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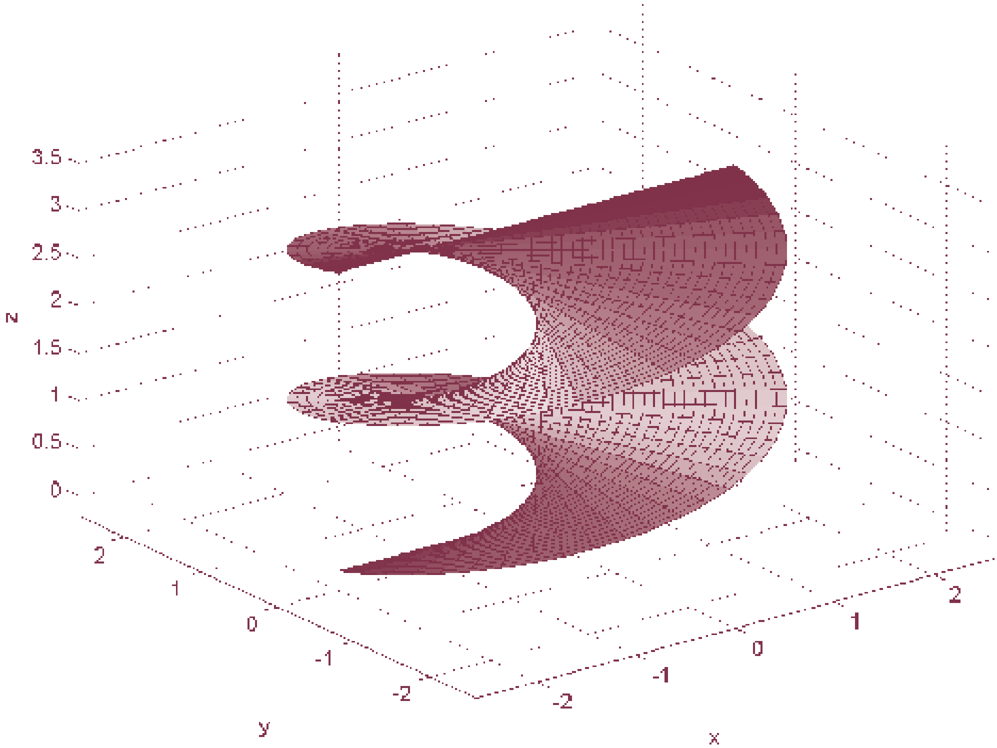

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A new member of the Pell sequences: The pseudo-Pell sequence

Hasan Gökbaş

Abstract. In this study, we define a new family of the Pell numbers and establish some properties of the relation to the ordinary Pell numbers. We give some identities the pseudo-Pell numbers. Moreover, we obtain the Binet's formula, generating function formula and some formulas for this new type numbers. Morever, we give the matrix representation of the pseudo-Pell numbers.

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Keywords: Pell and Pell-Lucas numbers, pseudo-Pell and Pell-Lucas numbers.

1. Introduction and preliminaries

Almost all branches of contemporary research, including computer sciences, physics, economics, architecture, geostatistics, art, color image processing, and music, employ a large number of integer sequences. One of mathematics' most well-known and intriguing number sequences, the Fibonacci sequence has been the subject of extensive research in the literature. The fascinating features of the Fibonacci sequence have delighted scientific enthusiasts for years. The Fibonacci sequence is generated by a recursive formula

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has many interesting properties. For example, the ratio $\dfrac{F_{n+1}}{F_n}$ converges to the golden ratio $\left(\dfrac{1+\sqrt{5}}{2}\right)$ as n tends to infinity. The Fibonacci sequence has been generalized in many ways, some

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by preserving the initial conditions and others by preserving the recurrence relation [1, 4, 6, 7, 10, 11, 14, 16, 19, 18, 21, 22, 27, 29].

The pseudo-Fibonacci and pseudo Lucas sequences was introduced by Ferns [8] as novel generalizations of the Fibonacci and Lucas sequences as follows

$$\Xi_{n+1} = \Xi_n + \Theta_n,$$

$$\Theta_{n+1} = \Xi_{n+1} + \xi\Xi_n$$

with initial conditions $\Xi_1 = 1$ and $\Theta_1 = 1$ in which ξ is a positive integer. We derive the recursion formula by elimination

$$\Xi_{n+2} = 2\Xi_{n+1} + \xi\Xi_n,$$

$$\Theta_{n+2} = 2\Theta_{n+1} + \xi\Theta_n$$

with initial conditions conditions $\Xi_0 = 0$, $\Xi_1 = 1$ and $\Theta_0 = \Theta_1 = 1$ respectively.
The Binet formula for each of the pseudo-Fibonacci and pseudo Lucas sequences is

$$\Xi_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$\Theta_n = \frac{\alpha^n + \beta^n}{2}$$

where $\alpha = 1 + \sqrt{1+\xi}$, $\beta = 1 - \sqrt{1+\xi}$.

The Pell sequence is one of the most famous and interesting numerical sequences in mathematics and has been widely studied in the literature. The Pell sequence is generated by a recursive formula

$$P_n = 2P_{n-1} + P_{n-2},$$

for ≥ 2 with $P_0 = 0$ and $P_1 = 1$. Similarly, the Pell sequence has many interesting properties. For example, the ratio $\dfrac{P_{n+1}}{P_n}$ converges to the silver ratio $(1+\sqrt{2})$ as n tends to infinity.

The two basic ways that the Pell sequence has been generalized are either by keeping the recurrence relation constant while changing the starting conditions or by changing the recurrence relation while keeping the beginning circumstances constant. A closed form for the nth term of the sequence, the sum of the first n terms of the sequence, the sum of the first n terms with odd (or even) indices of the sequence, an explicit sum formula, Catalan's identity, Cassini's identity, d'Ocagne's identity, Tagiuri's identity, and generating function are just a few of the properties that have been looked into by various researchers, among many others [2, 3, 12, 13, 15, 17, 20, 23, 24, 25, 26, 28].

In this work, a variety of algebraic properties of the pseudo-Pell and Pell-Lucas numbers will be presented. Some identities will be given for the pseudo-Pell and Pell-Lucas numbers sequences, such as Binet's formula, the generating function formula, and some sum formulas. A matrix representation of these sequences will also be given.

2. The pseudo-Pell sequence

Several integer sequences exist, many of them with charming shapes and several charming characteristics. For instance, two of the most well-known and attractive number sequences are the Pell and Pell-Lucas sequences. The mathematical community is still in awe of their universality and beauty. Additionally, there are countless possibilities to explore, locate, and estimate thanks to the Pell and Pell-Lucas numbers. Mathematical friends, the Pell and Pell-Lucas numbers share many similar characteristics. Our main aim here is to find alternatives to these two series of families.

In this section, a new generalization of the Pell and Pell-Lucas numbers is introduced. We give some properties of the pseudo-Pell and Pell-Lucas numbers.

Definition 2.1. Let $\xi > 0$ be integer number. The pseudo-Pell and Pell-Lucas numbers be recursively defined by

$$\mathbf{P}_{n+1} = 2\mathbf{P}_n + \mathbf{R}_n \tag{2.1}$$

$$\mathbf{R}_{n+1} = 2\mathbf{P}_{n+1} + \xi\mathbf{P}_n \tag{2.2}$$

with initial conditions $\mathbf{P}_1 = 1$ and $\mathbf{R}_1 = 2$. Actually, by eliminating first the $\mathbf{P}_n$'s and then the $\mathbf{R}_n$'s, from equations (2.1) and (2.2), following the pseudo-Pell and Pell-Lucas numbers are obtained

$$\mathbf{P}_{n+1} = 4\mathbf{P}_n + \xi\mathbf{P}_{n-1} \tag{2.3}$$

$$\mathbf{R}_{n+1} = 4\mathbf{R}_n + \xi\mathbf{R}_{n-1} \tag{2.4}$$

with initial conditions conditions $\mathbf{P}_0 = 0$, $\mathbf{P}_1 = 1$ and $\mathbf{R}_0 = 1$, $\mathbf{R}_1 = 2$ respectively.

From equations (2.3) and (2.4), the associated characteristic polynomial

$$p(x) = x^2 - 4x - \xi.$$

$p(x)$ has the roots $x_1 = 2 + \sqrt{4+\xi}$ and $x_2 = 2 - \sqrt{4+\xi}$. Thus, it is apparent that

$$x_1x_2 = -\xi,$$
$$x_1 + x_2 = 4,$$
$$x_1 - x_2 = 2\sqrt{4+\xi},$$
$$x_1^2 + x_2^2 = 16 + 2\xi,$$
$$x_1^2 - x_2^2 = 8\sqrt{4+\xi},$$
$$x_1^2 = 4x_1 + \xi,$$
$$x_2^2 = 4x_2 + \xi.$$

The first terms of the pseudo-Pell and Pell-Lucas numbers

n	$\mathbf{P}_n$	$\mathbf{R}_n$
0	0	1
1	1	2
2	4	$\xi + 8$
3	$\xi + 16$	$6\xi + 32$
4	$8\xi + 64$	$\xi^2 + 32\xi + 128$
5	$\xi^2 + 48\xi + 256$	$10\xi^2 + 160\xi + 512$
6	$12\xi^2 + 256\xi + 1024$	$\xi^3 + 72\xi^2 + 768\xi + 2048$

2.1. Binet's formula for the pseudo-Pell sequence

The Fibonacci numbers are among the brightest points within a wide range of integer sequences, according to Koshy. We can speculate that this sequence's abundance of intriguing features is one of the reasons it is referenced. Furthermore, Binet's formula can be used to obtain practically all of these features. We will state and prove a general closed formula for the pseudo-Pell sequence.

Theorem 2.2. *The Binet's formula for the pseudo-Pell and Pell-Lucas numbers are*

$$\mathbf{P}_n = \frac{x_1^n - x_2^n}{x_1 - x_2},$$

$$\mathbf{R}_n = \frac{x_1^n + x_2^n}{2}$$

where $x_1 = 2 + \sqrt{4+\xi}$, $x_2 = 2 - \sqrt{4+\xi}$ *and* ξ *is a positive integer.*

Proof. The pseudo-Pell sequence's characteristic equation $x^2 - 4x - \xi = 0$, and its real roots are $x_1 = 2 + \sqrt{4+\xi}$ and $x_2 = 2 - \sqrt{4+\xi}$. Then the sequences $\mathbf{P}_n = \eta(x_1)^n + \mu(x_2)^n$, for $n \geq 0$, and with η, μ real numbers are solutions of equation. Let us determine the constants η and μ, considering that $\mathbf{P}_0 = 0$ and $\mathbf{P}_1 = 1$, and we obtain the linear system,

$$\eta + \mu = 0$$

$$\eta x_1 + \mu x_2 = 1.$$

We find $\mu = -\frac{1}{x_1 - x_2}$ and $\eta = \frac{1}{x_1 - x_2}$. So we have that

$$\mathbf{P}_n = \frac{x_1^n - x_2^n}{x_1 - x_2}.$$

Similarly,

$$\mathbf{R}_n = \frac{x_1^n + x_2^n}{2}.$$ □

Corollary 2.3. *We let* $n = -m$ *where* m *is a positive integer. From Binet's formula, we find the negative subscript terms of the pseudo-Pell and Pell-Lucas numbers.*

$$\mathbf{P}_{-n} = -\frac{\mathbf{P}_n}{(-\xi)^n},$$

$$\mathbf{R}_{-n} = \frac{\mathbf{R}_n}{(-\xi)^n}.$$

Theorem 2.4. *For* $n \geq 0$*, the following identity holds*

$$\mathbf{R}_{n+1} - 2\mathbf{R}_n = (4+\xi)\mathbf{P}_n$$

where $\mathbf{P}_n$ *and* $\mathbf{R}_n$ *are the nth pseudo-Pell and Pell-Lucas numbers, respectively.*

Proof.

$$\mathbf{R}_{n+1} - 2\mathbf{R}_n = \frac{x_1^{n+1} + x_2^{n+1}}{2} - 2\frac{x_1^n + x_2^n}{2} = \frac{x_1^n\left[x_1 - 2\right] + x_2^n\left[x_2 - 2\right]}{2}$$

$$= \frac{\left[x_1^n - x_2^n\right]\sqrt{4+\xi}}{2} = (4+\xi)\mathbf{P}_n.$$ □

Corollary 2.5. *In (2.3) and (2.4) let $\xi = 4$. We get $\mathbf{P}_n = 2^{n-1}P_n$ and $\mathbf{R}_n = 2^{n-1}R_n$ where P_n and R_n are the nth Pell and Pell-Lucas numbers, respectively.*

2.2. Generating function for the pseudo-Pell sequence

A strong method for resolving linear homogeneous recurrence relations is offered by generating functions. We shall systematically apply generating functions for linear recurrence relations with nonconstant coefficients, even though they are usually employed in conjunction with linear recurrence relations with constant coefficients. Now, we consider the generating functions for the pseudo-Pell sequence.

Theorem 2.6. *The generating formula for the pseudo-Pell and Pell-Lucas numbers are*

$$\sum_{n=0}^{\infty} \mathbf{P}_n t^n = \frac{t}{1-4t-\xi t^2},$$

$$\sum_{n=0}^{\infty} \mathbf{R}_n t^n = \frac{1-2t}{1-4t-\xi t^2}.$$

Proof. Let $h(t)$ be the generating function for the pseudo-Pell numbers as $\sum_{n=0}^{\infty} \mathbf{P}_n t^n$. We get the following equations

$$4th(t) = 4\sum_{n=0}^{\infty} \mathbf{P}_n t^{n+1} \text{ and } \xi t^2 h(t) = \xi \sum_{n=0}^{\infty} \mathbf{P}_n t^{n+2}.$$

After the needed calculations, the generating function for the pseudo-Pell numbers is obtained as

$$\sum_{n=0}^{\infty} \mathbf{P}_n t^n = \frac{t}{1-4t-\xi t^2}.$$

Similarly,

$$\sum_{n=0}^{\infty} \mathbf{R}_n t^n = \frac{1-2t}{1-4t-\xi t^2}.$$ □

Theorem 2.7. *The following identities holds*

$$(4+\xi)\sum_{i=0}^{n} \mathbf{P}_i + \sum_{i=0}^{n} \mathbf{R}_i = \mathbf{R}_{n+1} - 1, \tag{2.5}$$

$$(2+\xi)\sum_{i=0}^{n+1} \mathbf{P}_i - \sum_{i=0}^{n+1} \mathbf{R}_i = \xi \mathbf{P}_{n+1} \tag{2.6}$$

where $\mathbf{P}_n$ and $\mathbf{R}_n$ are the nth pseudo-Pell and Pell-Lucas numbers, respectively.

Proof. The proof is carried out using elimination in the equations of theorem (2.4). □

Corollary 2.8. *From equations (2.5) and (2.6), the following equations are obtained,*

$$\sum_{i=0}^{n+1} \mathbf{P}_i = \frac{(4+\xi)\mathbf{P}_{n+1} + \xi \mathbf{P}_n - \frac{1}{2}}{3+\xi},$$

$$\sum_{i=0}^{n+1} \mathbf{R}_i = \frac{(8+3\xi)\mathbf{P}_{n+1} + \xi(2+\xi)\mathbf{P}_n - (2+\xi)\frac{1}{2}}{3+\xi}.$$

2.3. Matrix representation for the pseudo-Pell sequence

A close connection between matrices and Fibonacci numbers was shown in Charles'[9] work on what he called the Q matrix. An interesting pattern emerges from this work. The power of matrices was exploited to obtain new identities and results involving Fibonacci numbers. We will give the matrix representation of the pseudo-Pell sequence.

Definition 2.9. The basic matrix of the pseudo-Pell and pseudo-Pell-Lucas sequence is

$$Q = \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix}$$

where $\xi > 0$ is a integer. Based on the Cayley-Hamilton Theorem, the pseudo-Pell and pseudo-Pell-Lucas's characteristic polynomial is given as

$$p(\lambda) = det(\lambda I - Q).$$

$$p(\lambda) = det(\lambda I - Q) = \begin{vmatrix} \lambda - 4 & -\xi \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 4\lambda - \xi.$$

Then, if $p(\lambda) = \lambda^2 - 4\lambda - \xi$.

Theorem 2.10. *Let $n > 0$ be an integer. The following equality holds*

$$a) \quad \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_2 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{n+2} & \mathbf{P}_{n+1} \\ \mathbf{P}_{n+1} & \mathbf{P}_n \end{bmatrix}$$

$$b) \quad \begin{bmatrix} 0 & 1 \\ \frac{1}{\xi} & -\frac{4}{\xi} \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_2 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{-n+2} & \mathbf{P}_{-n+1} \\ \mathbf{P}_{-n+1} & \mathbf{P}_{-n} \end{bmatrix}$$

$$c) \quad \begin{bmatrix} 0 & 1 \\ \xi & 4 \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_n \\ \mathbf{P}_{n+1} \end{bmatrix}$$

$$d) \quad \begin{bmatrix} -\frac{4}{\xi} & \frac{1}{\xi} \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{-n} \\ \mathbf{P}_{-n+1} \end{bmatrix}$$

$$e) \quad \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ \xi & 0 \end{bmatrix}^n = \begin{bmatrix} \mathbf{P}_{n+1} & \mathbf{P}_n \end{bmatrix}$$

$$f) \quad \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\xi} \\ 1 & -\frac{4}{\xi} \end{bmatrix}^n = \begin{bmatrix} \mathbf{P}_{-n+1} & \mathbf{P}_{-n} \end{bmatrix}$$

Proof. For the prove, we utilize induction principle on n. The equality hold for $n = 1$. Now assume that the equality is true for $n > 1$. Then, we can verify for $n + 1$ as follows

$$a) \quad \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} \mathbf{P}_2 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_2 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_{n+2} & \mathbf{P}_{n+1} \\ \mathbf{P}_{n+1} & \mathbf{P}_n \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{n+3} & \mathbf{P}_{n+2} \\ \mathbf{P}_{n+2} & \mathbf{P}_{n+1} \end{bmatrix}.$$

Thus, the first step of the theorem can be proved easily. Similarly, the other steps of the proof are seen by induction on n. Matrix representations of the pseudo-Pell-Lucas numbers are proved similarly. □

2.4. Sums of the pseudo-Pell sequence

Finite sums of sequences have always been important for people working in this field. Some researchers examined the sum of product two consecutive terms, some researchers examined the sum of squares. In a new sequence found, the study of the sums of sequence terms became important. We present some results concerning sums of terms of the pseudo-Pell and Pell-Lucas sequence.

Theorem 2.11. *The following equalities hold*

$$a)\quad \sum_{i=0}^{n} \mathbf{P}_{2i} = \frac{\xi^2\mathbf{P}_{2n} - \mathbf{P}_{2n+2} + \mathbf{P}_2}{(\xi-5)(\xi+3)},$$

$$b)\quad \sum_{i=0}^{n} \mathbf{P}_{2i+1} = \frac{\xi^2\mathbf{P}_{2n+1} - \mathbf{P}_{2n+3} + (1-\xi)}{(\xi-5)(\xi+3)},$$

$$c)\quad \sum_{i=0}^{n} \mathbf{R}_{2i} = \frac{\xi^2\mathbf{R}_{2n} - \mathbf{R}_{2n+2} - \mathbf{R}_2 + 2}{(\xi-5)(\xi+3)},$$

$$d)\quad \sum_{i=0}^{n} \mathbf{R}_{2i+1} = \frac{\xi^2\mathbf{R}_{2n+1} - \mathbf{R}_{2n+3} + 2(1-\xi)}{(\xi-5)(\xi+3)}.$$

where $\mathbf{P}_n$ and $\mathbf{R}_n$ are the nth pseudo-Pell and Pell-Lucas numbers, respectively.

Proof.

$$a)\quad \sum_{i=0}^{n} \mathbf{P}_{2i} = \sum_{i=0}^{n} \frac{x_1^{2i} - x_2^{2i}}{x_1 - x_2} = \frac{1}{x_1 - x_2}\left[\sum_{i=0}^{n}(x_1^2)^i - \sum_{i=0}^{n}(x_2^2)^i\right]$$

$$= \frac{1}{x_1 - x_2}\left[\frac{x_1^{2n+2} - 1}{x_1^2 - 1} - \frac{x_2^{2n+2} - 1}{x_2^2 - 1}\right]$$

$$= \frac{1}{(x_1^2 - 1)(x_2^2 - 1)}\left[\frac{\xi^2(x_1^{2n} - x_2^{2n}) - (x_1^{2n+2} - x_2^{2n+2}) + (x_1^2 - x_2^2)}{x_1 - x_2}\right]$$

$$= \frac{\xi^2\mathbf{P}_{2n} - \mathbf{P}_{2n+2} + \mathbf{P}_2}{(\xi-5)(\xi+3)}$$

Other sums are shown in a similar way. □

Theorem 2.12. *The sum of squares of the first n terms and the sum of products of consecutive terms of the pseudo-Pell and Pell-Lucas sequence are*

$$a)\quad \sum_{i=0}^{n} \mathbf{P}_i^2 = \frac{\xi^2\mathbf{R}_{2n} - \mathbf{R}_{2n+2} - \xi - 7}{2(\xi-5)(\xi+3)(\xi+4)} + \frac{(-\xi)^{n+1} - 1}{2(\xi+1)(\xi+4)},$$

$$b)\quad \sum_{i=0}^{n} \mathbf{R}_i^2 = \frac{\xi^2\mathbf{R}_{2n} - \mathbf{R}_{2n+2} - \xi - 7}{2(\xi-5)(\xi+3)} - \frac{(-\xi)^{n+1} - 1}{2(\xi+1)},$$

$$c)\quad \sum_{i=0}^{n}\mathbf{P}_i\mathbf{P}_{i+1} = \frac{1}{4(\xi+4)}\left[\frac{\xi^2\mathbf{R}_{2n+1}-\mathbf{R}_{2n+3}+2(1-\xi)}{(\xi-5)(\xi+3)}+\frac{2(-\xi)^{n+1}-1}{(\xi+1)}\right],$$

$$d)\quad \sum_{i=0}^{n}\mathbf{R}_i\mathbf{R}_{i+1} = \frac{1}{2}\left[\frac{\xi^2\mathbf{R}_{2n+1}-\mathbf{R}_{2n+3}+2(1-\xi)}{(\xi-5)(\xi+3)}-\frac{2(-\xi)^{n+1}-1}{(\xi+1)}\right].$$

where $\mathbf{P}_n$ *and* $\mathbf{R}_n$ *are the nth pseudo-Pell and Pell-Lucas numbers, respectively.*

Proof.

$$a)\quad \sum_{i=0}^{n}\mathbf{P}_i^2 = \sum_{i=0}^{n}\left(\frac{x_1^i-x_2^i}{x_1-x_2}\right)^2 = \frac{1}{(x_1-x_2)^2}\left[\sum_{i=0}^{n}(x_1^2)^i+\sum_{i=0}^{n}(x_2^2)^i-2\sum_{i=0}^{n}(x_1x_2)^i\right]$$

$$= \frac{\xi^2\mathbf{R}_{2n}-\mathbf{R}_{2n+2}-\xi-7}{2(\xi-5)(\xi+3)(\xi+4)}+\frac{(-\xi)^{n+1}-1}{2(\xi+1)(\xi+4)}.$$

Sums are shown in a similar way. □

Some special equalities well-known for the Pell and Pell-Lucas sequences have also been calculated for the pseudo-Pell and Pell-Lucas numbers. The proofs of these equations are omitted. $\mathbf{P}_n$ and $\mathbf{R}_n$ be the *nth* pseudo-Pell and Pell-Lucas numbers such that $\xi > 0$ integer, respectively. Then, the following equalities hold:

a) Tagiuri's Identity:

$$\mathbf{P}_{m+k}\mathbf{P}_{n-k}-\mathbf{P}_m\mathbf{P}_n = -(\xi)^{n-k}\mathbf{P}_k\mathbf{P}_{m-n+k}$$

$$\mathbf{R}_{m+k}\mathbf{R}_{n-k}-\mathbf{R}_m\mathbf{R}_n = (\xi+4)(\xi)^{n-k}\mathbf{R}_k\mathbf{R}_{m-n+k}$$

b) d'Ocagne's Identity:

$$\mathbf{P}_{m+1}\mathbf{P}_{n-1}-\mathbf{P}_m\mathbf{P}_n = -(\xi)^{n-1}\mathbf{P}_{m-n+1}$$

$$\mathbf{R}_{m+1}\mathbf{R}_{n-1}-\mathbf{R}_m\mathbf{R}_n = 2(\xi+4)(\xi)^{n-1}\mathbf{R}_{m-n+1}$$

c) Catalan's Identity:

$$\mathbf{P}_{n+k}\mathbf{P}_{n-k}-\mathbf{P}_n\mathbf{P}_n = -(\xi)^{n-k}\mathbf{P}_k^2$$

$$\mathbf{R}_{n+k}\mathbf{R}_{n-k}-\mathbf{R}_n\mathbf{R}_n = (\xi+4)(\xi)^{n-k}\mathbf{R}_k^2$$

d) Cassini's Identity:

$$\mathbf{P}_{n+1}\mathbf{P}_{n-1}-\mathbf{P}_n\mathbf{P}_n = -(\xi)^{n-1}$$

$$\mathbf{R}_{n+1}\mathbf{R}_{n-1}-\mathbf{R}_n\mathbf{R}_n = 4(\xi+4)(\xi)^{n-1}$$

3. Some numerical examples

In this section, we show four numerical examples for verify our theoretical results. Let us examine the case $\xi = 3$ in some of the results we obtained. For $\xi = 3$, the pseudo-Pell sequence will be

$$\mathbf{P}_{n+1} = 4\mathbf{P}_n + 3\mathbf{P}_{n-1}$$

and the characteristic equation will be $p(x) = x^2 - 4x - 3$.

For $\xi = 3$, The basic matrix of the pseudo-Pell sequence is

$$Q = \begin{bmatrix} 4 & 3 \\ 1 & 0 \end{bmatrix}.$$

For $\xi = 3$, some sum formulas will be as follows

$$\sum_{i=0}^{n+1} \mathbf{P}_i = \frac{14\mathbf{P}_{n+1} + 6\mathbf{P}_n - 1}{12},$$

$$\sum_{i=0}^{n} \mathbf{P}_{2i} = -\frac{9\mathbf{P}_{2n} - \mathbf{P}_{2n+2} + \mathbf{P}_2}{12},$$

$$\sum_{i=0}^{n} \mathbf{P}_{2i+1} = -\frac{9\mathbf{P}_{2n+1} - \mathbf{P}_{2n+3} - 2}{12},$$

$$\sum_{i=0}^{n} \mathbf{P}_i^2 = -\frac{9\mathbf{R}_{2n} - \mathbf{R}_{2n+2} - 10}{168} + \frac{(-3)^{n+1} - 1}{56},$$

$$\sum_{i=0}^{n} \mathbf{P}_i\mathbf{P}_{i+1} = -\frac{1}{28}\left[\frac{9\mathbf{R}_{2n+1} - \mathbf{R}_{2n+3} - 4}{12} + \frac{2(-3)^{n+1} - 1}{4}\right].$$

Some well-known special equations for the pseudo-Pell sequence are obtained for $\xi = 3$ as follows

a) Tagiuri's Identity:

$$\mathbf{P}_{m+k}\mathbf{P}_{n-k} - \mathbf{P}_m\mathbf{P}_n = -(3)^{n-k}\mathbf{P}_k\mathbf{P}_{m-n+k}.$$

b) d'Ocagne's Identity:

$$\mathbf{P}_{m+1}\mathbf{P}_{n-1} - \mathbf{P}_m\mathbf{P}_n = -(3)^{n-1}\mathbf{P}_{m-n+1}.$$

c) Catalan's Identity:

$$\mathbf{P}_{n+k}\mathbf{P}_{n-k} - \mathbf{P}_n\mathbf{P}_n = -(3)^{n-k}\mathbf{P}_k^2.$$

d) Cassini's Identity:

$$\mathbf{P}_{n+1}\mathbf{P}_{n-1} - \mathbf{P}_n\mathbf{P}_n = -(3)^{n-1}.$$

4. Discussion and conclusions

Cigler[5] obtained the Fibonacci and Lucas polynomials, which are defined by

$$F_n(x,s) = xF_{n-1}(x,s) + sF_{n-2}(x,s),$$

$$L_n(x,s) = xL_{n-1}(x,s) + sL_{n-2}(x,s),$$

where $F_0(x,s) = 0$, $F_1(x,s) = 1$, $L_0(x,s) = 2$ and $L_1(x,s) = x$.

There is a relationship between the special cases of pseudo-Pell and Pell-Lucas numbers and the special cases of Fibonacci and Lucas polynomials defined by Cigler as follows

$$\mathbf{P}_n = F_n(4,\xi)$$

and

$$\mathbf{R}_n = \frac{L_n(4,\xi)}{2}.$$

This study presents the pseudo-Pell and Pell-Lucas sequences. We obtain this new sequence, which was not defined in the literature before. Some very important properties of sequence, such as characteristic equation, generating functions, and Binet's formula, are investigated. We obtain the matrix representation of the pseudo-Pell and Pell-Lucas numbers. There have been a large number of studies on numerical sequences in the literature lately, and these sequences have been employed extensively in a variety of academic fields, including biology, finance, physics, architecture, nature, and the arts. Since this study includes some new results, it contributes to the literature by providing essential information concerning the number sequences. Research in these fields can benefit from the pseudo-Pell and pseudo-Pell-Lucas sequences as well. Therefore, we hope that this new number system and properties that we have found will offer a new perspective to the researchers. Some further investigations are as follows:

- Studying the properties of the pseudo-Pell and pseudo-Pell-Lucas sequences quaternions (hybrid, octonion, sedenion, etc.) might be intriguing.
- Examining the partial infinite sum obtained from the pseudo-Pell numbers and pseudo-Pell-Lucas sequences' reciprocals would be fascinating.
- Non-positive values of the integer ξ may also be worth examining.

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On a unification of Mittag-Leffler function and Wright function

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Abstract. We introduce here a function that unifies Mittag-Leffler function and Wright function which is referred to here as an UMLW-function. This function turns out to be a solution of an infinite order differential equation. With the aid of this UMLW-function, an integral operator is constructed and shown that it is bounded in Lebesgue measurable space. Further an eigen function property is established for a particular UMLW-function with the help of hyper-Bessel operator and Caputo fractional derivative operator. Some well known functions occur in the illustrations of these properties. At the end, the graphs of this UMLW-function are plotted by suitably specializing the parameters and also compared with the graph of exponential as well as Mittag-Leffler function.

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1. Introduction and main results

Gosta Mittag-Leffler [15], introduced the function given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

where z is a complex variable and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$, which reduces to e^z when $\alpha = 1$.
After some decades, its importance was realized due to its occurrence in many problems of Physics, Chemistry, Biology, and Engineering as a solution of fractional order

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differential or integral equations. During the course of time, this function was generalized and studied by many researchers among them Wiman [22], Prabhakar [16], Kiryakova [11], Shukla and Prajapati [19], Srivastava and Tomovski [21], Garra and Polito [10] are worth mentioning.
With the aid of L-exponential function

$$e_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{k+1}} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma^{k+1}(n+1)} \tag{1.1}$$

(of order k) due to Ricci and Tavkhelidze [18], Garra and Polito [10] defined and studied a generalization of (1.1) in the form:

$$E_{\alpha;\nu,\gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma^{\alpha+1}(\nu n+\gamma)}, \tag{1.2}$$

wherein $x \in \mathbb{R}, \alpha > -1, \nu > 0$, and $\gamma \in \mathbb{R}$, which they called $\alpha-$Mittag-Leffler function. Noticing the *rapid* convergence of the series due to Sikkema [20]

$$\sum_{n=1}^{\infty} \frac{z^n}{n!^n} = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma^n(n+1)}, \tag{1.3}$$

we propose a more general series structure which would also encompass another function, namely the Wright function [12]

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n+\mu) n!}, \ \lambda > -1, \mu \in \mathbb{C}. \tag{1.4}$$

This was introduced and investigated by the eminent British mathematician E. Maitland Wright (in a series of notes starting from 1933) in the framework of the asymptotic theory of partitions [12].
Aiming at the unification of the series given by (1.2), (1.3) and (1.4), we introduce the function defined by the power series as follows.

Definition 1.1. For $Re(\alpha\delta) \geq 0, Re(\beta\delta + \sigma\gamma - \frac{\delta}{2} - r + 1) > 0,\ \alpha, \sigma \neq 0,$ and $\mu, z \in \mathbb{C}$,

$$\mathbb{E}^{\sigma,\nu,\gamma}_{\alpha,\beta,\delta}(\mu, r; z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}}{\Gamma^{\delta n}(\alpha n+\beta)\ \Gamma^{\gamma}(\sigma n+\nu)} \frac{z^n}{n!}, \tag{1.5}$$

where $(\mu)_{rn} = \frac{\Gamma(\mu+rn)}{\Gamma(\mu)}$ is generalized Pochhammer symbol.

We shall henceforth referred to this function as UMLW-function.

Remark 1.2. Chudasama M. H. and Dave B. I. studied ℓ-Hypergeometric function, its particular cases and their q-analogues in [4, 3, 7, 5, 6, 2]. In context of the study of these, when $r, \sigma, \gamma, \alpha \in \mathbb{N} \cup \{0\}$ with $z^* = \frac{r^r}{\sigma^{\sigma\gamma}} \frac{z}{\alpha^{\alpha}\ \Gamma^{\delta}(\beta)}$, we have

$$\mathbb{E}^{\sigma,\nu,\gamma}_{\alpha,\beta,\delta}(\mu, r; z) = \frac{1}{\Gamma^{\gamma}(\nu)}$$

$$\times\ {}_rH^{\alpha}_{\sigma\gamma}\left[\begin{matrix} \frac{\mu}{r}, & \frac{\mu+1}{r}, & \ldots, & \frac{\mu+r-1}{r}; & & z^* \\ \left(\frac{\nu}{\sigma}\right)^{\gamma}, & \left(\frac{\nu+1}{\sigma}\right)^{\gamma}, & \ldots, & \left(\frac{\nu+\sigma-1}{\sigma}\right)^{\gamma}; & \left(\frac{\beta}{\alpha}, \ \frac{\beta+1}{\alpha}, \ \ldots, \ \frac{\beta+\alpha-1}{\alpha} : \delta\right); & \end{matrix}\right].$$

The particular cases of (1.5) are worth mentioning; all of them are corresponding to the common choice $\delta = 0$. The substitutions for the other parameters involved are indicated in each special case below.

1. Exponential function : $(\gamma = \mu = r = \nu = \sigma = 1)$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \mathbb{E}_0^{1,1,1}(1,1;z).$$

2. Mittag-Leffler function [15] : $(\gamma = \mu = r = \nu = 1)$

$$E_\sigma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n+1)} = \mathbb{E}_0^{\sigma,1,1}(1,1;z).$$

3. Wiman function [22] : $(\gamma = \mu = r = 1)$

$$E_{\sigma,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n+\nu)} = \mathbb{E}_0^{\sigma,\nu,1}(1,1;z).$$

4. Wright function [12] : $(r = 0, \gamma = 1, \sigma \neq 0,\ \nu$ arbitrary$)$

$$W_{\sigma,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n+\nu)n!} = \mathbb{E}_0^{\sigma,\nu,1}(\mu,0;z).$$

5. Prabhakar's function [16] : $(r = \gamma = 1)$

$$E^{\mu}_{\sigma,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_n\ z^n}{\Gamma(\sigma n+\nu)n!} = \mathbb{E}_0^{\sigma,\nu,1}(\mu,1;z).$$

6. Cosine function : $(\gamma = \nu = \mu = r = 1, \sigma = 2,\ z$ is replaced by $-z^2)$

$$\cos(\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{\Gamma(2n+1)} = \mathbb{E}_0^{2,1,1}(1,1;-z).$$

7. Bessel Maitland function [14] : $(\gamma = 1, r = 0,\ \nu$ is replaced by $\nu+1$, z is replaced by $-z)$

$$J^{\mu}_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\sigma n+\nu+1)\ n!} = \mathbb{E}_0^{\sigma,\nu+1,1}(\mu,0;-z).$$

8. Mainardi's functions [12] : $(\gamma = 1, r = 0, \sigma$ is replaced by $-\sigma$ *with* $0 < \sigma < 1)$ $(\nu = 0$ in $F_\sigma(z))$ $(\nu = 1-\sigma$ in $M_\sigma(z))$

$$F_\sigma(z) = \sum_{n=1}^{\infty} \frac{(-z)^n}{\Gamma(-\sigma n)n!} = \mathbb{E}_0^{-\sigma,0,1}(\mu,0;z);$$

$$M_\sigma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(-\sigma n+1-\sigma)n!} = \mathbb{E}_0^{-\sigma,1-\sigma,1}(\mu,0;z).$$

9. Kiryakova's function [11] : ($\mu = r = 1, \gamma = m \in \mathbb{N}$, in (1.5); and putting $\mu_1 = \cdots = \mu_m = \nu, \frac{1}{\rho_1} = \cdots = \frac{1}{\rho_m} = \sigma$ in [11, Eq.(13)])

$$E_{\sigma,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma^m(\sigma n + \nu)} = \mathbb{E}_0^{\sigma,\nu,m}(1,1;z).$$

10. Garra and Polito's function [10] : ($\mu = r = 1, z = x, \nu = \gamma, \gamma = \alpha + 1, \sigma = \nu$)

$$E_{\alpha;\nu,\gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma^{\alpha+1}(\nu n + \gamma)} = \mathbb{E}_0^{\nu,\gamma,\alpha+1}(1,1;x).$$

11. Shukla and Prajapati's function [19] : ($\gamma = 1, r \in \mathbb{N} \cup (0,1)$)

$$E_{\sigma,\nu}^{\mu,r}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}\ z^n}{\Gamma(\sigma n + \nu)\ n!} = \mathbb{E}_0^{\sigma,\nu,1}(\mu,r;z).$$

12. Srivastava and Tomovski's function [21] : ($\gamma = 1, Re(r) > 0$)

$$E_{\sigma,\nu}^{\mu,r}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}\ z^n}{\Gamma(\sigma n + \nu)\ n!} = \mathbb{E}_0^{\sigma,\nu,1}(\mu,r;z).$$

Now as a main results, the domain of convergence, differential equation and the integral operator of the UMLW-function are discussed.

Theorem 1.3. *For $Re(\alpha\delta) \geq 0, Re(\beta\delta + \sigma\gamma - \frac{\delta}{2} - r + 1) > 0$, and $\alpha, \sigma \neq 0$, the UMLW-function (1.5) is an entire function.*

Proof. We have

$$\mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu,r;z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}}{\Gamma^{\delta n}(\alpha n + \beta)\Gamma^{\gamma}(\sigma n + \nu)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \varphi_n z^n \text{ (say)}. \tag{1.6}$$

Using Cauchy-Hadamard formula:

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|\varphi_n|} = \lim_{\mathrm{n}\to\infty} \sup \left| \frac{(\mu)_{\mathrm{rn}}}{\Gamma^{\delta \mathrm{n}}(\alpha \mathrm{n} + \beta)\Gamma^{\gamma}(\sigma \mathrm{n} + \nu)\ \mathrm{n}!} \right|^{\frac{1}{\mathrm{n}}},$$

and then applying Stirlng's asymptotic Formula [9]

$$\Gamma(an + b) \quad \sim \quad \sqrt{2\pi}\ e^{-(an+b)}\ (an + b)^{an+b-\frac{1}{2}} \tag{1.7}$$

for large n and for $a = r, \alpha, \sigma$ and $b = \mu, \beta, \nu$ respectively we have

$$\begin{aligned} \frac{1}{R} &= \lim_{n\to\infty} \sup \left| \frac{\Gamma(\mu + rn)}{\Gamma(\mu)\ \Gamma^{\delta n}(\alpha n + \beta)\ \Gamma^{\gamma}(\sigma n + \nu)\ \Gamma(n + 1)} \right|^{\frac{1}{n}} \qquad (1.8) \\ &\sim \lim_{n\to\infty} \sup \Bigg\{ \left| \frac{\sqrt{2\pi}\ e^{-(\mu + rn)}(\mu + rn)^{\mu + rn - \frac{1}{2}}}{\sqrt{2\pi}\ e^{-(\mu)}(\mu)^{\mu - \frac{1}{2}}} \right|^{\frac{1}{n}} \\ &\quad \times \left| \frac{1}{\sqrt{2\pi}\ e^{-(\alpha n + \beta)}(\alpha n + \beta)^{\alpha n + \beta - \frac{1}{2}}} \right|^{\delta} \end{aligned}$$

$$
\begin{aligned}
& \times\left|\frac{1}{\sqrt{2\pi}\ e^{-(\sigma n+\nu)}(\sigma n+\nu)^{\sigma n+\nu-\frac{1}{2}}}\right|^{\frac{\gamma}{n}}\left|\frac{1}{\sqrt{2\pi}\ e^{-(n+1)}(n+1)^{n+1-\frac{1}{2}}}\right|^{\frac{1}{n}}\Bigg\} \\
= \;& \lim_{n\to\infty}\sup\left\{|e^{-r}|\left|\frac{(rn)^{r+\frac{\mu}{n}-\frac{1}{2n}}\left(1+\frac{\mu}{rn}\right)^{\frac{\mu}{n}-\frac{1}{2n}}\left(1+\frac{\mu}{rn}\right)^{r}}{\mu^{\frac{\mu}{n}-\frac{1}{2n}}}\right|\right. \\
& \times\left|\frac{e^{\beta\delta}}{(\sqrt{2\pi})^{\delta}}\right|\left|\frac{e^{\alpha\delta n}}{(\alpha n)^{\delta(\alpha n+\beta-\frac{1}{2})}\left(1+\frac{\beta}{\alpha n}\right)^{\alpha\delta n}\left(1+\frac{\beta}{\alpha n}\right)^{\beta\delta-\frac{\delta}{2}}}\right| \\
& \times|e^{\sigma\gamma}|\left|\frac{e^{\frac{\gamma\nu}{n}}}{(\sqrt{2\pi})^{\frac{\gamma}{n}}(\sigma n)^{\sigma\gamma+\frac{\gamma\nu}{n}-\frac{\gamma}{2n}}\left(1+\frac{\nu}{\sigma n}\right)^{\sigma\gamma+\frac{\gamma\nu}{n}-\frac{\gamma}{2n}}}\right| \\
& \times\left.\left|\frac{e^{1+\frac{1}{n}}}{(\sqrt{2\pi})^{\frac{1}{n}}n^{1+\frac{1}{n}-\frac{1}{2n}}\left(1+\frac{1}{n}\right)^{1-\frac{1}{n}}}\right|\right\} \\
= \;& \left|\frac{e^{\beta\delta+\sigma\gamma-r+1}\ r^{r}}{(\sqrt{2\pi})^{\delta}\sigma^{\sigma\gamma}\alpha^{\beta\delta-\frac{\delta}{2}}}\right|\lim_{n\to\infty}\sup\left\{\left|\frac{n^{r}r^{\frac{\mu}{n}-\frac{1}{2n}}(n^{\frac{1}{n}})^{\mu-\frac{1}{2}}\left(1+\frac{\mu}{rn}\right)^{\frac{\mu}{n}-\frac{1}{2n}}\left(1+\frac{\mu}{rn}\right)^{r}}{\mu^{\frac{\mu}{n}-\frac{1}{2n}}}\right|\right. \\
& \times\left|\frac{\left(\frac{e}{\alpha}\right)}{n}\right|^{\alpha\delta n}\left|\frac{1}{n^{\beta\delta-\frac{\delta}{2}}\left(1+\frac{\beta}{\alpha n}\right)^{\alpha\delta n}\left(1+\frac{\beta}{\alpha n}\right)^{\beta\delta-\frac{\delta}{2}}}\right| \\
& \times\left|\frac{e^{\frac{\gamma\nu}{n}}}{(\sqrt{2\pi})^{\frac{\gamma}{n}}\sigma^{\frac{\gamma\nu}{n}-\frac{\gamma}{2n}}\ n^{\sigma\gamma}\ (n^{\frac{1}{n}})^{\gamma\nu-\frac{\gamma}{2n}}}\right| \\
& \times\left.\left|\frac{1}{\left(1+\frac{\nu}{\sigma n}\right)^{\sigma\gamma}\left(1+\frac{\nu}{\sigma n}\right)^{\frac{\sigma\gamma}{n}-\frac{\gamma}{n}}}\right|\left|\frac{e^{\frac{1}{n}}\left(1+\frac{1}{n}\right)^{\frac{1}{n}-1}}{(\sqrt{2\pi})^{\frac{1}{n}}\ n(n^{\frac{1}{n}})^{\frac{1}{2}}}\right|\right\} \qquad (1.9) \\
= \;& \left|\frac{e^{\beta\delta+\sigma\gamma-r+1}r^{r}}{(\sqrt{2\pi})^{\delta}\sigma^{\sigma\gamma}\alpha^{\beta\delta-\frac{\delta}{2}}}\right|\left|\frac{1}{e^{\beta\delta+1}}\right|\lim_{n\to\infty}\left\{\left|\frac{\left(\frac{e}{\alpha}\right)}{n}\right|^{\alpha\delta n}\left|\frac{1}{n^{\beta\delta+\sigma\gamma-\frac{\delta}{2}-r+1}}\right|\right\} \\
= \;& 0,
\end{aligned}
$$

provided that $Re(\alpha\delta)\geq 0, Re(\beta\delta+\gamma\sigma-\frac{\delta}{2}-r+1)>0$, and $\alpha,\sigma\neq 0$.
Thus, $R=\infty$. □

Remark 1.4.

1. We stick to the conditions proved in Theorem 1.3 throughout the article unless it is specified.
2. The series $\sum\varphi_n z^n$ thus, converges uniformly in any compact subset of $\mathbb{C}$.

Next, to obtain the differential equation we define an operator as follows.

Definition 1.5. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n,\ 0 \neq z \in \mathbb{C},\ p \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$. Define [4, 7]

$$ {}_p\Delta_{\alpha}^{\theta}(f(z)) = \begin{cases} \sum_{n=1}^{\infty} a_n (\alpha)_{n-1}^{p} (\theta + \alpha - 1)^{pn} z^n, & phif \quad p \in \mathbb{N} \\ f(z), & phif \quad p = 0 \end{cases}, \tag{1.10} $$

where $\theta = z\frac{d}{dz}$ and $(\theta + c)^n = \underbrace{(\theta + c)(\theta + c)\ldots(\theta + c)}_{\text{n times}}$, c is a constant.

Using this operator, we have now obtained the differential equation of the UMLW-function in the following theorem.

Theorem 1.6. *If* $\alpha = 1, \gamma \in \mathbb{N} \cup \{0\}, \sigma, r, \delta \in \mathbb{N}$ *and* $\beta \neq 0, -1, -2, \ldots$ *then*

$$ w = \mathbb{E}_{1,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}}{\Gamma^{\delta n}(n+\beta)\Gamma^{\gamma}(\sigma n + \nu)} \frac{z^n}{n!} $$

satisfies the differential equation :

$$ \left\{ \left[\{{}_\delta\Delta_{\beta}^{\theta}\} \left(\prod_{i=0}^{\sigma-1} \left(\theta + \frac{\nu + i}{\sigma} - 1 \right)^{\gamma} \right) \theta \right] - z_* \prod_{j=0}^{r-1} \left[\theta + \frac{\mu + j}{r} \right] \right\} w = 0, \tag{1.11} $$

where $z_* = \frac{r^r}{\sigma^{\sigma\gamma}} \frac{z}{\Gamma^{\delta}(\beta)}$.

In order to prove this theorem, we first prove the following lemma which allows us to actually apply the operator ${}_\delta\Delta_{\beta}^{\theta}$ onto the operand w.
For the sake of brevity, we put

$$ \{{}_\delta\Delta_{\beta}^{\theta}\} \prod_{i=0}^{\sigma-1} \left(\theta + \frac{\nu + i}{\sigma} - 1 \right)^{\gamma} \theta = {}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}. $$

In this notation, we have

Lemma 1.7. *If* $\alpha = 1, \gamma \in \mathbb{N} \cup \{0\}, \sigma, r, \delta \in \mathbb{N}$ *and* $\beta \neq 0, -1, -2, \ldots$ *with*

$$ w = \mathbb{E}_{1,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}}{\Gamma^{\delta n}(n+\beta)\Gamma^{\gamma}(\sigma n + \nu)} \frac{z^n}{n!} $$

and

$$ {}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w) = \sum_{n=0}^{\infty} f_n(\mu, r, \beta, \delta, \sigma, \nu, \gamma; z) \ \textit{(say)}, $$

then the operator ${}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}$ *is applicable to* w *provided that the series*

$$ \sum_{n=0}^{\infty} \varphi_n \ f_n(\mu, r, \beta, \delta, \sigma, \nu, \gamma; z) $$

converges (cf. [20, Definition 11, p.20]*).*

Proof. We first write

$$\frac{1}{\Gamma^{\gamma}(\sigma n+\nu)}=\frac{1}{\Gamma^{\gamma}(\nu)}\frac{\Gamma^{\gamma}(\nu)}{\Gamma^{\gamma}(\sigma n+\nu)}=\frac{1}{\Gamma^{\gamma}(\nu)}\left[\frac{1}{(\nu)_{\sigma n}}\right]^{\gamma},$$

then applying the formula [17, Lemma 6, p. 22]

$$(a)_{kn}=k^{kn}\left(\frac{a}{k}\right)_n\cdots\left(\frac{a+k-1}{k}\right)_n,$$

for $a=\mu,\nu$ and $k=r,\sigma$ respectively, we have

$$\begin{aligned} w &= \sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ z^n}{\Gamma^{\delta n}(n+\beta)\Gamma^{\gamma}(\sigma n+\nu)\ n!}\frac{\Gamma^{\delta n}(\beta)}{\Gamma^{\delta n}(\beta)} \\ &= \sum_{n=0}^{\infty}\frac{(\mu)_{rn}}{(\beta)_n^{\delta n}\Gamma^{\gamma}(\sigma n+\nu)\ n!}\frac{z^n}{\Gamma^{\delta n}(\beta)} \\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\sum_{n=0}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}\ n!}\frac{r^{rn}\ z^n}{\Gamma^{\delta n}(\beta)\sigma^{\sigma\gamma n}(\beta)_n^{\delta n}}. \end{aligned} \tag{1.12}$$

Now take

$$\frac{r^r\ z}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}=z_*,$$

then

$$w=\frac{1}{\Gamma^{\gamma}(\nu)}\sum_{n=0}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\frac{z_*^n}{(\beta)_n^{\delta n}\ n!}.$$

Now consider

$$\begin{aligned} &{}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w) \\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\left\{\left[\left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\}\left(\prod_{i=0}^{\sigma-1}\left(\theta+\frac{\nu+i}{\sigma}-1\right)^{\gamma}\right)\right]\right. \\ &\quad\left.\times\sum_{n=1}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}\right\} \\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\left\{\left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\}\left(\theta+\frac{\nu-\sigma}{\sigma}\right)^{\gamma}\left(\theta+\frac{\nu-\sigma+1}{\sigma}\right)^{\gamma}\cdots\left(\theta+\frac{\nu-1}{\sigma}\right)^{\gamma}\right\} \\ &\quad\times\sum_{n=1}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!} \\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\left\{\left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\}\left(\theta+\frac{\nu-\sigma}{\sigma}\right)^{\gamma}\left(\theta+\frac{\nu-\sigma+1}{\sigma}\right)^{\gamma}\cdots\left(\theta+\frac{\nu-2}{\sigma}\right)^{\gamma}\right\} \\ &\quad\times\sum_{n=1}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}\left[n+\frac{\nu-1}{\sigma}\right]^{\gamma} \\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\left\{\left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\}\left(\theta+\frac{\nu-\sigma}{\sigma}\right)^{\gamma}\left(\theta+\frac{\nu-\sigma+1}{\sigma}\right)^{\gamma}\cdots\left(\theta+\frac{\nu-2}{\sigma}\right)^{\gamma}\right\} \end{aligned}$$

$$\times \sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!} \left[\frac{\nu+\sigma n-1}{\sigma}\right]^{\gamma}.$$

Continuing in this manner, we finally arrive at

$$\begin{aligned}
&{}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w) \\
&= \frac{1}{\Gamma^{\gamma}(\nu)}\ \left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\} \left\{\sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ ((n-1)!}\right\} \\
&\quad \times \left(\frac{\nu+\sigma n-1}{\sigma}\right)^{\gamma} \left(\frac{\nu+\sigma n-2}{\sigma}\right)^{\gamma} \cdots \left(\frac{\nu+\sigma n-\sigma}{\sigma}\right)^{\gamma} \\
&= \frac{1}{\Gamma^{\gamma}(\nu)}\ \left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\} \left\{\sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_{n-1}^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_{n-1}^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}\right\} \\
&= \frac{1}{\Gamma^{\gamma}(\nu)} \sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_{n-1}^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_{n-1}^{\gamma}} \frac{(\beta)_{n-1}^{\delta}\ (\theta+\beta-1)^{\delta n}\ z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}. \qquad (1.13)
\end{aligned}$$

Now, observe that for $\theta = z\frac{d}{dz}$,

$$\begin{aligned}
(\theta+\beta-1)\ z_*^n &= (\theta+\beta-1)\left(\frac{r^r\ z}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}\right)^n \\
&= \left(\frac{r^r}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}\right)^n\ (\theta+\beta-1)z^n \\
&= \left(\frac{r^r}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}\right)^n\ (z\ nz^{n-1}+\beta z^n - z^n) \\
&= \left(\frac{r^r}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}\right)^n\ (n+\beta-1)\ z^n \\
&= (n+\beta-1)\ z_*^n.
\end{aligned}$$

Similarly $(\theta+\beta-1)^2\ z_*^n = (n+\beta-1)^2\ z_*^n$ and in general, for $\delta, n \in \mathbb{N}\cup\{0\}$, $(\theta+\beta-1)^{\delta n}\ z_*^n = (n+\beta-1)^{\delta n}\ z_*^n$. Using this in (1.13), we have

$$\begin{aligned}
{}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w) &= \frac{1}{\Gamma^{\gamma}(\nu)} \sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_{n-1}^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_{n-1}^{\gamma}} \frac{(\beta)_{n-1}^{\delta}(n+\beta-1)^{\delta n}\ z_*^n}{(\beta)_n^{\delta n}\ (n-1)!} \\
&= \frac{1}{\Gamma^{\gamma}(\nu)} \sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_{n-1}^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_{n-1}^{\gamma}} \frac{z_*^n}{(\beta)_{n-1}^{\delta n-\delta}\ (n-1)!} \\
&= \frac{z_*}{\Gamma^{\gamma}(\nu)} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\ \frac{(\mu+rn)_r}{r^r} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!} \qquad (1.14) \\
&= \sum_{n=0}^{\infty}\ f_n(\mu, r, \beta, \delta, \sigma, \nu, \gamma; z)\ \text{(say)}.
\end{aligned}$$

To complete the proof of lemma, it remains to show that

$$\sum_{n=0}^{\infty}\varphi_n\ f_n(\mu,r,\beta,\delta,\sigma,\nu,\gamma;z)=\sum_{n=0}^{\infty}\frac{(\mu)_{rn}^2\ (\mu+rn)_r}{\Gamma^{2\delta n}(n+\beta)\ \Gamma^{2\gamma}(\sigma n+\nu)}\ \frac{z_*^{n+1}}{(n!)^2}$$

is convergent.
For that take

$$\xi_n\quad=\quad\frac{(\mu)_{rn}^2\ (\mu+rn)_r}{\Gamma^{2\delta n}(n+\beta)\ \Gamma^{2\gamma}(\sigma n+\nu)\ (n!)^2}.$$

Using Cauchy Hadamard formula:

$$\frac{1}{R}=\lim_{n\to\infty}\sup\sqrt[n]{|\xi_n|}=\lim_{\mathrm{n}\to\infty}\sup\left|\frac{(\mu)_{\mathrm{rn}}^2\ (\mu+\mathrm{rn})_{\mathrm{r}}}{\Gamma^{2\delta\mathrm{n}}(\mathrm{n}+\beta)\ \Gamma^{2\gamma}(\sigma\mathrm{n}+\nu)\ (\mathrm{n}!)^2}\right|^{\frac{1}{\mathrm{n}}},$$

and then applying Stirlng's asymptotic formula (1.7), we have

$$\begin{aligned}\frac{1}{R}\quad&=\quad\lim_{n\to\infty}\sup\left|\frac{\Gamma^2(\mu+\mathrm{rn})\ \Gamma(\mu+\mathrm{rn}+\mathrm{r})}{\Gamma^2(\mu)\ \Gamma(\mathrm{r})\ \Gamma^{2\delta\mathrm{n}}(\mathrm{n}+\beta)\ \Gamma^{2\gamma}(\sigma\mathrm{n}+\nu)\ \Gamma^2(\mathrm{n}+1)}\right|^{\frac{1}{\mathrm{n}}}\\&=\quad\lim_{n\to\infty}\sup\left|\frac{\Gamma(\mu+\mathrm{rn})\ \Gamma^{\frac{1}{2}}(\mu+\mathrm{rn}+\mathrm{r})}{\Gamma(\mu)\ \Gamma^{\frac{1}{2}}(\mathrm{r})\ \Gamma^{\delta\mathrm{n}}(\mathrm{n}+\beta)\ \Gamma^{\gamma}(\sigma\mathrm{n}+\nu)\ \Gamma(\mathrm{n}+1)}\right|^{\frac{2}{\mathrm{n}}}.\end{aligned}$$

Proceeding in the similar manner from (1.8) to (1.9), we get

$$\begin{aligned}\frac{1}{R}\quad&\sim\quad\left|\frac{e^{2(\beta\delta+\sigma\gamma-r+1)}r^{2r}}{\Gamma^2(\mu)(\sqrt{2\pi})^{2\delta}\sigma^{2\sigma\gamma}}\right|\left|\frac{1}{e^{2\beta\delta+2}}\right|\lim_{n\to\infty}\left|\frac{e}{n}\right|^{2\delta n}\left|\frac{1}{n^{2(\beta\delta+\sigma\gamma-\frac{\delta}{2}-r+1)}}\right|\\&=\quad 0,\end{aligned}$$

provided that $Re(\beta\delta+\gamma\sigma-\frac{\delta}{2}-r+1)>0$, and $\alpha,\sigma\neq 0$.
This completes the proof of Lemma. □

Proof. (of Theorem 1.6) From (1.14), we have

$$_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w)\quad=\quad\frac{z_*}{\Gamma^{\gamma}(\nu)}\sum_{n=0}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\ \frac{(\mu+rn)_r}{r^r}\ \frac{z_*^n}{(\beta)_n^{\delta n}\ n!}.\qquad(1.15)$$

On the other hand,

$$\begin{aligned}&z_*\left\{\prod_{j=0}^{r-1}\left[\theta+\frac{\mu+j}{r}\right]\right\}w\\&=\quad\frac{z_*}{\Gamma^{\gamma}(\nu)}\left(\theta+\frac{\mu}{r}\right)\cdots\left(\theta+\frac{\mu+r-2}{r}\right)\sum_{n=0}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}\ (\beta)_n^{\delta n}\ n!}\\&\qquad\times\left(\theta+\frac{\mu+r-1}{r}\right)z_*^n\\&=\quad\frac{z_*}{\Gamma^{\gamma}(\nu)}\left(\theta+\frac{\mu}{r}\right)\cdots\left(\theta+\frac{\mu+r-2}{r}\right)\end{aligned}$$

$$\times \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!} \left(\frac{\mu+rn+r-1}{r}\right)$$

$$= \frac{z_*}{\Gamma^{\gamma}(\nu)} \left(\theta+\frac{\mu}{r}\right)\ldots\left(\theta+\frac{\mu+r-3}{r}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-2}{\sigma}\right)_n^{\gamma}\ (\beta)_n^{\delta n}\ n!}$$

$$\times \left(\frac{\mu+rn+r-1}{r}\right)\left(\theta+\frac{\mu+r-2}{r}\right) z_*^n$$

$$= \frac{z_*}{\Gamma^{\gamma}(\nu)} \left(\theta+\frac{\mu}{r}\right)\ldots\left(\theta+\frac{\mu+r-3}{r}\right)$$

$$\times \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!} \left(\frac{\mu+rn+r-1}{r}\right)\left(\frac{\mu+rn+r-2}{r}\right).$$

Proceeding in this way, we finally arrive at

$$z_* \left\{\prod_{j=0}^{r-1}\left[\theta+\frac{\mu+j}{r}\right]\right\} w$$

$$= \frac{z_*}{\Gamma^{\gamma}(\nu)\ r^r} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!}$$

$$\times(\mu+rn+r-1)(\mu+rn+r-2)\ldots(\mu+rn+1)(\mu+rn)$$

$$= \frac{z_*}{\Gamma^{\gamma}(\nu)} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{(\mu+rn)_r}{r^r} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!}. \tag{1.16}$$

The differential equation (1.11) now follows from (1.15) and (1.16). □

We next define an integral operator of $\mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; x)$ as follows.

Definition 1.8. For $Re(\nu) > 0$,

$$\mathcal{I}_{a+}\varphi(x) = \int_a^x (x-y)^{\nu-1}\mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; \lambda(x-y)^{\sigma})\varphi(y)dy. \tag{1.17}$$

For this operator, we prove

Theorem 1.9. *The operator $\mathcal{I}_{a+}$ defined in (1.17) is bounded in $L(a, b)$, the space of all Lebesgue measurable functions on finite interval (a, b) and*

$$\| \mathcal{I}_{a+}\ \varphi \|_1 \leq\ M\ \ \| \varphi \|_1,$$

where

$$M = \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}|\ |\lambda|^n\ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\delta n}(\alpha n+\beta)|\ |\Gamma^{\gamma}(\sigma n+\nu)|\ n!\ (Re(\nu)+Re(\sigma)n)}.$$

We need the following lemma for proving this theorem.

Lemma 1.10. *The series*

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}|\ |\lambda|^n\ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\delta n}(\alpha n+\beta)|\ |\Gamma^{\gamma}(\sigma n+\nu)|\ n!\ (Re(\nu)+Re(\sigma)n)}$$

converges absolutely under the convergence conditions as stated in Theorem 1.3.

The proof runs almost parallel to that of Theorem 1.3. Hence we omit the proof.

Proof. (of Theorem 1.9)
From the definition of integral operator (1.17), we have

$$\begin{aligned} \| \mathcal{I}_{a+}\varphi \|_1 &= \int_a^b \left| \int_a^x (x-y)^{\nu-1} \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu,r;\lambda(x-y)^{\sigma})\varphi(y)dy \right| dx \\ &\le \int_a^b \int_a^x (x-y)^{Re(\nu)-1} \left| \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu,r;\lambda(x-y)^{\sigma}) \right| |\varphi(y)|\, dy\ dx. \end{aligned}$$

Changing the order of integration, gives

$$\| \mathcal{I}_{a+}\varphi \|_1 \le \int_a^b \int_y^b (x-y)^{Re(\nu)-1} \left| \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu,r;\lambda(x-y)^{\sigma}) \right| dx\ \ |\varphi(y)|\, dy.$$

Now taking $x-y=u$, we get

$$\begin{aligned} \| \mathcal{I}_{a+}\varphi \|_1 &\le \int_a^b \int_0^{b-y} u^{Re(\nu)-1} \left| \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu,r;\lambda u^{\sigma}) \right| du\ \ |\varphi(y)|\, dy \\ &\le \int_a^b \left[\int_0^{b-a} u^{Re(\nu)-1} \left| \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu,r;\lambda u^{\sigma}) \right| du \right] |\varphi(y)|\, dy. \end{aligned}$$

Using the Definition 1.1 of $\mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu,r;x)$, we obtain

$$\begin{aligned} \| \mathcal{I}_{a+}\varphi \|_1 &\le \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}|\ |\lambda|^n}{|\Gamma^{\delta n}(\alpha n+\beta)|\ |\Gamma^{\gamma}(\sigma n+\nu)|\ n!} \\ &\quad \times \int_0^{b-a} u^{Re(\nu)+Re(\sigma)n-1} du \int_a^b |\varphi(y)|\, dy \\ &= \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}|\ |\lambda|^n\ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\delta n}(\alpha n+\beta)|\ |\Gamma^{\gamma}(\sigma n+\nu)|\ n!\ (Re(\nu)+Re(\sigma)n)} \| \varphi \|_1 . \end{aligned}$$

The series on the r.h.s. is of real constants which converges absolutely by Lemma 1.10. Hence denoting its sum by M, the theorem follows. □

2. Other results

In this section, we derive some results involving certain fractional order derivatives and obtain the eigen function property of the UMLW-function. At last, some special cases and graphs of the UMLW-function are compared.

Definition 2.1. The Riemann-Liouville fractional integral (RL-integral) operator of order $\alpha \in \mathbb{C}, \Re(\alpha) > 0$ is defined as [13]

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\ dt, x > a. \tag{2.1}$$

Definition 2.2. The Riemann-Liouville fractional derivative (RL-derivative) of order $\alpha \in \mathbb{C}, m-1 < Re(\alpha) \leq m, m \in \mathbb{N}$ is defined as [13]

$$D_a^{\alpha} f(x) = D_a^m I_a^{m-\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} D^m \left\{ \int_a^x (x-t)^{m-\alpha-1} f(t)\ dt \right\}, x > a, \tag{2.2}$$

where $D^m = \frac{d^m}{dx^m}$.

Definition 2.3. The Caputo derivative of order $\alpha \in \mathbb{C}, m-1 < Re(\alpha) \leq m, m \in \mathbb{N}$ is [13]

$$_cD_a^{\alpha} f(x) = I_a^{m-\alpha} D_a^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x D^m(f(t))(x-t)^{\alpha-1}\ dt, x > a, \tag{2.3}$$

where $D^m = \frac{d^m}{dx^m}$.

Then following hold true.

$$(1) \qquad I_a^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\beta+\alpha}, \beta > -1, \alpha \geq 0. \tag{2.4}$$

$$(2) \qquad D_a^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha-\beta+1)}(x-a)^{\beta-\alpha}, \beta > -1, \alpha \geq 0. \tag{2.5}$$

We take $a = 0$ now onwards. We define below hyper-Bessel type operators.

Definition 2.4. For $x \in \mathbb{R} \setminus \{0\}$ and $\ell \in \mathbb{N} \cup \{0\}$, the hyper-Bessel type operators denoted and defined by

$$(^x\mathbf{I}^{\alpha})^n = \underbrace{\mathbf{I}^{\alpha} x^{-\alpha} \mathbf{I}^{\alpha} \ldots \mathbf{I}^{\alpha} x^{-\alpha} \mathbf{I}^{\alpha}}_{(n+1)\ \text{integrals}}, \text{ for n} = 0, 1, 2, \ldots, \tag{2.6}$$

and

$$(^x\mathbf{D}^{\alpha})^{\ell n} = \underbrace{\mathbf{D}^{\alpha} x^{\alpha} \mathbf{D}^{\alpha} \ldots \mathbf{D}^{\alpha} x^{\alpha} \mathbf{D}^{\alpha}}_{(\ell n+1)\ \text{derivatives}}, \text{ for n} = 0, 1, 2, \ldots, \tag{2.7}$$

where $\mathbf{I}^{\alpha}$ denotes the RL-integral and $\mathbf{D}^{\alpha}$ will be either RL-derivative D^{α} or Caputo derivative $_cD^{\alpha}$ defined in (2.1), (2.2) and (2.3) respectively for $a = 0$.

Theorem 2.5. *For UMLW-function (1.5) with $\alpha, \beta, \delta, \sigma, \nu, \mu \in \mathbb{C}, \eta > 0, x \neq 0$ and $\gamma \in \mathbb{N}$, the hyper-Bessel type operators furnish*

$$({}^{x}\mathbf{I}^{\eta})^{\gamma}\left(x^{\nu-1}\mathbb{E}^{\sigma,\nu,\gamma+1}_{\alpha,\beta,\delta}(\mu, r; x^{\sigma})\right) = x^{\nu+\eta-1}\mathbb{E}^{\sigma,\nu+\eta,\gamma+1}_{\alpha,\beta,\delta}(\mu, r; x^{\sigma}) \tag{2.8}$$

and

$$({}^{x}\mathbf{D}^{\eta})^{\gamma}\left(x^{\nu-1}\mathbb{E}^{\sigma,\nu,\gamma+1}_{\alpha,\beta,\delta}(\mu, r; x^{\sigma})\right) = x^{\nu-\eta-1}\mathbb{E}^{\sigma,\nu-\eta,\gamma+1}_{\alpha,\beta,\delta}(\mu, r; x^{\sigma}), \tag{2.9}$$

where ${}^{x}I^{\eta}$ and ${}^{x}\mathbf{D}^{\eta}$ are as defined in the Definition 2.4.
That is, a fractional integration or differentiation transforms the function (1.5) with the ν-parameter is increased or decreased by the order of integration or differentiation respectively.

Proof. The equation (2.8) is proved below which uses (2.4). In fact,

$$\begin{aligned}
&({}^{x}\mathbf{I}^{\eta})^{\gamma}\left(x^{\nu-1}\mathbb{E}^{\sigma,\nu,\gamma+1}_{\alpha,\beta,\delta}(\mu, r; x^{\sigma})\right)\\
&= \underbrace{I^{\eta}x^{-\eta}I^{\eta}\dots I^{\eta}x^{-\eta}I^{\eta}}_{\gamma+1 \text{ integrals}}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\; x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\\
&= \underbrace{I^{\eta}x^{-\eta}I^{\eta}\dots I^{\eta}x^{-\eta}I^{\eta}}_{\gamma \text{ integrals}}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\; x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\frac{\Gamma(\sigma n+\nu)}{\Gamma(\sigma n+\nu+\eta)}\\
&= \dots = I^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\; x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\frac{\Gamma^{\gamma}(\sigma n+\nu)}{\Gamma^{\gamma}(\sigma n+\nu+\eta)}\\
&= x^{\nu+\eta-1}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\; x^{\sigma n}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu+\eta)n!}\\
&= x^{\nu+\eta-1}\mathbb{E}^{\sigma,\nu+\eta,\gamma}_{\alpha,\beta,\delta}(\mu, r; x^{\sigma}).
\end{aligned}$$

We next prove (2.9).
Observing that

$$\begin{aligned}
&({}^{x}\mathbf{D}^{\eta})^{\gamma}\left(x^{\nu-1}\mathbb{E}^{\sigma,\nu,\gamma+1}_{\alpha,\beta,\delta}(\mu, r; x^{\sigma})\right)\\
&= \underbrace{D^{\eta}x^{\eta}D^{\eta}\dots D^{\eta}x^{\eta}D^{\eta}}_{\gamma+1 \text{ derivatives}}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\; x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\\
&= \underbrace{D^{\eta}x^{\eta}D^{\eta}\dots D^{\eta}x^{\eta}D^{\eta}}_{\gamma \text{ derivatives}}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\; x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\frac{\Gamma(\sigma n+\nu)}{\Gamma(\sigma n+\nu-\eta)}\\
&= \dots = D^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\; x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)\; n!}\frac{\Gamma^{\gamma}(\sigma n+\nu)}{\Gamma^{\gamma}(\sigma n+\nu-\eta)}\\
&= x^{\nu-\eta-1}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\; x^{\sigma n}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu-\eta)n!}\\
&= x^{\nu-\eta-1}\mathbb{E}^{\sigma,\nu-\eta,\gamma}_{\alpha,\beta,\delta}(\mu, r; x^{\sigma}).
\end{aligned}$$

Hence the required result. □

For deriving the eigen function property, we first define the following operator.

Definition 2.6. Let $f(x) = \sum\limits_{n=1}^{\infty} a_n x^n$, $|x| < R,\ R > 0$. Define an operator for $x \neq 0$ as

$$\mathbf{I}_x f = \int\limits_0^{\infty} e^{-\frac{t}{x}} x^{-1} f(t)\ dt. \tag{2.10}$$

With the aid of this and the Caputo fractional derivative, we next define the eigen function operator below.

Definition 2.7. Let $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, $|x| < R,\ R > 0$ and $\ell, k \in \mathbb{N} \cup \{0\}$. Define an operator for $x \neq 0$ as

$$ {}^{\ell}_{D_x}\Omega^k_I \left(f(x^\eta)\right) = \underbrace{\mathbf{I}_x \mathbf{I}_x \cdots \mathbf{I}_x}_{\ell \text{ integrals}} \left({}^x\mathbf{D}^\eta\right)^k\ f\left(\left({}^x\mathbf{D}^\eta\right)^\ell x^\eta\right), \tag{2.11}$$

where ${}^x\mathbf{D}^\eta$ represents an operator defined in (2.7) with $\mathbf{D}^\alpha$ as the Caputo derivative and $\mathbf{I}_x$ is as defined in (2.10).

Theorem 2.8. *For $\beta = \sigma = \mu = r = 1$ and $\nu = \alpha, n-1 < Re(\alpha) < n, n \in \mathbb{N}$, the UMLW-function $\mathbb{E}^{\alpha,1,\gamma+1}_{\alpha,1,\delta}(1, 1; \lambda x^\alpha) = \sum\limits_{n=0}^{\infty} \frac{\lambda^n\ x^{\alpha n}}{\Gamma^{\delta n+\gamma+1}(\alpha n+1)} := \mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^\alpha)$, say, $\lambda \in \mathbb{C}, \alpha, x > 0$, is an eigen function of the operator ${}^{\ell}_{D_x}\Omega^{\gamma}_I$ in (2.11).*
That is,

$$ {}^{\delta}_{D_x}\Omega^{\gamma}_I \left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^\alpha)\right) = \lambda\ \mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^\alpha),\ \lambda \in \mathbb{R} - \{0\}.$$

Proof. Note that

$$\begin{aligned}
& {}^{\delta}_{D_x}\Omega^{\gamma}_I \left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^\alpha)\right) \\
&= \underbrace{\mathbf{I}_x \mathbf{I}_x \cdots \mathbf{I}_x}_{\delta \text{ integrals}} \left({}^x\mathbf{D}^\alpha\right)^\gamma\ \mathbb{E}^{\gamma}_{\alpha,\delta}\left(\lambda \left({}^x\mathbf{D}^\alpha\right)^\delta x^\alpha\right) \\
&= \underbrace{\mathbf{I}_x \mathbf{I}_x \cdots \mathbf{I}_x}_{\delta \text{ integrals}} \left({}^x\mathbf{D}^\alpha\right)^\gamma\ \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n+1)} \left({}^x\mathbf{D}^\alpha\right)^{\delta n}\ x^{\alpha n} \\
&= \underbrace{\mathbf{I}_x \mathbf{I}_x \cdots \mathbf{I}_x}_{\delta \text{ integrals}} \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n+1)} \left({}^x\mathbf{D}^\alpha\right)^{\delta n+\gamma}\ x^{\alpha n}.
\end{aligned} \tag{2.12}$$

For $n = 0$, $\left({}^x\mathbf{D}^\alpha\right)^{0+\gamma}\ x^0 = \underbrace{{}_cD^\alpha x^\alpha\, {}_cD^\alpha \ldots {}_cD^\alpha x^\alpha\, {}_cD^\alpha}_{\gamma+1 \text{ derivatives}}(1) = 0.$

For $n = 1$,

$$\left({}^x\mathbf{D}^\alpha\right)^{\delta+\gamma} x^\alpha = \underbrace{{}_cD^\alpha x^\alpha\, {}_cD^\alpha \ldots {}_cD^\alpha x^\alpha\, {}_cD^\alpha}_{\delta+\gamma+1 \text{ derivatives}}\ (x^\alpha). \tag{2.13}$$

Observing that

$$\begin{aligned} {}_cD^\alpha x^\alpha &= \frac{1}{\Gamma(m-\alpha)} \int_0^x \alpha(\alpha-1)\dots(\alpha-m+1)\ t^{\alpha-m}\ (x-t)^{m-\alpha-1} dt \\ &= \frac{\alpha(\alpha-1)\dots(\alpha-m+1)}{\Gamma(m-\alpha)} x^{m-\alpha-1} \int_0^x t^{\alpha-m} \left(1-\frac{t}{x}\right)^{m-\alpha-1} dt \end{aligned}$$

and substituting $t = xu$, we further have

$$\begin{aligned} {}_cD^\alpha x^\alpha &= \frac{\alpha(\alpha-1)\dots(\alpha-m+1)}{\Gamma(m-\alpha)} \int_0^1 u^{\alpha-m}\ (1-u)^{m-\alpha-1}\, du \\ &= \frac{\alpha(\alpha-1)\dots(\alpha-m+1)}{\Gamma(m-\alpha)}\ \Gamma(m-\alpha)\Gamma(\alpha-m+1) \\ &= \Gamma(\alpha+1). \end{aligned}$$

Using this repeatedly in (2.13), we finally arrive at

$$({}^x\mathbf{D}^\alpha)^{\delta+\gamma}\ x^\alpha \ = \ \Gamma^{\delta+\gamma+1}(\alpha+1).$$

Now for $n = 2$,

$$({}^x\mathbf{D}^\alpha)^{2\delta+\gamma}\ x^{2\alpha} = \underbrace{{}_cD^\alpha x^\alpha {}_cD^\alpha \dots {}_cD^\alpha x^\alpha {}_cD^\alpha}_{2\delta+\gamma+1 \text{ derivatives}}\ x^{2\alpha}.$$

We begin with

$$\begin{aligned} &{}_cD^\alpha(x^{2\alpha}) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (2\alpha)(2\alpha-1)\dots(2\alpha-m+1)\ t^{2\alpha-m}\ (x-t)^{m-\alpha-1} dt \\ &= \frac{(2\alpha)(2\alpha-1)\dots(2\alpha-m+1)}{\Gamma(m-\alpha)} x^{m-\alpha-1} \int_0^x t^{2\alpha-m} \left(1-\frac{t}{x}\right)^{m-\alpha-1} dt \\ &= \frac{(2\alpha)(2\alpha-1)\dots(2\alpha-m+1)}{\Gamma(m-\alpha)} x^{2\alpha-\alpha} \frac{\Gamma(m-\alpha)\ \Gamma(2\alpha-m+1)}{\Gamma(2\alpha-\alpha+1)} \\ &= \frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha-\alpha+1)} x^{2\alpha-\alpha}. \end{aligned}$$

Therefore,

$$({}^x\mathbf{D}^\alpha)^{2\delta+\gamma}\ x^{2\alpha} = \frac{\Gamma^{2\delta+\gamma+1}(2\alpha+1)}{\Gamma^{2\delta+\gamma+1}(2\alpha-\alpha+1)} x^{2\alpha-\alpha}.$$

In general,

$$({}^x\mathbf{D}^\alpha)^{\delta n+\gamma} x^{\alpha n} = \frac{\Gamma^{\delta n+\gamma+1}(\alpha n+1)}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)}\ x^{\alpha n-\alpha}. \tag{2.14}$$

Now substituting (2.14) in (2.12) and then applying the operator defined in (2.10), we find that

$$\begin{aligned}
&{}^{\delta}_{D_x}\Omega^{\gamma}_{I}\left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha})\right)\\
&= \underbrace{\mathbf{I}_x\mathbf{I}_x\cdots\mathbf{I}_x}_{\delta \text{ fold integrals}} \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n+1)} \frac{\Gamma^{\delta n+\gamma+1}(\alpha n+1)}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)} x^{\alpha n-\alpha}\\
&= \underbrace{\mathbf{I}_x\mathbf{I}_x\cdots\mathbf{I}_x}_{\delta-1 \text{ fold integrals}} \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)} \int_0^{\infty} e^{-\frac{t}{x}}\, x^{-1}\, t^{\alpha n-\alpha} dt\\
&= \underbrace{\mathbf{I}_x\mathbf{I}_x\cdots\mathbf{I}_x}_{\delta-1 \text{ fold integrals}} \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)} x^{\alpha n-\alpha-1} \int_0^{\infty} e^{-\frac{t}{x}} \left(\frac{t}{x}\right)^{\alpha n-\alpha} dt\\
&= \underbrace{\mathbf{I}_x\mathbf{I}_x\cdots\mathbf{I}_x}_{\delta-1 \text{ fold integrals}} \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)} x^{\alpha n-\alpha}\, \Gamma(\alpha n-\alpha+1).
\end{aligned}$$

Continuing in this way by applying the operator $\mathbf{I}_x$, $\delta-1$ times, we finally arrive at

$$\begin{aligned}
{}^{\delta}_{D_x}\Omega^{\gamma}_{I}\left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha})\right) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)} x^{\alpha n-\alpha}\, \Gamma^{\delta}(\alpha n-\alpha+1)\\
&= \sum_{n=1}^{\infty} \frac{\lambda^n\, x^{\alpha n-\alpha}}{\Gamma^{\delta n-\delta+\gamma+1}(\alpha n-\alpha+1)}.
\end{aligned}$$

Hence,

$$ {}^{\delta}_{D_x}\Omega^{\gamma}_{I}\left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha})\right) = \lambda\, \mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha}). \tag{2.15}$$

□

Remark 2.9. From the definition of $\mathbb{E}^{\gamma}_{\alpha,\delta}(x)$ in the Theorem 2.8, observe that $\mathbb{E}^{\gamma}_{\alpha,\delta}(0) = 1$ and from (2.15), ${}^{\delta}_{D_x}\Omega^{\gamma}_{I}\left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha})\right) = \lambda\, \mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha}),\ \ \lambda \in \mathbb{R}\setminus\{0\}$.

3. Application

In the view of [8], we now discuss the application of particular UMLW-function discussed in the Theorem 2.8. Let $\mathcal{D}$ be the bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\mathcal{D}$. We consider the infinite order fractional evolution type problem

$$ {}^{\ell}_{D_x}\Omega^{\gamma}_{I}\, u(x,t) = u_t(x,t),\ t \in [0,T],\quad T > 0; \tag{3.1}$$

$$ u(0,t) = f(t), \tag{3.2}$$

where the operator ${}^{\ell}_{D_x}\Omega^{\gamma}_{I}$ is as defined in (2.11) in L^{∞}-space and is operating only on the variable x and $f(t) \in \mathcal{C}[0,T]$.

Theorem 3.1. *If* $\delta, \gamma \in \mathbb{N} \cup \{0\}$, $n-1 < Re(\alpha) < n, n \in \mathbb{N}$ *then the solution of (3.1)-(3.2) is given by*

$$u(x,t) = \mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\ \frac{\partial}{\partial t}\right) f(t).$$

Proof. To prove the theorem, we prove that $u(x,t) = \mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\ \frac{\partial}{\partial t}\right) f(t)$ satisfies the problem described by (3.1)-(3.2).
Here, noticing that ${}^{\ell}_{D_x}\Omega^{\gamma}_{I}$ is the operator operating only on the variable x, we have from the Theorem 2.8,

$$\begin{aligned} {}^{\ell}_{D_x}\Omega^{\gamma}_{I}\ u(x,t) &= {}^{\ell}_{D_x}\Omega^{\gamma}_{I}\left\{\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\ \frac{\partial}{\partial t}\right) f(t)\right\} = \frac{\partial}{\partial t}\ \mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\ \frac{\partial}{\partial t}\right) f(t) \\ &= u_t(x,t). \end{aligned}$$

To complete the proof of the theorem it sufficient to prove that

$$\lim_{x\to 0} ||u(x,.) - f||_{\infty} = 0. \tag{3.3}$$

Observe that

$$\begin{aligned} &\lim_{x\to 0} ||u(x,.) - f||_{\infty} \\ &= \lim_{x\to 0}\left|\left|\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\ \frac{\partial}{\partial t}\right) f - f\right|\right|_{\infty} \\ &= \lim_{x\to 0}\ ||f||_{\infty}\ \left|\left|\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\ \frac{\partial}{\partial t}\right) - 1\right|\right|_{\infty}. \end{aligned}$$

But $\lim_{x\to 0}\left|\left|\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\ \frac{\partial}{\partial t}\right) - 1\right|\right|_{\infty} = 0$ by Remark 2.9 and $f \in \mathcal{C}[0,T]$ proves (3.3). □

Now, some of the special cases of the properties proved for UMLW-function are shown.

Differential equation:

We illustrate the reducibility of the differential equation of Theorem 1.6 corresponding to the special cases namely Garra and Polito's function and Srivastava and Tomovski's function as follows.

(i) By taking $\delta = 0, \mu = r = 1$ and replacing γ by $\gamma + 1$, z by x in (1.11) we obtain with $x^* = \frac{x}{\sigma^{\sigma(\gamma+1)}}$, the equation

$$\left\{\prod_{i=0}^{\sigma-1}\left[\theta + \tfrac{\nu+i}{\sigma} - 1\right]^{\gamma+1}\ \theta - x^*\left(\theta+1\right)\right\} w = 0,$$

where the solution $w = E_{\gamma;\sigma,\nu}(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{\Gamma^{\gamma+1}(\sigma n+\nu)}$ is Garra and Polito's function.

(ii) The Srivastava-Tomovski's function $w = E^{\mu,r}_{\sigma,\nu}(z) = \sum\limits_{n=0}^{\infty} \frac{(\mu)_{rn}\ (z)^n}{\Gamma(\sigma n+\nu)\ n!}$ satisfies the differential equation

$\left\{\prod\limits_{i=0}^{\sigma-1}\left[\theta + \tfrac{\nu+i}{\sigma} - 1\right]\ \theta - z^* \prod\limits_{j=0}^{r-1}\left[\theta + \tfrac{\mu+j}{r}\right]\right\} w = 0$, with substitutions $\delta = 0$ and $\gamma = 1$ in (1.11), and $z^* = \frac{r^r}{\sigma^{\sigma}} z$.

Integral Operator:
(i) By taking $\delta = 0, \mu = r = 1$ and replacing γ by $\gamma + 1$, z by x in Theorem 1.9, we obtain

$$M = \sum_{n=0}^{\infty} \frac{|\lambda|^n \ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\gamma+1}(\sigma n+\nu)| \ (Re(\nu)+Re(\sigma)n)},$$

which is a bound of integral operator of Garra and Polito's function in Lebesgue Measurable space.
(ii) In Theorem 1.9, on making substitutions $\delta = 0$ and $\gamma = 1$, we find

$$M = \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}| \ |\lambda|^n \ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\gamma}(\sigma n+\nu)| \ n! \ (Re(\nu)+Re(\sigma)n)},$$

which is nothing but the Integral operator of Srivastava and Tomovski's function [21, Theorem 2, Eq.(2.15)].
Eigen function property:
It is interesting to note that the substitutions $\delta = 0, \mu = r = 1$ and $z = x$ in Theorem 2.8, yields the eigen function property of the Garra and Polito's function [10, Theorem 3.6, p. 776]

$$\mathbb{E}^{\gamma}_{\alpha,0}(x^{\alpha}) = E_{\gamma;\alpha,1}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{\Gamma^{\gamma+1}(\alpha n+1)}$$

with respect to the operator $({}^{x}\mathbf{D}^{\alpha})^{\gamma} := \ {}^{0}_{D_x}\Omega^{\gamma}_{I}$.
Following are the graphs of UMLW-function for the specific values of the parameters involved.

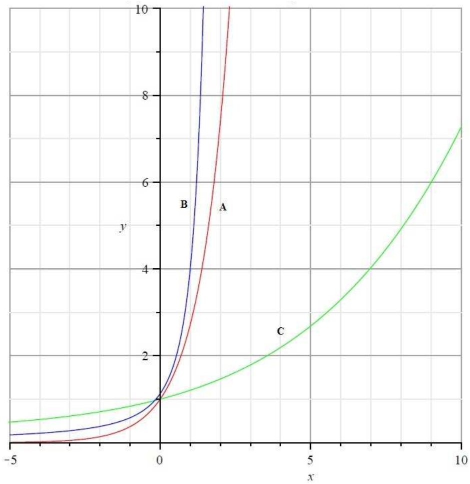

FIGURE 1. Graph A: $\exp(x)$, Graph B: $E_{\frac{1}{2},\frac{1}{3}}(x)$, Graph C : $E^{4,2,\frac{1}{2}}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(1,2;x)$

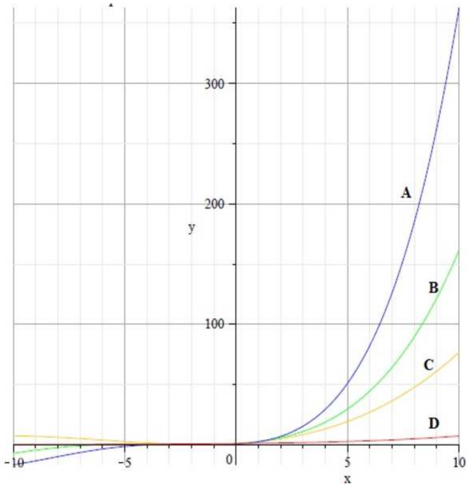

FIGURE 2. Graph A : $E^{\frac{1}{2},\frac{1}{2},1}_{\frac{1}{2},\frac{1}{2},1}(1,0;x)$, Graph B : $E^{\frac{1}{2},1,0}_{\frac{1}{2},1,1}(1,0;x)$, Graph C : $E^{\frac{1}{10},1,4}_{\frac{1}{2},1,2}(\frac{1}{2},\frac{1}{3};x)$, Graph D : $E^{4,2,\frac{1}{2}}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(1,2;x)$

In the Figure 1, a graph of particular UMLW-function $\left(E^{4,2,\frac{1}{2}}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(1,2;x)\right)$ is compared with that of exponential function and ML-function in two parameters indicated by Graph C, Graph A and Graph B respectively. And in the Figure 2, the graphs of certain specialized UMLW-functions are plotted. These are indicated in Figure 2 as Graph A, Graph B, Graph C and Graph D. Click: https://drive.google.com/file/d/0Bwly1qnYQNxZbEJnc0JQdW5tNE0/view?usp=sharing for more detail.

4. Conclusion

As a specific instance of the hypergeometric function ${}_pF_q$ if $\alpha \in \mathbb{N} \cup \{0\}$, the new function defined in (1.5) may clearly be viewed as an extension of the Mittag-Leffler and Wright functions (${}_1F_\alpha$ and ${}_0F_\alpha$), reduced to the hyper Bessel function ${}_0F_q$. However, in the power series, q in the second index and the summation index n both go to infinity at this point. It's also noteworthy to note that it solves an infinite order differential equation, which may arise in the turbulence field or in a system with an infinite number of degrees of freedom. Also, the integral involves this newly defined UMLW function as a kernel is bouned in $L(a,b)$. Notably, the specific instance of this new function possessing an eigen function characteristic concerning the hyper Bessel type fraction operators via which the infinite order evolution type problem is formulated is also intriguing.

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On some coefficient estimates for a class of p-valent functions

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Abstract. In this paper, we consider a class of p-valent functions. For functions in this class we find sharp estimates for their first three coefficients. Upper bound for the second order Hankel determinant is also obtained.

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Keywords: p-valent functions, coefficient estimates, Hankel determinant.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \tag{1.1}$$

defined on the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Note that for $p = 1$ we obtain $\mathcal{A}(1) = \mathcal{A}$ which is the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

Let $\mathcal{P}$ be the the well known Carathéodory class of functions consisting of functions q such that

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.3}$$

which are analytic in the unit disc $\mathcal{U}$ and satisfy $\Re q(z) > 0, z \in \mathcal{U}$ (see [2]).

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The Hankel determinant of a function f, for $q \geq 1, n \geq 1$ was defined by Pommerenke ([12]), [13]), as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

For our discussion in this paper , we consider the second order Hankel determinant for the case $q = 2$ and $n = p + 1$

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2.$$

Bounds for this determinant, for different classes of p-valent functions, has been investigated by several authors, see [1], [4], [5], [10] to mention only a few.

In a recent paper, Gupta et al. [3] extended Marx-Strohhäcker result [9], [14], to multivalent functions $f \in \mathcal{A}(p)$ $(p \geq 2)$, by finding β and γ such that

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \Longrightarrow \Re\sqrt{\frac{f'(z)}{pz^{p-1}}} > \beta \Longrightarrow \Re\frac{f(z)}{z^p} > \gamma, z \in \mathcal{U}. \tag{1.4}$$

Starting from Marx-Strohhäcker implication (1.4), we consider the following class of p-valent functions.

Definition 1.1. A function $f \in \mathcal{A}(p)$ $(p \geq 1)$ is said to be in the class $\mathcal{SQ}(p)$ if and only if

$$\Re\sqrt{\frac{f'(z)}{pz^{p-1}}} > 0, z \in \mathcal{U}. \tag{1.5}$$

In this paper, for the class $\mathcal{SQ}(p)$, we obtain sharp estimates for the coefficients $a_{p+1}, a_{p+2}, a_{p+3}$. We also find an upper bound for the second Hankel determinant $H_2(p+1)$.

In order to obtain our results we will need the next two lemmas.

Lemma 1.2. *[[6], [7]] If the function $p \in \mathcal{P}$ is given by (1.3), then*

$$|c_n| \leq 2, n \geq 1$$

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.6}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y \tag{1.7}$$

for some x, y with $|x| \leq 1$ and $|y| \leq 1$.

The second lemma is a special case of a more general result due to Ohno and Sugawa [11] (see also [8]).

Lemma 1.3. *For some given real numbers A, B, C, let*

$$Y(A, B, C) = \max_{z \in \overline{\mathbb{U}}}(|A + Bz + Cz^2| + 1 - |z|^2).$$

If $AC \geq 0$, *then*

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|\,, & |B| \geq 2(1-|C|) \\ 1 + |A| + \dfrac{B^2}{4(1-|C|)}\,, & |B| < 2(1-|C|). \end{cases}$$

If $AC < 0$, *then*

$$Y(A,B,C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leq B^2 \ \ and \ \ |B| < 2(1-|C|) \\ 1 + |A| + \dfrac{B^2}{4(1+|C|)}, & B^2 < \min\left\{4(1+|C|)^2, -4AC(C^{-2}-1)\right\} \\ R(A,B,C), & otherwise \end{cases}$$

where

$$R(A,B,C) = \begin{cases} |A| + |B| - |C|\,, & |C|(|B| + 4|A|) \leq |AB| \\ -|A| + |B| + |C|\,, & |AB| \leq |C|(|B| - 4|A|) \\ (|C| + |A|)\sqrt{1 - \dfrac{B^2}{4AC}}\,, & otherwise. \end{cases}$$

2. Coefficient estimates

In this section we obtain sharp inequalities for the coefficients a_{p+1}, a_{p+2} and a_{p+3}.

Theorem 2.1. *Let* $f \in \mathcal{SQ}(p)$ *be given be (1.1). Then*

$$|a_{p+1}| \leq \frac{4p}{p+1},$$

$$|a_{p+2}| \leq \frac{8p}{p+2},$$

$$|a_{p+3}| \leq \frac{12p}{p+3}.$$

Proof. Since $f \in \mathcal{SQ}(p)$, we have that $\sqrt{\frac{f'(z)}{pz^{p-1}}} \in \mathcal{P}$. It results that there exists a function $q \in \mathcal{P}$ such that

$$\sqrt{\frac{f'(z)}{pz^{p-1}}} = q(z), z \in \mathcal{U}. \tag{2.1}$$

Equating the coefficients in (2.1), we obtain

$$a_{p+1} = \frac{2p}{p+1}c_1,$$

$$a_{p+2} = \frac{2p}{p+2}(c_2 + \frac{c_1^2}{2}),$$

$$a_{p+3} = \frac{2p}{p+3}(c_3 + c_1 c_2).$$

Since $q \in \mathcal{P}$ we have $|c_1| \leq 2$ and thus $|a_{p+1}| \leq \dfrac{4p}{p+1}$. The inequality is sharp for $c_1 = 2$. In order to obtain $|a_{p+2}|$, making use of Lemma 1.2 , we replace the coefficient c_2 from (1.6) and we get

$$a_{p+2} = \frac{p}{p+2}(2c_1^2 + (4 - c_1^2)x), |x| \leq 1.$$

Suppose now that $c_1 = c$ and $0 \leq c \leq 2$. Then

$$|a_{p+2}| = \frac{p}{p+2}|2c_1^2 + (4 - c_1^2)x| \leq \frac{p}{p+2}(2c^2 + 4 - c^2) \leq \frac{8p}{p+2}.$$

The inequality is sharp for $c = 2$.
Since $a_{p+3} = \dfrac{2p}{p+3}(c_3 + c_1c_2)$, making use of Lemma 1.2 and replacing the coefficients c_2 and c_3, given by (1.6) and (1.7) respectively, we have

$$a_{p+3} = \frac{p}{p+3}\left[\frac{3c^3}{2} + 2cx(4 - c^2) - (4 - c^2)\frac{cx^2}{2} + (4 - c^2)(1 - |x^2|)y\right].$$

In view of triangle inequality, after some calculations, we obtain

$$|a_{p+3}| \leq \frac{p(4 - c^2)}{p+3}\left[|\frac{3c^3}{2(4 - c^2)} + 2cx - \frac{cx^2}{2}| + (1 - |x^2|)\right].$$

To obtain the upper bound of $|a_{p+3}|$ we use Lemma 1.3 with

$$A = \frac{3c^3}{2(4 - c^2)},\ B = 2c,\ C = -\frac{c}{2}.$$

It is easy to see that $AC > 0$ and $-4AC(C^{-2} - 1) \leq B^2$.
The inequality $|B| < 2(1 - |C|)$ holds true for $c < \frac{2}{3}$.
Thus, for the case $c \in [0, \frac{2}{3})$, we have

$$|a_{p+3}| \leq \frac{p(4 - c^2)}{p+3}Y(A, B, C) \ \text{ where } Y(A, B, C) = 1 - |A| + \frac{B^2}{4(1 - |C|)}.$$

By replacing A, B and C we obtain

$$Y(A, B, C) = \frac{c^3 + 6c^2 + 8}{2(4 - c^2)},$$

which implies

$$|a_{p+3}| \leq \frac{p}{2(p+3)}(c^3 + 6c^2 + 8).$$

Let $\varphi(c) = c^3 + 6c^2 + 8, c \in [0, \frac{2}{3})$ with $\varphi'(c) = 3c(c + 4)$. Since, $\varphi'(c) \geq 0, c \in [0, \frac{2}{3})$ we get $\varphi(c) < \frac{296}{27}$.
Therefore, if $c \in [0, \frac{2}{3})$, we have $|a_{p+3}| \leq \dfrac{148p}{27(p+3)}$.
We consider now the case $\frac{2}{3} \leq c \leq 2$ and we check the condition

$$B^2 < \min\left\{4(1 + |C|^2); -4AC(C^{-2} - 1)\right\} \tag{2.2}$$

from Lemma 1.3, which is equivalent to

$$4c^2 < \min\left\{4(1+c+\frac{c^2}{4}), 3c^2\right\}.$$

Hence, for $c \in [\frac{2}{3}, 2]$ the inequality (2.2) is not satisfied. We check now the conditions for $R(A,B,C)$ from the same Lemma 1.3.
It is easy to obtain that $|AB| \leq |C|(|B| - 4|A|)$ for $c \in [\frac{2}{3}, \frac{2}{\sqrt{7}}]$. For $c \in [\frac{2}{3}, \frac{2}{\sqrt{7}}]$ we have $Y(A,B,C) = R(A,B,C)$, where

$$R(A,B,C) = \frac{10c - 4c^3}{4 - c^2}.$$

In this case,

$$|a_{p+3}| \leq \frac{p}{p+3}(10c - 4c^3).$$

Let $\mu(c) = 10c - 4c^3, c \in [\frac{2}{3}, \frac{2}{\sqrt{7}}]$. Then $\mu^{'}(c) = 10 - 12c^2$. It follows that $\mu(c)$ is an increasing function, so $\mu(c) \leq \mu(\frac{2}{\sqrt{7}}) = \frac{108\sqrt{7}}{49}, c \in [\frac{2}{3}, \frac{2}{\sqrt{7}}]$. We obtain

$$|a_{p+3}| \leq \frac{p}{p+3}\frac{108\sqrt{7}}{49}.$$

Now, for $c \in (\frac{2}{\sqrt{7}}, 2]$ we get, $|a_{p+3}| \leq \frac{p(4-c^2)}{p+3}R(A,B,C)$, where

$$R(A,B,C) = (|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} = \frac{2+c^2}{4-c^2}\frac{\sqrt{16-c^2}}{\sqrt{3}}.$$

Then,

$$|a_{p+3}| \leq \frac{p}{p+3}(2+c^2)\frac{\sqrt{16-c^2}}{\sqrt{3}}.$$

We denote by $\eta(c) = (c^2+2)\sqrt{16-c^2}, c \in \left(\frac{2}{\sqrt{7}}, 2\right]$. Then

$$\eta'(c) = \frac{3c(10-c^2)}{\sqrt{16-c^2}} \geq 0, \ c \in \left(\frac{2}{\sqrt{7}}; 2\right],$$

which shows that $\eta(c)$ is an increasing function on $\left(\frac{2}{\sqrt{7}}; 2\right]$ and $\eta(c) \leq \eta(2) = 12\sqrt{3}$. Thus

$$|a_{p+3}| \leq \frac{12p}{p+3}.$$

Finally, we get

$$|a_{p+3}| \leq \max\left\{\frac{148p}{27(p+3)}; \frac{108\sqrt{7}}{49}\frac{p}{p+3}; \frac{12p}{p+3}\right\}, p \geq 1, c \in [0;2]$$

which implies

$$|a_{p+3}| \leq \frac{12p}{p+3}.$$

The last inequality is sharp for $c = 2$.
Now, the proof of our theorem is completed. □

3. Second Hankel determinant

In this section we find an upper bound for the second order Hankel determinant

$$H_2(p+1) = a_{p+1}a_{p+3} - a_{p+2}^2.$$

Theorem 3.1. *Let $f \in \mathcal{SQ}(p)$ be given be (1.1). Then*

$$|H_2(p+1)| \leq \frac{16p^2}{(p+1)(p+3)}.$$

Proof. Since $f \in \mathcal{SQ}(p)$, from the proof of Theorem 2.1, we have

$$a_{p+1} = \frac{2p}{p+1}c,$$

$$a_{p+2} = \frac{2p}{p+2}(c_2 + \frac{c^2}{2}),$$

$$a_{p+3} = \frac{2p}{p+3}(c_3 + c_2 c).$$

Then

$$H_2(p+1) = \frac{4p^2}{(p+1)(p+3)}c(c_3 + c_2 c) - \frac{4p^2}{(p+2)^2}(c_2 + \frac{c^2}{2})$$

$$= \frac{p^2}{(p+1)(p+2)^2(p+3)}[4c^2c_2 - c^4 - c^4(p+1)(p+3) + 4(p+2)^2]cc_3 - 4(p+1)(p+3)c_2^2].$$

Making use of Lemma 1.2, we get

$$4c^2c_2 = 2c^4 + 2c^2(4-c^2)x$$
$$4c_2^2 = c^4 + 2c^2(4-c^2)x + (4-c^2)^2x^2$$
$$4cc_3 = c^4 + 2c^2(4-c^2)x - c^2(4-c^2)x^2 + 2(4-c^2)c(1-|x|^2)y,$$

where $c \in [0,2]$, and $|x| \leq 1, |y| \leq 1$.
After lengthy calculations, we obtain

$$|H_2(p+1)| \leq \frac{p^2}{(p+1)(p+3)}2c(4-c^2)\left\{A + Bx + Cx^2 + (1-|x|^2)\right\},$$

where

$$A = \frac{-c^3(p^2+2p)}{2(p+2)^2(4-c^2)} < 0$$

$$B = \frac{2c}{(p+2)^2} > 0$$

$$C = -\frac{c^2 + 4(p+1)(p+3)}{2c(p+2)^2} < 0.$$

In order to obtain the upper bound of $|H_2(p+1)|$, we use Lemma 1.3 for the case $AC > 0$. Since the inequality $|B| < 2(1-|C|)$ is satisfied, then we have

$$Y(A,B,C) = 1 + |A| + \frac{B^2}{4(1-|C|)}$$

$$= 1+\frac{c^3}{2(p+2)^2(4-c^2)}\frac{c^2(p+2)^2-2c(p^2+4p)(p+2)^2+4(p^2+4p)(p+1)(p+3)-16}{c^2-2c(p+2)^2+4(p+1)(p+3)}.$$

It follows that

$$|H_2(p+1)| \le \frac{p^2}{(p+1)(p+3)}2c(4-c^2)Y(A,B,C)$$

$$= \frac{2p^2(4-c^2)c}{(p+1)(p+3)} + \frac{p^2c^4}{(p+1)(p+2)^2(p+3)}\frac{u(c)}{v(c)},$$

where

$$u(c) = c^2(p+2)^2 - 2c(p^2+4p)(p+2)^2 + 4(p^2+4p)(p+1)(p+3) - 16$$

and

$$v(c) = c^2 - 2c(p+2)^2 + 4(p+1)(p+3),\ \ c \in [0,2],\ \ p \ge 1.$$

We observe that $u(2) = 0$ and $u(c) = (c-2)[c-2(p^2+4p-1)](p+2)^2$. Also $v(2) = 0$ and $v(c) = (c-2)[c-2(p^2+4p+3)]$.
It follows that

$$\begin{aligned}
|H_2(p+1)| &\le \frac{2p^2(4-c^2)c}{(p+1)(p+3)} + \frac{p^2c^4}{(p+1)(p+3)}\frac{c-2(p^2+4p-1)}{c-2(p^2+4p+3)} \\
&= \frac{p^2}{(p+1)(p+3)}c\left[2(4-c^2) + c^3\frac{c-2(p^2+4p-1)}{c-2(p^2+4p+3)}\right] \\
&= \frac{p^2}{(p+1)(p+3)}\left\{2c(4-c^2) + c^4\left[1 + \frac{8}{c-2(p^2+4p+3)}\right]\right\} \\
&= \frac{p^2}{(p+1)(p+3)}\left[f_1(c) + 8f_2(c)\right],
\end{aligned}$$

where $f_1(c) = 2c(4-c^2) + c^4$ and $f_2(c) = \dfrac{c^4}{c-2(p^2+4p+3)}, c \in [0,2]$.
Since $f_1^{'}(c) = 2(2c^3-3c^2+4)$ for $c \in [0,2]$, we have that $f_1(c)$ is an increasing function and $f_1(c) \le f_1(2) = 16$.
Further $f_2{}'(c) = \dfrac{c^3[3c-8(p^2+4p+3)]}{[c-2(p^2+4p+3)]^2} \le 0$, which shows that $f_2(c)$ is a decreasing function on $[0,2]$ and $f_2(c) \le f_2(0) = 0$, $c \in [0,2]$.
Therefore

$$|H_2(p+1)| \le \frac{16p^2}{(p+1)(p+3)}.$$

The proof of theorem is now completed. □

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Some classes involving a convolution of analytic functions with some univalency conditions

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Abstract. In this paper, involving a convolution $f * g$, two classes of normalized analytic functions f are defined. Showing an inclusion relation between these classes, various sufficient conditions for functions to be in these classes are established. In particular, varied forms of univalency conditions of the convolution function $f * g$ are given which lead to some univalency conditions of several linear operators.

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1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the open unit disk

$$\mathbb{U} = \{z : |z| < 1\},$$

and for $k \in \mathbb{N} = \{1, 2, ...\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, k] = \left\{f \in \mathcal{H} : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + ...\right\}.$$

Let $\mathcal{A}$ denotes a class of functions in $\mathcal{H}[0, 1]$ of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}. \tag{1.1}$$

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A subclass of *univalent* functions in $\mathcal{A}$ is denoted by $\mathcal{S}$. Functions $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$, a class of *starlike* functions if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ in } \mathbb{U}.$$

A convolution (Hadamard product) $*$ of $f \in \mathcal{A}$ of the form (1.1) and $g \in \mathcal{A}$ of the form

$$g(z) = z + \sum_{k=1}^{\infty} b_{k+1} z^{k+1}, \tag{1.2}$$

is defined by

$$f(z) * g(z) = z + \sum_{k=1}^{\infty} a_{k+1} b_{k+1} z^{k+1} = g(z) * f(z). \tag{1.3}$$

Note that the convolution preserves the class $\mathcal{A}$.

Several linear operators have been studied in *Geometric Function Theory* so far, which are defined in the form of convolution, differential, integral, and fractional differintegral linear operators. Some of the known linear operators for the class $\mathcal{A}$, are the Dziok-Srivastava convolution operator [5], the Srivastava-Attiya linear operator [19], the Jung-Kim-Srivastava integral operator [7], a multiplier operator [16] and a fractional differintegral operator introduced by Owa and Srivastava [10]. The convolution representation of these operators may be given as follows:

The Dzoik-Srivastava operator [5]: ${}_pH_q\left([\alpha_1]\right) : \mathcal{A} \to \mathcal{A}$ is defined by

$${}_pH_q\left([\alpha_1]\right) f(z) = z \; {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z) * f(z) \tag{1.4}$$

where

$${}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z) = \sum_{k=0}^{\infty} \frac{\prod\limits_{i=1}^{p} (\alpha_i)_k}{\prod\limits_{i=1}^{q} (\beta_i)_k} \frac{z^k}{k!}$$

$$\left(p \leq q+1, p, q \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \alpha_i, \beta_i \in \mathbb{C} \; (\beta_i \neq 0, -1, -2, ...); z \in \mathbb{U}\right)$$

is the generalized hypergeometric function ([12, p. 19]). The symbol $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda+1)(\lambda+2)...(\lambda+k-1), k \in \mathbb{N}; (\lambda)_0 = 1.$$

The Srivastava-Attiya linear operator [19]: $J_{a,b} : \mathcal{A} \to \mathcal{A}$ is defined in terms of generalized Hurwitz-Lerch Zeta function $\phi(b, a, z)$ [20] by

$$J_{a,b} f(z) = G_{a,b}(z) * f(z), \tag{1.5}$$

where

$$G_{a,b}(z) = (b+1)^a \left(\phi(b, a, z) - b^{-a}\right) = z + \sum_{k=1}^{\infty} \left(\frac{b+1}{b+n}\right)^a z^{k+1}$$

$$\left(b \in \mathbb{C} \, (b \neq 0, -1, -2, ...), a \in \mathbb{C}; z \in \mathbb{U}\right).$$

The fractional integral operator $D_z^{-\mu}$ of order μ $(\mu > 0)$ for the function $f \in \mathcal{A}$ is defined by (see [9])

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int\limits_0^z \frac{f(t)}{(z-t)^{1-\mu}} \mathrm{d}t \quad (z \in \mathbb{U}),$$

where the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$. Also, the fractional derivative operator D_z^{λ} of order λ $(\lambda \geq 0)$ for the function $f \in \mathcal{A}$ is defined by

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int\limits_0^z \frac{f(t)}{(z-t)^{\lambda}} \mathrm{d}t \ (0 \leq \lambda < 1), \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) \ (n \leq \lambda < n+1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \end{cases}$$

where the multiplicity of $(z-t)^{-\lambda}$ is understood similarly.

Owa and Srivastava [10] introduced a fractional differintegral operator

$$\Omega_z^{\lambda} : \mathcal{A} \to \mathcal{A} \ (-\infty < \lambda < 2)$$

by

$$\Omega_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \ (z \in \mathbb{U}),$$

where $D_z^{\lambda} f(z)$ is, respectively, the fractional integral of order $\lambda(-\infty < \lambda < 0)$ and a fractional derivative of order λ $(0 \leq \lambda < 2)$. The operator Ω_z^{λ} for the function $f \in \mathcal{A}$ is given in the form of convolution by

$$\Omega_z^{\lambda} f(z) = z \ {}_2F_1 \ (2, 1; 2-\lambda; z) * f(z) \ (-\infty < \lambda < 2; z \in \mathbb{U}). \tag{1.6}$$

The Jung-Kim-Srivastava integral operator [7] $Q_{\gamma}^{\alpha} : \mathcal{A} \to \mathcal{A}$ ($\alpha > 0, \gamma > -1)$ is defined by

$$Q_{\gamma}^{\alpha} f(z) = \begin{pmatrix} \alpha+\gamma \\ \gamma \end{pmatrix} \frac{\alpha}{z^{\gamma}} \int\limits_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\gamma-1} f(t) \mathrm{d}t \quad (z \in \mathbb{U})$$

which can also be expressed as follows:

$$Q_{\gamma}^{\alpha} f(z) = z \ {}_2F_1 \ (\gamma+1, 1; \alpha+\gamma+1; z) * f(z). \tag{1.7}$$

The multiplier operator $\Im_{\lambda,\mu}^m : \mathcal{A} \to \mathcal{A}$, recently studied in [16] (see also [15, 18]) is defined for $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}, \mu > -1, \ \lambda > 0$, by

$$\mathcal{J}_{\lambda,\mu}^m f(z) = \begin{cases} f(z), & m = 0, \\ \frac{\mu+1}{\lambda} z^{1-\frac{\mu+1}{\lambda}} \int\limits_0^z t^{\frac{\mu+1}{\lambda}-2} \mathcal{J}_{\lambda,\mu}^{m+1} f(t) \mathrm{d}t, & m \in \mathbb{Z}^- = \{-1, -2, \ldots\}, \\ \frac{\lambda}{\mu+1} z^{2-\frac{\mu+1}{\lambda}} \frac{\mathrm{d}}{\mathrm{d}t} \left(z^{\frac{\mu+1}{\lambda}-1} \mathcal{J}_{\lambda,\mu}^{m-1} \ f(z) \right), & m \in \mathbb{Z}^+ = \{1, 2, \ldots\} \end{cases} \tag{1.8}$$

which may be given by

$$\mathcal{J}_{\lambda,\mu}^m f(z) = \Phi_{\lambda,\mu}^m (z) * f(z), \tag{1.9}$$

where

$$\Phi_{\lambda,\mu}^{m}(z) = \sum_{k=1}^{\infty} \left(1 + \frac{\lambda(k-1)}{\mu+1}\right)^{m} z^{k}.$$

Let f and g be analytic functions in the unit disc $\mathbb{U}$. Then we say that f is subordinate to g, and we write $f \prec g$ if there exists a function w analytic in unit disc $\mathbb{U}$, such that

$$w(0) = 0, \quad |w(z)| < 1 \;\; (z \in \mathbb{U})$$

and

$$f(z) = g(w(z)), \forall\; z \in \mathbb{U}.$$

In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In [6], Janowski introduced the class $\mathcal{S}^*[A,B]$ of functions $f \in \mathcal{A}$ satisfying the condition:

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}).$$

Geometrically, the above subordination condition means that the image of the unit disc $\mathbb{U}$ by the function $\frac{zf'(z)}{f(z)}$ is in the open disc whose endpoints of the diameter are $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$ (in case $B \neq -1$) and in the positive half plane $\Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1-A}{2}$ (in case $B = -1$).

For particular values of A, B, we get $S^*[1,-1] = S^*$, a class of starlike functions, $S^*[1-2\alpha,-1] = S^*(\alpha)$ $(0 \leq \alpha < 1)$, a class of starlike functions of order α; $S^*[1-\alpha, 0] = S^*_\alpha$ and $S^*[\alpha,-\alpha] = S^*[\alpha]$ (see [2]).

On using convolution, we define following subclasses of the class $\mathcal{A}$:

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*[\mu, g; A, B]$ if for $-1 \leq B < A \leq 1$, $\mu \geq -1$ and for some $g \in \mathcal{A}$ with $0 \neq \frac{(f*g)(z)}{z} \in \mathbb{C}$, it satisfies

$$\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) \prec \frac{1+Az}{1+Bz} \;\; (z \in \mathbb{U}), \tag{1.10}$$

where only principle values of the exponent function are considered.

Remark 1.2. Let $\mu = 0$ and $g(z) = \frac{z}{1-z}$ $(z \in \mathbb{U})$, we get $\mathcal{S}^*[\mu, g; A, B] = \mathcal{S}^*[A,B]$.

Remark 1.3. If we put $\mu = 0$, $g(z) = \frac{z}{(1-z)^2}$ $(z \in \mathbb{U})$ and $A = 1-2\alpha, B = -1$ in $\mathcal{S}^*[\mu, g; A, B]$ then we obtain the class $\mathcal{K}(\alpha)$ convex functions of order α studied by Robertson [17].

Definition 1.4. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{B}(g, \mu; \beta)$ if for $\frac{1}{2} < \beta \leq 1$, $\mu \geq -1$ and for some $g \in \mathcal{A}$ with $0 \neq \frac{(f*g)(z)}{z} \in \mathbb{C}$, it satisfies

$$\left|\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) - \beta\right| < \beta \;\; (z \in \mathbb{U}), \tag{1.11}$$

where only principle values of the exponent function are considered.

Example 1.5. The following example $\mu = 0$, $f(z) = \frac{z}{(1-z)^2}$ and $g(z) = \frac{z}{1-z}$ satisfies the conditions of Definitions 1.1 and 1.4.

In particular, $\mathcal{S}^*[\mu, g; 1, 0] \equiv \mathcal{B}(g, \mu; 1)$.

Remark 1.6. If $\beta = 1$ and $\mu = 1$, the class condition (1.11) for the class $\mathcal{B}(g, \mu; \beta)$ provides a univalency criterion for the functions $f*g$ according to Ozaki and Nunokawa [11], see also [1, 4].

In this paper, for a certain function $g \in \mathcal{A}$, involving a convolution $f * g$, two classes $\mathcal{S}^*[\mu, g; A, B]$ and $\mathcal{B}(g, \mu; \beta)$ of $f \in \mathcal{A}$, are defined. Showing an inclusion relation between these classes, various sufficient conditions for functions to be in these classes are established. In particular, varied sufficient conditions for univalency of the convolution function $f * g$ are given which lead to the univalency conditions of various known linear operators.

2. Main results

We first prove an inclusion result for the classes $\mathcal{S}^*[\mu, g; A, B]$ and $\mathcal{B}(g, \mu; \beta)$ which is as follows:

Theorem 2.1. *Let $f \in \mathcal{A}$ and $0 \leq B < A \leq 1$, $\frac{1}{2} < \beta \leq 1$ be such that*

$$A \leq 2B(1-\beta) + 2\beta - 1. \tag{2.1}$$

Let the classes $\mathcal{S}^*[\mu, g; A, B]$ and $\mathcal{B}(g, \mu; \beta)$ be defined, respectively, by Definitions 1.1 and 1.4.Then

$$\mathcal{S}^*[\mu, g; A, B] \subset \mathcal{B}(g, \mu; \beta).$$

Proof. If $f \in \mathcal{S}^*[\mu, g; A, B]$, then there is a Schwarz function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that

$$\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}). \tag{2.2}$$

Hence, for the given hypotheses (2.1) and for this Schwarz function w given by (2.2), we get

$$\left|\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) - \beta\right| = \left|1 + \frac{(A-B)\,w(z)}{1 + Bw(z)} - \beta\right|$$

$$< \quad 1 + \frac{A-B}{1-B} - \beta \leq \beta$$

which implies that $f \in \mathcal{B}(g, \mu; \beta)$. This proves Theorem 2.1. □

Example 2.2. The following example $\mu = 0$, $f(z) = z + \frac{z^2}{2}$ and $g(z) = \frac{z}{1-z}$ satisfies the condition of Theorem 2.1.

In view of Remark 1.6, for $\beta = 1$ and $\mu = 1$, Theorem 2.1 provides following univalency condition for the convolution $f * g$:

Corollary 2.3. *Let* $f \in \mathcal{A}$ *and let for some* $g \in \mathcal{A}$ *with* $0 \neq \frac{(f*g)(z)}{z} \in \mathbb{C}$,

$$\left(\frac{z}{(f*g)(z)}\right)^2 (f*g)'(z) \prec \frac{1+Az}{1+Bz} \ \ (0 \leq B < A \leq 1; z \in \mathbb{U}).$$

Then $f*g$ *is univalent in* $\mathbb{U}$.

Now, we prove certain sufficient conditions for functions to be in the class $\mathcal{S}^*[\mu, g; A, B]$, for this, we apply the method of admissible function used in the following lemma which is the special case of the result [8, (ii) Theorem 2.3h, p. 34].

Lemma 2.4. [8, (ii) Theorem 2.3h, p. 34] *Let* Ω *be a subset of the complex plane* $\mathbb{C}$ *and let an admissible function* $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ *satisfies the condition*

$$\psi(Me^{i\theta}, mMe^{i\theta}; z) \notin \Omega$$

for real $M > 0$ *and* $m \geq k \geq 1$ *and* $z \in \mathbb{U}$. *If the function* $w \in \mathcal{H}[a,k]$, *then*

$$\psi(w(z), zw'(z); z) \in \Omega \Rightarrow |w(z)| < M \ \ (z \in \mathbb{U}).$$

Theorem 2.5. *Let* $f \in \mathcal{A}$ *and let for some* $\theta \in \mathbb{R}, m \geq 1, -1 \leq B < A \leq 1$,

$$\left|1 + \frac{(A-B)\, me^{i\theta}}{(1+Ae^{i\theta})(1+Be^{i\theta})}\right| \geq 1. \tag{2.3}$$

If for some $g \in \mathcal{A}$ *with* $0 \neq (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C}$ *in* $\mathbb{U}$,

$$\left|1 + \frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right| < 1, \tag{2.4}$$

then $f \in \mathcal{S}^*[\mu, g; A, B]$.

Proof. Let

$$p(z) = \left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) \tag{2.5}$$

and $w \in \mathcal{H}[0,1]$ be defined by

$$p(z) = \frac{1+Aw(z)}{1+Bw(z)} \ \ (z \in \mathbb{U}), \tag{2.6}$$

then w is analytic in $\mathbb{U}$. To prove the theorem we only need to prove $|w(z)| < 1$. For this purpose, we define an admissible function $\Psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ by

$$\Psi(r, s; z) = 1 + \frac{(A-B)\, s}{(1+Ar)(1+Br)} \ \ (-1 \leq B < A \leq 1), \tag{2.7}$$

where $r \neq -\frac{1}{A}, -\frac{1}{B}$ (in case $A, B \neq 0$). Then, from (2.3), we have

$$\left|\Psi\left(e^{i\theta}, me^{i\theta}; z\right)\right| \geq 1. \tag{2.8}$$

Differentiating equations (2.6) and (2.5) logarithmically, we obtain

$$\begin{aligned} 1 + \frac{zp'(z)}{p(z)} &= 1 + \frac{(A-B)\, zw'(z)}{(1+Aw(z))(1+Bw(z))} \\ &= 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}. \end{aligned} \tag{2.9}$$

Let Ω be a subset of the complex plane $\mathbb{C}$ such that in $\mathbb{C}\setminus\Omega$, the admissible function Ψ satisfies (2.8). Hence, Lemma 2.4 for the case $M=1$ reveals in view of (2.7), (2.9) and (2.4), that

$$|\Psi(w(z), zw'(z); z)| < 1 \Rightarrow |w(z)| < 1 \ \ (z \in \mathbb{U}),$$

which proves that

$$p(z) \prec \frac{1+Az}{1+Bz},$$

and hence $f \in \mathcal{S}^*[\mu, g; A, B]$. □

Theorem 2.6. *Let $f \in \mathcal{A}$ and $-1 \le B < A \le 1$, let*

$$\lambda = \begin{cases} \frac{2\sqrt{A|B|}}{1-AB}, & \text{if } AB < 0 \text{ with } \left|(A+B)\left(1+\frac{1}{AB}\right)\right| \le 4, \\ \frac{A-B}{(1+|A|)(1+|B|)}, & \text{if } AB \ge 0. \end{cases} \tag{2.10}$$

If for some $g \in \mathcal{A}$ with $0 \ne (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C}$ in $\mathbb{U}$,

$$\left| \frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\} \right| < \lambda, \tag{2.11}$$

then $f \in \mathcal{S}^*[\mu, g; A, B]$.

Proof. To prove the result, we define an admissible function $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ by

$$\phi(r, s; z) = \Psi(r, s; z) - 1, \tag{2.12}$$

where $\Psi(r, s; z)$ is defined by (2.7). Then for some $\theta \in \mathbb{R}$ and for some $m \ge 1$

$$\begin{aligned} \left|\phi\left(e^{i\theta}, me^{i\theta}; z\right)\right| &= \left|\frac{(A-B)\, me^{i\theta}}{(1+Ae^{i\theta})(1+Be^{i\theta})}\right| \\ &= \frac{(A-B)\, m}{|(1+Ae^{i\theta})|\,|(1+Be^{i\theta})|} \\ &= \frac{(A-B)\, m}{\sqrt{1+A^2+2At} \cdot \sqrt{1+B^2+2Bt}} \\ &= \frac{(A-B)\, m}{h(t)}, \end{aligned}$$

where $t = \cos\theta \in [-1, 1]$. Observe that

$$\max_{-1 \le t \le 1} h(t) = \begin{cases} (1+A)(1+B), & \text{if } 0 \le B < A \le 1, \\ (1-A)(1-B), & \text{if } -1 \le B < A \le 0, \end{cases}$$

Hence,

$$\left|\phi\left(e^{i\theta}, me^{i\theta}; z\right)\right| \ge \frac{A-B}{(1+|A|)(1+|B|)}, \text{ if } AB \ge 0.$$

Further, if $-1 \le B < 0 < A \le 1$, i.e. if $AB < 0$, then the function $h(t)$ attains its maximum value at

$$t^* = -\frac{(A+B)(1+AB)}{4AB} \in [-1, 1].$$

Hence, if $AB<0$ with the condition: $4AB \le (A+B)(1+AB) \le -4AB$ or equivalently, $\left|(A+B)\left(1+\frac{1}{AB}\right)\right| \le 4$,

$$h(t^*) = \frac{(A-B)(1-AB)}{2\sqrt{A|B|}}.$$

So,

$$\left|\phi\left(e^{i\theta}, me^{i\theta}; z\right)\right| \ge \frac{2\sqrt{A|B|}}{1-AB}, \text{ if } AB<0 \text{ with } \left|(A+B)\left(1+\frac{1}{AB}\right)\right| \le 4.$$

Hence,

$$\left|\phi\left(e^{i\theta}, me^{i\theta}; z\right)\right| \ge \lambda, \tag{2.13}$$

where λ is given by (2.10). Thus, in view of (2.12) and for $p(z)$ defined by (2.5), we get from (2.9),

$$\begin{aligned} \left|\phi\left(w(z), zw'(z); z\right)\right| &= \left|\frac{zp'(z)}{p(z)}\right| \\ &= \left|\frac{z\left(f*g\right)^{"}(z)}{\left(f*g\right)'(z)} + (\mu+1)\left\{1-\frac{z\left(f*g\right)'(z)}{\left(f*g\right)(z)}\right\}\right|. \end{aligned} \tag{2.14}$$

Let Λ be a subset of the complex plane $\mathbb{C}$ such that in $\mathbb{C}\setminus\Lambda$, the admissible function ϕ satisfies (2.13). Hence, applying Lemma 2.4 (in case $M=1$), from (2.14) and (2.11)

$$\left|\phi\left(w(z), zw'(z); z\right)\right| < \lambda \Rightarrow |w(z)| < 1 \ \left(z\in\mathbb{U}\right),$$

which proves

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

This establishes Theorem 2.6. □

From Theorem 2.1 and Theorem 2.6, we obtain following result.

Corollary 2.7. *Let $f\in\mathcal{A}$ and $0\le B<A\le 1$, $\frac{1}{2}<\beta\le 1$ be such that*

$$A \le 2B\left(1-\beta\right)+2\beta-1. \tag{2.15}$$

If for some $g\in\mathcal{A}$ with $\frac{(f*g)(z)}{z}\neq 0$ in $\mathbb{U}$ and for $\mu\ge -1$,

$$\left|\frac{z\left(f*g\right)''(z)}{\left(f*g\right)'(z)} + (\mu+1)\left\{1-\frac{z\left(f*g\right)'(z)}{\left(f*g\right)(z)}\right\}\right| < \frac{A-B}{(1+A)(1+B)} \ \left(z\in\mathbb{U}\right), \tag{2.16}$$

then $f\in\mathcal{B}\left(g,\mu;\beta\right)$.

Proof. Applying Theorem 2.6 for $0\le B<A\le 1$, we get $f\in\mathcal{S}^*\left[\mu,g;A,B\right]$ if and condition (2.16) holds, and from Theorem 2.1, $\mathcal{S}^*\left[\mu,g;A,B\right]\subset\mathcal{B}\left(g,\mu;\beta\right)$ if (2.15) holds. Hence, this proves the result. □

Example 2.8. The following example $\mu=0$, $f(z)=z+\frac{z^n}{n}$ and $g\left(z\right)=\frac{z}{1-z}$ satisfies the condition of Corollary 2.7.

Again, in view of the Remark 1.6, for $\beta=1$ and $\mu=1$, above Corollary 2.7 provides the following univalency condition for the convolution $f*g$:

Corollary 2.9. *Let $f \in \mathcal{A}$ and $0 \le B < A \le 1$. If for some $g \in \mathcal{A}$ with*

$$0 \neq (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U},$$

$$\left| \frac{z(f*g)''(z)}{(f*g)'(z)} + 2\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\} \right| < \frac{A-B}{(1+A)(1+B)} \quad (z \in \mathbb{U}),$$

then $f*g$ is univalent in $\mathbb{U}$.

Also, for the special values: $A = 1-2\alpha$ $\left(0 \le \alpha < \frac{1}{2}\right)$ and $B = -1$, Theorem 2.6 provides following result:

Corollary 2.10. *Let $f \in \mathcal{A}$ and $0 \le \alpha < \frac{1}{2}$, $\mu \ge -1$. If for some $g \in \mathcal{A}$ with*

$$0 \neq (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U},$$

$$\left| \frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\} \right| < \frac{\sqrt{1-2\alpha}}{1-\alpha} \quad (z \in \mathbb{U}),$$

then

$$\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in \mathbb{U}).$$

In our next result we give some more sufficient conditions for the class $\mathcal{S}^*[\mu, g; A, B]$ in case $B = 0$.

Theorem 2.11. *Let $f \in \mathcal{A}$ and let $0 < A \le 1$. If for some $g \in \mathcal{A}$ with*

$$0 \neq (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C}$$

in $\mathbb{U}$, any one of the following conditions holds

$$\begin{aligned} &\left| \left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) \right. \\ &\left. \left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right] \right| \\ < \;& A \quad (z \in \mathbb{U}), \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\left| \left(\frac{(f*g)(z)}{z}\right)^{\mu+1} \frac{1}{(f*g)'(z)} \right. \\ &\left. \left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right] \right| \\ < \;& \frac{A}{(1+A)^2} \quad (z \in \mathbb{U}), \end{aligned} \tag{2.18}$$

$$\left| \frac{\left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right]}{\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) - 1} \right| < \frac{1}{1+A} \quad (z \in \mathbb{U}), \tag{2.19}$$

then $f \in \mathcal{S}^*[\mu, g; A, 0]$.

Proof. Let $p(z)$ be defined by (2.5). Then $p \in \mathcal{H}[1,1]$ and by the hypothesis $0 \neq p(z) \in \mathbb{C}$ in $\mathbb{U}$. Then from (2.9), we obtain

$$zp'(z) = \left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) \quad \times \left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right], \tag{2.20}$$

$$\frac{zp'(z)}{(p(z))^2} = \left(\frac{(f*g)(z)}{z}\right)^{\mu+1} \frac{1}{(f*g)'(z)} \quad \times \left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right], \tag{2.21}$$

and

$$\frac{zp'(z)}{p(z)(p(z)-1)} = \frac{\left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right]}{\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) - 1}, \tag{2.22}$$

where in (2.22) the singularity of the function at $z = 0$, is being removed by the numerator. To prove the result, we use the similar method used in the above proofs of Theorems 2.5 and 2.6 for the case if $B = 0$. Let $u(z)$ be defined by

$$p(z) = 1 + Au(z). \tag{2.23}$$

Then $u(0) = 0$ and now we prove $|u(z)| < 1$ in $\mathbb{U}$. For this, we may define admissible function $\eta_i : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ for each $i = 1, 2, 3$, by

$$\eta_1(r, s; z) = As, \tag{2.24}$$

$$\eta_2(r, s; z) = \frac{As}{(1+Ar)^2} \quad \left(r \neq -\frac{1}{A}\right),$$

and

$$\eta_3(r, s; z) = \frac{s}{r(1+Ar)} \quad \left(r \neq 0, -\frac{1}{A}\right).$$

Then for some $\theta \in \mathbb{R}$ and for some $m \geq 1$,

$$\left|\eta_1\left(e^{i\theta}, me^{i\theta}; z\right)\right| = Am \geq A, \tag{2.25}$$

$$\left|\eta_2\left(e^{i\theta}, me^{i\theta}; z\right)\right| = \frac{Am}{|1+Ae^{i\theta}|^2} \geq \frac{A}{(1+A)^2}, \tag{2.26}$$

and

$$|\eta_3(r, s; z)| = \frac{m}{|1+Ae^{i\theta}|} \geq \frac{1}{1+A}. \tag{2.27}$$

Then from (2.23)

$$\begin{aligned} zp'(z) &= zAu'(z), &(2.28)\\ \frac{zp'(z)}{(p(z))^2} &= \frac{zAu'(z)}{(1+Au(z))^2}, &(2.29)\\ \frac{zp'(z)}{p(z)(p(z)-1)} &= \frac{zu'(z)}{(1+Au(z))u(z)}. &(2.30)\end{aligned}$$

Let for each $i = 1, 2, 3$, Ω_i be a subset of the complex plane $\mathbb{C}$ such that in $\mathbb{C}\setminus\Omega_i$, the admissible function η_i satisfies for each $i = 1, 2, 3$, the conditions, (2.25), (2.26) and (2.27). Hence, by Lemma 2.4 for $M = 1$, in view of (2.20), (2.21) and (2.22), from the conditions (2.17), (2.18) and (2.19) and using the values (2.28), (2.29) and (2.30), we get

$$\begin{aligned} |\eta_1(u(z), zu'(z); z)| &= |zp'(z)| < A \Rightarrow |u(z)| < 1,\\ |\eta_2(u(z), zu'(z); z)| &= \left|\frac{zp'(z)}{(p(z))^2}\right| < \frac{A}{(1+A)^2} \Rightarrow |u(z)| < 1,\\ |\eta_3(u(z), zu'(z); z)| &= \left|\frac{zp'(z)}{p(z)(p(z)-1)}\right| < \frac{1}{1+A} \Rightarrow |u(z)| < 1.\end{aligned}$$

This proves the Theorem 2.11. □

Using Theorem 2.1 for the case $B = 0$, we obtain following result from Theorem 2.11.

Corollary 2.12. *Let $f \in \mathcal{A}$ and let $\frac{1}{2} < \beta \le 1, 0 < A \le 2\beta - 1$. If for some $g \in \mathcal{A}$ with $0 \ne (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C}$ in $\mathbb{U}$, any one of the conditions (2.17), (2.18) and (2.19) in Theorem 2.11 holds, then $f \in \mathcal{B}(g, \mu; \beta)$.*

In addition to the Corollaries 2.3 and 2.9, Corollary 2.12 provides for $\beta = 1$ and $\mu = 1$, the following univalency condition for the convolution function $f * g$.

Corollary 2.13. *Let $f \in \mathcal{A}$ and let $0 < A \le 1$. If for some $g \in \mathcal{A}$ with*

$$0 \ne (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U},$$

$f * g$ satisfies any one of the following conditions:

$$\left|\frac{z^2(f*g)'(z)}{((f*g)(z))^2}\left(\frac{z(f*g)''(z)}{(f*g)'(z)} + 2\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right)\right| < A \;\; (z \in \mathbb{U}),$$

$$\left|\frac{((f*g)(z))^2}{z^2(f*g)'(z)}\left(\frac{z(f*g)''(z)}{(f*g)'(z)} + 2\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right)\right| < \frac{A}{(1+A)^2} \;\; (z \in \mathbb{U}),$$

$$\left|\frac{\frac{z(f*g)''(z)}{(f*g)'(z)} + 2\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}}{\frac{z^2(f*g)'(z)}{((f*g)(z))^2} - 1}\right| < \frac{1}{1+A} \quad (z \in \mathbb{U}),$$

then $f * g$ is univalent in $\mathbb{U}$.

Remark 2.14. For $A = 1$ and $g(z) = \frac{z}{1-z}$ $(z \in \mathbb{U})$, above Corollary 229 coincides with the result [13, Corollary 3.2, p.361] for a function $f \in \mathcal{A}$, and with the result [14, Theorem 1, p. 2135] for the function $f(z) = K_{\nu,c}(z)$, where $K_{\nu,c}(z)$ is a normalized form of generalized Bessel function defined in [3, 21] by

$$K_{\nu,c}(z) = z + \sum_{k=2}^{\infty} \frac{\left(\frac{-c}{4}\right)^{k-1}}{(\nu+1)_{k-1}} \frac{z^k}{(k-1)!} \; (c \in \mathbb{C},\ \nu \in \mathbb{R},\ \nu \neq -1, -2, ...; z \in \mathbb{U}) .$$

3. Concluding remark

By considering special form of the function g, from our main results, we may obtain results involving several linear operators of the class $\mathcal{A}$, some of the known linear operators are mentioned in the *Introduction* section. We give here only the results giving varied univalency conditions of the Dzoik-Srivastava operator by taking $g(z) = z\ {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z)$. Results giving univalency conditions of other linear operators mentioned in the *Introduction* section may similarly be obtained by taking $g(z) = G_{a,b}(z)$, $z\ {}_2F_1\ (2, 1; 2-\lambda; z)$, $z\ {}_2F_1\ (\gamma+1, 1; \alpha+\gamma+1; z)$ and $\Phi^m_{\lambda,\mu}(z)$, respectively, in the Corollaries 2.3, 2.9 and 2.13.

Corollary 3.1. *Let $f \in \mathcal{A}$ and ${}_pH_q([\alpha_1]) f$ be defined by (1.4) with*

$$0 \neq \frac{{}_pH_q([\alpha_1]) f(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U}.$$

If

$$\left(\frac{z}{{}_pH_q([\alpha_1]) f(z)}\right)^2 \left({}_pH_q([\alpha_1]) f\right)'(z) \prec \frac{1+Az}{1+Bz} \quad (0 \leq B < A \leq 1; z \in \mathbb{U}),$$

then ${}_pH_q([\alpha_1]) f$ is univalent in $\mathbb{U}$.

Corollary 3.2. *Let $f \in \mathcal{A}$ and ${}_pH_q([\alpha_1]) f$ be defined by (1.4) with*

$$0 \neq \left({}_pH_q([\alpha_1]) f\right)'(z) \cdot \frac{{}_pH_q([\alpha_1]) f(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U}.$$

If

$$\left|\frac{z\left({}_pH_q([\alpha_1]) f\right)''(z)}{\left({}_pH_q([\alpha_1]) f\right)'(z)} + 2\left\{1 - \frac{z\left({}_pH_q([\alpha_1]) f\right)'(z)}{{}_pH_q([\alpha_1]) f(z)}\right\}\right| < \frac{A-B}{(1+A)(1+B)}$$

$$(0 \leq B < A \leq 1; z \in \mathbb{U}),$$

then ${}_pH_q([\alpha_1]) f$ is univalent in $\mathbb{U}$.

Corollary 3.3. *Let $f \in \mathcal{A}$ and ${}_pH_q([\alpha_1]) f$ be defined by (1.4) with*

$$0 \neq \left({}_pH_q([\alpha_1]) f\right)'(z) \cdot \frac{{}_pH_q([\alpha_1]) f(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U}.$$

If for $0 < A \leq 1; z \in \mathbb{U}$, any one of the following conditions:

$$\left|\frac{z^2\left({}_pH_q([\alpha_1]) f\right)'(z)}{\left({}_pH_q([\alpha_1]) f(z)\right)^2}\left(\frac{z\left({}_pH_q([\alpha_1]) f\right)''(z)}{\left({}_pH_q([\alpha_1]) f\right)'(z)} + 2\left\{1 - \frac{z\left({}_pH_q([\alpha_1]) f\right)'(z)}{{}_pH_q([\alpha_1]) f(z)}\right\}\right)\right| < A,$$

$$\left| \frac{\left({}_pH_q\left([\alpha_1]\right) f(z)\right)^2}{z^2 \left({}_pH_q\left([\alpha_1]\right) f\right)'(z)} \left(\frac{z\left({}_pH_q\left([\alpha_1]\right) f\right)''(z)}{\left({}_pH_q\left([\alpha_1]\right) f\right)'(z)} + 2\left\{1 - \frac{z\left({}_pH_q\left([\alpha_1]\right) f\right)'(z)}{{}_pH_q\left([\alpha_1]\right) f(z)}\right\}\right)\right|$$

$$< \frac{A}{(1+A)^2},$$

$$\left| \frac{\frac{z({}_pH_q([\alpha_1])f)''(z)}{({}_pH_q([\alpha_1])f)'(z)} + 2\left\{1 - \frac{z({}_pH_q([\alpha_1])f)'(z)}{{}_pH_q([\alpha_1])f(z)}\right\}}{\frac{z^2({}_pH_q([\alpha_1])f)'(z)}{({}_pH_q([\alpha_1])f(z))^2} - 1} \right| < \frac{1}{1+A},$$

holds, then ${}_pH_q\left([\alpha_1]\right) f$ *is univalent in* $\mathbb{U}$.

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Existence results for a coupled system of higher-order nonlinear differential equations with integral-multipoint boundary conditions

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Abstract. In this paper, we establish the existence and uniqueness criteria for solutions of an integral-multipoint coupled boundary value problem involving a system of nonlinear higher-order ordinary differential equations. We apply the Leray-Schauder's alternative to prove an existence result for the given problem, while the uniqueness of its solutions is accomplished with the aid of Banach's fixed point theorem. Examples are constructed for illustrating the obtained results.

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1. Introduction

The topic of boundary value problems is an important area of investigation in view of extensive occurrence of such problems in several diverse disciplines. Examples include conservation laws [8], nano boundary layer fluid flow [4], magnetohydrodynamic flow [18], magneto Maxwell nano-material [19], fluid flow problems [28], cellular systems and aging models [1], etc.

Much of the literature on boundary value problems includes classical boundary conditions. However, these conditions cannot model the physical and chemical processes taking place within the given domain. In order to cope with this situation, the concept of nonlocal conditions representing the changes happening at some interior

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points or sub-segments of the given domain was introduced. One can find the details and applications of nonlocal boundary conditions in the articles [14, 11, 22, 16, 24, 15] and the references cited therein.

Integral boundary conditions serve as an effective tool in the mathematical modeling of the problems arising in the flow and drag phenomena in arteries [27], heat conduction [9, 20, 10], biomedical CFD [23], etc. In fact, these conditions provide a practical approach to fluid flow problems with arbitrary shaped blood vessels, for instance, see [25]. For the boundary value problems involving integral boundary conditions, for instance, see the papers [26, 21, 3, 7, 2, 6, 5, 13].

In [3], the authors obtained some existence results for nth-order ordinary differential equations and inclusions supplemented with nonlocal multi-point integral boundary conditions:

$$\begin{cases} u^{(n)}(t) = f(t, u(t)),\ u^{(n)}(t) \in F(t, u(t))), t \in [0,1], \\ u(0) = \delta \displaystyle\int_0^{\xi} u(s)ds,\ u'(0) = 0,\ u''(0) = 0, \dots, u^{(n-2)}(0) = 0, \\ \alpha u(1) + \beta u'(1) = \displaystyle\sum_{i=1}^{m} \gamma_i \int_0^{\beta_i} u(s)ds,\ 0 < \xi < \beta_1 < \beta_2 < \dots < \beta_m < 1, \end{cases}$$

where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, $F : [0,1] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\alpha,\ \beta,\ \gamma_i,\ \delta,\ \xi,\ \beta_i\ (i = 1, 2, \dots, m)$ are appropriately chosen real constants.

In this paper, motivated by [3], we formulate and investigate a boundary value problem for a coupled system of higher-order nonlinear differential equations complemented with coupled integral-multipoint boundary conditions given by

$$\begin{cases} u^{(n)}(t) = f(t, u, v), \qquad v^{(m)}(t) = g(t, u, v), \qquad t \in [0, 1], \\ u(0) = \delta_1 \displaystyle\int_0^{\xi} v(s)\, ds, \qquad u'(0) = 0, \quad u''(0) = 0, \dots, \quad u^{(n-2)}(0) = 0, \\ v(0) = \delta_2 \displaystyle\int_0^{\xi} u(s)\, ds, \qquad v'(0) = 0, \quad v''(0) = 0, \dots, \quad v^{(m-2)}(0) = 0, \\ \epsilon_1 u(1) + \zeta_1 u'(1) = \displaystyle\sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} v(s)ds + \sum_{j=1}^{q} \omega_j v(\eta_j), \\ \epsilon_2 v(1) + \zeta_2 v'(1) = \displaystyle\sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} u(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} u(\eta_j), \end{cases} \tag{1.1}$$

where $0 < \xi < \beta_1 < \beta_2 < \dots < \beta_p < \eta_1 < \eta_2 < \dots < \eta_q < 1$, $\delta_1, \delta_2, \epsilon_1, \epsilon_2, \zeta_1, \zeta_2, \gamma_i, \widehat{\gamma_i}, \omega_j, \widehat{\omega_j} \in \mathbb{R}$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$ and $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.

The objective of the present work is to develop the existence theory for the problem (1.1) by applying the standard fixed point theorems. The outcome of the proposed work will be a useful contribution to the existing literature on nonlinear differential systems supplemented with coupled nonlocal integral boundary conditions.

The rest of the paper is arranged as follows. In Section 2, we prove an auxiliary lemma related to the linear variant of the problem (1.1). The main results for the given problem are proved in Section 3. Section 4 contains examples illustrating the main results.

2. An auxiliary lemma

In the following lemma, we solve a linear variant of the system (1.1) and use it to convert the problem (1.1) into a fixed point problem.

Lemma 2.1. *Let* $(J_1K_2 - J_2K_1) \neq 0$, $(1 - \delta_1\delta_2\xi^2) \neq 0$ *and* $y_1, y_2 \in C([0,1],\mathbb{R})$. *Then, the linear boundary value problem*

$$\begin{cases} u^{(n)}(t) = y_1(t), \qquad v^{(m)}(t) = y_2(t), \quad t \in [0,1], \\ u(0) = \delta_1 \int_0^{\xi} v(s)\,ds, \qquad u'(0) = 0, \quad u''(0) = 0, \ldots, \quad u^{(n-2)}(0) = 0, \\ v(0) = \delta_2 \int_0^{\xi} u(s)\,ds, \qquad v'(0) = 0, \quad v''(0) = 0, \ldots, \quad v^{(m-2)}(0) = 0, \\ \epsilon_1 u(1) + \zeta_1 u'(1) = \sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} v(s)ds + \sum_{j=1}^{q} \omega_j v(\eta_j), \\ \epsilon_2 v(1) + \zeta_2 v'(1) = \sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} u(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} u(\eta_j), \end{cases} \tag{2.1}$$

is equivalent to a pair of integral equations

$$\begin{aligned} u(t) \;=\; & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y_1(s)ds + N_1(t) \int_0^{\xi} \frac{(\xi - s)^m}{m!} y_2(s)ds \\ & + N_2(t) \int_0^{\xi} \frac{(\xi - s)^n}{n!} y_1(s)ds \\ & + N_3(t)\Big[\sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} \frac{(\beta_i - s)^m}{m!} y_2(s)ds + \sum_{j=1}^{q} \omega_j \int_0^{\eta_j} \frac{(\eta_j - s)^{m-1}}{(m-1)!} y_2(s)ds \\ & - \int_0^1 \frac{(1-s)^{n-2}}{(n-1)!} [\epsilon_1(1-s) + \zeta_1(n-1)] y_1(s)ds\Big] \\ & + N_4(t)\Big[\sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} \frac{(\beta_i - s)^n}{n!} y_1(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} \int_0^{\eta_j} \frac{(\eta_j - s)^{n-1}}{(n-1)!} y_1(s)ds \\ & - \int_0^1 \frac{(1-s)^{m-2}}{(m-1)!} [\epsilon_2(1-s) + \zeta_2(m-1)] y_2(s)ds\Big], \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v(t) &= \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} y_2(s)ds + N_5(t) \int_0^\xi \frac{(\xi-s)^m}{m!} y_2(s)ds \\ &+ N_6(t) \int_0^\xi \frac{(\xi-s)^n}{n!} y_1(s)ds \\ &+ N_7(t)\Big[\sum_{i=1}^p \gamma_i \int_0^{\beta_i} \frac{(\beta_i-s)^m}{m!} y_2(s)ds + \sum_{j=1}^q \omega_j \int_0^{\eta_j} \frac{(\eta_j-s)^{m-1}}{(m-1)!} y_2(s)ds \\ &- \int_0^1 \frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)] y_1(s)ds\Big] \\ &+ N_8(t)\Big[\sum_{i=1}^p \widehat{\gamma_i} \int_0^{\beta_i} \frac{(\beta_i-s)^n}{n!} y_1(s)ds + \sum_{j=1}^q \widehat{\omega_j} \int_0^{\eta_j} \frac{(\eta_j-s)^{n-1}}{(n-1)!} y_1(s)ds \\ &- \int_0^1 \frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)] y_2(s)ds\Big], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} N_1(t) &= \Delta_1 + \Delta_5 t^{n-1}, \quad N_2(t) = \Delta_2 + \Delta_6 t^{n-1}, \quad N_3(t) = \Delta_3 + \Delta_7 t^{n-1}, \\ N_4(t) &= \Delta_4 + \Delta_8 t^{n-1}, \quad N_5(t) = \Delta_9 + \Delta_{13} t^{m-1}, \quad N_6(t) = \Delta_{10} + \Delta_{14} t^{m-1}, \\ N_7(t) &= \Delta_{11} + \Delta_{15} t^{m-1}, \quad N_8(t) = \Delta_{12} + \Delta_{16} t^{m-1}, \end{aligned}$$

$$\begin{aligned} \Delta_1 &= \rho_1 + \frac{\rho_4(n\delta_1\xi^m K_1 + m\delta_1\delta_2\xi^{n+1}K_2) - \rho_6(n\delta_1\xi^m J_1 + m\delta_1\delta_2\xi^{n+1}J_2)}{M(1-\delta_1\delta_2\xi^2)mn}, \\ \Delta_2 &= \rho_2 + \frac{\rho_5(n\delta_1\xi^m K_1 + m\delta_1\delta_2\xi^{n+1}K_2) - \rho_7(n\delta_1\xi^m J_1 + m\delta_1\delta_2\xi^{n+1}J_2)}{M(1-\delta_1\delta_2\xi^2)n}, \\ \Delta_3 &= \frac{(n\delta_1\xi^m K_1 + m\delta_1\delta_2\xi^{n+1}K_2)}{M(1-\delta_1\delta_2\xi^2)mn}, \quad \Delta_4 = \frac{(n\delta_1\xi^m J_1 + m\delta_1\delta_2\xi^{n+1}J_2)}{M(1-\delta_1\delta_2\xi^2)mn}, \\ \Delta_5 &= \frac{(\rho_4 K_2 - \rho_6 J_2)}{M}, \quad \Delta_6 = \frac{(\rho_5 K_2 - \rho_7 J_2)}{M}, \quad \Delta_7 = \frac{K_2}{M}, \quad \Delta_8 = \frac{J_2}{M}, \\ \Delta_9 &= \rho_3 + \frac{\rho_4(n\delta_1\delta_2\xi^{m+1}K_1 + m\delta_1\xi^n K_2) - \rho_6(n\delta_1\delta_2\xi^{m+1}J_1 + m\delta_1\xi^n J_2)}{M(1-\delta_1\delta_2\xi^2)mn}, \\ \Delta_{10} &= \rho_1 + \frac{\rho_5(n\delta_1\delta_2\xi^{m+1}K_1 + m\delta_1\xi^n K_2) - \rho_7(n\delta_1\delta_2\xi^{m+1}J_1 + m\delta_1\xi^n J_2)}{M(1-\delta_1\delta_2\xi^2)mn}, \end{aligned}$$

$$
\begin{aligned}
\Delta_{11} &= \frac{(n\delta_1\delta_2\xi^{m+1}K_1+m\delta_1\xi^n K_2)}{M(1-\delta_1\delta_2\xi^2)mn}, \quad \Delta_{12}=\frac{(n\delta_1\delta_2\xi^{m+1}J_1+m\delta_1\xi^n J_2)}{M(1-\delta_1\delta_2\xi^2)mn},\\
\Delta_{13} &= \frac{(\rho_4K_1-\rho_6J_1)}{M}, \quad \Delta_{14}=\frac{(\rho_5K_1-\rho_7J_1)}{M}, \quad \Delta_{15}=\frac{K_1}{M}, \quad \Delta_{16}=\frac{J_1}{M},\\
\rho_1 &= \frac{1}{1-\delta_1\delta_2\xi^2}, \rho_2=\frac{A_2}{1-\delta_1\delta_2\xi^2}, \rho_3=\frac{B_2}{1-\delta_1\delta_2\xi^2}, \rho_4=\frac{B_2D_1-C_1}{1-\delta_1\delta_2\xi^2},\\
\rho_5 &= \frac{D_1-A_2C_1}{1-\delta_1\delta_2\xi^2}, \rho_6=\frac{B_2E_1-F_1}{1-\delta_1\delta_2\xi^2}, \rho_7=\frac{E_1-A_2F_1}{1-\delta_1\delta_2\xi^2},\\
J_1 &= \frac{\epsilon_1\delta_1\delta_2\xi^{n+1}-\delta_2\xi^n D_1}{n(1-\delta_1\delta_2\xi^2)}+\epsilon_1+\zeta_1(n+1),\\
J_2 &= \frac{\delta_1\xi^m(\delta_2\xi D_1-\epsilon_1)}{m(1-\delta_1\delta_2\xi^2)}+D_2, \; K_1=\frac{\delta_1\delta_2\xi^{n+1}F_1-\delta_2\xi^{n+1}\epsilon_2}{n(1-\delta_1\delta_2\xi^2)}+F_2,\\
K_2 &= \frac{\delta_1\xi^m(\delta_2\xi\epsilon_2-F_1)}{m(1-\delta_1\delta_2\xi^2)}+D_2, \; M=J_1K_2-J_2K_1,\\
D_1 &= \sum_{i=1}^{p}\gamma_i\beta_i+\sum_{j=1}^{q}\omega_j\eta_j, D_2=\sum_{i=1}^{p}\gamma_i\frac{\beta_i^m}{m}+\sum_{j=1}^{q}\omega_j\eta_j^{m-1},\\
F_1 &= \sum_{i=1}^{p}\widehat{\gamma_i}\beta_i+\sum_{j=1}^{q}\widehat{\omega_j}, \; F_2=\sum_{i=1}^{p}\widehat{\gamma_i}\frac{\beta_i^n}{n}+\sum_{j=1}^{q}\widehat{\omega_j}\eta_j^{n-1}.
\end{aligned} \tag{2.4}
$$

Proof. Solving the system of ordinary differential equations in (2.1), we get

$$
\begin{cases}
u(t)=\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y_1(s)ds+c_0+c_1t+\ldots+c_{n-1}t^{n-1},\\
v(t)=\int_0^t \frac{(t-s)^{m-1}}{(m-1)!}y_2(s)ds+b_0+b_1t+\ldots+b_{m-1}t^{m-1},
\end{cases} \tag{2.5}
$$

where $c_i, b_i \in \mathbb{R}$, $i=0,1,\ldots,n-1,m-1$, are arbitrary constants. Making use of the conditions $u'(0)=0$, $u''(0)=0,\ldots$, $u^{(n-2)}(0)=0$ and $v'(0)=0$, $v''(0)=0,\ldots$, $v^{(m-2)}(0)=0$ in (2.5), we get $c_1=c_2=\ldots,c_{n-2}=0, b_0=b_1=\ldots,b_{m-1}=0$. In consequence, (2.5) takes the form

$$
\begin{cases}
u(t)=\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y_1(s)ds+c_0+c_{n-1}t^{n-1},\\
v(t)=\int_0^t \frac{(t-s)^{m-1}}{(m-1)!}y_2(s)ds+b_0+b_{m-1}t^{m-1}.
\end{cases} \tag{2.6}
$$

Using (2.6) in the conditions $u(0)=\delta_1\int\limits_0^\xi v(s)ds$ and $v(0)=\delta_2\int\limits_0^\xi u(s)ds$, we get

$$
c_0=\delta_1\int_0^\xi \frac{(\xi-r)^m}{m!}y_2(r)dr+\delta_1b_0\xi+\delta_1b_{m-1}\frac{\xi^m}{m}, \tag{2.7}
$$

and

$$b_0 = \delta_2 \int_0^{\xi} \frac{(\xi - r)^n}{n!} y_1(r)dr + c_0\delta_2\xi + c_{n-1}\delta_2 \frac{\xi^n}{n}. \tag{2.8}$$

Now, inserting (2.6) in the conditions:

$$\epsilon_1 u(1) + \zeta_1 u'(1) = \sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} v(s)ds + \sum_{j=1}^{q} \omega_j v(\eta_j),$$

$$\epsilon_2 v(1) + \zeta_2 v'(1) = \sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} u(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} u(\eta_j),$$

we obtain

$$\begin{aligned} & c_0\epsilon_1 + c_{n-1}[\epsilon_1 + \zeta_1(n-1)] + \int_0^1 \frac{(1-s)^{n-2}[\epsilon_1(1-s) + \zeta_1(n-1)]}{(n-1)!} y_1(s)ds \\ & = b_0 \left[\sum_{i=1}^{p} \gamma_i\beta_i + \sum_{j=1}^{q} \omega_j\eta_j \right] + b_{m-1} \left[\sum_{i=1}^{p} \gamma_i \frac{\beta_i^m}{m} + \sum_{j=1}^{q} \omega_j \eta_j^{m-1} \right] \\ & \quad + \sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} \frac{(\beta_i - s)^m}{m!} y_2(s)ds + \sum_{j=1}^{q} \omega_j \int_0^{\eta_j} \frac{(\eta_j - s)^{m-1}}{(m-1)!} y_2(s)ds, \end{aligned} \tag{2.9}$$

$$\begin{aligned} & b_0\epsilon_2 + b_{m-1}[\epsilon_2 + \zeta_2(m-1)] + \int_0^1 \frac{(1-s)^{m-2}[\epsilon_2(1-s) + \zeta_2(m-1)]}{(m-1)!} y_2(s)ds \\ & = c_0 \left[\sum_{i=1}^{p} \widehat{\gamma_i}\beta_i + \sum_{j=1}^{q} \widehat{\omega_j} \right] + c_{n-1} \left[\sum_{i=1}^{p} \widehat{\gamma_i} \frac{\beta_i^n}{n} + \sum_{j=1}^{q} \widehat{\omega_j} \eta_j^{n-1} \right] \\ & \quad + \sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} \frac{(\beta_i - s)^n}{n!} y_1(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} \int_0^{\eta_j} \frac{(\eta_j - s)^{n-1}}{(n-1)!} y_1(s)ds. \end{aligned} \tag{2.10}$$

We can express equations (2.7)-(2.10) in the form

$$\begin{cases} c_0 - A_2 b_0 - A_3 b_{m-1} = A_1, \\ -B_2 c_0 + b_0 - B_3 c_{n-1} = B_1, \\ C_1 c_0 - D_1 b_0 + C_2 c_{n-1} - D_2 b_{m-1} = D_3 - C_3, \\ -F_1 c_0 + E_1 b_0 - F_2 c_{n-1} + E_2 b_{m-1} = F_3 - E_3, \end{cases} \tag{2.11}$$

where D_1, D_2, F_1 and F_2 are given in (2.4) and

$$A_1 = \delta_1 \Big[\int_0^{\xi} \frac{(\xi - r)^m}{m!} y_2(r)dr \Big], \qquad A_2 = \delta_1\xi, \qquad A_3 = \delta_1 \frac{\xi^m}{m},$$

$$B_1 = \delta_2 \Big[\int_0^{\xi} \frac{(\xi - r)^n}{n!} y_1(r)dr \Big], \qquad B_2 = \delta_2\xi, \qquad B_3 = \delta_2 \frac{\xi^n}{n},$$

$$
\begin{aligned}
C_1 &= \epsilon_1, \qquad C_2 = \epsilon_1 + \zeta_1(n-1),\\
C_3 &= \int_0^1 \frac{(1-s)^{n-2}[\epsilon_1(1-s)+\zeta_1(n-1)]}{(n-1)!} y_1(s)ds,\\
D_3 &= \sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} \frac{(\beta_i - s)^m}{m!} y_2(s)ds + \sum_{j=1}^{q} \omega_j \int_0^{\eta_j} \frac{(\eta_j - s)^{m-1}}{(m-1)!} y_2(s)ds,\\
E_1 &= \epsilon_2, \qquad E_2 = \epsilon_2 + \zeta_2(n-1),\\
E_3 &= \int_0^1 \frac{(1-s)^{m-2}[\epsilon_2(1-s)+\zeta_2(m-1)]}{(m-1)!} y_2(s)ds, \qquad (2.12)\\
F_3 &= \sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} \frac{(\beta_i - s)^n}{n!} y_1(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} \int_0^{\eta_j} \frac{(\eta_j - s)^{n-1}}{(n-1)!} y_1(s)ds.
\end{aligned}
$$

Solving the first two equations in (2.11) for c_0 and b_0 in term of c_{n-1} and $b_{m-1}$and using the notation in (2.12), we obtain

$$
\begin{cases}
c_0 = G_1 + G_2 b_{m-1} + G_3 c_{n-1},\\
b_0 = H_1 + H_2 b_{m-1} + H_3 c_{n-1},
\end{cases} \qquad (2.13)
$$

where

$$
\begin{aligned}
G_1 &= \frac{A_1 + A_2B_1}{r_1},\ G_2 = \frac{A_3}{r_1},\ G_3 = \frac{A_2B_3}{r_1},\ H_1 = \frac{A_1B_2 + B_1}{r_1},\\
H_2 &= \frac{A_3B_2}{r_1}, \qquad H_3 = \frac{B_3}{r_1}, \qquad r_1 = 1 - \delta_1\delta_2\xi^2.
\end{aligned} \qquad (2.14)
$$

Substituting the values of c_0 and b_0 from (2.13) in the last two equations of (2.11), we get

$$
\begin{cases}
c_{n-1}J_1 - b_{m-1}J_2 = J_3,\\
c_{n-1}K_1 - b_{m-1}K_2 = K_3,
\end{cases} \qquad (2.15)
$$

where J_1, J_2, K_1, K_2 are given in (2.4) and

$$
\begin{aligned}
J_3 &= \frac{A_1(B_2D_1 - C_1) + B_1(D_1 - A_2C_1)}{r_1} + D_3 - C_3,\\
K_3 &= \frac{A_1(B_2E_1 - F_1) + B_1(E_1 - A_2F_1)}{r_1} + E_3 - F_3.
\end{aligned} \qquad (2.16)
$$

Solving the system (2.15) for b_{m-1} and c_{n-1}, we find that

$$
\begin{cases}
c_{n-1} = \dfrac{J_3K_2 - J_2K_3}{J_1K_2 - J_2K_1},\\[2ex]
b_{m-1} = \dfrac{J_3K_1 - J_1K_3}{J_1K_2 - J_2K_1}.
\end{cases} \qquad (2.17)
$$

Inserting (2.17) in (2.13), we obtain

$$\begin{cases} c_0 = \dfrac{G_1(J_1K_2 - J_2K_1) + G_2(J_3K_1 - J_1K_3) + G_3(J_3K_2 - J_2K_3)}{J_1K_2 - J_2K_1}, \\ \\ b_0 = \dfrac{H_1(J_1K_2 - J_2K_1) + H_2(J_3K_1 - J_1K_3) + H_3(J_3K_2 - J_2K_3)}{J_1K_2 - J_2K_1}. \end{cases} \tag{2.18}$$

Substituting the above values of c_{n-1}, b_{m-1}, c_0 and b_0 into (2.6) together with the notation (2.4), we obtain the solution (2.2) and (2.3). One can obtain the converse of the lemma by direct computation. □

In the sequel, we set

$$\begin{aligned} &\bar{N}_i = \max_{t\in[0,1]} |N_i(t)|, \; i = 1, 2, \dots, 8, \\ &\sigma_1 = \frac{\xi^{m+1}}{(m+1)!}, \; \sigma_2 = \frac{\xi^{n+1}}{(n+1)!}, \\ &\sigma_3 = \sum_{i=1}^{p} |\gamma_i| \frac{\beta_i^{m+1}}{(m+1)!} + \sum_{i=1}^{p} |\omega_j| \frac{\eta_j^m}{m!}, \quad \sigma_4 = \left(\frac{|\epsilon_1|}{n!} + \frac{|\zeta_1|}{(n-1)!} \right), \\ &\sigma_5 = \sum_{i=1}^{p} |\widehat{\gamma}_i| \frac{\beta_i^{n+1}}{(n+1)!} + \sum_{i=1}^{p} |\widehat{\omega}_j| \frac{\eta_j^n}{n!}, \quad \sigma_6 = \left(\frac{|\epsilon_2|}{m!} + \frac{|\zeta_2|}{(m-1)!} \right), \end{aligned} \tag{2.19}$$

where $N_i(t), i = 1, 2, \dots, 8$, are given in (2.4).

3. Main results

In the forthcoming analysis, we need the assumptions:

(H_1) There exist real constants $\widehat{m}_i, \widehat{n}_i \geq 0, i = 1, 2$ and $\widehat{m}_0 > 0, \widehat{n}_0 > 0$ such that $\forall u, v \in \mathbb{R}$,

$$|f(t, u, v)| \leq \widehat{m}_0 + \widehat{m}_1|u| + \widehat{m}_2|v|, \quad |g(t, u, v)| \leq \widehat{n}_0 + \widehat{n}_1|u| + \widehat{n}_2|v|;$$

(H_2) There exist positive constants ℓ_1 and ℓ_2 such that, $\forall t \in [0, 1]$ and $u_i, v_i \in \mathbb{R}$, $i = 1, 2$,

$$\begin{aligned} &|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \ell_1(|u_1 - u_2| + |v_1 - v_2|), \\ &|g(t, u_1, v_1) - g(t, u_2, v_2)| \leq \ell_2(|u_1 - u_2| + |v_1 - v_2|). \end{aligned}$$

For the sake of convenience in the mathematical computations, we set

$$\begin{aligned} &Q_0 = \min\{1 - (Q_1\widehat{m}_1 + Q_2\widehat{n}_1), 1 - (Q_1\widehat{m}_2 + Q_2\widehat{n}_2)\}, \\ &Q_1 = q_1 + q_2, \; Q_2 = \bar{q_1} + \bar{q_2}, \\ &q_1 = \tfrac{1}{n!} + \bar{N}_2\sigma_2 + \bar{N}_3\sigma_4 + \bar{N}_4\sigma_5, \; \bar{q_1} = \bar{N}_1\sigma_1 + \bar{N}_3\sigma_3 + \bar{N}_4\sigma_6, \\ &q_2 = \bar{N}_6\sigma_2 + \bar{N}_7\sigma_4 + \bar{N}_8\sigma_5, \; \bar{q_2} = \frac{1}{m!} + \bar{N}_5\sigma_1 + \bar{N}_7\sigma_3 + \bar{N}_8\sigma_6. \end{aligned} \tag{3.1}$$

Let $\mathcal{X} = \{u(t) \mid u(t) \in C([a,b])\}$ be the space equipped with norm

$$\|u\| = \sup\{|u(t)|,\ t \in [a,b]\}.$$

Then, $(\mathcal{X}, \|\cdot\|)$ is a Banach space and consequently, the product space $(\mathcal{X}\times\mathcal{X}, \|(u,v)\|)$ is also a Banach space endowed with the norm $\|(u,v)\| = \|u\|+\|v\|$ for $(u,v) \in \mathcal{X}\times\mathcal{X}$. By Lemma 1, we define an operator $\mathcal{T} : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ associated with the problem (1.1) as

$$\mathcal{T}(u,v)(t) := (\mathcal{T}_1(u,v)(t), \mathcal{T}_2(u,v)(t)),$$

where

$$\begin{aligned}
&\mathcal{T}_1(u,v)(t)\\
=\;& \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u,v)ds + N_1(t)\int_0^\xi \frac{(\xi-s)^m}{m!} g(s,u,v)ds\\
&+N_2(t)\int_0^\xi \frac{(\xi-s)^n}{n!} f(s,u,v)ds + N_3(t)\Big[\sum_{i=1}^p \gamma_i \int_0^{\beta_i} \frac{(\beta_i-s)^m}{m!} g(s,u,v)ds\\
&+\sum_{j=1}^q \omega_j \int_0^{\eta_j} \frac{(\eta_j-s)^{m-1}}{(m-1)!} g(s,u,v)ds\\
&-\int_0^1 \frac{(1-s)^{n-2}}{(n-1)!}(\epsilon_1(1-s)+\zeta_1(n-1))f(s,u,v)ds\Big]\\
&+N_4(t)\Big[\sum_{i=1}^p \widehat{\gamma_i} \int_0^{\beta_i} \frac{(\beta_i-s)^n}{n!} f(s,u,v)ds + \sum_{j=1}^q \widehat{\omega_j} \int_0^{\eta_j} \frac{(\eta_j-s)^{n-1}}{(n-1)!} f(s,u,v)ds\\
&-\int_0^1 \frac{(1-s)^{m-2}}{(m-1)!}(\epsilon_2(1-s)+\zeta_2(m-1))g(s,u,v)ds\Big],
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\mathcal{T}_2(u,v)(t)\\
=\;& \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} g(s,u,v)ds + N_5(t)\int_0^\xi \frac{(\xi-s)^m}{m!} g(s,u,v)ds\\
&+N_6(t)\int_0^\xi \frac{(\xi-s)^n}{n!} f(s,u,v)ds + N_7(t)\Big[\sum_{i=1}^p \gamma_i \int_0^{\beta_i} \frac{(\beta_i-s)^m}{m!} g(s,u,v)ds\\
&+\sum_{j=1}^q \omega_j \int_0^{\eta_j} \frac{(\eta_j-s)^{m-1}}{(m-1)!} g(s,u,v)ds\\
&-\int_0^1 \frac{(1-s)^{n-2}}{(n-1)!}(\epsilon_1(1-s)+\zeta_1(n-1))f(s,u,v)ds\Big]
\end{aligned}$$

$$+N_8(t)\Big[\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}f(s,u,v)ds+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)!}f(s,u,v)ds$$
$$-\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}(\epsilon_2(1-s)+\zeta_2(m-1))g(s,u,v)ds\Big]. \tag{3.3}$$

3.1. Existence of solutions

In this subsection, we discuss the existence of solutions for the problem (1.1) by using Leray-Schauder's alternative [17], which is stated below.

Lemma 3.1. *Let $T : K \to K$ be a completely continuous operator (that is, a map restricted to any bounded set in K is compact). Let*

$$\psi(T) = \{x \in K : x = \varphi T(x) \text{ for some } 0 < \varphi < 1\}.$$

Then, either the set $\psi(T)$ is unbounded or T has at least one fixed point.

Theorem 3.2. *Let $f, g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous functions. Assume that condition (H_1) holds, and*

$$Q_1\widehat{m}_1 + Q_2\widehat{n}_1 < 1 \qquad Q_1\widehat{m}_2 + Q_2\widehat{n}_2 < 1,$$

where Q_1 and Q_2 are given by (3.1). Then, there exists at least one solution for the problem (1.1) on $[0,1]$.

Proof. First of all, we show that the operator $\mathcal{T} : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ is completely continuous. Notice that the operator $\mathcal{T}$ is continuous as the functions f and g are continuous. Let $\Psi = \{(u,v) \in \mathcal{X}\times\mathcal{X} : \|(u,v)\| \le \rho\}$. For any $u, v \in \Psi$ we have

$$\begin{aligned}|f(t,u,v)| &\le \widehat{m}_0 + \widehat{m}_1|u| + \widehat{m}_2|v| \le \widehat{m_0} + (\widehat{m_1}+\widehat{m_2})(\|u\|+\|v\|)\\ &\le \widehat{m_0} + (\widehat{m_1}+\widehat{m_2})\rho := \kappa_f,\end{aligned}$$

and similarly

$$|g(t,u,v)| \le \widehat{n_0} + (\widehat{n_1}+\widehat{n_2})\rho := \kappa_g.$$

Then, for any $(u,v) \in B_\rho$, we obtain

$$\begin{aligned}&|\mathcal{T}_1(u,v)(t)|\\ \le\ & \sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}|f(s,u,v)|ds + |N_1(t)|\int_0^\xi\frac{(\xi-s)^m}{m!}|g(s,u,v)|ds\\ &+|N_2(t)|\int_0^\xi\frac{(\xi-s)^n}{n!}|f(s,u,v)|ds\\ &+|N_3(t)|\Bigg[\int_0^1\frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]|f(s,u,v)|ds\\ &+\sum_{i=1}^{p}\gamma_i\int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}|g(s,u,v)|ds+\sum_{j=1}^{q}\omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)}|g(s,u,v)|ds\Bigg]\\ &+|N_4(t)|\Bigg[\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]|g(s,u,v)|ds\end{aligned}$$

$$\begin{aligned}
&+\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}|f(s,u,v)|ds+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)}|f(s,u,v)|ds\Bigg]\Bigg\}\\
\le\;& \kappa_f\left[\frac{1}{n!}+\bar{N}_2\sigma_2+\bar{N}_3\sigma_4+\bar{N}_4\sigma_5\right]+\kappa_g\left[\bar{N}_1\sigma_1+\bar{N}_3\sigma_3+\bar{N}_4\sigma_6\right]\\
\le\;& \kappa_f q_1+\kappa_g\bar{q_1},
\end{aligned}$$

which implies that $\|\mathcal{T}_1(u,v)\|\le\kappa_f q_1+\kappa_g\bar{q_1}$, where q_1 and $\bar{q_1}$ are given in (3.1). Similarly, one can obtain that $\|\mathcal{T}_2(u,v)\|\le\kappa_f q_2+\kappa_g\bar{q_2}$, where q_2 and $\bar{q_2}$ are defined in (3.1). From the forgoing inequalities, we get $\|\mathcal{T}(u,v)\|\le\kappa_f Q_1+\kappa_g Q_2$, where Q_1 and Q_2 are given in (3.1), which shows that the operator $\mathcal{T}$ is uniformly bounded. Next, we establish that $\mathcal{T}$ is equicontinuous. For $t_1,t_2\in[0,1]$ with $t_1<t_2$, we have

$$\begin{aligned}
&|\mathcal{T}_1(u_1,v_1)(t_2)-\mathcal{T}_1(u_2,v_2)(t_1)|\\
\le\;& \kappa_f\left|\int_0^{t_2}\frac{(t_2-s)^{n-1}}{(n-1)!}f(s,u,v)ds-\int_0^{t_1}\frac{(t_1-s)^{n-1}}{(n-1)!}f(s,u,v)ds\right|\\
&+|N_1(t_2)-N_1(t_1)|\int_0^{\xi}\frac{(\xi-s)^m}{m!}|g(s,u,v)|ds\\
&+|N_2(t_2)-N_2(t_1)|\int_0^{\xi}\frac{(\xi-s)^n}{n!}|f(s,u,v)|ds\\
&+|N_3(t_2)-N_3(t_1)|\Bigg[\int_0^1\frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]|f(s,u,v)|ds\\
&+\sum_{i=1}^{p}\gamma_i\int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}|g(s,u,v)|ds+\sum_{j=1}^{q}\omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)}|g(s,u,v)|ds\Bigg]\\
&+|N_4(t_2)-N_4(t_1)|\Bigg[\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]|g(s,u,v)|ds\\
&+\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}|f(s,u,v)|ds+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)}|f(s,u,v)|ds\Bigg]\Bigg\}\\
\le\;& \frac{\kappa_f}{n!}\left(2(t_2-t_1)^n+|t_2^n-t_1^n|\right)+|N_1(t_2)-N_1(t_1)|\kappa_g\sigma_1+|N_2(t_2)-N_2(t_1)|\kappa_f\sigma_2\\
&+|N_3(t_2)-N_3(t_1)|(\kappa_f\sigma_4+\kappa_g\sigma_3)+|N_4(t_2)-N_4(t_1)|(\kappa_f\sigma_5+\kappa_g\sigma_6),
\end{aligned}$$

which tends to zero as $(t_2-t_1)\to 0$ independent of $(u,v)\in\Psi$. In a similar manner, it can be shown that $|\mathcal{T}_2(u_1,v_1)(t_2)-\mathcal{T}_2(u_2,v_2)(t_1)|\to 0$ as $(t_2-t_1)\to 0$ independent of $(u,v)\in\Psi$. Thus, the operator $\mathcal{T}$ is equicontinuous.

Finally, it will be verified that the set $\psi=\{(u,v)\in\mathcal{X}\times\mathcal{X}|(u,v)=\varphi\mathcal{T}(u,v),\ 0<\varphi<1\}$ is bounded. Let $(u,v)\in\psi$. Then $(u,v)=\varphi\mathcal{T}(u,v)$ for any $t\in[0,1]$. Therefore, we have $u(t)=\varphi\mathcal{T}_1(u,v)(t)$, $v(t)=\varphi\mathcal{T}_2(u,v)(t)$. In consequence, it follows by the assumption (H_1) that

$$|u(t)|=q_1\widehat{m}_0+\bar{q_1}\widehat{n}_0+(q_1\widehat{m}_1+\bar{q_1}\widehat{n}_1)\|u\|+(q_1\widehat{m}_2+\bar{q_1}\widehat{n}_2)\|v\|,\tag{3.4}$$

and

$$|v(t)|=q_2\widehat{m}_0+\bar{q_2}\widehat{n}_0+(q_2\widehat{m}_1+\bar{q_2}\widehat{n}_1)\|u\|+(q_2\widehat{m}_2+\bar{q_2}\widehat{n}_2)\|v\|,\tag{3.5}$$

where $q_1, q_2.\bar{q}_1$, and $\bar{q}_2$ are given in (3.1). From (3.4) and (3.5), we have

$$\begin{aligned}\|u\|+\|v\| &\leq (q_1+q_2)\widehat{m}_0+(\bar{q_1}+\bar{q_2})\widehat{n}_0+[(q_1+q_2)\widehat{m}_1+(\bar{q_1}+\bar{q_2})\widehat{n}_1]\|u\| \\ &\quad +[(q_1+q_2)\widehat{m}_2+(\bar{q_1}+\bar{q_2})\widehat{n}_2]\|v\|,\end{aligned}$$

which, in view of (3.1), can be written as

$$\|(u,v)\| \leq \frac{Q_1\widehat{m}_0+Q_2\widehat{n}_0}{Q_0}.$$

This shows that the set ψ is bounded. Hence, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Therefore, the problem (1.1) has at least one solution on $[0,1]$. This completes the proof. □

3.2. Uniqueness of solutions

Here, we establish the uniqueness of solutions for the problem (1.1) by means of Banach's contractions mapping principle [12].

Theorem 3.3. *Suppose that $f, g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions, the assumption (H_2) and the following condition*

$$Q_1\ell_1+Q_2\ell_2<1, \tag{3.6}$$

hold, where Q_1 and Q_2 are given in (3.1). Then, the problem (1.1) has a unique solution on $[0,1]$.

Proof. Firstly, we show that $\mathcal{T}B_r \subset B_r$, where $B_r=\{(u,v)\in\mathcal{X}\times\mathcal{X} : \|(u,v)\|\leq r\}$ is a closed ball with

$$r \geq \frac{Q_1N_1+Q_2N_2}{1-(Q_1\ell_1+Q_2\ell_2)}. \tag{3.7}$$

Let us set $\sup_{t\in[0,1]}|f(t,0,0)|=\mu_1$ and $\sup_{t\in[0,1]}|g(t,0,0)|=\mu_2$. Then, by the assumption (H_2), we have

$$\begin{aligned}|f(s,u(s),v(s))| &= |f(s,u(s),v(s))-f(s,0,0)+f(s,0,0)| \\ &\leq |f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)| \\ &\leq \ell_1(\|u\|+\|v\|)+\mu_1 \leq \ell_1\|(u,v)\|+\mu_1 \leq \ell_1 r+\mu_1.\end{aligned}$$

Likewise, one can obtain that

$$|g(s,u(s),v(s))| \leq \ell_2 r+\mu_2.$$

For $(u,v)\in B_r$, we have

$$\begin{aligned}&|\mathcal{T}_1(u,v)(t)| \\ \leq\ & \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u,v)|ds+|N_1(t)|\int_0^\xi \frac{(\xi-s)^m}{m!}|g(s,u,v)|ds \\ &+|N_2(t)|\int_0^\xi \frac{(\xi-s)^n}{n!}|f(s,u,v)|ds \\ &+|N_3(t)|\Bigg[\int_0^1 \frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]|f(s,u,v)|ds\end{aligned}$$

$$
\begin{aligned}
& +\sum_{i=1}^{p}\gamma_i\int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}|g(s,u,v)|ds+\sum_{j=1}^{q}\omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)}|g(s,u,v)|ds\Bigg] \\
& +|N_4(t)|\Bigg[\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]|g(s,u,v)|ds \\
& +\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}|f(s,u,v)|ds+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)}|f(s,u,v)|ds\Bigg]\Bigg\} \\
\leq\ & [\ell_1 r+\mu_1]\Big[\frac{1}{n!}+\bar{N}_2\sigma_2+\bar{N}_3\sigma_4+\bar{N}_4\sigma_5\Big]+[\ell_2 r+\mu_2]\Big[\bar{N}_1\sigma_1+\bar{N}_3\sigma_3+\bar{N}_4\sigma_6\Big] \\
\leq\ & q_1(\ell_1 r+\mu_1)+\bar{q_1}(\ell_2 r+\mu_2),
\end{aligned}
$$

which implies that

$$\|\mathcal{T}_1(u,v))\|\leq q_1(\ell_1 r+\mu_1)+\bar{q_1}(\ell_2 r+\mu_2).$$

Similarly, we can get

$$\|\mathcal{T}_2(u,v))\|\leq q_2(\ell_1 r+\mu_1)+\bar{q_2}(\ell_2 r+\mu_2).$$

From the above estimates together with (3.7), it follows that $\|\mathcal{T}(u,v)\|\leq r$. Since $(u,v)\in B_r$ is an arbitrary element, therefore $\mathcal{T}B_r\subset B_r$.
Now, we show that the operator $\mathcal{T}$ is a contraction. For $(u_1,v_1),(u_2,v_2)\in\mathcal{X}\times\mathcal{X}$, we have

$$
\begin{aligned}
& |\mathcal{T}_1(u_1,v_1)(t)-\mathcal{T}_1(u_2,v_2)(t)| \\
\leq\ & \sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}|f(s,u_1,v_1)-f(s,u_2,v_2)|ds \\
& +|N_1(t)|\int_0^{\xi}\frac{(\xi-s)^m}{m!}|g(s,u_1,v_1)-g(s,u_2,v_2)|ds \\
& +|N_2(t)|\int_0^{\xi}\frac{(\xi-s)^n}{n!}|f(s,u_1,v_1)-f(s,u_2,v_2)|ds \\
& +|N_3(t)|\Bigg[\int_0^1\frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]|f(s,u_1,v_1)-f(s,u_2,v_2)|ds \\
& +\sum_{i=1}^{p}\gamma_i\int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}|g(s,u_1,v_1)-g(s,u_2,v_2)|ds \\
& +\sum_{j=1}^{q}\omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)}|g(s,u_1,v_1)-g(s,u_2,v_2)|ds\Bigg] \\
& +|N_4(t)|\Bigg[\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]|g(s,u_1,v_1)-g(s,u_2,v_2)|ds \\
& +\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}|f(s,u_1,v_1)-f(s,u_2,v_2)|ds \\
& +\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)}|f(s,u_1,v_1)-f(s,u_2,v_2)|ds\Bigg]\Bigg\}
\end{aligned}
$$

$$\begin{aligned}&\leq \ell_1\Big[\frac{1}{n!}+\bar{N}_2\sigma_2+\bar{N}_3\sigma_4+\bar{N}_4\sigma_5\Big](|u_1-u_2|+|v_1-v_2|)\\&\quad+\ell_2\Big[\bar{N}_1\sigma_1+\bar{N}_3\sigma_3+\bar{N}_4\sigma_6\Big](|u_1-u_2|+|v_1-v_2|)\\&\leq (\ell_1 q_1+\ell_2\bar{q}_1)(|u_1-u_2|+|v_1-v_2|),\end{aligned}$$

which implies that

$$\|\mathcal{T}_1(u_1,v_1)-\mathcal{T}_1(u_2,v_2)\| \leq (\ell_1 q_1+\ell_2\bar{q}_1)(|u_1-u_2|+|v_1-v_2|). \tag{3.8}$$

In a similar manners, we get

$$\|\mathcal{T}_2(u_1,v_1)-\mathcal{T}_2(u_2,v_2)\| \leq (\ell_1 q_2+\ell_2\bar{q}_2)(|u_1-u_2|+|v_1-v_2|). \tag{3.9}$$

From (3.8) and (3.9), we deduce that

$$\|\mathcal{T}(u_1,v_1)-\mathcal{T}(u_2,v_2)\| \leq (Q_1\ell_1+Q_2\ell_2)(\|u_1-u_2\|+\|v_1-v_2\|),$$

where Q_1 and Q_2 are given in (3.1). By the assumption (3.7), it follows from the above inequality that the operator $\mathcal{T}$ is a contraction. Thus, by the Banach's contraction mapping principle, the operator $\mathcal{T}$ has a unique fixed point, which corresponds to a unique solution to the problem (1.1) on $[0,1]$. □

4. Examples

Example 4.1. Consider the integral-multipoint boundary value problem of nonlinear differential equations

$$\begin{cases} u^{(3)}(t)=\dfrac{1}{t^2+9}+\dfrac{1}{\sqrt{t^2+4}}\dfrac{|u^2|}{(1+|u|)}+\dfrac{e^{-t}}{4+t^4}\sin v, & t\in[0,1],\\ v^{(4)}(t)=\dfrac{e^{-t}}{16}+\dfrac{u\cos v}{\sqrt{t^2+36}}+\dfrac{v}{(t^2+5)}\dfrac{|u^2|}{(1+|u|)}, & t\in[0,1],\\ u(0)=\delta_1\displaystyle\int_0^{\xi} v(s)ds,\ u'(0)=0,\ v(0)=\delta_2\int_0^{\xi} u(s)ds,\ v'(0)=0,\ v''(0)=0, \\ \epsilon_1 u(1)+\zeta_1 u'(1)=\displaystyle\sum_{i=1}^{4}\gamma_i\int_0^{\beta_i} v(s)ds+\sum_{j=1}^{3}\omega_j v(\eta_j), \\ \epsilon_2 v(1)+\zeta_2 v'(1)=\displaystyle\sum_{i=1}^{4}\widehat{\gamma_i}\int_0^{\beta_i} u(s)ds+\sum_{j=1}^{3}\widehat{\omega_j} u(\eta_j), \end{cases} \tag{4.1}$$

where $n=3$, $m=4$, $\delta_1=1.2$, $\delta_2=1.5$, $\epsilon_1=0.7$, $\epsilon_2=0.4$, $\zeta_1=2.6$, $\zeta_2=2.1$, $\xi=0.1$, $\beta_1=0.2$, $\beta_2=0.3$, $\beta_3=0.4$, $\beta_4=0.5$, $\eta_1=0.6$, $\eta_2=0.7$, $\eta_3=0.8$, $\gamma_1=0.325$, $\gamma_2=0.572$, $\gamma_3=0.811$, $\gamma_4=0.124$, $\omega_1=0.267$, $\omega_2=0.489$, $\omega_3=0.712$, $\widehat{\gamma}_1=0.452$, $\widehat{\gamma}_2=0.695$, $\widehat{\gamma}_3=0.831$, $\widehat{\gamma}_4=0.203$, $\widehat{\omega}_1=0.378$, $\widehat{\omega}_2=0.617$, $\widehat{\omega}_3=0.954$.

Using the given data in (2.4), (2.19) and (3.1), we find that $\bar{N}_1\approx 0.978509$, $\bar{N}_2\approx 0.3632664$, $\bar{N}_3\approx 0.172781$, $\bar{N}_4\approx 0.020315$, $\bar{N}_5\approx 0.621273$, $\bar{N}_6\approx 1.038184$, $\bar{N}_7\approx 0.035554$, $\bar{N}_8\approx 0.221574$, $\sigma_1\approx 0.0000008$, $\sigma_2\approx 0.000004$, $\sigma_3\approx 0.017157$, $\sigma_4\approx 1.416667$, $\sigma_5\approx 0.118329$, $q_1\approx 0.413846$, $q_2\approx 0.076591$, $\bar{q}_1\approx 0.010413$, $\bar{q}_2\approx 0.123521$, $Q_1\approx 0.490437$, $Q_2\approx 0.133934$.

Also it is easy to find that $|f(t,u,v)| \leq 1/9 + 1/2|u| + 1/4|v|$, $|g(t,u,v)| \leq 1/16+1/6|u|+1/5|v|$, $Q_1\widehat{m}_1+Q_2\widehat{n}_1 \approx 0.267541 < 1$ and $Q_1\widehat{m}_2+Q_2\widehat{n}_2 \approx 0.149396 < 1$. Clearly all the assumptions of Theorem 3.2 are satisfied. Therefore, there exists at least one solution to the problem (4.1).

Example 4.2. Consider the system of ordinary differential equations

$$\begin{cases} u^{(3)}(t) = \dfrac{1}{\sqrt{t^2+100}} \tan^{-1} u + \dfrac{1}{(t^2+10)} \dfrac{|v|}{(1+|v|)} + \dfrac{e^{-t}}{4}, & t \in [0,1], \\ v^{(4)}(t) = \dfrac{e^{-t}}{t^2+2} \sin u + \dfrac{1}{\sqrt{t^2+4}} \cos v + \dfrac{t^2+4}{\sqrt{t^3+4}}, & t \in [0,1], \end{cases} \tag{4.2}$$

subject to the boundary conditions in Example 4.1.

Observe that $\ell_1 = 1/10, \ell_2 = 1/2$ as

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq \frac{1}{10}(|u_1-u_2| + |v_1-v_2|),$$
$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \leq \frac{1}{2}(|u_1-u_2| + |v_1-v_2|).$$

Moreover, $Q_1\ell_1 + Q_2\ell_2 \approx 0.088091 < 1$. Thus, the hypotheses of Theorem 3.3 are satisfied and hence its conclusion applies to the problem (4.2).

5. Conclusions

We have developed the existence and uniqueness results for a new class of coupled systems of two nonlinear ordinary differential equations of order n and m subject to the coupled integral-multipoint boundary conditions. Our results are not only new in the given configuration but also yield some new ones by fixing the parameters involved in the given boundary data. In future, we plan to develop the multivalued version of the problem studied in this paper.

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On the stabilization of a thermoelastic laminated beam system with microtemperature effects

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Abstract. The present article investigates a one dimensional thermoelastic laminated beam with microtemperature effects. Using the energy method we prove in the case of zero thermal conductivity that the unique dissipation due to the microtemperatures is strong enough to exponentially stabilize the system if and only if the wave speeds of the system are equal. Our result is new and improves previous results in the literature.

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1. Introduction

In this paper, we address the following thermoelastic laminated beams with microtemperature effects

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ I_\rho\left(3s - \psi\right)_{tt} - D\left(3s - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G\left(\psi - \varphi_x\right) + 4\gamma s - \delta\theta + m\omega_x = 0, \\ c\theta_t + \kappa_1\omega_x + \delta s_t = 0, \\ \alpha\omega_t - \kappa_2\omega_{xx} + \kappa_3\omega + \kappa_1\theta_x + ms_{tx} = 0, \end{cases} \tag{1.1}$$

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for $(x,t) \in (0,1) \times \mathbb{R}_+$, system (1.1) is complemented with the following boundary conditions

$$
\begin{aligned}
&\varphi_x(0,t) = \psi(0,t) = s(0,t) = \theta(0,t) = \omega_x(0,t) = 0, && t > 0,\\
&\varphi_x(1,t) = \psi(1,t) = s(1,t) = \theta(1,t) = \omega_x(1,t) = 0, && t > 0,
\end{aligned} \tag{1.2}
$$

and the initial data

$$
\begin{aligned}
&\varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \quad s(x,0) = s_0(x), && x \in (0,1),\\
&\varphi_t(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \psi_1(x), \quad s_t(x,0) = s_1(x), && x \in (0,1),\\
&\theta(x,0) = \theta_0(x), \quad \omega(x,0) = \omega_0(x), && x \in (0,1),
\end{aligned} \tag{1.3}
$$

where the functions $\varphi(x,t)$ is the transversal displacement of the beam, ψ is the volume fraction difference, $(3s(x,t) - \psi(x,t))$ is the effective rotation angle, θ is the relative temperature and ω is the microtemperature difference and the coefficients, ρ, I_ρ, D, G, and γ are positive constant coefficients represent the density, the shear stiffness, the mass moment of inertia, the flexural rigidity, and the adhesive damping weight. And the coefficients γ, κ_1, κ_2, κ_3, c, m and α are positive constants represent the physical parameters describing the coupling between the various constituents of the materials.

The initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, s_0, s_1, \theta_0, \omega_0)$ are assumed to belong to a suitable functional space.

The laminated beam model describes a vibrating structure of an interfacial slip. It consists of two layered beams of uniform thickness which are attached by an adhesive layer of small thickness in such a way that small amount of slip is possible while they are continuously in contact with each other. And with the increasing demand of advanced performance, the vibration suppression of the laminated beams has been one of the main research topics in smart materials and structures, and these composite laminates usually have superior structural properties such as adaptability.

The laminated beam problem was first introduced by Hansen and Spies in [14]. In that paper, the authors derived the mathematical model for two-layered beams with structural damping due to the interfacial slip, namely

$$
\begin{cases}
\rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0,\\
I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \varphi_x) = 0,\\
3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - \varphi_x) + 4\gamma s + 4\alpha s_t = 0.
\end{cases} \tag{1.4}
$$

In recent years, researchers have focused on the study of the well-posedness and asymptotic stability properties of (1.4). With additional dampings on the first two equations or some sort of boundary damping mechanism, the authors [4, 5, 20, 21, 22, 27, 28, 32] showed that system (1.4) can be stabilized exponentially.

Regarding thermoelastic laminated-beam models, Apalara [2] analyzed a laminated beam system with thermal effect in the slip instead of the frictional damping

($4\alpha s_t$). More precisely, he studied the following laminated beam system

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \varphi_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - \omega_x) + 4\gamma s + \delta\theta_x = 0, \\ c\theta_t - k\theta_{xx} + \delta s_{tx} = 0, \end{cases}$$

and he came to the conclusion that an exponential stability result is achievable in the case of equal wave speeds, that is,

$$\frac{\rho}{G} = \frac{I_\rho}{D}.$$

We refer the reader to [1, 3, 7, 6, 10, 8, 9, 11, 13, 17, 18, 16, 25, 26, 23, 29] and the references cited therein for some other results.

In the matter of microtemperature effects, we bring up the study of Djeradi et al. [12] where they examined the joint of microtemperature, nonlinear structure damping, along with nonlinear time-varying delay term, and time- varying coefficient on a thermoelastic laminated beam. They examined the system

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ I_\rho\left(3s - \psi\right)_{tt} - D\left(3s - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G\left(\psi - \varphi_x\right) + 4\gamma s + \delta\theta_x + m\omega_x \\ \qquad\qquad + \beta\mathfrak{b}(t)\mathfrak{h}_1(s_t(x,t)) + \mu\mathfrak{b}(t)\mathfrak{h}_2(s_t(x, t - \varsigma(t))) = 0, \\ c\theta_t - \kappa_0\theta_{xx} + \kappa_1\omega_x + \delta s_{tx} = 0, \\ \alpha\omega_t - \kappa_2\omega_{xx} + \kappa_3\omega + \kappa_1\theta_x + ms_{tx} = 0, \end{cases}$$

and established a general decay result in the case of equal wave speeds and particular assumptions related to nonlinear terms.

The coupled system we've described involves several physical phenomena, including thermoelasticity, laminated beams, and microtemprature effects. For example, a laminated beam consists of multiple layers of different materials bonded together, thermoelasticity refers to the combined behavior of thermal and elastic properties of the materials, and microtemperature refers to the consideration of temperature variations at a very small scale, which can influence the overall behavior of the coupled system.

Taking the above observations into account, we consider the one-dimensional thermoelastic laminated beam problem with microtemperature effects and without thermal conductivity (1.1)-(1.3), and we establish that the dissipation due solely to microtemperature is adequate to stabilize the system exponentially in the case of equal wave speeds. i,e.

$$\chi = \frac{\rho}{G} - \frac{I_\rho}{D} = 0. \tag{1.5}$$

Concerning the stability of some thermoelastic systems with microtemperature effects and without thermal conductivity, we refer the reader to [15, 24, 31].

In order to be able to use Poincaré's inequality for φ and ω, we perform the following transformation. From the first equation in (1.1) and boundary conditions, it follows that

$$\frac{d^2}{dt^2}\int_0^1 \varphi(x,t)dx = 0, \quad \forall t \geq 0,$$

and therefore

$$\int_0^1 \varphi(x,t)dx = t\int_0^1 \varphi_1(x,t)dx + \int_0^1 \varphi_0(x,t)dx, \quad \forall t \geq 0.$$

Consequently, if we set

$$\overline{\varphi}(x,t) = \varphi(x,t) - t\int_0^1 \varphi_1(x)dx - \int_0^1 \varphi_0(x)dx, \quad t \geq 0,$$

we get

$$\int_0^1 \overline{\varphi}(x,t)dx = 0, \quad \forall t \geq 0.$$

Now, from the fifth equation of (1.1) and the boundry conditions, we get

$$\frac{d}{dt}\int_0^1 \omega(x,t)dx + \frac{\kappa_3}{\alpha}\int_0^1 \omega(x,t)dx = 0, \quad \forall t \geq 0,$$

thus

$$\int_0^1 \omega(x,t)dx = \left(\int_0^1 \omega_0(x)dx\right)e^{-\frac{\kappa_3}{\alpha}t},$$

so, if we put

$$\overline{\omega}(x,t) = \omega(x,t) - \left(\int_0^1 \omega_0(x)dx\right)e^{-\frac{\kappa_3}{\alpha}t}, \quad t \geq 0,$$

we obtain

$$\int_0^1 \overline{\omega}(x,t)dx = 0, \forall t \geq 0.$$

Clearly, the use of Poincaré's inequality for $\overline{\varphi}$ and $\overline{\omega}$ is justified, and $(\overline{\varphi}, \psi, s, \theta, \overline{\omega})$ satisfies the same equations in (1.1)-(1.3). Subsequently, we work with $\overline{\varphi}$ and $\overline{\omega}$ instead of φ and ω but write φ, ω for simplicity of notation.

For completeness we present a short discussion of the well-posedness and the semigroup formulation of (1.1)-(1.3). For this purpose, we denote by ξ the effective rotation angle, that is, $\xi = 3s - \psi$. Then, system (1.1)-(1.3) is equivalent to

$$\begin{cases} \rho\varphi_{tt} + G\left(3s - \xi - \varphi_x\right)_x = 0, \\ I_\rho \xi_{tt} - D\xi_{xx} - G\left(3s - \xi - \varphi_x\right) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G\left(3s - \xi - \varphi_x\right) + 4\gamma s - \delta\theta + m\omega_x = 0, \\ c\theta_t + \kappa_1\omega_x + \delta s_t = 0, \\ \alpha\omega_t - \kappa_2\omega_{xx} + \kappa_3\omega + \kappa_1\theta_x + ms_{tx} = 0, \\ \varphi_x(0,t) = \xi(0,t) = s(0,t) = \theta(0,t) = \omega_x(0,t) = 0, \\ \varphi_x(1,t) = \xi(1,t) = s(1,t) = \theta(1,t) = \omega_x(1,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \xi(x,0) = \xi_0(x),\ s(x,0) = s_0(x), \\ \varphi_t(x,0) = \varphi_1(x),\ s_t(x,0) = s_1(x),\ \xi_t(x,0) = \xi_1(x), \\ \theta(x,0) = \theta_0(x),\ \omega(x,0) = \omega_0(x). \end{cases} \tag{1.6}$$

Clearly, by introducing the vector function $U = (\varphi, \phi, \xi, u, s, v, \theta, \omega)^T$, where $\phi = \varphi_t$, $u = \xi_t$, and $v = s_t$, system (1.6) can be written as

$$\begin{cases} \frac{d}{dt}U(t) = \mathcal{A}U(t), \qquad t > 0, \\ U(0) = U_0 = \left(\varphi_0, \varphi_1, \xi_0, \xi_1, s_0, s_1, \theta_0, \omega_0\right)^T, \end{cases} \tag{1.7}$$

where $\mathcal{A}$ is a differential operator defined by

$$\mathcal{A}U = \begin{pmatrix} \phi \\ -\frac{G}{\rho}\left(3s - \xi - \varphi_x\right)_x \\ u \\ \frac{1}{I_\rho}\Big(D\xi_{xx} + G\left(3s - \xi - \varphi_x\right)\Big) \\ v \\ \frac{1}{3I_\rho}\Big(3Ds_{xx} - 3G\left(3s - \xi - \varphi_x\right) - 4\gamma s + \delta\theta - m\omega_x\Big) \\ -\frac{1}{c}\Big(\kappa_1\omega_x + \delta v\Big) \\ \frac{1}{\alpha}\Big(\kappa_2\omega_{xx} - \kappa_3\omega - \kappa_1\theta_x - mv_x\Big) \end{pmatrix}.$$

We consider the following spaces

$$\begin{aligned} &L^2_*(0,1) = \left\{\Psi \in L^2(0,1) : \int_0^1 \Psi(x)\,dx = 0\right\}, \\ &H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1), \\ &H^2_*(0,1) = \left\{\Psi \in H^2(0,1) :\ \Psi_x(0) = \Psi_x(1) = 0\right\}. \end{aligned}$$

The energy space

$$\mathcal{H} = H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1) \\ \times L^2(0,1) \times L^2_*(0,1)$$

is a Hilbert space with respect to the inner product

$$\begin{aligned}\langle U,\tilde{U}\rangle_{\mathcal{H}} =& \rho\int_0^1 \phi\tilde{\phi}dx + I_\rho\int_0^1 u\tilde{u}dx + 3I_\rho\int_0^1 v\tilde{v}dx + c\int_0^1 \theta\tilde{\theta}dx \\ &+ \alpha\int_0^1 \omega\tilde{\omega}dx + G\int_0^1 (3s-\xi-\varphi_x)\left(3\tilde{s}-\tilde{\xi}-\tilde{\varphi}_x\right)dx \\ &+ D\int_0^1 \xi_x\tilde{\xi}_x dx + 4\gamma\int_0^1 s\tilde{s}dx + 3D\int_0^1 s_x\tilde{s}_x dx,\end{aligned} \tag{1.8}$$

for $U = (\varphi,\phi,\xi,u,s,v,\theta,\omega)^T \in \mathcal{H}$ and $\tilde{U} = \left(\tilde{\varphi},\tilde{\phi},\tilde{\xi},\tilde{u},\tilde{s},\tilde{v},\tilde{\theta},\tilde{\omega}\right)^T \in \mathcal{H}$.

The domain of $\mathcal{A}$ is then

$$\mathcal{D}(\mathcal{A}) = \left\{ U\in\mathcal{H} \;\middle|\; \begin{array}{ll} \varphi,\omega \in H^2_*(0,1)\cap H^1_*(0,1); & \xi,s\in H^2(0,1)\cap H^1_0(0,1); \\ \phi\in H^1_*(0,1); & u,v,\theta\in H^1_0(0,1) \end{array}\right\}.$$

Using the standard semigroup method (see, for instance [19, 30]), one easily establishes the following well-posedness result:

Theorem 1.1. *Let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C(\mathbb{R}^+,\mathbb{H})$ of problem* (1.1)-(1.3). *Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$. Then $U \in C(\mathbb{R}^+,\mathcal{D}(\mathcal{A}))\cap C^1(\mathbb{R}^+,\mathcal{H})$.*

This paper is organized as follows. In section 2, we state and prove some technical lemmas needed in the proof of our main results. In section 3, we show that the system is exponentially stable under condition (1.5). In what follows, we use c_1 to denote a generic positive constant.

2. Technical lemmas

This section is devoted to the statements and proofs of some technical lemmas needed for the proof of our stability result.

Lemma 2.1. *Let $(\varphi,\psi,s,\theta,\omega)$ be the solution of* (1.1)-(1.3)*, then the energy functional defined by*

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 \Big[&\rho\varphi_t^2 + I_\rho(3s_t-\psi_t)^2 + 3I_\rho s_t^2 + D(3s_x-\psi_x)^2 \\ &+ 3Ds_x^2 + 4\gamma s^2 + G(\psi-\varphi_x)^2 + c\theta^2 + \alpha\omega^2\Big]dx,\ \forall t\geq 0,\end{aligned} \tag{2.1}$$

satisfies, along a strong solution of (1.1)-(1.3),

$$E'(t) = -\kappa_2\int_0^1 \omega_x^2 dx - \kappa_3\int_0^1 \omega^2 dx \leq 0, \qquad \forall t\geq 0. \tag{2.2}$$

Proof. Equation (2.2) follows by multiplying the five equations of system (1.1) by $\varphi_t, (3s_t-\psi_t), s_t, \theta$ and ω respectively, integrating by parts over $(0,1)$, boundary conditions (1.2) and summing up. □

Lemma 2.2. *The functional $F_1(t)$ defined by*

$$F_1(t) = \frac{3\alpha I_\rho}{m}\int_0^1 s_t\left(\int_0^x \omega(y)dy\right)dx + \frac{3\kappa_1 I_\rho}{m}\int_0^1 s\theta dx + \frac{3\kappa_1 I_\rho\delta}{2mc}\int_0^1 s^2 dx, \tag{2.3}$$

satisfies, for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, the estimate

$$\begin{aligned} F_1'(t) \leq & - I_\rho\int_0^1 s_t^2 dx + \varepsilon_1\int_0^1 s_x^2 dx + \varepsilon_2\int_0^1 (\psi-\varphi_x)^2 dx \\ & + \varepsilon_3\int_0^1 \theta^2 dx + c_1\left(1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)\int_0^1 \omega^2 dx \\ & + c_1\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \omega_x^2 dx. \end{aligned} \tag{2.4}$$

Proof. By taking the derivative of F_1, using (1.1), integrating by parts and the fact that $\int_0^1 \omega(x)\,dx = 0$, we get,

$$\begin{aligned} F_1'(t) = & -\frac{3\alpha D}{m}\int_0^1 s_x\omega dx - \frac{3\alpha G}{m}\int_0^1 (\psi-\varphi_x)\int_0^x \omega(y)dydx \\ & - \frac{4\alpha\gamma}{m}\int_0^1 s\int_0^x \omega(y)dydx + \frac{\alpha\delta}{m}\int_0^1 \theta\int_0^x \omega(y)dydx \\ & + \alpha\int_0^1 \omega^2 dx + \frac{3I_\rho\kappa_2}{m}\int_0^1 s_t\omega_x dx - 3I_\rho\int_0^1 s_t^2 dx \\ & - \frac{3I_\rho\kappa_3}{m}\int_0^1 s_t\int_0^x \omega(y)dydx - \frac{3I_\rho\kappa_1^2}{mc}\int_0^1 s\omega_x dx. \end{aligned} \tag{2.5}$$

Using Young's, Poincaré's and Cauchy-Schwarz inequalities, we have, for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$

$$-\frac{3\alpha D}{m}\int_0^1 s_x\omega dx \leq \frac{\varepsilon_1}{4}\int_0^1 s_x^2 dx + \frac{c_1}{\varepsilon_1}\int_0^1 \omega^2 dx, \tag{2.6}$$

$$\begin{aligned} & -\frac{3\alpha G}{m}\int_0^1 (\psi-\varphi_x)\int_0^x \omega(y)dydx \\ & \qquad \leq \varepsilon_2\int_0^1 (\psi-\varphi_x)^2 dx + \frac{c_1}{\varepsilon_2}\int_0^1 \left(\int_0^x \omega(y)dy\right)^2 dx \\ & \qquad \leq \varepsilon_2\int_0^1 (\psi-\varphi_x)^2 dx + \frac{c_1}{\varepsilon_2}\int_0^1 \omega^2 dx, \end{aligned} \tag{2.7}$$

similarly,

$$-\frac{4\alpha\gamma}{m}\int_0^1 s\int_0^x \omega(y)dydx \leq \frac{\varepsilon_1}{4}\int_0^1 s_x^2 dx + \frac{c_1}{\varepsilon_1}\int_0^1 \omega^2 dx, \tag{2.8}$$

$$\frac{\alpha\delta}{m}\int_0^1 \theta\int_0^x \omega(y)dydx \le \varepsilon_3\int_0^1 \theta^2 dx + \frac{c_1}{\varepsilon_3}\int_0^1 \omega^2 dx, \tag{2.9}$$

$$\frac{3I_\rho\kappa_2}{m}\int_0^1 s_t\omega_x dx \le I_\rho\int_0^1 s_t^2 dx + c_1\int_0^1 \omega_x^2 dx, \tag{2.10}$$

$$-\frac{3I_\rho\kappa_3}{m}\int_0^1 s_t\int_0^x \omega(y)dydx \le I_\rho\int_0^1 s_t^2 dx + c_1\int_0^1 \omega^2 dx, \tag{2.11}$$

$$-\frac{3I_\rho\kappa_1^2}{mc}\int_0^1 s\omega_x dx \le \frac{\varepsilon_1}{2}\int_0^1 s_x^2 dx + \frac{c_1}{\varepsilon_1}\int_0^1 \omega_x^2 dx. \tag{2.12}$$

Estimate (2.4) follows by substituting (2.6)(2.12) into (2.5). □

Lemma 2.3. *The functional $F_2(t)$ defined by*

$$F_2(t) = \frac{\alpha c}{\kappa_1}\int_0^1 \theta\left(\int_0^x \omega(y)dy\right)dx, \tag{2.13}$$

satisfies, the following estimate

$$F_2'(t) \le -\frac{c}{2}\int_0^1 \theta^2 dx + c_1\int_0^1 s_t^2 dx + c_1\int_0^1 \omega^2 dx + c_1\int_0^1 \omega_x^2 dx. \tag{2.14}$$

Proof. Direct computations, using (1.1), integrating by parts and the fact that $\int_0^1 \omega(x)\,dx = 0$, yield

$$\begin{aligned} F_2'(t) = &-c\int_0^1 \theta^2 dx + \alpha\int_0^1 \omega^2 dx - \frac{\alpha\delta}{\kappa_1}\int_0^1 s_t\int_0^x \omega(y)dydx \\ &+ \frac{c\kappa_2}{\kappa_1}\int_0^1 \theta\omega_x dx - \frac{c\kappa_3}{\kappa_1}\int_0^1 \theta\int_0^x \omega(y)dydx - \frac{mc}{\kappa_1}\int_0^1 \theta s_t dx. \end{aligned} \tag{2.15}$$

By virtue of Young's and Cauchy-Schwarz inequalities, we find

$$-\frac{\alpha\delta}{\kappa_1}\int_0^1 s_t\int_0^x \omega(y)dydx \le c_1\int_0^1 s_t^2 dx + c_1\int_0^1 \omega^2 dx, \tag{2.16}$$

$$\frac{c\kappa_2}{\kappa_1}\int_0^1 \theta\omega_x dx \le \frac{c}{8}\int_0^1 \theta^2 dx + c_1\int_0^1 \omega_x^2 dx, \tag{2.17}$$

$$-\frac{c\kappa_3}{\kappa_1}\int_0^1 \theta\int_0^x \omega(y)dydx \le \frac{c}{8}\int_0^1 \theta^2 dx + c_1\int_0^1 \omega^2 dx, \tag{2.18}$$

$$-\frac{mc}{\kappa_1}\int_0^1 \theta s_t dx \le \frac{c}{4}\int_0^1 \theta^2 dx + c_1\int_0^1 s_t^2 dx, \tag{2.19}$$

which yields the desired result (2.14), by inserting (2.16)(2.19) into (2.15). □

Lemma 2.4. *The functional $F_3(t)$ defined by*

$$F_3(t) = -\frac{D\rho}{G}\int_0^1 \varphi_t s_x dx + I_\rho \int_0^1 s_t (\psi - \varphi_x)\, dx, \tag{2.20}$$

satisfies, for any $\varepsilon_4 > 0$, the estimate

$$\begin{aligned} F_3'(t) \leq &-\frac{G}{2}\int_0^1 (\psi - \varphi_x)^2\, dx + \varepsilon_4 \int_0^1 (3s_t - \psi_t)^2\, dx \\ &+ c_1\left(1 + \frac{1}{\varepsilon_4}\right)\int_0^1 s_t^2 dx + c_1\int_0^1 s^2 dx + c_1 \int_0^1 \theta^2 dx \\ &+ c_1 \int_0^1 \omega_x^2 dx + D\chi \int_0^1 \varphi_{tx} s_t dx. \end{aligned} \tag{2.21}$$

Proof. Differentiating F_3, using (1.1) and intgrating by parts, we obtain

$$\begin{aligned} F_3'(t) = &-\frac{D\rho}{G}\int_0^1 \varphi_t s_{tx} dx - G\int_0^1 (\psi - \varphi_x)^2\, dx - \frac{4}{3}\gamma \int_0^1 s(\psi - \varphi_x)\, dx \\ &+ \frac{\delta}{3}\int_0^1 \theta(\psi - \varphi_x)\, dx - \frac{m}{3}\int_0^1 \omega_x (\psi - \varphi_x)\, dx \\ &+ I_\rho \int_0^1 s_t \psi_t dx - I_\rho \int_0^1 s_t \varphi_{tx} dx. \end{aligned}$$

Using the simple equality $\psi_t = -(3s_t - \psi_t) + 3s_t$, we arrive at

$$\begin{aligned} F_3'(t) = &- G\int_0^1 (\psi - \varphi_x)^2\, dx - \frac{4}{3}\gamma \int_0^1 s(\psi - \varphi_x)\, dx + 3I_\rho \int_0^1 s_t^2 dx \\ &+ \frac{\delta}{3}\int_0^1 \theta(\psi - \varphi_x)\, dx - \frac{m}{3}\int_0^1 \omega_x (\psi - \varphi_x)\, dx \\ &- I_\rho \int_0^1 s_t (3s_t - \psi_t)\, dx + D\chi \int_0^1 \varphi_{tx} s_t dx. \end{aligned} \tag{2.22}$$

Applying Young's and Poincaré's inequalities, for $\varepsilon_4 > 0$, we get

$$-\frac{4}{3}\gamma \int_0^1 s(\psi - \varphi_x)\, dx \leq \frac{G}{8}\int_0^1 (\psi - \varphi_x)^2\, dx + c_1 \int_0^1 s^2 dx, \tag{2.23}$$

$$\frac{\delta}{3}\int_0^1 \theta(\psi - \varphi_x)\, dx \leq \frac{G}{8}\int_0^1 (\psi - \varphi_x)^2\, dx + c_1 \int_0^1 \theta^2 dx, \tag{2.24}$$

$$-\frac{m}{3}\int_0^1 \omega_x (\psi - \varphi_x)\, dx \leq \frac{G}{4}\int_0^1 (\psi - \varphi_x)^2\, dx + c_1 \int_0^1 \omega_x^2 dx, \tag{2.25}$$

$$-I_\rho \int_0^1 s_t (3s_t - \psi_t)\, dx \leq \varepsilon_4 \int_0^1 (3s_t - \psi_t)^2\, dx + \frac{c_1}{\varepsilon_4}\int_0^1 s_t^2 dx. \tag{2.26}$$

By substituting (2.23)-(2.26) into (2.22), we obtain (2.21). □

Lemma 2.5. *The functional $F_4(t)$ defined by*

$$F_4(t) = -\frac{D\rho}{G}\int_0^1 \varphi_t \left(3s_x - \psi_x\right) dx + I_\rho \int_0^1 \left(\psi - \varphi_x\right)\left(3s_t - \psi_t\right) dx, \tag{2.27}$$

satisfies, the estimate

$$\begin{aligned} F_4'(t) \leq & -\frac{I_\rho}{2}\int_0^1 \left(3s_t - \psi_t\right)^2 dx + G\int_0^1 \left(\psi - \varphi_x\right)^2 dx \\ & + c_1 \int_0^1 s_t^2 dx + D\chi \int_0^1 \left(3s_t - \psi_t\right)\varphi_{tx} dx. \end{aligned} \tag{2.28}$$

Proof. Direct differentiation of F_4, using (1.1) and then integrating by parts, gives

$$\begin{aligned} F_4'(t) = & -\frac{D\rho}{G}\int_0^1 \varphi_t \left(3s_x - \psi_x\right)_t dx + G\int_0^1 \left(\psi - \varphi_x\right)^2 dx \\ & + I_\rho \int_0^1 \left(3s_t - \psi_t\right)\psi_t dx - I_\rho \int_0^1 \left(3s_t - \psi_t\right)\varphi_{tx} dx. \end{aligned}$$

By using the equality $\psi_t = -\left(3s_t - \psi_t\right) + 3s_t$, we obtain

$$\begin{aligned} F_4'(t) = & - I_\rho \int_0^1 \left(3s_t - \psi_t\right)^2 dx + 3I_\rho \int_0^1 s_t \left(3s_t - \psi_t\right) dx \\ & + G\int_0^1 \left(\psi - \varphi_x\right)^2 dx + D\chi \int_0^1 \left(3s_t - \psi_t\right)\varphi_{tx} dx. \end{aligned}$$

Estimate (2.28) follows thanks Youngs inequality. □

Lemma 2.6. *The functional $F_5(t)$ defined by*

$$F_5(t) = -\rho\int_0^1 \left(\int_0^x \varphi_t(y)dy\right) s dx + I_\rho \int_0^1 s_t s dx, \tag{2.29}$$

satisfies, for $\varepsilon_5 > 0$, the estimate

$$\begin{aligned} F_5'(t) \leq & -\frac{D}{2}\int_0^1 s_x^2 dx - \gamma\int_0^1 s^2 dx + \varepsilon_5 \int_0^1 \varphi_t^2 dx + c_1 \int_0^1 \theta^2 dx \\ & + c_1 \int_0^1 \omega_x^2 dx + c_1\left(1 + \frac{1}{\varepsilon_5}\right)\int_0^1 s_t^2 dx. \end{aligned} \tag{2.30}$$

Proof. The derivative of F_5, using (1.1), integration by parts and the boundary conditions, give

$$\begin{aligned} F_5'(t) = & - D\int_0^1 s_x^2 dx - \frac{4}{3}\gamma\int_0^1 s^2 dx + \frac{\delta}{3}\int_0^1 \theta s dx - \frac{m}{3}\int_0^1 \omega_x s dx \\ & + I_\rho \int_0^1 s_t^2 dx - \rho\int_0^1 s_t \left(\int_0^x \varphi_t(y)dy\right) dx. \end{aligned} \tag{2.31}$$

By using Young's, Poincaré's and Cauchy-Schwarz inequalities, for $\varepsilon_5 > 0$, we have

$$\frac{\delta}{3}\int_0^1 \theta s dx \leq \frac{\gamma}{3}\int_0^1 s^2 dx + c_1 \int_0^1 \theta^2 dx, \tag{2.32}$$

$$-\frac{m}{3}\int_0^1 \omega_x s dx \le \frac{D}{2}\int_0^1 s_x^2 dx + c_1\int_0^1 \omega_x^2 dx, \tag{2.33}$$

$$-\rho\int_0^1 s_t\left(\int_0^x \varphi_t(y)dy\right)dx \le \varepsilon_5\int_0^1 \varphi_t^2 dx + \frac{c_1}{\varepsilon_5}\int_0^1 s_t^2 dx. \tag{2.34}$$

Relation (2.30) follows by substituting (2.32)-(2.34) into (2.31). □

Lemma 2.7. *The functional F_6 defined by*

$$F_6(t) = -\rho\int_0^1 \varphi_t\varphi dx, \tag{2.35}$$

satisfies, the estimate

$$F_6'(t) \le -\rho\int_0^1 \varphi_t^2 dx + \frac{D}{4}\int_0^1 (3s_x - \psi_x)^2 dx + c_1\int_0^1 s_x^2 dx + c_1\int_0^1 (\psi - \varphi_x)^2 dx. \tag{2.36}$$

Proof. Direct differentiation of F_6, using (1.1) and then integrating by parts, gives

$$F_6'(t) = -G\int_0^1 \varphi_x(\psi - \varphi_x)dx - \rho\int_0^1 \varphi_t^2 dx.$$

Using the simple relation $\varphi_x = -(\psi - \varphi_x) - (3s - \psi) + 3s$, we get

$$F_6'(t) = G\int_0^1 (\psi - \varphi_x)^2 dx + G\int_0^1 (\psi - \varphi_x)(3s - \psi)dx - 3G\int_0^1 (\psi - \varphi_x)s\,dx - \rho\int_0^1 \varphi_t^2 dx.$$

Using Young's and Poincaré's inequalities, lead to the desired estimation. □

Lemma 2.8. *The functional F_7 defined by*

$$F_7(t) = I_\rho\int_0^1 (3s - \psi)(3s_t - \psi_t)dx, \tag{2.37}$$

satisfies, the estimate

$$F_7'(t) \le -\frac{D}{2}\int_0^1 (3s_x - \psi_x)^2 dx + I_\rho\int_0^1 (3s_t - \psi_t)^2 dx + c_1\int_0^1 (\psi - \varphi_x)^2 dx. \tag{2.38}$$

Proof. A simple differentiation of F_7, using (1.1) together with integration by parts, yield

$$F_7'(t) = I_\rho\int_0^1 (3s_t - \psi_t)^2 dx - D\int_0^1 (3s_x - \psi_x)^2 dx + G\int_0^1 (\psi - \varphi_x)(3s - \psi)dx.$$

The use of Young's and Poincaré's inequalities lead to (2.38). □

3. Stability result

In this section, we prove under the condition of equal wave-speed propagation (1.5) that the energy associated with (1.1)(1.3) is exponentially stable. To achieve this goal, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E.

Lemma 3.1. *Let* $(\varphi, \psi, s, \theta, \omega)$ *be the solution of* (1.1)-(1.3) *and assume* $\chi = 0$. *Then, for* N, $N_1, N_2, N_3, N_4, N_5 > 0$ *to be chosen appropriately later, the functional defined by*

$$\begin{aligned}\mathcal{L}(t) = NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + N_4F_4(t)\\ + N_5F_5(t) + F_6(t) + F_7(t),\end{aligned} \tag{3.1}$$

satisfies, for N *sufficiently large,*

$$\tau_1 E(t) \leq \mathcal{L}(t) \leq \tau_2 E(t), \qquad \forall t \geq 0, \tag{3.2}$$

and the estimate

$$\mathcal{L}'(t) \leq -\tau_3 E(t), \tag{3.3}$$

where τ_1, τ_2 *and* τ_3 *are positive constants.*

Proof. From (3.1) and the Lemmas in Section 2, it follows that

$$\begin{aligned}\Big|\mathcal{L}(t) - NE(t)\Big| \leq & \frac{3\alpha I_\rho}{m} N_1 \int_0^1 \left| s_t \int_0^x \omega(y)dy \right| dx + \frac{3\kappa_1 I_\rho}{m} N_1 \int_0^1 \Big| s\theta \Big| dx \\ &+ \frac{3\kappa_1 I_\rho \delta}{2mc} N_1 \int_0^1 s^2 dx + \frac{\alpha c}{\kappa_1} N_2 \int_0^1 \left| \theta \int_0^x \omega(y)dy \right| dx \\ &+ \frac{D\rho}{G} N_3 \int_0^1 \Big|\varphi_t s_x\Big| dx + I_\rho N_3 \int_0^1 \Big| s_t (\psi - \varphi_x) \Big| dx \\ &+ \frac{D\rho}{G} N_4 \int_0^1 \Big| \varphi_t (3s_x - \psi_x) \Big| dx \\ &+ I_\rho N_4 \int_0^1 \Big| (\psi - \varphi_x)(3s_t - \psi_t) \Big| dx \\ &+ \rho N_5 \int_0^1 \left| s \int_0^x \varphi_t(y)dy \right| dx + I_\rho N_5 \int_0^1 \Big| s_t s \Big| dx \\ &+ \rho \int_0^1 \Big| \varphi_t \varphi \Big| dx + I_\rho \int_0^1 \Big| (3s_t - \psi_t)(3s - \psi) \Big| dx.\end{aligned}$$

Exploiting Young's, Cauchy-Schwarz and Poincaré's inequalities, we get

$$\left|\mathcal{L}(t) - NE(t)\right| \leq c_1 \int_0^1 \bigg[\varphi_t^2 + (3s_t - \psi_t)^2 + s_t^2 + (3s_x - \psi_x)^2 + s_x^2 + s^2 + (\psi - \varphi_x)^2 + \theta^2 + \omega^2\bigg] dx.$$

Consequently, we have

$$\left|\mathcal{L}(t) - NE(t)\right| \leq c_1 E(t),$$

that is,

$$(N - c_1)\, E(t) \leq \mathcal{L}(t) \leq (N + c_1)\, E(t).$$

By choosing N large enough, (3.2) follows. Next, to prove (3.3), we take the derivative of $\mathcal{L}(t)$, use (2.2), (2.4), (2.14), (2.21), (2.28), (2.30), (2.36), (2.38), and set

$$\varepsilon_1 = \frac{DN_5}{4N_1}, \quad \varepsilon_2 = \frac{GN_3}{4N_1}, \quad \varepsilon_3 = \frac{cN_2}{4N_1}, \quad \varepsilon_4 = \frac{I_\rho N_4}{4N_3}, \quad \varepsilon_5 = \frac{\rho}{2N_5}.$$

So, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\frac{\rho}{2}\int_0^1 \varphi_t^2 dx - \left[\frac{I_\rho}{4}N_4 - I_\rho\right]\int_0^1 (3s_t - \psi_t)^2\, dx - \frac{D}{4}\int_0^1 (3s_x - \psi_x)^2\, dx \\
& - \left[I_\rho N_1 - c_1 N_2 - c_1 N_3\left(1 + \frac{N_3}{N_4}\right) - c_1 N_4 - c_1 N_5\,(1 + N_5)\right]\int_0^1 s_t^2 dx \\
& - \left[\frac{G}{4}N_3 - GN_4 - c_1\right]\int_0^1 (\psi - \varphi_x)^2\, dx - \left[\frac{D}{4}N_5 - c_1\right]\int_0^1 s_x^2 dx \\
& - [\gamma N_5 - c_1 N_3]\int_0^1 s^2 dx - \left[\frac{c}{4}N_2 - c_1 N_3 - c_1 N_5\right]\int_0^1 \theta^2 dx \\
& - \left[N\kappa_2 - c_1 N_1\left(1 + \frac{N_1}{N_5}\right) - c_1 N_2 - c_1 N_3 - c_1 N_5\right]\int_0^1 \omega_x^2 dx \\
& - \left[N\kappa_3 - c_1 N_1\left(1 + \frac{N_1}{N_5} + \frac{N_1}{N_3} + \frac{N_1}{N_2}\right) - c_1 N_2\right]\int_0^1 \omega^2 dx.
\end{aligned}$$

At this point, we choose the constants carefully. First, let us take $N_4 > 4$. We then choose N_3 large enough such that

$$\frac{G}{4}N_3 - GN_4 - c_1 > 0.$$

After that, we select N_5 large enough so that

$$\gamma N_5 - c_1 N_3 > 0 \qquad \text{and} \qquad \frac{D}{4}N_5 - c_1 > 0.$$

Next, we choose N_2 large enough such that

$$\frac{c}{4}N_2 - c_1 N_3 - c_1 N_5 > 0.$$

Then, we pick N_1 so large that

$$I_\rho N_1 - c_1 N_2 - c_1 N_3 \left(1 + \frac{N_3}{N_4}\right) - c_1 N_4 - c_1 N_5 \left(1 + N_5\right) > 0.$$

Finallay, we choose N very large enough (even larger so that (3.2) remains valid) such that

$$N\kappa_2 - c_1 N_1 \left(1 + \frac{N_1}{N_5}\right) - c_1 N_2 - c_1 N_3 - c_1 N_5 > 0,$$

and

$$N\kappa_3 - c_1 N_1 \left(1 + \frac{N_1}{N_5} + \frac{N_1}{N_3} + \frac{N_1}{N_2}\right) - c_1 N_2 > 0.$$

Therefore, we arrive at

$$\mathcal{L}'(t) \leq -\tau_4 \int_0^1 \Big[\varphi_t^2 + \left(3s_t - \psi_t\right)^2 + s_t^2 + \left(3s_x - \psi_x\right)^2 + s_x^2 + s^2 + \left(\psi - \varphi_x\right)^2 + \theta^2 + \omega_x^2 + \omega^2\Big] dx, \quad \tau_4 > 0.$$

We finally use Poincaré's inequality to substitute $-\int_0^1 \omega_x^2 dx$ by $-\int_0^1 \omega^2 dx$ and, hence, (3.3) is established. □

We are now ready to state and prove the following exponential stability result.

Theorem 3.2. *Let* $(\varphi, \psi, s, \theta, \omega)$ *be the solution of* (1.1)-(1.3) *and assume* (1.5). *Then, there exist two positive constants* λ_1, λ_2 *such that the energy functional satisfies*

$$E(t) \leq \lambda_1 e^{-\lambda_2 t}, \qquad \forall t \geq 0. \tag{3.4}$$

Proof. The combination of (3.2) and (3.3) gives

$$\mathcal{L}'(t) \leq -\lambda_2 \mathcal{L}(t), \qquad t \geq 0, \tag{3.5}$$

where $\lambda_2 = \frac{\tau_3}{\tau_2}$. A simple integration of (3.5) over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\lambda_2 t}, \qquad t \geq 0.$$

which yields the desired result (3.4) by using the other side of the equivalence relation (3.2) again. □

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Capacity solution for an elliptic coupled system with lower term in Orlicz spaces

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Abstract. In this paper, we will deal with the capacity solution for a nonlinear elliptic coupled system with a Leray-Lions operator $Au = -\text{div }\sigma(x, u, \nabla u)$ acting from Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$ into its dual, where M is an N-function.

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1. Introduction

Let Ω be an open bounded in $\mathbb{R}^N$, $N \geq 1$, and consider the coupled nonlinear elliptic system

$$-\text{div }\sigma(x, u, \nabla u) + \Phi(x, u) = \kappa(u) \mid \nabla u \mid^2 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\text{div}(\kappa(u)\nabla\varphi) = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\varphi = \varphi_0,\ u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

We assume that the following assumptions hold. Let M and P be two N-functions such that $P \ll M$ (P grows essentially less rapidly than Q) and $\overline{M}$ the N-function conjugate to M (see preliminaries).
$\sigma : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for almost every $x \in \Omega$ and for every $s, s_1, s_2 \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^N$,

$$\mid \sigma(x, s, \xi) \mid \leq \nu[a_0(x) + \overline{M}^{-1} P(k_1 \mid s \mid) + \overline{M}^{-1} M(k_2 \mid \xi \mid)], \tag{1.4}$$

$$\mid \sigma(x, s_1, \xi) - \sigma(x, s_2, \xi) \mid \leq \nu[a_1(x) + \mid s_1 \mid + \mid s_2 \mid + \overline{P}^{-1}(k_3 M(\mid \xi \mid))], \tag{1.5}$$

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$$(\sigma(x,s,\xi)-\sigma(x,s,\xi^*))(\xi-\xi^*)\geq \alpha M(\mid \xi-\xi^*\mid), \tag{1.6}$$

$$\sigma(x,s,0)=0, \tag{1.7}$$

where $a_0(.)\in E_{\overline{M}}(\Omega)$, $a_1(.)\in E_P(\Omega)$ ($E_{\overline{M}}(\Omega)$ and $E_P(\Omega)$ are specific Orlicz spaces) and α, ν, $k_i>0$ (i=1, 2, 3), are given real numbers.
Furthermore, let $\Phi:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, the growth condition

$$\mid \Phi(x,s)\mid\leq h(x)\overline{M}^{-1}M\left(\frac{\mid s\mid}{\lambda C_0}\right), \tag{1.8}$$

where $\lambda=diam(\Omega)$ and $\|h\|_{L^\infty(\Omega)}<\dfrac{\alpha}{2\lambda}$ and C_0 is a constant large enough.

$$\kappa\in C(\mathbb{R}) \text{ and there exists } \overline{\kappa}\in\mathbb{R} \text{ such that } 0<\kappa(s)\leq\overline{\kappa}, \text{ for all } s\in\mathbb{R}, \tag{1.9}$$

$$\varphi_0\in H^1(\Omega)\cap L^\infty(\Omega). \tag{1.10}$$

In this paper, we will introduce a solution of the coupled system (1.1)-(1.3) called the capacity solution. This type of solution will deal with the phenomena caused by the possible degeneration of the (1.1)-(1.3). Indeed, one cannot use the weak solution of (1.1) since κ can tend towards 0 when $|u|$ tends to infinity and consequently the equation becomes degenerate, no a priori estimates for $\nabla\varphi$ will be available and then φ may not belong to a Sobolev space. To overcome this obstacle, we use the entire function $\Phi=\kappa(u)\nabla\varphi$ instead of φ to show that $\Phi\in(L^2(\Omega))^N$.

The idea of capacity solution is inspired from the weak and renormalized solutions and X. Xu is the first author who introduced the concept in [13] where $\sigma:\mathbb{R}^N\to\mathbb{R}^N$ is a continuous function satisfying the conditions: $\exists\mu>0$, $\forall\mid\xi\mid\gg 1$ (i.e. $\mid\xi\mid$ is large enough), $\mid a(\xi)\mid\leq\mu\mid\xi\mid$, and $\exists\alpha>0$, $\forall\xi,\xi^*\in\mathbb{R}^N$, $\xi\neq\xi^*$, $(\sigma(\xi)-\sigma(\xi^*))(\xi-\xi^*)\geq\alpha\mid\xi-\xi^*\mid^2$. Also, he used this concept in other papers with various conditions (See [14]). Later, from other authors in [7], showed the existence of a capacity solution to the problem (1.1)-(1.3) where $\sigma=\sigma(x,\nabla u)$ is a Leray-Lions operator from $L^p(W^{1,p})$ into $L^{p'}(W^{-1,p'})$, $p\geq 2$, $\frac{1}{p}+\frac{1}{p'}=1$ and $\Phi=\Phi(x,s)$ satisfies the sign condition, and $\mid\Phi(x,s)\mid\leq h_r(x)$ with $h_r\in L^1(\Omega)$, for all $\mid s\mid\leq r$, $\forall r\geq 0$. For the parabolic, we refer the reader to [10]. Recently, the existence of a capacity solution in the context of Orlicz-Sobolev spaces with $\sigma=\sigma(x,u,\nabla u)$ and $H=0$ has been established in [12].

The motivation behind the study of differential equations comes from applications of non Newtonian mechanics turbulence modelling to as an example of an operator for which the present result can be applied, we give

$$\begin{cases} -\Delta_M u+h(x)M^{-1}M(\alpha u)=ru^\zeta e^{\frac{-s}{k_B u}}\mid\nabla u\mid^2 & \text{in } \Omega,\\ \operatorname{div}(\kappa(u)\nabla\varphi)=0 & \text{in } \Omega,\end{cases} \tag{1.11}$$

where $\Delta_M u=-\operatorname{div}\left((1+\mid u\mid)^2 Du\frac{\log(e+Du)}{|Du|}\right)$, $h(.)\in(L^\infty(Q_T))^N$ and $M(t)=t\log(e+t)$ is an N -function, φ represent the electric motive force, u the temperature inside the electrical conductor, and $\kappa(u)=ru^\zeta e^{\frac{-s}{k_B u}}$, the electrical conductivity where it means the ability of electrical material to pass charges, where $u>0$, $r,s\in\mathbb{R}^+$, $\zeta\in[-1,1)$ and k_B is the Boltzmann constant. Other applications of the stationary case of the thermostat problem can be found in [8, 15].

Our novelty in the present paper is to give the existence of a capacity solution of (1.1)-(1.3) in the framework of Orlicz spaces with the presence of a perturbation $\Phi(x,u)$. The difficulties encountered during the proof are that the term H satisfies neither the coercivity condition nor the monotony nor the sign condition and the nonlinearity described by N-functions M. The Δ_2-condition is not imposed on the N-functions M, we will lose the reflexivity of the space $L_M(\Omega)$ and $W_0^1L_M(\Omega)$. To overcome this difficulty, we will first introduce and prove the existence of the solution for the auxiliary elliptic problem (2.8) and by Schauder's fixed point theorem, we show the existence of the uniqueness of the weak solutions for two equations (1.2) and (1.1). Secondly, with adequate approximate problems we establish some a priori estimates for the approximate solution sequence. Finally, we draw a subsequence to obtain a limit function and prove this function is a capacity solution in the sense of Definition 4.1 by virtue of the convergence results of approximate solutions. Note that the second lower order term H is controlled by a non-polynomial growth (see (1.8)). It is similar to those in [2, 3]. Finally, it should be noted that this work is an extension of the results of [12].

The contents of this article are summarized as follows: Section 2 presents the mathematical preliminaries. In Section 3, we make precise all the basic assumptions on σ, H, κ, φ and some technical results. Finally, in Section 4, we give the definition of a capacity solution of (1.1)-(1.3) and we prove the main result (Theorem 4.2).

2. Preliminaries

Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be an N-function, that is, M is continuous, convex, with $M(t)>0$ for $t>0$, $\dfrac{M(t)}{t}\to 0$ as $t\to 0$, and $\dfrac{M(t)}{t}\to+\infty$ as $t\to+\infty$. Equivalently, M admits the representation $M(t)=\int_0^t a(s)ds$, where $a:\mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing, right continuous, with $a(0)=0, a(t)>0$ for $t>0$, and $a(t)\to+\infty$ as $t\to+\infty$. The N-function $\overline{M}$conjugate to M is defined by $\overline{M}(t)=\int_0^t\overline{a}(s)ds$, where $\overline{a}:\mathbb{R}^+\to\mathbb{R}^+$, is given by $\overline{a}(t)=\sup_{s\geq 0}\{s:a(s)\leq t\}$.
The N-function M is said to satisfy the Δ_2-condition if, for some k, $M(2t)\leq kM(t)$ for all $t\in IR^+$.

We will extend these N-functions into even functions on all $\mathbb{R}$. Let P and Q be two N-functions. $P\ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\epsilon>0$, $\dfrac{P(t)}{Q(\epsilon t)}\to 0$ as $t\to+\infty$. This is the case if and only if $\lim_{t\to+\infty}\dfrac{Q^{-1}(t)}{P^{-1}(t)}=0$.
The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$), is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(|u(x)|)dx<+\infty \left(\text{resp. } \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)dx<+\infty \quad \text{for some} \quad \lambda>0\right).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\},$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uvdx$ and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|u\|_{\overline{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$) [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [4]).

Lemma 2.1. ([11]) *For all $u \in W_0^1L_M(\Omega)$ with meas$(\Omega) < +\infty$ one has*

$$\int_\Omega M\left(\frac{|\, u \,|}{\lambda}\right) dx \leq \int_\Omega M(|\, \nabla u \,|)dx. \tag{2.1}$$

where $\lambda = diam(\Omega)$, is the diameter of Ω.

Statement of useful results.
We assume that there exists four positive constants γ_0 and γ_1 such that

$$|u|^2 \leq \gamma_0 M(u), \quad \text{and } |u|^2 \leq \gamma_1 P(u) \quad \text{for all} \quad u \geq 0, \tag{2.2}$$

Hence, the following continuous inclusions hold true:

$$L_M(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{M}}(\Omega), \quad \text{and } L_P(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{P}}(\Omega). \tag{2.3}$$

And we also deduce that

$$W_0^1L_M(\Omega) \hookrightarrow H_0^1(\Omega), \quad \text{and } H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\overline{M}}(\Omega). \tag{2.4}$$

Example 2.2. The N-function $M(t) = tlog(e+t)$ verifies the previous results.

Consider the following set $\mathbf{W} = \left\{\omega \in E_M(\Omega) : \int_\Omega M(\frac{|\,\omega\,|}{\lambda C_0})dx \leq 1\right\}$.
It is closed and convex. Indeed let $\omega_n \in \mathbf{W}$ such that $\omega_n \to \omega$ strongly in $E_M(\Omega)$, then for any $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$ we have $\|\omega_n - \omega\|_M \leq \epsilon$ and $\|\frac{\omega}{\lambda C_0}\|_M \leq \|\frac{\omega_n - \omega}{\lambda C_0}\|_M + \|\frac{\omega_n}{\lambda C_0}\|_M \leq \frac{\epsilon}{\lambda C_0} + 1$. Let tends as $\epsilon \to 0$ we have $\omega \in \mathbf{W}$; Thus $\mathbf{W}$ is closed. And since M is convex function, we deduce that $\mathbf{W}$ is also convex. Now, let is start by this first result that we will use later. Suppose that σ verifies the following strong hypothesis: There exists $a_3(.) \in E_{\overline{M}}(\Omega)$, and $\nu > 0$ and $k_4 \geq 0$, such that for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$|\, \sigma(x,s,\xi) \,| \leq \nu[a_3(x) + \overline{M}^{-1}(M(k_4 \,|\, \xi \,|))], \tag{2.5}$$

$\kappa \in C(\mathbb{R})$ and there exist κ_1 and $\kappa_2 \in \mathbb{R}$ such that

$$0 < \kappa_1 \leq \kappa(s) \leq \kappa_2, \text{ for all } s \in \mathbb{R}. \tag{2.6}$$

Then

$$\mathrm{div}(\kappa(\omega)\varphi\nabla\varphi) \in H^{-1}(\Omega). \tag{2.7}$$

Proof. Let $\omega \in \mathbf{W}$, we consider the elliptic problem

$$\begin{cases} \mathrm{div}(\kappa(\omega)\nabla\varphi) = 0 & \text{in}\,\Omega, \\ \varphi = \varphi_0 & \text{on}\,\partial\Omega. \end{cases} \tag{2.8}$$

By applying Lax-Milgram's theorem, we prove that there exists a unique solution $\varphi \in H^1(\Omega)$ to (2.8) and, by (1.10) and the maximum principle, we get

$$\|\varphi\|_{L^\infty(\Omega)} \leq \|\varphi_0\|_{L^\infty(\Omega)}. \tag{2.9}$$

Multiplying the first equation of (2.8) by $\varphi - \varphi_0 \in H_0^1(\Omega)$ we get

$$\int_\Omega \kappa(\omega)\nabla(\varphi - \varphi_0) = 0,$$

therefore

$$\kappa_1 \int_\Omega |\nabla\varphi|^2\, dx \leq \int_\Omega \kappa(\omega) |\nabla\varphi|| \nabla\varphi_0 |\, dx \leq \kappa_2 \int_\Omega |\nabla\varphi|| \nabla\varphi_0 |\, dx.$$

We deduce from the Cauchy-Schwarz inequality that

$$\int_\Omega |\nabla\varphi|^2\, dx \leq C = C(\kappa_1, \kappa_2, \varphi_0). \tag{2.10}$$

Notice that $\kappa(\omega) \mid \nabla\varphi \mid^2 \in L^1(\Omega)$, this term is also belongs to the space $H^{-1}(\Omega)$. Indeed, let $\psi \in \mathcal{D}(\Omega)$ and taking $\psi\varphi$ as a test function in (2.8), we have

$$\int_\Omega \kappa(\omega)\nabla\varphi\nabla(\psi\varphi)dx = 0,$$

then

$$\int_\Omega \kappa(\omega) |\nabla\varphi|^2 \psi dx = -\int_\Omega \kappa(\omega)\varphi\nabla\varphi\nabla\psi dx = \langle \mathrm{div}(\kappa(\omega)\varphi\nabla\varphi), \psi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Thus

$$\kappa(\omega) |\nabla\varphi|^2 = \mathrm{div}(\kappa(\omega)\varphi\nabla\varphi) \quad \text{in } \mathcal{D}'(\Omega). \tag{2.11}$$

Since $\kappa(\omega)\varphi\nabla\varphi \in L^2(\Omega)^N$, we deduce (2.7). □

3. Main result

Theorem 3.1. *Assume (1.4)-(2.3), with (2.5) and (2.6) instead of (1.4) and (1.9), respectively. Then there exists a weak solution (u, φ) to problem (1.1)-(1.3), that is,*

$$\begin{cases} u \in W_0^1 L_M(\Omega),\ \sigma(x, u, \nabla u) \in L_{\overline{M}}(\Omega)^N, \\ \varphi - \varphi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega \sigma(x, u, \nabla u)\nabla\psi + \int_\Omega \Phi(x, u)\psi = -\int_\Omega \kappa(u)\varphi\nabla\varphi\nabla\psi, \quad \textit{for all} \quad \psi \in W_0^1 L_M(\Omega), \\ \displaystyle\int_\Omega \kappa(u)\nabla\varphi\nabla\psi = 0, \quad \textit{for all} \quad \psi \in H_0^1(\Omega). \end{cases}$$

Proof. Let consider the following variational formulation problem

$$\begin{cases} u \in W_0^1 L_M(\Omega),\ \sigma(x,\omega,\nabla u) \in L_{\overline{M}}(\Omega),\ \Phi(x,\omega) \in L_{\overline{M}}(\Omega), \\ \displaystyle\int_\Omega \sigma(x,\omega,\nabla u)\nabla\phi + \int_\Omega \Phi(x,\omega)\phi = -\int_\Omega \kappa(\omega)\varphi\nabla\varphi\nabla\phi, \text{for all } \phi \in W_0^1 L_M(\Omega), \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

Notice that $\operatorname{div}(\kappa(\omega)\varphi\nabla\varphi) \in H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\overline{M}}(\Omega)$, and the existence of solution to (3.1) is derived by an application of the result obtained in [11]. Also we can check that the solution of (3.1) is unique [5].

Lemma 3.2. *Let u be a weak solution of problem (3.1). Then we have $|\nabla u| \in K_M(\Omega)$, and the estimates*

$$\int_\Omega M(|\nabla u|)dx \leq C_1, \tag{3.2}$$

$$\|\sigma(x,\omega,\nabla u)\|_{\overline{M},\Omega} \leq C_2, \tag{3.3}$$

Where C_1 and C_2 are two positive constants that do not depend on ω.

Proof. Let $\eta > 0$ such that $\dfrac{|\nabla u|}{\eta} \in K_M(\Omega)$. Since $\varphi \in H^1(\Omega) \subset W^1L_M(\Omega)$, there exist $\beta > 0$ such that $\dfrac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)}|\nabla\varphi|}{\beta} \in K_{\overline{M}}(\Omega)$, we take $\psi = u$ as a test function in (3.1). In view of (1.6), (1.7), (1.8), (2.6), (2.9) and Young's inequality, and Lemma 2.1, we get

$$\begin{aligned}
\frac{\alpha}{\eta\beta}\int_\Omega M(|\nabla u|)dx &\leq \frac{1}{\eta\beta}\int_\Omega \sigma(x,\omega,\nabla u)\nabla u dx \\
&\leq \frac{\lambda}{\beta}\int_\Omega |\Phi(x,\omega)|\frac{|u|}{\eta\lambda}dx + \int_\Omega \frac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)}|\nabla\varphi|}{\beta}\frac{|\nabla u|}{\eta}dx \\
&\leq \frac{\lambda}{\beta}\int_\Omega h(x)\overline{M}^{-1}M\left(\frac{|\omega|}{\lambda C_0}\right)\frac{|u|}{\eta\lambda}dx + \int_\Omega \frac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)}|\nabla\varphi|}{\beta}\frac{|\nabla u|}{\eta}dx \\
&\leq \frac{\lambda\|h\|_{L^\infty(\Omega)}}{\beta}\int_\Omega M\left(\frac{|\omega|}{\lambda C_0}\right)dx + \frac{\lambda\|h\|_{L^\infty(\Omega)}}{\beta}\int_\Omega M\left(\frac{|u|}{\eta\lambda}\right)dx \\
&+ \int_\Omega \overline{M}\left(\frac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)}|\nabla\varphi|}{\beta}\right)dx + \int_\Omega M\left(\frac{|\nabla u|}{\eta}\right)dx \\
&\leq \frac{\lambda\|h\|_{L^\infty(\Omega)}}{\beta} + \frac{\lambda\|h\|_{L^\infty(\Omega)}}{\beta}\int_\Omega M\left(\frac{|\nabla u|}{\eta}\right)dx \\
&+ \int_\Omega \overline{M}\left(\frac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)}|\nabla\varphi|}{\beta}\right)dx + \int_\Omega M\left(\frac{|\nabla u|}{\eta}\right)dx < \infty.
\end{aligned}$$

Then we deduce that $|\nabla u| \in K_M(\Omega)$. Let prove the estimate (3.2), by (1.6), (1.7), (1.8), (2.2) and Young's inequality, and Lemma 2.1, we obtain

$$\begin{aligned}
&\alpha \int_\Omega M(|\nabla u|)dx \leq \int_\Omega \sigma(x,\omega,\nabla u)\nabla u dx \\
&\leq \int_\Omega |\Phi(x,\omega)| \frac{|u|}{\lambda} dx + \int_\Omega \kappa_2 \|\varphi_0\|_{L^\infty(\Omega)} |\nabla\varphi| |\nabla u| \, dx \\
&\leq \lambda \int_\Omega h(x)\overline{M}^{-1} M\left(\frac{|\omega|}{\lambda C_0}\right)\frac{|u|}{\lambda}dx + \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon}\int_\Omega |\nabla\varphi|^2 + \frac{\alpha\epsilon}{2}\int_\Omega |\nabla u|^2\, dx \\
&\leq \lambda\|h\|_{L^\infty(\Omega)} \int_\Omega M\left(\frac{|\omega|}{\lambda C_0}\right)dx + \lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M\left(\frac{|u|}{\lambda}\right)dx \\
&+ \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon}\int_\Omega |\nabla\varphi|^2\, dx + \frac{\alpha\epsilon}{2}\int_\Omega |\nabla u|^2\, dx \\
&\leq \lambda\|h\|_{L^\infty(\Omega)} + \lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M(|\nabla u|)dx \\
&+ \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon} C(\kappa_1,\kappa_2,\varphi_0) + \frac{\alpha\epsilon\gamma_0}{2}\int_\Omega M(|\nabla u|)dx.
\end{aligned}$$

which implies, that

$$(\alpha - \lambda\|h\|_{L^\infty(\Omega)} - \frac{\alpha\epsilon\gamma_0}{2})\int_\Omega M(|\nabla u|)dx \leq \lambda\|h\|_{L^\infty(\Omega)} + \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon} C(\kappa_1,\kappa_2,\varphi_0).$$

Then by choosing ϵ such that $\alpha - 2\lambda\|h\|_{L^\infty(\Omega)} - \dfrac{\alpha\epsilon\gamma_0}{2} > 0$, as a consequence, we have the estimate (3.2).

Remark 3.3.

- We take the constant C_0 and C_1 such that

  $$\lambda\|h\|_{L^\infty(\Omega)} + \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon} C(\kappa_1,\kappa_2,\varphi_0) < C_0\left(\alpha - 2\lambda\|h\|_{L^\infty(\Omega)} - \frac{\alpha\epsilon\gamma_0}{2}\right)$$

  and

  $$C_0 < C_1. \tag{3.4}$$

- It is clear that u belongs also to **W** and do not depends on ω.

  On the other hand, from the previous prove and (1.6), we also have

  $$\int_\Omega \sigma(x,\omega,\nabla u)\nabla u dx \leq \frac{C_1}{\alpha}. \tag{3.5}$$

From (1.6), (3.5) and Young's inequality, we get

$$\int_\Omega \sigma(x,\omega,\nabla u)\nabla\psi dx \le \int_\Omega \sigma(x,\omega,\nabla u)\nabla u dx - \int_\Omega \sigma(x,\omega,\nabla\psi)(\nabla u - \nabla\psi)dx$$
$$\le \frac{C_1}{\alpha} + \int_\Omega \mid \sigma(x,\omega,\nabla\psi) \mid\mid \nabla u \mid dx + \int_\Omega \mid \sigma(x,\omega,\nabla\psi) \mid\mid \nabla\psi \mid dx$$
$$\le \frac{C_1}{\alpha} + 4\nu \int_\Omega \overline{M}\left(\frac{\sigma(x,\omega,\nabla\psi)}{2\nu}\right) + 2\nu \int_\Omega \left[M(\mid \nabla u \mid) + M(\mid \nabla\psi \mid)\right] dx.$$

by (2.5), we get

$$\int_\Omega \overline{M}\left(\frac{\sigma(x,\omega,\nabla\psi)}{2\nu}\right) dx \le \int_\Omega \frac{1}{2}(\overline{M}(a_3(x))dx + M(k_4 \mid \nabla\psi \mid))dx.$$

Choosing $\psi \in W_0^1 E_M(\Omega)$ such that $\|\nabla\psi\|_{M,\Omega} = \dfrac{1}{k_4+1}$, then

$$\int_\Omega \sigma(x,\omega,\nabla u)\nabla\psi dx \le C,$$

finally to deduce the estimate (3.3), we use the dual norm on $L_{\overline{M}}(\Omega)$. □

Now, let define the operator $T : \omega \in \mathbf{W} \longrightarrow u \in W_0^1 L_M(\Omega) \hookrightarrow E_M(\Omega)$, where u is the unique solution to (3.1), then due to the estimate (3.2), T is a compact operator. Moreover, from (3.2), (3.4) and Lemma 2.1, we have $T(\mathbf{W}) \subset \mathbf{W}$. And to satisfy the hypotheses of Schauder's fixed point theorem for T, it remains to be shown that T is a continuous operator. Indeed, taking a sequence $(\omega_n) \subset \mathbf{W}$ such that $\omega_n \to \omega$ strongly in $E_M(\Omega)$ and let $u_n = T(\omega_n)$, φ_n, $F_n = \kappa(\omega_n)\varphi_n\nabla\varphi_n$ and $F = \kappa(\omega)\varphi\nabla\varphi$. We have to show that

$$u_n \to u = T(\omega) \text{ strongly in } E_M(\Omega).$$

Owing to (3.2), we have $\nabla u \in L_M(\Omega)^N$. We also have $\omega_n \to \omega$ strongly in $L^2(\Omega)$ and thus, we may extract a subsequence, still denoted in the same way, such that $\omega_n \to \omega$ a.e. in Ω. Then it is easy task to show that $\varphi_n \to \varphi$ strongly in $H^1(\Omega)$ and, consequently, also for another subsequence denoted in the same way, $F_n \to F$ strongly in $L^2(\Omega)$.
Since $(\omega_n) \subset L_M(\Omega)$ is bounded, we deduce for a subsequence,

$$u_n \to U \text{ in } E_M(\Omega), \text{ for some } U \in E_M(\Omega), \tag{3.6}$$

$$\nabla u_n \to \nabla U \text{ weakly in } L^2(\Omega)^N. \tag{3.7}$$

By subtracting the respective equations of (3.1) for u_n and u, and taking $\phi = u_n - u$ as a test function, we obtain

$$\int_\Omega (\sigma(x,\omega_n,\nabla u_n) - \sigma(x,\omega,\nabla u))(\nabla u_n - \nabla u)dx + \int_\Omega (\Phi(x,\omega_n) - \Phi(x,\omega))(u_n - u)dx$$
$$= -\int_\Omega (F_n - F)(\nabla u_n - \nabla u)dx. \tag{3.8}$$

For the first term of the right hand-side of (3.8):
Using (1.6), we get

$$(\sigma(x,\omega_n,\nabla u_n)-\sigma(x,\omega,\nabla u))(\nabla u_n-\nabla u)\geq \alpha M(\mid \nabla(u_n-u)\mid)$$
$$+(\sigma(x,\omega_n,\nabla u)-\sigma(x,\omega,\nabla u))(\nabla u_n-\nabla u).$$

Let $B_n=\sigma(x,\omega_n,\nabla u)-\sigma(x,\omega,\nabla u)$, then $\mid B_n\mid\to 0$ a.e. in Ω. For a given positive number δ_0, to be chosen later, we have

$$\int_\Omega \mid B_n\nabla(u_n-u)\mid = \int_{\{\mid\nabla(u_n-u)\mid\leq\delta_0\}}\mid B_n\nabla(u_n-u)\mid + \int_{\{\mid\nabla(u_n-u)\mid>\delta_0\}}\mid B_n\nabla(u_n-u)\mid \tag{3.9}$$

For the first term of the right-hand side of (3.9), we have

$$\int_{\{\mid\nabla(u_n-u)\mid\leq\delta_0\}}\mid B_n\nabla(u_n-u)\mid\leq\delta_0\int_\Omega\mid B_n\mid$$
$$=\delta_0\int_{\{\mid B_n\mid\leq 4\nu\}}\mid B_n\mid+\delta_0\int_{\{\mid B_n\mid>4\nu\}}\mid B_n\mid.$$

The first of these integrals converges to zero. As for the second one, using the fact that $\dfrac{\mid B_n\mid}{4\nu}>1$ on the set $\{\mid B_n\mid>4\nu\}$ and (2.2), it yields

$$\delta_0\int_{\{\mid B_n\mid>4\nu\}}\mid B_n\mid\leq 4\nu\delta_0\int_{\{\mid B_n\mid>4\nu\}}\left(\frac{\mid B_n\mid}{4\nu}\right)^2\leq 4\nu\gamma_1\delta_0\int_\Omega P\left(\frac{\mid B_n\mid}{4\nu}\right).$$

In virtue of (1.5) and while $P\ll M$ for $\epsilon k_3\leq 1$, we deduce

$$P\left(\frac{\mid B_n\mid}{4\nu}\right)\leq\frac{1}{4}(P(a_1)+P(\omega)+P(\omega_n)+k_3M(\mid\nabla u\mid)),$$

and since $P(\omega_n)\to P(\omega)$ strongly in $L^1(\Omega)$, by Lebesgue's dominated theorem it yields that

$$\lim_{n\to\infty}\int_\Omega P\left(\frac{\mid B_n\mid}{4\nu}\right)=0,$$

consequently,

$$\lim_{n\to\infty}\int_{\{\mid\nabla(u_n-u)\mid\leq\delta_0\}}\mid B_n\nabla(u_n-u)\mid=0.$$

For the second term of the right-hand side of (3.9), we use Young's inequality and (2.2). It yields

$$\int_{\{\mid\nabla(u_n-u)\mid>\delta_0\}}\mid B_n\nabla(u_n-u)\mid\leq\frac{1}{\alpha\varepsilon_0}\int_E\mid B_n\mid^2+\frac{\alpha\varepsilon_0}{4}\int_{\{\mid\nabla(u_n-u)\mid>\delta_0\}}\mid\nabla(u_n-u)\mid^2$$
$$\leq\frac{16\gamma_1\nu^2}{\alpha\varepsilon_0}\int_\Omega P\left(\frac{\mid B_n\mid}{4\nu}\right)+\frac{\alpha\gamma_1\varepsilon_0}{4}\int_{\{\mid\nabla(u_n-u)\mid>\delta_0\}}P(\mid\nabla(u_n-u)\mid).$$

It has been already shown that the first of these terms converges to zero. As for the second one, since $P \ll M$, we fix $\delta_0 > 0$ such $P(s) \leq M(s)$ for all $s > \delta_0$. Then

$$\frac{\alpha\gamma_1\varepsilon_0}{4}\int_{\{|\nabla(u_n-u)|>\delta_0\}} P(|\ \nabla(u_n-u)\ |)dx \leq \frac{\alpha\gamma_1\varepsilon_0}{4}\int_{\{|\nabla(u_n-u)|>\delta_0\}} M(|\ \nabla(u_n-u)\ |)dx.$$

By taking $\varepsilon_0 = \dfrac{2}{\lambda_0}$, we obtain

$$\frac{\alpha}{2}\int_{\{|\nabla(u_n-u)|>\delta_0\}} P(|\ \nabla(u_n-u)\ |)dx \leq \frac{\alpha}{2}\int_{\{|\nabla(u_n-u)|>\delta_0\}} M(|\ \nabla(u_n-u)\ |)dx.$$

For the second term of the right hand-side of (3.8):

$$\begin{aligned}\int_\Omega (\Phi(x,\omega_n)-\Phi(x,\omega))(u_n-u)dx &\leq \int_\Omega |\ \Phi(x,\omega_n)-\Phi(x,\omega)\ ||\ u_n-u\ |\ dx\\ &\leq \int_\Omega \overline{M}\left(\frac{\lambda\ |\ \Phi(x,\omega_n)-\Phi(x,\omega)\ |}{\alpha_0}\right)dx + \alpha_0\int_\Omega M\left(\frac{|\ u_n-u\ |}{\lambda}\right)dx\\ &\leq \int_\Omega \overline{M}\left(\frac{\lambda\ |\ \Phi(x,\omega_n)-\Phi(x,\omega)\ |}{\alpha_0}\right)dx + \alpha_0\int_\Omega M(|\ \nabla(u_n-u)\ |)dx.\end{aligned} \tag{3.10}$$

From above, we deduce the following estimate, for some sequence (ϵ_n) such that $\epsilon_n \to 0$,

$$\begin{aligned}&(\frac{\alpha}{2}-\alpha_0)\int_\Omega M(|\ \nabla(u_n-u)\ |)dx\\ &\leq \int_\Omega |\ (F_n-F)\nabla(u_n-u)\ |\ dx + \alpha_0\int_\Omega \overline{M}\left(\frac{\lambda\ |\ \Phi(x,\omega_n)-\Phi(x,\omega)\ |}{\alpha_0}\right)dx + \epsilon_n.\end{aligned} \tag{3.11}$$

Choosing $\alpha_0 = \dfrac{\alpha}{4}$ and by (2.3), we obtain

$$\begin{aligned}\frac{\alpha}{4\gamma_0}\|\nabla(u_n-u)\|^2_{L^2(\Omega)} \leq &\int_\Omega |\ (F_n-F)\nabla(u_n-u)\ |\ dx\\ &+\alpha_0\int_\Omega \overline{M}\left(\frac{\lambda\ |\ \Phi(x,\omega_n)-\Phi(x,\omega)\ |}{\alpha_0}\right)dx + \epsilon_n.\end{aligned}$$

Using Poincare's inequality, we get

$$\begin{aligned}\|u_n-u\|^2_{L^2(\Omega)} \leq C&\int_\Omega |\ (F_n-F)\nabla(u_n-u)\ |\ dx\\ &+C\alpha_0\int_\Omega \overline{M}\left(\frac{\lambda\ |\ \Phi(x,\omega_n)-\Phi(x,\omega)\ |}{\alpha_0}\right)dx + \epsilon_n.\end{aligned} \tag{3.12}$$

We have $F_n \to F$ strongly in $L^2(\Omega)^N$ and $\nabla(u_n-u)$ is bounded in $L^2(\Omega)^N$. On the other hand $\omega_n \to \omega$ strongly in B_R, we may extract a subsequence, still denoted the same way, such that $\omega_n \to \omega$ a.e. in Ω. In addition, the function H is continuous with respect to its second argument, then from (1.8) and dominate convergence's theorem $\displaystyle\int_\Omega \overline{M}\left(\frac{\lambda\ |\ \Phi(x,\omega_n)-\Phi(x,\omega)\ |}{\alpha_0}\right)dx$ converges to 0.

Then the right-hand side in (3.12) converges to zero. In conclusion, $u_n \to u$ strongly in $L^2(\Omega)$. Since this limit does not depend upon the subsequence one may extract, it is in fact the whole sequence (u_n) which converges to u strongly in $L^2(\Omega)$. On other hand, in virtue of (3.6), we also have $u_n \to U$ strongly in $L^2(\Omega)$, so that $u = U$ and we can rewrite (3.6) to give $u_n \to u$ strongly in $E_M(\Omega)$. This shows that T is continuous and this ends the proof of theorem 3.1. □

4. An existence result

The definition of a capacity solution of (1.1)-(1.3) can be stated as follows.

Definition 4.1. A triplet (u, φ, Φ) is called a capacity solution of (1.1)-(1.3) if the following conditions are fulfilled:
(R_1) $u \in W_0^1 L_M(\Omega)$, $\sigma(x, u, \nabla u) \in L_{\overline{M}}(\Omega)^N$, $\Phi(x, u) \in L^1(\Omega)$, $\Phi(x, u).u \in L^1(\Omega)$, $\varphi \in L^\infty(\Omega)$, $\Phi \in L^2(\Omega)^N$.
(R_2) (u, φ, Φ) verifies the system of elliptic equations

$$\begin{cases} -\text{div}\, \sigma(x, u, \nabla u) + \Phi(x, u) = \text{div}(\varphi\Phi) & \text{in} \quad \Omega, \\ \text{div}(\Phi) = 0 & \text{in} \quad \Omega. \end{cases}$$

(R_3) For every $S \in C_0^1(\Omega) = \{\phi \in C^1(\Omega)/\text{supp}(\phi) \text{ is compact}\}$, one has

$$S(u)\varphi - S(0)\varphi_0 \in H_0^1(\Omega), \quad \text{and} \quad S(u)\Phi = \kappa(u)[\nabla(S(u)\varphi) - \varphi\nabla S(u)].$$

Our most general result reads as follows.

Theorem 4.2. *Assume that (1.4)-(2.3) hold true. Then there exists a capacity solution to problem (1.1)-(1.3).*

Proof of the theorem 4.2
Step 1: Approximative problem.
For every $n \in \mathbb{N}^*$, let us define the following approximation of κ, a and g:

$$\kappa_n(s) = \kappa(s) + \frac{1}{n}, \quad \sigma_n(x, s, \xi) = \sigma(x, T_n(s), \xi), \quad \Phi_n(x, s) = \frac{\Phi(x, s)}{1 + \dfrac{1}{n} \mid \Phi(x, s) \mid},$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$.
Let us now consider the approximate system

$$-\,\text{div}\; \sigma_n(x, u_n, \nabla u_n) + \Phi_n(x, u_n) = \kappa_n(u_n) \mid \nabla\varphi_n \mid^2 \;\text{ in } \Omega, \tag{4.1}$$

$$\text{div}(\kappa_n(u_n)\nabla\varphi_n) = 0 \text{ in } \Omega, \tag{4.2}$$

$$u_n = 0, \text{ on } \partial\Omega, \tag{4.3}$$

$$\varphi_n = \varphi_0, \text{ on } \partial\Omega. \tag{4.4}$$

From (1.4), we deduce

$$\mid \sigma(x, T_n(s), \xi) \mid \leq \nu \left[a_0(x) + \overline{M}^{-1}(P(k_1 \mid T_n(s) \mid)) + \overline{M}^{-1}(M(k_2 \mid \xi \mid)) \right],$$

where $\Big(a_0(x)+\overline{M}^{-1}(P(k_1n))\Big)\in E_{\overline{M}}(\Omega)$.
In view of (1.9), we have that

$$n^{-1}\leq \kappa_n(s)\leq \overline{\kappa}+1=\kappa_3,\ \text{for all } s\in\mathbb{R}. \tag{4.5}$$

We have also $\mid \Phi_n(x,s)\mid\leq\mid \Phi(x,s)\mid$ and $\mid \Phi_n(x,s)\mid\leq n$. Thus, we can apply Theorem 3.1 to deduce the existence of a weak solution (u_n,φ_n) to the system (4.1)-(4.4).
From the maximum principle

$$\|\varphi_n\|_{L^\infty(\Omega)}\leq\|\varphi_0\|_{L^\infty(\Omega)}; \tag{4.6}$$

hence, there exists a function $\varphi\in L^\infty(\Omega)$ and a subsequence, still denoted φ_n, such that

$$\varphi_n\to\varphi,\ \text{weakly-}*\ \text{in } L^\infty(\Omega). \tag{4.7}$$

Now let multiply (4.2) by $\varphi_n-\varphi_0\in H_0^1(\Omega)$ and integrate over Ω. We get

$$\int_\Omega \kappa_n(u_n)\nabla\varphi_n\nabla(\varphi_n-\varphi_0)dx=0;$$

hence

$$\int_\Omega \kappa_n(u_n)\mid\nabla\varphi_n\mid^2 dx\leq C_3,\ \text{for all } n\in\mathbb{N}^*, \tag{4.8}$$

where $C_3=C(\overline{\kappa},\|\varphi_0\|_{L^\infty(\Omega)})$. Consequently, the sequence $(\kappa_n(u_n)\nabla\varphi_n)$ is bounded in $L^2(\Omega)^N$. Thus, there exists a function $\phi\in L^2(\Omega)^N$ and a subsequence, still denoted in the same way, such that

$$\kappa_n(u_n)\nabla\varphi_n\to\phi \text{ weakly in } L^2(\Omega)^N. \tag{4.9}$$

This weak limit function $\phi\in L^2(\Omega)^N$ is in fact the third component of the triplet appearing in the defintion (4.1) of a capacity solution.
Taking u_n as a function test in (4.1), we obtain

$$\int_\Omega\sigma(x,T_n(u_n),\nabla u_n)\nabla u_n dx+\int_\Omega\Phi_n(x,u_n)u_n dx=-\int_\Omega\kappa_n(u_n)\varphi_n\nabla\varphi_n\nabla u_n dx. \tag{4.10}$$

Since $u_n\in W_0^1L_M(\Omega)$, and $\varphi_n\in H^1(\Omega)\subset W^1L_{\overline{M}}(\Omega)$, there exist η_n and $\beta_n>0$ such that $\frac{|\nabla u_n|}{\eta_n}\in K_M(\Omega)$ and $\frac{\kappa_3\|\varphi_0\|_{L^\infty(\Omega)}|\nabla\varphi_n|}{\beta_n}\in K_{\overline{M}}(\Omega)$.

From (1.6), (1.7), (4.4) and (4.6) and Young's inequality, and Lemma 2.1, we obtain

$$\begin{aligned}\alpha\int_\Omega M(\mid\nabla u_n\mid)dx &\le \int_\Omega \sigma(x,T_n(u_n),\nabla u_n)\nabla u_n dx\\ &\le 2\lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M\left(\frac{\mid u_n\mid}{\lambda C_0}\right)dx+\eta_n\beta_n\int_\Omega \overline{M}\left(\frac{\kappa_3\|\varphi_0\|_{L^\infty(\Omega)}\mid\nabla\varphi_n\mid}{\beta_n}\right)dx\\ &+\eta_n\beta_n\int_\Omega M\left(\frac{\mid\nabla u_n\mid}{\eta_n}\right)dx\\ &\le 2\lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M(\mid\nabla u_n\mid)dx+\eta_n\beta_n\int_\Omega \overline{M}\left(\frac{\kappa_3\|\varphi_0\|_{L^\infty(\Omega)}\mid\nabla\varphi_n\mid}{\beta_n}\right)dx\\ &+\eta_n\beta_n\int_\Omega M\left(\frac{\mid\nabla u_n\mid}{\eta_n}\right)dx.\end{aligned}$$

Therefore

$$\begin{aligned}(\alpha-2\lambda\|h\|_{L^\infty(\Omega)})\int_\Omega M(\mid\nabla u_n\mid)dx &\le \eta_n\beta_n\int_\Omega \overline{M}\left(\frac{\kappa_3\|\varphi_0\|_{L^\infty(\Omega)}\mid\nabla\varphi_n\mid}{\beta_n}\right)dx\\ &+\eta_n\beta_n\int_\Omega M\left(\frac{\mid\nabla u_n\mid}{\eta_n}\right)dx<\infty.\end{aligned}$$

and thus $\mid\nabla u_n\mid\in K_M(\Omega)$. On the other hand

$$\begin{aligned}\alpha\int_\Omega M(\mid\nabla u_n\mid)dx &\le \int_\Omega \sigma(x,T_n(u_n),\nabla u_n)\nabla u_n dx\\ &\le 2\lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M\left(\frac{\mid u_n\mid}{\lambda C_0}\right)dx+\frac{(\kappa_3\|\varphi_0\|_{L^\infty(\Omega)})^2}{\alpha\epsilon_1}\int_\Omega \kappa(u_n)\mid\nabla\varphi_n\mid^2 dx\\ &+\frac{\alpha\epsilon_1}{4}\int_\Omega \mid\nabla u_n\mid^2 dx\\ &\le 2\lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M(\mid\nabla u_n\mid)dx+\frac{(\kappa_3\|\varphi_0\|_{L^\infty(\Omega)})^2}{\alpha\epsilon_1}C_3+\frac{\alpha\epsilon_1\gamma_0}{4}\int_\Omega M(\mid\nabla u_n\mid)dx.\end{aligned}$$

Then

$$\left(\frac{\alpha\epsilon_1\gamma_0}{4}-2\lambda\|h\|_{L^\infty(\Omega)}\right)\int_\Omega M(\mid\nabla u_n\mid)dx\le\frac{(\kappa_3\|\varphi_0\|_{L^\infty(\Omega)})^2}{\alpha}C_3.$$

Taking $\epsilon_1=\frac{4}{\gamma_0}$, we obtain

$$\int_\Omega M(\mid\nabla u_n\mid)dx\le C_4. \tag{4.11}$$

It follows that (u_n) is bounded in $W_0^1L_M(\Omega)$. Consequently, there exists a subsequence, still denoted (u_n), and a function $u\in W_0^1L_M(\Omega)$ such that

$$u_n\rightharpoonup u \quad \text{in} \quad W_0^1L_M(\Omega) \quad \text{for} \quad \sigma(\Pi L_M,\Pi E_{\overline{M}}), \tag{4.12}$$

and since the embedding $W_0^1L_M(\Omega)\hookrightarrow E_M(\Omega)$ is compact, we also have

$$u_n\to u \quad \text{strongly in} \quad E_M(\Omega) \quad \text{and a.e. in} \quad \Omega, \tag{4.13}$$

On the other hand, let $\psi \in W_0^1 E_M(\Omega)^N$ be arbitrary with $\|\nabla \psi\|_{(M)} = \dfrac{1}{k_2+1}$.
In view of the monotonicity of a_n, one easily has

$$\begin{aligned}\int_\Omega \sigma(x,u_n,\nabla u_n)\nabla\psi dx &\leq \int_\Omega \sigma(x,u_n,\nabla u_n)\nabla u_n dx - \int_\Omega \sigma(x,u_n,\nabla\psi)(\nabla u_n - \nabla\psi)dx \\ &\leq C + \int_\Omega \mid \sigma(x,u_n,\nabla\psi) \mid\mid \nabla u_n \mid dx + \int_\Omega \mid \sigma(x,u_n,\nabla\psi) \mid\mid \nabla\psi \mid dx.\end{aligned} \tag{4.14}$$

For the first integral in the right side, we use the Young's inequality to have

$$\int_\Omega \mid \sigma(x,u_n,\nabla\psi) \mid\mid \nabla u_n \mid dx \leq 3\nu \int_\Omega \left[\overline{M}\left(\frac{\sigma(x,u_n,\nabla\psi)}{3\nu}\right) + M(\mid \nabla u_n \mid)\right] dx,$$

using (1.4) we have

$$3\nu\overline{M}\left(\frac{\sigma(x,T_n(u_n),\nabla\psi)}{3\nu}\right) \leq \nu(\overline{M}(a_0(x)) + P(k_1 T_n(u_n)) + M(k_2\nabla\psi)),$$

Since (u_n) is bounded in $W_0^1 L_M(\Omega)$, and owing to Poincare's inequality, there exist $\lambda > 0$ such that $\displaystyle\int_\Omega M\left(\frac{u_n}{\lambda}\right) dx \leq 1$ for all $n \in \mathbb{N}^*$. Also, since $P \ll M$, there exist $s_0 > 0$ such that $P(k_1 s) \leq P(k_1 s_0) + M\left(\dfrac{s}{\lambda}\right)$ for all $s \in \mathbb{R}$.
Consequently,

$$\begin{aligned}3\nu\int_\Omega \overline{M}\left(\frac{\sigma(x,T_n(u_n),\nabla\psi)}{3\nu}\right) dx &\leq \nu \int_\Omega \left(\overline{M}(a_0(x)) + P(k_1 T_n(u_n)) + M(k_2\nabla\psi)\right) dx \\ &\leq C,\end{aligned}$$

and thus $\displaystyle\int_\Omega \mid \sigma_n(x,u_n,\nabla\psi) \mid . \mid \nabla u_n \| dx \leq C$, for all $n \in \mathbb{N}^*$ and $\psi \in W_0^1 E_M(\Omega)^N$ such that $\|\nabla\psi\|_{(M)} = \dfrac{1}{k_2+1}$.
On the other hand, the second integral in (4.14), namely

$$\int_\Omega \mid \sigma_n(x,u_n,\nabla u_n) \mid . \mid \nabla u_n \mid dx \leq C$$

can be dealt in the same way so that it is easy to check that it is also bounded. Gathering all these estimates, and using the dual norm, one easily deduce that

$$(\sigma_n(x,u_n,\nabla u_n)) \quad \text{is bounded in} \quad L_{\overline{M}}(\Omega)^N. \tag{4.15}$$

Thus, up to a subsequence, still denoted in the same way, there exists $\varpi \in L_{\overline{M}}(\Omega)^N$ such that

$$(\sigma_n(x,u_n,\nabla u_n)) \rightharpoonup \varpi \quad \text{in} \quad L_{\overline{M}}(\Omega)^N \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}). \tag{4.16}$$

Step 2: Almost everywhere convergence of the gradient.
In this step, we may extract a subsequence of (u_n), still denoted the same way, such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to +\infty. \tag{4.17}$$

Let $v_j \in \mathcal{D}(\Omega)$ be a sequence that $v_j \to u$ in $W_0^1 L_M(\Omega)$ for the modular convergence see [9]. Setting for $s > 0$, $\Omega_s = \{x \in \Omega :\mid \nabla T_K(u) \mid \leq s\}$ and $\Omega_s^j = \{x \in \Omega : |\nabla T_K(v_j) \mid \leq s\}$ and denoting by χ^s and χ_s^j the characteristic functions of Ω_s and Ω_s^j respectively. And we denote by $\epsilon(i,j,\beta,n)$ the quantities such that

$$\lim_{i\to\infty}\lim_{j\to\infty}\lim_{\beta\to\infty}\lim_{n\to\infty}\epsilon(i,j,\beta,n) = 0.$$

For any $\eta > 0$ and $n, j \geq 1$, we may use the admissible test function

$$\varphi_{n,j}^{\eta} = T_\eta(u_n - T_K(v_j))$$

in (4.1). This leads to

$$\begin{aligned}\int_\Omega \sigma_n(x,u_n,\nabla u_n)\nabla T_\eta(u_n - T_K(v_j))dx + \int_\Omega \Phi_n(x,u_n)T_\eta(u_n - T_K(v_j))dx \\ = \int_\Omega \kappa_n(u_n) \mid \nabla\varphi_n \mid^2 \nabla T_\eta(u_n - T_K(v_j))dx.\end{aligned} \tag{4.18}$$

By Young's inequality and Lemma 2.1, we have

$$\begin{aligned}\int_\Omega |\Phi_n(x,u_n) \mid T_\eta(u_n - T_K(v_j))dx &\leq \int_\Omega \mid \Phi(x,u_n) \mid T_\eta(u_n - T_K(v_j))dx \\ &\leq \int_\Omega h(x)\overline{M}^{-1}M\left(\frac{\mid u_n \mid}{\lambda C_0}\right)T_\eta(u_n - T_K(v_j))dx \\ &\leq \eta\|h\|_{L^\infty(\Omega)}\int_\Omega M(\mid \nabla u_n \mid)dx + \eta\|h\|_{L^\infty(\Omega)}M(1)\mathrm{meas}(\Omega) \\ &\leq C\eta.\end{aligned}$$

Using (4.8) and above result, we get

$$\int_\Omega \sigma_n(x,u_n,\nabla u_n)\nabla T_\eta(u_n - T_K(v_j))dx \leq C\eta. \tag{4.19}$$

Let's study the left-hand side of (4.19). We have

$$\begin{aligned}&\int_\Omega \sigma_n(x,u_n,\nabla u_n)\nabla T_\eta(u_n - T_K(v_j))dx \\ &= \int_{\{|u_n - T_K(v_j)|\leq\eta\}} \sigma_n(x,u_n,\nabla u_n)\nabla(u_n - T_K(v_j))dx \\ &= \int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\leq\eta\}} \sigma_n(x,u_n,\nabla u_n)\nabla(u_n - T_K(v_j))dx \\ &+ \int_{\{|u_n|\leq K\}\cap\{|u_n - T_K(v_j)|\leq\eta\}} \sigma_n(x,u_n,\nabla u_n)\nabla(u_n - T_K(v_j))dx \\ &= \int_{\{|T_K(u_n) - T_K(v_j)|\leq\eta\}} \sigma_n(x,T_n(u_n),\nabla T_n(u_n))(\nabla T_n(u_n) - \nabla T_K(v_j))dx \\ &+ \int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\leq\eta\}} \sigma_n(x,u_n,\nabla u_n)\nabla u_n dx \\ &- \int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\leq\eta\}} \sigma_n(x,u_n,\nabla u_n)\nabla T_K(v_j)dx.\end{aligned}$$

which yields, thanks to (1.6) and (1.7),

$$\begin{cases} \displaystyle\int_\Omega \sigma_n(x,u_n,\nabla u_n)\nabla T_\eta(u_n - T_K(v_j))dx \\ \displaystyle\geq \int_{\{|u_n - T_K(v_j)|\leq\eta\}} \sigma_n(x,T_K(u_n),\nabla T_K(u_n))(\nabla T_n(u_n) - \nabla T_K(v_j))dx \\ \displaystyle-\int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\leq\eta\}} \sigma_n(x,u_n,\nabla u_n)\nabla T_K(v_j)dx. \end{cases} \tag{4.20}$$

Let $0 < \delta < 1$, we define

$$\Theta_{n,K} = (\sigma(x,T_K(u_n),\nabla T_K(u_n)) - \sigma(x,T_K(u_n),\nabla T_K(u)))(\nabla T_K(u_n) - \nabla T_K(u)).$$

Using the similar technic as in [1], we obtain,

$$\int_{\Omega_r} \Theta^\delta_{n,K}dx \leq C_1\text{meas}\{x\in\Omega :| T_K(u_n) - T_K(v_j) |> \eta\}^{1-\delta} + C_2(\epsilon(n,j,s,\eta))^\delta.$$

Which yields, by passing to the limit sup over n, j, s and η

$$\limsup_{n\to\infty}\int_{\Omega_r}\Big((\sigma(x,T_K(u_n),\nabla T_K(u_n)) - \sigma(x,T_K(u_n),\nabla T_K(u)))$$

$$\times(\nabla T_K(u_n) - \nabla T_K(u))\Big)^\delta dx = 0.$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Ω_r, and since r is arbitrary,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad \Omega.$$

Remark 4.3. A consequence of (4.17) is that,

$$\sigma(x,u_n,\nabla u_n) \rightharpoonup \sigma(x,u,\nabla u) \quad \text{in} \quad L_{\overline{M}}(\Omega)^N \quad \text{for} \quad \sigma(\Pi L_{\overline{M}},\Pi E_M). \tag{4.21}$$

Step 3: Equi-integrability of the nonlinearitie $\Phi_n(x,u_n)$.
We shall now prove that $\Phi_n(x,u_n) \to \Phi(x,u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem. Since $\Phi_n(x,u_n) \to \Phi(x,u)$ a.e in Ω, it is suffices to prove that $\Phi_n(x,u_n)$ are equi-integrable in Ω. Indeed, let ϵ and for any measurable subset $D\subset\Omega$. Using (1.8), Young's inequality and Lemma 2.1, we have

$$\int_D | \Phi_n(x,u_n) | dx \leq \int_D h(x)\overline{M}^{-1}M\left(\frac{| u_n |}{\lambda C_0}\right)dx$$

$$\leq \|h\|_{L^\infty(\Omega)}\int_D M(| \nabla u_n |)dx + \|h\|_{L^\infty(\Omega)}M(1)\text{meas}(D).$$

According to Lemma 3.2 in [6], we have $M(| \nabla u_n |) \to M(| \nabla u |)$ in $L^1(\Omega)$, and there exists $\eta(\epsilon) > 0$ such that

$$\|h\|_{L^\infty(\Omega)}\int_D M(| \nabla u_n |)dx \leq \frac{\epsilon}{2} \quad \text{and} \quad \|h\|_{L^\infty(\Omega)}M(1)\text{meas}(D) \leq \frac{\epsilon}{2},$$

such that $\text{meas}(D) < \eta(\epsilon)$. Then, by Vitali's theorem we conclude that

$$\Phi_n(x,u_n) \to \Phi(x,u) \quad \text{strongly in} \quad L^1(\Omega).$$

Using again (1.8), Young inequality and Lemma 2.1, we obtain

$$\int_D |\ \Phi_n(x,u_n)\ | \ . \ |\ u_n\ |\ dx \leq \lambda \int_D h(x)\overline{M}^{-1} M\left(\frac{|\ u_n\ |}{\lambda C_0}\right)\frac{|\ u_n\ |}{\lambda}dx$$
$$\leq 2\lambda \|h\|_{L^\infty(\Omega)} \int_D M(|\ \nabla u_n\ |)dx.$$

and in the same way, we show that

$$\Phi_n(x,u_n)u_n \to \Phi(x,u)u \quad \text{strongly in} \quad L^1(\Omega).$$

So that $\Phi(x,u)u \in L^1(\Omega)$.

Step 4: Passage to the limit.
The next result analyze the behavior of certain subsequences of (φ_n). They will allow us, to pass to the limit in the approximate system (4.1)-(4.4) to show the existence of a capacity solution to the system (1.1)-(1.3).

Lemma 4.4. [10] *Let (u_n,φ_n) be a weak solution to the system (4.1)-(4.4), $u \in E_M(\Omega)$ and $\varphi \in L^\infty(\Omega)$ the limits functions appearing, respectively in (4.7) and (4.13). Then for any function $S \in C_0^1(\mathbb{R})$,*

- *there exists a subsequence, still denoted in the same way, such that*
$$S(u_n)\varphi_n \rightharpoonup S(u)\varphi \quad \textit{weakly in} \quad H^1(\Omega). \tag{4.22}$$
- *Moreover, if $0 \leq S \leq 1$, then there exists a constant $C > 0$, independant of S, such that*
$$\limsup_{n\to\infty} \int_\Omega \kappa_n(u_n)\ |\ \nabla(S(u_n)\varphi_n - S(u)\varphi)\ |^2 \leq C\|S^{'}\|_\infty(1+\|S^{'}\|_\infty). \tag{4.23}$$
- *There exists a subsequence $(\varphi_{n_k}) \subset (\varphi_n)$ such that*
$$\lim_{n\to\infty} \int_\Omega |\ \varphi_{n_k} - \varphi\ | = 0. \tag{4.24}$$

Finally, the condition (R_1) and (R_2) of the Definition 4.1 are fulfilled. In order to obtain the condition (R_3), using (4.17), (4.22) and (4.24), it is enough to make $k \to +\infty$ in the expression

$$S(u_{n_k})\kappa_{n_k}(u_{n_k})\nabla\varphi_{n_k} = \kappa_{n_k}(u_{n_k})[\nabla(S(u_{n_k})\varphi_{n_k}) - \varphi_{n_k}\nabla S(u_{n_k})].$$

This completes the proof of theorem 4.2.

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Some saturation classes for deferred Riesz and deferred Nörlund means

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Abstract. One of main problem in approximation theory is determination a saturation class for given method. The problem of determining a saturation class has been considered by Zamanski, Sunouchi and Watari and others. Mohaparta and Russel have considered some direct and inverse theorems in approximation of functions. Sunouchi and Watari have studied the Riesz means of type n. In [5], Goel et al. have extended these results by considering Nörlund means. In this paper, we examine some direct and inverse theorems in approximation of functions under weaker conditions by considering Deferred Riesz means and Deferred Nörlund means. Also, we extent above mentioned results.

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1. Introduction

Let f be a 2π-periodic function and $f \in L_p := L_p[-\pi, \pi]$ for $p \geq 1$, where L_p consists of all measurable functions for which denote the L_p-norm with respect to x and defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^p dx \right\}^{\frac{1}{p}} .$$

$C_{2\pi}$ denote the set of all continuous functions defined on $[-\pi, \pi]$. For $p = \infty$, $L_p[-\pi, \pi]$ space replace by the space $C_{2\pi}$.

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Each $f \in L^1$ has the Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(x)\ . \tag{1.1}$$

The partial sum of the first $(n+1)$ terms of the Fourier series of f at a point x is defined by

$$S_n(f;x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{n} A_k(x)\ .$$

The conjugate series of the series (1.1) is

$$\sum_{k=1}^{\infty} B_k(x) = \sum_{k=1}^{\infty}(b_k \cos kx - a_k \sin kx)\ .$$

and also the conjugate function $\tilde{f}$ of f is given by

$$\tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\{f(x+t) - f(x-t)\}\cot\frac{t}{2}dt\ .$$

The integral is known as a Cauchy integral. Also, $\tilde{f}$ exists almost everywhere whenever f is integrable.

Moreover, if $\omega_p(\delta, f) = O(\delta^{\alpha})$, then $f \in Lip(\alpha, p)$, $(p \geq 1)$, where

$$\omega_p(\delta, f) = \sup_{|h|\leq\delta} \|f(x+h) - f(x)\|_p$$

is the integral modulus of continuity of $f \in L_p$. Clearly, if $f \in Lip(\alpha, p)$ for some $\alpha > 1$, then f must be constant. So it is interesting only in case of $0 < \alpha \leq 1$. Also for $p \geq 1$, the generalized Minkowski's inequality is given in [7] as follow

$$\left\|\int f(x,t)dt\right\|_p \leq \int \|f(x,t)\|_p dt\ .$$

Throughout the paper, we consider $K_p = \{f \in L_p : \tilde{f} \in Lip(1,p)\}$ for $1 \leq p < \infty$ and $K_{\infty} = \{f \in C_{2\pi} : \tilde{f} \in Lip1\}$ for $p = \infty$.

In 1932, Agnew [1] defined the Deferred Cesàro mean of the sequence $\{s_k\}$ by

$$(D_{a,b}, s)_n := \frac{s_{a_n+1} + s_{a_n+2} + \ldots + s_{b_n}}{b_n - a_n} = \frac{1}{b_n - a_n}\sum_{k=a_n+1}^{b_n} s_k$$

where $a = \{a_n\}$ and $b = \{b_n\}$ are sequences of non-negative integers satisfying

$$a_n < b_n\ , \quad n = 1, 2, 3, \ldots \quad \text{and} \quad \lim_{n\to\infty} b_n = \infty\ .$$

We note here that $D_{a,b}$ is clearly regular for any choice of $\{a_n\}$ and $\{b_n\}$.

Let $\{p_n\}$ be a sequence of non-negative real numbers. Deferred Riesz and Deferred Nörlund means of (1.1) are defined as follows

$$D_a^b R_n(f;x) := \frac{1}{P_{a_n+1}^{b_n}}\sum_{k=a_n+1}^{b_n} p_k S_k(f;x)$$

and

$$D_a^b N_n(f;x) := \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} S_k(f;x)$$

where

$$P_{a_n+1}^{b_n} = \sum_{k=a_n+1}^{b_n} p_k \neq 0 \ , \quad P_0^{b_n-a_n-1} = \sum_{k=0}^{b_n-a_n-1} p_k \neq 0$$

(see [11], [2] and [12]).

Taking $b_n = n$ and $a_n = 0$, Deferred Riesz and Deferred Nörlund means give us classically known Riesz and Nörlund means of the series (1.1), respectively. Also, in case $p_k = 1$ for all k, both of them yield Deferred Cesàro means of $S_k(f;x)$ as follows

$$D_a^b(f;x) := \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} S_k(f;x) \ .$$

Let $g_k(n)$ $k = 1, 2, \ldots$ be the summating function and consider a family of transform of (1.1) of a summability method G,

$$P_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} g_k(n)(a_k \cos kx + b_k \sin kx) \tag{1.2}$$

where the parameter n needs not be discrete.

If there are a positive non-increasing function $\phi(n)$ and a class K of functions in such a way that

$$\| f(x) - P_n(x) \| = \ o(\phi(n)) \ \text{ implies } f(x) = \text{constant}; \tag{1.3}$$

$$\| f(x) - P_n(x) \| = \ O(\phi(n)) \ \text{ implies } f(x) \in K; \tag{1.4}$$

$$\text{for every } f \in K, \quad \text{one has} \quad \| f(x) - P_n(x) \| = \ O(\phi(n)), \tag{1.5}$$

then it is said that the method of summation G is saturated with order $\phi(n)$ and its class of saturation in K ([3]).

Ever since the definition of saturation of summability methods was given by Favard [4] many authors have studied the saturation property of operators which are obtained as transforms of the n-th partial sum of the Fourier series by summability methods. Zamanski [14] have studied the notion of determining a saturation class by considering $(C, 1)$. Sunouchi and Watari [13] have obtained the saturation order and class for Cesàro, Abel and the (R, λ, k) method. Goel et al. [5] have examined order and class of saturation of Nörlund means with supremum norms. Mohapatra and Russell [10] have analyzed order and class of (N,c,d)-methods in the L_p spaces. Kuttner, Mohapatra and Sahney [8] have obtained results on saturation for a general class of summability methods in the supremum norm.

In this paper, our object is to extent some of these results under weaker conditions by considering Deferred Riesz means and Deferred Nörlund means.

We shall give some well-known results that we will use them to prove our theorems.

Lemma 1.1. [6] *If f belongs to $Lip(1,p)$, $1 < p \leq \infty$, then f is equivalent to the indefinite integral of a function belonging to L_p. Also, if $f \in Lip1$, then f is the indefinite integral of a bounded function.*

Lemma 1.2. [15] *If $f \in L^p$, $1 < p < \infty$, then $\tilde{f} \in L^p$. Moreover, $\tilde{S}[f] = S[\tilde{f}]$.*

Lemma 1.3. [5]

$$\left|\int_t^\pi \frac{\sin(k+1)u}{u^2} du\right| \leq \begin{cases} 2(k+1)\log\left(\frac{1}{(k+1)t}\right), & 0 < (k+1)t < \frac{1}{e} \\ \frac{2}{(k+1)t^2}\ , & k \geq 0,\ t > 0\ . \end{cases}$$

Lemma 1.4. [9] *Suppose that $d_{nk} \geq 0$ $(\forall n, k)$, $\sum_{k=0}^{\infty} d_{nk} = 1$ and*

$$\sum_{k=1}^{\infty} d_{nk} \log k < \infty\ .$$

Let $\phi(n)$ be a positive function. In order that D should be saturated with order $\phi(n)$ and some class, it is necessary and sufficient that

$$0 < \liminf_{n\to\infty} \frac{\phi(n)}{d_{n0}} < \infty\ .$$

2. Main results

If there are a positive non-increasing function $\phi(n)$ and a class of functions K with the following properties

$$\| f(x) - D_a^b R_n(f;x) \| = o(\phi(n)) \ \Rightarrow\ f \text{ is constant} \tag{2.1}$$

$$\| f(x) - D_a^b R_n(f;x) \| = O(\phi(n)) \ \Rightarrow\ f \in K \tag{2.2}$$

and

$$f \in K \ \Rightarrow \| f(x) - D_a^b R_n(f;x) \| = O(\phi(n)) \tag{2.3}$$

then we say that $D_a^b R_n(f;x)$ is saturated with the order $\phi(n)$ and class K.

Now, we give interesting results for Deferred Riesz means.

Lemma 2.1. *Let $1 \leq p \leq \infty$ and*

$$\frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \log k < \infty.$$

If

$$\| f(x) - D_a^b R_n(f;x) \|_p = o\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right)$$

then f is constant.

To proof the following lemma we use the same technique in [8].

Proof. Let us write $D_a^bR_n(x)$ instead of $D_a^bR_n(f;x)$. By definition of $D_a^bR_n(x)$, we get

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bR_n(x+t)\cos ktdt &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{P_{a_n+1}^{b_n}}\sum_{r=a_n+1}^{b_n} p_rS_r(x+t)\cos ktdt \\ &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{r=a_n+1}^{b_n} p_r\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt \\ &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k}^{b_n} p_rA_k(x),\end{aligned}$$

since hypothesis and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt = \begin{cases} A_k(x), & r\geq k \\ 0, & r<k\ .\end{cases}$$

Hence, we obtain

$$A_k(x) - \frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k}^{b_n} p_rA_k(x) = A_k(x)\left(\frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k+1}^{b_n} p_r\right).$$

$S_n(f)$ converges to f uniformly whenever f is continuous [15]. So, from hypothesis and generalized Minkowski's inequality we get

$$\begin{aligned}&\left\|\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bR_n(x+t)\cos ktdt\right\|_p \\ \leq\ &\left\|\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos ktdt\right\|_p \\ &+\left\|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bR_n(x+t)\cos ktdt\right\|_p \\ \leq\ &\frac{1}{\pi}\int_{-\pi}^{\pi}\|S_r-f\|_pdt + \frac{1}{\pi}\int_{-\pi}^{\pi}\|f-D_a^bR_n\|_pdt \\ =\ &o\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right).\end{aligned}$$

Therefore for all $k\geq 1$ we have

$$A_k(x)\left(\frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k+1}^{b_n} p_r\right) = o\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right)$$

i.e.,

$$A_k(x)\left(\frac{p_{k+1}+p_{k+2}\ldots+p_{b_n}}{p_{b_n}}\right) = o(1).$$

Because of $\left(\frac{p_{k+1}+p_{k+2}\ldots+p_{b_n}}{p_{b_n}}\right)\geq 1$ for each $r\geq 1$, we get $A_k(x)=0$. Consequently $f(x)=\frac{1}{2}a_0$ which is a constant. □

Lemma 2.2. *Let the limit*

$$\lim_{n\to\infty} \frac{p_r}{p_{b_n}} = 1$$

hold for a fixed $a_n + 1 \leq r < b_n$. *If the equation*

$$\| f(x) - D_a^b R_n(f;x) \|_p = 0 \left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}} \right)$$

is hold, then

$$\left\| \sum_{k=a_n+2}^{N} (k - a_n - 1) \left(1 - \frac{k-(a_n+2)}{N - a_n - 1} \right) A_k(x) \right\|_p = O(1).$$

Proof. Suppose that $\Delta_n(x) := f(x) - D_a^b R_n(f;x)$. In this case,

$$\Delta_n(x) \sim \sum_{k=a_n+2}^{b_n} \left(1 - \frac{P_k^{b_n}}{P_{a_n+1}^{b_n}} \right) A_k(x).$$

Let $N < b_n$, taking N-th arithmetic mean of $\Delta_n(x)$ we get

$$\sigma_N[x;\Delta_n] = \sum_{k=a_n+2}^{N} \left(1 - \frac{P_k^{b_n}}{P_{a_n+1}^{b_n}} \right) \left(1 - \frac{k - a_n - 2}{N - a_n - 1} \right) A_k(x).$$

On account of $\|\Delta_n\| \geq \|\sigma_N[x;\Delta_n]\|$, we obtain

$$\left\| \sum_{k=a_n+2}^{N} \left(1 - \frac{P_k^{b_n}}{P_{a_n+1}^{b_n}} \right) \left(1 - \frac{k - a_n - 2}{N - a_n - 1} \right) A_k(x) \right\|_p = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}} \right)$$

$$\Rightarrow \left\| \sum_{k=a_n+2}^{N} \left(\frac{P_{a_n+1}^{b_n} - P_k^{b_n}}{p_{b_n}} \right) \left(1 - \frac{k - a_n - 2}{N - a_n - 1} \right) A_k(x) \right\|_p = O(1)$$

$$\Rightarrow \left\| \sum_{k=a_n+2}^{N} \lim_{n\to\infty} \left(\frac{p_{a_n+1} + p_{a_n+2} + \ldots + p_{k-1}}{p_{b_n}} \right) \left(1 - \frac{k}{N+1} \right) A_k(x) \right\|_p = O(1)$$

$$\Rightarrow \left\| \sum_{k=a_n+2}^{N} (k - a_n - 1) \left(1 - \frac{k}{N+1} \right) A_k(x) \right\|_p = O(1).$$

This completes the proof. □

Lemma 2.3. *Let*

$$M_n(t) = \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{\cos(k+1/2)t}{\sin(t/2)}$$

and

$$G_n(t) = \int_t^{\pi} M_n(u)du. \tag{2.4}$$

If

$$\int_0^{\pi} |G_n(t)| = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right) \tag{2.5}$$

and $f \in K_p$ $(1 < p \leq \infty)$ *then* $\| f(x) - D_a^b R_n(f;x) \|_p = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right)$.

Proof. Let $\tilde{S}_k(\tilde{f};x)$ denote the partial sums of the conjugate series related to $\tilde{f}(x)$. So,

$$\tilde{S}_k(\tilde{f};x) = \frac{1}{2\pi}\int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \frac{\cos(t/2) - \cos(k+1/2)t}{\sin(t/2)} dt.$$

With a simple analysis, we get

$$\begin{aligned} D_a^b R_n(\tilde{S}_k(\tilde{f};x)) &= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \tilde{S}_k(\tilde{f};x) \\ &= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \cot(t/2) dt \\ &\quad - \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \frac{\cos(k+1/2)t}{\sin(t/2)} dt. \end{aligned}$$

By Lemma 1.2, $f \in L_p$ $(1 < p < \infty)$ implies $\tilde{f} \in L_p$. So $\tilde{\tilde{f}} \in L_p$, and thus we obtain $\tilde{S}(\tilde{f}) = S(\tilde{\tilde{f}})$. If $p = \infty$ then it means that $\tilde{f} \in Lip1$. Therefore, we say that $-f + \frac{1}{2}a_0$ is equal to $\tilde{\tilde{f}}$. As a result $\tilde{\tilde{f}} - D_a^b R_n(S_k(\tilde{\tilde{f}};x))$ is identical to $f(x) - D_a^b R_n(S_k(f;x))$. From hypothesis we get

$$\begin{aligned} f(x) - D_a^b R_n(f;x) &= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \tilde{\tilde{f}}(x) \\ &\quad - \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \cot(t/2) dt \\ &\quad + \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \frac{\cos(k+1/2)t}{\sin(t/2)} \\ &= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \cot(t/2) dt \\ &\quad - \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \cot(t/2) dt \\ &\quad + \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \frac{\cos(k+1/2)t}{\sin(t/2)} \end{aligned}$$

$$= \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t)-\tilde{f}(x-t)\}M_n(t). \tag{2.6}$$

As $f \in K_p$ by Lemma 1.1, we get $\tilde{f}' \in L_p$, $p > 1$. By integrating in (2.6) we have

$$f(x) - D_a^b R_n(f;x) = -\frac{1}{2\pi}\int_0^\pi \{\tilde{f}'(x+t)+\tilde{f}'(x-t)\}G_n(t)dt.$$

By generalized Minkowski's inequality, we obtain

$$\begin{aligned}\left\|f(x) - D_a^b R_n(f;x)\right\|_p &= \left\|-\frac{1}{2\pi}\int_0^\pi \{\tilde{f}'(x+t)+\tilde{f}'(x-t)\}G_n(t)dt\right\|_p \\ &\leq \int_0^\pi \left\|\{\tilde{f}'(x+t)+\tilde{f}'(x-t)\}\right\|_p |G_n(t)|dt \\ &\leq M\int_0^\pi |G_n(t)|dt = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right).\end{aligned}$$

This completes the proof of Lemma. □

Theorem 2.4. *Let* $1 < p \leq \infty$, (a_n) *and* (b_n) *be sequences of non-negative integers satisfying*

$$a_n < b_n, \quad \lim_{n\to\infty} b_n = \infty$$

and $\{p_n\}$ *be a sequence of non-negative real numbers such that*

$$\sum_{k=a_n+1}^{b_n} |p_k - p_{k+1}| = O(p_{b_n}) \tag{2.7}$$

$p_{a_n+1} = 0$, $p_{b_n+1} = 0$. *If* $f \in K_p$, $1 < p \leq \infty$, *then*

$$\| f(x) - D_a^b R_n(f;x) \|_p = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right). \tag{2.8}$$

Proof. Due to Lemma 2.3, it is enough to show (2.5). By Abel's transform, we get

$$M_n(t) = \frac{1}{2\sin^2(t/2)}\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_k-p_{k+1})\right\}$$

Since

$$\frac{1}{2\sin^2(t/2)} = \frac{2}{t^2} + O(1),$$

we have

$$\begin{aligned}M_n(t) &= \frac{2}{t^2}\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_k-p_{k+1})\right\} \\ &\quad + O\left(\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_k-p_{k+1})\right\}\right)\end{aligned}$$

$$= \frac{2}{t^2}\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_k-p_{k+1})\right\}+O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right).$$

From the last equation we obtain

$$\int_0^\pi |G_n(t)|dt \le \frac{1}{P_{a_n+1}^{b_n}}\sum_{k=a_n+1}^{b_n}|p_k-p_{k+1}|\int_0^\pi\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt.$$

To complete the proof, we shall show the following equation

$$I := \int_0^\pi\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt = O(1)\ . \tag{2.9}$$

By Lemma 1.3 we get

$$\begin{aligned} I := \int_0^\pi\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt &= \int_0^{1/e(k+1)}\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt \\ &\quad + \int_{1/e(k+1)}^\pi\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt \\ &\le \int_0^{1/e(b_k+1)} 2(k+1)\log\left(\frac{1}{(k+1)t}\right)dt \\ &\quad + \int_{1/e(b_k+1)}^\pi \frac{2}{(k+1)t^2}dt = O(1). \end{aligned}$$

This completes the proof. □

Now, we can give our results for Deferred Nörlund means.

If there are a positive non-increasing function $\phi(n)$ and a class of functions K with the following properties

$$\|\ f(x)-D_a^bN_n(f;x)\ \| = o(\phi(n)) \ \Rightarrow\ f \text{ is constant} \tag{2.10}$$

$$\|\ f(x)-D_a^bN_n(f;x)\ \| = O(\phi(n)) \ \Rightarrow\ f\in K \tag{2.11}$$

and

$$f\in K\ \Rightarrow \|\ f(x)-D_a^bN_n(f;x)\ \| = O(\phi(n)) \tag{2.12}$$

then we say that $D_a^bN_n(f;x)$ is saturated with the order $\phi(n)$ and class K.

Lemma 2.5. *Let* $1\le p\le\infty$ *and*

$$\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=a_n+1}^{b_n}p_{b_n-k}\log k<\infty.$$

If

$$\|\ f(x)-D_a^bN_n(f;x)\ \|_p = o\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right)$$

then f *is constant.*

Proof. Let us write $D_a^bN_n(x)$ instead of $D_a^bN_n(f;x)$. Now we get

$$\begin{aligned}
& \frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bN_n(x+t)\cos ktdt \\
= & \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{P_0^{b_n-a_n-1}}\sum_{r=a_n+1}^{b_n} p_{b_n-r}S_r(x+t)\cos ktdt \\
= & \frac{1}{P_0^{b_n-a_n-1}}\sum_{r=a_n+1}^{b_n} p_{b_n-r}\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt \\
= & \frac{1}{P_0^{b_n-a_n-1}}\sum_{r=k}^{b_n} p_{b_n-r}A_k(x)\ ,
\end{aligned}$$

since summation and integration can be replace by hypothesis and since

$$\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt = \begin{cases} A_k(x), & r \geq k \\ 0, & r < k. \end{cases}$$

Hence, we obtain

$$A_k(x) - \frac{1}{P_0^{b_n-a_n-1}}\sum_{r=k}^{b_n} p_{b_n-r}A_k(x) = A_k(x)\left(\frac{1}{P_0^{b_n-a_n-1}}\sum_{r=k+1}^{b_n} p_{b_n-r}\right)$$

$S_n(f)$ converges to f uniformly whenever f is continuous [15]. So, from hypothesis and generalized Minkowski's inequality we get

$$\begin{aligned}
& \left\|\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bN_n(x+t)\cos ktdt\right\|_p \\
\leq & \left\|\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos ktdt\right\|_p \\
& + \left\|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bN_n(x+t)\cos ktdt\right\|_p \\
\leq & \frac{1}{\pi}\int_{-\pi}^{\pi} \|S_r - f\|_p dt + \frac{1}{\pi}\int_{-\pi}^{\pi} \|f - D_a^bN_n\|_p dt \\
= & o\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right).
\end{aligned}$$

Hence we have

$$A_k(x)\left(\frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k+1}^{b_n} p_{b_n-r}\right) = o\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right)$$

for all $k \geq 1$, i.e.,

$$A_k(x)\left(\frac{p_{b_n-k-1} + p_{b_n-k-2}\cdots + p_0}{p_0}\right) = o(1).$$

Because of $\left(\frac{p_{b_n-k-1}+p_{b_n-k-2}\ldots+p_0}{p_0}\right) \geq 1$, $A_k(x) = 0$ for each $r \geq 1$. Consequently $f(x) = \frac{1}{2}a_0$ which is a constant. □

Lemma 2.6. *Let the limit*

$$\lim_{n\to\infty} \frac{p_{b_n-k+1}}{p_0} = 1$$

hold for a fixed $a_n + 2 \leq k < b_n$. If the equation

$$\| f(x) - D_a^b N_n(f;x) \|_p = O\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right)$$

is hold, then

$$\left\| \sum_{k=a_n+2}^{N} (k - a_n - 1)\left(1 - \frac{k-(a_n+2)}{N-a_n-1}\right) A_k(x) \right\|_p = O(1).$$

Proof. Suppose that $\Delta_n(x) := f(x) - D_a^b N_n(f;x)$. In this case,

$$\Delta_n(x) \sim \sum_{k=a_n+2}^{b_n} \left(1 - \frac{P_0^{b_n-k}}{P_0^{b_n-a_n-1}}\right) A_k(x).$$

Let $N < b_n$. Taking N-th arithmetic mean of $\Delta_n(x)$ we have

$$\sigma_N[x;\Delta_n] = \sum_{k=a_n+2}^{N} \left(1 - \frac{P_0^{b_n-k}}{P_0^{b_n-a_n-1}}\right)\left(1 - \frac{k-a_n-2}{N-a_n-1}\right) A_k(x).$$

Since $\|\Delta_n\| \geq \|\sigma_N[x;\Delta_n]\|$ we obtain

$$\left\| \sum_{k=a_n+2}^{N} \left(1 - \frac{P_0^{b_n-k}}{P_0^{b_n-a_n-1}}\right)\left(1 - \frac{k-a_n-2}{N-a_n-1}\right) A_k(x) \right\|_p = O\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right)$$

$$\left\| \sum_{k=a_n+2}^{N} \left(\frac{P_0^{b_n-a_n-1} - P_0^{b_n-k}}{p_0}\right)\left(1 - \frac{k-a_n-2}{N-a_n-1}\right) A_k(x) \right\|_p = O(1)$$

$$\left\| \sum_{k=a_n+2}^{N} \lim_{n\to\infty} \left(\frac{p_{b_n-k+1} + \ldots + p_{b_n-a_n-1}}{p_0}\right)\left(1 - \frac{k}{N+1}\right) A_k(x) \right\|_p = O(1)$$

$$\left\| \sum_{k=a_n+2}^{N} (k - a_n - 1)\left(1 - \frac{k}{N+1}\right) A_k(x) \right\|_p = O(1).$$

This completes the proof. □

Lemma 2.7. *Let*

$$M_n(t) = \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \frac{\cos(k+1/2)t}{\sin(t/2)}$$

and

$$G_n(t) = \int_t^\pi M_n(u)du. \tag{2.13}$$

If

$$\int_0^\pi |G_n(t)| = O\left(\frac{p_o}{P_0^{b_n - a_n - 1}}\right) \tag{2.14}$$

and $f \in K_p (1 < p \leq \infty)$ *then* $\| f(x) - D_a^b N_n(f;x) \|_p = O\left(\frac{p_o}{P_0^{b_n - a_n - 1}}\right)$.

Proof. Let $\tilde{S}_k(\tilde{f};x)$ denote the partial sums of the conjugate series related to $\tilde{f}(x)$. So,

$$\tilde{S}_k(\tilde{f};x) = \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(t/2) - \cos(k+1/2)t}{\sin(t/2)}dt.$$

With a simple analysis, we get

$$D_a^b N_n(\tilde{S}_k(\tilde{f};x)) = \frac{1}{P_0^{b_a - n - 1}}\sum_{a_n+1}^{b_n} p_{b_n - k}\tilde{S}_k(\tilde{f};x)$$

$$= \frac{1}{P_0^{b_a - n - 1}}\sum_{a_n+1}^{b_n} p_{b_n - k}\frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt$$

$$-\frac{1}{P_{a_n+1}^{b_n}}\sum_{a_n+1}^{b_n} p_{b_n - k}\frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(k+1/2)t}{\sin(t/2)}dt.$$

By Lemma 1.2, $f \in L_p$ $(1 < p < \infty)$ implies $\tilde{f} \in L_p$. So $\tilde{\tilde{f}} \in L_p$, and we get $\tilde{S}(\tilde{f}) = S(\tilde{\tilde{f}})$. If $p = \infty$ then it means that $\tilde{f} \in Lip1$. Therefore, $-f + \frac{1}{2}a_0$ is equal to $\tilde{\tilde{f}}$. Thus $\tilde{\tilde{f}} - D_a^b N_n(S_k(\tilde{\tilde{f}};x))$ is identical to $f(x) - D_a^b N_n(S_k(f;x))$. From hypothesis we get

$$f(x) - D_a^b N_n(f;x)$$

$$= \frac{1}{P_0^{b_a - n - 1}}\sum_{a_n+1}^{b_n} p_{b_n - k}\frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt$$

$$-\frac{1}{P_0^{b_a - n - 1}}\sum_{a_n+1}^{b_n} p_{b_n - k}\frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt$$

$$+\frac{1}{P_0^{b_a - n - 1}}\sum_{a_n+1}^{b_n} p_{b_n - k}\frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(k+1/2)t}{\sin(t/2)}dt$$

$$= \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}M_n(t). \tag{2.15}$$

As $f \in K_p$ by Lemma 1.1, we have $\tilde{f}' \in L_p$, $p > 1$. By integrating in (2.15), we get

$$f(x) - D_a^b N_n(f;x) = -\frac{1}{2\pi}\int_0^\pi \{\tilde{f}'(x+t) + \tilde{f}'(x-t)\}G_n(t)dt.$$

By generalized Minkowski's inequality, we obtain

$$\begin{aligned}\left\|f(x)-D_a^bN_n(f;x)\right\|_p &= \left\|-\frac{1}{2\pi}\int_0^\pi\{\tilde{f}'(x+t)+\tilde{f}'(x-t)\}G_n(t)dt\right\|_p\\ &\le \int_0^\pi\left\|\{\tilde{f}'(x+t)+\tilde{f}'(x-t)\}\right\|_p|G_n(t)|dt\\ &\le M.\int_0^\pi|G_n(t)|dt=O\left(\frac{p_o}{P_0^{b_n-a_n-1}}\right).\end{aligned}$$

This completes the proof of Lemma. □

Theorem 2.8. *Let $1<p\le\infty$, (a_n) and (b_n) be sequences of non-negative integers satisfying*

$$a_n<b_n\ ,\quad \lim_{n\to\infty}b_n=\infty$$

and $\{p_n\}$ be a sequence of non-negative real numbers such that

$$\sum_{k=a_n+1}^{b_n}|p_{b_n-k}-p_{b_n-k-1}|=O(p_0), \tag{2.16}$$

$p_{b_n-a_n-1}=0$ *and* $p_{-1}=0$*. If $f\in K_p(1<p\le\infty)$ then*

$$\|\ f(x)-D_a^bN_n(f;x)\ \|_p=O\left(\frac{p_o}{P_0^{b_n-a_n-1}}\right). \tag{2.17}$$

Proof. Due to the Lemma 2.7, it is enough to show (2.14). From Abel's transform, we get

$$M_n(t)=\frac{1}{2\sin^2(t/2)}\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_{b_n-k}-p_{b_n-k-1})\right\}.$$

Since

$$\frac{1}{2\sin^2(t/2)}=\frac{2}{t^2}+O(1),$$

we have

$$\begin{aligned}M_n(t) &= \frac{2}{t^2}\frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_{b_n-k}-p_{b_n-k-1})\right\}\\ &\quad+O\left(\frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_{b_n-k}-p_{b_n-k-1})\right\}\right)\\ &= \frac{2}{t^2}\frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_{b_n-k}-p_{b_n-k-1})\right\}\\ &\quad+O\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right).\end{aligned}$$

From the last equation we get

$$\int_0^\pi |G_n(t)|dt \le \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} |p_{b_n-k} - p_{b_n-k-1}| \int_0^\pi \left| \int_t^\pi \frac{\sin(k+1)u}{u^2} du \right| dt.$$

By Theorem 2.4 and hypothesis, we have

$$\int_0^\pi |G_n(t)| = O\left(\frac{p_o}{P_0^{b_n-a_n-1}}\right).$$

This completes the proof. □

If we take $p_k = 1$ for all k, both of them yield Deferred Cesàro means of the series (1.1). So we get following corollary.

Corollary 2.9. *Let*

$$M_n(t) = \frac{\cos[((b_n + a_n + 2)/2)t] \sin[((b_n - a_n)/2)t]}{(b_n - a_n) \sin^2(t/2)}$$

and

$$G_n(t) = \int_t^\pi M_n(u)du.$$

If

$$\int_0^\pi |G_n(t)| = O\left(\frac{1}{b_n - a_n}\right) \tag{2.18}$$

and $f \in K_p (1 < p \le \infty)$ *then* $\| f(x) - D_a^b(f;x) \|_p = O\left(\frac{1}{b_n - a_n}\right)$.

If we take $p_k = 1$ for all k, $a_n = 0$ and $b_n = \lambda(n)$, where $\lambda(n)$ is a strictly increasing sequence of positive integers, both of them yield C_λ-method. So, we immediately get following corollary.

Corollary 2.10. *Let*

$$M_n(t) = \frac{1}{\lambda(n)} \left(\frac{\cos((\lambda(n) + 2)/2)t. \sin((\lambda(n) - 1)/2)t}{\sin^2(t/2)} \right)$$

and

$$G_n(t) = \int_t^\pi M_n(u)du.$$

If

$$\int_0^\pi |G_n(t)| = O\left(\frac{1}{\lambda(n)}\right) \tag{2.19}$$

and $f \in K_p$ $(1 < p \le \infty)$ *then* $\| f(x) - \sigma_n^\lambda(f;x) \|_p = O\left(\frac{1}{\lambda(n)}\right)$.

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Study on interval Volterra integral equations via parametric approach of intervals

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Abstract. This work investigates the interval Volterra integral equation (IVIE) and its solution techniques through the parametric representation of intervals. First, the general form of the second-kind IVIE is expressed in both lower-upper bound format and its equivalent parametric form. Next, the methods of successive approximations and resolvent kernel are developed to solve the IVIE, utilizing parametric approaches and interval arithmetic. The solutions are presented in both parametric and lower-upper bound representations. Lastly, a series of numerical examples are provided to illustrate the application of these methods.

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Keywords: Interval IVP, interval integral equation, parametric approach, successive approximation method, resolvent kernel.

1. Introduction

Integral equations play a crucial role in various fields of applied mathematics, with numerous applications in real-world problems such as radioactive decay, diffusion and heat transfer analysis, energy systems, web security, and population growth models. In these cases, the parameters involved are often not fixed but fluctuate within certain ranges due to randomness or uncertainty, making the problem imprecise. Based on the nature of these problems, the theory of integral equations can be categorized into two types:

- Precise integral equations
- Imprecise integral equations

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Historically, integral equations have been studied in a crisp environment, where all variables (dependent and independent) are deterministic. Numerous works have focused on solving these crisp integral equations. However, as uncertainty in areas such as technology, energy, communications, and financial security continues to grow exponentially, scientists, engineers, and system analysts face increasing challenges in solving decision-making problems under inexact conditions. To address this uncertainty, researchers have introduced various approaches, such as stochastic, fuzzy, and interval methods. Imprecise integral equations, therefore, can be classified as:

- Stochastic integral equations
- Fuzzy integral equations
- Interval integral equations

In stochastic integral equation, all the imprecise known and unknown functions are represented by the random variables with suitable distribution functions. In this sector, many researchers and mathematicians contributed their works, among those some excellent pieces are reported here. Mao [15] investigated the results on existence of the solutions of a stochastic delay integral equation. Ogawa [20, 21]studied Fredholm stochastic integral equation in random environment whereas Mirzaee et al. [16], Yong et al. [30] derived computational method and backword method respectively for solving non-linear Volterra integral equations in stochastic environment. Recently, Mohammadi [17], Samadyar and Mirzaee [24, 25] contributed their works on stochastic integral equation.

In fuzzy integral equation, the imprecise functions are presented precisely by using fuzzy set having appropriate membership functions or fuzzy numbers. In this area, Subrahmanyam et al. [27], Agarwal et al. [2], Attari et al. [6] established different methods of solving Volterra integral equations in fuzzy environment. Later Mordeson and Newman [18] studied the different solution approaches of fuzzy integral equation. Babolian et al. [7], Abbasbandy et al. [1] established some numerical technique for solving Fredholm integral equation in fuzzy environment. Also, Bica and Popescu [8] together developed a methodology for approximate solution of nonlinear Hammerstein fuzzy integral equation. Recently, Zakeri et al. [31], Ziari et al. [32], Agheli and Firozja [3]and Noeiaghdam et al. [19] accomplished their works on different types of fuzzy integral equations.

Alternatively, if the known and unknown functions of an imprecise integral equation are presented in the form of intervals, then that imprecise integral equation is called as interval integral equation (or IIE). The general form of an interval integral equation (or IIE) is given below:

$$[g_L,\, g_U]\,(u)\,[y_L,\, y_U]\,(u) = [f_L,\, f_U]\,(u) + \lambda \int_{u_0}^{u \text{ or } u_1} K\,(u,z)\,[y_L,\, y_U]\,(z)\,dz$$

where $[f_L,\, f_U]\,,[g_L,\, g_U] : [u_0, u_1] \to K_c$ defined by

$$[f_L,\, f_U]\,(u) = [f_L\,(u)\,, f_U\,(u)]\ ,$$
$$[g_L,\, g_U]\,(u) = [g_L\,(u)\,, g_U\,(u)]$$

and

$$K : [u_0, u_1] \times [u_0, u_1] \to \mathbb{R}$$

which are known function whereas,

$$[y_L,\, y_U]\,(u) = [y_L\,(u)\,, y_U\,(u)]$$

is unknown function.

In the above-mentioned integral equation, if the upper limit is fixed or variable, then the integral equation is called interval Fredholm or interval Volterra integral equation respectively. With the help of this concept of interval environment in the mathematical lingua franca of these variables/parameters, the concepts of interval differential equation or interval system of differential equations have been formulated mathematically for those real-life problems. The interval differential equations have been studied by several researchers representing the imprecise parameters by random variables/fuzzy sets/intervals. Among of them, the contributions of Kaleva et al. [11], Buckley et al. [9], Vorobiev et al. [28], Stefanini et al. [26], Malinowski [13], Ramezanadeh et al. [23], Wang et al. [29], de Costa et al. [10] and Ahmady [4] are noteworthy. On the other hand, very few works on integral equations in interval environment were accomplished among which works of An et al. [5], Otadi and Mosleh [22], Lupulescu and Van [12] are worth mentioning. An et al. [5] in their work, studied the Fredholm integral equation in interval environment using interval arithmetic and Hukuhara differentiation. Otadi and Mosleh [22] developed simulation technique for the evaluation of linear fuzzy Fredholm type integral equations. Further, Lupulescu and Van [12] extended the theory of RiemannLiouville fractional integral to develop the theory of the Abel integral equation in interval environment.

In this work, the theory of interval Volterra integral equation is studied using parametric representation of intervals and parametric differentiation. To navigate the derivation of the theorems properly, the concepts of set of parameterizations of intervals, continuous parametric interval-valued functions along with metric with respect to which their different analytical properties (like continuity, differentiability, integrability etc.) are discussed. After that the class of all parametric interval valued L^2-functions is defined, over which all the discussions of interval Volterra interval equations have been performed. Beside these, two types of solution methodologies of interval Volterra integral equations named as successive approximation and Resolvent kernel theorems are developed in the parametric form of intervals.

2. Basic notations and definitions

Let $K_c = \{[\alpha_L, \alpha_U] : \alpha_L, \alpha_U \in \mathbb{R}\}$ the set of closed and bounded intervals.

Definition 2.1. Parametric representations of $[\alpha_L, \alpha_U]$ can be defined in the following manner:

- Increasing form (IF):

$$[\alpha_L, \alpha_U] = \{\alpha\,(\zeta) = \alpha_L + \zeta\,(\alpha_U - \alpha_L) : \; 0 \leqslant \zeta \leqslant 1\}$$

- Decreasing form (DF):

$$[\alpha_L, \alpha_U] = \{\alpha(\zeta) = \alpha_U + \zeta(\alpha_L - \alpha_U) : 0 \leqslant \zeta \leqslant 1\}$$

Therefore, the set of all parametric intervals K_P is defined as follows:

$$K_P = \{\alpha(\zeta) : \alpha(\zeta) \text{ is parametric form of the interval } [\alpha_L, \alpha_U] \in K_c\}.$$

Definition 2.2. Let $I_1 = \{\alpha(\zeta_1) : \zeta_1 \in [0,1]\}, I_2 = \{\beta(\zeta_2) : \zeta_2 \in [0,1]\} \in K_p$ be the parametric forms of two intervals $[\alpha_L, \alpha_U]$ and $[\beta_L, \beta_U]$ respectively and $\lambda \in \mathbb{R}$. Then,

- Addition:

$$I_1 + I_2 = \{\alpha(\zeta_1) + \beta(\zeta_2) : \zeta_1, \zeta_2 \in [0,1]\}$$

- Subtraction:

$$I_1 - I_2 = \{\alpha(\zeta_1) - \beta(\zeta_2) : \zeta_1, \zeta_2 \in [0,1]\}$$

- Parametric difference:

$$I_1 \ominus_p I_2 = \{\alpha(\zeta) - \beta(\zeta) : \zeta \in [0,1]\}$$

- Multiplication:

$$I_1 I_2 = \{\alpha(\zeta_1)\beta(\zeta_2) : \zeta_1, \zeta_2 \in [0,1]\}$$

- Division:

$$I_1/_{I_2} = \left\{\frac{\alpha(\zeta_1)}{\beta(\zeta_2)} : \zeta_1, \zeta_2 \in [0,1]\right\}$$

- Scalar multiplication:

$$\lambda I_1 = \{\lambda\alpha(\zeta) : \zeta \in [0,1]\}$$

- Equality of two intervals:

$$I_1 = I_2 \Leftrightarrow \alpha(\zeta) = \beta(\zeta), \ \forall\zeta \in [0,1]$$

Proposition 2.3. *Let* $I_1 = [\alpha_L, \alpha_U], I_2 = [\beta_L, \beta_U] \in K_c$, $\lambda \in \mathbb{R}$ *and their parametric representations be* $I_1 = \{\alpha(\zeta_1) : \zeta_1 \in [0,1]\}$ *and* $I_2 = \{\beta(\zeta_2) : \zeta_2 \in [0,1]\}$. *The different arithmetic operations on the set* K_c *can be obtained as follows:*

- *Addition:*

$$I_1 + I_2 = \left[\min_{\zeta_1,\zeta_2\in[0,1]} (\alpha(\zeta_1) + \beta(\zeta_2)), \max_{\zeta_1,\zeta_2\in[0,1]} (\alpha(\zeta_1) + \beta(\zeta_2))\right]$$

- *Subtraction:*

$$I_1 - I_2 = \left[\min_{\zeta_1,\zeta_2\in[0,1]} (\alpha(\zeta_1) - \beta(\zeta_2)), \max_{\zeta_1,\zeta_2\in[0,1]} (\alpha(\zeta_1) - \beta(\zeta_2))\right]$$

- *Parametric difference:*

$$I_1 \ominus_p I_2 = \left[\min_{\zeta\in[0,1]} (\alpha(\zeta) - \beta(\zeta)), \max_{\zeta\in[0,1]} (\alpha(\zeta) - \beta(\zeta))\right]$$

- *Multiplication:*

$$I_1 I_2 = \left[\min_{\zeta_1,\zeta_2\in[0,1]} (\alpha(\zeta_1)\beta(\zeta_2)), \max_{\zeta_1,\zeta_2\in[0,1]} (\alpha(\zeta_1)\beta(\zeta_2))\right]$$

- *Division:*

$$I_1/I_2 = \left[\min_{\zeta_1,\zeta_2\in[0,1]}\left(\frac{\alpha(\zeta_1)}{\beta(\zeta_2)}\right), \max_{\zeta_1,\zeta_2\in[0,1]}\left(\frac{\alpha(\zeta_1)}{\beta(\zeta_2)}\right)\right], 0 \notin I_2$$

- *Scalar multiplication:*

$$\lambda I_1 = \left[\min_{\zeta\in[0,1]}(\lambda\alpha(\zeta)), \max_{\zeta\in[0,1]}(\lambda\alpha(\zeta))\right]$$

Definition 2.4. The distance function on K_p is a function $\rho_p : K_p \times K_p \to \mathbb{R}^+ \cup \{0\}$ be a function defined by

$$\rho_p(\alpha(\zeta), \beta(\zeta)) = \sup_{\zeta\in[0,1]} |\alpha(\zeta) - \beta(\zeta)|, \ \forall\alpha(\zeta), \beta(\zeta) \in K_p.$$

Clearly ρ_p is a metric on K_p.

Proposition 2.5. *Let $\alpha(\zeta)$, $\beta(\zeta) \in K_p$, then*

$$\sup_{\zeta\in[0,1]} |\alpha(\zeta) - \beta(\zeta)| = \max_{\zeta\in\{0,1\}} |\alpha(\zeta) - \beta(\zeta)|.$$

Corollary 2.6. *Let $\rho_p^1 : K_p \times K_p \to \mathbb{R}^+ \cup \{0\}$ be defined by*

$$\rho_p^1(\alpha(\zeta), \beta(\zeta)) = \max_{\zeta\in\{0,1\}} |\alpha(\zeta) - \beta(\zeta)|, \ \forall\alpha(\zeta), \beta(\zeta) \in K_p.$$

Then ρ_p^1 is a metric on K_p.

Corollary 2.7. *Let $\rho_c : K_c \times K_c \to \mathbb{R}^+ \cup \{0\}$ be defined by*

$$\rho_c([\alpha_L, \alpha_U], [\beta_L, \beta_U]) = \max\{|\alpha_L - \beta_L|, |\alpha_U - \beta_U|\}, \ \forall [\alpha_L, \alpha_U], [\beta_L, \beta_U] \in K_c.$$

Then ρ_c is a metric on K_c.

Corollary 2.8. *The metrics ρ_p^1 and ρ_c are equivalent.*

2.1. Parametric form of interval valued functions (IVF)

Definition 2.9. An IVF is a function $[f_L, f_U] : I \subseteq \mathbb{R} \to K_c$ given by $[f_L, f_U](u) = [f_L(u), f_U(u)]$, where $f_L, f_U : I \to \mathbb{R}$ are real valued functions with $f_L(u) \leqslant f_U(u)$, $\forall u \in I$.

Definition 2.10. The parametric form (in IF) of IVF $[f_L, f_U](u)$ is denoted as $f_{\zeta\in[0,1]} : I \to K_p$ and it is defined by

$$f_{\zeta\in[0,1]}(u) = \{f_L(u) + \zeta(f_U(u) - f_L(u)) : \ \zeta \in [0,1]\}, \ \forall u \in I.$$

Let us consider an IVF in parametric form $f_{\zeta\in[0,1]} : I \to K_p$ defined by

$$f_{\zeta\in[0,1]}(u) = \{f_L(u) + \zeta(f_U(u) - f_L(u)) : \ \zeta \in [0,1]\}, \ \forall u \in I.$$

Definition 2.11. The IVF in parametric form $f_{\zeta\in[0,1]} : I \to K_p$ is called continuous at $u_0 \in I$ if the real valued function $\tilde{f} : I \times [0,1] \to \mathbb{R}$ defined by

$$\tilde{f}(u, \zeta) = f_L(u) + \zeta(f_U(u) - f_L(u))$$

is continuous at (u_0, ζ), $\forall\zeta \in [0,1]$.

Definition 2.12. The IVF in parametric form $f_{\zeta\in[0,1]} : I \to K_p$ is called differentiable at $u_0 \in I$ if the real valued function $\tilde{f} : I \times [0,1] \to \mathbb{R}$ defined by

$$\tilde{f}(u,\zeta) = f_L(u) + \zeta(f_U(u) - f_L(u))$$

is differentiable at (u_0, ζ), $\forall \zeta \in [0,1]$. And the derivative is obtained by the following limit:

$$\left.\frac{\partial \tilde{f}(u,\zeta)}{\partial u}\right|_{u=u_0} = \tilde{f}_u(u_0,\zeta) = \lim_{u\to u_0} \frac{\tilde{f}(u,\zeta) - \tilde{f}(u_0,\zeta)}{u-u_0}.$$

The parametric derivative of $f_{\zeta\in[0,1]}$ at u_0 is denoted by $f'_{\zeta\in[0,1]}(u_0)$.

Proposition 2.13.
1. The IVF $f_{\zeta\in[0,1]}$ is continuous at u_0 iff both the bounds f_L and f_U are continuous at u_0.
2. The IVF $f_{\zeta\in[0,1]}$ is differentiable at u_0 iff both the bounds f_L and f_U are differentiable at u_0.

Proposition 2.14. *If the IVF in parametric form $f_{\zeta\in[0,1]}$ is differentiable at u_0, then*

$$f'_{\zeta\in[0,1]}(u_0) = \left\{ {f_L}'(u_0) + \zeta\left({f_U}'(u_0) - {f'}_L(u_0)\right) : \zeta \in [0,1] \right\}.$$

Definition 2.15. Let I be a Lebesgue measurable set. The IVF in parametric form $f_{\zeta\in[0,1]}$ is said to be a Lebesgue measurable parametric interval valued function over I if for every fixed $\zeta^* \in [0,1]$, the function $\tilde{f}(u,\zeta^*)$ is a measurable function.

Definition 2.16. The IVF in parametric form $f_{\zeta\in[0,1]}$ is said to be integrable over I if for every fixed $\zeta^* \in [0,1]$, the function $\tilde{f}(u,\zeta^*)$ is integrable in over I and

$$\int_I f_{\zeta\in[0,1]}(u)\,du = \{I_L + \zeta(I_U - I_L) : \forall \zeta \in [0,1]\},$$

where

$$I_L = \int_I f_L(u)\,du\,,\; I_U = \int_I f_U(u)\,du.$$

Proposition 2.17. *The IVF $f_{\zeta\in[0,1]}$ is integrable over I iff both the bounds f_L and f_U are integrable over I.*

Definition 2.18. The IVF $f_{\zeta\in[0,1]}$ is said to be a parametric L^2- function over I if

$$\rho_p\left(\int_I f_{\zeta\in[0,1]}(u)\;du,\;0\right) < \infty.$$

Proposition 2.19. *The IVF $f_{\zeta\in[0,1]}$ is L^2- function over I iff both the bounds f_L and f_U are L^2- functions over I.*

Remark 2.20. The set of all parametric L^2- function over I is denoted by $L^2_p(I)$.

2.2. Interval initial value problem (Interval IVP)

Let $[y_L, y_U] : [u_0, u_1] \to K_c$ be a p-differentiable function and the IVF $[f_L, f_U] : [u_0, u_1] \to K_c$ be a continuous, then a second order interval valued initial value problem can be defined as follows:

$$\frac{d^2}{du^2}\left(\left[y_L(u), y_U(u)\right]\right) + a_1(u)\frac{d}{du}\left(\left[y_L(u), y_U(u)\right]\right) + a_2(u)\left[y_L(u), y_U(u)\right] = \left[f_L(u), f_U(u)\right] \quad (2.1)$$

with $[y_L(u_0), y_U(u_0)] = [y_{L0}, y_{U0}]$ and $\left.\frac{d}{du}[y_L(u), y_U(u)]\right|_{u=u_0} = [y_{L1}, y_{U1}]$

where $a_1(u)$, $a_2(u)$ are real valued continious functions over $[u_0, u_1]$.

The interval initial value problem (2.1) can be represented in parametric form as follows:

$$y''_{\zeta\in[0,1]}(u) + a_1(u)\, y'_{\zeta\in[0,1]}(u) + a_2(u)\, y_{\zeta\in[0,1]}(u) = f_{\zeta\in[0,1]}(u)$$

with $y_{\zeta\in[0,1]}(u_0) = \{y_0(\zeta) : \zeta \in [0,1]\}$ and $y'_{\zeta\in[0,1]}(u_0) = \{y_1(\zeta) : \zeta \in [0,1]\}$

where $y_0(\zeta) = y_{0L} + \zeta(y_{0U} - y_{0L})$ and $y_1(\zeta) = y_{1L} + \zeta(y_{1U} - y_{1L})$.

3. Interval Volterra integral equation (IVIE)

In this section, we have presented some theoretical aspects regarding an IVIE of second kind. Also, the different solution approaches viz. general solution method, method of series solutions and resolvent kernel for solving an IVIE of second kind are discussed.

The general form of an IVIE is

$$[g_L, g_U](u)\,[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)\,[y_L, y_U](z)\,dz \quad (3.1)$$

where $[f_L, f_U], [g_L, g_U] : [u_0, u_1] \to K_c$ are known functions. However, $[y_L, y_U](u)$ is an unknown function and λ is a non-zero real number. Here we discuss IVIE of second kind only.

An IVIE of second kind is defined as

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)\,[y_L, y_U](z)\,dz \quad (3.2)$$

The parametric form of (3.2) is of the following form:

$$y_{\zeta\in[0,1]}(u) = f_{\zeta_1\in[0,1]}(u) + \lambda \int_{u_0}^{u} K(u,z)\, y_{\zeta\in[0,1]}(z)\,dz \quad (3.3)$$

where $y_{\zeta\in[0,1]}$ and $f_{\zeta_1\in[0,1]}$ are respectively parametric forms of $[y_L, y_U]$ and $[f_L, f_U]$.

Remark 3.1. Here the real valued function $K(u,z)$ is a L^2- function and the parametric interval valued functions $y_{\zeta\in[0,1]}$ and $f_{\zeta_1\in[0,1]}$ are taken from $L_p^2[u_0,u_1]$.

Remark 3.2. For, $y_{\zeta\in[0,1]}$ and $f_{\zeta_1\in[0,1]}$ the parametric forms are given by

$$\tilde{y}(u,\zeta) = y_L(u) + \zeta(y_U(u) - y_L(u)) \text{ and } \tilde{f}(u,\zeta_1) = f_L(u) + \zeta_1(f_U(u) - f_L(u)).$$

Proposition 3.3. *The interval Volterra integral equation (3.2) is equivalent to its parametric form (3.3).*

Definition 3.4. The interval valued function $[y_L(u), y_U(u)]$ is called a solution of (3.2) if it satisfies the equation (3.2). Similarly, the solution of (3.3) can be defined.

Proposition 3.5. *The solutions of (3.2) and (3.3) are equivalent.*

Proof. Proof follows from the equality of two intervals in parametric form. □

Before to discuss the solution procedures of the IVIE, an important formula for converting multiple integrals into a single integral for integrable interval valued functions is presented in the next subsection.

3.1. Conversion of multiple integrals into a single integral for interval integrals

Theorem 3.6. *Let* $[y_L, y_U] : [u_0,u_1] \to K_c$ *be given by*

$$[f_L, f_U](u) = [f_L(u), f_U(u)], \quad \forall u \in [u_0,u_1]$$

be an integrable interval valued function. Then it satisfies the following integral formula:

$$\int_{u_0}^{u} [f_L(z), f_U(z)]\, dz^n = \int_{u_0}^{u} \frac{(u-z)^{n-1}}{(n-1)!} [f_L(z), f_U(z)]\, dz \tag{3.4}$$

To prove this theorem, we have required the following Lemma:

Lemma 3.7. *Let* $g, h : [u_0,u_1] \to \mathbb{R}$ *be two differentiable functions with non-negative derivatives over* $[u_0,u_1]$. *Then,*

$$\frac{d}{du}\left(\int_{g(u)}^{h(u)} [f_L(z), f_U(z)]\, dz\right) = [f_L(h(u)), f_U(h(u))]\frac{dh(u)}{du} \tag{3.5}$$
$$\ominus_p [f_L(g(u)), f_U(g(u))]\frac{dg(u)}{du}$$

Proof. From the parametric representation of $[f_L, f_U]$, it can written as

$$f_{\zeta\in[0,1]}(u) = \left\{\tilde{f}(u,\zeta) = f_L(u) + \zeta(f_U(u) - f_L(u)) : \zeta \in [0,1]\right\}, \ \forall u \in I.$$

From the Leibnitz's rule of differentiation under the sign of integration for real valued functions, it follows that

$$\frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} \tilde{f}(z,\zeta)\ dz\right) = \tilde{f}(h(u),\zeta)\frac{dh(u)}{du} - \tilde{f}(g(u),\zeta)\frac{dg(u)}{du},\ \ \forall\zeta\in[0,1]$$

$$\Rightarrow \left\{\frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} \tilde{f}(z,\zeta)\ dz\right) :\ \zeta\in[0,1]\right\}$$

$$= \left\{\tilde{f}(h(u),\zeta)\frac{dh(u)}{du} - \tilde{f}(g(u),\zeta)\frac{dg(u)}{du} :\ \zeta\in[0,1]\right\}$$

$$\Rightarrow \left\{\frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} \tilde{f}(z,\zeta)\ dz\right) :\ \zeta\in[0,1]\right\}$$

$$= \left\{\tilde{f}(h(u),\zeta)\frac{dh(u)}{du} :\ \zeta\in[0,1]\right\} \ominus_p \left\{\tilde{f}(g(u),\zeta)\frac{dg(u)}{du}\ :\zeta\in[0,1]\right\}$$

$$\Rightarrow \left\{\frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} \tilde{f}(z,\zeta)\ dz\right) :\ \zeta\in[0,1]\right\}$$

$$= \left\{\tilde{f}(h(u),\zeta) :\ \zeta\in[0,1]\right\}\frac{dh(u)}{du} \ominus_p \left\{\tilde{f}(g(u),\zeta)\ :\zeta\in[0,1]\right\}\frac{dg(u)}{du}$$

since, g', h' are non - negative

$$\Rightarrow \frac{d}{du}\left(\int\limits_{g(u)}^{h(u)} [f_L(z), f_U(z)]\ dz\right)$$

$$= [f_L(h(u)), f_U(h(u))]\frac{dh(u)}{du} \ominus_p [f_L(g(u)), f_U(g(u))]\frac{dg(u)}{du}.$$

This completes the proof. □

Now, we have proved the Theorem 3.6.

Proof of Theorem 3.6

Proof. Let us consider the interval integral

$$[J_{Ln}, J_{Un}](u) = \int\limits_{u_0}^{u} (u-z)^{n-1}[f_L(z), f_U(z)]\ dz \tag{3.6}$$

Differentiating (3.6) successively with respect to u for k times and using Lemma 3.7, we get

$$\frac{d^k\left[J_{Ln}(u), J_{Un}(u)\right]}{du^k} = (n-1)(n-2)\cdots(n-k)\left[J_{Ln-k}(u), J_{Un-k}(u)\right], \text{ for } n > k. \tag{3.7}$$

Therefore from (3.6) and (3.7), it follows that:

$$\frac{d^n\left[J_{Ln}(u), J_{Un}(u)\right]}{du^n} = (n-1)!\left[f_L(u), f_U(u)\right] \tag{3.8}$$

From (3.8), we get the following recurring integrals:

$$\left[J_{L1}(u), J_{U1}(u)\right] = \int_{u_0}^{u}\left[f_L(z_1), f_U(z_1)\right] dz_1$$

$$\left[J_{L2}(u), J_{U2}(u)\right] = \int_{u_0}^{u}\int_{u_0}^{z}\left[f_L(z_1), f_U(z_1)\right] dz_1\, dz$$

Proceeding similarly, one can get the following relation:

$$\left[J_{Ln}(u), J_{Un}(u)\right] = (n-1)!\int_{u_0}^{u}\int_{u_0}^{z}\cdots\int_{u_0}^{z_{n-1}}\left[f_L(z_{n-1}), f_U(z_{n-1})\right]\, dz_{n-1}dz_{n-2}\cdots dz \tag{3.9}$$

$$\Longrightarrow \left[J_{Ln}(u), J_{Un}(u)\right] = (n-1)!\int_{u_0}^{u}\left[f_L(u), f_U(u)\right] dz^n, \tag{3.10}$$

which is the required relation. □

3.2. General solution procedure for solving IVIE

Since the equations (3.2) and (3.3) are equivalent, to get the solution of (3.2), it is sufficient to solve (3.3). Also, since the equation (3.3) represents the crisp Volterra integral equation for each fixed $\zeta, \zeta_1 \in [0,1]$, (3.3) can be solved by using any existing method for solving the Volterra integral equation. Let $\tilde{y}(u,\zeta)$ be the solutions of (3.3). Then it satisfies (3.3). Thus,

$$\therefore \tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda\int_{u_0}^{u} K(u,z)\,\tilde{y}(z,\zeta)\, dz$$

Therefore, from IPF of intervals, it follows that

$$y_L(u) = \min_{\zeta\in[0,1]}\{\tilde{y}(u,\zeta)\} \quad \text{and} \quad y_U(u) = \max_{\zeta\in[0,1]}\{\tilde{y}(u,\zeta)\}$$

So, from Proposition 2.5, it follows that,

$$y_L(u) = \min_{\zeta\in\{0,1\}}\{\tilde{y}(u,\zeta)\} \quad \text{and} \quad y_U(u) = \max_{\zeta\in\{0,1\}}\{\tilde{y}(u,\zeta)\} \tag{3.11}$$

and $y(u) = [y_L(u), y_U(u)]$ is the desired solution of (3.2).

3.3. Solution of interval Volterra integral equation of second kind by iterative method

Theorem 3.8. *Let us consider an IVIE of second kind of the form*

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)[y_L, y_U](z)\,dz \tag{3.12}$$

satisfying the following conditions:

a) kernel K be a non-negative real valued continuous function on $[u_0, u_1] \times [u_0, u_1]$ and $\exists\alpha > 0$ such that

$$|K(u,z)| \leqslant \alpha, \ \ \forall (u,z) \in [u_0, u_1] \times [u_0, u_1]. \tag{3.13}$$

b)$[f_L, f_U]$ is an interval valued continuous function over $[u_0, u_1]$ and $\exists\beta > 0$ such that

$$\rho_c(f(u), 0) \leqslant \beta, \ \forall u \in [u_0, u_1]. \tag{3.14}$$

c) $\lambda > 0$ be a non-negative constant.
Then the IVIE (3.12) has a series solution as follows:

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)[f_L, f_U](z)\,dz \tag{3.15}$$

$$+\lambda^2 \int_{u_0}^{u} K(u,z) \int_{u_0}^{z} K(z,z_1)[f_L, f_U](z_1)\,dz_1 dz + \cdots$$

Proof. From Proposition 3.5, the given IVIE is equivalent to its parametric form

$$y_{\zeta\in[0,1]}(u) = f_{\zeta_1\in[0,1]}(u) + \lambda \int_{u_0}^{u} K(u,z)\, y_{\zeta\in[0,1]}(z)\,dz \tag{3.16}$$

Therefore for fixed $\zeta, \zeta_1 \in [0,1]$

$$\tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\,\tilde{y}(z,\zeta)\,dz \tag{3.17}$$

After nth substitution, the equation (3.17) gives

$$\tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\,\tilde{f}(z,\zeta_1)\,dz + \lambda^2 \int_{u_0}^{u} K(u,z) \int_{u_0}^{z} K(z,z_1)\tilde{f}(z_1,\zeta_1)\,dz_1 dz$$

$$+ \cdots + \lambda^n \int_{u_0}^{u} K(u,z) \int_{u_0}^{z} K(z,z_1) \cdots \int_{u_0}^{z_{n-2}} K(z_{n-2}, z_{n-1})\tilde{f}(z_{n-1},\zeta_1)\,dz_{n-1} \ldots dz_1 dz$$

$$+ \tilde{R}_{n+1}(u,\zeta) \tag{3.18}$$

where,

$$\tilde{R}_{n+1}(u,\zeta) = \lambda^{n+1}\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)\cdots\int_{u_0}^{z_{n-1}} K(z_{n-1},z_n)\tilde{y}(z_n,\zeta)\; dz_n \dots dz_1 dz.$$

Let

$$\tilde{M}_n(u,\zeta_1) = \lambda^n \int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)\cdots\int_{u_0}^{z_{n-2}} K(z_{n-2},z_{n-1})\tilde{f}(z_{n-1},\zeta_1)\; dz_{n-1}\dots dz_1 dz,$$

then from the conditions (a) and (b) and by the equivalency of the metrics $\rho_c$and ρ_p, we get

$$\left|\tilde{M}_n(z,\zeta_1)\right| \leqslant |\lambda|^n \alpha^n \frac{(b-a)^n}{n!}\beta,\;\; \forall \zeta_1 \in [0,1],\; \forall u \in [u_0,u_1] \tag{3.19}$$

Now $\sum_n |\lambda|^n \alpha^n \frac{(b-a)^n}{n!}\beta$ is convergent and hence $\sum_n \tilde{M}_n(u,\zeta_1)$ is uniformly convergent over $[u_0,u_1]$, for every choice of $\zeta_1 \in [0,1]$.
So, if (3.17) has a solution, clearly it can be expressed by (3.19). Therefore $\tilde{y}(u,\zeta)$ is continuous over $[u_0,u_1]$ and hence bounded.
Thus, let

$$|\tilde{y}(u,\zeta)| \leqslant \gamma(\zeta),\;\; \forall \zeta \in [0,1]. \tag{3.20}$$

Now,

$$\left|\tilde{R}_{n+1}(u,\zeta)\right| = \left|\lambda^{n+1}\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)\cdots\int_{u_0}^{z_{n-1}} K(z_{n-1},z_n)\tilde{y}(z_n,\zeta)\; dz_n\dots dz_1 dz\right|$$
$$\leqslant |\lambda|^{n+1}\alpha^{n+1}\frac{(u_1-u_0)^{n+1}}{(n+1)!}\max_{\zeta\in\{0,1\}}\gamma(\zeta) \to 0 \;\text{ as } n\to\infty.$$

Hence (3.17) has a series solution

$$\tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda\int_{u_0}^{u} K(u,z)\,\tilde{f}(z,\zeta_1)\,dz$$
$$+\lambda^2\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)\tilde{f}(z_1,\zeta_1)\; dz_1 dz + \cdots$$

Therefore,

$$y_{\zeta\in[0,1]}(u) = f_{\zeta_1\in[0,1]}(u) + \lambda\int_{u_0}^{u} K(u,z)\,f_{\zeta_1\in[0,1]}(z)\,dz$$
$$+\lambda^2\int_{u_0}^{u} K(u,z)\int_{u_0}^{z} K(z,z_1)f_{\zeta_1\in[0,1]}(z_1)\; dz_1 dz + \cdots.$$

Hence,

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)[f_L, f_U](z)\,dz$$
$$+ \lambda^2 \int_{u_0}^{u} K(u,z) \int_{u_0}^{z} K(z, z_1)\ [f_L, f_U](z_1)\,dz_1 dz + \cdots.$$

□

3.3.1. Solution of interval Volterra integral equation by the method of Resolvent Kernel.

Theorem 3.9. *Consider an IVIE of the form*

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} K(u,z)[y_L, y_U](z)\,dz. \tag{3.21}$$

Then it has a solution of the form

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} R(u, z; \lambda)[y_L(z), y_U(z)]\,dz, \tag{3.22}$$

where, $R(u, z; \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x, z)$ *is the resolvent kernel.*

Proof. Here the iterated kernel $K_n(u, z)$ is defined as

$$K_1(u,z) = K(u,z),\ \ K_n(u,z) = \int_{z}^{u} K(u, z_1)\, K_{n-1}(z_1, z)\ dt.$$

From Proposition 3.5, (3.21) is equivalent to

$$y_{\zeta\in[0,1]}(u) = f_{\zeta_1\in[0,1]}(u) + \lambda \int_{u_0}^{u} K(u,z)\, y_{\zeta\in[0,1]}(z)\,dz$$

Therefore for fixed $\zeta, \zeta_1 \in [0,1]$

$$\tilde{y}(u,\zeta) = \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\,\tilde{y}(z,\zeta)\,dz$$

Let

$$\tilde{y}_0(u,\zeta) = \tilde{f}(u,\zeta_1). \tag{3.23}$$

Then,

$$\begin{aligned}\tilde{y}_1(u,\zeta) &= \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\,\tilde{y}_0(z,\zeta)\,dz \\ &= \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\,\tilde{f}(z,\zeta_1)\,dz\end{aligned}$$

Proceeding in this way and using (3.23), we get

$$\begin{aligned}\tilde{y}_n(u,\zeta) &= \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} K(u,z)\,\tilde{f}(z,\zeta_1)\,dz + \lambda^2 \int_{u_0}^{u} K_2(u,z)\,\tilde{f}(z,\zeta_1)\,dz + \cdots \\ &\quad + \lambda^n \int_{u_0}^{u} K_n(u,z)\,\tilde{f}(z,\zeta_1)\,dz.\end{aligned}$$

Therefore, $\forall \zeta \in [0,1]$,

$$\begin{aligned}\tilde{y}(u,\zeta) = \lim_{n\to\infty} \tilde{y}_n(u,\zeta) &= \tilde{f}(u,\zeta_1) + \int_{u_0}^{u} \left(\sum_{n=1}^{\infty} \lambda^n K_n(u,z) \right) \tilde{f}(z,\zeta_1)\,dz \\ &= \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} R(u,z;\lambda)\tilde{f}(z,\zeta_1)\,dz\end{aligned}$$

This gives,

$$\{\tilde{y}(u,\zeta) : \zeta \in [0,1]\} = \left\{ \tilde{f}(u,\zeta_1) + \lambda \int_{u_0}^{u} R(u,z;\lambda)\tilde{f}(z,\zeta_1)\,dz : \zeta_1 \in [0,1] \right\}.$$

i.e.,

$$\tilde{y}_{\zeta\in[0,1]}(u) = \tilde{f}_{\zeta_1\in[0,1]}(u) + \lambda \int_{u_0}^{u} R(u,z;\lambda)\tilde{f}_{\zeta_1\in[0,1]}(z)\,dz.$$

Hence, by the equivalency of parametric form, we have

$$[y_L, y_U](u) = [f_L, f_U](u) + \lambda \int_{u_0}^{u} R(u,z;\lambda)\,[y_L(z), y_U(z)]\,dz.$$

This completes the proof. □

4. Illustrative examples

To validate all the methods, three numerical examples are considered and solved.

Example 4.1. Let us consider the interval IVP:

$$\frac{d^2}{du^2}\left[y_L\left(u\right), y_U\left(u\right)\right] + u\frac{d}{du}\left[y_L\left(u\right), y_U\left(u\right)\right] + \left[y_L\left(u\right), y_U\left(u\right)\right] = 0$$

with the initial conditions

$$\left[y_L\left(0\right), y_U\left(0\right)\right] = [1, 2] \;\text{ and }\; \left[{y'}_L\left(0\right), {y'}_U\left(0\right)\right] = [0, 1]\,. \tag{4.1}$$

Solution. The parametric form of (4.1) is

$$y''_{\zeta\in[0,1]}\left(u\right) + uy'_{\zeta\in[0,1]}\left(u\right) + y_{\zeta\in[0,1]}\left(u\right) = 0$$

with the initial conditions

$$y_{\zeta\in[0,1]}\left(0\right) = \{1+\zeta : \zeta\in[0,1]\} \;\text{ and }\; y'_{\zeta\in[0,1]}\left(0\right) = \{\zeta : \zeta\in[0,1]\}\,. \tag{4.2}$$

Therefore, for a fixed $\zeta \in [0, 1]$, we have

$$y''\left(u,\zeta\right) + uy'\left(u,\zeta\right) + y\left(u,\zeta\right) = 0.$$

Let us take,

$$v\left(u,\zeta_1\right) = y''\left(u,\zeta\right). \tag{4.3}$$

Integrating (4.3) from 0 to u and using the second initial condition of (4.2), we obtain

$$y'\left(u,\zeta\right) = \int_0^u v\left(z,\zeta_1\right)dz + \zeta. \tag{4.4}$$

Again, integrating (4.4) from 0 to u and using (4.2), it gives

$$y\left(u,\zeta\right) = \int_0^u \left(u-z\right)v\left(z,\zeta_1\right)dz + \zeta u + 1 + \zeta. \tag{4.5}$$

Now, multiplying (4.3) by 1, (4.4) by u and (4.5) by 1 and adding, we get,

$$v\left(u,\zeta_1\right) = -\int_0^u \left(2u-z\right)v\left(z,\zeta_1\right)dz - \left(1+\zeta\right) - 2\zeta u.$$

Therefore, the required interval integral equation is

$$\left[v_L\left(u\right), v_U\left(u\right)\right] = -\left[1, 2u+2\right] - \int_0^u \left(2u-z\right)\left[v_L\left(z\right), v_U\left(z\right)\right]dz.$$

This is the required interval Volterra integral equation.

Example 4.2. Consider the following interval Volterra integral equation:

$$\left[y_L, y_U\right]\left(u\right) = \left[e^{u^2}, 3e^{u^2}\right] + \int_0^u e^{u^2-z^2}\left[y_L\left(z\right), y_U\left(z\right)\right]\,dz \tag{4.6}$$

Solution. The parametric representation of the equation (4.6) is

$$\tilde{y}\left(u,\zeta\right) = \tilde{f}\left(u,\zeta_1\right) + \int_0^u e^{u^2-z^2}\tilde{y}\left(z,\zeta\right)\,dz\,, \quad \forall\zeta,\zeta_1\in[0,1] \tag{4.7}$$

where $\tilde{f}\left(u,\zeta_1\right) = \left(1+2\zeta_1\right)e^{u^2}$ and $\tilde{y}\left(u,\zeta\right) = y_L\left(u\right) + \zeta\left(y_U\left(u\right) - y_L\left(u\right)\right)$.

Therefore, by the method of successive approximation, the solution of (4.7) is

$$\tilde{y}(u,\zeta)=\tilde{f}(u,\zeta_1)+\int\limits_0^u K(u,z)\,\tilde{f}(z,\zeta_1)\,dz+\int\limits_0^u K(u,z)\int\limits_0^z K(z,z_1)\tilde{f}(z_1,\zeta_1)\,dz_1dz+\cdots$$

$$\Rightarrow \tilde{y}(u,\zeta)=(1+2\zeta_1)\,e^{u^2}+(1+2\zeta_1)\,e^{u^2}u+(1+2\zeta_1)\,e^{u^2}\frac{u^2}{2!}+\cdots$$

$$\Rightarrow \tilde{y}(u,\zeta)=(1+2\zeta_1)\,e^{u^2+u}$$

Therefore, the solution of the equation (4.6) is

$$[y_L(u)\,,\,y_U(u)]=\left[e^{u^2+u},\ 3e^{u^2+u}\right].$$

Example 4.3. Consider the following interval Volterra integral equation:

$$[y_L,y_U](u)=[1,2]+\int_0^u [y_L(z)\,,y_U(z)]\;dz. \tag{4.8}$$

Solution. The parametric representation of the equation (4.8) is

$$\begin{gathered}\tilde{y}(u,\zeta)=\tilde{f}(u,\zeta_1)+\int_0^u \tilde{y}(z,\zeta)\;dz\,,\quad \forall\zeta,\zeta_1\in[0,1]\\ \text{where } \tilde{f}(u,\zeta_1)=1+\zeta_1 \text{ and } \tilde{y}(u,\zeta)=y_L(u)+\zeta\,(y_U(u)-y_L(u))\end{gathered} \tag{4.9}$$

Here,

$$K_n(u,z)=\int_z^u K_1(u,z_1)\,K_{n-1}(z_1,z)\;dz_1=\frac{(u-z)^{n-1}}{(n-1)!},\quad n=1,2,3,\ldots$$

Therefore, the resolvent kernel for this problem is of the form

$$R(u,z;1)=\sum\nolimits_{n=1}^{\infty}K_n(u,z)=\sum\nolimits_{n=1}^{\infty}\frac{(u-z)^{n-1}}{(n-1)!}=e^{(u-z)}.$$

Therefore, the solution of the equation (4.9) is of the form

$$\tilde{y}(u,\zeta)=1+\zeta_1+\int_0^u (1+\zeta)\;e^{u-z}dz\quad\forall\zeta,\zeta_1\in[0,1]\,.$$

Hence, the required solution of the equation (4.8) is

$$[y_L(u)\,,\,y_U(u)]=[e^u,2e^u]\,.$$

5. Conclusion

In this work, the concept imprecise Volterra integral equation is introduced in the interval form with a brief motivation. Then the solution procedure of interval Volterra integral equation is derived in parametric form in a simple way. Then all the results including solution procedure regarding interval Volterra integral equation are derived in a simple way. In these derivations, all the results are presented in parametric form of intervals. After that, a set of examples have been solved for the illustration of the solution procedure. This concept of imprecise Volterra integral equation can be implemented in various real-life problems viz. analyses of diffusion and heat transferring, power sector, web-security problems in which fluctuation of parameters is occurred due to the uncertainty.

For future research, one may develop the same for nonlinear interval Volterra type integral equations. One can develop the numerical methods for solving an interval Volterra integral equation. Also, this concept can be extended by introducing interval-valued kernels etc.

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Approaching the split common solution problem for nonlinear demicontractive mappings by means of averaged iterative algorithms

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Abstract. We consider new iterative algorithms for solving split common solution problems in the class of demicontractive mappings. These algorithms are obtained by inserting an averaged term into the algorithms previously used in [He, Z. and Du, W-S., Nonlinear algorithms approach to split common solution problems, *Fixed Point Theory Appl.* **2012**, 2012:130, 14 pp] for the case of quasi-nonexpansive mappings. In this way, we are able to solve the split common solution problem in the larger class of demicontractive mappings, which strictly includes the class of quasi-nonexpansive mappings. Our investigation is based on the embedding of demicontractive operators in the class of quasi-nonexpansive operators by means of averaged mappings. For the considered algorithms we prove weak and strong convergence theorems in the setting of a real Hilbert space.

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1. Introduction

Let $\mathcal{C}$ and $\mathcal{D}$ be nonempty subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear bounded operator. Let also $f : C \times C \to \mathbb{R}$ and $F : D \times D \to \mathbb{R}$ be two bi-functions.

The *split equilibrium problem* (SEP), see [10], is asking to find a point $\overline{c} \in C$ such that

$$f(\overline{c}, c) \geq 0, \quad \text{for all } c \in C \tag{1.1}$$

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and

$$\overline{d} = A\overline{c} \in D \text{ is such that } F(\overline{d}, d) \geq 0, \text{ for all } d \in D. \tag{1.2}$$

Problem (1.1) alone is the classical equilibrium problem (EP) and its solution set is usually denoted by $EP(f)$.

Several important problems in nonlinear analysis, e.g., the optimization problems, variational inequalities problems, saddle point problems, the Nash equilibrium problems, fixed point problems, complementary problems, bilevel problems, and semi-infinite problems, are special cases of the classical equilibrium problem and have relevant applications in mathematical programming with equilibrium constraint, see [11] and references therein.

In turn, the split equilibrium problem (SEP) (1.1)+(1.2) defines a way to split the solution between two different subsets such that the solution of the equilibrium problem (1.1) and its image by the linear bounded operator A leads to the solution of the second equilibrium problem (1.2).

Let $G : \mathcal{C} \to \mathcal{C}$ be a mapping with $Fix(G) := \{v \in \mathcal{D} \ : \ Gv = v\} \neq \emptyset$ and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ a bi-function.

In this paper, our interest is to study the following *split common solution problem* (SCSP) for equilibrium problems and fixed point problems:

$$\text{find } u \in \mathcal{C} \text{ such that } u \in Fix\,(G)$$

and

$$Au \in \mathcal{D} \text{ with } f(Au, v) \geq 0, \quad \text{for all } \ v \in \mathcal{D}.$$

Denote the set of solutions of this problem by

$$\Omega := \{u \in Fix(G) : Au \in EP(f)\}.$$

Example 1.1. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, $\mathcal{C} = [0, 1]$ and $\mathcal{D} = [-100, -7/8]$. Let $Au = -u$ for all $u \in \mathbb{R}$ and

$$Gu = \begin{cases} 7/8, & \text{if } 0 \leq u < 1 \\ 1/4, & \text{if } u = 1. \end{cases} \tag{1.3}$$

The mapping G defined on $\mathcal{C}$ is $\frac{2}{3}$-demicontractive but it is neither quasi-nonexpansive nor nonexpansive [3]. Define $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by $f(u, v) = u - v$ for all $u, v \in \mathcal{D}$. It is clear that A is a linear bounded operator, $Fix(G) = \left\{\frac{7}{8}\right\}$ and $A\left(\frac{7}{8}\right) = -\frac{7}{8} \in EP(f)$. Thus, $\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset$.

Example 1.2. Let $\mathcal{H}_1 = \mathbb{R}^2$, $\mathcal{H}_2 = \mathbb{R}$ with the standard norms. Let $\mathcal{C} = \{u \in \mathbb{R}^2 : \|u\| \leq 1\}$ and $\mathcal{D} = [-100, -5/6]$. Let $Au = -u_2$ for all $u = (u_1, u_2) \in \mathbb{R}^2$ and

$$Gu = \begin{cases} (0, 5/6), & \text{if } u \neq (0, 1) \\ (0, 1/3), & \text{if } u = (0, 1). \end{cases} \tag{1.4}$$

It is easy to see that G defined on $\mathcal{C}$ is a $\frac{1}{2}$-demicontractive mapping.
Define $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by $f(v, w) = v - w$ for all $v, w \in \mathcal{D}$. It is clear that A is a linear bounded operator, $Fix(G) = \left\{(0, \frac{5}{6})\right\}$ and $A\left(0, \frac{5}{6}\right) = -\frac{5}{6} \in EP(f)$. Thus, $\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset$.

He and Du [11] presented some new iterative algorithms for solving the split common solution problems for equilibrium problems and fixed point problems of nonlinear quasi-nonexpansive mappings.

Our aim in this paper is to construct new averaged iterative algorithms for solving the split common solutions problem in the setting of Hilbert spaces for the the larger class of demicontractive mappings, thus extending the main results in He and Du [11].

Our results are obtained by considering new averaged iterative algorithms for which we prove weak and strong convergence theorems.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{D} \subset \mathcal{H}$ be a closed convex set, and consider the operator $G : \mathcal{D} \to \mathcal{D}$.

Recall that the mapping G is said to be

(a) *nonexpansive* if

$$\|Gu - Gv\| \leq \|u - v\|, \quad \text{for all } u, v \in \mathcal{D}; \tag{2.1}$$

(b) *quasi-nonexpansive* if $Fix(G) \neq \emptyset$ and

$$\|Gv - v^*\| \leq \|v - v^*\|, \quad \text{for all } v \in \mathcal{D} \text{ and } v^* \in Fix(G); \tag{2.2}$$

(c) α-*demicontractive* if $Fix(G) \neq \emptyset$ and there exists a positive number $\alpha < 1$ such that

$$\|Gv - v^*\|^2 \leq \|v - v^*\|^2 + \alpha\|v - Gv\|^2, \tag{2.3}$$

for all $v \in \mathcal{D}$ and $v^* \in Fix(G)$;

(d) *firmly nonexpansive* if

$$\|Gu - Gv\|^2 \leq \|u - v\|^2 - \|u - v - (Gu - Gv)\|^2, \tag{2.4}$$

for all $u, v \in \mathcal{D}$.

By the above definitions, it is clear that any firmly nonexpansive mapping is nonexpansive, any nonexpansive mapping G with $Fix(G) \neq \emptyset$ is demicontractive and that any quasi-nonexpansive mapping is demicontractive, too, but the reverses are no longer true, as illustrated by the previous Examples 1.1 and 1.2.

It is well known, see [15], that any Hilbert space H satisfies the Opial's condition, that is, if $\{u_p\}$ is a sequence in $\mathcal{H}$ which converges weakly to a point $u \in \mathcal{H}$, then we have

$$\liminf_{p\to\infty} \|u_p - u\| < \liminf_{p\to\infty} \|u_p - v\|, \quad \text{for all } v \in \mathcal{H}, v \neq u.$$

The following lemmas and proposition are very important in the proof our main results.

Lemma 2.1. [2] *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{D} \subset \mathcal{H}$ a closed and convex set. If $G : \mathcal{D} \to \mathcal{D}$ is α-demicontractive, then for any $\varphi \in (0, 1 - \alpha)$, the map*

$$G_\varphi = (1 - \varphi)I + \varphi G$$

is quasi-nonexpansive.

Lemma 2.2. [11] *Let $\mathcal{H}$ be a real Hilbert space, $D \subset \mathcal{H}$ a closed and convex set and $G : D \to D$ a mapping. Then, for any $\varphi \in (0,1)$, we have $Fix(G_\varphi) = Fix(G)$.*

Definition 2.3. [13] Let $\mathcal{D}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ and G a mapping from $\mathcal{D}$ into $\mathcal{D}$. The mapping G is said to be zero-demiclosed if, for any sequence $\{u_p\}$ which weakly converges to u, and if the sequence $\{Gu_p\}$ strongly converges to zero, then $Gu = 0$.

Proposition 2.4. [11] *Let $\mathcal{D}$ be a nonempty, closed, and convex subset of a real Hilbert space with zero vector $\mathbf{0}$ and G a mapping from $\mathcal{D}$ into $\mathcal{D}$. Then the following assertions hold.*

(i) *G is zero-demiclosed if and only if $I - G$ is demiclosed at $\mathbf{0}$;*
(ii) *If G is a nonexpansive mappings and there is a bounded sequence $\{u_p\} \subset \mathcal{H}$ such that $\|u_p - Gu_p\| \to 0$ as $p \to 0$, then G is zero-demiclosed.*

Lemma 2.5. [7] *Let D be a nonempty, closed, and convex subset of $\mathcal{H}$ and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-function that satisfies the following conditions.*

(C1) *$f(v, v) = 0$ for all $v \in \mathcal{D}$;*
(C2) *f is monotone, that is, $f(u, v) + f(v, u) \geq 0$;*
(C3) *for every $u, v, w \in \mathcal{D}$, $\limsup_{t \to 0} f(tw + (1-t)u, v) \leq f(u, v)$;*
(C4) *for every $u \in \mathcal{D}$, $F(v) \equiv f(u, v)$ is convex and lower semi-continuous.*

Let $\mu > 0$ and $u \in \mathcal{H}$. Then there exists $w \in \mathcal{D}$ such that

$$f(w, v) + \frac{1}{\mu}\langle v - w, w - u\rangle \geq 0$$

for all $v \in \mathcal{D}$.

Lemma 2.6. [9] *Let $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{H}$ and let f be a bi-function from $\mathcal{D} \times \mathcal{D}$ into $\mathbb{R}$ that satisfies* (C1)-(C3). *For $\mu > 0$ and $u \in \mathcal{H}$, define a mapping*

$$T_\mu^f(u) = \left\{ w \in \mathcal{D} : f(w, v) + \frac{1}{\mu}\langle v - w, w - u\rangle \geq 0, \text{ for all } v \in \mathcal{D} \right\}. \tag{2.5}$$

Then the following assertions hold:

(a) *T_μ^f is single-valued and $f(T_\mu^f) = EP(f)$ for any $\mu > 0$ and $EP(f)$ is closed and convex;*
(b) *T_μ^f is firmly nonexpansive.*

Lemma 2.7. [8] *The following assertions hold for all $u, v \in \mathcal{H}$.*

(a) *$\|u + v\|^2 \leq \|v\|^2 + 2\langle u, u + v\rangle$ and $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle$.*
(b) *$\|au + (1-a)v\|^2 = a\|u\|^2 + (1-a)\|v\|^2 - a(1-a)\|u - v\|^2$ for $a \in [0, 1]$.*

Lemma 2.8. [10] *Let T_μ^f be as in* (2.5). *Then for $\mu, \tau > 0$ and $u, v \in \mathcal{H}$,*

$$\|T_\mu^f u - T_\tau^f v\| \leq \|u - v\| + \frac{|\tau - \mu|}{\tau} \left\|T_\tau^f v - v\right\|.$$

In particular, T_μ^f is nonexpansive for any $\mu > 0$.

The next lemma is due to Li and He [12] and will be useful in proving our main results.

Lemma 2.9. [12] *Let $F_1, \cdots, F_n : \mathcal{H}_1 \to \mathcal{H}_1$ be quasi-nonexpansive mappings and set $T = \sum_{i=1}^{n} b_i F_{a_i}$, where $b_i \in (0,1)$ with $\sum_{i=1}^{n} b_i = 1$, and $F_{a_i} = (1-a_i)I + a_i F_i$ with $a_i \in (0,1), i = 1, 2, \cdots, n$. Then T is quasi-nonexpansive and*

$$Fix(T) = \bigcap_{i=1}^{n} Fix(F_i) = \bigcap_{i=1}^{n} Fix(F_{a_i}).$$

3. Split common solutions in the class of demicotractive mappings

In this section we prove convergence theorems for averaged algorithms used for finding split common solutions for demicontractive mappings. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces.

In the following theorem, we prove the weak convergence of an averaged algorithm used for solving the split common solution for equilibrium problems and fixed point problems of nonlinear demicontractive mappings.

Theorem 3.1. *Let $\mathcal{C} \subset \mathcal{H}_1$ and $\mathcal{D} \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let $G : \mathcal{C} \to \mathcal{C}$ be a zero-demiclosed α-demicontractive mapping and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-function with*

$$\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset,$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Consider the sequences $\{u_p\}$ and $\{v_p\}$ generated as follows:

$$\begin{cases} v_1 \in \mathcal{C}, & \\ u_p = T^f_{\mu_p} A v_p, & \{\mu_p\} \subset (0,\infty), \\ v_{p+1} = (1-a_p)w_p + a_p\left[(1-\varphi)w_p + \varphi G w_p\right], & \varphi \in (0, 1-\alpha), \\ w_p = P_{\mathcal{C}}(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p), & \beta \in \left(0, \frac{1}{\|A^*\|}\right), \quad \textit{for all } p \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\liminf_{p\to\infty} \mu_p > 0$, $P_{\mathcal{C}}$ is the projection operator from $\mathcal{H}_1$ onto $\mathcal{C}$ and $\{a_p\}$ is a sequence in $[\varepsilon, 1-\varepsilon]$ with $\varepsilon \in (0, \frac{1}{2})$. Then $\{v_p\}$ converges weakly to $v^* \in \Omega$, and $\{u_p\}$ converges weakly to $Av^* \in EP(f)$.

Proof. Since G is an α-demicontractive mapping, in view of Lemma 2.1 the averaged mapping

$$G_\varphi := (1-\varphi)I + \varphi G \tag{3.2}$$

is quasi-nonexpansive for $\varphi \in (0, 1-\alpha)$. Here I is the identity mapping. Hence, in Algorithm (3.1) we can write

$$v_{p+1} = (1-a_p)w_p + a_p G_\varphi w_p.$$

Let $\Omega_\varphi := \{u \in Fix(G_\varphi) : Au \in EP(f)\} \neq \emptyset$ and $u \in \Omega_\varphi$. Using Lemma 2.6 and Lemma 2.7, it is easy to see that for any $p \in \mathbb{N}$,

$$\|T^f_{\mu_p} Av_p - Au\|^2 \leq \|Av_p - Au\|^2 - \|T^f_{\mu_p} Av_p - Av_p\|^2. \tag{3.3}$$

We also obtain

$$\begin{aligned}
&2\beta\langle v_p - u, A^* \left(T^f_{\mu_p} - I\right) Av_p\rangle \\
&\qquad = 2\beta\langle A(v_p - u) + (T^f_{\mu_p} - I)Av_p - (T^f_{\mu_p} - I)Av_p, (T^f_{\mu_p} - I)Av_p\rangle \\
&\qquad \le 2\beta\left(\frac{1}{2}\|(T^f_{\mu_p} - I)Av_p\|^2 - \|(T^f_{\mu_p} - I)Av_p\|^2\right) \\
&\qquad = -\beta\|(T^f_{\mu_p} - I)Av_p\|^2.
\end{aligned}$$

Since for any $p \in \mathbb{N}$,

$$\|A^*(T^f_{\mu_p} - I)Av_p\|^2 \le \|A^*\|^2\|(T^f_{\mu_p} - I)Av_p\|^2, \tag{3.4}$$

and G_φ is quasi-nonexpansive, we have

$$\begin{aligned}
&\|v_{p+1} - u\|^2 \\
&\quad = (1 - a_p)\|w_p - u\|^2 + a_p\|G_\varphi w_p - u\|^2 - (1 - a_p)a_p\|w_p - G_\varphi w_p\|^2 \\
&\quad \le \|w_p - u\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \qquad (\text{Since } \varepsilon \in [a_p, 1 - a_p]) \\
&\quad = \|P_{\mathcal{C}}(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p) - P_{\mathcal{C}}u\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&\quad \le \|v_p + \beta A^*(T^f_{\mu_p} - I)Av_p - u\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&\quad = \|v_p - u\|^2 + \|\beta A^*(T^f_{\mu_p} - I)Av_p\|^2 + 2\beta\langle v_p - u, A^*(T^f_{\mu_p} - I)Av_p\rangle - \varepsilon^2\|w_p - G_\varphi w_p\| \\
&\quad \le \|v_p - u\|^2 + \beta^2\|A^*\|^2\|(T^f_{\mu_p} - I)Av_p\|^2 - \beta\|(T^f_{\mu_p} - I)Av_p\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\| \\
&\quad = \|v_p - u\|^2 - \beta(1 - \beta\|A^*\|^2)\|(T^f_{\mu_p} - I)Av_p\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|. \qquad (3.5)
\end{aligned}$$

Since $\beta \in (0, \frac{1}{\|A^*\|^2})$ and $\beta(1 - \beta\|A^*\|^2) > 0$, we have

$$\|v_{p+1} - u\| \le \|w_p - u\| \le \|v_p - u\| \tag{3.6}$$

and by (3.5),

$$\varepsilon^2\|w_p - G_\varphi w_p\|^2 + \beta(1 - \beta\|A^*\|^2)\|(T^f_{\mu_p} - I)Av_p\|^2 \le \|v_p - u\|^2 - \|v_{p+1} - u\|^2, \tag{3.7}$$

for any $p \in \mathbb{N}$. Note that since $u \in Fix(G_\varphi)$, it follows that the sequence $\{\|v_p - u\|\}$ is convergent. Inequalities (3.6) and (3.7) imply that

$$\lim_{p\to\infty} \|v_p - u\| = \lim_{p\to\infty} \|w_p - u\|, \tag{3.8}$$

$$\lim_{p\to\infty} \|w_p - G_\varphi w_p\| = 0$$

and

$$\lim_{p\to\infty} \|(T^f_{\mu_p} - I)Av_p\| = 0. \tag{3.9}$$

We obtain

$$\begin{aligned}
\|w_p - v_p\| &= \|P_{\mathcal{C}}\left(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p\right) - P_{\mathcal{C}}v_p\| \\
&\le \beta\|A^*(T^f_{\mu_p} - I)Av_p\| \to 0 \text{ as } p \to \infty.
\end{aligned}$$

Since $\lim_{p\to\infty}\|v_p-u\|$ exists, $\{v_p\}$ is bounded and thus, $\{v_p\}$ has a weakly convergence subsequence $\{v_{p_k}\}$. Let $v^*\in\mathcal{C}$ be the weak limit of $\{v_{p_k}\}$. Hence,

$$Av_{p_k}\rightharpoonup Av^*\in\mathcal{D},\quad y_{p_k}\rightharpoonup v^*$$

and

$$T^f_{\mu_{p_k}}Av_{p_k}\rightharpoonup Av^*.$$

Since G_φ is a zero-demiclosed mapping, and $y_{p_k}\rightharpoonup v^*$, we obtain $v^*\in Fix(G_\varphi)$. Applying Lemma 2.6, $EP(f)=Fix(T^f_\mu)$ for any $\mu>0$. We claim that $T^f_\mu Av^*=Av^*$. Suppose $T^f_\mu Av^*\neq Av^*$. Since $Av_p-T^f_{\mu_p}Av_p=(I-T^f_{\mu_p})Av_p\to 0$ as $p\to\infty$, applying the Opial's property and Lemma 2.8 yields

$$\begin{aligned}\liminf_{j\to\infty}\|Av_{p_k}-Av^*\| &< \liminf_{j\to\infty}\|Av_{p_k}-T^f_\mu Av^*\|\\ &\le \liminf_{j\to\infty}\left(\|Av_{p_k}-T^f_{\mu_{p_k}}Av_{p_k}\|+\|T^f_{\mu_{p_k}}Av_{p_k}-T^f_\mu Av^*\|\right)\\ &= \liminf_{j\to\infty}\|T^f_\mu Av^*-T^f_{\mu_{p_k}}Av_{p_k}\|\\ &\le \liminf_{j\to\infty}\left(\|Av_{p_k}-Av^*\|+\frac{|\mu_{p_k}-\mu|}{\mu_{p_k}}\|T^f_{\mu_{p_k}}Av_{p_k}-Av_{p_k}\|\right)\\ &= \liminf_{j\to\infty}\|Av_{p_k}-Av^*\|,\end{aligned}$$

which lead to a contradiction. So $Av^*\in Fix(T^f_\mu)=EP(f)$, and hence

$$v^*\in\Omega_\varphi=\{u\in Fix(G_\varphi):Au\in EP(f)\}.$$

Now we prove that $\{v_p\}$ converges weakly to $v^*\in\Omega_\varphi$. Otherwise, there exists a subsequence $\{v_{p_l}\}$ of $\{v_p\}$ such that $v_{p_l}\rightharpoonup u^*\in\Omega_\varphi$ with $u^*\neq v^*$. By Opial's condition,

$$\liminf_{l\to\infty}\|v_{p_l}-u^*\|<\liminf_{l\to\infty}\|v_{p_l}-v^*\|<\liminf_{l\to\infty}\|v_{p_l}-u^*\|.$$

This is a contradiction. Hence, $\{v_p\}$ converges weakly to an element $v^*\in\Omega_\varphi$.
Finally, we prove that $\{u_p\}$ converges weakly to $Av^*\in EP(f)$. Since $v_p\rightharpoonup v^*$, we have $Av_p\rightharpoonup Av^*$ as $p\to\infty$. Therefore, $u_p:=T^f_{\mu_p}Av_p\rightharpoonup Av^*\in EP(f)$. □

Corollary 3.2. *Let $\mathcal{C}\subset\mathcal{H}_1$ and $\mathcal{D}\subset\mathcal{H}_2$ be two nonempty closed convex sets. Let $G:\mathcal{C}\to\mathcal{C}$ be a zero-demiclosed α-demicontractive mapping and $f:\mathcal{D}\times\mathcal{D}\to\mathbb{R}$ be a bi-function with*

$$\Omega=\{u\in Fix(G):Au\in EP(f)\}\neq\emptyset,$$

where $A:\mathcal{H}_1\to\mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Consider the sequences $\{u_p\}$ and $\{v_p\}$ generated as follows:

$$\begin{cases} v_1\in\mathcal{C},\\ u_p=T^f_{\mu_p}Av_p, & \{\mu_p\}\subset(0,\infty),\\ v_{p+1}=(1-a_p)w_p+a_pGw_p,\\ w_p=P_{\mathcal{C}}(v_p+\beta A^*(T^f_{\mu_p}-I)Av_p), & \beta\in\left(0,\frac{1}{\|A^*\|}\right),\ p\in\mathbb{N},\end{cases}\tag{3.10}$$

where $\liminf_{p\to\infty} \mu_p > 0$, $P_{\mathcal{C}}$ *is the projection operator from* $\mathcal{H}_1$ *onto* $\mathcal{C}$ *and* $\{a_p\}$ *is a sequence in* $[\varepsilon, 1-\varepsilon]$ *with* $\varepsilon \in (0,1)$. *Then* $\{v_p\}$ *converges weakly to* $v^* \in \Omega$, *and* $\{u_p\}$ *converges weakly to* $Av^* \in EP(f)$.

Proof. Consider G_φ given in (3.2). By Lemma 2.2, for any $\varphi \in (0,1)$, we have $Fix(G_\varphi) = Fix(G)$. We have

$$\begin{aligned}(1-a_p)w_p + a_pG_\varphi w_p &= (1-a_p)w_p + a_p((1-\varphi)w_p + \varphi Gw_p)\\ &= (1-a_p\varphi)w_p + a_p\varphi Gw_p.\end{aligned}$$

To obtain exactly the iterative scheme (3.10), we simply denote $a_p := \varphi a_p \in (0,1)$ for all $p \in \mathbb{N}$. ☐

Next we prove a strong convergence theorem of an iterative method to split common solution for a demicontractive mapping.

Theorem 3.3. *Let* $\mathcal{C} \subset \mathcal{H}_1$ *and* $\mathcal{D} \subset \mathcal{H}_2$ *be two nonempty closed convex sets. Let* $G : \mathcal{C} \to \mathcal{C}$ *be a zero-demiclosed* α*-demicontractive mapping and* $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ *be a bi-function with* $\Omega = \{u \in Fix(G) : Au \in EP(f)\} \neq \emptyset$, *where* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *is a bounded linear operator with its adjoint* A^*. *Consider the sequence* $\{u_p\}$ *and* $\{v_p\}$ *generated as follows:*

$$\begin{cases} v_1 \in \mathcal{C}, \\ u_p = T^f_{\mu_p} Av_p, \quad \{r_n\} \subset (0,\infty), \\ y_p = (1-a_p)w_p + a_p[(1-\varphi)w_p + \varphi Gw_p], \quad \varphi \in (0, 1-\alpha), \\ w_p = P_C(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p), \quad \beta \in \left(0, \frac{1}{\|A^*\|^2}\right) \\ C_{p+1} = \{v \in C_p : \|y_p - v\| \le \|w_p - v\| \le \|v_p - v\|\}, \quad \textit{with } C_1 = C, \\ v_{p+1} = P_{C_{p+1}}(v_1), \quad p \in \mathbb{N}. \end{cases} \tag{3.11}$$

where $\liminf_{p\to\infty} r_n > 0$, P_C *is the projection operator from* $\mathcal{H}_1$ *onto* C, *and* $\{a_p\}$ *is a sequence in* $[\varepsilon, 1-\varepsilon], \varepsilon \in (0,1)$. *Then* $v_p \to v^* \in \Omega$ *and* $u_p \to Av^* \in EP(f)$.

Proof. Consider the mapping G_φ given in (3.2). Similar to the proof of Theorem 3.1, the sequence $\{y_n\}$ in Algorithm (3.11) can be written as

$$y_p = (1-a_p)w_p + a_pG_\varphi w_p.$$

Let $\Omega_\varphi = \{u \in Fix(G_\varphi) : Au \in EP(f)\} \neq \emptyset$. We claim that $\Omega_\varphi \subset C_p$ for $p \in \mathbb{N}$. In fact, let $u \in \Omega_\varphi$. Following the same argument as in the proof of Theorem 3.1, we have

$$2\beta\langle v_p - u, A^*(T^f_{\mu_p} - I)Av_p\rangle \le -\beta\|(T^f_{\mu_p} - I)Av_p\|^2, \tag{3.12}$$

and for any $p \in \mathbb{N}$,

$$\|A^*(T^f_{\mu_p} - I)Av_p\|^2 \le \|A^*\|^2\|(T^f_{\mu_p} - I)Av_p\|^2. \tag{3.13}$$

For any $p \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\|y_p - u\| \le \|w_p - u\|^2 - (1-a_p)a_p\|w_p - G_\varphi w_p\|^2 \\
&\le \|v_p + \beta A^*(T^f_{\mu_p} - I)Av_p - u\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&= \|v_p - u\|^2 + \|\beta A^*(T^f_{\mu_p} - I)Av_p\|^2 + 2\beta\langle v_p - u, A^*(T^f_{\mu_p} - I)Av_p\rangle - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&\le \|v_p - u\|^2 + \beta^2\|A^*\|^2\|(T^f_{\mu_p} - I)Av_p\|^2 - \beta\|(T^f_{\mu_p} - I)Av_p\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2 \\
&\le \|v_p - u\|^2 - \beta(1 - \beta\|A^*\|^2)\|(T^f_{\mu_p} - I)Av_p\|^2 - \varepsilon^2\|w_p - G_\varphi w_p\|^2.
\end{aligned}$$

Since $\beta \in \left(0, \frac{1}{\|A^*\|^2}\right)$, $\beta\left(1 - \beta\|A^*\|^2\right) > 0$, it follows that

$$\|y_p - u\| \le \|w_p - u\| \le \|v_p - u\|, \tag{3.14}$$

and thus $p \in C_p$ for all $p \in \mathbb{N}$. Hence, $\Omega \subset C_p$ and $C_p \neq \emptyset$ for all $p \in \mathbb{N}$.
Now we prove that C_p is a closed convex set for each $p \in \mathbb{N}$. It is not hard to verify that C_p is closed for each p, so it suffices to verify that C_p is convex for each $p \in \mathbb{N}$. Indeed, let $x_1, x_2 \in C_{p+1}$. For any $\gamma \in (0,1)$, since

$$\begin{aligned}
&\|y_p - (\gamma x_1 + (1-\gamma)x_2\|^2 \\
&\quad = \|\gamma(y_p - x_1) + (1-\gamma)(y_p - x_2)\|^2 \\
&\quad = \gamma\|y_p - x_1\|^2 + (1-\gamma)\|y_p - x_2\|^2 - \gamma(1-\gamma)\|x_1 - x_2\|^2 \\
&\quad \le \gamma\|w_p - x_1\|^2 + (1-\gamma)\|w_p - x_2\|^2 - \gamma(1-\gamma)\|x_1 - x_2\|^2 \\
&\quad = \|w_p - (\gamma x_1 + (1-\gamma)x_2\|^2,
\end{aligned}$$

the following inequality holds

$$\|y_p - (\gamma x_1 + (1-\gamma)x_2)\| \le \|w_p - (\gamma x_1 + (1-\gamma)x_2)\|.$$

Similarly, we also have

$$\|w_p - (\gamma x_1 + (1-\gamma)x_2)\| \le \|v_p - (\gamma x_1 + (1-\gamma)x_2)\|,$$

which implies that $\gamma x_1 + (1-\gamma)x_2 \in C_{p+1}$. Hence, C_{p+1} is convex.
Notice that $C_{p+1} \subset C_p$ and $v_{p+1} = P_{C_{p+1}}(v_1) \subset C_p$. Then $\|v_{p+1} - v_1\| \le \|v_p - v_1\|$ for $n > 2$. It follows that $\lim_{p\to\infty}\|v_p - v_1\|$ exists. Hence $\{v_p\}$ is bounded, which yields $\{w_p\}$ and $\{y_p\}$ are bounded. For any $k, p \in \mathbb{N}$ with $k > p$, from $v_k = P_{C_k}(v_1) \subset C_p$ and the character (iii) of the projection operator P, we have

$$\|v_p - v_k\|^2 + \|v_1 - v_k\|^2 = \|v_p - P_{C_k}(v_1)\|^2 + \|v_1 - P_{C_k}(v_1)\|^2 \le \|v_p - v_1\|^2. \tag{3.15}$$

Since $\lim_{p\to\infty}\|v_p - v_1\|$ exists, it follows that $\lim_{p\to\infty}\|v_p - v_k\| = 0$, which implies that $\{v_p\}$ is a Cauchy sequence.
Let $v_p \to v^*$. One can claim that $v^* \in \Omega$. Firstly, by the fact that

$$v_{p+1} = P_{C_{p+1}}(v_1) \in C_{p+1} \subset C_p,$$

we have

$$\|y_p - v_p\| \le \|y_p - v_{p+1}\| + \|v_{p+1} - v_p\| \le 2\|v_{p+1} - v_p\| \to 0, \quad \text{as } p \to \infty \tag{3.16}$$

and

$$\|w_p - v_p\| \le \|w_p - v_{p+1}\| + \|v_{p+1} - v_p\| \le 2\|v_{p+1} - v_p\| \to 0, \text{ as } p \to \infty. \tag{3.17}$$

Setting $\rho = \beta(1-\beta\|A^*\|^2)$, we obtain

$$\begin{aligned}\rho\|(T^f_{\mu_p}-I)Av_p\|^2+\varepsilon^2\|w_p-Tw_p\|^2 &\le \|v_p-v^*\|^2-\|y_p-v^*\|^2\\ &\le \|v_p-y_p\|\left(\|v_p-v^*\|+\|y_p-v^*\|\right).\end{aligned}$$

So

$$\lim_{p\to\infty}\|G_\varphi w_p-w_p\|=0$$

and

$$\lim_{p\to\infty}\|(T^f_{\mu_p}-I)Av_p\|=0.$$

Let $r>0$. Since $v_p\to v^*$ as $p\to\infty$, Lemma 2.8 implies that

$$\begin{aligned}\|T^f_{\mu_p}Av^*-Av^*\| &\le \|T^f_{\mu_p}Av^*-T^f_{\mu_p}Av_p\|+\|T^f_{\mu_p}Av_p-Av_p\|+\|Av_p-Av^*\|\\ &\le 2\|Av_p-Av^*\|+\left(1+\frac{|r_n-r|}{r_n}\right)\|T^f_{\mu_p}Av_p-Av_p\|\to 0,\ \text{as } p\to\infty.\end{aligned}$$

So $T^f_\mu Av^*=Av^*$, which says that $Av^*\in Fix(T^f_{\mu_p})=EP(f)$. On the other hand, since $v_p-w_p\to 0$ and $v_p\to v^*$, we conclude that $w_p\to v^*$. Notice that G_φ is zero-demiclosed quasi-nonexpansive, $G_\varphi v^*=v^*$. We also deduce that $\{u_p\}:=\{T^f_{\mu_p}Av_p\}$ converges strongly to $Av^*\in EP(f)$. □

Corollary 3.4. *Let $\mathcal{C}\subset\mathcal{H}_1$ and $\mathcal{D}\subset\mathcal{H}_2$ be two nonempty closed convex sets. Let $G:\mathcal{C}\to\mathcal{C}$ be a zero-demiclosed α-demicontractive mapping and $f:\mathcal{D}\times\mathcal{D}\to\mathbb{R}$ be a bi-function with $\Omega=\{u\in Fix(G):Au\in EP(f)\}\neq\emptyset$, where $A:\mathcal{H}_1\to\mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Consider the sequences $\{u_p\}$ and $\{v_p\}$ generated as follows:*

$$\begin{cases}v_1\in\mathcal{C},\\ u_p=T^f_{\mu_p}Av_p,\quad \{r_n\}\subset(0,\infty),\\ y_p=(1-a_p)w_p+a_pGw_p,\\ w_p=P_C(v_p+\beta A^*(T^f_{\mu_p}-I)Av_p),\quad \beta\in\left(0,\frac{1}{\|A^*\|^2}\right)\\ C_{p+1}=\{v\in C_p:\|y_p-v\|\le\|w_p-v\|\le\|v_p-v\|\},\quad \text{with } C_1=C,\\ v_{p+1}=P_{C_{p+1}}(v_1),\quad p\in\mathbb{N}.\end{cases}\tag{3.18}$$

where $\liminf_{p\to\infty}r_n>0$, P_C is the projection operator from $\mathcal{H}_1$ onto C, and $\{a_p\}$ is a sequence in $[\varepsilon,1-\varepsilon],\varepsilon\in(0,1)$. Then $v_p\to v^*\in\Omega$ and $u_p\to Av^*\in EP(f)$.

Proof. Consider the mapping G_φ given in (3.2). By Lemma 2.2, for any $\varphi\in(0,1)$, we have $Fix(G_\varphi)=Fix(G)$. We have

$$(1-a_p)w_p+a_pG_\varphi w_p=(1-a_p)w_p+a_p((1-\varphi)w_p+\varphi Gw_p)=(1-a_p\varphi)w_p+a_p\varphi Gw_p.$$

To obtain exactly the iterative scheme (3.18), we simply denote $a_p:=\varphi a_p\in(0,1)$ for all $p\in\mathbb{N}$. □

We close this section by stating the strong convergence of an iterative scheme for a split common solutions problem with a finite number of demicontractive mappings.

Theorem 3.5. *Let $\mathcal{C} \subset \mathcal{H}_1$ and $\mathcal{D} \subset \mathcal{H}_2$ be two nonempty closed convex sets. Let*

$$G_1, \cdots, G_n : \mathcal{C} \to \mathcal{C}$$

be a finite number of zero-demiclosed α-demicontractive mappings with

$$\bigcap_{i=1}^{n} Fix(G_i) \neq \emptyset$$

and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-function with

$$\Omega = \left\{ u \in \bigcap_{i=1}^{n} Fix(G_i) : Au \in EP(f) \right\} \neq \emptyset,$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint A^*. Consider the sequences $\{u_p\}$ and $\{v_p\}$ generated as follows:

$$\begin{cases} v_1 \in \mathcal{C}, \\ u_p = T^f_{\mu_p} Av_p, \quad \{r_n\} \subset (0, \infty), \\ y_p = (1 - a_p)w_p + a_p \sum_{i=1}^{n} c_i[(1 - \varphi_i)w_p + \varphi_i G w_p], \quad c_i, \varphi_i \in (0, 1), \quad \sum_{i=1}^{n} c_i = 1, \\ w_p = P_C(v_p + \beta A^*(T^f_{\mu_p} - I)Av_p), \quad \beta \in \left(0, \frac{1}{\|A^*\|^2}\right) \\ C_{p+1} = \{v \in C_p : \|y_p - v\| \leq \|w_p - v\| \leq \|v_p - v\|\}, \quad \textit{with } C_1 = C, \\ v_{p+1} = P_{C_{p+1}}(v_1), \quad p \in \mathbb{N}. \end{cases} \tag{3.19}$$

where $\liminf_{p\to\infty} r_n > 0$, *$P_C$ is the projection operator from $\mathcal{H}_1$ onto C, and $\{a_p\}$ is a sequence in $[\varepsilon, 1 - \varepsilon], \varepsilon \in (0, 1)$. Then $v_p \to v^* \in \Omega$ and $u_p \to Av^* \in EP(f)$.*

Proof. Let $F = \sum_{i=1}^{n} c_i G_{\varphi_i}$, where $G_{\varphi_i} = (1 - \varphi_i)I + \varphi_i G$. Lemma 2.9 implies that F is a quasi-nonexpansive mapping. Furthermore,

$$Fix(F) = \bigcap_{i=1}^{n} Fix(G_{\varphi_i}) = \bigcap_{i=1}^{n} Fix(G_i) \neq \emptyset.$$

It is straightforward to see that F is zero-demiclosed. The rest of the proof is similar to that of Theorem 3.3. □

4. Conclusion

(1) We have proven a weak convergence theorem for an iteration scheme used to approximate split common solutions for demicontractive mappings in Hilbert spaces, which is derived from an associated weak convergence theorem in the class of a quasi-nonexpansive operators.
(2) We also have established a strong convergence theorem for an iteration scheme used to approximate split common solutions of demicontractive mappings in Hilbert spaces, which is derived from a corresponding strong convergence theorem in the class of a quasi-nonexpansive operators.

(3) Our investigation is based on an embedding technique by means of an averaged mapping: if G is α-demicontractive, then for any $\varphi \in (0, 1-\alpha)$,

$$G_\varphi = (1-\varphi)I + \varphi G$$

is a quasi-nonexpansive mapping.

(4) For some very recent developments on related topics we refer the reader to Alakoya et al. [1], Berinde and Saleh [5] [6], Berinde and Păcurar [4], Onah et al. [14], Rathee and Swami [16], Wang and Pan [17],Yao et al. [18], Zhu et al. [19], etc., to which a similar approach seems to be applicable.

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An elastic-viscoplastic contact problem with internal state variable, normal damped response and unilateral constraint

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Abstract. In this manuscript, we study a contact problem between an elastic-viscoplastic body and an obstacle. The contact is quasistatic and it is described with a normal damped response condition with friction and unilateral constraint. Moreover, we use an elastic-viscoplastic constitutive law with internal state variable to model the material's behavior. We present the classical problem then we derive its variational formulation. Finally, we prove that the associated variational problem has a unique solution. The proof is based on arguments of quasivariational inequalities and fixed points.

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1. Introduction

Contact problems represent an important topic both in Applied Mathematics and Engineering Sciences. References in the field include [1, 5, 6, 11, 13, 12, 14, 15, 16, 17]. In this work, we deal with a model of the frictional contact between an elastic-viscoplastic body and an obstacle named foundation, for the purpose of modelling and establishing variational analysis of this one. This analysis is done within the infinitesimal strain theory. We model the material's behavior with the following

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constitutive law with internal state variable

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s))\, ds, \tag{1.1}$$

in which the viscosity operator $\mathcal{A}$ and the elasticity operator $\mathcal{B}$ are assumed to be nonlinear and $\mathcal{G}$ represents a nonlinear function. Also, $\mathbf{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linearized strain tensor. The internal state variable $\mathbf{k}$ is a vector-valued function whose evolution is governed by the following differential equation

$$\dot{\mathbf{k}}(t) = \varphi\left(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)\right), \tag{1.2}$$

in which φ is a nonlinear constitutive function with values in $\mathbb{R}^m$, m being a positive integer. Elastic-viscoplastic models can be found in [3, 4, 9, 10]. In particular, the reader can refer to [7, 8, 16] where he finds a detailed analysis of elastic-viscoplastic contact problems with internal state variables.

In this paper, we assume that the part of the body's boundary which will be in contact with the foundation is covered by a thin lubricant layer. Lubricants make sliding of rubbing surfaces easier by interposing a smooth film between these parts. We can find examples of lubrication in many fields such as oil rigs and car mechanics. To model lubrication, we usually use a normal damped response contact condition in which the normal stress on the contact surface depends on the normal velocity, see [1, 2]. However, in this manuscript, we model the contact with normal damped response and unilateral constraint for the velocity field, associated with a version of Coulomb's law of dry friction. These boundary conditions model the contact with a foundation in such a way that the normal velocity is restricted by a unilateral constraint. Also, when the body moves towards the obstacle, the contact is described with a normal damped response condition associated with the friction law. On the other hand, when the body moves in the opposite direction then the reaction of the foundation vanishes. More details on the normal damped response boundary condition with friction and unilateral constraint can be found in [1].

The main novelty of this paper is to describe a frictional contact with the normal damped response and unilateral constraint in velocity for elastic-viscoplastic materials with internal state variable.

The rest of the paper is divided into three sections. Section 2 contains both notations and preliminary material. In section 3, we list assumptions on the data that are required to solve the variational problem derived in the same section. Section 4 deals with different steps taken to prove the main existence and uniqueness result, Theorem 4.1.

2. Notations and preliminaries

In this short section, we make an overview of the notation we shall use and some preliminary material. The notation $\mathbb{N}$ is used to represent the set of positive integers. For $d \in \mathbb{N}$, we denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. We define the inner products and norms of $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$\mathbf{u}\ .\mathbf{v} = u_i v_i \ , \ \ \|\mathbf{v}\| = (\mathbf{v}\ .\mathbf{v})^{\frac{1}{2}} \ \ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\ .\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \ , \ \ \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\ .\boldsymbol{\tau})^{\frac{1}{2}} \ \ \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Note that the indices i and j run between 1 to d and that the summation convention over repeated indices is used. Also, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \delta u_i / \delta x_j$.

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded domain with a Lipschitz continuous boundary Γ and let Γ_1 be a measurable part of Γ such that *meas* $(\Gamma_1) > 0$. We use $\mathbf{x} = (x_i)$ for a generic point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. We use the standard notation for the Lebesgue and Sobolev spaces associated with Ω and Γ; moreover, we consider the spaces

$$\begin{cases} H = \left\{\mathbf{u} = (u_i)/u_i \in L^2(\Omega)\right\}, \\ \mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \\ H_1 = \left\{\mathbf{u} = (u_i)/u_i \in H^1(\Omega)\right\}, \\ \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H}/Div\boldsymbol{\sigma} \in H\}. \end{cases}$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces with the inner products

$$\begin{cases} (\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i v_i \, d\mathbf{x}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, d\mathbf{x}, \\ (\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, Div\ \boldsymbol{\tau})_H, \end{cases}$$

respectively, where $\boldsymbol{\varepsilon} : H_1 \longrightarrow \mathcal{H}$ and $Div : \mathcal{H}_1 \longrightarrow H$ are respectively the deformation and the divergence operators defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))\ , \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})\ , \qquad Div\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The associated norms on $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are denoted by $\|.\|_H, \|.\|_{\mathcal{H}}, \|.\|_{H_1}$ and $\|.\|_{\mathcal{H}_1}$ respectively.
Next, for the displacement field, we introduce the closed subspace V of H_1 defined as follows

$$V = \{\mathbf{v} \in H_1/\mathbf{v} = \mathbf{0} \ \text{ on}\, \Gamma_1\}.$$

We consider on V the inner product given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \qquad \forall \mathbf{u}, \mathbf{v} \in V,$$

and the associated norm

$$\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \ \ \forall \mathbf{v} \in V. \tag{2.1}$$

Completeness of the space $(V, \|.\|_V)$ follows from the assumption *meas* $(\Gamma_1) > 0$, since the use of Korn's inequality is allowed.
Moreover, for an element $\mathbf{v} \in V$, we still write $\mathbf{v}$ for the trace of $\mathbf{v}$ on the boundary.

In addition, v_ν and $\mathbf{v}_\tau$ denote the normal and the tangential components of $\mathbf{v}$ on the boundary Γ gave by

$$v_\nu = \mathbf{v}.\boldsymbol{\nu},\ \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}.$$

Let Γ_3 be a measurable part of Γ. We can see from the Sobolev trace theorem that there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_0\ \|\mathbf{v}\|_V \qquad \forall \mathbf{v} \in V. \tag{2.2}$$

For a regular function $\sigma \in \mathcal{H}$, σ_ν and σ_τ denote the normal and the tangential components of the vector $\boldsymbol{\sigma\nu}$ on Γ, respectively, and we recall that

$$\sigma_\nu = (\boldsymbol{\sigma\nu}).\boldsymbol{\nu},\ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}.$$

Moreover, we recall the following Green's formula,

$$\int_\Omega \boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\mathbf{v})\ d\mathbf{x} + \int_\Omega Div\ \boldsymbol{\sigma}.\mathbf{v}\ d\mathbf{x} = \int_\Gamma \boldsymbol{\sigma\nu}.\mathbf{v}\ da \qquad \forall\, \mathbf{v} \in V. \tag{2.3}$$

Furthermore, for the internal state variable, we introduce the notation

$$Y = L^2(\Omega)^m \quad m \in \mathbb{N}. \tag{2.4}$$

Finally, for a given Banach space X we use the notation $C\,(0,T;X)$ and $C^1\,(0,T;X)$ for the space of continuous and continuously differentiable functions defined on $[0,T]$ with values in X, respectively. The spaces $C\,(0,T;X)$ and $C^1\,(0,T;X)$ are Banach spaces endowed with the following norms

$$\|\mathfrak{v}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\mathfrak{v}\,(t)\|_X\,,$$

$$\|\mathfrak{v}\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|\mathfrak{v}\,(t)\|_X + \max_{t\in[0,T]} \|\dot{\mathfrak{v}}\,(t)\|_X\,.$$

The following fixed point result will be used in section 4 of the paper.

Theorem 2.1. *Let $(X, \|.\|_X)$ be a Hilbert space and let K be a nonempty closed subset of X. Let $\Lambda : C(0,T;K) \longrightarrow C(0,T;K)$ be a nonlinear operator. Assume that there exists $h \in \mathbb{N}$ with the following property: there exists $b \in [0,1)$ and $c \geq 0$ such that*

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^h \leq b\|\eta_1(t) - \eta_2(t)\|_X^h + c\int_0^t \|\eta_1(s) - \eta_2(s)\|_X^h\, ds,$$

$\forall \eta_1, \eta_2 \in C(0,T;K)$, $\forall t \in [0,T]$. *Then, there exists a unique element $\eta^* \in C(0,T;K)$ such that $\Lambda\eta^* = \eta^*$.*

Note that here and below, the notation $\Lambda\eta\,(t)$ means the value of the function $\Lambda\eta$, i.e. $\Lambda\eta\,(t) = (\Lambda\eta)\,(t)$.

Next, we recall a second result proved in [15] which will be used in section 4. To this end, we introduce the following setting. Let X be a real Hilbert space with the inner product $(.,.)_X$ and the associated norm $\|.\|_X$ and let K be a subset of X. Let consider the operator $A : K \to X$ and the functionals $j : K \times K \to \mathbb{R}$ and $f : [0,T] \to X$ such that

$$K \text{ is a nonempty closed convex subset of } X. \tag{2.5}$$

$$\begin{cases} \text{(a) There exists } M_A > 0 \text{ such that} \\ (A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_X \geq M_A \ \|\mathbf{u}_1 - \mathbf{u}_2\|_X^2 \quad \forall \mathbf{u}_1, \, \mathbf{u}_2 \in K. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \|A\mathbf{u}_1 - A\mathbf{u}_2\|_X \leq L_A \ \|\mathbf{u}_1 - \mathbf{u}_2\|_X \quad \forall \mathbf{u}_1, \, \mathbf{u}_2 \in K. \end{cases} \tag{2.6}$$

$$\begin{cases} \text{(a) The function } j(\mathbf{u}, .) \text{ is convex and lower} \\ \text{semicontinuous on } K, \text{ for all } \mathbf{u} \in X. \\ \text{(b) There exists } \alpha \geq 0 \text{ such that} \\ j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \ \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\ \mathbf{u}_2, \mathbf{v}_2) \\ \quad \leq \alpha \|\mathbf{u}_1 - \mathbf{u}_2\|_X \|\mathbf{v}_1 - \mathbf{v}_2\|_X , \\ \quad \forall \mathbf{u}_1, \, \mathbf{u}_2 \in X, \ \ \forall \mathbf{v}_1, \ \mathbf{v}_2 \in K. \end{cases} \tag{2.7}$$

$$f \in C(0, T; X). \tag{2.8}$$

Moreover, we assume that

$$M_A > \alpha, \tag{2.9}$$

where M_A and α are the constants in (2.6) and (2.7) respectively.

We have the following result.

Theorem 2.2. *Assume that* (2.5)-(2.9) *hold. Then there exists a unique function* $\mathbf{u} \in C(0, T; K)$ *such that*

$$\begin{aligned} &(A\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_X + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \mathbf{u}(t)) \\ &\quad \geq (f(t), \mathbf{v} - \mathbf{u}(t))_X \qquad \forall \mathbf{v} \in K. \end{aligned} \tag{2.10}$$

We can see that (2.10) is a time-dependent quasivariational inequality governed by the functional j which depends on the solution.

3. Problem statement and variational formulation

We consider an elastic-viscoplastic body that occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that *meas* $(\Gamma_1) > 0$. The body is acted upon by body forces of density $\mathbf{f}_0$ and surface tractions of density $\mathbf{f}_2$ act on Γ_2. We assume that the body is clamped on Γ_1, and therefore, the displacement field vanishes there. The body may come in contact over Γ_3 with an obstacle, the so-called foundation. Moreover, on Γ_3 we describe the contact with:
a) A unilateral constraint in velocity given by

$$\dot{u}_\nu \leq g,$$

where $g > 0$ is a given bound. Here we assume the nonhomogeneous case and, therefore, g is a function that depends on the spatial variable $\mathbf{x} \in \Gamma_3$.
b) A normal damped response condition associated to Coulomb's law of dry friction, as far as the normal velocity does not reach the bound g. When the normal velocity reaches the limit g, friction follows the Tresca law. Also, We assume a given compatibility condition to accommodate conditions in b) and to ensure the continuity of the friction bound when the normal velocity reaches its maximum value g. Therefore, we can see a natural transition from the Coulomb law (which is valid as far as $0 \leq \dot{u_\nu} \leq g$) to the Tresca friction law (which is valid when $\dot{u_\nu} = g$). Consequently, we obtain the

following frictional contact conditions with normal damped response and unilateral constraint

$$\begin{cases} \dot{u}_\nu(t) \le g, \;\; \sigma_\nu(t) + p(\dot{u}_\nu(t)) \le 0, \\ (\dot{u}_\nu(t) - g)(\sigma_\nu(t) + p(\dot{u}_\nu(t))) = 0, \end{cases} \qquad \text{on } \Gamma_3 \times [\,0, T\,]\,,$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau(t)\| \le \mu \; p(\dot{u}_\nu(t)) \\ -\,\boldsymbol{\sigma}_\tau(t) = \mu \; p(\dot{u}_\nu(t)) \dfrac{\dot{\mathbf{u}}_\tau(t)}{\|\dot{\mathbf{u}}_\tau(t)\|} \qquad \text{if } \dot{\mathbf{u}}_\tau(t) \neq 0, \end{cases} \qquad \text{on } \Gamma_3 \times [\,0, T\,]\,,$$

where p is a positive function such that $p(r) = 0$ for $r \le 0$ and μ denotes the coefficient of friction. More details on these contact conditions can be found in [1].
Furthermore, we assume that the process is quasistatic since the forces and tractions vary slowly in time and, therefore, we neglect the acceleration of the system. Hence, the classical formulation of the contact problem is as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}$: $\Omega \times [0,T] \to \mathbb{S}^d$ and an internal state variable $\mathbf{k} : \Omega \times [0,T] \to \mathbb{R}^m$ such that

$$\begin{aligned} \boldsymbol{\sigma}\,(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}\,(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}\,(t)) \\ &\quad + \int_0^t \mathcal{G}(\boldsymbol{\sigma}\,(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}\,(s)), \boldsymbol{\varepsilon}(\mathbf{u}\,(s)), \mathbf{k}\,(s))\; ds \quad \text{in } \Omega \times [0,T]\,, \end{aligned} \tag{3.1}$$

$$\dot{\mathbf{k}}\,(t) = \varphi\left(\boldsymbol{\sigma}\,(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}\,(t)), \boldsymbol{\varepsilon}(\mathbf{u}\,(t)), \mathbf{k}\,(t)\right) \quad \text{in } \Omega \times [0,T]\,, \tag{3.2}$$

$$Div\; \boldsymbol{\sigma}\,(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega \times [0,T]\,, \tag{3.3}$$

$$\mathbf{u}\,(t) = \mathbf{0} \quad \text{on } \Gamma_1 \times [0,T]\,, \tag{3.4}$$

$$\boldsymbol{\sigma}\,(t)\,.\boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2 \times [0,T]\,, \tag{3.5}$$

$$\begin{cases} \dot{u}_\nu(t) \le g, \;\; \sigma_\nu(t) + p(\dot{u}_\nu(t)) \le 0, \\ (\dot{u}_\nu(t) - g)(\sigma_\nu(t) + p(\dot{u}_\nu(t))) = 0, \end{cases} \qquad \text{on } \Gamma_3 \times [\,0, T\,]\,, \tag{3.6}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau(t)\| \le \mu \; p(\dot{u}_\nu(t)) \\ -\,\boldsymbol{\sigma}_\tau(t) = \mu \; p(\dot{u}_\nu(t)) \dfrac{\dot{\mathbf{u}}_\tau(t)}{\|\dot{\mathbf{u}}_\tau(t)\|} \qquad \text{if } \dot{\mathbf{u}}_\tau(t) \neq 0, \end{cases} \qquad \text{on } \Gamma_3 \times [\,0, T\,]\,, \tag{3.7}$$

$$\mathbf{u}\,(0) = \mathbf{u}_0, \, \mathbf{k}\,(0) = \mathbf{k}_0 \quad \text{in } \Omega. \tag{3.8}$$

Now, we describe the problem (3.1)-(3.8). First, equations (3.1) and (3.2) represent the elastic-viscoplastic constitutive law with internal state variable as well as the evolution equation of the latter. Equation (3.3) is the equilibrium equation while conditions (3.4)-(3.5) are the displacement-traction boundary conditions respectively.

The boundary conditions (3.6)-(3.7) describe the mechanical conditions on the contact surface Γ_3 that represents the frictional contact conditions with normal damped response and unilateral constraint in velocity. Finally, (3.8) represents the initial conditions in which $\mathbf{u}_0$ and $\mathbf{k}_0$ are the initial displacement and the initial state variable respectively.

We turn now to the variational formulation of the Problem P. To this end, we assume that the viscosity operator $\mathcal{A}$, the elasticity operator $\mathcal{B}$ and the nonlinear constitutive function $\mathcal{G}$ satisfy

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)\| \le L_{\mathcal{A}} \ \|\varepsilon_1 - \varepsilon_2\| \\ \quad \forall\ \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) There exists } M_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)) . (\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}} \ \|\varepsilon_1 - \varepsilon_2\|^2 \\ \quad \forall\ \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.9}$$

$$\begin{cases} \text{(a) } \mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \quad \|\mathcal{B}(\mathbf{x}, \varepsilon_1) - \mathcal{B}(\mathbf{x}, \varepsilon_2)\| \le L_{\mathcal{B}} \ \|\varepsilon_1 - \varepsilon_2\| \\ \quad \forall\ \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.10}$$

$$\begin{cases} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \varepsilon_1, \mathbf{k}_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \varepsilon_2, \mathbf{k}_2)\| \\ \quad \le L_{\mathcal{G}} \ (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\varepsilon_1 - \varepsilon_2\| + \|\mathbf{k}_1 - \mathbf{k}_2\|) \\ \quad \forall\ \boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d,\ \mathbf{k}_1,\ \mathbf{k}_2 \in \mathbb{R}^m, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \mathbf{k}) \text{ is measurable on } \Omega\ , \\ \quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d\ ,\ \mathbf{k} \in \mathbb{R}^m. \\ \text{(e) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.11}$$

Also, we assume that the constitutive function $\varphi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \ \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\varphi} > 0 \text{ such that} \\ \quad \|\varphi(\mathbf{x}, \boldsymbol{\sigma}_1,\ \varepsilon_1, \mathbf{k}_1) - \varphi(\mathbf{x}, \boldsymbol{\sigma}_2, \varepsilon_2, \mathbf{k}_2)\| \\ \quad \le L_{\varphi} \ (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\varepsilon_1 - \varepsilon_2\| + \|\mathbf{k}_1 - \mathbf{k}_2\|) \\ \quad \forall\ \boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d,\ \mathbf{k}_1,\ \mathbf{k}_2 \in \mathbb{R}^m, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) The mapping } \mathbf{x} \mapsto \varphi(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{k}) \text{ is measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d\ ,\ \mathbf{k} \in \mathbb{R}^m. \\ \text{(c) The mapping } \mathbf{x} \mapsto \varphi(\mathbf{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \text{ belongs to } Y. \end{cases} \tag{3.12}$$

The function $p : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}^+$ satisfies

$$\left\{\begin{array}{ll} \text{(a)} & \exists\, L_p > 0 \text{ such that} \\ & \quad |p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \leq L_p\, |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(b)} & (p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2))(r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e .}\mathbf{x} \in \Gamma_3. \\ \text{(c)} & \text{The mapping } \mathbf{x} \longmapsto p(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \\ & \text{for all } r \in \mathbb{R}. \\ \text{(d)} & p(\mathbf{x}, r) = 0 \quad \forall r \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{array}\right. \tag{3.13}$$

The friction coefficient μ satisfies

$$\mu \in L^\infty(\Gamma_3), \quad \mu \geq 0 \text{ a.e. } \mathbf{x} \in \Gamma_3. \tag{3.14}$$

The densities of body forces and surface tractions are such that

$$\mathbf{f}_0 \in C\ (0, T; H), \qquad \mathbf{f}_2 \in C\ (0, T; L^2(\Gamma_2)^d). \tag{3.15}$$

Finally, the initial data verify

$$\mathbf{u}_0 \in U. \tag{3.16}$$

$$\mathbf{k}_0 \in Y. \tag{3.17}$$

After that, we introduce the set of admissible velocities U defined by

$$U = \{\mathbf{v} \in V\ :\ v_\nu \leq g \text{ a.e. on} \,\Gamma_3\}. \tag{3.18}$$

We note that U is a nonempty, closed, convex subset of the space V and, on U, we use the inner product of V.

Next, we use the Green formula (2.3) to find that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t))_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_H = \int_\Gamma \boldsymbol{\sigma\nu}\,(t)\,(\mathbf{v} - \dot{\mathbf{u}}(t))\,da$$
$$= \int_{\Gamma_1} \boldsymbol{\sigma\nu}\,(t)\,(\mathbf{v} - \dot{\mathbf{u}}(t))\,da + \int_{\Gamma_2} \boldsymbol{\sigma\nu}\,(t)\,(\mathbf{v} - \dot{\mathbf{u}}(t))\,da + \int_{\Gamma_3} \boldsymbol{\sigma\nu}\,(t)\,(\mathbf{v} - \dot{\mathbf{u}}(t))\,da,$$

for all $\mathbf{v} \in U$. Since $\mathbf{v} - \dot{\mathbf{u}}(t) = 0$ on Γ_1, $\boldsymbol{\sigma\nu}(t) = \mathbf{f}_2(t)$ on Γ_2 and $Div\ \boldsymbol{\sigma}(t) = -\mathbf{f}_0(t)$ in Ω , we obtain

$$\begin{aligned} (\boldsymbol{\sigma}(t)\,,\, \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} &= (\mathbf{f}_0(t), \mathbf{v} - \dot{\mathbf{u}}(t))_H \\ &+ \int_{\Gamma_2} \mathbf{f}_2(t).(\mathbf{v} - \dot{\mathbf{u}}(t))\ da\ +\ \int_{\Gamma_3} \boldsymbol{\sigma\nu}(t).(\mathbf{v} - \dot{\mathbf{u}}(t))\,da. \end{aligned} \tag{3.19}$$

On the other hand, we use Riesz's theorem to define the element $\mathbf{f}(t) \in V$ by

$$(\mathbf{f}(t)\,,\,\mathbf{v})_V\ = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\ d\mathbf{x}\ + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\ da \quad \forall \mathbf{v} \in V, \tag{3.20}$$

where $\mathbf{f} : [0, T] \to V$. It follows from hypotheses (3.15) that the integral (3.20) is well-defined and we have

$$\mathbf{f} \in C\,(0, T; V)\,. \tag{3.21}$$

Now, we note that

$$\boldsymbol{\sigma\nu}(t)(\mathbf{v} - \dot{\mathbf{u}}(t))\ = \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t))\ +\ \boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t)) \quad \text{on}\,\Gamma_3 \times [0, T]\,.$$

We combine (3.19), (3.20) and the last equality to obtain

$$\begin{aligned} (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} &= (\mathbf{f}(t), \mathbf{v})_V \\ &+ \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t))\,da + \int_{\Gamma_3} \boldsymbol{\sigma_\tau}(t)(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t))\,da\ \forall \mathbf{v} \in U. \end{aligned} \tag{3.22}$$

Next, we write

$$\sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) = [\sigma_\nu(t) + p(\dot{u}_\nu(t))]\,(v_\nu - g)$$
$$+[\sigma_\nu(t) + p(\dot{u}_\nu(t))]\,(g - \dot{u}_\nu(t)) - p(\dot{u}_\nu(t))\,(v_\nu - \dot{u}_\nu(t)),$$

for all $\mathbf{v} \in U$. Moreover, (3.4), (3.6) and (3.18) show that

$$\dot{\mathbf{u}}(t) \in U\,,\ \mathbf{u}(t) \in V. \tag{3.23}$$

Thus, we deduce that $v_\nu - g \le 0$ and $\dot{u}_\nu - g \le 0$; in addition, we use the contact conditions (3.6) to obtain

$$\sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) \ge -p(\dot{u}_\nu(t))\,(v_\nu - \dot{u}_\nu(t)) \qquad \text{on } \Gamma_3.$$

We integrate the last inequality on Γ_3 to find that

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t))\,da \ge -\int_{\Gamma_3} p(\dot{u}_\nu(t))(v_\nu - \dot{u}_\nu(t))\,da. \tag{3.24}$$

Also, we use (3.7) to see that, if $\dot{\mathbf{u}}_\tau \ne \mathbf{0}$, we have

$$\boldsymbol{\sigma}_{\boldsymbol{\tau}}(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) = -\mu\, p(\dot{u}_\nu)\frac{\dot{\mathbf{u}}_\tau \mathbf{v}_\tau}{\|\dot{\mathbf{u}}_\tau\|} + \mu\, p(\dot{u}_\nu)\,\|\dot{\mathbf{u}}_\tau\|\,. \tag{3.25}$$

Using the Cauchy-Schwartz inequality, we obtain

$$-\mu\, p(\dot{u}_\nu)\frac{\dot{\mathbf{u}}_\tau \mathbf{v}_\tau}{\|\dot{\mathbf{u}}_\tau\|} + \mu\, p(\dot{u}_\nu)\,\|\dot{\mathbf{u}}_\tau\| \ \ge\ -\mu\, p(\dot{u}_\nu)\,\|\mathbf{v}_\tau\| + \mu\, p(\dot{u}_\nu)\,\|\dot{\mathbf{u}}_\tau\|\,.$$

Now, from (3.25) and the last inequality we find that

$$\boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) \ \ge\ \mu\, p(\dot{u}_\nu)(\|\dot{\mathbf{u}}_\tau\| \ -\ \|\mathbf{v}_\tau\|) \text{ if } \dot{\mathbf{u}}_\tau \ne \mathbf{0}. \tag{3.26}$$

On the other hand, if $\dot{\mathbf{u}}_\tau = \mathbf{0}$, then

$$\boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) = \boldsymbol{\sigma}_\tau \mathbf{v}_\tau.$$

From the Cauchy-Schwartz inequality and (3.7), we obtain

$$\begin{aligned} \boldsymbol{\sigma}_\tau.\mathbf{v}_\tau &\ge -\|\boldsymbol{\sigma}_\tau\|\,.\,\|\mathbf{v}_\tau\| \\ &\ge -\mu\, p(\dot{u}_\nu).\,\|\mathbf{v}_\tau\|\,. \end{aligned}$$

Since $\dot{\mathbf{u}}_\tau = \mathbf{0}$, the last inequality can be written as follows

$$\boldsymbol{\sigma}_\tau.\mathbf{v}_\tau - \boldsymbol{\sigma}_\tau.\dot{\mathbf{u}}_\tau \quad \ge \quad -\mu\, p(\dot{u}_\nu)\,\|\mathbf{v}_\tau\| + \mu\, p(\dot{u}_\nu)\,\|\dot{\mathbf{u}}_\tau\|\ ,$$

which yields

$$\boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) \quad \ge \quad \mu\, p(\dot{u}_\nu)(\|\dot{\mathbf{u}}_\tau\| - \|\mathbf{v}_\tau\|) \text{ if } \dot{\mathbf{u}}_\tau = \mathbf{0}. \tag{3.27}$$

We conclude from (3.26) and (3.27) that

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t))\,da \quad \ge \quad \int_{\Gamma_3} \mu\, p(\dot{u}_\nu(t))(\|\dot{\mathbf{u}}_\tau(t)\| - \|\mathbf{v}_\tau\|)\,da. \tag{3.28}$$

We gather (3.22),(3.24) and (3.28) to find that

$$\begin{aligned} (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} \ge (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V + \int_{\Gamma_3} p\,(\dot{u}_\nu(t))\,(\dot{u}_\nu(t) - v_\nu)\ da \\ + \int_{\Gamma_3} \mu\, p\,(\dot{u}_\nu(t))\,(\|\dot{\mathbf{u}}_\tau(t)\| - \|\mathbf{v}_\tau\|)\ da. \end{aligned} \tag{3.29}$$

To finalize the variational formulation of Problem P, we use again the Riesz's theorem to define the operator $P : V \to V$ by

$$(P\mathbf{u}, \mathbf{v})_V = \int_{\Gamma_3} p(u_\nu)\, v_\nu \, da \ \ \forall \, \mathbf{u}, \, \mathbf{v} \in V. \tag{3.30}$$

It follows from (2.2) and hypotheses (3.13) that

$$(P\mathbf{u} - P\mathbf{v}, \mathbf{u} - \mathbf{v})_V \geq 0, \ \|P\mathbf{u} - P\mathbf{v}\|_V \leq c_0^2 L_p \|\mathbf{u} - \mathbf{v}\|_V \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{3.31}$$

which means that $P : V \to V$ is monotone and Lipschitz continuous.
Finally, we define the function $j : U \times U \to \mathbb{R}^+$ by

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \mu \, p\,(u_\nu)\, \|\mathbf{v}_\tau\| \, da \ \ \forall \mathbf{u}, \mathbf{v} \in U. \tag{3.32}$$

We use now (3.30) and (3.32) to see that (3.29) becomes

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} \ + (P\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \\ &\quad + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \ \forall \mathbf{v} \in U. \end{aligned} \tag{3.33}$$

Lastly, we integrate (3.2) from 0 to t by using initial conditions (3.8) and we use (3.33) and (3.1) to obtain the following variational formulation of Problem P.
Problem PV. Find a displacement field $\mathbf{u} : [0, T] \to U$, a stress field $\boldsymbol{\sigma} : [0, T] \to \mathcal{H}$ and an internal state variable $\mathbf{k} : [0, T] \to Y$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)) \, ds, \tag{3.34}$$

$$\mathbf{k}(t) = \int_0^t \varphi(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)) \, ds + \mathbf{k}_0, \tag{3.35}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_{\mathcal{H}} + (P\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \\ &\quad + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \ \forall \mathbf{v} \in U, \end{aligned} \tag{3.36}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{3.37}$$

4. Existence and uniqueness result

In this section, we study the existence and the uniqueness of the solution of the variational problem PV introduced in section 3. We summarize this study in the following result.
Theorem 4.1. *Assume that hypotheses* (3.9) - (3.17) *are satisfied. Then, there exists a constant* $L_0 > 0$ *such that, if* $L_p < L_0$, *then the problem PV has a unique solution* $\{\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k}\}$. *Moreover , the solution satisfies*

$$\begin{aligned} &\mathbf{u} \in C^1(0, T; V), \\ &\boldsymbol{\sigma} \in C(0, T; \mathcal{H}_1), \\ &\mathbf{k} \in C^1(0, T; Y). \end{aligned} \tag{4.1}$$

Now let's move on to the proof of Theorem 4.1 which will be carried out in several steps. We assume in what follows that hypotheses (3.9)-(3.17) are satisfied. We use the product space $\mathcal{H}\times Y$ endowed with the norm

$$\|\boldsymbol{\eta}\|_{\mathcal{H}\times Y}=\|\boldsymbol{\eta}^{(1)}\|_{\mathcal{H}}+\|\boldsymbol{\eta}^{(2)}\|_{Y}\quad\forall\boldsymbol{\eta}=(\boldsymbol{\eta}^{(1)},\boldsymbol{\eta}^{(2)})\in\mathcal{H}\times Y. \tag{4.2}$$

Step 1. For all $\boldsymbol{\eta}=(\boldsymbol{\eta}^{(1)},\boldsymbol{\eta}^{(2)})\in C(0,T;\mathcal{H}\times Y)$, we consider the following intermediate variational problem.

Problem PV_η. Find a displacement field $\mathbf{u}_\eta:[0,T]\to U$ such that

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}}+(\boldsymbol{\eta}^{(1)}(t),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}_\eta(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t))_V+j(\dot{\mathbf{u}}_\eta(t),\mathbf{v})-j(\dot{\mathbf{u}}_\eta(t),\dot{\mathbf{u}}_\eta(t))\\&\quad\geq(\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t))_V\qquad\forall\mathbf{v}\in U.\end{aligned} \tag{4.3}$$

$$\mathbf{u}_\eta(0)=\mathbf{u}_0. \tag{4.4}$$

We have the following existence and uniqueness result.

Lemma 4.2. *If $L_p<L_0$, then there exists a unique solution $\mathbf{u}_\eta$ to Problem PV_η such that $\mathbf{u}_\eta\in C^1(0,T;V)$. Moreover, if $\mathbf{u}_i=\mathbf{u}_{\eta_i}$ are two solutions to Problem PV_η corresponding to $\boldsymbol{\eta}_i\in C(0,T;\mathcal{H}\times Y)$, $i=1,2$, then there exists a constant $c>0$ such that*

$$||\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)||_V\leq c\ \|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_{\mathcal{H}\times Y}\quad\forall t\in[0,T]. \tag{4.5}$$

Proof. First, we use Riesz's Theorem to define the operator $A:V\to V$ and the function $\mathbf{f}_\eta:[0,T]\to V$ by equalities

$$(A\mathbf{u},\mathbf{v})_V=(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}),\ \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}+(P\mathbf{u},\mathbf{v})_V, \tag{4.6}$$

$$(\mathbf{f}_\eta(t),\mathbf{v})_V=(\mathbf{f}(t),\mathbf{v})_V-(\boldsymbol{\eta}^{(1)}(t),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \tag{4.7}$$

for all $\mathbf{u},\mathbf{v}\in V$ and $t\in[0,T]$. Hence, (4.3) becomes

$$\begin{aligned}&(A\dot{\mathbf{u}}_\eta(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t))_V+j(\dot{\mathbf{u}}_\eta(t),\mathbf{v})-j(\dot{\mathbf{u}}_\eta(t),\dot{\mathbf{u}}_\eta(t))\\&\qquad\geq(\mathbf{f}_\eta(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t))_V\ \forall\mathbf{v}\in U,\end{aligned} \tag{4.8}$$

and using the notation $\mathbf{w}_\eta(t)=\dot{\mathbf{u}}_\eta(t)$, we can see that the last inequality can be written as follows

$$\begin{aligned}&(A\mathbf{w}_\eta(t),\mathbf{v}-\mathbf{w}_\eta(t))_V+j(\mathbf{w}_\eta(t),\mathbf{v})-j(\mathbf{w}_\eta(t),\mathbf{w}_\eta(t))\\&\qquad\geq(\mathbf{f}_\eta(t),\mathbf{v}-\mathbf{w}_\eta(t))_V\ \forall\mathbf{v}\in U,\end{aligned} \tag{4.9}$$

Next, we apply Theorem 2.2 for $K=U$ and $X=V$. First, we note that the space U defined in (3.18) satisfies conditions (2.5). Next, we consider $\mathbf{w_1},\mathbf{w_2}\in V$ and we use the monotonicity of the operator P expressed in (3.31) as well as (3.9)(c) and (2.1) to obtain

$$(A\mathbf{w}_1-A\mathbf{w}_2,\mathbf{w}_1-\mathbf{w}_2)_V\geq M_{\mathcal{A}}||\mathbf{w}_1-\mathbf{w}_2||_V^2.\quad\forall\mathbf{w}_1,\mathbf{w}_2\in V,$$

which shows that A is strongly monotone with constant $M_A=M_{\mathcal{A}}$.

On the other hand, for $\mathbf{w}_1,\mathbf{w}_2,\mathbf{v}\in V$, we use the Lipschitz continuity of P expressed in (3.31) as well as (3.9)(b) and (2.1) to find

$$(A\mathbf{w}_1-A\mathbf{w}_2,\mathbf{v})_V\leq(L_{\mathcal{A}}+c_0^2\,L_p)\,||\mathbf{w}_1-\mathbf{w}_2||_V||\mathbf{v}||_V.$$

By choosing $\mathbf{v}=A\mathbf{w}_1-A\mathbf{w}_2$ in the last inequality we obtain

$$||A\mathbf{w}_1-A\mathbf{w}_2||_V\leq(L_{\mathcal{A}}+c_0^2\,L_p)\,||\mathbf{w}_1-\mathbf{w}_2||_V,$$

which means that A is a Lipschitz continuous operator with constant $L_A = L_{\mathcal{A}} + c_0^2\, L_p$. We conclude that conditions (2.6) are satisfied.
Now we prove conditions (2.7) on the function j. First, it is easy to see that $j(\mathbf{w}, .)$ is a semi-norm on V, for all $\mathbf{w} \in V$. Moreover, we recall that $\|\mathbf{v}_\tau\| \leq \|\mathbf{v}\|$ and we use (3.13), (3.14) and (2.2) to see that for all $\mathbf{w} \in V$,

$$j(\mathbf{w}, \mathbf{v}) \leq c\, \|\mathbf{v}\|_V.$$

We conclude that $j(\mathbf{w}, .)$ is a continuous semi-norm on V and thus it is convex and lower semi-continuous on V, which means that it satisfies condition (2.7)(a) of Theorem 2.2. Now, for all $\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$, we use assumptions (3.13) (a) and (3.14), after a simple calculation we obtain

$$\begin{aligned} &j(\mathbf{w}_1, \mathbf{v}_2) - j(\mathbf{w}_1, \mathbf{v}_1) + j(\mathbf{w}_2, \mathbf{v}_1) - j(\mathbf{w}_2, \mathbf{v}_2) \\ &\qquad \leq \mu\, L_p \int_{\Gamma_3} |w_{1\nu} - w_{2\nu}|\ \big|\, \|\mathbf{v}_{1\tau}\| - \|\mathbf{v}_{2\tau}\|\, \big|\ da\ . \end{aligned}$$

Next, it is well known that $|w_{1\nu} - w_{2\nu}| \leq \|\mathbf{w}_1 - \mathbf{w}_2\|$, $\big|\, \|\mathbf{v}_{1\tau}\| - \|\mathbf{v}_{2\tau}\|\, \big| \leq \|\mathbf{v}_1 - \mathbf{v}_2\|$. Thus, we obtain

$$\begin{aligned} &j(\mathbf{w}_1, \mathbf{v}_2) - j(\mathbf{w}_1, \mathbf{v}_1) + j(\mathbf{w}_2, \mathbf{v}_1) - j(\mathbf{w}_2, \mathbf{v}_2) \\ &\qquad \leq \mu\, L_p\, \|\mathbf{w}_1 - \mathbf{w}_2\|_{L^2(\Gamma_3)^d}\, \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(\Gamma_3)^d}\ . \end{aligned}$$

Hence, inequality (2.2) yields

$$\begin{aligned} &j(\mathbf{w}_1, \mathbf{v}_2) - j(\mathbf{w}_1, \mathbf{v}_1) + j(\mathbf{w}_2, \mathbf{v}_1) - j(\mathbf{w}_2, \mathbf{v}_2) \\ &\leq c_0^2\ \mu\, L_p\, \|\mathbf{w}_1 - \mathbf{w}_2\|_V\, \|\mathbf{v}_1 - \mathbf{v}_2\|_V\ , \end{aligned} \tag{4.10}$$

for all $\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$. We note that the last inequality shows that the condition (2.7) (b) is satisfied for $\alpha = c_0^2\ \mu\, L_p$.
Moreover, we use (3.21) and (4.7) and we recall that $\boldsymbol{\eta} \in C(0, T; \mathcal{H} \times Y)$ to deduce that $\mathbf{f}_\eta \in C(0, T; V)$; i.e. $\mathbf{f}_\eta$ satisfies (2.8). Finally, for the condition (2.9) to be satisfied, we choose $L_0 = \dfrac{M_{\mathcal{A}}}{c_0^2\, \mu}$. As a consequence, if $L_p < L_0$, then $M_{\mathcal{A}} > c_0^2\ \mu\, L_p$, which means that condition (2.9) of Theorem 2.2 is now satisfied. We conclude that there exists a unique solution $\mathbf{w}_\eta \in C(0, T; U)$ to the quasivariational (4.9). Now, we use (4.4) and we define the displacement $\mathbf{u}_\eta$ by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{w}_\eta(s)\, ds + \mathbf{u}_0. \tag{4.11}$$

It results from the last equality and hypothesis (3.16) that $\mathbf{u}_\eta \in C^1(0, T; V)$ is the unique solution of the quasivariational inequality (4.8). Finally, we can see that, by substituting (4.6)-(4.7) in (4.8), we find that $\mathbf{u}_\eta \in C^1(0, T; V)$ is the unique solution of PV_η, which concludes the existence and uniqueness part of Lemma 4.2.
We turn now to the proof of estimate (4.5). To this end, let consider

$$\boldsymbol{\eta}_1 = (\boldsymbol{\eta}_1^{(1)}, \boldsymbol{\eta}_1^{(2)}),\ \boldsymbol{\eta}_2 = (\boldsymbol{\eta}_2^{(1)}, \boldsymbol{\eta}_2^{(2)}) \in C(0, T; \mathcal{H} \times Y)$$

and let use the notations $\mathbf{u}_1 = \mathbf{u}_{\eta_1}$, $\mathbf{u}_2 = \mathbf{u}_{\eta_2}$. We write (4.3) for $\mathbf{u}_\eta(t) = \mathbf{u}_1(t)$ and $\mathbf{v} = \dot{\mathbf{u}}_2\,(t)$ and then for $\mathbf{u}_\eta(t) = \mathbf{u}_2(t)$ with $\mathbf{v} = \dot{\mathbf{u}}_1\,(t)$, after a simple calculation we obtain

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}_1(t)-P\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t))_V\\&\qquad\leq(\boldsymbol{\eta}_1^{(1)}(t)-\boldsymbol{\eta}_2^{(1)}(t),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t)))_{\mathcal{H}}\\&\qquad+j(\dot{\mathbf{u}}_1(t),\dot{\mathbf{u}}_2(t))-j(\dot{\mathbf{u}}_1(t),\dot{\mathbf{u}}_1(t))+j(\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t))-j(\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_2(t)).\end{aligned}$$

On the one hand, we note that $\mathbf{u}_i\in C^1(0,T;V)$, $i=1,2$; this implies $\dot{\mathbf{u}}_i(t)\in V$, $i=1,2$. Then, we use (3.9) (c), the monotonicity of P expressed in (3.31) and (2.1) to find

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}_1(t)-P\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t))_V\\&\qquad\geq M_{\mathcal{A}}\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V^2 .\end{aligned}$$

On the othor hand, we use (4.10) to deduce that

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}_1(t)-P\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t))_V\\&\qquad\leq\left\|\boldsymbol{\eta}_1^{(1)}(t)-\boldsymbol{\eta}_2^{(1)}(t)\right\|_{\mathcal{H}}\left\|\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))\right\|_{\mathcal{H}}\\&\qquad+c_0^2\,\mu L_p\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V .\end{aligned}$$

We combine the two last inequalities and we recall (2.1) to find

$$M_{\mathcal{A}}\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V\leq\left\|\boldsymbol{\eta}_1^{(1)}(t)-\boldsymbol{\eta}_2^{(1)}(t)\right\|_{\mathcal{H}}+c_0^2\,\mu L_p\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V .$$

Now we use (4.2) to deduce that

$$\left\|\boldsymbol{\eta}_1^{(1)}(s)-\boldsymbol{\eta}_2^{(1)}(s)\right\|_{\mathcal{H}}\leq\left\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\right\|_{\mathcal{H}\times Y} .$$

Then, we combine the last two inequalities to obtain

$$(M_{\mathcal{A}}-c_0^2\,\mu\,L_p)\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V\leq\left\|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\right\|_{\mathcal{H}\times Y}.$$

Finally, we recall that the condition (2.9) of Theorem 2.2 is satisfied for $M_A=M_{\mathcal{A}}$ and $\alpha=c_0^2\mu\,L_p$, which yields $M_{\mathcal{A}}>c_0^2\,\mu\,L_p$; i.e. $M_{\mathcal{A}}-c_0^2\mu\,L_p>0$. Hence, we conclude that the estimate (4.5) is satisfied. □

Step 2. In the second step of the proof of Theorem 4.1, we denote by $\mathbf{k}_\eta\in C(0,T;Y)$ the function defined by

$$\mathbf{k}_\eta(t)=\int_0^t\boldsymbol{\eta}^{(2)}(s)\,ds+\mathbf{k}_0. \tag{4.12}$$

Step 3. The third step of the proof consists of using the displacement field $\mathbf{u}_\eta$ which was obtained in Lemma 4.2 and the function $\mathbf{k}_\eta$ defined in (4.12) to consider the following problem.

Problem Q_η. Find a stress field $\boldsymbol{\sigma}_\eta:[0,T]\to\mathcal{H}$ such that

$$\boldsymbol{\sigma}_\eta(t)=\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}_\eta(s),\boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)),\mathbf{k}_\eta(s))\,ds, \tag{4.13}$$

In the study of Problem Q_η we have the following result.

Lemma 4.3. *There exists a unique solution to Problem Q_η which satisfies $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H})$. Moreover, for all $\boldsymbol{\eta}_i \in C(0,T;\mathcal{H}\times Y)$, $i=1,2$, if $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{\eta_i}$ and $\mathbf{u}_i = \mathbf{u}_{\eta_i}$ represent the solutions of Problems Q_η and PV_η respectively and $\mathbf{k}_i = \mathbf{k}_{\eta_i}$, $i=1,2$ are defined by* (4.12) *then there exists $c>0$ such that*

$$\begin{aligned}&\| \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \leq \\ &c\left(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V + \textstyle\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V\,ds + \int_0^t \|\mathbf{k}_1(s)-\mathbf{k}_2(s)\|_Y\,ds\right),\end{aligned} \tag{4.14}$$

$\forall t\in[0,T]$.

Proof. We introduce the operator $\Lambda_\eta : C(0,T;\mathcal{H}) \longrightarrow C(0,T;\mathcal{H})$ defined by

$$\Lambda_\eta \boldsymbol{\sigma}(t) \;=\; \mathcal{B}\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s))\,ds. \tag{4.15}$$

First, we can see that assumptions (3.10) and (3.11) on $\mathcal{B}$ and $\mathcal{G}$ show that the operator Λ_η is well-defined. Next, for all $\boldsymbol{\sigma}\in C(0,T;\mathcal{H})$ and $t\in[0,T]$, we consider $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in C(0,T;\mathcal{H})$ and we use (4.15) and (3.11)(b) to obtain for all $t\in[0,T]$,

$$\|\Lambda_\eta\boldsymbol{\sigma}_1(t) - \Lambda_\eta\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \leq \quad L_{\mathcal{G}} \textstyle\int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\;ds\;.$$

The reiteration of the last inequality p times yields

$$\left\|\Lambda_\eta^p\boldsymbol{\sigma}_1(t) - \Lambda_\eta^p\boldsymbol{\sigma}_2(t)\right\|_{\mathcal{H}} \leq (L_{\mathcal{G}})^p \underbrace{\int_0^t\int_0^s ... \int_0^r}_{p \text{ times}} \|\boldsymbol{\sigma}_1(l) - \boldsymbol{\sigma}_2(l)\|_{\mathcal{H}}\;dl,$$

which implies

$$\left\|\Lambda_\eta^p\boldsymbol{\sigma}_1 - \Lambda_\eta^p\boldsymbol{\sigma}_2\right\|_{C(0,T;\mathcal{H})} \leq \frac{c^p\,T^p}{p!}\;\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{C(0,T;\mathcal{H})}\,.$$

It results from the last inequality that for p large enough, $\lim_{p\to+\infty}\dfrac{c^pT^p}{p!} = 0$; and therefore the operator Λ_η^p is a contraction on the Banach space $C(0,T;\mathcal{H})$. So we can deduce that there exists a unique function $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H})$ such that

$$\Lambda_\eta\boldsymbol{\sigma}_\eta \;=\; \boldsymbol{\sigma}_\eta.$$

The last equality combined with (4.15) shows that $\boldsymbol{\sigma}_\eta$ is a solution of Q_η. Its uniqueness follows from the uniqueness of the fixed point of the operator Λ_η.

Now, let consider $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;\mathcal{H}\times Y)$ and, for $i=1,2$, we use the notations $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$ and $\mathbf{k}_{\eta_i} = \mathbf{k}_i$. We use assumptions (3.10)(b) and (3.11)(b) on $\mathcal{B}$ and $\mathcal{G}$ as well as (2.1) to find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \leq c\,(&\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \textstyle\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V\;ds \\ &+ \textstyle\int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\;ds\; + \int_0^t \|\mathbf{k}_1(s) - \mathbf{k}_2(s)\|_Y\;ds),\end{aligned} \tag{4.16}$$

for all $t\in[0,T]$. We use now (4.16) and a Gronwell argument to deduce the estimate (4.14). □

Step 4. In this step, we use the properties of $\mathcal{B}$, $\mathcal{G}$ and φ to define the operator $\Lambda : C(0,T;\mathcal{H}\times Y) \to C(0,T;\mathcal{H}\times Y)$ which maps every element $\boldsymbol{\eta} = (\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}) \in C(0,T;\mathcal{H}\times Y)$ into the element $\Lambda\boldsymbol{\eta}$ given by

$$\Lambda\boldsymbol{\eta}(t) = \left(\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s)\right) ds, \varphi\left(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \mathbf{k}_\eta(t)\right)\right). \tag{4.17}$$

Here, for all $\boldsymbol{\eta} \in C(0,T;\mathcal{H}\times Y)$, $\mathbf{u}_\eta$ and $\boldsymbol{\sigma}_\eta$ represent respectively the displacement field and the stress field provided in Lemmas 4.2 and 4.3. Moreover, $\mathbf{k}_\eta$ is the internal state variable given by (4.12). We have the following result.

Lemma 4.4. *The operator Λ has a unique fixed point $\boldsymbol{\eta}^* \in C(0,T;\mathcal{H}\times Y)$.*

Proof. Let consider $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;\mathcal{H}\times Y)$ and let use the notations $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$, $\mathbf{k}_{\eta_i} = \mathbf{k}_i$, for $i = 1, 2$. We use (4.17),(4.2), (3.10)(b), (3.11)(b), (3.12)(a) and (2.1) to obtain

$$\begin{aligned} &\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}\times Y} \\ &\le c\left(\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \|\mathbf{k}_1(t) - \mathbf{k}_2(t)\|_Y\right) \\ &+ c\int_0^t \left(\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}} + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V + \|\mathbf{k}_1(s) - \mathbf{k}_2(s)\|_Y\right) ds\,. \end{aligned} \tag{4.18}$$

On the one hand, definition (4.12) yields

$$\|\mathbf{k}_1(t) - \mathbf{k}_2(t)\|_Y \le \int_0^t \left\|\boldsymbol{\eta}_1^{(2)}(s) - \boldsymbol{\eta}_2^{(2)}(s)\right\|_Y ds\,,$$

and, by using (4.2), we deduce that

$$\|\mathbf{k}_1(t) - \mathbf{k}_2(t)\|_Y \le \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}\times Y} ds \quad \forall t \in [0,T]. \tag{4.19}$$

On the other hand, we use the initial condition (3.8) to write

$$\mathbf{u}_i = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}_i(s)\, ds \qquad , i = 1, 2.$$

Hence,

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \le \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V ds\,.$$

The last inequality combined with the estimate (4.5) implies

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}\times Y} ds, \tag{4.20}$$

Now, we combine (4.18)-(4.20) and the estimate (4.14) to deduce that

$$\begin{aligned} &\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}\times Y} \\ &\le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}\times Y} ds + c\int_0^t \left(\int_0^s \|\boldsymbol{\eta}_1(r) - \boldsymbol{\eta}_2(r)\|_{\mathcal{H}\times Y} dr\right) ds \\ &+ c\int_0^t \left(\int_0^s \left(\int_0^r \|\boldsymbol{\eta}_1(l) - \boldsymbol{\eta}_2(l)\|_{\mathcal{H}\times Y} dl\right) dr\right) ds, \end{aligned}$$

which yields

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}\times Y} \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}\times Y} ds.$$

Finally, we apply Theorem 2.1 to conclude that there exists a unique fixed point $\boldsymbol{\eta}^* \in C(0,T;\mathcal{H}\times Y)$ of the operator Λ . $\square$

Now we have all the ingredients to prove Theorem 4.1.

Proof. **Existence.** Let $\boldsymbol{\eta}^* = (\boldsymbol{\eta}^{(1)*}, \boldsymbol{\eta}^{(2)*}) \in C(0,T;\mathcal{H}\times Y)$ be the fixed point of the operator Λ which is defined by (4.17). We use the notations

$$\mathbf{u}(t) = \mathbf{u}_{\boldsymbol{\eta}^*}(t) \tag{4.21}$$

$$\mathbf{k}(t) = \mathbf{k}_{\boldsymbol{\eta}^*}(t) \tag{4.22}$$

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \boldsymbol{\sigma}_{\boldsymbol{\eta}^*}(t) . \tag{4.23}$$

We prove that $(\mathbf{u},\boldsymbol{\sigma},\mathbf{k})$ is a solution of the Problem PV with regularity (4.1). In fact, we write (4.13) for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ and we use the notations (4.21)-(4.23) to obtain (3.34). Next, we write (4.3) for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ and we use (4.21) to find that

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_{\mathcal{H}}+\left(\boldsymbol{\eta}^{(1)*}(t),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))\right)_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}(t),\mathbf{v}-\dot{\mathbf{u}}(t))_V+j(\dot{\mathbf{u}}(t),\mathbf{v})-j(\dot{\mathbf{u}}(t),\dot{\mathbf{u}}(t))\geq(\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}(t))_V ,\end{aligned} \tag{4.24}$$

for all $\mathbf{v}\in V$ and $t\in[0,T]$. Now, we recall that $\Lambda\boldsymbol{\eta}^* = \boldsymbol{\eta}^* = (\boldsymbol{\eta}^{(1)*},\boldsymbol{\eta}^{(2)*})$. Hence, definition (4.17) and the notations (4.21)-(4.23) yield

$$\boldsymbol{\eta}^{(1)*}(t)=\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t))+\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s)-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)),\boldsymbol{\varepsilon}(\mathbf{u}(s)),\mathbf{k}(s))\,ds, \tag{4.25}$$

$$\boldsymbol{\eta}^{(2)*}(t)=\varphi(\boldsymbol{\sigma}(t)-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)),\boldsymbol{\varepsilon}(\mathbf{u}(t)),\mathbf{k}(t)) . \tag{4.26}$$

We use (4.26) and (4.12) to see that (3.35) is satisfied. Next, we substitute (4.25) in (4.24) and we use (3.34) to see that (3.36) is also satisfied.
Finally, (3.37) and the regularities of $\mathbf{u}$ and $\mathbf{k}$ which are given in (4.1) follow from the Lemma 4.2 and (4.12), combined with the fact that $\boldsymbol{\eta}^{(2)*}\in C(0,T;Y)$.
Moreover, for the stress tensor $\boldsymbol{\sigma}$, we use (4.23), (3.9) and we recall that from Lemma 4.3 we have $\boldsymbol{\sigma}_{\boldsymbol{\eta}^*}(t)\in\mathcal{H}$; hence, we deduce that $\boldsymbol{\sigma}(t)\in\mathcal{H}$. As for the regularity of $\boldsymbol{\sigma}$, we use again (4.23) to find that for all $t_1,t_2\in[0,T]$,

$$\|\boldsymbol{\sigma}(t_1)-\boldsymbol{\sigma}(t_2)\|_{\mathcal{H}}\leq\|\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t_1))-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t_2))\|_{\mathcal{H}}+\|\boldsymbol{\sigma}_{\eta^*}(t_1)-\boldsymbol{\sigma}_{\eta^*}(t_2)\|_{\mathcal{H}} .$$

Thus, hypothesis (3.9)(b) on the operator $\mathcal{A}$ and (2.1) yield

$$\|\boldsymbol{\sigma}(t_1)-\boldsymbol{\sigma}(t_2)\|_{\mathcal{H}}\leq L_{\mathcal{A}}\|\dot{\mathbf{u}}(t_1)-\dot{\mathbf{u}}(t_2)\|_V+\|\boldsymbol{\sigma}_{\eta^*}(t_1)-\boldsymbol{\sigma}_{\eta^*}(t_2)\|_{\mathcal{H}} .$$

The last inequality combined with regularities $\mathbf{u}\in C^1(0,T;V)$ and $\boldsymbol{\sigma}_{\eta^*}\in C(0,T;\mathcal{H})$ derived respectively from Lemmas 4.2 and 4.3 shows that $\boldsymbol{\sigma}\in C(0,T;\mathcal{H})$. In order to obtain the regularity $\boldsymbol{\sigma}\in C(0,T;\mathcal{H}_1)$, we test (3.36) with $\mathbf{v}=\dot{\mathbf{u}}+\boldsymbol{\varphi}$ and then with $\mathbf{v}=\dot{\mathbf{u}}-\boldsymbol{\varphi}$, where $\boldsymbol{\varphi}\in C_0^\infty(\Omega)^d$ and we recall that j is a positive function; after a simple calculation, we obtain

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_{\mathcal{H}}=(\mathbf{f}(t),\boldsymbol{\varphi})_V \quad \forall t\in[0,T],\ \ \forall\boldsymbol{\varphi}\in C_0^\infty(\Omega)^d .$$

Then we use (3.20) to deduce that

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{\varphi}))_{\mathcal{H}}=(\mathbf{f}_0(t),\boldsymbol{\varphi})_H .$$

Thus, from the definition of weak divergence we conclude that

$$(-Div\ \boldsymbol{\sigma}(t), \boldsymbol{\varphi})_H = (\mathbf{f}_0(t), \boldsymbol{\varphi})_H \ \ \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega)^d .$$

Since the space $C_0^\infty(\Omega)^d$ is dense in $L^2(\Omega)^d$ we deduce that

$$Div\ \boldsymbol{\sigma}(t) = -\ \mathbf{f}_0(t) \ \ \forall t \in [0,T]. \tag{4.27}$$

The last equality combined with the hypothesis (3.15) on $\mathbf{f_0}$ implies $Div\ \boldsymbol{\sigma}(t) \in H$ and, therefore, $\boldsymbol{\sigma}(t) \in \mathcal{H}_1$. Finally, we note that the norm on $\mathcal{H}_1$ allows us to write

$$\|\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2)\|_{\mathcal{H}_1}^2 \quad = \|\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2)\|_{\mathcal{H}}^2 + \|Div\ (\boldsymbol{\sigma}(t_1)) - Div\ (\boldsymbol{\sigma}(t_2))\|_H^2 \ .$$

Thus, (4.27), (3.15) and the regularity $\boldsymbol{\sigma} \in C(0,T;\mathcal{H})$ imply $\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1)$; which completes the proof of the existence of a solution $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k})$ to Problem PV with regularity (4.1).

Uniqueness. The uniqueness of the solution $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k})$ of Problem PV follows from the uniqueness of the fixed point of the operator Λ combined with the unique solvability of the intermediate problems PV_η and Q_η guaranteed by Lemmas 4.2 and 4.3. □

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On the stability of KdV equation with time-dependent delay on the boundary feedback in presence of saturated source term

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Abstract. The current paper investigate the question of stabilizability of the Korteweg-de Vries equation with time-varying delay on the boundary feedback in the presence of a saturated source term. Under specific assumptions regarding the time-varying delay, we have established that the studied system is well-posed. Moreover, using an appropriate Lyapunov functional, we prove the exponential stability result. Finally, we give some conclusions.

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1. Introduction

In recent years a lot of work has come out on the study of Korteweg-de Vries equation with time-delay (see e.g. [25, 2, 36]). The Korteweg-de Vries equation (KdV)

$$u_t + u_x + u_{xxx} + uu_x = 0 \tag{1.1}$$

is a nonlinear one dimensional equation, more precisely the KdV equation is a mathematical model of waves on shallow water surfaces. In recent decades, the study of the Korteweg-de Vries equation has yielded intriguing results, particularly with regard to its controllability and stabilizability properties. This studies can be attributed to the efforts made by Russell and Zhang in [32]. Subsequently, significant research efforts have been dedicated to the examination of both controllability and stabilizability. For a comprehensive review of these studies, interested readers can refer to various works

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(such as [31, 37]), as well as the following references [3, 9, 5]. In the majority of these papers, the following system have been studied:

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + f = 0, & t \geq 0, \ x \in [0,L]; \\ u(t,0) = u(t,L) = u_x(t,L) = 0, & t \geq 0; \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

In a general context, the feedback control f in (1.2) is chosen to fulfill specific objectives. As a result, it must consistently adhere to predefined constraints. In particular, equation (1.2) has been the subject of investigation, with two distinct approaches studied in the literature: one involving distributed control (as examined in [29, 27]) and another involving boundary control (as discussed in [18, 4]). Notably, in [29], the authors demonstrate that the linear feedback control $f(t,x) = a(x)u(t,x)$, where $a = a(x)$ is a nonnegative function that satisfies certain conditions, makes the origin exponentially stable. It's worth mentioning as well that when a equals zero, the authors also prove that the linear Korteweg-de Vries (KdV) equation without control is exponentially stable under the conditions $L \notin \left\{ 2\pi\sqrt{\frac{k^2+n^2+kn}{3}} \mid n,k \in \mathbb{N}^* \right\}$.

One of the most well-known constraints that can affect the proper functioning of the control system is the saturation constraint, which has been discussed in various works (see, for instance, [20, 24, 16, 17, 10, 6]. The issue of input saturation in the control system is inevitable. Physical constraints or practical limitations can cause the restriction of input signal amplitudes, leading to unfavorable and even catastrophic outcomes for the control system.

In the literature, there are several articles that studies the stability result of KdV equation with input saturation (see e.g. [20, 34, 19]. In particular, [34] looks at the study of the following KdV equation:

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + sat(a(x)u(t,x) = 0, & t \geq 0, \ x \in [0,L]; \\ u(t,0) = u(t,L) = u_x(t,L) = 0, & t \geq 0; \\ u(0,x) = u_0(x), \end{cases} \tag{1.3}$$

where $a = a(x) \in L^\infty([0,L])$ satisfying $a_1 \geq a = a(x) \geq a_0 > 0$ on $\omega \subseteq [0,L]$ (ω is a nonempty open subset of $[0,L]$), and the saturation function $sat(.)$ is defined as follows

$$sat(t) = \begin{cases} t, & \text{if } \|t\|_{L^2(0,L)} \leq 1 \\ \dfrac{t}{\|t\|_{L^2(0,L)}}, & \text{if } \|t\|_{L^2(0,L)} \geq 1 \end{cases} \tag{1.4}$$

The well-posedness of the closed-loop system for the linear KdV equation (1.3) has been proved through the application of nonlinear semigroup theory. Moreover, the authors have demonstrated that the origin of the KdV equation (1.3) in closed-loop system with the saturated control (1.4) is exponentially stable. The asymptotic stability of KdV equation with a saturated internal control has been studied by [19]. In their work , they considered the system (1.2) with

$$f(t,x) = asat(u),$$

where a is a positive constant and the saturation function is defined as follows

$$sat(t) = \begin{cases} -u_0, & \text{if } t \leq -u_0 \\ t, & \text{if } -u_0 \leq t \leq u_0 \\ u_0, & \text{if } t \geq u_0 \end{cases} \tag{1.5}$$

where u_0 represent a positive constant. The authors prove the well-posedness by applying nonlinear semigroup theory. Additionally, using Lyapunov theory for infinite-dimensional systems, they also establish that the origin is asymptotically stable.

In this paper, we are interested in the study of time-varying delay on the boundary of the Korteweg-de Vries equation in the presence of a saturated source term. That is to say, we consider the same problem as in [34] with time-varying delay on the boundary feedback.

In general, the presence of delay in scientific phenomena is a multifaceted consideration. It is widely acknowledged that even a minor delay in the feedback mechanism can potentially induce instability in a system, as discussed in various references [12, 7, 21]. Alternatively, delays can be used as a tool to improve performance by introducing beneficial phase shifts to optimize system behavior, as studied in references such as [1, 30]. When delays become time-varying, the complexity of analyzing system stability significantly increases. Several studies have examined the stability of partial differential equations (PDEs) involving time-varying delay, with notable references including [22, 8, 26, 13].

In recent years, researchers have shown increasing interest in solving stability and robustness problems related to constant delay for the Korteweg-de Vries equation. Notable contributions have been made by researchers such as Baudouin et al. and Parada et al., as mentioned in [2, 23], where they studied the Korteweg-de Vries equation with time-delay feedback, establishing the well-posedness and proving exponential stability through the use of the observability inequality. For more details on the KdV equation with time-delay, the readers can find more details in [35, 11, 15]. Concerning the Korteweg-de Vries equation with time-varying delay, there is a notably singular study conducted by Parada et al. [25]. This study examined the issue of time-varying delay both on the boundary or internal feedback. With specific assumptions concerning time-varying delay, they proved the well-posedness and the stability results is analyzed, using an appropriate Lyapunov functional. However, in the literature to the best of our knowledge, there has been no prior work addressing this issue in the context of the Korteweg-de Vries equation with a saturated source term.

In our paper, we focus on the Korteweg-de Vries equation with time-varying delay on the boundary feedback in presence of saturated source term. The equation under investigation is given as follows:

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = f(x,t), \quad t > 0, \quad x \in [0,L]; \\ u(0,t) = u(L,t) = 0, \quad t > 0; \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\theta(t)), \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in [0,L]; \\ u_x(0,t-\theta(0)) = z_0(t-\theta(0)), \quad 0 < t < \theta(0), \end{cases} \tag{1.6}$$

with

$$f(x,t) = -sat(a(x)u(t;x)), \tag{1.7}$$

where $a(\cdot) \in L^\infty([0,L])$ is a nonnegative function satisfying some conditions, and $sat(.)$ is the same given by (1.4). The main contribution of this paper is to study the well-posedness and exponential stability of the linear KdV equation with time-varying delay on the boundary feedback, as given in equation (1.6)-(1.7). The well-posedness of the system (1.6)-(1.7) is proven under some conditions. By using an appropriate Lyapunov functional, we demonstrate that the KdV equation (1.6)-(1.7) with the saturated source term (1.4) is exponentially stable.

Our paper is organized as follows. In the next section, we formulate our problem. In section 3, we examine the well-posedness of (1.6)-(1.7). Section 4 is dedicated to the exponential stabilization of (1.6)-(1.7). Finally, we present some conclusions in section 5.

2. Problem statement

The aim of this paper is to study the following KdV equation with time-varying delay

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = -sat(a(x)u(x,t)), \quad t>0, \quad x \in [0,L]; \\ u(0,t) = u(L,t) = 0, \quad t>0; \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\theta(t)), \quad t>0; \\ u(x,0) = u_0(x), \quad x \in [0,L]; \\ u_x(0,t-\theta(0)) = z_0(t-\theta(0)), \quad 0<t<\theta(0), \end{cases} \tag{2.1}$$

where where $a = a(x) \in L^\infty[0,L]$ satisfying

$$\begin{cases} a = a(x) \geq a_0 > 0 \quad \text{on} \quad \omega \subseteq [0,L], \\ \omega \quad \text{is a nonempty open subset of} \quad [0,L], \end{cases} \tag{2.2}$$

Moreover, suppose that the delay $\theta(\cdot) \in W^{2,\infty}[0,T]$ for all $T>0$ and satisfies the following conditions

$$0 < \theta_0 \leq \theta(t) \leq K, \text{ for all } t \geq 0, \tag{2.3}$$

and

$$\dot{\theta}(t) \leq d \leq 1, \text{ for all } t \geq 0, \tag{2.4}$$

where $d \geq 0$.
Furthermore, we define the matrix M_1 by

$$M_1 = \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 + |\beta|(d-1) \end{pmatrix} \tag{2.5}$$

Where α, β and d are real constants that satisfy the following inequality

$$|\alpha| + |\beta| + d < 1. \tag{2.6}$$

If (2.6) is satisfied, then the matrix M_1 is definite negative according to [25].
In this context, we introduce a new variable $z(\mu,t) = u_x(0, t-\theta(t)\mu)$ for $\mu \in [0,1]$ and $t > 0$. Then, $z(\cdot,\cdot)$ satisfies the following system

$$\begin{cases} \theta(t)z_t(\mu,t) + (1-\dot{\theta}(t)\mu)z_\mu(\mu,t) = 0, & t > 0, \mu \in [0,1]; \\ z(0,t) = u_x(0,t), & t > 0; \\ z(\mu,0) = z_0(-\theta(0)\mu), & \mu \in [0,1]. \end{cases} \tag{2.7}$$

For more detail about a new variable z that takes into account $\theta(\cdot)$ (see [21, 22]).
Therefore, we investigate the following semi-linear system

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = -sat(a(x)u(x,t)), & t > 0, x \in [0,L]; \\ \theta(t)z_t(\mu,t) + (1-\dot{\theta}(t)\mu)z_\mu(\mu,t) = 0, & t > 0, \mu \in [0,1]; \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0, t-\theta(t)), & t > 0; \\ u(x,0) = u_0(x), & x \in [0,L]; \\ u_x(0, t-\theta(0)) = z_0(t-\theta(0)), & 0 < t < \theta(0). \\ z(0,t) = u_x(0,t), & t > 0; \\ z(\mu,0) = z_0(-\theta(0)\mu), & \mu \in [0,1]. \end{cases} \tag{2.8}$$

Let $Y = \begin{pmatrix} u \\ z \end{pmatrix}$, then the system (2.8) can be rewritten as the following first-order system

$$\begin{cases} Y_t = A(t)Y(t) + \begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix}, & t > 0, \\ Y(0) = \begin{pmatrix} u_0, & z_0(-\theta(0)) \end{pmatrix}^T. \end{cases} \tag{2.9}$$

Where the operator $A(t)$ is defined by

$$\begin{aligned} D(A(t)) &= \{(u,z) \in H^3([0,L]) \times H^1([0,L]), u(0) = u(L) = 0 \\ &\qquad z(0) = u_x(0), u_x(L) = \alpha u_x(0,t) + \beta u_x(0, t-\theta(t))\} \end{aligned}$$

$$A(t)\begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} -u_x - u_{xxx} \\ \frac{\dot{\theta}(t)\mu - 1}{\theta(t)} z_\mu \end{pmatrix} \text{ for all } \begin{pmatrix} u \\ z \end{pmatrix} \in D(A(t)). \tag{2.10}$$

It has been proved in [25] that the domain of operator $A(t)$ is independent of t, i.e.

$$D(A(t)) = D(A(0)).$$

Let the Hilbert space $H = L^2[0,L] \times L^2[0,1]$ equipped with the following usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle_H = \int_0^L uu_1 dx + \int_0^1 zz_1 d\mu,$$

and its norm

$$\left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|_H^2 = \int_0^L u^2 dx + \int_0^1 z^2 d\mu$$

We introduce a new inner product on H. This inner product is dependent to time t and define as follows

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle_t = \int_0^L uu_1 dx + |\beta|\theta(t) \int_0^1 zz_1 d\mu,$$

with associated norm denoted by $\|\cdot\|_t$. Using (2.3), the norm $\|\cdot\|_t$ and $\|\cdot\|_H$ are equivalent in H. Indeed, for all $t \geq 0$, and all $\begin{pmatrix} u \\ z \end{pmatrix} \in H$, we have

$$(1+|\beta|\theta_0)\left\|\begin{pmatrix} u \\ z \end{pmatrix}\right\|_H^2 \leq \left\|\begin{pmatrix} u \\ z \end{pmatrix}\right\|_t^2 \leq (1+|\beta|K)\left\|\begin{pmatrix} u \\ z \end{pmatrix}\right\|_H^2 \tag{2.11}$$

Now, we recall the definition of mild solution.
Let us consider the abstract system in a Hilbert space Z

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t) + f(t), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{2.12}$$

where $\mathcal{A}$ is an infinitesimal generator of linear C_0-semigroup $(T(t))_{t\geq 0}$ defined on its domain $D(\mathcal{A}) \subseteq H$, where Z is a Hilbert space and $f \in L^1_{loc}([0,T], Z)$.

Definition 2.1. [28, Definition 2.3] Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Let $u_0 \in Z$ and $f \in L^1(0,T,Z)$. Then the function $u \in \mathcal{C}([0,T], Z)$ given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \quad 0 \leq t \leq T, \tag{2.13}$$

is the unique mild solution of the initial value problem (2.12) on $[0,T]$.

We recall that the saturation function is Lipschitzian in $L^2[0,L]$.

Lemma 2.2. [33, Theorem 5.1] *For all* $(u,v) \in L^2[0,L]$, *we have*

$$\|sat(u) - sat(v)\|_{L^2[0,L]} \leq 3\|u-v\|_{L^2[0,L]}$$

3. Well-posedness

Before stating the well-posedness result system (2.9), we recall the result of well-posedness of the following linear system without source term which has been treated by Parada et al. [25]

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = 0, & t>0, x\in[0,L]; \\ \theta(t)z_t(\mu,t) + (1-\theta_t(t)\mu)z_\mu(\mu,t) = 0, & t>0, \mu\in[0,1]; \\ u(0,t) = u(L,t) = 0, & t>0 \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\theta(t)), & t>0; \\ u(x,0) = u_0(x), & x\in[0,L]; \\ u_x(0,t-\theta(0)) = z_0(t-\theta(0)), & 0<t<\theta(0). \\ z(0,t) = u_x(0,t), & t>0; \\ z(\mu,0) = z_0(-\theta(0)\mu), & \mu\in[0,1]. \end{cases} \tag{3.1}$$

As previously, let $Y = \begin{pmatrix} u \\ z \end{pmatrix}$, then the system (3.1) can be rewritten as the following first-order system

$$\begin{cases} Y_t = A(t)Y(t), & t>0, \\ Y(0) = \begin{pmatrix} u_0, & z_0(-\theta(0)) \end{pmatrix}^T. \end{cases} \tag{3.2}$$

Where $A(t)$ is given by (2.10). The well-posedness of (3.2), is proved in [25, Theorem 2.2]. To prove the well-posedness of (3.2), they used the following Theorem

Theorem 3.1. *Assume that*

1. $D(A(0))$*, is a dense subset of* H.
2. $D(A(t)) = D(A(0)) \quad \forall t \geq 0.$
3. *For all* $t \in [0,T]$ $A(t)$ *generates a strongly continuous semigroup on* H *and the family* $A = \{A(t) : t \in [0,T]\}$ *is stable with stability constant* C *and* m *independent of* t*, i.e. the semigroup* $(T_t(s))_{s\geq 0}$ *generated by* $A(t)$ *satisfies*

$$\|T_t(s)Y\|_H \leq Ce^{ms}\|Y\|_H, \text{ for all } Y \in H, \text{ and } s \geq 0.$$

4. $\partial_t A(t)$ *belong to* $L^\infty_*([0,T], B(D(A(0))))$*, the space of equivalent of essentially classes bounded strongly measure functions from* $[0,T]$ *into the set* $B(D(A(0)), H)$ *of bounded operator from* $D(A(0)$ *to* H.

Then the system (3.2) *has a unique solution* $Y \in \mathcal{C}([0,T], D(A(0))) \cap \mathcal{C}^1([0,T], H)$

More precisely, in [25], the authors demonstrated that, if (2.3)-(2.6) holds, the operator $A(t)$ satisfy all assumptions of Theorem 3.1 and the system (3.2) has a unique solution $u \in \mathcal{C}([0,+\infty[, H)$. Moreover if $Y_0 \in D(A(0))$, then $Y \in \mathcal{C}([0,+\infty[, D(A(0))) \cap \mathcal{C}^1([0,+\infty[, H)$.

The following result gives the conditions for the existences and the uniqueness of the solution of (2.9).

Theorem 3.2. *Let* $(u_0, z_0) \in H$ *and suppose that* (2.3)-(2.6) *holds. Assume also that* $a = a(x) \in L^\infty[0,L]$ *satisfying* (2.2) *and* $u \in L^2(0,T,H^1[0,L])$*. Then, there exists a unique solution* $Y = (u,z) \in \mathcal{C}([0,+\infty[, H)$ *of* (2.9).

Proof. Let $G(u) = \begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix}$. By assumption $u \in L^2(0,T,H^1[0,L])$, hence $\begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$. Indeed, let $u_1, u_2 \in L^2(0,T,H^1[0,L])$, by using the Holder inequality, ([20, Proposition 3.4]) and ([33, Theorem 5.1]), we get

$$\begin{aligned}
\|G(u_1) - G(u_2)\|_{L^1(0,T,H)} &= \int_0^T \|G(u_1) - G(u_2)\|_H dt \\
&= \int_0^T \|sat(au_1) - sat(au_2)\|_{L^2[0,L])} + \|0\|_{L^2[0,1]} dt \\
&= \int_0^T \|sat(au_1) - sat(au_2)\|_{L^2[0,L])} dt \\
&\leq 3\int_0^T \|au_1 - au_2\|_{L^2[0,L])} dt \\
&\leq 3\|a\|_{L^\infty[0,L]} \int_0^T \|u_1 - u_2\|_{L^2[0,L])} dt \\
&\leq 3\|a\|_{L^\infty[0,L]} \sqrt{T}\sqrt{L}\|u_1 - u_2\|_{L^2(0,T,H^1[0,L])} < +\infty.
\end{aligned}$$

Therefore, $\begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$. Moreover, from [25, Theorem 2.2] the operator $A(t)$ satisfy all assumption of Theorem 3.1. Thus, since $\begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$, then if $Y_0 \in H$, the system (3.2) has a unique solution $Y = (u,z) \in \mathcal{C}([0,+\infty[,H)$, according to [14, Theorem 2].
Furthermore, $sat(\tilde{a}(x)u) \in L^1(0,T,L^2[0,L])$, hence if $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$, then from [2, Proposition 2] the solution of (2.9) is a regular solution. □

4. Exponential stability

Consider the following energy

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{\|\beta\|}{2}\theta(t)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu. \tag{4.1}$$

The following lemma proves that the energy (4.1) does not increase.

Lemma 4.1. *Assume that assumptions* (2.3), (2.4) *and* (2.6) *are satisfied. Moreover suppose also that* $u \in L^2(0,T,H^1[0,L])$ *and* $a = a(x) \in L^\infty[0,L]$, *satisfying* (2.2). *Then, for any regular solution of* (2.9), *the energy* (4.1) *satisfies the following inequality*

$$\begin{aligned} \frac{d}{dt}E(t) \leq & \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\ \leq & 0. \end{aligned} \tag{4.2}$$

Proof. Let u a regular solution of (2.1). By definition $z(\mu,t) = u_x(0,t-\theta(t)\mu)$, hence we rewrite the energy (4.1) as follows

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{\|\beta\|}{2}\theta(t)\int_0^1 z^2(\mu,t)d\mu.$$

Differentiating $E(\cdot)$, we get

$$\begin{aligned} \frac{d}{dt}E(t) =& \int_0^L uu_t dx + \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + |\beta|\theta(t)\int_0^1 zz_t d\mu \\ =& -\int_0^L uu_x dx - \int_0^L uu_{xxx}dx - \int_0^L sat(au)udx \\ &+ \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + |\beta|\theta(t)\int_0^1 zz_t d\mu \end{aligned} \tag{4.3}$$

After some integrations by parts, we obtain

$$-\int_0^L uu_x dx = 0; \quad -\int_0^L uu_{xxx}dx = \frac{1}{2}u_x^2(L,t) - \frac{1}{2}u_x^2(0,t),$$

and

$$|\beta|\theta(t)\int_0^1 zz_t d\mu = |\beta|\theta(t)\int_0^1 \frac{\dot{\theta}(t)\mu - 1}{\theta(t)} zz_\mu d\mu$$
$$= |\beta|\dot{\theta}(t)\int_0^1 \mu zz_\mu d\mu - |\beta|\int_0^1 zz_\mu d\mu.$$

Thus

$$\begin{aligned} |\beta|\dot{\theta}(t)\int_0^1 \mu zz_\mu d\mu &= \frac{|\beta|}{2}\dot{\theta}(t)\left[\mu z^2(\mu,t)\right]_0^1 - \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2(\mu,t)d\mu \\ &= \frac{|\beta|}{2}\dot{\theta}(t)z^2(1,t) - \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2(\mu,t)d\mu \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} -|\beta|\int_0^1 zz_\mu d\mu &= -\frac{|\beta|}{2}\left[z^2(\mu,t)\right]_0^1 \\ &= -\frac{|\beta|}{2}\left[z^2(1,t) - z^2(0,t)\right] \\ &= \frac{|\beta|}{2}u_x^2(0,t) - \frac{|\beta|}{2}z^2(1,t). \end{aligned} \tag{4.5}$$

Using (2.8), (4.3), (4.4) and (4.5), we get

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{1}{2}(\alpha u_x(0,t) + \beta z(1,t))^2 - \frac{1}{2}u_x^2(0,t) - \int_0^L sat(au)udx \\ &+ \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + |\beta|\dot{\theta}(t)\int_0^1 \mu zz_\mu d\mu - |\beta|\int_0^1 zz_\mu d\mu \\ &= \frac{1}{2}\alpha^2 u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) + \frac{1}{2}\beta^2 z^2(1,t) - \frac{1}{2}u_x^2(0,t) \\ &- \int_0^L sat(au)udx + \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + \frac{|\beta|}{2}\dot{\theta}(t)z^2(1,t) \\ &- \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + \frac{|\beta|}{2}u_x^2(0,t) - \frac{|\beta|}{2}z^2(1,t) \\ &= \frac{1}{2}(\alpha^2 - 1 + |\beta|)u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) \\ &+ \frac{1}{2}\left(\beta^2 + |\beta|(\dot{\theta}(t) - 1)\right)z^2(1,t) - \int_0^L sat(au)udx. \end{aligned}$$

Therefore using (2.4), we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) &+ \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(-\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\
&\leq \frac{1}{2}(\alpha^2 - 1 + |\beta|)u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) \\
&+ \frac{1}{2}\left(\beta^2 + |\beta|(\dot{\theta}(t) - 1)\right) z^2(1,t) - \int_0^L sat(au)udx \\
&+ \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(-\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\
&= -\int_0^L sat(au)udx \\
&\leq 0.
\end{aligned}$$

Because $\int_0^L sat(au)udx \geq 0$, indeed, if $\|au\|_{L^2} \leq 1$, then

$$sat(au)u = au^2 \geq 0.$$

If $\|au\|_{L^2} \geq 1$,

$$sat(au)u = \frac{au}{\|au\|_{L^2}}u = \frac{au^2}{\|au\|_{L^2}} \geq 0,$$

where $a = a(x)$ is a nonnegative function. Consequently, using (2.6), we have

$$\frac{d}{dt}E(t) \leq \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \leq 0. \qquad \square$$

The following lemmas play an important role to prove the exponential stability of (2.8). Before that, consider the following lyapunov function

$$V(t) = E(t) + \lambda V_1(t) + \gamma V(t)_2, \tag{4.6}$$

where $\lambda, \gamma \geq 0$, $E(\cdot)$ is given by (4.1), and

$$V_1(t) = \int_0^L xu^2(x,t)dx \tag{4.7}$$

$$V_2(t) = \theta(t)\int_0^1 (1-\mu)u^2(x, t-\theta(t)\mu)d\mu. \tag{4.8}$$

Lemma 4.2. *Assume that $a = a(ax) \in L^\infty[0,L]$ satisfies (2.2), $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ and $u \in L^2(0,T,H^1[0,L])$, then for any regular solution of (2.1), the following equation is satisfied*

$$\begin{aligned}
\dot{V}_1(t) =& L(\alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)u_x(0,t-\theta(t)) + \beta^2 u_x^2(0,t-\theta(t)) \\
&+ \int_0^L u^2dx - 3\int_0^L u_x^2dx - \int_0^L xsat(au)udx
\end{aligned} \tag{4.9}$$

Proof. Let us consider a regular solution, then Differentiate $V_1(\cdot)$ we have

$$\begin{aligned}\dot{V}_1(t) &= 2\int_0^L xuu_t dx \\ &= -2\int_0^L xuu_x dx - 2\int_0^L xuu_{xxx}dx - 2\int_0^L xsat(au)udx\end{aligned}$$

After some integrations by parts, we obtain

$$-2\int_0^L xuu_x dx = \int_0^L u^2 dx;$$

$$\begin{aligned}-2\int_0^L xuu_{xxx}dx &= Lu^2(L,t) - 3\int_0^L u_x^2 dx \\ &= L(\alpha u_x(0,t) + \beta u_x(0,t-\theta(t)))^2 - 3\int_0^L u_x^2 dx\end{aligned}$$

Using the last equations, we get

$$\begin{aligned}\dot{V}_1(t) &= \int_0^L u^2 dx + L(\alpha u_x(0,t) + \beta u_x(0,t-\theta(t)))^2 \\ &- 3\int_0^L u_x^2 dx - 2\int_0^L xsat(au)udx \\ &= L(\alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)u_x(0,t-\theta(t)) + \beta^2 u_x^2(0,t-\theta(t))) \\ &+ \int_0^L u^2 dx - 3\int_0^L u_x^2 dx - 2\int_0^L xsat(au)udx\end{aligned}$$

□

Lemma 4.3. *Assume that* (2.4) *is satisfied. Suppose also* $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ *and* $u \in L^2(0,T,H^1[0,L])$, *then for any regular solution of* (2.1), *the following inequality is satisfied*

$$\dot{V}_2(t) \leq -(1-d)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu + u_x^2(0,t). \tag{4.10}$$

Proof. Consider a regular solution, then differentiate $V_2(\cdot)$, we have

$$\begin{aligned}\dot{V}_2(t) =&\dot{\theta}(t)\int_0^1 (1-\mu)u_x^2(0,t-\theta(t)\mu)d\mu \\ &+2\theta(t)\int_0^1 (1-\mu)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu \\ =&\dot{\theta}(t)\int_0^1 (1-\mu)u_x^2(0,t-\theta(t)\mu)d\mu + 2\int_0^1 \theta(t)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu \\ &-2\int_0^1 \mu\theta(t)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu\end{aligned} \tag{4.11}$$

After some integration by parts and using the following equation

$$-\theta(t)\partial_t u_x(0,t-\theta(t)\mu) = (1-\dot{\theta}(t)\mu)\partial_\mu u_x(0,t-\theta(t)\mu),$$

we obtain

$$\begin{aligned}2\int_0^1 \theta(t)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu =& u_x^2(0,t) - (1-\dot{\theta}(t))u_x^2(0,t-\theta(t)) \\ & -\dot{\theta}(t)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}-2\int_0^1 \mu\theta(t)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu =& (1-\dot{\theta}(t))u_x^2(0,t-\theta(t)) \\ & -\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu \\ & +2\int_0^1 \mu\dot{\theta}(t)u_x^2(0,t-\theta(t)\mu)d\mu\end{aligned} \tag{4.13}$$

We deduce from (4.11), (4.12), (4.13) and (2.4) that

$$\begin{aligned}\dot{V}_2(t) &= -\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu \\ &+ \dot{\theta}(t)\int_0^1 \mu u_x^2(0,t-\theta(t)\mu)d\mu + u_x^2(0,t) \\ &\leq -\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu + d\int_0^1 \mu u_x^2(0,t-\theta(t)\mu)d\mu + u_x^2(0,t) \\ &= -(1-d)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu + u_x^2(0,t)\end{aligned}$$

□

Now, we are able to state and prove the main result of this section.

Theorem 4.4. *Assume that $a = a(x) \in L^\infty[0,L]$ satisfying (2.2), and $L < \pi\sqrt{3}$. Moreover suppose that the assumptions* (2.3) , (2.4) *and* (2.6) *are satisfied. Then, there exists $r > 0$ such that for every $(u_0, z_0) \in \mathcal{H}$ satisfying $\|(u_0,z_0)\|_{\mathcal{H}} \leq r$, there exists $\delta > 0$ and $M > 0$ such that*

$$E(t) \leq Me^{-2\delta t}E(0), \quad \forall t > 0. \tag{4.14}$$

where for λ and γ sufficiently small, the two positive constants δ and M satisfy the following inequality:

$$\delta \leq \min\left\{\frac{(9\pi^2 - 3L^2 - 2L^{\frac{3}{2}}r\pi^2)}{3L^2(1+2L\lambda)}\lambda, \frac{\gamma}{h(2\gamma+|\beta|)}\right\} \tag{4.15}$$

and

$$M \leq 1 + \max\left\{L\lambda, \frac{2\gamma}{|\beta|}\right\}.$$

Where λ *and* γ*, satisfying the following inequality*

$$\lambda \leq \left\{ \frac{(1-|\beta|)(1-|\beta|-d)+\alpha^2(d-1)+2\gamma(|\beta|+d-1)}{2L(|\beta|-\alpha^2(d-1)-\beta^2-2\gamma|\beta|)} \right. \\ \left. \frac{1-\alpha^2-\beta^2-|\beta|d+2\gamma}{2L(\alpha^2+\beta^2)} \right\}. \tag{4.16}$$

and

$$\gamma \leq \left\{ \frac{1-\alpha^2-\beta^2-|\beta|d}{2} \right. \\ \frac{(1-|\beta|)(1-|\beta|-d)+\alpha^2(d-1)}{2(1-|\beta|-d)} \\ \left. \frac{|\beta|-\alpha^2(d-1)-\beta^2}{2\beta} \right\}. \tag{4.17}$$

Remark 4.5. The Lyapunov function $V(\cdot)$ and the energy $E(\cdot)$ are equivalent. Indeed,

$$E(t) \leq V(t) \leq M_1 E(t) \quad \forall t > 0, \tag{4.18}$$

where $M_1 = 1 + \max\left\{L\lambda, \frac{2\gamma}{|\beta|}\right\} > 0$. Thanks to inequality (4.18), in order to prove the exponential stability of system (2.1), it is sufficient to show that for all $\delta > 0$,

$$\frac{d}{dt}V(t) + 2\delta V(t) \leq 0.$$

Proof. Let $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ such that $\|\begin{pmatrix} u_0 \\ z_0 \end{pmatrix}\|_0 \leq r$, with $r > 0$ chosen later. Using (4.2), (4.9) and (4.10), we get

$$\begin{aligned}
\dot{V}(t) \leq & \frac{1}{2} Y M_1 Y + L\lambda\alpha^2 u_x^2(0,t) + 2L\lambda\alpha\beta u_x(0,t)u_x(0,t-\theta(t)) \\
& + L\beta^2 u_x^2(0,t-\theta(t)) + \lambda\int_0^L u^2 dx - 3\lambda\int_0^L u_x^2 dx \\
& - \int_0^L x sat(au)u dx - \gamma(1-d)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu + \gamma u_x^2(0,t) \\
= & Y^T\left[\frac{1}{2}M_1 + M_2\right] Y + \lambda\int_0^L u^2 dx - 3\lambda\int_0^L u_x^2 dx \\
& - \int_0^L x sat(au)u dx - \gamma(1-d)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu,
\end{aligned}$$

where $Y = \begin{pmatrix} u_x(0,t) \\ u_x(0,t-\theta(t)) \end{pmatrix}$ and $M_2 = \begin{pmatrix} L\lambda\alpha^2+\gamma & L\lambda\alpha\beta \\ L\lambda\alpha\beta & L\lambda\beta^2 \end{pmatrix}$ and the matrix M_1 is given by (2.5).

Since $x \in [0, L]$ and $sat(au)u \geq 0$, we deduce that $\int_0^L xsat(au)udx \geq 0$. Consequently we deduce that

$$\begin{aligned}\dot{V}(t) \leq & Y^T \left[\frac{1}{2}M_1 + M_2\right] Y + \lambda \int_0^L u^2 dx - 3\lambda \int_0^L u_x^2 dx \\ & -\gamma(1-d) \int_0^1 u_x^2(0, t-\theta(t)\mu) d\mu.\end{aligned} \tag{4.19}$$

Now, we calculate $2\delta V(t)$, using (2.3), we have

$$\begin{aligned}2\delta V(t) =& 2\delta E(t) + 2\delta\lambda V_1(t) + 2\delta\gamma V_2(t) \\ =& \delta \int_0^L u^2 dx + \delta|\beta|\theta(t) \int_0^1 u_x^2(0, t-\theta(t)\mu) d\mu + 2\delta\lambda \int_0^L xu^2 dx \\ & + 2\delta\gamma\theta(t) \int_0^1 u_x^2(0, t-\theta(t)\mu) d\mu - 2\delta\gamma\theta(t) \int_0^1 \mu u_x^2(0, t-\theta(t)\mu) d\mu \\ \leq & \delta \int_0^L u^2 dx + \delta|\beta| K \int_0^1 u_x^2(0, t-\theta(t)\mu) d\mu \\ & + 2\delta\lambda L \int_0^L u^2 dx + 2\delta\gamma K \int_0^1 u_x^2(0, t-\theta(t)\mu) d\mu\end{aligned} \tag{4.20}$$

According to [25, Theorem 3.2], for λ and γ small enough, the matrix $\frac{1}{2}M_1 + M_2$ is definite negative, and from (4.19) and (4.20) we deduce that

$$\begin{aligned}\dot{V}(t) + 2\delta V(t) \leq & Y^T \left[\frac{1}{2}M_1 + M_2\right] Y + (\lambda + \delta + 2L\lambda\delta) \int_0^L u^2 dx - 3\lambda \int_0^L u_x^2 dx \\ & + (\delta|\beta|K + 2\gamma\delta K - \gamma(1-d)) \int_0^1 u_x^2(0, t-\theta(t)\mu) d\mu \\ \leq & (\lambda + \delta + 2L\lambda\delta) \int_0^L u^2 dx - 3\lambda \int_0^L u_x^2 dx \\ & + (\delta|\beta|K + 2\gamma\delta K - \gamma(1-d)) \int_0^1 u_x^2(0, t-\theta(t)\mu) d\mu\end{aligned} \tag{4.21}$$

By using the Poincaré inequality, we get

$$\begin{aligned}\dot{V}(t) + 2\delta V(t) \leq & \left(\frac{L^2}{\pi^2}(\lambda + \delta + 2L\lambda\delta) - 3\lambda\right) \int_0^L u_x^2 dx \\ & + (\delta|\beta|K + 2\gamma\delta K - \gamma(1-d)) \int_0^1 u_x^2(0, t-\theta(t)\mu) d\mu.\end{aligned} \tag{4.22}$$

□

By assumption $L < \pi\sqrt{3}$, then from [2], it is possible to choose r small enough to have $r < \dfrac{3(\pi^2 - L^2)}{2L^{\frac{3}{2}}\pi^2}$. Consequently, we can choose $\delta > 0$ such that (4.15) holds in order to obtain that

$$\frac{L^2}{\pi^2}(\lambda + \delta + 2L\lambda\delta) - 3\lambda \leq 0,$$

and

$$\delta|\beta|K + 2\gamma\delta K - \gamma(1-d) \leq 0,$$

therefore

$$\dot{V}(t) + 2\delta V(t) \leq 0 \quad \forall t \geq 0.$$

Hence, we deduce that

$$V(t) \leq Ce^{-2\delta t}V(0) \quad \forall t \geq 0.$$

By (4.18), we get

$$E(t) \leq Ce^{-2\delta t}E(0) \quad \forall t \geq 0.$$

Using the density of $D(A(0))$, we conclude the proof by extending the result to any initial condition within H.

5. Conclusion

In this work, we investigated the linear Korteweg-de Vries equation with a time-varying delay on the boundary feedback in the presence of a saturated source term. This study has illustrated that the incorporation of a time-varying delay in the Korteweg-de Vries equation, along with a saturated source term, leads to a well-posed system under some conditions. Using a suitable Lyapunov functional, we prove that the system (2.1) is locally exponentially stable. An inserting topic for further research is the analysis of exponential stability of the non-linear KdV equation with time-variyng delay in presence of non-linear source term.

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