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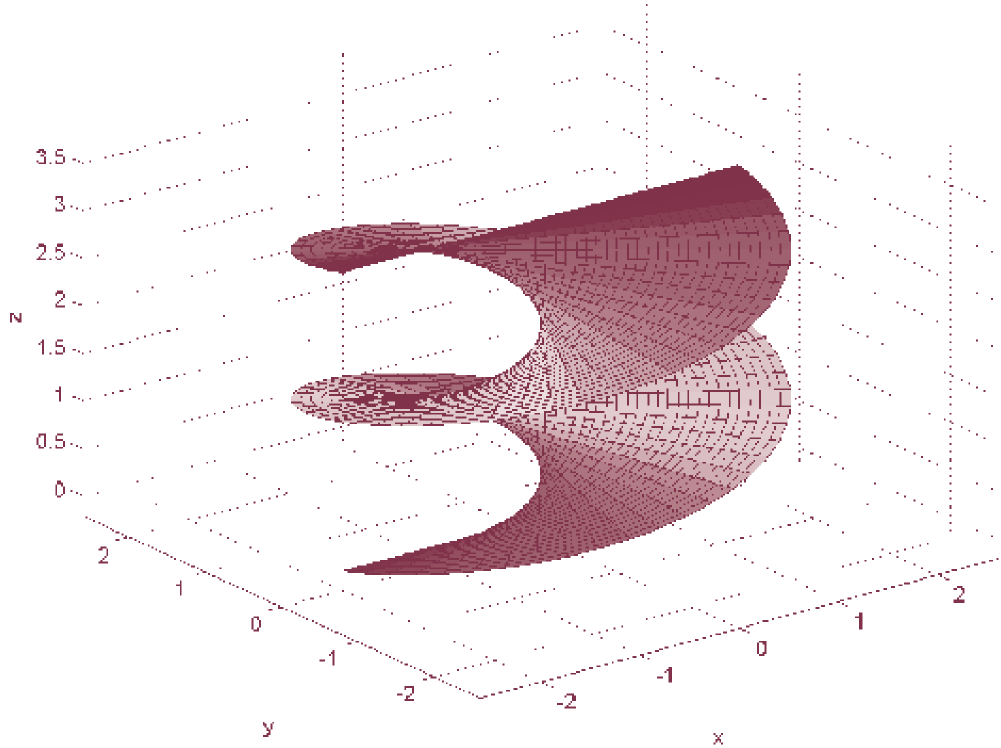

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CONTENTS

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New developments of fractional integral inequalities and their applications

Adrian Naço, Artion Kashuri and Rozana Liko

Abstract. In this paper, we propose the so-called higher order strongly m-polynomial exponentially type convex functions. Some of its algebraic properties are given and a new fractional integral identity is established. Applying the class of higher order strongly m-polynomial exponentially type convex functions, we deduce some fractional integral inequalities using the basic identity. Furthermore, we offer some applications to demonstrate the efficiency of our results. Our results not only generalize the known results but also refine them.

Mathematics Subject Classification (2010): 26A33, 26A51, 26D07, 26D10, 26D15, 26D20.

Keywords: Hermite-Hadamard inequality, Hölder's inequality, power mean inequality, higher order strongly m-polynomial exponentially type convex functions, Bessel functions, bounded functions.

1. Introduction

A set $\mathrm{T} \subset \mathbb{R}$ ($\mathbb{R}$ represents the set of real numbers) is said to be convex, if

$$\vartheta \flat_1 + (1-\vartheta)\flat_2 \in \mathrm{T}, \quad \forall \flat_1, \flat_2 \in \mathrm{T} \text{ and } \vartheta \in [0,1].$$

A function $\hbar : \mathrm{T} \to \mathbb{R}$ is called convex, if

$$\hbar(\vartheta \flat_1 + (1-\vartheta)\flat_2) \leq \vartheta\, \hbar(\flat_1) + (1-\vartheta)\hbar(\flat_2), \quad \forall \flat_1, \flat_2 \in \mathrm{T} \text{ and } \vartheta \in [0,1]. \tag{1.1}$$

Moreover, $\hbar$ is concave whenever $-\hbar$ is convex.

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For the convex function the Hermite-Hadamard type integral inequality (H-H), is given by [5]:

$$\hbar\left(\frac{\flat_1+\flat_2}{2}\right) \leq \frac{1}{\flat_2-\flat_1}\int_{\flat_1}^{\flat_2} \hbar(x)dx \leq \frac{\hbar(\flat_1)+\hbar(\flat_2)}{2}. \tag{1.2}$$

The H-H integral inequality (1.2) has been applied to different types of convex functions (see [3, 4, 9, 12, 13]).

Now, we recall some definitions of convex type functions.

Definition 1.1. [2] *A function $\hbar : \mathrm{T} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex, if*

$$\hbar(\vartheta\flat_1 + (1-\vartheta)\flat_2) \leq \vartheta\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}} + (1-\vartheta)\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}} \tag{1.3}$$

holds for all $\flat_1, \flat_2 \in \mathrm{T}$, $\vartheta \in [0,1]$ and $\varsigma \in \mathbb{R}$.

Toply *et al.* [10] introduced the class of m-polynomial convex functions as follows:

Definition 1.2. *Let $m \in \mathrm{N}$. A function $\hbar : \mathrm{T} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be m-polynomial convex, if*

$$\hbar(\vartheta\flat_1 + (1-\vartheta)\flat_2) \leq \frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\hbar(\flat_1) + \frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\hbar(\flat_2) \tag{1.4}$$

holds for all $\flat_1, \flat_2 \in \mathrm{T}$ and $\vartheta \in [0,1]$.

With the help of the above definitions, we introduce the following definition.

Definition 1.3. *Let $m \in \mathrm{N}$ and $\varsigma \in \mathbb{R}$. The function $\hbar : \mathrm{T} \to \mathbb{R}$ is called higher order strongly m-polynomial exponentially type convex, if there exists a constant $\zeta > 0$, such that*

$$\begin{aligned}\hbar(\vartheta\flat_1 + (1-\vartheta)\flat_2) \leq{}& \frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}} + \frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}} \\ &- \zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}\end{aligned} \tag{1.5}$$

holds for every $\flat_1, \flat_2 \in \mathrm{T}$, $\vartheta \in [0,1]$ and $p \geq 1$.

Remark 1.4. From Definition 1.3, we can observe that:

1. If $m = 1$ and $\zeta \to 0^+$, then Definition 1.3 reduces to Definition 1.1.
2. If $\varsigma = 0$ and $\zeta \to 0^+$, then Definition 1.3 reduces to Definition 1.2.

Definition 1.5. *Let $\ell > 0$, $\flat_1 < \flat_2$ and $\hbar \in \mathcal{L}[\flat_1, \flat_2]$. Then the Riemann–Liouville fractional integrals (R-L) of order ℓ are defined by*

$$\mathcal{J}^{\ell}_{\flat_1^+}\hbar(x) = \frac{1}{\Gamma(\ell)}\int_{\flat_1}^{x}(x-\vartheta)^{\ell-1}\hbar(\vartheta)d\vartheta, \quad \flat_1 < x$$

and

$$\mathcal{J}^{\ell}_{\flat_2^-}\hbar(x) = \frac{1}{\Gamma(\ell)}\int_{x}^{\flat_2}(\vartheta-x)^{\ell-1}\hbar(\vartheta)d\vartheta, \quad \flat_2 > x,$$

where $\Gamma(\cdot)$ is the gamma function.

The H-H type integral inequalities are involved in fractional calculus models and they has been applied for different types of convex functions (see [1, 6, 7]).

Motivated from above literatures our paper is organized as follows: In Section 2, we introduce the higher order strongly m-polynomial exponentially type convex function as a new class of convex functions with its algebraic properties. In Section 3, we derive new integral inequality of H-H by using the new introduced definition. In Section 4, we derive a generalized fractional identity and some related inequalities for the higher order strongly m-polynomial exponentially type convex functions. In Section 5, we give some applications of the Bessel functions and bounded functions to support the main results from previous section. Finally, conclusions and future research are drawn in Section 6.

2. Algebraic properties

Here we derive some algebraic properties of our new defined convex function.

Theorem 2.1. *Let $m \in \mathrm{N}$ and $\varsigma \in \mathbb{R}$. Assume that $\hbar$, $\hbar_1$, $\hbar_2 : \mathrm{T} \to \mathbb{R}$ are three higher order strongly m-polynomial exponentially type convex functions with respect to the constants, ζ, ζ_1 and ζ_2, respectively, then*

1. *$\hbar_1 + \hbar_2$ is higher order strongly m-polynomial exponentially type convex function, with respect to the constant $\zeta_1 + \zeta_2$.*
2. *For nonnegative real number c, $c\hbar$ is higher order strongly m-polynomial exponentially type convex function, with respect to the constant $c\zeta$.*

Proof. The proof is evident, so we omit here. □

Theorem 2.2. *Let $m \in \mathrm{N}, \varsigma \in \mathbb{R}$ and $\mathcal{U} = \{\varpi \in [\flat_1, \flat_2] : \hbar(\varpi) < +\infty\}$. Assume that $\hbar_j : [\flat_1, \flat_2] \to \mathbb{R}$ is a family of higher order strongly m-polynomial exponentially type convex functions with respect to the constant $\zeta > 0$ and $\hbar(\varpi) := \sup_j \hbar_j(\varpi)$. Then, $\hbar$ is an higher order strongly m-polynomial exponentially type convex function with respect to the constant ζ on $\mathcal{U}$.*

Proof. Let $\flat_1, \flat_2 \in \mathcal{U}$ and $\vartheta \in [0, 1]$, then we have

$$
\begin{aligned}
&\hbar(\vartheta\flat_1 + (1-\vartheta)\flat_2) = \sup_j \hbar_j(\vartheta\flat_1 + (1-\vartheta)\flat_2) \\
&\le \frac{1}{m}\sum_{\imath=1}^{m}[1-(1-\vartheta)^{\imath}]\frac{\sup_j \hbar_j(\flat_1)}{e^{\varsigma\flat_1}} + \frac{1}{m}\sum_{\imath=1}^{m}(1-\vartheta^{\imath})\frac{\sup_j \hbar_j(\flat_2)}{e^{\varsigma\flat_2}} \\
&\quad - \zeta\left[\vartheta^p(1-\vartheta) + \vartheta(1-\vartheta)^p\right]|\flat_2 - \flat_1|^p \\
&= \frac{1}{m}\sum_{\imath=1}^{m}[1-(1-\vartheta)^{\imath}]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}} + \frac{1}{m}\sum_{\imath=1}^{m}(1-\vartheta^{\imath})\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}} \\
&\quad - \zeta\left[\vartheta^p(1-\vartheta) + \vartheta(1-\vartheta)^p\right]|\flat_2 - \flat_1|^p < +\infty,
\end{aligned}
$$

which completes the proof. □

3. Main results

The aim of this section is to find some fractional integral inequalities of H-H type for higher order strongly m-polynomial exponentially type convex functions.

Theorem 3.1. *Let $\ell > 0$, $m \in \mathrm{N}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$. If $\hbar \in \mathcal{L}[\flat_1, \flat_2]$ and $\varsigma \in \mathbb{R}$, then we have*

$$m\left(\frac{2^m}{2^m(m-1)+1}\right)\left[\frac{1}{\ell}\hbar\left(\frac{\flat_1+\flat_2}{2}\right) - \frac{\zeta}{2^{p+\ell}}(\flat_2-\flat_1)^p\big(\beta(p+1,\ell)+\Im(p,\ell)\big)\right] \tag{3.1}$$
$$\leq \frac{1}{(\flat_2-\flat_1)^\ell}\left[A^\ell_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^\ell_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]$$
$$\leq \frac{1}{(\flat_2-\flat_1)^\ell}\Bigg\{\left[B^\ell_{1,m}(\varsigma;\flat_1,\flat_2)+B^\ell_{4,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}$$
$$+\left[B^\ell_{2,m}(\varsigma;\flat_1,\flat_2)+B^\ell_{3,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\Bigg\}$$
$$-\frac{\zeta}{(\flat_2-\flat_1)^{\ell+1}}\left[C^{\ell,\varsigma}_1(\flat_2,\flat_1,p)+C^{\ell,\varsigma}_2(\flat_2,\flat_1,p)\right],$$

where

$$A^\ell_{\hbar,1}(\varsigma;\flat_1,\flat_2) := \int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\frac{\hbar(x)}{e^{\varsigma x}}dx,\quad A^\ell_{\hbar,2}(\varsigma;\flat_1,\flat_2) := \int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\frac{\hbar(x)}{e^{\varsigma x}}dx$$

and

$$\Im(p,\ell) := \int_0^1 \vartheta^p(1+\vartheta)^{\ell-1}d\vartheta,$$

$$B^\ell_{1,m}(\varsigma;\flat_1,\flat_2) := \frac{1}{m}\sum_{\imath=1}^m\int_{\flat_1}^{\flat_2}\frac{(\flat_2-x)^{\ell-1}}{e^{\varsigma x}}\left[1-\left(\frac{x-\flat_1}{\flat_2-\flat_1}\right)^\imath\right]dx,$$

$$B^\ell_{2,m}(\varsigma;\flat_1,\flat_2) := \frac{1}{m}\sum_{\imath=1}^m\int_{\flat_1}^{\flat_2}\frac{(\flat_2-x)^{\ell-1}}{e^{\varsigma x}}\left[1-\left(\frac{\flat_2-x}{\flat_2-\flat_1}\right)^\imath\right]dx,$$

$$B^\ell_{3,m}(\varsigma;\flat_1,\flat_2) := \frac{1}{m}\sum_{\imath=1}^m\int_{\flat_1}^{\flat_2}\frac{(x-\flat_1)^{\ell-1}}{e^{\varsigma x}}\left[1-\left(\frac{\flat_2-x}{\flat_2-\flat_1}\right)^\imath\right]dx,$$

$$B^\ell_{4,m}(\varsigma;\flat_1,\flat_2) := \frac{1}{m}\sum_{\imath=1}^m\int_{\flat_1}^{\flat_2}\frac{(x-\flat_1)^{\ell-1}}{e^{\varsigma x}}\left[1-\left(\frac{x-\flat_1}{\flat_2-\flat_1}\right)^\imath\right]dx,$$

$$C^{\ell,\varsigma}_1(\flat_1,\flat_2,p) := \int_{\flat_1}^{\flat_2}\frac{(x-\flat_1)^{\ell-1}}{e^{\varsigma x}}\Big[(x-\flat_1)(\flat_2-x)^p+(x-\flat_1)^p(\flat_2-x)\Big]dx,$$

$$C^{\ell,\varsigma}_2(\flat_1,\flat_2,p) := \int_{\flat_1}^{\flat_2}\frac{(\flat_2-x)^{\ell-1}}{e^{\varsigma x}}\Big[(x-\flat_1)^p(\flat_2-x)+(x-\flat_1)(\flat_2-x)^p\Big]dx.$$

Here, $\beta(\cdot,\cdot)$ is the beta function.

Proof. Let $x, y \in [\flat_1, \flat_2]$. Applying definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ of $\hbar$ on $[\flat_1, \flat_2]$ and taking $\vartheta = 1/2$, we have

$$\hbar\left(\frac{x+y}{2}\right) \le \frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{1}{2}\right)^{\imath}\right]\left[\frac{\hbar(x)}{e^{\varsigma x}}+\frac{\hbar(y)}{e^{\varsigma y}}\right]-\frac{\zeta}{2^p}|y-x|^p. \tag{3.2}$$

By making use of inequality (3.2) with $x = \vartheta\flat_2 + (1-\vartheta)\flat_1$ and $y = \vartheta\flat_1 + (1-\vartheta)\flat_2$, we get

$$\hbar\left(\frac{\flat_1+\flat_2}{2}\right) \le \frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{1}{2}\right)^{\imath}\right]\left[\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}+\frac{\hbar(\vartheta\flat_1+(1-\vartheta)\flat_2)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}\right] \tag{3.3}$$
$$-\frac{\zeta}{2^p}(\flat_2-\flat_1)^p|1-2\vartheta|^p.$$

Multiplying both sides of (3.3) by $\vartheta^{\ell-1}$ and integrating the result with respect to ϑ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\ell}\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\\
&\le \frac{1}{m}\left(m-\frac{2^m-1}{2^m}\right)\left[\int_0^1 \vartheta^{\ell-1}\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}d\vartheta+\int_0^1 \vartheta^{\ell-1}\frac{\hbar(\vartheta\flat_1+(1-\vartheta)\flat_2)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}d\vartheta\right]\\
&\quad-\frac{\zeta}{2^p}(\flat_2-\flat_1)^p\int_0^1\vartheta^{\ell-1}|1-2\vartheta|^p d\vartheta\\
&=\frac{1}{m}\left(m-\frac{2^m-1}{2^m}\right)\frac{1}{(\flat_2-\flat_1)^\ell}\left[\int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\frac{\hbar(x)}{e^{\varsigma x}}dx+\int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\frac{\hbar(x)}{e^{\varsigma x}}dx\right]\\
&\quad-\frac{\zeta}{2^p}(\flat_2-\flat_1)^p\frac{1}{2^\ell}\left[\beta(p+1,\ell)+\Im(p,\ell)\right]\\
&=\frac{1}{m}\left(m-\frac{2^m-1}{2^m}\right)\frac{1}{(\flat_2-\flat_1)^\ell}\left[A^{\ell}_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^{\ell}_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]\\
&\quad-\frac{\zeta}{2^{p+\ell}}(\flat_2-\flat_1)^p\left[\beta(p+1,\ell)+\Im(p,\ell)\right],
\end{aligned}$$

which gives the left inequality of (3.5). In order to prove the right inequality of (3.5), we use the definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ of $\hbar$ to get

$$\begin{aligned}
\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}} &\le \frac{1}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}\\
&\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\right.\\
&\left.-\zeta\left[\vartheta^p(1-\vartheta)+\vartheta(1-\vartheta)^p\right]|\flat_2-\flat_1|^p\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\hbar(\vartheta\flat_1+(1-\vartheta)\flat_2)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}} &\leq \frac{1}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}} \\
&\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right. \\
&-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p},
\end{aligned}$$

where $\vartheta\in[0,1]$. Then, by adding the above inequalities, we have

$$\begin{aligned}
&\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}+\frac{\hbar(\vartheta\flat_1+(1-\vartheta)\flat_2)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}} \\
&\leq \frac{1}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}} \\
&\quad\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\right. \\
&\quad-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}+\frac{1}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}} \\
&\quad\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right. \\
&\quad-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}.
\end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $\vartheta^{\ell-1}$ and integrating the obtained inequality with respect to ϑ from 0 to 1 and making the change of the variables, we get

$$\begin{aligned}
&\int_0^1\vartheta^{\ell-1}\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}d\vartheta+\int_0^1\vartheta^{\ell-1}\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}d\vartheta \\
&\leq\int_0^1\vartheta^{\ell-1}\frac{1}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}} \\
&\quad\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left(1-\vartheta^{\imath}\right)\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\right. \\
&\quad-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}d\vartheta+\int_0^1\vartheta^{\ell-1}\frac{1}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}} \\
&\quad\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left(1-\vartheta^{\imath}\right)\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right. \\
&\quad-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}d\vartheta.
\end{aligned}$$

By simplifying it, we obtain

$$\begin{aligned}
&\frac{1}{(\flat_2-\flat_1)^{\ell}}\left[A^{\ell}_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^{\ell}_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right] \\
&\leq\frac{1}{(\flat_2-\flat_1)^{\ell}}\left\{\left[B^{\ell}_{1,m}(\varsigma;\flat_1,\flat_2)+B^{\ell}_{4,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right.
\end{aligned}$$

$$+ \left[B_{2,m}^{\ell}(\varsigma; \flat_1, \flat_2) + B_{3,m}^{\ell}(\varsigma; \flat_1, \flat_2)\right] \frac{\hbar(\flat_2)}{e^{\varsigma \flat_2}} \Bigg\}$$

$$- \frac{\zeta}{(\flat_2 - \flat_1)^{\ell+1}} \left[C_1^{\ell,\varsigma}(\flat_1, \flat_2, p) + C_2^{\ell,\varsigma}(\flat_1, \flat_2, p)\right].$$

The proof of Theorem 3.1 is completed. □

Corollary 3.2. *Theorem 3.1 with* $\varsigma = 0$ *becomes*

$$m\left(\frac{2^m}{2^m(m-1)+1}\right)\left[\frac{1}{\ell}\hbar\left(\frac{\flat_1+\flat_2}{2}\right) - \frac{\zeta}{2^{p+\ell}}(\flat_2-\flat_1)^p\big(\beta(p+1,\ell)+\Im(p,\ell)\big)\right] \quad (3.5)$$

$$\leq \frac{1}{(\flat_2 - \flat_1)^{\ell}}\left[A_{\hbar,1}^{\ell}(\flat_1, \flat_2) + A_{\hbar,2}^{\ell}(\flat_1, \flat_2)\right]$$

$$\leq \frac{1}{(\flat_2 - \flat_1)^{\ell}}\Bigg\{\left[B_{1,m}^{\ell}(\flat_1, \flat_2) + B_{4,m}^{\ell}(\flat_1, \flat_2)\right]\hbar(\flat_1)$$

$$+ \left[B_{2,m}^{\ell}(\flat_1, \flat_2) + B_{3,m}^{\ell}(\flat_1, \flat_2)\right]\hbar(\flat_2)\Bigg\}$$

$$- \frac{\zeta}{(\flat_2 - \flat_1)^{\ell+1}}\left[C_1^{\ell}(\flat_2, \flat_1, p) + C_2^{\ell}(\flat_2, \flat_1, p)\right],$$

where

$$A_{\hbar,1}^{\ell}(\flat_1, \flat_2) := \int_{\flat_1}^{\flat_2} (\flat_2 - x)^{\ell-1}\hbar(x)dx, \quad A_{\hbar,2}^{\ell}(\flat_1, \flat_2) := \int_{\flat_1}^{\flat_2} (x - \flat_1)^{\ell-1}\hbar(x)dx$$

and

$$B_{1,m}^{\ell}(\flat_1, \flat_2) := \frac{1}{m}\sum_{\imath=1}^{m}\int_{\flat_1}^{\flat_2}(\flat_2 - x)^{\ell-1}\left[1 - \left(\frac{x-\flat_1}{\flat_2-\flat_1}\right)^{\imath}\right]dx,$$

$$B_{2,m}^{\ell}(\flat_1, \flat_2) := \frac{1}{m}\sum_{\imath=1}^{m}\int_{\flat_1}^{\flat_2}(\flat_2 - x)^{\ell-1}\left[1 - \left(\frac{\flat_2-x}{\flat_2-\flat_1}\right)^{\imath}\right]dx,$$

$$B_{3,m}^{\ell}(\flat_1, \flat_2) := \frac{1}{m}\sum_{\imath=1}^{m}\int_{\flat_1}^{\flat_2}(x - \flat_1)^{\ell-1}\left[1 - \left(\frac{\flat_2-x}{\flat_2-\flat_1}\right)^{\imath}\right]dx,$$

$$B_{4,m}^{\ell}(\flat_1, \flat_2) := \frac{1}{m}\sum_{\imath=1}^{m}\int_{\flat_1}^{\flat_2}(x - \flat_1)^{\ell-1}\left[1 - \left(\frac{x-\flat_1}{\flat_2-\flat_1}\right)^{\imath}\right]dx,$$

$$C_1^{\ell}(\flat_1, \flat_2, p) := \int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\Big[((x-\flat_1)(\flat_2-x))^p + (x-\flat_1)^p(\flat_2-x)\Big]dx,$$

$$C_2^{\ell}(\flat_1, \flat_2, p) := \int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\Big[(x-\flat_1)^p(\flat_2-x)) + (x-\flat_1)(\flat_2-x)^p\Big]dx.$$

Corollary 3.3. *Theorem 3.1 with $\ell = 1$ leads to*

$$m\left(\frac{2^m}{2^m(m-1)+1}\right)\left[\hbar\left(\frac{\flat_1+\flat_2}{2}\right)-\frac{\zeta}{(p+1)2^p}(\flat_2-\flat_1)^p\right]$$
$$\leq \frac{1}{(\flat_2-\flat_1)}\left[A^1_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^1_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]$$
$$\leq \frac{1}{(\flat_2-\flat_1)}\left\{\left[B^1_{1,m}(\varsigma;\flat_1,\flat_2)+B^1_{4,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right.$$
$$\left.+\left[B^1_{2,m}(\varsigma;\flat_1,\flat_2)+B^1_{3,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\right\}$$
$$-\frac{\zeta}{(\flat_2-\flat_1)^2}\left[C_1^{1,\varsigma}(\flat_2,\flat_1,p)+C_2^{1,\varsigma}(\flat_2,\flat_1,p)\right].$$

Corollary 3.4. *Letting $\zeta \to 0^+$ in Theorem 3.1, we have*

$$\frac{m}{\ell}\left(\frac{2^m}{2^m(m-1)+1}\right)\hbar\left(\frac{\flat_1+\flat_2}{2}\right)$$
$$\leq \frac{1}{(\flat_2-\flat_1)^\ell}\left[A^\ell_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^\ell_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]$$
$$\leq \frac{1}{(\flat_2-\flat_1)^\ell}\left\{\left[B^\ell_{1,m}(\varsigma;\flat_1,\flat_2)+B^\ell_{4,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right.$$
$$\left.+\left[B^\ell_{2,m}(\varsigma;\flat_1,\flat_2)+B^\ell_{3,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\right\}.$$

Corollary 3.5. *Theorem 3.1 with $\varsigma = 0, \ell = 1$ and $\zeta \to 0^+$ becomes* [10, Theorem 4].

4. Further results

We need the following lemma in order to proceed with our next results.

Lemma 4.1. *Let $\hbar : \mathrm{T} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on T with $\flat_1, \flat_2 \in \mathrm{T}$ and $\flat_1 < \flat_2$. Also, let $\ell > 0$ and $m \in \mathbb{N}$. If $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$, then we have*

$$\mathrm{Q}^\ell_m(\hbar;\flat_1,\flat_2) := \left(\frac{\flat_2-\flat_1}{4m}\right)\Gamma(\ell+1)$$
$$\times\sum_{\jmath=0}^{m-1}\left\{\left(\frac{2m}{\flat_2-\flat_1}\right)^{\ell+1}\mathcal{J}^\ell_{\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)^-}\hbar\left(\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m}\right)\right.$$
$$\left.-\left(\frac{2m}{\flat_1-\flat_2}\right)^{\ell+1}\mathcal{J}^\ell_{\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)^-}\hbar\left(\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}\right)\right\}$$
$$-\sum_{\jmath=0}^{m-1}\hbar\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)$$

$$= \left(\frac{\flat_2 - \flat_1}{4m}\right) \cdot \sum_{\jmath=0}^{m-1} \left\{ \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2} \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} + \frac{(2-\vartheta)}{2} \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} \right) d\vartheta - \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2} \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} + \frac{(2-\vartheta)}{2} \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} \right) d\vartheta \right\}. \quad (4.1)$$

Proof. Setting

$$\mathcal{J}_1 := \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2} \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} + \frac{(2-\vartheta)}{2} \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} \right) d\vartheta, \quad (4.2)$$

and

$$\mathcal{J}_2 := \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2} \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} + \frac{(2-\vartheta)}{2} \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} \right) d\vartheta. \quad (4.3)$$

By applying integration by parts on equality (4.2), we have

$$\begin{aligned} \mathcal{J}_1 = &\left(\frac{2m}{\flat_1 - \flat_2}\right) \left[\vartheta^\ell \hbar \left(\frac{\vartheta}{2} \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} + \frac{(2-\vartheta)}{2} \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} \right) \Bigg|_0^1 \right. \\ &\left. - \ell \int_0^1 \vartheta^{\ell-1} \hbar \left(\frac{\vartheta}{2} \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} + \frac{(2-\vartheta)}{2} \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} \right) d\vartheta \right] \\ = &\left(\frac{2m}{\flat_1 - \flat_2}\right) \left[\hbar \left(\frac{(2(m-\jmath)-1)\flat_1 + (2\jmath+1)\flat_2}{2m} \right) \right. \\ &\left. - \left(\frac{2m}{\flat_1 - \flat_2}\right)^\ell \Gamma(\ell+1)\, \mathcal{J}^\ell_{\left(\frac{(2(m-\jmath)-1)\flat_1 + (2\jmath+1)\flat_2}{2m}\right)^-} \hbar \left(\frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} \right) \right]. \end{aligned} \quad (4.4)$$

Similarly, from equality (4.3), we obtain

$$\begin{aligned} \mathcal{J}_2 = &\left(\frac{2m}{\flat_2 - \flat_1}\right) \left[\hbar \left(\frac{(2(m-\jmath)-1)\flat_1 + (2\jmath+1)\flat_2}{2m} \right) \right. \\ &\left. - \left(\frac{2m}{\flat_2 - \flat_1}\right)^\ell \Gamma(\ell+1)\, \mathcal{J}^\ell_{\left(\frac{(2(m-\jmath)-1)\flat_1 + (2\jmath+1)\flat_2}{2m}\right)^-} \hbar \left(\frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} \right) \right], \end{aligned} \quad (4.5)$$

for all $\jmath = 0, 1, 2, \ldots, m-1$. Then, by subtracting equality (4.5) from (4.4), multiplying by the factor $\left(\frac{\flat_2 - \flat_1}{4m}\right)$ and summing over $\jmath$ from 0 to $m-1$, we can easily attain the desired identity (4.1). □

Remark 4.2. Lemma 4.1 with $m = 1$ leads to

$$\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2 - \flat_1)^\ell} \left\{ \mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^+} \hbar(\flat_2) + \mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^-} \hbar(\flat_1) \right\} - \hbar\left(\frac{\flat_1 + \flat_2}{2}\right)$$

$$= \frac{(\flat_2 - \flat_1)}{4} \left\{ \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2}\flat_1 + \frac{(2-\vartheta)}{2}\flat_2 \right) d\vartheta - \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2}\flat_2 + \frac{(2-\vartheta)}{2}\flat_1 \right) d\vartheta \right\}, \tag{4.6}$$

which is established in [8, Lemma 3].

Throughout the rest of this study, we consider

$$\mathbf{v}_{m,\jmath} := \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} \quad \text{and} \quad \mathbf{v}_{m,\jmath+1} := \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m}.$$

Theorem 4.3. *Let $\ell > 0$, $m \in \mathrm{N}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|$ is a higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ on $[\flat_1, \flat_2]$ and $\varsigma \in \mathbb{R}$, then we have*

$$\left|\mathrm{Q}_m^\ell(\hbar; \flat_1, \flat_2)\right| \le \left(\frac{\flat_2 - \flat_1}{4m}\right) [T_{m,\ell} + M_{m,\ell}] \sum_{\jmath=0}^{m-1} \left[\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} + \frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} \right] \tag{4.7}$$

$$- \frac{m\zeta}{2^p} \left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell+2, p+1) \right) \left| \frac{\flat_2 - \flat_1}{m} \right|^p,$$

where

$$T_{m,\ell} := \frac{1}{m} \sum_{\imath=1}^m \int_0^1 \vartheta^\ell \left[1 - \left(1 - \frac{\vartheta}{2}\right)^\imath \right] d\vartheta, \quad M_{m,\ell} := \frac{1}{m} \sum_{\imath=1}^m \int_0^1 \vartheta^\ell \left[1 - \left(\frac{\vartheta}{2}\right)^\imath \right] d\vartheta.$$

Here, $B_x(\cdot,\cdot)$ is the incomplete beta function for all $0 < x \le 1$.

Proof. By making use of Lemma 4.1 and properties of modulus, we can deduce

$$\left|\mathrm{Q}_m^\ell(\hbar; \flat_1, \flat_2)\right| \le \left(\frac{\flat_2 - \flat_1}{4m}\right)$$

$$\times \sum_{\jmath=0}^{m-1} \left\{ \int_0^1 \vartheta^\ell \left| \hbar' \left(\frac{\vartheta}{2} \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} + \frac{(2-\vartheta)}{2} \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} \right) \right| d\vartheta \right.$$

$$\left. + \int_0^1 \vartheta^\ell \left| \hbar' \left(\frac{\vartheta}{2} \frac{(m-\jmath-1)\flat_1 + (\jmath+1)\flat_2}{m} + \frac{(2-\vartheta)}{2} \frac{(m-\jmath)\flat_1 + \jmath\flat_2}{m} \right) \right| d\vartheta \right\}.$$

Using the definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ of $|\hbar'|$, we get

$$\left|\mathrm{Q}_m^\ell(\hbar; \flat_1, \flat_2)\right| \le \left(\frac{\flat_2 - \flat_1}{4m}\right) \sum_{\jmath=0}^{m-1} \left\{ \int_0^1 \vartheta^\ell \left[\frac{1}{m} \sum_{\imath=1}^m \left[1 - \left(1 - \frac{\vartheta}{2}\right)^\imath \right] \frac{|\hbar'(\mathbf{v}_{m,\jmath})|}{e^{\varsigma \mathbf{v}_{m,\jmath}}} \right. \right.$$

$$\left. + \frac{1}{m} \sum_{\imath=1}^m \left[1 - \left(\frac{\vartheta}{2}\right)^\imath \right] \frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}} - \zeta \left[(\frac{\vartheta}{2})^p (1 - \frac{\vartheta}{2}) + \frac{\vartheta}{2} (1 - \frac{\vartheta}{2})^p \right] \left| \frac{\flat_2 - \flat_1}{m} \right|^p \right] d\vartheta$$

$$+ \int_0^1 \vartheta^\ell \left[\frac{1}{m} \sum_{\imath=1}^m \left[1 - \left(1 - \frac{\vartheta}{2}\right)^\imath \right] \frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}} \right.$$

$$+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|}{e^{\varsigma\mathbf{v}_{m,\jmath}}}-\zeta\left[(\frac{\vartheta}{2})^{p}(1-\frac{\vartheta}{2})+\frac{\vartheta}{2}(1-\frac{\vartheta}{2})^{p}\right]\left|\frac{\flat_2-\flat_1}{m}\right|^{p}\Bigg]d\vartheta\Bigg\}$$

$$=\left(\frac{\flat_2-\flat_1}{4m}\right)\left[T_m^{\ell}+M_m^{\ell}\right]\sum_{\jmath=0}^{m-1}\left[\frac{|\hbar'(\mathbf{v}_{m,\jmath})|}{e^{\varsigma\mathbf{v}_{m,\jmath}}}+\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}\right]$$

$$-\frac{m\zeta}{2^{p}}\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}\left(\ell+2,p+1\right)\right)\left|\frac{\flat_2-\flat_1}{m}\right|^{p},$$

which completes the proof. □

Corollary 4.4. *Theorem 4.3 with $\varsigma=0$ leads to*

$$\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right|\leq\left(\frac{\flat_2-\flat_1}{4m}\right)\left[T_{m,\ell}+M_{m,\ell}\right]\sum_{\jmath=0}^{m-1}\left[|\hbar'(\boldsymbol{v}_{m,\jmath})|+|\hbar'(\boldsymbol{v}_{m,\jmath+1})|\right] \tag{4.8}$$

$$-\frac{m\zeta}{2^{p}}\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}\left(\ell+2,p+1\right)\right)\left|\frac{\flat_2-\flat_1}{m}\right|^{p}.$$

Corollary 4.5. *Theorem 4.3 with $m=1$ leads to*

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left\{\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^{+}}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^{-}}\hbar(\flat_1)\right\}-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right|$$

$$\leq\left(\frac{\flat_2-\flat_1}{4(\ell+1)}\right)\left[\frac{|\hbar'(\flat_1)|}{e^{\varsigma\flat_1}}+\frac{|\hbar'(\flat_2)|}{e^{\varsigma\flat_2}}\right] \tag{4.9}$$

$$-\frac{\zeta}{2^{p}}\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}\left(\ell+2,p+1\right)\right)|\flat_2-\flat_1|^{p}.$$

Moreover, if $\varsigma=0$ and $\zeta\to0^{+}$, we get

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left\{\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^{+}}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^{-}}\hbar(\flat_1)\right\}-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right|$$

$$\leq\left(\frac{\flat_2-\flat_1}{4(\ell+1)}\right)\left[|\hbar'(\flat_1)|+|\hbar'(\flat_2)|\right],$$

which is established in the first step of proof of [8, Theorem 5].

Theorem 4.6. *Let $\ell>0$, $m\in\mathrm{N}$ and $\hbar:[\flat_1,\flat_2]\to\mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$ such that $\hbar'\in\mathcal{L}[\flat_1,\flat_2]$. If $|\hbar'|^{q}$ is higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta>0$ on $[\flat_1,\flat_2]$ and $\varsigma\in\mathbb{R}$, then for $q>1$, and $\frac{1}{q}+\frac{1}{r}=1$, we have*

$$\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right|\leq\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}$$

$$\times\sum_{\jmath=0}^{m-1}\left\{\left(R_m\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|^{q}}{e^{\varsigma\boldsymbol{v}_{m,\jmath}}}+S_m\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^{q}}{e^{\varsigma\boldsymbol{v}_{m,\jmath+1}}}-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^{p}\right)^{\frac{1}{q}}\right.$$

$$+\left(R_m\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}}+S_m\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}}-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^p\right)^{\frac{1}{q}}\Bigg\}, \quad (4.10)$$

where

$$R_m:=\frac{1}{m}\sum_{\imath=1}^{m}\int_0^1\left[1-\left(1-\frac{\vartheta}{2}\right)^{\imath}\right]d\vartheta=1+\frac{2}{m}\sum_{\imath=1}^{m}\frac{1}{\imath+1}\left(\frac{1}{2^{\imath+1}}-1\right)$$

and

$$S_m:=\frac{1}{m}\sum_{\imath=1}^{m}\int_0^1\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]d\vartheta=1-\frac{1}{m}\sum_{\imath=1}^{m}\frac{1}{2^{\imath}(\imath+1)}.$$

Proof. By making use of Lemma 4.1, Hölder's inequality and properties of modulus, we can deduce

$$\begin{aligned}\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right| &\le \left(\frac{\flat_2-\flat_1}{4m}\right)\sum_{\jmath=0}^{m-1}\Bigg\{\int_0^1\vartheta^{\ell}\left|\hbar'\left(\frac{\vartheta}{2}\mathbf{v}_{m,\jmath}+\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath+1}\right)\right|d\vartheta\\ &\quad+\int_0^1\vartheta^{\ell}\left|\hbar'\left(\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath}+\frac{\vartheta}{2}\mathbf{v}_{m,\jmath+1}\right)\right|d\vartheta\Bigg\}\\ &\le\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\int_0^1\vartheta^{r\ell}d\vartheta\right)^{\frac{1}{r}}\\ &\quad\cdot\sum_{\jmath=0}^{m-1}\Bigg\{\left(\int_0^1\left|\hbar'\left(\frac{\vartheta}{2}\mathbf{v}_{m,\jmath}+\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath+1}\right)\right|^q d\vartheta\right)^{\frac{1}{q}}\\ &\quad+\left(\int_0^1\left|\hbar'\left(\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath}+\frac{\vartheta}{2}\mathbf{v}_{m,\jmath+1}\right)\right|^q d\vartheta\right)^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

Applying the definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta>0$ of $|\hbar'|^q$, we get

$$\begin{aligned}\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right| &\le \left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}\\ &\quad\times\sum_{\jmath=0}^{m-1}\Bigg\{\Bigg(\int_0^1\Bigg[\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(1-\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath}}}\\ &\quad+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}\\ &\quad-\zeta\left[\left(\frac{\vartheta}{2}\right)^p\left(1-\frac{\vartheta}{2}\right)+\frac{\vartheta}{2}\left(1-\frac{\vartheta}{2}\right)^p\right]\left|\frac{\flat_2-\flat_1}{m}\right|^p\Bigg]d\vartheta\Bigg)^{\frac{1}{q}}\\ &\quad+\Bigg(\int_0^1\Bigg[\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(1-\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}\end{aligned}$$

$$+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}$$

$$\left.\left.-\zeta\left[\left(\frac{\vartheta}{2}\right)^{p}\left(1-\frac{\vartheta}{2}\right)+\frac{\vartheta}{2}\left(1-\frac{\vartheta}{2}\right)^{p}\right]\left|\frac{\flat_2-\flat_1}{m}\right|^p\right]d\vartheta\right)^{\frac{1}{q}}\right\}$$

$$=\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}$$

$$\times\sum_{\jmath=0}^{m-1}\left\{\left(R_m\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}+S_m\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}}-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_1-1}{m}\right|^p\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(R_m\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}}+S_m\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^p\right)^{\frac{1}{q}}\right\},$$

which ends our proof. □

Corollary 4.7. *Theorem 4.6 with* $\varsigma=0$ *leads to*

$$\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right|\le\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}} \tag{4.11}$$

$$\times\sum_{\jmath=0}^{m-1}\left\{\left(R_m|\hbar'(\boldsymbol{v}_{m,\jmath})|^q+S_m|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^p\right)^{\frac{1}{q}}\right. \tag{4.12}$$

$$\left.+\left(R_m|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q+S_m|\hbar'(\boldsymbol{v}_{m,\jmath})|^q-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^p\right)^{\frac{1}{q}}\right\}. \tag{4.13}$$

Corollary 4.8. *Theorem 4.6 with* $m=1$ *leads to*

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left[\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^+}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^-}\hbar(\flat_1)\right]-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right| \tag{4.14}$$

$$\le\left(\frac{\flat_2-\flat_1}{4}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left\{\left(\frac{|\hbar'(\flat_1)|^q}{e^{\varsigma\flat_1}}+3\frac{|\hbar'(\flat_2)|^q}{e^{\varsigma\flat_2}}-\frac{\zeta}{(p+1)(p+2)}|\flat_2-\flat_1|^p\right)^{\frac{1}{q}}\right. \tag{4.15}$$

$$\left.+\left(3\frac{|\hbar'(\flat_1)|^q}{e^{\varsigma\flat_1}}+\frac{|\hbar'(\flat_2)|^q}{e^{\varsigma\flat_2}}-\frac{\zeta}{(p+1)(p+2)}|\flat_2-\flat_1|^p\right)^{\frac{1}{q}}\right\}. \tag{4.16}$$

Moreover, if $\varsigma=0$ *and* $\zeta\to 0^+$, *we get*

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left\{\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^+}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^-}\hbar(\flat_1)\right\}-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right|$$

$$\le\left(\frac{\flat_2-\flat_1}{4}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left\{(|\hbar'(\flat_1)|^q+3|\hbar'(\flat_2)|^q)^{\frac{1}{q}}+(3|\hbar'(\flat_1)|^q+|\hbar'(\flat_2)|^q)^{\frac{1}{q}}\right\},$$

which is established in [8, Theorem 6].

Theorem 4.9. *Let $\ell > 0$, $m \in \mathrm{N}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|^q$ is higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ on $[\flat_1, \flat_2]$ and $\varsigma \in \mathbb{R}$, then for $q \geq 1$, we have*

$$\begin{aligned}
\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right| &\leq \left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}}\sum_{\jmath=0}^{m-1}\left\{\left[T_{m,\ell}\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}}+M_{m,\ell}\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}}\right.\right.\\
&\left.-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\right]^{\frac{1}{q}}\\
&+\left[T_{m,\ell}\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}}+M_{m,\ell}\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}}\right.\\
&\left.\left.-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\right]^{\frac{1}{q}}\right\}, \quad (4.17)
\end{aligned}$$

where $T_{m,\ell}$ and $M_{m,\ell}$ are as given in Theorem 4.3.

Proof. By making use of Lemma 4.1, the power mean inequality and properties of modulus, we have

$$\begin{aligned}
\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right| &\leq \left(\frac{\flat_2-\flat_1}{4m}\right)\sum_{\jmath=0}^{m-1}\left\{\int_0^1 \vartheta^{\ell}\left|\hbar'\left(\frac{\vartheta}{2}\mathbf{v}_{m,\jmath}+\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath+1}\right)\right|d\vartheta\right.\\
&\left.+\int_0^1 \vartheta^{\ell}\left|\hbar'\left(\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath}+\frac{\vartheta}{2}\mathbf{v}_{m,\jmath+1}\right)\right|d\vartheta\right\}\\
\leq \left(\frac{\flat_2-\flat_1}{4m}\right)\left(\int_0^1\vartheta^{\ell}d\vartheta\right)^{1-\frac{1}{q}} &\sum_{\jmath=0}^{m-1}\left\{\left(\int_0^1 \vartheta^{\ell}\left|\hbar'\left(\frac{\vartheta}{2}\mathbf{v}_{m,\jmath}+\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath+1}\right)\right|^q d\vartheta\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 \vartheta^{\ell}\left|\hbar'\left(\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath}+\frac{\vartheta}{2}\mathbf{v}_{m,\jmath+1}\right)\right|^q d\vartheta\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By the definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ of $|\hbar'|^q$, we have

$$\begin{aligned}
&\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right| \leq \left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}}\\
&\times\sum_{\jmath=0}^{m-1}\left\{\left(\int_0^1\vartheta^{\ell}\left[\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(1-\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath}}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}\right.\right.\right.\\
&\left.\left.-\zeta\left[\left(\frac{\vartheta}{2}\right)^p\left(1-\frac{\vartheta}{2}\right)+\frac{\vartheta}{2}\left(1-\frac{\vartheta}{2}\right)^p\right]\left|\frac{\flat_2-\flat_1}{m}\right|^p\right]d\vartheta\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1 \vartheta^{\ell}\left[\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(1-\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}\right.\right.$$

$$\left.\left.-\zeta\left[\left(\frac{\vartheta}{2}\right)^p\left(1-\frac{\vartheta}{2}\right)+\frac{\vartheta}{2}\left(1-\frac{\vartheta}{2}\right)^p\right]\left|\frac{\flat_2-\flat_1}{m}\right|^p\right]d\vartheta\right)^{\frac{1}{q}}\right\}$$

$$=\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}}\sum_{\jmath=0}^{m-1}\left\{\left[T_{m,\ell}\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}+M_{m,\ell}\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}}\right.\right.$$

$$\left.-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\right]^{\frac{1}{q}}$$

$$+\left[T_{m,\ell}\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}}+M_{m,\ell}\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}\right.$$

$$\left.\left.-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\right]^{\frac{1}{q}}\right\},$$

which completes the proof. □

Corollary 4.10. *Theorem 4.9 with $\varsigma=0$ leads to*

$$\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right|\le\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}} \tag{4.18}$$

$$\times\sum_{\jmath=0}^{m-1}\left\{\left[T_{m,\ell}|\hbar'(\boldsymbol{v}_{m,\jmath})|^q+M_{m,\ell}|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q\right.\right.$$

$$\left.-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\right]^{\frac{1}{q}}$$

$$+\left[T_{m,\ell}|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q+M_{m,\ell}|\hbar'(\boldsymbol{v}_{m,\jmath})|^q\right.$$

$$\left.\left.-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\right]^{\frac{1}{q}}\right\}.$$

Corollary 4.11. *Theorem 4.9 with $m=1$ leads to*

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left\{\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^+}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^-}\hbar(\flat_1)\right\}-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right|$$

$$\le\left(\frac{\flat_2-\flat_1}{4}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{2(\ell+2)}\frac{|\hbar'(\flat_1)|^q}{e^{\varsigma\flat_1}}+\frac{\ell+3}{2(\ell+1)(\ell+2)}\frac{|\hbar'(\flat_2)|^q}{e^{\varsigma\flat_2}}\right.\right.$$

$$- \frac{\zeta}{2^{p+1}} |\flat_2 - \flat_1|^p \left(\frac{\ell + p + 3}{(\ell + p + 1)(\ell + p + 2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell + 2, p + 1) \right) \Bigg)^{\frac{1}{q}}$$

$$+ \Bigg(\frac{1}{2(\ell + 2)} \frac{|\hbar'(\flat_2)|^q}{e^{\varsigma \flat_2}} + \frac{\ell + 3}{2(\ell + 1)(\ell + 2)} \frac{|\hbar'(\flat_1)|^q}{e^{\varsigma \flat_1}}$$

$$- \frac{\zeta}{2^{p+1}} |\flat_2 - \flat_1|^p \left(\frac{\ell + p + 3}{(\ell + p + 1)(\ell + p + 2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell + 2, p + 1) \right) \Bigg)^{\frac{1}{q}} \Bigg\}. \quad (4.19)$$

Moreover, if $\varsigma = 0$ *and* $\zeta \to 0^+$*, we get*

$$\left| \frac{2^{\ell-1}\Gamma(\ell + 1)}{(\flat_2 - \flat_1)^\ell} \left\{ \mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^+} \hbar(\flat_2) + \mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^-} \hbar(\flat_1) \right\} - \hbar\left(\frac{\flat_1 + \flat_2}{2}\right) \right|$$

$$\leq \left(\frac{\flat_2 - \flat_1}{4(\ell + 1)} \right) \left\{ \left(\frac{\ell + 1}{2(\ell + 2)} |\hbar'(\flat_1)|^q + \frac{\ell + 3}{2(\ell + 2)} |\hbar'(\flat_2)|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{\ell + 1}{2(\ell + 2)} |\hbar'(\flat_2)|^q + \frac{\ell + 3}{2(\ell + 2)} |\hbar'(\flat_1)|^q \right)^{\frac{1}{q}} \right\}, \quad (4.20)$$

which is established in [8, Theorem 5].

5. Applications

5.1. Bessel functions

Consider the function $\mathrm{B}_\sigma : (0, +\infty) \to [1, +\infty)$ with $\sigma > -1$, given by

$$\mathrm{B}_\sigma(x) := 2^\sigma \Gamma(\sigma + 1) x^{-\sigma} \mathcal{P}_\sigma(x),$$

where $\mathcal{P}_\sigma$ is the modified Bessel function of the first kind defined by (see [11, on page 77]):

$$\mathcal{P}_\sigma(x) = \sum_{m=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^{\sigma+2m}}{m!\Gamma(\sigma + 1 + m)}, \quad x \in \mathbb{R}.$$

Following [11], we have

$$\mathrm{B}'_\sigma(x) = \frac{x}{2(\sigma + 1)} \mathrm{B}_{\sigma+1}(x), \quad (5.1)$$

$$\mathrm{B}''_\sigma(x) = \frac{x^2 \mathrm{B}_{\sigma+2}(x)}{4(\sigma + 1)(\sigma + 2)} + \frac{\mathrm{B}_{\sigma+1}(x)}{2(\sigma + 1)}. \quad (5.2)$$

Assume that all assumptions of the used corollaries in the following examples are satisfied.

Example 5.1. Let $0 < \flat_1 < \flat_2$ and $\sigma > -1$. Then, by using Corollary 4.5 with $\ell = 1$ for $\hbar(x) = \mathrm{B}'_\sigma(x)$ and the identities (5.1) and (5.2), we have

$$\left|\frac{B_\sigma(\flat_2)-B_\sigma(\flat_1)}{\flat_2-\flat_1}-\frac{(\flat_1+\flat_2)}{4(\sigma+1)}B_{\sigma+1}\left(\frac{\flat_1+\flat_2}{2}\right)\right|\leq\left(\frac{\flat_2-\flat_1}{16(\sigma+1)}\right)$$
$$\times\left[\frac{1}{e^{\varsigma\flat_1}}\left(\frac{\flat_1^2B_{\sigma+2}(\flat_1)}{2(\sigma+2)}+B_{\sigma+1}(\flat_1)\right)\right.$$
$$\left.+\frac{1}{e^{\varsigma\flat_2}}\left(\frac{\flat_2^2B_{\sigma+2}(\flat_2)}{2(\sigma+2)}+B_{\sigma+1}(\flat_2)\right)\right]$$
$$-\frac{\zeta}{2^p}\left(\frac{p+4}{(p+2)(p+3)}-2^{p+3}B_{\frac{1}{2}}(3,p+1)\right)|\flat_2-\flat_1|^p.$$

Example 5.2. Let $0<\flat_1<\flat_2$ and $\sigma>-1$. Then, by applying Corollary 4.8 with $\ell=1$, $\hbar(x)=B'_\sigma(x)$ and the identities (5.1) and (5.2), we get

$$\left|\frac{B_\sigma(\flat_2)-B_\sigma(\flat_1)}{\flat_2-\flat_1}-\frac{(\flat_1+\flat_2)}{4(\sigma+1)}B_{\sigma+1}\left(\frac{\flat_1+\flat_2}{2}\right)\right|\leq\left(\frac{\flat_2-\flat_1}{4\sqrt[q]{4}}\right)\left(\frac{1}{r+1}\right)^{\frac{1}{r}}$$
$$\times\left\{\left[\frac{1}{e^{\varsigma\flat_1}}\left(\frac{\flat_1^2B_{\sigma+2}(\flat_1)}{4(\sigma+1)(\sigma+2)}+\frac{B_{\sigma+1}(\flat_1)}{2(\sigma+1)}\right)^q+\frac{3}{e^{\varsigma\flat_2}}\left(\frac{\flat_2^2B_{\sigma+2}(\flat_2)}{4(\sigma+1)(\sigma+2)}+\frac{B_{\sigma+1}(\flat_2)}{2(\sigma+1)}\right)^q\right.\right.$$
$$\left.-\frac{\zeta}{(p+1)(p+2)}|\flat_2-\flat_1|^p\right]^{\frac{1}{q}}$$
$$+\left[\frac{3}{e^{\varsigma\flat_1}}\left(\frac{\flat_1^2B_{\sigma+2}(\flat_1)}{4(\sigma+1)(\sigma+2)}+\frac{B_{\sigma+1}(\flat_1)}{2(\sigma+1)}\right)^q+\frac{1}{e^{\varsigma\flat_2}}\left(\frac{\flat_2^2B_{\sigma+2}(\flat_2)}{4(\sigma+1)(\sigma+2)}+\frac{B_{\sigma+1}(\flat_2)}{2(\sigma+1)}\right)^q\right.$$
$$\left.\left.-\frac{\zeta}{(p+1)(p+2)}|\flat_2-\flat_1|^p\right]^{\frac{1}{q}}\right\}.$$

Example 5.3. Let $0<\flat_1<\flat_2$ and $\sigma>-1$. Then, by using Corollary 4.11 with $\ell=1$, $\hbar(x)=B'_\sigma(x)$ and the identities (5.1) and (5.2), we obtain

$$\left|\frac{B_\sigma(\flat_2)-B_\sigma(\flat_1)}{\flat_2-\flat_1}-\frac{(\flat_1+\flat_2)}{4(\sigma+1)}B_{\sigma+1}\left(\frac{\flat_1+\flat_2}{2}\right)\right|\leq\left(\frac{\flat_2-\flat_1}{4}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\frac{1}{6e^{\varsigma\flat_1}}\left(\frac{\flat_1^2B_{\sigma+2}(\flat_1)}{4(\sigma+1)(\sigma+2)}+\frac{B_{\sigma+1}(\flat_1)}{2(\sigma+1)}\right)^q+\frac{1}{3e^{\varsigma\flat_2}}\left(\frac{\flat_2^2B_{\sigma+2}(\flat_2)}{4(\sigma+1)(\sigma+2)}+\frac{B_{\sigma+1}(\flat_2)}{2(\sigma+1)}\right)^q\right.\right.$$
$$\left.-\frac{\zeta}{2^{p+1}}|\flat_2-\flat_1|^p\left(\frac{p+4}{(p+2)(p+3)}-2^{p+3}B_{\frac{1}{2}}(3,p+1)\right)\right]^{\frac{1}{q}}$$
$$+\left[\frac{1}{3e^{\varsigma\flat_1}}\left(\frac{\flat_1^2B_{\sigma+2}(\flat_1)}{4(\sigma+1)(\sigma+2)}+\frac{B_{\sigma+1}(\flat_1)}{2(\sigma+1)}\right)^q+\frac{1}{6e^{\varsigma\flat_2}}\left(\frac{\flat_2^2B_{\sigma+2}(\flat_2)}{4(\sigma+1)(\sigma+2)}+\frac{B_{\sigma+1}(\flat_2)}{2(\sigma+1)}\right)^q\right.$$
$$\left.\left.-\frac{\zeta}{2^{p+1}}|\flat_2-\flat_1|^p\left(\frac{p+4}{(p+2)(p+3)}-2^{p+3}B_{\frac{1}{2}}(3,p+1)\right)\right]^{\frac{1}{q}}\right\}.$$

5.2. Bounded functions

Proposition 5.4. *Let $\ell > 0$, $m \in \mathrm{N}$, $\varsigma \in \mathbb{R}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|$ is a higher order strongly m-polynomial exponentially type convex function, with respect to the constant $\zeta > 0$ and $|\hbar'| \leq \mathcal{K}$ on $[\flat_1, \flat_2]$, then we have*

$$\left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| \leq \mathcal{K}\left(\frac{\flat_2 - \flat_1}{4m}\right)[T_{m,\ell} + M_{m,\ell}] \sum_{\jmath=0}^{m-1}\left[\frac{1}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} + \frac{1}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}}\right] - \frac{m\zeta}{2^p}\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell+2, p+1)\right)\left|\frac{\flat_2 - \flat_1}{m}\right|^p, \tag{5.3}$$

where $T_{m,\ell}$ and $M_{m,\ell}$ are as given in Theorem 4.3.

Proposition 5.5. *Let $\ell > 0$, $m \in \mathrm{N}$, $\varsigma \in \mathbb{R}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|^q$ is higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ and $|\hbar'| \leq \mathcal{K}$ on $[\flat_1, \flat_2]$, then for $q > 1$ and $\frac{1}{q} + \frac{1}{r} = 1$, we have*

$$\left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| \leq \mathcal{K}\left(\frac{\flat_2 - \flat_1}{4m}\right)\left(\frac{1}{\ell r + 1}\right)^{\frac{1}{r}} \times \sum_{\jmath=0}^{m-1}\left\{\left(\frac{R_m}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} + \frac{S_m}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} - \frac{\zeta}{\mathcal{K}^q(p+1)(p+2)}\left|\frac{\flat_2 - \flat_1}{m}\right|^p\right)^{\frac{1}{q}} + \left(\frac{R_m}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} + \frac{S_m}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} - \frac{\zeta}{\mathcal{K}^q(p+1)(p+2)}\left|\frac{\flat_2 - \flat_1}{m}\right|^p\right)^{\frac{1}{q}}\right\}, \tag{5.4}$$

where R_m and S_m are as given in Theorem 4.6.

Proposition 5.6. *Let $\ell > 0$, $m \in \mathrm{N}, \varsigma \in \mathbb{R}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|^q$ is higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ and $|\hbar'| \leq \mathcal{K}$ on $[\flat_1, \flat_2]$, then for $q \geq 1$, we have*

$$\left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| \leq \mathcal{K}\left(\frac{\flat_2 - \flat_1}{4m}\right)\left(\frac{1}{\ell + 1}\right)^{1-\frac{1}{q}} \times \sum_{\jmath=0}^{m-1}\left\{\left[\frac{T_{m,\ell}}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} + \frac{M_{m,\ell}}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} - \frac{\zeta}{\mathcal{K}^q 2^{p+1}}\left|\frac{\flat_2 - \flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell+2, p+1)\right)\right]^{\frac{1}{q}} + \left[\frac{T_{m,\ell}}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} + \frac{M_{m,\ell}}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} - \frac{\zeta}{\mathcal{K}^q 2^{p+1}}\left|\frac{\flat_2 - \flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell+2, p+1)\right)\right]^{\frac{1}{q}}\right\}, \tag{5.5}$$

where $T_{m,\ell}$ and $M_{m,\ell}$ are as given in Theorem 4.3.

6. Conclusion

In this article, we proposed the higher order strongly m-polynomial exponentially type convex functions and some of its algebraic properties are given. Furthermore, we deduced some fractional integral inequalities using the basic identity for the new class of function. Moreover, we demonstrated the efficiency of our results via some applications. Our results not only generalized the previous known results but also refined them. For future research in this direction, we will offer several new inequalities pertaining to Hölder-İşcan, Chebyshev, Markov, Young and Minkowski type inequalities for this generic class of convex functions in fractional and quantum calculus.

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Existence, uniqueness and continuous dependence results of coupled system of Hilfer fractional stochastic pantograph equations with non-local integral conditions

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Abstract. This study explores the existence, uniqueness, and continuous dependence of solutions for coupled system of Hilfer fractional stochastic pantograph equations with nonlocal integral conditions. The existence of solutions is demonstrated using topological degree theory for condensing maps. The uniqueness is established via Banach's contraction principle. To address continuous dependence, the generalized Gronwall inequality is applied. Additionally, a numerical example is provided to illustrate and confirm the theoretical findings.

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1. Introduction

Fractional derivatives provide a flexible tool for modeling intricate processes in various fields by extending classical differentiation to non-integer orders. Several definitions of fractional derivatives exist, including the Riemann-Liouville (R-L) and Caputo derivatives. The R-L derivative offers a foundational approach to fractional differentiation [14], while the Caputo derivative is often used in practical applications due to its compatibility with standard initial conditions [14]. To unify and extend these approaches,

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Hilfer introduced a generalized fractional differential operator that combines the Caputo and R-L derivatives. This operator, called the fractional Hilfer derivative (HFD), has shown great promise in modeling systems with complex boundary conditions and temporal delays [3, 13, 20, 22, 30].

Fractional differential equations represent a substantial advancement in mathematical modeling, particularly in fields such as signal processing, biology, and engineering. By incorporating non-integer order derivatives, these equations capture complex system dynamics. A significant category within this domain is fractional pantograph delay differential equations, which integrate delays to model systems with memory effects. When combined with stochastic calculus, these equations evolve into stochastic fractional pantograph differential equations, which are valuable for describing systems influenced by both memory effects and random fluctuations [5, 7, 8, 21, 23, 24, 25, 28, 29, 31, 32, 34, 35].

The continuous dependence of stochastic fractional differential equations is crucial for ensuring that small changes in initial or nonlocal conditions lead to proportionally small variations in the solutions. Research has shown that mild solutions of mean-field stochastic functional differential equations exhibit sensitivity to initial data and coefficients within an appropriate topological framework [4, 27, 36, 37, 38]. Similarly, generalized Cauchy-type problems involving HFD demonstrate continuous dependence on the fractional order, supported by a generalization of Gronwall's inequality [1, 9, 11, 27, 33]. Solutions to random fractional-order differential equations with nonlocal conditions also maintain continuous dependence on initial conditions [15].

Coupled system with nonlocal conditions are particularly useful for modeling physical, chemical, or other processes that occur at multiple points within a domain rather than being restricted to boundary conditions. El-Sayed [16] explored the continuous dependence of solutions for stochastic differential equations with nonlocal conditions, while more recently, Arioui [6] studied the existence of coupled systems of fractional stochastic differential equations involving HFD. For more study about coupled systems, we refer to [2, 10, 17, 18, 19, 26, 39, 40]

To the best of our knowledge, no existing study has addressed the existence and continuous dependence of solutions for coupled systems of Hilfer fractional stochastic pantograph equations with nonlocal conditions. This paper aims to fill this gap by introducing a novel class of coupled system of Hilfer fractional stochastic pantograph equations with nonlocal integral conditions

$$\begin{cases} {}^{H}\mathfrak{D}^{p_1,q_1}_{0^+,\iota}\varrho(\iota) = \varpi_1\left(\iota, \varrho(\iota), \varrho\left(\kappa\iota\right), \xi(\iota)\right), \quad \iota \in J := (0,b], \\ {}^{H}\mathfrak{D}^{p_2,q_2}_{0^+,\iota}\xi(\iota) = \varpi_2\left(\iota, \varrho(\iota), \xi\left(\iota\right), \xi(\kappa\iota)\right)\dfrac{d\mathcal{W}(\iota)}{d\iota}, \\ \mathfrak{I}^{1-\gamma_1}_{0^+,\iota}\varrho(0) = \displaystyle\int_0^{\iota} g_1\left(s, \varrho(s), \xi(s)\right) d\mathcal{W}(s), \quad \gamma_1 = p_1 + q_1 - p_1 q_1, \\ \mathfrak{I}^{1-\gamma_2}_{0^+,\iota}\xi(0) = \displaystyle\int_0^{\iota} g_2\left(s, \varrho(s), \xi(s)\right) ds, \quad \gamma_2 = p_2 + q_2 - p_2 q_2, \end{cases} \tag{1.1}$$

where $\mathfrak{I}^{1-\gamma_j}_{0+,\iota}$ and ${}^{H}\mathfrak{D}^{p_j,q_j}_{0+,\iota}$ are the fractional integral of order $1-\gamma_j$ and the HFD of order p_j and type q_j, respectively, $j=1,2$. Here, $\frac{1}{2}<p_j<1, 0<q_j\leq 1$. Let $(W(\iota))_{\iota\geq 0}$ be 1-dimensional standard Brownian motion defined in the complete probability space $(\Omega,\mathcal{F}_\iota,\mathbb{P})$ with a normal filteration $(\mathcal{F}_\iota)_{\iota\geq 0}$. $\varpi_j, g_j : J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are measurable functions and $0<\kappa<1$.

2. Preliminaries

Let $\mathbb{L}^2(\Omega,\mathcal{F}_\iota,\mathbb{R})=\mathbb{L}^2(\Omega,\mathbb{R})$ is the Hilbert space of real-valued random variables that are square-integrable with respect to the probability measure on $(\Omega,\mathcal{F}_\iota)$. Let $C(J,\mathbb{L}^2(\Omega,\mathbb{R}))$ is the space of continuous time stochastic processes that are square-integrable with the norm $\|\varrho\|^2=\sup\left\{\mathbb{E}\|\varrho(\iota)\|^2 : \iota\in J\right\}$, where $\mathbb{E}$ is the mathematical expectation. On the other hand, define the Banach space

$$\begin{aligned}\mathcal{E}_j &= C_{1-\gamma_j}(J,L^2(\Omega,\mathbb{R}))\\ &=\left\{\varrho : J\to L^2(\Omega,\mathbb{R}) : \iota^{1-\gamma_j}\varrho(\iota)\in C(J,L^2(\Omega,\mathbb{R}))\right\}, 0<\gamma_j\leq 1, j=1,2,\end{aligned}$$

using the norm

$$\|\varrho\|^2_{\mathcal{E}_j}=\sup_{\iota\in J}\mathbb{E}\left\|\iota^{1-\gamma_j}\varrho(\iota)\right\|^2.$$

Furthermore, let $\mathcal{E}:=\mathcal{E}_1\times\mathcal{E}_2$ with the norm $\|(\varrho,\xi)\|_{\mathcal{E}}=\max\{\|\varrho\|_{\mathcal{E}_1},\|\xi\|_{\mathcal{E}_2}\}$. It is clear that $\mathcal{E}$ forms a Banach space.

Definition 2.1. [14] *For $p>0$, the fractional R-L integral with order p for a continuous function $\varrho : [a,\infty)\to\mathbb{R}$ can be written as*

$$\mathfrak{I}^{p}_{a+,\iota}\varrho(\iota)=\frac{1}{\Gamma(p)}\int_a^\iota(\iota-s)^{p-1}\varrho(s)ds.$$

Definition 2.2. [14] *For $n-1<p\leq n$, the fractional R-L derivative with order p for a continuous function ϱ is represented as*

$$\mathfrak{D}^{p}_{a+,\iota}\varrho(\iota)=D^n\mathfrak{I}^{n-p}_{a+,\iota}\varrho(\iota)=\frac{1}{\Gamma(n-p)}\left(\frac{d}{\mathrm{d}\iota}\right)^n\int_a^\iota(\iota-s)^{n-p-1}\varrho(s)\mathrm{d}s.$$

Definition 2.3. [20] *For $n-1<p\leq n$, the HFD with order p and type $0\leq q\leq 1$ of ϱ is represented as*

$${}^{H}\mathfrak{D}^{p,q}_{a+,\iota}\varrho(\iota)=\mathfrak{I}^{q(n-p)}_{a+,\iota}D^n\mathfrak{I}^{(1-q)(n-p)}_{a+,\iota}\varrho(\iota)=\mathfrak{I}^{q(n-p)}_{a+,\iota}\mathfrak{D}^{\theta}_{a+,\iota}\varrho(\iota),$$

where $D=\frac{d}{d\iota}$ and $\theta=p+q(n-p)$.

Lemma 2.4. [20] *For $n-1<p\leq n$, $f\in L^1(a,b)$, $0\leq\beta\leq 1$, and $\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}\varrho\in AC^k[a,b]$, then*

$$\mathfrak{I}^{p}_{a+,\iota}{}^{H}\mathfrak{D}^{p,q}_{a+,\iota}\varrho(\iota)=\varrho(\iota)-\sum_{k=1}^{n}\frac{(\iota-a)^{\theta-k}}{\Gamma(\theta+1-k)}\cdot\lim_{\iota\longrightarrow+a}\frac{d^k}{\mathrm{d}\iota^k}\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}\varrho(\iota).$$

Lemma 2.5. [20] *Let $p > 0$ and $q > 0$. Following that $\forall \iota \in J$ there is*

$$\left[\mathfrak{I}^{p}_{a^+,\iota}(\iota)^{q-1}\right](\iota) = \frac{\Gamma(q)}{\Gamma(q+p)}\iota^{q+p-1},$$

and

$$\left[\mathfrak{D}^{p}_{a^+,\iota}(\iota)^{p-1}\right](\iota) = 0, \quad 0 < p < 1.$$

Lemma 2.6. *A stochastic process $(\varrho, \xi) \in \mathcal{E}$ is called a solution of problem (1.1) if (ϱ, ξ) satisfies the following stochastic integral equation*

$$\begin{aligned}\varrho(\iota) =& \frac{\iota^{\gamma_1-1}}{\Gamma(\gamma_1)}\int_0^b g_1\left(s, \varrho(s), \xi(s)\right) d\mathcal{W}(s)\\ &+ \frac{1}{\Gamma(p_1)}\int_0^\iota (\iota - s)^{p_1-1}\varpi_1\left(s, \varrho(s), \varrho\left(\kappa s\right), \xi(s)\right) ds\end{aligned}$$

and

$$\begin{aligned}\xi(\iota) =& \frac{\iota^{\gamma_2-1}}{\Gamma(\gamma_2)}\int_0^b g_2\left(s, \varrho(s), \xi\left(s\right)\right) ds\\ &+ \frac{1}{\Gamma(p_2)}\int_0^\iota (\iota - s)^{p_2-1}\varpi_2\left(s, \varrho(s), \xi\left(s\right), \xi(\kappa s)\right) d\mathcal{W}(s).\end{aligned}$$

Definition 2.7. [12] *Let $\mathcal{A} : \mathcal{S} \longrightarrow \mathcal{E}$ be a bounded continuous map, where $\mathcal{S} \subseteq \mathcal{E}$. Then $\mathcal{A}$ is*

(i) *ϑ-Lipschitz if there exists $r \geq 0$ such that $\vartheta(\mathcal{A}(\mathcal{K})) \leq r\vartheta(\mathcal{K})$ for all bounded subsets $\mathcal{K} \subseteq \mathcal{S}$;*
(ii) *Strict ϑ-contraction if there exists $0 \leq r < 1$ such that $\vartheta(\mathcal{A}(\mathcal{K})) \leq r\vartheta(\mathcal{K})$;*
(iii) *ϑ-condensing if $\vartheta(\mathcal{A}(\mathcal{K})) < \vartheta(\mathcal{K})$ for all bounded subsets $\mathcal{K} \subseteq \mathcal{S}$ with $\vartheta(\mathcal{K}) > 0$,*

where ϑ is the Kuratowski measure of non-compactness.

Proposition 2.8. [21] *If $\mathcal{A}, \mathcal{B} : \mathcal{S} \longrightarrow \mathcal{E}$ are ϑ-Lipschitz with respective constants r_1 and r_2, then $\mathcal{A} + \mathcal{B}$ is ϑ-lipschitz with constant $r_1 + r_2$.*

Proposition 2.9. [21] *If $\mathcal{A} : \mathcal{S} \longrightarrow \mathcal{E}$ is Lipschitz with constant r, then $\mathcal{A}$ is ϑ-lipschitz with the same constant r.*

Proposition 2.10. [21] *If $\mathcal{A} : \mathcal{S} \longrightarrow \mathcal{E}$ is compact, then $\mathcal{Z}$ is ϑ-lipschitz with constant $r = 0$.*

Theorem 2.11. [21] *Let $\mathcal{C} : \mathcal{S} \longrightarrow \mathcal{E}$ is ϑ-condensing and*

$$\Gamma_\delta = \{\varrho \in \mathcal{C} : \textit{there exists } 0 \leq \delta \leq 1 \textit{ such that } \varrho = \delta\mathcal{C}\varrho\}.$$

If Γ_δ is a bounded set in $\mathcal{E}$, then there exists $a > 0$ such that $\Gamma_\delta \subset B_a(0)$ and

$$Deg\left(I - \delta\mathcal{C}, B_a(0), 0\right) = 1 \textit{ for all } \delta \in [0, 1].$$

Thus, $\mathcal{C}$ has at least one fixed point, and the set of all fixed points of $\mathcal{C}$ lies in $B_a(0)$.

3. Existence and uniqueness results

In this part we will use the degree theory to prove the existence of solutions to the problem (1.1).
First, we give the following essential hypotheses:
(H_1): For arbitrary $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, there exist positive constants $\mathcal{L}_{\varpi_1}, l_{\varpi_1}$, m_{ϖ_1} and $q_1, q_2 \in (0,1)$ such that

$$\begin{aligned}
&\left\|\varpi_1\left(\iota, \varrho_1(\iota), \varrho_1(\kappa\iota), \varrho_2(\iota)\right)-\varpi_1\left(\iota, \xi_1(\iota), \xi_1(\kappa\iota), \xi_2(\iota)\right)\right\|^2 \\
&\leq \mathcal{L}_{\varpi_1}\left(\iota^{2(1-\gamma_1)} 2\left\|\varrho_1-\xi_1\right\|^2+\iota^{2(1-\gamma_2)}\left\|\varrho_2-\xi_2\right\|^2\right), \\
&\left\|\varpi_1\left(\iota, \varrho_1(\iota), \varrho_1(\kappa\iota), \varrho_2(\iota)\right)\right\|^2 \\
&\leq l_{\varpi_1}\left(\iota^{2q_1(1-\gamma_1)} 2\left\|\varrho_1\right\|^{2q_1}+\iota^{2q_2(1-\gamma_2)}\left\|\varrho_2\right\|^{2q_2}\right)+m_{\varpi_1}.
\end{aligned}$$

(H_2): For arbitrary $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, there exist positive constants $\mathcal{L}_{\varpi_2}, l_{\varpi_2}$, m_{ϖ_2} and $q_1, q_2 \in (0,1)$ such that

$$\begin{aligned}
&\left\|\varpi_2\left(\iota, \varrho_1(\iota), \varrho_2(\iota), \varrho_2(\kappa\iota)\right)-\varpi_2\left(\iota, \xi_1(\iota), \xi_2(\iota), \xi_2(\kappa\iota)\right)\right\|^2 \\
&\leq \mathcal{L}_{\varpi_2}\left(\iota^{2(1-\gamma_1)}\left\|\varrho_1-\xi_1\right\|^2+2\iota^{2(1-\gamma_2)}\left\|\varrho_2-\xi_2\right\|^2\right), \\
&\left\|\varpi_2\left(\iota, \varrho_1(\iota), \varrho_2(\iota), \varrho_2(\kappa\iota)\right)\right\|^2 \\
&\leq l_{\varpi_2}\left(\iota^{2q_1(1-\gamma_1)}\left\|\varrho_1\right\|^{2q_1}+\iota^{2q_2(1-\gamma_2)} 2\left\|\varrho_2\right\|^{2q_2}\right)+m_{\varpi_2}.
\end{aligned}$$

(H_3): For arbitrary $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, there exist positive constants $\mathcal{L}_{g_j}, l_{g_j}$, m_{g_j} $(j=1,2)$ and $q_1, q_2 \in (0,1)$ such that

$$\begin{aligned}
&\left\|g_j\left(\iota, \varrho_1(\iota), \varrho_2(\iota)\right)-g_j\left(\iota, \xi_1(\iota), \xi_2(\iota)\right)\right\|^2 \\
&\leq \mathcal{L}_{g_j}\left(\iota^{2(1-\gamma_1)}\left\|\varrho_1-\xi_1\right\|^2+\iota^{2(1-\gamma_2)}\left\|\varrho_2-\xi_2\right\|^2\right), \\
&\left\|g_j\left(\iota, \varrho_1(\iota), \varrho_2(\iota)\right)\right\|^2 \leq l_{g_j}\left(\iota^{2q_1(1-\gamma_1)}\left\|\varrho_1\right\|^{2q_1}+\iota^{2q_2(1-\gamma_2)}\left\|\varrho_2\right\|^{2q_2}\right)+m_{g_j}.
\end{aligned}$$

To make clarity, we set the following notations:

$$\begin{aligned}
\Delta_{1_j} &= \frac{2b\mathcal{L}_{g_j}}{\Gamma^2(\gamma_j)}, j=1,2, \\
\Delta_{2_j} &= \frac{2bl_{g_j}}{\Gamma^2(\gamma_j)}, j=1,2, \\
\Delta_{3_j} &= \frac{bm_{g_j}}{\Gamma^2(\gamma_j)}, \\
\Delta_{4_j} &= \frac{3\mathcal{L}_{\varpi_j} b^{2-2\gamma_j+2p_j} l_{\varpi_j}}{\Gamma^2(p_j)2p_j-1}, j=1,2, \\
\Delta &= \max\left\{\Delta_{1_1}, \Delta_{1_2}\right\}, \quad \bar{\Delta}=\max\left\{\Delta_{4_1}, \Delta_{4_2}\right\}.
\end{aligned}$$

Based on Lemma 2.6, we let the operators $\mathcal{A}, \mathcal{B}, \mathcal{C} : \mathcal{E}_1 \times \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \times \mathcal{E}_2$ defined by

$$\mathcal{A}(\varrho,\xi)(\iota) = (\mathcal{A}_1(\varrho,\xi)(\iota), \mathcal{A}_2(\varrho,\xi)(\iota)), \quad \mathcal{B}(\varrho,\xi)(\iota) = (\mathcal{B}_1(\varrho,\xi)(\iota), \mathcal{B}_2(\varrho,\xi)(\iota)),$$
$$\mathcal{C}(\varrho,\xi)(\iota) = \mathcal{A}(\varrho,\xi)(\iota) + \mathcal{B}(\varrho,\xi)(\iota),$$

where

$$\begin{cases} \mathcal{A}_1(\varrho,\xi)(\iota) = \dfrac{\iota^{\gamma_1 - 1}}{\Gamma(\gamma_1)} \displaystyle\int_0^b g_1(s, \varrho(s), \xi(s))\, d\mathcal{W}(s), \\ \mathcal{A}_2(\varrho,\xi)(\iota) = \dfrac{\iota^{\gamma_2 - 1}}{\Gamma(\gamma_2)} \displaystyle\int_0^b g_2(s, \varrho(s), \xi(s))\, ds \end{cases}$$

and

$$\begin{cases} \mathcal{B}_1(\varrho,\xi)(\iota) = \dfrac{1}{\Gamma(p_1)} \displaystyle\int_0^\iota (\iota - s)^{p_1 - 1} \varpi_1(s, \varrho(s), \varrho(\kappa s), \xi(s))\, ds, \\ \mathcal{B}_2(\varrho,\xi)(\iota) = \dfrac{1}{\Gamma(p_2)} \displaystyle\int_0^\iota (\iota - s)^{p_2 - 1} \varpi_2(s, \varrho(s), \xi(s), \xi(\kappa s))\, d\mathcal{W}(s). \end{cases}$$

We shall now prove, step-by-step, that the proposed operators satisfy the conditions of Theorem 2.11.

Lemma 3.1. *The operator $\mathcal{A}$ is ϑ-Lipschitz with a constant Δ. Furthermore, $\mathcal{A}$ adheres to the inequality presented below*

$$\begin{aligned} \|\mathcal{A}(\varrho,\xi)\|_{\mathcal{E}}^2 \leq & \Lambda + \bar{\Lambda}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}, \text{ where} \\ & \Lambda = \max\{\Delta_{3_1}, \Delta_{3_2}\} \text{ and} \\ & \bar{\Lambda} = \max\{\Delta_{2_1}, \Delta_{2_2}\}. \end{aligned} \tag{3.1}$$

Proof. Let $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, we have

$$\begin{aligned} & \mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{A}_1(\varrho_1,\varrho_2)(\iota) - \mathcal{A}_1(\xi_1,\xi_2)(\iota)\right)\right\|^2 \\ \leq & \frac{1}{\Gamma^2(\gamma_1)} \mathbb{E}\left\|\int_0^b \left[g_1(s, \varrho_1(s), \varrho_2(s)) - g_1(s, \xi_1(s), \xi_2(s))\right] ds\right\|^2. \end{aligned}$$

By applying Ito isometry and (H_3), we arrive at

$$\begin{aligned} & \mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{A}_1(\varrho_1,\varrho_2)(\iota) - \mathcal{A}_1(\xi_1,\xi_2)(\iota)\right)\right\|^2 \\ \leq & \frac{\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)} \int_0^b \left[s^{2(1-\gamma)}\mathbb{E}\left\|\varrho_1(s) - \xi_1(s)\right\|^2 + s^{2(1-\gamma_2)}\mathbb{E}\left\|\varrho_2(s) - \xi_2(s)\right\|^2\right] ds. \end{aligned}$$

Therefore,

$$\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{A}_1(\varrho_1,\varrho_2)(\iota) - \mathcal{A}_1(\xi_1,\xi_2)(\iota)\right)\right\|^2 \leq \frac{b\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\left(\|\varrho_1 - \xi_1\|_{\mathcal{E}_1}^2 + \|\varrho_2 - \xi_2\|_{\mathcal{E}_2}^2\right).$$

Consequently,

$$\|\mathcal{A}_1(\varrho_1,\varrho_2) - \mathcal{A}_1(\xi_1,\xi_2)\|_{\mathcal{E}_1}^2 \leq \Delta_{1_1}\|(\varrho_1,\varrho_2) - (\xi_1,\xi_2)\|_{\mathcal{E}}^2.$$

Similar by the Cauchy-Schwartz (C-S) inequality, we can obtain

$$\|\mathcal{A}_2(\varrho_1,\varrho_2) - \mathcal{A}_2(\xi_1,\xi_2)\|_{\mathcal{E}_2}^2 \leq \Delta_{1_2}\|(\varrho_1,\varrho_2) - (\xi_1,\xi_2)\|_{\mathcal{E}}^2.$$

It follows that

$$\|\mathcal{A}\left(\varrho_1,\varrho_2\right)-\mathcal{A}\left(\xi_1,\xi_2\right)\|_{\mathcal{E}}^2 \leq \Delta\|\left(\varrho_1,\varrho_2\right)-\left(\xi_1,\xi_2\right)\|_{\mathcal{E}}^2.$$

Thus, $\mathcal{A}$ satisfies the Lipschitz condition with the constant Δ. By Proposition 2.9, $\mathcal{A}$ is also ϑ-Lipschitz with the same constant Δ.
For the growth condition.
Let $(\varrho,\xi)\in\mathcal{E}_1\times\mathcal{E}_2$. Under the Ito isometry and (H_3), we have

$$\mathbb{E}\left\|\iota^{1-\gamma_1}\mathcal{A}_1\left(\varrho,\xi\right)(\iota)\right\|^2 \leq \frac{1}{\Gamma^2(\gamma_1)}\int_0^\iota \Big[m_{g_1}+l_{g_1}s^{2q_1(1-\gamma_1)}\mathbb{E}\left\|\varrho(s)\right\|^{2q_1} +l_{g_1}s^{2q_2(1-\gamma_2)}\mathbb{E}\left\|\xi(s)\right\|^{2q_2}\Big]\,ds.$$

Therefore,

$$\mathbb{E}\left\|\iota^{1-\gamma_1}\mathcal{A}_1\left(\varrho,\xi\right)(\iota)\right\|^2 \leq \frac{b}{\Gamma^2(\gamma_1)}\left(m_{g_1}+l_{g_1}\|\varrho\|_{\mathcal{E}_1}^{2q_1}+l_{g_1}\|\xi\|_{\mathcal{E}_2}^{2q_2}\right).$$

Consequently,

$$\|\mathcal{A}_1\left(\varrho,\xi\right)\|_{\mathcal{E}_1}^2 \leq \Delta_{3_1}+\Delta_{2_1}\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^{2q},$$

where $q=\max\{q_1,q_2\}$.
Similarly, we find that

$$\|\mathcal{A}_2\left(\varrho,\xi\right)\|_{\mathcal{E}_2}^2 \leq \Delta_{3_2}+\Delta_{2_2}\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^{2q}.$$

It follows that

$$\|\mathcal{A}(\varrho,\xi)\|_{\mathcal{E}}^2 \leq \Lambda+\bar{\Lambda}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}.$$

□

Lemma 3.2. *The operator $\mathcal{B}$ is continuous. Furthermore, $\mathcal{B}$ satisfies the inequality*

$$\begin{aligned}
&\|\mathcal{B}(\varrho,\xi)\|_{\mathcal{E}}^2 \leq \Xi+\bar{\Xi}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}, \text{ where}\\
&\Xi=\max\{\frac{b^{2-2\gamma_1+2p_1}m_{\varpi_1}}{\Gamma^2(p_1)2p_1-1},\frac{b^{2-2\gamma_2+2p_2}m_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\} \text{ and}\\
&\bar{\Xi}=\max\{\frac{3b^{2-2\gamma_1+2p_1}l_{\varpi_1}}{\Gamma^2(p_1)2p_1-1},\frac{3b^{2-2\gamma_2+2p_2}l_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\}.
\end{aligned} \tag{3.2}$$

Proof. For the continuity of $\mathcal{B}$, let $(\varrho_n,\xi_n)\to(\varrho,\xi)$ in $\mathcal{E}$. From the fact that ϖ_1 and ϖ_2 are continuous functions linking with the Lebesgue dominated convergence theorem, we can obtain

$$\|\mathcal{B}(\varrho_n,\xi_n)-\mathcal{B}(\varrho,\xi)\|_{\mathcal{E}}^2\to 0 \quad \text{as} \quad n\to\infty.$$

Moreover, $\mathcal{B}$ satisfies the growth condition.
Using the C-S inequality, we have

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma_1}\mathcal{B}_1(\varrho,\xi)(\iota)\right\|^2\\
\leq&\iota^{2(1-\gamma_1)}\frac{\iota}{\Gamma^2(p_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\mathbb{E}\left\|\varpi_1\left(s,\varrho(s),\varrho(\kappa s),\xi\left(s\right)\right)\right\|^2 ds.
\end{aligned}$$

Based on the assumptions (H_1), it can be concluded that

$$\mathbb{E}\left\|\iota^{1-\gamma_1}\mathcal{B}_1(\varrho,\xi)(\iota)\right\|^2 \leq \frac{t^{2-2\gamma_1+2p_1}}{\Gamma^2(p_1)2p_1-1}\left(m_{\varpi_1}+2l_{\varpi_1}\|\varrho\|_{\mathcal{E}_1}^{2q_1}+l_{\varpi_1}\|\xi\|_{\mathcal{E}_2}^{2q_2}\right).$$

Subsequently,

$$\|\mathcal{B}_1(\varrho,\xi)\|_{\mathcal{E}_1}^2 \leq \frac{b^{2-2\gamma_1+2p_1}m_{\varpi_1}}{\Gamma^2(p_1)2p_1-1}+\frac{3b^{2-2\gamma_1+2p_1}l_{\varpi_1}}{\Gamma^2(p_1)2p_1-1}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}.$$

Similarly,

$$\|\mathcal{B}_2(\varrho,\xi)\|_{\mathcal{E}_2}^2 \leq \frac{b^{2-2\gamma_2+2p_2}m_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}+\frac{3b^{2-2\gamma_2+2p_2}l_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}.$$

Thus,

$$\begin{aligned}\|\mathcal{B}(\varrho,\xi)\|_{\mathcal{E}}^2 \leq& \Xi+\bar{\Xi}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}, \text{ where}\\ \Xi =& \max\{\frac{b^{2-2\gamma_1+2p_1}m_{\varpi_1}}{\Gamma^2(p_1)2p_1-1},\frac{b^{2-2\gamma_2+2p_2}m_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\} \text{ and}\\ \bar{\Xi} =& \max\{\frac{3b^{2-2\gamma_1+2p_1}l_{\varpi_1}}{\Gamma^2(p_1)2p_1-1},\frac{3b^{2-2\gamma_2+2p_2}l_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\}.\end{aligned}$$

□

Lemma 3.3. *$\mathcal{B}$ is compact; consequently, $\mathcal{B}$ is ϑ-Lipschitz with a zero constant.*

Proof. Let $B_\tau=\{(\varrho,\xi)\in\mathcal{E}_1\times\mathcal{E}_2:\|(\varrho,\xi)\|\leq\tau\}$ and consider a bounded set $\mathcal{K}$ such that $\mathcal{K}\subset\mathcal{B}_\tau$. It remains to demonstrate that $\mathcal{B}(\mathcal{K})$ is relatively compact in $\mathcal{E}$. For this purpose, let $(\varrho,\xi)\in\mathcal{K}\subset\mathcal{B}_\tau$ and by (3.2), we derive

$$\|\mathcal{B}(\varrho,\xi)\|_{\mathcal{E}}^2 \leq \Xi+\bar{\Xi}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q} := \Upsilon.$$

Thus, $\mathcal{B}(\mathcal{K})\subset\mathcal{B}_\tau$, and as a result, $\mathcal{B}(\mathcal{K})$ is bounded.
It remains to prove the equicontinuity of $\mathcal{B}$.
Let $0\leq\epsilon_1<\epsilon_2\leq b$ and $(\varrho,\xi)\in B_\tau$, then

$$\begin{aligned}&\mathbb{E}\left\|\epsilon_2^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_1)\right\|^2\\ \leq& 2\mathbb{E}\left\|\frac{1}{\Gamma(p_1)}\int_0^{\epsilon_1}\left[\epsilon_2^{2(1-\gamma_1)}(\epsilon_2-s)^{p_1-1}-\epsilon_1^{2(1-\gamma_1)}(\epsilon_1-s)^{p_1-1}\right]\right.\\ &\left.\times\varpi_1(s,\varrho(s),\varrho(\kappa s),\xi(s))ds\right\|^2\\ &+2\mathbb{E}\left\|\frac{1}{\Gamma(p_1)}\int_{\epsilon_1}^{\epsilon_2}\epsilon_2^{2(1-\gamma_1)}(\epsilon_2-s)^{p_1-1}\varpi_1(s,\varrho(s),\varrho(\kappa s),\xi(s))ds\right\|^2.\end{aligned}$$

By C-S inequality (H_1), we obtain

$$\begin{aligned}&\mathbb{E}\left\|\epsilon_2^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_1)\right\|^2\\ \leq& \frac{2\epsilon_1\left(m_{\varpi_1}+3l_{\varpi_1}\tau^q\right)}{\Gamma^2(p_1)}\int_0^{\epsilon_1}\left[\epsilon_2^{2(1-\gamma_1)}(\epsilon_2-s)^{p_1-1}-\epsilon_1^{2(1-\gamma_1)}(\epsilon_1-s)^{p_1-1}\right]^2ds\\ &+\frac{2(\epsilon_2-\epsilon_1)\left(m_{\varpi_1}+3l_{\varpi_1}\tau^q\right)\epsilon_2^{2(1-\gamma_1)}}{\Gamma^2(p_1)2p_1-1}(\epsilon_2-\epsilon_1)^{2p_1-1}\longrightarrow 0, \quad \text{as} \quad \epsilon_1\longrightarrow\epsilon_2.\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\mathbb{E}\left\|\epsilon_2^{1-\gamma_2}\left(\mathcal{B}_2(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_2}\left(\mathcal{B}_2(\varrho,\xi)\right)(\epsilon_1)\right\|^2\\
\leq&\frac{2\epsilon_1\left(m_{\varpi_2}+3l_{\varpi_2}\tau^q\right)}{\Gamma^2(p_2)}\int_0^{\epsilon_1}\left[\epsilon_2^{2(1-\gamma_2)}(\epsilon_2-s)^{p_1-1}-\epsilon_1^{2(1-\gamma_2)}(\epsilon_1-s)^{p_2-1}\right]^2 ds\\
&+\frac{2(\epsilon_2-\epsilon_1)\left(m_{\varpi_2}+3l_{\varpi_2}\tau^q\right)\epsilon_2^{2(1-\gamma_2)}}{\Gamma^2(p_2)2p_2-1}(\epsilon_2-\epsilon_1)^{2p_2-1}\longrightarrow 0,\quad \text{as}\quad \epsilon_1\longrightarrow\epsilon_2.
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
&\mathbb{E}\left\|\epsilon_2^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_1)\right\|^2,\\
&\mathbb{E}\left\|\epsilon_2^{1-\gamma_2}\left(\mathcal{B}_2(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_2}\left(\mathcal{B}_2(\varrho,\xi)\right)(\epsilon_1)\right\|^2
\end{aligned}$$

approaches zero as $\epsilon_1 \to \epsilon_2$. By applying the Arzelà-Ascoli theorem, it can be concluded that the operator $\mathcal{B}$ is compact. As a result of Proposition 2.10, $\mathcal{B}$ is ϑ-Lipschitz with a zero constant. □

Theorem 3.4. *Assume that $(H_1)-(H_3)$ hold and $0<\Delta<1$. Then the problem (1.1) has at least one solution on $\mathcal{E}$. Moreover, the set of the solutions of the problem (1.1) is bounded in $\mathcal{E}$.*

Proof. By Lemma 3.1, the operator $\mathcal{A}$ is shown to be ϑ-Lipschitz with a constant $\Delta\in(0,1)$. Similarly, from Lemma 3.3, the operator $\mathcal{B}$ is ϑ-Lipschitz with a constant equal to zero. Consequently, based on Proposition 2.8 and Definition 2.7, the operator $\mathcal{C}$ qualifies as a ϑ-contraction with the constant Δ. This implies that $\mathcal{C}$ is ϑ-condensing. Now, consider the following set

$$\Gamma_\delta=\{(\varrho,\xi)\in\mathcal{E}_1\times\mathcal{E}_2:(\varrho,\xi)=\delta\mathcal{C}\left(\varrho,\xi\right),\quad \text{for}\quad 0\leq\delta<1\}.$$

We need to demonstrate that Γ_δ is bounded in $\mathcal{E}_1\times\mathcal{E}_2$. Let $(\varrho,\xi)\in\Gamma_\delta$. Then, by Lemma 3.1 and 3.2, it follows that

$$\begin{aligned}
\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^2&=\delta^2\|\mathcal{A}\left(\varrho,\xi\right)+\mathcal{B}\left(\varrho,\xi\right)\|_{\mathcal{E}}^2\\
&\leq 2\delta^2\left(\|\mathcal{A}\left(\varrho,\xi\right)\|_{\mathcal{E}}^2+\|\mathcal{B}\left(\varrho,\xi\right)\|_{\mathcal{E}}^2\right)\\
&\leq 2\left(\Lambda+\Xi\right)+2\left(\bar\Lambda+\bar\Xi\right)\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^{2q}.
\end{aligned}$$

Thus, the set Γ_δ is bounded in $\mathcal{E}$. If this is not true, by dividing the above inequality by $\theta:=\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^2\to\infty$, we obtain

$$1\leq\lim_{\theta\to\infty}\frac{1}{\theta}\left[2\left(\Lambda+\Xi\right)+2\left(\bar\Lambda+\bar\Xi\right)\theta^q\right]=0,$$

which is a contradiction. Consequently, Theorem 2.11 ensures that $\mathcal{C}$ has at least one fixed point. Therefore, our problem (1.1) has at least one solution. □

Theorem 3.5. *Assume assumptions $(H_1)-(H_3)$ hold and $0<2\left(\Delta+\bar\Delta\right)<1$, it follows that the problem (1.1) has a unique solution.*

Proof. By applying the Banach contraction theorem, for any $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, it follows from the arguments presented in the proof of Lemma 3.3 that

$$\|\mathcal{A}(\varrho_1, \varrho_2) - \mathcal{A}(\xi_1, \xi_2)\|_{\mathcal{E}}^2 \leq \Delta \|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.$$

Next, by the C-S inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{B}_1(\varrho_1, \varrho_2)(\iota) - \mathcal{B}_1(\xi_1, \xi_2)(\iota)\right)\right\|^2 \\
&\leq \iota^{2(1-\gamma_1)} \frac{\iota}{\Gamma^2(p_1)} \int_0^\iota (\iota - s)^{2(p_1-1)} \mathbb{E}\left\|\varpi_1(s, \varrho_1(s), \varrho_1(\kappa s), \varrho_2(s))\right. \\
&\quad \left. - \varpi_1(s, \xi_1(s), \xi_1(\kappa s), \xi_2(s))\right\|^2 ds.
\end{aligned}$$

Based on the assumptions (H_1), it can be concluded that

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{B}_1(\varrho_1, \varrho_2)(\iota) - \mathcal{B}_1(\xi_1, \xi_2)(\iota)\right)\right\|^2 \\
&\leq \frac{\mathcal{L}_{\varpi_1} t^{2-2\gamma_1+2p_1}}{\Gamma^2(p_1) 2p_1 - 1}\left(2\|\varrho_1 - \xi_1\|_{\mathcal{E}_1}^2 + \|\varrho_2 - \xi_2\|_{\mathcal{E}_2}^2\right).
\end{aligned}$$

Subsequently,

$$\|\mathcal{B}_1(\varrho_1, \varrho_2) - \mathcal{B}_1(\xi_1, \xi_2)\|_{\mathcal{E}_1}^2 \leq \frac{3\mathcal{L}_{\varpi_1} b^{2-2\gamma_1+2p_1}}{\Gamma^2(p_1) 2p_1 - 1} \|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.$$

Similarly,

$$\|\mathcal{B}_2(\varrho_1, \varrho_2) - \mathcal{B}_2(\xi_1, \xi_2)\|_{\mathcal{E}_2}^2 \leq \frac{3\mathcal{L}_{\varpi_2} b^{2-2\gamma_2+2p_2}}{\Gamma^2(p_2) 2p_2 - 1} \|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.$$

Therefore,

$$\|\mathcal{B}(\varrho_1, \varrho_2) - \mathcal{B}(\xi_1, \xi_2)\|_{\mathcal{E}}^2 \leq \bar{\Delta} \|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.$$

Thus,

$$\begin{aligned}
\|\mathcal{C}(\varrho_1, \varrho_2) - \mathcal{C}(\xi_1, \xi_2)\|_{\mathcal{E}}^2 &\leq 2\left(\|\mathcal{A}(\varrho_1, \varrho_2) - \mathcal{A}(\xi_1, \xi_2)\|_{\mathcal{E}}^2 + \|\mathcal{B}(\varrho_1, \varrho_2) - \mathcal{B}(\xi_1, \xi_2)\|_{\mathcal{E}}^2\right) \\
&\leq 2\left(\Delta + \bar{\Delta}\right) \|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.
\end{aligned}$$

This implies that $\mathcal{C}$ is a contraction. As a result, the problem (1.1) has a unique solution.

□

4. Continuous dependence of solutions

Now, we study the continuous dependence on the nonlocal conditions of the solutions of problem (1.1).

Definition 4.1. *The solution $(\varrho, \xi) \in \mathcal{E}_1 \times \mathcal{E}_2$ of problem (1.1) is said to be continuously dependent on the nonlocal conditions g_1 and g_2 if for all $\epsilon > 0$, $\exists \delta > 0$ such that $\left\|g_j(s, \cdot, \cdot) - g_j^*(s, \cdot, \cdot)\right\|^2 \leq \delta$, $j = 1, 2$ implies that $\|(\varrho, \xi) - (\bar{\varrho}, \bar{\xi})\|_{\mathcal{E}}^2 \leq \epsilon$.*

Theorem 4.2. *Assume hypotheses (H_1)-(H_3) are fulfilled, then the solution of the problem (1.1) is continuously dependent on g_1 and g_2.*

Proof. Let $(\varrho,\xi),\left(\bar{\varrho},\bar{\xi}\right)$ be the solutions of problem (1.1) such that

$$\begin{cases}\varrho(\iota)=\dfrac{\iota^{\gamma_1-1}}{\Gamma(\gamma_1)}\displaystyle\int_0^{\iota}g_1\left(s,\varrho(s),\xi(s)\right)d\mathcal{W}(s)\\ \qquad+\dfrac{1}{\Gamma(p_1)}\displaystyle\int_0^{\iota}(\iota-s)^{p_1-1}\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right),\xi(s)\right)ds\\ \xi(\iota)=\dfrac{\iota^{\gamma_2-1}}{\Gamma(\gamma_2)}\displaystyle\int_0^{\iota}g_2\left(s,\varrho(s),\xi\left(s\right)\right)ds\\ \qquad+\dfrac{1}{\Gamma(p_2)}\displaystyle\int_0^{\iota}(\iota-s)^{p_2-1}\varpi_2\left(s,\varrho(s),\xi\left(s\right),\xi(\kappa s)\right)d\mathcal{W}(s),\end{cases}$$

and

$$\begin{cases}\bar{\varrho}(\iota)=\dfrac{\iota^{\gamma_1-1}}{\Gamma(\gamma_1)}\displaystyle\int_0^{\iota}g_1^*\left(s,\bar{\varrho}(s),\bar{\xi}(s)\right)d\mathcal{W}(s)\\ \qquad+\dfrac{1}{\Gamma(p_1)}\displaystyle\int_0^{\iota}(\iota-s)^{p_1-1}\varpi_1\left(s,\bar{\varrho}(s),\bar{\varrho}\left(\kappa s\right),\bar{\xi}(s)\right)ds\\ \bar{\xi}(\iota)=\dfrac{\iota^{\gamma_2-1}}{\Gamma(\gamma_2)}\displaystyle\int_0^{\iota}g_2^*\left(s,\bar{\varrho}(s),\bar{\xi}\left(s\right)\right)ds\\ \qquad+\dfrac{1}{\Gamma(p_2)}\displaystyle\int_0^{\iota}(\iota-s)^{p_2-1}\varpi_2\left(s,\bar{\varrho}(s),\bar{\xi}\left(s\right),\bar{\xi}(\kappa s)\right)d\mathcal{W}(s),\end{cases}$$

where

$$\left\|g_j\left(s,\cdot,\cdot\right)-g_j^*\left(s,\cdot,\cdot\right)\right\|^2\leq\delta,\quad j=1,2.$$

By the Ito isometry linking with the C-S inequality, we get

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(\iota)-\bar{\varrho}(\iota)\right)\right\|^2\\ \leq&\frac{2}{\Gamma^2(\gamma_1)}\int_0^{\iota}\mathbb{E}\left\|g_1\left(s,\varrho(s),\xi(s)\right)-g_1^*\left(s,\bar{\varrho}(s),\bar{\xi}(s)\right)\right\|^2ds\\ &+\frac{2\iota^{2(1-\gamma_1)}\iota}{\Gamma^2(\gamma_1)}\int_0^{\iota}(\iota-s)^{2(p_1-1)}\mathbb{E}\left\|\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right),\xi(s)\right)\right.\\ &\left.-\varpi_1\left(s,\bar{\varrho}(s),\bar{\varrho}\left(\kappa s\right),\bar{\xi}(s)\right)\right\|^2ds\\ \leq&\frac{4}{\Gamma^2(\gamma_1)}\int_0^{\iota}\mathbb{E}\left(\left\|g_1\left(s,\varrho(s),\xi(s)\right)-g_1^*\left(s,\varrho(s),\xi(s)\right)\right\|^2\right.\\ &\left.+\left\|g_1^*\left(s,\varrho(s),\xi(s)\right)-g_1^*\left(s,\bar{\varrho}(s),\bar{\xi}(s)\right)\right\|^2\right)ds\\ &+\frac{2\iota^{2(1-\gamma_1)}\iota}{\Gamma^2(\gamma_1)}\int_0^{\iota}(\iota-s)^{2(p_1-1)}\mathbb{E}\left\|\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right),\xi(s)\right)\right.\\ &\left.-\varpi_1\left(s,\bar{\varrho}(s),\bar{\varrho}\left(\kappa s\right),\bar{\xi}(s)\right)\right\|^2ds.\end{aligned}$$

By applying (H_1) and (H_3), we arrive at

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(\iota)-\bar{\varrho}(\iota)\right)\right\|^2\\
\leq&\frac{4b}{\Gamma^2(\gamma_1)}\delta+\frac{4\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\int_0^\iota\left[s^{2(1-\gamma_1)}\mathbb{E}\left\|\varrho(s)-\bar{\varrho}(s)\right\|^2+s^{2(1-\gamma_2)}\mathbb{E}\left\|\xi(s)-\bar{\xi}(s)\right\|^2\right]ds\\
&+\frac{2b^{3-2\gamma_1}\mathcal{L}_{\varpi_1}}{\Gamma^2(\gamma_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\left[s^{2(1-\gamma_1)}\mathbb{E}\left\|\varrho(s)-\bar{\varrho}(s)\right\|^2+s^{2(1-\gamma_1)}\mathbb{E}\left\|\varrho(\kappa s)-\bar{\varrho}(\kappa s)\right\|^2\right.\\
&\left.+s^{2(1-\gamma_2)}\mathbb{E}\left\|\xi(s)-\bar{\xi}(s)\right\|^2\right]ds.
\end{aligned}$$

Let
$\varphi_1(\iota)=\sup_{s\in(0,\iota)}\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(s)-\bar{\varrho}(s)\right)\right\|^2$ and
$\varphi_2(\iota)=\sup_{s\in(0,\iota)}\mathbb{E}\left\|\iota^{1-\gamma_2}\left(\xi(s)-\bar{\xi}(s)\right)\right\|^2$, for $\iota\in J$.
We have

$$\begin{cases}\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(s)-\bar{\varrho}(s)\right)\right\|^2\leq\varphi_1(s)\\ \mathbb{E}\left\|\iota^{1-\gamma_2}\left(\xi(s)-\bar{\xi}(s)\right)\right\|^2\leq\varphi_2(s),\end{cases}$$

and

$$\begin{cases}\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(\kappa s)-\bar{\varrho}(\kappa s)\right)\right\|^2\leq\varphi_1(s)\\ \mathbb{E}\left\|\iota^{1-\gamma_2}\left(\xi(\kappa s)-\bar{\xi}(\kappa s)\right)\right\|^2\leq\varphi_2(s).\end{cases}$$

Then, for $\iota\in J$, we get

$$\begin{aligned}\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(\iota)-\bar{\varrho}(\iota)\right)\right\|^2\leq&\frac{4b}{\Gamma^2(\gamma_1)}\delta+\frac{4\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\int_0^\iota\left(\varphi_1(s)+\varphi_2(s)\right)ds\\&+\frac{2b^{3-2\gamma_1}\mathcal{L}_{\varpi_1}}{\Gamma^2(\gamma_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\left(2\varphi_1(s)+\varphi_2(s)\right)ds.\end{aligned}$$

Then

$$\begin{aligned}\varphi_1(\iota)\leq&\frac{4b}{\Gamma^2(\gamma_1)}\delta+\frac{4\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\int_0^\iota\left(\varphi_1(s)+\varphi_2(s)\right)ds\\&+\frac{2b^{3-2\gamma_1}\mathcal{L}_{\varpi_1}}{\Gamma^2(\gamma_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\left(2\varphi_1(s)+\varphi_2(s)\right)ds.\end{aligned}$$

Similar, we find that

$$\begin{aligned}\varphi_2(\iota)\leq&\frac{4b}{\Gamma^2(\gamma_2)}\delta+\frac{4\mathcal{L}_{g_2}}{\Gamma^2(\gamma_2)}\int_0^\iota\left(\varphi_1(s)+\varphi_2(s)\right)ds\\&+\frac{2b^{3-2\gamma_2}\mathcal{L}_{\varpi_2}}{\Gamma^2(\gamma_2)}\int_0^\iota(\iota-s)^{2(p_2-1)}\left(\varphi_1(s)+2\varphi_2(s)\right)ds.\end{aligned}$$

Take now $\varphi=\max\{\varphi_1,\varphi_2\}$, it follows that

$$\varphi(\iota)\leq\wp\delta+\Re\int_0^\iota\varphi(s)ds+\Im\int_0^\iota(\iota-s)^{2(p-1)}\varphi(s)ds,$$

where

$$\begin{aligned}
&\wp = \max\{\tfrac{4b}{\Gamma^2(\gamma_1)}, \tfrac{4b}{\Gamma^2(\gamma_2)}\}, \quad \Re = \max\{\tfrac{8\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}, \tfrac{8\mathcal{L}_{g_2}}{\Gamma^2(\gamma_2)}\},\\
&\Im = \max\{\frac{6b^{3-2\gamma_1}\mathcal{L}_{\varpi_1}}{\Gamma^2(\gamma_1)}, \frac{6b^{3-2\gamma_2}\mathcal{L}_{\varpi_2}}{\Gamma^2(\gamma_2)}\}.\\
&p = \max\{p_1, p_2\}.
\end{aligned}$$

So,

$$\varphi(\iota) \leq \wp\delta + \Re\int_0^\iota \varphi(s)ds + \Im\int_0^\iota (\iota - s)^{2(p-1)}\varphi(s)ds.$$

By Generalised Gronwall inequality, we obtain

$$\begin{aligned}
\varphi(\iota) &\leq \left(\wp\delta + \Re\int_0^\iota \varphi(s)ds\right) E_{2p-1}\left(\Im\Gamma(2p-1)\iota^{2p-1}\right)\\
&\leq \aleph\delta + \hbar\int_0^\iota \varphi(s)ds,
\end{aligned}$$

where

$$\aleph = \wp E_{2p-1}\left(\Im\Gamma(2p-1)b^{2p-1}\right), \quad \hbar = \Re E_{2p-1}\left(\Im\Gamma(2p-1)b^{2p-1}\right).$$

By Gronwall inequality, we obtain

$$\varphi(\iota) \leq \aleph\delta e^{\hbar\iota}.$$

Hence,

$$\max\{\|\varrho - \bar{\varrho}\|^2_{\mathcal{E}_1}, \|\xi - \bar{\xi}\|^2_{\mathcal{E}_2}\} \leq \aleph\delta e^{\hbar b} = \epsilon.$$

We conclude that the solution of the problem (1.1) is continuously dependent on g_1 and g_2.

□

5. An example

Consider the following coupled system of Hilfer fractional stochastic pantograph equations with nonlocal integral conditions

$$\begin{cases}
{}^H\mathfrak{D}^{0.75,0.5}_{0^+,\iota}\varrho(\iota) = \varpi_1\left(\iota, \varrho(\iota), \varrho\left(0.5\iota\right), \xi(\iota)\right), \quad \iota \in (0,1],\\
{}^H\mathfrak{D}^{0.85,0.6}_{0^+,\iota}\xi(\iota) = \varpi_2\left(\iota, \varrho(\iota), \xi\left(\iota\right), \xi(0.5\iota)\right)\dfrac{d\mathcal{W}(\iota)}{d\iota},\\
\mathfrak{I}^{1-\gamma_1}_{0^+,\iota}\varrho(0) = \displaystyle\int_0^\iota g_1\left(\iota, \varrho(\iota), \xi(\iota)\right)d\mathcal{W}(s),\\
\mathfrak{I}^{1-\gamma_2}_{0^+,\iota}\xi(0) = \displaystyle\int_0^\iota g_2\left(\iota, \varrho(\iota), \xi(\iota)\right)ds,
\end{cases} \tag{5.1}$$

where

$$\varpi_1\left(\iota,\varrho(\iota),\varrho\left(0.5\iota\right),\xi(\iota)\right)=\frac{e^{-\pi\iota}}{\sqrt{13+\iota}}+\frac{1}{\sqrt{50+\iota}}\left(|\varrho(\iota)|+|\varrho(\kappa\iota)|+|\sin(\xi(\iota))|\right),$$

$$\varpi_2\left(\iota,\varrho(\iota),\xi\left(\iota\right),\xi(0.5\iota)\right)=\frac{1}{14}+\frac{e^{-\iota}}{4\sqrt{5}}\left(|\varrho(\iota)|+|\cos(\xi(\iota))|+\sqrt{|\xi(0.5\iota)|}\right),$$

$$g_1(\iota,\varrho(\iota),\xi(\iota))=\frac{1}{17}+\frac{1}{4\sqrt{2}}\left(|\varrho(\iota)|+|\sin(\xi(\iota))|\right),$$

$$g_1(\iota,\varrho(\iota),\xi(\iota))=\frac{e^{-\iota^2}}{3\iota+\sqrt{21}}+\frac{1}{\iota^2+\sqrt{31}}\left(|\cos(\varrho(\iota))|+|\xi(\iota)|\right).$$

Here, $p_1=0.75$, $p_2=0.85$, $q_1=0.5$, $q_2=0.6$, $\kappa=0.5$, $\gamma_1=0.875$, $\gamma_2=0.94$. The assumptions (H_1), (H_2) and (H_3) are satisfied with $\mathcal{L}_{\varpi_1}=l_{\varpi_1}=\frac{3}{25}$, $\mathcal{L}_{\varpi_2}=l_{\varpi_2}=\frac{3}{40}$, $m_{\varpi_1}=\frac{2}{13}$, $m_{\varpi_2}=\frac{1}{7}$, $\mathcal{L}_{g_1}=l_{g_1}=\frac{1}{8}$, $\mathcal{L}_{g_2}=l_{g_2}=\frac{4}{31}$, $m_{g_1}=\frac{2}{17}$ and $m_{g_2}=\frac{2}{21}$. Additionally, we find $\Delta=0.239329<1$. Theorem 3.4 shows that problem (5.1) has at least one solution. Further, $2\left(\Delta+\bar{\Delta}\right)=0.80914<1$. Thus by Theorem 3.5 the problem has a unique solution.

Next, we plot the approximate solution $(\varrho(\iota),\xi(\iota))$ of problem (5.1) for different values of p_1, p_2, q_1 and q_2.

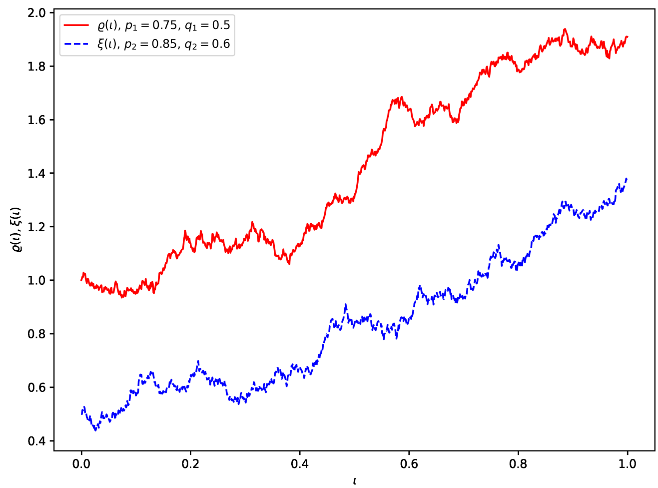

FIGURE 1. Solution $(\varrho(\iota),\xi(\iota))$ for $p_1=0.75$, $q_1=0.5$, $p_2=0.85$, $q_2=0.6$.

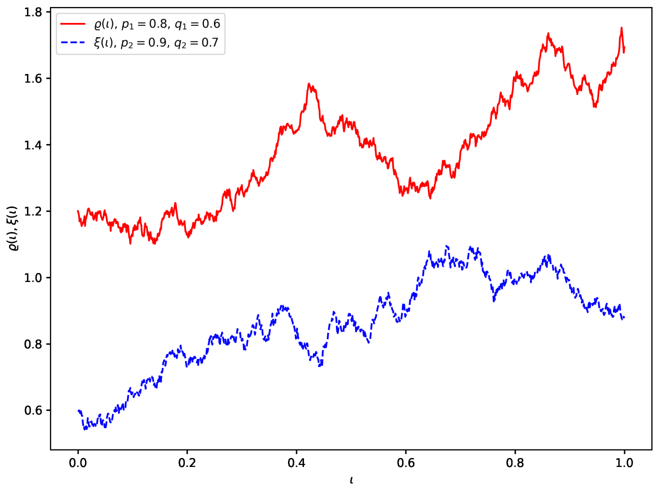

FIGURE 2. Solution $(\varrho(\iota), \xi(\iota))$ for $p_1 = 0.8$, $q_1 = 0.6$, $p_2 = 0.9$, $q_2 = 0.7$.

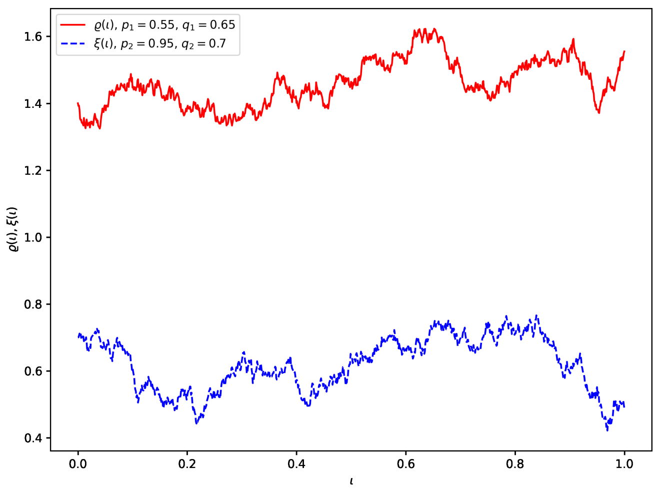

FIGURE 3. Solution $(\varrho(\iota), \xi(\iota))$ for $p_1 = 0.55$, $q_1 = 0.65$, $p_2 = 0.95$, $q_2 = 0.7$.

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Application of Riemann-Liouville fractional integral to fuzzy differential subordination of analytic univalent functions

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Abstract. This paper focuses on geometric function theory, a subfield of complex analysis that has been adapted for fuzzy set analysis. We construct new operator denoted by $\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}$, formed by applying Riemann-Liouville fractional integral to the linear combination of the Pascal and Catas operator. Using this operator, we describe a specific fuzzy class of analytic univalent functions, presented by $\mathcal{DN}Y^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$ in the open unit disk. A number of novel findings that are applicable to this class are found by applying the concept of fuzzy differential subordination. Interesting corollaries are discovered using specific functions, and an example illustrates the practical usage of the results.

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Keywords: Univalent function, differential subordination, fuzzy differential subordination, best fuzzy dominant, Pascal operator, Catas operator, Riemann-Liouville fractional integral.

1. Introduction

Lotfi A. Zadeh established the concept of fuzzy sets in 1965 [36], and it has seen remarkable development to become employed in numerous areas of science and technology nowadays. The constantly concerns of mathematicians about incorporating the concept of fuzzy sets into mathematical theories that were already well-established led to the combination of fuzzy sets theory and geometric function theory. The authors highlight Lotfi A. Zadeh's scholarly contributions in their 2017 review article [10] by going over the progress of the idea of a fuzzy set and its applications in numerous

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fields.

Differential subordination was first proposed by S.S. Miller and P.T. Mocanu in [17, 19]. These approaches made it easier to verify the conclusions that had previously been produced and inspired a great deal of new research using techniques specific to this theory. The book written by S.S. Miller and P.T. Mocanu[17] and released in 2000 contains the essential elements of the theory of differential subordination. It is effectively developed over subsequent decades by other authors [18, 12, 5, 6, 20]. There are a few instances of differential subordination in utilization [9, 33, 4].

The fuzzy differential subordination theory is based on the general theory of differential subordination and it evolves by incorporating the majority of the classical theory's concepts to provide novel outcomes. The notion of differential subordination was newly extended from fuzzy set theory to geometric function theory by authors G. I. Oros and Gh. Oros[22, 23, 24]. Numerous authors have further expanded it [32, 11, 25, 26, 14, 15, 21, 3, 13, 27], and they have produced findings using fuzzy differential subordination. The progress made possible by the incorporation of quantum calculus and elements of fractional calculus into geometric function theory.

Let $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(\mathscr{U})$ denote the class of analytic functions in $\mathscr{U}$. Denote

$$\mathcal{H}[c, n] = \{\mathfrak{t} : \mathfrak{t} \in \mathcal{H}(\mathscr{U}) \textit{ and } \mathfrak{t}(z) = c + c_n z^n + \cdots , z \in \mathscr{U}\},$$

$$\mathcal{A}_n = \{\mathfrak{t} : \mathfrak{t} \in \mathcal{H}(\mathscr{U}) \textit{ and } \mathfrak{t}(z) = z + c_{n+1} z^{n+1} + \cdots , z \in \mathscr{U}\} \text{ and } \mathcal{A}_1 = \mathcal{A}.$$

Definition 1.1. [23] *Consider, $\mathcal{X}$ be a non-empty set. An application $F : \mathcal{X} \to [0, 1]$ is called fuzzy subset. An alternate definition, more precise would be the following: A pair $(\mathcal{S}, F_{\mathcal{S}})$, where $F_{\mathcal{S}} : \mathcal{X} \to [0, 1]$ and $\mathcal{S} = \{x \in \mathcal{X} : 0 < F_{\mathcal{S}}(x) \leq 1\}$ is called fuzzy subset. The function $F_{\mathcal{S}}$ is called membership function of the fuzzy subset $(\mathcal{S}, F_{\mathcal{S}})$.*

Definition 1.2. [16] *Let $\mathcal{D}$ is a set in $\mathbb{C}$, $z_0 \in \mathcal{D}$ is a fixed point and let the functions $f, g \in \mathcal{H}(\mathcal{D})$. The function f is named a fuzzy subordinate to g and written as $f \prec_F g$ if*

1. $f(z_0) = g(z_0)$
2. $F_{f(\mathcal{D})} f(z) \leq F_{g(\mathcal{D})} g(z), z \in \mathcal{D}$.

Remark 1.3.
1. Let $\mathcal{D} \subset \mathbb{C}$, $z_0 \in \mathcal{D}$ be a fixed point, and the functions $f, g \in \mathcal{H}(\mathcal{D})$. If g is univalent function in $\mathcal{D}$ then $f \prec_F g$ if and only if $f(z_0) = g(z_0)$ and $f(\mathcal{D}) \subset g(\mathcal{D})$.
2. A function $F : \mathbb{C} \to [0, 1]$, can be defined as, for example $F(z) = \frac{|z|}{1+|z|}, F(z) = \frac{1}{1+|z|}, |\sin |z||, |\cos |z||$.
3. If $\mathcal{D} = \mathscr{U}$ then the conditions become $f(0) = g(0)$ and $f(\mathscr{U}) \subset g(\mathscr{U})$ which is same as the classical definition of subordination.

Definition 1.4. [35] *Let h be univalent in $\mathscr{U}$ and $\Psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $\mathcal{P}$ is analytic in $\mathscr{U}$ and satisfies the fuzzy differential subordination*

$$F_{\Psi(\mathbb{C}^3 \times \mathscr{U})}(\Psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)) \leq F_{h(\mathscr{U})} h(z) \tag{1.1}$$

$$i.e. \Psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec_F h(z), z \in \mathscr{U}$$

then $\mathcal{P}$ is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $\mathcal{P} \prec_F q$ for all $\mathcal{P}$ satisfying (1.1). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z), z \in \mathscr{U}$ for all fuzzy dominant q of (1.1) is said to be the best fuzzy dominant of (1.1).

Definition 1.5. [14] *Let $f(\mathcal{D}) = supp(f(\mathcal{D}), F_{f(\mathcal{D})})=\{z \in \mathcal{D} : 0 < F_{f(\mathcal{D})} \leq 1\}$, where $F_{f(\mathcal{D})}$ is the membership function of the fuzzy subset $f(\mathcal{D})$ associated to the function f.*
The membership function of the fuzzy set $(\mu f)(\mathcal{D})$ associated to the function μf coincides with the membership function of the fuzzy set $f(\mathcal{D})$ associated to the function f, i.e. $F_{(\mu f)\mathcal{D}} = F_{f(\mathcal{D})}, z \in \mathcal{D}$.
The membership function of the fuzzy set $(g+h)(\mathcal{D})$ associated to the function $g+h$ coincide with the half sum of the membership functions of the fuzzy sets $g(\mathcal{D})$, respectively $h(\mathcal{D})$, associated to the function g, respectively h,
i.e. $F_{(g+h)(\mathcal{D})}((g+h)z) = \dfrac{F_{g(\mathcal{D})}g(z) + F_{h(\mathcal{D})}h(z)}{2}, z \in \mathcal{D}$.

Definition 1.6. [34] *Let $\mathfrak{t} \in \mathcal{A}$ then a Pascal operator $Y_\vartheta^\eta : \mathcal{A} \to \mathcal{A}$ is given by*

$$Y_\vartheta^\eta \mathfrak{t}(z) = z + \sum_{r=2}^{\infty} \binom{r+\eta-2}{\eta-1} \vartheta^{r-1} c_r z^r;$$

$$(z \in \mathscr{U},\ \eta \geq 1, 0 \leq \vartheta < 1).$$

Definition 1.7. [7] *For $\mathfrak{t} \in \mathcal{A}$, Catas defined the operator as follow:*

$$\mathcal{N}_{b,v}^n \mathfrak{t}(z) = z + \sum_{r=2}^{\infty} \left\{ \frac{1+v+b(r-1)}{1+v} \right\}^n c_r z^r;$$

$$(n \in \mathbb{N}_0,\ z \in \mathscr{U},\ b, v \geq 0).$$

Now we define the linear operator $\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma} : \mathcal{A} \to \mathcal{A}$ as

$$\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma} \mathfrak{t}(z) = (1-\sigma)\mathcal{N}_{b,v}^n \mathfrak{t}(z) + \sigma Y_\vartheta^\eta \mathfrak{t}(z).$$

In series form, it is able to shown as

$$\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma} \mathfrak{t}(z) = z + \sum_{r=2}^{\infty} \Xi_r(n,\eta,\sigma,b,v,\vartheta) c_r z^r,$$

with $\Xi_r(n,\eta,\sigma,b,v,\vartheta) = \left[(1-\sigma) \left\{ \dfrac{1+v+b(r-1)}{1+v} \right\}^n + \sigma \dbinom{r+\eta-2}{\eta-1} \vartheta^{r-1} \right]$

$$(z \in \mathscr{U},\ \eta \geq 1, 0 \leq \vartheta < 1, n \in \mathbb{N}_0,\ b, v, \sigma \geq 0).$$

Definition 1.8. [8]*(see also* [1, 2]*) Given an analytical function $\mathfrak{t}$, the Riemann-Liouville fractional integral of order α is*

$$\mathfrak{D}_z^{-\alpha} \mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{\mathfrak{t}(t)}{(z-t)^{1-\alpha}} dt, \qquad \alpha > 0.$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ *with* $\Gamma(1) = 1, \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, *and* $\mathfrak{t}$ *is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of* $(z-t)^{1-\alpha}$ *is removed by requiring* $\log(z-t)$ *to be real when* $z-t>0$.

Applying the Riemann-Liouville fractional integral of order α to the linear operator $\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}$ yields the following:

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(t)}{(z-t)^{1-\alpha}}dt.$$

After simple calculation which yields the series form

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z) = \frac{z^{(1+\alpha)}}{\Gamma(2+\alpha)} + \sum_{r=2}^{\infty}\Xi_r(n,\eta,\sigma,b,v,\vartheta)\frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)}c_r z^{(r+\alpha)}.$$

This study focuses on recent work in fuzzy differential subordination that introduces new operators to construct and study a novel fuzzy class. Here, we discuss multiple findings related to fuzzy differential subordination connected to the Riemann-Liouville fractional integral from the linear combination of Pascal operator and Catas operator. Fuzzy differential subordinations have been obtained in order to identify the fuzzy best dominants. Specific functions are used to derive some corollaries of the primary findings. A few examples are provided to illustrate the main findings.

Previous studies [28, 29, 30, 31] served as inspiration for this work.

To support our primary findings, we shall use the following Lemmas.

Lemma 1.9. [17] *Let* $k \in \mathcal{A}$. *If* $\Re\{1 + \frac{zk''(z)}{k'(z)}\} > \frac{-1}{2}, z \in \mathscr{U}$, *then* $\frac{1}{z}\int_0^z k(t)dt$ *is convex function.*

Lemma 1.10. [23] *Let* h *be a convex function with* $h(0) = a$ *and* $\rho \in \mathbb{C}^*$ *such that* $\Re(\rho) \geq 0$. *If* $\mathcal{P} \in \mathcal{H}[a,n]$ *with* $\mathcal{P}(0) = a$ *and* $\Psi : \mathbb{C}^2 \times \mathscr{U} \to \mathbb{C}$, $\Psi(\mathcal{P}(z), z\mathcal{P}'(z))$ *is analytic in* $\mathscr{U}$, *then*

$$F_{\Psi(\mathbb{C}^2\times\mathscr{U})}\left[\mathcal{P}(z) + \frac{1}{\rho}z\mathcal{P}'(z)\right] \leq F_{h(\mathscr{U})}h(z),$$

implies

$$F_{\mathcal{P}(\mathscr{U})}\mathcal{P}(z) \leq F_{g(\mathscr{U})}g(z) \leq F_{h(\mathscr{U})}h(z)$$

with the convex function $g(z) = \frac{\rho}{nz^{\frac{\rho}{n}}}\int_0^z h(t)t^{\frac{\rho}{n}-1}dt, z \in \mathscr{U}$ *as the fuzzy best dominant.*

Lemma 1.11. [23] *Suppose that* g *be a convex function in* $\mathscr{U}$ *and* $h(z) = g(z) + n\lambda zg'(z), n \in \mathbb{N}, \lambda > 0$. *If* $\mathcal{P} \in \mathcal{H}[g(0), n]$ *and* $\Psi : \mathbb{C}^2 \times \mathscr{U} \to \mathbb{C}$, $\Psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \lambda z\mathcal{P}'(z)$ *is analytic in* $\mathscr{U}$, *then*

$$F_{\Psi(\mathbb{C}^2\times\mathscr{U})}\left[\mathcal{P}(z) + \lambda z\mathcal{P}'(z)\right] \leq F_{h(\mathscr{U})}h(z),$$

implies sharp result,

$$F_{\mathcal{P}(\mathscr{U})}\mathcal{P}(z) \leq F_{g(\mathscr{U})}g(z), \qquad z \in \mathscr{U}$$

and g *is fuzzy best dominant.*

We are going to define a new fuzzy class of univalent and analytic functions.

Definition 1.12. *If* $\mathfrak{t} \in \mathcal{A}$ *satisfies the following criteria, it is said to be in the class* $\mathcal{DNY}^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'(\mathscr{U})}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right)' > \varsigma;$$
$$(z\in\mathscr{U},\ \eta\geq 1, 0\leq\vartheta<1, n\in\mathbb{N}_0,\ b,v,\sigma\geq 0, \alpha>0, \varsigma\in[0,1)).$$

Remark 1.13. In particular, $t(z)=z\in\mathcal{A}$ with $F(z)=\frac{1}{1+|z|}$ belongs to the class $\mathcal{DNY}^F(n,\eta,\sigma,b,v,\vartheta,\alpha,0)$.

2. Main results

Theorem 2.1. *The class* $\mathcal{DNY}^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$ *is a convex set.*

Proof. Consider

$$\mathfrak{t}_j(z) = z + \sum_{r=2}^{\infty} c_{jr} z^r, \qquad j=1,2,$$

belongs to the class $\mathcal{DNY}^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$. We have to show that the function $h(z) = \beta_1\mathfrak{t}_1(z)+\beta_2\mathfrak{t}_2(z)$, $\beta_1,\beta_2\geq 0, \beta_1+\beta_2=1$, belongs to the class $\mathcal{DNY}^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$.

Now, $h'(z)=\beta_1\mathfrak{t}_1'(z)+\beta_2\mathfrak{t}_2'(z)$ and $\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h(z)}{z^\alpha}\right)'$
$=\beta_1\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1(z)}{z^\alpha}\right)' + \beta_2\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2(z)}{z^\alpha}\right)'.$

We have

$$\begin{aligned}
&F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h(z)}{z^\alpha}\right)'\\
&=F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}(\beta_1\mathfrak{t}_1(z)+\beta_2\mathfrak{t}_2))'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}(\beta_1\mathfrak{t}_1(z)+\beta_2\mathfrak{t}_2(z))}{z^\alpha}\right)'\\
&=F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}(\beta_1\mathfrak{t}_1+\beta_2\mathfrak{t}_2))'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\beta_1\mathfrak{t}_1(z)}{z^\alpha}\right)'\\
&\quad+F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}(\beta_1\mathfrak{t}_1+\beta_2\mathfrak{t}_2))'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\beta_2\mathfrak{t}_2(z)}{z^\alpha}\right)'\\
&=\frac{F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1(z)}{z^\alpha}\right)'+F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2(z)}{z^\alpha}\right)'}{2}.
\end{aligned}$$

As $\mathfrak{t}_1,\mathfrak{t}_2\in\mathcal{DNY}^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$, we have

$$\varsigma < F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1(z)}{z^\alpha}\right)' \leq 1,$$

$$\varsigma < F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2\right)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2(z)}{z^\alpha}\right)' \le 1.$$

This implies,

$$\varsigma < \frac{F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1\right)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1(z)}{z^\alpha}\right)' + F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2\right)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2(z)}{z^\alpha}\right)'}{2}$$
$$\le 1$$

i.e.

$$\varsigma < F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h\right)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h(z)}{z^\alpha}\right)' \le 1.$$

□

Theorem 2.2. *Considering g as a convex function and $h(z) = g(z) + \frac{1}{c+2}zg'(z), c > 0$. If $\mathfrak{t} \in \mathcal{D}\mathcal{N}Y^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$ and $G(z) = \frac{c+2}{z^{c+1}}\int_0^z t^c\mathfrak{t}(t)dt$, then*

$$F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}\right)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right)' \le F_{h(\mathscr{U})}h(z) \tag{2.1}$$

implies the next sharp result

$$F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G\right)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)' \le F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. Let $G(z) = \dfrac{c+2}{z^{c+1}}\displaystyle\int_0^z t^c\mathfrak{t}(t)dt$.
Differentiating w.r.t. z, we get

$$(c+1)G(z) + zG'(z) = (c+2)\mathfrak{t}(z),$$

$$(c+1)\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right) + z\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)'$$
$$= (c+2)\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right).$$

Again differentiating w.r.t z, we obtain

$$\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)' + \frac{1}{c+2}z\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)''$$
$$= \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right)'.$$

Now, the Inequality (2.1) becomes

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left[\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^{\alpha}}\right)'+\frac{1}{c+2}z\right.$$
$$\left.\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^{\alpha}}\right)''\right]\leq F_{g(\mathscr{U})}\left[g(z)+\frac{1}{c+2}zg'(z)\right].$$

Consider, $p(z)=F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^{\alpha}}\right)'$.

Here, $p\in\mathcal{H}[1,1]$ and we obtain

$$F_{p(\mathscr{U})}\left[p(z)+\tfrac{1}{c+2}zp'(z)\right]\leq F_{g(\mathscr{U})}\left[g(z)+\tfrac{1}{c+2}zg'(z)\right].$$

Employing Lemma 1.3, we have

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^{\alpha}}\right)'\leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

Theorem 2.3. *Consider that* $h(z)=\frac{1+(2\varsigma-1)z}{1+z}$ *and* $G(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c\mathfrak{t}(t)dt$, $\varsigma\in[0,1), c>0$ *then*

$$G[\mathcal{DN}Y^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)]\subset\mathcal{DN}Y^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma^*),$$

where $\varsigma^*=(2\varsigma-1)+2(c+2)(1-\varsigma)\int_0^1\frac{t^{c+1}}{t+1}dt$.

Proof. Given that $h(z)=\frac{1+(2\varsigma-1)z}{1+z}$ is convex function and following the same steps from Theorem (2.2), we conclude the following fuzzy differential subordination

$$F_{p(\mathscr{U})}\left[p(z)+\tfrac{1}{c+2}zp'(z)\right]\leq F_{h(\mathscr{U})}h(z)$$

where, $p(z)=\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^{\alpha}}\right)'$.

From Lemma 1.10, we may conclude that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^{\alpha}}\right)'\leq F_{g(\mathscr{U})}g(z)\leq F_{h(\mathscr{U})}h(z),$$

where

$$g(z)=\frac{c+2}{z^{c+2}}\int_0^z t^{c+1}\left[\frac{1+(2\varsigma-1)t}{1+t}\right]dt=(2\varsigma-1)+\frac{(c+2)(2-2\varsigma)}{z^{c+2}}\int_0^z\frac{t^{c+1}}{t+1}dt.$$

Since, g is convex function and $g(\mathscr{U})$ is symmetric with respect to real axis, we obtain

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^{\alpha}}\right)'\geq\min_{|z|=1}F_{g(\mathscr{U})}g(z)=F_{g(\mathscr{U})}g(1)$$

and $\varsigma^* = g(1) = 2\varsigma - 1 + (c+2)(2-2\varsigma)\int_0^1 \frac{t^{c+1}}{t+1}dt.$ □

Theorem 2.4. *Let's take g be a convex function such that $g(0) = 1$ and $h(z) = g(z) + zg'(z)$. If $\mathfrak{t} \in \mathcal{A}$, the fuzzy differential subordination is satisfied*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)' \leq F_{h(\mathscr{U})}h(z) \tag{2.2}$$

implies the sharp result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. The function $p(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)$ belongs to $\mathcal{H}[1,1]$. Furthermore, we may write

$$zp(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right).$$

Now, differentiating w.r.t. z, we have

$$p(z) + zp'(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'.$$

The Inequality (2.2), becomes

$$F_{p(\mathscr{U})}[p(z) + zp'(z)] \leq F_{h(\mathscr{U})}h(z).$$

Lemma 1.11 is applied, and we find that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

Example 2.5. Take $g(z) = \frac{1-z}{1+z}$ ans is convex in $\mathscr{U}$, with $g(0) = 1$, $g'(z) = \frac{-2}{(1+z)^2}$.

Now $h(z) = g(z) + zg'(z) = \frac{1 - z^2 - 2z}{(1+z)^2}$.

Taking $n = 0, \sigma = 0, \eta = 1$ and $\mathfrak{t}(z) = z + z^2$, then we find that $\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = z + z^2$.

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{(z-t)^{1-\alpha}}dt = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{z+z^2}{(z-t)^{1-\theta}}dt$$
$$= \frac{z^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{2z^{2+\alpha}}{\Gamma(3+\alpha)}.$$

Implies,

$$\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^\alpha} = z + \frac{2z^2}{(2+\alpha)}.$$

After differentiation, we get

$$\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^\alpha}\right)' = 1 + \frac{4z}{(2+\alpha)}.$$

Using Theorem 2.4 now, we can derive that the fuzzy subordination that follows

$$1 + \frac{4z}{2+\alpha} \prec_F \frac{1-z^2-2z}{(1+z)^2}$$

implies that

$$1 + \frac{2z}{2+\alpha} \prec_F \frac{1-z}{1+z}.$$

Theorem 2.6. *Let h be a analytic in $\mathscr{U}$ with $h(0)=1$ and $\Re\left(1+\frac{zh''(z)}{h'(z)}\right) > \frac{-1}{2}$. If $\mathfrak{t} \in \mathcal{A}$, the fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right)' \leq F_{h(\mathscr{U})}h(z) \tag{2.3}$$

implies that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \leq F_{q(\mathscr{U})}q(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and it is fuzzy best dominant.

Proof. Given that $\Re\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, z \in \mathscr{U}$, and from Lemma 1.9, we find that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex function and it is solution of Fuzzy differential subordination (2.3), $h(z) = q(z) + zq'(z)$, so it is fuzzy best dominant.
Let $zp(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\theta}\right)$
Differentiating w.r.t z, we get

$$p(z) + zp'(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right)'$$

The fuzzy differential subordination (2.3) is transformed into

$$F_{p(\mathscr{U})}[p(z) + zp'(z)] \leq F_{h(\mathscr{U})}h(z).$$

Using Lemma 1.11, we find that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \leq F_{q(\mathscr{U})}q(z).$$

$\square$

Corollary 2.7. *Assuming that* $h(z)=\frac{1+(2\xi-1)z}{1+z}, \xi\in[0,1)$ *is convex function in* $\mathscr{U}$*. If* $\mathfrak{t}\in\mathcal{A}$*, the following fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'\leq F_{h(\mathscr{U})}h(z) \tag{2.4}$$

implies that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)\leq F_{q(\mathscr{U})}q(z),$$

where $q(z)=(2\xi-1)+2(1-\xi)\dfrac{\ln(1+z)}{z}$ *is convex and fuzzy best dominant.*

Proof. Given $h(z)=\frac{1+(2\xi-1)z}{1+z}$ with $h(0)=1, h'(z)=\frac{2(\beta-1)}{(1+z)^2}, h''(z)=\frac{-4(\xi-1)}{(1+z)^3}$.
Consider
$\Re\left(1+\frac{zh''(z)}{h'(z)}\right)=\Re\left(\frac{1-z}{1+z}\right)=\Re\left(\frac{1-r\cos\phi-ir\sin\phi}{1+r\cos\phi+ir\sin\phi}\right)=\frac{1-r^2}{1+2r\cos\phi+r^2}>0>-\frac{1}{2}$.
Following the same steps from Theorem 2.6 with $p(z)=\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)$, the Inequality (2.4) becomes

$$F_{p(\mathscr{U})}[p(z)+zp'(z)]\leq F_{h(\mathscr{U})}h(z).$$

Employing Lemma 1.10 with $n=\rho=1$, we deduce that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)\leq F_{q(\mathscr{U})}q(z),$$

where $q(z)=\dfrac{1}{z}\displaystyle\int_0^z\frac{1+(2\xi-1)t}{1+t}dt=(2\xi-1)+2(1-\xi)\frac{\ln(1+z)}{z}$. $\square$

Example 2.8. Consider $h(z)=\dfrac{1-z}{1+z}$ is convex in $\mathscr{U}$.
Taking $n=0,\sigma=0,\eta=1$ and $\mathfrak{t}(z)=z+z^2$, then we find that $\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)=z+z^2$.

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)=\frac{1}{\Gamma(\alpha)}\int_0^z\frac{\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{(z-t)^{1-\alpha}}dt=\frac{1}{\Gamma(\alpha)}\int_0^z\frac{z+z^2}{(z-t)^{1-\theta}}dt$$
$$=\frac{z^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{2z^{2+\alpha}}{\Gamma(3+\alpha)}.$$

Hence,

$$\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}}=z+\frac{2z^2}{(2+\alpha)}.$$

After differentiation, we get

$$\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}}\right)'=1+\frac{4z}{(2+\alpha)}.$$

Also, $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = \frac{2\ln(1+z)}{z} - 1.$
Utilizing Theorem 2.6, we now possess the fuzzy differential subordination

$$1 + \frac{4z}{2+\alpha} \prec_F \frac{1-z}{1+z}$$

implies the result

$$1 + \frac{2z}{2+\alpha} \prec_F \frac{2\ln(1+z)}{z} - 1.$$

Theorem 2.9. *Letting g be a convex function and consider that $g(0) = 1$. If $\mathfrak{t} \in \mathcal{A}$, the fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left[\frac{z\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right]' \leq F_{h(\mathscr{U})}h(z) \tag{2.5}$$

implies the sharp result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right] \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. Suppose $p(z) = \left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right].$
Differentiating w.r.t. z, we have the relation

$$p(z) + zp'(z) = \left[\frac{z\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right]'.$$

Consequently, fuzzy differential subordination (2.5) turns into

$$F_{p(\mathscr{U})}[p(z) + zp'(z)] \leq F_{h(\mathscr{U})}h(z) = F_{g(\mathscr{U})}[g(z) + zg'(z)].$$

Now, applying Lemma 1.11, we have

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right] \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

Theorem 2.10. *Let g be a convex function and consider that $g(0) = 1$ and $h(z) = g(z) + \gamma z g'(z), \gamma, \lambda > 0$. If $\mathfrak{t} \in \mathcal{A}$ and the fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda-1}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'\right] \leq F_{h(\mathscr{U})}h(z) \tag{2.6}$$

implies the following sharp result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda} \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. Let $p(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda}$ belongs to $\mathcal{H}[1,1]$.
Differentiating w.r.t. z, we obtain

$$p'(z) = \lambda\left[\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right]^{\lambda-1}\left[\frac{\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)}{z}\right]'.$$

Following a little computation, we have

$$p(z)+\frac{1}{\lambda}zp'(z) = \left[\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda-1}\left(\frac{\Gamma(2+\theta)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'\right].$$

Therefore, fuzzy differential subordination (2.6), becomes

$$F_{p(\mathscr{U})}[p(z)+\frac{1}{\lambda}zp'(z)] \leq F_{h(\mathscr{U})}h(z) = F_{g(\mathscr{U})}[g(z)+\gamma zg'(z)].$$

Applying Lemma 1.11, we obtain that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda} \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

Example 2.11. Suppose $g(z) = \frac{1-z}{1+z}$ and $h(z) = g(z)+zg'(z) = \frac{1-2z-z^2}{(1+z)^2}$.
Take $n=0, \sigma=0, \eta=1$ and $\mathfrak{t}(z) = z+z^2$, then we find that $\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = z+z^2$.

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{(z-t)^{1-\alpha}}dt = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{z+z^2}{(z-t)^{1-\theta}}dt$$
$$= \frac{z^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{2z^{2+\alpha}}{\Gamma(3+\alpha)}.$$

Thus, we have

$$\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}} = z + \frac{2z^2}{(2+\alpha)}.$$

After differentiation, we get

$$\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}}\right)' = 1 + \frac{4z}{(2+\alpha)}.$$

The following fuzzy differential subordination is obtained by using Theorem 2.10

$$\left(1+\frac{2z}{2+\alpha}\right)^{\lambda-1}\left(1+\frac{4z}{2+\alpha}\right)' \prec_F \frac{1-2z-z^2}{(1+z)^2}$$

implies that

$$\left(1+\frac{2z}{2+\alpha}\right)^{\lambda} \prec_F \frac{1-z}{1+z}.$$

Theorem 2.12. *Considering h as a convex function with $h(0)=1, \lambda>0$. If $\mathfrak{t}\in\mathcal{A}$, the fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda-1}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'\right] \le F_{h(\mathscr{U})}h(z) \tag{2.7}$$

implies the result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda} \le F_{g(\mathscr{U})}g(z),$$

where $g(z)=\frac{1}{z}\int_0^z h(t)dt$ is convex and fuzzy best dominant.

Proof. Following the same technique of Theorem 2.10 and taking $p(z)=\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda}$, we have

$$F_{p(\mathscr{U})}\left[p(z)+\frac{1}{\lambda}zp'(z)\right] \le F_{h(\mathscr{U})}h(z).$$

Using Lemma 1.10, we deduce that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda} \le F_{g(\mathscr{U})}g(z),$$

where $g(z)=\frac{1}{z}\int_0^z h(t)dt$ is convex and fuzzy best dominant. □

Example 2.13. Considering $h(z)=\dfrac{1-z}{1+z}$ with $h(0)=1$ and it is convex function in $\mathscr{U}$.

Take $n=0, \sigma=0, \eta=1$ and $\mathfrak{t}(z)=z+z^2$, then we obtain $\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)=z+z^2$, then we find that

$$\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)=z+z^2.$$

Now,

$$\mathcal{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)=\frac{z^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{2z^{2+\alpha}}{\Gamma(3+\alpha)}.$$

Thus,

$$\frac{\Gamma(2+\alpha)\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}} = z + \frac{2z^2}{(2+\alpha)}.$$

After differentiation, we obtain

$$\left(\frac{\Gamma(2+\alpha)\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}}\right)' = 1 + \frac{4z}{(2+\alpha)}.$$

Additionally,

$$g(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{2\ln(1+z)}{z} - 1.$$

We have the fuzzy differential subordination described below using Theorem 2.12

$$\left(1 + \frac{2z}{2+\alpha}\right)^{\lambda-1}\left(1 + \frac{4z}{2+\alpha}\right) \prec_F \frac{1-z}{1+z}$$

implies the result

$$\left(1 + \frac{2z}{2+\alpha}\right)^{\lambda} \prec_F \frac{2\ln(1+z)}{z} - 1.$$

Theorem 2.14. *Let g is a convex function with $g(0) = 1$ and $h(z) = g(z) + zg'(z)$. If $\mathfrak{t} \in \mathcal{A}$, the fuzzy differential subordination*

$$F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}\right)\mathscr{U}}\left[1 - \frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)''}{\left[\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'\right]^2}\right] \leq F_{h(\mathscr{U})}h(z), \quad (2.8)$$

implies sharp result,

$$F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}\right)\mathscr{U}}\left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'}\right] \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. Suppose $p(z) = \dfrac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'}$ belongs to $\mathcal{H}[1,1]$ and $z \in \mathscr{U}$.
Differentiating w.r.t. z, we have the relation

$$p(z) + zp'(z) = 1 - \frac{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)''}{\left[\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'\right]^2}.$$

Inequality (2.8), becomes

$$F_{p(\mathscr{U})}[p(z) + zp'(z)] \leq F_{h(\mathscr{U})}h(z),$$

We now get the sharp fuzzy differential subordination using Lemma 1.11,

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'}\right]\leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

3. Conclusion

At this point, we discussed a number of fuzzy differential subordination results of analytic functions that are connected to the Riemann-Liouville fractional integral and the linear combination of the Pascal and Catas operator. A new fuzzy class was also developed, and fuzzy differential subordination results and a few examples were inferred.

With reference to this operator, further subclasses of analytic functions can be created, and some of their features, including coefficient estimates, distortion theorems, and closure theorems, can be examined. Also, New fuzzy class identification, fuzzy superordination results, higher-dimensional results extension, and the use of fuzzy differential subordination to address practical issues are important topics for development.

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On the coefficient estimates for a subclass of m-fold symmetric bi-univalent functions

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Abstract. In this work, we introduce and investigate a subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ of analytic and bi-univalent functions when both f and f^{-1} are m-fold symmetric in the open unit disk $\mathbb{U}$. Moreover, we find upper bounds for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to this subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\lambda,\gamma)$. The results presented in this paper would generalize and improve those that were given in several recent works.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the following normalized form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Also, we denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. The *Koebe One-Quarter Theorem* [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains

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a disk of radius $\frac{1}{4}$. Hence, every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$. The class consisting of bi-univalent functions are denoted by Σ.

Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient a_2 of functions $f \in \mathcal{S}$ gives the growth and distortion bounds as well as covering theorems.

Lewin [8] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [7] proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Tan [14] obtained the bound for $|a_2|$ namely $|a_2| \leq 1.485$ which is the best known estimate for functions in the class Σ. Recently there are interest to study the bi-univalent functions class Σ (see [5, 6, 16, 17]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{1, 2\})$ for each $f \in \Sigma$ given by (1.1) is still an open problem. For each function $f \in \mathcal{S}$ the function $h(z)$ given by

$$h(z) = \sqrt[m]{f(z^m)} \qquad (z \in \mathbb{U}, m \in \mathbb{N})$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is called m-fold symmetric (see[11, 12, 13]) if the function $f(z)$ has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \qquad (z \in \mathbb{U}, m \in \mathbb{N}) \tag{1.3}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$, which are normalized by the series expansion (1.3). In fact, the functions in the class $\mathcal{S}$ are one-fold symmetric, that is

$$\mathcal{S}_1 = \mathcal{S}$$

Analogous to the concept of m-fold symmetric univalent functions, we now introduce the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer$f \in N$. The normalized form of f is given as in (1.3). Furthermore, the series expansion for f^{-1} , which was recently proven by Srivastava et al. [13], is given as follows:

$$g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} -$$
$$[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \cdots,$$

where $g = f^{-1}$.
We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. In the special case when $m = 1$, the formula (1.4) for the class Σ_m coincides with the formula (1.2) for the class Σ. Some examples of m-fold symmetric bi-univalent functions are given below:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}} \quad \text{and} \quad [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions given by

$$\left(\frac{w^m}{1-w^m}\right)^{\frac{1}{m}} \quad \text{and} \quad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}}$$

respectively.

Quite recently, Wanas and Páll-Szabó [15] introduced two new general subclasses $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ and $\mathcal{AS}^*_{\Sigma_m}(\gamma, \lambda; \beta)$ of the m-fold symmetric bi-univalent function class Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in $\mathbb{U}$ and derived the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

Definition 1.1. [15] *A function $f \in \Sigma_m$ given by* (1.3) *is said to be in the class $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ if it satisfies the following conditions:*

$$\left|arg\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\gamma}\right| < \frac{\alpha\pi}{2}$$

and

$$\left|arg\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^{\gamma}\right| < \frac{\alpha\pi}{2},$$

where $z, w \in \mathbb{U},\ 0 < \gamma \le 1,\ 0 \le \lambda \le 1,\ 0 < \alpha \le 1,\ m \in \mathbb{N}$ *and* $g = f^{-1}$.

Theorem 1.2 ([15]). *Let $f \in \mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ be given by* (1.3)*. Then*

$$|a_{m+1}| \le \frac{2\alpha}{m\sqrt{2\alpha\gamma(1+\lambda m) + \gamma(\gamma-\alpha)(1+\lambda m)^2}}$$

and

$$|a_{2m+1}| \le \frac{2\alpha^2(m+1)}{m^2\gamma^2(1+\lambda m)^2} + \frac{\alpha}{m\gamma(1+2\lambda m)}.$$

Definition 1.3. [15] *A function $f \in \Sigma_m$ given by* (1.3) *is said to be in the class ${\mathcal{AS}^*}_{\Sigma_m}(\gamma, \lambda; \beta)$, if it satisfies the following conditions:*

$$\Re\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\gamma} > \beta$$

and

$$\Re\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\gamma}>\beta,$$

where $z, w \in \mathbb{U},\ 0<\gamma\leq 1,\ 0\leq\lambda\leq 1,\ 0\leq\beta<1,\ m\in\mathbb{N}$ *and* $g=f^{-1}$.

Theorem 1.4 (**[15]**). *Let* $f\in\mathcal{AS}^{*}{}_{\Sigma_m}(\gamma,\lambda;\beta)$ *be given by* (1.3). *Then*

$$|a_{m+1}|\leq\frac{2}{m}\sqrt{\frac{1-\beta}{2\gamma(1+\lambda m)+\gamma(\gamma-1)(1+\lambda m)^2}}$$

and

$$|a_{2m+1}|\leq\frac{2(m+1)(1-\beta)^2}{m^2\gamma^2(1+\lambda m)^2}+\frac{1-\beta}{m\gamma(1+2\lambda m)}.$$

The main objective of this paper is to present an elegant formula for computing the coefficients of the inverse functions for the class Σ_m of m-fold symmetric functions by means of the residue calculus. As an application, we introduce a new subclass of bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and obtain upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this subclass. Our results for the bi-univalent function class $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$, which we shall introduce in section 2, would generalize and improve some recent works by Wanas and Páll-Szabó [15] and some of other researchers[1, 9, 10]

2. Coefficient Estimates

In this section, we introduce and investigate the general subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$.

Definition 2.1. *Let* $h,p:\mathbb{U}\to\mathbb{C}$ *be analytic functions and*

$$\min\{\Re(h(z)),\Re(p(z))\}>0\ (z\in\mathbb{U})\ \textit{and}\ h(0)=p(0)=1.$$

A function f *given by* (1.3) *is said to be in the class* $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$, *if the following conditions are satisfied:*

$$\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\gamma}\in h(\mathbb{U}) \tag{2.1}$$

and

$$\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''w)}{g'(w)}\right)\right]^{\gamma}\in p(\mathbb{U}) \tag{2.2}$$

where $z,w\in\mathbb{U}$, $0<\gamma\leq 1$, $0\leq\lambda\leq 1$, $m\in\mathbb{N}$ *and* $g=f^{-1}$.

Remark 2.2. There are many choices of the functions h,p which would provide interesting subclasses of the general class $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$. For example, if we set $\gamma=1$, the subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$ reduces to the subclass $f\in\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,1)$ which was introduced by Motamednezhad et al. [10].

If we let

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots (0 < \alpha \leq 1),$$

it can easily be verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. Thus, if we have $f \in \mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$, then

$$\left| arg \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^\gamma \right| < \frac{\alpha\pi}{2}$$

and

$$\left| arg \left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda \left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^\gamma \right| < \frac{\alpha\pi}{2}.$$

In this case we say that f belongs to the subclass $f \in \mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$. If we put $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha$ and $\gamma = 1$, the subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$ reduces to the subclass $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, 1)$ which was considered by Motamednezad et al. [10].

Also, for $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha$, $\gamma = 1$ and $\lambda = 0$, the subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}^{\alpha}$ which was considered by Altinkaya and Yalcin [1].

On the other hand, if we take

$$h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots (0 \leq \beta < 1).$$

then the conditions of Definition 2.1 are satisfied for both functions $h(z)$ and $p(z)$. Thus, if $f \in \mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$; then

$$\Re \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^\gamma > \beta$$

and

$$\Re \left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda \left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^\gamma > \beta.$$

In this case we say that f belonges to the subclass $f \in \mathcal{AS^*}_{\Sigma_m}(\gamma, \lambda; \beta)$. If we put $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$ and $\gamma = 1$, the subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$ reduces to the subclass $\mathcal{M}_{\Sigma_m}(\beta, \lambda, 1)$ which was considered by Motamednezad et al. [10].

Also, for $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma = 1$ and $\lambda = 0$, the subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}^{\beta}$ which was considered by Altinkaya and Yalcin [1].

Remark 2.3. For one-Fold symmetric bi-univalent functions, we denote the subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda) = \mathcal{G}_{\Sigma}^{h,p}(\gamma, \lambda)$. Special cases of this subclass illustrated below:

(A) By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ and $\gamma = 1$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $M_\Sigma(\alpha, \lambda)$ studied by Li and Wang [9].

(B) By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $S_\Sigma^*(\alpha)$ studied by Brannan and Taha[3].

(C) By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, and $\lambda = \gamma = 1$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $M_{\Sigma}(\alpha, 1)$ studied by Li and Wang [9].

(D) By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ and $\gamma = 1$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $B_{\Sigma}(\beta, \lambda)$ studied by Li and Wang [9].

(E) By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$, $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $S_{\Sigma}^{*}(\beta)$ of bi-starlike functions of order $\beta(0 \leq \beta < 1)$ studied by Brannan and Taha[3].

(F) By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ and $\lambda = \gamma = 1$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $B_{\Sigma}(\beta, 1)$ of bi-convex functions of order $\beta(0 \leq \beta < 1)$ studied by Li and Wang [9].

We are now ready to express the bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for the subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$ of the normalized bi-univalent function class Σ_m.

Theorem 2.4. *Let the function f given by (1.3) be in the class $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma, \lambda)$. Then*

$$|a_{m+1}| \leq min\left\{\sqrt{\frac{|h_{2m}| + |p_{2m}|}{m^2\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}}, \sqrt{\frac{|h_m|^2 + |p_m|^2}{2[m\gamma(1+\lambda m)]^2}}\right\} \tag{2.3}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{|h_{2m}| + |p_{2m}|}{4\gamma m(1+2\lambda m)} + \frac{(m+1)(|h_m|^2 + |p_m|^2)}{4\gamma^2 m^2(1+\lambda m)^2}, \frac{\left|2m(1+\lambda m) + 2(m+1)(1+2\lambda m) + m(\gamma-1)(1+\lambda m)^2\right|}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}|h_{2m}| + \frac{\left|2(m+1)(1+2\lambda m) - 2m(1+\lambda m) - m(\gamma-1)(1+\lambda m)^2\right|}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}|p_{2m}|\right\}. \tag{2.4}$$

Proof. The main idea in the proof of Theorem 2.4 is to get the desired bounds for the coefficient $|a_{m+1}|$ and $|a_{2m+1}|$. Indeed, by considering the relations (2.1) and (2.2), we have

$$\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\gamma} = h(z) \tag{2.5}$$

and

$$\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^{\gamma} = p(z), \tag{2.6}$$

where each of the functions h and p satisfies the conditions of Definition 1.3. In light of the following Taylor-Maclaurin series expansions for the functions h and p, we get

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots . \tag{2.8}$$

By substituting the relations (2.7) and (2.8) into (2.5) and (2.6), respectively, we get

$$m\gamma(1+\lambda m)a_{m+1} = h_m, \tag{2.9}$$

$$\gamma m \left[\frac{(\gamma-1)}{2} m(1+\lambda m)^2 - (\lambda m^2 + 2\lambda m + 1) \right] a_{m+1}^2 + 2m\gamma(1+2\lambda m)a_{2m+1} = h_{2m}, \tag{2.10}$$

$$-m\gamma(1+\lambda m)a_{m+1} = p_m \tag{2.11}$$

and

$$\gamma m \left[\left(3\lambda m^2 + 2(\lambda+1)m + 1\right) + \frac{(\gamma-1)}{2} m(1+\lambda m)^2 \right] a_{m+1}^2 - 2m\gamma(1+2\lambda m)a_{2m+1} = p_{2m}. \tag{2.12}$$

Comparing the coefficients (2.9) and (2.11), we obtain

$$h_m = -p_m \tag{2.13}$$

and

$$2m^2\gamma^2(1+\lambda m)^2 a_{m+1}^2 = h_m^2 + p_m^2. \tag{2.14}$$

Now, if we add (2.10) and (2.12), we get the following relation

$$m^2\gamma \left[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2 \right] a_{m+1}^2 = h_{2m} + p_{2m}. \tag{2.15}$$

Therefore, from (2.14) and (2.15), we have

$$a_{m+1}^2 = \frac{h_m^2 + p_m^2}{2[m\gamma(1+\lambda m)]^2} \tag{2.16}$$

and

$$a_{m+1}^2 = \frac{h_{2m} + p_{2m}}{m^2\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}, \tag{2.17}$$

respectively.

Therefore, we find from the equations (2.16) and (2.17) that

$$|a_{m+1}|^2 \leq \frac{|h_m|^2 + |p_m|^2}{2\gamma^2 m^2 (1+\lambda m)^2}$$

and

$$|a_{m+1}|^2 \leq \frac{|h_{2m}| + |p_{2m}|}{m^2\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]},$$

respectively. We have thus derived the desired bound on the coefficient $|a_{m+1}|$.

The proof is completed by finding the bound on the coefficient $|a_{2m+1}|$. Upon subtracting (2.12) from (2.10), we get

$$a_{2m+1} = \frac{h_{2m} - p_{2m}}{4\gamma m(1+2\lambda m)} + \frac{(m+1)}{2} a_{m+1}^2. \tag{2.18}$$

Putting the value of a_{m+1}^2 from (2.16) into (2.18), it follows that

$$a_{2m+1} = \frac{h_{2m} - p_{2m}}{4\gamma m(1+2\lambda m)} + \frac{(m+1)h_{2m} + p_{2m}}{m^2\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}.$$

Therefore, we conclude the following bound:

$$|a_{2m+1}| \leq \frac{|h_{2m}| + |p_{2m}|}{4\gamma m(1+2\lambda m)} + \frac{(m+1)(|h_m|^2 + |p_m|^2)}{4[\gamma m(1+\lambda m)]^2}. \tag{2.19}$$

By substituting the value of a_{m+1}^2 from (2.17) into (2.18), we obtain

$$a_{2m+1} = \frac{m[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2](h_{2m} - p_{2m}) + (m+1)(1+2\lambda m)}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]} \tag{2.20}$$

$$\frac{(h_{2m} + p_{2m})}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}$$

which readily yields

$$|a_{2m+1}| \leq \frac{\left|2m(1+\lambda m) + 2(m+1)(1+2\lambda m) + m(\gamma-1)(1+\lambda m)^2\right|}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}|h_{2m}| + \tag{2.21}$$

$$\frac{\left|2(m+1)(1+2\lambda m) - 2m(1+\lambda m) - m(\gamma-1)(1+\lambda m)^2\right|}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}|p_{2m|}. \tag{2.22}$$

Finally, from (2.19) and (2.21), we get the desired estimate on the coefficient $|a_{2m+1}|$ as asserted in Theorem 2.4. The proof of Theorem 2.4 is thus completed. □

3. Corollaries and Consequences

If we put

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots,$$

in Theorem 2.4, then it can be obtained the following result.

Corollary 3.1. *Let the function f given by* (1.3) *be in the class $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$. Then*

$$|a_{m+1}| \leq min\left\{\frac{2\alpha}{m\gamma(1+\lambda m)}, \frac{2\alpha}{m\sqrt{\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}}\right\}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{\alpha^2}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2},\right.$$
$$\left|\frac{2m(1+\lambda m)+2(m+1)(1+2\lambda m)+m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}\right|\alpha^2 +$$
$$\left.\left|\frac{2(m+1)(1+2\lambda m)-2m(1+\lambda m)-m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}\right|\alpha^2\right\}.$$

Remark 3.2. For the coefficient $|a_{2m+1}|$ it is easily seen that

$$\frac{\alpha^2}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2} \leq \frac{\alpha}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2}.$$

Therefore, clearly, Corollary3.1 provides an improvement over Theorem1.2.

By setting $\gamma = 1$ in Corollary 3.1, we conclude the following result.

Corollary 3.3. *Let the function f given by* (1.3) *be in the subclass* $\mathcal{M}_{\Sigma_m}(\alpha,\lambda,1)$. *Then*

$$|a_{m+1}| \leq min\left\{\frac{2\alpha}{m(1+\lambda m)}, \frac{2\alpha}{m\sqrt{2(1+\lambda m)}}\right\} = \begin{cases} \frac{2\alpha}{m\sqrt{2(1+\lambda m)}}, & 0 \leq \lambda \leq \frac{1}{m} \\ \frac{2\alpha}{m(1+\lambda m)}, & \frac{1}{m} \leq \lambda < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{m(1+\lambda m)^2+2(m+1)(1+2\lambda m)}{m^2(1+2\lambda m)(1+\lambda m)^2}\alpha^2, \frac{(m+1)}{m^2(1+\lambda m)}\alpha^2\right\}.$$

By setting $\lambda = 0$ in Corollary 3.3, we conclude the following result.

Corollary 3.4. *Let the function f given by* (1.3) *be in the subclass* $\mathcal{S}^{\alpha}_{\Sigma_m}$. *Then*

$$|a_{m+1}| \leq min\left\{\frac{2\alpha}{m}, \frac{\sqrt{2}\alpha}{m}\right\} = \frac{\sqrt{2}\alpha}{m}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{(3m+2)\alpha^2}{m^2}, \frac{(m+1)\alpha^2}{m^2}\right\} = \frac{(m+1)\alpha^2}{m^2}.$$

Remark 3.5. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.4 are better than those given by Altinkaya and Yalcin [1, Corollary 6], because of

$$\frac{\sqrt{2}\alpha}{m} \leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$\frac{(m+1)\alpha^2}{m^2} \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}.$$

By setting $\gamma = 1$ and $m = 1$ in Corollary 3.1, we conclude the following result.

Corollary 3.6. *Let the function f given by (1.1) be in the subclass $\mathcal{M}_\Sigma(\alpha,\lambda)$. Then*

$$|a_2| \le min\left\{\frac{2\alpha}{1+\lambda}, \alpha\sqrt{\frac{2}{1+\lambda}}\right\} = \alpha\sqrt{\frac{2}{1+\lambda}}$$

and

$$|a_3| \le min\left\{\frac{\lambda^2+10\lambda+5}{(1+2\lambda)(1+\lambda)^2}\alpha^2, \frac{2\alpha^2}{1+\lambda}\right\} = \frac{2\alpha^2}{1+\lambda}.$$

Remark 3.7. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.6 are better than those given by Li and Wang [9, Theorem 2.2].

By setting $\lambda = 0$ in Corollary 3.6, we conclude the following result.

Corollary 3.8. *Let the function f given by (1.1) be in the subclass $S^*_\Sigma(\alpha)$. Then*

$$|a_2| \le \sqrt{2}\alpha \qquad and \qquad |a_3| \le 2\alpha^2.$$

Remark 3.9. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.10 are better than those given by Brannan and Taha [3].

By setting $\lambda = 1$ in Corollary 3.6, we conclude the following result.

Corollary 3.10. *Let the function f given by (1.1) be in the subclass $\mathcal{M}_\Sigma(\alpha,1)$. Then*

$$|a_2| \le \alpha \qquad and \qquad |a_3| \le \alpha^2.$$

Remark 3.11. The bound on $|a_3|$ given in Corollary 3.8 are better than those given by Li and Wang [9, Theorem 2.2] for $\lambda = 1$.

By letting

$$h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots (0 \le \beta < 1).$$

in Theorem 2.4, we deduce the following corollary.

Corollary 3.12. *Let the function f given by (1.3) be in the class $f \in \mathcal{AS^*}_{\Sigma_m}(\gamma,\lambda;\beta)$. Then*

$$|a_{m+1}| \le min\left\{\frac{2(1-\beta)}{m\gamma(1+\lambda m)}, \frac{2}{m}\sqrt{\frac{(1-\beta)}{\gamma[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}}\right\}$$

and

$$|a_{2m+1}| \le min\left\{\frac{1-\beta}{m\gamma(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{\gamma^2 m^2(1+\lambda m)^2},\right.$$

$$\left|\frac{2m(1+\lambda m)+2(m+1)(1+2\lambda m)+m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}\right|(1-\beta)+$$

$$\left.\left|\frac{2(m+1)(1+2\lambda m)-2m(1+\lambda m)-m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}\right|(1-\beta)\right\}.$$

Remark 3.13. Clearly, Corollary 3.12 provides an improvement over Theorem1.4.

By setting $\gamma = 1$ in Corollary 3.12, we conclude the following result.

Corollary 3.14. *Let the function f given by* (1.3) *be in the subclass $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, 1)$. Then*

$$|a_{m+1}| \leq min\left\{ \frac{2(1-\beta)}{m(1+\lambda m)}, \frac{2}{m}\sqrt{\frac{(1-\beta)}{2(1+\lambda m)}} \right\}$$

and

$$|a_{2m+1}| \leq min\left\{ \frac{1-\beta}{m(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{m^2(1+\lambda m)^2}, \frac{m+1}{m^2(1+\lambda m)}(1-\beta) \right\}.$$

By setting $\lambda = 0$ in Corollary 3.14, we conclude the following result.

Corollary 3.15. *Let the function f given by* (1.3) *be in the subclass $S^{\beta}_{\Sigma_m}$. Then*

$$|a_{m+1}| \leq min\left\{ \frac{2(1-\beta)}{m}, \frac{\sqrt{2(1-\beta)}}{m} \right\} = \begin{cases} \frac{\sqrt{2(1-\beta)}}{m}, & 0 \leq \beta \leq \frac{1}{2} \\ \frac{2(1-\beta)}{m}, & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq min\left\{ \frac{m(1-\beta) + 2(m+1)(1-\beta)^2}{m^2}, \frac{m+(1-\beta)}{m^2} \right\}$$

$$= \begin{cases} \frac{m+(1-\beta)}{m^2}, & 0 \leq \beta \leq \frac{1+2m}{2(1+m)} \\ \frac{m(1-\beta)+2(m+1)(1-\beta)^2}{m^2}, & \frac{1+2m}{2(1+m)} \leq \beta < 1. \end{cases}$$

Remark 3.16. Clearly,the bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.15 are better than those given by Altinkaya and Yalcin [1, Corolary 7].

By setting $\gamma = 1$ and $m = 1$ in Corollary 3.12, we conclude the following result.

Corollary 3.17. *Let the function f given by* (1.1) *be in the subclass $\mathcal{B}_{\Sigma}(\beta, \lambda)$. Then*

$$|a_2| \leq min\left\{ \frac{2(1-\beta)}{1+\lambda}, \sqrt{\frac{2(1-\beta)}{1+\lambda}} \right\} = \begin{cases} \sqrt{\frac{2(1-\beta)}{1+\lambda}}, & 0 \leq \beta \leq \frac{1-\lambda}{2} \\ \frac{2(1-\beta)}{1+\lambda}, & \frac{1-\lambda}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq min\left\{ \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}, \frac{2(1-\beta)}{1+\lambda} \right\}$$

$$= \begin{cases} \frac{2(1-\beta)}{1+\lambda}, & 0 \leq \beta \leq \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \\ \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}, & \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \leq \beta < 1. \end{cases}$$

.

Remark 3.18. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.17 is better than that given by Li and Wang [9, Theorem 3.2].

By setting $\lambda = 0$ in Corollary 3.17, we conclude the following result.

Corollary 3.19. *Let the function f given by* (1.1) *be in the subclass $S^*_\Sigma(\beta)$. Then*

$$|a_2| \leq min\left\{2(1-\beta), \sqrt{2(1-\beta)}\right\} = \begin{cases} \sqrt{2(1-\beta)}\ , & 0 \leq \beta \leq \frac{1}{2} \\ 2(1-\beta)\ , & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq min\left\{(1-\beta)(5-4\beta), 2(1-\beta)\right\} = \begin{cases} 2(1-\beta)\ ,\ 0 \leq \beta \leq \frac{3}{4} \\ (1-\beta)(5-4\beta)\ ,\ \frac{3}{4} \leq \beta < 1. \end{cases}$$

Remark 3.20. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.19 are better than those given by Brannan and Taha [3].

By setting $\lambda = 1$ in Corollary 3.17, we conclude the following result.

Corollary 3.21. *Let the function f given by* (1.1) *be in the subclass $B_\Sigma(\beta, 1)$. Then*

$$|a_2| \leq min\left\{1-\beta, \sqrt{1-\beta}\right\} = 1-\beta$$

and

$$|a_3| \leq min\left\{\frac{(1-\beta)+3(1-\beta)^2}{3}, 1-\beta\right\} = \begin{cases} 1-\beta\ , & 0 \leq \beta \leq \frac{1}{3} \\ \frac{(1-\beta)+3(1-\beta)^2}{3}\ , & \frac{1}{3} \leq \beta < 1. \end{cases}$$

Remark 3.22. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.21 are better than those given by Li and Wang [9, Theorem 3.2] for $\lambda = 1$.

4. Conclusions

In this paper, we introduce a new subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ of analytic functions, characterized by m-fold symmetric as a foundational framework. It is worth noting that this subclass is a generalization of many well-known or new subclasses, mentioned in section 2. Moreover, by Theorem 2.4, we obtained sharp bounds of the coefficients for many well-known subclasses as consequences. That in certain cases our data has improved the results of others.

Declaration of authorship

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Certain subclass of close-to-convex univalent functions defined with q-derivative operator

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Abstract. The objective of this paper is to introduce a new subclass of strongly close-to-convex functions defined with q-derivative operator and by subordinating to generalized Janowski function. We establish several useful properties such as coefficient estimates, distortion theorem, argument theorem, inclusion relations and radius of convexity for this class. Some relevant connections of the results investigated here with those derived earlier are mentioned.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disc $E = \{z :\mid z \mid< 1\}$ and having the Taylor-Maclaurin expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in E.

By $\mathcal{U}$, we denote the class of Schwarzian functions w satisfying $w(0) = 0$ and $|w(z)| \leq 1$, which are analytic in E and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, z \in E.$$

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For $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of starlike functions and convex functions of order α respectively and are defined as

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in E \right\}$$

and

$$\mathcal{K}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in E \right\}.$$

In particular, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ which is the class of starlike functions and $\mathcal{K}(0) = \mathcal{K}$, the class of convex functions. For $\alpha = \frac{1}{2}$, $\mathcal{S}^*(\frac{1}{2})$ is the class of starlike functions of order $\frac{1}{2}$.

For the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined in (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product(or convolution) of f and h is defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

A function $f \in \mathcal{A}$ is said to be close-to-convex function if there exists a function $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0 (z \in E).$$

The class of close-to-convex functions is denoted by $\mathcal{C}$ and was established by Kaplan [9].

Sakaguchi [18] established the class $\mathcal{S}_s^*$ of the functions $f \in \mathcal{A}$ which satisfy the condition

$$Re\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0.$$

The functions in the class $\mathcal{S}_s^*$ are called starlike functions with respect to symmetric points. Clearly, the class $\mathcal{S}_s^*$ is contained in the class $\mathcal{C}$ of close-to-convex functions, as $\frac{f(z) - f(-z)}{2}$ is a starlike function [3] in E.

Getting inspired from the class $\mathcal{S}_s^*$, Gao and Zhou [5] studied the class $\mathcal{K}_S$ given by

$$\mathcal{K}_s = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > 0, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

Kowalczyk and Les-Bomba [10] extended the class $\mathcal{K}_S$ by introducing the class $\mathcal{K}_S(\gamma)(0 \leq \gamma < 1)$ which is defined as

$$\mathcal{K}_s(\gamma) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > \gamma, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

For $\gamma = 0$, the class $\mathcal{K}_S(\gamma)$ reduces to the class $\mathcal{K}_S$.

Later on, Prajapat [14] established that, a function $f \in \mathcal{A}$ is said to be in the class $\chi_t(\gamma)(|t| \leq 1, t \neq 0, 0 \leq \gamma < 1)$, if there exists a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that

$$Re\left[\frac{tz^2 f'(z)}{g(z)g(tz)}\right] > \gamma.$$

In particular $\chi_{-1}(\gamma) \equiv \mathcal{K}_S(\gamma)$ and $\chi_{-1}(0) \equiv \mathcal{K}_S$.

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, Polatoglu et al. [13] introduced the class $\mathcal{P}(A, B; \alpha)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \dfrac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}$. Also for $\alpha = 0$, the class $\mathcal{P}(A, B; \alpha)$ agrees with $\mathcal{P}(A, B)$, which is a subclass of $\mathcal{A}$ introduced by Janowski [8].

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarzian function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$. Recently some subordination properties for certain classes of analytic functions were studied in [16].

Using the concept of subordination, Singh et al. [20] introduced the class $\chi_t(A, B)(|t| \leq 1, t \neq 0)$, which consists of functions $f \in \mathcal{A}$ with the conditions

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1, z \in E,$$

where $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$. The following observations are obvious:
(i) $\chi_t(1 - 2\gamma, -1) \equiv \chi_t(\gamma)$.
(ii)$\chi_{-1}(1 - 2\gamma, -1) \equiv \mathcal{K}_S(\gamma)$.
(iii)$\chi_{-1}(1, -1) \equiv \mathcal{K}_S$.

Raina et al. [15] defined the class of strongly close-to-convex functions of order β, as below:

$$\mathcal{C}'_\beta = \left\{f : f \in \mathcal{A}, \left|arg\left\{\frac{zf'(z)}{g(z)}\right\}\right| < \frac{\beta\pi}{2}, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E\right\},$$

or equivalently

$$\mathcal{C}'_\beta = \left\{f : f \in \mathcal{A}, \frac{zf'(z)}{g(z)} \prec \left(\frac{1 + z}{1 - z}\right)^\beta, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E\right\}.$$

Quantum calculus is ordinary classical calculus which introduces q-calculus, where q stands for quantum. Nowadays, q-calculus has attracted many researchers as it is widely useful in various branches of Mathematics and Physics. The application of q-calculus was initiated by Jackson [6, 7] and he developed q-integral and

q-derivative in a systematic way. For $0 < q < 1$, Jackson [6] defined the q-derivative of a function $f \in \mathcal{A}$ as

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases} \tag{1.2}$$

where $D_q^2 f(z) = D_q(D_q f(z))$.
From (1.2), it is obvious that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},$$

where $[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + ... + q^{k-1}$ and $[0]_q = 0$. If $q \to 1^-$, then $[k]_q \to k$. Further $D_q z^k = [k]_q z^{k-1}$ and $\lim_{q\to 1^-} D_q f(z) = f'(z)$. Recently a new subclass of analytic functions defined with q-derivative operator is studied in [22].

The q-shifted factorial is given by

$$(a;q)_n = \begin{cases} 1 & \text{for } n = 0, \\ (1-a)(1-aq)...(1-aq^{n-1}) & \text{for } n = 1, 2, ... \end{cases}$$

As a generalization of the hypergeometric series, Heine established the q-hypergeometric series as

$${}_2F_1[a, b; c : q, z] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n.$$

Generalising the Heine's series, we define the basic hypergeometric series ${}_r\phi_s$ as below:

$${}_r\phi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s : q, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n ... (a_r;q)_n}{(q;q)_n (b_1;q)_n ... (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1.3}$$

where $\binom{n}{2} = \frac{n(n-1)}{2}$, and $q \neq 0$ when $r > s + 1$. In (1.3), it is supposed that the parameters $b_1, b_2, ..., b_s$ are such that the denominator factors in the terms of the series are never zero. In basic hypergeometric series, q is a fixed parameter with $q \in \mathbb{C}$ and $|q| < 1$.

For complex parameteres $a_1, a_2, ..., a_r$ and $b_1, b_2, ..., b_s$, $(b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- = 0, -1, -2, ...; j = 1, 2, ..., s)$, the generalized q-hypergeometric function ${}_r\psi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s : q, z)$ is defined by

$${}_r\psi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n ... (a_r;q)_n}{(q;q)_n (b_1;q)_n ... (b_s;q)_n} z^n,$$

where $r = s + 1; r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in E$.

The function $\mathcal{G}_{r,s}(a_i, b_j; q, z)(i = 1, 2, ..., r; j = 1, 2, ..., s)$ is defined by

$$\mathcal{G}_{r,s}(a_i, b_j; q, z) := z_r\psi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s : q, z).$$

Now we define the operator $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f : E \to E$ as

$$\mathcal{J}_\lambda^0(a_1, b_1; q, z)f(z) = f(z) * \mathcal{G}_{r,s}(a_i, b_j; q, z),$$

$$\mathcal{J}_\lambda^1(a_1, b_1; q, z)f(z) = (1-\lambda)(f(z) * \mathcal{G}_{r,s}(a_i, b_j; q, z)) + \lambda z D_q(f(z) * \mathcal{G}_{r,s}(a_i, b_j; q, z), \tag{1.4}$$

$$\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z) = \mathcal{J}_\lambda^1(\mathcal{J}_\lambda^{m-1}(a_1, b_1; q, z)f(z)). \tag{1.5}$$

For $f \in \mathcal{A}$, it can be easily deduced from (1.4) and (1.5), that

$$\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z) = z + \sum_{n=2}^{\infty}[1 - \lambda + [n]_q\lambda]^m \Gamma_n a_n z^n,$$

where $\Gamma_n = \frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\cdots(a_r;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\cdots(b_s;q)_{n-1}}$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0$.
In particular
(i) For $m = 0$, the operator $\mathcal{J}_\lambda^m(a_1, b_1; q, z)$ agrees with the q-analogue of Dziok-Srivastava operator [4].
(ii) For $r = 2, s = 1, a_1 = b_1, a_2 = q, \lambda = 1$, the operator $\mathcal{J}_\lambda^m(a_1, b_1; q, z)$ reduces to the well known Sãlãgean operator [19].

Motivated by the above mentioned work, now we introduce the following subclass of close-to-convex functions defined by subordinating to generalized Janowski function.

Definition 1.1. *Let $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)(|t| \leq 1, t \neq 0, 0 \leq \alpha < 1, 0 < \beta \leq 1)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the conditions,*

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} \prec \left(\frac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}\right)^\beta,$$

where $\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$, $-1 \leq B < A \leq 1$ and $z \in E$.

The following observations are obvious:
(i) For $\alpha = 0, \beta = 1$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ reduces to $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, the class studied by Murugusundaramoorthy and Reddy [12].
(ii) For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1$ and $q \to 1^-$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ reduces to $\chi_t(A, B)$, the class studied by Singh et al. [20].
(iii) For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ and $q \to 1^-$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ agrees with $\chi_t(\gamma)$, the class established by Prajapat [14].
(iv) For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1, B = -1, t = -1$ and $q \to 1^-$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ reduces to K_s, the class introduced by Gao and Zhou [5].
(v) For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1 - 2\gamma, B = -1, t = -1$ and $q \to 1^-$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ agrees with $K_s(\gamma)$, the class studied by

Kowalczyk and Les Bomba [10].

As $f \in \mathcal{S}_{q,t}^{\lambda;m}(a_1, b_1; A, B; \alpha; \beta)$, therefore by the principle of subordination, it follows that

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} = \left(\frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}\right)^\beta, \tag{1.6}$$

where $w \in \mathcal{U}$.

In the present investigation, we obtain the coefficient estimates, inclusion relation, distortion theorem, argument theorem and radius of convexity for the functions in class $\mathcal{S}_{q,t}^{\lambda;m}(a_1, b_1; A, B; \alpha; \beta)$. Our results extend the known results due to various authors.
Throughout our discussion, we assume that $-1 \leq B < A \leq 1, 0 < |t| \leq 1, t \neq 0, 0 \leq \alpha < 1, 0 < \beta \leq 1, m \in \mathbb{N}_0, \lambda \geq 0, z \in E$.

2. Preliminary lemmas

Lemma 2.1. [1, 17] *Let,*

$$\left(\frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}\right)^\beta = (P(z))^\beta = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.1}$$

then

$$|p_n| \leq \beta(1 - \alpha)(A - B), n \geq 1.$$

Lemma 2.2. [21] *Let $g \in S^*\left(\frac{1}{2}\right)$, then $\frac{g(z)g(tz)}{tz} \in S^*$.*

On the lines of Lemma 2.2, the following result is obvious.

Lemma 2.3. *Let $\mathcal{J}_\lambda^m(a_1, b_1; q, z)g \in S^*\left(\frac{1}{2}\right)$, then for*

$$G(z) = \frac{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]}{tz} = z + \sum_{n=2}^{\infty} d_n z^n \in S^*, \tag{2.2}$$

we have, $|d_n| \leq n$.

Lemma 2.4. [15] *Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$ and $0 < \beta \leq 1$, then*

$$\left(\frac{1 + A_1 z}{1 + B_1 z}\right)^\beta \prec \left(\frac{1 + A_2 z}{1 + B_2 z}\right)^\beta.$$

3. Main results

Theorem 3.1. *If* $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, *then*

$$|a_n| \leq \frac{1}{|\Gamma_n|[1-\lambda+[n]_q\lambda]^m}\left[1+\frac{\beta(1-\alpha)(n-1)(A-B)}{2}\right]. \tag{3.1}$$

The result is sharp.

Proof. As $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, therefore from (1.6) and (2.1), we obtain

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1,b_1;q,z)g(z)][\mathcal{J}_\lambda^m(a_1,b_1;q,z)g(tz)]} = (P(z))^\beta. \tag{3.2}$$

Using (2.2), (3.2) takes the form

$$\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)} = (P(z))^\beta. \tag{3.3}$$

On expanding (3.3), it yields

$$1+\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n a_n z^{n-1} = \left(1+\sum_{n=2}^{\infty} d_n z^{n-1}\right)\left(1+\sum_{n=1}^{\infty} p_n z^n\right). \tag{3.4}$$

Equating the coefficients of z^{n-1} in (3.4), we have

$$n[1-\lambda+[n]_q\lambda]^m\Gamma_n a_n = d_n + d_{n-1}p_1 + d_{n-2}p_2 + ... + d_2p_{n-2} + p_{n-1}. \tag{3.5}$$

Using Lemma 2.1, Lemma 2.3 and applying triangle inequality in (3.5), it gives

$$n[1-\lambda+[n]_q\lambda]^m|\Gamma_n||a_n| \leq n+\beta(1-\alpha)(A-B)[(n-1)+(n-2)+...+2+1]. \tag{3.6}$$

After simplification, (3.1) can be easily obtained from (3.6).

Equality in (3.1) is attained for the function f defined by

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1,b_1;q,z)g(z)][\mathcal{J}_\lambda^m(a_1,b_1;q,z)g(tz)]} = \left(\frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz)}\right)^\beta.$$

□

For $\alpha = 0, \beta = 1$, Theorem 3.1 gives the following result due to Murugusundaramoorthy and Reddy [12].

Remark 3.2. If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then

$$|a_n| \leq \frac{1}{|\Gamma_n|[1-\lambda+[n]_q\lambda]^m}\left[1+\frac{(n-1)(A-B)}{2}\right].$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1$ and $q \to 1^-$, Theorem 3.1 agrees with the following result by Singh et al. [20].

Remark 3.3. If $f \in \chi_t(A, B)$, then

$$|a_n| \leq 1 + \frac{(n-1)(A-B)}{2}.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ and $q \to 1^-$, Theorem 3.1 yields the below mentioned result established by Prajapat [14].

Remark 3.4. If $f \in \chi_t(\gamma)$, then

$$|a_n| \leq 1 + (n-1)(1-\gamma).$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1, B = -1, t = -1$ and $q \to 1^-$, Theorem 3.1 gives the following result for the class $\mathcal{K}_s$.

Remark 3.5. If $f \in \mathcal{K}_s$, then

$$|a_n| \leq n.$$

Theorem 3.6. *If $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, then*

$$\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1; \alpha_1; \beta) \subset \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A_2, B_2; \alpha_2; \beta).$$

Proof. As $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1; \alpha_1; \beta)$, so

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} \prec \left(\frac{1 + [B_1 + (A_1 - B_1)(1-\alpha_1)]z}{1 + B_1 z}\right)^\beta.$$

As $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, we have

$$-1 \leq B_1 + (1-\alpha_1)(A_1 - B_1) \leq B_2 + (1-\alpha_2)(A_2 - B_2) \leq 1.$$

Thus by Lemma 2.4, it yields

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} \prec \left(\frac{1 + [B_2 + (A_2 - B_2)(1-\alpha_2)]z}{1 + B_2 z}\right)^\beta,$$

which implies $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A_2, B_2; \alpha_2; \beta)$.

□

Theorem 3.7. *If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^\beta \cdot \frac{1}{(1+r)^2}$$

$$\leq |[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'| \leq \left(\frac{1 + [B + (A-B)(1-\alpha)]r}{1 + Br}\right)^\beta \cdot \frac{1}{(1-r)^2} \tag{3.7}$$

and

$$\int_0^r \left(\frac{1-[B+(A-B)(1-\alpha)]t}{1-Bt}\right)^\beta \cdot \frac{1}{(1+t)^2} dt$$

$$\leq |\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)| \leq \int_0^r \left(\frac{1 + [B + (A-B)(1-\alpha)]t}{1 + Bt}\right)^\beta \cdot \frac{1}{(1-t)^2} dt. \tag{3.8}$$

Proof. From (3.3), we have

$$|[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'| = \frac{|G(z)|}{|z|}(P(z))^\beta. \tag{3.9}$$

Aouf [2] proved that

$$\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br} \le |P(z)| \le \frac{1+[B+(A-B)(1-\alpha)]r}{1+Br},$$

which implies

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^\beta \le |P(z)|^\beta \le \left(\frac{1+[B+(A-B)(1-\alpha)]r}{1+Br}\right)^\beta. \tag{3.10}$$

By Lemma 2.3, G is a starlike function and so due to Mehrok [11], we have

$$\frac{r}{(1+r)^2} \le |G(z)| \le \frac{r}{(1-r)^2}. \tag{3.11}$$

Using (3.10) and (3.11) in (3.9), (3.7) can be easily obtained. On integrating (3.7) from 0 to r, (3.8) follows.

□

On putting $\alpha = 0, \beta = 1$ in Theorem 3.7, the following result due to Murugusundaramoorthy and Reddy [12] can be easily obtained.

Remark 3.8. If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1,b_1;A,B)$, then for $|z|=r, 0<r<1$, we have

$$\left(\frac{1-Ar}{1-Br}\right)\cdot\frac{1}{(1+r)^2} \le |[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'| \le \left(\frac{1+Ar}{1+Br}\right)\cdot\frac{1}{(1-r)^2}$$

and

$$\int_0^r \left(\frac{1-At}{1-Bt}\right)\cdot\frac{1}{(1+t)^2}dt \le |\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)| \le \int_0^r \left(\frac{1+At}{1+Bt}\right)\cdot\frac{1}{(1-t)^2}dt.$$

For $m=0, r=2, s=1, a_1=b_1, a_2=q, \alpha=0, \beta=1$ and $q \to 1^-$, Theorem 3.7 gives the following result due to Singh et al. [20].

Remark 3.9. If $f \in \chi_t(A,B)$, then for $|z|=r, 0<r<1$, we have

$$\frac{1-Ar}{(1-Br)(1+r)^2} \le |f'(z)| \le \frac{1+Ar}{(1+Br)(1-r)^2}$$

and

$$\int_0^r \frac{1-At}{(1-Bt)(1+t)^2}dt \le |f(z)| \le \int_0^r \frac{1+At}{(1+Bt)(1-t)^2}dt.$$

For $m=0, r=2, s=1, a_1=b_1, a_2=q, \alpha=0, \beta=1, A=1-2\gamma, B=-1$ and $q \to 1^-$, Theorem 3.7 agrees with the following result established by Prajapat [14].

Remark 3.10. If $f \in \chi_t(\gamma)$, then for $|z| = r, 0 < r < 1$, we have

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)^3}$$

and

$$\int_0^r \frac{1-(1-2\gamma)t}{(1+t)^3}dt \leq |f(z)| \leq \int_0^r \frac{1+(1-2\gamma)t}{(1-t)^3}dt.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1, B = -1, t = -1$ and $q \to 1^-$, Theorem 3.7 gives the following result for the class $\mathcal{K}_s$.

Remark 3.11. If $f \in \mathcal{K}_s$, then for $|z| = r, 0 < r < 1$, we have

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$

and

$$\int_0^r \frac{1-t}{(1+t)^3}dt \leq |f(z)| \leq \int_0^r \frac{1+t}{(1-t)^3}dt.$$

Theorem 3.12. *If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$|arg[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'| \leq \beta sin^{-1}\left(\frac{(A-B)(1-\alpha)r}{1-[B+(A-B)(1-\alpha)]Br^2}\right) + 2sin^{-1}r.$$

Proof. From (3.3), we have

$$[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' = \frac{G(z)}{z}(P(z))^\beta,$$

which implies

$$|arg[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'| \leq \beta|argP(z)| + \left|arg\frac{G(z)}{z}\right|. \tag{3.12}$$

As G is a starlike function and so due to Mehrok [11], we have

$$\left|arg\frac{G(z)}{z}\right| \leq 2sin^{-1}r. \tag{3.13}$$

Aouf [1], established that,

$$|argP(z)| \leq sin^{-1}\left(\frac{(A-B)(1-\alpha)r}{1-[B+(A-B)(1-\alpha)]Br^2}\right). \tag{3.14}$$

Using (3.13) and (3.14) in (3.12), the proof is obvious.

□

On putting $\alpha = 0, \beta = 1$ in Theorem 3.12, the following result for the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ can be easily obtained.

Remark 3.13. If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then for $|z| = r, 0 < r < 1$, we have

$$|arg[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'| \leq sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) + 2sin^{-1}r.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1$ and $q \to 1^-$, Theorem 3.12 gives the following result due to Singh et al. [20].

Remark 3.14. If $f \in \chi_t(A, B)$, then for $|z| = r, 0 < r < 1$, we have

$$|argf'(z)| \leq sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) + 2sin^{-1}r.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ and $q \to 1^-$, Theorem 3.12 gives the following result for the class $\chi_t(\gamma)$.

Remark 3.15. If $f \in \chi_t(\gamma)$, then for $|z| = r, 0 < r < 1$, we have

$$|argf'(z)| \leq sin^{-1}\left(\frac{2(1-\gamma)r}{1+(1-2\gamma)r^2}\right) + 2sin^{-1}r.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1, B = -1, t = -1$ and $q \to 1^-$, Theorem 3.12 gives the following result for the class $\mathcal{K}_s$.

Remark 3.16. If $f \in \mathcal{K}_s$, then for $|z| = r, 0 < r < 1$, we have

$$|argf'(z)| \leq sin^{-1}\left(\frac{2r}{1+r^2}\right) + 2sin^{-1}r.$$

Theorem 3.17. *Let* $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$*, then* $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)$ *is convex in* $|z| < r_1$*, where* r_1 *is the smallest positive root in* $(0, 1)$ *of the equation*

$$B[B + (A-B)(1-\alpha)]r^3 - [B(B-2) + (A-B)(1-\alpha)(B-1-\beta)]r^2$$
$$-[(1-\beta)(A-B)(1-\alpha) + (2B-1)]r - 1 = 0. \quad (3.15)$$

Proof. As $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, we have

$$z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' = G(z)(P(z))^\beta.$$

On differentiating it logarithmically, we get

$$\frac{(z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]')'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'} = \frac{zG'(z)}{G(z)} + \beta\frac{zP'(z)}{P(z)}. \quad (3.16)$$

As $G \in \mathcal{S}^*$, from [11], we have

$$Re\left(\frac{zG'(z)}{G(z)}\right) \geq \frac{1-r}{1+r}. \quad (3.17)$$

Also it can be easily verified that

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{r(A-B)(1-\alpha)}{(1+Br)(1+[B+(A-B)(1-\alpha)]r)}. \quad (3.18)$$

(3.16) can be expressed as

$$Re\left(\frac{(z[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]')'}{[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]'}\right) \geq Re\left(\frac{zG'(z)}{G(z)}\right) - \beta\left|\frac{zP'(z)}{P(z)}\right|. \tag{3.19}$$

Using (3.17) and (3.18), (3.19) yields

$$Re\left(\frac{(z[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]')'}{[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]'}\right) \geq \frac{1-r}{1+r} - \beta\frac{r(A-B)(1-\alpha)}{(1+Br)(1+[B+(A-B)(1-\alpha)]r)}. \tag{3.20}$$

After some simplification, (3.20) takes the form

$$Re\left(\frac{(z[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]')'}{[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]'}\right) \geq \frac{-B[B+(A-B)(1-\alpha)]r^3}{(1+r)(1+Br)(1+[B+(A-B)(1-\alpha)]r)}$$
$$+\frac{[B(B-2)+(A-B)(1-\alpha)(B-1-\beta)]r^2}{(1+r)(1+Br)(1+[B+(A-B)(1-\alpha)]r)}$$
$$+\frac{[(1-\beta)(A-B)(1-\alpha)-(2B-1)]r+1}{(1+r)(1+Br)(1+[B+(A-B)(1-\alpha)]r)}.$$

Hence, the function $\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)$ is convex in $|z|<r_1$, where r_1 is the smallest positive root in $(0,1)$ of the equation

$$B[B+(A-B)(1-\alpha)]r^3 - [B(B-2)+(A-B)(1-\alpha)(B-1-\beta)]r^2$$
$$-[(1-\beta)(A-B)(1-\alpha)+(2B-1)]r-1=0.$$

□

On putting $\alpha=0,\beta=1$ in Theorem 3.17, the following result due to Murugusundaramoorthy and Reddy [12] can be easily obtained.

Remark 3.18. If $f\in\mathcal{S}_{q,t}^{\lambda,m}(a_1,b_1;A,B)$, then $\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)$ is convex in $|z|<r_2$, where r_2 is the smallest positive root in $(0,1)$ of the equation

$$ABr^3 - A(B-2)r^2 - (2B-1)r - 1 = 0.$$

For $m=0,r=2,s=1,a_1=b_1,a_2=q,\alpha=0,\beta=1$ and $q\to 1^-$, Theorem 3.17 gives the following result due to Singh et al. [20].

Remark 3.19. If $f\in\chi_t(A,B)$, then $f(z)$ is convex in $|z|<r_3$, where r_3 is the smallest positive root in $(0,1)$ of the equation

$$ABr^3 - A(B-2)r^2 - (2B-1)r - 1 = 0.$$

For $m=0,r=2,s=1,a_1=b_1,a_2=q,\alpha=0,\beta=1,A=1-2\gamma,B=-1$ and $q\to 1^-$, Theorem 3.17 gives the following result due to Prajapat [14].

Remark 3.20. If $f\in\chi_t(\gamma)$, then $f(z)$ is convex in $|z|<r_4=2-\sqrt{3}$.

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Results on $\phi-$like functions involving Hadamard product

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Abstract. In this paper, we derive a differential subordination theorem involving convolution of normalized analytic functions. By selecting different dominants to our main result, we find certain sufficient conditions for $\phi-$likeness and parabolic $\phi-$likeness of functions in class $\mathcal{A}$.

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1. Introduction

A function f is said to be analytic at a point z in a domain $\mathbb{D}$ if it is differentiable not only at z but also in some neighbourhood of the point z. A function f is said to be analytic in a domain $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$. Let $\mathcal{H}$ be the class of analytic functions in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots.$$

Let $\mathcal{A}$ be the class of functions f, analytic in the unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.
Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. The Taylor series expansion of any function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots.$$

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Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1,\ \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $\mathbb{E}$.

Let $f(z) = \displaystyle\sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \displaystyle\sum_{k=0}^{\infty} b_k z^k$ be two analytic functions, then the Hadamard product or convolution of f and g, written as $f * g$ is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Ronning [8] and Ma and Minda [6] studied the domain Ω and the function $q(z)$ defined below:

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

Clearly the function

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Let ϕ be analytic in a domain containing $f(\mathbb{E}),\ \phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0,\ z \in \mathbb{E}.$$

This concept was introduced by Brickman [4]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is $\phi-$ like for some analytic function ϕ. Later, Ruscheweyh [9] investigated the following general class of $\phi-$like functions:

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0,\ \phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E})\backslash\{0\}$, then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to a univalent function $q,\ q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|,\ z \in \mathbb{E}. \tag{1.2}$$

Equivalently, condition (1.2) can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

In 2007, Shanmugham et al. [10] proved the following result for $\phi-$like functions.

Theorem 1.1. *Let $q(z) \neq 0$ be analytic and univalent in $\mathbb{E}$ with $q(0) = 1$ such that $\dfrac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{E}$. Let $q(z)$ satisfy*

$$\Re\left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right] > 0.$$

Let

$$\Psi(\alpha,\gamma,g;z) := \alpha\left\{\frac{z(f*g)'(z)}{\phi(f*g)(z)}\right\} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z(\phi(f*g)(z))'}{\phi(f*g)(z)}\right\}.$$

If q satisfies

$$\Psi(\alpha,\gamma,g;z) \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},$$

then

$$\frac{z(f*g)'(z)}{\phi(f*g)(z)} \prec q(z)$$

and q is the best dominant.

Later in 2018, Brar and Billing [3] obtained the following result.

Theorem 1.2. *Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + (\gamma-1)\dfrac{zq'(z)}{q(z)}\right] > 0$ *and*

(ii) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + (\gamma-1)\dfrac{zq'(z)}{q(z)} + \dfrac{\beta(1-\alpha)}{\alpha}(q(z))^{\beta-\gamma} + \gamma\right] > 0.$

If f and $g \in \mathcal{A}$ satisfy

$$(1-\alpha)\left[\frac{z(f*g)'(z)}{\phi(f*g)(z)}\right]^\beta + \alpha\left[\frac{z(f*g)'(z)}{\phi(f*g)(z)}\right]^\gamma\left[2 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z(\phi((f*g)(z)))'}{\phi((f*g)(z))}\right]$$
$$\prec (1-\alpha)(q(z))^\beta + \alpha(q(z))^\gamma\left(1 + \frac{zq'(z)}{q(z)}\right),$$

then

$$\frac{z(f*g)'(z)}{\phi(f*g)(z)} \prec q(z),\ z \in \mathbb{E},$$

where $\alpha,\ \beta,\ \gamma$ are complex numbers such that $\alpha \neq 0$, and $q(z)$ is the best dominant.

In 2019, Adegani et al. [1] established sufficient subordination conditions for functions to be close-to-convex.
Moreover, this study is also motivated by the findings of Cho et al. [5] and Adegani et al. [2] who explored subordination conditions in geometric function theory.
The aim of the present investigation is to find sufficient conditions for parabolic $\phi-$likeness and $\phi-$likeness of analytic functions.
To prove our main result, we shall use the following lemma of Miller and Mocanu.

Lemma 1.3. *([7], Theorem 3.4h, p.132). Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\varphi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

1. *h is convex, or*
2. *Q is starlike.*
   In addition, assume that
3. $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ *for all* $z \in \mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0),\ p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)],\ z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. A subordination theorem

In what follows, all the powers taken are principal ones.

Theorem 2.1. *Let β and γ be complex numbers such that $\beta \neq 0$. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*
$(i)\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\gamma}{\beta} - 1\right)\dfrac{zq'(z)}{q(z)}\right] > 0$ *and*
$(ii)\Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \frac{a}{c}\left(\frac{\gamma}{\beta} + 1\right)q(z) + \frac{b}{c}\left(\frac{\gamma}{\beta} + 2\right)q^2(z)\right] > 0,$ *where $a,\ b$ and c are real numbers with $c \neq 0$. Let ϕ be analytic function in the domain containing $(f*g)(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in (f*g)(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{z(f*g)'(z)}{\phi((f*g)(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left[\frac{z(f*g)'(z)}{\phi((f*g)(z))}\right]^{\gamma} \cdot$$
$$\cdot\left\{a\frac{z(f*g)'(z)}{\phi((f*g)(z))} + b\left[\frac{z(f*g)'(z)}{\phi((f*g)(z))}\right]^2 + c\left[1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z(\phi((f*g)(z)))'}{\phi((f*g)(z))}\right]\right\}^{\beta}$$
$$\prec [q(z)]^{\gamma}\left[aq(z) + bq^2(z) + c\frac{zq'(z)}{q(z)}\right]^{\beta}, \tag{2.1}$$

then

$$\frac{z(f*g)'(z)}{\phi((f*g)(z))} \prec q(z),\ z \in \mathbb{E},$$

and $q(z)$ is the best dominant.

Proof. Define the function p by

$$p(z)=\frac{z(f*g)'(z)}{\phi((f*g)(z))},\ z\in\mathbb{E}.$$

Then the function p is analytic in $\mathbb{E}$ and $p(0)=1$.
Therefore, from equation (2.1), we get:

$$[p(z)]^{\gamma}\left[ap(z)+bp^2(z)+c\frac{zp'(z)}{p(z)}\right]^{\beta}\prec[q(z)]^{\gamma}\left[aq(z)+bq^2(z)+c\frac{zq'(z)}{q(z)}\right]^{\beta}$$

or

$$a\,[p(z)]^{\frac{\gamma}{\beta}+1}+b\,[p(z)]^{\frac{\gamma}{\beta}+2}+c\,[p(z)]^{\frac{\gamma}{\beta}-1}\,zp'(z)$$
$$\prec a\,[q(z)]^{\frac{\gamma}{\beta}+1}+b\,[q(z)]^{\frac{\gamma}{\beta}+2}+c\,[q(z)]^{\frac{\gamma}{\beta}-1}\,zq'(z)$$

Let the functions θ and φ be defined as:

$$\theta(w)=aw^{\frac{\gamma}{\beta}+1}+bw^{\frac{\gamma}{\beta}+2}\ \textit{and}\ \varphi(w)=cw^{\frac{\gamma}{\beta}-1}$$

Clearly, the functions θ and φ are analytic in domain $\mathbb{D}=\mathbb{C}\backslash\{0\}$ and $\varphi(w)\neq 0$ in $\mathbb{D}$. Therefore,

$$Q(z)=\varphi[q(z)]zq'(z)=c[q(z)]^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z)=\theta[q(z)]+Q(z)=a[q(z)]^{\frac{\gamma}{\beta}+1}+b[q(z)]^{\frac{\gamma}{\beta}+2}+c[q(z)]^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we get

$$\frac{zQ'(z)}{Q(z)}=1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q(z)}=1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}+\frac{a}{c}\left(\frac{\gamma}{\beta}+1\right)q(z)+\frac{b}{c}\left(\frac{\gamma}{\beta}+2\right)q^2(z).$$

In view of the given conditions (i) and (ii), we see that Q is starlike and $\Re\left(\frac{zh'(z)}{Q(z)}\right)>0$.
Therefore, the proof, now follows from Lemma [1.3]. □

For $g(z)=\dfrac{z}{1-z}$ in Theorem 2.1, we have

Theorem 2.2. *Let β and γ be complex numbers such that $\beta\neq 0$. Let $q(z)\neq 0$, be a univalent function in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 2.1. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$\left\{\frac{zf'(z)}{\phi(f(z))}\right\}^{\gamma}\left\{a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right\}^{\beta}$$
$$\prec(q(z))^{\gamma}\left\{aq(z)+bq^2(z)+c\frac{zq'(z)}{q(z)}\right\}^{\beta},$$

where a, b and c are real numbers with $c \neq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in \mathbb{E},$$

and $q(z)$ is the best dominant.

3. Applications to parabolic $\phi-$like functions

Remark 3.1. Selecting $q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, $\beta = \gamma = 1$ in Theorem 2.2, then after having some calculations,

$$q'(z) = \frac{4}{\pi^2\sqrt{z}(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

$$\frac{q'(z)}{q(z)} = \frac{\frac{4}{\pi^2\sqrt{z}(1-z)}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1+\frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2}$$

$$\frac{q''(z)}{q'(z)} = \frac{3z-1}{2z(1-z)} + \frac{1}{\sqrt{z}(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}.$$

Thus the conditions (i) and (ii) of Theorem 2.1 becomes

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)} = 1+\frac{zq''(z)}{q'(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}$$

and

$$\begin{aligned} &1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}+\frac{a}{c}\left(\frac{\gamma}{\beta}+1\right)q(z)+\frac{b}{c}\left(\frac{\gamma}{\beta}+2\right)q^2(z)\\ &= 1+\frac{zq''(z)}{q'(z)}+\frac{2a}{c}q(z)+\frac{3b}{c}q^2(z)\\ &= \frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}+\frac{2a}{c}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]\\ &+\frac{3b}{c}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2. \end{aligned}$$

Therefore, for real numbers $a,\ b,\ c$ with $c \neq 0$ *and* $\frac{a}{c},\ \frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we derive the following result from Theorem 2.2.

Theorem 3.2. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 + cz\left\{\frac{[\phi(f(z))]\,[zf''(z) + f'(z)] - zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\}$$
$$\prec a\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2 + b\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3 + \frac{4c\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

where a, b, c are real numbers such that $c \neq 0$ and $\dfrac{a}{c}$, $\dfrac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E}.$$

Hence f is parabolic ϕ-like.

Remark 3.3. Selecting $q(z) = 1 + \dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, then after having some calculations, we have

$$q'(z) = \frac{4}{\pi^2\sqrt{z}(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

$$\frac{q'(z)}{q(z)} = \frac{\dfrac{4}{\pi^2\sqrt{z}(1-z)}\left[\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1 + \dfrac{2}{\pi^2}\left[\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2}$$

$$\frac{q''(z)}{q'(z)} = \frac{3z-1}{2z(1-z)} + \frac{1}{\sqrt{z}(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}.$$

Thus the conditions (i) and (ii) of Theorem 2.1 becomes

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}$$
$$= \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\dfrac{4\sqrt{z}}{\pi^2(1-z)}\left[\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1 + \dfrac{2}{\pi^2}\left[\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2}$$

and

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}+\frac{a}{c}\left(\frac{\gamma}{\beta}+1\right)q(z)+\frac{b}{c}\left(\frac{\gamma}{\beta}+2\right)q^2(z)$$

$$=1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a}{c}q(z)+\frac{2b}{c}q^2(z)$$

$$=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}-\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1+\frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2}$$

$$+\frac{a}{c}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]+\frac{2b}{c}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2.$$

Therefore, for real numbers $a,\ b,\ c$ with $c\ \neq 0$ *and* $\frac{a}{c},\ \frac{b}{c}\geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we derive the following result from Theorem 2.2.

Theorem 3.4. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$

$$\prec a\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]+b\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2$$

$$+\frac{\frac{4c\sqrt{z}}{\pi^2(1-z)}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1+\frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2},$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0$ and $\frac{a}{c},\ \frac{b}{c}\geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

Hence f is parabolic ϕ-like.

4. Applications to $\phi-$like functions

Remark 4.1. By taking $q(z)=1+tz,\ 0<t\leq 1,\ \beta=\gamma=1$ in Theorem 2.2, then after having some calculations we have

$$1+\frac{zq''(z)}{q'(z)}=1$$

and

$$1+\frac{zq''(z)}{q'(z)}+\frac{2a}{c}q(z)+\frac{3b}{c}\;q^2(z)=1+\frac{2a}{c}(1+tz)+\frac{3b}{c}(1+tz)^2.$$

Thus for real numbers $a,\ b$ *and* $c\ (\neq 0)$ such that $0\leq\dfrac{a}{c}\leq 1$,

$0\leq\dfrac{b}{c}\leq 1$, we observe that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we immediately, arrive at the following result from Theorem 2.2.

Theorem 4.2. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 +cz\left\{\frac{[\phi(f(z))]\,[zf''(z)+f'(z)]-zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\} \prec a(1+tz)^2+b(1+tz)^3+ctz,$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0,\ 0\leq\dfrac{a}{c}\leq 1$ and $0\leq\dfrac{b}{c}\leq 1$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec 1+tz,\ 0<t\leq 1,\ z\in\mathbb{E}.$$

Therefore, f is ϕ-like in $\mathbb{E}$.

Remark 4.3. When we select $q(z)=e^z,\ \beta=\gamma=1$ in Theorem 2.2, a little calculation yields that

$$1+\frac{zq''(z)}{q'(z)}=1+z$$

and

$$1+\frac{zq''(z)}{q'(z)}+\frac{2a}{c}q(z)+\frac{3b}{c}\;q^2(z)=1+z+\frac{2a}{c}e^z+\frac{3b}{c}e^{2z}.$$

For real numbers $a,\ b,\ c$ such that $c\neq 0,\ \dfrac{a}{c}\geq 0.4$ and $\dfrac{b}{c}=1$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.4. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 +cz\left\{\frac{[\phi(f(z))]\,[zf''(z)+f'(z)]-zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\} \prec ae^{2z}+be^{3z}+cze^z,$$

where $a,\ b,\ c$ *are real numbers such that* $c \neq 0,\ \dfrac{a}{c} \geq 0.4$ *and* $\dfrac{b}{c} = 1$, *then*

$$\frac{zf'(z)}{\phi(f(z))} \prec e^z,\ z \in \mathbb{E},$$

i.e. f *is* ϕ*-like.*

Remark 4.5. By selecting $q(z) = 1 + \dfrac{2}{3}z^2,\ \beta = \gamma = 1$ in Theorem 2.2, we have

$$1 + \frac{zq''(z)}{q'(z)} = 2$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}\ q^2(z) = 2 + \frac{2a}{c}\left(1 + \frac{2}{3}z^2\right) + \frac{3b}{c}\left(1 + \frac{2}{3}z^2\right)^2.$$

For real numbers $a,\ b,\ c$ such that $c\ \neq 0,\ \dfrac{a}{c} \geq -0.6$ and $\dfrac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.6. *Let* ϕ *be analytic function in the domain containing* $f(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in f(\mathbb{E})\backslash\{0\}$. *If* $f \in \mathcal{A},\ \dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 + cz\left\{\frac{[\phi(f(z))]\,[zf''(z) + f'(z)] - zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\} \prec a\left(1 + \frac{2}{3}z^2\right)^2 + b\left(1 + \frac{2}{3}z^2\right)^3 + \frac{4}{3}cz^2,$$

where $a,\ b,\ c$ *are real numbers such that* $c \neq 0,\ \dfrac{a}{c} \geq -0.6$ *and* $\dfrac{b}{c} \geq 0$, *then*

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{3}z^2,\ z \in \mathbb{E}.$$

Thus f *is* ϕ*-like.*

Remark 4.7. By taking $q(z) = \left(\dfrac{1+z}{1-z}\right)^\delta;\ 0 < \delta \leq 1,\ \beta = \gamma = 1$ in Theorem 2.2, we get

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1 + 2\delta z + z^2}{1 - z^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}\ q^2(z) = \frac{1 + 2\delta z + z^2}{1 - z^2} + \frac{2a}{c}\left(\frac{1+z}{1-z}\right)^\delta + \frac{3b}{c}\left(\frac{1+z}{1-z}\right)^{2\delta}.$$

For real numbers $a,\ b,\ c$ such that $c \neq 0,\ b = 0$ *and* $\frac{a}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.8. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 + cz\left\{\frac{[\phi(f(z))]\,[zf''(z) + f'(z)] - zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\}$$
$$\prec a\left(\frac{1+z}{1-z}\right)^{2\delta} + b\left(\frac{1+z}{1-z}\right)^{3\delta} + cz\left(\frac{2\delta}{1-z^2}\right)\left(\frac{1+z}{1-z}\right)^{\delta},$$

where $a,\ b,\ c$ are real numbers such that $c \neq 0,\ b = 0$ and $\frac{a}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta};\ 0 < \delta \leq 1,\ z \in \mathbb{E}.$$

Remark 4.9. When we put $q(z) = \frac{1 + (1-2\eta)z}{1-z};\ 0 \leq \eta < 1,\ \beta = \gamma = 1$ in Theorem 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}\ q^2(z) = \frac{1+z}{1-z} + \frac{2a}{c}\left[\frac{1+(1-2\eta)z}{1-z}\right] + \frac{3b}{c}\left[\frac{1+(1-2\eta)z}{1-z}\right]^2.$$

For real numbers $a,\ b,\ c$ such that $c\ \neq 0,\ b = 0$ *and* $\frac{a}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we obtain the following result from Theorem 2.2.

Theorem 4.10. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3 + cz\left\{\frac{[\phi(f(z))]\,[zf''(z) + f'(z)] - zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\}$$

$$\prec a\left[\frac{1+(1-2\eta)z}{1-z}\right]^2+b\left[\frac{1+(1-2\eta)z}{1-z}\right]^3+cz\left[\frac{2(1-\eta)}{(1-z)^2}\right],$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0,\ b=0$ and $\frac{a}{c}\geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec\frac{1+(1-2\eta)z}{1-z},\ z\in\mathbb{E},\ 0\leq\eta<1,$$

i.e. f is ϕ-like in $\mathbb{E}$.

Remark 4.11. When we select $q(z)=\frac{\alpha'(1-z)}{\alpha'-z};\ \alpha'>1,\ \beta=\gamma=1$ in Theorem 2.2, after a little calculation, we obtain

$$1+\frac{zq''(z)}{q'(z)}=\frac{\alpha'+z}{\alpha'-z}$$

and

$$1+\frac{zq''(z)}{q'(z)}+\frac{2a}{c}q(z)+\frac{3b}{c}\ q^2(z)=\frac{\alpha'+z}{\alpha'-z}+\frac{2a}{c}\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]+\frac{3b}{c}\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^2.$$

For real numbers $a,\ b,\ c$ such that $c\neq 0,\ b=0$ *and* $\frac{a}{c}\geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we get the following Theorem from Theorem 2.2.

Theorem 4.12. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^3+cz\left\{\frac{[\phi(f(z))]\,[zf''(z)+f'(z)]-zf'(z)\,[\phi(f(z))]'}{[\phi(f(z))]^2}\right\}$$
$$\prec a\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^2+b\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^3+cz\left[\frac{\alpha'(1-\alpha')}{(\alpha'-z)^2}\right],$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0,\ b=0$ and $\frac{a}{c}\geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec\frac{\alpha'(1-z)}{\alpha'-z},\ z\in\mathbb{E},\ \alpha'>1,$$

i.e. f is ϕ-like.

Remark 4.13. By taking $q(z)=1+tz,\ 0<t\leq 0.8,\ \beta=1$ and $\gamma=0$ in Theorem 2.2, then after having some calculations we have

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1}{1+tz}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a}{c}q(z)+\frac{2b}{c}\ q^2(z)=\frac{1}{1+tz}+\frac{a}{c}\left(1+tz\right)+\frac{2b}{c}\left(1+tz\right)^2.$$

Thus for real numbers $a,\ b,\ c$ such that $c\neq 0$ *and* $\frac{a}{c},\ \frac{b}{c}\geq 0$, we observe that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we immediately, arrive at the following result from Theorem 2.2.

Theorem 4.14. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$
$$\prec a\left(1+tz\right)+b\left(1+tz\right)^2+\frac{ctz}{1+tz},$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0$ and $\frac{a}{c},\ \frac{b}{c}\geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec 1+tz,\ \ 0<t\leq 0.8,\ \ z\in\mathbb{E}.$$

Therefore, f is ϕ-like.

Remark 4.15. When we select $q(z)=e^z,\ \beta=1$ and $\gamma=0$ in Theorem 2.2, a little calculation yields that

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=1$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a}{c}q(z)+\frac{2b}{c}\ q^2(z)=1+\frac{a}{c}e^z+\frac{2b}{c}e^{2z}.$$

For real numbers $a,\ b,\ c$ such that $c\neq 0,\ \frac{a}{c}\geq 0$ and $0\leq\frac{b}{c}\leq 0.8$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.16. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$
$$\prec ae^z+be^{2z}+cz,$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0,\ \frac{a}{c}\geq 0$ and $0\leq\frac{b}{c}\leq 0.8$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec e^z,\ \ z\in\mathbb{E},$$

i.e. f is ϕ-like.

Remark 4.17. By selecting $q(z)=1+\frac{2}{3}z^2,\ \beta=1$ and $\gamma=0$ in Theorem 2.2, we have

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{6}{3+2z^2}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a}{c}q(z)+\frac{2b}{c}\ q^2(z)=\frac{6}{3+2z^2}+\frac{a}{c}\left(1+\frac{2}{3}z^2\right)+\frac{2b}{c}\left(1+\frac{2}{3}z^2\right)^2.$$

For real numbers $a,\ b,\ c$ such that $c\neq 0,\ \frac{a}{c}\geq 0.6$ and $0\leq\frac{b}{c}\leq 0.7$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.18. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in f(\mathbb{E})\backslash\{0\}$. If $f\in\mathcal{A},\ \frac{zf'(z)}{\phi(f(z))}\neq 0,\ z\in\mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))}+b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+c\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\prec a\left(1+\frac{2}{3}z^2\right)+b\left(1+\frac{2}{3}z^2\right)^2+\frac{4cz^2}{3+2z^2},$$

where $a,\ b,\ c$ are real numbers such that $c\neq 0,\ \frac{a}{c}\geq 0.6$ and $0\leq\frac{b}{c}\leq 0.7$, then

$$\frac{zf'(z)}{\phi(f(z))}\prec 1+\frac{2}{3}z^2,\ z\in\mathbb{E}.$$

Thus f is ϕ-like.

Remark 4.19. By taking $q(z)=\left(\frac{1+z}{1-z}\right)^\delta;\ 0<\delta\leq 0.5,\ \beta=1$ and $\gamma=0$ in Theorem 2.2, we get

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1+z^2}{1-z^2}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a}{c}q(z)+\frac{2b}{c}\ q^2(z)=\frac{1+z^2}{1-z^2}+\frac{a}{c}\left(\frac{1+z}{1-z}\right)^\delta+\frac{2b}{c}\left(\frac{1+z}{1-z}\right)^{2\delta}.$$

For real numbers $a,\ b,\ c$ such that $c\neq 0$ *and* $\frac{a}{c},\ \frac{b}{c}\geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.20. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + c\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$

$$\prec a\left(\frac{1+z}{1-z}\right)^{\delta} + b\left(\frac{1+z}{1-z}\right)^{2\delta} + \frac{2\delta cz}{1-z^2},$$

where a, b, c are real numbers such that $c \neq 0$ and $\dfrac{a}{c}$, $\dfrac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta};\ 0 < \delta \leq 0.5,\ z \in \mathbb{E}.$$

Remark 4.21. When we put $q(z) = \dfrac{1+(1-2\eta)z}{1-z}$; $0 \leq \eta < 1$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z} - \frac{2z(1-\eta)}{(1-z)\left[1+(1-2\eta)z\right]}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}\ q^2(z) = \frac{1+z}{1-z} - \frac{2z(1-\eta)}{(1-z)\left[1+(1-2\eta)z\right]}$$

$$+\frac{a}{c}\left[\frac{1+(1-2\eta)z}{1-z}\right] + \frac{2b}{c}\left[\frac{1+(1-2\eta)z}{1-z}\right]^2.$$

For real numbers a, b, c such that $c \neq 0$, $b = 0$ *and* $\dfrac{a}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we obtain the following result from Theorem 2.2.

Theorem 4.22. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + c\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$

$$\prec a\left[\frac{1+(1-2\eta)z}{1-z}\right] + b\left[\frac{1+(1-2\eta)z}{1-z}\right]^2 + cz\left[\frac{2(1-\eta)}{(1-z)(1+(1-2\eta)z)}\right],$$

where a, b, c are real numbers such that $c \neq 0$, $b = 0$ and $\dfrac{a}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+(1-2\eta)z}{1-z},\ z \in \mathbb{E},\ 0 \leq \eta < 1,$$

i.e. f is ϕ-like.

Remark 4.23. When we select $q(z) = \dfrac{\alpha'(1-z)}{\alpha'-z}$; $\alpha' > 1,\ \beta = 1$ and $\gamma = 0$ in Theorem 2.2, after a little calculation, we obtain

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{\alpha' - z^2}{(1-z)(\alpha'-z)}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}\, q^2(z) = \frac{\alpha' - z^2}{(1-z)(\alpha'-z)} + \frac{a}{c}\left[\frac{\alpha'(1-z)}{\alpha'-z}\right] + \frac{2b}{c}\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^2.$$

For real numbers $a,\ b,\ c$ such that $c \neq 0,\ \dfrac{a}{c} \geq 0$ and $\dfrac{b}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we get the following Theorem from Theorem 2.2.

Theorem 4.24. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + c\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec a\left[\frac{\alpha'(1-z)}{\alpha'-z}\right] + b\left[\frac{\alpha'(1-z)}{\alpha'-z}\right]^2 + \frac{(1-\alpha')cz}{(1-z)(\alpha'-z)},$$

where $a,\ b,\ c$ are real numbers such that $c \neq 0,\ \dfrac{a}{c} \geq 0$ and $\dfrac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{\alpha'(1-z)}{\alpha'-z},\ z \in \mathbb{E},\ \alpha' > 1,$$

i.e. f is ϕ-like.

5. Conclusion

Using the differential subordination technique involving convolution, we derived new conditions under which normalized analytic functions exhibit ϕ-likeness and parabolic ϕ-likeness. These results contribute to a deeper understanding of geometric function theory and open pathways for further applications.

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On a second order p-Laplacian impulsive boundary value problem on the half-line

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Abstract. In this article, we shall establish the existence of weak solutions for a p-Laplacian impulsive differential equation with Dirichlet boundary conditions on the half-line by using Browder theorem.

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Keywords: Monotone operator, Browder theorem, p-Laplacian, impulsive BVPs, uniqueness, monotone theory, half-line.

1. Introduction

In this paper, we consider the following second-order p-Laplacian impulsive difierential equation with Dirichlet boundary conditions on the half-line

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + |u|^{p-2}u &= f(x,u), \quad x \neq x_j, \quad \text{a.e. } x \geq 0, \\ \triangle(\rho(x_j)|u'(x_j)|^{p-2}u'(x_j)) &= g(x_j)I_j(u(x_j)), \quad j \in \mathbb{N}^*, \\ u(0) = u(\infty) &= 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\rho : [0, \infty) \to (0, \infty)$ satisfies $\rho^{-\frac{1}{p-1}} \in L^1[0, \infty)$ and

$$M_0 = \left(\int_0^\infty \left(\int_x^\infty \rho^{-\frac{1}{p-1}}(s)ds \right) dx \right) < \infty.$$

The functions $f \in C([0, \infty) \times \mathbb{R}, \mathbb{R})$, $I_j : \mathbb{R} \to \mathbb{R}$ and $g : [0, \infty) \to (0, \infty)$ are assumed to be continuous with $\sum_{j=1}^{\infty} g(x_j) < \infty$, $0 = x_0 < x_1 < x_2 < ... < x_j < ... < x_m \to \infty$, as $m \to \infty$, are the impulse points, and

$$\triangle(\rho(x_j)|u'(x_j)|^{p-2}u'(x_j)) = \rho(x_j^+)|u'(x_j^+)|^{p-2}u'(x_j^+) - \rho(x_j^-)|u'(x_j^-)|^{p-2}u'(x_j^-),$$

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such that $u'(x_j^{\pm}) = \lim_{x\to x_j^{\pm}} u'(x)$ for $j \in \mathbb{N}^*$.

Recently, there is increasing interest in the existence and multiplicity of solutions for several types of differential equations with a p-Laplacian operator by applying variational methods and critical point theory. Meanwhile, some people begin to study p-Laplacian differential equations with impulsive effects, for example, see [1, 2, 4, 7, 8, 9] and the references therein.

Motivated by the works cited above, in this paper, we shall discuss the existence of solutions for problem (1.1) on the half-line by adoptting Browder theorem. The results obtained here improve some existing results in the literature.

2. Variational structure

Let define the following reflexive Banach space

$$X = \left\{ u \in W^{1,p}(0,\infty) : \quad u(0) = u(\infty) = 0, \quad \rho^{\frac{1}{p}} u' \in L^p(0,\infty) \right\},$$

equipped with the norm

$$\|u\| = \left(\int_0^{+\infty} \rho(x)|u'(x)|^p dx + \int_0^{+\infty} |u(x)|^p dx \right)^{\frac{1}{p}},$$

or the equivalent norm

$$\|u\|_X = \|\rho^{\frac{1}{p}} u'\|_p + \|u\|_p.$$

Also consider the space

$$C_0[0,+\infty) = \left\{ u \in C([0,+\infty),\mathbb{R}) : \quad \lim_{x\to\infty} u(x) = 0 \right\},$$

endowed with the norm

$$\|u\|_\infty = \sup_{x\in[0,+\infty)} |u(x)|.$$

In what follows, we shall convert the problem (1.1) into an integral equation. Multiply the two sides of the equality

$$-(\rho(x)|u'|^{p-2}u')' + |u|^{p-2}u = f(x,u),$$

by $v \in X$ and integrate from 0 to ∞, to obtain,

$$-\int_0^{+\infty} (\rho(x)|u'(x)|^{p-2}u'(x))'v(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx = \int_0^{+\infty} f(x,u(x))v(x)dx.$$

Let consider the first term

$$-\int_0^{+\infty} (\rho(x)|u'(x)|^{p-2}u'(x))'v(x)dx = \sum_{j=0}^{\infty} \int_{x_j^+}^{x_{j+1}^-} -(\rho(x)|u'(x)|^{p-2}u'(x))'v(x)$$

$$= \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \sum_{j=1}^{\infty} \Big[\rho(x_j^+)|u'(x_j^+)|^{p-2}u'(x_j^+) - \rho(x_j^-)|u'(x_j^-)|^{p-2}u'(x_j^-)\Big]v(x_j)$$

$$= \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \sum_{j=1}^{\infty} \triangle(\rho(x_j)|u'(x_j)|^{p-2}u'(x_j))v(x_j)$$

$$= \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \sum_{j=1}^{\infty} g(x_j)I_j(u(x_j))v(x_j),$$

and then, we have

$$\int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx + \sum_{j=1}^{\infty} g(x_j)I_j(u(x_j))v(x_j) = \int_0^{+\infty} f(x,u(x))v(x)dx.$$

This leads us to introduce the following concept for the solution for (1.1).

Definition 2.1. *We say that a function $u \in X$ is a weak solution of the impulsive problem (1.1) if u satisfies*

$$\int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx + \sum_{j=1}^{\infty} g(x_j)I_j(u(x_j))v(x_j) - \int_0^{+\infty} f(x,u(x))v(x)dx = 0.$$

Concerning the previous spaces, we have the following vital embeddings.

Lemma 2.2. *Let $u \in X$. Then*

$$\|u\|_p^p \le M_0\|u\|^p, \tag{2.1}$$

where

$$M_0 = \int_0^{\infty} \left(\int_x^{\infty} \rho^{-\frac{1}{p-1}}(s)ds\right) dx.$$

Proof. For $u \in X$, we find

$$|u(x)| = \Big|\int_x^{\infty} u'(s)ds\Big| = \Big|\int_x^{\infty} \rho^{\frac{1}{p}}(s)u'(s)\rho^{-\frac{1}{p}}(s)ds\Big|.$$

Then, by the *Hölder* inequality, we obtain

$$\begin{aligned}|u(x)|^p &\le \left(\int_x^\infty \rho(s)|u'(s)|^p ds\right)\left(\int_x^\infty \rho^{-\frac{1}{p-1}}(s)ds\right)\\ &\le \left(\int_0^\infty \rho(s)|u'(s)|^p ds\right)\left(\int_x^\infty \rho^{-\frac{1}{p-1}}(s)ds\right).\end{aligned}$$

Hence,

$$\int_0^\infty |u(x)|^p dx \le \left(\int_0^\infty \left(\int_x^\infty \rho^{-\frac{1}{p-1}}(s)ds\right)dx\right)\left(\int_0^\infty \rho(s)|u'(s)|^p ds\right).$$

As a result we obtain (2.1). □

Lemma 2.3. *Let $u \in X$. Then*

$$\|u\|_\infty \le M\|u\|,$$

where $M = \|\rho^{-\frac{1}{p-1}}\|_1^{\frac{p-1}{p}}$.

Proof. For $u \in X$, we get

$$\begin{aligned}|u(x)| &= \left|\int_0^x u'(s)ds\right|\\ &\le \int_0^x \rho^{-\frac{1}{p}}(s)\rho^{\frac{1}{p}}(s)|u'(s)|ds\\ &\le \left(\int_0^\infty \rho^{-\frac{1}{p-1}}(s)ds\right)^{\frac{p-1}{p}}\left(\int_0^\infty \rho(s)|u'(s)|^p ds\right)^{\frac{1}{p}}\\ &\le \|\rho^{-\frac{1}{p-1}}\|_1^{\frac{p-1}{p}}\|u\|.\end{aligned}$$

Hence, $\|u\|_\infty \le M\|u\|$. □

To prove that X embeds compactly in $C_0[0,+\infty)$ we need the following Corduneanu compactness criterion.

Lemma 2.4. [5] *Let $D \subset C_0([0,+\infty),\mathbb{R})$ be a bounded set. Then D is relatively compact if the following conditions hold:*
(a) D is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, i.e.

$$\begin{aligned}&\forall\, J \subset [0,+\infty) \text{ compact}, \forall\, \varepsilon > 0,\, \exists\, \delta > 0,\, \forall\, x_1, x_2 \in J:\\ &\quad |x_1 - x_2| < \delta \Longrightarrow |u(x_1) - u(x_2)| \le \varepsilon, \forall\, u \in D;\end{aligned}$$

(b) D is equiconvergent at $+\infty$ i.e.,

$$\begin{aligned}&\forall\, \varepsilon > 0, \exists\, T = T(\varepsilon) > 0 \text{ such that}\\ &\forall\, x : x \ge T(\varepsilon) \Longrightarrow |u(x) - u(+\infty)| \le \varepsilon, \forall\, u \in D.\end{aligned}$$

Lemma 2.5. *The embedding $X \hookrightarrow C_0[0,\infty)$ is compact.*

Proof. Let $D \subset X$ be a bounded set. Then, D is bounded in $C_0[0,\infty)$ by Lemma 2.3. Let $R > 0$ be such that $\|u\| \le R$ for all $u \in D$. We will apply Lemma 2.4.

(a) D is equicontinuous on every compact interval of $[0,+\infty)$. Let $u \in D$ and $x_1, x_2 \in J \subset [0,+\infty)$ where J is a compact sub-interval. Using *Hölder* inequality, we have

$$\begin{aligned} |u(x_1) - u(x_2)| &= \Big| \int_{x_1}^{x_2} u'(s)ds \Big| \\ &= \Big| \int_{x_1}^{x_2} \rho^{-\frac{1}{p}}(s)\rho^{\frac{1}{p}}(s)u'(s)ds \Big| \\ &\le \Big(\int_{x_1}^{x_2} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \Big(\int_{x_1}^{x_2} \rho(s)|u'(s)|^p ds \Big)^{\frac{1}{p}} \\ &\le \Big(\int_{x_1}^{x_2} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \|u\| \le R\Big(\int_{x_1}^{x_2} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \longrightarrow 0, \end{aligned}$$

as $|x_1 - x_2| \to 0$.

(b) D is equiconvergent at $+\infty$. For $x \in [0,+\infty)$ and $u \in D$, using the fact that $u(\infty) = 0$ and by *Hölder* inequality, we have

$$\begin{aligned} |u(x) - u(\infty)| &= |u(x)| \\ &= \Big| \int_{x}^{\infty} u'(s)ds \Big| \\ &\le \Big(\int_{x}^{\infty} \rho^{\frac{1}{p}}(s)|u'(s)|ds \Big) \Big(\int_{x}^{\infty} \rho^{-\frac{1}{p-1}}(s) \Big)^{\frac{p-1}{p}} \\ &\le \Big(\int_{x}^{\infty} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \|u\| \\ &\le R\Big(\int_{x}^{\infty} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \xrightarrow{x \to \infty} 0. \end{aligned}$$

□

Finally, we present the Browder Theorem which will be needed in our argument.

Definition 2.6. [6] *Let X be a reflexive real Banach space and X^* its dual. The operator $\mathcal{L} : X \to X^*$ is called to be demicontinuous if $\mathcal{L}$ maps strongly convergent sequences in X to weakly convergent sequences in X^*.*

Lemma 2.7 (Browder theorem). [3], [6] *Let X be a reflexive real Banach space. Moreover, Let $\mathcal{L} : X \to X^*$ be an operator satisfying the following conditions:*
(i) $\mathcal{L}$ is bounded and demicontinuous;
(ii) $\mathcal{L}$ is coercive, that is, $\lim_{\|u\|\to\infty} \dfrac{\langle \mathcal{L}(u), u\rangle}{\|u\|} = +\infty$;
(iii) $\mathcal{L}$ is monotone on the space X; that is; for all $u, v \in X$; one has

$$\langle \mathcal{L}(u) - \mathcal{L}(v), u - v\rangle \ge 0. \tag{2.2}$$

Then the equation $\mathcal{L}(u) = f^*$ has at least one solution $u \in X$ for every $f^* \in X^*$. If, moreover, the inequality (2.2) is strict for all $u, v \in X$, $u \neq v$, then the equation $\mathcal{L}(u) = f^*$ has precisely one solution $u \in X$ for all $f^* \in X^*$.

3. Results

Suppose the following hypotheses hold:

(H1) The function $f(x,u)$ is decreasing about u, uniformly in $x \in [0,\infty)$; and $I_j(u)$ ($j \in \mathbb{N}^*$) are increased functions with u.

(H2) There exist $\alpha_j, \beta_j > 0$ and $\gamma \in [1,p)$ with $\sum_{j=1}^{\infty} \alpha_j g(x_j) < \infty$, $\sum_{j=1}^{\infty} \beta_j g(x_j) < \infty$, such that

$$|I_j(u)| \leq \alpha_j + \beta_j |u|^{\gamma-1}, \quad \text{for all } u \in \mathbb{R} \text{ and } j \in \mathbb{N}^*.$$

(H3) There exist positive functions $c_1, c_2 \in L^{\frac{p}{p-1}}[0,\infty)$ and a constant $\mu \in (0, p-1)$ such that

$$|f(x,u)| \leq c_1(x) + c_2(x)|u|^{\mu}, \quad \forall (x,u) \in [0,\infty) \times \mathbb{R}.$$

Let $\mathcal{L}$ be the operator defined from X into X^* by

$$\langle \mathcal{L}(u), v \rangle = \langle L_1(u), v \rangle + \langle L_2(u), v \rangle - \langle L_3(u), v \rangle, \qquad \forall u, v \in X,$$

where

$$\langle L_1(u), v \rangle = \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx,$$

$$\langle L_2(u), v \rangle = \sum_{j=1}^{\infty} g(x_j) I_j(u(x_j)) v(x_j),$$

$$\langle L_3(u), v \rangle = \int_0^{+\infty} f(x, u(x)) v(x) dx.$$

We search for a weak solution of problem (1.1) which is a solution for the operator equation $\mathcal{L}(u) = 0$.

Theorem 3.1. *Assume that $(H1)$-$(H3)$ hold. Then (1.1) has a unique weak solution.*

Proof. The proof consists of four steps:

Claim 1. $\mathcal{L}$ is bounded and demicontinuous.

It is sufficient to show that the operators $L_i (i = 1,2,3)$ are bounded and continuous. Firstly, we prove that $\mathcal{L}$ is bounded.

Using *Hölder* inequality, together with the following result

$$\forall a,b,c,d > 0 \quad \forall \beta \in (0,1): \quad (a+b)^{\beta}(c+d)^{1-\beta} \geq a^{\beta}c^{1-\beta} + b^{\beta}d^{1-\beta},$$

we obtain for all $u, v \in X$, (see [7]),

$$|\langle L_1(u), v \rangle| = \Big| \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx \Big|$$

$$
\begin{aligned}
&\leq \left(\int_0^{+\infty} \rho(x)|u'(x)|^p dx + \int_0^{+\infty} |u(x)|^p dx\right)^{\frac{p-1}{p}} \\
&\times \left(\int_0^{+\infty} \rho(x)|v'(x)|^p dx + \int_0^{+\infty} |v(x)|^p dx\right)^{\frac{1}{p}} \\
&\leq \|u\|^{p-1}\|v\| \\
&< \infty,
\end{aligned}
$$

as a result, L_1 is bounded.
Now, we prove the boundedness of L_2 and L_3 respectively. Using Lemma 2.3 and $(H2)$, gives

$$
\begin{aligned}
|\langle L_2(u), v\rangle| &= \Big|\sum_{j=1}^{\infty} g(x_j) I_j(u(x_j)) v(x_j)\Big| \\
&\leq \sum_{j=1}^{\infty} g(x_j)|I_j(u(x_j))||v(x_j)| \\
&\leq \sum_{j=1}^{\infty} g(x_j)(\alpha_j + \beta_j |u(x_j)|^{\gamma-1})|v(x_j)| \\
&\leq \sum_{j=1}^{\infty} (\alpha_j g(x_j) + \beta_j g(x_j)\|u\|_\infty^{\gamma-1})\|v\|_\infty \\
&\leq M\Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + M^{\gamma-1}\|u\|^{\gamma-1}\sum_{j=1}^{\infty}\beta_j g(x_j)\Big)\|v\| \\
&< \infty, \quad \forall u, v \in X,
\end{aligned}
$$

that implies L_2 is bounded.
From the condition (H_3), we get

$$
|\langle L_3(u), v\rangle| = \Big|\int_0^{+\infty} f(x, u(x)) v(x) dx\Big| \leq \int_0^{+\infty} (c_1(x) + c_2(x)|u(x)|^\mu)\,|v(x)| dx,
$$

by the *Hölder* inequality, Lemma 2.2 and Lemma 2.3, we arrive immediately at

$$
\begin{aligned}
|\langle L_3(u), v\rangle| \leq &\Big(\int_0^{+\infty} |c_1(x)|^{\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \Big(\int_0^{+\infty} |v(x)|^p dx\Big)^{\frac{1}{p}} + \|u\|_\infty^\mu \\
&\Big(\int_0^{+\infty} |c_2(x)|^{\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \Big(\int_0^{+\infty} |v(x)|^p dx\Big)^{\frac{1}{p}} \\
&\leq \|c_1\|_{\frac{p}{p-1}}\|v\|_p + \|u\|_\infty^\mu \|c_2\|_{\frac{p}{p-1}}\|v\|_p \\
&\leq M_0^{\frac{1}{p}}\left(\|c_1\|_{\frac{p}{p-1}} + M^\mu \|c_2\|_{\frac{p}{p-1}}\|u\|^\mu\right)\|v\| \\
&< \infty,
\end{aligned}
$$

as a consequence, L_3 is bounded. We deduce that $\mathcal{L}$ is a bounded operator.
Secondly, we prove that $\mathcal{L}$ is demicontinuous.
For $u_n \to u$ in X, we have

$$|\langle L_1(u_n) - L_1(u), u_n - u\rangle| \leq \Big(\int_0^{+\infty} \rho(x)\Big(|u_n'(x)|^{p-2}u_n'(x) - |u'(x)|^{p-2}u'(x)\Big)^{\frac{p}{p-1}} dx$$
$$+ \int_0^{+\infty} \Big(|u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x)\Big)^{\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \|u_n - u\|.$$

Since $\lim_{n\to\infty} \|u_n - u\| = 0$, the last integral tends to zero. We see that L_1 is continuous.
To show the continuity of L_2, we prove that L_2 is strongly continuous, that is, if $u_n \rightharpoonup u$ in X then $L_2(u_n) \to L_2(u)$, as $n \to \infty$.
Assume $u_n \rightharpoonup u$ in X, Lemma 2.5 guarantees that (u_n) converges uniformly to u on $[0, \infty)$, as $n \to \infty$. Since I_j are continuous, then

$$I_j(u_n(x_j)) \to I_j(u(x_j)), \quad n \to \infty, \quad j \in \mathbb{N}^*,$$

moreover, from $(H2)$ we get

$$\sum_{j=1}^{\infty} g(x_j)I_j(u_n(x_j)) < \infty,$$

by applying Lebesgue's dominated convergence theorem, we obtain

$$\sum_{j=1}^{\infty} g(x_j)I_j(u_n(x_j)) \to \sum_{j=1}^{\infty} g(x_j)I_j(u(x_j)) \quad \text{as} \quad n \to \infty,$$

concequently,

$$|\langle L_2(u_n) - L_2(u)\rangle| \to 0 \quad \text{as} \quad n \to \infty,$$

that means L_2 is strongly continuous and therefore it is continuous.
In what follows, we discuss the continuity of L_3.
Let (u_n) be such that $u_n \rightharpoonup u$ in X. So (u_n) is bounded in X and by Lemma 2.5, we have that (u_n) is bounded in $C_0[0, +\infty)$. By Lemma 2.5, $u_n \to u$ in $C_0[0, +\infty)$. We have

$$\begin{aligned}\|L_3(u_n) - L_3(u)\|_{X^*} &= \sup_{\|v\|\leq 1} |\langle L_3(u_n) - L_3(u)\rangle| \\ &= \sup_{\|v\|\leq 1} \Big| \int_0^{+\infty} \Big[f(x, u_n(x)) - f(x, u(x))\Big] dx\Big| \\ &\leq \sup_{\|v\|\leq 1} \Big(\int_0^{+\infty} |f(x, u_n(x))v(x)| dx\Big)\end{aligned}$$

$$
\begin{aligned}
&+ \sup_{\|v\|\le 1}\Big(\int_0^{+\infty} |f(x,u(x))v(x)|dx\Big)\\
&\le \sup_{\|v\|\le 1}\Big(\int_0^{+\infty} (c_1(x)|v(x)| + c_2(x)|u_n(x)|^{\mu}|v(x)|)dx\Big)\\
&+ \sup_{\|v\|\le 1}\Big(\int_0^{+\infty} (c_1(x)|v(x)| + c_2(x)|u(x)|^{\mu}|v(x)|dx\Big)\\
&\le 2\|c_1\|_{\frac{p}{p-1}} + \Big(\|u_n\|_{\infty}^{\mu} + \|u\|_{\infty}^{\mu}\Big)\\
&\le 2\|c_1\|_{\frac{p}{p-1}} + M^{\mu}\|c_2\|_{\frac{p}{p-1}}\Big(\|u_n\|^{\mu} + \|u\|^{\mu}\Big)\\
&\le 2\|c_1\|_{\frac{p}{p-1}} + CM^{\mu}\|c_2\|_{\frac{p}{p-1}}.
\end{aligned}
$$

for some constant $C > 0$. Since $u_n \to u$, $n \to \infty$ in $C_0[0,+\infty)$, we obtain

$$
\int_0^{+\infty} \left(f(x,u_n(x)) - f(x,u(x))\right)v(x)dx \to 0 \quad \text{as} \qquad n \to \infty,
$$

this implies that L_3 is continuous. Thus the operator $\mathcal{L}$ is continuous and hence it is demicontinuous. So assumption (i) of Lemma 2.7 holds.

Claim 2. $\mathcal{L}$ *is monotone.*

By $(H1)$, for all $u, v \in X$, we have

$$
\begin{aligned}
\langle \mathcal{L}(u) - \mathcal{L}(v), u - v\rangle &= \int_0^{+\infty} \rho(x)\Big[|u'(x)|^{p-2}u'(x) - |v'(x)|^{p-2}v'(x)\Big](u'(x) - v'(x))dx\\
&+ \int_0^{+\infty} \Big[|v(x)|^{p-2}v(x) - |v(x)|^{p-2}v(x)\Big](u(x) - v(x))dx\\
&- \int_0^{+\infty} \Big[f(x,u(x)) - f(x,v(x))\Big](u(x) - v(x))dx\\
&+ \sum_{j=1}^{\infty} \Big[g(x_j)I_j(u(x_j)) - g(x_j)I_j(v(x_j))\Big](u(x_j) - v(x_j))\\
&\ge \int_0^{+\infty} \rho(x)\Big[|u'(x)|^{p-2}u'(x) - |v'(x)|^{p-2}v'(x)\Big](u'(x) - v'(x))dx\\
&+ \int_0^{+\infty} \Big[|v(x)|^{p-2}v(x) - |v(x)|^{p-2}v(x)\Big](u(x) - v(x))dx
\end{aligned}
$$

$$\geq \left(\|u\|^{p-1} - \|v\|^{p-1}\right)\left(\|u\| - \|v\|\right)$$
$$\geq 0,$$

so, $\mathcal{L}$ is monotone.

Claim 3. $\mathcal{L}$ is coercive.

For all $u, v \in X$, we have

$$\langle \mathcal{L}(u), u\rangle = \|u\|^p + \sum_{j=1}^{\infty} g(x_j) I_j(u(x_j))u(x_j) - \int_0^{+\infty} f(x, u(x))u(x)dx.$$

From Lemma 2.2 and Lemma 2.3, combining assumption $(H2)$ and $(H3)$, we find

$$\begin{aligned}\langle \mathcal{L}(u), u\rangle &\geq \|u\|^p - \sum_{j=1}^{\infty} g(x_j) I_j(u(x_j))u(x_j) - \int_0^{+\infty} f(x, u(x))u(x)dx \\ &\geq \|u\|^p - \sum_{j=1}^{\infty} g(x_j)\left(\alpha_j + \beta_j |u(x_j)|^{\gamma-1}\right)u(x_j) \\ &\quad - \int_0^{+\infty} \left(c_1(x) + c_2(x)|u(x)|^{\mu}\right)|u(x)|dx\end{aligned}$$

hence,

$$\begin{aligned}\langle \mathcal{L}(u), u\rangle &\geq \|u\|^p - \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + \sum_{j=1}^{\infty} \beta_j g(x_j)\|u\|_{\infty}^{\gamma-1}\Big)\|u\|_{\infty} - \int_0^{+\infty} c_1(x)|u(x)|dx \\ &\quad - \int_0^{+\infty} c_2(x)|u(x)|^{\mu}|u(x)|dx \\ &\geq \|u\|^p - \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + \sum_{j=1}^{\infty} \beta_j g(x_j)\|u\|_{\infty}^{\gamma-1}\Big)\|u\|_{\infty} - \|c_1\|_{\frac{p}{p-1}}\|u\|_p \\ &\quad - \|u\|_{\infty}^{\mu}\|c_2\|_{\frac{p}{p-1}}\|u\|_p \\ &\geq \|u\|^p - \Big(M\sum_{j=1}^{\infty} \alpha_j g(x_j) + M_0^{\frac{1}{p}}\|c_1\|_{\frac{p}{p-1}}\Big)\|u\| - \Big(M^{\gamma}\sum_{j=1}^{\infty} \beta_j g(x_j)\Big)\|u\|^{\gamma} \\ &\quad - M_0^{\frac{1}{p}} M^{\mu}\|c_2\|_{\frac{p}{p-1}}\|u\|^{\mu+1},\end{aligned}$$

so $\lim_{\|u\|\to\infty} \frac{\langle \mathcal{L}(u), u\rangle}{\|u\|} = +\infty$.

Lemma 2.7 guarantees that problem (1.1) has a weak solution.

Claim 4. Uniqueness.

For all $u, v \in X$, $u \neq v$, we have

$$\langle \mathcal{L}(u) - \mathcal{L}(v), u - v\rangle \geq \left(\|u\|^{p-1} - \|v\|^{p-1}\right)\left(\|u\| - \|v\|\right) > 0,$$

so $\mathcal{L}$ is strictly monotone. □

Example 3.2. Let $p = 4$ and $\gamma = \frac{5}{2}$. Consider the problem

$$\begin{cases} -(e^{3x}|u'|u')' + |u|u &= e^{-x} - 2e^{-3x}u^2, \qquad \text{a.e. } x \neq x_j,\ x \geq 0, \\ \triangle(e^{3j}u'(j)) &= e^{-j}\left(\frac{1}{j} + \frac{1}{j^2}|u(j)|^{\frac{3}{2}}\right), \qquad j \in \mathbb{N}^*, \\ u(0) = u(\infty) &= 0, \end{cases}$$

where $c_1(x) = e^{-x}$, $c_2(x) = 2e^{-3x}$ and $g(x) = e^{-x}$.
It's clear that $(H1) - (H3)$ hold true. Hence, we may apply Lemma 2.7 and conclude that (1.1) has precisely a weak solution.

Next, we consider the limit case $\mu = p - 1$.

Theorem 3.3. *Assume that $(H1)$ and $(H2)$ are hold both with*

(H4) *There exist positive functions $c_1, c_2 \in L^{\frac{p}{p-1}}[0,\infty)$ such that*

$$|f(x,u)| \leq c_1(x) + c_2(x)|u|^{p-1}, \quad \forall (x,u) \in [0,\infty) \times \mathbb{R}.$$

with

$$M_0^{\frac{1}{p}} M^{p-1} \|c_2\|_{\frac{p}{p-1}} < 1.$$

Then (1.1) has a unique weak solution.

Proof. Arguing as in the proof of Theorem 3.1, we prove that $\mathcal{L}$ is bounded, demicontinuous and monotone.
We check that $\mathcal{L}$ is a coercive. Indeed, under $(H2)$, $(H4)$, in view of Lemma 2.2 and Lemma 2.3, it is easy to verify that

$$\begin{aligned} \langle \mathcal{L}(u), u\rangle &\geq \|u\|^p - \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + \sum_{j=1}^{\infty} \beta_j g(x_j) \|u\|_\infty^{\gamma-1}\Big)\|u\|_\infty - \int_0^{+\infty} c_1(x)|u(x)|dx \\ &\quad - \int_0^{+\infty} c_2(x)|u(x)|^{p-1}|u(x)|dx \\ &\geq \|u\|^p - \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + \sum_{j=1}^{\infty} \beta_j g(x_j) \|u\|_\infty^{\gamma-1}\Big)\|u\|_\infty \\ &\quad - \|c_1\|_{\frac{p}{p-1}} \|u\|_p - \|u\|_\infty^{p-1} \|c_2\|_{\frac{p}{p-1}} \|u\|_p \\ &\geq \|u\|^p - M\Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + M^{\gamma-1}\|u\|^{\gamma-1} \sum_{j=1}^{\infty} \beta_j g(x_j)\Big)\|u\| - M_0^{\frac{1}{p}} \|c_1\|_{\frac{p}{p-1}} \|u\| \\ &\quad - M_0^{\frac{1}{p}} M^{p-1} \|c_2\|_{\frac{p}{p-1}} \|u\|^p \\ &\geq \left(1 - M_0^{\frac{1}{p}} M^{p-1} \|c_2\|_{\frac{p}{p-1}}\right)\|u\|^p - \left(M \sum_{j=1}^{\infty} \alpha_j g(x_j) + M_0^{\frac{1}{p}} \|c_1\|_{\frac{p}{p-1}}\right)\|u\| \end{aligned}$$

$$- \Big(M^{\gamma}\sum_{j=1}^{\infty}\beta_j g(x_j)\Big)\|u\|^{\gamma},$$

we conclude that $\lim_{\|u\|\to\infty}\frac{\langle\mathcal{L}(u),u\rangle}{\|u\|}=+\infty$.
Theorem 3.3 guarantees that problem (1.1) has a unique weak solution. □

Example 3.4. Let $p=2$ and $\gamma=\frac{1}{2}$. Consider the problem

$$\begin{cases} -(e^x|u'|u')'+|u|u &= e^{-\frac{1}{2}x}-e^{-x}|u|, \quad \text{a.e. } x\neq x_j,\ x\geq 0,\\ \triangle(e^j u'(j)) &= e^{-j}\left(\frac{1}{j}+\frac{1}{j^2}|u(j)|^{\frac{1}{2}}\right), \qquad j\in\mathbb{N}^*,\\ u(0)=u(\infty) &= 0, \end{cases}$$

where $c_1(x)=e^{-\frac{1}{2}x}$, $c_2(x)=e^{-x}$, $\|c_2\|_2=\frac{1}{\sqrt{2}}$, $g(x)=e^{-x}$ and $M=M_0=1$.
It's clear that $(H1),(H2)$ and $(H4)$ hold true. Hence, from Lemma 2.7 we find that problem (1.1) has precisely a weak solution.

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Global well-posedness for the generalized Keller-Segel system in critical Besov-Morrey spaces with variable exponent

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Abstract. This article is devoted to studying the generalized Keller-Segel system (GKS) in homogeneous variable exponent Besov-Morrey spaces. By making use of the Littlewood-Paley theory and the Chemin mono-norm methods, we obtain, when $\frac{1}{2} < \beta \leq 1$, a global well-posedness result for GKS system with small initial data in the critical variable exponent Besov-Morrey spaces $\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}(\mathbb{R}^n)$ with $1 \leq r(\cdot) \leq q(\cdot) < \infty$, $1 \leq h \leq \infty$. In the limit case $\beta = \frac{1}{2}$, we show the global well-posedness for small initial data in $\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}(\mathbb{R}^n)$ with $1 \leq r(\cdot) \leq q(\cdot) < \infty$.

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1. Introduction

We are concerned with the generalized Keller-Segel system given by the following fractional diffusion:

$$\begin{cases} u_t + (-\Delta)^\beta u = -\nabla \cdot (u \nabla \psi) & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ -\Delta \psi = u & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1.1}$$

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where $n \geq 2,\ u = u(x,t)$ denotes the unknown density of cells, $\psi = \psi(x,t)$ represents the unknown concentration of the chemo-attractant, u_0 is the initial data, ∇ is the gradient operator, $(-\Delta)^\beta$ is the Laplacian operator, which is the Fourier multiplier with symbol $|\xi|^{2\beta}$, and $\frac{1}{2} \leq \beta \leq 1$, that is, the abnormal (normal) diffusion is modeled by a fractional power of the Laplacian.

Note that the function ψ, which is determined by the Poisson equation, is given by the second equation of (1.1) as the volume potential of v:

$$\psi(x,t) = (-\Delta)^{-1} u(x,t).$$

We can therefore eliminate ψ from the system (1.1) and get the following equivalent problem:

$$\begin{cases} u_t + (-\Delta)^\beta u = -\nabla \cdot (u \nabla (-\Delta)^{-1} u) & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{1.2}$$

For $\beta = 1$, (1.1) corresponds to the classical Keller-Segel equation which is a simplified system of

$$\begin{cases} u_t - \Delta u = -\nabla \cdot (u \nabla \psi) & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \psi_t - \Delta \psi = u - \psi & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad \psi(x,0) = \psi_0(x) & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{1.3}$$

The system (1.3) was introduced by Keller and Segel [17] in 1970. It describes a chemotaxis mathematical model, and it is also linked to astrophysical models of gravitational auto-interaction of huge particles in a cloud or nebula, the reader may refer to [6]. The well-posedness of classical Keller-Segel models has been studied by several researchers in various spaces. Recently, making use of the smoothing effect of the heat semigroup, Iwabuchi [16] proved the global well-posedness of the system (1.3) in $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ where $n \geq 1$ and $\max\{1, n/2\} < p < \infty$, under the condition of smallness of the initial data. Later, by the same method, Nogayama and Sawano [18] extended this well-posedness result, where they established global well-posedness in the Besov-Morrey spaces $\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}(\mathbb{R}^n)$ with $\max\left\{1, \frac{n}{2}\right\} < p < \infty$ and $1 \leq h \leq p$.

For the general case $\frac{1}{2} < \beta < 1$, (1.1) was initially considered by Escudero [11], in which it was utilized to characterize the spatio-temporal distribution of a population density of random walkers subjected to Lévy flights. Furthermore, in that paper, it has been established that (1.1) in this case, has global in time solutions. There are many studies on (1.1) by several researchers in various spaces. Recently, Zhao [21] obtained well-posedness results of (1.1) in the classical Besov spaces $\dot{B}^{-2\beta+\frac{n}{p}}_{p,r}(\mathbb{R}^n)$ with $\frac{1}{2} \leq \beta \leq 1$ and $1 \leq p, r \leq \infty$. We mention that certain aspects of these results were also extended to the fractional power bipolar type drift-diffusion system. Further information on this topic can be found in [14, 12] and the relevant references cited therein.

Inspired by this work, we aim to investigate, by making use of the Chemin mono-norm methods, global well-posedness of the generalized Keller-Segel system (1.1) with initial data in the critical variable exponent Besov-Morrey spaces $\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}(\mathbb{R}^n)$ with $\frac{1}{2} \leq \beta \leq 1,\ 1 \leq r(\cdot) \leq q(\cdot) < \infty$ and $1 \leq h \leq \infty$.

In general, variable exponent function spaces have garnered significant attention from researchers in recent times. This interest extends beyond theoretical aspects, encompassing their pivotal role in various applications, such as fluid dynamics [20] and resolving specific equations [10, 4]. Notably, variable exponent Besov-Morrey space, based on variable exponent Morrey spaces, is a new large framework compared to Besov space, i.e. variable exponent Besov-Morrey spaces are strictly broader than classical Besov spaces (also refer to Remark 2). However, there are many challenges in addressing the well-posedness of equations in these spaces. Replacing the L^p-norm by the $\mathcal{M}^{q(\cdot)}_{r(\cdot)}$-norm is not sufficient to ensure a direct transition from Besov spaces to variable exponent Besov-Morrey spaces. One of the main difficulties comes from the collapse of certain essential embedding features and the inapplicability of certain classical theories, like the multiplier theorem and Young's inequality, within Besov-Morrey spaces with variable exponents, unlike classical Besov spaces. To overcome these challenges, the present paper primarily relies on the properties described in Section 2 to look at the global well-posedness result. For an in-depth exploration of these variable exponent function spaces, we direct the reader to [1, 8, 9, 10, 19, 13, 20, 15, 2, 3] and the associated references therein.

To address the system (1.1), passing via (1.2), we think about the following equivalent integral equations:

$$u(t) = e^{-t(-\Delta)^\beta} u_0 - \int_0^t e^{-(t-t')(-\Delta)^\beta} \nabla \cdot \big(u \nabla (-\Delta)^{-1} u\big) \, \mathrm{d}t', \tag{1.4}$$

where $e^{-t(-\Delta)^\beta} := \mathcal{F}^{-1}(e^{-t|\xi|^{2\beta}} \mathcal{F})$ is the fractional heat semigroup operator.

Organization of the paper: In Section 2, we present some basic background information on the Littlewood-Paley theory and some different laws on products in variable exponent Besov-Morrey spaces, and then, in Section 3, we state and prove our main theorem.

2. Preliminaries

We introduce some background knowledge on Littlewood-Paley theory and variable exponent Besov-Morrey spaces, and present some propositions relevant to our objectives. Firstly, we start by introducing some of the notations used in the present paper, $E \lesssim H$ designates having a constant $C > 0$, which can be different at different places, such that $E \leq CH$ and $E \sim H$ designates having two constants $C_1, C_2 > 0$ such that $C_1 H \leq E \leq C_2 H$. We define, for two Banach spaces X and Y, and $u \in X \cap Y$, the norm $\|\cdot\|_{X \cap Y}$ as

$$\|u\|_{X \cap Y} := \|u\|_X + \|u\|_Y \,.$$

Definition 2.1. [3] *For the measurable function $r(\cdot)$, let*

$$\mathcal{P}_0(\mathbb{R}^n) := \left\{ r(\cdot) : \mathbb{R}^n \to (0, \infty]; 0 < r_- = \operatorname*{essinf}_{x \in \mathbb{R}^n} r(x), \operatorname*{esssup}_{x \in \mathbb{R}^n} r(x) = r_+ < \infty \right\}$$

The Lebesgue space with variable exponent is defined by

$$L^{r(\cdot)}(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable, } \int_{\mathbb{R}^n} |u(x)|^{r(x)} dx < \infty \right\},$$

with norm

$$\begin{aligned} \|u\|_{L^{r(\cdot)}} &:= \inf\left\{\lambda > 0 : \varrho_{r(\cdot)}(u/\lambda) \leq 1\right\} \\ &= \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|u(x)|}{\lambda}\right)^{r(x)} dx \leq 1\right\}. \end{aligned}$$

We use the following notation to separate variable exponents from constant exponents: $r(\cdot)$ for variable exponents, r for constant exponents. Also $(L^{r(\cdot)}(\mathbb{R}^n), \|u\|_{L^{r(\cdot)}})$ is a Banach space.

$L^{r(\cdot)}$ doesn't have the same features as L^r. Therefore, to assure the boundedness of the maximal Hardy-Littlewood operator M on $L^{r(\cdot)}(\mathbb{R}^n)$, the following standard conditions are assumed:

1. (Locally log-Hölder's continuous)[3] There exists a constant $C_{\log}(r)$ such that

$$|r(x) - r(y)| \leq \frac{C_{\log}(r)}{\log\left(e + |x-y|^{-1}\right)}, \text{ for all } x, y \in \mathbb{R}^n,\ x \neq y.$$

2. (Globally log-Hölder's continuous)[3] There exist two constants $C_{\log}(r)$ and r_∞ such that

$$|r(x) - r_\infty| \leq \frac{C_{\log}(r)}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$

$C_{\log}(\mathbb{R}^n)$ denotes the set of all functions $r(\cdot) : \mathbb{R}^n \to \mathbb{R}$ that satisfy 1 and 2.

Definition 2.2. [2] *Let $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < r_- \leq r(\cdot) \leq q(\cdot) \leq \infty$, the variable exponent Morrey space $\mathcal{M}^{q(\cdot)}_{r(\cdot)} := \mathcal{M}^{q(\cdot)}_{r(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions on $\mathbb{R}^n$ such that*

$$\|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} := \sup_{x_0 \in \mathbb{R}^n, R>0} \left\| R^{\frac{n}{q(x)} - \frac{n}{r(x)}} u \right\|_{L^{r(\cdot)}(B(x_0,R))} < \infty.$$

Here we give an important lemma.

Lemma 2.3. [2] *Let $r(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. Then for any measurable function u*

$$\sup_{x_0 \in \mathbb{R}^n, R>0} \varrho_{r(\cdot)}\left(u \chi_{B(x_0,R)}\right) = \varrho_{r(\cdot)}(u),$$

and $\|u\|_{\mathcal{M}^{r(\cdot)}_{r(\cdot)}} = \|u\|_{L^{r(\cdot)}}$.

We now recall the Littlewood-Paley decomposition (refer to [5] for further information). Consider $\varphi \in \mathcal{S}(\mathbb{R}^n)$ a smooth radial function such that

$$0 \le \varphi \le 1,$$
$$\operatorname{supp} \varphi \subset \left\{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\},$$
$$\sum_{j \in Z} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0,$$

and we denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Then for every $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the frequency localization operators for all $j \in \mathbb{Z}$, as follows

$$\Delta_j u = \mathcal{F}^{-1}\varphi_j * u \qquad \text{and} \qquad S_j u = \sum_{k \le j-1} \Delta_k u. \tag{2.1}$$

One observes here that $\dot{\Delta}_j$ has frequency $\{|\xi| \sim 2^j\}$ and that $\dot{S}_j$ has frequency $\{|\xi| \lesssim 2^j\}$, and one also notes that the quasi-orthogonality property holds for the Littlewood-Paley decomposition, that is, for every $u, v \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$,

$$\dot{\Delta}_i \dot{\Delta}_j u = 0 \quad \text{if} \quad |i-j| \ge 2, \quad \dot{\Delta}_i\left(\dot{S}_{j-1} u \dot{\Delta}_j v\right) = 0 \qquad \text{if} \quad |i-j| \ge 5, \tag{2.2}$$

with $\mathcal{P}(\mathbb{R}^n)$ denoting the collection of all polynomials over $\mathbb{R}^n$.

All through this document, we will use the following Bony paraproduct decomposition:

$$uv = T_u v + T_v u + R(u,v), \tag{2.3}$$

with

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \quad R(u,v) = \sum_j \sum_{|j-l| \le 1} \Delta_j u \Delta_l v.$$

Definition 2.4. [2] *Let $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $r(\cdot) \le q(\cdot)$, the mixed Morrey-sequence space $\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})$ is the set of all sequences $\{a_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that*

$$\|\{a_j\}_{j\in\mathbb{Z}}\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} := \inf\left\{\lambda > 0 : \varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\{a_j/\lambda\}_{j\in\mathbb{Z}}) \le 1\right\},$$

where

$$\varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\{a_j\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}} \inf\left\{\nu > 0 : \int_{\mathbb{R}^n}\left(\frac{|R^{\frac{n}{q(x)}-\frac{n}{r(x)}} a_j \chi_{B(x_0,R)}|}{\nu^{\frac{1}{h(x)}}}\right)^{r(x)} dx \le 1\right\}$$

Notice that if $h_+ < \infty$ and $r(\cdot) \le h(\cdot)$, then

$$\varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\{a_j\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}} \sup_{x_0\in\mathbb{R}^n, R>0} \left\|\left(R^{\frac{n}{q(x)}-\frac{n}{r(x)}} u\right)^{h(x)}\right\|_{L^{\frac{r(\cdot)}{h(\cdot)}}(B(x_0,R))}.$$

Definition 2.5. [2] *Let $s(\cdot) \in C_{\log}(\mathbb{R}^n)$ and $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ with $0 < r_- \leq r(\cdot) \leq q(\cdot) \leq \infty$. The variable exponent homogeneous Besov-Morrey space $\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$ is defined by*

$$\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)} := \left\{ u \in \mathcal{D}'(\mathbb{R}^n) : \|u\|_{\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} < \infty \right\},$$

with norm

$$\|u\|_{\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} := \left\| \left\{ 2^{js(\cdot)} \Delta_j u \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},$$

and $\mathcal{D}'(\mathbb{R}^n)$ represents the dual space of

$$\mathcal{D}(\mathbb{R}^n) = \{ u \in \mathcal{S}(\mathbb{R}^n) : (D^\alpha u)(0) = 0, \text{ for all multi-index } \alpha \}.$$

For $T > 0$ and $1 \leq h, \rho \leq \infty$. The mixed space-time space $\mathcal{L}^\rho \left(0, T; \mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)$ is the set of all tempered distribution u satisfying

$$\|u\|_{\mathcal{L}^\rho\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} := \left(\sum_{j\in\mathbb{Z}} \left\| 2^{js(\cdot)} \Delta_j u \right\|^h_{L^\rho_T\left(\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} \right)^{\frac{1}{h}} < \infty,$$

where

$$\|u\|_{L^\rho_T\left(\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} := \left(\int_0^T \|u(\cdot,t)\|^\rho_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \,\mathrm{nt} \right)^{\frac{1}{\rho}}.$$

With the standard modification if $h = \infty$ or $\rho = \infty$.

Proposition 2.6. *The following inclusions hold for variable exponent Morrey spaces.*

1. *(Hölder's inequality)*[1] *Let $r(\cdot), r_1(\cdot), r_2(\cdot), q(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ satisfying $r(\cdot) \leq q(\cdot)$, $r_i(\cdot) \leq q_i(\cdot) (i = 1, 2)$, $\frac{1}{r(\cdot)} = \frac{1}{r_1(\cdot)} + \frac{1}{r_2(\cdot)}$ and $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$. Then for all $u \in \mathcal{M}^{q_1(\cdot)}_{r_1(\cdot)}$ and $v \in \mathcal{M}^{q_2(\cdot)}_{r_2(\cdot)}$, there is a constant C depending only on r_- and r_+ such that*
$$\|uv\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C \|u\|_{\mathcal{M}^{q_1(\cdot)}_{r_1(\cdot)}} \|v\|_{\mathcal{M}^{q_2(\cdot)}_{r_2(\cdot)}}. \tag{2.4}$$
And for all $u \in L^\infty(\mathbb{R}^n)$ and $v \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, there is a constant C such that
$$\|uv\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C \|u\|_{L^\infty} \|v\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}. \tag{2.5}$$
2. *(Sobolev-type embedding)* [1] *Let $r_1(\cdot), r_2(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $0 < h < \infty$ and $s_1(\cdot), s_2(\cdot) \in L^\infty \cap C_{\log}(\mathbb{R}^n)$ with $s_1(\cdot) > s_2(\cdot)$. If $\frac{1}{h}$ and*
$$s_1(\cdot) - \frac{n}{r_1(\cdot)} = s_2(\cdot) - \frac{n}{r_2(\cdot)}$$
are locally log-Hölder continuous, then
$$\mathcal{N}^{s_1(\cdot)}_{r_1(\cdot),q_1(\cdot),h} \hookrightarrow \mathcal{N}^{s_2(\cdot)}_{r_2(\cdot),q_2(\cdot),h}. \tag{2.6}$$

3. *(Mollification inequality)* [2] *Let* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *and* $\phi \in L^1(\mathbb{R}^n)$, *suppose* $\Phi(y) = \sup_{x \notin B(0,|y|)} |\phi(x)|$ *is integrable. Then for all* $u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, *there is a constant* C *depending only on* d *such that*

$$\|u * \phi_\varepsilon\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C \|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \|\Phi\|_{L^1}, \tag{2.7}$$

where $\phi_\varepsilon = \frac{1}{\varepsilon^d}\phi(\varepsilon)$.

Lemma 2.7. [2] *Let* $\mathcal{C}$ *be a ring, and* $\mathcal{B}$ *a ball in* $\mathbb{R}^n$, *and let* $k \in \mathbb{N}$, $j \in \mathbb{Z}$, $\lambda > 0$, *and* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *with* $r(\cdot) \leq q(\cdot) < \infty$.

1. *Assume that* $u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$ *satisfying* supp$\mathcal{F}(u) \subset \lambda\mathcal{B}$, *then*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C^{k+1}\lambda^k \|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}.$$

2. *Assume that* $u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$ *satisfying* supp$\mathcal{F}(u) \subset \lambda^j\mathcal{B}$, *then*

$$\|u\|_{L^\infty} \leq C\lambda^{j\frac{n}{q(\cdot)}} \|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}.$$

where C *is a constant independent of* λ.

Lemma 2.8. *Let* $m \in \mathbb{R}$, $s(\cdot) \in C_{\log}(\mathbb{R}^n)$, $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *with* $r(\cdot) \leq q(\cdot)$, *and let* $0 < h < \infty$. *Then*

$$\partial^m_\xi : \mathcal{N}^{s(\cdot)+m}_{r(\cdot),q(\cdot),h} \to \mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}$$

is bounded.

Proof. For the proof, we can use the same idea as in [14, Lemma 2] □

Lemma 2.9. *(*[5, Lemma 5.5]*) Let* X *be a Banach space with norm* $\|\cdot\|_X$ *and* B *be a bounded bilinear operator from* $X \times X$ *to* X *satisfying*

$$\|B(x_1, x_2)\|_X \leq C_0 \|x_1\|_X \|x_2\|_X,$$

for all $x_1, x_2 \in X$ *and a constant* $C_0 > 0$. *Then for any* $a \in X$ *such that* $\|a\|_X < \frac{1}{4C_0}$, *the equation* $x = a + B(x, x)$ *has a solution* x *in* X. *Moreover, the solution is such that* $\|x\|_X \leq 2\|a\|_X$, *and it is the only one such that* $\|x\|_X < \frac{1}{2C_0}$.

3. Well-posedness

In this section, we state our main theorem, and then prove it for the case $\frac{1}{2} < \beta \leq 1$ and the case $\beta = \frac{1}{2}$ in Subsection 3.1 and in Subsection 3.2, respectively.

Theorem 3.1. *Let* $n \geq 2$, $1 \leq r(\cdot) \leq q(\cdot) < \infty$ *and* $1 \leq h \leq \infty$.

1. *Let* $\frac{1}{2} < \beta \leq 1$. *Then there exists a constant* $\varepsilon > 0$ *such that for any* $u_0 \in \mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}$ *satisfying* $\|u_0\|_{\dot{\mathcal{N}}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} \leq \varepsilon$, *the system* (1.1) *admits a unique time-global solution* $u \in \mathcal{X}_\varepsilon$, *where*

$$\mathcal{X}_\varepsilon := \left\{u \in \mathcal{X}^0 \cap \mathcal{X}^1 : \ \|u\|_{\mathcal{X}^0} < \infty, \ \|u\|_{\mathcal{X}^1} \lesssim \varepsilon\right\},$$

with

$$\mathcal{X}^0 := \mathcal{L}^\infty \left(\mathbb{R}_+; \mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right),$$

$$\mathcal{X}^1 := \mathcal{L}^{\gamma_1} \left(\mathbb{R}_+; \mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right) \cap \mathcal{L}^{\gamma_2} \left(\mathbb{R}_+; \mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right),$$

and

$$\gamma_1 = \frac{2\beta}{2\beta - 1 + \varepsilon}, \ \ \gamma_2 = \frac{2\beta}{2\beta - 1 - \varepsilon}, \ \ 0 < \varepsilon < 2\beta - 1,$$

$$s_1(\cdot) = -1 + \frac{n}{q(\cdot)} + \varepsilon, \ s_2(\cdot) = -1 + \frac{n}{q(\cdot)} - \varepsilon.$$

2. *Let $\beta = \frac{1}{2}$. Assume that $u_0 \in \mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}$ is small enough. Then the system* (1.1) *admits a unique global solution v satisfying*

$$u \in \mathcal{L}^\infty \left(\mathbb{R}_+; \mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right).$$

The above result requires some further comments.

Remark 3.2.

1. The results of this work remain valid if we take variable exponent Besov space $\mathcal{B}^{s(\cdot)}_{r(\cdot),h(\cdot)}$ instead of variable exponent Besov-Morrey space $\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$. Indeed, if we have $r(\cdot) = q(\cdot)$, then $\mathcal{N}^{s(\cdot)}_{r(\cdot),r(\cdot),h(\cdot)} = \mathcal{B}^{s(\cdot)}_{r(\cdot),h(\cdot)}$.
2. Theorem 3.1 extends the corresponding well-posedness results of [21], where the author considered the system (1.1) in Besov spaces, which is a particular case of our framework which is variable exponent Besov-Morrey spaces $\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$. Moreover, we have $\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)} \not\subset \dot{B}^{s'}_{p',r'}$ for any $s' \in \mathbb{R}$, $1 \leq p' < \infty$ and $1 \leq r' < \infty$.

In order to prove Theorem 3.1, we consider the following linear equation:

$$\begin{cases} u_t + (-\Delta)^\beta u = f & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{3.1}$$

for which we get the following linear estimate:

Proposition 3.3. *(Linear estimate) Let $0 < T \leq \infty$, $\frac{1}{2} \leq \beta \leq 1$, $s(\cdot) \in C_{\log}(\mathbb{R}^n)$, $1 \leq h, \gamma \leq \infty$, and let $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ with $1 \leq r(\cdot) \leq q(\cdot) < \infty$. Assume that $u_0 \in \mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}$ and $f \in \mathcal{L}^\gamma(0, T; \mathcal{N}^{s(\cdot)+\frac{2\beta}{\gamma}-2\beta}_{r(\cdot),q(\cdot),h})$. Then* (3.1) *has a unique solution v satisfying, for any $\rho \in [\gamma, \infty]$,*

$$\|u\|_{\mathcal{L}^\rho\left(0,T;\mathcal{N}^{s(\cdot)+\frac{2\beta}{\rho}}_{r(\cdot),q(\cdot),h}\right)} \leq C \left(\|u_0\|_{\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}} + \|f\|_{\mathcal{L}^\gamma\left(0,T;\mathcal{N}^{s(\cdot)+\frac{2\beta}{\gamma}-2\beta}_{r(\cdot),q(\cdot),h}\right)} \right), \tag{3.2}$$

where $C > 0$ is a constant depending only on β and d.

Before proving this proposition, we need to get estimates for the localisations of the fractional heat semigroup $\{e^{-t(-\Delta)^\beta}\}_{t \geq 0}$ in our framework.

Proposition 3.4. *Let $t > 0$, $j \in \mathbb{Z}$ and $1 \le r(\cdot) \le q(\cdot) < \infty$. Then for all $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $\Delta_j u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, we have*

$$\left\|\Delta_j(e^{-t(-\Delta)^\beta} v)\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \le \mathcal{K} e^{-\kappa t 2^{2\beta j}} \|\Delta_j v\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}},$$

where $\mathcal{K}$ and κ are tow constants independent of j and t.

Proof. Recalling that $supp(\mathcal{F}(\Delta_j v)) \subset 2^j\mathcal{C}$ (Δ_j is a frequency to $\{|\xi| \sim 2^j\}$), and considering a function $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with $\phi \equiv 1$ in a neighborhood of the ring $\mathcal{C}$, then one has

$$\begin{aligned}\Delta_j(e^{-t(-\Delta)^\beta} v) &= e^{-t(-\Delta)^\beta}\Delta_j v\\ &= \phi(2^j\cdot)e^{-t(-\Delta)^\beta}\Delta_j v\\ &= \mathcal{F}^{-1}\left(\phi(2^j\xi)e^{-t|\xi|^{2\beta}}\right) * \Delta_j v.\end{aligned}$$

Hence, Proposition 2.6, gives

$$\begin{aligned}\left\|\Delta_j(e^{-t(-\Delta)^\beta} v)\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} &\le \left\|\mathcal{F}^{-1}\left(\phi(2^j\xi)e^{-t|\xi|^{2\beta}}\right)\right\|_{L^1} \|\Delta_j v\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}\\ &\le \mathcal{K} e^{-\kappa t 2^{2\beta j}} \|\Delta_j v\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}},\end{aligned}$$

as desired. □

Proof of Proposition 3.3. Since $u_0 \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ and $f \in \mathcal{S}'([0,T) \times \mathbb{R}^n)/\mathcal{P}$, we can obtain $u \in \mathcal{S}'([0,T) \times \mathbb{R}^n)/\mathcal{P}$. And then, applying Δ_j to (3.1) and taking the $\mathcal{M}^{q(\cdot)}_{r(\cdot)}$-norm, we get

$$\|\Delta_j u(t)\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \le \left\|e^{-t(-\Delta)^\beta}\Delta_j u_0\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} + \int_0^t \left\|e^{-(t-t')(-\Delta)^\beta}\Delta_j f(t')\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} dt'.$$

According to Proposition 3.4, we obtain for some $\kappa > 0$,

$$\|\Delta_j u(t)\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \lesssim e^{-\kappa t 2^{2\beta j}} \|\Delta_j u_0\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} + \int_0^t e^{-\kappa 2^{2\beta j}(t-t')} \|\Delta_j f(t')\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} dt'.$$

Set $\frac{1}{\theta} = 1 + \frac{1}{\rho} - \frac{1}{\gamma}$. Young's inequality in L^ρ gives us,

$$\begin{aligned}&\|\Delta_j u(t)\|_{L^\rho_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim \left(\frac{1-e^{-\kappa T 2^{2\beta j}\rho}}{\kappa 2^{2\beta j}\rho}\right)^{\frac{1}{\rho}} \|\Delta_j u_0\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} + \left(\frac{1-e^{-\kappa T 2^{2\beta j}\theta}}{\kappa 2^{2\beta j}\theta}\right)^{\frac{1}{\theta}} \|\Delta_j f(t')\|_{L^\gamma_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim 2^{-\frac{2\beta}{\rho}j} \|\Delta_j u_0\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} + 2^{-2\beta(1+\frac{1}{\rho}-\frac{1}{\gamma})j} \|\Delta_j f(t')\|_{L^\gamma_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}.\end{aligned}$$

Finally, multiplying by $2^{(s(\cdot)+\frac{2\beta}{\rho})j}$, and taking l^h-norm of both sides in the above inequality, we obtain the desired estimate. And this completes the proof of Proposition 3.3. □

3.1. Proof of Theorem 3.1 (1) **(The case** $\frac{1}{2} < \beta \leq 1$**)**

In this part, we aim at proving global well-posedness for small initial data of the system (1.1) in critical variable exponent Besov-Morrey spaces $\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}$ with $\frac{1}{2} < \beta \leq 1$, $1 \leq r(\cdot) \leq q(\cdot) < \infty$ and $1 \leq h \leq \infty$. Firstly, we get the following key bilinear estimate.

Lemma 3.5. *Let* $0 < T \leq \infty$, $s(\cdot) \in C_{\log}(\mathbb{R}^n)$, $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ *with* $s(\cdot) > -1$, $1 \leq r(\cdot) \leq q(\cdot) < \infty$, *and let* $1 \leq h, \gamma, \gamma_1, \gamma_2 \leq \infty$ *satisfying* $\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$. *Then for any* $\varepsilon > 0$, *one has*

$$\begin{aligned}
&\left\| f\nabla(-\Delta)^{-1}g + g\nabla(-\Delta)^{-1}f \right\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\
&\quad + \|g\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.
\end{aligned} \tag{3.3}$$

Proof. Using the following paraproduct decomposition due to J. M. Bony [7],

$$f\nabla(-\Delta)^{-1}g + g\nabla(-\Delta)^{-1}f := J_1 + J_2 + J_3, \tag{3.4}$$

where,

$$\begin{aligned}
J_1 &:= \sum_{l\in\mathbb{Z}} \Delta_l f\nabla(-\Delta)^{-1}S_{l-1}g + \Delta_l g\nabla(-\Delta)^{-1}S_{l-1}f, \\
J_2 &:= \sum_{l\in\mathbb{Z}} S_{l-1}f\nabla(-\Delta)^{-1}\Delta_l g + S_{l-1}g\nabla(-\Delta)^{-1}\Delta_l f, \\
J_3 &:= \sum_{l\in\mathbb{Z}} \sum_{|l-l'|\leq 1} \Delta_l f\nabla(-\Delta)^{-1}\Delta_{l'}g + \Delta_l g\nabla(-\Delta)^{-1}\Delta_{l'}f.
\end{aligned}$$

Below, we estimate J_1, J_2 and J_3 separately. For J_1, we consider the estimate of its first term only, while the second one can be treated similarly. So, by the facts (2.1) and (2.2), Proposition 2.6, Hölder's inequality in L^p-space, and Lemmas 2.7 and 2.8, when $\varepsilon > 0$, one has

$$\begin{aligned}
&\left\| \Delta_j \sum_{l\in\mathbb{Z}} \Delta_l f\nabla(-\Delta)^{-1}S_{l-1}g \right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\leq 4} \left\| \left\|\mathcal{F}^{-1}\varphi_j\right\|_{L^1} \left\|\Delta_l f\nabla(-\Delta)^{-1}S_{l-1}g\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \right\|_{L^{\gamma}_T} \\
&\lesssim \sum_{|l-j|\leq 4} \|\Delta_l f\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-1}S_{l-1}g\right\|_{L^{\gamma_2}_T(L^\infty)} \\
&\lesssim \sum_{|l-j|\leq 4} \|\Delta_l f\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \sum_{k\leq l-2} 2^{k\left(-1+\frac{n}{q(\cdot)}\right)} \|\Delta_k g\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},
\end{aligned}$$

then,

$$\begin{aligned}&\left\|\Delta_j \sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-1} S_{l-1} g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim \sum_{|l-j|\le 4} \|\Delta_l f\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left(\sum_{k\le l-2} 2^{\varepsilon k h'}\right)^{1/h'} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}\\ &\lesssim 2^{-s(\cdot)j} \sum_{|l-j|\le 4} 2^{-s(\cdot)(l-j)} 2^{(s(\cdot)+\varepsilon)l} \|\Delta_l f\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.\end{aligned}$$

Multiplying by $2^{s(\cdot)j}$, and taking l^h-norm of both sides in the above estimate, we obtain

$$\begin{aligned}\left\|\sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-1} S_{l-1} g\right\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)}\\ &\quad\times \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)},\end{aligned}$$

which implies that

$$\begin{aligned}\|J_1\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}\\ &\quad + \|g\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}. \qquad (3.5)\end{aligned}$$

Similarly for J_2: By applying Hölder's inequality and Lemma 2.7, we get

$$\begin{aligned}&\left\|\Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1} \Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim \sum_{|l-j|\le 4} \left\|S_{l-1} f \nabla(-\Delta)^{-1} \Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim \sum_{|l-j|\le 4} \sum_{k\le l-2} 2^{k\frac{n}{q(\cdot)}} \|\Delta_k f\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-1}\Delta_l g\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},\end{aligned}$$

Lemma 2.8 gives us again,

$$\begin{aligned}&\left\|\Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1} \Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim \sum_{|l-j|\le 4} \sum_{k\le l-2} 2^{(-1+\frac{n}{q(\cdot)}-\varepsilon)k} 2^{(1+\varepsilon)k} \|\Delta_k f\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{-l} \|\Delta_l g\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim \sum_{|l-j|\le 4} 2^{-l} \|\Delta_l g\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left(\sum_{k\le l-2} 2^{(1+\varepsilon)kh'}\right)^{1/h'} \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.\end{aligned}$$

Since $\varepsilon > 0$, then

$$\left\| \Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1} \Delta_l g \right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \lesssim \sum_{|l-j|\leq 4} 2^{-sl} 2^{(s(\cdot)+\varepsilon)l} \|\Delta_l g\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.$$

Hence, we arrive at

$$\left\| \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-1} \Delta_l g \right\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim \|g\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \times \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.$$

Thus, we get

$$\begin{aligned} \|J_2\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim{}& \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\ &+ \|g\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|f\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}^{-1+\frac{n}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}. \end{aligned} \tag{3.6}$$

We are now moving on to the last term J_3. We use the following formula, based on an analysis of the algebraic structure of Equation (1.1) [21]:

$$(J_3)_i = \sum_{l\in\mathbb{Z}} \sum_{|l-l'|\leq 1} \Delta_l f \partial_i (-\Delta)^{-1} \Delta_{l'} g + \Delta_l g \partial_i (-\Delta)^{-1} \Delta_{l'} f = K^1_i + K^2_i + K^3_i,$$

for $i = 1, 2, ..., n$. Where $(J_3)_i$ is the i-th exponent of (J_3) and

$$\begin{aligned} K^1_i &:= \sum_{l\in\mathbb{Z}} \sum_{|l-l'|\leq 1} (-\Delta) \left[\left((-\Delta)^{-1} \Delta_l f \right) \left(\partial_i (-\Delta)^{-1} \Delta_{l'} g \right) \right], \\ K^2_i &:= \sum_{l\in\mathbb{Z}} \sum_{|l-l'|\leq 1} 2\nabla \cdot \left[\left((-\Delta)^{-1} \Delta_l f \right) \left(\partial_i \nabla (-\Delta)^{-1} \Delta_{l'} g \right) \right], \\ K^3_i &:= \sum_{l\in\mathbb{Z}} \sum_{|l-l'|\leq 1} \partial_i \left[\left((-\Delta)^{-1} \Delta_l f \right) \Delta_{l'} g \right]. \end{aligned}$$

In order to estimate the above three terms, we use Hölder's inequality in L^p-space and $\mathcal{M}^{q(\cdot)}_{r(\cdot)}$-space (2.5), and Lemma 2.8 as follows: From (2.2), there is $d_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\left\|\Delta_j K_i^1\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim 2^{2j} \sum_{l\geq j-d_0} \sum_{|l-l'|\leq 1} \left\|\left((-\Delta)^{-1}\Delta_l f\right)\left(\partial_i(-\Delta)^{-1}\Delta_{l'} g\right)\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim 2^{2j} \sum_{l\geq j-d_0} \sum_{|l-l'|\leq 1} 2^{-2l} \left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} 2^{(-1+\frac{n}{q(\cdot)})l'} \left\|\Delta_{l'} g\right\|_{L_T^{\gamma_2}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim \sum_{l\geq j-d_0} 2^{-s(\cdot)j} 2^{-(2+s(\cdot))(l-j)} 2^{(s(\cdot)+\varepsilon)l} \left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{-1+\frac{n}{q(\cdot)}-\varepsilon}\right)},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\left\|\Delta_j K_i^2\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim 2^{j} \sum_{l\geq j-d_0} \sum_{|l-l'|\leq 1} \left\|\left((-\Delta)^{-1}\Delta_l f\right)\left(\partial_i\nabla(-\Delta)^{-1}\Delta_{l'} g\right)\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim 2^{j} \sum_{l\geq j-d_0} \sum_{|l-l'|\leq 1} 2^{-2l} \left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} 2^{\frac{n}{q(\cdot)}l'} \left\|\Delta_{l'} g\right\|_{L_T^{\gamma_2}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim \sum_{l\geq j-d_0} 2^{-s(\cdot)j} 2^{-(1+s(\cdot))(l-j)} 2^{(s(\cdot)+\varepsilon)l} \left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{-1+\frac{n}{q(\cdot)}-\varepsilon}\right)},
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\left\|\Delta_j K_i^3\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim 2^{j} \sum_{l\geq j-d_0} \sum_{|l-l'|\leq 1} \left\|\left((-\Delta)^{-1}\Delta_l f\right)\Delta_{l'} g\right\|_{L_T^\gamma(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim 2^{j} \sum_{l\geq j-d_0} \sum_{|l-l'|\leq 1} 2^{-2l} \left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} 2^{\frac{n}{q(\cdot)}l'} \left\|\Delta_{l'} g\right\|_{L_T^{\gamma_2}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \\
&\lesssim \sum_{l\geq j-d_0} 2^{-s(\cdot)j} 2^{-(1+s(\cdot))(l-j)} 2^{(s(\cdot)+\varepsilon)l} \left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}_{r(\cdot)}^{q(\cdot)})} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{-1+\frac{n}{q(\cdot)}-\varepsilon}\right)}.
\end{aligned} \tag{3.9}$$

Thus, (3.7), (3.8) and (3.9) give us, when $s(\cdot)+1>0$,

$$\begin{aligned}
\|J_3\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{s(\cdot)}\right)} &\leq \sum_{i=1}^{n}\sum_{k=1}^{3} \left\|K_i^k\right\|_{\mathcal{L}^{\gamma}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{s(\cdot)}\right)} \\
&\lesssim \|f\|_{\mathcal{L}^{\gamma_1}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{s(\cdot)+\varepsilon}\right)} \|g\|_{\mathcal{L}^{\gamma_2}\left(0,T;\mathcal{N}_{r(\cdot),q(\cdot),h}^{-1+\frac{n}{q(\cdot)}-\varepsilon}\right)}.
\end{aligned} \tag{3.10}$$

Finally, by combining (3.5), (3.6) and (3.10) with (3.4), we get (3.3). This completes the proof of Lemma 3.5. □

Now, by using Lemma 2.9, we can start to prove the existence of local and global solutions of the system (1.1) in the case $\frac{1}{2} < \beta \leq 1$. We define

$$\mathcal{X}^1 := \mathcal{L}^{\gamma_1}\left(\mathbb{R}_+; \mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right) \cap \mathcal{L}^{\gamma_2}\left(\mathbb{R}_+; \mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right),$$

with

$$s_1(\cdot) = -1 + \frac{n}{q(\cdot)} + \varepsilon,\ s_2(\cdot) = -1 + \frac{n}{q(\cdot)} - \varepsilon,\ \ \gamma_1 = \frac{2\beta}{2\beta - 1 + \varepsilon},$$

$$\gamma_2 = \frac{2\beta}{2\beta - 1 - \varepsilon},\ \ 0 < \varepsilon < 2\beta - 1.$$

Due to Duhamel's principle, the solution of the system (1.1) can be written as

$$u(t) = e^{-t(-\Delta)^\beta} u_0 - \int_0^t e^{-(t-t')(-\Delta)^\beta} \nabla \cdot \left(u \nabla (-\Delta)^{-1} u\right) \mathrm{d}t'. \tag{3.11}$$

Set

$$\mathcal{A}(u,w) := \int_0^t e^{-(t-t')(-\Delta)^\beta} \nabla \cdot \left(u \nabla (-\Delta)^{-1} w\right) \mathrm{d}t'.$$

We note that $\mathcal{A}(u,w)$ can be considered as the solution of the dissipative equation (3.1) with $u_0 = 0$ and $f = \nabla \cdot \left(u\nabla(-\Delta)^{-1} w\right)$. Then by applying Proposition 3.3 and Lemma 3.5, with $\gamma = \frac{\beta}{2\beta-1}$, we see that

$$\begin{aligned}
\|\mathcal{A}(u,w)\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \left\|\nabla \cdot \left(u\nabla(-\Delta)^{-1}w\right)\right\|_{\mathcal{L}^{\frac{\beta}{2\beta-1}}\left(\mathbb{R}_+;\mathcal{N}^{-2+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \left\|u\nabla(-\Delta)^{-1}w\right\|_{\mathcal{L}^{\frac{\beta}{2\beta-1}}\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \|u\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \|w\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \\
&\qquad + \|w\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\mathcal{N}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \|u\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \|u\|_{\mathcal{X}^1} \|w\|_{\mathcal{X}^1},
\end{aligned}$$

and similarly,

$$\|\mathcal{A}(u,w)\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\mathcal{N}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim \|u\|_{\mathcal{X}^1} \|w\|_{\mathcal{X}^1}.$$

Thus,

$$\|\mathcal{A}(u,w)\|_{\mathcal{X}^1} \leq C \|u\|_{\mathcal{X}^1} \|w\|_{\mathcal{X}^1}. \tag{3.12}$$

On the other hand, $e^{-t(-\Delta)^\beta} u_0$ can also be considered as the solution of the dissipative equation (3.1) with $u_0 = u_0$ and $f = 0$. Then we can directly deduce from Proposition 3.3 that,

$$\left\|e^{-t(-\Delta)^\beta} u_0\right\|_{\mathcal{X}^1} \leq C \|u_0\|_{\dot{\mathcal{N}}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}}.$$

So, if $\|u_0\|_{\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} \leq \varepsilon$ with $\varepsilon = \frac{1}{4C^2}$, then by Lemma 2.9, the integral equation (3.11) admits a unique solution u such that $\|u\|_{\mathcal{X}^1} \leq 2C\varepsilon$, which is the unique solution of the system (1.1). Furthermore, Proposition 3.3 and Lemma 3.5 once again give us

$$\|u\|_{\mathcal{L}^\infty\left(0,T;\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} \lesssim \|u_0\|_{\mathcal{N}^{-2\beta+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} + \|u\|^2_{\mathcal{X}^1} < \infty.$$

Finally, $u \in \mathcal{X}_\varepsilon$. This completes the proof of the first assertion of Theorem 3.1.

3.2. Proof of Theorem 3.1 (2) (The case $\beta = \frac{1}{2}$)

In this part, we establish the global well-posedness for the system (1.1) in the limit case $\beta = \frac{1}{2}$, with initial data in critical variable exponent Besov-Morrey spaces $\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}$ with $1 \leq r(\cdot) \leq q(\cdot) < \infty$. Firstly, by making a slight modification to the proof of Lemma 3.5, we obtain the following estimate:

Lemma 3.6. *For any* $f, g \in \mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)$, *one has*

$$\left\|f\nabla(-\Delta)^{-1}g + g\nabla(-\Delta)^{-1}f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \times \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{3.13}$$

Proof. We estimate the first term of J_1 as follows:

$$\begin{aligned}
&\left\|\Delta_j \sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-1} S_{l-1} g\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\leq 4} \|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-1} S_{l-1} g\right\|_{L^\infty_\infty(L^\infty)} \\
&\lesssim \sum_{|l-j|\leq 4} \|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \sum_{k\leq l-2} 2^{k\left(-1+\frac{n}{q(\cdot)}\right)} \|\Delta_k g\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\leq 4} \|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.
\end{aligned}$$

Multiplying by $2^{(-1+\frac{n}{q(\cdot)})j}$, and taking l^1-norm of both sides in the above estimate, we obtain

$$\begin{aligned}
&\left\|\sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-1} S_{l-1} g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \\
&\qquad \lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.
\end{aligned}$$

And then,

$$\|J_1\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{3.14}$$

Similarly, for J_2,

$$\begin{aligned}&\left\|\Delta_j\sum_{l\in\mathbb{Z}}S_{l-1}f\nabla(-\Delta)^{-1}\Delta_l g\right\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim\sum_{|l-j|\le 4}\sum_{k\le l-2}2^{(-1+\frac{n}{q(\cdot)})k}2^k\|\Delta_k f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}2^{-l}\|\Delta_l g\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim\sum_{|l-j|\le 4}\|\Delta_l g\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)},\end{aligned}$$

which gives us that

$$\begin{aligned}&\left\|\sum_{l\in\mathbb{Z}}S_{l-1}f\nabla(-\Delta)^{-1}\Delta_l g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\\&\qquad\lesssim\|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.\end{aligned}$$

Thus, we get

$$\|J_2\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\lesssim\|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{3.15}$$

Moreover for the final term $J_3 = K^1 + K^2 + K^3$, and since K^3 is similar to K^2, we only estimate $K^1$and K^2 as follows:

$$\begin{aligned}&\|\Delta_j K^1_i\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim 2^{2j}\sum_{l\ge j-d_0}\sum_{|l-l'|\le 1}2^{-2l}\|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}2^{(-1+\frac{n}{q(\cdot)})l'}\|\Delta_{l'}g\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim 2^{(1-\frac{n}{q(\cdot)})j}\sum_{l\ge j-d_0}2^{-(1+\frac{n}{q(\cdot)})(l-j)}2^{(-1+\frac{n}{q(\cdot)})l}\|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\times\|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)},\end{aligned}$$

and

$$\begin{aligned}&\|\Delta_j K^2_i\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim 2^{j}\sum_{l\ge j-d_0}\sum_{|l-l'|\le 1}2^{-2l}\|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}2^{\frac{n}{q(\cdot)}l'}\|\Delta_{l'}g\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\&\lesssim 2^{(1-\frac{n}{q(\cdot)})j}\sum_{l\ge j-d_0}2^{-\frac{n}{q(\cdot)}(l-j)}2^{(-1+\frac{n}{q(\cdot)})l}\|\Delta_l f\|_{L^\infty_\infty(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\end{aligned} \tag{3.16}$$

Hence, from (**??**) and (3.16), we arrive at

$$\begin{aligned}\|J_3\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}&\le\sum_{i=1}^{n}\sum_{k=1}^{3}\|K^k_i\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\\&\lesssim\|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.\end{aligned} \tag{3.17}$$

Finally, putting the estimates (3.14), (3.15) and (3.17) together, we get (3.13). The proof of Lemma 3.6 is complete. □

We are now in a position to demonstrate the second assertion of Theorem 3.1. By considering the resolution space $\mathcal{L}^\infty(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1})$ and returning to the integral equation (3.11), Proposition 3.3 with $\beta=\frac{1}{2}$ and $\gamma=\infty$, and Lemma 3.6, give us

$$\begin{aligned}\|\mathcal{A}(u,w)\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} &\le C\left\|\nabla\cdot\left(u\nabla(-\Delta)^{-1}w\right)\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-2+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\\ &\le C\|u\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\|w\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.\end{aligned}$$

Applying Proposition 3.3 for $\beta=\frac{1}{2}$ again, we can obtain

$$\left\|e^{-t(-\Delta)^{\frac{1}{2}}}u_0\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \le C\|u_0\|_{\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}}.$$

If $\|u_0\|_{\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}}$ is sufficiently small, by using the fixed point argument as in Subsection 3.1, we get the global solution of the system (1.1) in $\mathcal{L}^\infty\left(\mathbb{R}_+;\mathcal{N}^{-1+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)$. The proof of Theorem 3.1 is complete, as desired.

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Threshold results of blow-up solutions to Kirchhoff equations with variable sources

Nadji Touil and Abita Rahmoune

Abstract. This paper analyzes an initial boundary value problem for variable source Kirchhoff-type parabolic equations. We aim to derive a new sub-critical energy threshold for finite-time blow-up, a new blow-up condition, and estimates for lifespan and upper bounds for blow-up time across various initial energy cases.

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Keywords: Kirchhoff, potential well method, arbitrary initial energy, blow-up, bounds of the blow-up time, $L^{p(\cdot)}(\Omega)$ Sobolev space.

1. Introduction

In recent years, there has been a significant interest among numerous mathematical researchers in examining the blow-up time properties of solutions to equations used for describing the transverse vibrations of a stretched string while taking into account the change in the string length. These equations, proposed by Kirchhoff [19], [26] are widely employed in engineering disciplines like automotive, aerospace, and large-scale structures. The extensive applications of these materials have led to a growing desire among researchers to establish findings related to the presence and control of elasticity problems. Almeida Junior et al. [25] studied polynomial stability for the equations of porous elasticity in one-dimensional bounded domains. Iesan et al. [16, 17, 18] studied the theory of thermoelastic materials with voids. Santos et al. [30] considered a porous elastic system with porous dissipation In recent years, there has been a significant amount of research focused on developing mathematical models for nonlocal diffusion. These models are formulated by using parabolic equations that combine linear or non-linear diffusion with a Kirchhoff term. The Kirchhoff problems are a type of problem

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that includes the term $M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)$, which causes the equation to no longer be a pointwise identity

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(x,u), & (x,t)\in\Omega\times(0,T),\\ u=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

The Kirchhoff problems are a type of problem that includes the term $M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)$, which causes the equation to no longer be a pointwise identity. The nonlinear Kirchhoff equation (NLKE) is a partial differential equation used to describe the transverse vibrations of a stretched string while taking into account the change in the string length [19]. It is also used to describe the movement of a semi-infinite string [26] and is an underlying equation of quantum mechanics. Partial differential equations have a wide range of applications, as listed in reference [33]. The study of Kirchhoff equations has a long history and was examined in detail in Lions research [23], where it became possible to investigate the existence, uniqueness, and regularity of the solutions in Kirchhoff's equations. For more information, interested readers can refer to [10, 11, 24] and the references therein. This paper studies a parabolic problem with a nonlocal diffusion coefficient, where a nonlinear source term modeled by an operator appears in the Kirchhoff equation.

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = |u|^{q(x)-1}u, & (x,t)\in\Omega\times(0,T),\\ u=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, we assume that $u_0\in H_0^1(\Omega)$ and $u_0(x)\not\equiv 0$, the diffusion coefficient has the specific form $M(s)=a+bs$ with positive parameters a,b, $\Omega\subset\mathbb{R}^n, q$ is constant and satisfy

$$(\mathcal{H}_1) 3<q_1\leq q(x)\leq q_2\leq \frac{n+2}{n-2}\text{if } \ n\geq 3,\ x\in\Omega; \tag{1.2}$$

$$(\mathcal{H}_2)\ \ 1<q_1\leq q(x)\leq q_2<3 \quad \text{if } n\geq 1,\ x\in\Omega.$$

We consider a mathematical model, where u_0 belongs to the Sobolev space $H_0^1(\Omega)$ and $-\Delta$ denotes the Laplace operator concerning the spatial variables. Our focus is on the explosion property in finite time. To this end, we use the potential well method and various inequality techniques to establish the the blow-up of weak solutions within a finite time and obtain a new blow-up criterion. Additionally, we determine the lifespan and an upper bounds for the blow-up time in different initial energy cases. It is important to note that the model (1.1) is called degenerate when $a=0$, and when $a>0$, we refer to it as a non-degenerate model. The exponent $q(.)$ is a measurable function on Ω that satisfies certain conditions.

$$1<q_1=ess\inf_{x\in\Omega} q(x)\leq q(x)\leq q_2=ess\sup_{x\in\Omega} q(x)<\infty, \tag{1.3}$$

and the following Zhikov–Fan uniform local continuity condition. There exist a constant $k>0$ such that for all points x, y in Ω with $0<|x-y|<\frac{1}{2}$, we have the inequality

$$|q(x) - q(y)| \leq k(|x-y|), \tag{1.4}$$

where $k(r)$ satisfies

$$\limsup_{r \to 0^+} k(r) \ln\left(\frac{1}{r}\right) = c < \infty.$$

This problem has its origin in the mathematical explanation of system in real world from the mathematical modeling for axially moving viscoelastic materials, they appear in numerous applications in the natural sciences, for instance models of flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media [3, 4, 28], and the processing of digital images [2, 9, 22], and can all be linked with problem (1.1), further details on the subject can be seen in [5, 6, 29] and the other references contained therein. In recent years, the study of mathematical nonlinear models with variable exponent nonlinearity has attracted the attention of many researchers. Let us highlight some of these issues. For example, Pinasco [27] established the local existence of positive solutions for the parabolic problem.

$$\begin{cases} u_t - \Delta u = f(u) \text{ in} & \Omega \times (0,T) \\ u = 0, \text{ in} & \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases}$$

where the source term is of the form

$$f(u) = a(x)u^{p(x)} \quad \text{or} \quad f(u) = a(x)\int_\Omega u^{q(y)} dy.$$

He also proved that with sufficiently large initial data, solutions blow up in a finite time. Alaoui et al. [21] considered the following nonlinear heat equation,

$$u_t - div\left(|\nabla u(x)|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u + f.$$

Under appropriate conditions on m and p, and with $f = 0$, they demonstrated that any solution with a nontrivial initial condition will experience a blow-up in finite time. Additionally, they provided numerical examples in two dimensions to illustrate their findings. Autuori et al. [7] investigated a nonlinear Kirchhoff system involving the $p(x,t)$-Laplace operator, a nonlinear force $f(t,x,u)$, and a nonlinear damping term $Q = Q(t,x,u,u\_t)$. They established a global nonexistence result under suitable conditions on f, Q, and p. In the classical case of constant exponent ($q(x)$ =constant= q), this equation has its origin in the nonlinear vibration of an elastic string, were the source term $u^{q-1}u$ forces the negative-energy solutions to explode in finite time. It's known that several authors have looked at problem (1.1) concerning the findings of the global existence and blow-up of solutions, and a powerful method for treating it is the "potential well method," which was founded by the first author Sattinger [31] in 1968 and later been enhanced by Liu and Zhao [32] by introducing the so-called family of potential wells which later became a significant technique for the study of nonlinear evolution equations and has also given many interesting results. Recently, authors of [14, 15] discussed in a bounded domain of $\mathbb{R}^n$ with $3 < q < \frac{n+2}{n-2}$ the global existence and finite time blow-up of solutions to problem

(1.1) when the initial data are at different energy levels $E(u_0) < d$, $E(u_0) = d$, and $E(u_0) > d$ respectively. If we know that the solutions of a given system explode in finite time, it is important to estimate the bounds of the explosion time from both above and below, which is the main goal of this work. We will expand the assumptions about the given q in the aforementioned works, assuming a new assumption on the critical exponent $q(.)$ such that $1 < q_1 \leq q(x) \leq q_2 < (n+2)/(n-2)$, under some sufficient conditions we giving a new blow-up criterion for problem (1.1) if the initial energy is not -negative, and derive the upper and lower bounds of this blow-up time. The table below provides a summary of the background for our work.

Table 1: Main results.

Main results	q	Initial data		Blow-up
Theorem 2	$(\mathcal{H}_1)$	$\left(\frac{B_1}{\sqrt{a}}\right)^{-q_1+1}$ (2.4) $\geq \left(a\|\nabla u_0\|_2^2 + \frac{b}{2}\|\nabla u_0\|_2^4\right)^{\frac{q_1+1}{2}} > \alpha_1$	$\mathrm{E}(u_0) < E_1$, E_1 as in (2.4)	Blow-up (2.6) $\lim_{t\to\widehat{T}} \|u(t)\|_2^2 = \infty$
Theorem 3	$(\mathcal{H}_1)$	$u_0 \in H_0^1(\Omega)$, $u_0 \neq 0$	$\mathrm{E}(u_0) < 0$ $\mathrm{E}(u_0) \leq \mathrm{E}_d$, E_d as in (1.11) $0 \leq \mathrm{E}(u_0) < C_0 \|u_0\|_2^2$ **(iii)**	Blow-up $\lim_{t\to T^*} \int_0^t \|u(\tau)\|_2^2 d\tau = \infty$
Theorem 3	$(\mathcal{H}_2)$	$u_0 \in H_0^1(\Omega)$	$\mathrm{E}(u_0) < -\frac{q_1+1}{q_1+5}\frac{b}{4\varepsilon}c(\varepsilon)$ (2.21),(2.22)	Blow-up $\lim_{t\to T} \|u(t)\|_2^2 = \infty$.

Table 2: The estimate of blow-up time.

$E(u_0)$	Upper bound estimate	Lower bound estimate
$\mathrm{E}(u_0) < E_1$	$\surd$	
$\mathrm{E}(u_0) < 0$	$\surd$	
$\mathrm{E}(u_0) = 0$	?	?
$\mathrm{E}(u_0) < E_d$	$\surd$	
$0 \leq \mathrm{E}(u_0) < C_0 \|u_0\|_2^2$	$\surd$	
$\mathrm{E}(u_0) < -\frac{q_1+1}{q_1+5}\frac{b}{4\varepsilon}c(\varepsilon)$	$\surd$	

1.1. Modified potential wells

For $u \in H_0^1(\Omega)$, we define the functionals

$$\begin{aligned}\mathrm{E}(u\,(t)) =: \mathrm{E}(t) =& \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x,\\ I(u\,(t)) =& a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \int_\Omega |u|^{q(x)+1}\mathrm{d}x. \qquad (1.5)\\ \mathrm{M}(u\,(t)) =: \mathrm{M}(t) =& \frac{1}{2}\|u\,(t)\,\|_2^2.\end{aligned}$$

and testing (1.1) by u_{t} we have $E(t)$ is nonincreasing, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E(t)} = -\,\|u_t(t)\|_2^2 \leq 0, \tag{1.6}$$

and

$$\mathrm{E}(t) + \int_0^t \|u_t(t)\|_2^2\,\mathrm{d}s \leq \mathrm{E}(u_0) \quad \text{a.e. } t \in (0,T), \tag{1.7}$$

$$L'(t) = -I(u\,(t)) \quad \text{a.e. } t \in (0,T). \tag{1.8}$$

We then have the following lemma.

Lemma 1.1. *For $q(x)$ be (1.4) and $u \in H_0^1(\Omega)\backslash\{0\}$. Let $F : [0,+\infty) \to \mathbb{R}$ the Euler functional defined by*

$$F(\lambda) = \frac{\lambda^2}{2}a\|\nabla u\|^2 + \frac{\lambda^4}{4}b\|\nabla u\|^4 - \int_\Omega \frac{\lambda^{q(x)+1}}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x,$$

then, F keeps the following properties:

- **(i) .** $\lim_{\lambda\to 0^+} F(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} F(\lambda) = -\infty$.
- **(ii).** *There is at least one solution to the equation $F'(\lambda) = 0$ on the interval $[\lambda_1, \lambda_2]$, where*

$$\lambda_1 = \min\left[\rho\,(u)^{\frac{-1}{1-q_2}}, \rho\,(u)^{\frac{-1}{3-q_1}}\right],\ \lambda_2 = \max\left[\rho\,(u)^{\frac{-1}{1-q_2}}, \rho\,(u)^{\frac{-1}{3-q_1}}\right], \tag{1.9}$$

and

$$\rho\,(u) := \frac{a\|\nabla u\|^2 + b\|\nabla u\|^4}{\int_\Omega |u|^{q(x)+1}\mathrm{d}x}.$$

- **(iii) .** *There exists a $\lambda^* = \lambda^*(u) > 0$ such that $F(\lambda)$ gets its maximum at $\lambda = \lambda^*$. Furthermore, we have that $0 < \lambda^* < 1$, $\lambda^* = 1$ and $\lambda^* > 1$ provided $I(u) < 0$, $I(u) = 0$ and $I(u) > 0$, respectively.*

Proof. Since $q(x) \in C_+(\bar\Omega) = \left\{p \in C(\bar\Omega) : \inf_{x\in\bar\Omega} q\,(x) > 3\right\}$, the assertion (i) is shown by the following:

$$F(\lambda) \leq \frac{\lambda^2}{2}a\|\nabla u\|^2 + \frac{\lambda^4}{4}b\|\nabla u\|^4 - \min\left\{\lambda^{q_1+1}, \lambda^{q_2+1}\right\}\int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x,$$

and

$$F(\lambda) \geq \frac{\lambda^2}{2}a\|\nabla u\|^2 + \frac{\lambda^4}{4}b\|\nabla u\|^4 - \max\left\{\lambda^{q_1+1}, \lambda^{q_2+1}\right\}\int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x,$$

For (ii). We have

$$F'(\lambda)=\lambda a\int_\Omega |\nabla u(x)|^2\mathrm{d}x+\lambda^3 b\|\nabla u\|^4-\int_\Omega \lambda^{q(x)}|u|^{q(x)+1}\mathrm{d}x,$$

which implies that $F'(\lambda)$ lies in the following two inequalities

$$\begin{aligned}F'(\lambda) &\geq \lambda a\int_\Omega |\nabla u(x)|^2\mathrm{d}x+\lambda^3 b\|\nabla u\|^4-\max\{\lambda^{q_1},\lambda^{q_2}\}\int_\Omega |u|^{q(x)+1}\mathrm{d}x\\ &= \max\{\lambda^{q_1},\lambda^{q_2}\}\left[\begin{array}{r}\min\left\{\lambda^{1-q_1},\lambda^{1-q_2}\right\}a\int_\Omega |\nabla u(x)|^2\mathrm{d}x\\ +\min\left\{\lambda^{3-q_1},\lambda^{3-q_2}\right\}a\int_\Omega |\nabla u(x)|^4\mathrm{d}x-\int_\Omega |u|^{q(x)+1}\mathrm{d}x\end{array}\right],\end{aligned}$$

and

$$\begin{aligned}F'(\lambda) &\leq \lambda a\int_\Omega |\nabla u(x)|^2\mathrm{d}x+\lambda^3 b\|\nabla u\|^4-\min\{\lambda^{q_1},\lambda^{q_2}\}\int_\Omega |u|^{q(x)+1}\mathrm{d}x\\ &= \min\{\lambda^{q_1},\lambda^{q_2}\}\left[\begin{array}{r}\max\left\{\lambda^{1-q_1},\lambda^{1-q_2}\right\}a\int_\Omega |\nabla u(x)|^2\mathrm{d}x\\ +\max\left\{\lambda^{3-q_1},\lambda^{3-q_2}\right\}a\int_\Omega |\nabla u(x)|^4\mathrm{d}x-\int_\Omega |u|^{q(x)+1}\mathrm{d}x\end{array}\right],\end{aligned}$$

Since $q_2\geq q_1>3$, we signify that $F'(\lambda)$ has at least one zero point λ satisfying (1.9). So we get (ii). The definition of λ^* and the relation $I(\lambda u)=\lambda F'(\lambda)$ and

$$F'(\lambda)\leq(\lambda-\lambda^{q_2})\,a\int_\Omega |\nabla u(x)|^2\mathrm{d}x+\left(\lambda^3-\lambda^{q_2}\right)b\int_\Omega |\nabla u(x)|^4\mathrm{d}x+\lambda^{q_2}I(u),\ \text{for}\ \ \lambda\in(0,1),$$

and

$$F'(\lambda)\geq(\lambda-\lambda^{q_2})\,a\int_\Omega |\nabla u(x)|^2\mathrm{d}x+\left(\lambda^3-\lambda^{q_2}\right)b\int_\Omega |\nabla u(x)|^4\mathrm{d}x+\lambda^{q_2}I(u),\ \text{for}\ \lambda\in(1,\infty),$$

lead to the last claim (iii). Completeness of the proof. □

1.2. Assumptions and main results

As E is the Fréchet-differentiable functional with derivative E', let suppose that $u\neq 0$ is a critical point of E, i.e., $E'(u)=0$. Then necessarily u is contained in the set

$$\mathcal{N}=\left\{u\in H_0^1(\Omega)\backslash\{0\}: I(u)=\langle \mathrm{E}'(u),u\rangle=0\right\},$$

so $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of E, $\mathcal{N}$ is called the Nehari manifold associated with the energy functional E. By Lemma 1.1 we know that $\mathcal{N}$ is not empty set. It is clear that $E(u)$ is coercive on $\mathcal{N}$. The depth of the potential well, denoted as d, characterized by

$$d=\inf_{u\in\mathcal{N}}\mathrm{E}(u). \tag{1.10}$$

Under the appropriate conditions we have d is a positive finite number and is therefore well-defined. For E_d is a constant given by

$$\mathrm{E}_d=\frac{q_1-1}{q_1+1}\frac{q_2+1}{q_2-1}d\leq d, \tag{1.11}$$

we define the modified stable and unstable sets as follows

$$\mathcal{W} = \left\{u \in H_0^1(\Omega) : \mathrm{E}(u) < \mathrm{E}_d,\ I(u) > 0\right\} \cup \{0\},$$
$$\mathcal{U} = \left\{u \in H_0^1(\Omega) : \mathrm{E}(u) < \mathrm{E}_d,\ I(u) < 0\right\}.$$

2. Blow-up and bounds of blow-up time

In this section, we get new bounds for the blow-up time to problem (1.1) if the variable exponent $q(.)$ and the initial data satisfy some conditions. Before stating our main results, without proof, we preferably give the following theorem of existence and uniqueness, as well as the regularity:

Definition 2.1 (Weak solution). [20]*A function $u(x,t)$ is said to be a weak solution of problem* (1.1) *defined on the time interval* $[0,T]$, *provide that* $u(x,t) \in L^\infty\left(0,T;H_0^1(\Omega)\right)$ *with* $u_t \in L^2\left(0,T;L^2(\Omega)\right)$, *if for every test-function* $\eta \in H_0^1(\Omega)$ *and a.e.* $t \in [0,T]$, *the following identity holds:*

$$(u_t,\eta)_\Omega + \left(a + b\|\nabla u\|_2^2\right)(\nabla u, \nabla\eta)_\Omega = \left(|u|^{q(x)-1}u,\eta\right)_\Omega, \ a.e.\ t \in (0,T), \tag{2.1}$$

with $u(x,0) = u_0 \in H_0^1(\Omega)$.

Without proof, we give the local existence of a solution of (1.1) that can be obtained by the Faedo-Galerkin methods together with the Banach fixed point theorem [1, 8].

Theorem 2.2. *Assume that* (1.3)-(1.4) *hold. Then the problem* (1.1) *for given* $u_0 \in H_0^1(\Omega)$ *admits a unique local solution*

$$u \in C\left([0,T_{\max});H_0^1(\Omega)\right),\ u_t \in C\left([0,T_{\max});L^2(\Omega)\right),$$

where $T_{\max} > 0$ *is the maximal existence time of* $u(t)$.

2.1. Function spaces and lemmas

In this section, we present some preliminary concepts and notations that we shall employ in our further analysis. Let us start by introducing the variable-order Lebesgue space $L^{p(.)}(\Omega)$, which is defined for all $p : \Omega \to [1,+\infty]$ a measurable function as

$$L^{p(.)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < +\infty\right\}.$$

We then know that $L^{p(.)}(\Omega)$ is a Banach space, equipped with the Luxemburg-type norm

$$\|u\|_{p(.)} := \inf\left\{\lambda > 0,\quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\}.$$

Next, we define the variable-order Sobolev space $W^{1,p(.)}(\Omega)$ as

$$W^{1,p(.)}(\Omega) := \left\{u \in L^{p(.)}(\Omega)\ :\ \nabla u \in L^{p(.)}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)}^2 + \|\nabla u\|_{p(.)}^2. \tag{2.2}$$

Moreover, in what follows we will need the following embedding result from [12, 13].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. It holds the following.*

1. *If $p \in C(\overline{\Omega})$ and $q : \Omega \to [1, +\infty)$ is a measurable function such that*
$$\underset{x\in\Omega}{ess\ inf}\big(p^*(x) - q(x)\big) > 0,$$
with p^* defined as in (1.2), then $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ with continuous and compact embedding.
2. *If p satisfy (1.3), then $\|u\|_{p(.)} \leq C\|\nabla u\|_{p(.)}$ for all $u \in W_0^{1,p(.)}(\Omega)$. In particular, $\|u\|_{1,p(.)} = \|\nabla u\|_{p(.)}$ defines a norm on $W_0^{1,p(.)}(\Omega)$ which is equivalent to (2.2).*

It is not difficult to set up the following lemma's, so we will ignore its proof here.

Lemma 2.4. *Allow (1.3)-(1.4) to apply. Let $u(t) := u(x, t)$ be a local solution to problem (1.1). Then, the following assertions hold:*

(i). *If there is a time $t_0 \in [0, T_{\max})$ such that $u(t_0) \in \mathcal{W}$ and $\mathrm{E}(t_0) < d$, then $u(t)$ stays within the set $\mathcal{W}$ for all $t \in [t_0, T_{\max})$.*

(ii). *If there is a time $t_0 \in [0, T_{\max})$ such that $u(t_0) \in \mathcal{U}$ and $\mathrm{E}(t_0) < d$, then $u(t)$ stays within the set $\mathcal{U}$ for all $t \in [t_0, T_{\max})$.*

Lemma 2.5. *Suppose that a positive, twice-differentiable function $\varphi(t)$ satisfies on $t \geq 0$ the inequality*
$$\varphi''\varphi - (1 + \alpha)(\varphi')^2 \geq 0, \ \alpha > 0.$$
If
$$\varphi(0) > 0, \ \text{and } \varphi'(0) > 0,$$
then, then there exists $t_1 \in \left(0, \frac{\varphi(0)}{\alpha\varphi'(0)}\right)$ such that
$$\varphi(t) \to \infty \ \text{as } t \to t_1.$$

Lemma 2.6. *Let Ω be a bounded domain of $\mathbb{R}^n$, $q(.)$ satisfies (1.2) and (1.4), then*
$$B\|\nabla u\|_2 \geq \|u\|_{q(.)+1}, \ \text{for all } u \in W_0^{1,2}(\Omega). \tag{2.3}$$
where the optimal constant of Sobolev embedding B is depends on $q_{1,2}$ and $|\Omega|$.

Lemma 2.7. *Assuming (u_0, u_1) are in $H_0^1(\Omega) \times L^2(\Omega)$ and that u_0 is an element of $\mathcal{U}$, the following holds:*
$$d \leq \left(\frac{1}{2} - \frac{1}{q_2 + 1}\right) a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4} - \frac{1}{q_2 + 1}\right) b\|\nabla u(t)\|_2^4, \quad \text{for } t \in [0, T_{\max}).$$

Proof. Because $u_0 \in \mathcal{U}$, according to Lemma 2.4 $u(t) \in \mathcal{U}$ for $t \in [0, T_{\max})$ and thus $I(u(t)) < 0$. By Lemma 1.1 there exists $\lambda^* \in (0, 1)$ such that $I(\lambda^* u) = 0$, i.e.
$$\int_\Omega (\lambda^*)^{q(x)+1} |u(t)|^{q(x)+1} \mathrm{d}x = a(\lambda^*)^2 \|\nabla u\|_2^2 + b(\lambda^*)^4 \|\nabla u\|_2^4$$

Thanks to $\lambda^* < 1$ we can derive from the definition of d

$$\begin{aligned}
d \leq \mathrm{E}\left(\lambda^* u(t)\right) &= a\frac{(\lambda^*)^2}{2}\|\nabla u(t)\|_2^2 + b\frac{(\lambda^*)^4}{4}\|\nabla u\|_2^4 - \int_\Omega \frac{(\lambda^*)^{q(x)+1}}{q(x)+1}|u(t)|^{p(x)}\mathrm{d}x \\
&\leq a\frac{(\lambda^*)^2}{2}\|\nabla u(t)\|_2^2 + b\frac{(\lambda^*)^4}{4}\|\nabla u\|_2^4 - \frac{1}{q_2+1}\int_\Omega (\lambda^*)^{q(x)+1}|u(t)|^{q(x)+1}\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{q_2+1}\right)(\lambda^*)^2 a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4} - \frac{1}{q_2+1}\right)(\lambda^*)^4 b\|\nabla u(t)\|_2^4 + \frac{1}{q_2+1}I\left(\lambda^* u(t)\right) \\
&\leq \left(\frac{1}{2} - \frac{1}{q_2+1}\right)a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4} - \frac{1}{q_2+1}\right)b\|\nabla u(t)\|_2^4.
\end{aligned}$$

The proof is completed. □

Suppose there are positive constants B_1, α_1, α_0, and E_1 that satisfy the following argument:

$$\begin{gathered}
B_1 = \max\left(1, B\right), \ \alpha_0 = \sqrt{a\|\nabla u_0\|_2^2 + \frac{b}{2}\|\nabla u_0\|_2^4}, \\
\alpha_1 = \left(\frac{B_1^2}{a}\right)^{-\frac{q_1+1}{2(q_1-1)}}, E_1 = \left(\frac{1}{2} - \frac{1}{q_1+1}\right)\alpha_1^2.
\end{gathered} \tag{2.4}$$

Based on equations (2.3) and (2.1), we can come to a conclusion that

$$\begin{aligned}
\mathrm{E}\left(t\right) \geq& \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{q_1+1}\max\left(\|u\|_{q(.)+1}^{q_2+1}, \|u\|_{q(.)+1}^{q_1+1}\right) \\
\geq& \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{q_1+1}\max\left(\left(B_1^2\|\nabla u\|_2^2\right)^{\frac{q_2+1}{2}}, \left(B_1^2\|\nabla u\|_2^2\right)^{\frac{q_1+1}{2}}\right) \\
\geq& \frac{1}{2}\alpha^2 - \frac{1}{q_1+1}\max\left(\left(\frac{B_1^2}{a}\right)^{\frac{q_2+1}{2}}\alpha^{q_2+1}, \left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha^{q_1+1}\right) := g\left(\alpha\right) \ \forall\alpha \geq 0,
\end{aligned} \tag{2.5}$$

where $\alpha = \sqrt{a\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4}$.

To the best of our knowledge, no evidence has been found regarding the blow-up of solutions to this equation in $\mathbb{R}^n$, given the initial data at a high energy level. This paper aims to investigate this matter by examining the finite-time explosion of weak solutions in the initial boundary value problem provided.

In the following sections, we will present our main theorems

For $3 < q_1 \leq q\left(x\right) \leq q_2 \leq \frac{n+2}{n-2}$, we have the following result

2.2. Results on the blow-up time

Theorem 2.8. *Supposed that q satisfies $(\mathcal{H}_1)$. If $u_0 \neq 0$ is chosen in such a way that $\mathrm{E}(u_0) < E_1$ and $\left(\frac{B_1}{\sqrt{a}}\right)^{-q_1+1} \geq \left(a\|\nabla u_0\|_2^2 + \frac{b}{2}\|\nabla u_0\|_2^4\right)^{\frac{q_1+1}{2}} > \alpha_1$. Then the solution of the problem (1.1) will eventually blow-up in finite time T. Moreover, the blow-up* *time T can be estimated from above by $\widehat{T}$, where*

$$\widehat{T}=\max\left(\begin{array}{c}\dfrac{(q_1+1)\,|\Omega|^{\frac{q_1-2}{2}}\left(\int_\Omega u_0^2\mathrm{d}x\right)^{\frac{1-q_1}{2}}}{(q_1-1)\,(q_1-3)\left(1-\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{-q_1-1}{q_1-1}}\right)},\\ \dfrac{(q_1+1)\,|\Omega|^{\frac{q_1-2}{2}}\left(\int_\Omega u_0^2\mathrm{d}x\right)^{\frac{1-q_2}{2}}}{(q_2-1)\,(q_1-3)\left(1-\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{-q_1-1}{q_1-1}}\right)}.\end{array}\right) \tag{2.6}$$

Lemma 2.9. *Let define $h:[0,+\infty)\to\mathbb{R}$ as*

$$h(\alpha)=\frac{1}{2}\alpha^2-\frac{1}{q_1+1}\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha^{q_1+1}. \tag{2.7}$$

Then, under the assumptions of Theorem 2.8, the following properties hold :

(i). *h is increasing for $0<\alpha\leq\alpha_1$ and decreasing for $\alpha\geq\alpha_1$;*
(ii). $\lim_{\alpha\to+\infty} h(\alpha)=-\infty$ *and $h(\alpha_1)=E_1$.*

Proof. By the assumption that $B_1>1$ and $p_1>1$, $h(\alpha)=g(\alpha)$, for $0<\alpha\leq\left(\frac{B_1}{\sqrt{a}}\right)^{-q_1+1}$. Moreover, $h(\alpha)$ is continuous and differentiable in $[0,+\infty)$.

$$h'(\alpha)=\alpha-\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha^{q_1},\ 0\leq\alpha<\left(\frac{B_1}{\sqrt{a}}\right)^{-q_1+1}.$$

Then **(i)** follows. Since $q_1-1>0$, we have $\lim_{\alpha\to+\infty} h(\alpha)=-\infty$. A typical computation yields to $h(\alpha_1)=E_1$. This means that (**ii**) is true. □

Lemma 2.10. *According to Theorem 2.8, it can be assumed that there is a positive constant $\alpha_2>\alpha_1$ such that*

$$\sqrt{a\|\nabla u\|_2^2+\frac{b}{2}\|\nabla u\|_2^4}\geq\alpha_2,\quad t\geq 0, \tag{2.8}$$

$$\int_\Omega\frac{1}{q(x)+1}\,|u(x,t)|^{q(x)+1}\,\mathrm{d}x\geq\frac{1}{q_1+1}\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha_2^{q_1+1}, \tag{2.9}$$

and

$$\frac{\alpha_2}{\alpha_1}\geq\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{1}{q_1-1}}>1. \tag{2.10}$$

Proof. According to Lemma 2.9, since $\mathrm{E}(u_0)<E_1$, there must be a positive constant $\alpha_2>\alpha_1$ such that $\mathrm{E}(u_0)=h(\alpha_2)$. Using equation (2.5), we can see that $h(\alpha_0)=g(\alpha_0)\leq\mathrm{E}(u_0)=h(\alpha_2)$. With the help of Lemma 2.9**(i)**, we can conclude that $\alpha_0\geq\alpha_2$, which proves that (2.8) holds for $t=0$. Now, to prove (2.8) by contradiction, let's assume that there exists a $t^*>0$ with $\sqrt{a\|\nabla u(t^*)\|_2^2+\frac{b}{2}\|\nabla u(t^*)\|_2^4}<\alpha_2$. By the continuity of $\sqrt{a\|\nabla u\left(.,t^*\right)\|_2^2+\frac{b}{2}\|\nabla u\left(.,t^*\right)\|_2^4}$ and $\alpha_2>\alpha_1$, we may take t^* such that $\alpha_2>\sqrt{a\|\nabla u\left(t^*\right)\|_2^2+\frac{b}{2}\|\nabla u\left(t^*\right)\|_2^4}>\alpha_1$, then it follows from (2.5) and (2.7) that

$$\mathrm{E}(u_0)=h(\alpha_2)<h\left(\sqrt{a\|\nabla u\left(t^*\right)\|_2^2+\frac{b}{2}\|\nabla u\left(t^*\right)\|_2^4}\right)\le \mathrm{E}\left(t^*\right),$$

which contradicts to (1.6), and (2.8) follows. By (2.1) and (**??**), we obtain

$$\begin{aligned}\int_\Omega \frac{1}{q\left(x\right)+1}\left|u\left(x,t\right)\right|^{q(x)+1}\mathrm{d}x \ge&\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{4}\|\nabla u\|_2^4-\mathrm{E}(u_0)\\ \ge&\frac{1}{2}\alpha_2^2-h(\alpha_2)=\frac{1}{q_1+1}\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha^{q_1+1}\end{aligned}\tag{2.11}$$

and (2.9) follows. Since $\mathrm{E}(u_0)<E_1$, by a simple calculation, we can check

$$\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{2}{q_1-1}}>1,$$

then the second inequality in (2.10) holds, and we only need to show the first inequality. Denote $\beta=\frac{\alpha_2}{\alpha_1}$, then $\beta>1$ by the fact that $\alpha_2>\alpha_1$. So it results from $\mathrm{E}(u_0)=h(\alpha_2)$, $B_1>1$ and (2.4) that

$$\begin{aligned}\mathrm{E}(u_0)=h(\alpha_2)=h\left(\beta\alpha_1\right)=&\alpha_1^2\left(\frac{1}{2}\beta^2-\frac{1}{q_1+1}\frac{1}{a^{\frac{q_1+1}{2}}}B_1^{q_1+1}\beta^{q_1+1}\alpha_1^{q_1-1}\right)\\ =\alpha_1^2\beta^2\left(\frac{1}{2}-\frac{1}{q_1+1}\beta^{q_1-1}\right)\ge&\alpha_1^2\left(\frac{1}{2}-\frac{1}{q_1+1}\beta^{q_1-1}\right),\end{aligned}$$

which implies that the first inequality in (2.10) holds. □

Consider $H(t)=E_1-\mathrm{E}(t)$ for $t\ge 0$, the following lemma holds.

Lemma 2.11. *According to Theorem 2.8, the functional $H(t)$ mentioned earlier has the following estimates:*

$$0<H(0)\le H(t)\le \int_\Omega \frac{1}{q\left(x\right)+1}\left|u\left(x,t\right)\right|^{q(x)+1}\mathrm{d}x,\quad t\ge 0.\tag{2.12}$$

Proof. By (1.6), $H(t)$ is nondecreasing in t. Thus

$$H(t)\ge H(0)=E_1-\mathrm{E}(u_0)>0,\quad t\ge 0.\tag{2.13}$$

Combining (2.1), (2.4), (2.8) and $\alpha_2>\alpha_1$, we have

$$\begin{aligned}H(t)-\int_\Omega \frac{1}{q\left(x\right)+1}\left|u\left(x,t\right)\right|^{q(x)+1}\mathrm{d}x =&E_1-\frac{1}{2}\left(a\|\nabla u\|_2^2+\frac{b}{2}\|\nabla u\|_2^4\right)\\ \le&\left(\frac{1}{2}-\frac{1}{q_1+1}\right)\alpha_1^2-\frac{1}{2}\alpha_1^2<0,\ t\ge 0.\end{aligned}\tag{2.14}$$

(2.12) follows from (2.13) and (2.14). □

With the three lemmas presented above, we can give the proof of the Theorem 2.8.

Proof of Theorem 2.8. Let define the function

$$\varphi(t)=\frac{1}{2}\int_{\Omega}u(x,t)^{2}\,\mathrm{d}x, \tag{2.15}$$

According to the definitions of E(t) and H(t), the derivative of $\varphi'(t)$ meets the requirements

$$\begin{aligned}
\varphi'(t)=&\int_{\Omega}u(x,t)\,u_t(x,t)\,\mathrm{d}x\\
=&\int_{\Omega}u(x,t)\left(M\left(\int_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u+|u|^{q(x)-1}u\right)\mathrm{d}x\\
=&-a\int_{\Omega}|\nabla u(x,t)|^2\,\mathrm{d}x-b\int_{\Omega}|\nabla u(x,t)|^4\,\mathrm{d}x+\int_{\Omega}|u|^{q(x)+1}\,\mathrm{d}x\\
\geq&\left(-4\mathrm{E}(t)+a\|\nabla u\|_2^2-4\int_{\Omega}\frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x\right)+\int_{\Omega}|u|^{q(x)+1}\,\mathrm{d}x\\
\geq&-4\left(E_1-H(t)\right)+\left(1-\frac{4}{q_1+1}\right)\int_{\Omega}|u(x,t)|^{q(x)+1}\,\mathrm{d}x\\
\geq&-4E_1+2H(t)+\frac{q_1-3}{q_1+1}\int_{\Omega}|u(x,t)|^{q(x)+1}\,\mathrm{d}x
\end{aligned} \tag{2.16}$$

By (2.4) and (2.8), we see

$$\begin{aligned}
4E_1=&4\frac{q_1-1}{2(q_1+1)}\left(\frac{B_1^2}{a}\right)^{-\frac{q_1+1}{q_1-1}}=2\frac{q_1-1}{q_1+1}\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha_1^{q_1+1}\\
=&2\frac{q_1-1}{q_1+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{q_1+1}\left(\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha_2^{q_1+1}\right)\\
\leq&2\frac{q_1-1}{q_1+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{q_1+1}\int_{\Omega}|u(x,t)|^{q(x)+1}\,\mathrm{d}x\\
\leq&\frac{q_1-3}{q_1+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{q_1+1}\int_{\Omega}|u(x,t)|^{q(x)+1}\,\mathrm{d}x.
\end{aligned} \tag{2.17}$$

According to Lemmas 2.11, (2.16) and (2.17), this result

$$\varphi'(t)\geq\gamma\int_{\Omega}|u(x,t)|^{q(x)+1}\,\mathrm{d}x \tag{2.18}$$

where

$$\gamma=\frac{q_1-3}{q_1+1}\left(1-\left(\frac{\alpha_1}{\alpha_2}\right)^{q_1+1}\right)>0$$

According to Hölder's inequality we have

$$\begin{aligned}
\varphi^{\frac{q_1+1}{2}}(t)\leq C_1\int_{\Omega}|u|^{q_1+1}\,\mathrm{d}x,\\
\varphi^{\frac{q_2+1}{2}}(t)\leq C_2\int_{\Omega}|u|^{q_2+1}\,\mathrm{d}x
\end{aligned} \tag{2.19}$$

where

$$C_1 = |\Omega|^{\frac{q_1-2}{2}} \left(\frac{1}{2}\right)^{\frac{q_1+1}{2}}, \text{ and } C_2 = |\Omega|^{\frac{q_2-1}{2}} \left(\frac{1}{2}\right)^{\frac{q_2+1}{2}}.$$

$|\Omega|$ is the Lebesgue measure of Ω. Then it follows from (2.18) and (2.19) that

$$\begin{aligned}\varphi'(t) \geq & \gamma \min\left(\int_\Omega |u(x,t)|^{q_1+1} \mathrm{d}x, \int_\Omega |u(x,t)|^{q_2+1} \mathrm{d}x\right) \\ \geq & \gamma \min\left(\frac{\varphi^{\frac{q_2+1}{2}}(t)}{C_2}, \frac{\varphi^{\frac{q_1+1}{2}}(t)}{C_1}\right),\end{aligned}$$

this implies

$$\varphi(t) \geq \min\left\{\begin{array}{l}\left(\left(\frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_1}{2}} - \frac{\gamma(q_1-1)}{2C_1} t\right)^{\frac{-2}{q_1-1}} \\ , \left(\left(\frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_2}{2}} - \frac{\gamma(q_2-1)}{2C_2} t\right)^{\frac{-2}{q_2-1}}\end{array}\right\}.$$

Now, let

$$0 < T^* := \max\left(\frac{2^{\frac{q_1}{2}} C_1}{\gamma(q_1-1)} \left(\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_1}{2}}, \frac{2^{\frac{q_2}{2}} C_2}{\gamma(q_2-1)} \left(\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_2}{2}}\right) < \infty, \tag{2.20}$$

then $\varphi(t)$ blows up at time T^*. Hence, $u(x,t)$ discontinues at some finite time $T \leq T^*$, that is to means, $u(x,t)$ blows up at a finite time T. Next, we estimate T. By (2.10) and the values of γ, C_1, C_2, we have

$$\begin{aligned}\frac{2^{\frac{q_1}{2}} C_1}{\gamma(q_1-1)} \quad & \leq \quad \frac{(q_1+1)|\Omega|^{\frac{q_1-2}{2}}}{(q_1-1)(q_1-3)\left(1-\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{-q_1-1}{q_1-1}}\right)}, \\ \frac{2^{\frac{p_2}{2}} C_2}{\gamma(q_2-1)} \quad & \leq \quad \frac{(q_1+1)|\Omega|^{\frac{q_1-2}{2}}}{(q_2-1)(q_1-3)\left(1-\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{-q_1-1}{q_1-1}}\right)}.\end{aligned}$$

The pair of inequalities shown above coupling (2.20) imply that $T \leq T^* \leq \widehat{T}$, with $\widehat{T}$ being a fixed in (2.6). □

For $1 < q_1 \leq q(x) \leq q_2 \leq \frac{n+2}{n-2}$, we have the following blow-up results

Theorem 2.12. *Let $u(x,t)$ the weak solution to problem* (1.1) *with the initial data $u_0 \in H_0^1(\Omega)$ are such that $u_0 \neq 0$.*

1. *Let q satisfy $(\mathcal{H}_1)$. Suppose that one of the following claims holds:*
   (i). $\mathrm{E}(u_0) < 0$,
   (ii). $E(0) \leq \mathrm{E}_d$,

(iii). $0 \le \mathrm{E}(u_0) < C_0 \|u_0\|_2^2 \triangleq \min\left(\frac{a(q_1-1)}{q_1+1}\lambda_1, \frac{b(q_1-3)}{2(q_1+1)}\lambda_1^2\|u_0\|_2^2\right)\|u_0\|_2^2$, *where* $\lambda_1 > 0$ *is the first eigenvalue of* $-\Delta$ *in* Ω *with homogeneous Dirichlet boundary condition.*

Then $u(x,t)$ *blows up in finite time. Moreover, the upper bound for* T *has the following proprieties:*

In case (i), $T \le \frac{\|u_0\|_2^2}{(1-q_1^2)\mathrm{E}(u_0)}$.

In case (ii), when $\mathrm{E}(u_0) < \mathrm{E}_d$, *then the* T *can be bounded above as:*

$$T \le \frac{4q_1 \|u_0\|_2^2}{(q_1-1)^2 (q_1+1)(\mathrm{E}_d - \mathrm{E}(0))}.$$

In case (iii), $T \le \frac{4q_1\|u_0\|_2^2}{(q_1-1)^2\left(\min\left(a(q_1-1)\lambda_1, \frac{b(q_1-3)}{2}\lambda_1^2\|u_0\|_2^2\right)\|u_0\|_2^2 - (q_1+1)\mathrm{E}(u_0)\right)}$.

2. *Let* q *satisfy* $(\mathcal{H}_2)$. *Suppose that the following claim holds:* $\mathrm{E}(u_0) < -\frac{q_1+1}{q_1+5}\frac{b}{4\varepsilon}c(\varepsilon)$, *where*

$$0 < c(\varepsilon) = \max\begin{pmatrix} \frac{3-q_1}{4}\left(B\varepsilon^{-\frac{q_1+1}{4}}\frac{q_1+1}{4}\right)^{\frac{4}{3-q_1}}, \\ \frac{3-q_2}{4}\left(B\varepsilon^{-\frac{q_2+1}{4}}\frac{q_2+1}{4}\right)^{\frac{4}{3-q_2}} \end{pmatrix}, \tag{2.21}$$

and

$$0 < \varepsilon \le \frac{b(q_1+1)^2}{16} \tag{2.22}$$

. *Then* $T < +\infty$, *which implies that* $u(x,t)$ *blows up in finite time. Moreover, the upper bound for* T *has the following form*

$$T \le \frac{\|u_0\|_2^2}{(1-q_1^2)\left(\frac{q_1+5}{q_1+1}\mathrm{E}(u_0) + \frac{b}{4\varepsilon}c(\varepsilon)\right)}.$$

Proof. 1. **(I)** Set

$$\mathrm{M}(t) = \frac{1}{2}\|u(t)\|_2^2, \quad \mathrm{J}(t) = -\mathrm{E}(u(t)) \triangleq -\mathrm{E}(u(x,t)),$$

then $\mathrm{M}(0) > 0$, $\mathrm{J}(0) > 0$. By (1.7) we have $\mathrm{J}'(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(u(t)) = \|u_t(t)\|_2^2 \ge 0$, which infers that $\mathrm{J}(t) \ge \mathrm{J}(0) > 0$ for all $t \in [0,T)$. Evoking (1.5), (1.8) and the fact that $q_1 > 3$, we gain, for any $t \in [0,T)$, that

$$\begin{aligned} \mathrm{M}'(t) = -I(u(t)) &\ge -(q_1+1)\mathrm{E}(u) + (q_1-1)\frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}(q_1-3)\|\nabla u\|_2^4 \\ &\ge (q_1+1)\mathrm{J}(t), \end{aligned} \tag{2.23}$$

This, when combined with the Cauchy-Schwarz inequality, results

$$\begin{aligned} \mathrm{M}(t)\mathrm{J}'(t) =& \frac{1}{2}\|u(t)\|_2^2\|u_t(t)\|_2^2 \ge \frac{1}{2}\|u(t)\|_2^2\|u_t(t)\|_2^2 \\ \ge& \frac{1}{2}(u,u_t)^2 = \frac{1}{2}(\mathrm{M}'(t))^2 \ge \frac{q_1+1}{2}\mathrm{M}'(t)\mathrm{J}(t). \end{aligned} \tag{2.24}$$

Based on direct calculations, it can be inferred from (2.24) that

$$\left(\mathrm{J}(t)\mathrm{M}^{-\frac{q_1+1}{2}}(t)\right)' = \mathrm{M}^{-\frac{q_1+3}{2}}(t)\left(\mathrm{J}'(t)\mathrm{M}(t) - \frac{q_1+1}{2}\mathrm{J}(t)\mathrm{M}'(t)\right) \geq 0.$$

Therefore,

$$\begin{aligned} 0 &< \kappa := \mathrm{J}(0)\mathrm{M}^{-\frac{q_1+1}{2}}(0) \leq \mathrm{J}(t)\mathrm{M}^{-\frac{q_1+1}{2}}(t) \\ &\leq \frac{1}{q_1+1}\mathrm{M}'(t)L^{-\frac{q_1+1}{2}}(t) = \frac{2}{1-q_1^2}\left(\mathrm{M}^{\frac{1-q_1}{2}}(t)\right)'. \end{aligned} \tag{2.25}$$

By integrating (2.25) over the interval $[0,t]$, where t belongs to the open interval $(0,T)$, and taking into consideration that $q_1 > 3$, we can derive the following result

$$\kappa t \leq \frac{2}{1-q_1^2}\left(\mathrm{M}^{\frac{1-q_1}{2}}(t) - \mathrm{M}^{\frac{1-q_1}{2}}(0)\right),$$

or equivalently

$$0 \leq \mathrm{M}^{\frac{1-q_1}{2}}(t) \leq \mathrm{M}^{\frac{1-q_1}{2}}(0) - \frac{q_1^2-1}{2}\kappa t, \quad t \in (0,T). \tag{2.26}$$

It is clear that (2.26) cannot hold for all $t > 0$, implying $T < +\infty$. Furthermore, it can be deduced from (2.26) that

$$T \leq \frac{2}{(q_1^2-1)\,\kappa}\mathrm{M}^{\frac{1-q_1}{2}}(0) = \frac{\|u_0\|_2^2}{(1-q_1^2)\,\mathrm{E}\,(u_0)}.$$

(II) Assuming the existence of $u(t)$ globally, we will use contradiction and define the following function:

$$\theta(t) = \int_0^t \|u(s)\|_2^2 \mathrm{d}s + (T_0 - t)\,\|u_0\|_2^2 + \beta\,(t+t_0)^2\,, \ t \in [0,T_0], \ t_0 > 0. \tag{2.27}$$

where t_0, T_0 and β are positive constants to be determined later. Then we have

$$\begin{aligned} \theta'(t) =&\|u(t)\|_2^2 - \|u_0\|_2^2 + 2\beta\,(t+t_0) \\ =&\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|_2^2 \mathrm{d}s + 2\beta\,(t+t_0) \\ =&2\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s + 2\beta\,(t+t_0)\,, \end{aligned} \tag{2.28}$$

and

$$\theta''(t) = 2\int_\Omega u_t(t)u(t)\mathrm{d}x + 2\beta. \tag{2.29}$$

Using (1.1), and (2.29) we deduce that

$$\theta''(t) = -a\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \int_\Omega |u|^{q(x)+1}\mathrm{d}x + 2\beta. \tag{2.30}$$

Based on (2.27), (2.28) and (2.30), it can be concluded that

$$
\begin{aligned}
\theta''(t)\theta(t)-&\frac{q_1+1}{2}\left(\theta'(t)\right)^2 \\
=&2\theta(t)\left[-a\|\nabla u\|_2^2-b\|\nabla u\|_2^4+\int_\Omega |u|^{q(x)+1}\mathrm{d}x+\beta\right] \\
&-\frac{q_1+1}{2}\left(2\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s+2\beta\left(t+t_0\right)\right)^2 \\
=&2\theta(t)\left[-a\|\nabla u\|_2^2-b\|\nabla u\|_2^4+\int_\Omega |u|^{q(x)+1}\mathrm{d}x+\beta\right] \\
&+2\left(q_1+1\right)\left[\eta(t)-\left(\theta(t)-(T-t)\left\|u_0\right\|_2^2\right)\left(\beta+\int_0^t\|u_t(s)\|_2^2\mathrm{d}s\right)\right]
\end{aligned}
\tag{2.31}
$$

where $\eta:[0,T]\to\mathbb{R}$ is the function given by

$$
\begin{aligned}
\eta(t) \quad = \quad & \left(\beta\left(t+t_0\right)^2+\int_0^t\|u(s)\|_2^2\mathrm{d}s\right)\left(\beta+\int_0^t\|u_t(s)\|_2^2\,\mathrm{d}s\right) \\
&-\left(\beta\left(t+t_0\right)+\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)^2.
\end{aligned}
\tag{2.32}
$$

By utilizing the Cauchy-Schwarz and Young's inequalities, we can ensure that:

$$
\begin{aligned}
\left(\int_\Omega u(t)u_t(t)\mathrm{d}x\right)^2 \leq&\|u(t)\|_2^2\left\|u_t(t)\right\|_2^2, \\
2\beta\left(t+\sigma\right)\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s \leq&\beta\left(t+t_0\right)^2\int_0^t\left\|u_t(s)\right\|_2^2\mathrm{d}s+\beta\int_0^t\|u(s)\|_2^2\mathrm{d}s
\end{aligned}
\tag{2.33}
$$

By (2.33), we get

$$
\begin{aligned}
\eta(t)\geq\beta\left(t+t_0\right)^2\int_0^t\left\|u_t(s)\right\|_2^2\mathrm{d}s+\beta\int_0^t\|u(s)\|_2^2\mathrm{d}s+\int_0^t\left\|u_t(s)\right\|_2^2\mathrm{d}s\int_0^t\|u(s)\|_2^2\mathrm{d}s \\
-2\beta\left(t+t_0\right)\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s-\left(\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)^2\geq 0,\quad \forall t\in[0,T].
\end{aligned}
\tag{2.34}
$$

From (2.31) and (2.34) we obtain

$$
\theta''(t)\theta(t)-\frac{q_1+1}{2}\left(\theta'(t)\right)^2\geq\theta(t)\zeta(t), \tag{2.35}
$$

where $\zeta(t)$ is given by

$$
\zeta(t)=-2a\|\nabla u\|_2^2-2b\|\nabla u\|_2^4+2\int_\Omega |u|^{q(x)+1}\mathrm{d}x+2\beta-2\left(q_1+1\right)\left(\beta+\int_0^t\|u_t(s)\|_2^2\mathrm{d}s\right)
\tag{2.36}
$$

We will now make an estimation of $\zeta(t)$, using equations (1.7), and (2.36) yields

$$\begin{aligned}\zeta(t) =& -2a\|\nabla u\|_2^2 - 2b\|\nabla u\|_2^4 + 2\int_\Omega |u|^{q(x)+1}\mathrm{d}x \\ &+ 2(q_1+1)\,\mathrm{E}(u) - 2(q_1+1)\,\mathrm{E}(u_0) - 2q_1\beta \\ \geq& -2a\|\nabla u\|_2^2 - 2b\|\nabla u\|_2^4 + (q_1+1)\,a\|\nabla u\|_2^2 \\ &+ \frac{b}{2}(q_1+1)\,\|\nabla u\|_2^4 - 2(q_1+1)\,\mathrm{E}(u_0) - 2q_1\beta \\ =& (q_1-1)\,a\|\nabla u\|_2^2 + \frac{q_1-3}{2}b\|\nabla u\|_2^4 - 2(q_1+1)\,\mathrm{E}(u_0) - 2q_1\beta \\ =& 2(q_1+1)\left[\begin{array}{r}\left(\frac{1}{2}-\frac{1}{q_1+1}\right)a\|\nabla u\|_2^2 + \left(\frac{1}{4}-\frac{1}{q_1+1}\right)b\|\nabla u\|_2^4 \\ -\mathrm{E}(u_0) - \frac{q_1}{q_1+1}\beta\end{array}\right]\end{aligned} \tag{2.37}$$

Let β be a positive value such that $\beta \in \left(0, \frac{q_1+1}{q_1}\left(\mathrm{E}_d - \mathrm{E}(u_0)\right)\right]$, and since $u_0 \in \mathcal{U}$ by Lemma 2.7, we have:

$$d \leq \left(\frac{1}{2} - \frac{1}{q_2+1}\right)a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4} - \frac{1}{q_2+1}\right)b\|\nabla u(t)\|_2^4. \tag{2.38}$$

And by assuming $\mathrm{E}(u_0) < \mathrm{E}_d$ we get

$$\begin{aligned}\mathrm{E}(u_0) <& \frac{q_1-1}{q_1+1}\frac{q_2+1}{q_2-1}d \leq d \\ &\leq \left(\frac{1}{2} - \frac{1}{q_2+1}\right)a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4} - \frac{1}{q_2+1}\right)b\|\nabla u(t)\|_2^4.\end{aligned} \tag{2.39}$$

If we connect (2.37) and (2.39) we obtain

$$\zeta(t) > \rho > 0. \tag{2.40}$$

From (2.35) and (2.40), we reach at

$$\theta''(t)\theta(t) - \frac{q_1+1}{2}\left(\theta'(t)\right)^2 \geq \rho\theta(t). \tag{2.41}$$

By the continuity of θ and equation (2.38), we can infer that there exists a positive constant c such that $\theta(t) \geq c$ for t in the interval $[0, T]$. Therefore, equation (2.41) produces

$$\theta''(t)\theta(t) - \frac{q_1+1}{2}\left(\theta'(t)\right)^2 \geq c\rho. \tag{2.42}$$

In this case, we prove that T cannot be infinite, meaning there is no weak solution at all times. We use Lemma 2.5 to infer that $\theta(t) \to \infty$ as $t \to T_*$, where

$$T_* \leq \frac{\theta(0)}{(q_1-1)\theta'(0)} = \frac{T_0\,\|u_0\|_2^2 + \beta t_0^2}{(q_1-1)\beta t_0},$$

there exists a $T^* < T_*$ which

$$\lim_{t\to T^*}\int_0^t \|u(s)\|_2^2 \mathrm{d}s + (T_0 - t)\|u_0\|_2^2 + \beta(t+t_0)^2 = +\infty.$$

Let's choose appropriate values for t_0 and T_0. We can set t_0 to any number that depends only on q_1, $d - E(0)$ and u_0

$$t_0 > \frac{\|u_0\|_2^2}{(q_1-1)\beta}$$

If t_0 is fixed, then T_0 can be chosen as

$$T_0 = \frac{T_0\|u_0\|_2^2 + \beta t_0^2}{(q_1-1)\beta t_0},$$

so that

$$T_0 = \frac{\beta t_0^2}{(q_1-1)\beta t_0 - \|u_0\|_2^2}.$$

The lifespan of the solution $u(x,t)$ is bounded by a certain number as

$$T_0 = \inf_{t\geq t_0}\frac{\beta t^2}{\left((q_1-1)\beta t - \|u_0\|_2^2\right)} = \frac{4\|u_0\|_2^2}{(q_1-1)^2\beta} = \frac{4q_1\|u_0\|_2^2}{(q_1-1)^2(q_1+1)(\mathrm{E}_d - \mathrm{E}(u_0))}.$$

Due to the arbitrariness of $T_0 < T$ it follows that

$$T \leq \frac{4q_1\|u_0\|_2^2}{(q_1-1)^2(q_1+1)(\mathrm{E}_d - \mathrm{E}(u_0))}.$$

(III) To deal with the case $0 \leq \mathrm{E}(u_0) < C_0\|u_0\|_2^2$, first, it follows from the definitions of $I(u)$, $\mathrm{E}(u)$ and the assumption (ii) that

$$\begin{aligned} I(u_0) &= (q_1+1)\mathrm{E}(u_0) - \frac{a(q_1-1)}{2}\|\nabla u_0\|_2^2 - \frac{b(q_1-3)}{4}\|\nabla u_0\|_2^4 \\ &= (q_1+1)\left(\mathrm{E}(u_0) - C_0\|u_0\|_2^2\right) - \frac{a(q_1-1)}{2}\left(\|\nabla u_0\|_2^2 - \lambda_1\|u_0\|_2^2\right) \\ &- \frac{b(q_1-3)}{4}\|\nabla u_0\|_2^4 < 0. \end{aligned}$$

We claim that for all $t \in [0,T)$, $I(u(t)) < 0$. Otherwise, there would exist a $t_0 \in (0,T)$ such that $I(u(t)) < 0$ for all $t \in [0,t_0)$ and $I(u(t_0)) = 0$. By (2.23), we have that $\|u(t)\|_2^2$ and $\|u(t)\|_2^4$ are strictly increasing in t for $t \in [0,t_0)$, and therefore

$$0 \leq \mathrm{E}(u_0) < C_0\|u_0\|_2^2 < C_0\|u(t_0)\|_2^2. \tag{2.43}$$

On the other hand, we can deduce from the monotonicity of $\mathrm{E}(u(t))$ and (1.5)

$$
\begin{aligned}
\mathrm{E}\left(u_0\right) \geq & \mathrm{E}\left(u\left(t_0\right)\right)=\frac{a(q_1-1)}{2(q_1+1)}\left\|\nabla u\left(t_0\right)\right\|_2^2+\frac{b(q_1-3)}{4(q_1+1)}\left\|\nabla u\left(t_0\right)\right\|_2^4+\frac{1}{q_1+1} I\left(u\left(t_0\right)\right) \\
& \geq \min \left(\frac{a(q_1-1)}{(q_1+1} \lambda_1, \frac{b(q_1-3)}{2(q_1+1)} \lambda_1^2\|u_0\|_2^2\right)\left\|u\left(t_0\right)\right\|_2^2=C_0\left\|u\left(t_0\right)\right\|_2^2,
\end{aligned}
$$

Therefore, since (2.43) is contradictory, we have $I(u(t)<0$ for all $t \in[0, T)$. Then, $\|u(t)\|_2^2$ is strictly increasing on $[0, T)$ and $\|u(t)\|_2^4$ is also strictly increasing on $[0, T)$. For any $T_0 \in(0, T)$, $\beta>0$, and $t_0>0$, we define

$$
F(t)=\int_0^t\|u(\tau)\|_2^2 \mathrm{d} \tau-\left(T_0-t\right)\left\|u_0\right\|_2^2+\beta(t+t_0)^2, \quad t \in\left[0, T_0\right]. \tag{2.44}
$$

Through a direct calculations

$$
\begin{aligned}
F^{\prime}(t)= & \|u(t)\|_2^2-\left\|u_0\right\|_2^2+2 \beta(t+t_0)=\int_0^t \frac{\mathrm{d}}{\mathrm{d} \tau}\|u(\tau)\|_2^2 \mathrm{d} \tau+2 \beta(t+t_0) \\
= & 2 \int_0^t\left(u, u_\tau\right) \mathrm{d} \tau+2 \beta(t+t_0), \\
F^{\prime \prime}(t)= & 2\left(u, u_t\right)+2 \beta=-2 I(u(t))+2 \beta \\
= & -2(q_1+1) \mathrm{E}(u(t))+a(q_1-1)\|\nabla u(t)\|_2^2+\frac{b(q_1-3)}{2}\|\nabla u(t)\|_2^4+2 \beta \\
= & -2(q_1+1) \mathrm{E}\left(u_0\right)+2(q_1+1) \int_0^t\left\|u_\tau(\tau)\right\|_2^2 \mathrm{~d} \tau+a(q_1-1)\|\nabla u(t)\|_2^2 \\
& +\frac{b(q_1-3)}{2}\|\nabla u(t)\|_2^4+2 \beta .
\end{aligned} \tag{2.45}
$$

For $t \in\left[0, T_0\right]$, set

$$
\begin{aligned}
\theta(t)= & \left(\int_0^t\|u(\tau)\|_2^2 \mathrm{d} \tau+\beta(t+t_0)^2\right)\left(\int_0^t\left\|u_\tau\right\|_2^2 \mathrm{~d} \tau+\beta\right) \\
& -\left(\int_0^t\left(u, u_\tau\right) \mathrm{d} \tau+\beta(t+t_0)\right)^2 .
\end{aligned}
$$

By applying Cauchy-Schwarz and Hölder's inequalities, we can show that $F(t)$ is non-negative on the interval $[0, T_0]$. As a result, we can use equation (2.44)-(2.45) and the monotonicity of $\|u(t)\|_2^2$ and $\|u(t)\|_2^4$ to conclude

$$\begin{aligned}
&F(t)F''(t) - \frac{q_1+1}{2}\left(F'(t)\right)^2 \\
=&F(t)F''(t) - 2(q_1+1)\left(\int_0^t (u,u_\tau)\,\mathrm{d}\tau + \beta(t+t_0)\right)^2 \\
=&F(t)F''(t) + 2(q_1+1)\left[\theta(t) - \left(F - (T-t)\|u_0\|_2^2\right)\left(\int_0^t \|u_\tau\|_2^2\,\mathrm{d}\tau + \beta\right)\right] \\
\geq&F(t)F''(t) - 2(q_1+1)F(t)\left(\int_0^t \|u_\tau\|_2^2\,\mathrm{d}\tau + \beta\right) \\
=&F(t)\Big[-2(q_1+1)\mathrm{E}(u_0) + 2(q_1+1)\int_0^t \|u_\tau\|_2^2\,\mathrm{d}\tau + a(q_1-1)\|\nabla u(t)\|_2^2 \\
&+\frac{b(q_1-3)}{2}\|\nabla u(t)\|_2^4 + 2\beta - 2(q_1+1)\int_0^t \|u_\tau\|_2^2\,\mathrm{d}\tau - 2(q_1+1)\beta\Big] \\
\geq&F(t)\left[-2(q_1+1)\mathrm{E}(u_0) + a(q_1-1)\lambda_1\|u(t)\|_2^2 + \frac{b(q_1-3)}{2}\lambda_1^2\|u(t)\|_2^4 - 2q_1\beta\right] \\
\geq&F(t)\left[-2(q_1+1)\mathrm{E}(u_0) + \min\left(a(q_1-1)\lambda_1, \frac{b(q_1-3)}{2}\lambda_1^2\|u_0\|_2^2\right)\|u_0\|_2^2 - 2q_1\beta\right] \\
=&2(q_1+1)F(t)\left[C_0\|u_0\|_2^2 - \mathrm{E}(u_0) - \frac{q_1\beta}{q_1+1}\right] \geq 0.
\end{aligned} \tag{2.46}$$

Choosing $0 < \beta < \frac{q_1+1}{q_1}\left(C_0\|u_0\|_2^2 - \mathrm{E}(u_0)\right)$. Then using Lemma 2.5, to infer $F(t) \to \infty$ as $t \to T^*$, where

$$T^* \leq \frac{F(0)}{(q_1-1)F'(0)} = \frac{T_0\|u_0\|_2^2 + \beta t_0^2}{(q_1-1)\beta t_0}. \tag{2.47}$$

Let's choose appropriate values for t_0 and T_0. We can set t_0 to any number that only depends on q_1, $d - E(0)$ and u_0 as

$$t_0 > \frac{\|u_0\|_2^2}{(q_1-1)\beta}.$$

Fix t_0, then T_0 can be picking a

$$T_0 = \frac{T_0\|u_0\|_2^2 + \beta t_0^2}{(q_1-1)\beta t_0},$$

so that

$$T_0 = \frac{\beta t_0^2}{(q_1-1)\beta t_0 - \|u_0\|_2^2}.$$

Therefore, the lifespan of the solution $u(x,t)$ is bounded by

$$T_0 = \inf_{t\geq t_0}\frac{\beta t^2}{(q_1-1)\beta t - \|u_0\|_2^2} = \frac{4q_1\|u_0\|_2^2}{(q_1-1)^2(q_1+1)\left(C_0\|u_0\|_2^2 - \mathrm{E}(u_0)\right)},$$

due to the arbitrariness of $T_0 < T$ it follows that

$$T_0 \leq \frac{4q_1 \|u_0\|_2^2}{(q_1-1)^2 (q_1+1) \left(\min\left(a(q_1-1)\lambda_1, \frac{b(q_1-3)}{2}\lambda_1^2 \|u_0\|_2^2\right) \|u_0\|_2^2 - \mathrm{E}(u_0)\right)}.$$

2. To handle the case where $1 < q_1 \leq q(x) \leq q_2 \leq 3$, we modify the energy functional E by setting

$$\begin{aligned}\mathrm{M}(t) &= \frac{1}{2}\|u(t)\|_2^2, \quad \mathrm{J}(t) = -\mathrm{E}(u(t)) - \left(\frac{4}{q_1+1}E(0) + \frac{b}{4\varepsilon}c(\varepsilon)\right) \\ &\triangleq -\mathrm{E}(u(x,t)) - \left(\frac{4}{q_1+1}\mathrm{E}(u_0) + \frac{b}{4\varepsilon}c(\varepsilon)\right),\end{aligned}$$

then $\mathrm{M}(0) > 0$, $\mathrm{J}(0) > 0$. By (1.7) we also have

$$\mathrm{J}'(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(u(t)) = \|u_t(t)\|_2^2 \geq 0.$$

It implies that $\mathrm{J}(t) \geq \mathrm{J}(0)$ for all $t \in [0,T)$. Additionally, Lemma 2.6 states that for any $\varepsilon > 0$

$$\begin{aligned}\int_\Omega |u|^{q(x)+1}\mathrm{d}x &\leq B\max\left(\|\nabla u\|_2^{q_1+1}, \|\nabla u\|_2^{q_2+1}\right) \\ &\leq \max\begin{pmatrix}\varepsilon\|\nabla u\|_2^4 + \frac{3-q_1}{4}\left(\frac{B}{\varepsilon^{\frac{q_1+1}{4}}}\frac{q_1+1}{4}\right)^{\frac{4}{3-q_1}}, \\ \varepsilon\|\nabla u\|_2^4 + \frac{3-q_2}{4}\left(\frac{B}{\varepsilon^{\frac{q_2+1}{4}}}\frac{q_2+1}{4}\right)^{\frac{4}{3-q_2}}\end{pmatrix} \\ &\leq \varepsilon\|\nabla u\|_2^4 + c(\varepsilon),\end{aligned}$$

which give

$$\|\nabla u\|_2^4 \geq \frac{1}{\varepsilon}\int_\Omega |u|^{q(x)+1}\mathrm{d}x - \frac{1}{\varepsilon}c(\varepsilon),$$

and from (1.5)

$$\begin{aligned}E(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x \\ &\geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\frac{1}{\varepsilon}\int_\Omega |u|^{q(x)+1}\mathrm{d}x - \frac{1}{q_1+1}\int_\Omega |u|^{q(x)+1}\mathrm{d}x - \frac{b}{4\varepsilon}c(\varepsilon),\end{aligned}$$

also using (1.5), and $(1.5)_2$ we have

$$b\|\nabla u\|_2^4 \leq 4E(0) + \frac{4}{q_1+1}\int_\Omega |u|^{q(x)+1}\mathrm{d}x,$$

in which $(1.5)_2$ becomes

$$\begin{aligned}I(u) \leq a\|\nabla u\|_2^2 + 4\mathrm{E}(u_0) + \frac{4}{q_1+1}\int_\Omega |u|^{q(x)+1}\mathrm{d}x \\ - \int_\Omega |u|^{q(x)+1}\mathrm{d}x + \frac{a(q_1-1)}{2}\|\nabla u_0\|_2^2\end{aligned}$$

thus we obtain, for any $t \in [0,T)$, that

$$\begin{aligned}
& (q_1+1)\,\mathrm{E}(u) - I(u) \\
\geq\ & \frac{q_1-1}{2} a\|\nabla u\|_2^2 + \frac{b\,(q_1+1)}{4\varepsilon}\int_\Omega |u|^{q(x)+1}\mathrm{d}x - \int_\Omega |u|^{q(x)+1}\mathrm{d}x \\
& -4\mathrm{E}(u_0) - \frac{b\,(q_1+1)}{4\varepsilon} c\,(\varepsilon) - \frac{4}{q_1+1}\int_\Omega |u|^{q(x)+1}\mathrm{d}x + \int_\Omega |u|^{q(x)+1}\mathrm{d}x \\
\geq\ & \left(\frac{b\,(q_1+1)}{4\varepsilon} - \frac{4}{q_1+1}\right)\int_\Omega |u|^{q(x)+1}\mathrm{d}x - 4\mathrm{E}(u_0) - \frac{b\,(q_1+1)}{4\varepsilon} c\,(\varepsilon) \\
\geq\ & -4\mathrm{E}(u_0) - \frac{b\,(q_1+1)}{4\varepsilon} c\,(\varepsilon)\,,
\end{aligned}$$

this meaning that

$$\mathrm{M}'(t) = -I(u) \geq -\,(q_1+1)\,\mathrm{E}(u) - 4\mathrm{E}(u_0) - \frac{b\,(q_1+1)}{4\varepsilon} c\,(\varepsilon) = -\,(q_1+1)\,\mathrm{J}(t),$$

which, together with Cauchy-Schwarz inequality, yields

$$\mathrm{M}(t)\mathrm{J}'(t) = \frac{1}{2}\|u(t)\|_2^2\,\|u_t(t)\|_2^2 \geq \frac{1}{2}\,(u,u_t)^2 = \frac{1}{2}\,(\mathrm{M}'(t))^2 \geq \frac{q+1}{2}\mathrm{M}'(t)\mathrm{J}(t).$$

By direct computations as previously, it follows that

$$0 \leq \mathrm{M}^{\frac{1-q_1}{2}}(t) \leq \mathrm{M}^{\frac{1-q_1}{2}}(0) - \frac{q_1^2-1}{2}\mathrm{J}(0)\mathrm{M}^{-\frac{q_1+1}{2}}(0)t, \quad t \in (0,T). \tag{2.48}$$

It is obvious to see that (2.48) cannot hold for all $t > 0$. Therefore, $T < +\infty$. Moreover, it can be inferred that

$$T \leq \frac{2}{(q_1^2-1)\,\mathrm{J}(0)L^{-\frac{q_1+1}{2}}(0)}\mathrm{M}^{\frac{1-q_1}{2}}(0) = \frac{\|u_0\|_2^2}{(1-q_1^2)\left(\frac{q_1+5}{q_1+1}\mathrm{E}(u_0) + \frac{b}{4\varepsilon}c\,(\varepsilon)\right)}.$$

□

Remark 2.13. It is not possible to compare the conditions in Theorem2.8 and Theorem2.12, which use E_d, E_1, and $\mathrm{E}(u_0)$. However, when $1 < q_1 \leq q\,(x) \leq q_2 \leq 3$, instead of $3 < q_1 \leq q\,(x) \leq q_2 \leq 2^*$, and $n \geq 3$, three new blow-up criteria are obtained which have not been addressed before in [14, 15].

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The Minty-Browder theorem for nonlinear elliptic equations involving p-Laplacian with singular coefficients under form boundary conditions

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Abstract. We consider the elliptic parabolic partial differential equation with singular coefficients under the rather general form boundary conditions. We proved that the bounded operator associated with the elliptic equation satisfies monotony, coercivity, and semicontinuity conditions. Employing Minty-Browder arguments, we establish the existence and uniqueness of the weak solution to the elliptic equation with singular coefficients under form-boundary conditions.

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1. Introduction

In this article, we consider the existence of the weak solution to a quasi-linear elliptic differential equation in the divergent form

$$\lambda u \left|u\right|^{p-2} - \frac{d}{dx_i} a_i \left(x,\ u,\ \nabla u\right) + b\left(x,\ u,\ \nabla u\right) = 0$$

with a positive parameter λ, where the divergent term is given by

$$\frac{d}{dx_i} a_i \left(x,\ u,\ \nabla u\right) = \sum_{i=1,\ \dots,\ l} \frac{\partial a_i \left(x,\ u,\ \nabla u\right)}{\partial x_i}$$

in domain $\Omega \subseteq \mathbb{R}^l$, $l \geq 3$. As a model example of the main term, we can consider the operator $\Delta_p u \equiv div\left(\nabla u \left|\nabla u\right|^{p-2}\right)$ and lower term $b\left(x,\ u,\ \nabla u\right) = c\left(x\right) u \left|u\right|^{p-2}$.

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Due to the plethora of applications of elliptic partial differential equations, the theory of the existence of solutions is well developed. There are many approaches to the solvability theory for elliptic equations, such as the mountain pass theorem, method of sub-super solutions, degree theory, and fixed point theory to name a few. A general version of the Minty-Browder theorem states that if an operator A from real, separable, reflexive Banach space X into its dual space X^* is semicontinuous, monotone, and coercive, then for each $\psi \in X^*$ there is a solution $f \in X$ to the equation $A(f) = \psi$. The classical results of the Minty-Browwder theorem can be found in [3 – 5, 15, 16], where the method of monotone operators was developed and its application to the Dirichlet problem for a quasi-linear elliptic partial differential equation in the divergence form was considered [16]. In [4, 5], nonlinear elliptic boundary value problems were considered in Hilbert spaces by the method of monotone operators, the semi-boundedness was employed instead of the positivity condition, also, the perturbation of such operators by compact operators was studied.

In this article, we consider elliptic differential equations in the divergent form under form-boundary conditions on its coefficients. The local singularities of the coefficients are supposed such that they belong to certain classes $PK(\beta)$.

Definition 1.1. *For a given number $\beta \in (0,1)$, the class of form-boundary functions $PK(\beta)$ consists of all functions $f \in L^1_{loc}(\Omega)$ such that the inequality*

$$\|f\phi\|^2_{L^2} \le \beta \|\nabla\phi\|^2_{L^2} + c(\beta)\|\phi\|^2_{L^2}, \tag{1.1}$$

holds with a positive constant $c(\beta)$ and for all $\phi \in W^2_1(\Omega)$.

Some additional information on this type of form-boundary condition can be founded in [22, 23, 24].

From the definition of form-boundary class, assuming $\gamma \ge 0$ and $\gamma^{\frac{1}{2}} \in PK(\beta)$, we obtain

$$\int_\Omega \gamma|\phi|^p\,dx \le \beta\frac{p^2}{4}\|\phi\|^{p-2}_{L^p}\|\nabla\phi\|^2_{L^p} + c(\beta)\|\phi\|^p_{L^p},$$

for all $\phi \in W^p_1(\Omega)$ and $p \ge 2$. Indeed, we estimate

$$\begin{aligned}
&\int_\Omega \gamma|\phi|^p\,dx = \int_\Omega \left(|\gamma|^{\frac12}|\phi|^{\frac p2}\right)^2 dx = \left\||\gamma|^{\frac12}|\phi|^{\frac p2}\right\|^2_{L^2}\\
&\le \beta\left\|\nabla\left(\phi^{\frac p2}\right)\right\|^2_{L^2} + c(\beta)\left\|\phi^{\frac p2}\right\|^2_{L^2}\\
&= \beta\int_\Omega\left(\nabla\left(\phi^{\frac p2}\right)\right)^2 dx + c(\beta)\int_\Omega\left(|\phi|^{\frac p2}\right)^2 dx\\
&= \beta\int_\Omega\left(\tfrac p2\phi^{\frac p2-1}\nabla\phi\right)^2 dx + c(\beta)\int_\Omega|\phi|^p\,dx\\
&= \beta\tfrac{p^2}{4}\int_\Omega\phi^{p-2}(\nabla\phi)^2\,dx + c(\beta)\|\phi\|^p_{L^p}.
\end{aligned}$$

Next, applying the Holder inequality, we obtain

$$\begin{aligned}
&\int_\Omega \gamma|\phi|^p\,dx \le \beta\tfrac{p^2}{4}\left\|\phi^{p-2}\right\|_{L^{\frac{p}{p-2}}}\left\|(\nabla\phi)^2\right\|_{L^{\frac p2}} + c(\beta)\|\phi\|^p_{L^p}\\
&= \beta\tfrac{p^2}{4}\|\phi\|^{p-2}_{L^p}\|\nabla\phi\|^2_{L^p} + c(\beta)\|\phi\|^p_{L^p}.
\end{aligned}$$

The form-boundary condition guarantees the coercitivity of the associated quadratic form in L^2, namely, the linear operator $-\Delta + \vec{f} \cdot \nabla$ is coercive in L^2 if $\left|\vec{f}\right| \in PK(\beta)$.

We proved that the operator $A : W_1^p(\Omega) \to W_1^p(\Omega)$ given in (2.1), satisfies the monotony, coercivity, and semi-continuity conditions. The existence and the uniqueness of the weak solution to the considered equation follow from the Minty-Browder theorem applied to the operator A.

2. An elliptic partial differential equation

2.1. Basic properties of Sobolev spaces

Let Ω be a smooth domain in R^l for $l \geq 3$. The Sobolev space $W_k^p(\Omega)$ is a Banach space consisting of all elements $u \in L^p(\Omega)$ such that for all multy-index α with $|\alpha| \leq k$, the distributional mixed partial derivative

$$D^\alpha u = u^{(\alpha)} = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_l^{\alpha_l}}$$

exists and belongs to $L^p(\Omega)$, i.e., $\left\|u^{(\alpha)}\right\|_{L^p} < \infty$. The norm in $W_k^p(\Omega)$ is defined by

$$\|u\|_{W_k^p} = \left(\int_\Omega \left(|u|^p + \sum_{m=1,\,...,\,k} \sum_{(m)} \left|D^{(m)} u\right|^p\right) dx\right)^{\frac{1}{p}},$$

or equivalent form in the sense of equivalence of norms

$$\|u\|_{W_k^p}^{\sim} = \|u\|_{L^p} + \sum_{m=1,\,...,\,k} \sum_{(m)} \left\|D^{(m)} u\right\|_{L^p},$$

where the symbol $\sum_{(m)}$ means summation by all possible derivatives of u up to order m. For the domains Ω with smooth enough boundaries $\partial\Omega$, the space $W_k^p(\Omega)$ coincides with the closure of the set $C^\infty(\Omega)$ of all infinitely differentiable functions in $clos(\Omega)$. In particular, the norm of $W_1^p(\Omega)$ is given by

$$\|u\|_{W_1^p} = \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{\frac{1}{p}},$$

or equivalent form in the sense of equivalence of norms

$$\|u\|_{W_1^p}^{\sim} = \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

Property. *For $p \in (1, \infty)$ and for each integer $m \geq 0$, the Sobolev space $W_k^p(\Omega)$ is a reflexive separable Banach space with the dual $W_{-k}^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The set $C^\infty(clos(\Omega))$ is dense subset of $W_k^p(\Omega)$. The subspace $W_{k,0}^p(\Omega)$ is dense in $W_k^p(\Omega)$.*

In $W_{1,0}^p$, the following inequality holds true (Poincare inequality)

$$\|u\|_{L^p} \leq c\,\|\nabla u\|_{L^p},$$

for all $u \in W_{1,0}^p(\Omega)$, where the constant c depends only on the domain Ω and exponent p.

The Sobolev embedding theorem establishes that if $m \geq s$ and $m - \frac{l}{p} \geq s - \frac{l}{r}$ then the embedding $W_k^p(\Omega) \subseteq W_s^r(\Omega)$ is continuous, and moreover, when $m - \frac{l}{p} > s - \frac{l}{r}$ then the embedding is completely continuous, i.e., each relatively weakly compact subset maps into a relatively compact subset.

In this paper, we use the following Holder inequality

$$\int_\Omega |fg\varphi|\, dx \leq \|f\|_{L^p} \|g\|_{L^q} \|\varphi\|_{L^r},$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Also, for all $x, y \geq 0$, we use the Young inequality

$$xy \leq \frac{1}{a}(\varepsilon x)^a + \frac{1}{b}\left(\frac{y}{\varepsilon}\right)^b$$

for all $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1$, and all $\varepsilon > 0$.

2.2. A nonlinear elliptic partial differential equation involving p-Laplace operator

Let Ω be a smooth domain in $\mathbb{R}^l$ for $l \geq 3$, which can coincide with whole $\mathbb{R}^l$. For some $\lambda > 0$, we consider a nonlinear elliptic partial differential equation

$$A(u) \equiv \lambda u |u|^{p-2} - div(a_i(x,\ u,\ \nabla u)) + b(x,\ u,\ \nabla u) = 0, \tag{2.1}$$

where $u(x)$ an unknown function in $\Omega \subseteq \mathbb{R}^l$.

Functions $a_i(x,\ u,\ \xi)$ and $b(x,\ u,\ \xi)$ are defined for all $x \in clos(\Omega)$ and all $u \in \mathbb{R},\ \xi \in \mathbb{R}^l$; $a_i(x,\ u,\ \xi)$ and $b(x,\ u,\ \xi)$ are continuous at u and ξ.

We assume the following conditions

$$\sum_i a_i(x,\ u,\ \xi)\,\xi_i \geq \nu |\xi|^p, \tag{2.2}$$

$$\sum_i (a_i(x,\ u,\ \xi) - a_i(x,\ v,\ \eta))(\xi_i - \eta_i) > \nu_1 |\xi - \eta|^p > 0, \tag{2.3}$$

$$|a_i(x,\ u,\ \xi)| \leq \mu |\xi|^{p-1} + \gamma_1(x)|u|^{p-1} + \gamma_2(x), \tag{2.4}$$

$$|a_i(x,\ u,\ \xi) - a_i(x,\ v,\ \eta)| \leq \mu_3 |\xi - \eta|^{p-1} + \gamma_6(x)|u - v|^{p-1}, \tag{2.5}$$

$$|b(x,\ u,\ \xi)| \leq \mu_1 |\xi|^{p-1} + \gamma_3(x)|u|^{p-1} + \gamma_4(x), \tag{2.6}$$

$$|b(x,\ u,\ \xi) - b(x,\ v,\ \eta)| \leq \mu_2 |\xi - \eta|^{p-1} + \gamma_5(x)|u - v|^{p-1}, \tag{2.7}$$

for all $\xi \in \mathbb{R}^l$. We assume

$${\gamma_1}^{\frac{q}{2}},\ {\gamma_3}^{\frac{q}{2}},\ {\gamma_5}^{\frac{q}{2}}, {\gamma_6}^{\frac{q}{2}} \in PK(\beta),\ {\gamma_3}^{\frac{1}{2}},\ {\gamma_5}^{\frac{1}{2}} \in PK(\beta),$$

and

$$\gamma_4 \in L^q(\Omega).$$

We remark that the inequalities

$$\left(\xi|\xi|^{p-2} - \eta|\eta|^{p-2},\ \xi - \eta\right) \geq c(p)|\xi - \eta|^p$$

and

$$\left|x|x|^{p-2} - y|y|^{p-2}\right| \leq (p-1)|x - y|\left(|x|^{p-2} + |y|^{p-2}\right)$$

hold for all $\xi,\ \eta \in \mathbb{R}^l$ and $x,\ y \in \mathbb{R}$ with the constant $c(p) = 2^{2-p}$. Employing this estimate, we obtain that p-Laplacian $a(u) = \Delta_p(u) = div\left(\nabla u \left|\nabla u\right|^{p-2}\right)$ satisfies our conditions.

Definition 2.1. *The function $u(x,\ t)$ is called a weak solution to the equation* (2.1) *if $u \in W_1^p(\Omega)$ and the identity*

$$\lambda \int_\Omega |u|^{p-2} \phi dx + \int_\Omega a_i(x,\ u,\ \nabla u) \nabla_i \phi dx + \int_\Omega b(x,\ u,\ \nabla u) \phi dx = 0 \tag{2.8}$$

holds for all $\phi \in W_{1,\,0}^p(\Omega)$. The solution u is called a bounded weak solution to the equation (2.1) *if $ess \max_\Omega |u| < \infty$.*

Definition 2.2. *The operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is called monotone if the inequality*

$$\langle A(u) - A(v),\ (u - v)\rangle \geq 0 \tag{2.9}$$

holds for all $u,\ v \in W_{1,0}^p(\Omega)$.

The operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is called strictly monotone if the inequality

$$\langle A(u) - A(v),\ u - v\rangle > 0 \tag{2.10}$$

holds for all $u,\ v \in W_{1,0}^p(\Omega)$, $u \neq v$.

Definition 2.3. *The operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is called coercive if the inequality*

$$\frac{\langle A(u),\ u\rangle}{\|u\|_{W_1^p}} \overset{\|u\|_{W_1^p} \to \infty}{\longrightarrow} \infty. \tag{2.11}$$

Definition 2.4. *The operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is called semicontinuous if the mapping $t \mapsto \langle A(u + tv),\ w\rangle$ is continuous for all $u,\ v,\ w \in W_{1,0}^p(\Omega)$.*

Below, we assume that the operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is associated with the elliptic equation (2.1).

Lemma 2.5. *Let $p \geq 2$ and let q be its conjugate, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Assume the conditions* (2.2)-(2.8) *are satisfied. Then, the operator*

$$A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$$

is bounded.

Proof. For all $u,\ v \in W_{1,0}^p(\Omega)$, we have

$$\begin{aligned}
&|\langle A(u),\ v\rangle| \leq \lambda \int_\Omega \lambda |u|^{p-1} |v| \, dx \\
&+ \int_\Omega \left(\mu |\nabla u|^{p-1} + \gamma_1(x) |u|^{p-1} + \gamma_2(x)\right) |\nabla v| \, dx \\
&+ \int_\Omega \left(\mu_1 |\nabla u|^{p-1} + \gamma_3(x) |u|^{p-1} + \gamma_4(x)\right) |v| \, dx \\
&\leq \lambda \|u\|_{L^p}^{p-1} \|v\|_{L^p} + \mu \|\nabla u\|_{L^p}^{p-1} \|\nabla v\|_{L^p} + \left\|\gamma_1{}^{\frac{1}{p-1}} u\right\|_{L^p}^{p-1} \|\nabla v\|_{L^p} \\
&+ \|\gamma_2\|_{L^q} \|\nabla v\|_{L^p} + \mu_1 \|\nabla u\|_{L^p}^{p-1} \|v\|_{L^p} \\
&+ \left\|\gamma_3{}^{\frac{1}{p-1}} u\right\|_{L^p}^{p-1} \|v\|_{L^p} + \|\gamma_4\|_{L^q} \|v\|_{L^p}.
\end{aligned}$$

Applying the Young inequality for $a = \frac{p}{p-2}$ and $b = \frac{p}{2}$, and the form-boundary condition, we have

$$\int_\Omega {\gamma_1}^q |u|^p \, dx \le \beta \int_\Omega \left(\nabla \left(|u|^{\frac{p}{2}} \right) \right)^2 dx + c(\beta) \int_\Omega |u|^p \, dx$$
$$\le \beta \frac{p^2}{4} \left(\frac{2}{\varepsilon^{\frac{p}{2}} p} \|\nabla u\|^p_{L^p} + (p-2) \frac{\varepsilon^{\frac{p}{p-2}}}{p} \|u\|^p_{L^p} \right) + c(\beta) \|u\|^p_{L^p},$$

and similarly, we obtain

$$\int_\Omega {\gamma_3}^q |u|^p \, dx$$
$$\le \beta \frac{p^2}{4} \left(\frac{2}{\varepsilon^{\frac{p}{2}} p} \|\nabla u\|^p_{L^p} + (p-2) \frac{\varepsilon^{\frac{p}{p-2}}}{p} \|u\|^p_{L^p} \right) + c(\beta) \|u\|^p_{L^p},$$

so we conclude

$$\begin{aligned} &|\langle A(u),\ v \rangle| \le \lambda \|u\|^{p-1}_{L^p} \|v\|_{L^p} + \mu \|\nabla u\|^{p-1}_{L^p} \|\nabla v\|_{L^p} \\ &+ \begin{pmatrix} \beta \frac{p^2}{4} \left(\frac{2}{\varepsilon^{\frac{p}{2}} p} \|\nabla u\|^p_{L^p} + (p-2) \frac{\varepsilon^{\frac{p}{p-2}}}{p} \|u\|^p_{L^p} \right) + \\ +c(\beta) \|u\|^p_{L^p} \end{pmatrix}^{p-1} \|\nabla v\|_{L^p} \\ &+ \|\gamma_2\|_{L^q} \|\nabla v\|_{L^p} + \mu_1 \|\nabla u\|^{p-1}_{L^p} \|v\|_{L^p} \\ &+ \begin{pmatrix} \beta \frac{p^2}{4} \left(\frac{2}{\varepsilon^{\frac{p}{2}} p} \|\nabla u\|^p_{L^p} + (p-2) \frac{\varepsilon^{\frac{p}{p-2}}}{p} \|u\|^p_{L^p} \right) + \\ +c(\beta) \|u\|^p_{L^p} \end{pmatrix}^{p-1} \|v\|_{L^p} \\ &+ \|\gamma_4\|_{L^q} \|v\|_{L^p}, \end{aligned}$$

thus, the operator A is bounded.

Lemma 2.6. *Let $p \ge 2$ and let q be its conjugate, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Assume the conditions (3)-(8) are satisfied. Then, the operator*

$$A \ : \ W^p_1(\Omega) \to W^q_{-1}(\Omega)$$

is monotone.

Proof. For all $u,\ v,\ w \in W^p_{1,0}(\Omega)$, we have

$$\begin{aligned} &\langle A(u) - A(v),\ u - v \rangle \\ &= \lambda \int_\Omega \left(u |u|^{p-2} - v |v|^{p-2} \right) (u - v) \, dx \\ &\quad + \int_\Omega (a_i(x,\ u,\ \nabla u) - a_i(x,\ v,\ \nabla v)) (\nabla_i u - \nabla_i v) \, dx \\ &\quad + \int_\Omega (b(x,\ u,\ \nabla u) - b(x,\ v,\ \nabla v)) (u - v) \, dx \\ &\ge \lambda c(p) \|u - v\|^p_{L^p} + \nu_1 \|\nabla (u - v)\|^p_{L^p} \\ &\quad - \int_\Omega \left(\mu_2 |\nabla (u - v)|^{p-1} + \gamma_5(x) |u - v|^{p-1} \right) (u - v) \, dx \\ &\ge \lambda c(p) \|u - v\|^p_{L^p} + \nu_1 \|\nabla (u - v)\|^p_{L^p} \\ &\quad - \left(\mu_2 \frac{1}{\varepsilon^p p} \|u - v\|^p_{L^p} + \mu_2 \frac{\varepsilon^q}{q} \|\nabla (u - v)\|^p_{L^p} \right) \\ &\quad - \int_\Omega \gamma_5(x) |u - v|^p \, dx. \end{aligned}$$

We assume $\gamma_5{}^{\frac{1}{2}} \in PK(\beta)$. Applying the form-boundary condition, we have

$$\begin{aligned}&\int_\Omega \gamma_5(x)\,|u-v|^p\,dx\\ &\le \beta\int_\Omega \left(\nabla\left(|u-v|^{\frac{p}{2}}\right)\right)^2 dx + c(\beta)\int_\Omega |u-v|^p\,dx\\ &\le \beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\,\|\nabla(u-v)\|_{L^p}^p + (p-2)\,\frac{\varepsilon^{\frac{p}{p-2}}}{p}\,\|u-v\|_{L^p}^p\right)\\ &\quad + c(\beta)\,\|u-v\|_{L^p}^p,\end{aligned}$$

so, we conclude

$$\begin{aligned}&\langle A(u)-A(v),\ u-v\rangle\\ &\ge \left(\lambda c(p) - \mu_2\frac{1}{\varepsilon^p p} - \beta(p-2)\,\varepsilon^{\frac{p}{p-2}}\frac{p}{4} - c(\beta)\right)\|u-v\|_{L^p}^p\\ &\quad + \left(\nu_1 - \mu_2\frac{\varepsilon^q}{q} - \beta\frac{p}{\varepsilon^{\frac{p}{2}}2}\right)\|\nabla(u-v)\|_{L^p}^p > 0,\end{aligned}$$

thus, the operator A is strictly monotone.

Lemma 2.7. *Let $p \ge 2$ and let q be its conjugate, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Assume the conditions (3)-(8) are satisfied. Then, the operator*

$$A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$$

is coercive.

Proof. From the definition, we obtain

$$\begin{aligned}&\langle A(u),\ u\rangle = \lambda\int_\Omega |u|^p\,dx\\ &+\int_\Omega a_i(x,\ u,\ \nabla u)\,\nabla_i u dx + \int_\Omega b(x,\ u,\ \nabla u)\,u dx\\ &\ge \lambda\,\|u\|_{L^p}^p + \nu\,\|\nabla u\|_{L^p}^p - \int_\Omega\left(\mu_1\,|\nabla u|^{p-1} + \gamma_3(x)\,|u|^{p-1} + \gamma_4(x)\right)u dx\\ &\ge \left(\lambda - \mu_1\frac{1}{\varepsilon_1{}^p p}\right)\|u\|_{L^p}^p + \left(\nu - \mu_1\frac{\varepsilon_1{}^q}{q}\right)\|\nabla u\|_{L^p}^p\\ &\quad - \int_\Omega \gamma_3(x)\,|u|^p\,dx - \int_\Omega \gamma_4\,|u|\,dx,\end{aligned}$$

for all $u \in W_{1,0}^p(\Omega)$. By form-boundary condition, we have

$$\int_\Omega \gamma_3(x)\,|u|^p\,dx \le \beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\,\|\nabla u\|_{L^p}^p + (p-2)\,\frac{\varepsilon^{\frac{p}{p-2}}}{p}\,\|u\|_{L^p}^p\right) + c(\beta)\,\|u\|_{L^p}^p,$$

thus, it follows that

$$\begin{aligned}\langle A(u),\ u\rangle &\ge \left(\lambda - \beta\frac{p}{4}(p-2)\,\varepsilon^{\frac{p}{p-2}} - \frac{\varepsilon^p}{p} - \mu_1\frac{1}{\varepsilon_1{}^p p}\right)\|u\|_{L^p}^p\\ &\quad + \left(\nu - \mu_1\frac{\varepsilon_1{}^q}{q} - \beta\frac{p}{\varepsilon^{\frac{p}{2}}2}\right)\|\nabla u\|_{L^p}^p - \frac{\varepsilon^q}{q}\,\|\gamma_4\|_{L^q}^q,\end{aligned}$$

and

$$\begin{aligned}\frac{\langle A(u),\,u\rangle}{\|u\|_{L^p}} &\ge \left(\lambda - \beta\frac{p}{4}(p-2)\,\varepsilon^{\frac{p}{p-2}} - \frac{\varepsilon^p}{p} - \mu_1\frac{1}{\varepsilon_1{}^p p}\right)\|u\|_{L^p}^{p-1}\\ &\quad + \left(\nu - \mu_1\frac{\varepsilon_1{}^q}{q} - \beta\frac{p}{\varepsilon^{\frac{p}{2}}2}\right)\frac{\|\nabla u\|_{L^p}^p}{\|u\|_{L^p}} - \frac{\varepsilon^q}{q}\,\frac{\|\gamma_4\|_{L^q}^q}{\|u\|_{L^p}} \overset{\|u\|_{L^p}\to\infty}{\longrightarrow} \infty,\end{aligned}$$

so, A is a coercive operator.

Lemma 2.8. *Let $p \geq 2$ and let q be its conjugate, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Assume the conditions (3)-(8) are satisfied. Then, the operator*

$$A \,:\, W_1^p(\Omega) \to W_{-1}^q(\Omega)$$

is semicontinuous.

Proof. Due to the arbitrarity of the element $v \in W_{1,0}^p(\Omega)$ in the definition of semicontinuity, we conclude that the operator $A \,:\, W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is semicontinuous if the limit

$$\langle A(u+tv),\, w\rangle \overset{t\to 0}{\longrightarrow} \langle A(u),\, w\rangle$$

holds for all $u,\ v,\ w \in W_{1,0}^p(\Omega)$. So, it is sufficient to show that

$$|\langle A(u+tv) - A(u),\, w\rangle| \overset{t\to 0}{\longrightarrow} 0,$$

for all $u,\ v,\ w \in W_{1,0}^p(\Omega)$.

For $u,\ v,\ w \in W_{1,0}^p(\Omega)$, we calculate

$$\begin{aligned}
&|\langle A(u+tv) - A(u),\, w\rangle| \\
&= \lambda \int_\Omega \left((u+tv)\,|u+tv|^{p-2} - u\,|u|^{p-2}\right) w dx \\
&\quad + \int_\Omega \left(a_i(x,\ u+tv,\ \nabla(u+tv)) - a_i(x,\ u,\ \nabla u)\right) \nabla_i w dx \\
&\quad + \int_\Omega \left(b(x,\ u+tv,\ \nabla(u+tv)) - b(x,\ u,\ \nabla u)\right) w dx \\
&\leq \lambda t(p-1) \int_\Omega |v| \left(|u+tv|^{p-2} + |u|^{p-2}\right) w dx \\
&\quad + t\int_\Omega \left(\mu_2 |\nabla v|^{p-1} + \gamma_6(x)\,|v|^{p-1}\right) |\nabla w|\, dx \\
&\quad + t\int_\Omega \left(\mu_3 |\nabla v|^{p-1} + \gamma_5(x)\,|v|^{p-1}\right) |w|\, dx.
\end{aligned}$$

Applying Holder inequality and form-boundary condition, we estimate

$$\begin{aligned}
&\int_\Omega \left(\mu_2 |\nabla v|^{p-1} + \gamma_6(x)\,|v|^{p-1}\right) |\nabla w|\, dx \\
&\quad + \int_\Omega \left(\mu_3 |\nabla v|^{p-1} + \gamma_5(x)\,|v|^{p-1}\right) |w|\, dx \\
&\leq \mu_2 \|\nabla v\|_{L^p}{}^{p-1} \|\nabla w\|_{L^p} \\
&\quad + \left(\beta \tfrac{p^2}{4}\left(\tfrac{2}{\varepsilon^{\frac{p}{2}} p}\|\nabla v\|_{L^p}^p + (p-2)\,\tfrac{\varepsilon^{\frac{p}{p-2}}}{p}\|v\|_{L^p}^p\right) + c(\beta)\,\|v\|_{L^p}^p\right)^{\frac{1}{q}} \|\nabla w\|_{L^p} \\
&\quad + \mu_3 \|\nabla v\|_{L^p}{}^{p-1} \|w\|_{L^p} \\
&\quad + \left(\beta \tfrac{p^2}{4}\left(\tfrac{2}{\varepsilon^{\frac{p}{2}} p}\|\nabla v\|_{L^p}^p + (p-2)\,\tfrac{\varepsilon^{\frac{p}{p-2}}}{p}\|v\|_{L^p}^p\right) + c(\beta)\,\|v\|_{L^p}^p\right)^{\frac{1}{q}} \|w\|_{L^p}\,.
\end{aligned}$$

Using the Holder inequality, we deduce

$$\begin{aligned}
&|\langle A(u+tv)-A(u),\ w\rangle| \\
&\le \lambda t(p-1)\left(\|v\|_{L^p}\|u+tv\|_{L^p}{}^{p-2}\|w\|_{L^p}+\|v\|_{L^p}\|u\|_{L^p}{}^{p-2}\|w\|_{L^p}\right) \\
&\quad +t\mu_2\|\nabla v\|_{L^p}{}^{p-1}\|\nabla w\|_{L^p}+t\mu_3\|\nabla v\|_{L^p}{}^{p-1}\|w\|_{L^p} \\
&\quad +t\left(\beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\|\nabla v\|_{L^p}^p+(p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|v\|_{L^p}^p\right)+c(\beta)\|v\|_{L^p}^p\right)^{\frac{1}{q}}\|\nabla w\|_{L^p} \\
&\quad +t\left(\beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\|\nabla v\|_{L^p}^p+(p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|v\|_{L^p}^p\right)+c(\beta)\|v\|_{L^p}^p\right)^{\frac{1}{q}}\|w\|_{L^p} \\
&\overset{t\to 0}{\longrightarrow} 0.
\end{aligned}$$

3. The Minty-Browder theorem

To complete our investigation, we present the scheme of the proof of the existence and uniqueness of the solution. The proof is based on Minty's ideas [9, 16] and on a variant of the Galerkin method, which are applied to the operator $A\ :\ W_1^p(\Omega)\to W_{-1}^q(\Omega)$, and employment of limits in weak topology.

Theorem 3.1 (Minty-Browder). *Let* $p\ge 2$ *and* q *its conjugate i.e.,* $\frac{1}{p}+\frac{1}{q}=1$. *Assume conditions (3) – (8) are satisfied. Then the elliptic equation (2) has a unique weak solution in the Sobolev space* $W_1^p(\Omega)$.

Proof. Let

$$A\ :\ W_1^p(\Omega)\to W_{-1}^q(\Omega)$$

be the operator associated with equation (2.1). Under conditions (2.2)-(2.8), the operator

$$A\ :\ W_1^p(\Omega)\to W_{-1}^q(\Omega)$$

is a bounded, monotone, coercive, and semi-continuous operator. Thus, we are going to show that there exists a solution $u\in W_1^p(\Omega)$ to the operator equation $A(u)=\varphi$ for each fixed $\varphi\in W_{-1,0}^q$.

Let $\{v_i\}$ be a basis in $W_{1,0}^p$ and X_k be a linear span of $\{v_1,\ ...,\ v_k\}$. We compose the nonlinear Galerkin approximation system

$$\langle A(u_k)-\varphi,\ v_i\rangle=0,$$

where $u_k\in X_k$, $i=1,\ ...,\ k$ so we denote $u_k=\sum_{i=1,\dots,k}c_{ik}v_i$ with coefficients c_{ik} to be calculated.

Since the operator A is coercive there exists a number $R>0$ such that

$$\langle A(u)-\varphi,\ u\rangle>0$$

for all $u\in W_{1,0}^p,\quad \|u\|\ge R$. Therefore, we have the system

$$\left\langle A\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle=0$$

with an unknown real vector $\{c_{1k},\ ...,\ c_{kk}\}$. The function

$$u\mapsto\langle A(u)-\varphi,\ v_i\rangle$$

is continuous on $W^p_{1,0}$ with respect to variables $\{c_{1k},\ ...,\ c_{kk}\}$. The system

$$\sum_{j=1,\dots,k}\left\langle A_\lambda^{p(\cdot)}\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle c_{jk}>0$$

has a solution for all $u_k\in X_k$, $i=1,\ ...,\ k$ such that $u_k\in W^p_{1,0},\quad \|u_k\|_{W^p_1}=R$.

The fixed point theorem states that: let function

$$\left\langle A\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle\ :\ clos\left(B\left(0,\ R\right)\right)\to\mathbb{R}$$

be continuous for each $j=1,\ ...,\ k$ and

$$\sum_{j=1,\dots,k}\left\langle A\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle c_{jk}>0$$

for all $u_k\in W^p_{1,0}$, $\|u_k\|_{W^p_1}=R$; then the system

$$\left\langle A\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle=0$$

has a solution for all $u_k\in W^p_{1,0},\quad \|u_k\|_{W^p_1}\le R$. □

From

$$\langle A\left(u\right)-\varphi,\ u\rangle=0$$

and statement that

$$\langle A\left(u\right)-\varphi,\ u\rangle>0$$

for all $u\in W^p_{1,0},\quad \|u\|_{W^p_1}\ge R$, we deduce that $\|u\|_{W^p_1}\le R$, which provides us with a priori solution estimate.

The sequences $\{u_k\}$ and $\{A\left(u_k\right)\}$ are bounded since the operator A is bounded, therefore, $A\left(u_k\right)\overset{weakly}{\longrightarrow}\varphi$ in $W^q_{-1,0}$ and there exists a subsequence $\left\{u_{\tilde{k}}\right\}\subset\{u_k\}$ such that $u_{\tilde{k}}\overset{weakly}{\longrightarrow}u$. Thus, we have

$$\left\langle A\left(u_{\tilde{k}}\right),\ u_{\tilde{k}}\right\rangle=\left\langle\varphi,\ u_{\tilde{k}}\right\rangle\overset{\tilde{k}\to\infty}{\longrightarrow}\langle\varphi,\ u\rangle\,.$$

In finite-dimensional Banach spaces, the strong and weak convergences coincide. We choose subsequence $\left\{u_{\tilde{k}}\right\}\subset\{u_k\}$ such that $u_{\tilde{k}}\overset{weakly}{\longrightarrow}u$ and

$$A\left(u_{\tilde{k}}\right)\overset{weakly}{\longrightarrow}\varphi,$$

and

$$\left\langle A\left(u_{\tilde{k}}\right),\ u_{\tilde{k}}\right\rangle\overset{\tilde{k}\to\infty}{\longrightarrow}\langle\varphi,\ u\rangle\,.$$

So, we have

$$u_{\tilde{k}}\overset{weakly}{\longrightarrow}u$$

and

$$A\left(u_{\tilde{k}}\right) \overset{weakly}{\longrightarrow} A\left(u\right).$$

Thus, the equation $A\left(u\right)=0$ has a solution in $W_1^p\left(\Omega\right)$, which proves the existence of a weak solution to elliptic equation (2.1) under the conditions (2.2)-(2.8).

Let $u \in W_1^p\left(\Omega\right)$ and $v \in W_1^p\left(\Omega\right)$ be two different solution to (2.1) so that $A\left(u\right)=0$ and $A\left(v\right)=0$. On another hand, the strict monotony yields that from

$$\langle A\left(u\right)-A\left(v\right),\ u-v\rangle = 0$$

follows $u=v$, thus we have proved the uniqueness of the solution.

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Existence results for nonlinear anisotropic elliptic partial differential equations with variable exponents

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Abstract. The focus of this paper will be on studying the existence of solutions in the sense of distribution, for a class of nonlinear partial differential equations defined by a variable exponent anisotropic elliptic operator with a growth conditions given by a strictly positive continuous real function. The functional setting involves variable exponents anisotropic Sobolev spaces.

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1. Introduction

Our goal is to prove the existence of at least one distributional solution to the $\overrightarrow{p}(\cdot)-$ nonlinear elliptic partial differential equations of the type :

$$\begin{cases} -\sum_{i=1}^{N} \partial_i\big(\sigma_i(x,u,\partial_i u)\big) + g(x,u) = f \ , \text{ in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad , \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Where, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open set with Lipschitz boundary $\partial\Omega$, $f \in L^{\overrightarrow{p'}(\cdot)}(\Omega)(= \bigcap_{i=1}^{N} L^{p'_i(\cdot)}(\Omega))$, with $p'_i(\cdot)(= \frac{p_i}{p_i-1}, i = 1,\dots,N)$ denotes the Hölder congugate of $p_i(\cdot)$, $\sigma_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1,\dots,N$, are Carathéodory functions

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fulfilling for almost everywhere $x \in \Omega$ and every $s, \eta, \eta' \in \mathbb{R}$, $(\eta, \eta') \neq (0,0)$, the following :

$$| \sigma_i(x,s,\eta) | \leq c_1 K(| s |) (| \eta | + | \vartheta_i |)^{p_i(x)-1}, \tag{1.2}$$

$$\sigma_i(x,s,\eta)\eta \geq c_2 K(| s |) | \eta |^{p_i(x)}, \tag{1.3}$$

$$(\sigma_i(x,s,\eta) - \sigma_i(x,s,\eta'))(\eta - \eta') \geq \Theta_i(x,\eta,\eta'), \tag{1.4}$$

$$\Theta_i(x,\eta,\eta') = \begin{cases} c_3 | \eta - \eta' |^{p_i(x)}, & \text{if } p_i(x) \geq 2 \\ c_4 \frac{|\eta-\eta'|^2}{(|\eta|+|\eta'|)^{2-p_i(x)}}, & \text{if } 1 < p_i(x) < 2 \end{cases}$$

where, $c_l,\ l = 1, \ldots, 4$ are positive constants, $\vartheta_i \in L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N$, and $K(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^*$ is a continuous function such that,

$$K(| \xi |) \geq \alpha | \xi |^{r(x)}, \quad \text{for all } | \xi | \geq \lambda, \tag{1.5}$$

with, $\alpha > 0$, $\lambda > 0$ and $r(\cdot) \in \mathcal{C}(\overline{\Omega})$, where $r(\cdot) > 0$ in $\overline{\Omega}$.
For some $\beta > 0$

$$K(| s |) \geq \beta, \quad \text{for all } s \in \mathbb{R}. \tag{1.6}$$

$g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies ;
a.e. $x \in \Omega$ the following conditions:

$$g(x,s)(s - s') \geq 0, \quad \forall s, s' \in \mathbb{R},\ | s | = | s' |, \tag{1.7}$$

$$\sup_{|s| \leq t} | g(x,s) | \in L^1(\Omega), \quad \forall s \in \mathbb{R} \text{ and } \forall t > 0, \tag{1.8}$$

$$|g(x,s)| \leq c \sum_{i=1}^{N} |s|^{p_i(x)-1}, \quad \forall s \in \mathbb{R}. \tag{1.9}$$

As a typical example, we can consider the following model equation $(\sigma_i(x,u,\partial_i u) = K(| u |)|\partial_i u|^{p_i(x)-2}\partial_i u,\ g(x,u) = u \sum_{i=1}^{N} |u|^{p_i(x)-2})$:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i\big(K(| u |)|\partial_i u|^{p_i(x)-2}\partial_i u\big) + u \sum_{i=1}^{N} |u|^{p_i(x)-2} & = f(x), \quad \text{in } \Omega, \\ & u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, and the continuous function $K(\cdot)$ is defined for any fixed $x \in \Omega$ as follows:

$$\forall \eta \geq 0 : K(\eta) = \begin{cases} 1 + \lambda^{r(x)}, & \text{if } \eta < \lambda, \\ 1 + \eta^{r(x)}, & \text{if } \eta \geq \lambda, \end{cases}$$

where $\lambda > 0$, and $r(\cdot) \in \mathcal{C}(\overline{\Omega})$ with $r(\cdot) > 0$ in $\overline{\Omega}$.

Our boundary-value problems entails an $\overrightarrow{p}(x)$-nonlinear elliptic differential operartor wherein those sorts of operators have many makes use of withinside the carried out discipline of diverse sciences, amongst them modeling of image processing and electro-rheological fluids(see [9, 21, 3]). From the theoretical side related to the existence of solutions, we can refer , without limitation, to [11, 12, 13, 10, 14, 15, 16, 17, 18, 19, 20].

This paper seeks to prove the existence results of ditributional solutions for a class of anisotropic nonlinear elliptic problems with variable exponents and growth conditions given by a real positive continuous function, will provide us regular solutions in the anisotropic space $W_0^{1,\overrightarrow{q}(\cdot)}(\Omega)$ such that, $\overrightarrow{q}(\cdot) = (q_1(\cdot), \ldots, q_N(\cdot))$, be restricted as in Theorem 3.2.

The proof of our main result requires proving the existence of a sequence of suitable approximate solutions (u_n) by applying the main Theorem of pseudo-monotone operators and the results obtained in [12, 13]. Prior estimates are then used to show the boundedness of the solutions u_n and the almost everywhere convergence of their partial derivatives $\partial_i u_n$, $i = 1, \ldots, N$, which can be converted into strong L^1-convergence. Through this, we can pass to the limit by L^1-strongly sense for $\sigma_i(x, T_n(u_n), \partial_i u_n)$, and for $g(x, u_n)$, then we conclude the convergence of u_n to the solution of (1.1).

The paper is divided into several sections, in Section 2 we discuss variable exponents anisotropic Lebesgue-Sobolev spaces and their key characteristics, as well as mentioning some embedding theorems. The main theorem and its proof can be found in Section 3.

2. Preliminaries and basic concepts

In this section, we will learn about anisotropic Lebesgue-Sobolev spaces with variable exponent and their most important distinctive properties, as explained, for example, in the papers [6, 4, 5].

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open subset, and let the following set be:

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R}, \quad p^-(= \min_{x \in \overline{\Omega}} p(x)) > 1\}.$$

Assume $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$. The variable exponent Lebesgue reflexive Banach space $\big(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\big)$ defined by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R}, \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

under the Luxemburg norm

$$u \mapsto \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{s > 0 : \varrho_{p(\cdot)}(\frac{u}{s}) \leq 1\right\}.$$

The function

$$\varrho_{p(\cdot)} : u \mapsto \int_\Omega |u(x)|^{p(x)} dx \text{ is called the convex modular.}$$

The variable exponents Sobolev Banach space $\big(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\big)$ defined as fellows

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

such that

$$u \mapsto \|u\|_{1,p(\cdot)} := \|\nabla u\|_{p(\cdot)}. \tag{2.1}$$

We define also the reflexive and separable Banach space $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ by

$$W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}.$$

The following Hölder type inequality holds :

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

where, $p'(\cdot)$ denotes the Hölder conjugate of $p(\cdot)$ (i.e. $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ in $\overline{\Omega}$).
Next results(see [4, 5]) we need to use them later. Let $u \in L^{p(\cdot)}(\Omega)$, then:

$$\min\left(\varrho_{p(\cdot)}^{\frac{1}{p^+}}(u), \rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\varrho_{p(\cdot)}^{\frac{1}{p^+}}(u), \varrho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right), \tag{2.2}$$

$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \varrho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \tag{2.3}$$

We will now define the variable exponents anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.
Let $p_i(\cdot) \in C\big(\overline{\Omega}, [1, +\infty)\big)$, $i \in \{1, \ldots, N\}$, and $\forall x \in \overline{\Omega}$ we set that

$$\overrightarrow{p}(x) = (p_1(x), \ldots, p_N(x)), \quad p_+(x) = \max_{1\leq i\leq N} p_i(x),\ p_-(x) = \min_{1\leq i\leq N} p_i(x),$$

$$\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i(x)}.$$

The Banach space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega) \text{ and } \partial_i u \in L^{p_i(\cdot)}(\Omega),\ i \in \{1, \ldots, N\}\right\},$$

equipped with the following norm :

$$u \mapsto \|u\|_{\overrightarrow{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^N \|\partial_i u\|_{p_i(\cdot)}. \tag{2.4}$$

The Banach space $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \|\cdot\|_{\overrightarrow{p}(\cdot)}\right)$ defined as follows

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)},$$

Let $p(\cdot) \in C_+(\overline{\Omega})$, the variable exponent Marcinkiewicz space $\mathcal{M}^{p(\cdot)}(\Omega)$ is defineed by

$$\mathcal{M}^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R};$$
$$\exists M > 0 : \int_{\{|u|>s\}} t^{p(x)}\,dx \leq M,\ \forall s > 0\}.$$

The truncation function $\forall t > 0$, $T_t : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$T_t(s) := \begin{cases} s, & \text{if } |s| \leq t, \\ \frac{s}{|s|}t, & \text{if } |s| > t, \end{cases} \tag{2.5}$$

and its derivative (see [11]) given by

$$(DT_t)(s) = \begin{cases} 1, & |s| < t, \\ 0, & |s| > t. \end{cases} \tag{2.6}$$

3. Statement of results and proof

Definition 3.1. *The function $u : \Omega \longrightarrow \mathbb{R}$ is a distributional solution for* (1.1) *if and only if $u \in W_0^{1,1}(\Omega)$, and $\forall \varphi \in C_c^\infty(\Omega)$:*

$$\int_\Omega \sum_{i=1}^N \sigma_i(x, u, \partial_i u)\partial_i \varphi \, dx + \int_\Omega g(x, u)\varphi \, dx = \int_\Omega f(x)\varphi \, dx.$$

The main result is that.

Theorem 3.2. *Let $p_i(\cdot) \in C\big(\overline{\Omega},]1, +\infty)\big)$, $i = 1, \ldots, N$, such that $\overline{p} < N$. Assume that $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, and g, σ_i, $i = 1, \ldots, N$, be Carathéodory functions that satisfy* (1.2)-(1.4), *and* (1.7)-(1.9). *If we have*

$$\frac{\overline{p}(\cdot)(N-1)}{N(r(\cdot)+1)(\overline{p}(\cdot)-1)} < p_i(\cdot) < \frac{\overline{p}(\cdot)(N-1)}{(r(\cdot)+1)(N-\overline{p}(\cdot))}, \ \text{in } \overline{\Omega}, \ i = 1, \ldots, N \tag{3.1}$$

where, $r(\cdot)$ defined in (1.5) *and satisfies*

$$r(\cdot) < \frac{N - \overline{p}(\cdot)}{N(\overline{p}(\cdot)-1)} \ \text{ in } \overline{\Omega}. \tag{3.2}$$

Then (1.1) *has at least one solution u in the sense of distributions in $W_0^{1,\overrightarrow{q}(\cdot)}(\Omega)$, where $\overrightarrow{q}(\cdot) = (q_1(\cdot), \ldots, q_N(\cdot))$, and*

$$\max\{1, (r(\cdot)+1)p_i(\cdot) - 1\} < q_i(\cdot) < \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)(r(\cdot)+1)}{\overline{p}(\cdot)(N-1)} \ \text{ in } \overline{\Omega}, \ i = 1, \ldots, N. \tag{3.3}$$

Remark 3.3. The assumption (3.2) ensure that

$$\frac{\overline{p}(\cdot)(N-1)}{N(r(\cdot)+1)(\overline{p}(\cdot)-1)} > 1 \ \text{ in } \overline{\Omega}, \ i = 1, \ldots, N.$$

Remark 3.4. The upper bound in (3.1) implies that

$$\frac{Np_i(\cdot)(\overline{p}(\cdot)-1)(r(\cdot)+1)}{\overline{p}(\cdot)(N-1)} > (r(\cdot)+1)p_i(\cdot) - 1 \ \text{ in } \overline{\Omega}, \ i = 1, \ldots, N. \tag{3.4}$$

3.1. Existence of approximate solutions

Let (f_n) be a sequence of bounded functions defined in Ω which converges to f in $L^{\overrightarrow{p}'(\cdot)}(\Omega)$.

It should be noted here that:

Since $f_n \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, then from (2.2), we obtain

$$\|f_n\|_{p_i'(\cdot)} \le 1 + \rho_{p_i'(\cdot)}^{\frac{1}{\overline{p'_i}(\cdot)}}(f_n) \le 2 + \rho_{p_i}^{\frac{1}{\overline{p'^-}}}(f_n) < \infty.$$

Through this, we conclude that

$$f_n \text{ is bounded in } L^{p_i'(\cdot)}(\Omega),\ i = 1, \ldots, N. \tag{3.5}$$

Lemma 3.5. *Let $p_i(\cdot) \in C\big(\overline{\Omega},]1, +\infty)\big)$, $i = 1, \ldots, N$, such that $\overline{p} < N$, and let f is in $L^{\overrightarrow{p}'(\cdot)}(\Omega)$. Let g, σ_i, $i = 1, \ldots, N$, be Carathéodory functions satisfying* (1.2)-(1.4), *and* (1.7)-(1.9). *Then, there exists at least one weak solution $u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to the approximated problems*

$$\begin{aligned} -\sum_{i=1}^{N} \partial_i\big(\sigma_i(x, T_n(u_n), \partial_i u_n)\big) + g(x, u_n)) = f_n, &\quad \textit{in } \Omega, \\ u_n = 0, &\quad \textit{on } \partial\Omega, \end{aligned} \tag{3.6}$$

in the sense that; for every $\varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega \sigma_i(x, T_n(u_n), \partial_i u_n)\partial_i\varphi\, dx + \int_\Omega g(x, u_n)\varphi\, dx = \int_\Omega f_n\varphi\, dx. \tag{3.7}$$

Proof. Consider the problem

$$\begin{aligned} -\sum_{i=1}^{N} \partial_i\big(\sigma_i(x, T_n(u_{n_k}), \partial_i u_{n_k})\big) + g_k(x, u_{n_k}) = f_n, &\quad \text{in } \Omega, \\ u_{n_k} = 0, &\quad \text{on } \partial\Omega, \end{aligned} \tag{3.8}$$

where,

$$g_k(x, \xi) = \frac{g(x, \xi)}{1 + \frac{|g(x,\xi)|}{k}}, \quad \forall k \in \mathbb{N}^*.$$

Note that,

$$\mid g_k(x, \xi) \mid \leq \mid g(x, \xi) \mid, \text{ and } \mid g_k(x, \xi) \mid \leq k.$$

In a similar manner to the results obtained in [12] or in [13] by applying the main Theorem on pseudo-monotone operators, we conclude that there exists a solution $u_{n_k} \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ to problem (3.8), which satisfies

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega \sigma_i(x, T_n(u_{n_k}), \partial_i u_{n_k})\partial_i\varphi\, dx + \int_\Omega g_k(x, u_{n_k})\varphi\, dx \\ = \int_\Omega f_n\varphi\, dx, \quad \forall\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega), \end{aligned} \tag{3.9}$$

And also in a similar way, we can obtain (3.7) by passing to the limit in (3.9). □

3.1.1. A priori estimates.

Lemma 3.6. *Let $u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a solution to problem* (3.6). *Then, there exists a constant C, such that*

$$\sum_{i=1}^{N} \int_{\{|u_n| \leq t\}} \mid \partial_i u_n \mid^{p_i(x)}\, dx \leq C(t + 1), \quad \forall t > 0, \tag{3.10}$$

$$\int_{\Omega} \mid g(x,u_n) \mid \, dx \leq C, \tag{3.11}$$

$$\sum_{i=1}^{N} \int_{\{K(|u_n|)<t\}} (t+1)^{-1} \mid \partial_i (K(\mid u_n \mid)) \mid^{p_i(x)} \, dx \leq C, \quad \forall t > 0. \tag{3.12}$$

Proof. By choosing $\varphi = T_t(u_n)$ in (3.7) and use (1.3), (1.6), (2.5), and (2.6), we find that, for all $t > 0$

$$C \sum_{i=1}^{N} \int_{|u_n|\leq t} \mid \partial_i u_n \mid^{p_i(x)} \, dx + t \int_{|u_n|>t} \frac{u_n}{\mid u_n \mid} g(x,u_n)\, dx$$
$$\leq ct + \int_{|u_n|<t} \mid u_n \mid\mid g(x,u_n) \mid \, dx. \tag{3.13}$$

Then, from (1.8), and the fact that

$$\frac{u_n}{\mid u_n \mid} g(x,u_n) \geq \mid g(x,u_n) \mid . \tag{3.14}$$

Which is produced by the following: due (1.7) we get
$\frac{u_n}{|u_n|} g(x,u_n) - \mid g(x,u_n) \mid = \frac{1}{|u_n|} g(x,u_n)\big(u_n - \mid u_n \mid \frac{g(x,u_n)}{|g(x,u_n)|}\big) \geq 0]$, we obtain

$$C \sum_{i=1}^{N} \int_{|u_n|\leq t} \mid \partial_i u_n \mid^{p_i(x)} \, dx + t \int_{|u_n|>t} \mid g(x,u_n) \mid \, dx \leq c' t. \tag{3.15}$$

So, (3.15) give us (3.10), and also, for all $t > 0$

$$\int_{|u_n|>t} \mid g(x,u_n) \mid \, dx \leq c'. \tag{3.16}$$

From (3.16) and (1.8) we get (3.11).
In order to prove (3.12), we choose $\varphi = T_t(K(\mid u_n \mid))$ in (3.7) with the use of (1.3), (1.6), (2.5), and (2.6), we can get for all $t > 0$

$$c_2 \beta \sum_{i=1}^{N} \int_{\{K(|u_n|)<t\}} \mid \partial_i (K(\mid u_n \mid)) \mid^{p_i(x)} \, dx + t \int_{K(|u_n|)>t} g(x,u_n)\, dx$$
$$+ \beta \int_{K(|u_n|)\leq t} g(x,u_n)\, dx \leq ct. \tag{3.17}$$

Then, we get

$$c_2 \beta \sum_{i=1}^{N} \int_{\{K(|u_n|)<t\}} \mid \partial_i (K(\mid u_n \mid)) \mid^{p_i(x)} \, dx$$
$$\leq ct + t \int_{\{(K(|u_n|)>t\}} \mid g(x,u_n) \mid \, dx + \beta \int_{\{K(|u_n|)\leq t\}} \mid g(x,u_n) \mid \, dx. \tag{3.18}$$

Let's simplify the second side to (3.18).
By (1.8) and (3.16), we obtain

$$\begin{aligned}\int_{\{K(|u_n|)>t\}} |\, g(x,u_n)\,|\ dx &= \int_{\{K(|u_n|)>t\}\cap\{|u_n|>t\}} |\, g(x,u_n)\,|\ dx \\ &+ \int_{\{K(|u_n|)>t\}\cap\{|u_n|\le t\}} |\, g(x,u_n)\,|\ dx \\ &\le \int_{\{|u_n|>t\}} |\, g(x,u_n)\,|\ dx + \int_{\{|u_n|\le t\}} |\, g(x,u_n)\,|\ dx \le C,\end{aligned}$$

and

$$\begin{aligned}\int_{\{(K(|u_n|)<t\}} |\, g(x,u_n)\,|\ dx &= \int_{\{(K(|u_n|)<t\}\cap\{|u_n|>t\}} |\, g(x,u_n)\,|\ dx \\ &+ \int_{\{(K(|u_n|)<t\}\cap\{|u_n|\le t\}} |\, g(x,u_n)\,|\ dx \\ &\le \int_{\{|u_n|>t\}} |\, g(x,u_n)\,|\ dx + \int_{\{|u_n|\le t\}} |\, g(x,u_n)\,|\ dx \le C'.\end{aligned}$$

Through this we find that, (3.18) gives us

$$\sum_{i=1}^{N} \int_{\{K(|u_n|)<t\}} |\, \partial_i(K(|\, u_n\,|))\,|^{p_i(x)}\ dx \le c(t+1). \tag{3.19}$$

Then, from (3.17) we obtain (3.12). □

Remark 3.7. (3.10) implies that

$$\int_{\{|u_n|\le t\}} (t+1)^{-1}\,|\,\partial_i u_n\,|^{p_i(x)}\ dx \le C, \quad \forall t>0. \tag{3.20}$$

Remark 3.8. The relationship (3.14) implies that

$$u_n g(x,u_n) \ge 0. \tag{3.21}$$

We need the following technical Lemma that came in [11] and scalar case in [2]

Lemma 3.9. *(see [2, 11]) Let* $\overrightarrow{p}(\cdot) = (p_1(\cdot),\ldots,p_N(\cdot)) \in \left(C_+(\overline{\Omega})\right)^N$ *with* $\overline{p}(\cdot) < N$, *and let f be a nonnegative function in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$. *Suppose that there exists a constant c such that*

$$\sum_{i=1}^{N} \int_{\{f\le t\}} |\,\partial_i f\,|^{p_i(x)}\ dx \le c(t+1), \quad \forall t>0. \tag{3.22}$$

Then there exists a constant C, depending on c, such that

$$\int_{\{f>t\}} t^{h(x)}\,dx \le C, \quad \forall t>0, \quad h(x) = \frac{N(\overline{p}(x)-1)}{N-\overline{p}(x)}, \quad \forall x\in\overline{\Omega}. \tag{3.23}$$

Lemma 3.10. *Let $u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a solution to problem (3.6). Then,*

$$\partial_i(K(|\, u_n \,|)) \text{ is bounded in } \mathcal{M}^{\frac{Np_i(\cdot)(\overline{p}(\cdot)-1)}{\overline{p}(\cdot)(N-1)}}(\Omega),\ i=1,\dots,N \tag{3.24}$$

$$\partial_i u_n \text{ is bounded in } \mathcal{M}^{\frac{Np_i(\cdot)(\overline{p}(\cdot)-1)(r(\cdot)+1)}{\overline{p}(\cdot)(N-1)}}(\Omega),\ i=1,\dots,N. \tag{3.25}$$

Proof. For all $i=1,\dots,N$ setting $\alpha_i(\cdot) = \frac{p_i(\cdot)}{h(\cdot)+1}$ where $h(x) = \frac{N(\overline{p}(x)-1)}{N-\overline{p}(x)}$, $\quad \forall x \in \overline{\Omega}$, then we have:
If $0 < t < 1$, we have trivially that

$$\int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}} t^{h(x)}\,dx \le |\, \Omega \,|\,.$$

If $t \ge 1$, using (3.12), Lemma 3.9 (due (3.10), and the fact that $|\, \partial_i \mid u_n \mid| \le |\, \partial_i u_n \mid$ $u_n \neq 0$, $i=1,\dots,N$), we get that

$$\begin{aligned}
\int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}} t^{h(x)}\,dx \le& \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)<t\}} t^{h(x)}\,dx \\
&+ \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)\ge t\}} t^{h(x)}\,dx \\
\le& \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)<t\}\cap\{|u_n|\le t\}} t^{h(x)}\,dx \\
&+ \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)<t\}\cap\{|u_n|>t\}} t^{h(x)}\,dx \\
&+ \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)\ge t\}\cap\{|u_n|\le t\}} t^{h(x)}\,dx \\
&+ \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)\ge t\}\cap\{|u_n|>t\}} t^{h(x)}\,dx \\
\le& 2\int_{\{K(|u_n|)<t\}} t^{h(x)}\,dx + 2\int_{\{|u_n|>t\}} t^{h(x)}\,dx \\
\le& 2\int_{\{K(|u_n|)<t\}} t^{h(x)} \left(\frac{|\, \partial_i(K(|\, u_n \,|)) \,|^{\alpha_i(x)}}{t}\right)^{\frac{p_i(x)}{\alpha_i(x)}} dx \\
&+ 2\int_{\{|u_n|>t\}} t^{h(x)}\,dx \\
\le& 2\int_{\{K(|u_n|)<t\}} t^{-1} \,|\, \partial_i(K(|\, u_n \,|)) \,|^{p_i(x)}\ dx + c \\
\le& 4\int_{\{K(|u_n|)<t\}} (2t)^{-1} \,|\, \partial_i(K(|\, u_n \,|)) \,|^{p_i(x)}\ dx + c \\
\le& 4\int_{\{K(|u_n|)<t\}} (t+1)^{-1} \,|\, \partial_i(K(|\, u_n \,|)) \,|^{p_i(x)}\ dx + c \le C'.
\end{aligned}$$

Then, for all $i = 1, \dots, N$, $| \partial_i(K(| u_n |)) |^{\alpha_i(\cdot)}$ is bounded in $\mathcal{M}^{h(\cdot)}(\Omega)$.
This gives us, for all $i = 1, \dots, N$, $| \partial_i(K(| u_n |)) |$ is bounded in $\mathcal{M}^{h(\cdot)\alpha_i(\cdot)}(\Omega)$ where

$$h(\cdot)\alpha_i(\cdot) = \frac{p_i(\cdot)h(\cdot)}{h(\cdot)+1} = \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)}{\overline{p}(\cdot)(N-1)}.$$

Now we will prove (3.25), For $\alpha_i(\cdot)$, $i = 1,,N$ and $h(\cdot)$ defined previously, we find that
If $0 < t < 1$, we get that

$$\int_{\{||\partial_i u_n|))|^{\alpha_i(x)(r(x)+1)}>t\}} t^{h(x)})\, dx \leq | \, \Omega \, | \, .$$

If $t \geq 1$, by (3.12), Lemma 3.9 (due (3.10), and the fact that $| \, \partial_i \, | \, u_n \, || \leq | \, \partial_i u_n \, |, \; u_n \neq 0, \, i = 1, \dots, N)$, we obtain that

$$\begin{aligned}
\int_{\{|\partial_i u_n|^{\alpha_i(x)(r(x)+1)}>t\}} t^{h(x)}\, dx &\leq \int_{\{|\partial_i u_n|^{\alpha_i(x)(r(x)+1)}>t\}\cap\{|u_n|\leq t\}} t^{h(x)}\, dx \\
&\quad + \int_{\{\{|\partial_i u_n|^{\alpha_i(x)(r(x)+1)}>t\}\cap\{|u_n|>t\}} t^{h(x)}\, dx \\
&\leq \int_{\{|u_n|\leq t\}} t^{h(x)} \left(\frac{| \, \partial_i u_n \, |^{\alpha_i(x)(r(x)+1)}}{t} \right)^{\frac{p_i(x)}{\alpha_i(x)(r(x)+1)}} dx \\
&\quad + \int_{\{|u_n|>t\}} t^{h(x)}\, dx \\
&\leq \int_{\{|u_n|\leq t\}} t^{h(x)r(x)-1} \, | \, \partial_i u_n \, |^{p_i(x)} \; dx + c. \qquad (3.26)
\end{aligned}$$

By noting that the assumption (3.2) is equivalent to:

$$h(\cdot)r(\cdot) - 1 < 0, \quad \text{in } \overline{\Omega},$$

and through the positivity of $r(\cdot)$ and $h(\cdot)$, we find that

$$h(\cdot)r(\cdot) - 1 \geq -1, \quad \text{in } \overline{\Omega}.$$

So, we get that

$$(h(\cdot)r(\cdot) - 1) \in [-1, 0), \quad \text{in } \overline{\Omega}. \qquad (3.27)$$

By using (3.27) in (3.26), and thanks to (3.20), we can obtain that

$$\begin{aligned}
\int_{\{|\partial_i u_n|^{\alpha_i(x)(r(x)+1)}>t\}} t^{h(x)}\, dx &\leq \int_{\{|u_n|\leq t\}} t^{-1} \, | \, \partial_i u_n \, |^{p_i(x)} \; dx + c \\
&\leq 2 \int_{\{|u_n|\leq t\}} (2t)^{-1} \, | \, \partial_i u_n \, |^{p_i(x)} \; dx + c \\
&\leq 2 \int_{\{|u_n|\leq t\}} (t+1)^{-1} \, | \, \partial_i u_n \, |^{p_i(x)} \; dx + c \leq C. \quad (3.28)
\end{aligned}$$

Hence, we can obtain,

$$| \, \partial_i u_n \, |^{\alpha_i(\cdot)(r(x)+1)} \;\; \text{is bounded in } \mathcal{M}^{h(\cdot)}(\Omega) \;\; i = 1, \dots, N.$$

From this we conclude that

$$|\ \partial_i u_n\ | \text{ is bounded in } \mathcal{M}^{h(\cdot)\alpha_i(\cdot)(r(\cdot)+1)}(\Omega) \ \ i = 1, \dots, N,$$

where,

$$h(\cdot)\alpha_i(\cdot)(r(\cdot)+1) = \frac{p_i(\cdot)h(\cdot)(r(\cdot)+1)}{h(\cdot)+1} = \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)(r(\cdot)+1)}{\overline{p}(\cdot)(N-1)}.$$

□

We need the following Lemma (see [22]) to prove the Lemma after it

Lemma 3.11 **([22]).** *Let $v(\cdot), w(\cdot) \in C(\overline{\Omega})$, such that $w^- > 0, (v-w)^- > 0$. If $u \in \mathcal{M}^{v(\cdot)}(\Omega)$, then $|\ u\ |^{w(\cdot)} \in L^1(\Omega)$. In addition to that, $\mathcal{M}^{v(\cdot)}(\Omega) \subset L^{w(\cdot)}(\Omega)$ for all $v(\cdot),\ w(\cdot) \geq 1$.*

Lemma 3.12. *Let f, g and p_i, σ_i, $i = 1, \dots, N$ be restricted as in Theorem 3.2. Then, for all $i = 1, \dots, N$,*

$$u_n \text{ is bounded in } L^{q_i(x)}(\Omega), \tag{3.29}$$

$$\partial_i u_n \text{ is bounded in } L^{q_i(x)}(\Omega), \tag{3.30}$$

where $q_i(\cdot)$, $i = 1, \dots, N$ satisfying (3.3).

Proof. From (3.24), thanks to Lemma 3.11, we deduce that

$$\partial_i(K(|\ u_n\ |)) \text{ is bounded in } L^{h_i(\cdot)}(\Omega),\ i = 1, \dots, N, \tag{3.31}$$

where, $1 < h_i(\cdot) < \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)}{\overline{p}(\cdot)(N-1)},\ i = 1, \dots, N$.
So, we can obtain

$$K(|\ u_n\ |) \text{ is bounded in } L^{h_i(\cdot)}(\Omega),\ i = 1, \dots, N, \tag{3.32}$$

where, $1 < h_i(\cdot) < \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)}{\overline{p}(\cdot)(N-1)},\ i = 1, \dots, N$.
By condition (1.5) we can get

$$C\ |\ K(|\ u_n\ |)\ | \geq |\ u_n\ |^{r(\cdot)+1},\ \ |\ u_n\ | \geq \lambda,\ i = 1, \dots, N. \tag{3.33}$$

Then, through (3.33) and (3.32) we obtain

$$\begin{aligned} \int_\Omega |\ u_n\ |^{h_i(\cdot)(r(\cdot)+1)}\, dx &= \int_{\{|u_n| \geq \lambda\}} |\ u_n\ |^{h_i(\cdot)(r(\cdot)+1)}\, dx \\ &\quad + \int_{\{|u_n| < \lambda\}} |\ u_n\ |^{h_i(\cdot)(r(\cdot)+1)}\, dx \\ &\leq c \int_\Omega |\ K(|\ u_n\ |)\ |^{h_i(\cdot)}\, dx + (1 + \lambda^{h_{i_+}^{+}(r_+^{+}+1)})\ |\ \Omega\ | \leq C. \end{aligned} \tag{3.34}$$

Then, (3.34) implies that, for all $i = 1, \dots, N$

$$u_n \in L^{q_i(\cdot)}(\Omega), \tag{3.35}$$

where, $q_i(\cdot)$, $i = 1, \dots, N$ satisfying (3.3).
From (2.2) and (3.35) we deduce (3.29).
Finally, by (3.25), Lemma 3.11, and (2.2), we can get (3.30). □

Remark 3.13. Lemma 3.12 implies that

$$u_n \text{ is bounded in } W_0^{1,\overrightarrow{q}(\cdot)}(\Omega), \tag{3.36}$$

where, $\overrightarrow{q}(\cdot) = (q_1(\cdot), \dots, q_N(\cdot))$, such that $q_i(\cdot)$, $i = 1, \dots, N$ satisfying (3.3).

Lemma 3.14. *For all* $i = 1, \dots, N$

$$\lim_{n \longrightarrow +\infty} \Delta_{i,n} = 0, \tag{3.37}$$

where,

$$\Delta_{i,n} = \int_\Omega \left(\sigma_i(x, T_n(u_n), \partial_i u_n) - \sigma_i(x, T_n(u_n), \partial_i u)\right)(\partial_i u_n - \partial_i u)\, dx.$$

Proof. From (3.36), we can conclude that the sequence (u_n) is bounded in

$$W_0^{1,q_-^-}(\Omega), \text{ where } q_-^- = \min_{1 \le i \le N} \min_{x \in \overline{\Omega}} q_i(x).$$

So, a sequence (still denoted by (u_n)) can be extracted from them, such that

$$u_n \longrightarrow u \quad \text{strongly in } W_0^{1,q_-^-}(\Omega) \text{ and a.e in } \Omega, \tag{3.38}$$

$$\text{and, } \partial_i u_n \rightharpoonup \partial_i u \text{ weakly in } L^{p_i(x)}(\Omega), \quad i = 1, \dots, N. \tag{3.39}$$

Note that, for all $i = 1, \dots, N$,

$$\Delta_{i,n} = \Delta_{i,n}^{(1)} - \Delta_{i,n}^{(2)}$$

where

$$\Delta_{i,n}^{(1)} = \int_\Omega \sigma_i(x, T_n(u_n), \partial_i u_n)(\partial_i u_n - \partial_i u)\, dx$$

$$\Delta_{i,n}^{(2)} = \int_\Omega \sigma_i(x, T_n(u_n), \partial_i u)(\partial_i u_n - \partial_i u)\, dx.$$

First, let's prove for all $i = 1, \dots, N$

$$\lim_{n \longrightarrow +\infty} \Delta_{i,n}^{(1)} = 0. \tag{3.40}$$

Choose $\varphi = u_n - u$ as a test function in (3.7), we get

$$\sum_{i=1}^N \int_\Omega \sigma_i(x, T_n(u_n), \partial_i u_n)(\partial_i u_n - \partial_i u)\, dx + \int_\Omega g(x, u_n)(u_n - u)\, dx = \int_\Omega f_n(u_n - u)\, dx. \tag{3.41}$$

By using (1.9), and that $u_n \in L^{p_i(\cdot)}(\Omega)$, $i = 1, \dots, N$, we can get

$$\int_\Omega |\, g(x, u_n) \,|^{p_i'(x)}\, dx \le \int_\Omega |\, u_n \,|^{p_i(x)}\, dx \le c, \tag{3.42}$$

then (3.42) implies that,

$$(g(x, u_n)) \text{ is bounded in } L^{p_i'(\cdot)}(\Omega). \tag{3.43}$$

So, from (3.43), and (3.38), we get

$$\lim_{n\longrightarrow+\infty}\int_\Omega g(x,u_n)(u_n-u)\,dx\,dx=0. \tag{3.44}$$

Now, by (1.2), Young's inequality, and that $\partial_i u_n$, $K(T_n(u_n)) \in L^{p_i(\cdot)}(\Omega)$, $i = 1,\dots,N$, we deduce that

$$\int_\Omega |\ \sigma_i(x,T_n(u_n),\partial_i u_n)\ |^{p_i'(\cdot)}\ dx \le c\int_\Omega |\ K(T_n(u_n))\ |^{p_i(x)}\ dx + c'\int_\Omega |\ \partial_i u_n\ |^{p_i(x)}\ dx + c'' \le C,$$

and this implies the boudedness of $(\sigma_i(x,T_n(u_n),\partial_i u_n))$ in $L^{p_i'(\cdot)}(\Omega)$.
Thanks to this and (3.39) we get

$$\lim_{n\longrightarrow+\infty}\int_\Omega \sigma_i(x,T_n(u_n),\partial_i u_n)(\partial_i u_n-\partial_i u)\,dx=0. \tag{3.45}$$

Then, from (3.38), (3.44), (3.45), and (3.5), we get (3.40).
Now, from (1.2), and that $\partial_i u\in L^{p_i(\cdot)}$, we obtain for all $i=1,\dots,N$

$$\int_\Omega |\ \sigma_i(x,T_n(u_n),\partial_i u)\ |^{p_i'(\cdot)}\ dx \le c\int_\Omega |\ \partial_i u\ |^{p_i(x)}\ dx + c' \le C.$$

And therefore

$$(\sigma_i(x,T_n(u_n),\partial_i u)) \text{ is bounded in } L^{p_i'(\cdot)}(\Omega),\quad i=1,\dots,N. \tag{3.46}$$

Then, (3.46) and (3.39) implies that

$$\lim_{n\longrightarrow+\infty}\Delta^{(2)}_{i,n}=0. \tag{3.47}$$

So, by (3.40) and (3.47), we derive (3.37). □

Lemma 3.15. *For all $i=1,\dots,N$*

$$\partial_i u_n \longrightarrow \partial_i u,\quad a.e.\ in\ \overline{\Omega}. \tag{3.48}$$

Proof. Through (1.4) we conclude that, for all $i=1,\dots,N$

$$(\sigma_i(x,T_n(u_n),\partial_i u_n)-\sigma_i(x,T_n(u_n),\partial_i u))\,(\partial_i u_n-\partial_i u)>0. \tag{3.49}$$

Then, (3.49) and (3.37) gives us, for all $i=1,\dots,N$

$$(\sigma_i(x,T_n(u_n),\partial_i u_n)-\sigma_i(x,T_n(u_n),\partial_i u))\,(\partial_i u_n-\partial_i u)\longrightarrow 0,\text{strongly in } L^1(\Omega). \tag{3.50}$$

Extracting a subsequence (still denoted by (u_n)), we have for all $i=1,\dots,N$

$$(\sigma_i(x,T_n(u_n),\partial_i u_n)-\sigma_i(x,T_n(u_n),\partial_i u))\,(\partial_i u_n-\partial_i u)\longrightarrow 0\quad \text{a.e. in } \Omega. \tag{3.51}$$

Then, by the same techniques used in [11, 8] we can obtain (3.48). □

3.2. Proof of the Theorem 3.2 :

By (3.38), we have

$$g(x,u_n) \longrightarrow g(x,u) \quad \text{a.e. in } \Omega. \tag{3.52}$$

Let $E \subset \Omega$ be any measurable set, we write

$$\int_E \mid g(x,u_n) \mid \, dx = \int_{E\cap\{|u_n|\le t\}} \mid g(x,u_n) \mid \, dx + \int_{E\cap\{|u_n|>t\}} \mid g(x,u_n) \mid \, dx.$$

Let $0 < M < t$, and observe that

$$|T_t(u_n)| \le |T_t(u_n)|\mathbf{1}_{\{|u_n|\le M\}} + |T_t(u_n)|\mathbf{1}_{\{|u_n|>M\}} \le M + t\mathbf{1}_{\{|u_n|>M\}}. \tag{3.53}$$

Then, after taking $\varphi = T_t(u_n)$ in (3.7) and using (3.53), yields

$$t\int_{\{|u_n|>t\}} |g(x,u_n)|\,dx \le M\int_\Omega |f_n|\,dx + t\int_{\{|u_n|>M\}} |f_n|\,dx. \tag{3.54}$$

From (3.54) and (3.11), we conclude the equi-integrability of $g(x,u_n)$ in $L^1(\Omega)$. Through this, (3.52), and Vitali's theorem we get

$$g(x,u_n) \longrightarrow g(x,u) \quad \text{strongly in } L^1(\Omega). \tag{3.55}$$

From (3.29) and (3.48), we have

$$\sigma_i\big(x,T_n(u_n),\partial_i u_n\big) \longrightarrow \sigma_i\big(x,u,\partial_i u\big) \ \text{ a.e. in } \Omega. \tag{3.56}$$

Now, we prove that

$$\sigma_i\left(x,T_n(u_n),\partial_i u_n\right) \longrightarrow \sigma_i\left(x,u,\partial_i u\right) \quad \text{strongly in } L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega),$$

where $q_i(\cdot)$, $i=1,\dots,N$ are a continuous functions on $\overline{\Omega}$ satisfying (3.3).
Then, we have, for all $x\in\overline{\Omega}$

$$1 < \frac{q_i(x)}{p_i(x)-1} < \frac{N(\overline{p}(x)-1)p_i(x)(r(x)+1)}{\overline{p}(x)(N-1)(p_i(x)-1)}, \quad i=1,\dots,N. \tag{3.57}$$

The choice of $\frac{q_i(x)}{p_i(x)-1} > 1$ is possible since we have (3.4).
Using (1.2), and (3.30), we get that,

$$\int_\Omega \mid \sigma_i(x,T_n(u_n),\partial_i u_n) \mid^{\frac{q_i(x)}{p_i(x)-1}} \, dx \le c\int_\Omega \mid \partial_i u_n \mid^{q_i(x)} + C\,dx \le C',\ i=1,\dots,N. \tag{3.58}$$

Then, by (3.58) and using (2.2), we conclude that, for all $i=1,\dots,N$,

$$\big(\sigma_i\big(x,T_n(u_n),\partial_i u_n\big)\big) \quad \text{uniformly bounded in} \quad L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega).$$

So, by (3.56) and Vitali's theorem, we derive, for all $i=1,\dots,N$,

$$\sigma_i\big(x,T_n(u_n),\partial_i u_n\big) \longrightarrow \sigma_i\big(x,u,\partial_i u\big) \ \text{ strongly in } L^1(\Omega). \tag{3.59}$$

So, by passing to the limit in (3.7), we have completed the proof.

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