -x, x^2\}$, and let (2.2) hold. Then, \mathfrak{B} -statistical convergence holds for all nonnegative functions $h \in C([0, 1])$. If all the operators U_j are also translatable, then \mathfrak{B} -statistical convergence holds for all $f \in C([0, 1])$.

Proof. Let $\varepsilon > 0$ and let f be non-negative. Due to the uniform continuity of f , we can choose a sufficiently small $\varepsilon_0 > 0$ such that $\varepsilon_0 < \varepsilon$. Then, there exists a $\delta_0 > 0$, such that the following condition holds

$$|f(t) - f(x)| \leq \varepsilon_0 + 2\|f\| \frac{(t-x)^2}{\delta_0^2}, \text{ for all } t, x \in [0, 1],$$

where $\|f\| = \sup_{x \in [0,1]} |f(x)|$. Therefore, we have

$$\begin{aligned} \left| B_k^{(n)}(f)(x) - f(x) \right| &\leq \varepsilon_0 + \frac{2\|f\|}{\delta_0^2} B_k^{(n)}((t-x)^2)(x) \\ &\leq \varepsilon_0 + \frac{2\|f\|}{\delta_0^2} (B_k^{(n)}(e_2)(x) + 2xB_k^{(n)}(-e_1)(x) + x^2). \end{aligned}$$

This leads to the following inclusions:

$$\begin{aligned} &\left\{ x \in [0, 1] : \left| B_k^{(n)}(f)(x) - f(x) \right| \geq \varepsilon \right\} \\ &\subset \left\{ x \in [0, 1] : \varepsilon_0 + \frac{2\|f\|}{\delta_0^2} (B_k^{(n)}(e_2)(x) + 2xB_k^{(n)}(-e_1)(x) + x^2) \geq \varepsilon \right\} \\ &= \left\{ x \in [0, 1] : B_k^{(n)}(e_2)(x) + 2xB_k^{(n)}(-e_1)(x) + x^2 \geq (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{2\|f\|} \right\} \\ &= \left\{ x \in [0, 1] : \left(B_k^{(n)}(e_2)(x) - x^2 \right) + 2x \left(B_k^{(n)}(-e_1)(x) + x \right) \geq (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{2\|f\|} \right\} \\ &\subset \left\{ x \in [0, 1] : \left| B_k^{(n)}(e_2)(x) - x^2 \right| \geq (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{4\|f\|} \right\} \\ &\cup \left\{ x \in [0, 1] : \left| B_k^{(n)}(-e_1)(x) + x \right| \geq (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{4\|f\|} \right\}. \end{aligned}$$

From this, we deduce the following inequality:

$$\begin{aligned} &m \left(\left\{ x \in [0, 1] : \left| B_k^{(n)}(f)(x) - f(x) \right| \geq \varepsilon \right\} \right) \\ &\leq m \left(\left\{ x \in [0, 1] : \left| B_k^{(n)}(e_2)(x) - x^2 \right| \geq (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{4\|f\|} \right\} \right) \\ &\quad + m \left(\left\{ x \in [0, 1] : \left| B_k^{(n)}(-e_1)(x) + x \right| \geq (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{4\|f\|} \right\} \right). \end{aligned}$$

For any $\eta > 0$, define the sets as

$$D := \left\{ k \in \mathbb{N} : m \left(\left\{ x \in [0, 1] : \sup_n \left| B_k^{(n)}(f)(x) - f(x) \right| \geq \varepsilon \right\} \right) \geq \eta \right\}$$

$$D_1 := \left\{ k \in \mathbb{N} : m \left(\left\{ x \in [0, 1] : \sup_n \left| B_k^{(n)}(e_2)(x) - x^2 \right| \geq (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{4\|f\|} \right\} \right) \geq \frac{\eta}{2} \right\}$$

and

$$D_2 := \left\{ k \in \mathbb{N} : m \left(\left\{ x \in [0, 1] : \sup_n \left| B_k^{(n)}(-e_1)(x) + x \right| \geq (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{4\|f\|} \right\} \right) \geq \frac{\eta}{2} \right\}.$$

It is evident that $D \subset D_1 \cup D_2$.

Recall that $\varepsilon > \varepsilon_0$. Now, let $\gamma = (\varepsilon - \varepsilon_0) \frac{\delta_0^2}{4\|f\|}$, and since $\varepsilon - \varepsilon_0$ can be chosen arbitrarily close to 0, $\gamma > 0$ can also be made arbitrarily close to 0.

The remainder of the proof proceeds as in the proof of Theorem 3.1, and we omit it here.


Finally, assume the operators U_j are translatable too and $f \in C([0, 1])$. Since $f + \|f\| \geq 0$, by the first part of the proof, we conclude that $st_{\mathfrak{B}} - \lim B_k^{(n)}(f + \|f\|) = f + \|f\|$ in measure and uniformly in n . Additionally, since $B_k^{(n)}(f + \|f\|) = B_k^{(n)}(f) + \|f\|$, it follows that $st_{\mathfrak{B}} - \lim B_k^{(n)}(f) = f$ in measure and uniformly in n . \square

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Fixed points and dynamic programming in complex-valued controlled metric spaces

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Abstract. In this paper, we introduce the concepts of $(\alpha - \Theta)$ -contraction and Reich-type contraction within the framework of complex-valued controlled metric spaces (CVCMS). We also present related fixed point theorems for CVCMS, building on the works considered in the literature review for controlled metric type spaces. To demonstrate the practical implications and significance of our results, we provide several examples and an application in dynamic programming.

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1. Introduction and preliminaries

Fixed point theory has numerous applications across various fields of mathematics, engineering, and physics. It provides foundational tools for solving equations and modeling dynamic systems. One of the most significant contributions to this field was made by Stefan Banach [7], who introduced the Banach contraction theorem. This theorem not only established a method for proving the existence and uniqueness of fixed points in complete metric spaces but also laid the groundwork for further advancements in analysis and topology. Over the years, many researchers have extended Banach's theorem in diverse ways, exploring its implications in more complex and abstract settings, such as in non-linear analysis, differential equations, and optimization problems. These extensions have broadened the scope of fixed point theory, making it a versatile and powerful tool in both theoretical and applied mathematics.

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Azam et al. [6] gave the notion named complex-valued metric space (CV - metric space). Rao et al. [22] generalized CV - metric space as complex-valued b-metric space(CVBMS), which was later extended by Ullah et al. [23] by introducing a metric and naming this notion as complex valued-extended *b*-metric space (CVEb-metric space), further discussed by Belhenniche et al. [9]). In 2018 Mlaiki et al. [19] presented a controlled metric space that behaves as a different extension of *b* - metric space. In [20] Mlaiki et al. extended the results in [19]. Many mathematicians and researchers are working on this idea. Al-Mazrooei et al. [3] presented common fixed-point results for generalized contactions, Hussain et al. [15] discussed Fixed Point results For Nonlinear Contraction, Panday et al. [21] discussed Rational Type Contraction and Durdana Lateef presented Fisher [17] and Kannan [18] types contractions for controlled metric spaces.

Recently, the idea of the controlled metric space was further generalized by Aslam et al. [5], by introducing the new generalized space with the name of complex-valued controlled metric type space (CVCMS). They also presented different contraction theorems to illustrate the new concept. Ahmad et al. [2] presented rational type $(\alpha-\Theta)$ - contraction and Ahmad et al. [1] presented Reich type contraction for controlled metric spaces. In this new paper our aim is to define few type of contractions for the case of CVCMS. Moreover, several examples are provided, along with an application to dynamic programming.

Suppose we represent \mathbb{C} for the complex numbers set with $\epsilon_1, \epsilon_2 \in \mathbb{C}$. To compare two complex numbers we will use “ \lesssim ” symbol as the *partial order* on the set \mathbb{C} , which is called in related literature as *lexicographic order*,

“ $\epsilon_1 \lesssim \epsilon_2$ ” if and only if $Re(\epsilon_1) \leq Re(\epsilon_2)$ or $(Re(\epsilon_1) = Re(\epsilon_2) \text{ and } Im(\epsilon_1) \leq Im(\epsilon_2))$.

Taking under consideration the previous definition of partial order, we can say that “ $\epsilon_1 \lesssim \epsilon_2$ ” if from following conditions any one holds or satisfied:

- (P₁) $Re(\epsilon_1) < Re(\epsilon_2)$ together with $Im(\epsilon_1) < Im(\epsilon_2)$;
- (P₂) $Re(\epsilon_1) < Re(\epsilon_2)$ together with $Im(\epsilon_1) = Im(\epsilon_2)$;
- (P₃) $Re(\epsilon_1) < Re(\epsilon_2)$ together with $Im(\epsilon_1) > Im(\epsilon_2)$;
- (P₄) $Re(\epsilon_1) = Re(\epsilon_2)$ together with $Im(\epsilon_1) < Im(\epsilon_2)$.

From literature, we start from extension of *b*-metric and recall the concept of CVEb-metric space which was presented by N. Ullah et al. [23] in 2019.

Definition 1.1. [23] Consider \mathbb{H} as non empty set with a mapping $\zeta : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty)$. Then the functional defined as $h : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ is known to be CVEb-metric if all following described axioms holds good:

- (CEB₁) $0 \lesssim h(\epsilon, f)$ and $h(\epsilon, f) = 0$ if and only if $\epsilon = f$,
 - (CEB₂) $h(\epsilon, f) = h(f, \epsilon)$,
 - (CEB₃) $h(\epsilon, g) \lesssim \zeta(\epsilon, g)[h(\epsilon, f) + h(f, g)]$,
- for all $\epsilon, f, g \in \mathbb{H}$. A pair (\mathbb{H}, h) is called a CVEb-metric space.

Mlaiki et al.[19] gave the definition of the controlled metric type and there gave some fixed point results for this newly introduced type of metric.

Definition 1.2. ([19]) Consider \mathbb{H} as non empty set with a mapping $\zeta : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty)$. The functional $h : \mathbb{H} \times \mathbb{H} \rightarrow [0, \infty)$ is known to be controlled metric if the following conditions are satisfied:

$$(CMT_1) \quad 0 \leq h(\mathbf{e}, f) \text{ and } h(\mathbf{e}, f) = 0 \text{ if and only if } \mathbf{e} = f,$$

$$(CMT_2) \quad h(\mathbf{e}, f) = h(f, \mathbf{e}),$$

$$(CMT_3) \quad h(\mathbf{e}, g) \leq \zeta(\mathbf{e}, f)h(\mathbf{e}, f) + \zeta(f, g)h(f, g),$$

for all $\mathbf{e}, f, g \in \mathbb{H}$. A pair (\mathbb{H}, h) is said to be a controlled metric type space .

Recently Aslam et al. [5] introduced the concept of CVCMS defined as follows. As we observe, the CVC-metric is a symmetric norm, which is an important detail in order to obtain new fixed point results.

Definition 1.3. Consider \mathbb{H} as a non empty set with a mapping $\zeta : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty)$. The functional $h : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ is called as CVC- metric if following described axioms holds good:

$$(CCMT_1) \quad 0 \lesssim h(\mathbf{e}, f) \text{ and } h(\mathbf{e}, f) = 0 \text{ if and only if } \mathbf{e} = f,$$

$$(CCMT_2) \quad h(\mathbf{e}, f) = h(f, \mathbf{e}),$$

$$(CCMT_3) \quad h(\mathbf{e}, g) \lesssim \zeta(\mathbf{e}, f)h(\mathbf{e}, f) + \zeta(f, g)h(f, g),$$

for all $\mathbf{e}, f, g \in \mathbb{H}$. A pair (\mathbb{H}, h) is called a CVCMS.

Example 1.4. [5] Let $\mathbb{H} = [0, \infty)$ and $\zeta : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty)$ be described as

$$\zeta(\mathbf{e}, f) = \begin{cases} 1, & \text{if } \mathbf{e}, f \in [0, 1], \\ 1 + \mathbf{e} + f, & \text{otherwise.} \end{cases}$$

and $h : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ defined by the following

$$h(\mathbf{e}, f) := \begin{cases} 0, & \mathbf{e} = f \\ i, & \mathbf{e} \neq f \end{cases}$$

Then (\mathbb{H}, h) is a CVCMS.

Definition 1.5. We consider Θ as the set of the functionals $\Theta : (0, \infty) \rightarrow (1, \infty)$ which satisfy the axioms described as under:

(Θ_1) Θ be a non-decreasing function.

(Θ_2) For each of the sequence $\{l_n\} \subseteq l^+$, $\lim_{n \rightarrow \infty} \Theta(l_n) = 1 \iff \lim_{n \rightarrow \infty} l_n = 0^+$;

(Θ_3) \exists a constant $0 < k < 1$ and $\vartheta \in (0, \infty]$, so that, $\lim_{l \rightarrow 0^+} \frac{\Theta(l)-1}{l^k} = \vartheta$.

Definition 1.6. A self map $T : \mathbb{H} \rightarrow \mathbb{H}$ is known as Θ -contraction, if there exists a Θ which satisfy conditions (Θ_1) – (Θ_3) and $k \in (0, 1)$, such that:

$$h(T\mathbf{e}, Tf) \neq 0 \implies \Theta(h(T\mathbf{e}, Tf)) \lesssim [\Theta(h(\mathbf{e}, f))]^k$$

for all $\mathbf{e}, f \in \mathbb{H}$.

Considering the last section, we will present one application in dynamic programming, domain highly used in economics and financial phenomena. Taking into account the economics direction, during a simultaneous auction, each agent calculates their demand for a product at every possible price and puts it up to an auctioneer, which is known as a Walrasian auction. The price is then determined in such a way that the total demand of all agents is equal to the total amount of the property. The perfect match between supply and demand is achieved in a Walrasian auction.

The presence of a Walrasian auctioneer is the reason why most heterogeneous agent models in economics have an obvious symmetry. The only way to maintain symmetry is by agents acting in accordance with identical state values, which can only be maintained by them.

In this paper, we introduce the $(\alpha - \Theta)$ - contraction and Reich type contraction for complex-valued controlled metric spaces (CVCMS). Also related fixed point theorems for CVCMS are presented, as presented in [1] and [2] for controlled metric type space. To illustrate the results and their significance some examples and an application is also presented for the dynamic programming.

2. New fixed point results in CVCMS

In this section of the paper we give some new fixed point result for CVCMS with illustrative examples. First fixed point result is the following.

2.1. (α, Θ) – type fixed point results in CVCMS

In this subsection, we will present some new fixed points results corresponding to the (α, Θ) – contraction principle on the settings of the CVCMS. Further, in our theorems, we will consider (\mathbb{H}, h) as a complete complex-valued controlled metric spaces and we denote it by CVCMS. By CVEb-metric space we will denote a complex-valued extended b -metric space.

Definition 2.1. Assume (\mathbb{H}, h) be CVCMS. A functional $T : \mathbb{H} \rightarrow \mathbb{H}$ is denoted as rational type (α, Θ) - contraction, if there exists $\alpha : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}^+$, where $\kappa \in (0, 1)$ with $\Theta \in \Omega$, so that

$$\alpha(\mathbf{e}, f)\Theta(h(T\mathbf{e}, Tf)) \lesssim \Theta(M(\mathbf{e}, f))^\kappa \tag{2.1}$$

here

$$M(\mathbf{e}, f) = \max\{h(\mathbf{e}, f), h(\mathbf{e}, T\mathbf{e}), h(f, Tf), \frac{h(\mathbf{e}, T\mathbf{e})h(f, Tf)}{1 + h(\mathbf{e}, f)}\} \tag{2.2}$$

for all $\mathbf{e}, f \in \mathbb{H}$ with $h(T\mathbf{e}, Tf) \gtrsim 0$.

Theorem 2.2. Consider (\mathbb{H}, h) be a CVCMS. A function $T : \mathbb{H} \rightarrow \mathbb{H}$ be rational type (α, Θ) –contraction satisfying:

1. T is an α –admissible;
2. there exist $\mathbf{e}_0 \in \mathbb{H}$, so that the $\alpha(\mathbf{e}_0, T\mathbf{e}_0) \geq 0$;
3. T is continuous;
4. $\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\zeta(\mathbf{e}_{i+1}, \mathbf{e}_{i+2})\zeta(\mathbf{e}_{i+1}, \mathbf{e}_m)}{\zeta(\mathbf{e}_i, \mathbf{e}_{i+1})} < 1$.

Assume also, for every $e \in \mathbb{H}$, we have $\lim_{n \rightarrow \infty} \zeta(\mathbf{e}_n, \mathbf{e})$ and $\lim_{n \rightarrow \infty} \zeta(\mathbf{e}, \mathbf{e}_n)$ exist and are finite. Then there exist $\mathbf{e}^* \in \mathbb{H}$ such that the $\mathbf{e}^* = T\mathbf{e}^*$.

Proof. Suppose $\mathbf{e}_0 \in \mathbb{H}$ such that $\alpha(\mathbf{e}_0, T\mathbf{e}_0) \geq 1$. Considering a sequence $\{\mathbf{e}_n\}$ in \mathbb{H} by $\mathbf{e}_{n+1} = T\mathbf{e}_n$, for all $n \in N$. If there exists $n_0 \in N$ for which $\{\mathbf{e}_{n_0+1} = \mathbf{e}_{n_0}\}$, then $T\mathbf{e}_0 = \mathbf{e}_0$ and hence the proof is completed. Thus, we assume that the sequence $\{\mathbf{e}_{n+1} \neq \mathbf{e}_n\}$, for all $n \in N$. By using (1) and (2), it's obvious

$$\alpha(\mathbf{e}_n, \mathbf{e}_{n+1}) \geq 1,$$

for all $n \in N$. So, by (2.1), we obtain

$$1 \lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1})) = \Theta(h(\mathbf{T}\mathbf{e}_{n-1}, \mathbf{T}\mathbf{e}_n)) \lesssim \alpha(\mathbf{e}_n, \mathbf{e}_{n+1})\Theta(h(\mathbf{T}\mathbf{e}_{n-1}, \mathbf{T}\mathbf{e}_n)).$$

Since \mathbf{T} is a (α, Θ) -contraction, so for all $n \in N$, we have

$$\begin{aligned} 1 &\lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1})) \lesssim \alpha(\mathbf{e}_n, \mathbf{e}_{n+1})\Theta(h(\mathbf{T}\mathbf{e}_{n-1}, \mathbf{T}\mathbf{e}_n)) \\ &\lesssim \Theta(M(\mathbf{e}_{n-1}, \mathbf{e}_n))^\kappa \\ &= \Theta(\max\{h(\mathbf{e}_{n-1}, \mathbf{e}_n), h(\mathbf{e}_{n-1}, \mathbf{T}\mathbf{e}_{n-1}), h(\mathbf{e}_n, \mathbf{T}\mathbf{e}_n), \frac{h(\mathbf{e}_{n-1}, \mathbf{T}\mathbf{e}_{n-1})h(\mathbf{e}_n, \mathbf{T}\mathbf{e}_n)}{1 + h(\mathbf{e}_{n-1}, \mathbf{e}_n)}\})^\kappa \\ &= \Theta(\max\{h(\mathbf{e}_{n-1}, \mathbf{e}_n), h(\mathbf{e}_{n-1}, \mathbf{T}\mathbf{e}_n), h(\mathbf{e}_n, \mathbf{e}_{n+1}), \frac{h(\mathbf{e}_{n-1}, \mathbf{e}_n)h(\mathbf{e}_n, \mathbf{e}_{n+1})}{1 + h(\mathbf{e}_{n-1}, \mathbf{e}_n)}\})^\kappa \\ &\lesssim \Theta(\max\{h(\mathbf{e}_{n-1}, \mathbf{e}_n), h(\mathbf{e}_{n-1}, \mathbf{T}\mathbf{e}_n), h(\mathbf{e}_n, \mathbf{e}_{n+1}), h(\mathbf{e}_n, \mathbf{e}_{n+1})\})^\kappa \\ &= \Theta(\max\{h(\mathbf{e}_{n-1}, \mathbf{e}_n), h(\mathbf{e}_n, \mathbf{e}_{n+1})\})^\kappa. \end{aligned}$$

Thus

$$1 \lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1})) \lesssim \Theta(\max\{h(\mathbf{e}_{n-1}, \mathbf{e}_n), h(\mathbf{e}_n, \mathbf{e}_{n+1})\})^\kappa. \quad (2.3)$$

If there exists $n \in N$ (a natural number), such that

$$\Theta(\max\{h(\mathbf{e}_{n-1}, \mathbf{e}_n), h(\mathbf{e}_n, \mathbf{e}_{n+1})\})^\kappa = h(\mathbf{e}_n, \mathbf{e}_{n+1}),$$

then (2.3) becomes

$$1 \lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1})) \lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1}))^\kappa \lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1})).$$

Which is obviously a contradiction.

So $\Theta(\max\{h(\mathbf{e}_{n-1}, \mathbf{e}_n), h(\mathbf{e}_n, \mathbf{e}_{n+1})\})^\kappa = h(\mathbf{e}_{n-1}, \mathbf{e}_n)$ for all natural numbers $n \in N$.

Therefore by (2.3), we have

$$\begin{aligned} 1 &\lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1})) \lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1}))^\kappa \\ &\lesssim \Theta(h(\mathbf{e}_{n-1}, \mathbf{e}_n))^{\kappa^2} \\ &\lesssim \Theta(h(\mathbf{e}_{n-2}, \mathbf{e}_{n-1}))^{\kappa^3} \\ &\dots \\ &\lesssim \Theta(h(\mathbf{e}_0, \mathbf{e}_1))^{\kappa^n}. \end{aligned}$$

Thus by (2.3), we have

$$1 \lesssim \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1})) \lesssim \Theta(h(\mathbf{e}_0, \mathbf{e}_1))^{\kappa^n}. \quad (2.4)$$

Taking $n \rightarrow \infty$ in (2.4), we get

$$\lim_{n \rightarrow \infty} \Theta(h(\mathbf{e}_n, \mathbf{e}_{n+1})) = 1.$$

By (Θ_2) we get

$$\lim_{n \rightarrow \infty} h(\mathbf{e}_n, \mathbf{e}_{n+1}) = 0.$$

By (Θ_3) , there exists a $0 < v < 1$ with $\zeta \in (0, \infty]$ such that

$$\lim_{n \rightarrow \infty} \frac{\Theta(h(\mathbf{T}\mathbf{e}_n, \mathbf{T}\mathbf{e}_{n+1})) - 1}{(h(\mathbf{e}_n, \mathbf{e}_{n+1}))^v} = \zeta.$$

Suppose that for $\zeta < \infty$. In this case, consider $\lambda = \frac{\zeta}{2} > 0$. So by definition, there exists $n_1 \in N$, such that

$$\left| \frac{\Theta(h(\mathbf{T}\mathbf{e}_n, \mathbf{T}\mathbf{e}_{n+1})) - 1}{(h(\mathbf{e}_n, \mathbf{e}_{n+1}))^v} - \zeta \right| \leq \lambda, \text{ for all } n > n_1.$$

This implies that

$$\frac{\Theta(h(\mathbf{T}\mathbf{e}_n, \mathbf{T}\mathbf{e}_{n+1})) - 1}{(h(\mathbf{e}_n, \mathbf{e}_{n+1}))^v} \geq \zeta - \lambda = \frac{\zeta}{2} = \lambda, \forall n > n_1.$$

Then

$$nh(\mathbf{e}_n, \mathbf{e}_{n+1})^h \lesssim \mu n[\Theta(h(\mathbf{T}\mathbf{e}_n, \mathbf{T}\mathbf{e}_{n+1})) - 1], \text{ for all } n > n_1,$$

with $\mu = \frac{1}{\lambda}$. Now we suppose that $\zeta = \infty$ with $\lambda > 0$. By the definition, there exists $n_1 \in N$, such that

$$\lambda \leq \frac{\Theta(h(\mathbf{T}\mathbf{e}_n, \mathbf{T}\mathbf{e}_{n+1})) - 1}{h(\mathbf{e}_n, \mathbf{e}_{n+1})^v}, \text{ for all } n > n_1.$$

This implies

$$nh(\mathbf{e}_n, \mathbf{e}_{n+1})^v \lesssim \mu n[\Theta(h(\mathbf{T}\mathbf{e}_n, \mathbf{T}\mathbf{e}_{n+1})) - 1], \text{ for all } n > n_1,$$

where we have $\mu = \frac{1}{\lambda}$. Hence, in all type of situations, there exists $\mu > 0$ together with $n_1 \in N$ such that

$$nh(\mathbf{e}_n, \mathbf{e}_{n+1})^v \lesssim \mu n[\Theta(h(\mathbf{T}\mathbf{e}_n, \mathbf{T}\mathbf{e}_{n+1})) - 1], \quad (2.5)$$

for all $n > n_1$. So by (2.4) and (2.5), we get

$$nh(\mathbf{e}_n, \mathbf{e}_{n+1})^v \lesssim \mu n([\Theta(h(\mathbf{e}_0, \mathbf{e}_1))]^{k^n} - 1).$$

Applying $n \rightarrow \infty$, we get the following

$$\lim_{n \rightarrow \infty} nh(\mathbf{e}_n, \mathbf{e}_{n+1})^v = 0.$$

So, there exists $n_2 \in N$, such that

$$h(\mathbf{e}_n, \mathbf{e}_{n+1}) \lesssim \frac{1}{n^{1/v}}, \forall n > n_2.$$

Further, considering the triangular inequality for $p \geq 1$, we get

$$\begin{aligned} h(\mathbf{e}_n, \mathbf{e}_{n+p}) &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+p})h(\mathbf{e}_{n+1}, \mathbf{e}_{n+p}) \\ &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+p})\zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+2})h(\mathbf{e}_{n+1}, \mathbf{e}_{n+2}) \\ &\quad + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+p})\zeta(\mathbf{e}_{n+2}, \mathbf{e}_{n+p})h(\mathbf{e}_{n+2}, \mathbf{e}_{n+p}) \\ &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+p})\zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+2})h(\mathbf{e}_{n+1}, \mathbf{e}_{n+2}) \\ &\quad + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+p})\zeta(\mathbf{e}_{n+2}, \mathbf{e}_{n+p})\zeta(\mathbf{e}_{n+2}, \mathbf{e}_{n+3})h(\mathbf{e}_{n+2}, \mathbf{e}_{n+3}) \\ &\quad + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+p})\zeta(\mathbf{e}_{n+2}, \mathbf{e}_{n+p})\zeta(\mathbf{e}_{n+3}, \mathbf{e}_{n+p})h(\mathbf{e}_{n+3}, \mathbf{e}_{n+p}) \end{aligned}$$

...

$$\begin{aligned} \lesssim & \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \sum_{i=n+1}^{n+p-2} \left[\prod_{j=n+1}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})h(\mathbf{e}_i, \mathbf{e}_{i+1}) \\ & + \prod_{i=n+1}^{n+p-1} \zeta(\mathbf{e}_i, \mathbf{e}_{n+p})h(\mathbf{e}_{n+p-1}, \mathbf{e}_{n+p}), \end{aligned}$$

which leads to the following

$$\begin{aligned} h(\mathbf{e}_n, \mathbf{e}_{n+p}) & \lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \sum_{i=n+1}^{n+p-2} \left(\prod_{j=n+1}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right) \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})h(\mathbf{e}_i, \mathbf{e}_{i+1}) \\ & + \prod_{i=n+1}^{n+p-1} \zeta(\mathbf{e}_{n+p-1}, \mathbf{e}_{n+p})h(\mathbf{e}_{n+p-1}, \mathbf{e}_{n+p}) \\ & = \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \sum_{i=n+1}^{n+p-1} \left[\prod_{j=n+1}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})h(\mathbf{e}_i, \mathbf{e}_{i+1}) \\ & \lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \sum_{i=n+1}^{n+p-1} \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})h(\mathbf{e}_i, \mathbf{e}_{i+1}) \\ & \lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \sum_{i=n+1}^{n+p-1} \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1}) \frac{1}{i^{1/k}}. \end{aligned}$$

Thus,

$$h(\mathbf{e}_n, \mathbf{e}_{n+1}) \lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \sum_{i=n+1}^{n+p-1} \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1}) \frac{1}{i^{1/k}}. \quad (2.6)$$

Now, consider

$$\begin{aligned} \sum_{i=n+1}^{n+p-1} \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1}) \frac{1}{i^{1/k}} & = \sum_{i=n+1}^{n+p-1} \frac{1}{i^{1/k}} \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1}) \\ & \lesssim \sum_{i=n+1}^{\infty} \frac{1}{i^{1/k}} \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1}) = \sum_{i=n+1}^{\infty} X_i Y_i, \end{aligned}$$

where

$$X_i = \frac{1}{i^{1/k}},$$

and

$$Y_i = \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p}) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1}).$$

Since, $\frac{1}{k} > 0$, and $\sum_{i=n+1}^{\infty} (\frac{1}{i^{1/k}})$ converges and here $Y_i = [\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p})]\zeta(\mathbf{e}_i, \mathbf{e}_{i+1})$ is also increasing and bounded above, we have $\lim_{i \rightarrow \infty} = \sup(Y_i)$, exists and non zero.

Hence, it is obvious that the product $[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p})]\zeta(\mathbf{e}_i, \mathbf{e}_{i+1})$ converges. Thus $\sum_{i=n+1}^{\infty} X_i Y_i$ converges.

Further, let us consider the partial sum

$$R_p = \sum_{i=0}^p [\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_{n+p})]\zeta(\mathbf{e}_i, \mathbf{e}_{i+1}) \frac{1}{i^{1/k}}.$$

Now from (2.6), we have

$$h(\mathbf{e}_n, \mathbf{e}_{n+1}) \lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + [R_{n+p-1} - R_n]. \tag{2.7}$$

By applying the ratio test and then using the above mentioned condition (2.2), we have the guarantee of the existence of the $\lim_{n \rightarrow \infty} R_n$. So it can be said that $\{R_n\}$ is Cauchy. Now apply the limit $n \rightarrow +\infty$ in (2.7), we get

$$\lim_{n \rightarrow \infty} h(\mathbf{e}_n, \mathbf{e}_{n+p}) = 0.$$

which shows that $\{\mathbf{e}_n\}$ is the Cauchy sequence in (\mathbb{H}, h) , so $\{\mathbf{e}_n\}$ is convergent and converges to $u \in \mathbb{H}$. Now we show that $u = Tu$.

Since $\mathbf{e}_n \rightarrow u$ when $n \rightarrow \infty$ and also the mapping T is continuous, so we get $T\mathbf{e}_n \rightarrow Tu$ when we apply the limit $n \rightarrow \infty$.

So we have

$$h(u, Tu) = \lim_{n \rightarrow \infty} h(\mathbf{e}_{n+1}, Tu) = \lim_{n \rightarrow \infty} h(T\mathbf{e}_n, Tu) = 0$$

and hence it is clear that $u = Tu$. □

Further, we will give the above result for the cases of CVEb-metric space, complex-valued complete b -metric space (CVb-metric space), complex-valued complete metric space (CV-metric space). We will observe that our space is more general than the spaces enumerated here.

Theorem 2.3. *Let (\mathbb{H}, h) be a CVEb-metric space with a rational type (α, Θ) -Contraction, as $T : \mathbb{H} \rightarrow \mathbb{H}$ such that,*

$$\Theta(h(T\mathbf{e}, T\mathbf{f})) \lesssim \Theta(M(\mathbf{e}, f))^k, \tag{2.8}$$

where

$$M(\mathbf{e}, f) = \max\{h(\mathbf{e}, f), h(\mathbf{e}, T\mathbf{e}), h(f, Tf), \frac{h(\mathbf{e}, T\mathbf{e})h(f, cf)}{1 + h(\mathbf{e}, f)}\}, \text{ for all } \mathbf{e}, f \in \mathbb{H}, \tag{2.9}$$

with $h(T\mathbf{e}, Tf) \gtrsim 0$. Assume also that the following assertions also hold

1. T is continuous.
2. $\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\zeta(\mathbf{e}_{i+1}, \mathbf{e}_{i+2})\zeta(\mathbf{e}_{i+1}, \mathbf{e}_m)}{\zeta(\mathbf{e}_i, \mathbf{e}_{i+1})} < 1$.

In addition to this assume also that, for every $l \in \mathbb{H}$, we get $\lim_{i \rightarrow \infty} \zeta(\epsilon_n, \epsilon)$ and $\lim_{i \rightarrow \infty} \zeta(\epsilon, \epsilon_n)$ exist and are finite. Then, there exists $\epsilon^* \in \mathbb{H}$ such that $\epsilon^* = T\epsilon^*$.

Corollary 2.4. Let (\mathbb{H}, h) be a CVEb-metric space and with a rational type (α, Θ) -Contraction, as $T : \mathbb{H} \rightarrow \mathbb{H}$ such that,

1. T is α -admissible;
2. there exists $\epsilon_0 \in \mathbb{H}$ such that $\alpha(\epsilon_0, T\epsilon_0) \geq 0$;
3. T is continuous;
4. $\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\zeta(\epsilon_{i+1}, \epsilon_{i+2})\zeta(\epsilon_{i+1}, \epsilon_m)}{\zeta(\epsilon_i, \epsilon_{i+1})} < 1$.

In addition to this assume also that, for every $l \in \mathbb{H}$, we get $\lim_{i \rightarrow \infty} \zeta(\epsilon_n, \epsilon)$ and $\lim_{i \rightarrow \infty} \zeta(\epsilon, \epsilon_n)$ exist and are finite. Then, there exists $\epsilon^* \in \mathbb{H}$ such that $\epsilon^* = T\epsilon^*$.

Proof. If we take $\zeta(\epsilon, f) = \zeta(f, g)$ in above theorem, Theorem (2.2), we get the conclusion. □

Corollary 2.5. Let (\mathbb{H}, h) be a CV-complete b-MS and with a rational type (α, Θ) -Contraction, as $T : \mathbb{H} \rightarrow \mathbb{H}$ such that,

1. T is α -admissible;
2. there exists a $\epsilon_0 \in \mathbb{H}$ so that $\alpha(\epsilon_0, T\epsilon_0) \geq 0$;
3. T is continuous;

Then there exists $\epsilon^* \in \mathbb{H}$ such that $T\epsilon^* = \epsilon^*$.

Proof. If we take $\zeta(\epsilon, f) = \zeta(f, g) = b \geq 1$ in the above theorem, Theorem (2.2). □

Corollary 2.6. Let (\mathbb{H}, h) be a CV-complete MS and $T : \mathbb{H} \rightarrow \mathbb{H}$ be a rational type (α, Θ) -Contraction such that:

1. T is α -admissible;
2. there exists a $\epsilon_0 \in \mathbb{H}$ so that $\alpha(\epsilon_0, T\epsilon_0) \geq 0$;
3. T is continuous;

Then there exists $\epsilon^* \in \mathbb{H}$ such that $T\epsilon^* = \epsilon^*$.

Proof. If we take $\zeta(\epsilon, f) = \zeta(f, g) = 1$ in the above theorem, Theorem (2.2). □

Example 2.7. Let $\mathbb{H} = \{0, 1, 2\}$. Define $\zeta : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty]$ and $h : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty]$ as $\zeta(\epsilon, f) = 1 + \epsilon + f$ and

$$\begin{aligned} h(2, 2) &= h(0, 0) = h(1, 1) = 0 \\ h(2, 0) &= h(0, 2) = 1 + 2i, \\ h(1, 0) &= h(0, 1) = 4 + 3i, \\ h(1, 2) &= h(2, 1) = 2 - 2i \end{aligned}$$

We define next $T : \mathbb{H} \rightarrow \mathbb{H}$ by $T(0) = 0; T(1) = 2; T(2) = 0$.

For $k = \frac{9}{10}$ and by defining $\Theta(\kappa) = \mathbf{e}^{\sqrt{\kappa}}$, we checked multiple cases to sustain the assumptions of the main result.

Case 1: If $\mathbf{e} = 0, f = 1$, we have

$$\begin{aligned} \Theta(h(\mathbf{T}\mathbf{e}, \mathbf{T}f)) &= \Theta(h(\mathbf{T}0, \mathbf{T}1)) = \Theta(h(0, 2)) = \mathbf{e}^{\sqrt{\sqrt{5}}} \\ &\lesssim (\mathbf{e}^{\sqrt{5}})^{\frac{9}{10}} \\ &= [\Theta(\max\{5, 0, \sqrt{8}, 0\})]^{\frac{9}{10}} \\ &= [\Theta(\max\{ h(0, 1), h(0, \mathbf{T}0), h(1, \mathbf{T}1), \frac{h(0, \mathbf{T}0) h(1, \mathbf{T}1)}{1 + h(0, 1)} \})]^{\frac{9}{10}} \\ &= [\Theta(\max\{ h(\mathbf{e}, f), h(\mathbf{e}, \mathbf{T}\mathbf{e}), h(f, \mathbf{T}f), \frac{h(\mathbf{e}, \mathbf{T}\mathbf{e}) h(f, \mathbf{T}f)}{1 + h(\mathbf{e}, f)} \})]^{\frac{9}{10}}. \end{aligned}$$

Case 2: If $\mathbf{e} = 1, f = 2$, we have

$$\begin{aligned} \Theta(h(\mathbf{T}\mathbf{e}, \mathbf{T}f)) &= \Theta(h(\mathbf{T}1, \mathbf{T}2)) = \Theta(h(1, 2)) = \mathbf{e}^{\sqrt{\sqrt{5}}} \\ &\lesssim (\mathbf{e}^{\sqrt{\sqrt{8}}})^{\frac{9}{10}} \\ &= [\Theta(\max\{\sqrt{8}, \sqrt{8}, \sqrt{5}, \frac{\sqrt{8}\sqrt{5}}{1 + \sqrt{8}}\})]^{\frac{9}{10}} \\ &= [\Theta(\max\{ h(1, 2), h(1, \mathbf{T}1), h(2, \mathbf{T}2), \frac{h(1, \mathbf{T}1) h(2, \mathbf{T}2)}{1 + h(1, 2)} \})]^{\frac{9}{10}} \\ &= [\Theta(\max\{ h(\mathbf{e}, f), h(\mathbf{e}, \mathbf{T}\mathbf{e}), h(f, \mathbf{T}f), \frac{h(\mathbf{e}, \mathbf{T}\mathbf{e}) h(f, \mathbf{T}f)}{1 + h(\mathbf{e}, f)} \})]^{\frac{9}{10}}. \end{aligned}$$

Case 3: If $\mathbf{e} = 0, f = 2$, we have

$$\begin{aligned} \Theta(h(\mathbf{T}\mathbf{e}, \mathbf{T}f)) &= \Theta(h(\mathbf{T}0, \mathbf{T}2)) = \Theta(h(0, 0)) = \mathbf{e}^0 \\ &\lesssim (\mathbf{e}^{\sqrt{\sqrt{5}}})^{\frac{9}{10}} \\ &= [\Theta(\max\{\sqrt{5}, 0, \sqrt{5}, 0\})]^{\frac{9}{10}} \\ &= [\Theta(\max\{ h(0, 2), h(0, \mathbf{T}0), h(2, \mathbf{T}2), \frac{h(0, \mathbf{T}0) h(2, \mathbf{T}2)}{1 + h(0, 2)} \})]^{\frac{9}{10}} \\ &= [\Theta(\max\{ h(\mathbf{e}, f), h(\mathbf{e}, \mathbf{T}\mathbf{e}), h(f, \mathbf{T}f), \frac{h(\mathbf{e}, \mathbf{T}\mathbf{e}) h(f, \mathbf{T}f)}{1 + h(\mathbf{e}, f)} \})]^{\frac{9}{10}}. \end{aligned}$$

Case 4: If $\mathbf{e} = f = 0, \mathbf{e} = f = 1, \mathbf{e} = f = 2$, we have

$$\begin{aligned} \Theta(h(\mathbf{T}\mathbf{e}, \mathbf{T}f)) &= \mathbf{e}^0 \\ &\lesssim [\Theta(\max\{ h(\mathbf{e}, f), h(\mathbf{e}, \mathbf{T}\mathbf{e}), h(f, \mathbf{T}f), \frac{h(\mathbf{e}, \mathbf{T}\mathbf{e}) h(f, \mathbf{T}f)}{1 + h(\mathbf{e}, f)} \})]^{\frac{9}{10}} \\ &= [\Theta(\max\{ h(\mathbf{e}, f), h(\mathbf{e}, \mathbf{T}\mathbf{e}), h(f, \mathbf{T}f), \frac{h(\mathbf{e}, \mathbf{T}\mathbf{e}) h(f, \mathbf{T}f)}{1 + h(\mathbf{e}, f)} \})]^{\kappa}. \end{aligned}$$

So, all assumptions of the above said theorem are satisfied. So, \mathbf{T} has $\mathbf{e} = 0$ as unique fixed point.

We can also satisfy many other results as a special cases of the main Theorem 2.2.

2.2. Reich type contractions on CVCMS

Theorem 2.8. *Suppose (\mathbb{H}, h) be a CVCMS by the functions $\zeta : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty)$. Suppose also, the mapping $T : \mathbb{H} \times \mathbb{H}$ be such that there are $\alpha, \beta, \gamma \in (0, 1)$ and $k = \frac{\alpha + \beta}{1 - \gamma} < 1$,*

$$h(T\mathbf{e}, Tf) \lesssim \alpha h(\mathbf{e}, f) + \beta h(\mathbf{e}, T\mathbf{e}) + \gamma h(f, Tf), \quad (2.10)$$

for all $\mathbf{e}, f \in \mathbb{H}$. For $\mathbf{e}_0 \in \mathbb{H}$, take $\mathbf{e}_n = T^n \mathbf{e}_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\zeta(\mathbf{e}_{i+1}, \mathbf{e}_{i+2})}{\zeta(\mathbf{e}_i, \mathbf{e}_{i+1})} \zeta(\mathbf{e}_{i+1}, \mathbf{e}_m) < \frac{1}{k}. \quad (2.11)$$

Suppose that

$\lim_{n \rightarrow \infty} \zeta(\mathbf{e}, \mathbf{e}_n)$ and $\lim_{n \rightarrow \infty} \zeta(\mathbf{e}_n, \mathbf{e})$ exist and are finite, and $\gamma \lim_{n \rightarrow \infty} \zeta(\mathbf{e}_n, \mathbf{e}) < 1$ for every $e \in \mathbb{H}$.

Then, T possess a unique fixed point.

Proof. The assumed sequence $\{\mathbf{e}_n\}$ verifies $\mathbf{e}_{n+1} = T\mathbf{e}_n$ for all $n \in \mathbb{N}$. It is obvious that if there exists $n_0 \in \mathbb{N}$ for which $\mathbf{e}_{n_0+1} = T\mathbf{e}_{n_0}$, then $T\mathbf{e}_{n_0} = \mathbf{e}_{n_0}$, and the proof is trivially finished. So here we assume that $\{\mathbf{e}_{n+1}\} \neq \{\mathbf{e}_n\}$ for every $n \in \mathbb{N}$. So, by (2.10), we get

$$\begin{aligned} h(\mathbf{e}_n, \mathbf{e}_{n+1}) &= h(T\mathbf{e}_{n-1}, T\mathbf{e}_n) \lesssim \alpha h(\mathbf{e}_{n-1}, \mathbf{e}_n) + \beta h(\mathbf{e}_{n-1}, T\mathbf{e}_{n-1}) + \gamma h(\mathbf{e}_n, T\mathbf{e}_n) \\ &= \alpha h(\mathbf{e}_{n-1}, \mathbf{e}_n) + \beta h(\mathbf{e}_{n-1}, \mathbf{e}_n) + \gamma h(\mathbf{e}_n, \mathbf{e}_{n+1}), \end{aligned} \quad (2.12)$$

which implies that

$$h(\mathbf{e}_n, \mathbf{e}_{n+1}) \lesssim \frac{\alpha + \beta}{1 - \gamma} h(\mathbf{e}_{n-1}, \mathbf{e}_n) = kh(\mathbf{e}_{n-1}, \mathbf{e}_n). \quad (2.13)$$

Thus, we have

$$h(\mathbf{e}_n, \mathbf{e}_{n+1}) \lesssim kh(\mathbf{e}_{n-1}, \mathbf{e}_n) \lesssim k^2 h(\mathbf{e}_{n-2}, \mathbf{e}_{n-1}) \lesssim \dots \lesssim k^n h(\mathbf{e}_0, \mathbf{e}_1).$$

For all $n, m \in \mathbb{N}$ with $n < m$, we have,

$$\begin{aligned}
 h(\mathbf{e}_n, \mathbf{e}_m) &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_m)h(\mathbf{e}_{n+1}, \mathbf{e}_m) \\
 &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_m)\zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+2})h(\mathbf{e}_{n+1}, \mathbf{e}_{n+2}) \\
 &\quad + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_m)\zeta(\mathbf{e}_{n+2}, \mathbf{e}_m)h(\mathbf{e}_{n+2}, \mathbf{e}_m) \\
 &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_m)\zeta(\mathbf{e}_{n+1}, \mathbf{e}_{n+2})h(\mathbf{e}_{n+1}, \mathbf{e}_{n+2}) \\
 &\quad + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_m)\zeta(\mathbf{e}_{n+2}, \mathbf{e}_m)\zeta(\mathbf{e}_{n+2}, \mathbf{e}_{n+3})h(\mathbf{e}_{n+2}, \mathbf{e}_{n+3}) \\
 &\quad + \zeta(\mathbf{e}_{n+1}, \mathbf{e}_m)\zeta(\mathbf{e}_{n+2}, \mathbf{e}_m)\zeta(\mathbf{e}_{n+3}, \mathbf{e}_m)h(\mathbf{e}_{n+3}, \mathbf{e}_m) \\
 &\lesssim \dots \\
 &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \sum_{i=n+1}^{m-2} \left[\prod_{j=n+1}^i \zeta(\mathbf{e}_j, \mathbf{e}_m) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})h(\mathbf{e}_i, \mathbf{e}_{i+1}) \\
 &\quad + \prod_{i=n+1}^{m-1} \zeta(\mathbf{e}_i, \mathbf{e}_m)h(\mathbf{e}_{m-1}, \mathbf{e}_m).
 \end{aligned}$$

Which implies that

$$\begin{aligned}
 h(\mathbf{e}_n, \mathbf{e}_m) &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})h(\mathbf{e}_n, \mathbf{e}_{n+1}) + \sum_{i=n+1}^{m-2} \left[\prod_{j=n+1}^i \zeta(\mathbf{e}_j, \mathbf{e}_m) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})h(\mathbf{e}_i, \mathbf{e}_{i+1}) \\
 &\quad + \left[\prod_{i=n+1}^{m-1} \zeta(\mathbf{e}_j, \mathbf{e}_m) \right] \zeta(\mathbf{e}_{m-1}, \mathbf{e}_m)h(\mathbf{e}_{m-1}, \mathbf{e}_m) \\
 &\lesssim \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})k^n h(\mathbf{e}_0, \mathbf{e}_1) + \sum_{i=n+1}^{m-2} \left[\prod_{j=n+1}^i \zeta(\mathbf{e}_j, \mathbf{e}_m) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})k^i h(\mathbf{e}_0, \mathbf{e}_1) \\
 &\quad + \left[\prod_{i=n+1}^{m-1} \zeta(\mathbf{e}_j, \mathbf{e}_m) \right] \zeta(\mathbf{e}_{m-1}, \mathbf{e}_m)k^{m-1} h(\mathbf{e}_0, \mathbf{e}_1) \\
 &= \zeta(\mathbf{e}_n, \mathbf{e}_{n+1})k^n h(\mathbf{e}_0, \mathbf{e}_1) + \sum_{i=n+1}^{m-1} \left[\prod_{j=n+1}^i \zeta(\mathbf{e}_j, \mathbf{e}_m) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})k^i h(\mathbf{e}_0, \mathbf{e}_1).
 \end{aligned} \tag{2.14}$$

Let

$$R_g = \sum_{i=0}^g \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_m) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})k^i h(\mathbf{e}_0, \mathbf{e}_1).$$

Consider

$$u_i = \left[\prod_{j=0}^i \zeta(\mathbf{e}_j, \mathbf{e}_m) \right] \zeta(\mathbf{e}_i, \mathbf{e}_{i+1})k^i h(\mathbf{e}_0, \mathbf{e}_1).$$

We have

$$\frac{u_{i+1}}{u_i} = \zeta(\mathbf{e}_{i+1}, \mathbf{e}_m) \frac{\zeta(\mathbf{e}_{i+1}, \mathbf{e}_{i+2})}{\zeta(\mathbf{e}_i, \mathbf{e}_{i+1})} k.$$

Considering the condition (2.11) and applying the ratio test, we can see that the series $\sum_i u_i$ converges. Thus, $\lim_{n \rightarrow \infty} R_n$ exists. So, we say the real sequence $\{R_n\}$ is a Cauchy sequence.

Now, using (2.14), we get

$$h(\mathbf{e}_n, \mathbf{e}_m) \lesssim h(\mathbf{e}_0, \mathbf{e}_1)[k^n \zeta(\mathbf{e}_n, \mathbf{e}_{n+1}) + (R_{m-1}, R_n)]. \quad (2.15)$$

Above, we used $\zeta(\mathbf{e}, f) \geq 1$. Assuming $n, m \rightarrow \infty$ in (2.15), we obtain $\lim_{n, m \rightarrow \infty} h(\mathbf{e}_n, \mathbf{e}_m) = 0$.

Thus, the sequence $\{\mathbf{e}_n\}$ is Cauchy in the CVCMS (\mathbb{H}, h) . So, there is some $\mathbf{e}^* \in \mathbb{H}$ such that

$$\lim_{n \rightarrow \infty} h(\mathbf{e}_n, \mathbf{e}^*) = 0.$$

that is, $\mathbf{e}_n \rightarrow \mathbf{e}^*$ as $n \rightarrow \infty$.

In the following we will show that \mathbf{e}^* is a fixed point of T. By (2.10) and by condition (3), we have

$$\begin{aligned} h(\mathbf{e}^*, T\mathbf{e}^*) &\lesssim \zeta(\mathbf{e}^*, \mathbf{e}_{n+1})h(\mathbf{e}^*, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, T\mathbf{e}^*)h(\mathbf{e}_{n+1}, T\mathbf{e}^*) \\ &= \zeta(\mathbf{e}^*, \mathbf{e}_{n+1})h(\mathbf{e}^*, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, T\mathbf{e}^*)h(T\mathbf{e}_n, T\mathbf{e}^*) \\ &\lesssim \zeta(\mathbf{e}^*, \mathbf{e}_{n+1})h(\mathbf{e}^*, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, T\mathbf{e}^*)[\alpha h(\mathbf{e}_n, \mathbf{e}^*) + \beta h(\mathbf{e}_n, T\mathbf{e}_n) + \gamma h(\mathbf{e}^*, T\mathbf{e}^*)] \\ &= \zeta(\mathbf{e}^*, \mathbf{e}_{n+1})h(\mathbf{e}^*, \mathbf{e}_{n+1}) + \zeta(\mathbf{e}_{n+1}, T\mathbf{e}^*)[\alpha h(\mathbf{e}_n, \mathbf{e}^*) + \beta h(\mathbf{e}_n, T\mathbf{e}_{n+1}) + \gamma h(\mathbf{e}^*, T\mathbf{e}^*)]. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (2.12), (2.13) and the fact that $\lim_{n \rightarrow \infty} h(\mathbf{e}_n, \mathbf{e})$ and $\lim_{n \rightarrow \infty} h(\mathbf{e}, \mathbf{e}_n)$ exist, and are finite, we obtain that

$$h(\mathbf{e}^*, T\mathbf{e}^*) \lesssim [\gamma \lim_{n \rightarrow \infty} \zeta(\mathbf{e}_{n+1}, T\mathbf{e}^*)]h(\mathbf{e}^*, T\mathbf{e}^*).$$

Suppose that $\mathbf{e}^* \neq T\mathbf{e}^*$, keeping in the mind that $\gamma \lim_{n \rightarrow \infty} \zeta(\mathbf{e}_{n+1}, T\mathbf{e}^*) < 1$, so

$$0 < h(\mathbf{e}^*, T\mathbf{e}^*) \lesssim [\gamma \lim_{n \rightarrow \infty} \zeta(\mathbf{e}_{n+1}, T\mathbf{e}^*)]h(\mathbf{e}^*, T\mathbf{e}^*) < h(\mathbf{e}^*, T\mathbf{e}^*).$$

It is a clear contradiction which yields the result that $\mathbf{e}^* = T\mathbf{e}^*$. So easily the uniqueness of the fixed point follows and it also completes the proof. \square

Example 2.9. Assume that $\mathbb{H} = \{0, 1, 2\}$. And define $\zeta : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty]$ and $h : \mathbb{H} \times \mathbb{H} \rightarrow [1, \infty]$ as $\zeta(\mathbf{e}, f) = 1 + \mathbf{e} + f$ and

$$\begin{aligned} h(2, 2) &= h(0, 0) = h(1, 1) = 0 \\ h(2, 0) &= h(0, 2) = 1 + 2i, \\ h(1, 0) &= h(0, 1) = 4 + 3i, \\ h(1, 2) &= h(2, 1) = 2 - 2i \end{aligned}$$

We define the mapping $T : \mathbb{H} \rightarrow \mathbb{H}$ by $T(0) = 1, T(1) = 1, T(2) = 1$. For $\alpha = \frac{1}{11}, \beta = \frac{3}{11}, \gamma = \frac{3}{11}$ and setting $\mathbf{e}_0 = 0, \mathbf{e}_1 = 2$ and $\mathbf{e}_n = 1$, for all $n \geq 2$, condition (2.11) is satisfied. For (2.10) we check different cases to sustain the assumptions of

the main result.

Case 01: If $\mathbf{e} = 0, f = 1$, we have

$$\begin{aligned} h(\mathbf{T}\mathbf{e}, \mathbf{T}f) &= h(\mathbf{T}0, \mathbf{T}1) = 0 \\ &\lesssim \alpha h(p, \mathbf{e}) + \beta h(p, \mathbf{T}p) + \gamma h(\mathbf{e}, \mathbf{T}\mathbf{e}). \end{aligned}$$

Case 02: If $\mathbf{e} = 1, f = 2$, we have

$$\begin{aligned} h(\mathbf{T}\mathbf{e}, \mathbf{T}f) &= h(\mathbf{T}1, \mathbf{T}2) = 0 \\ &\lesssim \alpha h(p, \mathbf{e}) + \beta h(p, \mathbf{T}p) + \gamma h(\mathbf{e}, \mathbf{T}\mathbf{e}). \end{aligned}$$

Case 03: If $\mathbf{e} = 0, f = 2$, we have

$$\begin{aligned} h(\mathbf{T}\mathbf{e}, \mathbf{T}f) &= h(\mathbf{T}0, \mathbf{T}2) = 0 \\ &\lesssim \alpha h(p, \mathbf{e}) + \beta h(p, \mathbf{T}p) + \gamma h(\mathbf{e}, \mathbf{T}\mathbf{e}). \end{aligned}$$

Case 04: If $\mathbf{e} = f = 0, \mathbf{e} = f = 1, \mathbf{e} = f = 2$, we have

$$\begin{aligned} h(\mathbf{T}\mathbf{e}, \mathbf{T}f) &= 0 \\ &\lesssim \alpha h(p, \mathbf{e}) + \beta h(p, \mathbf{T}p) + \gamma h(\mathbf{e}, \mathbf{T}\mathbf{e}). \end{aligned}$$

So, all assumptions of the above said theorem (2.8) are accomplished. So, \mathbf{T} has $\mathbf{e} = 1$ as a unique fixed point. We can also satisfy many results as special case of main Theorem (2.8).

3. Application to dynamic programming

The idea of *Dynamic programming* was given by Richard Bellman (see [10]). This idea was used for solving a large decision issues by splitting them into simpler fitted sub-issues which are solved iteratively over time. Dynamic programming techniques were highly used in optimization, control issues, remarkably, impulsive control issues.

In [13] Bertsekas and Tsitsikls gave a new class of RL algorithms for example Value Iteration (VI) and Policy Iteration (PI) process. Then, in [12], Bertsekas and Ioffe gave a new iteration scheme, the Temporal Differences policy iteration $TD(\lambda)$ scheme, and they proven that the $TD(\lambda)$ scheme can be resumed to a PI scheme, appointed by λ -PIR. Some details about are given in the following studies [4, 8, 11, 14, 16].

This section is dedicated to analyse the application of Reich fixed-point theorem in CVCMS in order to prove the results of existence and uniqueness for the solution of the *dynamic programming Bellman's equation*.

Let H and Q be two CVCMS with $\tilde{S} \subseteq H$ be the state space with $\tilde{D} \subseteq Q$ be the decision space. Let $\pi : \tilde{S} \times \tilde{D} \rightarrow \tilde{S}$ be the process transformation and $\tilde{\delta} : \tilde{S} \times \tilde{D} \rightarrow \mathbb{C}$ with $G : \tilde{S} \times \tilde{D} \times \mathbb{C} \rightarrow \mathbb{C}$ be some given functional, where \mathbb{C} is the set of complex numbers. The optimal return map $F : \tilde{S} \rightarrow \mathbb{C}$ of the process of continuous decision is represented by the following functional equation of the form

$$F(\mathbf{e}) = \sup_{f \in \tilde{D}} \{ \tilde{\delta}(\mathbf{e}, f) + G(\mathbf{e}, f, F(\pi(\mathbf{e}, f))) \}, \text{ with } \mathbf{e}, f \in \tilde{S}. \tag{3.1}$$

Let $\mathfrak{B}(\tilde{S})$ be collection of all complex-valued mappings, bounded on \tilde{S} .

Theorem 3.1. *Consider the previous instances true and we suppose further the following*

1. G and $\bar{\delta}$ are two bounded mappings;

2. $|G(\mathbf{e}, f, \mathfrak{h}z) - G(\mathbf{e}, f, \mathfrak{k}z)| \lesssim \alpha|\mathfrak{h}z - \mathfrak{k}z| + \beta|\mathfrak{h}z - T\mathfrak{k}z| + \gamma|\mathfrak{k}z - T\mathfrak{k}z|$,

for all $\mathfrak{h}, \mathfrak{k} \in \mathfrak{B}(\tilde{S})$ and $(\mathbf{e}, f, \mathfrak{h}z) \in \tilde{S} \times \tilde{D} \times \mathbb{C}$, where $z \in \tilde{S}$ and $\alpha, \beta, \gamma \in (0, 1)$ together with $\alpha + \beta + \gamma < 1$ and $\kappa = \frac{\alpha + \beta}{1 - \gamma} < 1$.

Then the functional equation (3.1) has a unique bounded solution on \tilde{S} .

Proof. We take $\mathfrak{B}(\tilde{S})$ the set of all complex-valued mappings, bounded on \tilde{S} , endowed with the norm $h : \mathfrak{B}(\tilde{S}) \times \mathfrak{B}(\tilde{S}) \rightarrow \mathbb{C}$ defined as $h(\mathbf{e}, f) = \|\mathbf{e} - f\|_\infty = |\mathbf{e} - f|e^{-i\theta}$, where $|p| = \sqrt{s^2 + t^2}$ with $p = s + it$, for $s, t \in \mathbb{R}$, $\theta > 0$ and $i = \sqrt{-1} \in \mathbb{C}$.

Let $\varsigma : \mathfrak{B}(\tilde{S}) \times \mathfrak{B}(\tilde{S}) \rightarrow [1, \infty)$ be a functional defined as

$$\varsigma(p, \mathbf{e}) = \begin{cases} 1, & \text{if } \mathbf{e}, f \in (0, 1], \\ \frac{1 + \max\{\mathbf{e}, f\}}{\min\{\mathbf{e}, f\}}, & \text{otherwise.} \end{cases}$$

We remark that $(\mathfrak{B}(\tilde{S}), h)$ is a complete CVCMS. We consider the operator $T : \mathfrak{B}(\tilde{S}) \rightarrow \mathfrak{B}(\tilde{S})$ defined by $T\mathfrak{h} = \varphi$, where

$$\varphi(\mathbf{e}) = \sup_{y \in \tilde{D}} \{\bar{\delta}(\mathbf{e}, f) + G(\mathbf{e}, f, \mathfrak{h}(\pi(\mathbf{e}, f)))\}.$$

We know that G and g are bounded, then $\varphi \in \mathfrak{B}(\tilde{S})$. Then, to find a bounded solution of the functional equation (3.1) is the same with getting a fixed point of T .

Let $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{B}(\tilde{S})$ and $T\mathfrak{h}_1 = \varphi_1, T\mathfrak{h}_2 = \varphi_2$. Then we have

$$\varphi_1(p) = \sup_{y \in \tilde{D}} \{\bar{\delta}(\mathbf{e}, f) + G(\mathbf{e}, f, \mathfrak{h}_1(\pi(\mathbf{e}, f)))\}.$$

$$\varphi_2(p) = \sup_{y \in \tilde{D}} \{\bar{\delta}(\mathbf{e}, f) + G(\mathbf{e}, f, \mathfrak{h}_2(\pi(\mathbf{e}, f)))\}.$$

For $e \in \tilde{S}$ and $\mathbf{e}_1, \mathbf{e}_2 \in Q$ and $\epsilon > 0$ we get

$$T\mathfrak{h}_1e \prec \bar{\delta}(\mathbf{e}, f_1) + G(\mathbf{e}, f_1, \mathfrak{h}_1(\pi(\mathbf{e}, f_1))) + \epsilon. \tag{3.2}$$

$$T\mathfrak{h}_2e \prec \bar{\delta}(\mathbf{e}, f_2) + G(\mathbf{e}, f_2, \mathfrak{h}_2(\pi(\mathbf{e}, f_2))) + \epsilon. \tag{3.3}$$

Also, we have

$$T\mathfrak{h}_1e \succ \bar{\delta}(\mathbf{e}, f_2) + G(\mathbf{e}, f_2, \mathfrak{h}_1(\pi(\mathbf{e}, f_2))). \tag{3.4}$$

$$T\mathfrak{h}_2e \succ \bar{\delta}(\mathbf{e}, f_1) + G(\mathbf{e}, f_1, \mathfrak{h}_2(\pi(\mathbf{e}, f_1))). \tag{3.5}$$

From (3.2) and (3.5) we get

$$\begin{aligned} |T\mathfrak{h}_1e - T\mathfrak{h}_2e| &\lesssim |G(\mathbf{e}, f, \mathfrak{h}_1(\pi(\mathbf{e}, f_1))) - G(\mathbf{e}, f, \mathfrak{h}_2(\pi(\mathbf{e}, f_2)))| + \epsilon \\ &\lesssim \alpha|\mathfrak{h}_1e - \mathfrak{h}_2e| + \beta|\mathfrak{h}_1e - T\mathfrak{h}_1e| + \gamma|\mathfrak{h}_2e - T\mathfrak{h}_2e|. \end{aligned} \tag{3.6}$$

From (3.3) and (3.4) we get

$$\begin{aligned} |T\mathfrak{h}_2e - T\mathfrak{h}_1e| &\lesssim |G(\mathbf{e}, f, \mathfrak{h}_2(\pi(\mathbf{e}, f_2))) - G(\mathbf{e}, f, \mathfrak{h}_1(\pi(\mathbf{e}, f_1)))| + \epsilon \\ &\lesssim \alpha|\mathfrak{h}_2e - \mathfrak{h}_1e| + \beta|T\mathfrak{h}_1e - \mathfrak{h}_1e| + \gamma|T\mathfrak{h}_2e - \mathfrak{h}_2e|. \end{aligned} \tag{3.7}$$

Further, from (3.6) and (3.7), and multiplying on both sides with $e^{-i\theta}$ (exponential function) we get

$$|\text{Th}_1\epsilon - \text{Th}_2\epsilon|e^{-i\theta} \lesssim \alpha|\mathfrak{h}_1\epsilon - \mathfrak{h}_2\epsilon|e^{-i\theta} + \beta|\mathfrak{h}_1\epsilon - \text{Th}_1\epsilon|e^{-i\theta} + \gamma|\mathfrak{h}_2\epsilon - \text{Th}_2\epsilon|e^{-i\theta}. \quad (3.8)$$

Then, we have

$$\begin{aligned} h(\text{Th}_1\epsilon, \text{Th}_2\epsilon) &= \|\text{Th}_1\epsilon - \text{Th}_2\epsilon\|_\infty \lesssim \alpha\|\mathfrak{h}_1\epsilon - \mathfrak{h}_2\epsilon\|_\infty \\ &\quad + \beta\|\mathfrak{h}_1\epsilon - \text{Th}_1\epsilon\|_\infty + \gamma\|\mathfrak{h}_2\epsilon - \text{Th}_2\epsilon\|_\infty \\ &= \alpha h(\mathfrak{h}_1\epsilon, \mathfrak{h}_2\epsilon) + \beta h(\mathfrak{h}_1\epsilon, \text{Th}_1\epsilon) + \gamma h(\mathfrak{h}_2\epsilon, \text{Th}_2\epsilon). \end{aligned} \quad (3.9)$$

It is easy to see, that for both cases of the functional $\zeta(\epsilon, f)$, for $\epsilon, f \in [0, 1]$ and otherwise. Moreover, the condition (2.10) and the related limits are verified. Then, for $\alpha, \beta, \gamma \in (0, 1)$, with $\kappa = \frac{\alpha+\beta}{1-\gamma} < 1$, all the conditions of the Theorem 2.8 are satisfied.

Then, we have proved the existence and the uniqueness of a bounded solution of the equation (3.1) in $\mathfrak{B}(\tilde{S})$. □

4. Conclusions

Fixed point theory it is a main tool in topology and nonlinear analysis, with the aim to obtain existence, uniqueness results and solution approximation schemes. Moreover, has a lot application in different fields as, dynamic optimization, optimal control, partial or common differential equations, differential inclusions and fractional equations.

Our paper is fully dedicated for the study of the existence and uniqueness of a fixed point for the case of a new type of space, CVCMS. Contractions presented by [1] and [2] for controlled metric spaces are now defined for CVCMS. Also, with the help of examples sustainability of results is verified. Moreover, we discussed the existence and the uniqueness of a bounded solution of the Bellman equation from dynamic programming field, applying our fixed point results.

The claim for these results is particularly evident in the fields of engineering, biological systems, management, economy, finance and information technology.

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A new computational method based on the Picard iteration method for solving boundary optimal control problems governed by PDEs with two-point boundary conditions

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Abstract. This paper presents a new computational method based on the Picard iteration method for solving boundary optimal control problems governed by parabolic partial differential equations with two-point boundary conditions. The proposed approach adapts the Picard iteration method to solve the necessary optimality conditions derived from Pontryagin’s minimum principle, yielding a solution expressed as a truncated power series. To evaluate the effectiveness of the proposed method, a numerical example is provided, and the obtained results are compared with those derived from an alternative approach, demonstrating the accuracy and reliability of the method.

Mathematics Subject Classification (2010): 65K10, 49K25.


Keywords: Optimal control problem, Picard iteration method, Pontryagin minimum principle.

1. Introduction and problem statement

Optimal control problems, particularly those governed by partial differential equations (PDEs), constitute an important area of research with numerous practical applications, such as chemical reactions, diffusion–reaction phenomena, gas dynamics, blood cancer diagnosis, and many others, see, e.g., [3, 4, 5, 6, 8, 9, 10, 15, 16, 18, 21]. The objective of such problems is to determine the control signal that optimizes a

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cost functional subject to the governing PDE constraints. Due to the importance in applications, several numerical and semi-analytical methods have been proposed in the literature to approximate the solution, including the shooting method [7], the variational iteration method (VIM) [9], and the Picard iteration method (PIM) [19]. Comparative studies in the literature have demonstrated their efficiency (see, e.g. [6, 9, 20]).

The Picard iteration method (PIM) [19, 20] is a well-established approach for solving a wide range of differential equations, including homogeneous, nonhomogeneous, linear, and nonlinear cases. It has proven to be a powerful tool in addressing various problems. For example, PIM has been applied to time-space fractional differential equations in [12], to the numerical solution of two-point nonlinear boundary value problems in [2], and to fractional differential-algebraic systems in [14]. The PIM is based on representing the solution of an ordinary differential equation (ODE) as an infinite series, which is then approximated by truncating the series after a finite number of terms. A key advantage of PIM is that it provides solutions without imposing restrictive assumptions, and, unlike the variational iteration method (VIM), it does not require the computation of Lagrange multipliers. In fact, several studies documented in the literature demonstrate favourable convergence results for the PIM, as seen in the reference [2, 19]. However, this method has yet to be implemented in optimal control problems governed by PDEs due to its complexity, which serves as the impetus for this work.

In this paper, we focus on using and evaluating the numerical performance of the Picard iteration method to solve an optimal control problem governed by a one-dimensional parabolic-type partial differential equation (PDE) with two-point boundary conditions, formulated as follows

$$\min_{u_0, u_\ell} J = \int_0^{t_f} \int_0^\ell y^2(x, t) dx dt + \int_0^{t_f} [q_0 u_0^2(t) + q_\ell u_\ell^2(t)] dt, \quad (1.1)$$

subject to the one-dimensional parabolic differential equation

$$\frac{\partial y}{\partial t}(x, t) = \frac{\partial^2 y}{\partial x^2}(x, t) \quad \text{in } Q, \quad (1.2)$$

with initial and final conditions

$$y(x, 0) = y_0(x), \quad y(x, t_f) = y_{t_f}(x) \quad \text{in } \Omega, \quad (1.3)$$

and with two-points boundary conditions

$$y(0, t) = u_0(t), \quad y(\ell, t) = u_\ell(t) \quad \text{on } \Sigma, \quad (1.4)$$

where $Q = \Omega \times (0, t_f)$, $\Sigma = \partial\Omega \times (0, t_f)$ with $\Omega = (0, \ell)$ is a segment of the real axis, q_0 and q_ℓ are positive weighting factors. The function $u_0(t)$ and $u_\ell(t)$ are the control variables on the two-point boundary conditions.

In this paper, we propose a new computational procedure based on the PIM to determine the control variable $u(x, t)$ that minimizes the objective function (1.1) subject to the PDE (1.2) and the constraints (1.3)-(1.4). First, by using finite-difference approximation [1], we transform the problem into a control problem governed by a system of ordinary differential equations (ODEs). Then the Pontryagin minimum

principle [17] is applied to derive the necessary optimality conditions, which are expressed by the Hamilton–Pontryagin (HP) equations. Next, the PIM is adapted to iteratively obtain an approximate analytical solution of these equations in the form of a truncated power series.

The rest of the paper is organized as follows. In Section 2, we present a numerical method based on the PIM method for solving the control problem (1.1)-(1.4). This section also discusses Pontryagin’s minimum principle and examines the convergence of the PIM when applied to the HP equations. In Section 3, we summarize the proposed algorithm for solving the problem using the PIM. A numerical example is presented in Section 4 to demonstrate the effectiveness of the proposed approach. Finally, Section 5 concludes the paper.

2. Numerical analysis of the proposed method

2.1. Finite difference approximation

In this section, the finite difference method is employed to transform problem (1.1)-(1.4) into an optimal control problem governed by ODEs. First, the parabolic equation (1.2) is discretized in the spatial variable x , resulting in a system of ODEs. Then, the trapezoidal rule is applied to the objective function (1.1) to derive a weighted quadratic objective function expressed in terms of the state and control variables.

By dividing the interval $[0, \ell]$ into $n \in \mathbb{N}$ intervals, and define the step size $h = \frac{\ell}{n}$, the problem (1.1)-(1.4) becomes

$$\min_{u_0, u_\ell} J = \left(\frac{h}{2} + q_0\right) \int_0^{t_f} u_0^2(t) dt + h \sum_{k=1}^{n-1} \int_0^{t_f} y_k^2(t) dt + \left(\frac{h}{2} + q_l\right) \int_0^{t_f} u_\ell^2(t) dt, \quad (2.1)$$

subject to the system of ODEs

$$\begin{cases} \dot{y}_1(t) = \frac{1}{h^2}(y_2(t) - 2y_1(t) + u_0(t)), \\ \dot{y}_k(t) = \frac{1}{h^2}(y_{k+1}(t) - 2y_k(t) + y_{k-1}(t)), \quad k = 2, \dots, n-2 \\ \dot{y}_{n-1}(t) = \frac{1}{h^2}(u_\ell(t) - 2y_{n-1}(t) + y_{n-2}(t)), \end{cases} \quad (2.2)$$

with boundary conditions

$$y(x_k, 0) = y_k(0), \quad y(x_k, t_f) = y_k(t_f), \quad k = 1, \dots, n-1. \quad (2.3)$$

2.2. Pontryagin’s Minimum Principle

The minimum principle of Pontryagin developed by the Russian mathematician Lev Pontryagin and his colleagues [17], is a fundamental concept in optimal control theory. It provides the necessary conditions for optimality in solving extremum problems in optimal control. According to this principle, the solution of (2.1)-(2.3) is obtained by minimizing the Hamiltonian function \mathcal{H} , which is defined as follows

$$\mathcal{H} = \left(\frac{h}{2} + q_0\right)u_0^2(t) + h \sum_{k=1}^{n-1} y_k^2(t) + \left(\frac{h}{2} + q_l\right)u_\ell^2(t) + p(t)^T f(y(t), u(t)), \quad (2.4)$$

where $p(t) \in \mathbb{R}^{n-1}$ is the adjoint vector, $f : \mathbb{R}^{n-1} \times \mathbb{R}^2 \rightarrow \mathbb{R}^{n-1}$ is a vector function, and the system (2.2) can be written as

$$\dot{y}(t) = f(y(t), u(t)), \quad y(t) \in \mathbb{R}^{n-1}, \quad u(t) \in \mathbb{R}^2.$$

The optimal control law is then obtained by minimizing the Hamiltonian (2.4) as follows

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial u_0}(y(t), u(t), p(t)) = 0, \\ \frac{\partial \mathcal{H}}{\partial u_\ell}(y(t), u(t), p(t)) = 0, \end{cases} \quad (2.5)$$

which yields

$$\begin{cases} u_0(t) = -\frac{p_1(t)}{h^3 + 2h^2q_0}, \\ u_\ell(t) = -\frac{p_{n-1}(t)}{h^3 + 2h^2q_\ell}. \end{cases} \quad (2.6)$$

By substituting (2.6) into the Hamiltonian (2.4) and applying the minimum principle of Pontryagin, the Hamilton–Pontryagin (HP) equations are obtained as follows

$$\begin{cases} \dot{y}(t) = \frac{\partial \mathcal{H}}{\partial p(t)}(y(t), p(t)), \\ \dot{p}(t) = -\frac{\partial \mathcal{H}}{\partial y(t)}(y(t), p(t)), \end{cases} \quad (2.7)$$

which yields

$$\begin{cases} \dot{y}_1(t) = \frac{1}{h^2} \left(y_2(t) - 2y_1(t) - \frac{p_1(t)}{h^3 + 2h^2q_0} \right), \\ \dot{y}_k(t) = \frac{1}{h^2} \left(y_{k+1}(t) - 2y_k(t) + y_{k-1}(t) \right), \quad k = 2, \dots, n-2, \\ \dot{y}_{n-1}(t) = \frac{1}{h^2} \left(y_{n-2}(t) - 2y_{n-1}(t) - \frac{p_{n-1}(t)}{h^3 + 2h^2q_\ell} \right), \end{cases} \quad (2.8)$$

$$\begin{cases} \dot{p}_1(t) = -\left(2hy_1(t) - 2\frac{p_1(t)}{h^2} + \frac{p_2(t)}{h^2} \right), \\ \dot{p}_k(t) = -\left(2hy_k(t) + \frac{p_{k-1}(t)}{h^2} - 2\frac{p_k(t)}{h^2} + \frac{p_{k+1}(t)}{h^2} \right), \quad k = 2, \dots, n-2, \\ \dot{p}_{n-1}(t) = -\left(2hy_{n-1}(t) + \frac{p_{n-2}(t)}{h^2} - 2\frac{p_{n-1}(t)}{h^2} \right), \end{cases} \quad (2.9)$$

subject to the following boundary conditions:

$$y(0) \in E_0, \tag{2.10}$$

$$y(t_f) \in E_f, \tag{2.11}$$

$$p(0) = \sum_{i=1}^d \mu_i \nabla E_{0i}(y(0)), \tag{2.12}$$

$$p(t_f) = - \sum_{j=1}^b \nu_j \nabla E_{fj}(y(t_f)), \tag{2.13}$$

where E_0 and E_f are manifolds in \mathbb{R}^{n-1} given as follows:

$$E_0 = \{y(t) \in \mathbb{R}^{n-1} \mid E_{01}(y(t)) = E_{02}(y(t)) = \dots = E_{0d}(y(t)) = 0\},$$

$$E_f = \{y(t) \in \mathbb{R}^{n-1} \mid E_{f1}(y(t)) = E_{f2}(y(t)) = \dots = E_{fb}(y(t)) = 0\},$$

where the functions $E_{0i} : \mathbb{R}^{n-1} \mapsto \mathbb{R}$, $i = 1, \dots, d$, (with $d \leq n - 1$) and $E_{fj} : \mathbb{R}^{n-1} \mapsto \mathbb{R}$, $j = 1, \dots, b$, (with $b \leq n - 1$) are assumed to be continuously differentiable. $\mu = [\mu_1, \mu_2, \dots, \mu_d]^T$ and $\nu = [\nu_1, \nu_2, \dots, \nu_b]^T$ are the vectors of additional Lagrange multipliers associated with $E_{0i} = (E_{01}, E_{02}, \dots, E_{0d})$ and $E_{fj} = (E_{f1}, E_{f2}, \dots, E_{fb})$ respectively.

Remark 2.1. From [13, 22], we can observe that the following property holds:

1. If E_0 is reduced to a single point, that is $E_0 = \{y(0) = y_0\}$, then condition (2.12) becomes vacuous.
2. If $E_0 = \mathbb{R}^{n-1}$ meaning that the initial point is not specified, we obtain $p(0) = 0$.
3. If E_f is reduced to a single point, that is $E_f = \{y(t_f) = y_{t_f}\}$, then condition (2.13) becomes vacuous.
4. If $E_f = \mathbb{R}^{n-1}$ meaning that the final point is free, we obtain $p(t_f) = 0$.

2.3. Application of the Picard iteration method

2.3.1. A brief description. Before applying the Picard iteration method to solve the HP equations (2.8)–(2.9), it is important to first illustrate its fundamental principle. Consider the following differential equation

$$\dot{y}(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t > t_0, \tag{2.14}$$

where f is a continuous function. By integrating both sides of (2.14) with respect to t , we get the following integral equation

$$y(t) = y_0 + \int_{t_0}^t f(\tau, y(\tau)) d\tau. \tag{2.15}$$

To solve (2.14) using the PIM method, we generate a sequence of functions $y_k(t)$ iteratively as

$$y_{k+1}(t) = y_0 + \int_{t_0}^t f(\tau, y_k(\tau)) d\tau. \tag{2.16}$$

where $y_0(t)$ is the initial guess, which can be chosen based on the initial condition of problem (2.14), and each approximation $y_{k+1}(t)$, $k \geq 0$ is defined in terms of the previous approximation $y_k(t)$ using the iterative formula (2.16). Therefore, the solution of problem (2.10) is given as the limit of the sequence of functions $\{y_k\}$ generated by the formula (2.16), that is

$$y(t) = \lim_{k \rightarrow \infty} y_k(t). \tag{2.17}$$

2.3.2. Convergence analysis. This section presents the convergence analysis of the PIM method for solving the HP equations (2.8)–(2.9). We set

$$\begin{cases} v(t) = \begin{pmatrix} y(t) \\ p(t) \end{pmatrix}, \\ \varphi(t, v(t)) = \begin{pmatrix} \dot{y}(t) \\ \dot{p}(t) \end{pmatrix}, \end{cases} \tag{2.18}$$

where $\varphi(t, v(t)) : [0, t_f] \times \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$ is assumed to be continuous in its arguments for all $t \in [0, t_f]$. Thus, the HP equations (2.8)–(2.9) can be conveniently rewritten as follows

$$\dot{v}(t) = \varphi(t, v(t)), \quad v(0) = v_0 = (y_0, p_0). \tag{2.19}$$

Applying the PIM method to the IVP (2.19), we immediately obtain the following integral equation

$$v(t) = v_0 + \int_0^t \varphi(\tau, v(\tau)) d\tau, \tag{2.20}$$

and the sequence of Picard iterates $\{v_k\}$ of the IVP (2.19) is given as

$$v_{k+1}(t) = v_0 + \int_0^t \varphi(\tau, v_k(\tau)) d\tau, \quad k \geq 0. \tag{2.21}$$

The following lemma establishes the equivalence between the IVP (2.15) and the integral equation (2.20).

Lemma 2.2. *Assume that \mathcal{D} is an open subset of $[0, t_f] \times \mathbb{R}^{2n}$, $\varphi(t, v(t)) : \mathcal{D} \mapsto \mathbb{R}^{2n}$ is continuous, and $(0, v_0) \in \mathcal{D}$; then $v(t)$ is a solution of the IPV (2.19) on $[0, t_f]$ if and only if $v(t)$ is a solution of (2.20).*

Proof. We refer to the proof of [11, Lemma 8.6] for details. □

Definition 2.3. *A vector function $\varphi(t, v(t)) : \mathcal{D} \mapsto \mathbb{R}^{2n}$ is said to satisfy a uniform Lipschitz condition with respect to $v(t)$ on $\mathcal{D} \subset [0, t_f] \times \mathbb{R}^{2n}$ if there exists a positive constant C such that*

$$\|\varphi(t, v(t)) - \varphi(t, u(t))\| \leq C \|(v(t) - u(t))\|, \tag{2.22}$$

for all $(t, v(t)), (t, u(t)) \in \mathcal{D}$. The constant C is called a Lipschitz constant for $\varphi(t, v(t))$ with respect to $v(t)$ on the open set \mathcal{D} .

According to the Picard-Lideloof theorem [11, Theorem 8.13], we obtain the following result regarding the sufficient conditions for the convergence of the PIM method applied to the HP equations (2.18).

Theorem 2.4. *Assume that $\varphi(t, v(t)) : \mathcal{D} \mapsto \mathbb{R}^{2n}$ is a continuous $2n$ -dimensional vector function on the parallelepiped*

$$\Gamma = \{(t, v(t)) : 0 \leq t \leq s, \|v(t) - v_0(t)\| \leq l\}, l > 0, \tag{2.23}$$

and assume that $\varphi(t, v(t))$ satisfies a uniform Lipschitz condition with respect to $v(t)$ on Γ , and let

$$\lambda = \max\{\|\varphi(t, v(t))\| : (t, v(t)) \in \Gamma\}, \gamma = \min\left\{s, \frac{l}{\lambda}\right\}. \tag{2.24}$$

Then the sequence of Picard iterates $\{v_k(t)\}$ given by (2.21) converges to the exact solution v of the IVP (2.19) on $[0, \gamma]$, such that

$$\|v(t) - v_0(t)\| \leq l, \forall t \in [0, \gamma]. \tag{2.25}$$

Proof. The proof is inspired by [11]. We first prove by induction that each Picard iterate v_k from the sequence of Picard iterates $\{v_k(t)\}$, $k \geq 0$, defined by (2.21), is continuous on the interval $[0, \gamma]$ with respect to $v(t)$ on Γ . It is clear that $v_0(t)$ satisfies these conditions. Suppose now that $v_k(t)$ is well defined and continuous on $[0, \gamma]$, and satisfies the required assumptions

$$\|v_k(t) - v_0\| \leq l, \quad t \in [0, \gamma]. \tag{2.26}$$

Using Lemma 2.2, it follows that

$$v_{k+1}(t) = v_0 + \int_0^t \psi(\tau, v_k(\tau)) d\tau, \quad t \in [0, \gamma], \tag{2.27}$$

is well defined and continuous on $[0, \gamma]$.

Moreover, for all $t \in [0, \gamma]$ we have

$$\begin{aligned} \|v_{k+1}(t) - v_0\| &\leq \int_0^t |\psi(\tau, v_k(\tau))| d\tau \\ &\leq \lambda(t) \leq \lambda\gamma \leq l, \end{aligned}$$

and the induction is complete.

Let C be a Lipschitz constant for $\varphi(t, v(t))$ with respect to $v(t)$ on Γ , and we consider the following hypothesis:

$$\|v_{k+1}(t) - v_k(t)\| \leq \frac{\lambda C^k (t)^{k+1}}{(k+1)!}, \quad k \geq 0, t \in [0, \gamma]. \tag{2.28}$$

Let us verify this hypothesis by induction. The result is trivially true for $k = 0$. Now, assume that it holds for $k - 1$. Then, using the Lipschitz condition (2.22) together with the induction assumption, we obtain

$$\begin{aligned}
 \|v_{k+1}(t) - v_k(t)\| &= \left\| \int_0^t [\varphi(\tau, v_k(\tau)) - \varphi(\tau, v_{k-1}(\tau))] d\tau \right\| \\
 &\leq \int_0^t \|\varphi(\tau, v_k(\tau)) - \varphi(\tau, v_{k-1}(\tau))\| d\tau \\
 &\leq C \int_0^t \|v_k(\tau) - v_{k-1}(\tau)\| d\tau \\
 &\leq \lambda C^k \int_0^t \frac{(\tau)^k}{k!} d\tau \\
 &= \frac{\lambda C^k (t)^{k+1}}{(k+1)!}, \quad \text{holds for all } t \in [0, \gamma].
 \end{aligned}$$

Hence the induction is complete.

We now proceed to prove the convergence of the sequence of Picard iterates $\{v_k(t)\}$. For this purpose, we rewrite $\{v_k(t)\}$ as the sequence of partial sums of an infinite series, given by:

$$\{v_k(t)\}_{k=1}^\infty = \{\varphi_0(t) + \sum_{m=0}^{k-1} (v_{m+1}(t) - v_m(t))\}. \tag{2.29}$$

Obviously, we have

$$\|v_{m+1}(t) - v_m(t)\| \leq \frac{\lambda(C\gamma)^{m+1}}{C(m+1)!}, \quad \text{holds for all } t \in [0, \gamma], \tag{2.30}$$

and

$$\sum_{m=0}^\infty \frac{\lambda(C\gamma)^{m+1}}{C(m+1)!} \text{ converge.}$$

Consequently, by applying the Weierstrass M-test [23], it follows that the sequence of Picard iterates $\{v_k(t)\}$ converges uniformly on $[0, \gamma]$. Therefore, we can write:

$$v(t) = \lim_{k \rightarrow \infty} v_k(t), \quad \forall t \in [0, \gamma]. \tag{2.31}$$

It follows that (2.25) holds for all $t \in [0, \gamma]$. Moreover, since the Lipschitz condition (2.22) is satisfied,

$$\|\varphi(t, v_k(t)) - \varphi(t, v(t))\| \leq C \|v_k(t) - v(t)\|, \quad t \in [0, \gamma], \tag{2.32}$$

we deduce that

$$\lim_{k \rightarrow \infty} \varphi(t, v_k(t)) = \psi(t, v(t)), \tag{2.33}$$

uniformly on $[0, \gamma]$.

Now, applying the limits (2.31) and (2.33) to both sides of Eq. (2.27), we obtain

$$v(t) = v_0 + \int_0^t \varphi(\tau, v(\tau)) d\tau, \quad t \in [0, \gamma], \tag{2.34}$$

which means that (2.34) is a solution of the IVP (2.19).

To prove the uniqueness, suppose that g is another solution of the IVP (2.15) on $[0, \mu]$, where $0 \leq \mu \leq \gamma$, and we prove that $g = v$. From Lemma 2.2, it follows that g satisfies the following integral equation:

$$g(t) = v_0 + \int_0^t \varphi(\tau, g(\tau)) d\tau, \quad t \in [0, \mu]. \tag{2.35}$$

Similarly, we can prove by induction that

$$\|g(t) - v_k(t)\| \leq \frac{\lambda C^k (t)^{k+1}}{(k+1)!}, \quad k \geq 0, \quad t \in [0, \mu]. \tag{2.36}$$

Consequently, we get

$$g(t) = \lim_{k \rightarrow \infty} v_k(t) = v_k(t), \quad t \in [0, \mu],$$

and this completes the proof. □

The following result follows immediately from Theorem 2.4.

Corollary 2.5. *Let $\{v_k(t)\}$ be the sequence of the Picard iterates generated by the iterative formula (2.13) of the IVP (2.15), and assume that the hypotheses of Theorem 2.4 hold. If φ is the exact solution of the (2.15), then*

$$\|v(t) - v_k(t)\| \leq \frac{\lambda C^k (t)^{k+1}}{(k+1)!}, \quad k \geq 0, \quad t \in [0, \gamma], \tag{2.37}$$

where C be a Lipschitz constant for $\varphi(t, v(t))$ with respect to $v(t)$ on Γ .

3. Algorithm for the proposed method

In this section a proposed algorithm based on the PIM to approximate the solution of the optimal control problem (1.1)–(1.4) is proposed. The different steps of the proposed approach can be summarized as follows

Algorithm 3.1. Algorithm for the proposed approach

Step 1. Transform the optimal control problem (1.1)–(1.4) into an optimal control problem governed by a system of ODEs using the finite difference approximation.

Step 2. Determine the necessary optimality conditions for the an approximation of $J(u_0(t), u_\ell(t))$ using the PMP.

Step 3. Establish the sequence of Picard iterates $y_{k+1}(t)$ and $p_{k+1}(t)$ of the IVP (2.8)–(2.9) using the PIM method.

Step 4. Choose a desired threshold $\epsilon > 0$, set $k = 0$, and let initial approximations $y_0(t) = y(0)$, $t \in [0, t_f]$ and $p_0(t) = \Lambda$, where Λ is a vector which can be determined by using boundary conditions.

Step 5. Determine the approximate solution $y_{k+1}(t)$ and $p_{k+1}(t)$ using the iterative formula from step 3.

Step 6. Deduce the optimal control $u_0^k(t)$ and $u_\ell^k(t)$ using expression (2.6), and evaluate the corresponding objective function.

Step 7. Stopping criterion. if

$$|J(u_0^{k+1}(t), u_\ell^{k+1}(t)) - J(u_0^k(t), u_\ell^k(t))| \leq \epsilon,$$

stop the iteration process. Otherwise, set $k = k + 1$, and go to step 6.

4. Numerical example and discussions

In this section, the proposed method is examined through numerical experiments. To demonstrate its efficiency, the results obtained using the proposed method are compared with those obtained using the variational iteration method (VIM) [3]. In the calculations, the following data is used:

$\ell = \pi, t_f = 1, q_0 = q_\ell = 0, y(0) = 0, y(t_f) = \frac{3}{2}, n = 50$ and the step size $h = \frac{2500}{\pi^2}$.

Therefore, according to the PIM we construct the following iterative formulas:

$$\begin{cases} y_1^{(k+1)}(t) = y_1(0) + \int_0^t \frac{1}{h^2} \left(y_2^{(k)}(\tau) - 2y_1^{(k)} - \frac{p_1^{(k)}(\tau)}{h^3} \right) d\tau, \\ y_m^{(k+1)}(t) = y_m(0) + \int_0^t \frac{1}{h^2} \left(y_{m+1}^{(k)}(\tau) - 2y_m^{(k)} - y_{m-1}^{(k)} \right) d\tau, \quad m = 2, \dots, 48, \\ y_{49}^{(k+1)}(t) = y_{49}(0) + \int_0^t \frac{1}{h^2} \left(y_{48}^{(k)}(\tau) - 2y_{49}^{(k)} - \frac{p_{49}^{(k)}(\tau)}{h^3} \right) d\tau. \end{cases} \quad (4.1)$$

$$\begin{cases} p_1^{(k+1)}(t) = p_1(0) - \int_0^t \left(2h y_1^{(k)}(\tau) - 2\frac{p_1^{(k)}(\tau)}{h^2} + \frac{p_2^{(k)}(\tau)}{h^2} \right) d\tau, \\ p_m^{(k+1)}(t) = p_m(0) - \int_0^t \left(2h y_m^{(k)}(\tau) + \frac{p_{m-1}^{(k)}}{h^2} - 2\frac{p_m^{(k)}}{h^2} + \frac{p_{m+1}^{(k)}}{h^2} \right) d\tau, \\ \text{for, } m = 2, \dots, 48, \\ p_{49}^{(k+1)}(t) = p_{49}(0) - \int_0^t \left(2h y_{49}^{(k)}(\tau) + \frac{p_{48}^{(k)}(\tau)}{h^2} - 2\frac{p_{49}^{(k)}(\tau)}{h^2} \right) d\tau. \end{cases} \quad (4.2)$$

From (2.5) and (2.6), the optimal control law is given as:

$$\begin{cases} u_0^*(t) = y_0(t) = -\frac{p_1(t)}{h^3}, \\ u_\ell^*(t) = y_{50}(t) = -\frac{p_{49}(t)}{h^3}, \end{cases} \quad (4.3)$$

where, the zeroth approximation can be chosen as

$$y_m(0) = 0 \text{ and } p_m(0) = \lambda_m, \forall m = 1, \dots, 49,$$

where λ_m are unknown parameters to be determined by imposing the final condition $y(t_f)$. The results of the iterative process are presented in Table 1.

TABLE 1. Iterations result

k	$ J^{k+1} - J^k $	k	$ J^{k+1} - J^k $
0	/	6	0.02877604
1	1.67698260	7	0.01246203
2	1.04756755	8	0.00409854
3	0.10723897	9	0.00083735
4	0.10156208	10	0.00009973
5	0.08039717	11	0.00000073

Based on the obtained results, the optimal control is determined using a convergence threshold of $\epsilon = 10^{-6}$. Therefore, we conclude that the proposed method converges after 11 iterations, and yields the following approximate optimal control law:

$$\begin{aligned}
 u_0^*(t) = & \left(-\frac{50}{\pi}\right)^3 \left[\left(\frac{1.65611e+12 t^{11}}{\pi^{22}}\right) + \left(\frac{1.94656e+11 t^{10}}{\pi^{20}}\right) + \left(\frac{5.18143e+10 t^9}{\pi^{18}}\right) \right. \\
 & + \left(\frac{7.8365e+9 t^8}{\pi^{16}}\right) + \left(\frac{8.30182e+8 t^7}{\pi^{14}}\right) + \left(\frac{1.0815e+8 t^6}{\pi^{12}}\right) + \left(\frac{6.7437e+6 t^5}{\pi^{10}}\right) \\
 & \left. + \left(\frac{6.3835e+5 t^4}{\pi^8}\right) + \left(\frac{2.1615e+4 t^3}{\pi^6}\right) + \left(\frac{1.2509e+3 t^2}{\pi^4}\right) + \left(\frac{15.15 t}{\pi^2}\right) + 0.987 \right]
 \end{aligned}$$

$$\begin{aligned}
 u_l^*(t) = & \left(-\frac{50}{\pi}\right)^3 \left[\left(\frac{6.1665e+12 t^{11}}{\pi^{22}}\right) + \left(\frac{1.6388e+12 t^{10}}{\pi^{20}}\right) + \left(\frac{1.1178e+11 t^9}{\pi^{18}}\right) \right. \\
 & + \left(\frac{2.3576e+10 t^8}{\pi^{16}}\right) + \left(\frac{1.4122e+9 t^7}{\pi^{14}}\right) + \left(\frac{2.2684e+8 t^6}{\pi^{12}}\right) + \left(\frac{1.1203e+7 t^5}{\pi^{10}}\right) \\
 & \left. + \left(\frac{1.2864e+6 t^4}{\pi^8}\right) + \left(\frac{4.7711e+4 t^3}{\pi^6}\right) + \left(\frac{3.5206e+3 t^2}{\pi^4}\right) + \left(\frac{79.75 t}{\pi^2}\right) + 2.846 \right]
 \end{aligned}$$

In Figure 1, the approximate optimal trajectories of the control variables obtained using the proposed approach are plotted alongside those obtained with the Variational Iteration Method (VIM)[3]. The plots show a high level of agreement between the two methods.

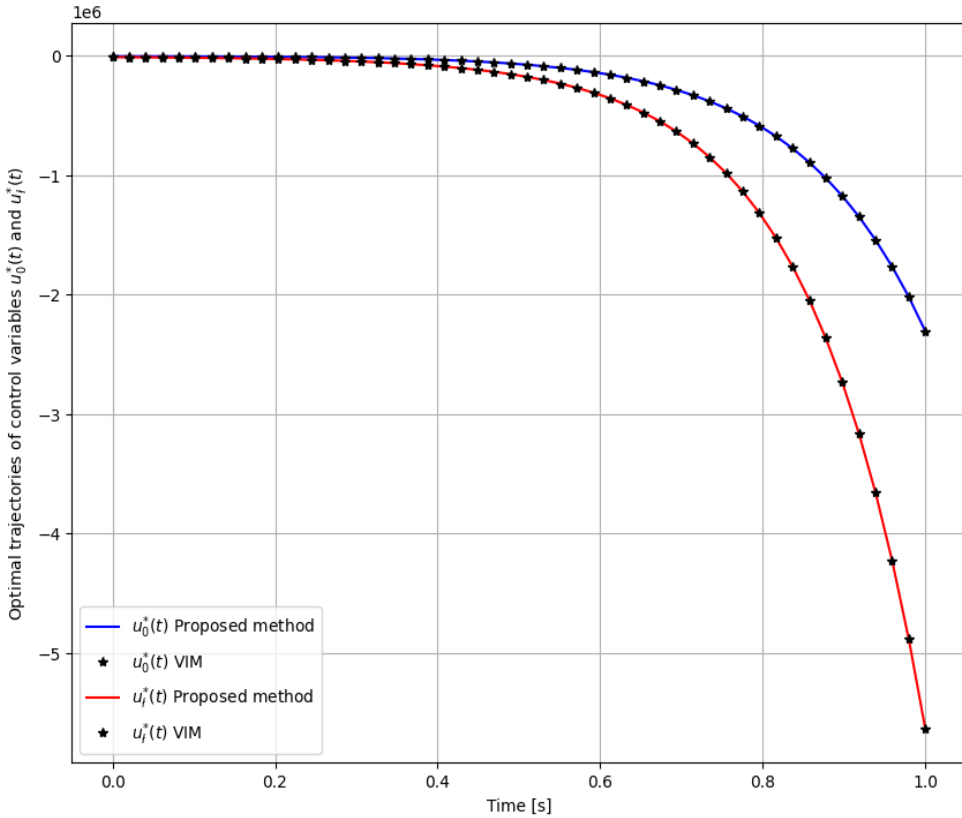


FIGURE 1. Optimal trajectories of control variables

5. Conclusion

In this work, a new approach based on the Picard iteration method is successfully employed to obtain an approximate solution to an optimal control problem governed by a one-dimensional parabolic partial differential equation with two-point boundary conditions. In fact, the Picard iteration method is adapted to solve the necessary optimality conditions derived from Pontryagin's minimum principle. The resulting solution is expressed as a truncated power series.

The proposed approach is illustrated by a numerical example. To demonstrate its efficiency, a comparison is made between the obtained results and those obtained using the variational iteration method (VIM), showing that the results are very close. The main advantage of the proposed method over the variational iteration method is

that it provides a solution to the problem without requiring any restrictive assumptions. Furthermore, unlike the variational iteration method, it does not require the computation of a Lagrange multiplier.

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