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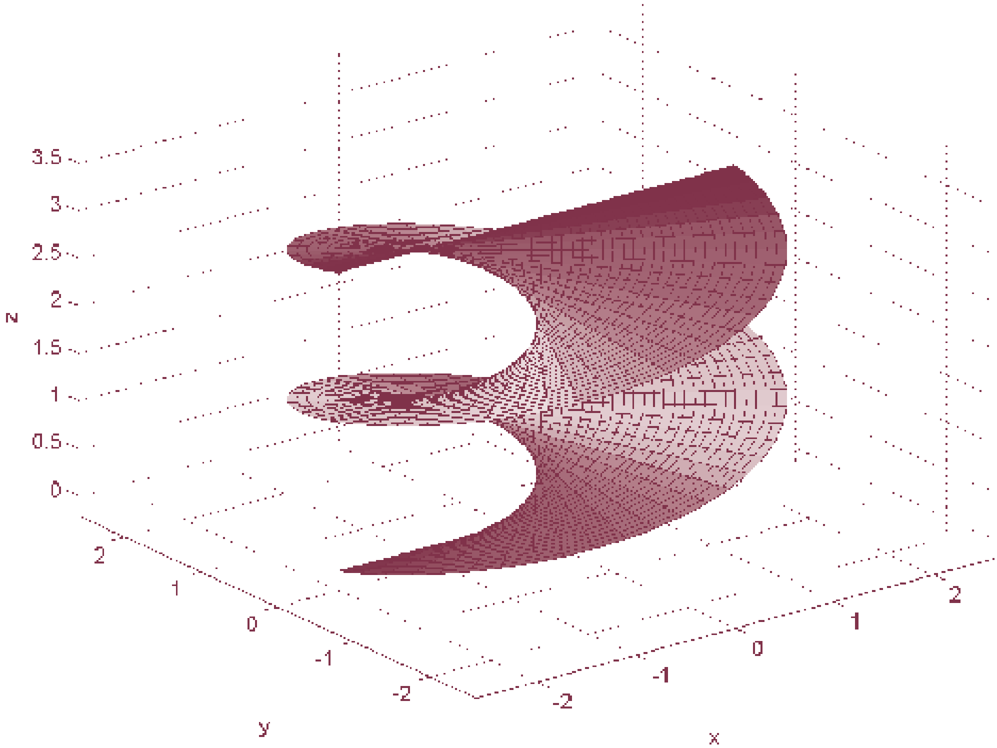

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Depth and sdepth of powers of the path ideal of a cycle graph. II

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Abstract. Let $J_{n,m} := (x_1x_2\cdots x_m,\ x_2x_3\cdots x_{m+1},\ \ldots,\ x_{n-m+1}\cdots x_n,\ x_{n-m+2}\cdots x_nx_1, \ldots, x_nx_1\cdots x_{m-1})$ be the m-path ideal of the cycle graph of length n, in the ring of polynomials $S = K[x_1,\ldots,x_n]$.

As a continuation of our previous paper [2], we prove several new results regarding $\operatorname{depth}(S/J_{n,m}^t)$ and $\operatorname{sdepth}(S/J_{n,m}^t)$, where $t \geq 1$.

Mathematics Subject Classification (2010): 13C15, 13P10, 13F20.

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1. Introduction

Let $n \geq 1$ be an integer. We denote $[n] = \{1,2,\ldots,n\}$. Let K be a field and let $S = K[x_1,\ldots,x_n]$ be the polynomial ring in n variables over K. Given a simple graph $G = (V,E)$, on the vertex set $V = [n]$ with the edge set E, the edge ideal associated to G is

$$I(G) = (x_ix_j\ :\ \{i,j\} \in E) \subset S.$$

Note that $I(G)$ is a monomial ideal generated in degree 2. The study of the algebraic properties of $I(G)$ is a well established topic in combinatorial commutative algebra. Conca and De Negri generalized the definition of an edge ideal and first introduced the notion of a m-path ideal in [6], that is

$$I_m(G) = (x_{i_1}x_{i_2}\cdots x_{i_m}\ :\ i_1i_2\cdots i_m \text{ is a path in } G) \subset S.$$

In the recent years, several algebraic and combinatorial properties of path ideals have been studied. However, the study of powers of path ideals is a relatively new area of research.

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The path graph of length $n-1$ is $P_n = (V(P_n), E(P_n))$, where

$$V(P_n) = [n] \text{ and } E(P_n) = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}.$$

Let $1 \leq m \leq n$. One can easily check that the *m-path ideal of the path graph* of length $n-1$ is

$$I_{n,m} = (x_1x_2\cdots x_m,\ x_2x_3\cdots x_{m+1},\ \ldots,\ x_{n-m+1}\cdots x_n) \subset S.$$

The cycle graph of length n is $C_n = (V(C_n), E(C_n))$, where

$$V(C_n) = [n] \text{ and } E(C_n) = E(P_n) \cup \{\{1,n\}\}.$$

Also, for $2 \leq m < n$, the *m-path ideal of the cycle graph* of length n is

$$J_{n,m} = I_{n,m} + (x_{n-m+2}\cdots x_nx_1, x_{n-m+3}\cdots x_nx_1x_2, \ldots, x_nx_1\cdots x_{m-1}).$$

In [1] we studied the depth and Stanley depth of $S/I_{n,m}^t$, where $t \geq 1$. Also, in [2] we studied the depth and Stanley depth of $S/J_{n,m}^t$, where $t \geq 1$. Following [2], the aim of our paper is to further investigate the depth and Stanley depth of the quotient rings associated to powers of the m-path ideal of a cycle.

In Theorem 3.2, we reprove and also extend some results from [2]. As part of our new results, we show that for any $n \geq 5$:

$$\operatorname{sdepth}(S/J_{n,n-2}^t), \operatorname{depth}(S/J_{n,n-2}^t) > 0 \text{ if } n \text{ is odd, and } t < \frac{n-1}{2},$$
$$\operatorname{sdepth}(S/J_{n,n-2}^t), \operatorname{depth}(S/J_{n,n-2}^t) > 0 \text{ if } n \text{ is even, and } t \geq 1,$$
$$\frac{n}{2} \geq \operatorname{sdepth}(S/J_{n,n-2}^t) \geq \operatorname{depth}(S/J_{n,n-2}^t) = 1 \text{ if } n \text{ is even, and } t \geq n-1.$$

In Theorem 3.6, we show that if $n \geq 2m+1$ then

$$\operatorname{depth}(S/J_{n,m}^t) \leq \operatorname{depth}(S/I_{n,m}^t),$$

which improves the upper bound for $\operatorname{depth}(S/J_{n,m}^t)$ given in [2, Theorem 2.10].

Finally, in Section 4, we make some small steps in tackling the problem of computing $\operatorname{depth}(S/J_{n,m}^t)$, for any $t \geq 1$, see Proposition 4.2 and Proposition 4.3. Also, we illustrate, in two examples, the technical difficulties that arise, see Example 4.4 and Example 4.5.

2. Preliminaries

First, we recall the well known Depth Lemma, see for instance [12, Lemma 2.3.9].

Lemma 2.1. *(Depth Lemma) If $0 \to U \to M \to N \to 0$ is a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local, then*

(1) $\operatorname{depth} M \geq \min\{\operatorname{depth} N, \operatorname{depth} U\}$.
(2) $\operatorname{depth} U \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.
(3) $\operatorname{depth} N \geq \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}$.

The following result, which will be used later on, is an easy application of the Depth Lemma:

Lemma 2.2. *Let $d \geq 1$ and $Z_1 \cup Z_2 \cup \cdots \cup Z_d = \{x_1, \ldots, x_n\}$ be a partition, i.e. $|Z_i| > 0$ and $Z_i \cap Z_j = \emptyset$ for all $i \neq j$. Let $P_i = (Z_i) \subset S$ for $1 \leq i \leq d$ and $U := P_1 \cap \cdots \cap P_d$. Then* $\operatorname{depth}(S/U) = d - 1$.

Now, we briefly recall the definition of the Stanley depth invariant, for a quotient of monomial ideals.

Let $0 \subset I \subsetneq J \subset S$ be two monomial ideals and $M = J/I$. A *Stanley decomposition* of M is the decomposition of M as a direct sum

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i],$$

of $\mathbb{Z}^n$-graded K-vector spaces, where $m_i \in S$ are monomials and $Z_i \subset \{x_1, \ldots, x_n\}$. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) |\ \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

Herzog, Vlădoiu and Zheng show in [9] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps. We say that M satisfies the Stanley inequality, if

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M).$$

Stanley [11] conjectured that any quotient of monomial ideals $M = J/I$ satisfies the Stanley inequality, a conjecture which proves to be false in general for $M = J/I$, where $I \neq 0$; see Duval et al. [8].

The explicit computation of the Stanley depth is a difficult task, both from a theoretical and practical point of view. Also, although the Stanley conjecture was disproved in the most general setting, it is interesting to find large classes of quotients of monomial ideals which satisfy the Stanley inequality.

In [10], Asia Rauf proved the analog of Lemma 2.1 for sdepth:

Lemma 2.3. *If $0 \to U \to M \to N \to 0$ is a short exact sequence of $\mathbb{Z}^n$-graded S-modules, then*

$$\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}.$$

We also recall the following well known results. See for instance [10, Corollary 1.3], [4, Proposition 2.7], [3, Theorem 1.1], [9, Lemma 3.6] and [10, Corollary 3.3].

Lemma 2.4. *Let $I \subset S$ be a monomial ideal and let $u \in S$ a monomial such that $u \notin I$. Then*

(1) $\operatorname{sdepth}(S/(I : u)) \geq \operatorname{sdepth}(S/I)$.
(2) $\operatorname{depth}(S/(I : u)) \geq \operatorname{depth}(S/I)$.

Lemma 2.5. *Let $I \subset S$ be a monomial ideal and let $u \in S$ a monomial such that $I = u(I : u)$. Then*

(1) $\operatorname{sdepth}(S/(I : u)) = \operatorname{sdepth}(S/I)$.
(2) $\operatorname{depth}(S/(I : u)) = \operatorname{depth}(S/I)$.

Lemma 2.6. *Let $I \subset S$ be a monomial ideal and $S' = S[x_{n+1}]$. Then*
(1) $\operatorname{sdepth}_{S'}(S'/IS') = \operatorname{sdepth}_S(S/I) + 1$,
(2) $\operatorname{depth}_{S'}(S'/IS') = \operatorname{depth}_S(S/I) + 1$.

Lemma 2.7. *Let $I \subset S$ be a monomial ideal. Then the following assertions are equivalent:*
(1) $\mathfrak{m} := (x_1, \ldots, x_n) \in \operatorname{Ass}(S/I)$.
(2) $\operatorname{depth}(S/I) = 0$.
(3) $\operatorname{sdepth}(S/I) = 0$.

We will use later on the following result from [5]:

Theorem 2.8. *(see* [5, Theorem 2.11]*) Let $1 \leq p \leq n-1$, $S' = K[x_1, \ldots, x_p]$ and $S'' = K[x_{p+1}, \ldots, x_n]$. Let $I \subset S'$ be a monomial ideal and let $L \subset S''$ be a complete intersection monomial ideal. We denote by $I+L$, the ideal $IS+LS$ of $S = S' \otimes_K S'' = K[x_1, \ldots, x_n]$. Then, for all $t \geq 1$, we have that*
(1) $\operatorname{depth}(S/(I+L)^t) = \min\limits_{1\leq i\leq t}\{\operatorname{depth}_{S'}(S'/I^i)\} + \dim(S''/L)$.
(2) $p + \dim(S''/L) \geq \operatorname{sdepth}(S/(I+L)^t) \geq \min\limits_{1\leq i\leq t}\{\operatorname{sdepth}_{S'}(S'/I^i)\} + \dim(S''/L)$.

Let $1 \leq m \leq n$ be two integers. The m-path ideal of the path graph P_n is

$$I_{n,m} = (x_1 \cdots x_m,\ x_2 \cdots x_{m+1},\ \ldots,\ x_{n-m+1} \cdots x_n) \subset S.$$

We denote

$$\varphi(n,m,t) := \begin{cases} n-t+2-\left\lfloor \frac{n-t+2}{m+1} \right\rfloor - \left\lceil \frac{n-t+2}{m+1} \right\rceil, & t \leq n+1-m \\ m-1, & t > n+1-m \end{cases}.$$

We recall the main result of [1]:

Theorem 2.9. *(See* [1, Theorem 2.6]*) With the above notations, we have that*
(1) $\operatorname{sdepth}(S/I_{n,m}^t) \geq \operatorname{depth}(S/I_{n,m}^t) = \varphi(n,m,t)$, *for all* $t \geq 1$.
(2) $\operatorname{sdepth}(S/I_{n,m}^t) \leq \operatorname{sdepth}(S/I_{n,m}) = \varphi(n,m,1)$.

Let $2 \leq m < n$ be two integers. The m-path ideal of the cycle graph C_n is

$$J_{n,m} = I_{n,m} + (x_{n-m+2} \cdots x_n x_1,\ x_{n-m+3} \cdots x_n x_1 x_2,\ \ldots,\ x_n x_1 \cdots x_{m-1}).$$

Let $d = \gcd(n,m)$ and let $t_0 := t_0(n,m)$ be the maximal integer such that $t_0 \leq n-1$ and there exists a positive integer α such that

$$mt_0 = \alpha n + d.$$

Let $t \geq t_0$ be an integer. Let $w = (x_1 x_2 \cdots x_n)^\alpha$, $w_t = w \cdot (x_1 \cdots x_m)^{t-t_0}$, $r := \frac{n}{d}$ and $s := \frac{m}{d}$. If $d > 1$, we consider the ideal

$$U_{n,d} = (x_1, x_{d+1}, \cdots, x_{d(r-1)+1}) \cap (x_2, x_{d+2}, \cdots, x_{d(r-1)+2}) \cap \cdots \cap (x_d, x_{2d}, \ldots, x_{rd}).$$

We recall the following results from [2]:

Lemma 2.10. *(*[2, Lemma 2.2]*) With the above notations, we have:*
(1) *If $d = 1$ then $(J_{n,m}^t : w_t) = \mathfrak{m}$ for all $t \geq t_0$.*
(2) *If $d > 1$ then $(J_{n,m}^t : w_t) = U_{n,d}$ for all $t \geq t_0$.*

We also recall the following result of [2]:

Theorem 2.11. *([2, Theorem 2.10]) With the above notations, we have that*

$$\operatorname{depth}(S/J_{n,m}^t) \leq \varphi(n-1,m,t)+1, \text{ for all } t \geq 1.$$

3. Main results

At the beginning of this section, we prove the following lemma:

Lemma 3.1. *We have that:*

(1) $\mathfrak{m} \in \operatorname{Ass}(S/J_{n,n-1}^t)$, *for all* $n \geq 2$ *and* $t \geq n-1$.
(2) *If* $n \geq 3$ *is odd then* $\mathfrak{m} \notin \operatorname{Ass}(S/J_{n,n-2}^t)$, *for all* $t < \frac{n-1}{2}$.
(3) *If* $n \geq 3$ *is odd then* $\mathfrak{m} \in \operatorname{Ass}(S/J_{n,n-2}^t)$, *for all* $t \geq \frac{n-1}{2}$.
(4) *If* $n \geq 3$ *is even then* $\mathfrak{m} \notin \operatorname{Ass}(S/J_{n,n-2}^t)$, *for all* $t \geq 1$.

Proof. (1) The result follows from Lemma 2.10(1). However, we present here a new proof: Let $w_t := x_1^{t-1}\cdots x_{n-1}^{t-1}x_n^{n-2}$. Note that $J_{n,n-1}^t$ is minimally generated by monomials of degree $(n-1)t$, while $\deg(w_t) = (n-1)t-1$. Thus, $w_t \notin J_{n,n-1}^t$. We claim that

$$(J_{n,n-1}^t : w_t) = \mathfrak{m}. \tag{3.1}$$

Since $(x_1\cdots x_{n-1})^{t-n+1} \in J_{n,n-1}^{t-n+1}$, $w_t = (x_1\cdots x_{n-1})^{t-n+1}w_{n-1}$ and $w_t \notin J_{n,n-1}^t$ it follows that

$$(J_{n,n-1}^{n-1} : w_{n-1}) \subseteq (J_{n,n-1}^t : w_t) \subsetneq S.$$

Therefore, as $\mathfrak{m}$ is maximal, it is enough to prove (3.1) for $t = n-1$.

Note that $G(J_{n,n-1}) = \{u_1,\ldots,u_n\}$, where $u_i = \prod_{j\neq i} x_j$ for all $1 \leq j \leq n$. It is easy to see that:

$$x_jw_{n-1} = \prod_{k\neq j} u_k \in J_{n,n-1}^{n-1} \text{ for all } 1 \leq j \leq n,$$

hence (3.1) is true.

(2) Assume by contradiction that $\mathfrak{m} \in \operatorname{Ass}(S/J_{n,n-2}^t)$. Then there exists a monomial $w \in S$ with $w \notin J_{n,n-2}^t$ such that $(J_{n,n-2}^t : w) = \mathfrak{m}$. For degree reasons, we must have $\deg(w) \geq t(n-2)-1$. Without any loss of generality, we may assume that $w = x_1^{a_1}\cdots x_n^{a_n}$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then, we deduce that $a_1 \geq t$. Since $x_1w \in J_{n,n-2}^t$ it follows that $w \in J_{n,n-2}^t$, a contradiction.

(3) The result follows from Lemma 2.10(1), since $\gcd(n,n-2) = 1$ and $t_0(n,n-2) = \frac{n-1}{2}$ for n odd.

(4) First, note that $d = \gcd(n,n-2) = 2$. Assume by contradiction that there exists a monomial $w \in S$ with $w \notin J_{n,n-2}^t$ such that $(J_{n,n-2}^t : w) = \mathfrak{m}$. It follows that $x_jw \in J_{n,n-2}^t$ for all $1 \leq j \leq n$. Since $x_1w \in J_{n,n-2}^t$ and $w \notin J_{n,n-2}^t$ it follows that $w = u_1\cdots u_{t-1}v$, where $u_j \in G(J_{n,n-2})$, $v \notin J_{n,n-2}$ and $x_1v \in J_{n,n-2}$. This implies

(i) $\operatorname{supp}(v) = \{x_1,x_2,\ldots,x_n\} \setminus \{x_1,x_j,x_{j+1}\}$, where $2 \leq j \leq n-1$, or
(ii) $\operatorname{supp}(v) = \{x_1,x_2,\ldots,x_n\} \setminus \{x_1,x_j\}$, where $3 \leq j \leq n-1$.

Note that, if $x_2(u_1 \cdots u_{t-1}) = x_\ell(u'_1 \cdots u'_{t-1})$ for some $u'_j \in G(J_{n,n-2})$, then it follows that ℓ is even, since u_j's and u'_j's are products of $\frac{n-2}{2}$ variables with odd indices and $\frac{n-2}{2}$ variables with even indices. Therefore, if $x_2 w \in J^t_{n,n-2}$ then $x_\ell v \in J_{n,n-2}$ for some even index ℓ.

In the case (i), it follows that $\operatorname{supp}(x_\ell v) = \{x_1, x_2, \ldots, x_n\} \setminus \{x_1, x_j\}$ with $j \neq 1$ and j odd. But this contradicts the fact that $x_\ell v \in J_{n,n-2}$.

In the case (ii), if j is odd, then $\operatorname{supp}(x_\ell v) = \{x_1, x_2, \ldots, x_n\} \setminus \{x_1, x_j\}$ and again we get a contradiction. It follows that

$$\operatorname{supp}(v) = \{x_1, x_2, \ldots, x_n\} \setminus \{x_1, x_{2k}\}, \text{ where } 2 \leq k \leq \frac{n-2}{2}. \tag{3.2}$$

We claim that $w \in J^t_{n,n-2}$. Indeed, since $x_2 w \in J^t_{n,n-2}$, from (3.2) it follows that there exist $u'_j \in G(J_{n,n-2})$ with $1 \leq j \leq t-1$ such that $x_2(u_1 \cdots u_{t-1}) = x_{2k}(u'_1 \cdots u'_{t-1})$. Therefore,

$$w = (u_1 \cdots u_{t-1})v = x_2(u_1 \cdots u_{t-1})\frac{v}{x_2} = x_{2k}(u'_1 \cdots u'_{t-1})\frac{v}{x_2}.$$

Denoting $v' = v \cdot \frac{x_{2k}}{x_2}$, it is easy to see that $\operatorname{supp}(v') = \{x_3, \ldots, x_n\}$. Hence, $v' \in J_{n,n-2}$ and thus $w \in J^t_{n,n-2}$, a contradiction. □

Theorem 3.2. *We have that:*

1. $\operatorname{sdepth}(S/J^t_{n,n-1}) = \operatorname{depth}(S/J^t_{n,n-1}) = 0$, *for all* $n \geq 2$ *and* $t \geq n-1$.
2. *If* $n \geq 3$ *is odd then* $\operatorname{sdepth}(S/J^t_{n,n-2}) = \operatorname{depth}(S/J^t_{n,n-2}) = 0$, *for all* $t \geq \frac{n-1}{2}$.
3. *If* $n \geq 3$ *is odd then* $\operatorname{sdepth}(S/J^t_{n,n-2}), \operatorname{depth}(S/J^t_{n,n-2}) > 0$, *for all* $t < \frac{n-1}{2}$.
4. *If* $n \geq 3$ *is even then* $\operatorname{sdepth}(S/J^t_{n,n-2}), \operatorname{depth}(S/J^t_{n,n-2}) > 0$, *for all* $t \geq 1$.
5. *If* $n \geq 3$ *is even then* $\frac{n}{2} \geq \operatorname{sdepth}(S/J^t_{n,n-2}) \geq \operatorname{depth}(S/J^t_{n,n-2}) = 1$, *for all* $t \geq n-1$.

Proof. (1), (2), (3) and (4) follows immediately from Lemma 3.1 and Lemma 2.7.

(5) follows from (4) and [2, Corollary 2.8]. □

Remark 3.3. Note that (1) from Theorem 3.2 was proved in [2, Theorem 3.1] and (2) from Theorem 3.2 was proved in [2, Corollary 2.8(1)]. The results (3), (4) and (5) from Theorem 3.2 are new.

Lemma 3.4. *Let* $n > m \geq 2$ *and* $t \geq 1$ *be some integers. Then*

$$(J^t_{n,m} : (x_{n-m+1}x_{n-m+2} \cdots x_{n-1})^t) = \begin{cases} (x_{n-m}, x_n)^t, & n \leq 2m \\ (I_{n-m-1,m}S + (x_{n-m}, x_n))^t, & n \geq 2m+1 \end{cases}.$$

Proof. It is easy to check that

$$(J_{n,m} : (x_{n-m+1} \cdots x_{n-1})) = \begin{cases} (x_{n-m}, x_n), & n \leq 2m \\ I_{n-m-1,m}S + (x_{n-m}, x_n), & n \geq 2m+1 \end{cases}. \tag{3.3}$$

Since $(J_{n,m} : (x_{n-m+1} \cdots x_{n-1}))^t \subset (J^t_{n,m} : (x_{n-m+1} \cdots x_{n-1})^t)$, in order to complete the proof, by (3.3), it suffices to show that for any monomial v with

$(x_{n-m+1}\cdots x_{n-1})^t v \in J_{n,m}^t$, there exists some monomials

$$v_1,\ldots,v_t \in \begin{cases} (x_{n-m},x_n), & n\leq 2m \\ I_{n-m-1,m}S+(x_{n-m},x_n), & n\geq 2m+1 \end{cases},$$

such that $v_1\cdots v_t \mid v$.

Indeed, let v be a monomial as above. Let $a=\deg_{x_{n-m}}(v)$ and $b=\deg_{x_n}(v)$. If $a\geq t$ then we can choose $v_1=\cdots=v_t=x_{n-m}$ and we are done. Also, if $a<t$ and $a+b\geq t$, we can choose $v_1=\cdots=v_a=x_{n-m}$, $v_{a+1}=\cdots=v_t=x_n$ and we are also done.

Now, assume $a+b<t$. Let $v_1=\cdots=v_a=x_{n-m}$ and $v_{a+1}=\cdots=v_{a+b}=x_n$. Since $(x_{n-m+1}\cdots x_{n-1})^t v\in J_{n,m}^t$, there are $g_1,\ldots,g_t\in G(J_{n,m})$ such that $g_1\cdots g_t \mid (x_{n-m+1}\cdots x_{n-1})^t v$. It is clear that at most $a+b$ of the monomials $g_1,\ldots,g_t$ are divisible by x_{n-m} or x_n. Hence, there are $t-a-b$ such monomials, let's say $g_{a+b+1},\ldots,g_t$ which are divisible neither by x_{n-m}, neither by x_n. In particular, it follows that

$$g_{a+b+1},\ldots,g_t\in K[x_1,\ldots,x_{n-m-1},x_{n-m+1},\ldots,x_{n-1}],$$

which leads to a contradiction if $n\leq 2m$. On the other hand, if $n\geq 2m+1$, then we get $g_{a+b+1},\ldots,g_t\in G(I_{n-m-1,m})$. We let $v_{a+b+1}=g_{a+b+1},\ldots,v_t=g_t$ and we are done. □

Lemma 3.5. *Let $n>m\geq 2$ and $t\geq 1$ be some integers with $n\geq 2m+1$. Let $V:=(J_{n,m}^t:(x_{n-m+1}x_{n-m+2}\cdots x_{n-1})^t)$. Then:*

$$\operatorname{sdepth}(S/V)\geq \operatorname{depth}(S/V)=\begin{cases} \varphi(n,m,t), & t\leq n-2m \\ 2(m-1), & t\geq n-2m+1 \end{cases}.$$

Proof. Let $S':=K[x_1,\ldots,x_{n-m-1}]$ and $S'':=K[x_{n-m},\ldots,x_n]$. We consider the ideals $I=I_{n-m-1,m}\subset S'$ and $L=(x_{n-m},x_n)\subset S''$. According to Lemma 3.4, we have that $V=(IS+LS)^t$. It is clear that

$$\dim(S''/L)=m+1-2=m-1. \tag{3.4}$$

Also, from Theorem 2.9, we have that

$$\operatorname{sdepth}(S'/I^t)\geq \operatorname{depth}(S'/I^t)=\varphi(n-m-1,m,t). \tag{3.5}$$

Since $V=(IS+LS)^t$, from Theorem 2.8, (3.4), (3.5) and the fact that $t\mapsto\varphi(n-m-1,m,t)$ is nonincreasing, it follows that

$$\operatorname{sdepth}(S/V)\geq \operatorname{depth}(S/V)=\varphi(n-m-1,m,t)+m-1.$$

The required formula follows by straightforward computations. □

The following result is an improvement of Theorem 2.11 for $t\leq n-2m$.

Theorem 3.6. *Let $n>m\geq 2$ and $t\geq 1$ be some integers. If $n\geq 2m+1$ then*

$$\operatorname{depth}(S/J_{n,m}^t)\leq \begin{cases} \varphi(n,m,t), & t\leq n-2m \\ \varphi(n-1,m,t)+1, & n-2m+1\leq t\leq n-m \\ m, & t\geq n-m+1 \end{cases}.$$

Proof. From Lemma 3.4, Lemma 3.5 and Lemma 2.4(2) it follows that

$$\operatorname{depth}(S/J_{n,m}^t) \leq \begin{cases} \varphi(n,m,t), & t \leq n-2m \\ 2(m-1), & t \geq n-2m+1 \end{cases}. \tag{3.6}$$

On the other hand, according to Theorem 2.11 we have that

$$\operatorname{depth}(S/J_{n,m}^t) \leq \varphi(n-1,m,t)+1. \tag{3.7}$$

Also, it is easy to check that

$$\varphi(n-1,m,t)+1 \leq 2m-2 \text{ for all } t \geq n-2m+1 \text{ and}$$

$$\varphi(n-1,m,t) = m-1 \text{ for all } t \geq n-m+1. \tag{3.8}$$

The required conclusion follows from (3.6), (3.7) and (3.8). □

Remark 3.7. If $m < n \leq 2m$ and $t \geq 1$ then Lemma 3.4 and Lemma 2.4 imply

$$\operatorname{depth}(S/J_{n,m}^t) \leq \operatorname{depth}(S/(x_{n-m},x_n)^t) = n-2 \text{ and}$$
$$\operatorname{sdepth}(S/J_{n,m}^t) \leq \operatorname{sdepth}(S/(x_{n-m},x_n)^t) = n-2.$$

However, the above inequalities are trivial, since the ideal $J_{n,m}^t$ is not principal.

4. Remarks in the general case

Let $n > m \geq 2$ be two integers. In [2] we studied the functions $t \mapsto \operatorname{depth}(S/J_{n,m}^t)$ and $t \mapsto \operatorname{sdepth}(S/J_{n,m}^t)$ for $t \geq t_0$, where t_0 was defined in Section 2. However, for $2 \leq t \leq t_0 - 1$, this problem is much harder. In the following, we present a possible way of tackling it and we point out the difficulties which appear.

For convenience, we introduce the following notations: We let $J = J_{n,m} \subset S$, $S' = K[x_1,\dots,x_{n-1}]$, $J' = (J : x_n) \cap S'$ and $I = I_{n-1,m} \subset S'$.

Lemma 4.1. *With the above notations, we have that:*

1. $(J^t, x_n^k) = (I^{t+1-k}J^{k-1}, x_n^k)$, *for all* $1 \leq k \leq t$.
2. $((J^t : x_n^{k-1}), x_n) = (I^{t+1-k}J'^{k-1}, x_n)$, *for all* $1 \leq k \leq t$.
3. $(J^t, x_n) = (I^t, x_n)$ *and* $(J^t : x_n^t) = J'^t S$.
4. $\operatorname{depth}(S/J^t) \leq \operatorname{depth}(S'/J'^t) + 1$ *and* $\operatorname{sdepth}(S/J^t) \leq \operatorname{sdepth}(S'/J'^t) + 1$

Proof. (1) Since $IS \subset J$, the inclusion "$\supseteq$" is clear. In order to prove the other inclusion, it is enough to note that if $u \in G(J^t)$ such that $x_n^k \nmid u$, then $u \in G(I^{t+1-k}J^{k-1})$.

(2) Since $((J^t : x_n^{k-1}), x_n) = ((J^t, x_n^k) : x_n^{k-1})$, the conclusion follows from (1).

(3) $(J^t, x_n) = (I^t, x_n)$ follows from (1), for $k = 1$. The second equality is trivial.

(4) From Lemma 2.4 and Lemma 2.6 and (3) it follows that

$$\operatorname{depth}(S/J^t) \leq \operatorname{depth}(S/(J^t : x_n)) \leq \cdots \leq \operatorname{depth}(S/(J^t : x_n^t)) = \operatorname{depth}(S'/J'^t) + 1 \text{ and}$$
$$\operatorname{sdepth}(S/J^t) \leq \operatorname{sdepth}(S/(J^t : x_n)) \leq \cdots \leq \operatorname{sdepth}(S/(J^t : x_n^t)) = \operatorname{sdepth}(S'/J'^t) + 1.$$

Hence, we get the required conclusion. □

With the notations from Lemma 4.1, let

$$d_k := \operatorname{depth}(S'/I^{t+1-k}J'^{k-1}) \text{ and } s_k = \operatorname{sdepth}(S'/I^{t+1-k}J'^{k-1}) \text{ for } 1 \le k \le t.$$

Note that, according to Theorem 2.9, we have that $s_1 \ge d_1 = \varphi(n-1, m, t)$.

Proposition 4.2. *We have that:*

1. $d_k \ge \operatorname{depth}(S/(J^t : x_n^{k-1})) - 1$ *for all* $1 \le k \le t$.
2. *If* $\operatorname{depth}(S/(J^t : x_n^k)) > \operatorname{depth}(S/(J^t : x_n^{k-1}))$ *then* $d_k = \operatorname{depth}(S/(J^t : x_n^{k-1}))$.
3. *If* $\operatorname{sdepth}(S/(J^t : x_n^k)) > \operatorname{sdepth}(S/(J^t : x_n^{k-1}))$ *then* $s_k \le \operatorname{sdepth}(S/(J^t : x_n^{k-1}))$.

Proof. We fix some k with $1 \le k \le t$. We have the short exact sequence

$$0 \to S/(J^t : x_n^k) \to S/(J^t : x_n^{k-1}) \to S/((J^t : x_n^{k-1}), x_n) \cong S'/I^{t+1-k}J'^{k-1} \to 0, \tag{4.1}$$

where the isomorphism is given by Lemma 4.1(2). Thus, from (4.1), Lemma 2.1 (Depth lemma), Lemma 2.4 and Lemma 2.3 it follows that

$$\begin{aligned}
&\operatorname{depth}(S/(J^t : x_n^k)) \ge \operatorname{depth}(S/(J^t : x_n^{k-1})),\\
&\operatorname{depth}(S/(J^t : x_n^{k-1})) \ge \min\{\operatorname{depth}(S/(J^t : x_n^k)), d_k\},\\
&d_k \ge \min\{\operatorname{depth}(S/(J^t : x_n^k)) - 1, \operatorname{depth}(S/(J^t : x_n^{k-1}))\} \text{ and}\\
&\operatorname{sdepth}(S/(J^t : x_n^{k-1})) \ge \min\{\operatorname{sdepth}(S/(J^t : x_n^k)), s_k\}.
\end{aligned}$$

Now, we get the required conclusions (1-3). □

Proposition 4.3. *With the above notations, we have that:*

1. $\operatorname{depth}(S/(J^t, x_n^t)) \ge \min\{\varphi(n-1, m, t), d_2, \ldots, d_t\}$.
2. $\operatorname{sdepth}(S/(J^t, x_n^t)) \ge \min\{\varphi(n-1, m, t), s_2, \ldots, s_t\}$.
3. $\operatorname{depth}(S/J^t) \le \operatorname{depth}(S/(J^t, x_n^t)) + 1$.
4. *If* $\operatorname{depth}(S/(J^t : x_n^t)) > \operatorname{depth}(S/J^t)$ *then* $\operatorname{depth}(S/J^t) \ge \operatorname{depth}(S/(J^t, x_n^t))$ *and similarly for the Stanley depth.*

Proof. (1) We consider the following short exact sequences

$$0 \to S/((J^t : x_n^{k-1}), x_n) \to S/(J^t, x_n^k) \to S/(J^t, x_n^{k-1}) \to 0 \text{ for } 2 \le k \le t. \tag{4.2}$$

Since, by Lemma 4.1, we have that $S'/I^{t+1-k}J'^{k-1} \cong S/((J^t : x_n^{k-1}), x_n)$, the conclusion follows from (4.2), Lemma 2.1 (Depth lemma) and Theorem 2.9.

(2) The proof is similar to the proof of (1), using Lemma 2.3 instead of Lemma 2.1.

(3) follows from Lemma 2.1, Lemma 2.4 and the short exact sequence

$$0 \to S/(J^t : x_n^t) \to S/J^t \to S/(J^t, x_n^t) \to 0. \tag{4.3}$$

(4) follows from (4.3), Lemma 2.1 and Lemma 2.3. □

The following examples illustrate the computational difficulties which appear, when we apply the previous results in order to compute $\operatorname{depth}(S/J^t)$ and $\operatorname{sdepth}(S/J^t)$.

Example 4.4. Let $J = J_{6,3} \subset S = K[x_1, \ldots, x_6]$. Let $I = I_{5,3} \subset S' = K[x_1, \ldots, x_5]$. We have

$$J' = (J : x_6) \cap S' = (x_1x_2, x_2x_3x_4, x_4x_5, x_5x_1).$$

As in the proof of Proposition 4.2, we have the short exact sequences:

$$0 \to S/(J^2 : x_6) \to S/J^2 \to S/(J^2, x_6) \cong S'/I^2 \to 0 \text{ and} \tag{4.4}$$

$$0 \to S/(J^2 : x_6^2) \to S/(J^2 : x_6) \to S/((J^2 : x_6), x_6) \cong S'/IJ' \to 0. \tag{4.5}$$

From Theorem 2.9 it follows that:

$$\operatorname{sdepth}(S'/I^2) \geq \operatorname{depth}(S'/I^2) = \varphi(5, 3, 2) = 2.$$

Note that $IJ' = x_3L$, where $L = (x_1x_2, x_2x_4, x_4x_5)(x_1x_2, x_2x_3x_4, x_4x_5, x_5x_1) \subset S'$, and $(J^2 : x_6^2) = J'^2S = (x_1x_2, x_2x_3x_4, x_4x_5, x_5x_1)^2S$. From Lemma 2.5, it follows that

$$\operatorname{depth}(S'/IJ') = \operatorname{depth}(S'/L) \text{ and } \operatorname{sdepth}(S'/IJ') = \operatorname{sdepth}(S'/L). \tag{4.6}$$

We have $(L : x_2x_3x_4) = (x_1, x_4) \cap (x_2, x_5)$. By straightforward computations we get:

$$\operatorname{sdepth}(S'/(L : x_2x_3x_4)) = \operatorname{depth}(S'/(L : x_2x_3x_4)) = 2.$$

Also, we have that

$$W := (L, x_2x_3x_4) = (x_2x_3x_4, x_1x_2x_4x_5, x_1^2x_2x_5, x_1^2x_2^2, x_2x_4^2x_5, x_1x_2^2x_4, x_4^2x_5^2, x_1x_4x_5^2).$$

Note that $(W : x_2x_4x_5) = (x_3, x_4, x_1)$. Therefore we get

$$\operatorname{sdepth}(S'/(W : x_2x_4x_5)) = \operatorname{depth}(S'/(W : x_2x_4x_5)) = 2.$$

Also, $(W, x_2x_4x_5) = (x_2x_4x_5, x_2x_3x_4, x_1x_4x_5^2, x_4^2x_5^2, x_1x_2^2x_4, x_1^2x_2^2, x_1^2x_2x_5)$. By continuing the computations, we can deduce that

$$\operatorname{sdepth}(S'/(W, x_2x_4x_5)) = \operatorname{depth}(S'/(W, x_2x_4x_5)) = 2.$$

From the short exact sequence

$$0 \to S'/(W : x_2x_4x_5) \to S'/W \to S'/(W, x_2x_4x_5) \to 0,$$

using Lemma 2.1, Lemma 2.3 and Lemma 2.4, we deduce that

$$\operatorname{depth}(S'/W) = \operatorname{sdepth}(S'/W) = 2.$$

From the short exact sequence

$$0 \to S'/(L : x_2x_3x_4) \to S'/L \to S'/W \to 0,$$

and (4.6), using Lemma 2.1, Lemma 2.3 and Lemma 2.4, we deduce that:

$$d_2 := \operatorname{depth}(S'/IJ') = \operatorname{depth}(S'/L) = 2 \text{ and}$$
$$s_2 := \operatorname{sdepth}(S'/IJ') = \operatorname{sdepth}(S'/L) = 2.$$

Using similar computations, as above, one can deduce that

$$d_1 = \operatorname{depth}(S'/J'^2) \geq 2 \text{ and } s_1 := \operatorname{sdepth}(S'/J'^2) \geq 2.$$

As in the proof of Proposition 4.2, we can deduce from the short exact sequences (4.4) that:

$$\operatorname{depth}(S/J^2) \geq 2 \text{ and } \operatorname{sdepth}(S/J^2) \geq 2.$$

We mention that, according to Cocoa [7], $\operatorname{sdepth}(S/J^2) = \operatorname{depth}(S/J^2) = 3$.

Example 4.5. Let $J = J_{6,4} \subset S = K[x_1, \dots, x_6]$. Let $I = I_{5,4} \subset S' = K[x_1, \dots, x_5]$. We have $J' = (J : x_6) \cap S' = (x_1x_2x_3, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2)$, and the short exact sequences

$$0 \to S/(J^2 : x_6) \to S/J^2 \to S/(J^2, x_6) \cong S'/I^2 \to 0 \text{ and} \tag{4.7}$$
$$0 \to S/(J^2 : x_6^2) \to S/(J^2 : x_6) \to S/((J^2 : x_6), x_6) \cong S'/IJ' \to 0. \tag{4.8}$$

From Theorem 2.9 it follows that

$$\operatorname{sdepth}(S'/I^2) \geq \operatorname{depth}(S'/I^2) = \varphi(5, 4, 2) = 3.$$

Note that $IJ' = x_2x_3x_4L$, where $L = (x_1, x_5)(x_1x_2x_3, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2) \subset S'$. From Lemma 2.5, it follows that

$$\operatorname{depth}(S'/IJ') = \operatorname{depth}(S'/L) \text{ and } \operatorname{sdepth}(S'/IJ') = \operatorname{sdepth}(S'/L). \tag{4.9}$$

We have $(L : x_3) = (x_1, x_5)(x_1x_2, x_4x_5)$ and $(L, x_3) = x_1x_5(x_1, x_5)(x_2, x_4) + (x_3)$. By straightforward computation we have

$$\operatorname{sdepth}\left(\frac{K[x_1, x_2, x_4, x_5]}{(x_1, x_5)(x_2, x_4)}\right) = \operatorname{depth}\left(\frac{K[x_1, x_2, x_4, x_5]}{(x_1, x_5)(x_2, x_4)}\right) = 1$$
$$\operatorname{sdepth}\left(\frac{K[x_1, x_2, x_4, x_5]}{(x_1, x_5)(x_1x_2, x_4x_5)}\right) = \operatorname{depth}\left(\frac{K[x_1, x_2, x_4, x_5]}{(x_1, x_5)(x_1x_2, x_4x_5)}\right) = 2.$$

Therefore, using Lemma 2.5, we obtain $\operatorname{sdepth}(S/(L, x_3)) = \operatorname{depth}(S/(L, x_3)) = 1$. Also $\operatorname{sdepth}(S/(L : x_3)) = \operatorname{depth}(S/(L : x_3)) = 3$. From the short exact sequence

$$0 \to S/(L : x_3) \to S/L \to S/(L, x_3) \to 0$$

we deduce that $\operatorname{depth}(S/L), \operatorname{sdepth}(S/L) \geq 1$.

Hence, from (4.9) it follows that $\operatorname{depth}(S'/IJ'), \operatorname{sdepth}(S'/IJ') \geq 1$. Using similar computations, one can deduce that $\operatorname{depth}(S'/J'^2), \operatorname{sdepth}(S'/J'^2) \geq 2$.

Therefore, from (4.7), we deduce that $\operatorname{depth}(S/J^2), \operatorname{sdepth}(S/J^2) \geq 1$. Note that

$$(J^2 : x_1x_2x_3x_4x_5x_6) = (x_1, x_3, x_5) \cap (x_2, x_4, x_6).$$

Therefore, according to Lemma 2.2, it follows that

$$\operatorname{depth}(S/(J^2 : x_1x_2x_3x_4x_5x_6)) = 1.$$

Consequently, we get $\operatorname{depth}(S/J^2) = 1$.

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On nilpotent matrices that are unit-regular

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Abstract. In this paper, we characterize regular nilpotent 2×2 matrices over Bézout domains and prove that they are unit-regular. We also demonstrate that nilpotent $n \times n$ matrices over division rings are unit-regular.

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1. Introduction

The genuine ring-theoretic definition of exchange elements and exchange rings (also referred to as "suitable" in [4]) is as follows: An element $a \in R$ is called *left exchange* if there exists an idempotent $e \in R$ such that $e \in Ra$ and $1 - e \in R(1 - a)$. This definition is left-right symmetric, and a ring is called *exchange* if every element satisfies this property.

Nicholson ([4], Theorem 2.1) proved that a ring R is exchange if and only if the full $n \times n$ matrix ring $\mathbb{M}_n(R)$ is exchange.

A remarkable result first proved by P. Ara (1996), with a simplified proof by D. Khurana (2016), states that the regular nilpotents in any exchange ring are unit-regular.

This result motivated us to search for other classes of rings where this property holds even if the ring itself is not exchange.

In this note, we show that this property holds in the matrix ring $\mathbb{M}_2(R)$ for any Bézout domain R. Specifically, we determine all the regular nilpotents in such matrix rings. Since Bézout domains need not be exchange rings (for example, $\mathbb{Z}$ is not exchange), our result identifies a new class of rings in which regular nilpotents are unit-regular.

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Recall that an element a in a ring R is (von Neumann) *regular* if there exists $b \in R$ such that $a = aba$. Such an element b is called an *inner inverse* of a. If b is a unit, then a is said to be *unit-regular*, and b is called a *unit inner inverse* of a.

Throughout this note, we assume that all rings are associative, nonzero, and possess a multiplicative identity. We use the well-known abbreviations UFD (unique factorization domain) and GCD (greatest common divisor), assuming that GCD exists.

2. Preliminaries

Since we intend to work with 2×2 matrices over commutative domains, we begin by recalling some well-known results.

Lemma 2.1. *Over any ring, both nilpotent and regular properties are invariant under conjugation.*

Since this will be used in the proof of Theorem 3.4, we supply a proof for the next special case.

Lemma 2.2. *If a is conjugate to b and b is unit-regular then a is also unit-regular.*

Proof. Assume $b = u^{-1}au$ and $b = bvb$ for $u, v \in U(R)$. Then $a(uvu^{-1})a = a$. □

Lemma 2.3. *Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any matrix over a commutative ring R.*

(i) If $\det(T) = Tr(T) = 0$ then $T^2 = 0_2$.
(ii) If R is a domain, the converse holds.

Proof. For a 2×2 matrix over a commutative ring, if $\det(T) = Tr(T) = 0$, by Cayley-Hamilton theorem, it follows that $T^2 = 0_2$.

Conversely, also by Cayley-Hamilton theorem, if $T^2 = 0_2$ then $Tr(T)T = \det(T)I_2$ and so, denoting $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a^2 = -bc = d^2$ and $b(a+d) = c(a+d) = 0$.

In the absence of zero divisors, we have $Tr(T) = a + d = 0$ or $b = c = 0$.

In the first case, $\det(T) = 0$ and in the second $T = 0_2$. □

The nonzero nilpotent $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ (with $a^2 + bc = 0$), is also regular if there is $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ with $T = TXT$. This equality is equivalent to the linear system

$$(S)\begin{cases} a^2x + acy + abz + bcw &= a \\ abx - a^2y + b^2z - abw &= b \\ acx + c^2y - a^2z - acw &= c \\ bcx - acy - abz + a^2w &= -a \end{cases}.$$

As T is supposed to be nonzero, there are two cases to consider: $a = 0$, then T can be either $T = bE_{12}$ or $T = cE_{21}$ (with b or c nonzero), or else $a, b, c \neq 0$. In the first case, since cE_{21} is similar to cE_{12}, the case is settled as follows.

Lemma 2.4. *Over any commutative domain, a nonzero nilpotent* $rE_{12} = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$ *is regular if and only if* $r \in U(R)$. *In this case,* rE_{12} *is unit-regular.*

Proof. One way, $(rE_{12})\begin{bmatrix} x & y \\ z & w \end{bmatrix}(rE_{12}) = rE_{12}$ is equivalent to $r^2z = r$. Hence r^2 and r are associates, whence $r \in U(R)$. Conversely, if r is a unit, $r^{-1}(E_{12} + E_{21})$ is a unit inner inverse for rE_{12}. □

In the remaining case, we focus on matrices with only nonzero entries. Recall that elements a, b, c in a ring R are said to form a *unimodular row* if $aR+bR+cR = R$.

Theorem 2.5. *Over any commutative domain, the nilpotent matrix* $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ *with nonzero entries is regular if and only if* a, b, c *form a unimodular row.*

Proof. In the linear system (S) above, proceed as follows. In the first three equations, replace a^2 with $-bc$. In the last equation, replace bc with $-a^2$. Then, in each of the four equations, cancel a, b, c, and $-a$ respectively. As a result, all equations reduce to $a(x - w) + cy + bz = 1$. Hence the system is solvable if and only if a, b, c form a unimodular row. □

Over Bézout domains, the precise form of these regular nilpotents is given in Theorem 3.3.

Remarks. 1) In a UFD (such domains are GCD), if $a^2+bc = 0$ then a, b, c cannot be pairwise coprime, unless $a^2 = 1$.

Indeed, since a^2 divides $-bc$, *any prime* dividing a (if any !) divides bc. But then $\gcd(a, b) = \gcd(a, c) = 1$ fails and so a, b, c cannot be pairwise coprime.

As mentioned, the $a^2 = 1$ case is excepted. For example,

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} E_{11} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

2) There are many triples (a, b, c) that satisfy $\gcd(a, b, c) = 1$ and $a^2 + bc = 0$.

A trivial example is $(0, 1, 0)$, but easy examples are found taking $b = 1$ (i.e., $(a, 1, -a^2)$) or $c = 1$ (i.e., $(a, -a^2, 1)$).

3) The determinant Δ of the system matrix is

$$\Delta = \begin{vmatrix} a^2 & ac & ab & bc \\ ab & -a^2 & b^2 & -ab \\ ac & c^2 & -a^2 & -ac \\ bc & -ac & -ab & a^2 \end{vmatrix} = (a^2 + bc)^4 = 0, \tag{2.1}$$

and the other four minors of the augmented matrix also vanish. However, applying the Kronecker–Capelli theorem (also known as the Rouché–Capelli theorem) to analyze the solvability of a system of linear equations can be quite challenging.

Example. The nonzero nilpotent $A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$ is regular only if $4 \mid 2$.

Indeed, for $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, $A = AXA$ reduces to $4[a(x-w)+cy+bz] = 2$. Over any commutative domain r, this amounts to $2 \in U(R)$.

Since division rings are (trivially) clean, and hence exchange, it follows from Ara's result that regular nilpotent matrices over division rings are unit-regular. In closing this section, we show that this result still holds even when the regularity hypothesis is dropped.

Recall (a Jordan canonical form for matrices) that every nilpotent matrix over a field is similar to a block diagonal matrix $\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{bmatrix}$, where each block B_i is a shift matrix (possibly of different sizes). A *shift* matrix has 1's along the superdiagonal and 0's everywhere else, i.e. $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, as $n \times n$ matrix. When $n = 1$, $S = 0$.

Proposition 2.6. *Over any ring, the shift matrices are unit-regular.*

Proof. The $n \times n$ shift matrix is $S = E_{12} + E_{23} + ... + E_{n-1,n}$. Consider

$$U = E_{21} + E_{32} + ... + E_{n,n-1} + E_{1n},$$

that is, the nonzero entries are on the subdiagonal and in the NE corner and all are equal to 1. Clearly U is invertible and $SUS = S$. □

Theorem 2.7. *Every block diagonal matrix where each block is a shift matrix (possibly of different sizes) is unit-regular.*

Proof. Such block diagonal matrices have nonzero entries (equal to 1) only on the superdiagonal, but some entries on the superdiagonal may be zero (if there are at least two blocks). We can use the same unit inner inverse as in the proof of the previous proposition. To be more specific, assume $S' = E_{12} + 0_{23} + E_{34} + .. + E_{n-1,n}$ (i.e., the first block has size two). Then $S'U = E_{11} + 0_{22} + E_{33} + ... + E_{n-1,n-1}$ and so $S'US' = S'$. □

Using Theorem 3.3 in [2], it follows that *over any division ring R, the nilpotent matrices of $\mathbb{M}_n(R)$ are unit-regular.*

3. Over Bézout domains

Proposition 3.1. *Over any Bézout domain R, every nonzero nilpotent 2×2 matrix is similar to rE_{12}, for some $r \in R$.*

Proof. Take $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 + bc = 0$. We construct an invertible matrix $U = (u_{ij})$ such that $TU = U(rE_{12})$ with a suitable $r \in R$.

Let $d = \gcd(a, b)$ and denote $a = da_1$, $b = db_1$ with $\gcd(a_1, b_1) = 1$. Then $d^2a_1^2 = -db_1c$ and since $\gcd(a_1, b_1) = 1$ implies $\gcd(a_1^2, b_1) = 1$, it follows b_1 divides d. Set $d = b_1u_2$ and so $T = \begin{bmatrix} a_1b_1u_2 & b_1^2u_2 \\ -a_1^2u_2 & -a_1b_1u_2 \end{bmatrix} = u_2 \begin{bmatrix} a_1b_1 & b_1^2 \\ -a_1^2 & -a_1b_1 \end{bmatrix} = u_2T'$.

Since $\gcd(a_1, b_1) = 1$ there exist $s, t \in R$ such that $sa_1 + tb_1 = 1$. If we take $U = \begin{bmatrix} b_1 & s \\ -a_1 & t \end{bmatrix}$, which is invertible (indeed, $U^{-1} = \begin{bmatrix} t & -s \\ a_1 & b_1 \end{bmatrix}$), one can check $T'U = \begin{bmatrix} 0 & b_1 \\ 0 & -a_1 \end{bmatrix} = UE_{12}$, so $r = u_2$. □

Corollary 3.2. *Over Bézout domains, the nonzero nilpotents* 2×2 *matrices that are regular are those similar to unit ring multiples of* E_{12}.

As this result is not explicit enough, we will elaborate further.

For the first main result, we revisit the proof of the Proposition 3.1 and use Lemma 2.4.

Theorem 3.3. *Up to association, over Bézout domains, the nonzero nilpotent matrices that are regular are of form* $\begin{bmatrix} b_1a_1 & b_1^2 \\ -a_1^2 & -b_1a_1 \end{bmatrix}$ *with* $\gcd(a_1, b_1) = 1$.

Proof. Following the proof of the Proposition 3.1, for $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ we take $d = \gcd(a, b)$ and write $a = da_1$, $b = db_1$. From $a^2 = -bc$ (i.e., $d^2a_1^2 = -db_1c$) using $\gcd(a_1, b_1) = 1$ (and so $\gcd(a_1^2, b_1) = 1$) we get $b_1 \mid d$. If $d = b_1u_2$ then for $r = u_2$ we have T similar to rE_{12}.

According to Lemma 2.4, for T to be regular, u_2 should be a unit, that is, b_1 and d are associates. Then up to association, $b = b_1^2$ and $a = b_1a_1$, that is, b is a square $(= b_1^2)$ and $b_1 \mid a$.

These are precisely the nilpotent matrices $\begin{bmatrix} b_1a_1 & b_1^2 \\ -a_1^2 & -b_1a_1 \end{bmatrix}$. If $sa_1 + tb_1 = 1$ then for $U = \begin{bmatrix} b_1 & s \\ -a_1 & t \end{bmatrix}$ we have $TU = UE_{12}$ (here $r = u_2 = 1$). □

Interestingly (see Theorem 4.4, [2]), over GCD domains, these coincide with the nonzero nilpotents that are *fine* - namely, those that can be expressed as the sum of a unit and a nilpotent. It would be worthwhile to explore whether any relationship exists between regular nilpotents and fine nilpotents.

We can now state and prove the second main result of this section.

Theorem 3.4. *Over any Bézout domain, the regular nilpotents* 2×2 *matrices are unit-regular.*

Proof. This follows at once from Proposition 3.1, Lemma 2.2 and the unit inner inverse $E_{12}+E_{21}$ for E_{12}. Indeed, $E_{12}=E_{12}(E_{12}+E_{21})E_{12}$.

An *explicit* unit inner inverse for $T=\begin{bmatrix} b_1a_1 & b_1^2 \\ -a_1^2 & -b_1a_1 \end{bmatrix}$ (see Theorem 3.3) with $sa_1+tb_1=1$ is $U(E_{12}+E_{21})U^{-1}=\begin{bmatrix} st+a_1b_1 & -s^2+b_1^2 \\ t^2-a_1^2 & -st-a_1b_1 \end{bmatrix}$ (see the proof of Lemma 2.2) where $U=\begin{bmatrix} b_1 & s \\ -a_1 & t \end{bmatrix}$ and $U^{-1}=\begin{bmatrix} t & -s \\ a_1 & b_1 \end{bmatrix}$. □

Example. For $T=\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$, an unit inner inverse is $\begin{bmatrix} 5 & 8 \\ -3 & -5 \end{bmatrix}$.

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Local fractal functions on Orlicz-Sobolev spaces

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Abstract. In these notes we consider a class of iterated function systems whose attractors are the graphs of local fractal functions belonging to Orlicz and to Orlicz-Sobolev spaces. We prove that these maps are in correspondence with the fixed points of the Read-Bajraktarević operator. Our method extends a number of outcomes on the existence of local fractal functions of the Lebesgue and Sobolev classes, to more general function spaces where the role of the norm is now played by a Young function. The existence of local fractal functions of the Orlicz and of the Orlicz-Sobolev classes is demonstrated through an intermediate result. The realization of a contractive iterated function system in the (previously untreated) multidimensional case is obtained via a stronger version of the finite increments theorem. Our results somewhat show that it would be natural to extend the Read-Bajraktarević functional to other function spaces on subdomains of differentiable and real analytic manifolds. Other questions, such as the existence of fixed points in higher-orders etc., remain open as well. Our generalizations may be useful in applications.

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1. Introduction

An iterated function system (IFS) consists of a finite set S of continuous functions defined on a complete metric space, with the images in the same space. Hutchinson and Mandelbrot introduced IFSs on the real plane in the literature in the early 1980s and their applications were widely popularized by Barnsley in the 1990s. A result due to Hutchinson [11] asserts that there exists a unique nonempty compact subset of the plane which is equal to the union of its images under the corresponding elements in S.

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Such subset is called an attractor of the IFS. Conversely, Barnsley demonstrated that it is possible to construct an IFS with prescribed attractor. The method is known as the *random iteration algorithm,* dubbed the chaos game by Barnsley himself [5]. During the last decades, the theory has been extended to other spaces and contracting maps [20].

Iterated function systems are tightly connected with the Read-Bajraktarević equation, introduced by M. Bajraktarević in the 1950s [4]. In its elementary form the Read-Bajraktarević equation corresponds to $u(x) = \nu(x, u(b(x)))$, where $b : I \to I$ and $\nu : I \times \mathbb{R} \to \mathbb{R}$, together with the unknown function $u : I \to \mathbb{R}$, are defined on a closed real interval I. Similarly, the Read-Bajraktarević operator

$$\mathbf{T}u(x) = \nu(x, u(b(x))) \tag{1.1}$$

appeared later (1980s) in the works of C.J. Read in the context of the invariant subspace problem [19]. Initially defined on the space $C^\infty(I)$ of infinitely differentiable functions on I, the operator (1.1) is closely related to Bajraktarević equation, as follows. It is known that if b, ν are continuous, b is surjective, and if there exists $c \in (0, 1)$ such that $|\nu(x, y_1) - \nu(x, y_2)| \leq c|y_1 - y_2|$ for $x \in I$ and $y_1, y_2 \in \mathbb{R}$, then (1.1) is contractive [16]. It follows from the Banach theorem that $\mathbf{T}$ admits a unique fixed point $u^* \in C^\infty(I)$, which is actually the limit of the iterates $u_k(x) = \nu(x, u_{k-1}(b(x)))$, $k = 1, 2, \ldots$, where the initial condition u_0 is any element in $C^\infty(I)$. The fixed point u^* is called a *smooth local fractal function*, or a local fractal function of the smooth class. Different choices of the source space of (1.1) give rise to a number of corresponding categories of fixed points. Local fractal functions in the bounded, Hölder, smooth, Lebesgue and Sobolev spaces of functions (on a subset X of the real line) are well characterized and their properties are extensively examined in [17, 16]. For example, it is known that the graph of a local fractal function in any of these categories is the local attractor (a self-similar fractal) of an associated local IFS, and that the set of the discontinuities of a bounded local fractal function in 1 dimension is at most countably infinite [6].

These notes are strongly based on the pioneering paper by Massopust [17]. However, a remark made in the introduction of the reference [6] leads us to believe that the multidimensional case has not been previously treated. The purpose of this note is then twofold. On the one hand, we examine the multidimensional case, in which the underlying set X is assumed to be a nonempty connected and bounded subset of $\mathbb{R}^N$, $N \geq 1$. We employ a stronger version of the finite increments theorem in $\mathbb{R}^N$ to solve the *realization problem.* More specifically, we construct explicit contractive local IFSs whose attractors are, respectively, the graphs of local fractal functions in the Orlicz and in the (order one) Orlicz-Sobolev spaces of functions on X. On the other hand, we prove that local fractal functions in these categories are exactly the fixed points of the Read-Bajraktarević functional. Our results thus generalize previous theorems on the existence of local fractal functions in the Lebesgue and in the Sobolev classes, to a broader category of spaces, where the role of the norm is played now by a Young function. The problem whether our conclusions actually extend to higher orders is still open. The generalized Faà di Bruno formula [9] seems to be instrumental in this direction.

2. Iterated function systems

Let (X, d) denote a complete metric space and $\{w_i : X \to X\}_{i=1}^n$ a set of n continuous maps. Then the tuple $\mathcal{F} = (X, w_1, \ldots, w_n)$ is called an n-map iterated function system, or IFS.[1]. We say that the functions w_i belong to the IFS $\mathcal{F}$, and we write $w_i \in \mathcal{F}$, $1 \le i \le n$. These structures were introduced in the works of Hutchinson (1981), Mandelbrot (1982) and Barnsley (1993), as follows. Let $\mathcal{H}(X)$ denote the set of nonempty compact subsets of X. Associated with the IFS $\mathcal{F}$ is the set-valued map $\mathbf{w} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$\mathbf{w}(S) = \bigcup_{i=1}^{n} w_i(S) \quad S \subset \mathcal{H}(X).$$

The IFS $\mathcal{F}$ is called contractive if there exists a metric $\tilde{d}$, which is equivalent to d, such that each $w_i \in \mathcal{F}$ is a contraction with respect to $\tilde{d}$. That is, for each $1 \le i \le n$ there exists $c_i \in [0, 1)$ such that $\tilde{d}(w_i(x), w_i(y)) \le c_i\, \tilde{d}(x, y)$ for all $x, y \in X$. Hutchinson demonstrated [11] that in this case $\mathbf{w}$ is itself a contraction on $\mathcal{H}(X)$:

$$d_{\mathcal{H}}(\mathbf{w}(A), \mathbf{w}(B)) \le c\, d_{\mathcal{H}}(A, B),$$

where

$$A, B \in \mathcal{H}(X), c = \max_{1 \le i \le n} \{c_i\}$$

and

$$d_{\mathcal{H}}(A, B) = \max \left\{ \max_{a\in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(a, b) \right\}$$

is the Hausdorff distance between compact sets. In this case, the unique fixed point (Banach theorem) $A \in \mathcal{H}(X)$ is called the attractor of the IFS. By definition the attractor satisfies $A = \mathbf{w}(A)$ and is self-similar, since it may be expressed as a union of n contracted copies of itself. It is also known that non-contractive IFSs (i.e., such that the maps w_i are not contractions with respect to any topologically equivalent metric in X) can yield attractors [14]. For more details and examples we refer the reader to [5, 13].

The following notion is due to Barnsley and Hurd [7]. Let $\{X_1, \ldots, X_n\} \subset (X, d)$ be a family of nonempty subsets, equipped with a family of continuous maps $w_i : X_i \to X$, $1 \le i \le n$. Then $\mathcal{F}_{\text{loc}} = \{(X_1, w_1), \ldots, (X_n, w_n)\}$ is called a local iterated function system. A local IFS is called contractive if there exists a metric, equivalent to d, for which every $w \in \mathcal{F}_{\text{loc}}$ is contractive. Associated with any local IFS $\mathcal{F}_{\text{loc}} = \{(X_1, w_1), \ldots, (X_n, w_n)\}$ is the operator $\mathbf{w}_{\text{loc}} : 2^X \to 2^X$,

$$\mathbf{w}_{\text{loc}}(S) = \bigcup_{i=1}^{n} w_i(S \cap X_i), \tag{2.1}$$

where 2^X is the power set of X.

[1]The letter N is commonly used in the literature to denote the number of maps in the definition of the IFS. We will use n instead, and we will employ N to rather denote the dimension of the domain that appears later.

Definition 2.1. *A subset $A \in 2^X$ is called a local attractor of $\mathcal{F}_{\text{loc}}$ if $A = \mathbf{w}_{\text{loc}}(A)$.*

The local attractor of a contractive local IFS is unique. Moreover, suppose that $\mathcal{F} = (X, w_1, \dots, w_n)$ and $\mathcal{F}_{\text{loc}} = \{(X_1, w_1), \dots, (X_n, w_n)\}$ are both contractive, where $X_1, \dots, X_n \subset X$ are nonempty. It is well known [17, Proposition 1] that if X is compact and X_i is closed for $1 \le i \le n$, then the attractor A of $\mathcal{F}_{\text{loc}}$ is a subset of the attractor of $\mathcal{F}$.

3. Orlicz and Orlicz-Sobolev spaces

This section is brief summary on Orlicz and Orlicz-Sobolev spaces. For further details we refer the reader to [8, 18]. A function $\Phi : [0, \infty) \to [0, \infty)$ which is increasing, continuous, unbounded and such that $\Phi(0) = 0$ is called a φ-function [15, p.11]. If any such a Φ is, moreover, convex then it has the integral representation

$$\Phi(t) = \int_0^t \phi(s)\, ds, \tag{3.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing, right-continuous function satisfying $\phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. The function ϕ is called the right derivative of Φ. A convex φ-function Φ satisfying

$$\frac{\Phi(t)}{t} \to 0 \text{ as } t \to 0 \quad \text{and} \quad \frac{\Phi(t)}{t} \to \infty \text{ as } t \to \infty,$$

is denominated a Young function [15, p.47]. Young functions are sometimes called N-functions; however, to avoid confusion we will not employ that denomination in this article[2]. For example, if $\phi(t) = pt^{p-1}$, $t \ge 0$ and $1 \le p < \infty$, then $\Phi(t) = t^p$ and

$$\|u\|_p = \Phi^{-1}\left(\int_X \Phi(|u(x)|)dx\right)$$

for $u \in L^p(X)$ and where $\Phi^{-1}(t) = t^{1/p}$ is the inverse function.

Given a Young function Φ with the integral representation (3.1), the right-inverse function of ϕ is defined for $s \ge 0$ by $\psi(s) = \sup\{t : \phi(t) \le s\}$. If ϕ is continuous and increasing then ψ is the ordinary inverse ϕ^{-1}. The function ψ has the same properties as ϕ : it is positive for $s > 0$, right-continuous for $s \ge 0$ and satisfies $\psi(0) = 0$ and $\psi(s) \to \infty$ as $s \to \infty$. Hence the integral

$$\int_0^t \psi(s)\, ds$$

is a Young function as well, called the conjugate (or complementary) of Φ. The complementary Young function is usually denoted by $\overline{\Phi}(t)$.

Let $X \subset \mathbb{R}^N$ (with $N \ge 1$) be a bounded subset and let Φ be a Young function. The Orlicz class $\mathcal{L}_\Phi(X)$ is the set of (equivalence classes of) real-valued measurable functions u such that $\Phi(u) \in L^1(X)$. In general, $\mathcal{L}_\Phi(X)$ is not a vector space [12].

[2]This definition is ambiguous. Some texts denominate $\Phi : [0, \infty) \to [0, \infty]$ a Young function if Φ is not identically zero, convex and $\lim_{t\to 0^+} \Phi(t) = \Phi(0) = 0$.

However, the linear hull $L_\Phi(X)$ of the Orlicz class $\mathcal{L}_\Phi(X)$ is a Banach space when equipped with the Luxemburg norm

$$\|u\|_\Phi = \inf\left\{\tau > 0 : \int_X \Phi\left(\frac{|u|}{\tau}\right) dx \le 1\right\} \quad u \in L_\Phi(X).$$

We denote by $E_\Phi(X)$ the closure (for the norm-topology) of $L^\infty(X)$ in $L_\Phi(X)$. The space $E_\Phi(X)$ is separable and Banach for the inherited norm. In general, $E_\Phi(X) \subset \mathcal{L}_\Phi(X) \subset L_\Phi(X)$ and $E_\Phi(X) = L_\Phi(X)$ if and only if Φ satisfies a Δ_2-condition (at infinity). This means that for $r > 1$ there exists $\gamma(r) > 0$ such that

$$\Phi(rt) \le \gamma(r)\,\Phi(t) \quad t \ge T,$$

where T is also positive. (The Δ_2-condition is global if $T \ge 0$). It is known that if Φ and $\overline{\Phi}$ satisfy a Δ_2-condition at infinity then $L_\Phi(X)$ and $L_{\overline{\Phi}}(X)$ are reflexive and separable. It follows that one can identify the dual space of $E_\Phi(X)$ with $L_{\overline{\Phi}}(X)$ and the dual space of $E_{\overline{\Phi}}(X)$ with $L_\Phi(X)$, cf. [1, 8].

The Orlicz-Sobolev space $W^1L_\Phi(X)$. The Orlicz-Sobolev space $W^1L_\Phi(X)$ is the vector subspace of functions in $L_\Phi(X)$ with first distributional derivatives in $L_\Phi(X)$. Likewise, the Orlicz-Sobolev space $W^1E_\Phi(X)$ is the vector subspace of functions in $E_\Phi(X)$ with first distributional derivatives in $E_\Phi(X)$. The spaces $W^1L_\Phi(X)$ and $W^1E_\Phi(X)$ are Banach when endowed with the norm

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \|\nabla u\|_\Phi = \|u\|_\Phi + \sum_{i=1}^N \|\partial_{x_i} u\|_\Phi, \tag{3.2}$$

where ∂_{x_i} is the partial derivative $\partial/\partial x_i$. Usually, $W^1L_\Phi(X)$ and $W^1E_\Phi(X)$ are identified with subspaces of the products $L_\Phi(X)^{N+1} = \Pi L_\Phi(X)$ and $E_\Phi(X)^{N+1} = \Pi E_\Phi(X)$, respectively. The natural embedding of $W^1E_\Phi(X)$ into $E_\Phi(X)^{N+1}$ proves that $W^1E_\Phi(X)$ is separable since $E_\Phi(X)$ is itself separable. The space $W^1L_\Phi(X)$ is not separable in general, and $W^1L_\Phi(X) = W^1E_\Phi(X)$ if Φ satisfies a Δ_2-condition.

4. Local fractal functions of Orlicz and Orlicz-Sobolev classes

Let $X \subset \mathbb{R}^N$ ($N \ge 1$) be a nonempty connected and bounded subset and let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function with the integral form (3.1). We assume that there exist p_Φ, $q_\Phi > 0$ such that

$$p_\Phi \le \frac{t\,\phi(t)}{\Phi(t)} \le q_\Phi < \infty \quad t \ne 0. \tag{4.1}$$

These numbers appear in the characterization of the properties of Young functions in connection with Orlicz and Orlicz-Sobolev spaces [2, 3, 10]. For example, these inequalities ensure that Φ satisfies a global Δ_2-condition [1]. The complementary $\overline{\Phi}$ satisfies a global Δ_2-condition if and only if $p_\Phi > 1$, etc. A quick integration yields

$$\min\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\,\Phi(t) \le \Phi(\rho t) \le \max\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\,\Phi(t) \quad \rho, t \ge 0. \tag{4.2}$$

Let $\{X_i\}_{i=1}^n$ be a family of nonempty and connected subsets of X. In what follows $\{\alpha_i : X_i \to X\}_{i=1}^n$ will denote a collection of bounded diffeomorphisms such that

$$\alpha_i(X_i) \cap \alpha_{i'}(X_{i'}) = \emptyset \quad \text{for } i \neq i', \text{ and } \quad X = \dot{\bigcup}_{i=1}^n \alpha_i(X_i). \tag{4.3}$$

For $1 \le j \le N$ let ∂_{x_j} be the x_j-th partial derivative $\partial/\partial x_j$, and let $\partial_x \alpha_i$ denote the Jacobian matrix of α_i at $x \in X_i$.

Finally, let $\{\lambda_i : X_i \to \mathbb{R}\}_{i=1}^n \subset L_\Phi(X)$, and choose $S_1, \ldots, S_n$ with $S_i \in (L^\infty(X_i), \|\cdot\|_{i,\infty})$. Define the Read-Bajraktarević functional $\mathbf{T} : L_\Phi(X) \to \mathbb{R}^X$ by

$$\mathbf{T}u(x) = \sum_{i=1}^n \Big((\lambda_i \circ \alpha_i^{-1})(x) + (S_i \circ \alpha_i^{-1})(x)(u|_{X_i} \circ \alpha_i^{-1})(x) \Big) \mathbb{1}_{\alpha_i(X_i)}(x), \tag{4.4}$$

where $\mathbb{1}_{\alpha_i(X_i)}$ is the characteristic function of $\alpha_i(X_i)$ (i.e., $\mathbb{1}_{\alpha_i(X_i)}(x) = 1$ if $x \in \alpha_i(X_i)$ and it is zero otherwise) and $u|_{X_i}$ is the restriction of $u \in L_\Phi(X)$ to X_i. Note that if u^* is a fixed point of $\mathbf{T}$ then

$$u^* \circ \alpha_i = \lambda_i + S_i \, u^*|_{X_i}, \tag{4.5}$$

for all indices $1 \le i \le n$.

4.1. Existence of fixed points

In this paragraph we tackle the problem on the existence of fixed points of (4.4), and of its restriction to the Orlicz-Sobolev space $W^1 L_\Phi(X)$. We set for every $1 \le i \le n$

$$a_i = \sup_{x \in X_i} |\det \partial_x \alpha_i| \quad \text{and} \quad r_i = \max\{1, \|S_i\|_{i,\infty}\}. \tag{4.6}$$

Lemma 4.1. *The Read-Bajraktarević operator is well defined on $L_\Phi(X)$ and sends this space into itself. Moreover,*

$$\|\mathbf{T}u - \mathbf{T}v\|_\Phi \le \Big(\sum_{i=1}^n a_i r_i^{q_\Phi} \Big) \|u - v\|_\Phi$$

for all $u, v \in L_\Phi(X)$.

Proof. Since α_i is a diffeomorphism and $\lambda_i \in L_\Phi(X)$, $1 \le i \le n$, the functional (4.4) is well defined and indeed $\mathbf{T}(L_\Phi(X)) \subset L_\Phi(X)$. Let τ be positive and $u, v \in L_\Phi(X)$. A change of coordinates $y = \alpha_i^{-1}(x)$ and subsequent re-labeling $x \mapsto y$ produce

$$\begin{aligned}\int_X \Phi\Big(\frac{1}{\tau}|\mathbf{T}u(x) - \mathbf{T}v(x)|\Big) dx &\le \sum_{i=1}^n \int_{\alpha_i(X_i)} \Phi\Big(\frac{1}{\tau}|S_i \circ \alpha_i^{-1}(x)| \big|u|_{X_i} \circ \alpha_i^{-1}(x) - v|_{X_i} \circ \alpha_i^{-1}(x)\big|\Big) dx \\ &= \sum_{i=1}^n \int_{X_i} \Phi\Big(\frac{1}{\tau}|S_i(x)| \big|u|_{X_i}(x) - v|_{X_i}(x)\big|\Big) |\det \partial_x \alpha_i| dx.\end{aligned}$$

Therefore, (4.2) implies

$$\int_X \Phi\Big(\frac{1}{\tau}|\mathbf{T}u(x) - \mathbf{T}v(x)|\Big) dx \le \Big(\sum_{i=1}^n a_i r_i^{q_\Phi} \Big) \int_X \Phi\Big(\frac{1}{\tau}|u(x) - v(x)|\Big) dx.$$

The conclusion follows by the definition of the Luxemburg norm. □

In particular, the functional (4.4) is a contraction on $L_\Phi(X)$ provided

$$\sum_{i=1}^{n} a_i r_i^{q_\Phi} < 1. \tag{4.7}$$

In this case, the fixed point of $\mathbf{T} : L_\Phi(X) \to L_\Phi(X)$ is called a local fractal function of the Orlicz class. A local fractal function evidently depends on the choice of the functions λ_i, S_i and on the diffeomorphisms $\alpha_1, \ldots, \alpha_n$. Define

$$b_i = \max\Big\{1, \sup_{x\in X_i} \max_{1\le k,j\le N} \big|\big[(\partial_{\alpha_i^{-1}(x)}\alpha_i)^{-1}\big]_{kj}\big|\Big\} \quad 1 \le i \le n, \tag{4.8}$$

where $\big[(\partial_{\alpha_i^{-1}(x)}\alpha_i)^{-1}\big]_{kj}$ is the (k,j)-entry of the inverse of the Jacobian matrix of α_i at $y = \alpha_i^{-1}(x)$, and $y = (y_1, \ldots, y_N)$ parametrizes the source space of α_i. The next result asserts that if (4.7) is slightly modified then the contraction of (4.4) happens on $W^1L_\Phi(X)$.

Theorem 4.2. *Assume that $S_i(x) = s_i \in \mathbb{R}$, $1 \le i \le n$, $x \in X_i$ and $\{\lambda_1, \ldots, \lambda_n\} \subset W^1L_\Phi(X)$. Then the restriction of the Read-Bajraktarević operator to $W^1L_\Phi(X)$ is well defined and sends this space into itself. Moreover, if the condition*

$$\sum_{i=1}^{n} a_i (r_i b_i)^{q_\Phi} < \frac{1}{N^{\frac{q_\Phi}{2}+1}} \tag{4.9}$$

is fulfilled then (4.4) *is a contraction on $W^1L_\Phi(X)$.*

Proof. Since $\lambda_i \in W^1L_\Phi(X)$, $1 \le i \le n$, the functional (4.4) is well defined on $W^1L_\Phi(X)$ and clearly sends this space into itself. Note that if $u \in W^1L_\Phi(X)$ then the composite $u\circ\alpha_i^{-1}$ is x_j-differentiable and $\partial_{x_j}(u\circ\alpha_i^{-1}) = \nabla(u\circ\alpha_i^{-1})\cdot\partial_{x_j}\alpha_i^{-1}$, for $1 \le j \le N$. The chain rule yields for $u, v \in W^1L_\Phi(X)$ and $\tau > 0$,

$$\begin{aligned}\int_X \Phi\Big(\frac{1}{\tau}\big|\partial_{x_j}\mathbf{T}u(x) - \partial_{x_j}\mathbf{T}v(x)\big|\Big)dx &\le \sum_{i=1}^{n}\int_{\alpha_i(X_i)} \Phi\Big(\frac{|s_i|}{\tau}\big|\partial_{x_j}\{u|_{X_i}\circ\alpha_i^{-1}(x) - v|_{X_i}\circ\alpha_i^{-1}(x)\}\big|\Big)dx\\ &= \sum_{i=1}^{n}\int_{\alpha_i(X_i)} \Phi\Big(\frac{|s_i|}{\tau}\big|\nabla(u|_{X_i} - v|_{X_i})(\alpha_i^{-1}(x))\cdot\partial_{x_j}\alpha_i^{-1}(x)\big|\Big)dx.\end{aligned}$$

Cauchy-Schwarz inequality implies that the integrand on the right is bounded by

$$\Phi\Big(\frac{r_i}{\tau}\big|\nabla(u|_{X_i} - v|_{X_i})(\alpha_i^{-1}(x))\big|\sqrt{N}\max_{1\le k,j\le N}\big|\big[(\partial_{\alpha_i^{-1}(x)}\alpha_i)^{-1}\big]_{kj}\big|\Big)dx. \tag{4.10}$$

The maximum above is bounded by b_i. The change of coordinates $y = \alpha_i^{-1}(x)$ (and re-labeling $x \mapsto y$) along with (4.2) applied with $\rho = \sqrt{N}r_ib_i$ altogether yield

$$\int_X \Phi\Big(\frac{1}{\tau}\big|\partial_{x_j}\mathbf{T}u(x) - \partial_{x_j}\mathbf{T}v(x)\big|\Big)dx \;\le\; N^{\frac{q_\Phi}{2}}\Big(\sum_{i=1}^{n} a_i(r_ib_i)^{q_\Phi}\Big)\int_X \Phi\Big(\frac{1}{\tau}\big|\nabla u(x) - \nabla v(x)\big|\Big)dx,$$

for $1 \le j \le N$. The definition of the Luxemburg norm and (3.2) thus produce

$$\|\nabla\mathbf{T}u - \nabla\mathbf{T}v\|_\Phi \le N^{\frac{q_\Phi}{2}+1}\Big(\sum_{i=1}^{n} a_i(r_ib_i)^{q_\Phi}\Big)\|\nabla u - \nabla v\|_\Phi.$$

Since $b_i^{q_\Phi} \geq 1$ for $1 \leq i \leq n$, Lemma 4.1 implies

$$\|\mathbf{T}u - \mathbf{T}v\|_{1,\Phi} = \|\mathbf{T}u - \mathbf{T}v\|_\Phi + \|\nabla\mathbf{T}u - \nabla\mathbf{T}v\|_\Phi \leq N^{\frac{q_\Phi}{2}+1}\Big(\sum_{i=1}^n a_i(r_ib_i)^{q_\Phi}\Big)\|u-v\|_{1,\Phi}.$$

The theorem is proved. □

If (4.9) is satisfied then the fixed point u^* of $\mathbf{T} : W^1L_\Phi(X) \to W^1L_\Phi(X)$ is called a local fractal function of the Orlicz-Sobolev class $W^1L_\Phi(X)$.

Observation. A coarser (more restrictive) condition substitutes (4.9) if the convexity of Φ is utilized. Indeed, we note that bound (4.10) can be replaced by

$$\Phi\bigg(\frac{r_i}{\tau}\max_{1\leq k,j\leq N}\big|\big[(\partial_{\alpha_i^{-1}(x)}\alpha_i)^{-1}\big]_{kj}\big|\sum_{k=1}^N|\partial_{y_k}(u|_{X_i} - v|_{X_i})(\alpha_i^{-1}(x))|\bigg)dx.$$

Jensen inequality ensures that this term is in turn less than or equal to

$$\frac{1}{N}\sum_{k=1}^N\Phi\bigg(\frac{1}{\tau}Nr_ib_i\big|\partial_{y_k}(u|_{X_i} - v|_{X_i})(\alpha_i^{-1}(x))\big|\bigg)dx.$$

Therefore, (4.2) applied with $\rho = Nr_ib_i$ yields

$$\int_X\Phi\bigg(\frac{1}{\tau}|\partial_{x_j}\mathbf{T}u(x) - \partial_{x_j}\mathbf{T}v(x)|\bigg)dx \leq N^{q_\Phi}\Big(\sum_{i=1}^n a_i(r_ib_i)^{q_\Phi}\Big)\int_X\Phi\bigg(\frac{1}{\tau}\big|\nabla u(x) - \nabla v(x)\big|\bigg)dx,$$

which entails $\|\nabla\mathbf{T}u - \nabla\mathbf{T}v\|_\Phi \leq N^{q_\Phi+1}\Big(\sum_{i=1}^n a_i(r_ib_i)^{q_\Phi}\Big)\|\nabla u - \nabla v\|_\Phi$.

4.2. Realization of a local contractive IFS in N dimensions

Suppose that $\mathbf{T} : L_\Phi(X) \to \mathbb{R}^X$ has a fixed point $u^* \in W^1L_\Phi(X)$. We would like to construct a contractive local IFS whose attractor is exactly the graph of u^*, namely $G(u^*) = \{(x, u^*(x)) : x \in X\}$. In dimension $N = 1$ the solution to this problem is due to Massopust *et al.* and can be found in [6]. We adapt the procedure to the case of N dimensions via a general version of the finite increments theorem.

Recall that if $x', x'' \in \mathbb{R}^N$ then $[x', x''] = \{(1-t)x' + tx'' : 0 \leq t \leq 1\}$ is the closed segment from x' to x'', and the set $(x', x'') = \{(1-t)x' + tx'' : 0 < t < 1\}$ is the corresponding open segment.

Lemma 4.3. *Let* $\mathrm{int}(\Omega)$ *denote the interior of a nonempty set* $\Omega \subset \mathbb{R}^N$ *and choose* $x', x'' \in \Omega$ *such that* $[x', x''] \subset \mathrm{int}(\Omega)$. *Let* $\alpha : \Omega \to \mathbb{R}^N$ *be continuous on* $[x', x'']$ *and differentiable on* (x', x''). *Then*

$$\|\alpha(x'') - \alpha(x')\|_1 \leq N\Big(\max_{1\leq k,j\leq N}\sup_{x\in(x',x'')}|[\partial_x\alpha]_{kj}|\Big)\|x'' - x'\|_1,$$

where $[\partial_x\alpha]_{kj}$ *is the* (k,j)*-entry of the Jacobian matrix of* α *at* $x \in \Omega$ *and*

$$\|x\|_1 = \sum_{k=1}^N |x_k|.$$

This result follows from the finite increments theorem for real-valued functions.[3] With the maps and notation from the previous section, we assume that X is a closed subset of a complete metric space and that X_i is convex, $1 \le i \le n$. We require that

$$M = \max_{1 \le i \le n} \|S_i\|_{i,\infty} < 1.$$

In this case the functions $\nu_i : X_i \times \mathbb{R} \to \mathbb{R}$,

$$\nu_i(x, y) = \lambda_i(x) + S_i(x)y \quad 1 \le i \le n, \tag{4.11}$$

are uniformly contractive in the second variable: $|\nu_i(x, y_1) - \nu_i(x, y_2)| \le M|y_1 - y_2|$, $x \in X_i$ and $y_1, y_2 \in \mathbb{R}$. Furthermore, we assume that for $1 \le i \le n$ the function λ_i is so chosen that ν_i is uniformly Lipschitz continuous in the first variable. That is, there exists $\tilde{M} > 0$ such that

$$|\nu_i(x_1, y) - \nu_i(x_2, y)| \le \tilde{M}\|x_1 - x_2\|_1, \quad x_1, x_2 \in X_i \text{ and } y \in \mathbb{R}.$$

(This happens e.g., when the λ_i are uniformly Lipschitz continuous and the S_i are constants). Finally, for $1 \le i \le n$ define $w_i : X_i \times \mathbb{R} \to X \times \mathbb{R}$ by

$$w_i(x, y) = (\alpha_i(x), \nu_i(x, y))$$

and let $d_\theta : (X \times \mathbb{R}) \times (X \times \mathbb{R}) \to \mathbb{R}$ be the map

$$d_\theta((x_1, y_1), (x_2, y_2)) = \|x_1 - x_2\|_1 + \theta|y_1 - y_2|,$$

where $\theta = (1 - Nm)/2\tilde{M}$, and

$$m = \max_{\substack{1 \le i \le n \\ 1 \le k,j \le N}} \sup_{x \in X_i} |[\partial_x \alpha_i]_{kj}|.$$

(Evidently, d_θ is a metric on the product space, provided $m < 1/N$. In dimension $N = 1$ the latter occurs when each α_i is contractive).

Theorem 4.4. *If $m < 1/N$ then $\mathcal{F}_{\text{loc}} = \{(X_i \times \mathbb{R}, w_i)\}_{i=1}^n$ is a contractive local* IFS *for the metric d_θ, and $G(u^*)$ is a local attractor of $\mathcal{F}_{\text{loc}}$, i.e. $\mathbf{w}_{\text{loc}}(G(u^*)) = G(u^*)$, where $\mathbf{w}_{\text{loc}}$ is the map* (2.1) *associated to $\mathcal{F}_{\text{loc}}$.*

Proof. Let $i \in \{1, \cdots, n\}$ and choose two points (x_1, y_1) and (x_2, y_2) in the product $X_i \times \mathbb{R}$. Since X_i is convex, Lemma 4.3 implies

$$\begin{aligned} d_\theta(w_i(x_1, y_1), w_i(x_2, y_2)) &\le Nm\|x_1 - x_2\|_1 + \theta|\nu_i(x_1, y_1) - \nu_i(x_2, y_1)| + \theta|\nu_i(x_2, y_1) - \nu_i(x_2, y_2)| \\ &\le (Nm + \theta\tilde{M})\|x_1 - x_2\|_1 + \theta M|y_1 - y_2| \\ &\le \epsilon\, d_\theta((x_1, y_1), (x_2, y_2)), \end{aligned}$$

where $\epsilon = \max\{Nm + \theta\tilde{M}, M\} < 1$, and the first assertion follows.

[3] If $\alpha(x) = (\alpha^1(x), \ldots, \alpha^N(x))$, $\alpha^j : \mathbb{R}^N \to \mathbb{R}$, then for $1 \le j \le N$ there exists $c_j \in (x', x'')$ such that $\alpha^j(x'') - \alpha^j(x') = \sum_{k=1}^N \partial_{x_k}\alpha^j(c_j)(x''_k - x'_k)$. Hence,

$$\max_{1 \le j \le N} |\alpha^j(x'') - \alpha^j(x')| \le \max_{1 \le k,j \le N} \sup_{x \in (x', x'')} |[\partial_x \alpha]_{kj}|\, \|x'' - x'\|_1.$$

On the other hand, from (4.5) and (4.11) the fixed point of (4.4) satisfies the equation $u^*(\alpha_i(x)) = \nu_i(x, u^*(x))$, for all $x \in X_i$. Therefore,

$$\mathbf{w}_{\rm loc}(G(u^*)) = \bigcup_i \{\big(\alpha_i(x), \nu_i\big(x, u^*(x)\big)\big) : x \in X_i\} = \bigcup_i \{\big(\alpha_i(x), u^*(\alpha_i(x))\big) : x \in X_i\},$$

which is equal to $G(u^*)$. The two equalities above follow from (4.3). □

5. Examples

The following cases were treated in [6, 17]. In one dimension these examples are easily retrieved from Lemma 4.1 and Theorem 4.2, respectively, as we demonstrate below.

5.1. Lebesgue space $L^p[0,1]$ with $p \in (0,\infty]$.

Let $\{X_i\}_{i=1}^n$ be a family of connected semi-open intervals of $[0,1]$ and let $\{x_0 = 0 < \cdots < x_n = 1\}$ be a partition of the closed unit interval. The map $\alpha_i : X_i \to [0,1]$ is chosen to be linear, from X_i onto $[x_{i-1}, x_i)$, $1 \le i \le n$, and where $\alpha_i^{-1}([x_{n-1}, x_n) \cup \{x_n\})$ is denoted by X_n^+. Then (4.3) is satisfied in this case. If we choose $\lambda_i \in L^p(X_i)$ and $S_i \in L^\infty(X_i)$, $1 \le i \le n$, then the image of $\mathbf{T} : L^p[0,1] \to \mathbb{R}^{[0,1]}$ belongs to $L^p[0,1]$. Let d_i denote the ordinary derivative (slope) of α_i. If

$$\begin{cases} \displaystyle\sum_{i=1}^n d_i \|S_i\|_{\infty,X_i}^p < 1, & p > 0; \\ \displaystyle\max_{1\le i\le n} \|S_i\|_{\infty,X_i} < 1, & p = \infty, \end{cases} \tag{5.1}$$

are fulfilled, then (4.4) is contractive on $L^p[0,1]$ and there exists a unique local fractal function $u^* \in L^p[0,1]$ satisfying (4.5). Evidently, this function depends on the choice of λ_i, S_i and α_i, $1 \le i \le n$. Note that for values $1 < p < \infty$, the first condition in (5.1) is a particular case of (4.7). The existence of a fractal function itself is consequence of Lemma 4.1 applied with the Young function $\Phi(t) = t^p$ in (3.1) and therefore $p_\Phi = q_\Phi = p$ in (4.1) while $a_i = d_i$ and $b_i = 1/d_i$ in (4.6) and (4.8), respectively.

5.2. The Sobolev space $W^{m,p}(0,1)$ with $1 < p \le \infty$.

This case is also well documented in the literature. Let $X_1, \ldots, X_n$ be nonempty connected open subintervals of $X = (0,1)$ and let $\{x_1 < \cdots < x_{n-1}\}$ be a partition of X. The map $\alpha_i : X_i \to [0,1]$ is chosen linear and increasing such that $\alpha_i(X_i) = (x_{i-1}, x_i)$, where $x_0 = 0$ and $x_n = 1$. Let m be a positive integer. We choose $S_i \equiv s_i \in \mathbb{R}$ and $\lambda_i \in W^{m,p}(X_i)$, $1 \le i \le n$. The operator $\mathbf{T} : W^{m,p}(0,1) \to \mathbb{R}^{(0,1)}$ is then well defined and sends $W^{m,p}(0,1)$ into itself. Let $d_i > 0$ denote the slope of α_i. If

$$\begin{cases} \displaystyle\max_{q=0,1,\ldots,m} \sum_{i=1}^n |s_i|^p / d_i^{qp-1} < 1, & 1 \le p < \infty; \\ \displaystyle\max_{q=0,1,\ldots,m} \sum_{i=1}^n |s_i| / d_i^q < 1, & p = \infty, \end{cases} \tag{5.2}$$

then $\mathbf{T}$ is contractive on $W^{m,p}(0,1)$. The unique fixed point $u^* \in W^{m,p}(0,1)$ is called a local fractal function of class $W^{m,p}(0,1)$. In the case $N = m = 1$ and for values

$1 < p < \infty$ the first condition in (5.2) is a particular case of our general requirement (4.9). In this case, Theorem 4.2 is applied to the Young function $\Phi(t) = t^p$ so that again, $p_\Phi = q_\Phi = p$ in (4.1) while $a_i = d_i$ in (4.6) and $b_i = 1/d_i$ in (4.8).

6. Concluding remarks and future research

In one dimension the two examples from the previous section also hold true under the assumption that each function α_i is either a smooth bounded diffeomorphism, or a bounded invertible real-analytic map, from X_i onto the semi-open interval $[x_{i-1}, x_i)$. It would be natural to extend these results to underlying sets X belonging to subdomains of differentiable and real analytic manifolds. The geometric properties of these spaces (such as the existence of Riemann-Schwarz reflections and complex conjugations in some sense) may eventually reflect in the particular form of the Read-Bajraktarević operator itself. The problem whether this functional extends to these finer spaces is, to our knowledge, untreated. In addition, it seems that the proof of Theorem 4.2 can be modified in the obvious way so that this result be valid as well in the Orlicz-Sobolev space $W^m L_\Phi(X)$, $m \geq 2$. The generalization would require an expression for the higher-order weak derivatives of the composite $(u \circ \alpha_i^{-1})$. Arguably, condition (4.9) should be modified as well so as to ensure that (4.4) be a contraction on $W^m L_\Phi(X)$. These generalizations may be useful in applications. We look forward to addressing these questions in future publications.

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On some univalence criteria for certain integral operators

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Abstract. For analytic functions in the open unit disk, we define new general integral operators. The aim of this paper is to study these new operators and related univalence criteria. First of all, we recall some classes of functions defined on the unit disk. We will use functions from these classes to construct our integral operators. Secondly, we recall the univalence criteria that we use in the proofs of our results. Finally, we use the univalence criteria to establish univalence conditions related to our general integral operators.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk

$$\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\},$$

which are given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

for all $z \in \mathcal{U}$, that satisfy the following normalization condition

$$f(0) = f'(0) - 1 = 0.$$

Furthermore, let us denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ that consists of univalent functions (i.e., injective functions).

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In the paper [11], J. Liu and D. Yang introduced the class $\mathcal{S}(\alpha)$ as follows (see [11, Relations (1.4)-(1.5)]).

Definition 1.1. *For $\alpha \in (0,2]$, define the class $\mathcal{S}(\alpha)$ as the class of all functions $f \in \mathcal{A}$ that satisfy the following conditions:*

(i) $f(z) \neq 0$, *for* $0 < |z| < 1$ *and*

(ii)

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq \alpha,$$

for all $z \in \mathcal{U}$.

We also have the following result (see [11, Theorem 3, Relation (3.14)]).

Lemma 1.2. *Let $\alpha \in (0,2]$. If $f \in \mathcal{S}(\alpha)$ then*

$$\left| \frac{zf'(z)}{f^2(z)} - 1 \right| \leq \alpha |z|^2,$$

for all $z \in \mathcal{U}$.

Next, we recall the notions of starlikeness of order k and convexity of order k (see, e.g., [8, Def. 2.3.1]).

We say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(k)$ (i.e., the class of starlike functions of order k in $\mathcal{U}$), for $k \in [0,1)$, if it satisfies the inequality

$$\text{Re} \left[\frac{zf'(z)}{f(z)} \right] > k,$$

for $z \in \mathcal{U}$.

For $k \in [0,1)$, we denote by $\mathcal{K}(k)$ the class of convex functions f of order k in $\mathcal{U}$, i.e., the class of analytic functions that satisfy

$$\text{Re} \left[\frac{zf''(z)}{f'(z)} + 1 \right] > k,$$

for $z \in \mathcal{U}$.

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}(k)$, for $k \in [0,1)$, if

$$\text{Re}|f'(z)| > k,$$

for $z \in \mathcal{U}$.

Frasin and Jahangiri (see [7, Relation (1.5)]) have studied the class $\mathcal{B}(\mu, k)$, for $\mu \geq 0$ and $k \in [0,1)$. This class contains all functions $f \in \mathcal{A}$, that satisfy the condition

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| < 1 - k,$$

for $z \in \mathcal{U}$.

The class $\mathcal{B}(\mu, k)$ is a comprehensive class of normalized analytic functions in $\mathcal{U}$ that contains other classes of analytic and univalent functions in $\mathcal{U}$, such as

$$\mathcal{B}(1,k) = \mathcal{S}^*(k),\ \mathcal{B}(0,k) = \mathcal{R}(k).$$

A great deal of researchers have devoted themselves to the study of sufficient conditions for the univalence of various integral operators (see, e.g., [1], [2], [3], [4], [5], [6], [8], [9], [10], [12], [17]).

The goal of this present paper is to study the univalence conditions for the integral operators

$$F_{n,\beta}(z) := \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \frac{e^{f_i(t)}}{g_i'(t)} \mathrm{d}t\right]^{\frac{1}{\beta}}, \tag{1.1}$$

for $\beta \in \mathbb{C} \setminus \{0\}$, $f_i, g_i \in \mathcal{A}$, $i = \overline{1,n}$, and

$$G_{\beta,\gamma}(f,g)(z) := \left[\beta \int_0^z t^{\beta-1} \left[\frac{e^{f(t)}}{g'(t)}\right]^{\gamma} \mathrm{d}t\right]^{\frac{1}{\beta}}, \tag{1.2}$$

for $\beta \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{C}$, $f, g \in \mathcal{A}$.

Remark 1.3. In the case $f_i = f$, $g_i = g$, for $i = \overline{1,n}$, then

$$F_{n,\beta} = G_{\beta,n}(f,g).$$

Remark 1.4. For $\beta = 1$, the operator (1.2) reduces to the operator $G_\gamma(f,g)$, given by

$$G_\gamma(f,g)(z) := \int_0^z \left[\frac{e^{f(t)}}{g'(t)}\right]^{\gamma} \mathrm{d}t.$$

For additional details on the operator $G_\gamma(f,g)$, we refer the reader to [2].

In order to derive our main results, we recall the following univalence criteria (see also, [14]).

Theorem 1.5. ([15, Theorem 2]) *Let $f \in \mathcal{A}$ and $\beta \in \mathbb{C}$. If* $\mathrm{Re}(\beta) > 0$ *and*

$$\frac{1 - |z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)} \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq 1.$$

for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta$ defined by

$$\mathcal{F}_\beta(z) := \left[\beta \int_0^z t^{\beta-1} f'(t)\mathrm{d}t\right]^{\frac{1}{\beta}}, \tag{1.3}$$

belongs to the class $\mathcal{S}$ of analytic and univalent functions in $\mathcal{U}$.

Theorem 1.6. ([16]) *Let $c, \alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$ and $|c| \leq 1$, $c \neq -1$. If the function $f \in \mathcal{A}$,*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

for all $z \in \mathcal{U}$, satisfies the condition

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1,$$

for $z \in \mathcal{U}$, then we have that the function $\mathcal{F}_\alpha$ given by relation (1.3) belongs to the class $\mathcal{S}$.

In addition, we recall the generalized Schwarz lemma (see, e.g., [8, p. 35], [9], [13]).

Lemma 1.7. *(Generalized Schwarz Lemma) Let $R > 0$ be a constant. Let f be an analytic function in the disk*

$$\mathcal{U}_R := \{z \in \mathbb{C} : |z| < R\},$$

satisfies the property that there is a fixed constant $M > 0$ such that

$$|f(z)| < M,$$

for $|z| < R$. If f has at $z = 0$ a root with its multiplicity order greater than some m, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \tag{1.4}$$

for all $z \in \mathcal{U}_R$.

The equality in relation (1.4) holds for $z \neq 0$ if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

for all $z \in \mathcal{U}_R$, where θ is a constant.

2. Main results

In this section, we aim to use the univalence criteria presented in the previous section (see Theorem 1.5, Theorem 1.6) in order to give univalence conditions for the integral operators $F_{n,\beta}$ and $G_{\beta,\gamma}$, given by relations (1.1) and (1.2), respectively.

In the latter, let us denote the set of non-negative real numbers by $\mathbb{R}_+$.

The following theorem gives us sufficient conditions for the univalence of the integral operators $F_{n,\beta}$ (see relation (1.1)) and $G_{\beta,\gamma}$ (see relation (1.2)), whenever the functions involved in the definitions of these integral operators belong to the class $\mathcal{A}$.

Theorem 2.1. *Let $f_i \in \mathcal{A}$, $i = \overline{0,n}$, such that*

$$\left| \frac{z^2 f_i'(z)}{f_i^2(z)} - 1 \right| \leq 1, \tag{2.1}$$

for $i = \overline{0,n}$, and for all $z \in \mathcal{U}$.

Let $M_i, N_i \in \mathbb{R}_+$, $i = \overline{0,n}$ and let $g_i \in \mathcal{A}$, $i = \overline{0,n}$, such that

$$|f_i(z)| < M_i, \quad \left| \frac{g_i''(z)}{g_i'(z)} \right| \leq N_i, \tag{2.2}$$

for $i = \overline{0,n}$, and for all $z \in \mathcal{U}$.

Let $\gamma \in \mathbb{C}$ and let $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) =: a > 0$.

Denote

$$c := \frac{2}{2a+1} \left(\frac{1}{2a+1} \right)^{\frac{1}{2a}}. \tag{2.3}$$

(i) If

$$c \sum_{i=1}^{n} (2M_i^2 + N_i) \leq 1, \tag{2.4}$$

then the integral operator $F_{n,\beta}$ given by relation (1.1) belongs to the class $\mathcal{S}$.

(ii) If

$$c|\gamma|(2M_0^2 + N_0) \leq 1, \tag{2.5}$$

then the integral operator $G_{\beta,\gamma}(f_0, g_0)$ given by relation (1.2) belongs to the class $\mathcal{S}$.

Proof. We will prove statement (i).

To achieve this, we consider the function $h_n : \mathcal{U} \to \mathbb{C}$, defined by

$$h_n(z) := \int_0^z \prod_{i=1}^{n} \frac{e^{f_i(t)}}{g_i'(t)} \mathrm{d}t. \tag{2.6}$$

The function h_n is holomorphic in the unit disk. By differentiation, we get

$$h_n'(z) = \prod_{i=1}^{n} \frac{e^{f_i(z)}}{g_i'(z)}, \tag{2.7}$$

and

$$h_n''(z) = \sum_{i=1}^{n} \frac{e^{f_i(z)} f_i'(z) g_i'(z) - e^{f_i(z)} g_i''(z)}{g_i^2(z)} \cdot \prod_{k=1, k\neq i}^{n} \frac{e^{f_k(z)}}{g_k'(z)}. \tag{2.8}$$

Using relations (2.7) and (2.8) we obtain

$$\frac{h_n''(z)}{h_n'(z)} = \sum_{i=1}^{n} \left(f_i'(z) - \frac{g_i''(z)}{g_i'(z)} \right). \tag{2.9}$$

We multiply the modulus of (2.9) by $\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)} > 0$ and by $|z|$ and we get

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(z)}{h_n'(z)}\right| &= \frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}|z|\sum_{i=1}^{n}\left|f_i'(z)-\frac{g_i''(z)}{g_i'(z)}\right| \\ &\le \frac{1-|z|^{2a}}{a}|z|\sum_{i=1}^{n}\left(|f_i'(z)|+\left|\frac{g_i''(z)}{g_i'(z)}\right|\right).\end{aligned} \tag{2.10}$$

Taking into account the inequality

$$\frac{1-|z|^{2a}}{a}|z| \le \frac{2}{2a+1}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}} = c, \tag{2.11}$$

for all $|z|<1$, and the relation (2.10), we obtain

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le c\sum_{i=1}^{n}\left(|f_i'(z)|+\left|\frac{g_i''(z)}{g_i'(z)}\right|\right). \tag{2.12}$$

Now, we apply Lemma 1.7 to the functions f_i, $i=\overline{1,n}$, and we get

$$|f_i(z)| \le M_i|z|, \tag{2.13}$$

for all $i=\overline{1,n}$ and for all $z\in\mathcal{U}$.

Using relations (2.12) and (2.13) we deduce that

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(z)}{h_n'(z)}\right| &\le c\sum_{i=1}^{n}\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\left|\frac{f_i(z)}{z}\right|^2+\left|\frac{g_i''(z)}{g_i'(z)}\right|\right) \\ &\le c\sum_{i=1}^{n}\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1+1\right|M_i^2+N_i\right) \\ &\le c\sum_{i=1}^{n}(2M_i^2+N_i) \le 1.\end{aligned} \tag{2.14}$$

Now, by applying Theorem 1.5, it follows that $F_{n,\beta}\in\mathcal{S}$.

The proof of statement (ii) follows similar arguments to those presented in the proof of statement (i).

In the case of statement (ii), one considers the function $h:\mathcal{U}\to\mathbb{C}$,

$$h(z) := \int_0^z\left(\frac{e^{f_0(t)}}{g_0'(t)}\right)^{\gamma}\mathrm{d}t. \tag{2.15}$$

The proof of this statement is omitted for the sake of brevity.

This concludes our proof. □

In the following result, we show that the operators $F_{n,\beta}$ (see relation (1.1)) and $G_{\beta,\gamma}(f_0,g_0)$ (see relation (1.2)) belong to the class of univalent functions in $\mathcal{U}$, in the case that the functions $f_i\in\mathcal{S}(\alpha_i)$, for $\alpha_i\in(0,2]$, $i=\overline{0,n}$ (see Definition 1.1).

Theorem 2.2. *Let $\alpha_i \in (0,2]$ and let $f_i \in \mathcal{S}(\alpha_i)$, for $i = \overline{0,n}$. Let $M_i, N_i \in \mathbb{R}_+$ and $g_i \in \mathcal{A}$ for $i = \overline{0,n}$, such that*

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \le N_i, \tag{2.16}$$

for $i = \overline{0,n}$ and for all $z \in \mathcal{U}$.

Let $\gamma \in \mathbb{C}$ and let $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) =: a > 0$.

Denote

$$c := \frac{2}{2a+1}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}. \tag{2.17}$$

(i) If

$$c\sum_{i=1}^{n}((\alpha_i+1)M_i^2 + N_i) \le 1, \tag{2.18}$$

then the integral operator $F_{n,\beta}$ given by relation (1.1) belongs to the class $\mathcal{S}$.

(ii) If

$$c|\gamma|((\alpha_0+1)M_0^2 + N_0) \le 1, \tag{2.19}$$

then the integral operator $G_{\beta,\gamma}(f_0, g_0)$ given by relation (1.2) belongs to the class $\mathcal{S}$.

Proof. We will prove statement (i). To this end, we consider the function $h_n : \mathcal{U} \to \mathbb{C}$ given by relation (2.6). Recall that the function h_n is holomorphic in the unit disk.

Recall that, in the proof of Theorem 2.1, we have obtained the inequality (2.14).

Now, let $|z| < 1$. If we take into account relations (2.11), (2.14) and Lemma 1.7 and Lemma 1.2, we obtain

$$\begin{aligned}
\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(z)}{h_n'(z)}\right| &\le c\sum_{i=1}^{n}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1\right| M_i^2 + M_i^2 + N_i\right)\\
&\le c\sum_{i=1}^{n}(\alpha_i|z|^2 M_i^2 + M_i^2 + N_i)\\
&\le c\sum_{i=1}^{n}((\alpha_i+1)M_i^2 + N_i)\\
&\le 1.
\end{aligned} \tag{2.20}$$

By applying Theorem 1.5, we deduce that $F_{n,\beta} \in \mathcal{S}$.

The proof of statement (ii) is similar to that of statement (i). In this case, one must consider the function $h : \mathcal{U} \to \mathbb{C}$ given by relation (2.15). We omit the proof for the sake of brevity.

This concludes our proof. □

Next, we analyze univalence conditions for the operators $F_{n,\beta}$ (see relation (1.1)) and $G_{\beta,\gamma}$ (see relation (1.2)) in the case that the functions $f_i \in \mathcal{B}(\mu_i, \alpha_i)$, $\mu_i \ge 0$, $\alpha_i \in [0,1)$, for $i = \overline{0,n}$.

Theorem 2.3. *Let $\mu_i \geq 0$, $\alpha_i \in [0,1)$, and let $f_i \in \mathcal{B}(\mu_i, \alpha_i)$, for $i = \overline{0,n}$.*

Let $M_i, N_i \in \mathbb{R}_+$ and $g_i \in \mathcal{A}$, for $i = \overline{0,n}$, such that

$$M_i \geq 1, \tag{2.21}$$

for $i = \overline{0,n}$ and

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq N_i, \tag{2.22}$$

for $i = \overline{0,n}$ and for all $z \in \mathcal{U}$.

Let $\gamma \in \mathbb{C}$ and let $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) =: a > 0$.

Denote

$$c := \frac{2}{2a+1}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}. \tag{2.23}$$

(i) If

$$c\sum_{i=1}^{n}((2-\alpha_i)M_i^{\mu_i} + N_i) \leq 1, \tag{2.24}$$

then the integral operator $F_{n,\beta}$ given by relation (1.1) belongs to the class $\mathcal{S}$.

(ii) If

$$c|\gamma|((2-\alpha_0)M_0^{\mu_0} + N_0) \leq 1, \tag{2.25}$$

then the integral operator $G_{\beta,\gamma}(f_0, g_0)$ given by relation (1.2) belongs to the class $\mathcal{S}$.

Proof. In order to prove statement (i), we consider the function $h_n : \mathcal{U} \to \mathbb{C}$, given by relation (2.6).

We also recall the fact that, in the proof of Theorem 2.1, we have established relation (2.10).

Furthermore, if we now apply similar arguments to those in the proof of Theorem 2.2 and take into account the inequality (2.11), the hypotheses of Theorem 2.3, Lemma 1.7 (for f_i, $i = \overline{1,n}$), we obtain

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(n)}{h_n'(z)}\right| \leq c\sum_{i=1}^{n}((2-\alpha_i)M_i^{\mu_i} + N_i) \leq 1. \tag{2.26}$$

It remains now to apply Theorem 1.5 in order to show that, indeed, $F_{n,\beta} \in \mathcal{S}$.

Now, to prove statement (ii), we consider the function $h : \mathcal{U} \to \mathbb{C}$ given by relation (2.15). Using the same reasoning as in the proof of Theorem 2.1, one is able to show that

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(n)}{h_n'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}|z||\gamma|\left(|f_0'(z)| + \left|\frac{g_0''(z)}{g_0'(z)}\right|\right). \tag{2.27}$$

We proceed with the computations and we get

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(n)}{h_n'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}|z||\gamma| \cdot \\ \left(\left|f_0'(z)\left(\frac{z}{f_0(z)}\right)^{\mu} - 1\right|\left|\left(\frac{f_0(z)}{z}\right)^{\mu}\right| + \left|\left(\frac{f_0(z)}{z}\right)^{\mu}\right| + \left|\frac{g_0''(z)}{g_0'(z)}\right|\right).\end{aligned} \tag{2.28}$$

We make use now of the inequality (2.11), the hypotheses of Theorem 2.3 and Lemma 1.7 applied to the function f_0 and we get

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(n)}{h_n'(z)}\right| \leq c|\gamma|((2-\alpha_0)M_0^{\mu_0}+N_0) \leq 1. \tag{2.29}$$

Now, by Theorem 1.5, it follows that $G_{\beta,\gamma}(f_0,g_0) \in \mathcal{S}$.
This concludes our proof. □

We end this section by providing a theorem that gives different conditions for the univalence of the operators $F_{n,\beta}$ and $G_{\beta,\gamma}$ (see relations (1.1) and (1.2)), for $f_i \in \mathcal{B}(\mu_i,\alpha_i)$, $\mu_i \geq 0$, $\alpha_i \in [0,1)$, for $i=\overline{0,n}$.

Theorem 2.4. *Let $\mu_i \geq 0$, $\alpha_i \in [0,1)$, and let $f_i \in \mathcal{B}(\mu_i,\alpha_i)$, for $i=\overline{0,n}$.*
Let $M_i, N_i \in \mathbb{R}_+$ and $g_i \in \mathcal{A}$, for $i=\overline{0,n}$, such that

$$M_i \geq 1, \tag{2.30}$$

for $i=\overline{0,n}$ and

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq N_i, \tag{2.31}$$

for $i=\overline{0,n}$ and for all $z\in\mathcal{U}$.
Let $\gamma\in\mathbb{C}$ and let $\beta\in\mathbb{C}$ such that $\mathrm{Re}(\beta)>0$.
In addition, let $c\in\mathbb{C}$ such that $|c|\leq 1$, $c\neq -1$.

(i) If

$$\mathrm{Re}(\beta) \geq \sum_{i=1}^{n}\left((2-\alpha_i)M_i^{\mu_i}+N_i\right), \tag{2.32}$$

and

$$|c| \leq 1-\frac{1}{\mathrm{Re}(\beta)}\sum_{i=1}^{n}\left((2-\alpha_i)M_i^{\mu_i}+N_i\right) \tag{2.33}$$

then the integral operator $F_{n,\beta}$ given by relation (1.1) belongs to the class $\mathcal{S}$.

(ii) If

$$\mathrm{Re}(\beta) \geq |\gamma|\left((2-\alpha_0)M_0^{\mu_0}+N_0\right), \tag{2.34}$$

and

$$|c| \leq 1-\frac{|\gamma|}{\mathrm{Re}(\beta)}\left((2-\alpha_0)M_0^{\mu_0}+N_0\right) \tag{2.35}$$

then the integral operator $G_{\beta,\gamma}(f_0,g_0)$ given by relation (1.2) belongs to the class $\mathcal{S}$.

Proof. In order to prove statement (i), we use the function $h_n:\mathcal{U}\to\mathbb{C}$ given by relation (2.6). We start by recalling that we have shown that relation (2.9) holds.

For $c \in \mathbb{C}$, $|c| \leq 1$, $c \neq -1$, we have

$$\begin{aligned}
&\left|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh_n''(z)}{\beta h_n'(z)}\right| = \left|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{z}{\beta}\sum_{i=1}^{n}\left(f_i'(z) - \frac{g_i''(z)}{g_i'(z)}\right)\right| \\
&\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\left(|f_i'(z)| + \left|\frac{g_i''(z)}{g_i'(z)}\right|\right) \\
&\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i} - 1\right|\left|\left(\frac{f_i(z)}{z}\right)^{\mu_i}\right| + \left|\left(\frac{f_i(z)}{z}\right)^{\mu_i}\right| + \left|\frac{g_i''(z)}{g_i'(z)}\right|\right)
\end{aligned} \tag{2.36}$$

Now, we take into account the hypotheses of Theorem 2.4, Lemma 1.7 (applied to the functions f_i, for $i = \overline{1,n}$) and relation (2.36) becomes

$$\begin{aligned}
\left|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh_n''(z)}{\beta h_n'(z)}\right| &\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}((2-\alpha_i)M_i^{\mu_i} + N_i) \\
&\leq |c| + \frac{1}{\mathrm{Re}(\beta)}\sum_{i=1}^{n}((2-\alpha_i)M_i^{\mu_i} + N_i) \\
&\leq 1.
\end{aligned} \tag{2.37}$$

Now, we apply Theorem 1.6, which shows that $F_{n,\beta} \in \mathcal{S}$.

For the second part, namely statement (ii), one considers the function $h : \mathcal{U} \to \mathbb{C}$ given by relation (2.15) and uses similar arguments to those in the proof of statement (i) in order to show that $G_{\beta,\gamma}(f_0, g_0) \in \mathcal{S}$. The proof is omitted for the sake of brevity.

This concludes the proof of our result. □

3. Conclusion

In this work, we have considered two new general integral operators. We have employed established univalence criteria to determine conditions under which our new integral operators are univalent. These univalence conditions were obtained for different choices of the functions that are involved in the definitions of the considered integral operators.

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On implicit φ-Hilfer fractional differential equations with the p-Laplacian operator

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Abstract. In this paper, we establish the existence and uniqueness of solutions for a new class of nonlocal boundary implicit φ-Hilfer fractional differential equations involving the p-Laplacian operator. The existence results are derived using the topological degree method for condensing maps and the Banach contraction principle. Moreover, we investigate the Ulam-Hyers and generalized Ulam-Hyers stability of our main problem. To illustrate the applicability of our theoretical results, we provide an example.

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1. Introduction

Fractional differential equations have gained significant importance due to their ability to accurately model a wide range of physical phenomena across various fields, including chemistry, physics, biology, engineering, viscoelasticity, electrical engineering, signal processing, electrochemistry, and controllability (see [21, 22]). The motivation for introducing new fractional derivatives is twofold: first, to capture certain dynamic behaviors of physical systems that are not well represented by existing fractional derivatives, and second, to preserve key properties of the standard derivative. Numerous types of fractional derivatives have been proposed, such as the Riemann-Liouville, Caputo, Hadamard, Hilfer, and Katugampola derivatives [1, 11, 13, 17], each offering unique advantages in capturing specific phenomenological behaviors.

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Recently, Sousa and Oliveira [18] introduced a new class of fractional derivatives known as the φ-Hilfer fractional derivative. This derivative generalizes several well-known fractional derivatives through appropriate choices of the function φ. For instance, it reduces to the Hilfer fractional derivative ${}^{H}\mathfrak{D}^{\alpha,\beta}(\theta)$ when $\varphi(\theta)=\theta$; the Caputo fractional derivative ${}^{C}\mathcal{D}^{\alpha}(\theta)$ and the Riemann-Liouville fractional derivative ${}^{RL}\mathcal{D}^{\alpha}(\theta)$ when $\varphi(\theta)=\theta$ with $\beta\to 1$ and $\beta\to 0$, respectively; the Hadamard fractional derivative ${}^{H}\mathcal{D}^{\alpha}(\theta)$ when $\varphi(\theta)=\ln(\theta)$ and $\beta\to 0$; and the Katugampola fractional derivative ${}^{\rho}\mathcal{D}^{\alpha}(\theta)$ when $\varphi(\theta)=\theta^{\rho}$ and $\beta\to 0$, while taking the φ-Caputo fractional derivative ${}^{C}\mathcal{D}^{\alpha;\varphi}(\theta)$ when $\beta\to 1$ [1, 6, 7, 8, 11, 13, 17]. Thus, this provides a flexible framework for modeling different physical phenomena depending on the choice of φ. Due to this versatility, it has attracted considerable attention and has been extensively studied by various researchers (see [2, 3, 5, 19, 20]).

In [15], Mali et al. investigated the existence and Ulam-Hyers stability of implicit φ-Hilfer fractional differential equations of the form

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}z(\theta)=h(\theta,z(\theta),{}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}z(\theta)), & \theta\in(a,b] \\ z(a)=0, \quad z(b)=\displaystyle\sum_{i=1}^{m}a_i\mathfrak{I}^{\nu_i,\varphi}_{0^+}z(\xi_i), \end{cases}$$

where ${}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}$ is the φ-Hilfer fractional derivative of order $\alpha\in(1,2)$ with type $\beta\in[0,1]$, $\xi_i\in(a,b]$, $a_i\in\mathbb{R}$, $\mathfrak{I}^{\nu_i,\varphi}_{0^+}$ is the φ-Riemann-Liouville fractional integral of order ν_i, and $h:(a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous function.

In [4], Alsaedi et al. established the existence and uniqueness of solutions for the following implicit φ-Hilfer differential equation involving the p-Laplacian operator

$$\begin{cases} \left(\Phi_p{}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}z(\theta)\right)'+h(\theta,z(\theta))=0, & \theta\in[0,1], \\ z(0)=0, \quad {}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}z(0)=0, \quad z(T)=\displaystyle\sum_{i=1}^{m}a_i\mathfrak{I}^{\nu_i,\varphi}_{0^+}z(\xi_i), \end{cases}$$

where ${}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}$ is the $\varphi-$Hilfer fractional derivative of orders $\alpha\in(1,2]$ with type $\beta\in[0,1]$, $\xi_i\in(0,1)$, $a_i\in\mathbb{R}$, Φ_p denotes the p-Laplacian operator such that $\Phi_p(\zeta)=|\zeta|^{p-2}\zeta$, $p>1$ and $h:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous function.

Motivated by the aforementioned works, we establish the existence and stability of solutions to the following implicit φ-Hilfer fractional differential equation

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}\Phi_p{}^{H}\mathfrak{D}^{\alpha',\beta',\varphi}_{0^+}z(\theta)=h\Big(\theta,z(\theta),{}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}\Phi_p{}^{H}\mathfrak{D}^{\alpha',\beta',\varphi}_{0^+}z(\theta)\Big), & \theta\in\Delta \\ z(0)=0, \quad {}^{H}\mathfrak{D}^{\alpha',\beta',\varphi}_{0^+}z(0)=0, \quad z(T)=\displaystyle\sum_{i=1}^{m}a_i\mathfrak{I}^{\nu_i,\varphi}_{0^+}z(\xi_i), \end{cases} \tag{1.1}$$

where $\Delta=[0,T]$ and ${}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}$ and ${}^{H}\mathfrak{D}^{\alpha',\beta',\varphi}_{0^+}$ are the φ-Hilfer fractional derivatives of orders $\alpha\in(0,1)$ with types $\beta\in[0,1]$, $\alpha'\in(1,2)$ with types $\beta,\beta'\in[0,1]$ respectively, $\xi_i\in(0,T)$, $a_i\in\mathbb{R}$ and $h:\Delta\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function such that

($\mathcal{A}_1$) There exist constants $\kappa_1>0$ and $0<\kappa_2<1$ such that

$$|h(\theta,z_1,\eta_1)-h(\theta,z_1,\eta_2)|\le\kappa_1\mid z_1-z_2\mid+\kappa_2|\eta_1-\eta_2|,$$

for any z_1, η_1, z_2, $\eta_2 \in \mathbb{R}$ and $\theta \in \Delta$.

$(\mathcal{A}_2)$ There exist constants $\tau_1, \tau_2, \tau_3 > 0$ with $0 < \tau_2 < 1$ such that

$$|h(\theta, z, \eta)| \leq \tau_1|z| + \tau_2|\eta| + \tau_3,$$

for any z, $\eta \in \mathbb{R}$ and $\theta \in \Delta$.

This paper is organized as follows: In Section 2, we introduce the necessary definitions and results required for proving the existence theorems. In Section 3 we investgate the existence and uniqueness of solutions for our main problem. In Section 4, we analyze the Ulam-Hyers-type stability results. Finally, we provide an example to validate the obtained results.

2. Preliminaries

Let $C(\Delta, \mathbb{R})$ be the Banach space of all continuous functions $z : \Delta \to \mathbb{R}$ equipped with the norm $\|z\| = \sup_{\theta\in\Delta} |z(\theta)|$, and let $B_r(0)$ denote the closed ball centered at 0 with radius r. We introduce the space

$$\mathcal{A}l^n(\Delta) = \{\varphi \in C^n(\Delta, \mathbb{R}) \text{ such that } \varphi'(\theta) > 0 \text{ for all } \theta \in \Delta\}.$$

Definition 2.1. [18] *Let $\alpha > 0$, $h : \Delta \longrightarrow \mathbb{R}$ an integrable function and $\varphi \in \mathcal{A}l^1(\Delta)$. Then the φ-Riemann-Liouville fractional integral of order α of the function h is given by*

$$\mathfrak{I}_{0^+}^{\alpha,\varphi} h(\theta) = \frac{1}{\Gamma(\alpha)} \int_0^\theta \varphi'(s)\Omega_\varphi^{\alpha-1}(\theta, s)h(s)ds,$$

where $\Omega_\varphi^\alpha(\theta, s) = (\varphi(\theta) - \varphi(s))^\alpha$ and $\Gamma(.)$ is the Gamma function.

Definition 2.2. [18] *Let $\alpha > 0$, $h \in C^n(\Delta, \mathbb{R})$ and $\varphi \in \mathcal{A}l^n(\Delta)$. Then the φ-hilfer fractional derivative of order α and type $\beta \in [0, 1]$ of the function h is given by*

$$^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi} h(\theta) = \mathfrak{I}_{0^+}^{\beta(n-\alpha),\varphi} \left(\frac{1}{\varphi'(\theta)}\frac{d}{d\theta}\right)^n \mathfrak{I}_{0^+}^{(1-\beta)(n-\alpha),\varphi} h(\theta),$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the part integer of α.

Proposition 2.3. [18] *Let $h \in C^n(\Delta, \mathbb{R})$, then the following holds*

1) $^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi}\mathfrak{I}_{0^+}^{\alpha,\varphi} h(\theta) = h(\theta)$

2) $\mathfrak{I}_{0^+}^{\alpha,\varphi}\ {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi} h(\theta) = h(\theta) - \sum_{i=1}^{n} \frac{\Omega_\varphi^{\gamma-1}(\theta, 0)}{\Gamma(\gamma - i + 1)} h_\varphi^{[n-i]}\mathfrak{I}_{0^+}^{(1-\beta)(n-\alpha),\varphi} h(0),$

where $h_\varphi^{[n]} h(\theta) = \left(\frac{1}{\varphi'(\theta)}\frac{d}{d\theta}\right)^n h(\theta)$ *and* $\gamma = \alpha + \beta(n - \alpha)$.

Lemma 2.4. [14] *Let Φ_p be the p-Laplacian operator. Then*

1. *If $1 < p < 2$, $z_1 z_2 > 0$ and $|z_1|, |z_2| \geq m > 0$, we have*

$$|\Phi_p(z_1) - \Phi_p(z_2)| \leq (p - 1)m^{p-2}|z_1 - z_2|.$$

2. *If $p > 2$, $z_1 z_2 > 0$ and $|z_1|, |z_2| \leq M$, we have*

$$|\Phi_p(z_1) - \Phi_p(z_2)| \leq (p - 1)M^{p-2}|z_1 - z_2|.$$

3. Φ_p *is invertible with* $\Phi_p^{-1}(z) = \Phi_{p'}(z)$, *such that* $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2.5. [9] *Let E be a Banach space and O a bounded subset of E. The Kuratowski measure of noncompactness is the map μ: $O \longrightarrow \mathbb{R}_+$ given by*

$$\mu(O) = \inf\{\rho > 0 : \ O \subseteq \bigcup_{i=1}^{n} O_i \ \textit{and} \ diam(O_i) \leq \rho\}.$$

Proposition 2.6. [9] *Let O, O_1, O_2 be bounded subsets of E. Then the Kuratowski measure of noncompactness μ satisfies the following*

1. *O is relatively compact* $\Leftrightarrow \mu(O) = 0$
2. $\mu(\kappa O) = |\kappa|\mu(O), \quad \kappa \in \mathbb{R}$.
3. $\mu(O_1 + O_2) \leq \mu(O_1) + \mu(O_2)$.
4. $O_1 \subset O_2 \Rightarrow \mu(O_1) \leq \mu(O_2)$.
5. $\mu(O_1 \cup O_2) = \max\{\mu(O_1), \ \mu(O_2)\}$.
6. $\mu(O) = \mu(\overline{O}) = \mu(convO)$ *where $\overline{O}$ and $convO$ represent the closure and the convex hull of the set O, respectively.*

Definition 2.7. [9] *Let $\mathcal{F} : \mathcal{W} \subset E \to E$ be a continuous bounded map. Then $\mathcal{F}$ is called*

1. *μ-Lipschitz if there exists $\kappa \geq 0$ such that*
$$\mu(\mathcal{F}(O)) \leq \kappa\mu(O) \quad \textit{for all } O \subset \mathcal{W} \textit{ bounded.}$$
Furthermore, if $\kappa < 1$ it is called a strict μ-contraction.
2. *μ-condensing if*
$$\mu(\mathcal{F}(O)) < \mu(O) \quad \textit{for all } \ O \subset \mathcal{W} \ \textit{bounded with} \ \mu(O) > 0.$$

Definition 2.8. [9] *Let $\mathcal{Q} : \mathcal{W} \subset E \to E$. The map $\mathcal{F}$ is called Lipschitz if there exists a constant $\kappa \geq 0$ such that*

$$\|\mathcal{F}w_1 - \mathcal{F}w_2\| \leq \kappa\|w_1 - w_2\|, \quad \textit{for all } w_1, w_2 \in \mathcal{W}.$$

Furthermore, $\mathcal{F}$ is called a strict contraction if $\kappa < 1$.

Lemma 2.9. [9] *If $\mathcal{F} : \mathcal{W} \subset E \to E$ is Lipschitz having constant κ, then $\mathcal{F}$ is μ-Lipschitz having the constant κ.*

Lemma 2.10. [9] *If $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{W} \subset E \to E$ are μ-Lipschitz maps having constants κ_1, κ_2 respectively, then $\mathcal{F}_1 + \mathcal{F}_2 : \mathcal{W} \to E$ is μ-Lipschitz map having constant $\kappa_1 + \kappa_2$.*

Lemma 2.11. [9] *If $\mathcal{F} : \mathcal{W} \subset E \to E$ is compact, then $\mathcal{F}$ is μ- Lipschitz having constant $\kappa = 0$.*

Theorem 2.12. [12] *Let $\mathcal{F} : E \to E$ be μ-condensing, consider the set*

$$\mathbb{S}_\epsilon = \big\{z \in E : \ \textit{there exist} \ 0 \leq \epsilon \leq 1 \ \textit{such that} \ z = \epsilon\mathcal{F}(z)\big\}.$$

If $\mathbb{S}_\epsilon$ is a bounded set in E, then there exists $r > 0$ such that $\mathbb{S}_\epsilon \subset B_r(0)$, then

$$deg(I_d - \epsilon\mathcal{F}, B_r(0), 0) = 1, \quad \textit{for all} \ \epsilon \in [0, 1].$$

Consequently, the operator $\mathcal{F}$ has at least one fixed point and the set of the fixed points of $\mathcal{F}$ lies in $B_r(0)$.

3. Existence results

In this section under certain hypotheses we investigate the existence and uniqueness of solutions for the problem (1.1). To simplify, we introduce the following notation.

$$\begin{aligned}
h_z(\theta) &= h(\theta, z(\theta), \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\Phi_p \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi} z(\theta)), \quad \gamma' = \alpha' + \beta'(2-\alpha'),\\
\mathbb{A}_{h_z} &= \frac{\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(T) - \mathfrak{I}_{0^+}^{\alpha'+\nu_i,\beta',\varphi}\Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\xi_i)}{\varpi\Gamma(\gamma')},\\
\Lambda_1 &= \frac{(p'-1)M^{p'-2}\Omega_\varphi^{\gamma'-1}(T,0)}{|\varpi|\Gamma(\gamma')}\Big(\frac{\Omega_\varphi^{\alpha'+\alpha}(T,0)}{\Gamma(\alpha'+\alpha+1)} + \sum_{i=1}^m |a_i| \frac{\Omega_\varphi^{\nu_i+\alpha'+\alpha}(\xi_i,0)}{\Gamma(\nu_i+\alpha'+\alpha+1)}\Big),\\
\Lambda_2 &= (p'-1)M^{p'-2}\frac{\Omega_\varphi^{\alpha'+\alpha-1}(T,0)}{\Gamma(\alpha'+\alpha)}, \ \varpi = \sum_{i=1}^m \frac{a_i\Omega_\varphi^{\gamma'+\nu_i-1}(\xi_i,0)}{\Gamma(\gamma'+\nu_i)} - \frac{\Omega_\varphi^{\gamma'-1}(T,0)}{\Gamma(\gamma')}.
\end{aligned}$$

Lemma 3.1. *A function $z \in C^2(\Delta,\mathbb{R})$ is a solution of problem* (1.1) *if and only if it satisfies the following fractional integral equation*

$$z(\theta) = \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\theta) + \mathbb{A}_{h_z}\Omega_\varphi^{\gamma'-1}(\theta,0). \tag{3.1}$$

Proof. Let $z \in C^2(\Delta,\mathbb{R})$ be a solution of the problem (1.1). By applying fractional integral $\mathfrak{I}_{0^+}^{\alpha,\varphi}$ to both sides of the first equation in (1.1) and Proposition 2.3, we obtain

$$\Phi_p\big({}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi} z(\theta)\big) = d_0\Omega_\varphi^{\gamma-1}(\theta,0) + \mathfrak{I}_{0^+}^{\alpha,\varphi} h_z(\theta),$$

where

$$d_0 \in \mathbb{R}.$$

Since

$${}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi} z(0) = 0$$

then

$$d_0 = 0,$$

it follows that

$$\Phi_p\big({}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi} z(\theta)\big) = \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\theta). \tag{3.2}$$

By applying the operator $\Phi_{p'}$ the inverse operator of Φ_p on both sides of (3.2), we get

$${}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi} z(\theta) = \Phi_{p'}\left(\mathfrak{I}_{0^+}^{\alpha,\varphi} h_z(\theta)\right). \tag{3.3}$$

Applying the operator $\mathfrak{I}_{0^+}^{\alpha',\varphi}$ on both sides of (3.3), we obtain

$$z(\theta) = d_1\Omega_\varphi^{\gamma'-1}(\theta,0) + d_2\Omega_\varphi^{\gamma'-2}(\theta,0) + \mathfrak{I}_{0^+}^{\alpha',\varphi}\Phi_{p'}\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\theta), \text{ where } d_1,\ d_2 \in \mathbb{R}.$$

Using the condition $z(0) = 0$, we conclude that $d_2 = 0$. It follows that

$$z(\theta) = d_1\Omega_\varphi^{\gamma'-1}(\theta,0) + \mathfrak{I}_{0^+}^{\alpha',\varphi}\ \Phi_{p'}\mathfrak{I}_{0^+}^{\alpha,\varphi} h_z(\theta). \tag{3.4}$$

By the condition $z(T)=\sum_{i=1}^{m}a_i\mathfrak{I}_{0^+}^{\nu_i,\varphi}z(\xi_i)$, we get

$$d_1=\frac{\mathfrak{I}_{0^+}^{\alpha',\varphi}\Phi_{p'}\mathfrak{I}_{0^+}^{\alpha,\varphi}h_z(T)-\sum_{i=1}^{m}a_i\mathfrak{I}_{0^+}^{\nu_i+\alpha',\varphi}\Phi_{p'}\mathfrak{I}_{0^+}^{\alpha,\varphi}h_z(\xi_i)}{\Gamma(\gamma')\sum_{i=1}^{m}\frac{a_i\Omega_{\varphi}^{\gamma'+\nu_i-1}(\xi_i,0)}{\Gamma(\gamma'+\nu_i)}-\Omega_{\varphi}^{\gamma'-1}(T,0)}.$$

By substituting d_1 in (3.4), we get the integral equation (3.1).

Conversely, a direct computation shows that if z satisfies the integral equation (3.1), the φ-Hilfer problem (1.1) holds, completing the proof. □

We define the operators $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}:C^2(\Delta,\mathbb{R})\to C^2(\Delta,\mathbb{R})$ as follows

$$\mathcal{T}_1z(\theta)=\mathbb{A}_{h_z}\Omega_{\varphi}^{\gamma'-1}(\theta,0),$$
$$\mathcal{T}_2z(\theta)=\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_z(\theta),$$

and

$$\mathcal{T}z(\theta)=\mathcal{T}_1z(\theta)+\mathcal{T}_2z(\theta). \tag{3.5}$$

From (3.5), we can deduce that the problem (1.1) is equivalent to the following operator equation

$$\mathcal{T}z(\theta)=z(\theta),\quad \theta\in\Delta. \tag{3.6}$$

Hence, the problem (1.1) has a solution if only if $\mathcal{T}$ has a fixed point.
For this purpose we shall demonstrate that $\mathcal{T}$ satisfies all the conditions given in Theorem 2.12.

Lemma 3.2. *The operator $\mathcal{T}_1$ is Lipschitz with constant $\frac{\kappa_1\Lambda_1}{1-\kappa_2}$. Furthermore, satisfies the following growth condition*

$$\|\mathcal{T}_1z\|\le\Lambda_1\Big(\frac{\tau_1}{1-\tau_2}\|z\|+\frac{\tau_3}{1-\tau_2}\Big),\quad \text{for all } z\in C^2(\Delta,\ \mathbb{R}). \tag{3.7}$$

Proof. Let $z_1,z_2\in C^2(\Delta,\mathbb{R})$, then

$$|\mathcal{T}_1z_1(\theta)-\mathcal{T}_1z_2(\theta)|=\big|\Omega_{\varphi}^{\gamma'-1}(\theta,0)(\mathbb{A}_{h_{z_1}}-\mathbb{A}_{h_{z_2}})\big|\le\Omega_{\varphi}^{\gamma'-1}(T,0)|\mathbb{A}_{h_{z_1}}-\mathbb{A}_{h_{z_2}}|.$$

By using Lemma 3.5, we have

$$\begin{aligned}|\mathbb{A}_{h_{z_1}}-\mathbb{A}_{h_{z_2}}|&\le\frac{1}{|\varpi|\Gamma(\gamma')}\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Big|\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{z_1}(T)-\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{z_2}(T)\Big|\\&\quad+\frac{1}{|\varpi|\Gamma(\gamma')}\mathfrak{I}_{0^+}^{\alpha'+\nu_i,\beta',\varphi}\big|\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{z_1}(\xi_i)-\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{z_2}(\xi_i)\big|\\&\le\frac{(p'-1)M^{p'-2}}{|\varpi|\Gamma(\gamma')}\mathfrak{I}_{0^+}^{\alpha'+\alpha,\beta',\varphi}|h_{z_1}(T)-h_{z_2}(T)|\\&\quad+\frac{(p'-1)M^{p'-2}}{|\varpi|\Gamma(\gamma')}\mathfrak{I}_{0^+}^{\alpha'+\alpha+\nu_i,\beta',\varphi}|h_{z_1}(\xi_i)-h_{z_2}(\xi_i)|.\end{aligned}$$

By using the assumption $(\mathcal{A}_1)$, we obtain

$$|h_{z_1}(\theta) - h_{z_2}(\theta)| \leq \frac{\kappa_1}{1-\kappa_2}|z_1 - z_2|, \quad \text{for all } \theta \in \Delta.$$

Thus, we get

$$|\mathcal{T}_1 z_1(\theta) - \mathcal{T}_1 z_2(\theta)| \leq \frac{\kappa_1 (p'-1) M^{p'-2} \Omega_\varphi^{\gamma'-1}(T,0)}{(1-\kappa_2)|\varpi|\Gamma(\gamma')} \times \Big(\frac{\Omega_\varphi^{\alpha'+\alpha}(T,0)}{\Gamma(\alpha'+\alpha+1)} + \sum_{i=1}^{m} |a_i| \frac{\Omega_\varphi^{\nu_i+\alpha'+\alpha}(\xi_i,0)}{\Gamma(\nu_i+\alpha'+\alpha+1)} \Big) \|z_1 - z_2\|.$$

Taking supremum over θ we get

$$\|\mathcal{T}_1 z_1 - \mathcal{T}_1 z_2\| \leq \frac{\kappa_1 \Lambda_1}{1-\kappa_2} \|z_1 - z_2\|.$$

Thus, $\mathcal{T}_1$ is Lipschitz having constant $\dfrac{\kappa_1 \Lambda_1}{1-\kappa_2}$.
To demonstrate the growth condition (3.7), we proceed as follows:

$$\begin{aligned} |\mathcal{T}_1 z(\theta)| &\leq \Omega_\varphi^{\gamma'-1}(T,0) |\mathbb{A}_{h_z}| \\ &\leq \frac{\Omega_\varphi^{\gamma'-1}(T,0)}{|\varpi|\Gamma(\gamma')} \big| \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi} \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(T) - \mathfrak{I}_{0^+}^{\alpha'+\nu_i,\beta',\varphi} \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\xi_i) \big|. \end{aligned}$$

By Lemma (3.5), we get

$$|\mathcal{T}_1 z(\theta)| \leq \frac{(p'-1) M^{p'-2} \Omega_\varphi^{\gamma'-1}(T,0)}{|\varpi|\Gamma(\gamma')} \times \Big(\mathfrak{I}_{0^+}^{\alpha'+\alpha,\beta',\varphi} |h_z(T)| + \mathfrak{I}_{0^+}^{\alpha'+\alpha+\nu_i,\beta',\varphi} |h_z(\xi_i)| \Big).$$

By using $(\mathcal{A}_2)$, we have

$$|h_z(\theta)| \leq \frac{1}{1-\tau_2}(\tau_1 |z| + \tau_3), \quad \theta \in \Delta.$$

Thus, we obtain

$$|\mathcal{T}_1 z(\theta)| \leq \frac{(p'-1) M^{p'-2} \Omega_\varphi^{\gamma'-1}(T,0)}{|\varpi|\Gamma(\gamma')} \times \Big(\mathfrak{I}_{0^+}^{\alpha'+\alpha,\beta',\varphi} \Big(\frac{\tau_1 \|z\| + \tau_3}{1-\tau_2} \Big) + \mathfrak{I}_{0^+}^{\alpha'+\alpha+\nu_i,\beta',\varphi} \Big(\frac{\tau_1 \|z\| + \tau_3}{1-\tau_2} \Big) \Big).$$

Taking supremum over θ we get

$$\|\mathcal{T}_1 z\| \leq \Lambda_1 \Big(\frac{\tau_1}{1-\tau_2} \|z\| + \frac{\tau_3}{1-\tau_2} \Big).$$

□

Lemma 3.3. *The operator $\mathcal{T}_2$ is continuous. Furthermore, $\mathcal{T}_2$ satisfies the following growth condition*

$$\|\mathcal{T}_2 z\| \leq \Lambda_2 (\tau_1 \|z\| + \tau_3), \quad \textit{for all} \ \ z \in C^2(\Delta, \mathbb{R}). \tag{3.8}$$

Proof. Let $z_n \in C^2(\Delta, \mathbb{R})$ a sequence that converges to $z \in C^2(\Delta, \mathbb{R})$ for each $t \in \Delta$. Then we have

$$\begin{aligned}|\mathcal{T}_2 z(\theta) - \mathcal{T}_2 z_n(\theta)| &= \left|\mathfrak{I}_{0+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0+}^{\alpha,\beta,\varphi}h_z(\theta) - \mathfrak{I}_{0+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0+}^{\alpha,\beta,\varphi}h_{z_n}(\theta)\right| \\ &\leq \mathfrak{I}_{0+}^{\alpha',\beta',\varphi}\left|\Phi_q\mathfrak{I}_{0+}^{\alpha,\beta,\varphi}h_z(\theta) - \Phi_q\mathfrak{I}_{0+}^{\alpha,\beta,\varphi}h_{z_n}(\theta)\right|.\end{aligned}$$

By Lemma 3.5 and $(\mathcal{A}_1)$, we get

$$\begin{aligned}|\mathcal{T}_2 z(\theta) - \mathcal{T}_2 z_n(\theta)| &\leq (p'-1)M^{p'-2}\mathfrak{I}_{0+}^{\alpha'+\alpha,\beta',\varphi}|h_z(\theta) - h_{z_n}(\theta)| \\ &\leq (p'-1)M^{p'-2}\frac{\kappa_1}{1-\kappa_2}\Omega_\varphi^{\alpha'+\alpha-1}(T,0)\|z - z_n\|.\end{aligned}$$

By taking supremum over θ, we obtain

$$\|\mathcal{T}_2 z - \mathcal{T}_2 z_n\| \leq \frac{\kappa_1}{1-\kappa_2}\Lambda_2\|z - z_n\|,$$

it follows that

$$\|\mathcal{T}_2 z - \mathcal{T}_2 z_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $\mathcal{T}_2$ is a continuous.
Now let us demonstrate (3.8). By using Lemma 3.5, we get

$$\begin{aligned}|\mathcal{T}_2 z(\theta)| &= \left|\mathfrak{I}_{0+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0+}^{\alpha,\beta,\varphi}h_z(\theta)\right| \leq \mathfrak{I}_{0+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0+}^{\alpha,\beta,\varphi}|h_z(\theta)| \\ &\leq (p'-1)M^{p'-2}\mathfrak{I}_{0+}^{\alpha',\beta',\varphi}\mathfrak{I}_{0+}^{\alpha,\beta,\varphi}|h_z(\theta)|.\end{aligned}$$

By using $(\mathcal{A}_2)$, we get

$$|\mathcal{T}_2 z(\theta)| \leq (p'-1)M^{p'-2}\frac{\Omega_\varphi^{\alpha'+\alpha-1}(T,0)}{\Gamma(\alpha'+\alpha)}\Big(\frac{\tau_1}{1-\tau_2}\|z\| + \frac{\tau_3}{1-\tau_2}\Big).$$

Hence

$$\|\mathcal{T}_2 z\| \leq \Lambda_2\Big(\frac{\tau_1}{1-\tau_2}\|z\| + \frac{\tau_3}{1-\tau_2}\Big).$$

□

Lemma 3.4. *The operator $\mathcal{T}_2$ is a compact. Consequently $\mathcal{T}_2$ is μ-Lipschitz with zero constant.*

Proof. Let $z \in B_r$, then by (3.8) we get

$$\|\mathcal{T}_2 z\| \leq \frac{\Lambda_2}{1-\tau_2}(\tau_1 r + \tau_3).$$

It follows that $\mathcal{T}_2(B_r)$ is uniformly bounded.
Let $\theta_1, \theta_2 \in \Delta$ such that $0 < \theta_1 < \theta_2 < T$. Then by Lemma 3.5 and $(\mathcal{A}_2)$, we get

$$\begin{aligned}
|\mathcal{T}_2 z(\theta_2) - \mathcal{T}_2 z(\theta_1)| \leq & \Big| \int_{\theta_1}^{\theta_2} \frac{\varphi'(s)\Omega_{\varphi}^{\alpha'-1}(\theta_2, s)}{\Gamma(\alpha')} \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} \frac{(\tau_1 r + \tau_3)}{(1-\tau_2)} ds \Big| \\
& + \Big| \int_0^{\theta_1} \frac{\varphi'(s)(\Omega_{\varphi}^{\alpha'-1}(\theta_2, s) - \Omega_{\varphi}^{\alpha'-1}(\theta_1, s))}{\Gamma(\alpha')} \\
& \times \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} \frac{(\tau_1 r + \tau_3)}{(1-\tau_2)} ds \Big| \\
\leq & \frac{(p'-1)M^{p'-2}(\tau_1 r + \tau_3)}{(1-\tau_2)\Gamma(\alpha'+\alpha+1)} \\
& \times \Big(\Omega_{\varphi}^{\alpha'+\alpha}(\theta_2, 0) - \Omega_{\varphi}^{\alpha'+\alpha}(\theta_2, \theta_1) + \Omega_{\varphi}^{\alpha'+\alpha}(\theta_1, 0) \Big).
\end{aligned}$$

Thus

$$|\mathcal{T}_2 z(\theta_1) - \mathcal{T}_2 z(\theta_2)| \to 0 \quad \text{as} \quad \theta_1 \to \theta_2,$$

which implies that the set $\mathcal{T}_2(B_r)$ is equicontinuous. By applying the Arzelà–Ascoli Theorem [10], $\mathcal{T}_2$ is compact. Consequently, by Lemma 2.11, the operator $\mathcal{T}_2$ is μ-Lipschitz having constant zero. □

Theorem 3.5. *Assume that* $(\mathcal{A}_1) - (\mathcal{A}_2)$ *hold, if*

$$\frac{\kappa_1 \Lambda_1}{1-\kappa_2} < 1. \tag{3.9}$$

Then the problem (1.1) *has at least one solution in* $C^2(\Delta, \mathbb{R})$. *Moreover, the set of solutions is bounded in* $C^2(\Delta, \mathbb{R})$.

Proof. The operators $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}$ are bounded and continuous. Moreover, $\mathcal{T}_1$ is μ-Lipschitz having constant $\frac{\kappa_1 \Lambda_1}{1-\kappa_2}$ and $\mathcal{T}_2$ is μ-Lipschitz having zero constant. By the condition 3.9 and both Lemma 2.10 and Lemma 2.9, we deduce that $\mathcal{T}$ is μ-condensing.
Now, we consider the following set

$$\mathbb{S}_{\varepsilon} = \big\{ z \in C^2(\Delta, \mathbb{R}) : \text{ there exists } \varepsilon \in [0,1] \text{ such that } z = \varepsilon \mathcal{T} z \big\}.$$

Let $z \in \mathbb{S}_{\epsilon}$, then $z = \varepsilon \mathcal{T} z = \varepsilon(\mathcal{T}_1 z + \mathcal{T}_2 z)$, it follows

$$\begin{aligned}
\|z\| = & \leq \varepsilon(\|\mathcal{T}_1 z\| + \|\mathcal{T}_2 z\|) \\
& \leq \frac{\Lambda_1 + \Lambda_2}{1-\tau_2}(\tau_1 \|z\| + \tau_3).
\end{aligned}$$

Hence, the set $\mathbb{S}_{\varepsilon}$ is bounded. Since all conditions of Theorem 2.12 hold. Then the operator $\mathcal{T}$ has at least one fixed point, which represents a solution to the problem (1.1). □

Theorem 3.6. *Assume that* $(\mathcal{A}_1) - (\mathcal{A}_2)$ *hold, if*

$$\frac{\kappa_1(\Lambda_1 + \Lambda_2)}{1-\kappa_2} < 1. \tag{3.10}$$

Then the problem (1.1) has a unique solution $z \in C^2(\Delta, \mathbb{R})$.

Proof. Let $z_1, z_2 \in C^2(\Delta, \mathbb{R})$, then by $(\mathcal{A}_1)$ we have

$$\begin{aligned}
|\mathcal{T}_2 z_1(\theta) - \mathcal{T}_2 z_2(\theta)| &= \Big| \int_0^\theta \frac{\varphi'(s)\Omega_\varphi^{\alpha'-1}(\theta,s)}{\Gamma(\alpha')} \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}(h_{z_1}(s) - h_{z_2}(s))ds \Big| \\
&\leq (p'-1)M^{p'-2} \int_0^\theta \frac{\varphi'(s)\Omega_\varphi^{\alpha'-1}(\theta,s)}{\Gamma(\alpha')} \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} |h_{z_1}(s) - h_{z_2}(s)|ds \\
&\leq (p'-1)M^{p'-2} \int_0^T \frac{\varphi'(s)\Omega_\varphi^{\alpha'-1}(T,s)}{\Gamma(\alpha')} \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} \frac{\kappa_1 \|z_1 - z_2\|}{1-\kappa_2} ds \\
&\leq \frac{\kappa_1}{1-\kappa_2} \Lambda_2 \|z_1 - z_2\|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\mathcal{T} z_1 - \mathcal{T} z_2| &\leq |\mathcal{T}_1 z_1 - \mathcal{T}_1 z_2| + |\mathcal{T}_2 z_1 - \mathcal{T}_2 z_2| \\
&\leq \frac{\kappa_1}{1-\kappa_2} \Lambda_1 \|z_1 - z_2\| + \frac{\kappa_1}{1-\kappa_2} \Lambda_2 \|z_1 - z_2\|.
\end{aligned}$$

It follows

$$\|\mathcal{T} z_1 - \mathcal{T} z_2\| \leq \frac{\kappa_1(\Lambda_1 + \Lambda_2)}{1-\kappa_2} \|z_1 - z_2\|.$$

Thus, $\mathcal{T}$ is a contraction. By Banach's contraction principle, we deduce that $\mathcal{T}$ has a unique fixed point which corresponds to the unique solution of the problem (1.1). □

4. Ulam Stability Results

In this section, we investigate the Ulam-Hyers ($\mathbb{UH}$) and generalized Ulam-Hyers ($\mathbb{GUH}$) stability of our problem (1.1) under certain conditions.

Definition 4.1. [16] *The problem* (1.1) *is said to be Ulam-Hyers stable if there is a constant $c_h > 0$ such that for each $\delta > 0$ and each solution $\bar{z} \in C^2(\Delta, \mathbb{R})$ of the inequality*

$$\left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi} \Phi_p\left({}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi} \bar{z}(\theta)\right) - h_{\bar{z}}(\theta)\right| \leq \delta, \quad \theta \in \Delta, \tag{4.1}$$

there exists a solution $z \in C^2(\Delta, \mathbb{R})$ of problem (1.1) *with*

$$|\bar{z}(\theta) - z(\theta)| \leq c_h \delta, \quad \theta \in \Delta.$$

Definition 4.2. [16] *The problem* (1.1) *is said to be generalized Ulam-Hyers stable if there exists a function $\Psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Psi(0) = 0$ such that for each solution $\bar{z} \in C^2(\Delta, \mathbb{R})$ of the inequality (4.1), there exists a solution $z \in C^2(\Delta, \mathbb{R})$ of problem* (1.1) *with*

$$|\bar{z}(\theta) - z(\theta)| \leq \Psi(\delta), \quad \theta \in \Delta.$$

Remark 4.3. Definition 4.1 ⇒ Definition 4.2.

Remark 4.4. A function $\bar{z} \in C^2(\Delta, \mathbb{R})$ is a solution of the inequality (4.1) if and only if there exists a function $\Psi_1 \in C(\Delta, \mathbb{R})$ such that

1. $|\Psi_1(\theta)| \leq \delta, \quad$ for all $\theta \in \Delta$

2. ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi}\Phi_p\big({}^{H}\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi}\bar{z}(\theta)\big) = h_{\bar{z}}(\theta) + \Psi_1(\theta), \quad \theta \in \Delta.$

Lemma 4.5. *If $\bar{z} \in C^2(\Delta, \mathbb{R})$ is a solution to the inequality* (4.1) *then it satisfies*

$$|\bar{z} - \mathcal{T}\bar{z}| \leq (\Lambda_1 + \Lambda_2)\delta.$$

Proof. By Remark 4.4 and Lemma 3.1, we have

$$\bar{z}(\theta) = \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\big(h_{\bar{z}}(\theta) + \Psi(\theta)\big) + \Omega_{\varphi}^{\gamma'-1}(\theta, 0)\mathbb{A}_{h_{\bar{z}}+\Psi}.$$

Then, we obtain

$$\begin{aligned}
|\bar{z}(\theta) - \mathcal{T}\bar{z}(\theta)| &= \Big|\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\big(h_{\bar{z}}(\theta) + \Psi(\theta)\big) + \Omega_{\varphi}^{\gamma'-1}(\theta, 0)\mathbb{A}_{h_{\bar{z}+\Psi}} \\
&\qquad - \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{\bar{z}}(\theta) - \Omega_{\varphi}^{\gamma'-1}(\theta, 0)\mathbb{A}_{h_{\bar{z}}}\Big| \\
&\leq \big|\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\Psi(\theta) + \Omega_{\varphi}^{\gamma'-1}(\theta, 0)\mathbb{A}_{\Psi}\big| \\
&\leq \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\delta + \frac{\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\delta + \mathfrak{I}_{0^+}^{\alpha'+\nu_i,\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\delta}{|\varpi|\Gamma(\gamma')} \\
&\leq (\Lambda_1 + \Lambda_2)\delta.
\end{aligned}$$

□

Theorem 4.6. *Assume that the assumptions $(\mathcal{A}_1) - (\mathcal{A}_2)$ and the condition* (3.10) *are satisfied. Then the problem (1.1) is Ulam–Hyers stable as well as generalized Ulam–Hyers stable.*

Proof. Let $\bar{z} \in C^2(\Delta, \mathbb{R})$ be a solution of the inequality (4.1) and $z \in C^2(\Delta, \mathbb{R})$ be a solution of (1.1). Then, we have

$$\begin{aligned}
|\bar{z}(\theta) - z(\theta)| &= |\bar{z}(\theta) - \mathcal{T}z(\theta)| \\
&\leq |\bar{z}(\theta) - \mathcal{T}\bar{z}(\theta)| + |\mathcal{T}\bar{z}(\theta) - \mathcal{T}z(\theta)| \\
&\leq (\Lambda_1 + \Lambda_2)\delta + \frac{\kappa_1}{1-\kappa_2}(\Lambda_1 + \Lambda_2)|\bar{z}(\theta) - z(\theta)|.
\end{aligned}$$

Thus

$$|\bar{z} - z| \leq c_h\delta, \qquad \text{where} \;\; c_h = \frac{(\Lambda_2 + \Lambda_1)}{1 - \frac{\kappa_1}{1-\kappa_2}(\Lambda_2 + \Lambda_1)}.$$

Hence, the problem (1.1) is Ulam-Hyers stable. On the other hand, by choosing $\Psi(\delta) = c_h\delta, \;\; \Psi(0) = 0$, we obtain that the problem (1.1) is generalized Ulam-Hyers stable. □

5. An illustrative example

In this section, we consider the following φ-Hilfer fractional differential equation with p-Laplacian operator

$$\begin{cases} {}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z(\theta) = \dfrac{\sin(z(\theta))+1}{19+e^\theta} + \dfrac{|{}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z(\theta)|}{(\theta+4)^2}, \\ z(0)=0, \quad {}^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z(0)=0, \quad z(1) = \mathfrak{I}_{0^+}^{\frac{1}{4},\theta+1}z\left(\frac{1}{4}\right) + 3\mathfrak{I}_{0^+}^{\frac{5}{4},\theta+1}z\left(\frac{3}{4}\right). \end{cases} \tag{5.1}$$

We observe that problem (5.1) is a special case of problem (1.1) when

$$\alpha = \frac{1}{2},\ \beta = \frac{1}{2},\ \alpha' = \frac{3}{2},\ \beta' = \frac{1}{2},\ \gamma = \frac{3}{4},\ \gamma' = \frac{7}{4},\ T = 1,\ \varphi(\theta) = \theta + 1,$$

$$p = 2,\ p' = 2,\ M = 1,\ m = 2,\ a_1 = 1,\ a_2 = 3,\ \nu_1 = \frac{1}{4},\ \nu_2 = \frac{5}{4},$$

$\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{3}{4}$ and

$$h_z(\theta) = \frac{\sin(z(\theta))+1}{19+e^\theta} + \frac{|{}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z(\theta)|}{(\theta+4)^2}.$$

For all $z_1, z_2, \bar{z}_1, \bar{z}_2 \in \mathbb{R}$, and $\theta \in [0,1]$, we have

$$\begin{aligned} &|h_{z_1}(\theta) - h_{z_2}(\theta)| \\ &\leq \frac{1}{20}|z_1 - z_2| + \frac{1}{16}\left|{}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z_1 - {}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{5}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{5}{2},\theta+1}z_2\right|. \end{aligned}$$

Thus, the assumption $(\mathcal{A}_1)$ holds, with $\kappa_1 = \dfrac{1}{20}$ and $\kappa_2 = \dfrac{1}{16}$.
On other hand,

$$|h_z(\theta)| \leq \frac{1}{20} + \frac{1}{20}|z| + \frac{1}{16}\left|{}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{5}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{5}{2},\theta+1}z\right|.$$

Hence, the assumption $(\mathcal{A}_2)$ satisfied, with $\tau_1 = \tau_2 = \dfrac{1}{20}$ and $\tau_3 = \dfrac{1}{16}$.
With simple calculations, we get

$$\frac{\kappa_1(\Lambda_1 + \Lambda_2)}{1 - \kappa_2} = 0.0666666666666667 < 1.$$

Then by virtue of Theorem 3.6, the problem (5.1) has a unique solution in $C^2([0,1],\mathbb{R})$. Moreover, by Theorem 4.6, the problem (5.1) is $\mathbb{UH}$ and $\mathbb{GUH}$ stable.

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On a Riemann-Liouville fractional anti-periodic boundary value problem in a weighted space

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Abstract. This paper is concerned with the existence of solutions for a fractional anti-periodic boundary value problem of order $\alpha \in (2, 3]$ involving Riemann–Liouville fractional derivative and integral operators in a weighted space. The existence of solutions for the given problem is shown by means of the Leray-Schauder's alternative, while the uniqueness of its solutions is established with the aid of the Banach's fixed point theorem. We also discuss the Ulam–Hyers stability for the problem at hand. Examples are presented for illustration of the main results.

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1. Introduction

Riemann–Liouville fractional differential equations have been extensively studied in the literature. It has been mainly due to the ability of fractional operators to describe memory and hereditary characteristics of several materials and processes. One can find applications of Riemann–Liouville fractional derivative operators in the study of complex systems like polymers, biological tissue and self-similar protein dynamics [27], real materials [31], projectile motion [5], electrical circuits [9], backward diffusion problems [32], viscoelasticity [25], bioengineering [26], fractional dynamics and control [11], modeling framework [35], etc.

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Influenced by the occurrence of Riemann–Liouville fractional operators in the mathematical modeling of several real-world phenomena, many researchers have shown a keen interest in developing theoretical aspects of Riemann–Liouville type fractional boundary value problems, for instance, see [23, 2, 3, 33, 7, 12, 28]. Anti-periodic boundary conditions form a special case of non-separated boundary conditions and have been extensively studied in the literature, for instance, see the papers [1, 19, 15, 8]. In [4], the authors solved a fractional integro-differential equation equipped with a new class of dual anti-periodic boundary conditions. In [6], the authors studied a Riemann-Liouville fractional differential equation of order $\alpha \in (1, 2]$ with fractional anti-periodic boundary conditions.

In 1940, Ulam [34] introduced the notion of stability for a functional equation to find a set of conditions ensuring an approximate solution of this equation to be close to its exact solution. Hyers [16] discussed the Ulam's idea of stability more rigorously in the context of Banach spaces in 1941. Later, it was known as the Ulam–Hyers stability. Then, Rassias [29] applied the idea of Ulam–Hyers stability to a wide class of functional equations, which is now referred to as Ulam–Hyers-Rassias stability [17]. The Ulam-Hyers stability for Black-Scholes equation was studied in [24]. For some recent results on Ulam–Hyers stability for fractional differential equations, for instance, see [37, 20, 22, 13, 10, 14, 36].

In this paper, motivated by the foregoing discussion, we discuss the existence, uniqueness and Ulam-Hyers stability for solutions of a nonlinear Riemann–Liouville fractional differential equation equipped with fractional anti-periodic boundary conditions in a weighted space. Precisely, we consider a nonlinear Riemann–Liouville fractional differential equation

$$D^{\alpha}x(t) = f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)), \ 2 < \alpha \leq 3, \ t \in \mathcal{J} = [0, T], \ T > 0, \tag{1.1}$$

complemented with fractional anti-periodic boundary conditions

$$\begin{cases} D^{\alpha-1}x(0^+) + D^{\alpha-1}x(T^-) = 0, \\ D^{\alpha-2}x(0^+) + D^{\alpha-2}x(T^-) = 0, \\ D^{\alpha-3}x(0^+) + D^{\alpha-3}x(T^-) = 0, \end{cases} \tag{1.2}$$

where D^{α} denote the Riemann–Liouville fractional derivative operator of order α, $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an appropriate continuous function and

$$(\lambda_1 x)(t) = \int_0^t \Psi_1(t, s)\, x(s)\, ds, \quad (\lambda_2 x)(t) = \int_0^t \Psi_2(t, s)\, x(s)\, ds, \tag{1.3}$$

with Ψ_1 and Ψ_2 being continuous functions on $\mathcal{J} \times \mathcal{J}$. The relationship between the Green's functions of lower- and higher-order anti-periodic fractional boundary value problems is also described.

We derive the existence and uniqueness results for the problem (1.1)-(1.2) with the aid of the Leray-Schauder's alternative and Banach's contraction mapping principle.

We arrange the remaining content of the paper as follows. Section 2 contains some preliminary concepts and a subsidiary result related to the linear version of the given nonlinear problem. An interesting discussion concerning the Green's function is given in Section 3. We prove the existence and uniqueness results for the problem (1.1)-(1.2) in Section 4, which are well-illustrated with examples in Section 5. We discuss the Ulam–Hyers stability for the problem at hand in Section 6.

2. A subsidiary result

Let us first recall some basic concepts of fractional calculus from the text [21].

Definition 2.1. *For $\varphi \in L_1[a,b]$, the (left) Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$, denoted by $I_{a+}^{\alpha}\varphi$, is defined as*

$$I_{a+}^{\alpha}\varphi(t) = (\varphi * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)ds,$$

where $K_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, Γ denotes the Euler gamma function.

Definition 2.2. *Let $\varphi, \varphi^{(m)} \in L_1[a,b], a,b \in \mathbb{R}$ and $\alpha \in (m-1,m]$, $m \in \mathbb{N}$. The Riemann–Liouville fractional derivative of order α, denoted by $D_{a+}^{\alpha}\varphi$, is defined as*

$$D_{a+}^{\alpha}\varphi(t) = \frac{d^m}{dt^m} I_{a+}^{1-\alpha}\varphi(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-1-\alpha}\varphi(s)ds.$$

In the current work, we write the Riemann–Liouville fractional integral and derivative operators I_{a+}^{q} and D_{a+}^{q} as I^q and D^q when $a=0$, respectively.

Lemma 2.3. *Let p and q be positive reals. If φ is a continuous function, then*
(i) $I^p I^q \varphi(t) = I^{p+q}\varphi(t)$,
(ii) $D^p I^q \varphi(t) = I^{q-p}\varphi(t)$ *for* $q > p > 0$.

Note that $D^p t^{p-i} = 0,\ i = 1,2,\ldots,[p]+1$, where $[p]$ is the largest integer less than p and

$$D^p t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-p+1)}t^{\lambda-p},\ \lambda > -1,\ \lambda \neq p-1, p-2, \ldots, p-n.$$

In the following lemma, we solve the linear version of the equation (1.1) complemented with the boundary data (1.2).

Lemma 2.4. *For $g \in C(\mathcal{J},\mathbb{R})$, the unique solution of the linear equation*

$$D^\alpha x(t) = g(t),\ 2 < \alpha \leq 3,\ t \in \mathcal{J}, \tag{2.1}$$

subject to the boundary conditions (1.2) *is given by*

$$\begin{aligned} x(t) &= \int_0^T \left[\mu_1(t) + \mu_2(t)(T-s) + \mu_3(t)\frac{(T-s)^2}{2}\right] g(s)ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds, \end{aligned} \tag{2.2}$$

where

$$\mu_1(t) = \frac{-t^{\alpha-1}}{2\Gamma(\alpha)} + \frac{Tt^{\alpha-2}}{4\Gamma(\alpha-1)}, \ \mu_2(t) = \frac{-t^{\alpha-2}}{2\Gamma(\alpha-1)} + \frac{Tt^{\alpha-3}}{4\Gamma(\alpha-2)},$$
$$\mu_3(t) = \frac{-t^{\alpha-3}}{2\Gamma(\alpha-2)}. \tag{2.3}$$

Proof. Operating the integral operator I^α to (2.1), we get

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + I^\alpha g(t), \tag{2.4}$$

where $c_1, c_2, c_3 \in \mathbb{R}$ are unknown arbitrary constants. From (2.4), we have

$$\begin{cases} D^{\alpha-1}x(t) = c_1\Gamma(\alpha) + I^1 g(t), \\ D^{\alpha-2}x(t) = c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) + I^2 g(t), \\ D^{\alpha-3}x(t) = c_1\dfrac{\Gamma(\alpha)}{2}t^2 + c_2\Gamma(\alpha-1)t + c_3\Gamma(\alpha-2) + I^3 g(t). \end{cases} \tag{2.5}$$

Using (2.5) in the boundary condition (1.2), we obtain

$$\begin{cases} 2c_1\Gamma(\alpha) + I^1 g(T) = 0, \\ 2c_2\Gamma(\alpha-1) + I^2 g(T) + c_1\Gamma(\alpha)T = 0, \\ 2c_3\Gamma(\alpha-2) + I^3 g(T) + \dfrac{c_1\Gamma(\alpha)T^2}{2} + c_2\Gamma(\alpha-1)T = 0. \end{cases} \tag{2.6}$$

From (2.6), we get

$$c_1 = \frac{-1}{2\Gamma(\alpha)} I^1 g(T), \ c_2 = \frac{1}{2\Gamma(\alpha-1)}\left(\frac{T}{2}I^1 g(T) - I^2 g(T)\right),$$
$$c_3 = \frac{1}{2\Gamma(\alpha-2)}\left(\frac{T}{2}I^2 g(T) - I^3 g(T)\right). \tag{2.7}$$

Inserting the above values of c_1, c_2 and c_3 in (2.4) together with the notation (2.3), we obtain the solution (2.2). The converse of the lemma follows by direct computation. The proof is completed. □

Remark 2.5. The solution (2.2) of the equation (2.1) subject to the boundary conditions (1.2) can be expressed in terms of the Green's function as

$$x(t) = \int_0^T G(t, s, \alpha)x(s)ds,$$

where

$$G(t,s,\alpha) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t^{\alpha-1}}{2\Gamma(\alpha)} + \dfrac{t^{\alpha-2}}{4\Gamma(\alpha-1)}(2s-T) \\ \quad + \underbrace{\dfrac{st^{\alpha-3}(T-s)}{4\Gamma(\alpha-2)}}, & s \le t, \\ \dfrac{-t^{\alpha-1}}{2\Gamma(\alpha)} + \dfrac{t^{\alpha-2}}{4\Gamma(\alpha-1)}(2s-T) + \underbrace{\dfrac{st^{\alpha-3}(T-s)}{4\Gamma(\alpha-2)}}, & t \le s. \end{cases} \tag{2.8}$$

3. An interesting analogy

It is interesting to note that the Green's function (2.8) contains the expressions for the Green's function for the problem

$$\begin{cases} D^{\alpha}x(t) = g(t),\ 1 < \alpha \leq 2,\ t \in \mathcal{J}, \\ D^{\alpha-1}x(0^+) = -D^{\alpha-1}x(T^-),\ D^{\alpha-2}x(0^+) = -D^{\alpha-2}x(T^-), \end{cases} \tag{3.1}$$

given by

$$G(t,s,\alpha) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t^{\alpha-1}}{2\Gamma(\alpha)} + \underbrace{\dfrac{t^{\alpha-2}}{4\Gamma(\alpha-1)}(2s-T)}, & s \leq t, \\ \dfrac{-t^{\alpha-1}}{2\Gamma(\alpha)} + \underbrace{\dfrac{t^{\alpha-2}}{4\Gamma(\alpha-1)}(2s-T)}, & t \leq s. \end{cases} \tag{3.2}$$

Likewise, the Green's function (3.2) contains the one for the problem

$$\begin{cases} D^{\alpha}x(t) = g(t),\ 0 < \alpha \leq 1,\ t \in \mathcal{J}, \\ D^{\alpha-1}x(0^+) = -D^{\alpha-1}x(T^-), \end{cases} \tag{3.3}$$

given by

$$G(t,s,\alpha) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t^{\alpha-1}}{2\Gamma(\alpha)}, & s \leq t, \\ \dfrac{-t^{\alpha-1}}{2\Gamma(\alpha)}, & t \leq s. \end{cases} \tag{3.4}$$

Thus, we deduce that the Green's function for a higher order fractional boundary value problem involving a Riemann-Liouville fractional differential equation contains the expressions for the Green's functions associated with lower order Riemann-Liouville fractional anti-periodic boundary value problems. The additional terms in (2.8) and (3.2) with reference to the Green's functions for the lower order problems (3.1) and (3.3) respectively are indicated with the under-braces.

4. Main Results

Let $C(\mathcal{J},\mathbb{R})$ be the Banach space of all continuous real-valued functions from $\mathcal{J} \to \mathbb{R}$ endowed with the supremum norm $\|x\| = \sup_{t\in\mathcal{J}} |x(t)|$. For $t \in \mathcal{J}$, we define $x_{\tilde{r}}(t) = t^{\tilde{r}}x(t),\ \tilde{r} > 0$, and let $C_{\tilde{r}}(\mathcal{J},\mathbb{R})$ be the space of all functions $x_{\tilde{r}}$ such that $x \in C(\mathcal{J},\mathbb{R})$ which turns out to be a Banach space when endowed with the norm $\|x\|_{\tilde{r}} = \sup_{t\in\mathcal{J}}\{t^{\tilde{r}}|x(t)|\}$.

By Lemma 2.4, we transform the problem (1.1)-(1.2) into a fixed point problem as

$$x = \mathcal{T}x,$$

where $\mathcal{T} : C_{3-\alpha}(\mathcal{J}, \mathbb{R}) \to C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ is an operator defined by

$$\begin{aligned}(\mathcal{T}x)(t) &= \int_0^T \Big[\mu_1(t) + \mu_2(t)(T-s) + \mu_3(t)\frac{(T-s)^2}{2}\Big] \times \\ &\quad \times f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))ds. \qquad (4.1)\end{aligned}$$

Observe that the fixed points of the operator $\mathcal{T}$ are solution to the problem (1.1)-(1.2).

Lemma 4.1. *Assume that $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then, the operator $\mathcal{T} : C_{3-\alpha}(\mathcal{J}, \mathbb{R}) \to C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ is compact.*

Proof. Let us first note that continuity of $\mathcal{T}$ follows from that of f. Let $\mathcal{G}$ be a bounded set in $C_{3-\alpha}(\mathcal{J}, \mathbb{R})$. Then, there exists a positive constant N_f such that $|f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t))| \le N_f$, $\forall x \in \mathcal{G}$, $t \in \mathcal{J}$. In consequence, we have

$$\begin{aligned}&\|\mathcal{T}x\|_{3-\alpha} \\ = &\sup_{t\in\mathcal{J}}\Big\{ t^{3-\alpha}\Big| \int_0^T \Big[\mu_1(t) + \mu_2(t)(T-s) + \mu_3(t)\frac{(T-s)^2}{2}\Big] \times \\ &\times f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))ds\Big|\Big\} \\ \le &\sup_{t\in\mathcal{J}}\Big\{ t^{3-\alpha}\Big[N_f \int_0^T \Big[|\mu_1(t)| + |\mu_2(t)|(T-s) + |\mu_3(t)|\frac{(T-s)^2}{2}\Big]ds \\ &+ N_f \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big]\Big\} \\ \le &N_f\Big[\sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_1(t)|T + \sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_2(t)|\frac{T^2}{2} + \sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_3(t)|\frac{T^3}{6}\Big] \\ &+ N_f \sup_{t\in\mathcal{J}}\Big\{ t^{3-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big\} \\ = &\Big[\delta_1 T + \delta_2\frac{T^2}{2} + \delta_3\frac{T^3}{6} + \frac{T^3}{\Gamma(\alpha+1)}\Big]N_f,\end{aligned}$$

where

$$\delta_i = \sup_{t\in\mathcal{J}}|t^{3-\alpha}\mu_i(t)|,\ i = 1, 2, 3, \qquad (4.2)$$

and $\mu_i, i = 1, 2, 3$, are given in (2.3). Thus, we have that $\mathcal{T}(\mathcal{G}) < \infty$. Hence $\mathcal{T}(\mathcal{G})$ is uniformly bounded. For verifying that $\mathcal{T}(\mathcal{G})$ is equicontinuous, we take $\tau_1, \tau_2 \in \mathcal{J}$ with $\tau_1 < \tau_2$. Then, we obtain

$$|\tau_2^{3-\alpha}(\mathcal{T}x)(\tau_2) - \tau_1^{3-\alpha}(\mathcal{T}x)(\tau_1)|$$

$$
\begin{aligned}
= & \Bigg| \tau_2^{3-\alpha} \Bigg[\int_0^T \Big[\mu_1(\tau_2) + \mu_2(\tau_2)(T-s) + \mu_3(\tau_2) \frac{(T-s)^2}{2} \Big] \times \\
& \times f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \\
& + \int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \Bigg] \\
& - \tau_1^{3-\alpha} \Bigg[\int_0^T \Big[\mu_1(\tau_1) + \mu_2(\tau_1)(T-s) + \mu_3(\tau_1) \frac{(T-s)^2}{2} \Big] \times \\
& \times f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \\
& + \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \Bigg] \Bigg| \\
\leq & \Big[|\tau_2^{3-\alpha} \mu_1(\tau_2) - \tau_1^{3-\alpha} \mu_1(\tau_1)| T + |\tau_2^{3-\alpha} \mu_2(\tau_2) - \tau_1^{3-\alpha} \mu_2(\tau_1)| \frac{T^2}{2} \\
& + |\tau_2^{3-\alpha} \mu_3(\tau_2) - \tau_1^{3-\alpha} \mu_3(\tau_1)| \frac{T^3}{6} \Big] N_f + \frac{2N_f}{\Gamma(\alpha+1)} \tau_2^{3-\alpha} (\tau_2 - \tau_1)^\alpha \\
& + \frac{N_f}{\Gamma(\alpha+1)} |\tau_2^3 - \tau_1^3| \\
\leq & \frac{TN_f}{2\Gamma(\alpha)} |\tau_2^2 - \tau_1^2| + \frac{TN_f}{4\Gamma(\alpha-1)} |\tau_2 - \tau_1| (T+1) + \frac{2N_f}{\Gamma(\alpha+1)} \tau_2^{3-\alpha} (\tau_2 - \tau_1)^\alpha \\
& + \frac{N_f}{\Gamma(\alpha+1)} |\tau_2^3 - \tau_1^3|, \qquad (4.3)
\end{aligned}
$$

which tends to zero as $\tau_2 \to \tau_1$ independent of $x \in \mathcal{G}$. Thus, $\mathcal{T}(\mathcal{G})$ is equicontinuous. From the preceding steps, it follows that $\mathcal{T}$ is compact. □

Now, we prove our first existence result for the problem (1.1)-(1.2) by applying the Leray-Schauder alternative [18, Theorem 2.4, p.4].

Theorem 4.2. *Assume that $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. In addition, we suppose that there exist a constant $N_f > 0$ such that $|f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t))| \leq N_f$, $\forall x \in \mathbb{R}$, $t \in \mathcal{J}$. Then, the problem (1.1)-(1.2) has at least one solution on $\mathcal{J}$.*

Proof. From Lemma 4.1, we know that $\mathcal{T}$ is completely continuous. So, the conclusion of the Leray-Schauder alternative will be applicable when it is shown that the set $V = \{t^{3-\alpha} x \in \mathbb{R} : t^{3-\alpha} x = \xi t^{3-\alpha} \mathcal{T} x, 0 < \xi < 1\}$ is bounded. For $x \in V$, we have $|t^{3-\alpha} x(t)| = |\xi t^{3-\alpha} \mathcal{T} x(t)| < t^{3-\alpha} |\mathcal{T} x(t)|$. Following the method of proof of Lemma 4.1, we obtain

$$
\|x\|_{3-\alpha} < \Big[\delta_1 T + \delta_2 \frac{T^2}{2} + \delta_3 \frac{T^3}{6} + \frac{T^3}{\Gamma(\alpha+1)} \Big] N_f < \infty, \qquad (4.4)
$$

which implies that the set V is bounded. Thus, by the Leray-Schauder alternative, we deduce that the operator $\mathcal{T}$ has at least one fixed point, which is indeed a solution of the problem (1.1)-(1.2). □

Next, we establish the existence of a unique solution to the problem (1.1)–(1.2) with the aid of Banach's fixed point theorem.

Theorem 4.3. *Let $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition:*

(H_1) *There exist positive functions $\mathbb{L}_1(t), \mathbb{L}_2(t), \mathbb{L}_3(t)$ such that*

$$\begin{aligned} & |f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)) - f(t, y(t), (\lambda_1 y(t), (\lambda_2 y)(t))| \\ \leq\ & \mathbb{L}_1(t)|x - y| + \mathbb{L}_2(t)|\lambda_1 x - \lambda_1 y| + \mathbb{L}_3(t)|\lambda_2 x - \lambda_2 y|,\ \forall t \in \mathcal{J},\ x, y \in \mathbb{R}. \end{aligned}$$

Then, the problem (1.1)–(1.2) has a unique solution in $\mathcal{J}$, provided that

$$\Lambda = \left(1 + \frac{\psi_1 + \psi_2}{(\alpha - 2)}\right)\left(\delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3 + T^{3-\alpha} e_4\right) < 1, \tag{4.5}$$

where $\delta_m,\ m = 1, 2, 3,$ are given by (4.2),

$\psi_1 = \sup_{t,s\in\mathcal{J}} |\Psi_1(t, s)|,\ \psi_2 = \sup_{t,s\in\mathcal{J}} |\Psi_2(t, s)|,$
$e_1 = \max\{|I\mathbb{L}_1(T)T^{\alpha-3}|, |I\mathbb{L}_2(T)T^{\alpha-2}|, |I\mathbb{L}_3(T)T^{\alpha-2}|\},$
$e_2 = \max\{|I^2\mathbb{L}_1(T)T^{\alpha-3}|, |I^2\mathbb{L}_2(T)T^{\alpha-2}|, |I^2\mathbb{L}_3(T)T^{\alpha-2}|\},$
$e_3 = \max\{|I^3\mathbb{L}_1(T)T^{\alpha-3}|, |I^3\mathbb{L}_2(T)T^{\alpha-2}|, |I^3\mathbb{L}_3(T)T^{\alpha-2}|\},$
$e_4 = \sup_{t\in\mathcal{J}}\{|I^\alpha\mathbb{L}_1(t)t^{\alpha-3}|, |I^\alpha\mathbb{L}_2(t)t^{\alpha-2}|, |I^\alpha\mathbb{L}_3(t)t^{\alpha-2}|\},$

I^α denotes the Riemann-Liouville integral operator of order α and

$$I^n\phi(t) = \frac{1}{\Gamma(n)}\int_0^t (t - s)^{n-1}\phi(s)ds,\ n = 1, 2, 3.$$

Proof. For verifying the hypotheses of Banach's fixed point theorem, we consider a closed ball $\mathcal{E}_\rho = \{x \in C_{3-\alpha}(\mathcal{J}, \mathbb{R}) : \|x\|_{3-\alpha} \leq \rho\}$ with

$$\rho \geq (\overline{f}\ \overline{\delta})(1 - \Lambda)^{-1}, \tag{4.6}$$

where $\sup_{t\in\mathcal{J}} |f(t, 0, 0, 0)| = \overline{f}$,

$$\overline{\delta} = \delta_1 T + \delta_2\frac{T^2}{2} + \delta_3\frac{T^3}{6} + \frac{T^3}{\Gamma(\alpha + 1)}, \tag{4.7}$$

$\delta_m,\ m = 1, 2, 3,$ are given by (4.2). Now, we establish that $\mathcal{T}\mathcal{E}_\rho \subset \mathcal{E}_\rho$, where $\mathcal{T} : \mathcal{E}_\rho \longrightarrow C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ is given by (4.1). By (H_1), we have

$$\begin{aligned} & |f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t))| \\ \leq\ & |f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)) - f(t, 0, 0, 0)| + |f(t, 0, 0, 0)| \\ \leq\ & \mathbb{L}_1(t)|x| + \mathbb{L}_2(t)|\lambda_1 x| + \mathbb{L}_3(t)|\lambda_2 x| + \overline{f}. \end{aligned} \tag{4.8}$$

For $x \in \mathcal{E}_\rho$, it follows by using (4.8) that

$$\begin{aligned} & \|\mathcal{T}x\|_{3-\alpha} \\ =\ & \sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg|\int_0^T \Big[\mu_1(t) + \mu_2(t)(T - s) + \mu_3(t)\frac{(T - s)^2}{2}\Big] \times \\ & \times f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))ds \end{aligned}$$

$$
\begin{aligned}
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s)) ds \Bigg| \Bigg\} \\
\leq \ & \sup_{t\in\mathcal{J}} \Bigg\{ t^{3-\alpha} \Bigg[\int_0^T \left[|\mu_1(t)| + |\mu_2(t)|(T-s) + |\mu_3(t)| \frac{(T-s)^2}{2} \right] \\
& \times \Big(\mathbb{L}_1(s)|x| + \mathbb{L}_2(s)|\lambda_1 x| + \mathbb{L}_3(s)|\lambda_2 x| + \overline{f} \Big) ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\mathbb{L}_1(s)|x| + \mathbb{L}_2(s)|\lambda_1 x| + \mathbb{L}_3(s)|\lambda_2 x| + \overline{f} \Big) ds \Bigg] \Bigg\} \\
\leq \ & \int_0^T \left[\sup_{t\in\mathcal{J}} |t^{3-\alpha}\mu_1(t)| + \sup_{t\in\mathcal{J}} |t^{3-\alpha}\mu_2(t)|(T-s) + \sup_{t\in\mathcal{J}} |t^{3-\alpha}\mu_3(t)| \frac{(T-s)^2}{2} \right] \\
& \times \Big(\mathbb{L}_1(s)|x| + \mathbb{L}_2(s)|\lambda_1 x| + \mathbb{L}_3(s)|\lambda_2 x| + \overline{f} \Big) ds \\
& + \sup_{t\in\mathcal{J}} \left\{ t^{3-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\mathbb{L}_1(s)|x| + \mathbb{L}_2(s)|\lambda_1 x| + \mathbb{L}_3(s)|\lambda_2 x| + \overline{f} \Big) ds \right\} \\
\leq \ & \int_0^T \left[\delta_1 + \delta_2(T-s) + \delta_3 \frac{(T-s)^2}{2} \right] \\
& \times \left(\left(\mathbb{L}_1(s)s^{\alpha-3} + \mathbb{L}_2(s)\frac{\psi_1 s^{\alpha-2}}{(\alpha-2)} + \mathbb{L}_3(s)\frac{\psi_2 s^{\alpha-2}}{(\alpha-2)} \right) \|x\|_{3-\alpha} + \overline{f} \right) ds \\
& + \sup_{t\in\mathcal{J}} \Bigg\{ t^{3-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Bigg(\Bigg(\mathbb{L}_1(s)s^{\alpha-3} + \mathbb{L}_2(s)\frac{\psi_1 s^{\alpha-2}}{(\alpha-2)} \\
& + \mathbb{L}_3(s)\frac{\psi_2 s^{\alpha-2}}{(\alpha-2)} \Bigg) \|x\|_{3-\alpha} + \overline{f} \Bigg) ds \Bigg\} \\
\leq \ & \left[\left(1 + \frac{\psi_1}{(\alpha-2)} + \frac{\psi_2}{(\alpha-2)} \right) \left(\delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3 + T^{3-\alpha} e_4 \right) \right] \rho \\
& + \overline{f} \left[\delta_1 T + \delta_2 \frac{T^2}{2} + \delta_3 \frac{T^3}{6} + \frac{T^3}{\Gamma(\alpha+1)} \right] \\
\leq \ & \Lambda\rho + \overline{f}\,\overline{\delta}.
\end{aligned} \tag{4.9}
$$

Combining (4.9) with (4.6), we obtain

$$
\|\mathcal{T}x\|_{3-\alpha} \leq \Lambda\rho + \overline{f}\,\overline{\delta} \leq \rho,
$$

which shows that $\mathcal{T}x \in \mathcal{E}_\rho$. Hence, $\mathcal{T}\mathcal{E}_\rho \subset \mathcal{E}_\rho$ since $x \in \mathcal{E}_\rho$ is an arbitrary element.

Next, we show that the operator $\mathcal{T}$ is a contraction. For that, let $x, y \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$. Then, for any $t \in \mathcal{J}$ and using (H_1) together with the relation

$$
\begin{aligned}
& \mathbb{L}_1(t)|x-y| + \mathbb{L}_2(t)|\lambda_1 x - \lambda_1 y| + \mathbb{L}_3(t)|\lambda_2 x - \lambda_2 y| \\
\leq \ & \left(\mathbb{L}_1(t)t^{\alpha-3} + \mathbb{L}_2(t)\frac{\psi_1 t^{\alpha-2}}{(\alpha-2)} + \mathbb{L}_3(s)\frac{\psi_2 t^{\alpha-2}}{(\alpha-2)} \right) \|x-y\|_{3-\alpha},
\end{aligned} \tag{4.10}
$$

we obtain

$$\begin{aligned}
&\|\mathcal{T}x-\mathcal{T}y\|_{3-\alpha}\\
\leq\ &\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[\int_0^T\Big[|\mu_1(t)|+|\mu_2(t)|(T-s)+|\mu_3(t)|\frac{(T-s)^2}{2}\Big]\\
&\times|f(s,x(s),(\lambda_1x)(s),(\lambda_2x)(s))-f(s,y(s),(\lambda_1y)(s),(\lambda_2y)(s))|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s),(\lambda_1x)(s),(\lambda_2x)(s))\\
&-f(s,y(s),(\lambda_1y)(s),(\lambda_2y)(s))|ds\Bigg]\Bigg\}\\
\leq\ &\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[\int_0^T\Big[|\mu_1(t)|+|\mu_2(t)|(T-s)+|\mu_3(t)|\frac{(T-s)^2}{2}\Big]\\
&\times\Big(\mathbb{L}_1(s)|x-y|+\mathbb{L}_2(s)|\lambda_1x-\lambda_1y|+\mathbb{L}_3(s)|\lambda_2x-\lambda_2y|\Big)ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\mathbb{L}_1(s)|x-y|+\mathbb{L}_2(s)|\lambda_1x-\lambda_1y|\\
&+\mathbb{L}_3(s)|\lambda_2x-\lambda_2y|\Big)ds\Bigg]\Bigg\}\\
\leq\ &\int_0^T\Big[\delta_1+\delta_2(T-s)+\delta_3\frac{(T-s)^2}{2}\Big]\\
&\times\Big(\mathbb{L}_1(s)s^{\alpha-3}+\mathbb{L}_2(s)\frac{\psi_1s^{\alpha-2}}{(\alpha-2)}+\mathbb{L}_3(s)\frac{\psi_2s^{\alpha-2}}{(\alpha-2)}\Big)\|x-y\|_{3-\alpha}ds\\
&+\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\mathbb{L}_1(s)s^{\alpha-3}+\mathbb{L}_2(s)\frac{\psi_1s^{\alpha-2}}{(\alpha-2)}\\
&+\mathbb{L}_3(s)\frac{\psi_2s^{\alpha-2}}{(\alpha-2)}\Big)\|x-y\|_{3-\alpha}ds\Bigg\}\\
&\leq\Lambda\|x-y\|_{3-\alpha},
\end{aligned}$$

which shows that the operator $\mathcal{T}$ is a contraction as $\Lambda<1$ by the assumption (4.5). Thus, the hypotheses of Banach's fixed point theorem is verified and hence its conclusion implies that the operator $\mathcal{T}$ has a unique fixed point. Therefore, there exists a unique solution to the problem (1.1)-(1.2) on $\mathcal{J}$. This finish the proof. □

As a special case of Theorem 4.3, by taking $\Psi_1(t,s)=\dfrac{(t-s)^{p-1}}{\Gamma(p)}$, $\Psi_2(t,s)=\dfrac{(t-s)^{q-1}}{\Gamma(q)}$, $0<p,q$, in (1.1), we get a nonlinear fractional differential equation involving both Riemann-Liouville derivative and integral operators given by

$$D^{\alpha}x(t)=f(t,x(t),I^px(t),I^qx(t)). \tag{4.11}$$

Now we present a uniqueness result for fractional differential equation (4.1) subject to the boundary conditions (1.2).

Theorem 4.4. *Assume that* $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and the following condition holds:*

(H_2) *There exist positive functions* $\overline{\nu}_1(t)$, $\overline{\nu}_2(t)$ *and* $\overline{\nu}_3(t)$, *such that:*

$$\begin{aligned} & |f(t, x(t), (I^p x)(t), (I^q x)(t)) - f(t, y(t), (I^p y)(t), (I^q y)(t))| \\ \leq \; & \overline{\nu}_1(t)|x - y| + \overline{\nu}_2(t) I^p |x - y| + \overline{\nu}_3(t) I^q |x - y|, \ \forall \ t \in \mathcal{J}, x, y \in \mathbb{R}. \quad (4.12) \end{aligned}$$

Then, there exists a unique solution to the Riemann-Liuoville fractional differential equation (4.11) *subject to the boundary conditions* (1.2) *on* $\mathcal{J}$, *provided that*

$$\begin{aligned} \Lambda_1 \; = \; & \left(1 + \frac{\Gamma(\alpha - 2)}{\Gamma(p + \alpha - 2)} + \frac{\Gamma(\alpha - 2)}{\Gamma(q + \alpha - 2)}\right)\Big(\delta_1 k_1 + \delta_2 k_2 + \delta_3 k_3 \\ & + k_4 T^{3-\alpha}\Big) < 1, \quad (4.13) \end{aligned}$$

where $\delta_m, m = 1, 2, 3,$ *are given in* (4.2),

$k_1 = \max\{|I\overline{\nu}_1(T)T^{\alpha-3}|, |I\overline{\nu}_2(T)T^{p+\alpha-3}|, |I\overline{\nu}_3(T)T^{q+\alpha-3}|\}$,
$k_2 = \max\{|I^2\overline{\nu}_1(T)T^{\alpha-3}|, |I^2\overline{\nu}_2(T)T^{p+\alpha-3}|, |I^2\overline{\nu}_3(T)T^{q+\alpha-3}|\}$,
$k_3 = \max\{|I^3\overline{\nu}_1(T)T^{\alpha-3}|, |I^3\overline{\nu}_2(T)T^{p+\alpha-3}|, |I^3\overline{\nu}_3(T)T^{q+\alpha-3}\}$,
$k_4 = \sup_{t\in\mathcal{J}}\{|I^\alpha\overline{\nu}_1(t)t^{\alpha-3}|, |I^\alpha\overline{\nu}_2(t)t^{p+\alpha-3}|, |I^\alpha\overline{\nu}_3(t)t^{q+\alpha-3}|\}$.

Proof. We omit the proof as it is similar to that of Theorem 4.3. □

5. Examples

In this section, we present examples illustrating the results obtained in the last section.

Example 5.1. Let us consider the nonlinear fractional differential equation

$$D^\alpha x(t) = f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)), \ t \in [0, 1], \quad (5.1)$$

subject to the Riemann–Liouville boundary conditions

$$\begin{cases} D^{\alpha-1}x(0^+) + D^{\alpha-1}x(T^-) = 0, \\ D^{\alpha-2}x(0^+) + D^{\alpha-2}x(T^-) = 0, \\ D^{\alpha-3}x(0^+) + D^{\alpha-3}x(T^-) = 0. \end{cases} \quad (5.2)$$

Here, $\alpha = 8/3, \mathcal{J} = [0, 1], T = 1$, and

$$\begin{aligned} & f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)) \\ = \; & \frac{\sin x(t)}{\sqrt{t^2 + 144}} + \int_0^t \frac{e^{-(s^2-t)}}{600} x(s) ds + \int_0^t \frac{\sin(s - t)}{(t + 225)^2} x(s) ds. \end{aligned}$$

Using the given data, it is found that $\delta_1 \approx 0.05538661, \delta_2 \approx 0.36924406, \delta_3 \approx 0.36924406, \psi_1 \approx 0.00045350, \psi_2 \approx 0.00001662, e_1 \approx 0.06136364, e_2 \approx 0.22500000, e_3 \approx 0.60000000, e_4 \approx 0.09748313$. Clearly, (H_1) is satisfied with $L_1(t) = \frac{1}{\sqrt{t^2 + 144}}$, $L_2(t) = L_3(t) = 1$ and $\Lambda \approx 0.23661991 < 1$, that is, the condition (4.5) is verified. As the hypothesis of Theorem 4.4 holds true, so its conclusion implies that the boundary value problem (5.1)-(5.2) has a unique solution on $[0, 1]$.

Example 5.2. Consider the nonlinear equation

$$D^{\alpha}x(t) = f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)),\ t \in [0, 1], \tag{5.3}$$

subject to the Riemann–Liouville boundary conditions in (5.2), where $\alpha = 8/3$, and

$$\begin{aligned} & f(t, x(t), (\lambda_1 x)(t), (\lambda_2 x)(t)) \\ = & \frac{\tan^{-1} x(t)}{\sqrt{t^2 + 144}} + \frac{1}{6}\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} x(s)ds + \frac{1}{25}\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} x(s)ds, \end{aligned}$$

with $p = 0.6,\ q = 0.8$.

Using the given values, we find that $\delta_1 \approx 0.05538661,\ \delta_2 \approx 0.36924406,\ \delta_3 \approx 0.36924406,\ k_1 \approx 0.13157895,\ k_2 \approx 0.07497047,\ k_3 \approx 0.02811867,\ k_4 \approx 0.04060714$, and the assumption (H_2) holds true with $\overline{\nu}_1(t) = \dfrac{1}{\sqrt{t^2 + 144}}, \overline{\nu}_2(t) = 1/6,\ \overline{\nu}_3(t) = 1/25$, and the condition (4.13) is satisfied as $\Lambda_1 \approx 0.34628001 < 1$. Thus, by the conclusion of Theorem 4.4, the equation (5.3) with the boundary conditions (5.2) has a unique solution on $[0, T]$.

6. Ulam–Hyers Stability Analysis

Let us first develop the arguments for the Ulam–Hyers stability [30] of the problem (1.1)-(1.2).

For $\epsilon > 0$ and $t \in \mathcal{J}$, let us consider the inequality

$$\left| D^{\alpha}\widehat{x}(t) - f(t, \widehat{x}(t), (\lambda_1 \widehat{x})(t)(\lambda_2 \widehat{x})(t)) \right| \leq \epsilon, \tag{6.1}$$

with boundary conditions (1.2).

If $\widehat{x} \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ is a solution to the inequality (6.1) with boundary conditions (1.2), then there exists a function $\kappa \in C(\mathcal{J}, \mathbb{R})$ such that $|\kappa(t)| \leq \epsilon$, $t \in \mathcal{J}$, and the function $\widehat{x}$ satisfies the Riemann-Liouville fractional differential equation

$$D^{\alpha}\widehat{x}(t) = f(t, \widehat{x}(t), (\lambda_1 \widehat{x})(t)(\lambda_2 \widehat{x})(t)) + \kappa(t),$$

with boundary conditions (1.2). Thus, we consider the boundary value problem

$$\begin{cases} D^{\alpha}\widehat{x}(t) = f(t, \widehat{x}(t), (\lambda_1 \widehat{x})(t)(\lambda_2 \widehat{x})(t)) + \kappa(t),\ t \in \mathcal{J}, \\ D^{\alpha-1}\widehat{x}(0^+) + D^{\alpha-1}\widehat{x}(T^-) = 0, \\ D^{\alpha-2}\widehat{x}(0^+) + D^{\alpha-2}\widehat{x}(T^-) = 0, \\ D^{\alpha-3}\widehat{x}(0^+) + D^{\alpha-3}\widehat{x}(T^-) = 0. \end{cases} \tag{6.2}$$

Definition 6.1. *The problem (1.1)-(1.2) is called Ulam–Hyers stable if we can find $c_f > 0$, such that, for each solution $\widehat{x} \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ of (6.2), there exists a unique solution $x \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ of the problem (1.1)-(1.2) satisfying*

$$\|\widehat{x} - x\|_{3-\alpha} \leq c\epsilon,\ t \in \mathcal{J}.$$

Definition 6.2. *If there exists* $\Phi \in C_{3-\alpha}(\mathbb{R}^+, \mathbb{R}^+)$, *with* $\Phi(0) = 0$, *such that, for each solution* $u \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ *of* (6.2), *there exists a unique solution* $x \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ *of the problem* (1.1)–(1.2) *satisfying*

$$\|\widehat{x} - x\|_{3-\alpha} \leq \Phi(\epsilon), \ t \in \mathcal{J}.$$

Then, the problem (1.1)–(1.2) *is generalized Ulam–Hyers stable.*

Theorem 6.3. *If the assumption* (H_1) *and the condition (4.5) are satisfied, then the problem* (1.1)–(1.2) *is Ulam–Hyers stable and hence generalized Ulam–Hyers stable in* $C_{3-\alpha}(\mathcal{J}, \mathbb{R})$.

Proof. By Lemma 2.4, the solution of (6.2) can be written as

$$\begin{aligned} \widehat{x}(t) &= \int_0^T \left[\mu_1(t) + \mu_2(t)(T-s) + \mu_3(t)\frac{(T-s)^2}{2}\right] \times \\ &\quad \times \Big(f(s, \widehat{x}(s), (\lambda_1 \widehat{x})(s), (\lambda_2 \widehat{x})(s)) + \kappa(s)\Big) ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(s, \widehat{x}(s), (\lambda_1 \widehat{x})(s), (\lambda_2 \widehat{x})(s)) + \kappa(s)\Big) ds. \end{aligned}$$

Using $|\kappa| < \epsilon$ and (4.7), we get

$$\begin{aligned} &\sup_{t\in\mathcal{J}} \left\{ t^{3-\alpha} \left| \widehat{x}(t) - \int_0^T \left[\mu_1(t) + \mu_2(t)(T-s) + \mu_3(t)\frac{(T-s)^2}{2}\right] \times \right.\right. \\ &\times f(s, \widehat{x}(s), (\lambda_1 \widehat{x})(s), (\lambda_2 \widehat{x})(s)) ds \\ &\left.\left. - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \widehat{x}(s), (\lambda_1 \widehat{x})(s), (\lambda_2 \widehat{x})(s)) ds \right| \right\} \leq \overline{\delta}\epsilon. \end{aligned}$$

It follows by the assumption (H_1) together with (4.10) that

$$\begin{aligned} & \|\widehat{x} - x\|_{3-\alpha} \\ = \ & \sup_{t\in\mathcal{J}} \{t^{3-\alpha} |\widehat{x}(t) - x(t)|\} \\ \leq \ & \overline{\delta}\epsilon + \sup_{t\in\mathcal{J}} \left\{ t^{3-\alpha} \left[\int_0^T \left[|\mu_1(t)| + |\mu_2(t)|(T-s) + |\mu_3(t)|\frac{(T-s)^2}{2}\right] \right.\right. \\ & \times |f(s, \widehat{x}(s), (\lambda_1 \widehat{x})(s), (\lambda_2 \widehat{x})(s)) - f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))| ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, \widehat{x}(s), (\lambda_1 \widehat{x})(s), (\lambda_2 \widehat{x})(s)) \\ & \left.\left. - f(s, x(s), (\lambda_1 x)(s), (\lambda_2 x)(s))| ds \right] \right\} \\ \leq \ & \overline{\delta}\epsilon + \Lambda \|\widehat{x} - x\|_{3-\alpha}, \end{aligned}$$

which can alternatively be written as

$$\|\widehat{x} - x\|_{3-\alpha} \leq \frac{\overline{\delta}\epsilon}{1 - \Lambda}.$$

Letting $c = c_f = \dfrac{\overline{\delta}}{1-\Lambda}$, we get $\|\widehat{x} - x\|_{3-\alpha} \leq c\epsilon$. Hence, the problem (1.1)–(1.2) is Ulam–Hyers stable. Moreover, it is generalized Ulam–Hyers stable as $\|\widehat{x} - x\|_{3-\alpha} \leq \Phi(\epsilon)$, whit $\Phi(\epsilon) = c\epsilon,\ \Phi(0) = 0$. □

Example 6.4. The problems (5.1)-(5.2) and (5.3)-(5.2) are Ulam–Hyers stable, and generalized Ulam–Hyers stable as $\Lambda \approx 0.23661991 < 1$ and $\Lambda_1 \approx 0.34628001 < 1$, respectively.

7. Conclusions

We explored the criteria ensuring the existence and uniqueness of solutions for nonlinear Riemann–Liouville fractional integro-differential equations of order $\alpha \in (2, 3]$ subject to fractional anti-periodic boundary conditions in a weighted space. We applied the Leray-Schauder's alternative and Banach's fixed point theorem to accomplish the desired results. A special case for the nonlinearity of (1.1) depending upon the Riemann–Liouville fractional integrals is also discussed. We also studied the Ulam–Hyers stability for the problem at hand. It is imperative to point out that the solution of the problem (1.1)–(1.2) contains the solution of the Riemann-Liouville fractional differential equation of order $\alpha \in (1, 2]$ complemented with fractional anti-periodic boundary conditions investigated in [6] (for details, see Section 3). Thus, our results are novel in the given configuration and contribute to the known literature on fractional anti-periodic boundary value problems of nonlinear Riemann–Liouville fractional differential equations in the weighted space.

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Integral solution to a parabolic equation involving the fractional p-Laplacian operator

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Abstract. The aim of this work is to study the existence and uniqueness of integral solutions for a class of non-local parabolic equations. There are two main results. First, we use a subdifferential technique to verify the existence and uniqueness of weak solutions when the initial data belong to L^2. Secondly, the existence and uniqueness of an integral solution is demonstrated by extending the study to initial data in L^1 space. To overcome the difficulties caused by non-local terms, the proposed strategy combines new approaches with sophisticated strategies derived from the theory of accretive operators. These results contribute to a better understanding of nonlocal evolution equations and their applications.

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1. Introduction

The main purpose of our work is to study a parabolic equation that includes the fractional p-Laplacian operator, which is modeled as follows:

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_p^s u = f & \text{in } Q_T :=]0,T[\times\Omega \\ u = 0 & \text{on }]0,T[\times(\mathbb{R}^N \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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with $s \in (0,1)$, $p > 1$, $ps < N$, $T > 0$, Ω is a bounded open subset of $\mathbb{R}^N$ and $(-\Delta)^s_p$ is the fractional p-Laplacian operator defined as follows

$$\begin{aligned}(-\Delta)^s_p u(x) := P.V \int_{\mathbb{R}^N} \frac{\Theta(u(x)-u(y))}{\mid x-y \mid^{N+sp}}\,dy \\ = \lim_{\sigma\to 0}\int_{\mathbb{R}^N\setminus B_\sigma(x)} \frac{\Theta(u(x)-u(y))}{\mid x-y \mid^{N+sp}}\,dy,\end{aligned}$$

where $\Theta(\xi) =\mid \xi \mid^{p-2} \xi$, P.V. often abbreviated to signify "in the principal value sense" and $B_\sigma(x) = \{z \in \mathbb{R}^N : \mid x - z \mid< \sigma\}$. Additionally, we consider that u_0 and f are nonnegative functions, satisfying the conditions

$$f \in L^1(Q_T) \quad \text{and} \quad u_0 \in L^1(\Omega). \tag{1.2}$$

There has been a growing interest in the study of nonlocal operators due to their relevance in both pure mathematics and practical applications. These operators naturally emerge in various fields, such as water waves [14, 15, 31], crystal dislocations [33], elasticity problems [29], game theory [11], phase transition [3, 10, 30], Lévy processes [5], flame propagation [12] and quasi-geostrophic flows [13, 21]. A key example of these nonlocal operators is the fractional p-Laplacian operator, which is crucial in various physical and mathematical contexts, including its application in image processing [6, 20, 17].

The value of $(-\Delta)^s_p u(x)$ for each $x \in \Omega$ is not solely dependent on the values of the Ω, but also on the entire $\mathbb{R}^N$, which is a typical characteristic of this operator. The reason is that $u(x)$ is the expected value for a random variable that is linked to a jump process that can leap far from the point x without any prior knowledge. Brownian motion's continuity properties lead to landing on $\partial\Omega$ upon exiting Ω in the classical case. However, exiting can result in landing anywhere outside of Ω as a result of the jump process's nature. In this regard, assigned values of u in $\mathbb{R}^N \setminus \Omega$ rather than on $\partial\Omega$ is how the non-homogeneous Dirichlet boundary condition works.

Numerous references in the literature address the nonlocal Laplacian operator (when $p = 2$). For example, in Ref. [18], Leonori et al. considered problem (1.1) and established both the existence and uniqueness of the solution using duality and approximation techniques, showing that it resides in an appropriate fractional Sobolev space.

In Ref. [35], Vázquez studied problem (1.1) with $f = 0$. He proved the existence of a solution to problem (1.1), referred to as the friendly giant, which takes the form $U(t,x) = t^{\frac{-1}{p-2}}F(x)$, with $F(x)$ is a positive function in Ω, C^α-Hölder continuous and solves an interesting nonlocal elliptic problem. In [34], Vázquez worked on problem (1.1) with $f = 0$ and with the initial condition $\lim\limits_{t\to 0} u(t,x) = u_0(x)$. He demonstrated that for each mass $K > 0$, there is only one self-similar solution with initial data $K\delta(x)$ taking the form

$$V(t,x,K) = K^{sp\gamma}t^{-N\gamma}G(K^{(2-p)\gamma}xt^{-\gamma}), \tag{1.3}$$

where $\gamma = \dfrac{1}{sp - N(2-p)}$ and $G(r)$ is a positive, continuous, radially symmetric function with $G(r) \underset{+\infty}{\sim} r^{-(N+sp)}$. He also showed in the case $u_0 \in L^1(\mathbb{R}^N)$, the estimate $\lim_{t\to+\infty} \| u(t) - V_K(t) \|_1 = 0$, with V_K defined by (1.3).

In Ref.[22], Mazón et al. considered the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = (-\Delta)_p^s u & \text{in }]0,T[\times\Omega \\ u = 0 & \text{on }]0,T[\times(\mathbb{R}^N \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.4}$$

The authors showed that problem (1.4) has a unique strong solution when $u_0 \in L^2(\Omega)$, using Nonlinear Semigroup Theory.

In [32], Teng et al. establish the existence and uniqueness of nonnegative renormalized and entropy solutions to problem (1.1) with L^1 data. They also demonstrate the equivalence of these renormalized and entropy solutions and provide a comparison result.

In Ref. [19], Liao established the local Hölder regularity for weak solutions. The author's proof employs DeGiorgi's iterative technique while enhancing DiBenedetto's intrinsic scaling method. In Ref. [7], Brasco et al. showed space-time Hölder continuity estimates for weak solutions, specifying the exponents explicitly. Their approach relies on a method of iterated discrete differentiation inspired by Moser's technique.

Drawing from the existing literature, we examine the existence and uniqueness of integral solutions for problem (1.1). As far as we are aware, there have been no prior results concerning this specific type of solution for problem (1.1). For this argument, our work is original.

Demonstrating the existence and uniqueness of an integral solution to problem (1.1) presents several challenges. One significant challenge is the involvement of nonlocal and nonlinear fractional p-Laplacian operator. Another issue arises when considering the condition (1.2). Fortunately, innovative methods introduced by Akagi et al. [1], Alaa et al. [2], [25] and Roubicek [28] offer solutions to this challenge. We will be employing these techniques.

The outline of this paper is: Section 2 provides a review notations and definitions for the pertinent fractional Sobolev-type spaces, along with some subdifferential calculus tools. Section 3 focuses on the solvability of problem (1.1) when the data belongs to L^2, we demonstrate the existence and uniqueness of a weak solution using maximal monotone operator theory. In section 4, we prove that problem (1.1) admits a unique integral solution when the data are only in L^1 performing accretive operator theory.

2. Foundational concepts and intermediate outcomes

The main purpose of this section is to provide a summary of certain definitions and fundamental properties of fractional Sobolev spaces, which are essential for the discussion of Problem (1.1). Additionally, we will revisit some characteristics of the subdifferentials of lower semi-continuous, convex and proper functionals within a Hilbert space.

2.1. Fractional Sobolev spaces

We now recall the primary notations and definitions for the pertinent fractional Sobolev-type spaces that will be used throughout this paper. Let Ω be an open subset in $\mathbb{R}^N$, $p \in [1,\infty)$ and $s \in (0,1)$, we define the fractional Sobolev space $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) = \left\{ \phi \in L^p(\Omega) \text{ such that } \frac{\mid \phi(x) - \phi(z) \mid}{\mid x - z \mid^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

which is a Banach space furnished with the following norm

$$\| \phi \|_{W^{s,p}(\Omega)} = \left(\| \phi \|^p_{L^p(\Omega)} + [\phi]^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}},$$

where

$$[\phi]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{\mid \phi(x) - \phi(z) \mid^p}{\mid x - z \mid^{N+sp}} \, dx \, dz \right)^{\frac{1}{p}}.$$

$$W_0^{s,p}(\Omega) = \left\{ \phi \in W^{s,p}(\mathbb{R}^N) \text{ such that } \phi = 0 \ in \ \mathbb{R}^N \setminus \Omega \right\},$$

provided with the following norm

$$\| \phi \|_{W_0^{s,p}(\Omega)} = [\phi]_{W^{s,p}(\Omega)}.$$

To learn more about fractional Sobolev spaces, the reader is recommended to read papers [8, 24, 26].

Lemma 2.1. *([16], Lemma 2.3) for all* $u, v \in W_0^{s,p}(\Omega)$*, we have*

$$\langle (-\Delta)^s_p u, v \rangle = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\Theta(u(x) - u(y))(v(x) - v(y))}{\mid x - y \mid^{N+sp}} \, dx \, dy, \tag{2.1}$$

where $\langle ., . \rangle$ denote the duality pairing betwen $W_0^{s,p}(\Omega)$ and $W^{-s,p'}(\Omega)$.

Proposition 2.2. *([26], Proposition 2.1)*
The energy functional $\mathcal{K}^s_p : W_0^{s,p}(\Omega) \to \mathbb{R}$ *defined by*

$$\mathcal{K}^s_p(u) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mid u(x) - u(y) \mid^p}{\mid x - y \mid^{N+sp}} \, dx \, dy,$$

is well defined, bounded and possesses the Gâteaux derivative

$$\langle (\mathcal{K}^s_p)'(u), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\Theta(u(x) - u(y))(v(x) - v(y)}{\mid x - y \mid^{N+sp}} \, dx \, dy, \tag{2.2}$$

for any $u, v \in W_0^{s,p}(\Omega)$.

2.2. Subdifferentials

Consider the Hilbert space $\mathbf{H}$ endowed with the inner product denoted by $(.,.)_{\mathbf{H}}$ and let $\Psi : \mathbf{H} \to [0,+\infty]$ a lower semi-continuous, proper and convex functional defined on the domain

$$\mathcal{D}(\Psi) := \{\vartheta \in \mathbf{H} \ : \ \Psi(\vartheta) < +\infty\}.$$

The functional Ψ has a subdifferential $\partial\Psi$ that is defined through

$$\begin{cases} \mathcal{D}(\partial\Psi) := \{\vartheta \in \mathcal{D}(\Psi) : \exists \xi \in \mathbf{H}, \forall \nu \in \mathbf{H} : \Psi(\nu) - \Psi(\vartheta) \geq (\xi, \nu - \vartheta)_{\mathbf{H}}\}, \\ \partial\Psi(\vartheta) := \{\xi \in \mathbf{H} : \forall \nu \in \mathbf{H} : \Psi(\nu) - \Psi(\vartheta) \geq (\xi, \nu - \vartheta)_{\mathbf{H}}\}. \end{cases}$$

For additional information on the concept of subdifferentials, readers are encouraged to consult [9, 27].

Definition 2.3. *Consider $\mathfrak{B}$ as a Banach space and $x, \tilde{x} \in \mathfrak{B}$. The directional derivative of $||\cdot||$ at x along the vector $\tilde{x}$ is*

$$[x, \tilde{x}] := \lim_{\varepsilon \to 0^+} \frac{\| x + \varepsilon\tilde{x} \| - \| x \|}{\varepsilon}.$$

Proposition 2.4. *(see Proposition 57.2 page 582 in [36])*

i) *For any $x, \varpi_1, \varpi_2 \in L^1(\Omega)$, we have*

$$-[x, \varpi_1 + \varpi_2] \leq [x, -\varpi_1] - [x, \varpi_2] \tag{2.3}$$

$$[x, \varpi_1] \leq \| \varpi_1 \|_{L^1(\Omega)} . \tag{2.4}$$

ii) *For all $\tilde{v} \in \mathcal{Z}$, it follows that*

$$\frac{d}{dt} \|\tilde{v}\|_{L^1(\Omega)} = - \left[\tilde{v}, -\frac{\partial \tilde{v}}{\partial t}\right], \tag{2.5}$$

where $\mathcal{Z} = \{\tilde{v} : Q_T \to \mathbb{R}, \quad \tilde{v}(t) \in L^1(\Omega), \ \text{and} \ \dfrac{\partial \tilde{v}(t)}{\partial t} \in L^1(\Omega)\}$

3. Existence and Uniqueness of a Weak Solution with L^2 Data

In this section, we show the existence and uniqueness of weak solutions of problem (1.1) with $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$ by using the theory of evolution equations guided by subdifferential operators. The following definition presents the concept of a weak solution to our problem.

Definition 3.1. *A function $u \in C\left([0,T]; L^2(\Omega)\right)$ is said to be a weak solution of problem* (1.1)*, if the following conditions are all satisfied:*

- *$u \in L^2(Q_T)$, $\frac{\partial u}{\partial t} \in L^2(Q_T)$ and $(-\Delta)^s_p u \in L^2(Q_T)$,*
- *$u(0) = u_0$,*
- *for all $w \in W^{s,p}_0(Q_T) \cap L^2(Q_T)$, it holds that*

$$\int_0^T \langle \frac{\partial u}{\partial t}, w\rangle dt + \int_0^T \langle(-\Delta)^s_p u, w\rangle dt = \int_0^T \langle f, w\rangle dt. \tag{3.1}$$

Theorem 3.2. *Assume that $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$. Then there exists a unique weak solution $u = u(t,x)$ of the problem* (1.1)*that fulfills the conditions specified in the Definition* (3.1)*.*

Proof. To prove Theorem 3.2, we integrate the theory of evolution equations (see Chapter III in [9]) with the subdifferential approach. To this end, let us consider the energy functional $\Phi_p^s : L^2(\Omega) \to [0,\infty]$ defined by

$$\Phi_p^s(u) = \begin{cases} \dfrac{1}{2p}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{| u(x)-u(y) |^p}{| x-y |^{N+sp}}\,dx\,dy \\ \text{if } u \in W_0^{s,p}(\Omega)\cap L^2(\Omega) + \infty \text{ otherwise.} \end{cases} \tag{3.2}$$

Using Fatou's Lemma, we can conclude that Φ_p^s is lower semicontinuous in the space $L^2(\Omega)$. Furthermore, due to the fact that Φ_p^s is proper and convex, we can deduce that the subdifferential $\partial\Phi_p^s$ is identified as a maximal monotone operator in $L^2(\Omega)$. Next, we will demonstrate that

$$\partial\Phi_p^s(u) = (-\Delta)_p^s u.$$

Let $u \in W_0^{s,p}(\Omega)\cap L^2(\Omega)$ and set $h := \partial\Phi_p^s(u)$. Then for all $v \in W_0^{s,p}(\Omega)\cap L^2(\Omega)$, we have

$$\Phi_p^s(v) - \Phi_p^s(u) \geq (h, v-u).$$

Then

$$\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{| v(x)-v(y) |^p - | u(x)-u(y) |^p}{| x-y |^{N+sp}}\,dx\,dy \geq \int_\Omega h(v-u)dx. \tag{3.3}$$

Taking $v = \varepsilon w + (1-\varepsilon)u$ in (3.3) with $w \in W_0^{s,p}(\Omega)\cap L^2(\Omega)$ and $\varepsilon \in [0,1]$, we obtain

$$\begin{aligned}\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{| (\varepsilon w+(1-\varepsilon)u)(x) - (\varepsilon w+(1-\varepsilon)u)(y) |^p - | u(x)-u(y) |^p}{| x-y |^{N+sp}}\,dx\,dy \\ &\geq \varepsilon\int_\Omega h(w-u)dx.\end{aligned}$$

Consequently

$$\lim_{\varepsilon\to 0}\frac{\Phi_p^s(u+\varepsilon(w-u)) - \Phi_p^s(u)}{2\varepsilon} \geq \int_\Omega h(w-u)dx.$$

Hence

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{| u(x)-u(y) |^{p-2}\,(u(x)-u(y))((w-u)(x)-(w-u)(y))}{| x-y |^{N+sp}}\,dx\,dy \\ &\geq \int_\Omega h(w-u)dx.\end{aligned} \tag{3.4}$$

Replacing w by $w+u$ in (3.4), we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{| u(x)-u(y) |^{p-2}\,(u(x)-u(y))(w(x)-w(y))}{| x-y |^{N+sp}}\,dx\,dy \\ &\geq \int_\Omega hw dx.\end{aligned} \tag{3.5}$$

As (3.5) is valid when w is replaced by $-w$, we deduce

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))(w(x)-w(y))}{\mid x-y\mid^{N+sp}}\,dx\,dy=\int_{\Omega}hwdx, \quad (3.6)$$

for all $w\in W_0^{s,p}(\Omega)\cap L^2(\Omega)$. As a result

$$h=(-\Delta)_p^s u.$$

Consequently, our problem (1.1) is transformed into the following Cauchy problem

$$\begin{cases}\frac{\partial u(t)}{\partial t}+\partial\Phi_p^s u(t)=f(t) & \text{in } L^2(\Omega) \text{ for all } t\in]0,T[,\\ u(0)=u_0.\end{cases}$$

This type of abstract evolution equation was extensively studied by H. Brézis (refer to Chapter III of [9]). Hence, we deduce that for any $f\in L^2(Q_T)$ and $u_0\in L^2(\Omega)$, problem (1.1) admits a unique weak solution as described in Definition (3.1). □

4. Existence and uniquness of integral solutions

In this section, we prove the existence and uniqueness of integral solutions to problem (1.1) when $(f,u_0)\in L^1(Q_T)\times L^1(\Omega)$. To begin, we outline the concept of accretive operators in the context of Banach spaces.

Definition 4.1. *An operator* $\mathbf{A}$ *defined on a subset* $\mathcal{D}(\mathbf{A})$ *of a Banach space* $\mathfrak{B}$ *with values in* $\mathfrak{B}$ *is said to be accretive if*

$$\textit{for all}\, w_1,w_2\in\mathcal{D}(\mathbf{A}):\ [w_1-w_2,\mathbf{A}(w_1)-\mathbf{A}(w_2)]\geq 0. \quad (4.1)$$

By ([36], Section 57.2), we know that (4.1) is equivalent to

$$\text{for all}\,\lambda>0; w_1,w_2\in\mathcal{D}(\mathbf{A}):\ \parallel w_1-w_2\parallel\leq\parallel w_1+\lambda\mathbf{A}(w_1)-w_2-\lambda\mathbf{A}(w_2)\parallel. \quad (4.2)$$

In this section, we will use $\mathfrak{B}=L^1(\Omega)$ and we introduce the operator $\mathbf{A}$, which is defined as follows

$$\mathcal{D}(\mathbf{A}):=\{w\in W_0^{s,p}(\Omega)\cap L^2(\Omega);\ (-\Delta)_p^s w\in L^2(\Omega)\} \quad (4.3)$$

$$\mathbf{A}(w)=(-\Delta)_p^s w. \quad (4.4)$$

Lemma 4.2. *The operator* $\mathbf{A}$ *specified by* (4.3)-(4.4) *is accretive with respect to the* $L^1(\Omega)$ *norm.*

Proof. To establish Lemma (4.2), we consider a strictly positive value λ and verify (4.2). for this purpose, let us consider for $i=1,2, f_i\in L^2(\Omega)$ and we consider the respective weak solution $u_i\in W_0^{s,p}(\Omega)\cap L^2(\Omega)$ of the following boundary-value problem

$$\begin{cases}u_i+\lambda(-\Delta)_p^s u_i=f_i & \text{in } \Omega\\ u_i=0 & \text{on } \mathbb{R}^N\setminus\Omega.\end{cases} \quad (4.5)$$

Remark 4.3. By results obtained in Section 3, we have Φ_p^s is proper, convex and lower semicontinuous. In addition, we know that $\mathbf{A}=\Phi_p^s$. Hence, $\mathbf{A}$ is a maximal monotone operator. Thus, the operator $I+\lambda\mathbf{A}$ is bijective from $\mathcal{D}(\mathbf{A})$ into $L^2(\Omega)$. Consequently, for any $f_i\in L^2(\Omega)$, there exist a unique weak solution $u_i\in W_0^{s,p}(\Omega)\cap L^2(\Omega)$ to problem (4.5).

To verify (4.2), we show that

$$\|u_1 - u_2\|_{L^1(\Omega)} \leq \|f_1 - f_2\|_{L^1(\Omega)} . \tag{4.6}$$

The weak solution to (4.5) satisfies for any $v \in W^{s,p}(\Omega) \cap L^2(\Omega)$, the equality

$$\int_\Omega u_i(x)v(x) + \lambda \int_\Omega \langle (-\Delta)_p^s u_i, v\rangle = \int_\Omega f_i(x)v(x)\ \mathrm{d}x. \tag{4.7}$$

To demonstrate (4.6), we subtract the equation (4.7) for $i = 2$ from the equation (4.7) for $i = 1$, and choose
$v = \mathrm{sgn}_\varepsilon (u_1 - u_2) \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$ where

$$\mathrm{sgn}_\varepsilon(\zeta) = \begin{cases} -1 & \text{for } \zeta \leq -\varepsilon, \\ \zeta/\varepsilon & \text{for } -\varepsilon \leq \zeta \leq \varepsilon, \\ 1 & \text{for } \zeta \geq \varepsilon. \end{cases}$$

Thus we obtain

$$\int_\Omega (u_1(x) - u_2(x))\, \mathrm{sgn}_\varepsilon (u_1 - u_2)\ \mathrm{d}x + \lambda \int_\Omega \langle (-\Delta)_p^s u_1 - (-\Delta)_p^s u_2, \mathrm{sgn}_\varepsilon (u_1 - u_2)\rangle$$
$$= \int_\Omega (f_1(x) - f_2(x))\, \mathrm{sgn}_\varepsilon (u_1 - u_2)\ \mathrm{d}x \leq \int_\Omega |f_1 - f_2|\, \mathrm{d}x = \|f_1 - f_2\|_{L^1(\Omega)} .$$

The term multiplied by λ is positive, moreover

$$\lim_{\varepsilon\to 0} \int_\Omega (u_1 - u_2)\, \mathrm{sgn}_\varepsilon (u_1 - u_2)\, \mathrm{d}x = \int_\Omega |u_1 - u_2|\, \mathrm{d}x = \|u_1 - u_2\|_{L^1(\Omega)} .$$

Thus, (4.6) is proved. □

We will also need the subsequent result. For simplicity, we use the notation $u(t) = u(t, x)$, $f(t) = f(t, x)$, and so forth. The notation $[.,.]$ will be used to denote the derivative in a given direction of the norm in $L^1(\Omega)$.

Lemma 4.4. *Let $f, g \in L^2(Q_T)$, $u_0, w_0 \in L^2(\Omega)$ and let u, w be two weak solutions to the problem* (1.1)*, hence for all $0 \leq \sigma \leq t \leq T$ we get*

$$\| u(t) - w(t) \|_{L^1(\Omega)} \leq \| u(\sigma) - w(\sigma) \|_{L^1(\Omega)} + \int_\sigma^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)]d\varsigma. \tag{4.8}$$

Proof. By Lemma 4.2, we have the operator defined by (4.3)-(4.4) is accretive, hence (4.1) holds. Then for $u_1, u_2 \in \mathcal{D}(\mathbf{A})$ we obtain

$$[u_1 - u_2, (-\Delta)_p^s u_1 - (-\Delta)_p^s u_2] \geq 0.$$

We have $(-\Delta)_p^s u \in L^2(Q_T)$. Hence, we can take $u_1 := u(t) \in \mathcal{D}(\mathbf{A})$ for a.e $t \in [0, T]$. similarly, we choose $u_2 := w(t) \in \mathcal{D}(\mathbf{A})$ for a.e $t \in [0, T]$. Utilizing these substitutions and considering (1.1), $(-\Delta)_p^s u(t)$ can be substituted by $f(t) - \frac{\partial u(t)}{\partial t}$. This leads to the result that, for almost every $t \in [0, T]$,

$$-[u(t) - w(t), -\frac{\partial u(t)}{\partial t} + f(t) - (-\Delta)_p^s w(t)] \leq 0. \tag{4.9}$$

Using successively (2.3), (2.5) and the inequality (4.9), we deduce the following estimate,

$$\begin{aligned}
&\| u(t) - w(t) \|_{L^1(\Omega)} - \| u(\sigma) - w(\sigma) \|_{L^1(\Omega)} \\
&= \int_\sigma^t \frac{d}{dt} \| u(\varsigma) - w(\varsigma) \|_{L^1(\Omega)} \, d\varsigma \\
&= \int_\sigma^t -[u(\varsigma) - w(\varsigma), \frac{\partial w(\varsigma)}{\partial \varsigma} - \frac{\partial u(\varsigma)}{\partial \varsigma}] d\varsigma \\
&\leq \int_\sigma^t [u(\varsigma) - w(\varsigma), -\frac{\partial w(\varsigma)}{\partial \varsigma} + f(\varsigma) - (-\Delta)_p^s w(\varsigma)] \\
&\quad - [u(\varsigma) - w(\varsigma), -\frac{\partial u(\varsigma)}{\partial \varsigma} + f(\varsigma) - (-\Delta)_p^s w(\varsigma)] \\
&\leq \int_\sigma^t [u(\varsigma) - w(\varsigma), -\frac{\partial w(\varsigma)}{\partial \varsigma} + f(\varsigma) - (-\Delta)_p^s w(\varsigma)].
\end{aligned}$$

We have

$$-\frac{\partial w(\varsigma)}{\partial \varsigma} - (-\Delta)_p^s w(\varsigma) = -g(\varsigma),$$

consequently

$$\| u(t) - w(t) \|_{L^1(\Omega)} \leq \| u(\sigma) - w(\sigma) \|_{L^1(\Omega)} + \int_\sigma^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)].$$

□

Lemma 4.5. *Let $f, g \in L^2(Q_T)$, $u_0, v_0 \in L^2(\Omega)$ and let u, w be two weak solutions to Problem* (1.1)*, hence*

$$\| u - w \|_{C([0,T];L^1(\Omega))} \quad \leq \| f - g \|_{L^1(Q_T)} + \| u_0 - w_0 \|_{L^1(\Omega)} . \tag{4.10}$$

Proof. Using (4.8) with $\sigma = 0$, moreover taking into account (2.4), we obtain

$$\begin{aligned}
&\| u(t) - w(t) \|_{L^1(\Omega)} - \| u_0 - w_0 \|_{L^1(\Omega)} \\
&\leq \int_0^t [u - w, f - g] \\
&\leq \int_0^t \| f - g \|_{L^1(\Omega)} \\
&\leq \| f - g \|_{L^1(Q_T)} .
\end{aligned} \tag{4.11}$$

□

Remark 4.6. For any $(f, u_0) \in L^1(Q_T) \times L^1(\Omega)$, it is possible to construct a sequence of smooth data $(f_n, u_{0n}) \in L^2(Q_T) \times L^2(\Omega)$ such that

$$f_n \to f \textit{ in } L^1(Q_T) \text{ and } u_{0n} \to u_0 \textit{ in } L^1(\Omega). \tag{4.12}$$

Let (u_n, u_m) be the weak solutions to problem (1.1) corresponding to the data sets (f_n, u_{0n}) and (f_m, u_{0m}), by Lemma 4.5, it can be inferred that

$$\| u_n - u_m \|_{C([0,T];L^1(\Omega))} \quad \leq \| f_n - f_m \|_{L^1(Q_T)} + \| u_{0n} - u_{0m} \|_{L^1(\Omega)} . \tag{4.13}$$

Therefore, every weak solution u_n with data that satisfies (4.12), constitutes a Cauchy sequence within $C([0,T];L^1(\Omega))$.

The remark 4.6 prompts the introduction of a new type of solution for problem (1.1), which we will term a generalized solution to (1.1).

Definition 4.7. *We call u a generalized solution to* (1.1) *when it is the limit in $C([0,T];L^1(\Omega))$ of a sequence of weak solutions whose corresponding data satisfies the requirements of* (4.12)*.*

Utilizing Remark 4.6 along with Definition 4.7, it is possible to rigorously demonstrate the subsequent proposition.

Proposition 4.8. *Suppose that $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$, hence*

(i) (1.1) *possesses only one generalized solution as described in Definition 4.7. Moreover, this generalized solution is independent of the specific sequences chosen to satisfy condition* (4.12)*.*

(ii) *every weak solution to problem* (1.1) *is the generalized solution.*

Definition 4.9. *Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$, we call $u \in C([0,T];L^1(\Omega))$ an integral solution of* (1.1) *if*

$$\| u(t) - w(t) \|_{L^1(\Omega)} \leq \| u_0 - w_0 \|_{L^1(\Omega)} + \int_0^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)] d\varsigma \tag{4.14}$$

holds for any $t \in [0,T]$, in which w is a generalized solution to

$$\begin{cases} \frac{\partial w}{\partial t} + (-\Delta)_p^s w = g & \textit{in }]0,T[\times\Omega \\ w = 0 & \textit{on }]0,T[\times(\mathbb{R}^N \setminus \Omega) \\ w(0,x) = w_0(x) & \textit{in } \Omega, \end{cases} \tag{4.15}$$

for $g \in L^1(Q_T)$ and $w_0 \in L^1(\Omega)$.

In the following theorem, we present the second main result of our paper.

Theorem 4.10. *Suppose that $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. Then problem* (1.1) *admits a unique integral solution.*

Proof. We propose dividing the demonstration of Theorem 4.10 into two phases: the first phase will concentrate on establishing the existence of an integral solution to (1.1), and the second phase will focus on proving the uniqueness of the integral solution derived.

Existence of an integral solution:

Consider w as a generalized solution to Problem (4.15), meaning there exists a sequence $(w_n) \in L^2(Q_T)$, such that

$$w_n \longrightarrow w \textit{ in } C([0,T];L^1(\Omega))$$

with w_n being the weak solution that fulfills the equation

$$\int_0^T \langle \frac{\partial w_n}{\partial t}, v \rangle dt + \int_0^T \langle (-\Delta)_p^s w_n, v \rangle dt = \int_0^T \langle g_n, v \rangle dt, \tag{4.16}$$

for all $v \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$ with $(g_n, w_{0n}) \in L^2(Q_T) \times L^2(\Omega)$ converging to $(g, w_0) \in L^1(Q_T) \times L^1(\Omega)$.
Furthermore, let u a generalized solution to (1.1).
According to definition 4.9, this implies the existence of a sequence $(u_n) \in L^2(Q_T)$ in such a manner that

$$u_n \longrightarrow u \ in \ C([0,T]; L^1(\Omega))$$

and u_n is a weak solution satisfies

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, v \rangle dt + \int_0^T \langle (-\Delta)_p^s u_n, v \rangle dt = \int_0^T \langle f_n, v \rangle dt, \tag{4.17}$$

for all $v \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$, and

$$f_n \in L^2(Q_T); \ f_n \longrightarrow f \ in \ L^1(Q_T)$$
$$u_{0n} \in L^2(\Omega); \ u_{0n} \longrightarrow u_0 \ in \ L^1(\Omega)$$

holds, then (4.8) with $\sigma = 0$ yields that

$$\| u_n(t) - w_n(t) \|_{L^1(\Omega)} \leq \| u_{0n} - w_{0n} \|_{L^1(\Omega)} + \int_0^t [u_n(\varsigma) - w_n(\varsigma), f_n(\varsigma) - g_n(\varsigma)] \tag{4.18}$$

pass to limit with $n \longrightarrow +\infty$, we have obviously

$$\| u_n(t) - w_n(t) \|_{L^1(\Omega)} \longrightarrow \| u(t) - w(t) \|_{L^1(\Omega)}$$

and

$$\| u_{0n} - w_{0n} \|_{L^1(\Omega)} \longrightarrow \| u_0 - w_0 \|_{L^1(\Omega)}$$

and

$$\limsup_{n \longrightarrow +\infty} \int_0^t [u_n(\varsigma) - w_n(\varsigma), f_n(\varsigma) - g_n(\varsigma)]_\Omega d\varsigma \leq \int_0^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)] d\varsigma.$$

Altogether, we get

$$\| u(t) - w(t) \|_{L^1(\Omega)} \leq \| u_0 - w_0 \|_{L^1(\Omega)} + \int_0^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)] d\varsigma \tag{4.19}$$

according to the definition (4.9), We have demonstrated that the generalized solution is, in fact, also an integral solution. This confirms that an integral solution exists.
Uniqueness of integral solution:
Let u and $\tilde{w}$ be two integral solutions to problem (1.1). Then, it follows that

$$\| \tilde{w}(t) - w(t) \|_{L^1(\Omega)} \leq \| u_0 - w_0 \|_{L^1(\Omega)} + \int_0^t [\tilde{w} - w, f - g] d\varsigma. \tag{4.20}$$

Since u satisfies (4.19), we can substitute w with u, then

$$\int_0^t [\tilde{w} - w, f - g] d\varsigma = 0.$$

Consequently, we obtain

$$\| \tilde{w}(t) - u(t) \|_{L^1(\Omega)} \leq \| u_0 - u_0 \|_{L^1(\Omega)} = 0.$$

The result is that $\tilde{w}(t) = u(t) \ for \ all \ t \in [0,T]$. □

Conclusion

In this paper, we have investigated a class of non-local parabolic equations involving a fractional p-Laplacian operator. By employing subdifferential techniques and the theory of accretive operators, we established the existence and uniqueness of weak solutions for initial data in L^2, and extended the analysis to prove the existence and uniqueness of integral solutions when the initial data belong to L^1.

The results obtained contribute to a deeper understanding of regularity and solution structure in non-local evolution equations. They also highlight the effectiveness of combining classical nonlinear analysis tools with methods adapted to non-local operators.

As a direction for future research, it would be interesting to consider the case where the source term f depends on the unknown function u. This would involve studying nonlinearities of the type $f = f(u)$, which may require the use of fixed-point techniques or variational methods. Such an extension could provide further insight into nonlinear dynamics driven by non-local operators and broaden the applicability of the current theory.

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Some existence results for a class of parabolic equations with nonlinear boundary conditions

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Abstract. In this paper, we are interested in studying the weak solutions for the following nonlinear parabolic problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \ t > 0, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega, \ t > 0, \\ u(x;0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Using the Galerkin approximation and a family of potential wells, we establish the existence of a global weak solution under appropriate conditions. Additionally, we provide a result on the blow-up and asymptotic behavior of certain solutions with positive initial energy.

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Keywords: Parabolic problem, global existence, blow-up.

1. Introduction and main results

Parabolic problems are an important class of partial differential equations (PDEs) that frequently appear in models of physical phenomena, such as heat diffusion (see [2, 4, 6, 8, 5]), diffusion of substances in fluids, and population dynamics (see [13, 12, 11, 3]). The motivation for studying parabolic problems lies in their ability to describe time-evolving processes in which a quantity diffuses or propagates in a medium. For example, in engineering, parabolic PDEs are used to design heat management systems, such as cooling systems in electronic devices, or to study the properties of materials under various temperature conditions (see [23, 19, 21, 1]).

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In this paper, we deal with the following nonlinear parabolic problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega,\ t > 0, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega,\ t > 0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and $g(u)$ satisfies the following conditions:

$$\text{(C)} \begin{cases} g \in C^1 \text{ and } g(0) = g'(0) = 0; \\ g(u) \text{ is monotone, concave for } u < 0 \text{ and convex for } u > 0; \\ (q+1)G(u) \leqslant ug(u), |ug(u)| \leqslant \mu|G(u)|; \end{cases}$$

where

$$G(u) = \int_0^u g(s)\mathrm{d}s,$$

$$\begin{cases} 2 < q+1 \leqslant \mu < \infty \text{ if } n = 2, \\ 2 < q+1 \leqslant \mu \leqslant \frac{2(n-1)}{n-2} \text{ if } n \geqslant 3, \end{cases} \quad \text{and} \quad \begin{cases} 1 \leq \mu \leq p^{\partial} \text{ if } p \neq n, \\ 1 \leq \mu < \infty \text{ if } p = n, \end{cases}$$

with

$$p^{\partial} := \begin{cases} \frac{p(n-1)}{n-p} \text{ if } 1 < p < n, \\ \infty \text{ if } p \geq n. \end{cases}$$

Equations of the form

$$u_t - \Delta_p u = u^q,$$

are also called the non-Newtonian filtration equations, which are known as fast diffusive for $1 < p < 2$, and as slow diffusive for $p > 2$.
From a physical point of view, the nonlinear boundary value condition:

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u),$$

can be physically interpreted as the nonlinear radial law (see [9, 16]).

In the literature, many works have been devoted to nonlinear parabolic problems (see [17, 10, 15]). For example, Y. Li and C. Xie [17] have considered the following Dirichlet problem:

$$\begin{cases} u_t - \Delta_p u = \lambda|u|^{q-2}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \text{in } \Omega. \end{cases}$$

The authors have given a complete overview of the explosion criteria for the above problem. More specifically, in the critical case $q = p > 2$, they proved that if $\lambda > \lambda_1$, there are no global non-trivial weak solutions, and that if $\lambda \leq \lambda_1$, all weak solutions are global, where λ_1 is the first eigenvalue of the following eigenvalue problem:

$$\begin{cases} -\Delta_p \varphi = \lambda \mid \varphi \mid^{p-2} \varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, Y. Jingxue et al. in [10] studied the following nonlinear parabolic problem:

$$\begin{cases} u_t - \Delta_q u = u^{q_1} & \text{in } \Omega,\ t > 0, \\ |\nabla u|^{q-2} \frac{\partial u}{\partial \nu} = u^{q_2} & \text{on } \partial\Omega,\ t > 0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $q > 1, q_1, q_2 > 0$ are all constants and $u_0 \in L^\infty(\Omega) \cap W^{1,q}(\Omega)$. The authors studied the critical exponents of the explosion of the evolutionary q-Laplacian with multiple sources, and showed that the critical values of (q_1, q_2) are $q_1^* = 1$ and $q_2^* = \min\{1, 2(q-1)/q\}$.
Recently, in [15], A. Lamaizi et al. have considered the following problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega,\ t > 0, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^q u & \text{on } \partial\Omega,\ t > 0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain for $n \geq 2$ with smooth boundary $\partial\Omega$, $\lambda > 0$, p and q satisfy

$$(H) \qquad \frac{2n}{n+2} \leq p < +\infty\ ,\ p < 2 + q \quad \text{and} \quad \begin{cases} 1 \leq q + 2 \leq p^\partial \text{ if } p \neq n, \\ 1 \leq q + 2 < \infty \text{ if } p = n. \end{cases}$$

By using the Galerkin approximation, they established the existence of global weak solution and finite time blow-up under some suitable conditions. So a natural question arises, can we obtain some qualitative results such as the existence and blow up of solutions if we replace the term $\lambda |u|^q u$ by the function $g(u)$ which satisfies condition (C) ? Then, the goal of this article is to give a positive answer to this question, more precisely, we will establish existence and blow up results by applying Galerkin approximation and similar techniques to those used in [15].

In order to investigate our problem, it is necessary to define some functionals and sets:

$$B(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_{\partial\Omega} G(u)d\rho,$$

$$A(u) = \|u\|_{1,p}^p - \int_{\partial\Omega} ug(u)d\rho,$$

$$S = \left\{u \in W^{1,p}(\Omega) \mid A(u) > 0, B(u) < h\right\} \cup \{0\},$$

where

$$h = \inf_{u \in Y} B(u),$$

$$Y = \left\{u \in W^{1,p}(\Omega) \mid A(u) = 0, \|u\|_{1,p} \neq 0\right\},$$

and

$$U = \left\{u \in W^{1,p}(\Omega) \mid A(u) < 0, B(u) < h\right\}.$$

We are now in a position to state the main results of this paper.

Theorem 1.1. (**Global Existence**)
Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy (C). Suppose that $0 < B(u_0) < h$ and $A(u_0) > 0$. Then, problem (1.1) admits a global weak solution $u(t) \in L^\infty\left(0, \infty; W^{1,p}(\Omega)\right) \cap C\left([0,T]; L^2(\Omega) \times L^2(\partial\Omega, \rho)\right)$ with $u_t(t) \in L^2\left(0, \infty; L^2(\Omega)\right)$ and $u(t) \in S$ for $0 \leqslant t < \infty$.

Theorem 1.2. (**Finite Time Blow-up**)
Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy (C). Suppose that $B(u_0) < h$ and $A(u_0) < 0$. Then, the weak solution of problem (1.1) must blow up in finite time i.e. there exists a $T > 0$ such that

$$\lim_{t \to T} \int_0^t \|u\|_2^2 \, \mathrm{d}\tau = +\infty. \tag{1.2}$$

Theorem 1.3. (**Asymptotic Behavior**)
Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy (C). Suppose also that $B(u_0) < h$ and $A(u_0) > 0$. Then, for the weak global solution $u(t)$ of problem (1.1) there exists a constant $\omega > 0$ such as:

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 \, \mathrm{e}^{-\omega t}, \quad 0 \leq t < \infty. \tag{1.3}$$

2. Preliminaries

Throughout this work, we denote the Lebesgue space $L^p(\Omega)$ by :

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^p dx < +\infty \right\},$$

endowed with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, we denote

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \operatorname{ess-}\sup_\Omega |u| < +\infty \right\},$$

equipped with the norm

$$\operatorname{ess-}\sup_\Omega |u| = \inf\{C > 0 \text{ such that } |u(x)| \leq C \text{ a.e. } \Omega\}.$$

Next, for simplicity, we denote

$$\langle u, v \rangle = \int_\Omega uv \, dx \text{ and } \langle u, v \rangle_0 = \oint_{\partial\Omega} uv \, d\rho.$$

Moreover, we denote the usual Sobolev space on Ω

$$W^{1,p}(\Omega) = \{ u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega) \},$$

endowed with the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p,$$

or to the equivalent norm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}}, \text{ if } 1 \le p < +\infty.$$

Proposition 2.1. (See [2]). The trace operator $u: W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ is continuous if and only if

$$\begin{cases} 1 \le q \le p^\partial \text{ if } p \ne n, \\ 1 \le q < \infty \text{ if } p = n, \end{cases}$$

Let X be a Banach space and $T > 0$. Denote the following spaces:

$$C([0,T];X) = \{u : [0,T] \longrightarrow X \text{ continue } \},$$

$$L^p(0,T;X) = \left\{ u : [0,T] \longrightarrow X \text{ is a measurable such that } \int_0^T \|u(t)\|_X^p dt < \infty \right\},$$

equipped with the norms

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}},$$

and

$$L^\infty(0,T;X) = \{u : [0,T] \longrightarrow X \text{ is a measurable such that } : \exists C > 0; \|u(t)\|_X < C \text{ a.e.t}\},$$

endowed with the norm

$$\|u\|_{L^\infty(0,T;X)} = \inf\{C > 0; \|u(t)\|_X < C \text{ a.e.t}\}.$$

In addition, for $\theta > 0$ we define

$$A_\theta(u) = \theta\|u\|_{1,p}^p - \int_{\partial\Omega} ug(u)\mathrm{d}\rho,$$

$$h(\theta) = \inf_{u \in Y_\theta} B(u),$$

$$Y_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) = 0, \|u\|_{1,p} \ne 0\right\},$$

$$S_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) > 0, B(u) < h(\theta)\right\} \cup \{0\},$$

and

$$U_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) < 0, B(u) < h(\theta)\right\}.$$

3. Proof of main results

3.1. Proof of Theorem 1.1

In this part, we prove the global existence result. Firstly, we give the definition of the weak solution and some lemmas which will be used later.

Definition 3.1. Let $u(x,0) = u_0 \in W^{1,p}(\Omega)$. A **weak solution** of problem (1.1) is a function $u : \Omega \times (0;T) \to \mathbb{R}$ such that:
(i) $u = u(x,t) \in L^\infty\left(0,\infty; W^{1,p}(\Omega)\right) \cap C\left([0,T]; L^p(\Omega) \times L^p(\partial\Omega,\rho)\right);$
(ii) $u_t(t) \in L^2\left(0,\infty; L^2(\Omega)\right);$
(iii) for any $v \in W^{1,p}(\Omega)$, and for almost all $t \in [0,T]$ it holds

$$\langle u_t, v\rangle + \langle |u|^{p-2}u, v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle = \langle g(u), v\rangle_0; \tag{3.1}$$

(iv)

$$\int_0^t \|u_t\|_2^2 \, \mathrm{d}\tau + B(u) \leqslant B(u_0), \quad \forall t \in [0,T). \tag{3.2}$$

Lemma 3.2. ([20]). Let $g(u)$ satisfy (C). Then,

1. $|G(u)| \leqslant M|u|^\mu$ for some $M > 0$ and all $u \in \mathbb{R}$.
2. $G(u) \geqslant N|u|^{p+1}$ for some $N > 0$ and $|u| \geqslant 1$.
3. The equality $u\left(ug'(u) - g(u)\right) \geqslant 0$ holds only for $u = 0$.

Corollary 3.3. ([20]). Let $g(u)$ satisfy (C). Then,

1. $|ug(u)| \leqslant \mu M|u|^\mu, |g(u)| \leqslant \mu M|u|^{\mu-1}$ for all $u \in \mathbb{R}$.
2. $ug(u) \geqslant (p+1)N|u|^{p+1}$ for $|u| \geqslant 1$.

Lemma 3.4. Let $\theta_1 < \theta_2$ be the two roots of equation $h(\theta) = B(u)$. Then, the sign of $A_\theta(u)$ does not change for $\theta \in (\theta_1, \theta_2)$, provided $0 < B(u) < h$ for some $u \in W^{1,p}(\Omega)$.

Proof. If this were false, there would exist a $\theta_0 \in (\theta_1, \theta_2)$ such as $A_{\theta_0}(u) = 0$. By $B(u) > 0$, we have $\|u\|_{1,p} \neq 0$, consequently $u \in Y_{\theta_0}$. Then, $B(u) \geq h(\theta_0)$, which contradicts

$$B(u) = h(\theta_1) = h(\theta_2) < h(\theta_0).$$

□

Lemma 3.5. Let $g(u)$ satisfy (C), $u_0(x) \in W^{1,p}(\Omega), 0 < e < h$ and $\theta_1 < \theta_2$ be the two roots of equation $h(\theta) = e$. Assume that $A(u_0) > 0$, then all weak solutions $u(t)$ of problem (1.1) with $B(u_0) = e$ belong to S_θ for $\theta \in (\theta_1, \theta_2)$ and $0 \leq t < T$.

Proof. By $B(u_0) = e, A(u_0) > 0$ and Lemma 3.4 we obtain $A_\theta(u_0) > 0$ and $B(u_0) < h(\theta)$ i.e. $u_0(x) \in S_\theta$ for $\theta \in (\theta_1, \theta_2)$. Let $u(t)$ be any weak solution of problem (1.1) with $A(u_0) > 0$ and $B(u_0) = e$, and T be the maximal existence time of $u(t)$. Next, we show that $u(t) \in S_\theta$ for $\theta_1 < \theta < \theta_2$ and $0 < t < T$. If it is false, so it must exists a $\theta_0 \in (\theta_1, \theta_2)$ and $t_0 \in (0,T)$ such as

$$A_{\theta_0}(u(t_0)) = 0, \|u(t_0)\|_{1,p} \neq 0 \text{ or } B(u(t_0)) = h(\theta_0).$$

From (3.2), it follows that

$$\int_0^t \|u_\tau\|_2^2 \, \mathrm{d}\tau + B(u) \leq B(u_0) < h(\theta), \quad \theta_1 < \theta < \theta_2, \ 0 \leq t < T. \tag{3.3}$$

As a result $B(u(t_0)) \neq h(\theta_0)$. If $A_{\theta_0}(u(t_0)) = 0, \|u(t_0)\|_{1,p} \neq 0$, thus the definition of $h(\theta)$ means that $B(u(t_0)) \geq h(\theta_0)$, which contradicts (3.3). □

Proof of Theorem 1.1. The idea of the proof is classical, for more details see [7, 14, 22]. Let $w_j(x)$ be a system of base functions in $W^{1,p}(\Omega)$. Define the approximate solutions $u_m(x,t)$ of problem (1.1)

$$u_m(x,t) = \sum_{j=1}^{m} \Phi_{jm}(t) w_j(x), \quad m = 1, 2, \dots$$

verifying

$$\langle u_{mt}, w_s\rangle + \langle |u_m|^{p-2}u_m, w_s\rangle + \langle |\nabla u_m|^{p-2}\nabla u_m, \nabla w_s\rangle = \langle g(u_m), w_s\rangle_0, \quad s = 1, 2, \dots, m \tag{3.4}$$

and

$$u_m(x,0) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0(x) \quad \text{in } W^{1,p}(\Omega). \tag{3.5}$$

Multiplying (3.4) by $\Phi'_{sm}(t)$ and summing for s yields

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau + B(u_m) \leqslant B(u_0) < h, \quad \forall t \in [0, T), \tag{3.6}$$

and $u_m \in S$ for $0 \leqslant t < \infty$ (see the proof of Lemma 3.5).
Combining (3.6) and

$$\begin{aligned} B(u_m) = \frac{1}{p}\|u_m\|_{1,p}^p - \int_{\partial\Omega} G(u_m)\,\mathrm{d}\rho &\geqslant \frac{1}{p}\|u_m\|_{1,p}^p - \frac{1}{q+1}\int_{\partial\Omega} u_m g(u_m)\,\mathrm{d}\rho \\ &= \left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_m\|_{1,p}^p + \frac{1}{q+1}A(u_m) \\ &\geqslant \frac{q-p+1}{p(q+1)}\|u_m\|_{1,p}^p, \end{aligned}$$

we obtain

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau + \frac{q-p+1}{p(q+1)}\|u_m\|_{1,p}^p < h, \quad 0 \leqslant t < \infty. \tag{3.7}$$

From (3.7), we get

$$\|u_m\|_{1,p}^p < \frac{p(q+1)}{q-p+1}h, \quad 0 \leqslant t < \infty, \tag{3.8}$$

$$\||u_m|^{p-2}u_m\|_s^s = \|u_m\|_p^p < \frac{p(q+1)}{q-p+1}h, \quad s = \frac{p}{p-1}, \ 0 \leqslant t < \infty,$$

$$\|u_m\|_{\mu,\partial\Omega} \leqslant C_*\|u_m\|_{1,p} < C_*\left(\frac{p(q+1)}{q-p+1}h\right)^{\frac{1}{p}}, \quad 0 \leqslant t < \infty, \tag{3.9}$$

$$\begin{aligned} \|g(u_m)\|_{r,\partial\Omega}^r &\leqslant \int_{\partial\Omega} \left(\mu M |u_m|^{\mu-1}\right)^r \, \mathrm{d}\rho \\ &= (\mu M)^r \|u_m\|_{\mu,\partial\Omega}^{\mu} \\ &\leqslant (\mu M)^r C_*^{\mu}\left(\frac{p(q+1)}{q-p+1}h\right)^{\frac{\mu}{p}}, \quad r = \frac{\mu}{\mu-1}, \quad 0 \leqslant t < \infty, \end{aligned} \tag{3.10}$$

where C_* is the embedding constant from $W^{1,p}(\Omega)$ into $L^\mu(\partial\Omega)$.
Furthermore

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau < h, \quad 0 \leqslant t < \infty. \tag{3.11}$$

Therefore, there exist u, ϕ and a subsequence $\{u_v\}$ of $\{u_m\}$ such as

$$\begin{aligned}
&u_v \to u \ \text{ in } L^\infty\left(0,\infty; W^{1,p}(\Omega)\right) \ \text{ weakly star},\\
&u_{vt} \to u_t \ \text{ in } L^2\left(0,\infty; L^2(\Omega)\right) \ \text{ weakly },\\
&|u_v|^{p-2}u_v \to |u|^{p-2}u \ \text{ in } L^\infty\left(0,\infty; L^s(\Omega)\right) \ \text{ weakly star},\\
&g\left(u_v\right) \to \phi \ \text{ in } L^\infty\left(0,\infty; L^r(\partial\Omega)\right) \ \text{ weakly star, and a.e. in } \partial\Omega \times [0,\infty).
\end{aligned}$$

Consequently, from Lemma (1.3) in [18], we deduce $\phi = g(u)$. In (3.4) for fixed s letting $m = v \to \infty$, we have

$$\langle u_t, w_s\rangle + \langle |u|^{p-2}u, w_s\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla w_s\rangle = \langle g\left(u\right), w_s\rangle_0, \quad \forall s,$$

and

$$\langle u_t, v\rangle + \langle |u|^{p-2}u, v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle = \langle g\left(u\right), v\rangle_0, \ \forall v \in W^{1,p}(\Omega).$$

By (3.5), we obtain $u(x,0) = u_0(x)$ in $W^{1,p}(\Omega)$. Then $u(t)$ is a global weak solution of problem (1.1). Finally, by applying Lemma 3.5 we deduce that the solution $u(t) \in S$.

3.2. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following auxiliary lemmas.

Lemma 3.6. Let $g(u)$ satisfy (C), and $A_\theta(u) < 0$. Then, $\|u\|_{1,p} > z(\theta)$. In particular, if $A(u) < 0$, then $\|u\|_{1,p} > z(1)$.

Where

$$z(\theta) = \left(\frac{\theta}{aC_*^\mu}\right)^{1/(\mu-2)},$$

with C_* is the embedding constant form $W^{1,p}(\Omega)$ into $L^\mu(\partial\Omega)$, and

$$a = \sup \frac{ug(u)}{|u|^\mu}.$$

Proof. $A_\delta(u) < 0$ gives

$$\theta\|u\|_{1,p}^2 < \int_{\partial\Omega} ug(u)\mathrm{d}\rho \leqslant a\|u\|_{\mu,\partial\Omega}^\mu \leqslant aC_*^\mu\|u\|_{1,p}^{\mu-2}\|u\|_{1,p}^2, \tag{3.12}$$

then $\|u\|_{1,p} > z(\theta)$. □

Lemma 3.7. Let $g(u)$ satisfy (C), and $u_0(x) \in W^{1,p}(\Omega)$. Suppose that $0 < e < h$, $\theta_1 < \theta_2$ are the two roots of equation $h(\theta) = e$. Then, all weak solutions of problem (1.1) with $B\left(u_0\right) = e$ belong to $\theta \in (\theta_1, \theta_2)$, provided $A\left(u_0\right) < 0$.

Proof. Let $u(t)$ be any solution of problem (1.1) with $B(u_0)=e$, $A(u_0)<0$ and T be the existence time of $u(t)$. First from $B(u_0)=e$, $A(u_0)<0$ and Lemma 3.4 we can deduce $A_\theta(u_0)<0$ and $B(u_0)<h(\theta)$, i.e. $u_0(x)\in U_\theta$ for $\theta\in(\theta_1,\theta_2)$.
Next, we prove $u(t)\in U_\theta$ for $\theta_1<\theta<\theta_2$ and $0<t<T$. If it is false, let $t_0\in(0,T)$ be the first time such that $u(t)\in U_\theta$ for $0\leqslant t<t_0$ and $u(t_0)\in\partial U_\theta$, i.e. $A_\theta(u(t_0))=0$ or $B(u(t_0))=h(\theta)$ for some $\theta\in(\theta_1,\theta_2)$. So (3.3) implies $B(u(t_0))=h(\theta)$ is impossible. If $A_\theta(u(t_0))=0$, thus $A_\theta(u(t))<0$ for $t\in(0,t_0)$ and Lemma 3.6 yield $\|u(t)\|_{1,p}>z(\theta)$ and $\|u(t_0)\|_{1,p}\geqslant z(\theta)$. Therefore by the definition of $h(\theta)$ we have $B(u(t_0))\geqslant h(\theta)$ which contradicts (3.3). □

Proof of Theorem 1.2. Let $u(t)$ be any weak solution of problem (1.1) with $B(u_0)<h$ and $A(u_0)<0$. We consider the auxiliary function

$$\varphi_1(t)=\int_0^t\|u\|_2^2\,\mathrm{d}\tau.$$

A direct calculation gives

$$\dot{\varphi}_1(t)=\|u\|_2^2,$$

and

$$\ddot{\varphi}_1(t)=2\langle u_t,u\rangle=2\left(\langle g(u),u\rangle-\|u\|_{1,p}^p\right)=-2A(u). \tag{3.13}$$

By (3.13), (3.2) and

$$\int_{\partial\Omega}ug(u)\mathrm{d}\rho\geqslant(q+1)\int_{\partial\Omega}G(u)\mathrm{d}\rho$$

we can deduce

$$\begin{aligned}\ddot{\varphi}_1(t)&\geqslant 2(q+1)\int_0^t\|u_t\|_2^2\,\mathrm{d}\tau+(q-1)\|u\|_{1,p}^2-2(q+1)B(u_0)\\&\geqslant 2(q+1)\int_0^t\|u_t\|_2^2\,\mathrm{d}\tau+(q-1)\dot{\varphi}_1(t)-2(q+1)B(u_0),\end{aligned}$$

and

$$\begin{aligned}\varphi_1\ddot{\varphi}_1-\frac{q+1}{2}(\dot{\varphi}_1)^2\geqslant&2(q+1)\left[\int_0^t\|u\|_2^2\,\mathrm{d}\tau\int_0^t\|u_t\|_2^2\,\mathrm{d}\tau-\left(\int_0^t\langle u,u_t\rangle\mathrm{d}\tau\right)^2\right]\\&+(q-1)\varphi_1\dot{\varphi}_1-(q+1)\|u_0\|_2^2\dot{\varphi}_1\\&-2(q+1)B(u_0)\varphi_1+\frac{q+1}{2}\|u_0\|_2^2.\end{aligned}$$

Making use of the Hölder inequality, we get

$$\begin{aligned}\varphi_1\ddot{\varphi}_1-\frac{q+1}{2}(\dot{\varphi}_1)^2\geqslant&(q-1)\varphi_1\dot{\varphi}_1-(q+1)\|u_0\|_2^2\dot{\varphi}_1\\&-2(q+1)B(u_0)\varphi_1+\frac{q+1}{2}\|u_0\|_2^2.\end{aligned}\tag{3.14}$$

1. If $B(u_0) \leqslant 0$, then

$$\varphi_1\ddot{\varphi}_1 - \frac{q+1}{2}(\dot{\varphi}_1)^2 \geqslant (q-1)\varphi_1\dot{\varphi}_1 - (q+1)\|u_0\|_2^2\,\dot{\varphi}_1.$$

The next task is to prove that $A(u) < 0$ for $t > 0$. Otherwise, we assume the existence of a $t_0 > 0$ so that $A(u(t_0)) = 0$.
Next, let $t_0 > 0$ be the first time such as $A(u(t)) = 0$, thus $A(u(t)) < 0$ for $t \in [0, t_0)$. From Lemma 3.6 we obtain $\|u\|_{1,p} > z(1)$ for $t \in (0, t_0)$. Consequently, we obtain $\|u(t_0)\|_{1,p} \geqslant z(1)$ and $B(u(t_0)) \geqslant h$ which contradicts (3.2). Then, from (3.13) we have $\ddot{\varphi}_1(t) > 0$ for $t > 0$. By this and $\dot{\varphi}_1(0) = \|u_0\|_2^2 \geqslant 0$, then there exists a $t_0 \geqslant 0$ such as $\dot{\varphi}_1(t_0) > 0$ and

$$\varphi_1(t) \geqslant \dot{\varphi}_1(t_0)(t-t_0) + \varphi_1(t_0) \geqslant \dot{\varphi}_1(t_0)(t-t_0), \quad t \geqslant t_0.$$

Therefore for sufficiently large t we can deduce

$$(q-1)\varphi_1 > (q+1)\|u_0\|_2^2,$$

and

$$\varphi_1(t)\,\ddot{\varphi}_1(t) - \frac{q+1}{2}(\dot{\varphi}_1(t))^2 > 0. \tag{3.15}$$

Since, for $t > 0$

$$\left(\varphi_1^{-\beta}(t)\right)'' = -\frac{\beta}{\varphi_1^{\beta+2}(t)}\left(\varphi_1(t)\,\ddot{\varphi}_1(t) - (\beta+1)\dot{\varphi}_1(t)^2\right),$$

we see that for $\beta = \frac{q-1}{2}$ we have $\left(\varphi_1^{-\beta}(t)\right)'' < 0$. Therefore $\varphi_1^{-\beta}(t)$ is concave for sufficiently large t, and there exists a finite time T for which $\varphi_1^{-\beta}(t) \to 0$. In other words,

$$\lim_{t\to T^-} \varphi_1(t) = +\infty.$$

2. If $0 < B(u_0) < h$, thus by Lemma 3.7, we have $u(t) \in U_\theta$ for $1 < \theta < \theta_2$ and $t > 0$, where θ_2 is the larger root of equation $h(\theta) = B(u_0)$. Therefore $A_\theta(u) < 0$ and from Lemma 3.6 we deduce $\|u\|_{1,p} > z(\theta)$ for $1 < \theta < \theta_2$ and $t > 0$. Then, we have $A_{\theta_2}(u) \leqslant 0$ and $\|u\|_{1,p} \geqslant z(\theta_2)$ for $t > 0$. Thus (3.13) gives

$$\ddot{\varphi}_1(t) = -2A(u) = 2(\theta_2 - 1)\|u\|_{1,p}^p - 2A_{\theta_2}(u) \geqslant 2(\theta_2-1)z^p(\theta_2) > 0, \quad t \geqslant 0,$$
$$\dot{\varphi}_1(t) \geqslant 2(\theta_2-1)z^p(\theta_2)\,t + \dot{\varphi}_1(0) \geqslant 2(\theta_p-1)z^p(\theta_2)\,t, \quad t \geqslant 0,$$
$$\varphi_1(t) \geqslant (\theta_2-1)z^p(\theta_2)\,t^2 + \varphi_1(0) = (\theta_2-1)z^p(\theta_2)\,t^2, \quad t \geqslant 0.$$

Therefore for sufficiently large t we get

$$\frac{1}{2}(q-1)\varphi_1(t) > (q+1)\|u_0\|_2^2,$$
$$\frac{1}{2}(q-1)\dot{\varphi}_1(t) > 2(q+1)B(u_0).$$

Hence from (3.14) we again obtain (3.15) for sufficiently large t. The remainder of the proof is similar to that in the proof of (i).

3.3. Proof of Theorem 1.3

Under the conditions of Theorem 1.3, and according to Theorem 1.1, the problem (1.1) has a global weak solution. Next, multiplying (3.1) by any $h(t) \in C[0,\infty)$, we have

$$\langle u_t, h(t)v\rangle + \langle |u|^{p-2}u, h(t)v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla(h(t)v)\rangle = \langle g(u), h(t)v\rangle_0,\ \forall v \in W^{1,p}(\Omega),$$

and $t \in (0,T)$, consequently

$$\langle u_t, \varphi\rangle + \langle |u|^{p-2}u, \varphi\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle = \langle g(u), \varphi\rangle_0, \quad \forall \varphi \in L^\infty\left(0,\infty; W^{1,p}(\Omega)\right), \tag{3.16}$$

and $t \in (0,T)$.

Setting $\varphi = u$, (3.16) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + A(u) = 0, \quad 0 \le t < \infty. \tag{3.17}$$

By $0 < B(u_0) < h, A(u_0) > 0$ and Lemma 3.5, we obtain $u(t) \in S_\theta$ for $\theta_1 < \theta < \theta_2$ and $0 \le t < \infty$, where $\theta_1 < \theta_2$ are the two roots of equation $h(\theta) = B(u_0)$. Consequently, we get $A_\theta(u) \ge 0$ for $\theta_1 < \theta < \theta_2$ and $A_{\theta_1}(u) \ge 0$ for $0 \le t < \infty$. Then, (3.17) leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + (1-\theta_1)\|u\|_{1,p}^p + A_{\theta_1}(u) = 0, \quad 0 \le t < \infty,$$

accordingly

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + (1-\theta_1)\|u\|_2^2 \le 0, \quad 0 \le t < \infty.$$

Finally, Gronwall's inequality, leads to

$$\|u\|_2^2 \le \|u_0\|_2^2\,\mathrm{e}^{-2(1-\theta_1)t}, \quad 0 \le t < \infty.$$

This completes the proof of the Theorem 1.3.

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Existence and multiplicity of solutions to an abstract Cauchy problem for an evolution equation involving fractional dissipation term of Caputo type

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Abstract. The aim of this work is to investigate the existence and multiplicity of solutions to a nonhomogenious quasi-linear second order evolution equation involving a fractional dissipation term of Caputo type in abstract framework. Some criteria on the existence of at least one or two solutions are obtained by using some well known fixed point theorems for the sum of two operators. An example is presented to validate our analysis.

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1. Introduction

Evolution problems have been an area of wide research for a decade. Their role in modeling real world scenarios is crucial in sciences, physics and engineering. However, the loss of energy often produces friction effects in real world models [14, 19, 20]. Fractional dissipation which extend the classical dissipation mechanisms where fractional derivatives are involved provide an appropriate description of memory effects.
In recent years, special attention has been focused on fractional derivatives, both in their interpretation and as a nonlocal dissipation. More details can be found in

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[4, 13, 22, 23, 24, 25].
The following abstract system has been extensively studied in the literature

$$u'' + \phi(\|\mathcal{A}^{\frac{1}{2}}u\|^2)\mathcal{A}u + V(t) = U(t). \tag{1.1}$$

Many authors have developed several methods and different techniques to study this kind of problems in the last decade. See for instance [2, 7, 11, 12, 16, 17, 18].
It is remarkable that no paper in the aforementioned literature discussed such models by means of topological methods which offer a key tool to overcome many analysis difficulties and guarantee the well-posedness of solutions. The use of topological methods has been extensively researched in both qualitative and quantitative analysis of nonlinear boundary value problems, see for instance [5, 15, 26] and the references therein. Figueiredo et al. in [9] used Leray-Schauder degree and the bifurcation method to study the existence and uniqueness of positive solution for the following non-homogeneous elliptic Kirchhoff problem with nonlinear reaction term

$$\begin{aligned} -\phi\left(x, \|u\|^2\right)\Delta \quad u &= \varrho u^q \quad in\ \Omega \\ u &= 0 \quad on\ \partial\Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^N, N \geq 1$ is a bounded regular domain, $0 < q \leq 1, \varrho \in \mathbb{R}$ and

$$\phi(x,s) = a(x) + b(x)s, \quad \|u\|^2 = \int_\Omega |\nabla u|^2 dx,$$

with $a, b \in C^\gamma(\overline{\Omega}), \gamma \in (0,1)$ and $a(x) \geq a_0 > 0, b \geq 0$.

The authors in [1] applied sup- and super-solutions methods to investigate the existence of positive solutions for the following Kirchhoff type problem

$$\begin{gathered} -\phi\left(\int_\Omega |\nabla u|^p dx\right)\Delta_p u = \varrho f(u) \quad in\ \Omega \\ u > 0 \quad in\ \Omega, \quad u = 0 \quad on\ \partial\Omega, \end{gathered}$$

where $1 < p < N, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function, ϱ is a parameter and $f : [0, +\infty) \to \mathbb{R}$ is a C^1 semipositone nondecreasing function.

The authors in [28] used the method of upper and lower solutions to investigate the existence and multiplicity of solutions for the following Kirchhoff type problem

$$\begin{gathered} -\phi\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u(x) = f(x, u(x), \nabla u(x)) - g(x, u(x), \nabla u(x)), \quad in\ \Omega \\ u = 0 \quad on\ \partial\Omega, \end{gathered}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded and smooth domain and $\phi \in C([0, +\infty), (0, +\infty))$ with $\phi(t)$ nondecreasing on $[0, +\infty)$ and $\phi(t) \geq \phi(0) > 0, \forall t \geq 0$.

Very recently Precup and Stan [27] discussed by means of fixed point theorems of Banach and Schaefer, the existence of solutions to the following stationary Kirchhoff problem with reaction terms

$$\begin{gathered} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f + g(x, u, \nabla u), \quad in\ \Omega \\ u|_{\partial\Omega} = 0. \end{gathered}$$

In this paper, we investigate the following second order evolution problem

$$u''(t) + \phi\left(\|\mathcal{G}(t)u(t)\|^2\right)\mathcal{A}(t)u(t) + \gamma\partial_t^{\alpha,\eta}u(t) = f(u(t)), \quad t \in [0, \varpi], \tag{1.2}$$

$$u(0) = u_0, \quad u'(0) = u_1,$$

where $0 \leq u_0, u_1 \leq \chi$ where χ is a nonnegative constant, $\gamma > 0$, ϕ is is a function that will be specialized in the sequel and $\mathcal{A}(t), \mathcal{G}(t) : \mathcal{H} \to \mathcal{H}$are two families of operators. When $\mathcal{A}(t) = \mathcal{A}$ is a positive self-adjoint operator and $\mathcal{G}(t) = \mathcal{A}^{\frac{1}{2}}$ the equation above coincides with a class of Kirchhoff equations. We begin by introducing some assumptions.

(A1). $\mathcal{H}$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$, f is a nonlinear operator with domain

$$\mathcal{D}(f) = \{u \in \mathcal{H} : f(u) \in \mathcal{H}\}$$

and

$$\|f(u)\|_{\mathcal{H}} \leq \theta_0(t) + \sum_{k=1}^{r} \theta_k(t) \|u\|_{\mathcal{H}}^{p_k},$$

where $r \in N^*$, $p_k \geq 0$, $k \in \{1, \dots, r\}$, $\theta_k \in \mathcal{C}([0, \varpi])$, $0 \leq \theta_k \leq \chi$ on $[0, \varpi]$ for some nonnegative constant χ,

(A2). $\mathcal{A}(t)$ and $\mathcal{G}(t)$ verify

$$\|\mathcal{A}(t)u\|_{\mathcal{H}} \leq \zeta_0(t) + \sum_{k=1}^{n} \zeta_k(t) \|u\|_{\mathcal{H}}^{s_k}, \quad t \in [0, \varpi],$$

where $n \in N^*$, $s_k \geq 0$, $k \in \{1, \dots, n\}$, $\zeta_k \in \mathcal{C}([0, \varpi])$, $0 \leq \zeta_k \leq \chi$ on $[0, \varpi]$, $k \in \{1, \dots, n\}$, and

$$\|\mathcal{G}(t)u\|_{\mathcal{H}} \leq \varsigma_0(t) + \sum_{k=1}^{l} \varsigma_k(t) \|u\|_{\mathcal{H}}^{r_k}, \quad t \in [0, \varpi],$$

where $l \in N^*$, $r_k \geq 0$, $k \in \{1, \dots, l\}$, $\varsigma_k \in \mathcal{C}([0, \varpi])$, $0 \leq \varsigma_k \leq \chi$ on $[0, \varpi]$, $k \in \{1, \dots, l\}$,

(A3). $\phi \in \mathcal{C}([0, \infty), [0, \infty))$ verifies

$$|\phi(z)| \leq \vartheta_0(t) + \sum_{k=1}^{m} \vartheta_k(t) |z|^{q_k}, \quad t \in [0, \varpi],$$

where $m \in N^*$, $q_k \geq 0$, $k \in \{1, \dots, m\}$, $\vartheta_k \in \mathcal{C}([0, \varpi])$, $0 \leq \vartheta_k \leq \chi$ on $[0, \varpi]$ for some nonnegative constant χ,

(A4). $\partial_t^{\alpha,\eta}$ is the generalized Caputo fractional differential operator of order $\alpha \in (0, 1)$ (see [6] and [3]). It is given by

$$\partial_t^{\alpha,\eta} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-\tau)} \frac{d}{d\tau} u(\tau) d\tau, \quad \eta \geq 0.$$

In this paper under hypotheses $(A1) - (A4)$ we prove that the problem (1.2) has at least one or two bounded solutions.

Let

$$\mathcal{E} = \mathcal{C}^1([0, \varpi], \mathcal{H})$$

be endowed with the norm

$$\|u\| = \max\left\{ \sup_{t\in[0,\varpi]} \|u\|_{\mathcal{H}},\ \sup_{t\in[0,\varpi]} \|u'\|_{\mathcal{H}} \right\}$$

provided it exists. For $u \in \mathcal{H}$, define $\mathcal{J}$ as the identity operator in $\mathcal{H}$.

The paper is organized as follows. In the next section, we give some preliminary definitions, lemmas and theorems. Section 3 is devoted to our main findings. In section 4, we illustrate the main results with an example. Finally, the last section provides a conclusion summarizing the study.

2. Preliminaries

First, we recall the following definitions.

Definition 2.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be real Banach spaces. A map $\mathcal{T} : \mathcal{E} \to \mathcal{F}$ is called expansive if there exists a constant $h > 1$ for which the following inequality holds*

$$\|\mathcal{T}x - \mathcal{T}y\|_{\mathcal{F}} \geq h\|x - y\|_{\mathcal{E}},$$

for any $x, y \in \mathcal{E}$.

Definition 2.2. *A closed, convex set $\mathcal{Q}$ in a Banach space $\mathcal{E}$ is said to be a cone if*

1. *$\sigma x \in \mathcal{Q}$ for any $\sigma \geq 0$ and for any $x \in \mathcal{Q}$,*
2. *$x, -x \in \mathcal{Q}$ implies $x = 0$.*

The following fixed point theorems are the main tools to prove our results. To prove our first existence result we will use the following fixed point theorem.

Theorem 2.3. *([10], [21]) Let $\mathcal{E}$ be a Banach space, $\mathcal{F}$ a closed, convex subset of $\mathcal{E}$, $\mathcal{U}$ be any open subset of $\mathcal{F}$ with $0 \in \mathcal{U}$. Consider two operators $\mathcal{T}$ and $\mathcal{S}$, where*

$$\mathcal{T}x = \varepsilon x,\ x \in \overline{\mathcal{U}},$$

for some $\varepsilon > 1$ and $\mathcal{S} : \overline{\mathcal{U}} \to \mathcal{E}$ be such that

(i). *$\mathcal{I} - \mathcal{S} : \overline{\mathcal{U}} \to \mathcal{F}$ continuous, compact and*
(ii). *$\{x \in \partial\mathcal{U} : x = \lambda(\mathcal{I} - \mathcal{S})x\} = \emptyset$, for any $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.*

Then there exists $x^* \in \overline{\mathcal{U}}$ such that

$$\mathcal{T}x^* + \mathcal{S}x^* = x^*.$$

The next fixed point theorem will allow us to prove existence of at least two nonnegative global classical solutions of the IVP (1.2).

Theorem 2.4. *([8], [29]) Let $\mathcal{Q}$ be a cone of a Banach space $\mathcal{E}$; Ω a subset of $\mathcal{Q}$ and $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_3$ three open bounded subsets of $\mathcal{Q}$ such that $\overline{\mathcal{U}}_1 \subset \overline{\mathcal{U}}_2 \subset \mathcal{U}_3$ and $0 \in \mathcal{U}_1$. Assume that $\mathcal{T} : \Omega \to \mathcal{E}$ is an expansive mapping, $\mathcal{S} : \overline{\mathcal{U}}_3 \to \mathcal{E}$ is a completely continuous map and $\mathcal{S}(\overline{\mathcal{U}}_3) \subset (\mathcal{I}-\mathcal{T})(\Omega)$. Suppose that $(\mathcal{U}_2\backslash\overline{\mathcal{U}}_1)\cap\Omega \neq \emptyset$, $(\mathcal{U}_3\backslash\overline{\mathcal{U}}_2)\cap\Omega \neq \emptyset$, and there exist $u_0 \in \mathcal{Q}^* = \mathcal{Q}\backslash\{0\}$ and $\epsilon > 0$ such that the following conditions hold:*

(i). *$\mathcal{S}x \neq (\mathcal{I} - \mathcal{T})(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial\mathcal{U}_1 \cap (\Omega + \lambda u_0)$,*
(ii). *$\mathcal{S}x \neq (\mathcal{I} - \mathcal{T})(\lambda x)$, for all $\lambda \geq 1 + \epsilon$, $x \in \partial\mathcal{U}_2$ and $\lambda x \in \Omega$,*

(iii). $\mathcal{S}x \neq (\mathcal{I}-\mathcal{T})(x-\lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial\mathcal{U}_3 \cap (\Omega + \lambda u_0)$.

Then $\mathcal{T}+\mathcal{S}$ *has at least two non-zero fixed points* $x_1, x_2 \in \mathcal{Q}$ *such that*

$$x_1 \in \partial\mathcal{U}_2 \cap \Omega \text{ and } x_2 \in (\overline{\mathcal{U}}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega$$

or

$$x_1 \in (\mathcal{U}_2 \setminus \mathcal{U}_1) \cap \Omega \text{ and } x_2 \in (\overline{\mathcal{U}}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega.$$

We give also some auxiliary results which will be used in the sequel. To this end, for any $u \in \mathcal{E}$, let us define the operators

$$\begin{aligned}\mathcal{N}u(t) &= u(t) - u_0 - u_1 t \\ &\quad - \int_0^t (t-s)\left(\phi\left(\|\mathcal{G}(t)u(s)\|^2\right)\mathcal{A}(t)u(s) + \gamma\partial_s^{\alpha,\eta}u(s) - f(u(s))\right) ds,\end{aligned}$$

for all $t \in [0,\varpi]$. Set

$$\begin{aligned}\vartheta_1 &= 2\chi + \chi\varpi \\ &\quad + \left(\chi^2\left(1+\sum_{k=1}^{m}\left(\chi\left(1+\sum_{k=1}^{l}\chi^{r_k}\right)\right)\right)\right)^{q_k} \\ &\quad \times \left(1+\sum_{k=1}^{n}\chi^{s_k}\right) + \chi\left(1+\sum_{k=1}^{r}\chi^{p_k}\right)\varpi^2 + \frac{\gamma\chi}{\Gamma(2-\alpha)}\varpi^{3-\alpha}.\end{aligned}$$

The following lemma is going to be needed in what follows.

Lemma 2.5. *Suppose that* $(A1)-(A4)$ *hold. If* $u \in \mathcal{E}$ *and* $\|u\| \leq \chi$, *then*

$$\|\mathcal{N}u(t)\|_{\mathcal{H}} \leq \vartheta_1, \quad t \in [0,\varpi].$$

Proof. We have

$$\begin{aligned}\|\mathcal{G}(t)u(t)\|_{\mathcal{H}} &\leq \varsigma_0(t) + \sum_{k=1}^{l}\varsigma_k(t)\|u(t)\|_{\mathcal{H}}^{r_k} \\ &\leq \chi\left(1+\sum_{k=1}^{l}\chi^{r_k}\right), \quad t \in [0,\varpi],\end{aligned}$$

$$\begin{aligned}\left|\phi\left(\|\mathcal{G}(t)u(t)\|^2\right)\right| &\leq \vartheta_0(t) + \sum_{k=1}^{m}\vartheta_k(t)\|\mathcal{G}(t)u(t)\|_{\mathcal{H}}^{q_k} \\ &\leq \chi\left(1+\sum_{k=1}^{m}\left(\chi\left(1+\sum_{k=1}^{l}\chi^{r_k}\right)\right)^{q_k}\right), \quad t \in [0,\varpi],\end{aligned}$$

$$\begin{aligned} \|\partial_t^{\alpha,\eta} u(t)\|_{\mathcal{H}} &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} e^{-\eta(t-\tau)} \|u'(\tau)\|_{\mathcal{H}} d\tau \\ &\leq \frac{\chi}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} d\tau \\ &= \frac{\chi}{\Gamma(2-\alpha)} t^{1-\alpha}, \quad t \in [0, \varpi], \end{aligned}$$

$$\begin{aligned} \|f(u(t))\|_{\mathcal{H}} &\leq \theta_0(t) + \sum_{k=1}^{r} \theta_k(t) \|u(t)\|_{\mathcal{H}}^{p_k} \\ &\leq \chi \left(1 + \sum_{k=1}^{r} \chi^{p_k}\right), \quad t \in [0, \varpi], \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{A}(t)u(t)\|_{\mathcal{H}} &\leq \zeta_0(t) + \sum_{k=1}^{n} \zeta_k(t) \|u(t)\|_{\mathcal{H}}^{s_k} \\ &\leq \chi \left(1 + \sum_{k=1}^{n} \chi^{s_k}\right), \quad t \in [0, \varpi]. \end{aligned}$$

Hence,

$$\begin{aligned} &\|\mathcal{N}u(t)\|_{\mathcal{H}} \\ &\leq \|u(t)\|_{\mathcal{H}} + u_0 + u_1 t \\ &\quad + \int_0^t (t-s) \left(\left| \phi\left(\|\mathcal{G}(t)u(s)\|^2\right) \right| \|\mathcal{A}(t)u(s)\|_{\mathcal{H}} + \gamma \|\partial_s^{\alpha,\eta} u(s)\|_{\mathcal{H}} + \|f(u(s))\|_{\mathcal{H}} \right) ds \\ &\leq 2\chi + \chi t + \int_0^t (t-s) \Bigg(\chi^2 \left(1 + \sum_{k=1}^{m} \left(\chi \left(1 + \sum_{k=1}^{l} \chi^{r_k}\right)\right)^{q_k}\right) \left(1 + \sum_{k=1}^{n} \chi^{s_k}\right) \\ &\quad + \gamma \frac{\chi}{\Gamma(2-\alpha)} s^{1-\alpha} + \chi \left(1 + \sum_{k=1}^{r} \chi^{p_k}\right) \Bigg) ds \\ &\leq 2\chi + \chi t \\ &\quad + \left(\chi^2 \left(1 + \sum_{k=1}^{m} \left(\chi \left(1 + \sum_{k=1}^{l} \chi^{r_k}\right)\right)\right)^{q_k} \left(1 + \sum_{k=1}^{n} \chi^{s_k}\right) + \chi \left(1 + \sum_{k=1}^{r} \chi^{p_k}\right) \right) t^2 \\ &\quad + \frac{\gamma\chi}{\Gamma(2-\alpha)} t^{3-\alpha} \end{aligned}$$

$$
\begin{aligned}
&\leq 2\chi + \chi\varpi \\
&\quad + \left(\chi^2\left(1+\sum_{k=1}^{m}\left(\chi\left(1+\sum_{k=1}^{l}\chi^{r_k}\right)\right)\right)^{q_k}\left(1+\sum_{k=1}^{n}\chi^{s_k}\right)+\chi\left(1+\sum_{k=1}^{r}\chi^{p_k}\right)\right)\varpi^2 \\
&\quad + \frac{\gamma\chi}{\Gamma(2-\alpha)}\varpi^{3-\alpha} \\
&= \vartheta_1, \quad t\in[0,\varpi].
\end{aligned}
$$

□

Let

$$\varpi_1 = \max\{\varpi, \varpi^2\}.$$

For $u \in \mathcal{E}$, define the operator

$$\mathcal{S}_2 u(t) = \frac{A}{\varpi_1}\int_0^t (t-s)\mathcal{N}u(s)ds, \quad t\in[0,\varpi],$$

where A is a positive constant.

Lemma 2.6. *For $u \in \mathcal{E}$ and $\|u\| \leq \chi$, one has $\|\mathcal{S}_2 u\| \leq A\vartheta_1$.*

Proof. We have

$$
\begin{aligned}
\|\mathcal{S}_2 u\|_{\mathcal{H}} &\leq \frac{A}{\varpi_1}\int_0^t (t-s)\|\mathcal{N}u(s)\|_{\mathcal{H}}ds \\
&\leq \frac{A}{\varpi_1}\vartheta_1\int_0^t (t-s)ds \\
&\leq \frac{A}{\varpi_1}\vartheta_1\varpi^2 \\
&\leq A\vartheta_1, \quad t\in[0,\varpi],
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\frac{d}{dt}\mathcal{S}_2 u\right\|_{\mathcal{H}} &\leq \frac{A}{\varpi_1}\int_0^t \|\mathcal{N}u(s)\|_{\mathcal{H}}ds \\
&\leq \frac{A}{\varpi_1}\vartheta_1\varpi \\
&\leq A\vartheta_1, \quad t\in[0,\varpi].
\end{aligned}
$$

□

Lemma 2.7. *If $u \in \mathcal{E}$ verifies the equation*

$$\mathcal{S}_2 u(t) = D, \quad t\in[0,\varpi],$$

for some constant D, then u is a solution to the problem (1.2).

Proof. We have

$$\int_0^t (t-s)\mathcal{N}u(s)ds = D, \quad t \in [0, \varpi],$$

so two times differentiation with respect to t leads to

$$\mathcal{N}u(t) = 0, \quad t \in [0, \varpi].$$

□

3. Main results

Theorem 3.1. *Assume that assumptions* $(A1)-(A4)$ *are satisfied, then the problem* (1.2) *has at least one bounded solution in* $\mathcal{E}$.

Proof. In the sequel, $\widetilde{\mathcal{F}}$ denotes the set of all equi-continuous and relatively compact in $\mathcal{H}$ families of functions in $\mathcal{E}$ with respect to the norm $\|\cdot\|$. Let, $\mathcal{F} = \overline{\widetilde{\mathcal{F}}}$ and

$$\mathcal{U} = \left\{u \in \mathcal{F} : \|u\| < \chi \quad \text{and if} \quad \|u\| \geq \frac{\chi}{2}, \quad \text{then} \quad \|u(0)\| > \frac{\chi}{2}\right\}.$$

For $u \in \overline{\mathcal{U}}$, $\epsilon > 1$ and $t \in [0, \varpi]$, define the operators $\mathcal{T}$ and $\mathcal{S}$ as follows

$$\mathcal{T}u(t) = \epsilon u(t),$$

$$\mathcal{S}u(t) = u(t) - \epsilon u(t) - \epsilon \mathcal{S}_2 u(t).$$

For $u \in \overline{\mathcal{U}}$, we have

$$\begin{aligned} \|(\mathcal{I}-\mathcal{S})u\| &= \|\epsilon u + \epsilon \mathcal{S}_2 u\| \\ &\leq \epsilon\|u\| + \epsilon\|\mathcal{S}_2 u\| \\ &\leq \epsilon\vartheta_1 + \epsilon A \vartheta_1. \end{aligned}$$

Thus, $\mathcal{S} : \overline{\mathcal{U}} \to \mathcal{E}$ is continuous and $\overline{\mathcal{U}}$ is compact, then $(\mathcal{I}-\mathcal{S})(\overline{\mathcal{U}})$ resides in a compact subset of $\mathcal{F}$. Now, suppose that there is a $u \in \partial\mathcal{U}$ such that

$$u = \lambda(\mathcal{I}-\mathcal{S})u$$

or

$$u = \lambda\epsilon\,(u + \mathcal{S}_2 u),$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, using that $\mathcal{S}_2 u(0) = 0$ and $\|u\| = \chi > \frac{\chi}{2}$, we have $\|u(0)\| > \frac{\chi}{2}$ and

$$u(0) = \lambda\epsilon u(0),$$

and

$$\|u(0)\| = \lambda\epsilon\|u(0)\|$$

whereupon $\lambda\epsilon = 1$, which is a contradiction. Consequently

$$\{u \in \partial\mathcal{U} : u = \lambda_1(\mathcal{I}-\mathcal{S})u\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 2.3, it follows that the operator $\mathcal{T} + \mathcal{S}$ has a fixed point $u^* \in \mathcal{F}$. Therefore

$$\begin{aligned} u^*(t) &= \mathcal{T}u^*(t) + \mathcal{S}u^*(t) \\ &= \epsilon u^*(t) - u^*(t) + \epsilon u^*(t) - \epsilon \mathcal{S}_2 u^*(t), \quad t \in [0, \varpi], \end{aligned}$$

whereupon

$$0 = \mathcal{S}_2 u^*(t), \quad t \in [0, \varpi].$$

Hence, we conclude that u^* is a solution to the problem (1.2).

□

Theorem 3.2. *Assume that hypotheses* (A1) – (A4) *hold, then the problem* (1.2) *has at least two nonnegative bounded solutions in* $\mathcal{E}$.

Proof. Let r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \leq \chi, \ A\vartheta_1 < \frac{L}{5}.$$

Set

$$\widetilde{\mathcal{Q}} = \{u \in \mathcal{E} : \langle uw, w \rangle_{\mathcal{H}} \geq 0, \quad w \in \mathcal{H}, \quad \text{on} \quad [0, \varpi]\}.$$

With $\mathcal{Q}$ we will denote the set of all equi-continuous families in $\widetilde{\mathcal{Q}}$. For $v \in \mathcal{E}$ and $t \in [0, \varpi]$, define the operators $\mathcal{T}_1$ and $\mathcal{S}_3$ as follows

$$\mathcal{T}_1 v(t) = (1 + m\epsilon)v(t) - \epsilon \frac{L}{10}\mathcal{J},$$

$$\mathcal{S}_3 v(t) = -\epsilon \mathcal{S}_2 v(t) - m\epsilon v(t) - \epsilon \frac{L}{10}\mathcal{J}.$$

Note that any fixed point $v \in \mathcal{E}$ of the operator $\mathcal{T}_1 + \mathcal{S}_3$ is a solution to the IVP (1.2). Define

$$\begin{aligned} \mathcal{U}_1 &= \mathcal{Q}_r = \{v \in \mathcal{Q} : \|v\| < r\}, \\ \mathcal{U}_2 &= \mathcal{Q}_L = \{v \in \mathcal{Q} : \|v\| < L\}, \\ \mathcal{U}_3 &= \mathcal{Q}_{R_1} = \{v \in \mathcal{Q} : \|v\| < R_1\}, \\ \Omega &= \mathcal{Q}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

   $$\|\mathcal{T}_1 v_1 - \mathcal{T}_1 v_2\| = (1 + m\epsilon)\|v_1 - v_2\|,$$

   whereupon $\mathcal{T}_1 : \Omega \to \mathcal{E}$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.
2. For $v \in \overline{\mathcal{Q}_{R_1}}$, we get

   $$\begin{aligned} \|\mathcal{S}_3 v\| &\leq \epsilon\|\mathcal{S}_2 v\| + m\epsilon\|v\| + \epsilon\frac{L}{10} \\ &\leq \epsilon\left(A\vartheta_1 + mR_1 + \frac{L}{10}\right). \end{aligned}$$

   Therefore $\mathcal{S}_3(\overline{\mathcal{Q}_{R_1}})$ is uniformly bounded. Since $\mathcal{S}_3 : \overline{\mathcal{Q}_{R_1}} \to \mathcal{E}$ is continuous, we have that $\mathcal{S}_3(\overline{\mathcal{Q}_{R_1}})$ is equi-continuous. Consequently $\mathcal{S}_3 : \overline{\mathcal{Q}_{R_1}} \to \mathcal{E}$ is completely continuous.

3. Let $v_1 \in \overline{\mathcal{Q}_{R_1}}$. Set
$$v_2 = v_1 + \frac{1}{m}\mathcal{S}_2 v_1 + \frac{L}{5m}\mathcal{J}.$$
Take $w \in \mathcal{H}$ arbitrarily. Then
$$\langle v_1 w, w\rangle_{\mathcal{H}} \geq 0,$$
and
$$\begin{aligned}\langle v_2 w, w\rangle_{\mathcal{H}} &= \langle\left(v_1 + \frac{1}{m}\mathcal{S}_2 v_1 + \frac{L}{5m}\mathcal{J}\right) w, w\rangle_{\mathcal{H}} \\ &= \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{1}{m}\langle \mathcal{S}_2 v_1 w, w\rangle_{\mathcal{H}} \\ &\quad + \frac{L}{5m}\langle w, w\rangle_{\mathcal{H}} \\ &\geq \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{L}{5m}\langle w, w\rangle_{\mathcal{H}} \\ &\quad - \frac{1}{m}\|\mathcal{S}_2 v_1 w\|_{\mathcal{H}}\|w\|_{\mathcal{H}} \\ &\geq \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{L}{5m}\|w\|^2_{\mathcal{H}} - \frac{1}{m}\|\mathcal{S}_2 v_1\|\|w\|^2_{\mathcal{H}} \\ &\geq \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{L}{5m}\|w\|^2_{\mathcal{H}} - \frac{A\vartheta_1}{m}\|w\|^2_{\mathcal{H}} \\ &= \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{1}{m}\left(\frac{L}{5} - A\vartheta_1\right)\|w\|^2_{\mathcal{H}} \\ &\geq 0.\end{aligned}$$
Therefore $v_2 \in \Omega$ and
$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon \mathcal{S}_2 v_1 - \epsilon\frac{L}{10}\mathcal{J} - \epsilon\frac{L}{10}\mathcal{J}$$
or
$$\begin{aligned}(\mathcal{I} - \mathcal{T}_1)v_2 &= -\epsilon m v_2 + \epsilon\frac{L}{10}\mathcal{J} \\ &= \mathcal{S}_3 v_1.\end{aligned}$$
Consequently $\mathcal{S}_3(\overline{\mathcal{Q}_{R_1}}) \subset (\mathcal{I} - \mathcal{T}_1)(\Omega)$.
4. Assume that for any $v_0 \in \mathcal{Q}^* = \mathcal{Q}\backslash\{0\}$, there exist $\lambda > 0$ and $v \in \partial\mathcal{Q}_r \cap (\Omega + \lambda v_0)$ or $v \in \partial\mathcal{Q}_{R_1} \cap (\Omega + \lambda v_0)$ such that
$$\mathcal{S}_3 v = (\mathcal{I} - \mathcal{T}_1)(v - \lambda v_0).$$
Then
$$-\epsilon\mathcal{S}_2 v - m\epsilon v - \epsilon\frac{L}{10}\mathcal{J} = -m\epsilon(v - \lambda v_0) + \epsilon\frac{L}{10}\mathcal{J}$$
or
$$-\mathcal{S}_2 v = \lambda m v_0 + \frac{L}{5}\mathcal{J}.$$
Hence,
$$\|\mathcal{S}_2 v\| = \left\|\lambda m v_0 + \frac{L}{5}\mathcal{J}\right\| > \frac{L}{5}.$$

This is a contradiction.

5. Let $\epsilon_1 = \frac{2}{5m}$. Suppose that there exist a $v_1 \in \partial \mathcal{Q}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that
$$\mathcal{S}_3 v_1 = (\mathcal{I} - \mathcal{T}_1)(\lambda_1 v_1). \tag{3.1}$$
Moreover,
$$-\epsilon \mathcal{S}_2 v_1 - m\epsilon v_1 - \epsilon \frac{L}{10}\mathcal{J} = -\lambda_1 m \epsilon v_1 + \epsilon \frac{L}{10}\mathcal{J},$$
or
$$\mathcal{S}_2 v_1 + \frac{L}{5}\mathcal{J} = (\lambda_1 - 1)m v_1.$$
From here,
$$2\frac{L}{5} > \left\| \mathcal{S}_2 v_1 + \frac{L}{5}\mathcal{J} \right\| = (\lambda_1 - 1)m\|v_1\| = (\lambda_1 - 1)mL$$
and
$$\frac{2}{5m} + 1 > \lambda_1,$$
which is a contradiction.

Therefore all conditions of Theorem 2.4 hold. Hence, the problem (1.2) has at least two solutions u_1 and u_2 so that
$$\|u_1\| = L < \|u_2\| \leq R_1$$
or
$$r \leq \|u_1\| < L < \|u_2\| \leq R_1.$$
□

4. Example

As an illustration of our main results, we consider the following equation
$$u''(t) + \|u'(t)\|^2 u''(t) + \frac{2}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-s)^{-\frac{1}{2}} e^{-3(t-\tau)} u'(\tau) d\tau = \|u(t)\|^2 w,$$
for $t \in [0,1]$, and a fixed non zero $w \in \mathcal{H}$, subject to the initial conditions
$$u(0) = u'(0) = 1.$$
Here
$$\begin{aligned} r = m = n = l = s_1 = \varpi = \chi = 1, \\ p_1 = q_1 = r_1 = \gamma = 2, \\ \theta_0(t) = \vartheta_0(t) = \zeta_0(t) = \varsigma_0(t) = 0, \\ \theta_1(t) = \vartheta_1(t) = \zeta_1(t) = \varsigma_1(t) = 1, \quad t \in [0,1], \end{aligned}$$
and $\vartheta_1 = 23 + \frac{4}{\sqrt{\pi}}$. Take
$$R_1 = \frac{9}{10}, \quad L = \frac{3}{5}, \quad r = \frac{2}{5}, \quad A = \frac{L}{10\vartheta_1}.$$
One can check that all conditions of Theorem 3.1 and Theorem 3.2 are fulfilled.

5. Conclusion

We discussed the existence of at least one or two global solutions to an abstract quasi-linear nonhomogeneous evolution equation involving fractional dissipation term subject to initial boundary conditions. We used well known fixed point theorems for the sum of two operators to achieve existence and multiplicity criteria.

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On a class of nonlinear discrete problems of Kirchhoff type

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Abstract. In view of variational methods and critical points theory, we study the existence of solutions for a discrete boundary value problem, which is a discrete variant of a continuous $(p_1(x), p_2(x))$-Kirchhoff-type problem, with a real parameter $\lambda > 0$.

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1. Introduction

In this paper, we are concerned with the following anisotropic discrete problem of Kirchhoff-type

$$(P) \quad \begin{cases} -\displaystyle\sum_{i=1}^{2} M_i\left(\phi_i(u(r))\right)\Delta(|\triangle u(r-1)|^{p_i(r-1)-2}) = \lambda f(r, u(r)), & r \in [1, N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases}$$

where $N \geq 2$ is a positive integer, $[1, N]_{\mathbb{Z}}$ is the discrete interval $[1, N]_{\mathbb{Z}} := \{1, 2, \ldots, N\}$, Δ denotes the forward difference operator defined by $\Delta u(r) = u(r+1) - u(r)$, and for $i = 1, 2$

$$\phi_i(u) = \sum_{r=1}^{N+1} \frac{|\Delta u(r-1)|^{p_i(r-1)}}{p_i(r-1)}.$$

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For every fixed $r \in [0, N]_{\mathbb{Z}}$, the function $f(r, .) : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $M_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $i = 1, 2$, are continuous functions that satisfy some conditions which will be stated later on and $p_1, p_2 : [0, N+1]_{\mathbb{Z}} \to [2; +\infty)$ are two given functions.

Put

$$p_i^+ := \max_{r\in[0,N+1]_{\mathbb{Z}}} p_i(r), \quad p_i^- := \min_{r\in[0,N+1]_{\mathbb{Z}}} p_i(r),$$
$$P_M^+ := \max\{p_1^+, p_2^+\}, \quad p_m^- := \min\{p_1^-, p_2^-\}.$$

In recent years, a great attention has been focused on studying nonlocal equations and corresponding problems involving fractional Sobolev spaces. To be more precise, the Kirchhoff type equations involving variable exponent growth conditions have been studied in many papers and by many authors, this interest is justified by its various applications in many fields of research. In fact, there are applications concerning image restoration [8], electro-rheological fluids and stationary thermo-rheological viscous flows of non-Newtonian fluids [16, 17]. For some interesting results we refer to [1, 9] and the references therein. Problem (P) is related to the stationary problem of a model introduced by Kirchhoff [14]. More precisely, Kirchhoff presented a model given by

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

which extends the classical D'Alembert's wave equation by taking into consideration the effects of the changes in the length of the strings during the vibrations. On the other hand, stationary counterpart of (1.1) is given as

$$\begin{cases} \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which has received much attention from many authors, where new methods and a new functional analysis framework for the problem were proposed (see, e.g.,) [2, 7, 10], for some interesting results. Later, the study of Kirchhoff type equations has been extended to the case of nonlocal elliptic boundary value problem driving by the $p(x)$-Laplacian like [11],

$$-M\left(\int_\Omega |\nabla u|^{p(x)} dx\right)\Delta_{p(x)} u = f(x, u) \text{ in } \Omega. \tag{1.2}$$

Problem (P) may be viewed as a discretization of the nonlocal equation ([4])

$$\begin{aligned} &-M_1\left(\int_\Omega \frac{|\nabla u|^{p_1(x)} dx}{p_1(x)}\right)\operatorname{div}\left(|\nabla u|^{p_1(x)-2}\nabla u\right) \\ &-M_2\left(\int_\Omega \frac{|\nabla u|^{p_2(x)} dx}{p_2(x)}\right)\operatorname{div}\left(|\nabla u|^{p_2(x)-2}\nabla u\right) \\ &\quad = f(x, u) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega. \end{aligned}$$

There are few papers concerning existence results for anisotropic discrete boundary-value problems of Kirchhoff type. To the best of our knowledge, the first study dealing with this class of problems was conducted by Z. Yucedag (see [19]). In this work, we are inspired by the results in [4, 5, 20] and the ideas introduced in the above mentioned papers, we employ a mountain pass lemma in Theorem 4.2 and

Ekeland's variational principle [12] in Theorem 4.5 to get our main results. More precisely, we consider the previous problem ([4]) and investigate its parametric version in the discrete case. We are proving existence of solutions for appropriate value of parameter λ and under suitable assumptions on the nonlinear term and the Kirchhoff functions M_1, M_2.

The rest of this article is structured as follows, in section 2, we introduce some basic properties of the investigated space of solutions and provide several inequalities useful in our approach. After we give the variational framework in section 3, and we state and prove the main results in the fourth section.

2. Preliminaries

Solutions to (P) will be considered in a space E defined as

$$E = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R};\ \text{such that } u(0) = u(N+1) = 0\}$$

which is a N-dimensional Hilbert space (see [1]) equipped with the inner product

$$\langle u, v\rangle = \sum_{i=1}^{N+1} \Delta u(i-1)\Delta v(i-1),\ \forall u, v \in E$$

and the corresponding norm is defined by

$$\|u\| = \left(\sum_{i=1}^{N+1} |\Delta u(i-1)|^2\right)^{1/2}. \tag{2.1}$$

We also consider other normes on E, denoted by $|u|_m$ and is namely,

$$|u|_m = \left(\sum_{i=1}^{N} |u(i)|^m\right)^{1/m},\ \forall u \in E \text{ and } m \geq 2. \tag{2.2}$$

It is easy to verify that (see [6])

$$N^{(2-m)/2m}|u|_2 \leq |u|_m \leq N^{1/m}|u|_2,\ \forall u \in E \text{ and } m \geq 2 \tag{2.3}$$

Lemma 2.1. [13] *For any function $p : [0, N+1]_{\mathbb{Z}} \to [2; +\infty)$ and $u \in E$*

$$p^+ := \max_{r\in[0,N+1]_{\mathbb{Z}}} p(r)\ p^- := \min_{r\in[0,N+1]_{\mathbb{Z}}} p(r),$$

we have the following assertions:

(A.1) *If $\|u\| > 1$, we have*

$$\sum_{i=1}^{N+1} \frac{|\Delta u(i-1)|^{p(i-1)}}{p(i-1)} \geq \frac{(\sqrt{N})^{2-p^-}}{p^+}\|u\|^{p^-} - N.$$

(A.2) *If $\|u\| < 1$, we have*

$$\sum_{i=1}^{N+1} \frac{|\Delta u(i-1)|^{p(i-1)}}{p(i-1)} \geq \frac{(\sqrt{N})^{p^+-2}}{p^+}\|u\|^{p^+}.$$

(A.3) *For any* $m \geq 2$, *there exist a positive constant* c_m *such that*

$$\sum_{i=1}^{N} |u(i)|^m \leq c_m \sum_{i=1}^{N+1} |\Delta u(i-1)|^m. \tag{2.4}$$

Moreover, from (2.3) *and* (*A*.3), *we have*

$$|u|_m^m \leq N|u|_2^m \leq c_m N \left(\sum_{i=1}^{N+1} |\Delta u(i-1)|^2 \right)^{\frac{m}{2}} = c_m N \|u\|^m. \tag{2.5}$$

Definition 2.2. *Let* $(X, \|.\|)$ *be a Banach space and* $J \in C^1(X, \mathbb{R})$, *we say that* J *satisfies the Palais-Smale condition (we denote* (PS) *condition), if any sequence* $(u_n) \subset X$ *such that* $\{J(u_n)\}$ *bounded and* $J'(u_n) \longrightarrow 0$, *there exists a subsequence of* (u_n) *which is convergent in* X.

Proposition 2.3. *(Mountain Pass Lemma* [18]*). Let* $(X, \|.\|)$ *be a Banach space and* $J \in C^1(X, \mathbb{R})$ *satisfies* (PS) *condition with*

(*i*) *There exist* $\varrho, \gamma > 0$ *such that* $J(u) \geq \gamma$, $\forall u \in X$ *with* $\|u\| = \varrho$.
(*ii*) *There exists* $e \in X$ *with* $\|e\| > \varrho$ *such that* $J(e) < 0$.

Then J *possesses a critical value* $c \geq \gamma$ *with*

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t)),$$

where

$$\Gamma := \Big\{ g \in C\left([0,1], X\right) \mid \ g(0) = 0, \ g(1) = e \Big\}.$$

3. Variational framework

Related to problem (P), let us define the following functional $J_\lambda : E \to \mathbb{R}$ by the formula

$$J_\lambda(u) = \Phi(u) - \lambda \Psi(u), \text{ for all } u \in E,$$

where

$$\Phi(u) = \sum_{i=1}^{2} \widehat{M_i}\left(\phi_i(u)\right) \text{ and } \Psi(u) = \sum_{r=1}^{N} F(r, u(r)),$$

such that $\widehat{M_i}(t) = \displaystyle\int_0^t M_i(s)ds$ for $i = 1, 2$ and $F(r, s) = \displaystyle\int_0^s f(r, t)dt$.

The functional J_λ is well-defined on E and is of class $C^1(E,\mathbb{R})$ with derivative given by

$$\begin{aligned}\langle J'_\lambda(u), v\rangle &= \sum_{i=1}^{2} M_i\left(\phi_i(u)\right)\langle \phi'_i(u), v\rangle - \lambda\sum_{r=1}^{N} f(r,u(r))v(r)\\ &= \sum_{i=1}^{2}\left(M_i\left(\phi_i(u)\right)\sum_{r=1}^{N+1}|\Delta u(r-1)|^{p_i(r-1)-2}\Delta u(r-1)\Delta v(r-1)\right)\\ &\quad - \lambda\sum_{r=1}^{N} f(r,u(r))v(r)\end{aligned}$$

for all $u, v \in E$.

Definition 3.1. *A solution of (P) is a function $u \in E$ such that*

$$\sum_{i=1}^{2}\left(M_i\left(\sum_{r=1}^{N+1}\frac{|\Delta u(r-1)|^{p_i(r-1)}}{p_i(r-1)}\right)\sum_{r=1}^{N+1}|\Delta u(r-1)|^{p_i(r-1)-2}\Delta u(r-1)\Delta\varphi(r-1)\right)$$
$$= \lambda\sum_{r=1}^{N} f(r,u(r))\varphi(r),$$

for any $\varphi \in E$.

It is clear that a function $u \in E$ is a solution of the problem (P) if and only if u is a critical point to the functional J_λ.

We impose the following assumptions on the functions M_1, M_2 and the nonlinear term:

(f_0) There exists a function $q : [1,N]_{\mathbb{Z}} \to [2,\infty)$ such that

$$|f(r,t)| \le C_0\left(1 + |t|^{q(r)-1}\right), \quad \text{for all } (r,t) \in [1,N]_{\mathbb{Z}} \times \mathbb{R},$$

where C_0 is a positive constant.

(M_0) $M_1, M_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions and satisfy the conditions

$$a_1 t^{\alpha-1} \le M_1(t),$$
$$a_2 t^{\alpha-1} \le M_2(t),$$

for all $t > 0$, where a_1 and a_2 are positive constants and $\alpha > 1$.

(M'_0) $M_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $i = 1, 2$, are continuous functions such that

$$m_1 t^{\alpha-1} \le M_i(t) \le m_2 t^{\alpha-1},$$

for all $t > 0$, where m_1, m_2 and α are real numbers such that $0 < m_1 \le m_2$ and $\alpha > 1$.

(f_1) $\displaystyle\lim_{t\to 0}\frac{f(r,t)}{|t|^{\alpha P_M^+ - 1}} = 0$, uniformly for all $r \in [0,N]_{\mathbb{Z}}$.

(AR) $\exists s_* > 0,\ \kappa > \dfrac{m_2}{m_1}\alpha P_M^+$ such that

$$0 \le \kappa F(r,s) \le f(r,s)s,\ |s| \ge s_*, \quad \forall r \in [1,N]_{\mathbb{Z}}.$$

Throughout the sequel, the letters $C,\ \tilde{c},\ c_i,\ i=1,2,...$ stand for positive constants which may differ from line to line.

4. Main results and proofs

Our main results are the following.

Theorem 4.1. *Assume that the assumptions* (f_0) *and* (M_0) *are fulfilled and*

$$q^+ < \alpha p_m^-.$$

Then, for all $\lambda > 0$ *the problem* (P) *has a solution* u_λ *in* E. *Moreover, if* $f(r,t) > 0$ *for all* $r \in [1,N]_{\mathbb{Z}}$ *and* $t > 0$, *there exists* $\lambda^* > 0$ *such that for all* $\lambda > \lambda^*$, *the solution* u_λ *is nontrivial.*

Proof. Let $u \in E$ such that $\|u\| > 1$. We point out that

$$|u(r)|^{q(r)} \leq |u(r)|^{q^-} + |u(r)|^{q^+},\ \forall r \in [1,N]_{\mathbb{Z}},\ u \in E.$$

Relations (2.2) and (2.3) give

$$\begin{aligned} |u(r)|^{q^-} + |u(r)|^{q^+} &\leq c_{q^+} \sum_{r=1}^{N+1} |\Delta u(r-1)|^{q^+} + c_{q^-} \sum_{r=1}^{N+1} |\Delta u(r-1)|^{q^-} \\ &\leq c_{q^+} N \|u\|^{q^+} + c_{q^-} N \|u\|^{q^-} \\ &\leq c_1 N \|u\|^{q^+}, \end{aligned}$$

which implies that

$$\begin{aligned} \Psi(u) = \sum_{r=1}^{N} F(r,u(r)) &\leq \sum_{r=1}^{N} C_0 \frac{|u(r)|^{q(r)}}{q(r)} + c_2 \\ &\leq \frac{c_3 N}{q^-} \|u\|^{q^+} + c_2. \end{aligned} \tag{4.1}$$

On the other hand, in view of condition (M_0), we have

$$\begin{aligned} \Phi(u) &= \sum_{i=1}^{2} \widehat{M_i}\left(\phi_i(u)\right) \\ &\geqslant \sum_{i=1}^{2} \frac{a_i}{\alpha} \left(\sum_{r=1}^{N+1} \frac{|\Delta u(r-1)|^{p_i(r-1)}}{p(r-1)} \right)^{\alpha} \\ &\geqslant \sum_{i=1}^{2} \frac{a_i}{\alpha \left(P_M^+\right)^{\alpha}} \left(\sum_{r=1}^{N+1} |\Delta u(r-1)|^{p_i(r-1)} \right)^{\alpha} \\ &\geqslant \frac{2A}{\alpha \left(P_M^+ (\sqrt{N})^{p_m^- - 2} \right)^{\alpha}} \|u\|^{\alpha p_m^-} - c_4,\ \text{with } A = max\{a_1; a_2\}, \end{aligned} \tag{4.2}$$

for $u \in E$ with $\|u\| > 1$.
We combine the inequalities (4.1) and (4.2) with each other, this fact gives

$$J_\lambda(u) \geqslant \frac{2A}{\alpha\left(P_M^+(\sqrt{N})^{p_m^- - 2}\right)^\alpha}\|u\|^{\alpha p_m^-} - \lambda\frac{c_3 N}{q^-}\|u\|^{q^+} - c_5. \tag{4.3}$$

Since $\alpha p_m^- > q^+ \geq q^-$, we infer that $J_\lambda(u) \to +\infty$ as $\|u\| \to +\infty$.

J_λ is coercive, weakly lower semicontinuous on E. Therefore, it has a minimum point $u_\lambda \in E$ and thus, a solution of (P).

To complete the proof, it is enough to show that u_λ is not trivial.
For this reason, letting $s > 1$ be a fixed real and $r_0 \in [1, N]_{\mathbb{Z}}$. Defining the function $u_0 : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}$ by

$$\begin{cases} u_0(r_0) = s, \\ u_0(r) = 0 \quad \text{for } r \in [0, N+1]_{\mathbb{Z}} \setminus \{r_0\}. \end{cases}$$

We have

$$\begin{aligned} J_\lambda(u_0) &= \sum_{i=1}^{2} \widehat{M}i\left(\frac{s^{p_i(r_0-1)}}{p_i(r_0-1)} + \frac{s^{p_i(r_0)}}{p_i(r_0)}\right) - \lambda F(r_0, s) \\ &\leq \widehat{M}_1\left(2\frac{s^{P_M^+}}{p_m^-}\right) + \widehat{M}_2\left(2\frac{s^{P_M^+}}{p_m^-}\right) - \lambda F(r_0, s). \end{aligned}$$

Since $F(r_0, s) > 0$, so there exists $\lambda^* > 0$ large enough such that $J_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^*, +\infty)$. It follows that for any $\lambda \geq \lambda^*$, $J_\lambda(u_\lambda) \leq J_\lambda(u_0) < 0$. Then, u_λ is nontrivial because $J_\lambda(0) = 0$.

□

Theorem 4.2. *Assume that the assumptions (f_0), (f_1), (M_0') and (AR) hold. Suppose additionally that the function q satisfies*

$$\alpha P_M^+ < q^-. \tag{4.4}$$

Then, for any $\lambda \in (0, +\infty)$ the problem (P) admits at least a nontrivial solution.

In order to prove existence of solution, we shall use the mountain pass theorem, so we start by proving that J_λ satisfies (PS) condition.

Lemma 4.3. *Under assumptions (M_0') and (AR), for any $\lambda > 0$, the functional J_λ satisfies the (PS) condition*

Proof. Let $(u_n) \subset E$ be a sequence such that $|J_\lambda(u_n)| \leq \tilde{c}$ and $J_\lambda'(u_n) \to 0$. As E is a finite dimensional space, it is enough to show that (u_n) is bounded. If not, we can find a subsequence, still denoted (u_n) such that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\| > 1$ for any integer n. Then, we get

$$\begin{aligned}\tilde{c}+\|u_n\| \geq & J_\lambda(u_n)-\frac{1}{\kappa}\left\langle J_\lambda'(u_n), u_n\right\rangle \\ = & \sum_{i=1}^{2}\widehat{M}_i(\phi_i(u_n))-\lambda\sum_{r=1}^{N}F(r,u_n(r)) \\ & -\frac{1}{\kappa}\sum_{i=1}^{2}M_i(\phi_i(u_n))\sum_{r=1}^{N+1}|\Delta u_n(r-1)|^{p_i(r-1)}+\frac{\lambda}{\kappa}\sum_{r=1}^{N}f(r,u_n(k))\,u_n(r).\end{aligned}$$

By (AR), (A_1) and (M_0'), we infer that

$$\begin{aligned}\tilde{c}+\|u_n\| \geq & \sum_{i=1}^{2}\widehat{M}_i(\phi_i(u_n))-\frac{P_M^+}{\kappa}\sum_{i=1}^{2}M_i(\phi_i(u_n))(\phi_i(u_n)) \\ & +\frac{\lambda}{\kappa}\sum_{r=1}^{N}\left(f(r,u_n(r))\,u_n(r)-\kappa F(r,u_n(r))\right) \\ \geq & \left(\frac{m_1}{\alpha}-\frac{m_2P_M^+}{\kappa}\right)\sum_{i=1}^{2}\left(\sum_{r=1}^{N+1}\frac{|\Delta u_n(r-1)|^{p_i(r-1)}}{p_i(r-1)}\right)^{\alpha}-c_6 \\ \geq & \left(\frac{m_1}{\alpha}-\frac{m_2P_M^+}{\kappa}\right)\frac{2}{\left(P_M^+(\sqrt{N})^{p_m^- -2}\right)^{\alpha}}\|u\|^{\alpha p_m^-}-c_7.\end{aligned}$$

Dividing the above inequality by $\|u_n\|^{\alpha p_m^-}$, and using the fact that $\kappa>\frac{m_2}{m_1}\alpha P_M^+$ we pass to the limit as $n\to\infty$, to obtain a contradiction. So, (u_n) is bounded in E and thus, J_λ satisfies the Palais-Smale condition. □

Lemma 4.4. *Suppose that the hypotheses of Theorem 4.2 hold. Then,*

(i) *There exist $\gamma,\varrho>0$ such that $J_\lambda(u)\geq\gamma>0,\ u\in E$ with $\|u\|=\varrho$.*
(ii) *There exists $e\in E$ such that $\|e\|>\varrho$ and $J_\lambda(e)<0$.*

Proof. (i) Taking $\|u\|<1$ and by condition $(M_0'),(A.2)$ and $(A.3)$, it follows that

$$\begin{aligned}J_\lambda(u) = & \sum_{i=1}^{2}\widehat{M}_i\left(\sum_{r=1}^{N+1}\frac{|\Delta u(r-1)|^{p_i(r-1)}}{p_i(r-1)}\right)-\lambda\sum_{r=1}^{N}F(r,u(r)) \\ \geq & \sum_{i=1}^{2}\frac{m_1}{\alpha\left(p_i^+\right)^{\alpha}}N^{\left(\frac{P_i^+-2}{2}\right)\alpha}\|u\|^{\alpha p_i^+}-\lambda\sum_{r=1}^{N}F(r,u(r)) \\ \geq & \frac{2m_1}{\alpha\left(P_M^+\right)^{\alpha}}N^{\left(\frac{P_M^+-2}{2}\right)\alpha}\|u\|^{\alpha P_M^+}-\lambda\sum_{r=1}^{N}F(r,u(r)).\end{aligned}$$

From (f_0) and (f_1), it follows that

$$F(r,s)\leq\varepsilon|s|^{\alpha P_M^+}+c_8|s|^{q(r)},\ \text{for all } (r,s)\in[1,N]_{\mathbb{Z}}\times\mathbb{R} \quad , \tag{4.5}$$

where $\varepsilon > 0$ and thus

$$\begin{aligned}\Psi(u) &\leq \varepsilon \sum_{r=1}^{N} |u(r)|^{\alpha P_M^+} + c_8 \sum_{r=1}^{N} |u(r)|^{q(r)} \\ &\leq \varepsilon c_{\alpha P_M^+} N \|u\|^{\alpha P_M^+} + c_9 \left(c_{q^+} N \|u\|^{q^+} + c_{q^-} N \|u\|^{q^-} \right) \\ &\leq \varepsilon c_{\alpha P_M^+} N \|u\|^{\alpha P_M^+} + c_{10} N \|u\|^{q^-}.\end{aligned}$$

Let taking $\varepsilon > 0$ be sufficiently small such that $\lambda \varepsilon c_{\alpha P_M^+} N \leq \dfrac{m_1}{\alpha \left(P_M^+\right)^{\alpha}} N^{(\frac{P_M^+ - 2}{2})\alpha}$. Also, we get

$$\begin{aligned}J_\lambda(u) &\geq 2 \frac{m_1}{\alpha \left(P_M^+\right)^{\alpha}} N^{(\frac{P_M^+ - 2}{2})\alpha} \|u\|^{\alpha P_M^+} - \lambda \varepsilon c_{\alpha p^+} N \|u\|^{\alpha P_M^+} - \lambda c_3 N \|u\|^{q^-} \\ &\geq \frac{m_1}{\alpha \left(P_M^+\right)^{\alpha}} N^{(\frac{P_M^+ - 2}{2})\alpha} \|u\|^{\alpha P_M^+} - \lambda c_3 N \|u\|^{q^-}.\end{aligned}$$

From (4.4) the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(s) = \frac{m_1}{\alpha \left(P_M^+\right)^{\alpha}} N^{(\frac{P_M^+ - 2}{2})\alpha} - \lambda c_3 N s^{q^- - \alpha P_M^+},$$

is positive in a neighborhood of the origin, the proof of (i) is completed.

(ii) From (AR), one can deduce

$$F(r, su) \geq s^{\kappa} F(r, u), \forall r \in [1, N]_{\mathbb{Z}} \text{ and } s \geq 1.$$

Therefore, for any $v \in E, v \neq 0$ and $t > 1$, we have

$$\begin{aligned}J_\lambda(tv) &= \sum_{i=1}^{2} \widehat{M}_i \left(\sum_{r=1}^{N+1} \frac{|\Delta tv(r-1)|^{p_i(r-1)}}{p_i(r-1)} \right) - \lambda \sum_{r=1}^{N} F(r, tv(r)) \\ &\leq \frac{m_2}{\alpha \left(p_m^-\right)^{\frac{m_2}{m_1}\alpha}} t^{\frac{m_2}{m_1}\alpha P_M^+} \sum_{i=1}^{2} \sum_{r=1}^{N+1} |\Delta v(r-1)|^{p_i(r-1)} - \lambda t^{\kappa} \sum_{r=1}^{N} F(r, v(r)).\end{aligned}$$

Then, $\lim_{t\to\infty} J_\lambda(tv) = -\infty$, because $\kappa > \dfrac{m_2}{m_1} \alpha P_M^+$. So, we can take $e = tv$ such that $\|e\| > \varrho$ and $J_\lambda(e) < 0$ for some t large enough.

□

Proof. Consequently, Since $J_\lambda(0) = 0$ and from the above lemmas, applying proposition 2.3, we conclude that the problem (P) admits at least a nontrivial solution.

□

Theorem 4.5. *Under assumptions (M_0') and (f_0), there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ the problem (P) has a nontrivial solution provided*

$$q^- < \alpha p_m^-.$$

To give the proof of this result, To prove this result, we apply Ekeland's variational principle, which requires the following two auxiliary lemmas.

Lemma 4.6. *There exist $\lambda_* > 0$ and $\rho,\ a > 0$ such that for all $\lambda \in (0, \lambda_*)$, we have $J_\lambda(u) \geq a > 0,\ \forall u \in E$ with $\|u\| = \rho$.*

Proof. Let us fix $\rho \in (0,1)$, so by conditions (M_0'), (f_0) and relations $(A.2)$, (2.3) we state that for any $u \in E$ with $\|u\| = \rho$, the following holds

$$\begin{aligned} J_\lambda(u) &= \sum_{i=1}^{2} \widehat{M}_i\left(\sum_{r=1}^{N+1} \frac{|\Delta u(r-1)|^{p_i(r-1)}}{p_i(r-1)}\right) - \lambda \sum_{r=1}^{N} F(r, u(r)) \\ &\geq \sum_{i=1}^{2} \frac{m_1}{\alpha\left(p_i^+\right)^\alpha} N^{(\frac{P_i^+-2}{2})\alpha} \|u\|^{\alpha p_i^+} - \lambda \frac{c_{11} c_{q^-}}{q^-} N \|u\|^{q^-} \\ &\geq \frac{2m_1}{\alpha\left(P_M^+\right)^\alpha} N^{(\frac{P_M^+-2}{2})\alpha} \|u\|^{\alpha P_M^+} - \lambda \frac{c_{11} c_{q^-}}{q^-} N \|u\|^{q^-} \\ &= \rho^{q^-}\left(\frac{2m_1}{\alpha\left(P_M^+\right)^\alpha} N^{(\frac{P_M^+-2}{2})\alpha} \rho^{\alpha P_M^+ - q^-} - \lambda c_{12}\right). \end{aligned}$$

Choosing λ_* as

$$\lambda_* = \frac{m_1 \rho^{\alpha P_M^+ - q^-}}{\alpha\left(P_M^+\right)^\alpha c_{12}} N^{(\frac{P_M^+-2}{2})\alpha}, \tag{4.6}$$

then for any $\lambda \in (0, \lambda_*)$ and $u \in E$ with $\|u\| = \rho$, there exists $a > 0$ such that $J_\lambda(u) \geq a$. □

Lemma 4.7. *For any $\lambda \in (0, \lambda_*)$ with λ_* given by (4.6), there exists $v \in E$ such that $v \neq 0$ and $J_\lambda(tv) < 0$, for $t > 0$ sufficiently small.*

Proof. Let $t \in (0,1)$ and take $v \in E$ such that $v(\tilde{r}) = 1$ and $v(r) = 0$ for $r \in [1, N]_{\mathbb{Z}} \setminus \{\tilde{r}\}$, with $r_0 \in [1, N]_{\mathbb{Z}}$ such that $q(\tilde{r}) = q^-$.
Using hypothesis (M_0'), (f_0) and relation $(A.2)$, we can write

$$\begin{aligned} J_\lambda(tv) &= \sum_{i=1}^{2} \widehat{M}_i\left(\phi_i(tv)\right) - \lambda \sum_{r=1}^{N} F(r, tv(r)) \\ &\leq \sum_{i=1}^{2} \frac{m_2}{\alpha} t^{\alpha p_m^-} \left(\phi_i(v)\right)^\alpha - \lambda F(\tilde{r}, t) \\ &\leq \frac{4m_2}{\alpha p_m^-} t^{\alpha p_m^-} - \lambda\left(c_{13} + c_{14} \frac{t^{q(\tilde{r})}}{q(\tilde{r})}\right) \\ &\leq \left(\frac{4m_2}{\alpha p_m^-} t^{\alpha p_m^- - q^-} - \frac{\lambda c_{14}}{q^-}\right) t^{q^-} - \lambda c_{13} \end{aligned}$$

So, for all $t < \delta^{\frac{1}{\alpha p_m^- - q^-}}$ such that

$$0 < \delta < \min\left\{1, \frac{\lambda \alpha p_m^- c_{14}}{4m_2 q^-}\right\},$$

we conclude that $J_\lambda(tv) < 0$. □

Proof. From Lemma 4.6 it follows that on the boundary of the ball centered at the origin and of radius ρ in E

$$\inf_{\partial B_\rho(0)} J_\lambda(u) > 0.$$

On the other hand, by Lemma 4.7, there exists $v \in E$ such that $J_\lambda(tv) < 0$ for $t > 0$ small enough. As a result, we have

$$-\infty < \bar{c} = \inf_{B_\rho(0)} J_\lambda(u) < 0.$$

Let $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$, let's use the Ekeland variational principle [12] to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, there exists $u_\epsilon \in \overline{B_\rho(0)}$, such that

$$J_\lambda(u_\epsilon) < \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon$$

$$J_\lambda(u) > J_\lambda(u_\epsilon) - \epsilon \|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

Since

$$J_\lambda(u_\epsilon) \leq \inf_{B_\rho(0)} J_\lambda + \epsilon \leq \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$.

Let us now define $L_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $L_\lambda(u) = J_\lambda(u) + \epsilon \|u - u_\epsilon\|$. It is obvious that u_ϵ is a minimum point of L_λ and thus

$$\frac{L_\lambda(u_\epsilon + t\varphi) - L_\lambda(u_\epsilon)}{t} \geq 0, \tag{4.7}$$

for $t > 0$ small enough and $\varphi \in B_\rho(0)$. From (4.7) , we have

$$\frac{J_\lambda(u_\epsilon + t\varphi) - J_\lambda(u_\epsilon)}{t} + \epsilon\|\varphi\| \geq 0.$$

Taking the limit as $t \to 0$, we infer that $\langle J'_\lambda(u_\epsilon), \varphi\rangle + \epsilon\|\varphi\| > 0$. Then, $\|J'_\lambda(u_\epsilon)\| \leq \epsilon$. So, there exists a sequence $(\varphi_n) \subset B_\rho(0)$ such that

$$J_\lambda(\varphi_n) \to \bar{c}, \quad J'_\lambda(\varphi_n) \to 0$$

It is clear that (φ_n) is bounded in E. Thus, there exists $\varphi_0 \in E$ and a subsequence, still denoted (φ_n) converges to φ_0 in E.

$$J_\lambda(\varphi_0) = \bar{c} < 0, \quad J'_\lambda(\varphi_0) = 0$$

Consequently, the problem (P) possesses a nontrivial solution. □

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Characterization and stability of essential pseudospectra by measure of polynomially inessential operators

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Abstract. In this article, we investigate the essential pseudospectra through the framework of polynomially inessential operators, which extends the class of polynomially strictly singular operators and provides a broader setting for Fredholm-type perturbations. We establish new results on the behavior of the essential pseudospectrum of closed linear operators on Banach spaces under perturbations by polynomially inessential operators. Moreover, we apply these results to study the influence of such perturbations on the left (resp. right) Weyl essential pseudospectra and the left (resp. right) Fredholm essential pseudospectra. In addition, we give a description of the essential pseudospectrum of the sum of two bounded linear operators. Finally, an application is provided to characterize the pseudo left (resp. right) Fredholm spectra of 2×2 block operator matrices.

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1. Introduction

Eigenvalue problems play a crucial role in many areas of science and engineering. The main objectives in such problems are to determine and localize the eigenvalues of a given operator. However, classical spectral theory is often insufficient to achieve both of these goals, as it merely identifies the eigenvalues without providing information about their localization or stability. To address this limitation, researchers introduced the concept of the *pseudospectrum*, first proposed by Varah [29] and Schechter and

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[27, 28]. Since then, pseudospectral analysis has been widely applied in various domains of mathematical physics, such as electrical engineering, aeronautics, ecology, and chemistry (see [9, 17, 28]). For instance, in electrical engineering, eigenvalues determine the stability and accuracy of an amplifier's frequency response; in aerodynamics, they indicate whether airflow over a wing remains laminar or becomes turbulent; in ecology, they govern the stability of equilibria in population dynamics; and in chemistry, they describe the energy levels of atomic systems. In summary, the concept of pseudospectrum has proven to be an effective tool in solving eigenvalue problems, allowing researchers to both locate and interpret spectral behavior, and thus contribute to major advances across scientific and engineering disciplines.

Motivated by the importance of pseudospectra, F. Abdmouleh *et al.* [2] introduced the notion of the pseudo-Browder essential spectrum for densely defined closed linear operators on Banach spaces. Later, F. Abdmouleh and B. Elgabeur in their works [3, 4, 11, 12] introduced and studied the pseudo left (resp. right) Fredholm and pseudo left (resp. right) Browder operators, together with their associated essential pseudospectra for bounded linear operators on Banach spaces. Among the principal results obtained in these works are stability theorems of the pseudo-essential spectra under Riesz operator perturbations. Moreover, the authors characterized the pseudo left (resp. right) Fredholm and pseudo left (resp. right) Browder essential pseudospectra for the sum of two bounded linear operators. Ammar and Jeribi in their contributions [5, 6] extended these investigations by developing the theory of essential pseudospectra of bounded linear operators and introducing the pseudo-Fredholm operator and its essential pseudospectrum.

In the present paper, we continue the study of essential pseudospectra in Banach spaces by considering a broader class of perturbations known as polynomially inessential operators. This family extends the well-known classes of compact, strictly singular, and polynomially strictly singular operators and can be viewed as a generalization of several classical Fredholm perturbations. This class of operators has drawn considerable interest due to its ability to unify several spectral perturbation frameworks and to yield new insights into Fredholm-type spectral stability. The reader may find further details and related results in [13, 14, 19]. The first objective of this paper is to extend the stability results of essential pseudospectra obtained in [2, 3, 4, 5, 6] to perturbations by polynomially inessential operators for closed, densely defined linear operators. The second goal is to describe the essential pseudospectrum of the sum of two bounded linear operators within the setting of polynomially inessential operators.

1.1. Organization of the paper

In Section 2, we recall some basic definitions and notations concerning Fredholm operators and their essential spectra. We also introduce and study the main properties of polynomially inessential operators. In Section 3, we establish stability results and provide a new characterization of the left (resp. right) Weyl and left (resp. right) Fredholm essential pseudospectra within this class of operators. Section 4 is devoted to the essential pseudospectra of the sum of two bounded linear operators, motivated by the concept of polynomially inessential operators. Finally, we extend the obtained results to the pseudo left (resp. right) Fredholm spectra for 2×2 block operator matrices.

2. Notations and definitions

Let X and Y be two Banach spaces. By an operator A from X into Y we mean a linear operator with domain $\mathcal{D}(A) \subseteq X$ and range contained in Y. We denote by $\mathcal{C}(X,Y)$ (resp., $\mathcal{L}(X,Y)$) the set of all closed, densely defined (resp., bounded) linear operators from X to Y. The subset of all compact operators of $\mathcal{L}(X,Y)$ is designated by $\mathcal{K}(X,Y)$. If $A \in \mathcal{C}(X,Y)$, we write $N(A) \subset X$ and $R(A) \subset Y$ for the null space and the range of A. We set $\alpha(A) := \dim N(A)$ and $\beta(A) := \operatorname{codim} R(A)$. Let $A \in \mathcal{C}(X,Y)$ with closed range. Then A is a Φ_+-operator ($A \in \Phi_+(X,Y)$) if $\alpha(A) < \infty$, and then A is a Φ_--operator ($A \in \Phi_-(X,Y)$) if $\beta(A) < \infty$.$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$ is the class of Fredholm operators while $\Phi_\pm(X,Y)$ denotes the set $\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y)$. For $A \in \Phi(X,Y)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. If $X = Y$, then $\mathcal{L}(X,Y), \mathcal{K}(X,Y), \mathcal{C}(X,Y), \Phi_+(X,Y), \Phi_\pm(X,Y)$ and $\Phi(X,Y)$ are replaced, respectively, by $\mathcal{L}(X), \mathcal{K}(X), \mathcal{C}(X), \Phi_+(X), \Phi_\pm(X)$ and $\Phi(X)$. Let $A \in \mathcal{C}(X)$, the spectrum of A will be denoted by $\sigma(A)$. The resolvent set of A, $\rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in $\Phi_{+A}, \Phi_{-A}, \Phi_{\pm A}$ or Φ_A if $\lambda - A$ is in $\Phi_+(X), \Phi_-(X), \Phi_\pm(X)$ or $\Phi(X)$, respectively. Let $F \in \mathcal{L}(X,Y)$.F is called a Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$. F is called an upper (resp., lower) Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ (resp., $U + F \in \Phi_-(X,Y)$) whenever $U \in \Phi_+(X,Y)$ (resp., $U \in \Phi_-(X,Y)$). The set of Weyl operators is defined as $\mathcal{W}(X,Y) = \{A \in \Phi(X,Y) : i(A) = 0\}$. Sets of left and right Fredholm operators, respectively, are defined as:

$$\Phi_l(X) := \big\{A \in \mathcal{L}(X) : R(A) \text{ is a closed and complemented subspace of } X, \text{ and } \alpha(A) < \infty\big\},$$

$$\Phi_r(X) := \big\{A \in \mathcal{L}(X) : N(A) \text{ is a closed and complemented subspace of } X, \text{ and } \beta(A) < \infty\big\}.$$

An operator $A \in \mathcal{L}(X)$ is left (right) Weyl if A is left (right) Fredholm operator and $i(A) \leq 0 (i(A) \geq 0)$. We use $\mathcal{W}_l(X)(\mathcal{W}_r(X))$ to denote the set of all left(right) Weyl operators. It is known that the sets $\Phi_l(X)$ and $\Phi_r(X)$ are open satisfying the following inclusions:

$$\Phi(X) \subset \mathcal{W}_l(X) \subset \Phi_l(X) \text{ and } \Phi(X) \subset \mathcal{W}_r(X) \subset \Phi_r(X).$$

The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y), \mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively. In general, we have

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}_+(X,Y) \subseteq \mathcal{F}(X,Y)$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{F}_-(X,Y) \subseteq \mathcal{F}(X,Y).$$

If $X = Y$ we write $\mathcal{F}(X), \mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ for $\mathcal{F}(X,X), \mathcal{F}_+(X,X)$ and $\mathcal{F}_-(X,X)$, respectively. Let $\Phi^b(X,Y), \Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ denote the sets $\Phi(X,Y) \cap \mathcal{L}(X,Y), \Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_-(X,Y) \cap \mathcal{L}(X,Y)$, respectively. If in Definition 1.1 we replace $\Phi(X,Y), \Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by $\Phi^b(X,Y), \Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ we obtain the sets $\mathcal{F}^b(X,Y), \mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$. These classes of operators were introduced and investigated in [6]. In particular, $\mathcal{F}^b(X,Y)$ is shown to be a closed subset of $\mathcal{L}(X,Y)$ and $\mathcal{F}^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. In general we have

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}^b_+(X,Y) \subseteq \mathcal{F}^b(X,Y)$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{F}^b_-(X,Y) \subseteq \mathcal{F}^b(X,Y)$$

Let $A \in \mathcal{C}(X)$. It follows from the closeness of A that $\mathcal{D}(A)$ endowed with the graph norm $\|\cdot\|_A$ ($\|x\|_A = \|x\| + \|Ax\|$) is a Banach space denoted by X_A. Clearly, for $x \in \mathcal{D}(A)$ we have $\|Ax\| \leqslant \|x\|_A$, so $A \in \mathcal{L}(X_A, X)$. Furthermore, we have Basic properties

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \quad \beta(\hat{A}) = \beta(A), \quad R(\hat{A}) = R(A) \\ \alpha(\hat{A}+\hat{B}) = \alpha(A+B), \\ \beta(\hat{A}+\hat{B}) = \beta(A+B) \text{ and } R(\hat{A}+\hat{B}) = R(A+B) \end{cases} \tag{2.1}$$

In this paper we are concerned with the following essential spectra of $A \in C(X)$:

$\sigma_e(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi(X)\}$: the Fredholm spectrum of A.

$\sigma_e^l(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_l(X)\}$: the left Fredholm spectrum of A.

$\sigma_e^r(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_r(X)\}$: the right Fredholm spectrum of A.

$\sigma_w(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}(X)\}$: the Weyl spectrum of A.

$\sigma_w^l(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}_l(X)\}$: the left Weyl spectrum of A.

$\sigma_w^r(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}_r(X)\}$: the right Weyl spectrum of A.

$\sigma_{\text{eap}}(A) := \mathrm{C} \backslash \rho_{\text{eap}}(A)$: the essential approximate point spectrum of A.

$\sigma_{e\delta}(A) := \mathrm{C} \backslash \rho_{e\delta}(T)$: the essential defect spectrum of A.

where

$$\rho_{\text{eap}}(A) := \{\lambda \in \mathrm{C} \text{ such that } \lambda - A \in \Phi_+(X) \text{ and } i(\lambda - A) \leq 0\},$$

and

$$\rho_{e\delta}(A) := \{\lambda \in \mathrm{C} \text{ such that } \lambda - A \in \Phi_-(X) \text{ and } i(\lambda - A) \geq 0\}$$

The definition of pseudo spectrum of a closed densely linear operator A for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{\lambda \in \mathbb{C} : \left\|(\lambda - A)^{-1}\right\| > \frac{1}{\varepsilon}\right\}.$$

By convention, we write $\left\|(\lambda - A)^{-1}\right\| = \infty$ if $(\lambda - A)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(A)$. In [9], Davies defined another equivalent of the pseudo spectrum, one that is in terms of perturbations of the spectrum. In fact for $A \in C(X)$, we have

$$\sigma_\varepsilon(A) := \bigcup_{\|D\|<\varepsilon} \sigma(A + D).$$

Inspired by the notion of pseudospectra, Ammar and Jeribi in their works [5, 6], aimed to extend these results for the essential pseudo-spectra of bounded linear operators on a Banach space and give the definitions of pseudo-Fredholm operator as follows: for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo-upper (resp. lower) semi-Fredholm operator if $A+D$ is an upper (resp. lower) semi-Fredholm operator and it is called a pseudo semi-Fredholm operator if $A+D$ is a semi-Fredholm operator. A is called a pseudo-Fredholm operator if $A + D$ is a Fredholm operator. They are noted by $\Phi^\varepsilon(X)$ the set of pseudo-Fredholm operators, by $\Phi^\varepsilon_\pm(X)$ the set of pseudo-semi-Fredholm operator and by $\Phi^\varepsilon_+(X)$ (resp. $\Phi^\varepsilon_-(X)$) the set of pseudo-upper semi-Fredholm operators (resp. lower semi-Fredholm). A complex number λ is in $\Phi^\varepsilon_{\pm A}$, Φ^ε_{+A}, Φ^ε_{-A} or Φ^ε_A if $\lambda - A$ is in $\Phi^\varepsilon_\pm(X), \Phi^\varepsilon_+(X), \Phi^\varepsilon_-(X)$ or $\Phi^\varepsilon(X)$.
F. Abdmouleh and B. Elgabeur in [4] defining the concept of pseudo left (resp. right) Fredholm, for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo left (resp. right) Fredholm operator if $A + D$ is a left (resp. right) Fredholm operator, they are denoted by $\Phi^\varepsilon_l(X)$ (resp. $\Phi^\varepsilon_r(X)$).

In this paper, we are concerned with the following essential pseudospectra of $A \in C(X)$:

$$\begin{aligned}
\sigma_{e1,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_+(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{+A},\\
\sigma_{e2,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_-(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{-A},\\
\sigma_{e3,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_\pm(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{\pm A},\\
\sigma_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon(X)\} = \mathbb{C}\backslash\Phi^\varepsilon_A,\\
\sigma_{eap,\varepsilon}(A) &:= \sigma_{e1,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) > 0, \forall\|D\| < \varepsilon\},\\
\sigma_{e\delta,\varepsilon}(A) &:= \sigma_{e2,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) < 0, \forall\|D\| < \varepsilon\},\\
\sigma^l_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_l(X)\},\\
\sigma^r_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_r(X)\},\\
\sigma^l_{w,\varepsilon}(A) &:= \sigma^l_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) > 0, \forall\|D\| < \varepsilon\},\\
\sigma^r_{w,\varepsilon}(A) &:= \sigma^r_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) < 0, \forall\|D\| < \varepsilon\},\\
\sigma_{w,\varepsilon}(A) &:= \sigma_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) = 0, \forall\|D\| < \varepsilon\}.
\end{aligned}$$

Note that if ε tends to 0, we recover the usual definition of the essential spectra of a closed operator A. The subsets σ_{e1} and σ_{e2} are the Gustafson and Weidmann essential spectra [16], σ_{e3} is the Kato essential spectrum,[19] σ_e is the Wolf essential spectrum[16], σ_{e5} is the Schechter essential spectrum[26], σ_{eap} is the essential approximate point spectrum [24], $\sigma_{e\delta}$ is the essential defect spectrum [25], $\sigma_e^l(A)$(resp.$\sigma_e^r(A)$) is the left (resp. right) Fredholm essential spectra and $\sigma_w^l(A)$(resp.$\sigma_w^r(A)$) is the left (resp. right) Weyl essential spectra [15, 30, 31].
As a concept, pseudospectra and essential pseudospectra are interesting because they offer more information than spectra, especially about transients rather than just asymptotic behavior. Moreover, they perform more efficiently than spectra in terms of convergence and approximation. These include the existence of approximate eigenvalues far from the spectrum and the instability of the spectrum even under small perturbations. Various applications of the pseudospectra and essential pseudospectra have been developed as a result of the analysis of the pseudospectra and essential pseudospectra.
We now list some of the known facts about left and right Fredholm operators in Banach space which will be used in the sequel.

Proposition 2.1. [18, propositon 2.3] *Let X, Y and Z be three Banach spaces.*

(i) *If $A \in \Phi^b(Y,Z)$ and $T \in \Phi_l^b(X,Y)$ (resp. $T \in \Phi_r^b(X,Y)$), then $AT \in \Phi_l^b(X,Z)$ (resp. $AT \in \Phi_r^b(X,Z)$).*

(ii) *If $A \in \Phi^b(Y,Z)$ and $T \in \Phi_l^b(X,Y)$ (resp. $T \in \Phi_r^b(X,Y)$), then $TA \in \Phi_l^b(X,Z)$ (resp. $TA \in \Phi_r^b(X,Z)$).* $\diamondsuit$

Theorem 2.2. [22, 26] *Let X, Y and Z be three Banach spaces, $A \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(X,Y)$.*

(i) *If $A \in \Phi^b(Y,Z)$ and $T \in \Phi^b(X,Y)$, then $AT \in \Phi^b(X,Z)$ and $i(AT) = i(A) + i(T)$.*

(ii) *If $X = Y = Z$, $AT \in \Phi^b(X)$ and $TA \in \Phi^b(X)$, then $A \in \Phi^b(X)$ and $T \in \Phi^b(X)$.* $\diamondsuit$

Lemma 2.3. [15, Theorem 2.3] *Let $A \in \mathcal{L}(X)$, then*

(i) $A \in \Phi_l^b(X)$ if and only if there exist $A_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $A_lA = I - K$.

(ii) $A \in \Phi_r^b(X)$ if and only if there exist $A_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $AA_r = I - K$. $\diamondsuit$

Lemma 2.4. [15, Theorem 2.7] *Let $A \in \mathcal{L}(X)$.*

If $A \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) and $K \in \mathcal{K}(X)$, then $A + K \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) and $i(A+K) = i(A)$. $\diamondsuit$

Lemma 2.5. [15, Theorem 2.5] *Let $A, B \in \mathcal{L}(X)$, If $A \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) and $B \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) then $AB \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) and,*

$$i(A+B) = i(A) + i(B). \qquad \diamondsuit$$

We close with the following Lemma.

Lemma 2.6. [8, Lemma 3.4] *Let* $A \in \mathcal{L}(X)$.

(i) If $AB \in \Phi_l^b(X)$ *then* $B \in \Phi_l^b(X)$.

(ii) If $AB \in \Phi_r^b(X)$ *then* $A \in \Phi_r^b(X)$.

Definition 2.7. *Let* X *be a Banach space.*

(i) *An operator* $A \in \mathcal{L}(X)$ *is said to have a left Fredholm inverse if there exists* $A_l \in \mathcal{L}(X)$ *such that* $I - A_l A \in \mathcal{K}(X)$.

(ii) *An operator* $A \in \mathcal{L}(X)$ *is said to have a right Fredholm inverse if there exists* $A_r \in \mathcal{L}(X)$ *such that* $I - AA_r \in \mathcal{K}(X)$. ♢

We know by the classical theory of Fredholm operators, see for example [19], that A belong to $\Phi(X)$ if it possesses a left, right or two-sided Fredholm inverse, respectively.

We define these sets $\mathcal{I}nvF_A^l(X)$ and $\mathcal{I}nvF_A^r(X)$ by:

$$\mathcal{I}nv_{A,l}^F(X) := \{A_l \in \mathcal{L}(X) : A_l \text{ is a left Fredholm inverse of } A\},$$

$$\mathcal{I}nv_{A,r}^F(X) := \{A_r \in \mathcal{L}(X) : A_r \text{ is a right Fredholm inverse of } A\}.$$

Definition 2.8. *An operator* $S \in \mathcal{L}(X, Y)$ *is to be strictly singular if for every infinite dimensional subspace* M *of* X*, the restriction of* S *to* M *is not a homeomorphism.*

Let $\mathcal{S}(X, Y)$ denote the set of strictly singular operators from X into Y. Note that $\mathcal{S}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$. In general, strictly singular operators are not compact (see [13, 14]) and if $X = Y, \mathcal{S}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $\mathcal{K}(X) = \mathcal{S}(X)$. For basic properties of strictly singular operators, we refer to [14, 19].

Definition 2.9. *An minimal polynomial* P *is the unitary polynomial of smaller degree that cancels an endomorphism, that is to say a linear application of a vector space in itself.*

In the following, we define the set of polynomially strict singular operators will denote by $\mathcal{P}_\mathcal{S}$, as follows:

$\mathcal{P}_\mathcal{S} = \{A \in \mathcal{L}(X)$, such that there exists a nonzero complex polynomial $P(z) := \sum_{k=0}^{p} a_k z^k$, satisfying $P\left(\frac{1}{n}\right) \neq 0, \forall n \in \mathbb{Z}^*$ and $P(A) \in \mathcal{S}(X)\}$.

Polynomially Inessential Operators

We now introduce and study the class of *polynomially inessential operators*, which provides a natural polynomial extension of the ideal of inessential operators. This class contains, as particular cases, both compact and polynomially strictly singular operators.

Definition 2.10 (Polynomially Inessential Operators). *Let X be a complex Banach space. We denote by $\mathcal{IIO}(X)$ the ideal of inessential operators on X, that is,*

$$\mathcal{IIO}(X) = \{T \in \mathcal{L}(X) : I - ST \text{ is Fredholm for every } S \in \mathcal{L}(X)\}.$$

An operator $A \in \mathcal{L}(X)$ is said to be polynomially inessential *if there exists a nonzero complex polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ such that*

$$P(A) \in \mathcal{IIO}(X).$$

The set of all polynomially inessential operators on X is denoted by

$$\mathcal{PIIO}(X) := \{A \in \mathcal{L}(X) : \exists P \in \mathbb{C}(X) \setminus \{0\},\ P(A) \in \mathcal{IIO}(X)\}.$$

Remark 2.11. By definition, we clearly have

$$\mathcal{P}_{\mathcal{S}}(X) \subset \mathcal{PIIO}(X),$$

where $PS(X)$ denotes the set of polynomially strictly singular operators. Consequently, $\mathcal{PIIO}(X)$ also contains $S(X)$ and $K(X)$. Thus, $\mathcal{PIIO}(X)$ is a strictly larger ideal than these classes in general Banach spaces.

Theorem 2.12 (Basic Properties of $\mathcal{PIIO}(X)$). *Let X be a complex Banach space. Then the following properties hold:*

1. *$\mathcal{PIIO}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.*
2. *The following inclusions hold:*

$$K(X) \subset S(X) \subset \mathcal{P}_{\mathcal{S}}(X) \subset \mathcal{PIIO}(X) \subset \mathcal{IIO}(X) \subset \mathcal{L}(X).$$

3. *If X is a Hilbert space, then*

$$K(X) = S(X) = \mathcal{P}_{\mathcal{S}}(X) = \mathcal{PIIO}(X) = \mathcal{IIO}(X).$$

4. *$\mathcal{PIIO}(X)$ is norm-closed: if $A_n \in \mathcal{PIIO}(X)$ and $A_n \to A$ in operator norm, then $A \in \mathcal{PIIO}(X)$.*
5. *$\mathcal{PIIO}(X)$ is stable under compact perturbations: if $A \in \mathcal{PIIO}(X)$ and $K \in K(X)$, then $A + K \in \mathcal{PIIO}(X)$.*

Proof. (1) Let $A \in \mathcal{PIIO}(X)$ and $B, C \in \mathcal{L}(X)$. By definition, there exists $P \in \mathbb{C}[z] \setminus \{0\}$ such that $P(A) \in \mathcal{IIO}(X)$. Since $\mathcal{IIO}(X)$ is a two-sided ideal of $\mathcal{L}(X)$, we have $BP(A)C \in \mathcal{IIO}(X)$. As $BP(A)C = Q(BAC)$ for some polynomial Q, it follows that $BAC \in \mathcal{PIIO}(X)$.

(2) The inclusion chain follows directly from the definitions and the fact that $\mathcal{IIO}(X)$ contains both $S(X)$ and $K(X)$ (see [23, 21]).

(3) If X is a Hilbert space, it is known that $K(X) = S(X) = \mathcal{IIO}(X)$ (see [21]). Hence, all intermediate polynomial extensions coincide.

(4) The norm-closedness follows from the closedness of $\mathcal{IIO}(X)$ in $\mathcal{L}(X)$ and the continuity of the polynomial functional calculus.

(5) Compact perturbations preserve inessentiality, i.e., if $T \in \mathcal{IIO}(X)$ and $K \in K(X)$, then $T + K \in \mathcal{IIO}(X)$ (see [1]). Applying this to $P(A)$ yields $P(A + K) \in \mathcal{IIO}(X)$, so $A + K \in \mathcal{IIO}(X)$. ☐

Remark 2.13. The class $\mathcal{PIIO}(X)$ can be seen as a *polynomial extension of the inessential ideal*, and therefore as a natural setting for the study of *polynomial perturbations of Fredholm operators*. In particular, if $A \in \mathcal{PIIO}(X)$ and $P(A) \in \mathcal{IIO}(X)$, then $0 \in \sigma_e(P(A))$, which implies that the essential and Browder essential spectra of $P(A)$ coincide at 0.

Proposition 2.14. *Let $A \in \mathcal{PIIO}(X)$, that is, there exists a nonzero complex polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_p z^p$$

such that $P(A) \in \mathcal{IIO}(X)$, where $\mathcal{IIO}(X)$ denotes the ideal of inessential operators on the Banach space X. Let $\lambda \in \mathbb{C}$ be such that $P(\lambda) \neq 0$. Then the following hold:

(i) *The element $\pi_{\mathcal{IIO}}(A) - \lambda I$ is invertible in the quotient algebra $\mathcal{L}(X)/\mathcal{IIO}(X)$, where $\pi_{\mathcal{IIO}} : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{IIO}(X)$ is the canonical projection.*

(ii) *Consequently, $A - \lambda I$ is* invertible modulo *$\mathcal{E}(X)$, that is, $A - \lambda I$ is a Fredholm element relative to the ideal $\mathcal{IIO}(X)$, and its relative index is zero.*

(iii) *If, in addition, $\mathcal{IIO}(X) \subset K(X)$ (for instance, if X is a Hilbert space), then $A - \lambda I$ is a Fredholm operator in the classical sense and its Fredholm index is zero.*

Proof. Let $\lambda \in \mathbb{C}$ with $P(\lambda) \neq 0$. Using the standard polynomial division, we can write

$$P(z) = P(\lambda) + (z - \lambda)Q(z),$$

where Q is a polynomial of degree $p - 1$.

Applying the polynomial functional calculus to A, we get

$$P(A) = P(\lambda)I + (A - \lambda I)Q(A).$$

Since $P(A) \in \mathcal{IIO}(X)$ by hypothesis, its class in the quotient algebra

$$B := \mathcal{L}(X)/\mathcal{IIO}(X)$$

vanishes:

$$\pi_{\mathcal{IIO}}(P(A)) = 0.$$

Hence, in B, we have the relation

$$0 = P(\lambda)I_B + (\pi_{\mathcal{IIO}}(A) - \lambda I_B)Q(\pi_{\mathcal{IIO}}(A)).$$

Since $P(\lambda) \neq 0$ is an invertible scalar, we can rearrange this equation to obtain

$$(\pi_{\mathcal{IIO}}(A) - \lambda I_B)\left(-\frac{1}{P(\lambda)}Q(\pi_{\mathcal{IIO}}(A))\right) = I_B.$$

Thus, $\pi_{\mathcal{IIO}}(A) - \lambda I_B$ admits a right inverse in B. Repeating the argument with the polynomial identity

$$P(z) = P(\lambda) + Q_1(z)(z - \lambda)$$

provides a left inverse as well. Therefore, $\pi_{\mathcal{IIO}}(A) - \lambda I_B$ is invertible in the quotient algebra B. This proves (i).

Statement (ii) follows immediately, since invertibility in the quotient algebra $IIO(X)/\mathcal{E}(X)$ means that there exists $B \in \mathcal{L}(X)$ such that

$$(A - \lambda I)B - I \in \mathcal{IIO}(X) \quad \text{and} \quad B(A - \lambda I) - I \in \mathcal{IIO}(X).$$

By definition, this means that $A - \lambda I$ is invertible modulo $\mathcal{IIO}(X)$, or equivalently, that it is a Fredholm element relative to $\mathcal{IIO}(X)$. The index of a Fredholm element in a quotient algebra is always zero.

Finally, for (iii), if $\mathcal{IIO}(X) \subset K(X)$, then invertibility modulo $\mathcal{IIO}(X)$ implies invertibility modulo $K(X)$, since any inverse modulo $\mathcal{IIO}(X)$ also works modulo the smaller ideal $K(X)$. Hence, $A - \lambda I$ is Fredholm in the classical sense and $\text{ind}(A - \lambda I) = 0$. In particular, this holds whenever X is a Hilbert space, where $\mathcal{IIO}(X) = K(X)$. □

Remark 2.15. The above result generalizes the classical statement for polynomially strictly singular operators (see [7, Corollary 2.1]), where the ideal $S(X)$ is replaced by $\mathcal{IIO}(X)$. In this more general setting, the conclusion concerns invertibility modulo $\mathcal{IIO}(X)$, which coincides with the usual Fredholm property only when $\mathcal{IIO}(X) = K(X)$, for instance on Hilbert spaces.

3. Stability of essential pseudospectra by means of polynomially inessential perturbations of operators

The following theorem provides a practical criterion for the stability of some essential pseudospectra for perturbed linear operators.

Theorem 3.1. *Let $\varepsilon > 0$ and consider $A, B \in \mathcal{C}(X)$. Assume that there are $A_0,\ B_0 \in \mathcal{L}(X)$ and $S_1,\ S_2 \in \mathcal{PIIO}(X)$ such that*

$$AA_0 = I - S_1, \tag{3.1}$$

$$BB_0 = I - S_2. \tag{3.2}$$

(i) *If $0 \in \Phi_A \cap \Phi_B, A_0 - B_0 \in \mathcal{F}_+(X)$ and $i(A) = i(B)$ then*

$$\sigma_{\text{eap},\varepsilon}(A) = \sigma_{\text{eap},\varepsilon}(B). \tag{3.3}$$

(ii) *If $0 \in \Phi_A \cap \Phi_B, A_0 - B_0 \in \mathcal{F}_-(X)$ and $i(A) = i(B)$ then*

$$\sigma_{e\delta,\varepsilon}(A) = \sigma_{e\delta,\varepsilon}(B). \tag{3.4}$$

(iii) *If $A_0 - B_0 \in \mathcal{F}(X)$, then*

$$\sigma_{e,\varepsilon}(A) = \sigma_{e,\varepsilon}(B).$$

If, further, $0 \in \Phi_A \cap \Phi_B$ such that $i(A) = i(B)$, then

$$\sigma_{w,\varepsilon}(A) = \sigma_{w,\varepsilon}(B). \tag{3.5}$$

Proof. Let λ be a complex number, Equations (3.1) and (3.2) imply

$$(\lambda - A - D)A_0 - (\lambda - B - D)B_0 = S_1 - S_2 + (\lambda - D)(A_0 - B_0). \tag{3.6}$$

(i) Let $\lambda \notin \sigma_{\text{eap},\varepsilon}(B)$, then $\lambda \in \Phi^{\varepsilon}_{+B}$ such that $i(\lambda - B - D) \leq 0$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since $B + D$ is closed and $\mathcal{D}(B + D) = \mathcal{D}(B)$ endowed with the graph norm is a Banach space denoted by X_{B+D}. We can regard $B+D$ as an operator from X_{B+D} into X. This will be denoted by $\widehat{B+D}$. Using Equation (2.1) we can show that

$$\lambda - \widehat{B+D} \in \Phi_+^b(X_B, X) \text{ and } i(\lambda - \widehat{B+D}) \leq 0.$$

Moreover, since $S_2 \in \mathcal{PIIO}(X)$, applying Proposition 2.14, we obtain $I - S_2 \in \Phi(X)$. Applying [26], Theorem 2.7, p.171 and Equation (3.2), we get $B_0 \in \Phi^b(X, X_B)$. That is $(\lambda - \widehat{B+D})B_0 \in \Phi_+^b(X)$. This asserts that $A_0 - B_0 \in \mathcal{F}_+(X)$ and taking into account Equation (3.6), $(\lambda - \widehat{A+D})A_0 \in \Phi_+^b(X)$ and

$$i((\lambda - \widehat{A+D})A_0) = i((\lambda - \widehat{B+D})B_0). \tag{3.7}$$

A similar reasoning as before combining Equations (2.1) and (3.1), Proposition 2.14 and [26], Corollary 1.6, p. 166, [26], Theorem 2.6, p. 170 shows that $A_0 \in \Phi^b(X, X_A)$ where $X_A := (\mathcal{D}(A), \|.\|_A)$. By [26], Theorem 1.4, p. 108 one sees that

$$A_0 S = I - F \text{ on } X_A,$$

where $S \in \mathcal{L}(X_A, X)$ and $F \in \mathcal{K}(X_A)$, by Equation (3.2) we have

$$(\lambda - \widehat{B+D})A_0 S = (\lambda - \widehat{A+D}) - (\lambda - \widehat{A+D})F.$$

Combining the fact that $S \in \Phi^b(X_A, X)$ with [[26], Theorem 6.6, p. 129], we show that
$(\lambda - \widehat{A+D})A_0 S \in \Phi_+^b(X_A, X)$. Following [[26], Theorem 6.3, p. 128], we derive $(\lambda - \widehat{A+D}) \in \Phi_+^b(X_A, X)$. Thus, Equation (2.1) asserts that

$$(\lambda - A - D) \in \Phi_+(X). \tag{3.8}$$

On the other hand, the assumptions $S_1, S_2 \in \mathcal{PIIO}(X)$, Equations (3.1), (3.2) and Proposition 2.1, [[26], Theorem 2.3 , p. 111] reveals that

$$i(A) + i(A_0) = i(I - S_1) = 0 \text{ and } i(B) + i(B_0) = i(I - S_2) = 0,$$

since $i(A) = i(B)$. That is $i(A_0) = i(B_0)$.

Using Equation (3.7) and [[22], Theorem 2.3, p. 111], we can write

$$i(\lambda - A - D) + i(A_0) = i(\lambda - B - D) + i(B_0).$$

Therefore

$$i(\lambda - A - D) \leq 0, \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{3.9}$$

Using Equations (3.8) and (3.9), we conclude that

$$\lambda \notin \sigma_{eap,\varepsilon}(A).$$

Therefore we prove the inclusion

$$\sigma_{eap,\varepsilon}(A) \subset \sigma_{eap,\varepsilon}(B).$$

The opposite inclusion follows from symmetry and we obtain Equation (3.3).

(ii) The proof of Equation (3.4) may be checked in a similar way to that in *(i)*. It suffices to replace $\sigma_{eap,\varepsilon}(.)$, $\Phi_+(.)$, $i(.) \leq 0$, [26], Theorem 6.6, p. 129, [26], Theorem 6.3, p. 128 by $\sigma_{e\delta,\varepsilon}(.)$, $\Phi_-(.)$, $i(.) \geq 0$, [22], Theorem 5 (i), p. 150, [26], Theorem 6.7, p. 129 respectively. The details are therefore omitted.

(iii) If $\lambda \notin \sigma_{e,\varepsilon}(B)$, then $\lambda - B - D \in \Phi(X)$. Since B is closed, its domain $\mathcal{D}(B)$ becomes a Banach space X_B for the graph norm $\|.\|_B$. The use of Equation (2.1) leads to $\lambda - \widehat{B+D} \in \Phi^b(X_B, X)$. Moreover, Equation (3.2), Proposition 2.1 and [26], Theorem 5.13 reveals that $B_0 \in \Phi^b(X, X_B)$ and consequently $(\lambda - \widehat{B+D})B_0 \in \Phi^b(X)$. Following with the assumption, Equation (3.6) and [26], Theorem 5.13, leads to estimate $(\lambda - \widehat{A+D})A_0 \in \Phi^b(X)$ with

$$i[(\lambda - \widehat{A+D})A_0] = i[(\lambda - \widehat{B+D})B_0]. \tag{3.10}$$

Since $A \in \mathcal{C}(X)$, proceeding as above, Equation (3.1) implies that $A_0 \in \Phi^b(X, X_A)$. By [26], Theorem 5.4 we can write

$$A_0 S = I - F \ on\ X_A, \tag{3.11}$$

where $S \in \mathcal{L}(X_A, X)$ and $F \in \mathcal{F}(X_A)$. Taking into account Equation (3.11) we infer that

$$(\lambda - \widehat{A+D})A_0 S = (\lambda - \widehat{A+D}) - (\lambda - \widehat{A+D})F.$$

Therefore, since $S \in \Phi^b(X_A, X)$, the use of [26], Theorem 6.6 amounts to

$$(\lambda - \widehat{A+D})A_0 S \in \Phi^b(X_A, X).$$

Applying [26],Theorem 6.3, we prove that $(\lambda - \widehat{A+D}) \in \Phi^b(X_A, X)$ and consequently

$$(\lambda - A - D) \in \Phi(X).$$

Thus $\lambda \notin \sigma_{e,\varepsilon}(A)$. This implies that $\sigma_{e,\varepsilon}(A) \subset \sigma_{e,\varepsilon}(B)$. Conversely, if $\lambda \notin \sigma_{e,\varepsilon}(A)$, we can easily derive the opposite inclusion.

Now, we prove Equation (3.5). If $\lambda \notin \sigma_{w,\varepsilon}(B)$, then, $\lambda \in \Phi_B^{\varepsilon}$ and $i(\lambda - B - D) = 0$, for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$. On the other hand, since $S_1, S_2 \in \mathcal{PIIO}(X)$ and $i(A) = i(B) = 0$, using the Atkinson theorem, we obtain $i(A_0) = i(B_0) = 0$. This together with Equation (3.10) gives $i(\lambda - \widehat{A+D}) = i(\lambda - \widehat{B+D})$. Consequently $i(\lambda - A - D) = 0$, for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$.. Hence $\lambda \notin \sigma_{w,\varepsilon}(A)$, which proves the inclusion $\sigma_{w,\varepsilon}(A) \subset \sigma_{w,\varepsilon}(B)$. The opposite inclusion follows by symmetry.

□

In the following theorems, we give some perturbation results of the pseudo left, pseudo right Fredholm and pseudo left, pseudo right Weyl spectra for a bounded linear operator in a Banach space.

Theorem 3.2. *Let A and B be two operators in $\mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. The following statements hold:*

(i) *Assume that $\lambda - A \in \Phi_l(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_l \in \mathcal{I}nv^F_{\lambda-A-D,l}(X)$ such that $BA_l \in \mathcal{PIIO}(X)$, then*

$$\sigma^l_{e,\varepsilon}(A+B) \subseteq \sigma^l_{e,\varepsilon}(A).$$

(ii) *Assume that* $\lambda - A \in \Phi_r(X)$ *and for all* $D \in \mathcal{L}(X)$ *with* $\|D\| < \varepsilon$, *there exists* $A_r \in \mathcal{I}nv^F_{\lambda-A-D,r}(X)$ *such that* $A_rB \in \mathcal{PIIO}(X)$, *then*

$$\sigma^r_{e,\varepsilon}(A+B) \subseteq \sigma^r_{e,\varepsilon}(A).$$

Proof. (i) Let $\lambda \notin \sigma^{\text{left}}_{\text{e},\varepsilon}(A)$, $\lambda - A - D \in \Phi^\varepsilon_l(X)$. As A_l is a left Fredholm inverse of $\lambda - A - D$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. then by Lemma 2.3 there exists a compact operator $K \in \mathcal{K}(X)$ such that

$$A_l(\lambda - A - D) + K = I.$$

Then, we can write

$$\lambda - A - B - D = (I - BA_l)(\lambda - A - D) - BK. \tag{3.12}$$

Using the fact that $BA_l \in \mathcal{PIIO}(X)$ and according to Proposition 2.14, we have $I - BA_l \in \Phi(X)$. Consequently, by Lemma 2.5 we get

$$(I - BA_l)(\lambda - A - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

Thus, combining the fact that $BK \in \mathcal{K}(X)$ with the use of Equation (3.12) and Lemma 2.4, we have $\lambda-A-B-D \in \Phi_l(X)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Therefore, $\lambda \notin \sigma^l_{\text{e},\varepsilon}(A+B)$ as required.

(ii) Let $\lambda \notin \sigma^r_{\text{e},\varepsilon}(A)$, then $\lambda - A - D \in \Phi_r(X)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since A_r is a right Fredholm inverse of $\lambda - A - D$. From Lemma 2.3 we infer there exists a compact operator $K \in \mathcal{K}(X)$ such that

$$(\lambda - A - D)A_\text{r} = I - K\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

Then, we can write $\lambda - A - B - D$ with the following form

$$\lambda - A - B - D = (\lambda - A - D)(I - A_\text{r}B) - KB,\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon. \tag{3.13}$$

Since $A_\text{r}B \in \mathcal{PIIO}(X)$ then, according to Proposition 2.14, we have $I - A_\text{r}B \in \Phi(X)$. Consequently, by Lemma 2.5, we get

$$(\lambda - A - D)(I - A_\text{r}B) \in \Phi_\text{r}(X),\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

On the other hand, from Equation (3.13) and Lemma 2.4 and the fact $BK \in \mathcal{K}(X)$ we show that $\lambda - A - B - D \in \Phi_\text{r}(X)$, for all $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$. We deduce that, $\lambda \notin \sigma^r_{\text{e},\varepsilon}(A+B)$.

□

Theorem 3.3. *Let* A *and* B *be two operators in* $\mathcal{L}(X)$ *and* $\lambda \in \mathbb{C}$. *The following statements hold:*

(i) *Assume that* $\lambda - A \in \Phi_l(X)$ *and for all* $D \in \mathcal{L}(X)$ *with* $\|D\| < \varepsilon$, *there exists* $A_l \in \mathcal{I}nv^F_{\lambda-A-D,l}(X)$ *such that* $BA_l \in \mathcal{PIIO}(X)$, *then*

$$\sigma^l_{e,\varepsilon}(A+B) \subseteq \sigma^l_{e,\varepsilon}(A).$$

(ii) *Assume that* $\lambda - A \in \Phi_r(X)$ *and for all* $D \in \mathcal{L}(X)$ *with* $\|D\| < \varepsilon$, *there exists* $A_r \in \mathcal{I}nv^F_{\lambda-A-D,r}(X)$ *such that* $A_rB \in \mathcal{PIIO}(X)$, *then*

$$\sigma^r_{e,\varepsilon}(A+B) \subseteq \sigma^r_{e,\varepsilon}(A).$$

Proof. (i) Assume that $\lambda \notin \sigma^l_{\mathrm{w},\varepsilon}(A)$, then we have $\lambda - A - D \in \Phi_l(X)$ and $i(\lambda - A - D) \leq 0$. A similar reasoning as above gives $\lambda - A - B - D \in \Phi_l(X)$ and it suffices to prove that $i(\lambda - A - B - D) \leq 0$. Since $BK \in \mathcal{K}(X)$ then, Using Equation 3.12 together with Lemmas 2.4 and 2.5, we obtain that

$$i(\lambda - A - B - D) = i\,(I - BA_l) + i(\lambda - A - D).$$

Now, Since $BA_l \in \mathcal{PIIO}(X)$, we get by Proposition 2.14, that $i\,(I - BA_l) = 0$. We deduce that

$$i(\lambda - A - B - D) = i\,(\lambda - A - D) \leq 0.$$

Finally, we conclude that $\lambda - A - B - D \in \mathcal{W}_l(X)$, which entails that $\lambda \notin \sigma^{\mathrm{l}}_{\mathrm{w},\varepsilon}(A + B)$.

(ii) with the same reasoning of (i). Let $\lambda \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(A)$, then we have $\lambda - A - D \in \Phi_r(X)$ and $i(\lambda - A - D) \geq 0$. Proceeding as the proof above, we establish that $\lambda - A - B - D \in \Phi_r(X)$ and $i(\lambda - A - B - D) \geq 0$. Therefore, $\lambda - A - B - D \in \mathcal{W}_r(X)$ and we deduce that $\lambda \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(A + B)$.

□

Remark 3.4. The results of Theorems 3.1, 3.2 and 3.3 are extensions and an improvements of the results of in [2, 3, 4, 5, 6] to a large class of polynomially strict singular operators. ♢

4. Characterization essential spectrum of two linear bounded operators

This section aims to carry out a new criterion allowing us to investigate some spectral analysis of the sum of two linear bounded operators. We begin by giving the following lemma when we need it in the sequel.

Lemma 4.1. [8, Lemma 4.1] *Let $A \in \mathcal{L}(X)$.*

(i) If $C\sigma^l_e(A)$ is connected, then

$$\sigma^l_e(A) = \sigma^l_w(A).$$

(ii) If $C\sigma^r_e(A)$ is connected, then

$$\sigma^r_e(A) = \sigma^r_w(A).$$

Theorem 4.2. *Let A, $B \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:*

(i) *Assume that the subsets $C\sigma^l_e(A)$ and $C\sigma^l_e(B)$ are connected, $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$ and $-\lambda^{-1}BAQ_l \in \mathcal{PIIO}(X)$, for every $Q_l \in \mathcal{I}nv^F_{\lambda - A - B - D,l}(X)$, then we have:*

$$\left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma^l_{w,\varepsilon}(A + B) \setminus \{0\}.$$

(ii) *Assume that the subsets $C\sigma^r_e(A)$ and $C\sigma^r_e(B)$ are connected, $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ and $-\lambda^{-1}Q_rBA \in \mathcal{PIIO}(X)$, for every $Q_r \in \mathcal{I}nv^F_{\lambda - A - B - D,r}(X)$, then we have:*

$$\left[\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}^r(A+B) \setminus \{0\}.$$

(iii) *Assume that the subsets* $C\sigma_e^l(A)$, $C\sigma_e^l(B)$, $C\sigma_e^r(A)$ *and* $C\sigma_e^r(B)$ *are connected,* $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$, $-\lambda^{-1}BAQ_l \in \mathcal{PIIO}(X)$, $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ *and* $-\lambda^{-1}Q_rBA \in \mathcal{PIIO}(X)$, *for* $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$ *and* $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, *then we have:*

$$\left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A+B) \setminus \{0\}. \qquad \diamondsuit$$

Proof.

Firstly we note two equality which is used repeatedly

$$(\lambda - A)(\lambda - B - D) = A(B + D) + \lambda(\lambda - A - B - D). \tag{4.1}$$

$$(\lambda - B - D)(\lambda - A) = (B + D)A + \lambda(\lambda - A - B - D). \tag{4.2}$$

(i) Let $\lambda \notin \sigma_{w,\varepsilon}^l(A+B)\cup\{0\}$ so we have $\lambda-A-B-D \in \Phi_l(X)$ and $i(\lambda-A-B-D) \leq 0$. Then following to the Lemma 2.3 there exist $Q_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\lambda - A - B - D) = I - K$.

So when we use Equation (4.1) we obtain

$$\begin{aligned}
(\lambda - A)(\lambda - B - D) &= A(B + D) + \lambda(\lambda - A - B - D). \\
&= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D). \\
&= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK. \\
&= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK,
\end{aligned}$$

Since $\lambda[\lambda^{-1}ABQ_l+I] \in \Phi(X)$ and $(\lambda-A-B-D) \in \Phi_l(X)$ it follows from Proposition 2.1 that $\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) \in \Phi_l(X)$. Since $ABK \in \mathcal{K}(X)$, this implies by the use of Lemma 2.4 that

$$\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABQ_lK \in \Phi_l(X).$$

So $(\lambda - A)(\lambda - B - D) \in \Phi_l(X)$ and as a direct consequence of Lemma 2.6 we obtain

$$\lambda - B - D \in \Phi_l(X), \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{4.3}$$

In the other hand, when we use the Equation (4.2) we have

$$\begin{aligned}
(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D), \\
&= BA[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D), \\
&= [BAQ_l + \lambda I](\lambda - A - B - D) + BAK, \\
&= \lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) + BAK.
\end{aligned}$$

Since $\lambda[\lambda^{-1}BAQ_l+I] \in \Phi(X)$ and $(\lambda-A-B-D) \in \Phi_l(X)$ it follows from Proposition 2.1 that

$$\lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) \in \Phi_l(X).$$

Obviously, since $BAK \in \mathcal{K}(X)$ and applying Lemma 2.4, we find that

$$\lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) + BAK \in \Phi_l(X).$$

So $(\lambda - B - D)(\lambda - A) \in \Phi_l(X)$. Therefore using Lemma 2.6 we obtain

$$\lambda - A \in \Phi_l(X). \tag{4.4}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$i(\lambda - A) + i(\lambda - B - D) = i(\lambda - A - B - D) \leq 0.$$

Case1: If $i(\lambda - A) \leq 0$

Using Lemma 4.1 the index $i(\lambda - B - D)$ must be negative.Therefore adding this condition to Equations (4.3) and (4.4) we obtain

$$\lambda \notin \left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case2: If $i(\lambda - B - D) \leq 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ must be negative.

Then adding this condition to Equations (4.3) and (4.4) we assert

$$\lambda \notin \left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case3: If $i(\lambda - A) > 0$.

Following to Lemma 4.1 the index $i(\lambda - B - D)$ should be positif which contradicts the fact that $i(\lambda - A - B - D) \leq 0$.

Case4: If $i(\lambda - B - D) > 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ must be positif which contradicts the fact that $i(\lambda - A - B - D) \leq 0$.

(ii) Let $\lambda \notin \sigma^r_{w,\varepsilon}(A+B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_r(X)$ and $i(\lambda - A - B - D) \leq 0$. So by Lemma 2.3 there exist $Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $(\lambda - A - B - D)Q_r = I - K$

So following to the Equation (4.1) we have

$$\begin{aligned} (\lambda - A)(\lambda - B - D) &= AB + \lambda(\lambda - A - B - D), \\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D), \\ &= (\lambda - A - B - D)[Q_r AB + \lambda I] + ABK, \\ &= \lambda(\lambda - A - B - D)[\lambda^{-1} Q_r AB + I] + ABK. \end{aligned}$$

Since $\lambda[\lambda^{-1} Q_r AB + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_r(X)$ it follows by Proposition 2.1 that

$$\lambda[\lambda^{-1} Q_r AB + I](\lambda - A - B - D) \in \Phi_r(X).$$

Since $ABK \in \mathcal{K}(X)$ then

$$\lambda[\lambda^{-1} Q_r AB + I](\lambda - A - B - D) + ABK \in \Phi_r(X).$$

So $(\lambda - A)(\lambda - B - D) \in \Phi_r(X)$, following to Lemma 2.6 we infer that

$$\lambda - A \in \Phi_r(X). \tag{4.5}$$

In the other hand, the use of Equation (4.2) assert

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D), \\ &= BA[(\lambda - A - B - D)Q_r + K]BA + \lambda(\lambda - A - B - D), \\ &= (\lambda - A - B - D)[Q_rBA + \lambda I] + KBA, \\ &= \lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] + KBA.\end{aligned}$$

Since by hypothesis $[\lambda^{-1}Q_rBA + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_r(X)$ we have by Proposition 2.1

$$\lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] \in \Phi_r(X).$$

Since $KBA \in \mathcal{K}(X)$ we obtain

$$\lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] + KBA \in \Phi_r(X).$$

So $(\lambda - B - D)(\lambda - A) \in \Phi_r(X)$ then the use of Lemma 2.6 infer that

$$\lambda - B - D \in \Phi_r(X), \ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.6}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$i(\lambda - A) + i(\lambda - B - D) = i(\lambda - A - B - D) \geq 0.$$

Case 1: If $i(\lambda - A) \geq 0$

Using Lemma 4.1 the index $i(\lambda - B - D)$ must be positif. Therefore adding this condition to Equations (4.5) and (4.6) we get

$$\lambda \notin \left[\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right] \cup \{0\}.$$

Case 2: If $i(\lambda - B - D) \geq 0$.

Following to Lemma 4.1 the index $i(\lambda - A)$ must be positif.

Then adding this condition to Equations (4.3) and (4.4) we obtain

$$\lambda \notin \left[\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right] \cup \{0\}.$$

Case 3: If $i(\lambda - A) < 0$

Following to Lemma 4.1 the index $i(\lambda - B - D)$ should be negative which contradicts the fact that $i(\lambda - A - B - D) \geq 0$.

Case 4: If $i(\lambda - B - D) < 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ should be negative which contradicts the fact that $i(\lambda - A - B - D) \geq 0$.

(iii) Let $\lambda \notin \sigma_{w,\varepsilon}(A+B) \cup \{0\}$ therefore $\lambda - A - B - D \in \Phi(X)$ and $i(\lambda - A - B - D) = 0$ then there exist $Q_l, Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\lambda - A - B - D) = I - K$ and $(\lambda - A - B - D)Q_r = I - K$.

Now, according to items (i) and (ii) we get

$$\left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A+B) \setminus \{0\}. \qquad \square$$

Theorem 4.3. *Let A, $B \in \mathcal{L}(X)$ such that $AB = BA$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:*

(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, such that $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$ then*

$$\sigma^l_{e,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma^l_e(A) \cup \sigma^l_{e,\varepsilon}(B)\big] \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ then*

$$\sigma^r_{e,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma^r_e(A) \cup \sigma^r_{e,\varepsilon}(B)\big] \setminus \{0\}.$$

(iii) *If there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}QAB \in \mathcal{PIIO}(X)$ and $-\lambda^{-1}ABQ \in \mathcal{PIIO}(X)$ then*

$$\sigma_{e,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\big] \setminus \{0\}. \qquad \diamondsuit$$

Proof.

(i) Let $\lambda \notin \sigma^l_{e,\varepsilon}(A+B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_l(X)$.

We assume there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, thus, using Equation (4.1) we have

$$\begin{aligned}(\lambda - A)(\lambda - B - D) &= A(B+D) + \lambda(\lambda - A - B - D),\\ &= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\ &= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK,\\ &= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK.\end{aligned}$$

Obviously, $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$ then by Proposition 2.14 we infer that $\lambda^{-1}ABQ_l + I \in \Phi(X)$. Therefore, by Lemma 2.5 we obtain $[\lambda^{-1}ABQ_l + \lambda I](\lambda - A - B - D) \in \Phi_l(X)$.
Since $ABK \in \mathcal{K}(X)$ and by applying Lemma 2.4 we obtain

$$\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK \in \Phi_l(X).$$

We conclude that

$$(\lambda - A)(\lambda - B - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon.$$

Hence, by Lemma 2.6 we deduce that

$$(\lambda - B - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.7}$$

On the other hand, using the fact that $AB = BA$ and according to the Equation (4.2) we observe that

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= AB + \lambda(\lambda - A - B - D),\\ &= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\ &= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK,\\ &= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK.\end{aligned}$$

Using the same reasoning we conclude that $(\lambda - B - D)(\lambda - A) \in \Phi_l(X)$. Therefore, by Lemma 2.6 we deduce that

$$(\lambda - A) \in \Phi_l(X). \tag{4.8}$$

Finally, the two Equations (4.7) and (4.8) imply that $\lambda \notin \left[\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right] \cup \{0\}$. So, we obtain

$$\left[\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right] \setminus \{0\} \subset \sigma_{e,\varepsilon}^l(A+B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [8, Theorem 4.3].

(ii) Let $\lambda \notin \sigma_{e,\varepsilon}^r(A+B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_r(X)$, for all $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$.

We assume there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$ thus,

$$\begin{aligned}
(\lambda - A)(\lambda - B - D) &= AB + \lambda(\lambda - A - B - D),\\
&= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\
&= (\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB.
\end{aligned}$$

Evidently, $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ and by applying Proposition 2.14 we deduce that $\lambda^{-1}Q_rAB + I \in \Phi(X)$. Since, KAB is compact, then by Lemma 2.4 we obtain

$$(\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB \in \Phi_l(X).$$

Consequently, we have $(\lambda - A)(\lambda - B - D) \in \Phi_r(X)$ and by Lemma 2.6 we infer that

$$(\lambda - A) \in \Phi_r(X). \tag{4.9}$$

Further, we have $AB = BA$ so,

$$\begin{aligned}
(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\
&= AB + \lambda(\lambda - A - B - D),\\
&= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\
&= (\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB.
\end{aligned}$$

Using the same reasoning we conclude that $(\lambda - B - D)(\lambda - A) \in \Phi_r(X)$. Then, by Lemma 2.6 we deduce that

$$(\lambda - B - D) \in \Phi_r(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.10}$$

Finally, the two Equations (4.9) and (4.10) imply that

$$\lambda \notin \left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \cup \{0\}.$$

So, we obtain

$$\left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \setminus \{0\} \subset \sigma_{e,\varepsilon}^r(A+B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [8, Theorem 4.3].

(iii) Let $\lambda \notin \sigma_{e,\varepsilon}(A+B) \cup \{0\}$. Then $\lambda - A - B - D \in \Phi(X)$ means that $\lambda - A - B - D \in \Phi_l(X) \cap \Phi_r(X)$.

Now, by the hypothesis there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, and by applying the results in statements *(i)* and *(ii)* we infer that $(\lambda - A - B - D) \in \Phi_r(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$, therefore $(\lambda - A - B - D) \in \Phi(X)$.

Also, using the hypothesis that $-\lambda^{-1}QAB \in \mathcal{PIIO}(X))$, $-\lambda^{-1}ABQ \in \mathcal{PIIO}(X)$ and $AB = BA$ we give us this two condition:

$$(\lambda - A)(\lambda - B - D) \in \Phi(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \Phi(X).$$

Therefore, following Theorem 2.2 we obtain $(\lambda - A) \in \Phi(X)$ and $(\lambda - B - D) \in \Phi(X)$ means that $\lambda \notin \left[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\right] \cup \{0\}$. Then we get the following inclusion

$$\left[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{e,\varepsilon}(A+B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [8, Theorem 4.3]. □

The same reasoning of the above theorem, we allow to obtain the result of the following result.

Theorem 4.4. *Let A, $B \in \mathcal{L}(X)$ such that $AB = BA$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:*

(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, such that $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$ then*

$$\sigma^l_{w,\varepsilon}(A+B) \setminus \{0\} = \left[\sigma^l_{w,\varepsilon}(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ then*

$$\sigma^r_{w,\varepsilon}(A+B) \setminus \{0\} = \left[\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right] \setminus \{0\}.$$

(iii) *If there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}QAB \in \mathcal{PIIO}(X)$ and $-\lambda^{-1}ABQ \in \mathcal{PIIO}(X)$ then*

$$\sigma_{w,\varepsilon}(A+B) \setminus \{0\} = \left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\}. \qquad \diamondsuit$$

5. Application to bounded 2×2 block operator matrices forms

The objective of this section is to utilize Theorem 4.3 from Section 4 in order to analyze the pseudo left (right)-Fredholm essential spectra of the given operator matrix.

Let X_1 and X_2 be two Banach spaces and consider the 2×2 block operator matrices defined on $X_1 \times X_2$ by:

$$\mathcal{M} := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

where $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$, $C \in \mathcal{L}(X_2, X_1)$ and $D \in \mathcal{L}(X_1, X_2)$.

Next, we define the following matrix:

$$\mathfrak{D} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where $D_1 \in \mathcal{L}(X_1)$, $D_2 \in \mathcal{L}(X_2)$ and $\|\mathfrak{D}\| = \max\{\|D_1\|, \|D_2\|\}$.

In the following theorem, we seek the pseudo left (right)-Fredholm essential spectra of Matrix $\mathcal{M}_C$.

Theorem 5.1. *Let the* 2×2 *block operator matrix* $\mathcal{M}_C$ *and* $\varepsilon > 0$*. In all that follows we will make the following assumptions:*

$$\mathcal{H} : \begin{cases} \|\mathfrak{D}\| < \varepsilon, \\ AC = CB, \\ A \in \Phi(X), B \in \Phi(X), \\ CB \in \mathcal{IIO}(X_1 \times X_2). \end{cases}$$

Then, we have that

(*i*) $\sigma^{left}_{e,\varepsilon}(\mathcal{M}_C)\backslash\{0\} \subseteq \left[\sigma^{left}_{e,\varepsilon}(A) \cup \sigma^{left}_{e,\varepsilon}(B)\right]\backslash\{0\}$.

(*ii*) $\sigma^{right}_{e,\varepsilon}(\mathcal{M}_C)\backslash\{0\} \subseteq \left[\sigma^{right}_{e,\varepsilon}(A) \cup \sigma^{right}_{e,\varepsilon}(B)\right]\backslash\{0\}$.

Proof. We begin by presenting the polynomial P in the specified format:

$$\begin{array}{ccccc} P & : & \mathbb{R}^2 & \rightarrow & \mathbb{R} \\ & & (x, y) & \mapsto & P(x, y) = x.y \end{array}$$

We can write

$$\mathcal{M} := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$= \ \mathcal{M}_{\mathcal{C}} \ + \ \mathcal{M}_{A,B}.$

We have:

$$P(\mathcal{M}_{\mathcal{C}}, \mathcal{M}_{A,B}) = \mathcal{M}_{\mathcal{C}}.\mathcal{M}_{A,B} = \begin{pmatrix} 0 & CB \\ 0 & 0 \end{pmatrix}.$$

it follows from the hypothesis (H) that:

$$P(\mathcal{M}_{\mathcal{C}}, \mathcal{M}_{A,B}) \in \mathcal{IIO}(X_1 \times X_2), \text{and } \mathcal{M}_{\mathcal{C}}.\mathcal{M}_{A,B} \in \mathcal{PIIO}_T(X).$$

Moreover we have $A + B \in \Phi(X)$ then there exist $A_0 \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $A_0(A + B) = I - K$. Then

$$A_0(A + B + D) = I - K', \text{with } K' \in \mathcal{K}(X).$$

Using Theorem 4.3, we obtain that

(*i*)

$$\begin{aligned} \sigma^{left}_{e,\varepsilon}(\mathcal{M}) \setminus \{0\} &= \sigma^{left}_{e,\varepsilon}(\mathcal{M}_C + \mathcal{M}_{A,B}) \setminus \{0\} \\ &= \left[\sigma^{left}_{e}(\mathcal{M}_C) \cup \sigma^{left}_{e,\varepsilon}(\mathcal{M}_{A,B})\right] \setminus \{0\}. \end{aligned}$$

(*ii*)

$$\begin{aligned} \sigma^{right}_{e,\varepsilon}(\mathcal{M}) \setminus \{0\} &= \sigma^{right}_{e,\varepsilon}(\mathcal{M}_C + \mathcal{M}_{A,B}) \setminus \{0\} \\ &= \left[\sigma^{right}_{e}(\mathcal{M}_C) \cup \sigma^{right}_{e,\varepsilon}(\mathcal{M}_{A,B})\right] \setminus \{0\}. \end{aligned}$$

Furthermore, we can readily demonstrate $\sigma_e^{left}(\mathcal{M}_\mathcal{C}) = \sigma_e^{right}(\mathcal{M}_\mathcal{C}) = \{0\}$. Consequently, applying [[3], Theorem 4 (i)], we show that

$$\begin{aligned}\sigma_{e,\varepsilon}^{left}(\mathcal{M}) \setminus \{0\} &= \left[\sigma_e^{left}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right] \setminus \{0\} \\ &= \left[\{0\} \cup \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right] \setminus \{0\} \\ &= \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B}) \\ &\subseteq \left[\sigma_{e,\varepsilon}^{left}(A) \cup \sigma_{e,\varepsilon}^{left}(B)\right] \setminus \{0\}.\end{aligned}$$

Similarly, we have:

$$\begin{aligned}\sigma_{e,\varepsilon}^{right}(\mathcal{M}) \setminus \{0\} &= \left[\sigma_e^{right}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right] \setminus \{0\} \\ &= \left[\{0\} \cup \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right] \setminus \{0\} \\ &= \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B}) \\ &\subseteq \left[\sigma_{e,\varepsilon}^{right}(A) \cup \sigma_{e,\varepsilon}^{right}(B)\right] \setminus \{0\}.\end{aligned}$$

□

Conclusion

In this article, we have explored the behavior of essential pseudospectra within the framework of polynomially inessential operators. Building on foundational concepts from Fredholm theory, we have developed new stability results and characterizations for the left and right Weyl and Fredholm essential pseudospectra. These insights not only deepen the understanding of spectral properties for such operators but also provide tools for analyzing operator sums. Furthermore, the extension of these results to 2×2 block operator matrices highlights the robustness of the theory and its applicability in more structured operator settings. These findings open new avenues for future research in the spectral analysis of operator matrices and perturbation theory.

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Lefschetz admissible dominated spaces for maps with an inclusion property

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Abstract. We consider the notion of a Lefschetz admissible dominated space and we present some fixed point results for compact maps with a selection property.

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Keywords: Fixed points, set–valued maps, admissible spaces.

1. Introduction

In this paper we consider two general classes of maps, namely the $HYAd$ and $HYAdC$ maps which have a very general selection property (motivated by KLU [10], $HLPY$ [11], Scalzo [17] and Wu [18] maps). Using a result that NES(compact) and $SANES$(compact) spaces are Lefschetz spaces (see [6, 13]) we establish new Lefschetz fixed point theorems for NES, $SANES$ and Lefschetz admissible and admissible dominated spaces. Our results improve and complement those in the literature (see [2, 3, 6, 13, 14, 16] and the references therein).

First we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

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Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\left(p^{-1}(x)\right) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [7]. An upper semicontinuous map $\phi : X \to 2^Y$ (nonempty subsets of Y) with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. Another example is an acyclic map which we now describe. Let X and Z be subsets of Hausdorff topological spaces and let $F : X \to K(Z)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now we consider maps $F : X \to Ac(Z)$ i.e. $F : X \to K(Z)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X, Z)$ (i.e. F is an acyclic map) if $F : X \to Ac(Z)$ is upper semicontinuous.

Next we consider a general class of maps, namely the PK maps of Park (which include Kak and Ad maps). Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : \; Fix\, F \neq \emptyset \;\text{ for all }\; F \in \mathbf{X}(Z, Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:
(i). $\mathbf{U}$ contains the class C of single valued continuous functions;
(ii). each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and
(iii). $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \; \|x\| \leq 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Next we recall the following fixed point result [15] for PK maps. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E

there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 1.1. *Let X be an admissible convex set in a Hausdorff topological vector space and $F \in PK(X,X)$ be a closed compact map. Then F has a fixed point in X.*

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. Of course, given two single valued maps $f, g : X \to Y$ and $\alpha \in Cov(Y)$, then f and g are α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say f and g are α-homotopic if there is a homotopy $h_t : X \to Y$ $(t \in [0,1])$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0,1]$. We say f and g are homotopic if there is a homotopy $h_t : X \to Y$ $(t \in [0,1])$ joining f and g. We recall the following result [2].

Theorem 1.2. *Let X be a regular topological space, $F : X \to 2^X$ an upper semicontinuous map with closed values and suppose there exists a cofinal covering $\theta \subseteq Cov_X(\overline{F(X)})$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.3. From Theorem 1.2 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [3, page 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [9, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular [5]). Also note in Theorem 1.2 if F is compact valued, then the assumption that X is regular can be removed. We note here that when we apply Theorem 1.2 we will assume the space is uniform. Of course one could consider other appropriate spaces (like regular (Hausdorff) spaces) as well.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X,K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X,K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0. A space Y is a neighborhood extension space for Q (written $Y \in NES(Q)$) if $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$ there exists a continuous extension $f : U \to Y$ of f_0 over a neighborhood U of K in X. A space Y is an approximate neighborhood extension space for Q (written $Y \in ANES(Q)$) if $\forall \alpha \in Cov(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function

$f_0 : K \to Y$ there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α–close. A space Y is a strongly approximate neighborhood extension space for Q (written $Y \in SANES(Q)$) if $\forall \alpha \in Cov(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$ there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α–close and homotopic.

Next we describe the maps due to Wu [18]. Let X and Y be subsets lying in Hausdorff topological vector spaces and we say $\Phi \in W(X,Y)$ if $\Phi : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Next we recall a selection theorem [1] (see the proof in Theorem 1.1 there) for Wu maps.

Theorem 1.4. *Let X be a paracompact subset of a Hausdorff topological vector space and Y a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X,Y)$ and let $\theta : X \to 2^Y$ be a lower semicontinuous map with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists an upper semicontinuous map $\Psi : X \to CK(Y)$ (collection of nonempty convex compact subsets of Y) with $\Psi(x) \subseteq \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Remark 1.5. Let X be paracompact and Y a metrizable subset of a complete Hausdorff locally convex linear topological space E and $\Phi \in W(X,Y)$ with $\theta : X \to 2^Y$ a lower semicontinuous map and $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [12] that $\overline{co}\,\theta : X \to 2^Y$ (since $\overline{co}\,(\theta(x)) \subseteq \Phi(x) \subseteq Y$ for $x \in X$) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in \overline{co}\,(\theta(x))$ for $x \in X$, so consequently $f(x) \in \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z,W)$ [11] if W is convex and there exists a map $S : Z \to W$ (i.e. $S : Z \to P(W)$ (collection of subsets of W)) with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\,int\,S^{-1}(w) :\ w \in W\}$; here $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\,S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below see [11].

Theorem 1.6. *Let X be a paracompact subset of a Hausdorff topological space, Y a convex subset of a Hausdorff topological vector space and $F \in HLPY(X,Y)$ (let $S : X \to 2^Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\,int\,S^{-1}(w) :\ w \in Y\}$). Then there exists a continuous (single–valued) map $f : X \to Y$ with $f(x) \in co\,S(x) \subseteq F(x)$ for all $x \in X$.*

Remark 1.7. These maps are related to the DKT maps in the literature and $F \in DKT(Z,W)$ [4] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the Fan maps.

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \to 2^Y$ has the strong continuous inclusion property (SCIP) [10] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \to 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co\,F^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X,Y)$ if T has the SCIP at every $x \in X$.

In this paper our map T will be a compact map so T has the SCIP is equivalent to T has the CIP [8].

Remark 1.8. These maps contain as a special case the Scalzo maps [17] in the literature (see [10, p. 12]).

Next we recall a selection theorem [10].

Theorem 1.9. *Let X be a paracompact subset of a Hausdorff topological space, Y a subset of a Hausdorff topological vector space and $T \in KLU(X,Y)$. Then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(w) \subseteq co\,T(w)$ for all $w \in X$.*

Finally we present some preliminaries on the Lefschetz set for Ad maps needed in Section 2. Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E :\ f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\,\tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q\, Tr\,(f_q).$$

With Čech homology functor extended to a category of morphisms we have the following well known result [7] (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$). If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then $(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star$. Two morphisms $\phi, \psi \in M(X,Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x,0) = \phi(x)$, $\chi(x,1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$). Recall the following result [10]: If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

A map $\phi \in Ad(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ the linear map $q_\star p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism. If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}(\phi)$ (or $\mathbf{\Lambda}_X(\phi)$) by

$$\mathbf{\Lambda}(\phi) = \left\{ \Lambda(q_\star p_\star^{-1}) : \ (p,q) \subset \phi \right\}.$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class Ad) provided every compact $\phi \in Ad(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Now we recall the fixed point results in [6, 13] needed in Section 2. Let X be a subset of a Hausdorff topological vector space. We say $F \in HY Ad(X,X)$ if $F : X \to 2^X$ and there exists a map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$.

Let $X \in NES(\text{compact})$ (and X a subset of a Hausdorff topological vector space) and $F \in HY Ad(X,X)$ with $co\,F$ a compact map. Then there exists a compact map $\Phi \in Ad(X,Y)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$ (note Φ is a compact map since $co\,F$ is a compact map). In [6, 13] we showed that the Lefschetz set $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.

Let X be a subset of a Hausdorff topological vector space. We say $F \in HY AdC(X,X)$ if $F : X \to 2^X$ and there exists a map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$.

Let $X \in NES(\text{compact})$ and $F \in HY AdC(X,X)$ with F a compact map. Then there exists a compact map $\Phi \in Ad(X,Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$. In [6, 13] we showed that the Lefschetz set $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.

A special case of the above (when $\Phi = F$) is the following.

Theorem 1.10. *Let $X \in NES(compact)$ and $F \in Ad(X,X)$ a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

Let $X \in SANES(\text{compact})$ (and X a subset of a Hausdorff topological vector space) and $F \in HY Ad(X,X)$ with $co\,F$ a compact map. Then there exists a compact map $\Phi \in Ad(X,Y)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$. In [13] we showed that the Lefschetz set $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.

Let $X \in SANES$(compact) and X a uniform space and $F \in HYAdC(X, X)$ with F a compact map. Then there exists a compact map $\Phi \in Ad(X, Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$. In [13] we showed that the Lefschetz set $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.

A special case of the above (when $\Phi = F$) is the following.

Theorem 1.11. *Let $X \in SANES(compact)$, X a uniform space and $F \in Ad(X, X)$ a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

2. Lefschetz Fixed Point Theory

The results in this section are motivated by admissibility in Section 1, Schauder projections and dominating space [6, 7]. Let W be a subset of a Hausdorff topological space.

Definition 2.1. *We say W is NES admissible if for all compact subsets K of W, all $\alpha \in Cov_W(K)$, there exists a continuous function $\pi_\alpha : K \to W$ such that*

(i). π_α and $i : K \hookrightarrow W$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$ and $C_\alpha \in NES(compact)$;

(iii). π_α and $i : K \hookrightarrow W$ are homotopic.

Definition 2.2. *We say W is $SANES$ admissible if for all compact subsets K of W, all $\alpha \in Cov_W(K)$, there exists a continuous function $\pi_\alpha : K \to W$ such that*

(i). π_α and $i : K \hookrightarrow W$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$, $C_\alpha \in SANES(compact)$ and C_α is a uniform space;

(iii). π_α and $i : K \hookrightarrow W$ are homotopic.

Indeed one could combine the above two definitions and consider the more general situation.

Definition 2.3. *We say W is Lefschetz admissible if for all compact subsets K of W, all $\alpha \in Cov_W(K)$, there exists a continuous function $\pi_\alpha : K \to W$ such that*

(i). π_α and $i : K \hookrightarrow W$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$ and C_α is a Lefschetz space;

(iii). π_α and $i : K \hookrightarrow W$ are homotopic.

Remark 2.4. In Definition 2.2 if W is a subset of a Hausdorff topological vector space then W is a uniform space and so automatically C_α is a uniform space (recall a subset of a uniform space is a uniform space). Thus C_α is a uniform space is redundant in Definition 2.2 if W is a subset of a Hausdorff topological vector space or more generally if W is a uniform space.

Theorem 2.5. *Let X be NES admissible (and X a subset of a Hausdorff topological vector space) and $F \in HY\,Ad(X,X)$ with $co\,F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Then $\mathbf{\Lambda}\,(\Phi)$ is well defined and if $\mathbf{\Lambda}\,(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.*

Proof. Let Φ be as in the statement above. Let $p,\, q : \Gamma \to X$ be a selected pair of Φ and let $K = \overline{\Phi(X)}$. Also let $\alpha \in Cov_X(K)$. Then there exists a continuous function $\pi_\alpha : K \to X$ and a subset C of X with $\pi_\alpha(K) \subseteq C$, $C \in NES(\text{compact})$ and π_α and $i : K \hookrightarrow X$ are α–close. Let $q_\alpha = \pi_\alpha\, q$. Note q_α is a compact map and $q \sim q_\alpha$; to see this note there exists a continuous map $h : K \times [0,1] \to X$ with $h(x,0) = \pi_\alpha(x)$ and $h(x,1) = i(x)$ and with $\eta(x,t) = h(q(x),t)$ we have $h : \Gamma \times [0,1] \to X$ (note p is surjective so $p^{-1}(X) = \Gamma$ and so $q : \Gamma \to K$) with $\eta(x,0) = \pi_\alpha q(x) = q_\alpha(x)$ and $\eta(x,1) = i(g(x)) = g(x)$.

Also note $q_\alpha(\Gamma) = \pi_\alpha q(\Gamma) \subseteq \pi_\alpha(K) \subseteq C$. Let

$$p_\alpha : p^{-1}(C) \to C,\ \overline{q_\alpha} : p^{-1}(C) \to C,\ q'_\alpha : \Gamma \to C$$

denote contractions of the appropriate maps and note p_α is a Vietoris map (see [6, pp 214]). Also let

$$i_C : C \hookrightarrow X \quad \text{and} \quad j : p^{-1}(C) \hookrightarrow \Gamma$$

be inclusions. Note $(p_\alpha, \overline{q_\alpha})$ is a selected pair of $\pi_\alpha\Phi$ (note $p_\alpha : p^{-1}(C) \to C$ and $\overline{q_\alpha} : p^{-1}(C) \to C$) and $\pi_\alpha\Phi \in Ad(C,C)$ is a compact map (note the composition of Ad maps is an Ad map [6] and note $\Phi \in Ad(C,K)$ [7, 13]). Now since $C \in NES(\text{compact})$ then Theorem 1.10 guarantees that $(\overline{q_\alpha})_\star (p_\alpha^{-1})_\star$ is a Leray endomorphism. Next we note (recall $i_C q'_\alpha = q_\alpha$)

$$(i_C)_\star (q'_\alpha)_\star p_\star^{-1} = (i_C q'_\alpha)_\star p_\star^{-1} = (q_\alpha)_\star p_\star^{-1} = q_\star p_\star^{-1}$$

since $q \sim q_\alpha$ (so [7] $q_\star = (q_\alpha)_\star$). Also we have (note [16] $p_\star p_\star^{-1} = i_\star$ and $i_C p_\alpha = p\,j$ and $q'_\alpha j = \overline{q_\alpha}$)

$$(q'_\alpha)_\star p_\star^{-1} (i_C)_\star = (q'_\alpha)_\star j_\star (p_\alpha^{-1})_\star = (\overline{q_\alpha})_\star (p_\alpha^{-1})_\star.$$

Now [6, pp 214 (see (1.3))] (here $E' = H(C)$, $E'' = H(X)$, $u = (i_C)_\star$, $v = (q'_\alpha)_\star p_\star^{-1}$, $f' = (\overline{q_\alpha})_\star (p_\alpha^{-1})_\star$, $f'' = q_\star p_\star^{-1}$ and note $u\, f' = (i_C)_\star (\overline{q_\alpha})_\star (p_\alpha^{-1})_\star = (i_C)_\star (q'_\alpha)_\star p_\star^{-1} (i_C)_\star = q_\star p_\star^{-1} (i_C)_\star = f''\, u$) guarantees that $q_\star p_\star^{-1}$ is a Leray endomorphism and $\mathbf{\Lambda}\,(q_\star p_\star^{-1}) = \mathbf{\Lambda}\,((\overline{q_\alpha})_\star (p_\alpha^{-1})_\star)$ i.e. $\mathbf{\Lambda}\,(\Phi)$ is well defined and $\mathbf{\Lambda}\,(\Phi) \subseteq \mathbf{\Lambda}\,(\pi_\alpha\Phi)$.

Now assume $\mathbf{\Lambda}\,(\Phi) \neq \{0\}$. Then $\mathbf{\Lambda}\,(\pi_\alpha\Phi) \neq \{0\}$. Since $C \in NES(\text{compact})$ then Theorem 1.10 guarantees that there exists a $x_\alpha \in C$ with $x_\alpha \in \pi_\alpha\Phi(x_\alpha)$. Since π_α and i are α–close then Φ has an α–fixed point of Φ; note $x_\alpha = \pi_\alpha(y_\alpha)$ where $y_\alpha \in \Phi(x_\alpha)$ so there exists a $U_\alpha \in \alpha$ with $\pi_\alpha(y_\alpha) \in U_\alpha$ and $y_\alpha \in U_\alpha$ i.e. $x_\alpha \in U_\alpha$ and $y_\alpha \in U_\alpha$ i.e. $x_\alpha \in U_\alpha$ and $\Phi(x_\alpha) \cap U_\alpha \neq \emptyset$ (note $y_\alpha \in U_\alpha$ and $y_\alpha \in \Phi(x_\alpha)$). Thus Φ has an α–fixed point (for each $\alpha \in Cov_X(K)$). Now Theorem 1.2 and Remark 1.3 (note Hausdorff topological vector spaces are uniform spaces) guarantee that Φ (so consequently $co\,(F)$) has a fixed point. □

The same argument as in Theorem 2.5 establishes the following result (here X is a subset of a Hausdorff topological space).

Theorem 2.6. *Let X be NES admissible with X a uniform space and $F \in HYAdC(X,X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

The same argument as in Theorem 2.5 (with Theorem 1.10 replaced by Theorem 1.11) yields the following results.

Theorem 2.7. *Let X be $SANES$ admissible (and X a subset of a Hausdorff topological vector space) and $F \in HYAd(X,X)$ with $co\, F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently $co\, F$) has a fixed point.*

Theorem 2.8. *Let X be $SANES$ admissible with X a uniform space and $F \in HYAdC(X,X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

Also the same argument as in Theorem 2.5 establishes the following results.

Theorem 2.9. *Let X be Lefschetz admissible (and X a subset of a Hausdorff topological vector space) and $F \in HYAd(X,X)$ with $co\, F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently $co\, F$) has a fixed point.*

Theorem 2.10. *Let X be Lefschetz admissible with X a uniform space and $F \in HYAdC(X,X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

A special case of Theorem 2.10 (with $\Phi = F$) is the following.

Theorem 2.11. *Let X be Lefschetz admissible with X a uniform space and let $F \in Ad(X,X)$ be a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

We will now define admissible dominating spaces. Again we could consider NES admissible dominated, $SANES$ admissible dominated and Lefschetz admissible dominated. The argument presented below is exactly the same for the three situations so we will just consider Lefschetz admissible dominated (which of course contains NES admissible dominated and $SANES$ admissible dominated).

Let X be a Hausdorff topological space and let $\alpha \in Cov\,(X)$. X is said to be Lefschetz admissible α–dominated (respectively, NES admissible α–dominated; respectively $SANES$ admissible α–dominated) if there exists a Lefschetz admissible space X_α (respectively, a NES admissible space X_α; respectively a $SANES$ admissible space X_α) with X_α a uniform space, and two continuous mappings $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and the identity $i_X : X \to X$ are α–close and

$r_\alpha\, s_\alpha \sim i_X$ (i.e. there exists a homotopy h joining $r_\alpha s_\alpha$ and i_X). X is said to be Lefschetz admissible dominated (respectively, NES admissible dominated; respectively, $SANES$ admissible dominated) if X is Lefschetz admissible α–dominated (respectively, NES admissible α–dominated; respectively $SANES$ admissible α–dominated) for every $\alpha \in Cov\,(X)$.

Theorem 2.12. *Let X be Lefschetz admissible dominated (and X a subset of a Hausdorff topological vector space) and $F \in HYAd(X,X)$ with $co\,F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Then $\mathbf{\Lambda}\,(\Phi)$ is well defined and if $\mathbf{\Lambda}\,(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.*

Proof. Let Φ be as in the statement above. Let $\alpha \in Cov\,(X)$ and let $p,\, q : \Gamma \to X$ be a selected pair of Φ. Now since X is Lefschetz admissible dominated then there exists a Lefschetz space X_α with X_α a uniform space and two continuous mappings $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and the identity $i_X : X \to X$ are α–close and $r_\alpha\, s_\alpha \sim i_X$ (note in particular [7] that $(r_\alpha)_\star (s_\alpha)_\star = (i_X)_\star$). Let $\Phi_\alpha = s_\alpha \Phi\, r_\alpha$ and note $\Phi \in Ad(X_\alpha, X_\alpha)$. Also from [7, Section 40] there exists a selected pair (p_α, q_α) of Φ_α with $(q_\alpha)_\star (p_\alpha^{-1})_\star = (s_\alpha)_\star q_\star p_\star^{-1} (r_\alpha)_\star$. Since X_α is Lefschetz admissible then (see Theorem 2.11) $(q_\alpha)_\star (p_\alpha^{-1})_\star$ is a Leray endomorphism. Next note since $(r_\alpha)_\star (s_\alpha)_\star = (i_X)_\star$ that

$$(r_\alpha)_\star (s_\alpha)_\star q_\star p_\star^{-1} = (i_X)_\star q_\star p_\star^{-1} = q_\star p_\star^{-1}$$

and from above

$$(s_\alpha)_\star q_\star p_\star^{-1} (r_\alpha)_\star = (q_\alpha)_\star (p_\alpha^{-1})_\star .$$

Now [6, pp 214 (see (1.3))] (here $E' = H(X_\alpha)$, $E'' = H(X)$, $u = (r_\alpha)_\star$, $v = (s_\alpha)_\star q_\star p_\star^{-1}$, $f' = (q_\alpha)_\star (p_\alpha^{-1})_\star$, $f'' = q_\star p_\star^{-1}$ and note $u\,f' = (r_\alpha)_\star (q_\alpha)_\star (p_\alpha^{-1})_\star = (r_\alpha)_\star (s_\alpha)_\star q_\star p_\star^{-1} (r_\alpha)_\star = q_\star p_\star^{-1} (r_\alpha)_\star = f''\,u$) guarantees that $q_\star p_\star^{-1}$ is a Leray endomorphism and $\mathbf{\Lambda}\,(q_\star p_\star^{-1}) = \mathbf{\Lambda}\,((q_\alpha)_\star (p_\alpha^{-1})_\star)$ i.e. $\mathbf{\Lambda}\,(\Phi)$ is well defined and $\mathbf{\Lambda}\,(\Phi) \subseteq \mathbf{\Lambda}\,(\Phi_\alpha)$.

Now assume $\mathbf{\Lambda}\,(\Phi) \neq \{0\}$. Then $\mathbf{\Lambda}\,(\Phi_\alpha) \neq \{0\}$. Now since X_α is Lefschetz admissible then Theorem 2.11 guarantees that there exists a $x_\alpha \in X_\alpha$ with $x_\alpha \in \Phi_\alpha(x_\alpha)$. Since $r_\alpha s_\alpha$ and i_X are α–close then Φ has an α–fixed point of Φ; note $x_\alpha \in s_\alpha \Phi r_\alpha(x_\alpha)$ so $x_\alpha = s_\alpha(y)$ for some $y \in \Phi r_\alpha(x_\alpha)$, so with $z = r_\alpha(x_\alpha)$ then $z = r_\alpha s_\alpha(y)$ for some $y \in \Phi(z)$ and now since $r_\alpha\, s_\alpha$ and i_X are α–close then there exists a $U \in \alpha$ with $r_\alpha s_\alpha(y) \in U$ and $i_X(y) \in U$ i.e. $z \in U$ and $y \in U$ i.e. $z \in U$ and $\Phi(z) \cap U \neq \emptyset$ (note $y \in U$ and $y \in \Phi(z)$). Thus Φ has an α–fixed point (for each $\alpha \in Cov_X(X)$). Now Theorem 1.2 and Remark 1.3 guarantee that Φ (so consequently $co\,(F)$) has a fixed point. □

Remark 2.13. (i). From the above analysis note that one could replace the condition $r_\alpha\, s_\alpha \sim i_X$ (in the definition of Lefschetz admissible α–dominated) with any condition that guarantees $(r_\alpha)_\star (s_\alpha)_\star = (i_X)_\star$.

(ii). Also note from the above analysis we only need X to be Lefschetz admissible α–dominated for every $\alpha \in Cov_X\,(\overline{\Phi(X)})$ (here Φ is as in the proof of Theorem 2.12) to deduce Theorem 2.12.

The same argument as in Theorem 2.12 establishes the following result.

Theorem 2.14. *Let X be Lefschetz admissible dominated with X a uniform space and $F \in HYAdC(X,X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

A special case of Theorem 2.14 (with $\Phi = F$) is the following.

Theorem 2.15. *Let X be Lefschetz admissible dominated with X a uniform space and let $F \in Ad(X,X)$ be a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

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