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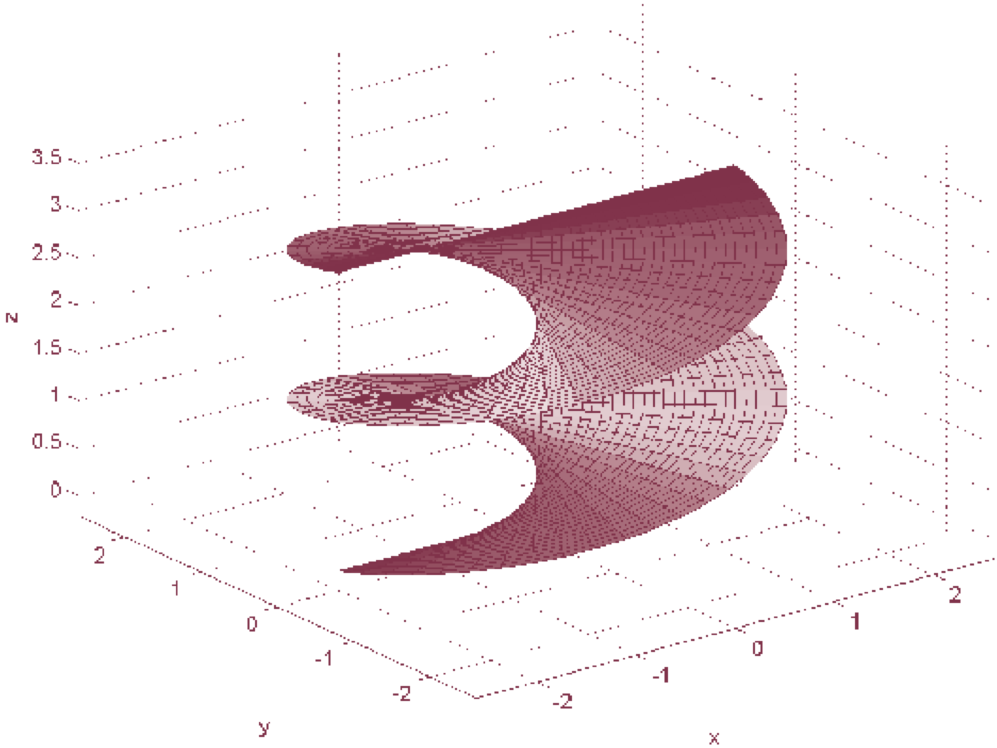

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CONTENTS

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On the class of analytic functions defined by Robertson associated with nephroid domain

Kavitha Sivasubramanian

Abstract. The primary focus of this article is to explore a novel subclass, denoted as $\mathscr{G}_{\mathcal{N}}$, of analytic functions. These functions exhibit starlike properties concerning a boundary point within a nephroid domain. The author obtains representation theorems, establishes growth and distortion theorems, and investigates various implications related to differential subordination. In addition to the investigation of coefficient estimates, the study also explores specific consequences of differential subordination.

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1. Introduction

Let $\mathscr{H}$ be the class of all holomorphic functions in the open unit disc $\mathscr{D} := \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathscr{A}$ represent the subclass of $\mathscr{H}$ entailing of functions h with the normalization $h(0) = h'(0) - 1 = 0$. Hence, the class of all functions $h \in \mathscr{A}$ will be of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathscr{D}.$$

By $\mathscr{S}$, we mean the subclass of $\mathscr{A}$ comprising of univalent functions. A function $f \in \mathscr{H}$ is subordinate to another function $g \in \mathscr{H}$ written as $f \prec g$ if there exists a function $\omega \in \mathscr{H}$ satisfying $\omega(0) = 0$, $\omega(\mathscr{D}) \subset \mathscr{D}$ and such that $f(z) = g(\omega(z))$ for every $z \in \mathscr{D}$. In precise, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathscr{D}) \subset g(\mathscr{D})$.

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Two well-established subclasses of $\mathscr{A}$ are the starlike functions and convex functions of order γ $(0 \leq \gamma < 1)$, which were introduced by Robertson [20]. These classes are defined analytically as follows:

The starlike functions of order γ, denoted as $\mathscr{S}^*(\gamma)$, consist of functions in $\mathscr{A}$ for which $\Re\left(\dfrac{zh'(z)}{h(z)}\right) > \gamma$ for all $z \in \mathscr{D}$. The convex functions of order γ, denoted as $\mathscr{C}(\gamma)$, comprise functions in $\mathscr{A}$ satisfying $\Re\left(1 + \dfrac{zh''(z)}{h'(z)}\right) > \gamma$ for all $z \in \mathscr{D}$.

It is well-known that $\mathscr{S}^*(\gamma) \subset \mathscr{S}$ and $\mathscr{C}(\gamma) \subset \mathscr{S}$. Additionally, based on Alexander's relation, if $h \in \mathscr{C}(\gamma)$, then $zh'(z)$ belongs to $\mathscr{S}^*(\gamma)$ for each $0 \leq \gamma < 1$. For $\gamma = 0$, $\mathscr{S}^*$ corresponds to the normalized starlike univalent functions, and $\mathscr{C}$ represents the normalized convex univalent functions.

A function $h \in \mathscr{H}$ is said to be close-to-convex if and only if there exists a function $\psi \in \mathscr{C}$ and $\beta \in (-\pi/2, \pi/2)$ such that

$$\Re\left(\frac{e^{i\beta} h'(z)}{\psi'(z)}\right) > 0, \quad z \in \mathscr{D}.$$

The class of close-to-convex functions was defined in [11]. Further, it is also known that the class of close-to-convex functions generally are normalized.Let $\mathscr{P}$ denote the class of functions p holomorphic in $\mathscr{D}$, satisfying $p(0) = 1$ and $\Re\,(p(z)) > 0$ for $z \in \mathscr{D}$. This class is referred to as the class of functions with a positive real part, commonly known as Class of Caratheodory functions.

A significant development emerged in [21], where a novel class $\mathscr{G}$ of functions $G(z)$ was introduced. These functions are analytic within $|z| < 1$, normalized such that $G(0) = 1$, $G(1) = \lim_{r \to 1^-} G(r) = 0$, and they satisfy the condition that $\Re(e^{i\delta} G(z)) > 0$ for $z \in \mathscr{D}$. Additionally, $G(z)$ maps $\mathscr{D}$ univalently to a domain that is starlike with respect to $G(1)$. Notably, the constant function 1 is included in the class $\mathscr{G}$.

A significant conjecture proposed by Robertson [21] is that the class $\mathscr{G}$ of functions f of the form:

$$f(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \tag{1.1}$$

holomorphic and nonvanishing in $\mathscr{D}$ and such that

$$\Re\left\{\frac{2zf'(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0, z \in \mathscr{D} \tag{1.2}$$

coincides with $\mathscr{G}^*$. The above hypothesis was confirmed by Lyzzaik [17] in 1984. Robertson [21] also proved that if the function $f \in \mathscr{G}$ and $g \neq 1$ then f is close-to-convex and univalent in $\mathscr{D}$. It is worth to be mentioned here that the analytic condition (1.2) was known to Styer [24] earlier. In [10], a class closely related to $\mathscr{G}$ denoted by $\mathscr{G}(M)$, $M > 1$, of functions g of the form (1.1) holomorphic and nonvanishing in $\mathscr{D}$ was introduced and such that

$$\Re\left\{\frac{2zf'(z)}{g(z)} + z\frac{P_M'(z)}{P_M(z)}\right\} > 0, z \in \mathscr{D},$$

where $P_M(z)$ denotes the Pick function. The class

$$\mathscr{G}(1) = \left\{ f \text{ of the form (1.1)} : f(z) \neq 0 \text{ and } \Re\left\{2z\frac{f'(z)}{f(z)} + 1\right\} > 0, z \in \mathscr{D}\right\}$$

was also considered in [10]. In an another investigation, Obradovic and Owa [19] investigated the class $\mathscr{G}(\gamma)$, $0 \leq \gamma < 1$, of functions g of the form (1.1) holomorphic in the disc $\mathscr{D}$, $g(z) \neq 0$ for $z \in \mathscr{D}$ and satisfying the condition

$$\Re\left\{\frac{zf'(z)}{f(z)} + (1-\gamma)\frac{1+z}{1-z}\right\} > 0, \ z \in \mathscr{D}.$$

Todorov [25] established a connection between the class $\mathscr{G}$ and a functional expression $\dfrac{f(z)}{1-z}$, resulting in a well-structured formula and precise coefficient estimates. Silverman and Silvia [22] offered a comprehensive exploration of the class of univalent functions within the domain $\mathscr{D}$ whose images take on a star-like configuration concerning a boundary point. Subsequently, this category of functions exhibiting star-like behavior with respect to boundary points has garnered significant interest among geometric function theorists and researchers from diverse backgrounds. Among the works in this direction, Abdullah et al. [1] obtained certain properties of functions belonging to $\mathscr{G}$ and derived a set of inequalities pertaining to functional coefficients. Distortion results associated with star-like functions concerning boundary points were further examined, and were presented in both [3] and [6]. Moreover, dynamic characterizations of functions demonstrating star-like characterizations concerning boundary points can be found in [8].

Lecko [13] introduced an alternative representation of functions manifesting star-like qualities concerning a boundary point. Additionally, Lecko and Lyzzaik [14] contributed diverse characterizations of the class $\mathscr{G}$. Furthermore, Aharonov et al. [2] provided a definition for spiral-shaped domains concerning a boundary point, outlined as follows:

Let $\mathscr{G}_\mu$ denote the class of functions $f \in H(\mathscr{D})$, and non-vanishing in $\mathscr{D}$ with $f(0) = 1$, and for $\mu \in \mathbb{C}$, $\left|\frac{\mu}{\pi} - 1\right| \leq 1$ satisfying

$$\Re\left\{\frac{2\pi zf'(z)}{\mu f(z)} + \frac{1+z}{1-z}\right\} > 0, z \in \mathscr{D}.$$

In the work by Elin [8], a set of fundamental properties and several equivalent descriptions of the class $\mathscr{G}\mu$ are formulated (also see [7]). When μ is chosen to be π, the class $\mathscr{G}\mu$ aligns with the class initially introduced by Robertson [21], who sparked interest in this class and related categories. It's worth noting that functions within $\mathscr{G}_\mu$ are either close-to-convex or simply the constant function 1.

In recent times, the investigation of star-like functions concerning boundary points has attracted attention from researchers such as Cho et al. [4], Dhurai et al. [5], Lecko et al. [15, 16, 9], and Sivasubramanian [23] (also see [12]). The purpose of this paper is to introduce and investigate a new class of the aforesaid type involving nephroid domains with respect to a boundary point.

Definition 1.1. Let $\mathscr{G}_{\mathscr{N}}$, denote the class of functions f of the form (1.1) holomorphic and nonvanishing in disc $\mathscr{D}$ and such that

$$\Re\left\{\frac{2zf'(z)}{f(z)}+1+z-\frac{z^3}{3}\right\}>0, z\in\mathscr{D}, \tag{1.3}$$

which can be rewritten as

$$\Re\left\{\frac{2zf'(z)}{f(z)}+Q_{\mathscr{N}}(z)\right\}>0, z\in\mathscr{D}, \tag{1.4}$$

where

$$Q_{\mathscr{N}}(z)=1+z-\frac{z^3}{3}, z\in\mathscr{D}. \tag{1.5}$$

It is to be observed that the function $Q_{\mathscr{N}}$ of the form (1.5) maps $\mathscr{D}$ onto the region bounded by the nephroid $\left((u-1)^2+v^2-\frac{4}{9}\right)^3-\frac{4v^2}{3}=0$ which is symmetric about the real axis and lies completely inside the right-half plane $u>0$. The nephroid domain was introduced and studied by Wani and Swaminathan [26]. Let us first construct few examples for the new class of functions to show that the class is non empty.

Examples. The functions

$$f_0(z)=\exp\left(\frac{1}{2}\left(-z+\frac{z^3}{9}\right)\right),\ z\in\mathscr{D}, \tag{1.6}$$

and

$$f_1(z)=\frac{\exp\left(\frac{1}{2}\left(-z+\frac{z^3}{9}\right)\right)}{1-z},\ z\in\mathscr{D} \tag{1.7}$$

belong to the class $\mathscr{G}_{\mathscr{N}}$.

To see this, one may compute

$$\Re\left\{\frac{2zf_0'(z)}{f_0(z)}+1+z-\frac{z^3}{3}\right\}=1>0, z\in\mathscr{D},$$

and

$$\Re\left\{\frac{2zf_1'(z)}{f_1(z)}+1+z-\frac{z^3}{3}\right\}=1>0, z\in\mathscr{D},$$

A straight forward computations will show that both the functions $f_0(z)$ and $f_1(z)$ belong to the class $\mathscr{G}_{\mathscr{N}}$. However, it is of interest to observe that although the functions f_0 and f_1 belong to $\mathcal{G}_N$, the functions f_0 and f_1 need not be necessarily univalent and hence $\mathcal{G}_N \not\subseteq \mathcal{G}$.

2. Main results

We start this section with the following representation theorem

Theorem 2.1. *Let f be a holomorphic function in $\mathscr{D}$ such that $f(0) = 1$. Then $f \in \mathscr{G}_{\mathscr{N}}$ if and only if there exists a function $h \in \mathscr{S}^*$ such that*

$$f(z) = \sqrt{\frac{h(z)}{z}} \exp\left(\frac{1}{2}\left(-z + \frac{z^3}{9}\right)\right), z \in \mathscr{D}, \tag{2.1}$$

Proof. Let F be a function satisfying the relation

$$\frac{zF'(z)}{F(z)} = 1 + z - \frac{z^3}{3}, z \in \mathscr{D}. \tag{2.2}$$

Then $F \in \mathscr{S}^*_{\mathscr{N}}$, where

$$\mathscr{S}^*_{\mathscr{N}} = \left\{F : F(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathscr{D} \text{ and } \frac{zF'(z)}{F(z)} \prec Q_{\mathscr{N}}(z)\right\}.$$

From (2.2), one may easily see after a simple calculation that

$$F(z) = z \cdot \exp\left(\int_0^z \frac{Q_{\mathscr{N}}(\zeta) - 1}{\zeta} d\zeta\right), z \in \mathscr{D},. \tag{2.3}$$

and therefore

$$F(z) = z \cdot \exp\left(z - \frac{z^3}{9}\right), z \in \mathscr{D}. \tag{2.4}$$

From (1.4) and (1.5), it can be easily seen that for some function $f \in \mathscr{G}_{\mathscr{N}}$ there exists a starlike function h of the class $\mathscr{S}^*$ such that $(f(z))^2 F(z) = h(z), z \in \mathscr{D}$ and conversely. □

Remark 2.2. Let us consider the function f_3, $f_3(0) = 1$, satisfying the equation

$$\frac{2zf_3'(z)}{f_3(z)} + 1 + z - \frac{z^3}{3} = \frac{1 + z^2}{1 - z^2}, z \in \mathscr{D}.$$

In view of (1.3) and (1.5) it is obvious that $f_3 \in \mathscr{G}_{\mathscr{N}}$.

Theorem 2.3. *Let f be a holomorphic function in $\mathscr{D}$ such that $f(0) = 1$. Then $f \in \mathscr{G}_{\mathscr{N}}$ if and only if there exists a function $h \in \mathscr{S}^*(1/2)$ such that*

$$f(z) = \frac{h(z)}{z} \exp\left(\frac{1}{2}\left(-z + \frac{z^3}{9}\right)\right), z \in \mathscr{D}. \tag{2.5}$$

Proof. It is a known that,

$$h \in \mathscr{S}^*(1/2) \Leftrightarrow h = \frac{f^2}{I}, I(z) \equiv z.$$

An application of (2.1) essentially completes the proof of Theorem 2.3. □

Remark 2.4. It follows immediately from the Herglotz representation that for $\mathscr{S}^*\left(\frac{1}{2}\right)$ that $g \in \mathscr{G}_{\mathscr{N}}$ if and only if

$$f(z) = \exp\left(-z + \frac{z^3}{6}\right)\left(\frac{\mu}{\pi}\int_{-\pi}^{\pi} \log\left(\frac{1}{1-ze^{-it}}\right) d\mu(t)\right),\ z \in \mathscr{D}. \tag{2.6}$$

where $\mu(t)$ is a probability measure on $[-\pi, \pi]$.

From Theorem 2.1 and from the known estimates of the respective functionals in the class $\mathscr{S}^*$ we have the following theorem.

Theorem 2.5. *If $f \in \mathscr{G}_{\mathscr{N}}$, then the following sharp estimate*

$$\frac{1}{1+|z|}\exp\left(-\frac{1}{2}\Re\left(z - \frac{z^3}{9}\right)\right) \leq |f(z)| \leq \frac{1}{1-|z|}\exp\left(-\frac{1}{2}\Re\left(z - \frac{z^3}{9}\right)\right) \tag{2.7}$$

hold. The extremal function for the upper estimate (2.7) is the function f_ε^* of the form

$$f_\varepsilon^*(z) = \exp\left(-\frac{1}{2}\Re\left(z - \frac{z^3}{9}\right)\right)\sqrt{\frac{k_\varepsilon(z)}{z}},$$

where $\varepsilon = e^{-i\varphi}$, while for the lower estimate is the function g_ε^* for $\varepsilon = -e^{-i\varphi}$ with

$$k(z) = \frac{z}{(1-z)^2}.$$

Theorem 2.6. *If $f(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \in \mathscr{G}_{\mathscr{N}}$, then the coefficients d_n satisfy the following sharp coefficient inequalities*

$$|d_1| \leq \frac{3}{2} \tag{2.8}$$

$$|2d_1 + 1| \leq 2 \tag{2.9}$$

$$\left|2d_2 - d_1^2\right| \leq 1 \tag{2.10}$$

$$\left|6(3d_3 - 3d_1d_2 + d_1^3) - 1\right| \leq 6 \tag{2.11}$$

$$\left|4d_2 - 2d_1^2(1+2\gamma) - 4\gamma d_1 - \gamma\right| \leq \begin{cases} 2 - \gamma\left|2d_1+1\right|^2 & \left(\gamma \leq \frac{1}{2}\right) \\ 2 - (1-\gamma)\left|2d_1+1\right|^2 & \left(\gamma \geq \frac{1}{2}\right). \end{cases} \tag{2.12}$$

and finally

$$\left|6\left(3d_3 - 7d_1d_2 - 2d_2 + 3d_1^3 + d_1^2\right) - 1\right| \leq 6. \tag{2.13}$$

Proof. Let $d_0 = 1$ and $p(z) = \dfrac{2zg'(z)}{g(z)} + 1 + z - \dfrac{z^3}{3}$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots.$$

On expanding the right hand side of the above function and comparing with $p(z)$ we get,

$$1+p_1z+p_2z^2+\cdots = 1+(2d_1+1)\,z+\left(2(2d_2-d_1^2)\right)z^2+\left(2(3d_3-3d_1d_2+d_1^3)-\frac{1}{3}\right)z^3+\cdots.$$

Hence,

$$2d_1 + 1 = p_1$$
$$2(2d_2 - d_1^2) = p_2$$

$$2(3d_3 - 3d_1d_2 + d_1^3) - \frac{1}{3} = p_3$$

It is a known fact that if $p(z) = 1+p_1z+p_2z^2+\cdots \in \mathscr{P}$, then $|p_i| \le 2, i = 1, 2\cdots$. By virtue of this inequality one may easily get (2.8),(2.9), (2.10) and (2.11). Inequality (2.12) follows from the fact that

$$|p_2 - \gamma p_1^2| \le \begin{cases} 2 - \gamma|p_1|^2 & \left(\gamma \le \frac{1}{2}\right) \\ 2 - (1-\gamma)|p_1|^2 & \left(\gamma \ge \frac{1}{2}\right). \end{cases} \tag{2.14}$$

By applying a less unknown familiar inequality $|p_3 - p_1p_2| \le 2$ and performing a computation yields the inequality (2.13). □

It is known that for each function $h \in \mathscr{S}^*$ the functions

$$z \to \frac{1}{\rho}h(\rho z), z \to e^{i\varphi}h(e^{-i\varphi}z), 0 < \rho < 1, \varphi \in \mathbb{R}, z \in \mathscr{D}, \tag{2.15}$$

also belong to $\mathscr{S}^*$. From Theorem 2.1 and estimation (2.9) we obtain:

Theorem 2.7. *The region of values of the coefficient d_1, i.e. $\{d_1 : g \in \mathscr{G}_{\mathscr{N}}, g(z) = 1 + d_1z + \cdots\}$ has the form*

$$\left\{w \in \mathbb{C} : \left|w + \frac{1}{2}\right| \le 1\right\}.$$

In this section, we establish specific differential subordination findings related to the class $\mathscr{G}_N$.

To prove differential subordination results, we recall the following lemma (see [18, Theorem 3.4h, p. 132]).

Lemma 2.8. *Let q be univalent in $\mathscr{D}$, θ and φ be holomorphic in a domain D containing $q(\mathscr{D})$ with $\varphi(w) \neq 0$ when $w \in q(\mathscr{D})$. Let $Q(z) := zq'(z)\varphi(q(z))$ and $h(z) := \theta(q(z)) + Q(z)$ for $z \in \mathscr{D}$. Suppose that either*

(i) *Q is starlike univalent in $\mathscr{D}$, or*
(ii) *h is convex univalent in $\mathscr{D}$.*

Assume also that

(iii)

$$\Re\frac{zh'(z)}{Q(z)} > 0, \quad z \in \mathscr{D}.$$

If $p \in \mathscr{H}$ with $p(0) = q(0)$, $p(\mathscr{D}) \subset D$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \quad z \in \mathscr{D},$$

then $p \prec q$ and q is the best dominant.

Theorem 2.9. *Let $f \in \mathscr{H}$ and $f(0) = 1$. If f satisfies the subordination condition*

$$\frac{2zf'(z)}{f(z)} + 1 + z - \frac{z^3}{3} \prec \frac{1+z}{1-z}, \quad z \in \mathscr{D}, \tag{2.16}$$

then

$$\frac{(f(z))^2}{\exp\left(-z+\frac{z^3}{9}\right)} \prec \frac{1}{(1-z)^2}, \quad z \in \mathscr{D}. \tag{2.17}$$

That is, if $f \in \mathscr{G}_{\mathscr{N}}$, *then* $\dfrac{(f(z))^2}{\exp\left(-z+\frac{z^3}{9}\right)} \prec \dfrac{1}{(1-z)^2}, \quad z \in \mathscr{D}.$

Proof. Let us define a function $p(z) = \dfrac{(f(z))^2}{\exp\left(-z+\frac{z^3}{9}\right)}$. Let $q(z) = \dfrac{1}{(1-z)^2}, z \in \mathscr{D}$. Subsequently, one can readily notice that $p(0) = q(0) = 1$, $p(z) \neq 0$ for $z \in \mathscr{D}$, and p is holomorphic. Further,

$$1 + \frac{zp'(z)}{p(z)} = \frac{2zf'(z)}{f(z)} + 1 + z - \frac{z^3}{3}, \quad z \in \mathscr{D}.$$

Let $f \in \mathscr{H}$ with $f(0) = 1$ and $f(z)$ be nonzero for $z \in \mathscr{D}$ satisfying (2.16). Let a function p be defined as in (2.17). Let $D := \mathbb{C} \setminus \{0\}$. Let $\theta(w) := 1,\ w \in \mathbb{C}$, and $\varphi(w) := 1/w,\ w \in D$. Note that $q(\mathscr{D}) \subset D$ and θ and φ are holomorphic in D. Thus

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{zq'(z)}{q(z)} = \frac{2z}{1-z}, \quad z \in \mathscr{D}, \tag{2.18}$$

is well defined and holomorphic. Clearly, Q is a univalent. Additionally, a straightforward calculation will demonstrate that Q is a starlike function as well. Hence for a function $h(z) := \theta(q(z)) + Q(z) = \dfrac{1+z}{1-z}$, $z \in \mathscr{D}$, we obtain

$$\Re\frac{zh'(z)}{Q(z)} = \Re\frac{zQ'(z)}{Q(z)} = \frac{1}{1-z} > 0, \quad z \in \mathscr{D}.$$

Therefore, for any given function p belonging to $\mathscr{H}$ with $p(0) = q(0) = 1$ such that $p(\mathscr{D}) \subset D$, i.e., for p non-vanishing in $\mathscr{D}$, by applying Lemma 2.8 we infer that from the subordination

$$1 + \frac{zp'(z)}{p(z)} \prec 1 + \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z}, \quad z \in \mathscr{D}, \tag{2.19}$$

implies the subordination $p \prec \dfrac{1}{(1-z)^2}$ is true.

□

Theorem 2.10. *Let* $f \in \mathscr{H}$ *with* $f(0) = 1$. *If* f *satisfies*

$$\frac{2zf'(z)}{f(z)} + 1 + z - \frac{z^3}{3} \prec \frac{1+z}{1-z}, \quad z \in \mathscr{D}, \tag{2.20}$$

then

$$p(z) := z\left(\frac{f(z)}{1-z}\right)^2 \left(\int_0^z \left(\frac{f(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \prec \frac{1}{(1-z)^2}, \quad z \in \mathscr{D}. \tag{2.21}$$

That is $f \in \mathscr{G}_N$ *then* $z\left(\dfrac{f(z)}{1-z}\right)^2 \left(\displaystyle\int_0^z \left(\frac{f(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \prec \dfrac{1}{(1-z)^2}.$

Proof. Let $D := \mathbb{C} \setminus \{0\}$. Let $\phi(w) := w,\ w \in \mathbb{C}$, and $\psi(w) := 1/w,\ w \in D$. Note that $q(\mathscr{D}) \subset D$ and ϕ and ψ are holomorphic in D. Thus the function Q defined by (2.18), i.e., the identity function, is univalent starlike. Hence for a function

$$h(z) := \theta(q(z)) + Q(z) = q(z) + Q(z),\ z \in \mathscr{D},$$

we obtain

$$\Re\frac{zh'(z)}{Q(z)} = \Re\frac{zq'(z)}{Q(z)} + \Re\frac{zQ'(z)}{Q(z)} = \Re q(z) + \Re\frac{zQ'(z)}{Q(z)} > 0, \quad z \in \mathscr{D}.$$

Thus for any function $p \in \mathscr{H}$ with $p(0) = q(0) = 1$ such that $p(\mathscr{D}) \subset D$, i.e., $p(z) \neq 0$ for $z \in \mathscr{D}$, by applying Lemma 2.8 we can conclude that from the subordination

$$p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z}, \quad z \in \mathscr{D}, \tag{2.22}$$

it follows the subordination $p \prec \dfrac{1}{(1-z)^2}$.

Let now take $f \in \mathscr{H}$ with $f(0) = 1$ and $f(z) \neq 0$ for $z \in \mathscr{D}$ satisfying (2.16). Define a function p as in (2.21). We see that

$$\begin{aligned} p(0) &= \lim_{z\to 0} z \left(\frac{f(z)}{1-z}\right)^2 \left(\int_0^z \left(\frac{f(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \\ &= (f(0))^2 \lim_{z\to 0} z \left(\int_0^z \left(\frac{f(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} = 1 = q(0), \end{aligned}$$

$p(z) \neq 0$ for $z \in \mathscr{D}$ and p is holomorphic. Since

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{2zf'(z)}{f(z)} + \frac{1+z}{1-z}, \quad z \in \mathscr{D},$$

from (2.22), (2.20) follows which completes the proof. □

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Ma-Minda starlikeness of certain analytic functions

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Abstract. A normalized analytic function defined on the open unit disc $\mathbb{D}$ is called Ma-Minda starlike if $zf'(z)/f(z)$ is subordinate to the function φ. For a normalized convex function f defined on $\mathbb{D}$ and $\alpha > 0$, we determine the radius of Ma-Minda starlikeness of the function g defined as $g(z) = (zf'(z)/f(z))^{\alpha} f(z)$ for certain choices of φ. In particular, we investigate the radius of Janowski starlikeness of the function g.

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1. Introduction and preliminaries

Let $\mathbb{C}$ denote the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ represent the open unit disc, and $\mathcal{A}$ denote the class of analytic functions defined on $\mathbb{D}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Additionally, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent (one-to-one) functions. A function $f \in \mathcal{A}$ is considered starlike if it maps $\mathbb{D}$ onto a domain that is starlike with respect to the origin. Similarly, a function $f \in \mathcal{A}$ is said to be convex if $f(\mathbb{D})$ is a convex set. Let $\mathcal{ST}$ and $\mathcal{CV}$ denote the subclasses of $\mathcal{A}$ respectively consisting of starlike and convex functions. Analytically, we have: $\mathcal{ST} := \{f \in \mathcal{A} : \operatorname{Re}(zf'(z)/f(z)) > 0\}$ and $\mathcal{CV} := \{f \in \mathcal{A} : 1+\operatorname{Re}(zf''(z)/f'(z)) > 0\}$. Alexander's theorem [4] establishes a relationship between these two classes, stating that $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{ST}$. For two analytic functions f and g, we say that f is subordinate to g, written $f \prec g$, if there exists an analytic function w satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. This relationship

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implies that $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Moreover, if the function $g(z)$ is univalent (one-to-one), then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. The function w is commonly known as the Schwarz function. Using subordination, Ma and Minda [12] investigated growth, distortion and covering theorems for the class $\mathcal{ST}(\varphi)$ consisting of starlike functions that satisfy the subordination

$$\frac{zf'(z)}{f(z)} \prec \varphi(z),$$

where $\varphi : \mathbb{D} \to \mathbb{C}$ is an analytic function that is univalent with a positive real part, $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0) = 1$, symmetric about the real axis, and $\varphi'(0) > 0$. Different subclasses of starlike and convex functions are obtained for various choices of φ. For instance, when $\varphi(z) = (1 + Az)/(1 + Bz)$, where $-1 \leqslant B < A \leqslant 1$, the class $\mathcal{ST}(\varphi)$ is the class $\mathcal{ST}[A, B]$ of Janowski starlike functions [8]. An analytic function $p : \mathbb{D} \to \mathbb{C}$ is known as a Carathéodory function if $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ for every $z \in \mathbb{D}$. The class of all Carathéodory functions is denoted as $\mathcal{P}$. For $-1 \leqslant B < A \leqslant 1$ and $p(z) = 1 + c_1 z + \cdots$ with positive real part, we say that $p \in \mathcal{P}[A, B]$ if $p(z) \prec (1 + Az)/(1 + Bz), z \in \mathbb{D}$.

Lemma 1.1. [16] *If* $p \in \mathcal{P}[A, B]$, *then*

$$\left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leqslant \frac{(A - B)r}{1 - B^2r^2} \quad (|z| \leqslant r < 1).$$

The class of functions $f \in \mathcal{A}$ with the property that $zf'(z)/f(z)/ \in \mathcal{P}[A, B]$ is denoted by $\mathcal{ST}[A, B]$. In this manuscript, we are interested in the class $\mathcal{J}_1^\alpha$ defined as follows:

$$\mathcal{J}_1^\alpha := \left\{ g \in \mathcal{A} : g(z) = \left(\frac{zf'(z)}{f(z)}\right)^\alpha f(z), \quad f \in \mathcal{CV}, \alpha > 0 \right\}.$$

We determine $\mathcal{ST}(\varphi)$ radius of the class $\mathcal{J}_1^\alpha$ for various choices of φ. In particular, we consider the following classes of starlike functions:

1. Mendiratta et al. [14] introduced the class consisting of all functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec e^z$ or equivalently $|\log(zf'(z)/f(z))| < 1$.
2. Sharma et al. [18] studied the class $\mathcal{ST}_C = \mathcal{ST}(\varphi_C)$, where $\varphi_C(z) = 1 + (4/3)z + (2/3)z^2$. The boundary of $\varphi_C(\mathbb{D})$ is a cardiod.
3. Raina and Sokól [15] considered the class $\mathcal{ST}_m = \mathcal{ST}(\varphi_m)$, where $\varphi_m(z) = z + \sqrt{1 + z^2}$ and proved that $f \in \mathcal{ST}_m$ if and only if $zf'(z)/f(z) \in \Omega_m := \{w \in \mathbb{C} : |w^2 - 1| < 2|w|\}$ which is the interior of a lune.
4. Kumar and Kamaljeet [20] introduced the class $\mathcal{ST}_\wp = \mathcal{ST}(\varphi_\wp)$, where $\varphi_\wp(z) = 1 + ze^z$. The boundary of $\varphi_\wp(\mathbb{D})$ is a cardiod.
5. The class of starlike functions associated with a nephroid domain, given by $\mathcal{ST}_{Ne} = \mathcal{ST}(\varphi_{Ne})$ where $\varphi_{Ne}(z) = 1 + z - (z^3/3)$ was studied by Wani and Swaminathan [22]. The function φ_{Ne} maps the unit circle onto a 2-cusped curve, $\left((u - 1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4v^2}{3} = 0$.
6. The class $\mathcal{ST}_{SG} = \mathcal{ST}(\varphi_{SG})$ where $\varphi_{SG}(z) = 2/(1 + e^{-z})$ was introduced by Goel and Kumar [7]. The boundary of $\varphi_{SG}(\mathbb{D})$ is a modified sigmoid.
7. Cho et al. [3] introduced the class $\mathcal{ST}_{\sin} = \mathcal{ST}(\varphi_{\sin})$, where $\varphi_{\sin}(z) = 1 + \sin z$.

8. Kumar and Arora [2] defined the class $\mathcal{ST}_h = \mathcal{ST}(\varphi_h)$ where $\varphi_h(z) = 1 + \sinh^{-1}(z)$. The boundary of $\varphi_h(\mathbb{D})$ is petal shaped.

These functions behave like the identity function for small values of α and hence belong to the classes of our interest. However, for $B = -1$, the range of $zg'(z)/g(z)$ is unbounded, and therefore these classes are not contained in various subclasses obtained for special choices of the function φ. When the inclusion fails, we are interested in the corresponding radius problem. For two subclasses $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$, the largest number $\mathcal{R} \in (0,1]$ such that for $0 < r < \mathcal{R}$, $f(rz)/r \in \mathcal{F}$ for every $f \in \mathcal{G}$ is called the $\mathcal{F}$-radius of the class $\mathcal{G}$ and is denoted by $\mathcal{R}_{\mathcal{F}}(\mathcal{G})$. Many radius problems have been extensively explored in recent times [1, 9, 10, 11, 13, 17]. In Theorem 2.1, we obtain the Janowski starlikeness of the class $\mathcal{J}_1^\alpha$ and, in particular, the radius of starlikeness of order β. Theorem 2.2 gives $\mathcal{ST}(\varphi)$ radius of the class $\mathcal{J}_1^\alpha$ for various choices of φ discussed above. To obtain the radii, we find the largest positive number $\mathcal{R}$ less than 1 such that the image of the disc $\mathbb{D}_{\mathcal{R}} := \{z \in \mathbb{C} : |z| < \mathcal{R}\}$ under the mapping $zg'(z)/g(z)$, for g in the classes defined, lie inside the image of the corresponding superordinate functions and the radii obtained are sharp.

2. Radius estimates of various starlikeness for the class $\mathcal{J}_1^\alpha$

Our first theorem gives the radius of Janowski starlikeness of functions in the class $\mathcal{J}_1^\alpha$ and, in particular, the radius of starlikeness of order β (see (2.3)). It follows that the class $\mathcal{J}_1^\alpha$ is a subclass of starlike functions.

Theorem 2.1. *The $\mathcal{ST}[A,B]$ radius of the class $\mathcal{J}_1^\alpha$, $\alpha > 0$, is given by*

$$\mathcal{R}_{\mathcal{ST}[A,B]} = \frac{A-B}{1+\alpha+|A+\alpha B|}.$$

Proof. Let $g \in \mathcal{J}_1^\alpha$. Then there is a function $f \in \mathcal{CV}$ satisfying

$$g(z) = \Big(\frac{zf'(z)}{f(z)}\Big)^\alpha f(z).$$

A computation shows that

$$\frac{zg'(z)}{g(z)} = \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\left(\frac{zf'(z)}{f(z)}\right). \tag{2.1}$$

Since f is convex, it is starlike of order 1/2 and therefore we have $1 + zf''(z)/f'(z) \in \mathcal{P} = \mathcal{P}_1[1,-1]$ and $zf'(z)/f(z) \in \mathcal{P}(1/2) := \mathcal{P}_1[0,-1]$. Using the Lemma 1.1, we get

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{2r}{1-r^2} \quad (|z| \leqslant r < 1)$$

and

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-r^2}\right| \leqslant \frac{r}{1-r^2} \quad (|z| \leqslant r < 1).$$

These inequalities together with (2.1) immediately yield

$$\left|\frac{zg'(z)}{g(z)} - \frac{1+\alpha r^2}{1-r^2}\right| \leqslant \frac{(1+\alpha)r}{1-r^2} \quad (|z| \leqslant r < 1). \tag{2.2}$$

1. We first prove the result in the case when $B = -1$. In this case, we write A as $A = 1 - 2\beta$, where $0 \leqslant \beta < 1$ so that $\mathcal{ST}[A,B]$ radius is the same as $\mathcal{ST}(\beta)$ radius. A simple calculation shows that the result in this becomes

$$\mathcal{R}_{\mathcal{ST}(\beta)} = \min\left(1, \frac{1-\beta}{\beta+\alpha}\right) = \begin{cases} 1 & \beta \leqslant \frac{1-\alpha}{2}, \\ \dfrac{1-\beta}{\beta+\alpha} & \beta \geqslant \frac{1-\alpha}{2}. \end{cases} \tag{2.3}$$

With $R = \mathcal{R}_{\mathcal{ST}(\beta)}$, our aim is to show that $\operatorname{Re}(zg'(z)/g(z)) > \beta$ for $|z| = r \leqslant R$ for every $g \in \mathcal{J}_1^\alpha$. The inequality (2.2) shows that

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geqslant \frac{1+\alpha r^2}{1-r^2} - \frac{(1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r} := \phi(r). \tag{2.4}$$

Since $\phi'(r) = -(1+\alpha)/(1+r)^2$, the function ϕ is decreasing for $0 \leqslant r < 1$. For $\beta \leqslant (1-\alpha)/2$, the inequality (2.4) gives

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geqslant \phi(r) \geqslant \phi(1) = \frac{1-\alpha}{2} \geqslant \beta$$

and so $g \in \mathcal{ST}(\beta)$. For $\beta > (1-\alpha)/2$, the inequality (2.4) gives

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geqslant \phi(r) \geqslant \phi(R) = \beta$$

for $r \leqslant R$. This shows that $\mathcal{ST}(\beta)$ radius of $\mathcal{J}_1^\alpha$ is at least R. To show that the result is sharp, we consider the function $\tilde{g} : \mathbb{D} \to \mathbb{C}$ is given by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. This function corresponds to the function $\tilde{f} \in \mathcal{CV}$ given by

$$\tilde{f}(z) = \frac{z}{1-z}. \tag{2.5}$$

The function $\tilde{g}$ is clearly starlike of order $(1-\alpha)/2$. The result is therefore sharp for $\beta \leqslant (1-\alpha)/2$. Note that

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z}. \tag{2.6}$$

For $\beta > (1-\alpha)/2$ and $z = R$, using (2.6), we see that

$$\operatorname{Re}\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right) = \frac{1-\alpha R}{1+R} = \beta,$$

which proves the sharpness of R.

2. Now we assume that $B \neq -1$. Let $f \in \mathcal{J}_1^\alpha$. Then, by (2.2), we see that

$$g(\mathbb{D}_r) \subset \{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$$

where

$$c_1(\alpha, r) := \frac{1+\alpha r^2}{1-r^2} \quad \text{and} \quad d_1(\alpha, r) := \frac{(1+\alpha)r}{1-r^2}.$$

We show that, for $r \leqslant R = \mathcal{R}_{\mathcal{ST}[A,B]}$, the inclusion

$$\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\} \subseteq \{w : |w - a| \leqslant b\}$$

holds where

$$a = \frac{1-AB}{1-B^2} \quad \text{and} \quad b = \frac{A-B}{1-B^2}.$$

Since $\{w : |w-c| \leqslant d\} \subseteq \{w : |w-a| \leqslant b\}$ if and only if $|a-c| \leqslant b-d$ (see [19] and [6]), it is enough to show that, for $r \leqslant R$, the inequality $|a-c_1(\alpha,r)| \leqslant b-d_1(\alpha,r)$ holds. The inequality $|a-c_1(\alpha,r)| \leqslant b-d_1(\alpha,r)$ is equivalent to the inequalities

$$c_1(\alpha,r) + d_1(\alpha,r) \leqslant a+b \tag{2.7}$$

and

$$a-b \leqslant c_1(\alpha,r) - d_1(\alpha,r). \tag{2.8}$$

The inequality (2.7) becomes

$$\frac{1+A}{1+B} \geqslant \frac{1+\alpha r^2 + (1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r}.$$

This inequality holds for

$$0 \leqslant r \leqslant \frac{A-B}{1+\alpha+A+\alpha B} := \rho_2.$$

Similarly, the inequality (2.8) becomes

$$\frac{1-A}{1-B} \leqslant \frac{1+\alpha r^2 - (1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r}$$

or

$$0 \leqslant r \leqslant \frac{A-B}{1+\alpha-A-\alpha B} := \rho_3.$$

Since

$$\min[\rho_2, \rho_3] = \frac{A-B}{1+\alpha+|A+\alpha B|} = R,$$

it follows that the inequalities (2.7) and (2.8) holds for $0 \leqslant r \leqslant R$. This shows that $\mathcal{ST}[A,B]$ radius of $\mathcal{J}_1^\alpha$ is at least R.

To prove the sharpness of R , we again consider the function $\tilde{f} \in \mathcal{CV}$ defined by (2.5). When $A+\alpha B > 0$, then $R = \rho_2$. For $z = \rho_2$, the equation (2.6) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+A}{1+B},$$

which proves the sharpness for ρ_2. When $A+\alpha B < 0$, then $R = \rho_3$. For $z = -\rho_3$, the equation (2.6) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1-A}{1-B},$$

which proves the sharpness for ρ_3. □

Theorem 2.2. *Let $\alpha > 0$. For the class $\mathcal{J}_1^\alpha$, the following radius results hold:*

1. *The $\mathcal{ST}_e$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_e} = \begin{cases} \frac{e-1}{e\alpha+1} & \text{if} \quad \alpha \geqslant 1 \\ \frac{e-1}{e+\alpha} & \text{if} \quad \alpha \leqslant 1. \end{cases}$$

2. The $\mathcal{ST}_c$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_c} = \begin{cases} \frac{2}{3\alpha+1} & \text{if} \quad \alpha \geqslant 1 \\ \frac{2}{\alpha+3} & \text{if} \quad \alpha \leqslant 1. \end{cases}$$

3. The $\mathcal{ST}_m$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_m} = \begin{cases} \frac{2-\sqrt{2}}{\alpha-(1-\sqrt{2})} & \text{if} \quad \alpha \geqslant 1 \\ \frac{\sqrt{2}}{\alpha+(1+\sqrt{2})} & \text{if} \quad \alpha \leqslant 1. \end{cases}$$

4. The $\mathcal{ST}_\wp$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_\wp} = \begin{cases} \frac{1}{e(1+\alpha)-1} & \text{if} \quad \alpha \geqslant \frac{2}{e-e^{-1}} - 1 \\ \frac{e}{\alpha+e+1} & \text{if} \quad \alpha \leqslant \frac{2}{e-e^{-1}} - 1. \end{cases}$$

5. The $\mathcal{ST}_{Ne}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_{Ne}} = \frac{2}{3\alpha+5}.$$

6. $\mathcal{ST}_{SG}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_{SG}} = \frac{e-1}{(e+1)\alpha+2e}.$$

7. The $\mathcal{ST}_{\sin}$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_{\sin}} = \frac{\sin 1}{(1+\alpha)+\sin 1}.$$

8. The $\mathcal{ST}_h$ radius is given by

$$\mathcal{R}_{\mathcal{ST}_g} = \frac{\sinh^{-1}(1)}{(1+\alpha)+\sinh^{-1}(1)}.$$

Proof. Let $g \in \mathcal{J}_1^\alpha$. For various choices of φ, we are interested in computing $\mathcal{ST}(\varphi)$ radius of the function g. To do this, we first note that, by (2.2), we have $g(\mathbb{D}_r) \subset \{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$, where

$$c_1(\alpha, r) := \frac{1+\alpha r^2}{1-r^2} \quad \text{and} \quad d_1(\alpha, r) := \frac{(1+\alpha)r}{1-r^2}. \tag{2.9}$$

We compute the largest R, such that, for $0 \leqslant r \leqslant R$, the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in $\varphi(\mathbb{D})$. For this purpose, we use the formula for the radius r_a of the largest disc centered at a contained in $\varphi(\mathbb{D})$ obtained by various authors. We also need the fact that the center $c_1(\alpha,r)$ is an increasing function of r which follows easily from the equation

$$c_1'(\alpha,r)=\frac{2(1+\alpha)r}{(1-r^2)^2}.$$

One immediate consequence is that $c_1(\alpha,r)\geqslant c_1(\alpha,0)=1$.

1. Let Ω_e be the image of the unit disc $\mathbb{D}$ under the exponential function $\varphi(z)=e^z$. Mendiratta et al. [14] proved that the inclusion $\{w:|w-a|<r_a\}\subseteq\Omega_e:=\{w:|\log w|<1\}$ holds when

$$r_a=\begin{cases} a-\frac{1}{e} & \text{if } \frac{1}{e}<a\leqslant\frac{e+e^{-1}}{2}\\ e-a & \text{if } \frac{e+e^{-1}}{2}\leqslant a<e.\end{cases}$$

Using this inclusion result, we now show that, for $0\leqslant r\leqslant R:=\mathcal{R}_{\mathcal{ST}_e}$, the disc $\{w:|w-c_1(\alpha,r)|\leqslant d_1(\alpha,r)\}$ is contained in Ω_e where $c_1(\alpha,r)$ and $d_1(\alpha,r)$ given by (2.9).

First, we consider the case $\alpha\geqslant1$. Let the number

$$\rho_1:=\sqrt{\frac{e+e^{-1}-2}{2\alpha+e+e^{-1}}}<1$$

be the unique root of the equation $c_1(\alpha,r)=(e+e^{-1})/2$. Let the number

$$\rho_2:=\frac{e-1}{\alpha e+1}<1$$

be the positive root of the equation $d_1(\alpha,r)=c_1(\alpha,r)-1/e$ or

$$\frac{1+\alpha r^2}{1-r^2}-\frac{(1+\alpha)r}{1-r^2}=\frac{1-\alpha r}{1+r}=\frac{1}{e}. \tag{2.10}$$

A computation shows that $\rho_2\leqslant\rho_1$ for $\alpha\geqslant1$. We shall show that $R=\mathcal{R}_{\mathcal{ST}_e}=\rho_2$.

Since $c_1(\alpha,r)\geqslant1$, it follows that $c_1(\alpha,r)>1/e$ for $0\leqslant r\leqslant\rho_2<1$. Since $c_1(\alpha,r)$ is an increasing function, for $r\leqslant\rho_1$, we have $c_1(\alpha,r)\leqslant c_1(\alpha,\rho_1)=(e+e^{-1})/2$. Since $c_1(\alpha,r)-d_1(\alpha,r)$ is a decreasing function of r, it follows, for $0\leqslant r\leqslant\rho_2$, that

$$c_1(\alpha,r)-d_1(\alpha,r)\geqslant c_1(\alpha,\rho_2)-d_1(\alpha,\rho_2)=1/e$$

and hence

$$d_1(\alpha,r)\leqslant c_1(\alpha,r)-\frac{1}{e}. \tag{2.11}$$

Therefore, for $0\leqslant r\leqslant R=\rho_2$, we have, using (2.2) and (2.11)

$$\left|\frac{zg'(z)}{g(z)}-c_1(\alpha,r)\right|\leqslant c_1(\alpha,r)-\frac{1}{e}.$$

Therefore, the inclusion $\{w:=zg'(z)/g(z):|w-a|<r_a\}\subseteq\Omega_e:=\{w:|\log w|<1\}$ holds which proves that $\mathcal{ST}_e$ radius of $\mathcal{J}_1^\alpha$ is at least $R=\rho_2$.

To prove the sharpness, consider the function $\tilde g\in\mathcal{J}_1^\alpha$ defined by $\tilde g(z)=z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde g'(z)}{\tilde g(z)}=\frac{1+\alpha z}{1-z},$$

we have, for $z=-\rho_2$,

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{1-\alpha\rho_2}{1+\rho_2}\right)\right| = \left|\log\left(\frac{1}{e}\right)\right| = 1,$$

which proves the sharpness for ρ_2.

We now consider the case when $\alpha \leqslant 1$. Let the number

$$\rho_3 := \frac{e-1}{e+\alpha} < 1$$

be the positive root of the equation $d_1(\alpha,r) = e - c_1(\alpha,r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = e. \tag{2.12}$$

A computation shows that $\rho_3 \geqslant \rho_1$ for $\alpha \leqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_e} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c_1(\alpha,R) < e$. Since $c_1(\alpha,r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha,r) \leqslant c_1(\alpha,\rho_1) = (e+e^{-1})/2$. Since $c_1(\alpha,r)+d_1(\alpha,r)$ is an increasing function of r, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha,r)+d_1(\alpha,r) \leqslant c_1(\alpha,\rho_3)+d_1(\alpha,\rho_3) = e$$

and hence

$$d_1(\alpha,r) \leqslant e - c_1(\alpha,r). \tag{2.13}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.13)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha,r)\right| \leqslant e - c_1(\alpha,r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \Omega_e := \{w : |\log w| < 1\}$ holds which proves that $\mathcal{ST}_e$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = \rho_3$,

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{1+\alpha\rho_3}{1-\rho_3}\right)\right| = |\log e| = 1,$$

proving the sharpness for ρ_3.

2. Let Ω_C be the image of the unit disc $\mathbb{D}$ under the function $\varphi_C(z) = 1+(4/3)z+(2/3)z^2$. Sharma et al. [18] proved that the inclusion $\{w : |w-a| < r_a\} \subset \varphi_C(\mathbb{D}) = \Omega_C$ holds when

$$r_a = \begin{cases} a - \frac{1}{3} & \text{if } \frac{1}{3} < a \leqslant \frac{5}{3} \\ 3-a & \text{if } \frac{5}{3} \leqslant a < 3. \end{cases}$$

Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha,r)| \leqslant d_1(\alpha,r)\}$ is contained in Ω_C where $c_1(\alpha,r)$ and $d_1(\alpha,r)$ given by (2.9).

We first consider the case $\alpha \geqslant 1$. Let the number

$$\rho_1 := \sqrt{\frac{2}{3\alpha+5}} < 1$$

be the unique root of the equation $c_1(\alpha, r) = 5/3$. Let the number

$$\rho_2 := \frac{2}{3\alpha+1} < 1$$

be the positive root of the equation $d_1(\alpha, r) = c_1(\alpha, r) - 1/3$ or

$$\frac{1+\alpha r^2}{1-r^2} - \frac{(1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r} = \frac{1}{3}. \tag{2.14}$$

A computation shows that $\rho_2 \leqslant \rho_1$ for $\alpha \geqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_C} = \rho_2$.

Since $c_1(\alpha, r) \geqslant 1$, it follows that $c_1(\alpha, r) > 1/3$ for $0 \leqslant r \leqslant \rho_2 < 1$. Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha, r) \leqslant c_1(\alpha, \rho_1) = 5/3$. Since $c_1(\alpha, r) - d_1(\alpha, r)$ is a decreasing function of r, it follows, for $0 \leqslant r \leqslant \rho_2$, that

$$c_1(\alpha, r) - d_1(\alpha, r) \geqslant c_1(\alpha, \rho_2) - d_1(\alpha, \rho_2) = 1/3$$

and hence

$$d_1(\alpha, r) \leqslant c_1(\alpha, r) - \frac{1}{3}. \tag{2.15}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_2$, we have, using (2.2) and (2.15)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant c_1(\alpha, r) - \frac{1}{3}.$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_C(\mathbb{D}) = \Omega_C$ holds which proves that $\mathcal{ST}_C$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_2$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1-\alpha\rho_2}{1+\rho_2} = \frac{1}{3} = \varphi_C(-1),$$

which proves the sharpness for ρ_2.

We now consider the case when $\alpha \leqslant 1$. Let the number

$$\rho_3 := \frac{2}{\alpha+3} < 1$$

be the positive root of the equation $d_1(\alpha, r) = 3 - c_1(\alpha, r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = 3. \tag{2.16}$$

A computation shows that $\rho_3 \geqslant \rho_1$ for $\alpha \leqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_C} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c_1(\alpha, R) < 3$. Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha, r) \leqslant c_1(\alpha, \rho_1) = 5/3$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of r, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 3$$

and hence

$$d_1(\alpha, r) \leqslant 3 - c_1(\alpha, r). \tag{2.17}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.17)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant 3 - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_C(\mathbb{D}) = \Omega_C$ holds which proves that $\mathcal{ST}_C$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha\rho_3}{1 - \rho_3} = 3 = \varphi_C(1),$$

proving the sharpness for ρ_3.

3. Let Ω_m be the image of the unit disc $\mathbb{D}$ under the function $\varphi_m(z) = z + \sqrt{1 + z^2}$. Gandhi and Ravichandran [5] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_m(\mathbb{D}) = \Omega_m := \{w : \left|w^2 - 1\right| < 2|w|\}$ holds when

$$r_a = 1 - |\sqrt{2} - a|$$

for $\sqrt{2} - 1 < a \leqslant \sqrt{2} + 1$. Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in Ω_m where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

First, we consider the case $\alpha \geqslant 1$. Let the number

$$\rho_1 := \sqrt{\frac{\sqrt{2} - 1}{\alpha + \sqrt{2}}} < 1$$

be the unique root of the equation $c_1(\alpha, r) = \sqrt{2}$. Let the number

$$\rho_2 := \frac{2 - \sqrt{2}}{\alpha - (1 - \sqrt{2})} < 1$$

be the positive root of the equation $d_1(\alpha, r) = c_1(\alpha, r) - (\sqrt{2} - 1)$ or

$$\frac{1 + \alpha r^2}{1 - r^2} - \frac{(1 + \alpha)r}{1 - r^2} = \frac{1 - \alpha r}{1 + r} = \sqrt{2} - 1. \tag{2.18}$$

A computation shows that $\rho_2 \leqslant \rho_1$ for $\alpha \geqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_m} = \rho_2$.

Since $c_1(\alpha, r) \geqslant 1$, it follows that $c_1(\alpha, r) > \sqrt{2} - 1$ for $0 \leqslant r \leqslant \rho_2 < 1$. Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha, r) \leqslant c_1(\alpha, \rho_1) = \sqrt{2}$.

Since $c_1(\alpha, r) - d_1(\alpha, r)$ is a decreasing function of r, it follows, for $0 \leqslant r \leqslant \rho_2$, that

$$c_1(\alpha, r) - d_1(\alpha, r) \geqslant c_1(\alpha, \rho_2) - d_1(\alpha, \rho_2) = \sqrt{2} - 1$$

and hence

$$d_1(\alpha, r) \leqslant c_1(\alpha, r) - (\sqrt{2} - 1). \tag{2.19}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_2$, we have, using (2.2) and (2.19)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant c_1(\alpha, r) - (\sqrt{2} - 1).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_m(\mathbb{D}) = \Omega_m$ holds which proves that $\mathcal{ST}_m$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_2$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 - \alpha\rho_2}{1 + \rho_2} = \sqrt{2} - 1 = \varphi_m(-1),$$

which proves the sharpness for ρ_2.

We now consider the case when $\alpha \leqslant 1$. Let the number

$$\rho_3 := \frac{\sqrt{2}}{\alpha + (1 + \sqrt{2})} < 1$$

be the positive root of the equation $d_1(\alpha, r) = \sqrt{2} + 1 - c_1(\alpha, r)$ or

$$\frac{1 + \alpha r^2}{1 - r^2} + \frac{(1 + \alpha) r}{1 - r^2} = \frac{1 + \alpha r}{1 - r} = \sqrt{2} + 1. \tag{2.20}$$

A computation shows that $\rho_3 \geqslant \rho_1$ for $\alpha \leqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_m} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c_1(\alpha, R) < \sqrt{2} + 1$. Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha, r) \leqslant c_1(\alpha, \rho_1) = \sqrt{2}$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of r, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = \sqrt{2} + 1$$

and hence

$$d_1(\alpha, r) \leqslant \sqrt{2} + 1 - c_1(\alpha, r). \tag{2.21}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.21)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant \sqrt{2} + 1 - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_m(\mathbb{D}) = \Omega_m$ holds which proves that $\mathcal{ST}_m$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha\rho_3}{1-\rho_3} = \sqrt{2}+1 = \varphi_m(1),$$

proving the sharpness for ρ_3.

4. Let $\Omega_\wp$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_\wp(z) = 1+ze^z$. Kumar and Kamaljeet [20] proved that the inclusion $\{w : |w-a| < r_a\} \subset \varphi_\wp(\mathbb{D}) = \Omega_\wp$ holds when

$$r_a = \begin{cases} (a-1)+\frac{1}{e} & \text{if } 1-\frac{1}{e} < a \leqslant 1+\frac{e-e^{-1}}{2} \\ e-(a-1) & \text{if } 1+\frac{e-e^{-1}}{2} \leqslant a < 1+e. \end{cases}$$

Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w-c_1(\alpha,r)| \leqslant d_1(\alpha,r)\}$ is contained in $\Omega_\wp$ where $c_1(\alpha,r)$ and $d_1(\alpha,r)$ given by (2.9).

First, we consider the case $\alpha \geqslant 1$. Let the number

$$\rho_1 := \sqrt{\frac{e-e^{-1}}{2(1+\alpha)+e-e^{-1}}} < 1$$

be the unique root of the equation $c_1(\alpha,r) = 1+(e-e^{-1})/2$. Let the number

$$\rho_2 := \frac{1}{e(1+\alpha)-1} < 1$$

be the positive root of the equation $d_1(\alpha,r) = c_1(\alpha,r)-1+(1/e)$ or

$$\frac{1+\alpha r^2}{1-r^2} - \frac{(1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r} = \frac{1}{e}-1. \tag{2.22}$$

A computation shows that $\rho_2 \leqslant \rho_1$ for $\alpha \geqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_\wp} = \rho_2$.

Since $c_1(\alpha,r) \geqslant 1$, it follows that $c_1(\alpha,r) > 1-(1/e)$ for $0 \leqslant r \leqslant \rho_2 < 1$. Since $c_1(\alpha,r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha,r) \leqslant c_1(\alpha,\rho_1) = 1+(e-e^{-1})/2$. Since $c_1(\alpha,r)-d_1(\alpha,r)$ is a decreasing function of r, it follows, for $0 \leqslant r \leqslant \rho_2$, that

$$c_1(\alpha,r)-d_1(\alpha,r) \geqslant c_1(\alpha,\rho_2)-d_1(\alpha,\rho_2) = \frac{1}{e}-1$$

and hence

$$d_1(\alpha,r) \leqslant c_1(\alpha,r)-1+\frac{1}{e}. \tag{2.23}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_2$, we have, using (2.2) and (2.23)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha,r)\right| \leqslant c_1(\alpha,r)-1+\frac{1}{e}.$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \varphi_\wp(\mathbb{D}) = \Omega_\wp$ holds which proves that $\mathcal{ST}_\wp$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_2$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1-\alpha\rho_2}{1+\rho_2} = 1 - e^{-1} = \varphi_\wp(-1),$$

which proves the sharpness for ρ_2.

We now consider the case when $\alpha \leqslant 1$. Let the number

$$\rho_3 := \frac{e}{\alpha+e+1} < 1$$

be the positive root of the equation $d_1(\alpha, r) = e + 1 - c_1(\alpha, r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = e+1. \tag{2.24}$$

A computation shows that $\rho_3 \geqslant \rho_1$ for $\alpha \leqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_\wp} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c_1(\alpha, R) < e+1$. Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha, r) \leqslant c_1(\alpha, \rho_1) = 1 + (e - e^{-1})/2$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of r, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = e + 1$$

and hence

$$d_1(\alpha, r) \leqslant e + 1 - c_1(\alpha, r). \tag{2.25}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.25)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant e + 1 - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_\wp(\mathbb{D}) = \Omega_\wp$ holds which proves that $\mathcal{ST}_\wp$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha\rho_3}{1-\rho_3} = 1 + e = \varphi_\wp(1),$$

proving the sharpness for ρ_3.

5. Let Ω_{Ne} be the image of the unit disc $\mathbb{D}$ under the function $\varphi_{Ne}(z) = 1 + z - (z^3/3)$. Wani and Swaminathan [21] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_{Ne}(\mathbb{D}) = \Omega_{Ne}$ holds when

$$r_a = \begin{cases} a - \frac{1}{3} & \text{if } \frac{1}{3} < a \leqslant 1 \\ \frac{5}{3} - a & \text{if } 1 \leqslant a < \frac{5}{3}. \end{cases}$$

Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in Ω_{Ne} where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

Let the number

$$\rho_3 := \frac{2}{3\alpha + 5} < 1$$

be the positive root of the equation $d_1(\alpha, r) = (5/3) - c_1(\alpha, r)$ or

$$\frac{1 + \alpha r^2}{1 - r^2} + \frac{(1 + \alpha)r}{1 - r^2} = \frac{1 + \alpha r}{1 - r} = \frac{5}{3}. \tag{2.26}$$

We shall show that $R = \mathcal{R}_{\mathcal{ST}_\varphi} = \rho_3$. For $0 \leqslant r \leqslant R < 1$ it follows that $1 \leqslant c_1(\alpha, r) \leqslant c_1(\alpha, R) < 5/3$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of r, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = \frac{5}{3}$$

and hence

$$d_1(\alpha, r) \leqslant \frac{5}{3} - c_1(\alpha, r). \tag{2.27}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.27)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant \frac{5}{3} - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_{Ne}(\mathbb{D}) = \Omega_{Ne}$ holds which proves that $\mathcal{ST}_{Ne}$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1 - z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha \rho_3}{1 - \rho_3} = \frac{5}{3} = \varphi_{Ne}(1),$$

proving the sharpness for ρ_3.

6. Let Ω_{SG} be the image of the unit disc $\mathbb{D}$ under the function $\varphi_{SG}(z) = 2/(1+e^{-z})$. Goel and Kumar [7] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_{SG}(\mathbb{D}) = \Omega_{SG} := \{w : |\log w/(2 - w)| < 1\}$ holds when

$$r_a = \frac{e - 1}{e + 1} - |a - 1|$$

for $2/(1 + e) < a < 2e/(1 + e)$. Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in Ω_{SG} where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

Let the number

$$\rho_3 := \frac{e - 1}{(e + 1)\alpha + 2e} < 1$$

be the positive root of the equation $d_1(\alpha,r)=(e-1)/(e+1)+1-c_1(\alpha,r)$ or

$$\frac{1+\alpha r^2}{1-r^2}+\frac{(1+\alpha)r}{1-r^2}=\frac{1+\alpha r}{1-r}=\frac{e-1}{e+1}+1. \tag{2.28}$$

We shall show that $R=\mathcal{R}_{\mathcal{ST}_{SG}}=\rho_3$. For $0\leqslant r\leqslant R<1$ it follows that $2/(1+e)<1\leqslant c_1(\alpha,r)\leqslant c_1(\alpha,R)\leqslant c_1(\alpha,\rho_3)+d_1(\alpha,\rho_3)=2e/(1+e)$. Since $c_1(\alpha,r)+d_1(\alpha,r)$ is an increasing function of r, it follows for $0\leqslant r\leqslant\rho_3$, that

$$c_1(\alpha,r)+d_1(\alpha,r)\leqslant c_1(\alpha,\rho_3)+d_1(\alpha,\rho_3)=\frac{e-1}{e+1}+1$$

and hence

$$d_1(\alpha,r)\leqslant\frac{e-1}{e+1}+1-c_1(\alpha,r). \tag{2.29}$$

Therefore, for $0\leqslant r\leqslant R=\rho_3$, we have, using (2.2) and (2.29)

$$\left|\frac{zg'(z)}{g(z)}-c_1(\alpha,r)\right|\leqslant\frac{e-1}{e+1}+1-c_1(\alpha,r).$$

Therefore, the inclusion $\{w:=zg'(z)/g(z):|w-a|<r_a\}\subseteq\varphi_{SG}(\mathbb{D})=\Omega_{SG}:=\{w:|\log w/(2-w)|<1\}$ holds which proves that $\mathcal{ST}_{SG}$ radius of $\mathcal{J}_1^\alpha$ is at least $R=\rho_3$.

To prove the sharpness, consider the function $\tilde{g}\in\mathcal{J}_1^\alpha$ defined by $\tilde{g}(z)=z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)}=\frac{1+\alpha z}{1-z},$$

we have, for $z=\rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)}=\frac{1+\alpha\rho_3}{1-\rho_3}=\frac{2e}{e+1}=\varphi_{SG}(1),$$

proving the sharpness for ρ_3.

7. Let $\Omega_{\sin}$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_{\sin}(z)=1+\sin z$. Cho et al. [3] proved that the inclusion $\{w:|w-a|<r_a\}\subset\varphi_{\sin}(\mathbb{D})=\Omega_{\sin}$ holds when

$$r_a=\sin 1-|a-1|$$

for $1-\sin 1<a<1+\sin 1$. Using the inclusion result, we now show that, for $0\leqslant r\leqslant R$ the disc $\{w:|w-c_1(\alpha,r)|\leqslant d_1(\alpha,r)\}$ is contained in $\Omega_{\sin}$ where $c_1(\alpha,r)$ and $d_1(\alpha,r)$ given by (2.9).

Let the number

$$\rho_3:=\frac{\sin 1}{(1+\alpha)+\sin 1}<1$$

be the positive root of the equation $d_1(\alpha,r)=(\sin 1)+1-c_1(\alpha,r)$ or

$$\frac{1+\alpha r^2}{1-r^2}+\frac{(1+\alpha)r}{1-r^2}=\frac{1+\alpha r}{1-r}=1+\sin 1. \tag{2.30}$$

We shall show that $R = \mathcal{R}_{\mathcal{ST}_{\sin}} = \rho_3$. For $0 \leqslant r \leqslant R < 1$ it follows that $1 - \sin 1 < 1 \leqslant c_1(\alpha, r) \leqslant c_1(\alpha, R) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 1 + \sin 1$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of r, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 1 + \sin 1$$

and hence

$$d_1(\alpha, r) \leqslant 1 + \sin 1 - c_1(\alpha, r). \tag{2.31}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.31)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant 1 + \sin 1 - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_{\sin}(\mathbb{D}) = \Omega_{\sin}$ holds which proves that $\mathcal{ST}_{\sin}$ radius of $\mathcal{J}_1^{\alpha}$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^{\alpha}$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha \rho_3}{1 - \rho_3} = 1 + \sin 1 = \varphi_{\sin}(1),$$

proving the sharpness for ρ_3.

8. Let Ω_h be the image of the unit disc $\mathbb{D}$ under the function $\varphi_h(z) = 1 + \sinh^{-1}(z)$. Kumar and Arora [2] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_h(\mathbb{D}) = \Omega_h$ holds when

$$r_a = \begin{cases} a - (1 - \sinh^{-1}(1)) & \text{if } \ 1 - \sinh^{-1}(1) < a \leqslant 1 \\ 1 + \sinh^{-1}(1) - a & \text{if } \ 1 \leqslant a < 1 + \sinh^{-1}(1). \end{cases}$$

Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in Ω_h where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

Let the number

$$\rho_3 := \frac{\sinh^{-1}(1)}{(1 + \alpha) + \sinh^{-1}(1)} < 1$$

be the positive root of the equation $d_1(\alpha, r) = 1 + \sinh^{-1}(1) - c_1(\alpha, r)$ or

$$\frac{1 + \alpha r^2}{1 - r^2} + \frac{(1 + \alpha) r}{1 - r^2} = \frac{1 + \alpha r}{1 - r} = 1 + \sinh^{-1}(1). \tag{2.32}$$

We shall show that $R = \mathcal{R}_{\mathcal{ST}_{\sin}} = \rho_3$. For $0 \leqslant r \leqslant R < 1$ it follows that $1 - \sinh^{-1}(1) < 1 \leqslant c_1(\alpha, r) \leqslant c_1(\alpha, R) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 1 + \sinh^{-1}(1)$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of r, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 1 + \sinh^{-1}(1)$$

and hence

$$d_1(\alpha, r) \leqslant 1 + \sinh^{-1}(1) - c_1(\alpha, r). \tag{2.33}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.33)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant 1 + \sinh^{-1}(1) - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_h(\mathbb{D}) = \Omega_h$ holds which proves that $\mathcal{ST}_h$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha\rho_3}{1 - \rho_3} = 1 + \sinh^{-1}(1) = \varphi_h(1),$$

proving the sharpness for ρ_3. □

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Harmonic close-to-convex mappings associated with Sălăgean q-differential operator

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Abstract. In this paper, we define a new subclass $\mathcal{W}(n,\alpha,q)$ of analytic functions and a new subclass $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ of harmonic functions $f=h+\overline{g}\in\mathcal{H}^0$ associated with Sălăgean q-differential operator. We prove that a harmonic function $f=h+\bar{g}$ belongs to the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ if and only if the analytic functions $h+\epsilon g$ belong to $\mathcal{W}(n,\alpha,q)$ for each ϵ ($|\epsilon|=1$), and using a method by Clunie and Sheil-Small, we determine a sufficient condition for the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ to be close-to-convex. We provide sharp coefficient estimates, sufficient coefficient condition, and convolution properties for such functions classes. We also determine several conditions of partial sums of $f\in\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$.

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1. Introduction

Quantum calculus is the calculus without use of the limits. The history of quantum calculus dates back to the studies of Leonhard Euler (1707-1783) and Carl Gustav Jacobi (1804-1851). Later, geometrical interpretation of the q-calculus has been applied in studies of quantum groups. The great interest to quantum calculus is due to its applications in various branches of mathematics and physics; as for example, in quantum mechanics, analytic number theory, sobolev spaces, group representation theory, theta functions, gamma functions, operator theory and several other areas. For the definitions and properties of q-calculus, one may refer to the books [5] and

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[14]. Jackson [10, 11] was the first who gave some applications of q-calculus by introducing the q-analogues of derivative and integral. The $q-$derivative (or $q-$difference operator) of a function h, defined on a subset of $\mathbb{C}$, is given by

$$(D_q h)(z) = \begin{cases} \frac{h(z)-h(qz)}{(1-q)z}, & z \neq 0 \\ \\ h'(0), & z = 0, \end{cases}$$

where $q \in (0,1)$. Note that $\lim_{q\to 1^-}(D_q h)(z) = h'(z)$ if h is differentiable at z ([10]).

For a function $h(z) = z^k$ $(k \in \mathbb{N})$, we observe that

$$D_q z^k = [k]_q z^{k-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + ...q^{k-1}$$

is the $q-$number of k. Clearly, $\lim_{q\to 1^-}[k]_q = k$. For more details, one may refer to [14] and references therein.

Connection of q-calculus with geometric function theory was first introduced by Ismail *et al.* [9]. Recently, q-calculus is involved in the theory of analytic functions [7, 8, 21]. But research on q-calculus in connection with harmonic functions is fairly new and not much published (see [12, 23, 22, 28]).

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ denote an open disk with $r > 0$. The open unit disk will be denoted by $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{H}$ denote the class of complex-valued functions $f = u + iv$ which are harmonic in the open unit disk $\mathbb{D}$, where u and v are real-valued harmonic functions in $\mathbb{D}$. Functions $f \in \mathcal{H}$ can also be expressed as $f = h + \overline{g}$, where h the analytic and g the co-analytic parts of f, respectively. A subclass of functions $f = h + \overline{g} \in \mathcal{H}$ with the additional condition $g'(0) = 0$ is denoted by $\mathcal{H}^0$. According to the Lewy's Theorem [15], every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if the Jacobian of f, given by $J_f(z) = |h'(z)|^2 - |\ g'(z)|^2$, is positive in $\mathbb{D}$. This case is equivalent to the existence of an analytic function $\omega(z) = g'(z)/h'(z)$ in $\mathbb{D}$, which is called as the dilatation of f such that

$$|\omega(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}.$$

Clunie and Sheil-Small [3] introduced the class of all univalent, sense preserving harmonic functions $f = h + \overline{g}$, denoted by $\mathcal{S}_{\mathcal{H}}$, with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$. If the function $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, then

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad (z \in \mathbb{D}). \tag{1.1}$$

A subclass of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ with the condition $g'(0) = 0$ is denoted by $\mathcal{S}_{\mathcal{H}}^0$. Further, the subclass of functions f in $\mathcal{S}_{\mathcal{H}}$ $\left(\mathcal{S}_{\mathcal{H}}^0\right)$, denoted by $\mathcal{K}_{\mathcal{H}}$ $\left(\mathcal{K}_{\mathcal{H}}^0\right)$ consists of functions f that map the unit disk $\mathbb{D}$ onto a convex region, the subclass $\mathcal{S}_{\mathcal{H}}^*$ $\left(\mathcal{S}_{\mathcal{H}}^{*0}\right)$ consists of functions f that are starlike, and the subclass $\mathcal{C}_{\mathcal{H}}^*$ $\left(\mathcal{C}_{\mathcal{H}}^{*0}\right)$ consists of functions f which are close-to-convex. Also, if $g(z) \equiv 0$, the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of univalent functions in the class $\mathcal{A}$. Here, $\mathcal{A}$ is the class of all analytic functions of the form $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For more details, we refer [4].

Let $f \in \mathcal{S}$ and be given by $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then the l^{th} section (partial sum) of f is defined by

$$s_l(f)(z) = \sum_{k=0}^{l} a_k z^k, \quad (l \in \mathbb{N})$$

where $a_0 = 0$ and $a_1 = 1$. For a harmonic function $f = h + \overline{g} \in \mathcal{H}$, where h and g of the form (1.1), the sequences of sections (partial sums) of f is defined by

$$s_{i,j}(f)(z) = s_i(h)(z) + \overline{s_j(g)(z)},$$

where $s_i(h)(z) = \sum_{k=1}^{i} a_k z^k$ and $s_j(g)(z) = \sum_{k=1}^{j} b_k z^k$, $i, j \geq 1$ with $a_1 = 1$.

In [32], it is noted that the partial sums of univalent functions is univalent in the disk $\mathbb{D}_{1/4}$. Starlikeness and convexity of the partial sums of univalent functions was discussed in [29, 30].

The convolution or Hadamard product of two analytic functions

$$f_1(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} b_k z^k$$

is defined by

$$(f_1 * f_2)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{D}).$$

The convolution of two harmonic functions $f = h + \overline{g}$ and $F = H + \overline{G}$ is defined by

$$(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)}, \quad (z \in \mathbb{D}).$$

In 2013, Li and Ponnusamy [16] investigated properties of functions given by

$$\mathcal{P}_{\mathcal{H}}^{0} = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z)) > |g'(z)|, \quad z \in \mathbb{D}\}$$

The class $\mathcal{P}_{\mathcal{H}}^{0}$ is harmonic analogue of the class $\mathcal{R} = \{f \in \mathcal{S} : \Re(f'(z)) > 0, \ z \in \mathbb{D}\}$ introduced by MacGregor [20]. It is known that a harmonic function $f = h + \overline{g}$ belongs to the class $\mathcal{P}_{\mathcal{H}}^{0}$ if and only if the analytic function $h + \epsilon g$ belongs to $\mathcal{R}$ for each ϵ ($|\epsilon| = 1$).

In 1977, Chichra [2] studied the class $\mathcal{W}(\alpha)$ consisting of functions $f \in \mathcal{A}$ such that $\Re(f'(z) + \alpha z f''(z)) > 0$ for $\alpha \geq 0$ and $z \in \mathbb{D}$. Later, Nagpal and Ravichandran [24] studied the following class

$$\mathcal{W}_{\mathcal{H}}^{0} = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) + zh''(z)) > |g'(z) + zg''(z)|, \ z \in \mathbb{D}\},$$

which is harmonic analogue of $\mathcal{W}(1)$. Recently, Ghosh and Vasudevarao [6] defined the class $\mathcal{W}_{\mathcal{H}}^{0}(\alpha)$ for $\alpha \geq 0$ by

$$\mathcal{W}_{\mathcal{H}}^{0}(\alpha) = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) + \alpha z h''(z)) > |g'(z) + \alpha z g''(z)|, \ z \in \mathbb{D}\}.$$

In [2], Chichra also studied the class $\mathcal{G}(\alpha)$ of an analytic function f for $\alpha \geq 0$ such that

$$\Re\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] > 0$$

for $|z| < r$ with $r \in (0, 1]$. In 2018, Liu ang Yang [19] defined the class

$$\mathcal{G}^k_{\mathcal{H}}(\alpha) = \left\{ f = h + \bar{g} \in \mathcal{H}^0 : \Re\big((1-\alpha)\frac{h(z)}{z} + \alpha h'(z)\big) > \big|(1-\alpha)\frac{g(z)}{z} + \alpha g'(z)\big| \right\},$$

where $\alpha \geq 0$, $k \geq 1$ and $|z| < r$ with $r \in (0, 1]$.

For an analytic function $h \in \mathcal{A}$, let the Sălăgean q-differential operator be defined by ([7]);

$$\mathcal{D}^0_q h(z) = h(z), \quad \mathcal{D}^1_q h(z) = zD_q h(z), ..., \; \mathcal{D}^n_q h(z) = zD_q(\mathcal{D}^{n-1}_q h(z)),$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Making use of h given by (1.1), and simple calculations yield

$$\mathcal{D}^n_q h(z) = h(z) * \mathcal{F}_{q,n}(z) = z + \sum_{k=2}^{\infty} [k]^n_q a_k z^k, \quad (z \in \mathbb{D}) \tag{1.2}$$

where

$$\mathcal{F}_{q,n}(z) = z + \sum_{k=2}^{\infty} [k]^n_q z^k,$$

and $[k]^n_q = \left(\frac{1-q^k}{1-q}\right)^n$, $q \in (0, 1)$. The operator (1.2) easily reduces to the well-known Sălăgean differential operator as $q \to 1^-$ (see [27]).

For a harmonic function $f = h + \overline{g}$ given by (1.1) and the operator $\mathcal{D}^n_q$ defined by (1.2), the harmonic Sălăgean q-differential operator is defined by ([12]);

$$\begin{aligned} \mathcal{D}^n_q f(z) &= \mathcal{D}^n_q h(z) + (-1)^n \overline{\mathcal{D}^n_q g(z)} \\ &= z + \sum_{k=2}^{\infty} [k]^n_q a_k z^k + (-1)^n \overline{\sum_{k=1}^{\infty} [k]^n_q b_k z^k}. \end{aligned}$$

As $q \to 1^-$, the operator $\mathcal{D}^n_q f$ reduces to the Sălăgean differential operator $\mathcal{D}^n f$ for a harmonic function $f = h + \bar{g}$ ([13]).

Motivated by the Sălăgean q-differential operator, we define a new subclass $\mathcal{W}(n, \alpha, q)$ of analytic functions as follows:

Definition 1.1. An analytic function $f \in \mathcal{A}$ is in the class $\mathcal{W}(n, \alpha, q)$ if it satisfies the condition

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}^n_q f(z) + \alpha \mathcal{D}^{n+1}_q f(z)}{z}\right) > 0, \tag{1.3}$$

where $\mathcal{D}^n_q f(z)$ is the Sălăgean q-differential operator defined by (1.2), and where $\alpha \geq 0$, $n \in \mathbb{N}_0$, $q \in (0, 1)$ and $|z| < r$ with $0 < r \leq 1$.

Remark 1.2. i) Letting $q \to 1^-$, $n = 0$ we get the class $\mathcal{W}(0, \alpha, q) := \mathcal{G}(\alpha)$ introduced by Chichra [2].

ii) Letting $q \to 1^-$, $n = 1$ we get the class $\mathcal{W}(1, \alpha, q) := \mathcal{W}(\alpha)$ introduced by Chichra [2].

iii) Letting $q \to 1^-$, $n = 1$, $\alpha = 0$ we get the class $\mathcal{W}(1, 0, q) := \mathcal{R}$ introduced by MacGregor [20].

Making use of the harmonic Sălăgean q-differential operator, we also define the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ of harmonic functions as follows:

Definition 1.3. A harmonic function $f = h + \overline{g} \in \mathcal{H}^0$ with $h(0) = g(0) = g'(0) = h'(0) - 1 = 0$ is in the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ if it satisfies the condition

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}^n_q h(z) + \alpha\mathcal{D}^{n+1}_q h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}^n_q g(z) + \alpha\mathcal{D}^{n+1}_q g(z)}{z}\right|, \tag{1.4}$$

where $\mathcal{D}^n_q f(z)$ is the harmonic Sălăgean q-differential operator, and where $\alpha \geq 0$, $n \in \mathbb{N}_0$, $q \in (0,1)$ and $|z| < r$ with $0 < r \leq 1$.

Remark 1.4. i) Letting $q \to 1^-$, $n = 0$ we get the class $\mathcal{W}^0_{\mathcal{H}}(0, \alpha, q) := \mathcal{G}^1_{\mathcal{H}}(\alpha)$ introduced by Liu ang Yang [19].

ii) Letting $q \to 1^-$, $n = 1$ we get the class $\mathcal{W}^0_{\mathcal{H}}(1, \alpha, q) := \mathcal{W}^0_{\mathcal{H}}(\alpha)$ introduced by Ghosh and Vasudevarao [6].

iii) Letting $q \to 1^-$, $n = 1$, $\alpha = 1$ we get the class $\mathcal{W}^0_{\mathcal{H}}(1, 1, q) := \mathcal{W}^0_{\mathcal{H}}$ introduced by Nagpal and Ravichandran in [24].

iv) Letting $q \to 1^-$, $n = 1$, $\alpha = 0$ we get the class $\mathcal{W}^0_{\mathcal{H}}(1, 0, q) := \mathcal{P}^0_{\mathcal{H}}$ introduced by Li and Ponnusamy [16].

In this paper, we define a new subclass $\mathcal{W}(n, \alpha, q)$ of analytic functions and a new subclass $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ of harmonic functions $f = h + \overline{g} \in \mathcal{H}^0$ associated with Sălăgean q-differential operator. In Section 2, we prove that a harmonic function $f \in \mathcal{H}^0$ belongs to the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ if and only if the analytic functions $h + \epsilon g$ belong to $\mathcal{W}(n, \alpha, q)$ for each ϵ with $|\epsilon| = 1$, and by a method of Clunie and Sheil-Small, we obtain a sufficient condition for the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ to be close-to-convex. We also provide sharp coefficient estimates and sufficient coefficient condition for such functions classes. In Section 3, we examine that the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ is closed under convex combinations and convolutions of its members. In Section 4, we determine several conditions of partial sums of $f \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$.

2. Coefficient bounds

Clunie and Sheil-Small proved the following result, which gives a sufficient condition for a harmonic function f to be close-to-convex.

Lemma 2.1. [3] *If h and g are analytic in $\mathbb{D}$ satisfies $|g'(0)| < |h'(0)|$ and the function $f_\epsilon = h + \epsilon g$ is close-to-convex for all complex number ϵ with $|\epsilon| = 1$, then $f = h + \overline{g}$ is close-to-convex..*

Theorem 2.2. *A harmonic mapping $f = h + \overline{g}$ is in $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ if and only if the analytic function $f_\epsilon = h + \epsilon g$ belongs to $\mathcal{W}(n, \alpha, q)$ for each complex number ϵ with $|\epsilon| = 1$.*

Proof. If $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$, then for each complex number ϵ with $|\epsilon| = 1$

$$\begin{aligned}
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n f_\epsilon(z) + \alpha\mathcal{D}_q^{n+1} f_\epsilon(z)}{z}\right)\\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h(z)+\epsilon g(z)) + \alpha\mathcal{D}_q^{n+1} (h(z)+\epsilon g(z))}{z}\right)\\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&> \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) - \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| > 0,
\end{aligned}$$

thus $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$ for each ϵ with $|\epsilon| = 1$.

Conversely, if $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$, then

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right) > 0, \;\; (z \in \mathbb{D}_r)$$

or equivalently

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) > -\Re\left(\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right), (z \in \mathbb{D}_r).$$

Since $|\epsilon| = 1$ is arbitrary, for an appropriate choice of ϵ we obtain

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right|, \;\; (z \in \mathbb{D}_r)$$

Hence, $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$. □

Theorem 2.3. *The functions in the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ are close-to-convex in $\mathbb{D}$.*

Proof. Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$, and let $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$ where $|\epsilon| = 1$. By the method used by Ponnusammy *et al.* [25, Theorem 1.3], if $f_\epsilon \in \mathcal{W}(n,\alpha,q)$, then q-derivative of f_ϵ is positive; that is, $\Re\{\mathcal{D}_q^n f_\epsilon\} > 0$, and hence f_ϵ is analytic and close-to-convex function. Therefore,

$$\begin{aligned}
&\Re\{\mathcal{D}_q^n f_\epsilon\} =\\
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&> \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| + \Re\left(\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&\geq \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| - \left|\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right| = 0,
\end{aligned}$$

showing that f_ϵ is analytic and close-to-convex function. Thus according to Lemma 2.1 and Theorem 2.2, it follows that the harmonic function $f \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ is also close-to-convex in $\mathbb{D}$. □

We now establish the sharp coefficient bounds for functions in the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$.

Theorem 2.4. *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ be of the form (1.1) with $b_1 = 0$. Then for any $k \geq 2$*

$$|b_k| \leq \frac{1}{[k]_q^n (1 + \alpha([k]_q - 1))}. \tag{2.1}$$

The result is sharp when f is given by $f(z) = z + \frac{1}{[k]_q^n(1+\alpha([k]_q-1))}\overline{z}^k$.

Proof. Let $f \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$. Then

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right|$$

and

$$\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z} = \sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) b_k z^{k-1}.$$

Using the series expansion of g, we derive

$$\begin{aligned} r^{k-1}[k]_q^n(1+\alpha([k]_q-1))|b_k| &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(re^{i\theta}) + \alpha\mathcal{D}_q^{n+1} g(re^{i\theta})}{re^{i\theta}}\right| d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(re^{i\theta}) + \alpha\mathcal{D}_q^{n+1} h(re^{i\theta})}{re^{i\theta}}\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left(1 + [k]_q^n\big(1+\alpha([k]_q-1)\big)a_k r^{k-1}\right) d\theta \\ &= 1. \end{aligned}$$

Letting $r \to 1^-$ gives the desired bound. □

Remark 2.5. (i) When $q \to 1^-$, $n = 0$ we get the result by Liu ang Yang [19, Corollary 3.2].

(ii) When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.2].

Theorem 2.6. *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ be of the form (1.1) with $b_1 = 0$. Then for any $k \geq 2$*

(i) $|a_k| + |b_k| \leq \frac{2}{[k]_q^n(1+\alpha([k]_q-1))}$
(ii) $||a_k| - |b_k|| \leq \frac{2}{[k]_q^n(1+\alpha([k]_q-1))}$
(iii) $|a_k| \leq \frac{2}{[k]_q^n(1+\alpha([k]_q-1))}$

The results are sharp and the equality is held for the function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n(1+\alpha([k]_q-1))} z^k.$$

Proof. Suppose that $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$, then from Theorem 2.2 $f_\epsilon = h + \epsilon g \in \mathcal{W}(n, \alpha, q)$ for ϵ with $|\epsilon| = 1$. Thus for any $|\epsilon| = 1$, we have

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(h(z) + \epsilon g(z)) + \alpha\mathcal{D}_q^{n+1}(h(z) + \epsilon g(z))}{z}\right) > 0, \quad |z| < r.$$

Then there exists an analytic function p of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ with $\Re(p(z)) > 0$ in $\mathbb{D}$ such that

$$\frac{(1-\alpha)\,\mathcal{D}_q^n(h(z)+\epsilon g(z)) + \alpha\mathcal{D}_q^{n+1}(h(z)+\epsilon g(z))}{z} = p(z). \tag{2.2}$$

Comparing coefficients on both sides of (2.2), we have

$$[k]_q^n(1+\alpha([k]_q-1))(a_k+\epsilon b_k) = p_{k-1}, \quad k \geq 2. \tag{2.3}$$

Since $|p_k| \leq 2$ for $k \geq 1$ and ϵ ($|\epsilon| = 1$) is arbitrary, from (2.3) we get

$$[k]_q^n(1+\alpha([k]_q-1))(|a_k|+|b_k|) \leq 2,$$

which proves (i). The last two inequalities are consequences of the first inequality. □

Remark 2.7. (i) When $q \to 1^-$, $n = 0$ we get the result by Liu ang Yang [19, Corollary 3.4].

(ii) When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.3].

The following result gives a sufficient condition for a function to be belong to $\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$.

Theorem 2.8. *Let $f = h + \overline{g} \in \mathcal{H}^0$ be of the form (1.1) with $b_1 = 0$. If*

$$\sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1))(|a_k|+|b_k|) \leq 1, \tag{2.4}$$

then $f \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$.

Proof. Let $f = h+\overline{g} \in \mathcal{H}^0$. Using the series representation of h given by (1.1), we get

$$\begin{aligned}
\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1}h(z)}{z}\right) &= \Re\left(1 + \sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1)a_k z^{k-1}\right) \\
&> 1 - \sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1)|a_k| \\
&\geq \sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1)|b_k| \\
&> \left|\sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1)b_k z^{k-1}\right| \\
&= \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1}g(z)}{z}\right|,
\end{aligned}$$

therefore $f \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$. □

Remark 2.9. When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.5].

3. Convex combinations and convolutions

In this section, we prove that the class $\mathcal{W}_{\mathcal{H}}^{0}(n,\alpha,q)$ is closed under convex combinations and convolutions of its members.

Theorem 3.1. *The class $\mathcal{W}_{\mathcal{H}}^{0}(n,\alpha,q)$ is closed under convex combinations.*

Proof. Suppose $\mathcal{D}_q^n f_i = \mathcal{D}_q^n h_i + (-1)^n \overline{\mathcal{D}_q^n g_i} \in \mathcal{W}_{\mathcal{H}}^{0}(n,\alpha,q)$ for $i = 1,2,...,k$ and $\sum_{i=1}^{k} t_i = 1$ $(0 \le t_i \le 1)$. The convex combination of functions $\mathcal{D}_q^n f_i$ can be written as

$$\mathcal{D}_q^n f(z) = \sum_{i=1}^{k} t_i \mathcal{D}_q^n f_i(z) = \mathcal{D}_q^n h(z) + (-1)^n \mathcal{D}_q^n \overline{g(z)}$$

where $\mathcal{D}_q^n h(z) = \sum_{i=1}^{k} t_i \mathcal{D}_q^n h_i(z)$ and $\mathcal{D}_q^n g(z) = \sum_{i=1}^{k} t_i \mathcal{D}_q^n g_i(z)$. Then h and g both are analytic in $\mathbb{D}$ with $h(0) = g(0) = h'(0) - 1 = g'(0) = 0$. A simple computation yields

$$\begin{aligned}
\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha \mathcal{D}_q^{n+1} h(z)}{z}\right) &= \Re\left(\sum_{i=1}^{k} t_i \frac{(1-\alpha)\,\mathcal{D}_q^n h_i(z) + \alpha \mathcal{D}_q^{n+1} h_i(z)}{z}\right) \\
&> \left|\sum_{i=1}^{k} t_i \frac{(-1)^n\,(1-\alpha)\,\mathcal{D}_q^n g_i(z) + (-1)^{n+1} \alpha \mathcal{D}_q^{n+1} g_i(z)}{z}\right| \\
&\ge \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha \mathcal{D}_q^{n+1} g(z)}{z}\right|.
\end{aligned}$$

This shows that $f \in \mathcal{W}_{\mathcal{H}}^{0}(n,\alpha,q)$ □

A sequence $\{c_k\}_{k=0}^{\infty}$ of non-negative real numbers is said to be a convex null sequence if $c_k \to 0$ as $k \to \infty$, and

$$c_0 - c_1 \ge c_1 - c_2 \ge c_2 - c_3 \ge ... \ge c_{k-1} - c_k \ge ... \ge 0.$$

To prove the convolution results, we need the following lemmas.

Lemma 3.2. [31] *Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function*

$$s(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^k$$

is analytic, and $\Re(s(z)) > 0$ in $\mathbb{D}$.

Lemma 3.3. [31] *Let the function p be analytic in $\mathbb{D}$ with $p(0) = 1$ and $\Re(p(z)) > 1/2$ in $\mathbb{D}$. Then for any analytic function F in $\mathbb{D}$, the function $p * F$ takes values in the convex hull of the image of $\mathbb{D}$ under F.*

Using Lemmas 3.2 and 3.3, we prove the following lemma.

Lemma 3.4. *Let $F \in \mathcal{W}(n,\alpha,q)$, then $\Re\left(\frac{F(z)}{z}\right) > \frac{1}{2}$.*

Proof. Suppose $F \in \mathcal{W}(n,\alpha,q)$ be given by $F(z) = z + \sum_{k=2}^{\infty} A_k z^k$, then

$$\Re\left(1 + \sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}\right) > 0,$$

which is equivalent to $\Re(p(z)) > 1/2$ in $\mathbb{D}$, where

$$p(z) = 1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}.$$

Now consider a sequence $\{c_k\}_{k=0}^{\infty}$ defined by

$$c_0 = 1 \quad \text{and} \quad c_{k-1} = \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1))} \quad \text{for} \quad k \geq 2.$$

It can be easily seen that the sequence $\{c_k\}_{k=0}^{\infty}$ is convex null sequence and using Lemma 3.2, the function

$$s(z) = 1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}$$

is analytic with $\Re(s(z)) > \frac{1}{2}$ in $\mathbb{D}$. Hence

$$\frac{F(z)}{z} = p(z) * \left(1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}\right)$$

$$= \left(1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}\right) * \left(1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}\right)$$

and making use of Lemma 3.3 we observe that $\Re\left(\frac{F(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$. □

Lemma 3.5. *Let F_1 and F_2 belong to $\mathcal{W}(n,\alpha,q)$. Then $F = F_1 * F_2 \in \mathcal{W}(n,\alpha,q)$.*

Proof. Let $F_1(z) = z + \sum_{k=2}^{\infty} A_k z^k$ and $F_2(z) = z + \sum_{k=2}^{\infty} B_k z^k$. Then the convolution of F_1 and F_2 is given by

$$F(z) = (F_1 * F_2)(z) = z + \sum_{k=2}^{\infty} A_k B_k z^k.$$

To prove that $F \in \mathcal{W}(n,\alpha,q)$, we have to show that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n F(z) + \alpha \mathcal{D}_q^{n+1} F(z)}{z}\right) > 0,$$

which is equivalent to

$$\Re\left(1 + \sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k B_k z^{k-1}\right) > 0$$

or

$$\Re\left(1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k B_k z^{k-1}\right) > \frac{1}{2}. \tag{3.1}$$

Since $F_1 \in \mathcal{W}(n,\alpha,q)$ we have

$$\Re\left(1+\frac{1}{2}\sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1))A_k z^{k-1}\right)>\frac{1}{2}$$

and by Lemma 3.4, $F_2 \in \mathcal{W}(n,\alpha,q)$ implies $\Re\left(\frac{F_2(z)}{z}\right)>\frac{1}{2}$ in $\mathbb{D}$ or

$$\Re\left(1+\frac{1}{2}\sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1))B_k z^{k-1}\right)>\frac{1}{2}.$$

By applying Lemma 3.3, we conclude we have (3.1). Hence, $F = F_1 * F_2 \in \mathcal{W}(n,\alpha,q)$. □

Now using Lemma 3.5, we prove that the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ is closed under convolutions of its members.

Theorem 3.6. *If f_1 and f_2 belong to $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$, then $f_1 * f_2 \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$.*

Proof. Let $f_1 = h_1 + \overline{g}_1$ and $f_2 = h_2 + \overline{g}_2$ be two functions in $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$. Then the convolution of f_1 and f_2 is defined as $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$. In order to prove that $f_1 * f_2 \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$, we need to prove that $F = h_1 * h_2 + \epsilon(g_1 * g_2) \in \mathcal{W}(n,\alpha,q)$ for each ϵ ($|\epsilon| = 1$). By Lemma 3.5, the class $\mathcal{W}(n,\alpha,q)$ is closed under convolutions for each ϵ ($|\epsilon| = 1$), $h_i + \epsilon g_i \in \mathcal{W}(n,\alpha,q)$ for $i = 1, 2$. Then both F_1 and F_2 given by

$$F_1 = (h_1 - g_1) * (h_2 - \epsilon g_2)$$

and

$$F_2 = (h_1 + g_1) * (h_2 + \epsilon g_2)$$

belong to $\mathcal{W}(n,\alpha,q)$. Since $\mathcal{W}(n,\alpha,q)$ is closed under convex combinations, then the function

$$F = \frac{1}{2}(F_1 + F_2) = h_1 * h_2 + \epsilon(g_1 * g_2)$$

belongs to $\mathcal{W}(n,\alpha,q)$. Thus $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ is closed under convolution. □

4. Partial sums

In this section, we examine sections (partial sums) of functions in the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$.

Theorem 4.1. *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ with $\alpha \geq 0$. Then for each ϵ ($|\epsilon| = 1$) and $|z| < 1/2$, we have*

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\right)>\frac{1}{4}.$$

Proof. Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$. Then by Theorem 2.2, $h + \epsilon g \in \mathcal{W}(n,\alpha,q)$ for ϵ ($|\epsilon| = 1$), so $\Re f_\epsilon(z) > 0$, where

$$f_\epsilon(z) = \frac{(1-\alpha)\,\mathcal{D}_q^n(h(z)+\epsilon g(z))+\alpha\mathcal{D}_q^{n+1}(h(z)+\epsilon g(z))}{z} = 1+\sum_{k=1}^{\infty}p_k z^k.$$

Moreover

$$\begin{aligned}
&\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\\
&=1+[2]_q^n(1+\alpha([2]_q-1)(a_2+\epsilon b_2)z+[3]_q^n(1+\alpha([3]_q-1)(a_3+\epsilon b_3)z^2\\
&=1+p_1z+p_2z^2.
\end{aligned}$$

It is easy to see that

$$|2p_2-p_1^2|\leq 4-|p_1|^2$$

Let $2p_2-p_1^2=p$. Then $p_2=p/2+p_1^2/2$ and $|p|\leq 4-|p_1|^2$. Also, let $p_1z=\gamma+i\beta$ and $\sqrt{pz}=\eta+i\delta$ where β,γ,δ,η are real numbers. Then for $|z|<1/2$

$$\gamma^2+\beta^2=|p_1|^2|z|^2\leq\frac{|p_1|^2}{4}$$

and

$$\delta^2=|p||z|^2-\eta^2\leq\frac{|p|}{4}-\eta^2\leq\frac{4-|p_1|^2}{4}-\eta^2\leq 1-(\gamma^2+\beta^2)-\eta^2$$

so that

$$\begin{aligned}
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\right)\\
&=\Re(1+p_1z+p_2z^2)\\
&=\Re(1+p_1z+\frac{p}{2}z^2+\frac{p_1^2}{2}z^2)\\
&=1+\gamma+\left(\frac{\eta^2}{2}-\frac{\delta^2}{2}\right)+\left(\frac{\gamma^2}{2}-\frac{\beta^2}{2}\right)\\
&=1+\gamma+\frac{\eta^2}{2}-\frac{1-\gamma^2-\beta^2-\eta^2}{2}+\frac{\gamma^2}{2}-\frac{\beta^2}{2}\\
&=\frac{1}{4}+\left(\gamma+\frac{1}{2}\right)^2+\eta^2\geq\frac{1}{4},
\end{aligned}$$

which gives the result. □

Theorem 4.2. *Let $f=h+\overline{g}\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, where h and g given by (1.1) with $b_1=0$. Then for each $j\geq 2$, $s_{1,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ for $|z|<1/2$.*

Proof. Let $f=h+\overline{g}\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$. It is clear that

$$s_{1,j}(f)(z)=s_1(h)(z)+\overline{s_j(g)(z)}=z+\sum_{k=2}^{j}\overline{b_kz^k}$$

It follows from Theorem 2.4 that for all $|z| < 1/2$,

$$\begin{aligned}
&\left|\frac{(1-\alpha)\,\mathcal{D}_q^n s_j(g)(z) + \alpha\mathcal{D}_q^{n+1} s_j(g)(z)}{z}\right| \\
&= \left|\sum_{k=2}^{j} [k]_q^n (1+\alpha([k]_q - 1)) b_k z^{k-1}\right| \\
&\leq \sum_{k=2}^{j} [k]_q^n (1+\alpha([k]_q - 1)) |b_k| |z^{k-1}| \\
&\leq \sum_{k=2}^{j} |z|^{k-1} = \frac{|z|(1-|z|^{j-1})}{1-|z|} < \frac{|z|}{1-|z|} \\
&< 1 = \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n s_1(h)(z) + \alpha\mathcal{D}_q^{n+1} s_1(h)(z)}{z}\right).
\end{aligned}$$

This implies that $s_{1,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z| < 1/2$. □

Theorem 4.3. *Let $f = h + \overline{g} \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, where h and g given by (1.1) with $b_1 = 0$, and let i and j satisfy of the following conditions:*

(i) $3 \leq i < j$,
(ii) $i = j \geq 2$,
(iii) $i = 3$ *and* $j = 2$.

Then $s_{i,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z| < 1/2$.

Proof. Let $\vartheta_i(h)(z) = \sum_{k=i+1}^{\infty} a_k z^k$ and $\vartheta_j(g)(z) = \sum_{k=j+1}^{\infty} b_k z^k$. Then

$$h = s_i(h) + \vartheta_i(h) \quad \text{and} \quad g = s_j(g) + \vartheta_j(g).$$

To prove $s_{i,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, it suffices to prove that $s_i(h) + \epsilon s_j(g) \in \mathcal{W}(n,\alpha,q)$ for ϵ ($|\epsilon| = 1$). If $f \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, then

$$\begin{aligned}
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (s_i(h) + \epsilon s_j(g)) + \alpha\mathcal{D}_q^{n+1}(s_i(h) + \epsilon s_j(g))}{z}\right) \\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h+\epsilon g) + \alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\right. \\
&\qquad \left. - \frac{(1-\alpha)\,\mathcal{D}_q^n (\vartheta_i(h) + \epsilon\vartheta_j(g)) + \alpha\mathcal{D}_q^{n+1}(\vartheta_i(h) + \epsilon\vartheta_j(g))}{z}\right) \\
&\geq \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h+\epsilon g) + \alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\right) \\
&\qquad - \left|\frac{(1-\alpha)\,\mathcal{D}_q^n (\vartheta_i(h) + \epsilon\vartheta_j(g)) + \alpha\mathcal{D}_q^{n+1}(\vartheta_i(h) + \epsilon\vartheta_j(g))}{z}\right|.
\end{aligned} \tag{4.1}$$

By assumption, we see that

$$\frac{(1-\alpha)\,\mathcal{D}_q^n(h+\epsilon g)+\alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\prec\frac{1+z}{1-z},$$

where $\prec$ is the subordination symbol. From the last relation, we conclude that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(h+\epsilon g)+\alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\right)\geq\frac{1-|z|}{1+|z|}. \tag{4.2}$$

Case (i): $3\leq i<j$

Applying Theorems 2.4 and 2.6, we observe that

$$\begin{aligned}&\left|\frac{(1-\alpha)\,\mathcal{D}_q^n(\vartheta_i(h)+\epsilon\vartheta_j(g))+\alpha\mathcal{D}_q^{n+1}(\vartheta_i(h)+\epsilon\vartheta_j(g))}{z}\right|\\&=\left|\sum_{k=i+1}^{j}[k]_q^n(1+\alpha([k]_q-1))a_kz^{k-1}+\sum_{k=j+1}^{\infty}[k]_q^n(1+\alpha([k]_q-1))(a_k+\epsilon b_k)z^{k-1}\right|\\&\leq\sum_{k=i+1}^{j}2|z|^{k-1}+\sum_{k=j+1}^{\infty}2|z|^{k-1}=2\frac{|z|^i}{1-|z|}\end{aligned}\tag{4.3}$$

Using (4.1), (4.2) and (4.3), we obtain

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right)\geq\frac{1-|z|}{1+|z|}-2\frac{|z|^i}{1-|z|}. \tag{4.4}$$

For $4\leq i<j$ and $|z|=1/2$, the inequality (4.4) gives

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right)\geq\frac{1}{3}-\frac{1}{4}>0.$$

Since $\Re\left(\frac{(1-\alpha)\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right)$ is harmonic, it assumes its minimum value on the circle $|z|=1/2$. Hence, if $4\leq i<j$ then $s_{i,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$.

If $i=3<j$, then in view of Theorem 2.4 and Theorem 4.1, we attain

$$\begin{aligned}&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_j(g))}{z}\right)\\&=\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\right.\\&\qquad\left.+\epsilon\sum_{k=4}^{j}[k]_q^n(1+\alpha([k]_q-1))b_kz^{k-1}\right)\\&\geq\frac{1}{4}-\sum_{k=4}^{j}[k]_q^n(1+\alpha([k]_q-1))|b_kz^{k-1}|\\&\geq\frac{1}{4}-\frac{|z|^3}{1-|z|}\end{aligned}$$

so that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_j(g))}{z}\right)>0$$

for $|z|<1/2$, and thus $s_{3,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$.

Case (ii): $i=j\geq 2$

If $i=j\geq 4$, then the inequality (4.4) gives $s_{i,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. For $i=j=2$, $s_{2,2}(f)(z)=z+a_2z^2+\overline{b_2z^2}$. Using Theorem 2.6, we get

$$\begin{aligned}&\Re\big(1+[2]_q^n(1+\alpha([2]_q-1))(a_2+\epsilon b_2)z\big)\\&\geq 1-[2]_q^n(1+\alpha([2]_q-1))|a_2+\epsilon b_2||z|\\&\geq 1-2|z|>0\end{aligned}$$

in $|z|<1/2$.

If $i=j=3$, then Theorem 4.1 shows that $s_{3,3}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. Therefore, we prove that for $i=j\geq 2$, $s_{i,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$.

Case (iii): $i=3$ and $j=2$.

In view of Theorems 2.4 and 4.1, we have

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_2(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_2(g))}{z}\right)$$

$$=\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}-\epsilon[3]_q^n(1+\alpha([3]_q-1))b_3z^2\right)$$

$$\geq\frac{1}{4}-|z|^2=\frac{1}{4}-\frac{1}{2^2}=0$$

for $|z|<1/2$. Thus $s_{3,2}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. □

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Strongly nonlinear periodic parabolic equation in Orlicz spaces

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Abstract. In this paper, we prove the existence of a weak solution to the following nonlinear periodic parabolic equations in Orlicz-spaces:

$$\frac{\partial u}{\partial t} - div(a(x,t,\nabla u)) = f(x,t)$$

where $-div(a(x,t,\nabla u))$ is a Leray-Lions operator defined on a subset of $W_0^{1,x}L_M(Q)$. The Δ_2-condition is not assumed and the data f belongs to $W^{-1,x}E_{\overline{M}}(Q)$.
The Galerkin method and the fixed point argument are employed in the proof.

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1. Introduction

Let Ω be a bounded subset of $\mathbb{R}^N$, and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. In this paper we deal with the following periodic parabolic boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - div(a(x,t,\nabla u)) = f(x,t) \text{ in } Q\ , \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T)\ , \\ u(x,0) = u(x,T) \text{ in } \Omega\ , \end{cases} \tag{1.1}$$

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where A is a second-order operator in divergence form

$$A(u) = -div(a(x,t,\nabla u)),$$

with the coefficient a satisfying Leray-Lions conditions related to some N-function.
The study of nonlinear partial differential equations in Orlicz-spaces is motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field (see for examples [1], [10], [14], [15], [16] and [21]).
Consider first the case where a have polynomial growth with respect to u and ∇u. Therefore A is a bounded operator from $L^p(0,T,W^{1,p}(\Omega))$, $1 < p < \infty$, into its dual. In this setting, Brézis and Browder in cite16 proved the existence of problem (1) when $p > 2$ and the periodic condition is replaced by the initial one, and by Landes and Mustonen when $1 < p < 2$ [19].
Specifically, when we have the periodicity condition Boldrini and Crema in [4] studied the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = m(t)g(u) + h(x,t) \text{ in } Q_T; \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T); \\ u(x,T) = u(x,0) \text{ in } \Omega; \end{cases} \tag{1.2}$$

g is a continuous function such that $|g(v)| \leq a(|v|s + 1)$, where s and a are positive constants. The existence of a solution to this problem is established under the condition $0 \leq s < p-1$, and for $s = p-1$ by using Schauder's fixed point theorem. Related topics can be found in [7], [8], [9]. However, when attempting to relax the restriction on a, we replace the space $L^p(0,T,W_0^{1,p})$ with an inhomogeneous Orlicz-Sobolev space $W_0^{1,x}L_M(Q)$, constructed from an Orlicz space L_M instead of L^p, where the N-function M is related to the actual growth of a. Several studies have explored this setting, considering $u(x,0) = u_0$ and a depending on u and ∇u, see for instance, the works of Donaldson in [6] and Robert in [20], who proved the existence of a solution for a nonlinear parabolic problem under the Δ_2 condition, $u^2 \leq cM(ku)$, with c and k are positive constants, and A is monotone. Additionally, in cases where the Δ_2 condition is not assumed and under various assumptions, other authors have demonstrated the existence of solutions to diverse parabolic problems (see [2], [14], [17], [19]).
The objective of this paper is to establish the existence of a solution to problem (1.1) when f belongs to $W^{-1,x}E_{\overline{M}}(Q)$, without assuming the Δ_2 condition. Moreover, we consider the periodicity condition instead of the initial one, which necessitates demonstrating the existence of the approximate problem once more. To achieve this, we assume that $u^2 \leq cM(ku)$ with c and k are positive constants.
We employ the Galerkin method due to Landes and Mustonen, along with the fixed point argument due to Schauder.
The paper is structured as follows: In Section 2, we provide a review of some preliminary concepts concerning Orlicz-Sobolev spaces, along with various inequalities and compactness results. Section 3 is dedicated to stating the assumptions and presenting the main result. In the fourth section, we prove the existence theorem. In the appendix we prove the existence of a solution to the approximate problem.

2. Preliminaries

2.1. Orlicz-Sobolev Spaces-Notations and Properties

- let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e continuous, convex, with $M(t) > 0$ for $t > 0, M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$.
  Equivalently, M admits the representation: $M(t) = \int_0^t m(\tau)d\tau$ where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $m(0) = 0,\ m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$.
  The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{m}(\tau)d\tau$ where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $m(t) = \sup\{s : m(s) \leq t\}$.
  The N-function M is said to satisfy a Δ_2 condition if, for some $k > 0$:
  $$M(2t) \leq kM(t) \quad \forall t \geq 0$$
  When this inequality holds only for $t \geq t_0 > 0$, M is said to satisfy the Δ_2-condition near infinity.
- Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))dx < +\infty$ (resp. $\int_\Omega M(u(x)/\lambda)dx < +\infty$ for some $\lambda > 0$).
  $L_M(\Omega)$ is a Banach space under the norm:
  $$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\}$$
  and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
  The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.
  The dual of $E_M(\Omega)$ can be identified with $L_M(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.
  The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition (near infinity only if Ω has finite measure).

- We now turn to the Orlicz-Sobolev spaces. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm:
  $$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}\,.$$
  Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspace of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right)$ and $\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)$.
  The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

- We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$
$$\int_\Omega M\left(\left(D^\alpha u_n - D^\alpha u\right)/\lambda\right)dx \to 0 \text{ for all } |\alpha| \leq 1$$
This implies convergence for $\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)$. Note that, if $u_n \to u$ in $L_M(\Omega)$ for the modular convergence and $v_n \to v$ in $L_M(\Omega)$ for the modular convergence, we have
$$\int_\Omega u_n v_n dx \to \int_\Omega uv dx \quad \text{ as } n \to \infty$$
If M satisfies the Δ_2-condition on $\mathbb{R}^+$, then modular convergence coincides with norm convergence.

- Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$ denote the space of distributions on Ω which can be written as sums of derivatives of order at most 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm.

- If the open set Ω has the segment property then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [13], [19]). Consequently, the action of a distribution S in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined, it will be noted by $< S, u >$.

2.2. The Inhomogeneous Orlicz-Sobolev

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega\times]\,0, T[$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces of order 1 are defined as follows
$$W^{1,x}L_M(Q) = \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q), \forall|\alpha| \leq 1\}$$
and
$$W^{1,x}E_M(Q) = \{u \in E_M(Q) : D_x^\alpha u \in E_M(Q), \forall|\alpha| \leq 1\}$$
The last space is a subspace of the former. Both are Banach spaces under the norm
$$\|u\| = \sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{M,Q} \, .$$
The space $W_0^{1,x}L_M(Q)$ is defined as the (norm) closure in $W^{1,x}L_M(Q)$ of $\mathcal{D}(Q)$ and we have.
$$W_0^{1,x}L_M(Q) = \overline{\mathcal{D}(Q)}^{\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)}.$$
We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which has (N+1) copies. We shall also consider the weak topologies $\sigma(\Pi L_M \Pi E_M)$ and $\sigma(\Pi L_M, \Pi L_M)$. If $u \in W^{1,x}L_M(Q)$, then the function: $t \mapsto u(t) = u(., t)$ is defined on $(0, T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$, then $u(., t)$ is $W^1E_M(\Omega)$-valued and is strongly measurable.
Furthermore, the following continuous imbedding holds: $W^{1,x}E_M(Q) \subset L^1(0,T)$,

$W^1E_M(\Omega)$. The space $W^{1,x}L_M(Q)$ is not in general separable; if $u \in W^{1,x}L_M(Q)$, we cannot conclude that the function $u(t)$ is measurable from $(0,T)$ into $W^{1,x}L_M(\Omega)$. However the scalar function $t \mapsto \|D_x^\alpha u(t)\|_{M,\Omega}$ is in $L^1(0,T)$ for all $|\alpha| \leq 1$. Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \Pi E_M$. Poincare's inequality also holds in $W_0^{1,x}L_M(Q)$ and then there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has

$$\sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{M,Q} \leq C \sum_{|\alpha|=1} \|D_x^\alpha u\|_{M,Q}$$

thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix}$$

F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, up to an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q)$ and it is shown that

$$W^{-1,x}L_M(Q) = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha \ : \ f_\alpha \in L_{\overline{M}}(Q) \right\}$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{M},Q}$$

where the infinum is taken on all possible decompositions

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, f_\alpha \in L_{\overline{M}}(Q)$$

The space F_0 is then given by

$$F_0 = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

2.3. Some inequalities

Lemma 2.1. [17] *Let M be an N-function, we have the following inequality:*

$$st \leq M(s) + \overline{M}(t)$$

called Young inequality.

Lemma 2.2. [17] *The generalized Holder inequality*

$$\left| \int_\Omega u(x)v(x)|dx \right| \leq 2\|u\|_M \|v\|_{\overline{M}}$$

hold for any pair function $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$.

Proof. The proof of this inequalities is detailed in [17] (see pages 18 for the first one 111 for the second). □

2.4. Approximation theorem and trace result

Let Ω is an open subset of $\mathbb{R}^N$ with the segment property and I is a sub-interval of $\mathbb{R}$ (both possibly unbounded) and $Q = \Omega \times I$. It is easy to see that Q also satisfies the segment property.

Definition 2.3. [12] We say that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha|\leq 1} D_x^\alpha u_n^\alpha + u_n^0$$

$$u = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$$

with $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q)$ for the modular convergence for all $|\alpha| \leq 1$ and $u_n^0 \to u^0$ strongly in $L^2(Q)$.

This implies, in particular, that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$ for the weak topology $\sigma(\Pi L_M + L^2, \Pi L_M \cap L^2)$ in the sense that $< u_n, v > \to < u, v >$ for all $v \in W_0^{1,x}L_M(Q) \cap L^2(Q)$, where here and throughout the paper, $< ., . >$ means either the pairing between $W_0^{1,x}L_M(Q)$ and $W^{-1,x}L_{\overline{M}}(Q)$, or the pairing between $W_0^{1,x}L_M(Q) \cap L^2(Q)$ and $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$. Indeed,

$$\langle u_n, v\rangle = \sum_{|\alpha|\leqslant 1} (-1)^{|\alpha|} \int_Q u_n^\alpha D_x^\alpha v \, dx \, dt + \int_Q u_n^0 v \, dx \, dt$$

and since for all $|\alpha| \leqslant 1, u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q)$ for the modular convergence, and so for $\sigma\left(L_{\overline{M}}, L_M\right)$, we have

$$\sum_{|\alpha|\leqslant 1} (-1)^{|\alpha|} \int_Q u_n^\alpha D_x^\alpha v \, dx \, dt + \int_Q u_n^0 v \, dx \, dt \to \sum_{|\alpha|\leqslant 1} (-1)^{|\alpha|} \int_Q u^\alpha D_x^\alpha v \, dx \, dt + \int_Q u^0 v \, dx \, dt = \langle u, v\rangle.$$

Moreover, if $v_n \to v$ in $W_0^{1,x}L_M(Q) \cap L^2(Q)$ for the modular convergence (i.e. $v_n \to v$ in $W_0^{1,x}L_M(Q)$ for the modular convergence and in $L^2(Q)$ strong), we have $\langle u_n, v_n\rangle \to \langle u, v\rangle$ as $n \to \infty$.

Theorem 2.4. [12] *If $u \in W^{1,x}L_M(\Omega) \cap L^2(\Omega)$ (respectively $W_0^{1,x}L_M(\Omega) \cap L^2(\Omega)$) and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$, then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q})$ (respectively $\mathcal{D}(I, \mathcal{D}(\Omega))$) such that*

$$v_j \to u \text{ in } W^{1,x}L_M(\Omega) \cap L^2(\Omega)$$

$$\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$$

for the modular convergence.

Remark 2.5. If in the statement of theorem (2.4), one considers $\Omega \times \mathbb{R}$ instead of Q we have $\mathcal{D}(\Omega \times \mathbb{R})$ is dense in

$$\{u \in W_0^{1,x}(\Omega \times \mathbb{R}) \cap L^2(\Omega \times \mathbb{R}) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times \mathbb{R}) + L^2(\Omega \times \mathbb{R})\}$$

for the modular convergence.

A first application of Theorem (2.4) is the following trace result generalizing a classical result which states that if u belongs to $L^2(a,b;H_0^1(\Omega))$ and $\frac{\partial u}{\partial t}$ belongs to $L^2(a,b;H^{-1}(\Omega))$, then u is in $C(a,b;L^2(\Omega))$.

Lemma 2.6. [12] *Let $a < b \in \mathbb{R}$ and let Ω be a bounded subset of $\mathbb{R}^N$ with the segment property, then*

$$\{u \in W_0^{1,x}L_M(\Omega\times(a,b))\cap L^2(\Omega\times(a,b)); \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega\times(a,b))+L^2(\Omega\times(a,b))\}$$

is a subset of $C([a,b],L^2(\Omega))$.

3. Existence result

3.1. Assumption and statement of main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with the segment property, and Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. Let M be an N-function.
Consider the second order operator $A : D(A) \subset W_0^{1,x}L_M(Q) \rightarrow W^{-1,x}L_{\overline{M}}(Q)$ of the form:

$$A(u) = -div(a(x,t,\nabla u))$$

where $a : \Omega \times (0,T) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ are a Carateodory function satisfying for almost every $(x,t) \in \Omega \times (0,T)$ and all $\xi \neq \xi^* \in \mathbb{R}^N$ we have the following assumptions:

$$|a(x,t,\xi)| \leq \beta(h_1(x,t) + \overline{M}^{-1}M(\delta|\xi|))\ ; \tag{3.1}$$

$$[a(x,t,\xi) - a(x,t,\xi^*)][\xi - \xi^*] > 0\ ; \tag{3.2}$$

$$a(x,t,\xi)\xi \geq \alpha M\big(\frac{|\xi|}{\lambda}\big)\ ; \tag{3.3}$$

$$f \in W^{-1,x}E_{\overline{M}}(Q)\ ; \tag{3.4}$$

where $h_1 \in L^1(Q)$, and $\beta, \delta, \alpha, \lambda > 0$.
and suppose that there exist $s^{'} > 0$ and c, k two positive constant such that for all $s \geq s^{'}$:

$$s^2 \leq cM(ks) \tag{3.5}$$

We shall prove the following existence theorem

Theorem 3.1. *Assume that (3.1)-(3.5) hold true then there exist a unique solution $u \in D(A) \cap W_0^{1,x}L_M(Q) \cap C(0,T,L^2(\Omega))$ of (1.1) in the following sense:*

$$< \frac{\partial u}{\partial t}, \varphi >_Q + \int_Q a(x,t,\nabla u)\nabla\varphi dxdt =< f, \varphi >_Q; \tag{3.6}$$

for every $\varphi \in W_0^{1,x}L_M(Q) \cap L^2(Q)$ *with* $\frac{\partial \varphi}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$*. where here* $< .,. >$ *means for either the pairing between* $W_0^{1,x}L_M(Q)$ *and* $W^{-1,x}L_{\overline{M}}(Q)$*, or between* $W_0^{1,x}L_M(Q) \cap L^2(Q)$ *and* $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$*.*

Integrating by part and using the periodicity condition equation (3.6) can be written as:

$$-\int_Q \frac{\partial \varphi}{\partial t} u dxdt + \int_Q a(x,t,\nabla u)\nabla \varphi dxdt =< f, \varphi >_Q \tag{3.7}$$

Remark 3.2. Note that all term in (3.7) are well defined, and by the trace result of lemma (2.6) we have that $u \in C([0,T), L^2(\Omega))$ wish make sense of the periodicity condition.

4. The proof of the main result

The proof of theorem (3.1) is divided into five steps:

Proof. **Step 1**: Firstly we have to prove that the solution u is unique. For that we suppose that there exist another solution v of problem (1.1) then v satisfy also (3.6), then by taking $\varphi = u(t) - v(t)$ we can easily see that

$$\frac{1}{2}\frac{d}{dt}\int_Q (u(t)-v(t))^2 dx + \int_Q (a(x,t,\nabla u) - a(x,t,\nabla v))(\nabla u - \nabla v) dxdt = 0 \tag{4.1}$$

Using periodicity condition and (3.2) we get $\nabla u = \nabla v$, then we have by (4.1) that $u(t) = v(t)$ for almost every $t \in (0,T)$, finally we deduce that $u = v$.
Step 2: *Approximate problem:* As in [12] we will use Galerkin method due to Landes and Mustonen [19]. For that we choose a sequence $\{w_1, w_2, w_3, \cdots\}$ in $\mathcal{D}(\Omega)$ such that $\bigcup_{n=1}^{\infty} V_n$ with

$$V_n = span\{w_1, w_2, w_3, \cdots\}$$

is dense in $H_0^m(\Omega)$with m large enough such that $H_0^m(\Omega)$ is continuously embedded in $C^1(\Omega)$. For any $v \in H_0^m(\Omega)$, there exists a sequence $(v_k) \subset \bigcup_{n=1}^{\infty} V_n$ such that $v_k \to v$ in $H_0^m(\Omega)$and in $\mathcal{C}^1(\overline{\Omega})$ too.
We denote further $\mathcal{V}_n = C([0,T], V_n)$. We have that the closure of $\bigcup_{n=1}^{\infty} \mathcal{V}_n$ with respect to the norm:

$$\|v\|_{C^{1,0}(Q)} = \sup_{|\alpha|\le 1} \{|D_x^{|\alpha|} v(x,t)| : (x,t) \in Q\}$$

contains $\mathcal{D}(Q)$, for more detail see [11] and [18]).
This implies that, for any $f \in W^{-1,x}E_{\overline{M}}(Q)$, there exists a sequence $(f_k) \subset \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that $f_k \to f$ strongly in $W^{-1,x}E_{\overline{M}}(Q)$. Indeed, let $\varepsilon > 0$ be given. Writing

$$f = \sum_{|\alpha|\le 1} D_x^{\alpha} f^{\alpha}$$

for all $|\alpha| \leq 1$, there exists $g^\alpha \in \mathcal{D}(Q)$ such that, $\|f^\alpha - g^\alpha\|_{\overline{M},Q} \leq \frac{\varepsilon}{2N+2}$. Moreover, by setting $g = \sum_{|\alpha|\leq 1} D_x^\alpha g^\alpha$, we see that for any $g \in \mathcal{D}(Q)$, and so there exists $\varphi \in \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that $\|g - \varphi\|_{\infty,Q} \leq \frac{\varepsilon}{2meas(Q)}$. We deduce then

$$\|f^\alpha - g^\alpha\|_{W^{-1,x}E_{\overline{M}}(Q)} \leq \sum_{|\alpha|\leq 1} \|f^\alpha - g^\alpha\|_{\overline{M},Q} + \|g - \varphi\|_{\infty,Q}$$

Now, let us consider the following approximate problem:

$$\begin{cases} u_n \in \mathcal{V}_n; \frac{\partial u_n}{\partial t} \in L^1(0, T, V_n); \\ u_n(x, 0) = u_n(x, T) \ ; \\ \text{and for all } \varphi \in \mathcal{V}_n \\ \int_Q \frac{\partial u_n}{\partial t} \varphi dxdt + \int_Q a(x, t, \nabla u_n) \nabla \varphi dxdt = \int_Q f_n \varphi dxdt \end{cases} \tag{4.2}$$

See the appendix for the prove of the existence of $u_n \in \mathcal{V}_n$.

Step 3: a priori estimates

Let as prove that:

$$\|u_n\|_{W_0^{1,x}L_M(Q)} \leq C \quad ; \int_Q a\left(x, t, \nabla u_n\right) \nabla u_n \ \mathrm{d}x \leq C' \tag{4.3}$$

and

$$a\left(x, t, \nabla u_n\right) \text{ is bounded in } \left(L_{\overline{M}}(Q)\right)^N \tag{4.4}$$

where here C, C' are a positives constants not depending on n.

Proof. Taking u_n as a test function in (4.2), then using periodicity condition and young inequality we have

$$\int_Q a(x, t, \nabla u_n) \nabla u_n dxdt \leq \frac{1}{\epsilon} \|f_n\|_{\overline{M},Q} + \epsilon \|u_n\|_{M,Q}.$$

By using (3.2) and applying Poinccare inequality there exist $C_1 > 0$ such that

$$\alpha \int_Q M\left(\frac{|\nabla u_n|}{\lambda}\right) dxdt \leq \|f_n\|_{\overline{M},Q} + \epsilon C_1 \int_Q M\left(\frac{|\nabla u_n|}{\lambda}\right) dxdt.$$

By a choice of ϵ and the fact that $\|f_n\|_{\overline{M},Q} \leq C$ we obtain

$$\int_Q M\left(\frac{|\nabla u_n|}{\lambda}\right) dxdt \leq C. \tag{4.5}$$

This implies that (u_n) is bounded in $W_0^{1,x}L_M(Q)$ and so in $L^2(Q)$. By using (3.1) and (4.5) we can conclude that there exist a constant $C' > 0$ such that

$$\int_Q a(x, t, \nabla u_n) \nabla u_n dxdt \leq C'; \tag{4.6}$$

To prove that $a(x, t, \nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$, let $\varphi \in (E_{\overline{M}}(Q))^N$ with $\|\varphi\|_{M,Q} = 1$. By (3.2) we have

$$\int_Q (a(x, t, \nabla u_n) - a(x, t, \varphi))(\nabla u_n, \varphi) dxdt > 0$$

which gives

$$\int_Q a(x,t,\nabla u_n)\varphi < \int_Q a(x,t,\nabla u_n)\nabla u_n dxdt - \int_Q a(x,t,\varphi)(\nabla u_n - \varphi)dxdt$$

Using (3.1) and (4.3) we can easily see that

$$\int_Q a(x,t,\nabla u_n)\varphi < C$$

and so $a(x,t,\nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$.
Thus for a subsequence still denote u_n and for some $h \in (L_{\overline{M}}(Q))^N$:

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right), \qquad (4.7)$$

$$\textit{and} \text{ weakly in } L^2(Q).$$

$$a(x,t,\nabla u_n) \rightharpoonup h \text{ weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\left(\Pi L_{\overline{M}}, \Pi E_{\overline{M}}\right) \qquad (4.8)$$

□

Step 4: Almost everywhere convergence of the gradient.
For all $\varphi \in C^1(0,T,\mathcal{D}(\Omega))$, we get by (4.2) and (4.8) that

$$-\int_Q u\frac{\partial \varphi}{\partial t} + \int_Q h\nabla\varphi dxdt = \int_Q f\nabla\varphi dxdt. \qquad (4.9)$$

We can see by taking φ arbitrary in $\mathcal{D}(Q)$ that $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q)$, then by theorem (2.4) there exist a subsequence denote $v_k \in \mathcal{D}(Q)$ such that:

$$v_k \to u \text{ in } W_0^{1,x}L_M(Q) \cap L^2(Q) \text{ and } \frac{\partial v_k}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$$

for the modular convergence, then by lemma (2.6), we have $v_k \to u$ in $C([0,T],L^2(\Omega))$ and so $u \in C([0,T],L^2(\Omega))$.
From (4.2), (3.7) we have

$$\begin{aligned}
&\limsup_{n\to\infty}\int_Q a(x,t,\nabla u_n)\nabla u_n - h\nabla v_k dxdt \\
&\leq \limsup_{n\to\infty}\Big(-\int_Q \frac{\partial u_n}{\partial t}u_n dxdt\Big) + \int_Q \frac{\partial v_k}{\partial t}u dxdt \\
&\quad + \limsup_{n\to\infty}\int_Q \big(f_n u_n dxdt - \int_Q f_n v_k\big)dxdt \\
&= \limsup_{n\to\infty}\Big(\int_Q \frac{\partial u_n}{\partial t}(v_k - u_n)dxdt\Big) + \int_Q f(u - v_k)dxdt
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
-\int_Q \frac{\partial v_k}{\partial t}u dxdt &= \lim_{n\to\infty} -\int_Q \frac{\partial v_k}{\partial t}u_n dxdt \\
&= \lim_{n\to\infty} -\int_Q \frac{\partial u_n}{\partial t}v_k dxdt + \int_\Omega \big[u_n(t)v_k(t)\big]_0^T dx
\end{aligned}$$

then the periodicity condition imply

$$-\int_Q \frac{\partial v_k}{\partial t} u dxdt = \lim_{n\to\infty} -\int_Q \frac{\partial u_n}{\partial t} v_k dxdt.$$

For the first term in the right hand sand we have

$$\begin{aligned}
\limsup_{n\to\infty} \int_Q \frac{\partial u_n}{\partial t}(v_k - u_n)dxdt &= \limsup_{n\to\infty} \big(-\frac{1}{2}\frac{d}{dt}\int_Q (u_n(t) - v_k(t))^2 dxdt\big) \\
&+ \limsup_{n\to\infty} \int_Q \frac{\partial v_k}{\partial t}(v_k - u_n)dxdt \\
&= \limsup_{n\to\infty} \big(-\frac{1}{2}\int_\Omega \big[u_n(t) - v_k(t))^2\big]_0^T dx\big) \\
&+ \limsup_{n\to\infty} \int_Q \frac{\partial v_k}{\partial t}(v_k - u_n)dxdt
\end{aligned}$$

the fact that $\frac{\partial v_k}{\partial t} \in E_{\overline{M}}(Q)$ and $v_k \to u$ gives

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_Q \frac{\partial v_k}{\partial t}(v_k - u_n)dxdt = 0.$$

By periodicity condition we have

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_Q \frac{\partial u_n}{\partial t}(v_k - u_n)dxdt = 0.$$

Then we obtain

$$\limsup_{n\to\infty} \int_Q a(x,t,\nabla u_n)\nabla u_n dxdt = \int_Q h\nabla v_k dxdt + \int_Q f(u - v_k)dxdt$$

Having in mind that v_k converge strongly to u in $W_0^{1,x}L_M(Q)$ for the modular convergence, we can pass to the limit sup in k, to deduce

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_Q a(x,t,\nabla u_n)\nabla u_n = \int_Q h\nabla v dxdt. \tag{4.10}$$

Fix a real number $r > 0$ and any $k \in \mathbb{N}$, we denote by χ_k^r and χ^r the characteristic functions of $Q_k^r = \{(x,t) \in Q : |\nabla v_k| \leqslant r\}$ and $Q^r = \{(x,t) \in Q\colon |\nabla u| \leqslant r\}$, respectively. We also denote by $\varepsilon(n,k,s)$ all quantities (possibly different)such that

$$\lim_{s\to\infty} \lim_{k\to\infty} \lim_{n\to\infty} \varepsilon(n,k,s) = 0,$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then k, and finally s. Similarly, we will write only $\varepsilon(n)$, or $\varepsilon(n,k), \ldots$ to mean that the limits are only on the specified parameters.

Taking $s \geq r$ one has

$$\begin{aligned} 0 &\leq \int_{Q^r} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big(\nabla u_n - \nabla u\big)dxdt \\ &\leq \int_{Q^s} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big(\nabla u_n - \nabla u\big)dxdt \\ &= \int_{Q^s} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u\chi^s)\big(\nabla u_n - \nabla u\chi^s\big)dxdt \\ &\leq \int_{Q} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u\chi^s)\big(\nabla u_n - \nabla u\chi^s\big)dxdt. \end{aligned}$$

On the other hand

$$\begin{aligned} &\int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla u\chi^s\right)\right]\left[\nabla u_n - \nabla u\chi^s\right]dxdt \\ &\quad= \int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k\chi_k^s\right)\right] \\ &\qquad\times\left[\nabla u_n - \nabla v_k\chi_k^s\right]dxdt \\ &\quad+ \int_Q a\left(x,t,\nabla v_k\chi_k^s\right)\left[\nabla u_n - \nabla v_k\chi_k^s\right]dxdt \\ &\quad+ \int_Q a\left(x,t,\nabla u_n\right)\left[\nabla v_k\chi_k^s - \nabla u\chi^s\right]dxdt \\ &\quad+ \int_Q a\left(x,t,\nabla u\chi^s\right)\left[\nabla u\chi^s - \nabla u_n\right]dxdt \\ &\quad= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We shall go to the limit in all integrals I_i (for i=1, 2, 3, 4) as first n, then k, and finally s tend to infinity.
Starting with I_2 and letting $n \to \infty$, since $\nabla u_n \rightharpoonup \nabla u$ in $L_{\overline{M}}(Q)^N$ by Lebesgue theorem we get that

$$I_2 = \int_Q a\left(x,t,\nabla v_k\chi_k^s\right)\left[\nabla u - \nabla v_k\chi_k^s\right]dxdt + \varepsilon(n).$$

Letting then $k \to \infty$ this imply

$$I_2 = \int_{\{|\nabla u| > s\}} a\left(x,t,0\right)\nabla u dxdt + \varepsilon(n,k).$$

Finally we deduce when s tends to infinity that

$$I_2 = \varepsilon(n,k,s). \tag{4.11}$$

For I_3 we have by letting $n \to \infty$ and using (4.8) that

$$I_3 = \int_Q h(\nabla v_k\chi_k^s - \nabla u\chi^s)dxdt$$

and so, by letting $k \to \infty$ in the integral of the last side and using the fact that $\nabla v_k \chi_k^s \to \nabla u \chi^s$ strongly in $(E_M(Q))^N$, we deduce that $I_2 = \varepsilon(n,k)$. For the fourth term I_4, we have, by letting $n \to \infty$,

$$I_4 = -\int_{\{|\nabla u|>s\}} a(x,t,0)\nabla u dxdt + \varepsilon(n),$$

and since the first term of the last side tends to zero as $s \to \infty$, we obtain $I_4 = \varepsilon(n,k,s)$. We have then proved that

$$\begin{aligned}&\int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla u\chi^s\right)\right]\left[\nabla u_n - \nabla u\chi^s\right] dxdt \\ &= \int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k\chi_k^s\right)\right]\left[\nabla u_n - \nabla v_k\chi_k^s\right] dxdt \\ &\quad + \varepsilon(n,k,s).\end{aligned}$$

Finally we can deduce that

$$0 \le \int_{Q^r} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\big(\nabla u_n - \nabla u\big) dxdt \tag{4.12}$$

$$\le \int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k\chi_k^s\right)\right]\left[\nabla u_n - \nabla v_k\chi_k^s\right] dxdt + \varepsilon(n,k,s)$$

we can write

$$\begin{aligned}\int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k\chi_k^s\right)\right]\left[\nabla u_n - \nabla v_k\chi_k^s\right] dxdt &= \int_Q a(x,t,\nabla u_n)\nabla u_n dxdt \\ &- \int_Q (a(x,t,\nabla u_n) - a(x,t,\nabla v_k\chi_k^s)\nabla v_k\chi_k^s dxdt \\ &= J_1 + J_2 + J_3.\end{aligned} \tag{4.13}$$

First all we have by using (4.10) that

$$\limsup_{k\to\infty}\limsup_{n\to\infty} J_1 = \int_Q h\nabla u dxdt. \tag{4.14}$$

For J_2, letting first $n \to \infty$ then k, and using Lebesgue theorem hence $\nabla v_k\chi_k^s \to \nabla u\chi^s$ strongly in $(E_M(Q))^N$ we get

$$J_2 = -\int_Q (h - a(x,t,\nabla u\chi^s))\nabla u\chi^s dxdt + \varepsilon(n,k).$$

We can easily see that

$$J_2 = -\int_{Q^s} (h - a(x,t,\nabla u))\nabla u dxdt + \varepsilon(n,k). \tag{4.15}$$

Letting $n \to \infty$ on J_3 we have

$$J_3 = -\int_{Q^s} a(x,t,\nabla u)\nabla u dxdt - \int_{\{|\nabla u|>s\}} a(x,t,0)\nabla u dxdt. \tag{4.16}$$

Finally by combining (4.13), (4.14), (4.15), (4.16) we conclude that

$$\int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k \chi_k^s\right)\right]\left[\nabla u_n - \nabla v_k \chi_k^s\right] dxdt = \tag{4.17}$$

$$= -\int_{\{|\nabla u|>s\}} a(x,t,0)\nabla u dxdt + \varepsilon(n,k)$$

So, when s tend to infinity (4.12) and(4.17) gives

$$\lim_{n\to\infty}\int_{Q^r} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\big(\nabla u_n - \nabla u\big) dxdt = 0$$

and thus, as in the elliptic case see [3], we deduce that, for a subsequence still denoted by u_n,

$$\nabla u_n \to \nabla u \quad \text{a.e. in } Q \tag{4.18}$$

Since $a(x,t,.)$ is continuous then

$$a(x,t,\nabla u_n) \to a(x,t,\nabla u) \text{ a.e in } Q$$

If we take in consideration that $a(x,t,\nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$ we have by lemma (4.4) of [19] that

$$a(x,t,\nabla u_n) \rightharpoonup a(x,t,\nabla u) \text{ weakly in } (L_{\overline{M}}(Q))^N.$$

Therefore, we get for all $\varphi \in C^1([0,T],\mathcal{D}(\Omega))$,

$$-\int_Q u\frac{\partial\varphi}{\partial t} dxdt + \int_Q a(x,t,\nabla u)\nabla\varphi dxdt = < f,\varphi >_Q \tag{4.19}$$

Step 5: Passage to the limit

Going back to the approximating equations (4.2), then we obtain in the sense of distribution when n tend to infinity that

$$\frac{\partial u}{\partial t} - div(a(x,t,\nabla u)) = f(x,t) \text{ and } u(x,t) = 0$$

Furthermore, by the fact that $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$ for the modular convergence and we have already that $u_n \to u$ in $W_0^{1,x}L_M(Q)\cap L^2(Q)$ for the modular convergence, then by lemma (2.6) we get $u_n \to u$ in $C([0,T],L^2(\Omega)$, so using the periodicity condition, since

$$< \frac{\partial u}{\partial t}, u > = \lim_{n\to\infty} < \frac{\partial u_n}{\partial t}, u_n > = \frac{1}{2}[u_n(T)^2 - u_n(0)^2] = 0$$

we deduce finally

$$u(x,0) = u(x,T) \text{ in } \Omega.$$

Then the proof of theorem (3.1) is completed.

5. Appendix

let us consider the following approximate problem:

$$\begin{cases} u_n \in \mathcal{V}_n; \frac{\partial u_n}{\partial t} \in L^1(0,T,V_n); \\ u_n(x,0) = u_n(x,T)\ ; \\ \text{and for all } \varphi \in \mathcal{V}_n \\ \int_Q \frac{\partial u_n}{\partial t}\varphi dxdt + \int_Q a(x,t,\nabla u_n)\nabla\varphi dxdt = \int_Q f_n\varphi dxdt \end{cases} \tag{5.1}$$

we will use the point fixed theorem due to Leray-Schauder to prove the existence of solution, for that let us consider the following initial boundary value problem

$$\begin{cases} \frac{\partial u_n}{\partial t} + A(u_n) = f_n \\ u_n(x,t) = 0 \\ u_n(0) = u_{0n} \end{cases} \tag{5.2}$$

where u_{0n} in V_n. And let $\overline{B}_n(0,R)$ be a closed ball in the space V_n with the norm $\|.\|$. We define the Poincarré operator by

$$P : \overline{B}_n(0,R) \to \overline{B}_n(0,R)$$

$$u_{0n} \mapsto u_n(T)$$

We have to prove that P is continuous and relatively compact (i.e find the existence of a constant $R > 0$ such that $\|u_{0n}\| \le R \to \|u_n(T)\| \le R$.
let consider $\varphi = u_n$ in (4.2) we have

$$\int_\Omega \frac{\partial u_n}{\partial t} u_n dx + \int_\Omega a(x,t,\nabla u_n)\nabla u_n dx = \int_\Omega f_n u_n dx.$$

Using Hölder inequality to the term in the left hide sand we get

$$\int_\Omega \frac{\partial u_n}{\partial t} u_n dx + \int_\Omega a(x,t,\nabla u_n)\nabla u_n dx \le 2\|f_n\|_{\overline{M},\Omega}\|u_n\|_{M,\Omega}.$$

Then we can easily see that for $\varepsilon > 0$ there exist a constant $c(\varepsilon)$ such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2 dx + \int_\Omega a(x,t,\nabla u_n)\nabla u_n dx \le C(\varepsilon)\|f_n\|^2_{\overline{M},\Omega} + \varepsilon\|u_n\|^2_{M,\Omega}$$

Using (3.2) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2 dx + \alpha\int_\Omega M(\frac{|\nabla u_n|}{\lambda})dx \le C(\varepsilon)\|f_n\|^2_{\overline{M},\Omega} + \varepsilon\|u_n\|^2_{M,\Omega}.$$

By lemma 5.7 of [19] there exist two positive constants δ, λ such that

$$\int_Q M(v)dxdt \le \delta\int_Q M(\lambda|\nabla v|)dxdt \quad \text{for all } v \in W_0^{1,x}L_M(Q).$$

Then for $c_1 > 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2 dx + \alpha c_1\int_\Omega M(|u_n|)dx \le C(\varepsilon)\|f_n\|^2_{\overline{M},\Omega} + \varepsilon\|u_n\|^2_{M,\Omega}$$

Using now (3.5), and by the choice of ε we can easily see that there exist $c_2 > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_n(t))^2dx + c_2\|u_n\|^2 \leq C(\varepsilon)\|f_n\|_{\overline{M},\Omega}$$

Multiplying by $e^{c_2 t}$ and integrating by part we obtain

$$e^{c_2 T}\|u_n(T)\|^2 \leq 2\|f_n\|_{\overline{M},Q} + R^2$$

we choice R such that $R^2 > \frac{2e^{-c_2 T}}{1-e^{-c_2 T}}$ we deduce the existence of $R > 0$.
Now we pass to prove the continuity of P, for that we consider u_{0n} and ν_{0n} two sequences in $\overline{B}_n(0,R)$, by taking $\varphi = u_n - \nu_n$ such that u_n and ν_n satisfy (4.2) we get

$$\frac{1}{2}\frac{d}{dt}\int_{Q}(u_n(t) - \nu_n(t))^2dxdt + \int_{Q}(a(x,t,\nabla u_n) - a(x,t,\nabla \nu_n)(\nabla u_n - \nabla \nu_n)dx = 0$$

then using (3.2), we can write

$$\|u_n(T) - \nu_n(T)\|^2 \leq \|u_{0n} - \nu_{0n}\|^2.$$

Finally we deduce the continuity of P, hence by the point fixed argument there exist u_n solution of (4.2) satisfy $u_n(T) = u_n(0)$. □

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Existence of periodic solutions to fractional $p(z)$-Laplacian parabolic problems

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Abstract. We consider a class of nonlinear parabolic initial boundary value problems having the fractional $p(z)$-Laplacian operator. By combining variable exponent fractional Sobolev spaces with topological degree theory, we establish the existence of a time-periodic non-trivial weak solution.

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1. Introduction and motivation

The main objective of this investigation is to analyse the existence of a time-periodic non-trivial weak solution for a nonlinear parabolic equation containing a fractional $p(z)$-Laplacian operator. The model for this investigation can be described as follows

$$\begin{cases} \dfrac{\partial w}{\partial t} + (-\triangle)^{\sigma}_{p(z)} w = \xi(z,t) & \text{in} \quad Q := \Omega \times (0,T), \\ u(z,0) = u(z,T) & \text{on} \quad \Omega \\ w(z,t) = 0 & \text{on} \quad \partial Q := \big(\mathbb{R}^N \backslash \Omega\big) \times (0,T), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open set with smooth boundary $\partial\Omega$, $T > 0$ is the period, $\sigma \in (0,1)$, $\xi \in \mathcal{V}^* := L^{(p^-)'}\big(0,T;\mathcal{W}^*\big)$, with $\mathcal{V} := L^{p^-}\big(0,T;\mathcal{W}\big)$, and let $p \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$1 < p^- = \min_{(z,y)\in\overline{\Omega}\times\overline{\Omega}} p(z,y) \leq p(z,y) \leq p^+ = \max_{(z,y)\in\overline{\Omega}\times\overline{\Omega}} p(z,y) < +\infty, \tag{1.2}$$

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p is symmetric i.e.

$$p(z,y)=p(y,z), \quad \text{for all } (z,y)\in\overline{\Omega}\times\overline{\Omega},$$

we denote by

$$\widetilde{p}(z)=p(z,z), \quad \text{for every } z\in\overline{\Omega}. \tag{1.3}$$

Here, the main operator $(-\Delta)^{\sigma}_{p(z)}$ is the fractional $p(z)$-Laplacian which is non-local operator described on smooth functions by

$$(-\Delta)^{\sigma}_{p(z)}w(z)=p.v.\int_{\mathbb{R}^N\setminus B_\varepsilon(z)}\frac{|w(z)-w(y)|^{p(z,y)-2}(w(z)-w(y))}{|z-y|^{N+\sigma p(z,y)}}dy,$$

where $z\in\mathbb{R}^N$, $p.v.$ is a commonly used abbreviation in the principal value sense, $B_\varepsilon(z):=\{y\in\mathbb{R}^N \ : \ |z-y|<\varepsilon\}$. As far as we know, the introduction of this operator can be credited to Kaufmann et al. [26]. In their work, the authors extended the Sobolev spaces with variable exponents to the fractional case and demonstrated a compact embedding theorem. Additionally, they applied this development to establish the existence and uniqueness of weak solutions for the following fractional $p(x)$-Laplacian problem

$$\begin{cases} (-\Delta)^{s}_{p(z)}u+|u|^{q(z)-2}u=f(z) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\end{cases}$$

where $f\in L^{a(z)}(\Omega)$ for some $a(x)>1$.

Recently, considerable attention has been paid to the study of fractional $p(z)$-Laplacian and non-local differential problems. The importance of studying equations similar to (1.1) goes beyond mathematical interests and finds applications in various fields of modern applied science, including phase continuum mechanics, fluid dynamics, image processing, game theory, transition phenomena and population dynamics. These problems arise as a natural consequence of the stochastic stabilization of Lévy processes, as evidenced by the works of [7, 28, 32] and other relevant references.

Of particular note is the seminal work of Caffarelli and Silvestre [18], who introduced the concept of the σ-harmonic extension to describe the fractional Laplacian operator. This development has led to significant progress in the understanding of elliptic problems associated with the fractional Laplacian. Notable advances in this context can be found in references such as [23, 36] and related sources. Furthermore, for hyperbolic problems, important contributions have been made by [14, 33], while the Camassa-Holm system has been treated by [30]. Taken together, these studies emphasise the relevance and wide-ranging applications of such non-local operators in various scientific fields.

The study of parabolic equations involving fractional Laplacian operators has attracted considerable interest in recent years, mainly due to their prevalence in various phenomena in physics, ecology, biology, geophysics, finance and other fields characterized by non-Brownian scaling.

An essential and highly recommended work in this area is the book by Bisci et al. [15], which provides a comprehensive and in-depth introduction to the study of fractional problems. Building on this foundation, several previous studies have focused on the study of specific instances of the problem (1.1). In particular, we will now review some of the key results of previous research on parabolic problems (1.1) with initial conditions w_0.

In [16], Boudjeriou consider a non-local diffusion equation involving the fractional $p(z)$-Laplacian with nonlinearities of the variable exponent type. By employing the subdifferential approach, the author ensured the existence of local solutions. Thereafter, there obtained the existence of global solutions and the explosion of solutions in finite time via the potential well theory and the Nehari manifold. there then studed the asymptotic stability of global solutions when time goes to infinity in certain Lebesgue spaces with variable exponents. The case of $p = p(z)$ and $s \to 1^-$, problem (1.1), with an initial data $w_0 \in L^2$, was studied by Hammou [25] by applying the theory of topological degrees. In this direction, we also refer to [24, 29] and references therein for the interested reader.

Concurrently, there was comprehensive research in the literature on periodic solutions. In the book by Lions [29], a qualitative investigation of periodic solutions to the problem (1.1) was conducted when $p(z, y) = p$ and $\sigma = 1$ is an integer. The author explored the existence, regularity, and uniqueness of weak periodic solutions to (1.1) under the condition that $f \in L^{p'}\big(0, T; W^{-1,p'}(\Omega)\big)$, where p' is the conjugate exponent of p.

In a more recent study, Pucci, Xiang, and Zhang [35] employed standard techniques to demonstrate the existence of periodic solutions for a similar initial-boundary value problem involving fractional p-Laplacian parabolic equations (1.1), but with an additional Kirchhoff term. For further details, interested readers can also refer to [37] and [39]. In [31], by means of the sub-differential approach, Mazăn et al. established the existence and uniqueness of strong solutions for the following diffusion problems implying a nonlocal fractional p-Laplacian operator

$$\frac{\partial w}{\partial t} + (-\Delta)_p^\sigma w = 0, \qquad \text{in } \Omega,\ t > 0, \tag{1.4}$$

with $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, $\sigma \in (0, 1)$ and $p \in (1, \infty)$. In addition, the authors also prove that when $p \neq 2$ and s tends to 1^-, after inserting a normalizing constant, the equation (1.4) is simplified to the following evolution equation $w_t - \Delta_p w = 0$ implying the p-Laplacian. For all $2 < p \leq 2^*_\sigma = 2N/(N - 2\sigma)$ and $\sigma \in (0, 1), N > 2\sigma$, Fu and Pucci [22] tackled the following problem

$$\begin{cases} \dfrac{\partial w(z,t)}{\partial t} + (-\Delta)^\sigma w(z,t) = |w(z,t)|^{p-2} w(z,t), & z \in \Omega,\ t > 0, \\ u(z, 0) = w_0(z), & z \in \Omega, \\ w(z,t) = 0, & z \in \mathbb{R}^N \backslash \Omega,\ t > 0. \end{cases}$$

Basing on the potential well theory, they showed the existence of global weak solutions to the considered problem. Thereafter, they obtained the vacuum isolating and blow-up of strong solutions.

Taking inspiration from previous research, we employ topological degree theory to establish the existence of periodic weak solutions for the nonlinear parabolic problems (1.1), which involve the fractional $p(z)$-Laplacian operator. To the best of our knowledge, these problems have not been addressed in earlier studies. We transform this fractional parabolic problem into a new problem governed by an operator equation of the form $\mathcal{N}w + \Phi w = \xi$, where $\mathcal{N}$ is a densely defined monotone maximal linear operator, and Φ is a demicontinuous bounded map of type (S_+) with respect to the domain of $\mathcal{N}$.

The topological degree theory for perturbations of linear maximal monotone mappings and its application to a class of parabolic problems were proposed by Berkovits and Mustonen in 1991, as described in [13]. This method has been extensively employed by various authors to study nonlinear parabolic problems and has proven to be a highly effective tool. For more details, interested readers can refer to the works [4, 5, 8, 9]. For additional background information and applications of this theory, readers can consult the articles [3, 1, 2, 6, 12, 20].

The organization of this paper is as follows: Section 2 presents essential preliminary results and related lemmas that will be utilized in subsequent sections. In Section 3, we provide the proof of the main results of this paper.

2. Preliminary results

In this section, we initiate by introducing the necessary functional framework to explore the problem (1.1). Additionally, we provide essential explanations and characteristics of topological degree theory that are pertinent to our objective.

Let Ω be a bounded open subset of $\mathbb{R}^N, N \geq 2$, we denote

$$C_+(\overline{\Omega}) = \left\{p(\cdot) : \Omega \to \mathbb{R} \text{ such that } p^- \leq p(z) \leq p^+ < +\infty\right\},$$

where

$$p^- := \operatorname*{ess\,inf}_{z\in\overline{\Omega}} p(z); \quad p^+ := \operatorname*{ess\,sup}_{z\in\overline{\Omega}} p(z).$$

We define the Lebesgue space with variable exponents $L^{p(z)}(\Omega)$, as follows

$$L^{p(z)}(\Omega) = \{w : \Omega \to \mathbb{R}, \text{ measurable} : \int_\Omega |w|^{p(z)} dz < \infty\},$$

endowed with the norm

$$\|w\|_{p(z)} = \inf\left\{\lambda > 0 \mid \varrho_{p(\cdot)}\left(\frac{z}{\lambda}\right) \leq 1\right\}$$

where

$$\varrho_{p(\cdot)}(w) = \int_\Omega |w(z)|^{p(z)} dz, \qquad \text{for all } w \in L^{p(z)}(\Omega)$$

$\left(L^{p(z)}(\Omega), \|\cdot\|_{p(z)}\right)$ is a Banach space, separable and reflexive. Its dual space is $L^{p'(z)}(\Omega)$, where $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ for all $z \in \Omega$. We have also the following result

Proposition 2.1. [21] *For any $w \in L^{p(z)}(\Omega)$ we have*

1. $\|w\|_{p(z)} < 1 \ (= 1; > 1) \Leftrightarrow \varrho_{p(\cdot)}(w) < 1 \ (= 1; > 1)$,
2. $\|w\|_{p(z)} \geq 1 \Rightarrow \|w\|_{p(z)}^{p^-} \leq \varrho_{p(\cdot)}(w) \leq \|w\|_{p(z)}^{p^+}$,
3. $\|w\|_{p(z)} \leq 1 \Rightarrow \|w\|_{p(z)}^{p^+} \leq \varrho_{p(\cdot)}(w) \leq \|w\|_{p(z)}^{p^-}$.

From this statement, we can infer the following inequalities

$$\|w\|_{p(z)} \leq \varrho_{p(\cdot)}(w) + 1 \quad \text{and} \quad \varrho_{p(\cdot)}(w) \leq \|w\|_{p(z)}^{p^-} + \|w\|_{p(z)}^{p^+}$$

If p, $q \in \mathcal{C}_+(\overline{\Omega})$ such that $p(z) \leq q(z)$ for any $z \in \overline{\Omega}$, then there exists the continuous embedding $L^{q(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega)$.

Following this, we will provide an introduction to fractional Sobolev spaces with variable exponents, as introduced in the references [11, 26].

Suppose σ is a constant real number, where $0 < \sigma < 1$, and let p and q be two continuous functions mapping from the closed set $\overline{\Omega}$ to the interval $(0, \infty)$. Additionally, assuming that the conditions (1.2) and (1.3) hold, we proceed to define the fractional Sobolev space with a variable exponent using the Gagliardo approach in the following manner

$$\mathcal{W} = W^{\sigma,q(z),p(z,y)}(\Omega) = \Big\{ w \in L^{q(z)}(\Omega) : \int_{\Omega\times\Omega} \frac{|w(z)-w(y)|^{p(z,y)}}{\lambda^{p(z,y)}|z-y|^{N+\sigma p(z,y)}} dzdy < +\infty \text{ for some } \lambda > 0 \Big\}$$

We endow the space $\mathcal{W}$ with the norm given by

$$\|w\|_{\mathcal{W}} = \|w\|_{q(z)} + [w]_{\sigma,p(z,y)},$$

where $[\,\cdot\,]_{\sigma,p(z,y)}$ is a Gagliardo seminorm with variable exponent, which is defined as follows

$$[w]_{\sigma,p(z,y)} = \inf \Big\{ \lambda > 0 : \int_{\Omega\times\Omega} \frac{|w(z)-w(y)|^{p(z,y)}}{\lambda^{p(z,y)}|z-y|^{N+\sigma p(z,y)}} dzdy \leq 1 \Big\}.$$

The space $\big(\mathcal{W}, \|\cdot\|_{\mathcal{W}}\big)$ is a Banach space (as referenced in [19]), and it possesses the properties of separability and reflexivity (as mentioned in [11, Lemma 3.1]).

We define $\mathcal{W}_0$ as a subspace of $\mathcal{W}$, obtained by taking the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\mathcal{W}}$. This construction is based on [10, Theorem 2.1 and Remark 2.1].

$$\|\cdot\|_{\mathcal{W}_0} := [\,\cdot\,]_{\sigma,p(z,y)}$$

is a norm on $\mathcal{W}_0$ which is equivalent to the norm $\|\cdot\|_{\mathcal{W}}$, and we have the compact embedding $\mathcal{W}_0 \hookrightarrow\hookrightarrow L^{q(z)}(\Omega)$. Consequently, the space $\big(\mathcal{W}_0, \|\cdot\|_{\mathcal{W}_0}\big)$ is a Banach space that is also separable and reflexive.

We define the modular $\varrho_{p(\cdot,\cdot)} : \mathcal{W}_0 \to \mathbb{R}$ by

$$\varrho_{p(\cdot,\cdot)}(w) = \int_{\Omega\times\Omega} \frac{|w(z)-w(y)|^{p(z,y)}}{|z-y|^{N+\sigma p(z,y)}} dzdy$$

The modular ϱ_p checks the following results

Proposition 2.2. [27] *For any $w \in \mathcal{W}_0$ we have*

(i) $\|w\|_{\mathcal{W}_0} \geq 1 \Rightarrow \|w\|_{\mathcal{W}_0}^{p^-} \leq \varrho_{p(\cdot,\cdot)}(w) \leq \|w\|_{\mathcal{W}_0}^{p^+}$,

(ii) $\|w\|_{\mathcal{W}_0} \leq 1 \Rightarrow \|w\|_{\mathcal{W}_0}^{p^+} \leq \varrho_{p(\cdot,\cdot)}(w) \leq \|w\|_{\mathcal{W}_0}^{p^-}$.

Afterwards, we present the devised approach to address the problem (1.1). For a given time interval $0 < T < \infty$, we consider the functional space

$$\mathcal{V} := L^{p^-}(0, T; \mathcal{W}_0),$$

that is a separable and reflexive Banach space with the norm

$$|w|_{\mathcal{V}} = \Big(\int_0^T \|w\|_{\mathcal{W}}^{p^-} dt \Big)^{\frac{1}{p^-}}.$$

In view of [38], we can clearly establish that the norm $|w|_{\mathcal{V}}$ is equivalent to the following standard norm

$$\|w\| := \|w\|_{\mathcal{V}} = \Big(\int_0^T \|w\|_{\mathcal{W}_0}^{p^-} dt \Big)^{\frac{1}{p^-}}.$$

For reader's convenience, we start by recalling some results and properties from the Berkovits and Mustonen degree theory for demicontinuous operators of generalized (S_+) type in real separable reflexive Banach $\mathcal{Z}$.
In what follows, We respectively denote by $\mathcal{Z}^*$ the topological dual of the Banach space $\mathcal{Z}$ with continuous dual pairing $\langle \cdot , \cdot \rangle$ and $\rightharpoonup$ represents the weak convergence. Given a nonempty subset Ω of $\mathcal{Z}$.
Let $\mathcal{A}$ from $\mathcal{Z}$ to $2^{\mathcal{Z}^*}$ be a multi-values mapping. We designate by $Gr(\mathcal{A})$ the graph of $\mathcal{A}$, i.e.

$$Gr(\mathcal{A}) = \big\{ (w, v) \in \mathcal{Z} \times \mathcal{Z}^* \ : \ v \in \mathcal{A}(w) \big\}.$$

Definition 2.3. The multi-values mapping $\mathcal{A}$ is called

1. monotone, if for each pair of elements $(\eta_1, \theta_1), (\eta_2, \theta_2)$ in $Gr(\mathcal{A})$, we have the inequality
$$\langle \theta_1 - \theta_2, \eta_1 - \eta_2 \rangle \geq 0.$$
2. maximal monotone, if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from $\mathcal{Z}$ to $2^{\mathcal{Z}^*}$. The last clause has an analogous variant in that, for each $(\eta_0, \theta_0) \in \mathcal{Z} \times \mathcal{Z}^*$ for which $\langle \theta_0 - \theta, \eta_0 - \eta \rangle \geq 0$, for all $(\eta, \theta) \in Gr(\mathcal{A})$, we have $(\eta_0, \theta_0) \in Gr(\mathcal{A})$.

Let $\mathcal{Y}$ be another real Banach space.

Definition 2.4. A mapping $\Phi : D(\Phi) \subset \mathcal{Z} \to \mathcal{Y}$ is said to be

1. demicontinuous, if for each sequence $(w_n) \subset \Omega$, $w_n \to w$ implies $\Phi(w_n) \rightharpoonup \Phi(w)$.
2. of type (S_+), if for any sequence $(w_n) \subset D(\Phi)$ such that $w_n \rightharpoonup w$ and $\limsup_{n\to\infty} \langle \Phi w_n, w_n - w \rangle \leq 0$, we have $w_n \to w$.

Let $\mathcal{N} : \mathcal{D}(\mathcal{N}) \subset \mathcal{Z} \to \mathcal{Z}^*$ be a linear maximal monotone map such that $\mathcal{D}(\mathcal{N})$ is dense in $\mathcal{Z}$.
In the following, for each open and bounded subset $\mathcal{O}$ on $\mathcal{Z}$, we consider classes of operators :

$$\mathcal{F}_{\mathcal{O}}(\Omega) := \big\{ \mathcal{N} + \Phi : \overline{\mathcal{O}} \cap \mathcal{D}(\mathcal{N}) \to \mathcal{Z}^* \mid \Phi \quad \text{is bounded, demicontinuous}$$
$$\text{and of type } (S_+) \text{ with respect to } \mathcal{D}(\mathcal{N}) \text{ from } \mathcal{O} \text{ to } \mathcal{Z}^* \big\},$$

$$\mathcal{H}_{\mathcal{O}} := \big\{\mathcal{N} + \Phi(t) \,:\, \overline{\mathcal{O}} \cap D(L) \to \mathcal{Z}^* \mid \Phi(t) \text{ is a bounded homotopy of type } (S_+) \text{ with respect to } \mathcal{D}(\mathcal{N}) \text{ from } \overline{\mathcal{O}} \text{ to } \mathcal{Z}^*\big\}.$$

Remark 2.5. [13] Remark that the class $\mathcal{H}_{\mathcal{O}}$ contains all affine homotopy

$$\mathcal{N} + (1-t)\Phi_1 + t\Phi_2 \quad \text{with} \quad (\mathcal{N} + \Phi_i) \in \mathcal{F}_{\mathcal{O}}, \ \ i = 1, 2.$$

The following theorem provides the notion of the Berkovits and Mustonen topological degree for a class of demicontinuous operators satisfying the condition (S_+), which is the main key to the existence proof, for more details see [13].

Theorem 2.6. *Let $\mathcal{N}$ be a linear maximal monotone densely defined map from $\mathcal{D}(\mathcal{N}) \subset \mathcal{Z}$ to $\mathcal{Z}^*$ and*

$$\mathcal{M} = \Big\{(F, G, \psi) \,:\, F \in \mathcal{F}_{\mathcal{O}},\ \mathcal{O} \text{ an open bounded subset in } \mathcal{Z},\ \psi \notin F\big(\partial\mathcal{O} \cap \mathcal{D}(\mathcal{N})\big)\Big\}.$$

There is a unique degree function $d \,:\, \mathcal{M} \longrightarrow \mathbb{Z}$ which satisfies the following properties :

1. *(Normalization) $\mathcal{N} + \mathcal{J}$ is a normalising map, where $\mathcal{J}$ is the duality mapping of $\mathcal{Z}$ into $\mathcal{Z}^*$, that is, $d(\mathcal{N} + \mathcal{J}, \mathcal{O}, \psi) = 1$, when $\psi \in (\mathcal{N} + \mathcal{J})(\mathcal{O} \cap \mathcal{D}(\mathcal{N}))$.*
2. *(Additivity) Let $\Phi \in \mathcal{F}_{\mathcal{O}}$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two disjoint open subsets of $\mathcal{O}$ such that $\psi \notin \Phi\big((\overline{\mathcal{O}} \backslash (\mathcal{O}_1 \cup \mathcal{O}_2)) \cap D(L)\big)$ then we have*
$$d(\Phi, \mathcal{O}, \psi) = d(\Phi, \mathcal{O}_1, \psi) + d(\Phi, \mathcal{O}_2, \psi).$$
3. *(Homotopy invariance) If $\Phi(t) \in \mathcal{H}_{\mathcal{O}}$ and $\psi(t) \notin \Phi(t)(\partial\mathcal{O} \cap \mathcal{D}(\mathcal{N}))$ for every $t \in [0,1]$, where $\psi(t)$ is a continuous curve in $\mathcal{Z}^*$, then*
$$d(\Phi(t), \mathcal{O}, \psi(t)) = \text{ constant}, \qquad \text{for all } t \in [0,1].$$
4. *(Existence) if $d(\Phi, \mathcal{O}, \psi) \neq 0$, then the equation $\Phi w = \psi$ has a solution in $\mathcal{O} \cap \mathcal{D}(\mathcal{N})$.*

Lemma 2.7. *Let $\mathcal{N} + \Phi \in \mathcal{F}_{\mathcal{Z}}$ and $\psi \in \mathcal{Z}^*$. Suppose that there is $R > 0$ such as*

$$\langle \mathcal{N}w + \Phi w - \psi, w\rangle > 0, \tag{2.1}$$

for each $w \in \partial B_R(0) \cap \mathcal{D}(\mathcal{N})$. Hence

$$(\mathcal{N} + \Phi)(\mathcal{D}(\mathcal{N})) = \mathcal{Z}^*. \tag{2.2}$$

Proof. Let $\varepsilon > 0,\ \theta \in [0,1]$ and

$$\mathcal{S}_\varepsilon(\theta, w) = \mathcal{N}w + (1-\theta)\mathcal{J}w + \theta(\Phi w + \varepsilon\mathcal{J}w - \psi).$$

As $0 \in \mathcal{N}(0)$ and applying the boundary condition (2.1), we have

$$\begin{aligned}
\langle \mathcal{S}_\varepsilon(\theta, w), w\rangle &= \big\langle \theta(\mathcal{N}w + \Phi w - \psi, w\big\rangle + \big\langle (1-\theta)\mathcal{N}w + (1-\theta+\varepsilon)\mathcal{J}w, w\big\rangle \\
&\geq \big\langle (1-\theta)\mathcal{N}w + (1-\theta+\varepsilon)\mathcal{J}w, w\big\rangle \\
&= (1-\theta)\big\langle \mathcal{N}w, w\big\rangle + (1-\theta+\varepsilon)\big\langle \mathcal{J}w, w\big\rangle \\
&\geq (1-\theta+\varepsilon)\|w\|^2 = (1-\theta+\varepsilon)R^2 > 0.
\end{aligned}$$

Which means that $0 \notin \mathcal{S}_\varepsilon(\theta, w)$. As $\mathcal{J}$ and $\Phi+\varepsilon\mathcal{J}$ are bounded, continuous and of type (S_+), $\{\mathcal{S}_\varepsilon(\theta,\cdot)\}_{\theta\in[0,1]}$ is an admissible homotopy. Hence, by using the normalisation and invariance under homotopy, we get

$$d(\mathcal{S}_\varepsilon(\theta,\cdot), B_R(0), 0) = d(\mathcal{N}+\mathcal{J}, B_R(0), 0) = 1.$$

As a result, there is $w_\varepsilon \in \mathcal{D}(\mathcal{N})$ such that $0 \in \mathcal{S}_\varepsilon(\theta,\cdot)$.
If we take $\theta = 1$ and when $\varepsilon \to 0^+$, then we have $\psi = \mathcal{N}w + \Phi w$ for certain $w \in \mathcal{D}(\mathcal{N})$. As $\psi \in \mathcal{Z}^*$ is of any kind, we deduce that $(\mathcal{N}+\Phi)(\mathcal{D}(\mathcal{N})) = \mathcal{Z}^*$. □

3. Main result

To demonstrate the existence of a weak periodic solution for (1.1), we employ compactness methods. Initially, we transform this nonlinear parabolic problem into a new one governed by an operator equation of the type $\mathcal{N}w + \Phi w = \xi$. Subsequently, we employ the theory of topological degrees to further investigate the problem.

In this context, we take into consideration the mapping $\Phi\colon \mathcal{V} \longrightarrow \mathcal{V}^*$, where

$$\langle \Phi w, \varphi\rangle = \int_0^T \int_{\Omega\times\Omega} |w(z,t)-w(y,t)|^{p(z,y)-2}(w(z,t)-w(y,t)) \\ \times (\varphi(z,t)-\varphi(y,t))\mathcal{L}(z,y)dzdydt, \quad (3.1)$$

for all $v \in \mathcal{V}$, with $\mathcal{L}(z,y) = |z-y|^{-N-\sigma p(z,y)}$.
The central outcome of this investigation is encapsulated in the subsequent theorem.

Theorem 3.1. *Assuming that $\xi \in \mathcal{V}^*$ and $u(z,0) = u(z,T) \in L^2(\Omega)$ are satisfied, the problem (1.1) admits at least one weak periodic solution $u \in \mathcal{D}(\mathcal{N})$ in the following sense*

$$-\int_Q u\varphi_t dzdt + \langle \Phi w, \varphi\rangle = \int_Q h\varphi dzdt, \quad (3.2)$$

for each $\varphi \in \mathcal{V}$.

To prove Theorem 3.1, we initially relied on the subsequent technical lemma

Lemma 3.2. *For $0 < \sigma < 1$ and $2 < p^- \leq p(z,y) < +\infty$, the operator Φ defined in (3.1) possesses the following properties*

(i) *It is bounded and demicontinuous.*
(ii) *It is strictly monotone.*
(iii) *It is of type (S_+).*

Proof. (i) As in [23], the operator given by

$$\langle \mathcal{B}u, \varphi\rangle = \int_{\Omega\times\Omega} |w(z,t)-w(y,t)|^{p(z,y)-2}(w(z,t)-w(y,t)) \\ \times (\varphi(z,t)-\varphi(y,t))\mathcal{L}(z,y)dzdy, \quad \forall w,\varphi \in \mathcal{W}_0$$

is well defined, bounded, continuous.
Furthermore, the form $\mathcal{B}$ gives rise to a Nemytskii operator that inherits the aforementioned properties, implying that the nonlinear operator Φ is bounded and demicontinuous.
(ii) Thanks to Perera et al. [34, Lemma 6.3], It is sufficient to show that

$$\langle \Phi w, \varphi\rangle \leq \|w\|^{p^{\pm}-1}\|\varphi\| \qquad \text{for all } \; w, \varphi \in \mathcal{V}$$

Additionally, the equality holds if and only if $\delta w = \gamma\varphi$ for some $\delta, \gamma \geq 0$, with both not being zero simultaneously.
Applying Hölder's inequality, we obtain (without loss of generality, we can assume that $\beta(z,y) = p(z,y) - 1$)

$$\langle \Phi w, \varphi\rangle \leq \int_0^T \int_{\Omega\times\Omega} |w(z,t) - w(y,t)|^{p(z,y)-1} |\varphi(z,t) - \varphi(y,t)| \times \mathcal{L}(z,y) dz dy dt \leq \|w\|^{p^{\pm}-1}\|\varphi\|,$$

where $p^{\pm} = \begin{cases} p^{+} & \text{if } \|w\| \geq 1 \\ p^{-} & \text{if } \|w\| < 1 \end{cases}$
The equivalence becomes apparent when $\delta w = \gamma\varphi$ for any $\delta, \gamma \geq 0$, with both not being zero simultaneously. Conversely, if $\langle \Phi w, v\rangle = \|w\|^{p^{\pm}-1}\|\varphi\|$, equality occurs in both inequalities. Consequently, the equality in the second inequality results in

$$\delta|w(z,t) - w(y,t)| = \gamma|\varphi(z,t) - \varphi(y,t)| \quad \text{a.e. in } \mathbb{R}^N \times \mathbb{R}^N \times (0,T)$$

for each $\delta, \gamma \geq 0$, not both null. Therefore, the equality in the first inequality implies

$$\delta(w(z,t) - w(y,t)) = \gamma(\varphi(z,t) - \varphi(y,t)) \quad \text{a.e. in } \mathbb{R}^N \times \mathbb{R}^N \times (0,T).$$

Since w and φ disappear a.e. in $\mathbb{R}^N\backslash\Omega \times (0,T)$, it results that $\delta w = \gamma\varphi$ a.e. in Q.
(iii) We still need to demonstrate that the operator Φ is of type (S_+).
Let $(w_n)_n$ be a sequence in $D(\Phi)$ such that

$$\begin{cases} w_n \rightharpoonup w & \text{in } \mathcal{V} \\ \limsup_{n\to\infty} \langle \Phi w_n, w_n - w\rangle \leq 0. \end{cases}$$

We want to demonstrate that $w_n \to w$ in $\mathcal{V}$.
From the weak convergence $w_n \rightharpoonup w$, $\limsup_{n\to\infty}\langle \Phi w_n - \Phi w, w_n - w\rangle \leq 0$ and (ii), we infer

$$\lim_{n\to+\infty} \langle \Phi w_n, w_n - w\rangle = \lim_{n\to+\infty} \langle \Phi w_n - \Phi w, w_n - w\rangle = 0. \tag{3.3}$$

According to the compact embedding $\mathcal{W}_0 \hookrightarrow\hookrightarrow L^{p(z)}(\Omega)$ and [29, Theorem 5.1] informs us that $\mathcal{V} \hookrightarrow\hookrightarrow L^{p^-}(Q)$. Consequently, there is a subsequence still referred to as (w_n), such that

$$w_n(z,t) \to w(z,t), \text{ a.e. } (z,t) \in Q. \tag{3.4}$$

Thus, we have from the Fatou lemma and (3.4)

$$\liminf_{n\to+\infty}\int_0^T\int_{\Omega\times\Omega}|w_n(z)-w_n(y)|^{p(z,y)}\mathcal{L}(z,y)dzdydt$$
$$\geq\int_0^T\int_{\Omega\times\Omega}|w(z)-w(y)|^{p(z,y)}\mathcal{L}(z,y)dzdydt, \quad (3.5)$$

On the other hand, By using the Young inequality, there is a positive constant C such that

$$\begin{aligned}\langle\Phi w_n, w_n-w\rangle &= \int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt\\ &-\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)-2}(w_n(z,t)-w_n(y,t))\\ &\qquad\times(w(z,t)-w(y,t))\mathcal{L}(z,y)dzdydt \qquad (3.6)\\ &\geq\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt\\ &-\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)-1}|w(z,t)-w(y,t)|\mathcal{L}(z,y)dzdydt\\ &\geq C\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt\\ &-C\int_0^T\int_{\Omega\times\Omega}|w(z,t)-w(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt,\end{aligned}$$

by (3.5), (3.3) and (3.6), we drive

$$\lim_{n\to+\infty}\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt$$
$$=\int_0^T\int_{\Omega\times\Omega}|w(z,t)-w(y,t)|^{p}\mathcal{L}(z,y)dzdydt. \quad (3.7)$$

Combining (3.4), (3.7) with the Brezis-Lieb lemma [17], our conclusion has been in place. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. To demonstrate the existence of a weak solution to (1.1), we aim to utilize the topological degree methods. To accomplish this objective, we introduce

$$\mathcal{D}(\mathcal{N})=\{\varphi\in\mathcal{V} \,:\, \varphi'\in\mathcal{V}^*,\ \varphi(0)=0\},$$

By exploiting the density property of $\mathcal{C}_c^\infty(Q_T)$ within $\mathcal{V}$ and considering that $\mathcal{C}_c^\infty(Q_T)\subset\mathcal{D}(\mathcal{N})$, we can conclude that $\mathcal{D}(\mathcal{N})$ densely exists in $\mathcal{V}$. Now, let us examine the operator $\mathcal{N} : \mathcal{D}(\mathcal{N}) \subset \mathcal{V} \longrightarrow \mathcal{V}^*$ defined as follows

$$\langle \mathcal{N}w, \varphi\rangle = -\int_Q w\varphi dzdt, \qquad \text{for all } w \in \mathcal{D}(\mathcal{N}),\ \varphi \in \mathcal{V}.$$

Thereby, the operator $\mathcal{N}$ is generated by $\partial/\partial t$ by making of the relation

$$\langle \mathcal{N}w, \varphi\rangle = \int_0^T \Big\langle \frac{\partial w(t)}{\partial t}, \varphi(t)\Big\rangle dt, \text{ for each } w \in \mathcal{D}(\mathcal{N}),\ \varphi \in \mathcal{V}.$$

Thanks to the outcome presented in [29, Lemma 1.1, p. 313], it can be deduced that $\mathcal{L}$ qualifies as a maximal monotone operator. For further details, one may refer to the comprehensive information provided in [40].
On a separate note, the fact that $\mathcal{N}$ is a monotone operator (i.e., $\langle \mathcal{N}w, w\rangle \geq 0$ for all $u \in \mathcal{D}(\mathcal{N})$) ensures that

$$\begin{aligned}
&\langle \mathcal{N}w + \Phi w, w\rangle \geq \langle \Phi w, w\rangle \\
&= \int_0^T \int_{\Omega\times\Omega} |w(z,t) - w(y,t)|^{p(z,y)-2}(w(z,t) - w(y,t))^2 \mathcal{L}(z,y) dzdydt \\
&\qquad = \int_0^T \int_{\Omega\times\Omega} |w(z,t) - w(y,t)|^{p(z,y)} \mathcal{L}(z,y) dzdydt \\
&\qquad\qquad \geq \min\{\|w\|^{p^-}, \|w\|^{p^+}\} \qquad (3.8)
\end{aligned}$$

for all $w \in \mathcal{V}$.
From (3.8) the right hand side goes to infinity as $\|w\| \to \infty$, as for each $\xi \in \mathcal{V}^*$ there exists $R = R(h)$ for which

$$\langle \mathcal{N}w + \Phi w - \xi, w\rangle > 0 \qquad \text{for all } w \in B_R(0) \cap \mathcal{D}(\mathcal{N}).$$

By relying on the principles established in Lemma 2.7, it follows that there exists an element $w \in \mathcal{D}(\mathcal{N})$ that serves as a solution to the operator equation $\mathcal{N}w + \Phi w = \xi$. Consequently, this result indicates the existence of a weak periodic solution to the problem (1.1). □

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Existence and asymptotic stability for a semilinear damped wave equation with dynamic boundary conditions involving variable nonlinearity

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Abstract. We investigate the solvability of a class of quasilinear elliptic equations characterized by a $(p(x), k(x))$ growth structure and nonlinear boundary conditions, specifically in the context of Kelvin-Voigt damping with arbitrary data. Our approach involves analyzing the problem within appropriate functional spaces, utilizing Lebesgue and Sobolev spaces with variable exponents. In the first step, we establish the existence and uniqueness of results for solutions to the model, provided the data meet certain regularity conditions. Our methodology primarily relies on fixed-point theory and Faedo-Galerkin techniques, incorporating some novel strategies. In the second part, we consider scenarios with sufficiently large data sets and show that the system's energy grows exponentially.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. In this work, we deal with the existence and asymptotic behavior of weak solutions of a weakly damped wave equation with dynamic boundary conditions and source terms

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involving nonlinearities with variable exponents. More specifically, let's look at the problem

$$\begin{array}{ll} u_{tt}\left(x,t\right)-\Delta u\left(x,t\right)-\gamma\Delta u_t\left(x,t\right)=\left|u\right|^{p(x)-2}u\left(x,t\right), & x\in\Omega,\ t>0,\\ u\left(x,t\right)=0, & x\in\Gamma_0,\ t>0,\\ u_{tt}\left(x,t\right)=-a\left(\frac{\partial u}{\partial\nu}\left(x,t\right)+\gamma\frac{\partial u_t}{\partial\nu}\left(x,t\right)+r\left|u_t\right|^{k(x)-2}u_t\left(x,t\right)\right), & x\in\Gamma_1,\ t>0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \end{array} \tag{1.1}$$

where $u=u\left(x,t\right),\ t\geq 0$, γ, a and r are positive real numbers and $-\Delta$ represent the Laplace operator with respect to the spatial variable. The boundary Γ of Ω is assumed to be regular and the union of two closed and disjoint parts Γ_0, Γ_1, where $\Gamma_0\neq\emptyset$. $\frac{\partial u}{\partial\nu}$ denotes the unit of the exterior normal derivative, u_0, u_1 are given functions and the exponents $k(.)$ and $p(.)$ are given measurable functions on Ω to satisfy

$$\begin{cases} 2\leq p_1\leq p\left(x\right)\leq p_2<\infty,\\ 2\leq k_1\leq k\left(x\right)\leq k_2<\infty, \end{cases} \tag{1.2}$$

where we fix q on Ω for any given measurable function:

$$q_2=ess\sup_{x\in\Omega}q\left(x\right),\quad q_1=ess\inf_{x\in\Omega}q\left(x\right). \tag{1.3}$$

We also assume that the following uniform Zhikov-Fan local continuity condition holds

$$\begin{aligned} \left|p\left(x\right)-p\left(y\right)\right|+\left|k\left(x\right)-k\left(y\right)\right| &\leq \frac{M}{\left|\log\left|x-y\right|\right|},\ \text{for all } x,\ y \text{ in } \Omega, \\ \text{with } 0 &< \left|x-y\right|<\frac{1}{2},\ M>0. \end{aligned} \tag{1.4}$$

In recent years, many authors have engaged in the study of nonlinear hyperbolic, parabolic and elliptic equations with a non-standard growth condition, since they are applicable to real problems and many physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media [3, 18], and the processing of digital images [2, 7], and can all be associated with problem (1.1), more details on the subject can be found in [19] and the other references contained therein. In the classical case of constant exponent ($k(x)=$constant $=p$, $p(x)=$constant $=p$), this equation has its origin in the nonlinear dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end [5, 14, 13], where the source term $\left|u\right|^{p-2}u$ forces the negative-energy solutions to explode in finite time, and the dissipation term $\left|u_t\right|^{k-2}u_t$ assures the existence (in time) of global solutions. The dynamic boundary conditions represent Newton's law for the attached mass [5, 4]. In two-dimensional space, as shown in [15], boundary conditions of this kind appear when we consider the transverse motion of a flexible membrane, the boundary of which is only allowed to be affected by vibrations in one region. For other applications and related results, we refer the reader to [9, 16, 1, 17]. The aim of this article is to consider a class of nonlinear damped wave equations with dynamic boundary conditions and source terms with variable exponents and to prove a local existence theorem and sufficient conditions and initial data for the exponential energy increase to appear, indicate that this study is through the presence of the strong damping term $-\Delta u_t$ and the variable exponents differs from those previously considered. For this reason, extensive changes in the approaches are required.

2. Preliminaries

2.1. Function spaces

Throughout this paper, we assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\Gamma = \partial\Omega$. Let $p(x) \geq 2$ be a measurable bounded function defined in Ω. We introduce the set of functions

$$L^{p(.)}(\Omega) = \left\{ u(x) : u \text{ is measurable in } \Omega,\ \varrho_{p(.)}(u) \equiv \int_{\Omega} |u(x)|^{p(x)}\, \mathrm{d}x < \infty \right\}.$$

The set $L^{p(.)}(\Omega)$ equipped with the norm (Luxemburg norm)

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0,\ \int_{\Omega} \left| \frac{u}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\},$$

becomes a Banach space. The set $C^{\infty}(\Omega)$ is dense in $L^{p(.)}(\Omega)$, provided that the exponent $p(x) \in C^0(\Omega)$. Hölder's inequality holds for the elements of these spaces in the following form:

$$\int_{\Omega} |u(x)\, v(x)|\, \mathrm{d}x \leq \left(\frac{1}{p_1} + \frac{1}{q_1} \right) \|u\|_{p(x)} \|v\|_{q(x)},$$

for all $u \in L^{p(.)}(\Omega)$, $v \in L^{q(.)}(\Omega)$ with $p(x) \in [p_1, p_2] \subset (1, \infty)$, $q(x) = \frac{p(x)}{p(x)-1} \in [q_1, q_2] \subset (1, \infty)$. With $W_0^{1,p(.)}(\Omega)$ we denote the Banach space

$$W_0^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \mid\ |\nabla u|^{p(x)} \in L^1(\Omega),\ u = 0 \text{ on } \Gamma = \partial\Omega \right\}.$$

An equivalent norm of $W_0^{1,p(.)}(\Omega)$ is given by

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)} = \sum_i \|D_i u\|_{p(.)} + \|u\|_{p(.)},$$

and $W^{-1,p'(.)}(\Omega)$ is defined in the same way as the usual Sobolev spaces (see [8]). Here we note that the space $W_0^{1,p(.)}(\Omega)$ is usually defined differently for the variable exponent case. The $\left(W_0^{1,p(.)}(\Omega) \right)'$ is the dual space of $W_0^{1,p(.)}(\Omega)$ with respect to the inner product in $L^2(\Omega)$ and is defined as $W^{-1,q(.)}(\Omega)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$. If $p \in C\left(\overline{\Omega}\right)$, $q : \Omega \to [1, +\infty)$ is a measurable function and $\underset{x\in\Omega}{ess\inf}\,(p^*(x) - q(x)) > 0$ with $p^*(x) = \frac{np(x)}{(n-p(x))_2}$, then $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Lemma 2.1. *([8]) If Ω is a bounded domain of $\mathbb{R}^n$, $p(.) \subset (1,\infty)$ is a measurable function on $\overline{\Omega}$, then*

$$\min\left(\varrho_{p(.)}(u)^{\frac{1}{p_1}}, \varrho_{p(.)}(u)^{\frac{1}{p_2}}\right) \le \|u\|_{p(.)} \le \max\left(\varrho_{p(.)}(u)^{\frac{1}{p_1}}, \varrho_{p(.)}(u)^{\frac{1}{p_2}}\right), \qquad (2.1)$$

for any $u \in L^{p(\cdot)}(\Omega)$.

Proposition 2.2. *(See [12]) If Ω is a bounded domain in $\mathbb{R}^n$, $p \in C^{0,1}\left(\overline{\Omega}\right)$, $1 < p_1 \le p(x) \le p_2 < n$. Then, for every $q \in C(\Gamma)$ with $1 \le q(x) \le \frac{(n-1)p(x)}{n-p(x)}$, there is a continuous trace $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Gamma)$, when $1 \le q(x) << \frac{(n-1)p(x)}{n-p(x)}$, the trace is compact, and in particular, the continuous trace $W^{1,p(x)}(\Omega) \to L^{p(x)}(\Gamma)$ is compact.*

2.2. Mathematical Hypotheses

We start this section by introducing some hypotheses and our main result. In this paper we use standard function spaces and denote that $\|.\|_{q,\Gamma_1}$, $\|\cdot\|_{p(.),\Gamma_1}$ are the $L^q(\Gamma_1)$ norm and the $L^{p(\cdot)}(\Gamma_1)$ norm such that

$$\|u\|_{p(.),\Gamma_1} = \int_{\Gamma_1} |u(x)|^{p(x)}\, d\Gamma.$$

And we define $(u,v) = \int_\Omega u(x)v(x)\,dx$ and $(u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)\,d\Gamma$. Furthermore, we use standard functional spaces and denote that $(.,.)$, $\|.\|$ the inner products and norms are represented in $L^2(\Omega)$ and $H_0^1(\Omega)$ and they are given by

$$(u,v) = \int_\Omega u(x)v(x)\,dx \text{ and } \|u\|^2_{L^2(\Omega)} = \int_\Omega |u|^2\,dx.$$

We adopt the fixed definition of the $H_0^1(\Omega)$ norm as

$$\|u\|^2_{H_0^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Next we give the assumptions for the problem (1.1).

(H) Hypotheses on $p(.)$, $k(.)$**.** Let $k(.)$ and $p(.)$ be measurable functions on $\overline{\Omega}$ that satisfy the following condition:

$$2 < p_1 \le p(x) \le p_2 < \infty, \text{ and } 2 \le k_1 \le k(x) \le k_2 < \infty. \qquad (2.2)$$

We will use the embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1)$, $2 \le q \le \overline{q}$, where $\overline{q} = \frac{2n-2}{n-2}$, $n > 2$ and $1 \le \overline{q} < \infty$ if $n = 2$ where

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) : u\,|_{\Gamma_0} = 0\right\},$$

equipped with the Hilbert structure induced by $H^1(\Omega)$ is a Hilbert space.

3. Existence of weak solutions

In this section we prove the existence of weak solutions to our problem (1.1). Our proof method is based on the Faedo-Galerkin approximation, the fixed point theory in Banach spaces, and the concept of compactness, which we discussed in this section. For the sake of simplicity, $a = 1$.

Theorem 3.1. *Let* $2 \leq p_1 \leq p(x) \leq p_2 \leq \overline{q}$ *and* $\max\left(2, \frac{\overline{q}}{\overline{q}+1-p_2}\right) \leq k_1 \leq k(x) \leq k_2 \leq \overline{q}$. *Then given* $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, *there exists* $T > 0$ *and a unique solution* u *of the problem* (1.1) *on* $(0, T)$ *such that*

$$\begin{gathered} u \in C\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left(0, T; L^2(\Omega)\right), \\ u_t \in L^2\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^{k(.)}\left((0, T) \times \Gamma_1\right). \end{gathered}$$

In order to prove the main theorem, we need the local existence and uniqueness of the solution to the following related problem:

$$\begin{gathered} v_{tt}(x,t) - \Delta v(x,t) - \gamma \Delta v_t(x,t) = |u|^{p(x)-2} u(x,t) \text{ in } \Omega \times \mathbb{R}^+, \\ v = 0 \text{ on } \Gamma_0 \times (0, +\infty), \\ v_{tt}(x,t) = -\left[\frac{\partial v}{\partial \nu}(x,t) + \gamma \frac{\partial v_t}{\partial \nu}(x,t) + r|v_t|^{k(x)-2} v_t(x,t)\right] \\ \text{on } \Gamma_1 \times (0, +\infty), \\ v(x,0) = u_0(x), \ v_t(x,0) = u_1(x), \ x \in \Omega. \end{gathered} \tag{P4}$$

We now have to give the following existence result of the local solution of problem (P4) for an arbitrary initial value $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$.

Lemma 3.2. *Let* $2 \leq p_1 \leq p(x) \leq p_2 \leq \overline{q}$ *and* $\max\left(2, \frac{\overline{q}}{\overline{q}+1-p_2}\right) \leq k_1 \leq k(x) \leq k_2 \leq \overline{q}$. *Then given* $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *there exists* $T > 0$ *and a unique solution* v *of the problem* (P4) *on* $(0, T)$ *such that*

$$\begin{gathered} v \in C\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left(0, T; L^2(\Omega)\right), \\ v_t \in L^2\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^{k(.)}\left((0, T) \times \Gamma_1\right). \end{gathered}$$

To justify Lemma (3.2), we first investigate the following problem for every $T > 0$ and $f \in H^1(0, T; L^2(\Omega))$

$$\begin{gathered} v_{tt}(x,t) - \Delta v(x,t) - \gamma \Delta v_t(x,t) = f(x,t) \text{ in } \Omega \times \mathbb{R}^+, \\ v(x,t) = 0 \text{ on } \Gamma_0 \times (0, +\infty), \\ v_{tt}(x,t) = -\left[\frac{\partial v}{\partial \nu}(x,t) + \gamma \frac{\partial v_t}{\partial \nu}(x,t) + r|v_t|^{k(x)-2} v_t(x,t)\right] \\ \text{on } \Gamma_1 \times (0, +\infty), \\ v(x,0) = u_0(x), \ v_t(x,0) = u_1(x), \ x \in \Omega. \end{gathered} \tag{P5}$$

At this point, as reported by Doronin et al. [11], we need to know exactly what kind of solutions to problem (P5) we need

Definition 3.3. We say that a function v is a local generalized solution to problem (P5) if

- **(i).** $v \in L^\infty\left(0, T; H^1_{\Gamma_0}(\Omega)\right)$,
- **(ii).** $v_t \in L^2\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^{k(.)}\left((0, T) \times \Gamma_1\right) \cap L^\infty\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Gamma_1)\right)$,
- **(iii).** $v_{tt} \in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Gamma_1)\right)$,
- **(iv).** $v(x,0) = u_0(x), \ v_t(x,0) = u_1(x)$,

(v). for all $\varphi \in H^1_{\Gamma_0}(\Omega) \cap L^{k(.)}(\Gamma_1)$ and a.e. $t \in [0,T]$ with $\phi \in C(0,T)$ and $\phi(T) = 0$, the following identity hold:

$$\begin{aligned}\int_0^T (f,\varphi)(t)\,\phi(t)\,\mathrm{d}t = & \int_0^T [(v_{tt},\varphi)(t) + (\nabla v, \nabla\varphi)(t) + \gamma(\nabla v_t, \nabla\varphi)(t)]\,\phi(t)\,\mathrm{d}t \\ & + \int_0^T \phi(t) \int_{\Gamma_1} \left[v_{tt}(t) + r\,|v_t|^{k(x)-2}\, v_t(t)\right] \varphi \mathrm{d}\Gamma \mathrm{d}t.\end{aligned}$$

Using the Galerkin arguments, we prove the following lemma on the existence and uniqueness of a local solution of (P5) in time.

Lemma 3.4. *Let* $2 \leq p_1 \leq p(x) \leq p_2 \leq \overline{q}$ *and* $2 \leq k_1 \leq k(x) \leq k_2 \leq \overline{q}$. *Then, for all* $(u_0, u_1) \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \cap L^{k(.)}(\Gamma_1) \times H^2(\Omega)$ *and* $f \in H^1(0,T;L^2(\Omega))$, *there is a unique solution* v *of problem* (P5) *in the sense of definition* (3.3).

The proof of the above lemma depends on the Faedo-Galerkin method, which consists of constructing approximations of the solution. Then we get the necessary a priori estimates to ensure the convergence of these approximations. It seems difficult to get second-order estimates for $v_{tt}(0)$. To obtain them we relied on the ideas of Doronin and Larkin in [10] and Cavalcanti et al. [6] be inspired.

Proof of Lemma (3.4). We propose the following modification of variables:

$$\widetilde{v}(t,x) = v(t,x) - \omega(t,x) \text{ with } \omega(t,x) = u_0(x) + tu_1(x).$$

Hence we have the following problem with the unknown $\widetilde{v}(t,x)$ and null initial conditions

$$\begin{cases} \widetilde{v}_{tt} - \Delta\widetilde{v} - \gamma\Delta\widetilde{v}_t = f(x,t) + \Delta\omega + \gamma\Delta\omega_t \text{ in } \Omega \times \mathbb{R}^+, \\ \widetilde{v} = 0 \text{ on } \Gamma_0 \times (0,+\infty), \\ \widetilde{v}(x,t) = -\left[\begin{array}{c} \frac{\partial(\widetilde{v}+\omega)}{\partial\nu}(x,t) + \gamma\frac{\partial(\widetilde{v}_t+\omega_t)}{\partial\nu}(x,t) \\ +r\,|\widetilde{v}_t + \omega_t|^{k(x)-2}(\widetilde{v}_t + \omega_t)(x,t) \end{array}\right] \text{ on } \Gamma_1 \times (0,+\infty), \\ \widetilde{v}(x,0) = 0, \ \ v_t(x,0) = 0, \ x \in \Omega. \end{cases} \tag{P6}$$

Therefore we first prove the existence and uniqueness of the local solution for (P5). Let (w_j), $j = 1,2,...,$ be a complete orthonormal system in $L^2(\Omega) \cap L^2(\Gamma_1)$ with the following properties:

$*$ $\forall j; w_j \in H^1_{\Gamma_0}(\Omega) \cap L^{k(.)}(\Gamma_1)$;

$*$ The family $\{w_1, w_2, ..., w_m\}$ is linearly independent;

$*$ V_m the space generated by $\{w_1, w_2, ..., w_m\}$, $\underset{m}{\cup} V_m$ is dense in $H^1_{\Gamma_0}(\Omega) \cap L^{k(.)}(\Gamma_1)$.

We construct approximate solutions, $\widetilde{v}_m$ $(m = 1,2,3,...)$ in V_m in the form

$$\widetilde{v}_m(t) = \sum_{i=1}^{m} K_{jm}(t) w_i, \ \ m = 1,2,..., \tag{3.1}$$

where $K_{jm}(t)$ are determined by the following ordinary differential equation:

$$\begin{aligned}&\left(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{v}_m(t), w_j\right) + (\nabla(\widetilde{v}_m + \omega), \nabla w_j) + \gamma(\nabla(\widetilde{v}_m + \omega)_t, \nabla w_j) \\ &+ \left(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{v}_m(t) + r\,|(\widetilde{v}_m + \omega)_t|^{k(x)-2}(\widetilde{v}_m + \omega)_t, w_j\right)_{\Gamma_1} = (f(t), w_j), \ \ j = 1,2,...,\end{aligned}$$

and is completed by the following initial conditions $v_m(0)$, $v_{tm}(0)$ that satisfy

$$\widetilde{v}_m(0) = \widetilde{v}_{tm}(0) = 0. \tag{3.2}$$

Then

$$\begin{aligned}&\left(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{v}_m(t), v\right) + \left(\nabla\left(\widetilde{v}_m+\omega\right), \nabla v\right) + \gamma\left(\nabla\left(\widetilde{v}_m+\omega\right)_t, \nabla v\right)\\ &+\left(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{v}_m(t) + r\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t, v\right)_{\Gamma_1} = \left(f\left(t\right), v\right),\end{aligned} \tag{3.3}$$

it holds for any given $v \in Span\left\{w_1, w_2, ..., w_m\right\}$, due to the theory of ordinary differential equations, the system (3.1)-(3.3) has a unique local solution, which is extended to maximal intervals $[0, t_m]$.

A solution $\widetilde{v}$ of problem (1.1) in an interval $[0, t_m]$ is obtained as the limit of $\widetilde{v}_m$ as $m \to \infty$. Then, as a consequence of the a priori estimates to be proved in the next step, this solution can be extended to the entire interval $[0, T]$ for all $T > 0$. In this section, $C > 0$ and $c_* > 0$ denote various positive constants that vary from line to line, are independent of the natural number m, and only (possibly) depend on the initial value.

Estimates for $\widetilde{v}_{tm}(t)$

By taking $v = \widetilde{v}_{tm}(t)$ in (3.3), we have for $t \in (0, t_m)$

$$\begin{aligned}&\tfrac{1}{2}\left(\int_\Omega\left|\widetilde{v}_{tm}\right|^2\mathrm{d}x + \int_\Omega\left|\nabla\widetilde{v}_m\right|^2\mathrm{d}x + \left\|\widetilde{v}_{tm}\right\|^2_{\Gamma_1}\right) + \gamma\int_0^t\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x\mathrm{d}s\\ &\qquad+\int_0^t\left(\nabla\omega, \nabla\widetilde{v}_{tm}\right)\mathrm{d}s + \gamma\int_0^t\left(\nabla\omega_t, \nabla\widetilde{v}_{tm}\right)\mathrm{d}s\\ &+r\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t, \widetilde{v}_{tm}\right)_{\Gamma_1}\mathrm{d}s = \int_0^t\left(f, \widetilde{v}_{tm}\right)\mathrm{d}s.\end{aligned} \tag{3.4}$$

Using Young's inequality, there are $\delta_1 > 0$ (actually small enough) so they hold

$$\gamma\int_0^t\left(\nabla\omega_t, \nabla\widetilde{v}_{tm}\right)\mathrm{d}s \le \delta_1\int_0^t\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x\mathrm{d}s + \frac{1}{4\delta_1}\int_0^t\int_\Omega\left|\nabla\omega_t\right|^2\mathrm{d}x\mathrm{d}s, \tag{3.5}$$

and

$$\int_0^t\left(\nabla\omega, \nabla\widetilde{v}_{tm}\right)\mathrm{d}s \le \delta_1\int_0^t\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x\mathrm{d}s + \frac{1}{4\delta_1}\int_0^t\int_\Omega\left|\nabla\omega\right|^2\mathrm{d}x\mathrm{d}s. \tag{3.6}$$

By the inequalities of Hölder and Young there is $C > 0$ such that

$$\int_0^t\left(f, \widetilde{v}_{tm}\right)\mathrm{d}s \le C\int_0^t\int_\Omega\left(|f|^2 + \left|\widetilde{v}_{tm}\left(s\right)\right|^2\right)\mathrm{d}x\mathrm{d}s. \tag{3.7}$$

The last term on the left of Equation (3.4) can be written as follows

$$\begin{aligned}&\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t, \widetilde{v}_{tm}\right)_{\Gamma_1}\mathrm{d}s\\ &=\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t, \left(\widetilde{v}_m+\omega\right)_t\right)_{\Gamma_1}\mathrm{d}s\\ &-\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t, \omega_t\right)_{\Gamma_1}\mathrm{d}s\\ &=\int_0^t\int_{\Gamma_1}\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)}\mathrm{d}\Gamma\mathrm{d}s - \int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t, \omega_t\right)_{\Gamma_1}\mathrm{d}s.\end{aligned}$$

Therefore, Young's inequality grants us for $\delta_2 > 0$

$$\begin{aligned}\left|\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t, \omega_t\right)_{\Gamma_1}\mathrm{d}s\right| &\le \tfrac{1}{k_1}\int_0^t\int_{\Gamma_1}\delta_2^{k(x)}\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)}\mathrm{d}\Gamma\mathrm{d}s\\ &+\tfrac{k_2-1}{k_1}\int_0^t\int_{\Gamma_1}\delta_2^{-\frac{k(x)}{k(x)-1}}\left|\omega_t\right|^{k(x)}\mathrm{d}\Gamma\mathrm{d}s.\end{aligned} \tag{3.8}$$

So if we apply inequalities (3.5), (3.6), (3.7) and (3.8) to Equation (3.4) and make δ_1 and δ_2 small enough, we can conclude

$$\int_\Omega |\widetilde{v}_{tm}|^2\,\mathrm{d}x + \int_\Omega |\nabla\widetilde{v}_m|^2\,\mathrm{d}x + \|\widetilde{v}_{tm}\|^2_{\Gamma_1} + \int_0^t |\nabla\widetilde{v}_{tm}|^2\,\mathrm{d}s + \int_0^t\int_{\Gamma_1} |(\widetilde{v}_m+\omega)_t|^{k(x)}\,\mathrm{d}\Gamma\mathrm{d}s \le C_T, \tag{3.9}$$

where C_T is a positive constant independent of m. Thus the solution can be extended to $[0,T)$ and also hold

$$(\widetilde{v}_m)\ \text{ is a bounded sequence in } L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right), \tag{3.10}$$

$$(\widetilde{v}_{tm})\ \text{ is a bounded sequence in } L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1_{\Gamma_0}(\Omega)\right)\cap L^\infty\left(0,T;L^2(\Gamma_1)\right), \tag{3.11}$$

Now applying the following algebraic inequality:

$$A^\lambda = (A+B-B)^\lambda \le 2^{\lambda-1}\left((A+B)^\lambda + B^\lambda\right),\ \ A,\ B>0 \text{ and } \lambda\ge 1,$$

there are $C_1 >$ and $C_2>0$ such that

$$\int_0^t\int_{\Gamma_1} |\widetilde{v}_{tm}|^{k(x)}\,\mathrm{d}\Gamma\mathrm{d}s = \int_0^t\int_{\Gamma_1} |(\widetilde{v}_m+\omega)_t-\omega_t|^{k(x)}\,\mathrm{d}\Gamma_1\mathrm{d}s$$
$$\le C_1\int_0^t\int_{\Gamma_1} |(\widetilde{v}_m+\omega)_t|^{k(x)}\,\mathrm{d}\Gamma\mathrm{d}s + C_2\int_0^t\int_{\Gamma_1}|\omega_t|^{k(x)}\,\mathrm{d}\Gamma\mathrm{d}s.$$

Hence, from inequalities (3.9) and (3.6), there are $C_T'>0$ such that

$$\int_0^t\int_{\Gamma_1} |\widetilde{v}_{tm}|^{k(x)}\,\mathrm{d}\Gamma\mathrm{d}s \le C_T'.$$

Thus

$$(\widetilde{v}_{tm})\ \text{ is a bounded sequence in } L^{k(.)}\left((0,T)\times\Gamma_1\right). \tag{3.12}$$

Estimates for $\widetilde{v}_{ttm}(t)$

First we estimate $\widetilde{v}_{ttm}(0)$. Putting $t=0$ and $v=\widetilde{v}_{ttm}(0)$ in (3.3) and considering (3.11), we get

$$\int_\Omega |\widetilde{v}_{ttm}(0)|^2\,\mathrm{d}x + \|\widetilde{v}_{ttm}(0)\|^2_{2,\Gamma_1} + (\nabla\omega(0),\nabla\widetilde{v}_{ttm}(0))$$
$$+\gamma(\nabla\omega_t(0),\nabla\widetilde{v}_{ttm}(0)) + \left(r|\omega_t(0)|^{k(x)-2}\omega_t(0),\widetilde{v}_{ttm}(0)\right)_{\Gamma_1} = (f(0),\widetilde{v}_{ttm}(0)).$$

Knowing that the following inequalities hold:

$$(\nabla\omega_t(0),\nabla\widetilde{v}_{ttm}(0)) = -(\Delta\omega_t(0),\widetilde{v}_{ttm}(0)) + \left(\omega_t(0),\frac{\partial\widetilde{v}_{ttm}(0)}{\partial\nu}\right)_{\Gamma_1},$$

$$(\nabla\omega(0),\nabla\widetilde{v}_{ttm}(0)) = -(\Delta\omega(0),\widetilde{v}_{ttm}(0)) + \left(\omega(0),\frac{\partial\widetilde{v}_{ttm}(0)}{\partial\nu}\right)_{\Gamma_1},$$

and from $2(k_1-1)\le 2(k_2-1)\le \frac{2n}{n-2}$,

$$\begin{aligned}
&\left|\left(r\left|\omega_t(0)\right|^{k(x)-2}\omega_t(0),\widetilde{v}_{ttm}(0)\right)_{\Gamma_1}\right| \\
&\le r\max\begin{pmatrix} \int_{\Gamma_1}\left|\omega_t(0)\right|^{k_2-2}\left|\omega_t(0)\right|\left|\widetilde{v}_{ttm}(0)\right|\mathrm{d}\Gamma, \\ \int_{\Gamma_1}\left|\omega_t(0)\right|^{k_1-2}\left|\omega_t(0)\right|\left|\widetilde{v}_{ttm}(0)\right|\mathrm{d}\Gamma \end{pmatrix} \\
&\le r\max\begin{pmatrix} \|\omega_t(0)\|_{(k_2-2)n,\Gamma_1}^{k_2-2}\|\omega_t(0)\|_{\frac{2n}{n-2},\Gamma_1}\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1}, \\ \|\omega_t(0)\|_{(k_1-2)n,\Gamma_1}^{k_1-2}\|\omega_t(0)\|_{\frac{2n}{n-2},\Gamma_1}\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1} \end{pmatrix} \\
&\le Cr\max\begin{pmatrix} \|\nabla\omega_t(0)\|_{2,\Gamma_1}^{k_2-2}\|\nabla\omega_t(0)\|_{2,\Gamma_1}, \\ \|\nabla\omega_t(0)\|_{2,\Gamma_1}^{k_1-2}\|\nabla\omega_t(0)\|_{2,\Gamma_1} \end{pmatrix}\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1} \\
&\le Cr\max\left(\left|\omega_t(0)\right|^{k_2-1},\left|\omega_t(0)\right|^{k_1-1}\right)\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1}.
\end{aligned} \tag{3.13}$$

Then from $(u_0,u_1)\in H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\cap L^{k(\cdot)}(\Gamma_1)\times H^2(\Omega)$ and $f\in H^1(0,T;L^2(\Omega))$, by applying Young's inequality and embedding $H^1(\Omega)\hookrightarrow L^{k_2}(\Gamma_1)$ and $H^1(\Omega)\hookrightarrow L^{k_1}(\Gamma_1)$ we conclude that there is $C>0$ independent of m such that

$$\int_\Omega\left|\widetilde{v}_{ttm}(0)\right|^2\mathrm{d}x+\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1}^2\le C. \tag{3.14}$$

By differentiating equation (3.3) with respect to t and replacing v with $\widetilde{v}_{ttm}(t)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\left|\widetilde{v}_{ttm}\right|^2\mathrm{d}x+\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x+\|\widetilde{v}_{ttm}\|_{2,\Gamma_1}^2\right)+\gamma\int_\Omega\left|\nabla\widetilde{v}_{ttm}\right|^2\mathrm{d}x+\left(\nabla\omega_t,\nabla\widetilde{v}_{ttm}\right) \\
&+r\int_{\Gamma_1}(k(x)-1)\left|(\widetilde{v}_m+\omega)_t\right|^{k(x)-2}(\widetilde{v}_m+\omega)_{tt}\widetilde{v}_{ttm}\mathrm{d}\Gamma=\left(f_t,\widetilde{v}_{ttm}\right).
\end{aligned} \tag{3.15}$$

Since $\omega_{tt}=0$, the last term on the left-hand side of Equation (3.15) can be expressed as follows

$$\begin{aligned}
&\int_{\Gamma_1}\left|(\widetilde{v}_m+\omega)_t\right|^{k(x)-2}(\widetilde{v}_m+\omega)_{tt}\widetilde{v}_{ttm}\mathrm{d}\Gamma \\
&=\int_{\Gamma_1}\frac{4}{k^2(x)}\left(\frac{\partial}{\partial t}\left(\left|\widetilde{v}_{tm}(t)+\omega_t\right|^{\frac{k(x)-2}{2}}\left(\widetilde{v}_{tm}(t)+\omega_t\right)\right)\right)^2\mathrm{d}\Gamma.
\end{aligned}$$

Now Equation (3.15) is integrated over $(0,t)$ using estimate (3.14) and the Young and Poincare's inequalities (as in (3.8)) there is $\widetilde{C}_T>0$ such that

$$\begin{aligned}
&\left(\int_\Omega\left|\widetilde{v}_{ttm}\right|^2\mathrm{d}x+\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x+\|\widetilde{v}_{ttm}\|_{2,\Gamma_1}^2\right)+\gamma\int_0^t\left|\nabla\widetilde{v}_{ttm}\right|^2\mathrm{d}s \\
&+r\frac{4(k_1-1)}{(k_2)^2}\int_0^t\int_{\Gamma_1}\left(\frac{\partial}{\partial t}\left(\left|\widetilde{v}_{tm}(t)+\omega_t\right|^{\frac{k(x)-2}{2}}\left(\widetilde{v}_{tm}(t)+\omega_t\right)\right)\right)^2\mathrm{d}\Gamma\mathrm{d}s\quad\le\widetilde{C}_T.
\end{aligned}$$

Consequently we come to the following results:

$$\begin{aligned}
&(\widetilde{v}_{ttm})\text{ is a bounded sequence in }L^\infty\left(0,T;L^2(\Omega)\right), \\
&(\widetilde{v}_{ttm})\text{ is a bounded sequence in }L^\infty\left(0,T;L^2(\Gamma_1)\right), \\
&(\widetilde{v}_{tm})\text{ is a bounded sequence in }L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right).
\end{aligned} \tag{3.16}$$

From (3.10), (3.11), (3.12) and (3.16), we have that $(\widetilde{v}_m)$ is bounded in $L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)$. Then, $(\widetilde{v}_m)$ is bounded in $L^2\left(0,T;H^1_{\Gamma_0}(\Omega)\right)$. Since $(\widetilde{v}_{tm})$ is bounded in $L^\infty\left(0,T;L^2(\Omega)\right)$, $(\widetilde{v}_{tm})$ is bounded in $L^2\left(0,T;L^2(\Omega)\right)$. Thus, $(\widetilde{v}_m)$ is bounded in $H^1(Q)$. Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, by the Aubin-Lions theorem we have that there is a subsequence of $(\widetilde{v}_m)$, still denoted by $(\widetilde{v}_m)$, so that

$$\widetilde{v}_m \to v \text{ strongly in } L^2(Q).$$

Therefore

$$\widetilde{v}_m \to v \text{ strongly and a.e. on } (0,T) \times \Omega.$$

Using Lion's Lemma, we get

$$|\widetilde{v}_m|^{p(.)-2} \widetilde{v}_m \to |\widetilde{v}|^{p(.)-2} \widetilde{v} \text{ strongly and a.e. on } (0,T) \times \Omega.$$

On the other hand, we have from (3.11)

$$(\widetilde{v}_{tm}) \text{ is a bounded sequence in } L^\infty\left(0,T; L^2\left(\Gamma_1\right)\right).$$

From (3.10) and (3.16), since

$$\|\widetilde{v}_m\|_{H^{\frac{1}{2}}(\Gamma_1)} \leq C\|\nabla \widetilde{v}_m\|_2 \text{ and } \|\widetilde{v}_{tm}\|_{H^{\frac{1}{2}}(\Gamma_1)} \leq C\|\nabla \widetilde{v}_{tm}\|_2,$$

we derive that

$$\begin{aligned}
&(\widetilde{v}_m) \text{ is a bounded sequence in } L^2\left(0,T; H^{\frac{1}{2}}\left(\Gamma_1\right)\right),\\
&(\widetilde{v}_{tm}) \text{ is a bounded sequence in } L^2\left(0,T; H^{\frac{1}{2}}\left(\Gamma_1\right)\right),\\
&(\widetilde{v}_{ttm}) \text{ is a bounded sequence in } L^2\left(0,T; L^2\left(\Gamma_1\right)\right).
\end{aligned}$$

Since the embedding $H^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ is compact, again using the Aubin-Lions theorem, we conclude that we can extract a subsequence of $(\widetilde{v}_m)$ still denoted by $(\widetilde{v}_m)$ so that

$$\widetilde{v}_{tm} \to v_t \text{ strongly in } L^2\left(0,T; L^2\left(\Gamma_1\right)\right). \tag{3.17}$$

So we get that from (3.12)

$$|\widetilde{v}_{tm}|^{k(.)-2} \widetilde{v}_{tm} \to \varkappa \text{ weakly in } L^{\frac{k(.)}{k(.)-1}}\left((0,T) \times \Gamma_1\right).$$

It is enough to prove that $\varkappa = |\widetilde{v}_t|^{k(.)-2} \widetilde{v}_t$.
Clearly, from (3.17) we get

$$|\widetilde{v}_{tm}|^{k(.)-2} \widetilde{v}_{tm} \to |\widetilde{v}_t|^{k(.)-2} \widetilde{v}_t \text{ strongly and a.e. on } (0,T) \times \Gamma_1.$$

Again, using the Lions lemma, we get $\varkappa = |\widetilde{v}_t|^{k(.)-2} \widetilde{v}_t$. The proof can now be completed as follows

Proof of uniqueness:

Let u_1 and u_2 be two solutions of the problem (P5) with the same initial data. Let us denote $w = u_1 - u_2$. It is easy to see that w satisfies

$$\begin{aligned}
&\left(\int_\Omega |w_t|^2 \mathrm{d}x + \int_\Omega |\nabla w|^2 \mathrm{d}x + \|w_t\|_{\Gamma_1}^2\right) + 2\gamma \int_0^t |\nabla w_t|^2 \mathrm{d}s\\
&+2r \int_0^t \int_{\Gamma_1} \left(|u_{1t}|^{k(x)-2} u_{1t} - |u_{2t}|^{k(x)-2} u_{2t}\right) w_t(s) \mathrm{d}\Gamma \mathrm{d}s \quad = 0.
\end{aligned}$$

By using the inequality

$$\left(|a|^{k(x)-2} a - |b|^{k(x)-2} b\right) . (a-b) \geq 0, \tag{3.18}$$

for all a, $b \in \mathbb{R}^n$ and a.e. $x \in \Omega$, we have

$$\int_\Omega |w_t|^2 \,\mathrm{d}x + \int_\Omega |\nabla w|^2 \,\mathrm{d}x + \|w_t\|_{2,\Gamma_1}^2 = 0,$$

which implies that $w = C = 0$. Hence, the uniqueness follows.

This completes the proof of the lemma (3.4). □

Proof of lemma (3.2). First we approximate $u \in \left(C[0,T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1([0,T], L^2(\Omega))$ equipped with the norm $\|u\| = \max\limits_{t\in[0,T]} \left(\|u_t\|_2 + \|u\|_{H^1(\Omega)}\right)$, by a sequence $(u^\mu) \in C^\infty\left(([0,T]) \times \overline{\Omega}\right)$ by standard convolution arguments. It is clear that $|u^\mu|^{p_1-2} u^\mu$ and $|u^\mu|^{p_2-2} u^\mu \in H^1([0,T], L^2(\Omega))$, since $2(p_1-1) \le 2(p_2-1) \le \frac{2n}{n-2}$. Next, we approximate the initial data $u_1 \in L^2(\Omega)$ by a sequence (u_1^μ) in $C_0^\infty(\Omega)$ since the space $H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \cap L^{k(\cdot)}(\Gamma_1)$ is dense in $H^1_{\Gamma_0}(\Omega)$ for the H^1 endowed norm we approximate $u_0 \in H^1_{\Gamma_0}(\Omega)$ by a sequence (u_0^μ) in $H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \cap L^{k(\cdot)}(\Gamma_1)$. We examine the set of the following approximation problems:

$$\begin{gathered}
v_{tt}^\mu - \Delta v^\mu - \gamma \Delta v_t^\mu = |u^\mu|^{p(x)-2} u^\mu \text{ in } \Omega \times \mathbb{R}^+,\\
v^\mu = 0 \text{ on } \Gamma_0 \times (0,+\infty),\\
v_{tt}^\mu(x,t) = -\left[\frac{\partial v^\mu}{\partial \nu}(x,t) + \gamma \frac{\partial v_t^\mu}{\partial \nu}(x,t) + r\left|v_t^\mu\right|^{k(x)-2} v_t^\mu(x,t)\right]\\
\text{on } \Gamma_1 \times (0,+\infty),\\
v^\mu(x,0) = u_0^\mu(x),\ \ v_t^\mu(x,0) = u_1^\mu(x),\ \ x \in \Omega.
\end{gathered} \tag{3.19}$$

Since Lemma (3.4) is hypothesized, we can find a sequence of unique solutions (v^μ) to problem (3.19). We will show that the sequence $\{(v^\mu, v_t^\mu)\}$ is a Cauchy sequence in space

$$\mathcal{W}_T = \left\{ \begin{array}{c} (v, v_t) \mid v \in C\left([0,T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1([0,T], L^2(\Omega)),\\ v_t \in L^\infty\left(0,T; H^1_{\Gamma_0}(\Omega)\right) \cap L^{k(\cdot)}((0,T)\times\Gamma_1), \end{array} \right\}$$

endowed with the norm

$$\begin{aligned}
\|(v,v_t)\|_{\mathcal{W}_T}^2 &= \max_{t\in[0,T]} \left(\|v_t\|_2^2 + \|\nabla v\|_2^2\right) + \|v_t\|_{L^{k(\cdot)}((0,T)\times\Gamma_1)}^2\\
&+ \textstyle\int_0^t \|\nabla v_t(s)\|_2^2 \,\mathrm{d}s.
\end{aligned}$$

For this purpose we set $U = u^\mu - u^\tau$, $V = v^\mu - v^\tau$. It is easy to see that V satisfies

$$\begin{gathered}
V_{tt} - \Delta V - \gamma \Delta V_t = |u^\mu|^{p(x)-2} u^\mu - |u^\tau|^{p(x)-2} u^\tau \text{ in } \Omega \times \mathbb{R}^+,\\
V = 0 \text{ on } \Gamma_0 \times (0,+\infty),\\
V_{tt}(x,t) = -\left[\begin{array}{c} \frac{\partial V}{\partial \nu}(x,t) + \gamma \frac{\partial V_t}{\partial \nu}(x,t)\\ +r\left(\left|v_t^\mu\right|^{k(x)-2} v_t^\mu(x,t) - \left|v_t^\tau\right|^{k(x)-2} v_t^\tau(x,t)\right) \end{array}\right]\\
\text{on } \Gamma_1 \times (0,+\infty),\\
V(x,0) = u_0^\mu(x) - u_0^\tau(x),\ \ V_t(x,0) = u_1^\mu(x) - u_1^\tau(x),\ \ x \in \Omega.
\end{gathered}$$

Multiply the above differential equations by V_t for all $t \in (0,T)$ and integrate over $(0,t)\times\ \Omega$ we get

$$\begin{aligned}
&\left(|V_t|^2+|\nabla V|^2+\|V_t\|_{2,\Gamma_1}^2\right)+2\gamma\int_0^t\int_\Omega|\nabla V_t|^2\,\mathrm{d}x\mathrm{d}s\\
&+2r\int_0^t\int_{\Gamma_1}\left(|v_t^\mu|^{k(x)-2}v_t^\mu(x,t)-|v_t^\tau|^{k(x)-2}v_t^\tau(x,t)\right)(v_t^\mu-v_t^\tau)(s)\,\mathrm{d}\Gamma\mathrm{d}s\\
&=\left(|V_t(0)|^2+|\nabla V(0)|^2+\|V_t(0)\|_{2,\Gamma_1}^2\right)\\
&+2\int_0^t\int_\Omega\left(|u^\mu|^{p(x)-2}u^\mu-|u^\tau|^{p(x)-2}u^\tau\right)(v_t^\mu-v_t^\tau)(s)\,\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Using the inequality (3.18), we get

$$\begin{aligned}
&\left(|V_t|^2+|\nabla V|^2+\|V_t\|_{\Gamma_1}^2\right)+2\gamma\int_0^t\int_\Omega|\nabla V_t|^2\,\mathrm{d}x\mathrm{d}s\\
&\le\left(|V_t(0)|^2+|\nabla V(0)|^2+\|V_t(0)\|_{\Gamma_1}^2\right)\\
&+2\int_0^t\int_\Omega\left(|u^\mu|^{p(x)-2}u^\mu-|u^\tau|^{p(x)-2}u^\tau\right)(v_t^\mu-v_t^\tau)(s)\,\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{3.20}$$

Let's estimate the last term of the second member of the above inequality

$$\begin{aligned}
&\int_\Omega\left(|u^\mu|^{p(x)-2}u^\mu-|u^\tau|^{p(x)-2}u^\tau\right)(v_t^\mu-v_t^\tau)\,\mathrm{d}x\\
&\le(p_2-1)\int_\Omega\sup\left(|u^\mu|^{p(x)-2},|u^\tau|^{p(x)-2}\right)|u^\mu-u^\tau|\,|v_t^\mu-v_t^\tau|\,\mathrm{d}x\\
&\le c\max\begin{pmatrix}\left(||u^\mu(t)||_{(p_2-2)n}^{p_2-2}+||u^\mu(t)||_{(p_1-2)n}^{p_1-2}\right)||U||_{\frac{2n}{n-2}}\|V_t\|_2\,,\\ \left(||u^\tau(t)||_{(p_2-2)n}^{p_2-2}+||u^\tau(t)||_{(p_1-2)n}^{p_1-2}\right)||U||_{\frac{2n}{n-2}}\|V_t\|_2\end{pmatrix}\\
&\le cc_*\max\begin{pmatrix}\int_\Omega\left(|\nabla u^\mu(t)|^{p_2-2}+|\nabla u^\mu(t)|^{p_1-2}\right)\mathrm{d}x,\\ \int_\Omega\left(|\nabla u^\tau(t)|^{p_2-2}+|\nabla u^\tau(t)|^{p_1-2}\right)\mathrm{d}x\end{pmatrix}||\nabla U||_2\,\|V_t\|_2\,.
\end{aligned}\tag{3.21}$$

Then the estimate (3.20) takes the form

$$\begin{aligned}
&\left(\int_\Omega|V_t|^2\,\mathrm{d}x+\int_\Omega|\nabla V|^2\,\mathrm{d}x+\|V_t\|_{2,\Gamma_1}^2\right)+2\gamma\int_0^t\int_\Omega|\nabla V_t|^2\,\mathrm{d}x\mathrm{d}s\\
&\le\left(\int_\Omega|V_t(0)|^2\,\mathrm{d}x+\int_\Omega|\nabla V(0)|^2\,\mathrm{d}x+\|V_t(0)\|_{2,\Gamma_1}^2\right)\\
&+2cc_*\int_0^t\max\begin{pmatrix}\int_\Omega\left(|\nabla u^\mu(t)|^{p_2-2}+|\nabla u^\mu(t)|^{p_1-2}\right)\mathrm{d}x,\\ \int_\Omega\left(|\nabla u^\tau(t)|^{p_2-2}+|\nabla u^\tau(t)|^{p_1-2}\right)\mathrm{d}x\end{pmatrix}||\nabla U||_2\,\|V_t\|_2\,\mathrm{d}s.
\end{aligned}\tag{3.22}$$

From (3.10) , (3.22) becomes

$$\begin{aligned}
&\left(\int_\Omega|V_t|^2\,\mathrm{d}x+\int_\Omega|\nabla V|^2\,\mathrm{d}x+\|V_t\|_{\Gamma_1}^2\right)+2\gamma\int_0^t\|\nabla V_t\|_2^2\,\mathrm{d}s\\
&\le\left(\int_\Omega|V_t(0)|^2\,\mathrm{d}x+\int_\Omega|\nabla V(0)|^2\,\mathrm{d}x+\|V_t(0)\|_{2,\Gamma_1}^2\right)\qquad+C\int_0^t||\nabla U||_2\,\|V_t\|_2\,\mathrm{d}s.
\end{aligned}$$

Thus, applying Young's and Gronwall inequalities, there is C that depending only on Ω, p_1 and p_2 such that

$$\|V\|_{\mathcal{W}_T}\le C\left(\int_\Omega|V_t(0)|^2\,\mathrm{d}x+\int_\Omega|\nabla V(0)|^2\,\mathrm{d}x+\|V_t(0)\|_{2,\Gamma_1}^2\right)+CT\,\|U\|_{\mathcal{W}_T}\,.$$

Since $\{(u_0^\mu)\}$, $\{(u_1^\mu)\}$ and $\{(u^\mu)\}$ Cauchy in $H^1_{\Gamma_0}(\Omega)$, $L^2(\Omega)$ and $\left(C[0,T],H^1_{\Gamma_0}(\Omega)\right)\cap C^1([0,T],L^2(\Omega))$ we vonclude that $\{v_t^\mu\}$ and $\{v^\mu\}$ are Cauchy in $\mathcal{W}_T$. Thus, (v^μ,v_t^μ) converges to a limit $(v,v_t)\in\mathcal{W}_T$.

We now prove that the limit $(v(x,t),v_t(x,t))$ is a weak solution of (P4).

To this end, we multiply equation (3.19) by ψ in $D(\Omega)$ and integrate over Ω; then, we get

$$\begin{aligned}&\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Omega v^\mu\psi\mathrm{d}x+\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1}v_t^\mu\psi\mathrm{d}\Gamma+\int_\Omega\nabla v^\mu\nabla\psi\mathrm{d}x+\gamma\int_\Omega\nabla v_t^\mu\nabla\psi\mathrm{d}x\\&+r\int_{\Gamma_1}|v_t^\mu|^{k(x)-2}v_t^\mu(t)\psi\mathrm{d}\Gamma=\int_\Omega|u^\mu|^{p(x)-2}u^\mu\psi\mathrm{d}x.\end{aligned}$$

As $\mu\to\infty$, the followings hold in $C([0,T])$:

$$\begin{aligned}&\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Omega v^\mu\psi\mathrm{d}x\to\int_\Omega v\psi\mathrm{d}x; && \int_\Omega\nabla v^\mu\nabla\psi\mathrm{d}x\to\int_\Omega\nabla v\nabla\psi\mathrm{d}x;\\&\int_\Omega\nabla v_t^\mu\nabla\psi\mathrm{d}x\to\int_\Omega\nabla v_t\nabla\psi\mathrm{d}x; && \int_{\Gamma_1}v_t^\mu\psi\mathrm{d}\Gamma\to\int_{\Gamma_1}v_t\psi\mathrm{d}\Gamma;\\&\int_\Omega|u^\mu|^{p(x)-2}u^\mu\psi\mathrm{d}x\to\int_\Omega|u|^{p(x)-2}u\psi\mathrm{d}x; && \int_{\Gamma_1}|v_t^\mu|^{k(x)-2}v_t^\mu(t)\psi\mathrm{d}\Gamma\to\\& && \int_{\Gamma_1}|v_t|^{k(x)-2}v_t(t)\psi\mathrm{d}\Gamma.\end{aligned}$$

It follows that $\int_\Omega v_{tt}\psi\mathrm{d}x=\lim_{\mu\to\infty}\int_\Omega v_{tt}^\mu\psi\mathrm{d}x$ is an absolutely continuous function on $[0,T]$, hence $(v(x,t),v_t(x,t))$ is a weak solution to the problem (P4) for almost all $t\in[0,T]$.

Remaining to prove uniqueness, we denote that v^μ, v^ν are the corresponding solutions of problem (P4) to u^μ, u^ν, respectively. Then obviously $V=v^\mu-v^\nu$ satisfies

$$\left(\int_\Omega|V_t|^2\mathrm{d}x+\int_\Omega|\nabla V|^2\mathrm{d}x+\|V_t\|^2_{2,\Gamma_1}\right)+2\gamma\int_0^t\|\nabla V_t\|_2^2\mathrm{d}s\le C\int_0^t\|\nabla U\|_2\|V_t\|_2\mathrm{d}s.$$

This shows that $V=0$ for $u^\mu=u^\nu$ which implies the uniqueness. □

Proof of theorem (3.1). Let us define for $T>0$ the convex closed subset of $\mathcal{W}_T$

$$Y_T=\{(v,v_t)\in\mathcal{W}_T\ \text{ such that } v(0)=u_0 \text{ and } v_t(0)=u_1\}.$$

Let's denote

$$B_R(Y_T)=\left\{(v,v_t)\in\mathcal{W}_T \text{ such that } \|(v,v_t)\|_{\mathcal{W}_T}\le R\right\}.$$

Then Lemma (3.2) implies that for every $u\in Y_T$ we define $v=\Phi(u)$ as the unique solution of problem (P4) corresponding to u. We want to show that this is a satisfying contractive map

$$\Phi(B_R(Y_T))\subset B_R(Y_T).$$

Let $u\in B_R(Y_T)$ and $v=\Phi(u)$. Then for all $t\in[0,T]$

$$\begin{aligned}&\left(\int_\Omega|v_t|^2\mathrm{d}x+\int_\Omega|\nabla v|^2\mathrm{d}x+\|v_t\|^2_{2,\Gamma_1}\right)+2\gamma\int_0^t|\nabla v_t|^2\mathrm{d}s+2r\int_0^t\int_{\Gamma_1}|v_t|^{k(x)}\mathrm{d}\Gamma\mathrm{d}s\\&=\left(\int_\Omega|v_t(0)|^2\mathrm{d}x+\int_\Omega|\nabla v(0)|^2\mathrm{d}x+\|v_t(0)\|^2_{2,\Gamma_1}\right)+2\int_0^t\int_\Omega|u|^{p(x)-2}uv_t(s)\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{3.23}$$

Using Hölder's inequality, we can examine the last term on the right-hand side of inequality (3.23) as follows

$$\begin{aligned}\int_0^t\int_\Omega |u|^{p(x)-2}uv_t(s)\,\mathrm{d}x\mathrm{d}s &\le (p_2-1)\int_\Omega \max\left(|u|^{p_2-2},|u|^{p_1-2}\right)|u|\,|v_t|\,\mathrm{d}x\\ &\le c\max\left(\left(||u(t)||^{p_2-2}_{(p_2-2)n},||u(t)||^{p_1-2}_{(p_1-2)n}\right)||u||_{\frac{2n}{n-2}}\|v_t\|_2\right)\\ &\le cc_*\max\left(\int_\Omega|\nabla u(t)|^{p_2-2}\mathrm{d}x,\int_\Omega|\nabla u(t)|^{p_1-2}\mathrm{d}x\right)||\nabla u||_2\,\|v_t\|_2\\ &= cc_*\max\left(||\nabla u||_2^{p_2-1},||\nabla u||_2^{p_1-1}\right)\|v_t\|_2\,.\end{aligned}$$

Since $(p_1-2)\,n\le(p_2-2)\,n\le\frac{2n}{n-2}$. Thus, by Young's and Sobolev's inequalities, we get $\forall\delta>0$, $\exists C(\delta)>0$, such that $\forall t\in(0,T)$,

$$\int_0^t\int_\Omega |u|^{p(x)-2}uv_t(s)\,\mathrm{d}x\mathrm{d}s\le C(\delta)t\max\left(R^{2(p_2-1)},R^{2(p_1-1)}\right)+\delta\int_0^t|\nabla v_t|^2\,\mathrm{d}s.$$

Plugging the last estimate into inequality (3.23) and choosing δ small enough we get

$$\begin{aligned}\|v\|^2_{Y_T}\ &\le\left(\int_\Omega|v_t(0)|^2\mathrm{d}x+\int_\Omega|\nabla v(0)|^2\mathrm{d}x+\|v_t(0)\|^2_{2,\Gamma_1}\right)\\ &+CT\max\left(R^{2(p_2-1)},R^{2(p_1-1)}\right).\end{aligned}\tag{3.24}$$

By choosing R large enough so that

$$\int_\Omega|v_t(0)|^2\,\mathrm{d}x+\int_\Omega|\nabla v(0)|^2\,\mathrm{d}x+\|v_t(0)\|^2_{2,\Gamma_1}\le\frac{1}{2}R^2,$$

then T sufficiently small so that $CT\max\left(R^{2(p_2-1)},R^{2(p_1-1)}\right)\le\frac{1}{2}R^2$, it follows that $\|v\|_{Y_T}\le R$ from (3.24), hence $v\in B_R(Y_T)$. Next, we have to check that it is a contraction. To this point, we set $U=u-\overline{u}$, $V=v-\overline{v}$ where $v=\Phi(u)$ and $\overline{v}=\Phi(\overline{u})$

$$\begin{array}{c} V_{tt}-\Delta V-\gamma\Delta V_t=|u|^{p(x)-2}u-|\overline{u}|^{p(x)-2}\overline{u}\text{ in }\Omega\times\mathbb{R}^+,\\ V=0\text{ on }\Gamma_0\times(0,+\infty),\\ V_{tt}(x,t)=-\left[\begin{array}{c}\frac{\partial V}{\partial\nu}(x,t)+\gamma\frac{\partial V_t}{\partial\nu}(x,t)\\ +r\left(|v_t|^{k(x)-2}v_t(x,t)-|\overline{v}_t|^{k(x)-2}\overline{v}_t(x,t)\right)\end{array}\right]\\ \text{on }\Gamma_1\times(0,+\infty),\\ V(x,0)=0,\ \ V_t(x,0)=0,\ \ x\in\Omega.\end{array}\tag{3.25}$$

Multiplying the first equation in (3.25) by V_t, integrating over $(0,t)\times\Omega$, and using the algebraic inequality in (3.18) and the estimate (3.21) yields

$$\begin{aligned}&\left(\int_\Omega|V_t|^2\mathrm{d}x+\int_\Omega|\nabla V|^2\mathrm{d}x+\|V_t\|^2_{2,\Gamma_1}\right)+2\gamma\int_0^t|\nabla V_t|^2\mathrm{d}s\\ &\le 2cc_*\int_0^t\max\left(\begin{array}{c}\int_\Omega\left(|\nabla u(t)|^{p_2-2}+|\nabla u(t)|^{p_1-2}\right)\mathrm{d}x,\\ \int_\Omega\left(|\nabla\overline{u}(t)|^{p_2-2}+|\nabla\overline{u}(t)|^{p_2-2}\right)\mathrm{d}x\end{array}\right)||\nabla U||_2\,||V_t||_2\,\mathrm{d}s.\end{aligned}$$

So

$$\|V\|^2_{Y_T}\le 4cc_*T\left(R^{p_2-2}+R^{p_1-2}\right)\|U\|^2_{Y_T}\le CTR^{p_2-2}\|U\|^2_{Y_T}\,.\tag{3.26}$$

If one chooses T small enough to have $CTR^{p_2-2} < 1$, estimate (3.26) shows that Φ is a contraction. The contraction mapping theorem guarantees the existence of a unique solution v that satisfies $v = \Phi(v)$. This completes the proof of Theorem (3.1). □

4. Exponential growth

In this section we consider the problem (1.1) from an energetic point of view: The energy grows exponentially and with it the L^{p_1} and L^{p_2} norms. To state and prove the result, we declare the following notations. From Corollary 3.3.4 in [8] we know $L^{p_2}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$. Hence it is a consequence of embedding $H_0^1(\Omega) \hookrightarrow L^{p_2}(\Omega)$ and Poincar's inequality

$$\|u\|_{p(.)} \leq B \|\nabla u\|_2, \tag{4.1}$$

where B is the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ determined by

$$B^{-1} = \inf\left\{\|\nabla u\|_2 : u \in H_0^1(\Omega), \ \ \|u\|_{p(.)} = 1\right\}.$$

We also define the following constant which will play an important role in the proof of our result.

Let B_1, α_1, α_0, E_1, and $E(0)$ be satisfying constants

$$\begin{aligned} B_1 = \max(1, B), \quad \alpha_1 = B_1^{\frac{-2p_1}{p_1-2}}, \\ \alpha_0 = \|\nabla u_0\|_2^2, \quad E_1 = \left(\tfrac{1}{2} - \tfrac{1}{p_1}\right)\alpha_1. \\ E(0) = \tfrac{1}{2}\|u_1\|_2^2 + \tfrac{1}{2}\|\nabla u_0\|_2^2 + \tfrac{1}{2}\|u_1\|_{2,\Gamma_1}^2 - \int_\Omega \tfrac{1}{p(x)} |u_0|^{p(x)}\,\mathrm{d}x. \end{aligned} \tag{4.2}$$

For the sake of simplicity, we also write $\varrho(u)$ instead of $\varrho_{p(.)}(u)$.

For this purpose we start with the following lemma, which defines the energy of the solution.

Lemma 4.1. *We define the energy of a solution u of (1.1) as:*

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 - \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\,\mathrm{d}x. \tag{4.3}$$

If we multiply the first equation in (1.1) by u_t and integrate over Ω and with respect to t, we get

$$E(t) - E(s) = -\int_s^t \left(\gamma\|\nabla u_t(\tau)\|_2^2 + r\|u_t(\tau)\|_{k(.),\Gamma_1}\right)\mathrm{d}\tau \leq 0, \ \forall\ 0 < s \leq t < T. \tag{4.4}$$

Thus the function E is decrease along the trajectories.

Theorem 4.2. *Let $k_2 < p_1 \leq p(x) \leq p_2$ with $2 < p_1 \leq p(x) \leq p_2 \leq \overline{q}$. Assume that the initial value u_0 is chosen suvh that $E(0) < E_1$ and $B_1^{-2} \geq \|\nabla u_0\|_2^2 > \alpha_1$ hold. Then, under the above conditions, the solution to problem (1.1) will grow exponentially in the norms L^{p_1} and L^{p_2}.*

We conclude from (4.3) and (4.1)

$$\begin{aligned} E(t) &\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{p_1}\max\left(\|u\|_{p(.)}^{p_2}, \|u\|_{p(.)}^{p_1}\right) \\ &\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{p_1}\max\left((B_1\|\nabla u\|_2)^{p_2}, (B_1\|\nabla u\|_2)^{p_1}\right) \\ &= \tfrac{1}{2}\alpha - \tfrac{1}{p_1}\max\left(\left(B_1^2\alpha\right)^{\frac{p_2}{2}}, \left(B_1^2\alpha\right)^{\frac{p_1}{2}}\right) := g(\alpha) \ \forall\alpha \in [0,+\infty), \end{aligned} \tag{4.5}$$

where $\alpha = \|\nabla u\|_2^2$.

Lemma 4.3. *Let $h : [0,+\infty) \to \mathbb{R}$ be defined by*

$$h(\alpha) = \frac{1}{2}\alpha - \frac{1}{p_1}\left(B_1^2\alpha\right)^{\frac{p_1}{2}}. \tag{4.6}$$

Then the following claims hold under the hypotheses of Theorem (4.2):

(i). *h is increasing for $0 < \alpha \leq \alpha_1$ and decreasing for $\alpha \geq \alpha_1$;*

(ii). $\lim_{\alpha\to+\infty} h(\alpha) = -\infty$ *and* $h(\alpha_1) = E_1$.

Proof. By the assumption that $B_1 > 1$ and $p_1 > 2$, one can see that $h(\alpha) = g(\alpha)$, for $0 < \alpha \leq B_1^{-2}$. Furthermore, $h(\alpha)$ is differentiable and continuous in $[0,+\infty)$. We can see that

$$h'(\alpha) = \frac{1}{2} - \frac{1}{2}B_1^{p_1}\alpha^{\frac{p_1-2}{2}}, \quad 0 \leq \alpha < B_1^{-2}.$$

Then follows **(i)**. Since $p_1 - 2 > 0$, we have $\lim_{\alpha\to+\infty} h(\alpha) = -\infty$. A common calculation gives $h(\alpha_1) = E_1$. Then **(ii)** holds. □

Lemma 4.4. *Under the assumptions of Theorem (4.2), there exists a positive constant $\alpha_2 > \alpha_1$ such that*

$$\|\nabla u\|_2^2 \geq \alpha_2, \ t \geq 0, \tag{4.7}$$

$$\int_\Omega |u(x,t)|^{p(x)}\,\mathrm{d}x \geq \left(B_1^2\alpha_2\right)^{\frac{p_1}{2}}. \tag{4.8}$$

Proof. Since $E(0) < E_1$, Lemma (4.3) implies that there is a positive constant $\alpha_2 > \alpha_1$ such that $E(0) = h(\alpha_2)$. By (4.5) we have $h(\alpha_0) = g(\alpha_0) \leq E(0) = h(\alpha_2)$, from Lemma (4.3)**(i)** it follows that $\alpha_0 \geq \alpha_2$ so (4.7) holds for $t = 0$. Now we prove (4.7) by contradiction. Suppose $\|\nabla u(t^*)\|_2^2 < \alpha_2$ for some $t^* > 0$. Suppose that $\|\nabla u(t^*)\|_2^2 < \alpha_2$ for some $t^* > 0$. By the continuity of $\|\nabla u(.,t)\|_2$ and $\alpha_2 > \alpha_1$, we can assume t^* such that $\alpha_2 > \|\nabla u(t^*)\|_2^2 > \alpha_1$, then (4.5) yildes

$$E(0) = h(\alpha_2) < h\left(\|\nabla u(t^*)\|_2^2\right) \leq E(t^*),$$

which contradicts to Lemma (3.2), and (4.7) holds.

By (4.3) and (4.4), we obtain

$$\begin{aligned} \tfrac{1}{p_1}\textstyle\int_\Omega |u(x,t)|^{p(x)}\,\mathrm{d}x &\geq \textstyle\int_\Omega \tfrac{1}{p(x)}|u(x,t)|^{p(x)}\,\mathrm{d}x \geq \tfrac{1}{2}\|\nabla u\|_2^2 - E(0) \\ &\geq \tfrac{1}{2}\alpha_2 - E(0) = \tfrac{1}{2}\alpha_2 - h(\alpha_2) = \tfrac{1}{p_1}\left(B_1^2\alpha_2\right)^{\frac{p_1}{2}}, \end{aligned} \tag{4.9}$$

and (4.8) follows. □

Let $H(t) = E_1 - E(t)$ for $t \geq 0$, we have the following lemma:

Lemma 4.5. *Under the assumptions of Theorem* (4.2) *the function* $H(t)$ *presented above gives the following estimates:*

$$0 < H(0) \leq H(t) \leq \int_{\Omega} \frac{1}{p(x)} |u(x,t)|^{p(x)} \mathrm{d}x, \quad t \geq 0. \tag{4.10}$$

Proof. By Lemma (3.2), $H(t)$ is nondecreasing in t thus

$$H(t) \geq H(0) = E_1 - E(0) > 0, \quad t \geq 0. \tag{4.11}$$

If we combine (4.3), (4.2), (4.7) and $\alpha_2 > \alpha_1$, we get

$$\begin{aligned} H(t) - \int_{\Omega} \tfrac{1}{p(x)} |u(x,t)|^{p(x)} \mathrm{d}x \quad & \leq E_1 - \tfrac{1}{2} \|\nabla u\|_2^2 \\ & \leq \left(\tfrac{1}{2} - \tfrac{1}{p_1}\right) \alpha_1 - \tfrac{1}{2}\alpha_1 < 0, \ t \geq 0, \end{aligned} \tag{4.12}$$

and (4.10) follows from (4.11) and (4.12). □

Based on the above three lemmas, we can provied the proof of Theorem (4.2).

Proof of Theorem (4.2). We then define the auxiliary function for the value $\varepsilon > 0$ small to be selected later

$$L(t) = H(t) + \varepsilon \int_{\Omega} u_t u \mathrm{d}x + \varepsilon \int_{\Gamma_1} u_t u \mathrm{d}\Gamma + \frac{1}{2}\varepsilon\gamma \int_{\Omega} |\nabla u|^2 \mathrm{d}x. \tag{4.13}$$

Let's consider that L is a small perturbation of the energy. Taking the time derivative of (4.13), we get

$$\begin{aligned} \tfrac{\mathrm{d}L(t)}{\mathrm{d}t} \quad & = \gamma \int_{\Omega} |\nabla u_t|^2 \mathrm{d}x + \varepsilon \int_{\Omega} |u_t|^2 \mathrm{d}x + \varepsilon \|u_t\|_{\Gamma_1}^2 + r \int_{\Gamma_1} |u_t|^{k(x)} \mathrm{d}\Gamma \\ & + \varepsilon \int_{\Omega} u_{tt} u \mathrm{d}x + \varepsilon \int_{\Gamma_1} u_{tt} u \mathrm{d}\Gamma + \varepsilon\gamma \int_{\Omega} \nabla u \nabla u_t \mathrm{d}x. \end{aligned} \tag{4.14}$$

Using problem (1.1), we get from equation (4.14)

$$\begin{aligned} \tfrac{\mathrm{d}L(t)}{\mathrm{d}t} \quad & = \gamma \int_{\Omega} |\nabla u_t|^2 \mathrm{d}x + \varepsilon \int_{\Omega} |u_t|^2 \mathrm{d}x + \varepsilon \|u_t\|_{2,\Gamma_1}^2 + r \int_{\Gamma_1} |u_t|^{k(x)} \mathrm{d}\Gamma \\ & - \varepsilon \int_{\Omega} |\nabla u|^2 \mathrm{d}x + \varepsilon \int_{\Omega} |u(t)|^{p(x)} \mathrm{d}x - \varepsilon r \int_{\Gamma_1} |u_t|^{k(x)} u_t u \mathrm{d}\Gamma. \end{aligned} \tag{4.15}$$

To estimate the last term on the right-hand side of the previous equation, let $\delta > 0$ shall be determined later. Young's inequality drives

$$\int_{\Gamma_1} |u_t|^{k(x)} u_t u \mathrm{d}\Gamma \leq \frac{1}{k_1} \int_{\Gamma_1} \delta^{k(x)} |u|^{k(x)} \mathrm{d}\Gamma + \frac{k_2 - 1}{k_1} \int_{\Gamma_1} \delta^{-\frac{k(x)}{k(x)-1}} |u_t|^{k(x)} \mathrm{d}\Gamma.$$

This is obtained by substituting in (4.15)

$$\begin{aligned} \tfrac{\mathrm{d}L(t)}{\mathrm{d}t} \quad & \geq \gamma \int_{\Omega} |\nabla u_t|^2 \mathrm{d}x + \varepsilon \int_{\Omega} |u_t|^2 \mathrm{d}x + \varepsilon \|u_t\|_{2,\Gamma_1}^2 + r \int_{\Gamma_1} |u_t|^{k(x)} \mathrm{d}\Gamma \\ & - \varepsilon \int_{\Omega} |\nabla u|^2 \mathrm{d}x + \varepsilon \int_{\Omega} |u(t)|^{p(x)} \mathrm{d}x - \varepsilon r \tfrac{1}{k_1} \int_{\Gamma_1} \delta^{k(x)} |u|^{k(x)} \mathrm{d}\Gamma \\ & - \varepsilon r \tfrac{k_2 - 1}{k_1} \int_{\Gamma_1} \delta^{-\frac{k(x)}{k(x)-1}} |u_t|^{k(x)} \mathrm{d}\Gamma. \end{aligned} \tag{4.16}$$

Let us evoke the inequality concerning the continuity of the trace operator

$$\int_{\Gamma_1} |u|^{k(x)} \mathrm{d}\Gamma \leq \max\left(\int_{\Gamma_1} |u|^{k_2} \mathrm{d}\Gamma, \int_{\Gamma_1} |u|^{k_1} \mathrm{d}\Gamma\right) \leq C \|u\|_{H^s(\Omega)},$$

who works for

$$k_1 \geq 1 \text{ and } 0 < s < 1, \quad s \geq \tfrac{n}{2} - \tfrac{n-1}{k_2} \geq \tfrac{n}{2} - \tfrac{n-1}{k_1} > 0, \\ \text{because } k_1 \leq k_2 \leq \tfrac{2n-2}{n-2},$$

and the interpolation and Poincaré's inequalities

$$\|u\|_{H^s(\Omega)} \leq C \|u\|_2^{1-s} \|\nabla u\|_2^s \leq C \|u\|_{p(.)}^{1-s} \|\nabla u\|_2^s, \text{ according to } L^{p(.)}(\Omega) \hookrightarrow L^2(\Omega).$$

So we have the following inequality:

$$\begin{aligned} \textstyle\int_{\Gamma_1} |u|^{k(x)} \, \mathrm{d}\Gamma &\leq C \|u\|_{p(.)}^{1-s} \|\nabla u\|_2^s \\ &\leq C \max\left(\varrho(u)^{\frac{1-s}{p_1}}, \varrho(u)^{\frac{1-s}{p_2}} \right) \|\nabla u\|_2^s, \text{ (see (2.1))}. \end{aligned}$$

If $s < \frac{2}{k_2}$ and we use the Young's inequality again, we get

$$\begin{aligned} \textstyle\int_{\Gamma_1} |u|^{k(x)} \, \mathrm{d}\Gamma &\leq C \|u\|_{p(.)}^{1-s} \|\nabla u\|_2^s \\ &\leq C \left[\max\left(\varrho(u)^{\frac{(1-s)k_2\mu}{p_1}}, \varrho(u)^{\frac{(1-s)k_2\mu}{p_2}} \right) + \left(\|\nabla u\|_2^2 \right)^{\frac{k_2 s\theta}{2}} \right]. \end{aligned} \tag{4.17}$$

for $1/\mu + 1/\theta = 1$. Here we choose $\theta = \frac{2}{k_2 s}$ to get $\mu = 2/(2-k_2 s)$. Therefore, the previous inequality becomes

$$\int_{\Gamma_1} |u|^{k(x)} \, \mathrm{d}\Gamma \leq C \left[\max\left(\varrho(u)^{\frac{2(1-s)k_2}{(2-k_2 s)p_1}}, \varrho(u)^{\frac{2(1-s)k_2}{(2-k_2 s)p_2}} \right) + \|\nabla u\|_2^2 \right]. \tag{4.18}$$

Chose s such that

$$0 < s \leq \min\left(\frac{2(p_1 - k_2)}{k_2(p_1 - 2)}, \frac{2(p_2 - k_2)}{k_2(p_2 - 2)} \right),$$

we get

$$\frac{2k_2(1-s)}{(2-k_2 s)p_2} \leq \frac{2k_2(1-s)}{(2-k_2 s)p_1} \leq 1. \tag{4.19}$$

If inequality (4.19) is satisfied, we apply the classical algebraic inequality

$$z^d \leq (z+1) \leq \left(1 + \frac{1}{\omega}\right)(z + \omega), \ \forall z \geq 0, \ 0 < d \leq 1, \ \omega \geq 0,$$

to get the following estimate:

$$\begin{aligned} &\max\left(\varrho(u)^{\frac{2(1-s)k_2}{(2-k_2 s)p_1}}, \varrho(u)^{\frac{2(1-s)k_2}{(2-k_2 s)p_2}} \right) \\ &\quad \leq \left(1 + H(0)^{-1}\right)(\varrho(u) + H(0)) \\ &\quad \leq C(\varrho(u) + H(t)) \quad \forall t \geq 0 \end{aligned} \tag{4.20}$$

Inserting estimate (4.20) into (4.18), we get the following inequality:

$$\int_{\Gamma_1} |u|^{k(x)} \, \mathrm{d}\Gamma \leq C \left(\varrho(u) + 2H(t) + \|\nabla u\|_2^2 \right). \tag{4.21}$$

which eventually gives

$$\begin{aligned} \int_{\Gamma_1} |u|^{k(x)} \,\mathrm{d}\Gamma &\leq C \left(\varrho(u) + 2E_1 - \int_\Omega |u_t|^2 \,\mathrm{d}x - \|u_t\|_{2,\Gamma_1}^2 + \int_\Omega \tfrac{2}{p(x)} |u|^{p(x)} \,\mathrm{d}x \right) \\ &\leq C \left(2E_1 - \int_\Omega |u_t|^2 \,\mathrm{d}x - \|u_t\|_{2,\Gamma_1}^2 + \left(1 + \tfrac{2}{p_1}\right) \int_\Omega |u|^{p(x)} \,\mathrm{d}x \right). \end{aligned} \tag{4.22}$$

Therefore, by injecting inequality (4.22) into inequality (4.16), we get

$$\begin{aligned} \tfrac{\mathrm{d}L(t)}{\mathrm{d}t} &\geq \gamma \int_\Omega |\nabla u_t|^2 \,\mathrm{d}x + \varepsilon \left(1 + \tfrac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right)\right) \int_\Omega |u_t|^2 \,\mathrm{d}x \\ &+\varepsilon \left(1 + \tfrac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right)\right) \|u_t\|_{2,\Gamma_1}^2 \\ &-2\tfrac{\varepsilon r}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) CE_1 \\ &-\varepsilon \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \varepsilon \left(1 - \tfrac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) \left(1 + \tfrac{2}{p_1}\right)\right) \int_\Omega |u(t)|^{p(x)} \,\mathrm{d}x \\ &+r \left(1 - \tfrac{\varepsilon(k_2-1)}{k_1} \max\left(\delta^{-\frac{k_2}{k_1-1}}, \delta^{-\frac{k_1}{k_2-1}}\right)\right) \int_{\Gamma_1} |u_t|^{k(x)} \,\mathrm{d}\Gamma. \end{aligned} \tag{4.23}$$

Of inequality

$$2H(t) = -\left(\int_\Omega |u_t|^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \|u_t\|_{\Gamma_1}^2 - \int_\Omega \frac{2}{p(x)} |u|^{p(x)} \,\mathrm{d}x \right),$$

we have

$$\begin{aligned} -\int_\Omega |\nabla u|^2 \,\mathrm{d}x &= 2H(t) + \int_\Omega |u_t|^2 \,\mathrm{d}x + \|u_t\|_{2,\Gamma_1}^2 - \int_\Omega \tfrac{2}{p(x)} |u|^{p(x)} \,\mathrm{d}x \\ &\geq 2H(t) - 2E_1 + \int_\Omega |u_t|^2 \,\mathrm{d}x + \|u_t\|_{2,\Gamma_1}^2 - \tfrac{2}{p_1} \int_\Omega |u|^{p(x)} \,\mathrm{d}x. \end{aligned} \tag{4.24}$$

So if we inject it into (4.23) we get the following inequality:

$$\begin{aligned} \tfrac{\mathrm{d}L(t)}{\mathrm{d}t} &\geq \gamma \int_\Omega |\nabla u_t|^2 \,\mathrm{d}x + \varepsilon \left(2 + \tfrac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right)\right) \int_\Omega |u_t|^2 \,\mathrm{d}x \\ &+\varepsilon \left(2 + \tfrac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right)\right) \|u_t\|_{2,\Gamma_1}^2 \\ &+\varepsilon \left(1 - \tfrac{2}{p_1} - \tfrac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) \left(1 + \tfrac{2}{p_1}\right)\right) \int_\Omega |u|^{p(x)} \,\mathrm{d}x \\ &+\varepsilon \left(2H(t) - 2\left(1 + \tfrac{r}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) C\right) E_1\right) \\ &+r \left(1 - \tfrac{\varepsilon(k_2-1)}{k_1} \max\left(\delta^{-\frac{k_2}{k_1-1}}, \delta^{-\frac{k_1}{k_2-1}}\right)\right) \int_{\Gamma_1} |u_t|^{k(x)} \,\mathrm{d}\Gamma. \end{aligned} \tag{4.25}$$

Using the definition of α_2 and E_1 (see Equation (4.2) and Lemma (4.4)), we have

$$\begin{aligned} &-2E_1 - 4\tfrac{r}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) CE_1 \\ &= -2E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} \left(B_1^2 \alpha_2\right)^{\frac{p_1}{2}} \\ &-4\tfrac{r}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) CE_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} \left(B_1^2 \alpha_2\right)^{\frac{p_1}{2}} \\ &\geq \left(-2E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} - 4\tfrac{r}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) CE_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} \right) \int_\Omega |u|^{p(x)} \,\mathrm{d}x. \end{aligned}$$

Finally we get

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &\geq \gamma \int_\Omega |\nabla u_t|^2 \mathrm{d}x + \varepsilon \left(2 + \frac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right)\right) \int_\Omega |u_t|^2 \mathrm{d}x \\ &+\varepsilon \left(2 + \frac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right)\right) \|u_t\|^2_{2,\Gamma_1} \\ &+\varepsilon \left(\begin{array}{c} 1 - \frac{2}{p_1} - 2E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} \\ -\frac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) \left[\left(1 + \frac{2}{p_1}\right) + 4E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}}\right] \end{array}\right) \int_\Omega |u|^{p(x)} \mathrm{d}x \\ &+2\varepsilon \left(H(t) + \frac{r}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) C E_1\right) \\ &+r \left(1 - \frac{\varepsilon(k_2 - 1)}{k_1} \max\left(\delta^{-\frac{k_2}{k_1 - 1}}, \delta^{-\frac{k_1}{k_2 - 1}}\right)\right) \int_{\Gamma_1} |u_t|^{k(x)} \mathrm{d}\Gamma, \end{aligned} \tag{4.26}$$

because

$$1 - \frac{2}{p_1} - 2E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} > 0 \text{ since } \alpha_2 > B_1^{-\frac{2p_1}{p_1 - 2}},$$

we can now choose δ small enough such that

$$\left(\begin{array}{c} 1 - \frac{2}{p_1} - 2E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} \\ -\frac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) \left[\left(1 + \frac{2}{p_1}\right) + 4E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}}\right] \end{array}\right) > 0.$$

Once δ is fixed, let's select ε small enough

$$\left(1 - \frac{\varepsilon(k_2 - 1)}{k_1} \max\left(\delta^{-\frac{k_2}{k_1 - 1}}, \delta^{-\frac{k_1}{k_2 - 1}}\right)\right) > 0 \text{ and } L(0) > 0.$$

Hence the inequality (4.26) becomes

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} \geq \varepsilon\eta \left[H(t) + \int_\Omega |u_t|^2 \mathrm{d}x + \|u_t\|^2_{2,\Gamma_1} + \int_\Omega |u|^{p(x)} \mathrm{d}x + E_1\right] \quad \text{for some } \eta > 0. \tag{4.27}$$

Next it is clear that by Young's inequality and Poincaré's inequality we obtain

$$L(t) \leq \lambda \left[H(t) + \int_\Omega |u_t|^2 \mathrm{d}x + \|u_t\|^2_{2,\Gamma_1} + \int_\Omega |\nabla u|^2 \mathrm{d}x\right] \text{ for some } \lambda > 0. \tag{4.28}$$

From (4.12), we have

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \leq 2E_1 + \frac{2}{p_1} \int_\Omega |u(x,t)|^{p(x)} \mathrm{d}x, \quad t \geq 0.$$

So the inequality (4.28) becomes

$$L(t) \leq \zeta \left[H(t) + \int_\Omega |u_t|^2 \mathrm{d}x + \|u_t\|^2_{2,\Gamma_1} + \int_\Omega |u|^{p(x)} \mathrm{d}x + E_1\right] \quad \text{for some } \zeta > 0. \tag{4.29}$$

From the two inequalities (4.27) and (4.29), we finally get the differential inequality

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} \geq \mu L(t) \text{ for some } \mu > 0. \tag{4.30}$$

Integrating the previous differential inequality (4.30) on $(0,t)$ gives the following estimate for the function L:

$$L(t) \geq L(0)\, e^{\mu t}. \tag{4.31}$$

On the other hand, from the definition of the function L (and for small values of the parameter ε) follows

$$\begin{aligned} L(0)\, e^{\mu t} &\leq L(t) \leq \frac{1}{p_1}\int_\Omega |u|^{p(x)}\,\mathrm{d}x \\ &\leq \frac{1}{p_1}\max\left(\int_\Omega |u|^{p_2}\,\mathrm{d}x, \int_\Omega |u|^{p_1}\,\mathrm{d}x\right). \end{aligned} \tag{4.32}$$

From the two inequalities (4.31) and (4.32) we derive the exponential growth of the solution in the L^{p_2} and L^{p_1} norms. □

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Hyers-Ulam stability of some positive linear operators

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Abstract. The present article deals with the Hyers-Ulam stability of positive linear operators in approximation theory. We discuss the HU-stability of Bernstein-Schurer type operators, Bernstein-Durrmeyer operators and find the HU-stability constant for these operators. Also, we show that the beta operators with Jacobi weights are HU-unstable.

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1. Introduction

In a conference at the University of Wisconsin, Madison, Ulam asked a question regarding the stability of an equation in a metric group. The question posed by Ulam was whether,
"Given a metric group $(G, ., \rho)$, a number $\epsilon > 0$, and a mapping $f : G \to G$ that satisfies the inequality

$$\rho(f(xy), f(x)f(y)) < \epsilon \text{ for all } x, y \in G,$$

does there exists a homomorphism a of G and a constant $k > 0$ (dependent only on G) such that

$$\rho(a(x), f(x)) \leq k\epsilon \text{ for all } x \in G?\text{"}$$

This question is concerned with finding an exact solution close to every approximate solution. If the answer to this question is positive, then the equation $a(xy) = a(x)a(y)$ is called HU-stable, indicating the existence of a unique exact solution close to the

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approximate solution.
In 1941, Hyers [8] provided a proof for a specific equation of the form

$$f(x+y) = f(x) + f(y)$$

in Banach spaces, known as the Cauchy functional equation. This equation is fundamental in Mathematics. Further developments in this field can be found in the references: [3, 4, 9, 10, 12, 15, 16, 21]. For unbounded Cauchy difference equations, Aoki [1] and Rassias [19] introduced another type of stability for functional equations, where the parameter ϵ is replaced by a function depending on x and y.

The Hyers-Ulam stability of linear operators was first observed in the papers by Miura et al. [2, 6, 7, 13], who provided characterizations of HU-stability and its constants for linear operators.

To the best of our knowledge, the HU-stability of positive linear operators in approximation theory was first investigated by Popa and Raşa [17], who examined the HU-stability of both discrete and integral operators. They established the general result that every positive linear operator with finite-dimensional range is HU-stable. Additionally, they determined the HU-stability constant for Bernstein operators and showed that Szász-Mirakyan and beta operators are unstable. In another article [18], the authors obtained stability constants for more general operators and improved the constant for Bernstein operators.

In 2015, Mursaleen and Ansari [14] found the best constant in terms of HU-stability for Kantorovich-Stancu and King's operators. They also demonstrated the unstability of Szász-Mirakyan type operators.

Positive linear operators have many applications in various areas of Mathematics, including functional analysis, approximation theory, and numerical analysis. Hyers-Ulam stability helps us to see the change in behavior of positive linear operators under perturbations. This implies that the operators T has a stable behavior with respect to small perturbations in the function it operates on.

Motivated by the applications of positive linear operators and behavior of their solution with Hyers-Ulam stabiility, in the present article, we determine the stability and the best constant for Bernstein-Stancu type operators and Bernstein-Durrmeyer operators. We also establish the unstability of beta operators with Jacobi weights.
The paper is organized as: Section 1 includes introduction that provides an overview of the problem and the motivation behind studying HU-stability of operators in approximation theory. In section 2, we provide basic definitions and results useful in the subsequent sections. In next section, we discuss the HU-stability of two specific types of operators: Bernstein-Schurer type operators and Bernstein-Durrmeyer operators and determine the HU-stability constants for these operators, which quantify how close the approximate solutions are to the exact solutions. Section 4 investigates the unstability of beta operators with Jacobi weights.

2. Basic definitions and results

Definition 2.1. [20] Let X and Y are two normed spaces and $L : X \to Y$ is a mapping. We say that L has the Hyers-Ulam stability property or L is HU-stable if there exists

a constant K such that
(i) for any $g \in L(X)$, $\epsilon > 0$ and $f \in X$ with $\| Lf - g \| \leq \epsilon$, there exists a $f_0 \in X$ such that $Lf_0 = g$ and $\| f - f_0 \| \leq K\epsilon$.
The condition expresses the Hyers-Ulam stabiliy of the equation

$$Lf = g,$$

where $g \in R(L)$ is given and $f \in X$ is unknown. The number K is called Hyers-Ulam stability (HUS) constant of L, and the infimum of all HUS contants is denoted by K_L, which, in general, is not a HUS constant.

For any bounded linear operator L with kernel $N(L)$ and the range space $R(L)$, we can consider a one-to-one operator $\tilde{L}$ from the quotient space $X/N(L)$ into Y defined as:

$$\tilde{L}(f + N(L)) = Lf, \; f \in X.$$

The inverse of this operator is $\tilde{L}^{-1} : R(L) \to X/N(L)$.

Theorem 2.2. [20] *Let X and Y be Banach spaces and $L : X \to Y$ be a bounded linear operator. Then the following statements are equivalent:*
(I) *L is HU-stable;*
(II) *$R(L)$ is closed;*
(III) *$\tilde{L}^{-1}$ is bounded.*
Moreover, if any of the above conditions are satisfied, then $K_L = \| \tilde{L}^{-1} \|$.

Remark 2.3. If $L : X \to Y$ is bounded linear operator, then (i) in Definition 2.1 is equivalent to:
for any $f \in X$ with $\| Lf \| \leq 1$ there exists an $f_0 \in N(L)$ such that

$$\| f - f_0 \| \leq K. \tag{2.1}$$

It is clear from Remark 2.3 that, to study the HU-stability of a bounded linear operator $L : X \to Y$, we need to show either the existence of a constant K for (2.1) or the boundedness of the operators $\tilde{L}^{-1}$.
Let $g \in \Pi_n$, where Π_n is the set of all polynomials of degree at most n with real coefficients. Then g has a unique Lorentz representation of the form

$$g(x) = \sum_{k=0}^{n} c_k x^k (1-x)^{n-k}, \tag{2.2}$$

where $c_k \in \mathbb{R}, \; k = 0, 1, \cdots, n$.
Let T_n denotes the usual nth degree Chebyshev polynomial of the first kind. Then the following representation [11] holds:

$$T_n(2x-1) = \sum_{k=0}^{n} d_{n,k} \, x^k (1-x)^{n-k} (-1)^{n-k}, \tag{2.3}$$

where

$$d_{n,k} := \sum_{j=0}^{\min\{k,n-k\}} \binom{n}{2j} \binom{n-2j}{k-j} 4^j, \; k = 0, 1, \cdots, n.$$

It is proved in [17] that

$$d_{n,k} = \binom{2n}{2k}, \ k = 0, 1, \cdots, n. \tag{2.4}$$

Therefore,

$$T_n(2x-1) = \sum_{k=0}^{n} \binom{2n}{2k} x^k (1-x)^{n-k} (-1)^{n-k}.$$

Theorem 2.4. [11] *Let $g(x)$ has the representation* (2.2) *and* $0 \leq k \leq n$. *Then*

$$| \, c_k \, | \leq d_{n,k}. \parallel g \parallel_\infty,$$

where equality holds if and only if gis a constant multiple of $T_n(2x-1)$.

3. HU-stability of Bernstein-Schurer type Operators and Bernstein-Durrmeyer Operators

3.1. Bernstein-Schurer type operators

For any integer $n \geq 1$. Let Π_n denote the space of all polynomials of degree $\leq n$, which is a subspace of $C[0,1]$, a space consisting all continuous functions on $[0,1]$. Consider $C[0,1+p]$ be the linear space of all continuous functions $f : [0, 1+p] \to \mathbb{R}$ having supremum norm. Let $0 \leq a \leq b$, the Bernstein-Schurer type operators $S_{n,p} : C[0,1+p] \to \Pi_{n+p}$ are defined by

$$S_{n,p}(f;x) = \sum_{k=0}^{n+p} \binom{n+p}{k} x^k (1-x)^{n+p-k} f\left(\frac{k+a}{n+b}\right).$$

These operators are HU-stable being finite dimensional operators. Here, we find the HUS constant for Bernstein-Schurer type operators.
The kernel of $S_{n,p}$ is given as:

$$N(S_{n,p}) = \left\{ f \in C[0,1+p]; f\left(\frac{k+a}{n+b}\right) = 0, 0 \leq k \leq n+p \right\}.$$

$N(S_{n,p})$ is closed subspace of $C[0,1+p]$ and $R(S_{n,p}) = \Pi_{n+p}$.

Thus, $S_{n,p} : \dfrac{C[0,1+p]}{N(S_{n,p})} \to \Pi_{n+p}$ is bijective. Hence, $\tilde{S}_{n,p}^{-1} : \Pi_{n+p} \to \dfrac{C[0,1+p]}{N(S_{n,p})}$ exists and bijective.
Now, to find the HUS constant, we need to find the $\|\tilde{S}_{n,p}^{-1}\|$.
Let $g \in \Pi_{n+p}$with $\|g\| \leq 1$ has its Lorentz representation as

$$g(x) = \sum_{k=0}^{n+p} c_k(g) x^k (1-x)^{n+p-k}, \quad x \in [0,1].$$

Consider a piecewise function

$$f_g(x)=\begin{cases} c_0(g), & x\in\left[0,\dfrac{a}{n+b}\right) \\ \dfrac{c_k(g)}{\binom{n+p}{k}}, & x\in\left[\dfrac{k+a}{n+b},\dfrac{k+a+1}{n+b}\right)\ \ 0\le k\le n-1 \\ c_{n+p}(g), & x\in\left[\dfrac{n+a}{n+b},1\right]. \end{cases} \tag{3.1}$$

Clearly, $S_{n,p}(f_g;x)=g(x)$ that is $\tilde{S}^{-1}_{n,p}(g(x))=f_g+N(S_{n,p})$.
Thus,

$$\begin{aligned} \|\tilde{S}^{-1}_{n,p}\| &= \sup_{\|g\|\le 1}\|\tilde{S}^{-1}_{n,p}(g)\| = \sup_{\|g\|\le 1}\inf_{h\in N(S_{n,p})}\|f_g+h\| \\ &= \sup_{\|g\|\le 1}\|f_g\| = \sup_{\|g\|\le 1}\max_{0\le k\le n+p}\frac{|c_k(g)|}{\binom{n+p}{k}} \\ &\le \sup_{\|g\|\le 1}\max_{0\le k\le n+p}\frac{d_{n+p,k}\,\|g\|}{\binom{n+p}{k}} \quad \text{[Using Theorem 2.4]} \\ &\le \max_{0\le k\le n+p}\frac{d_{n+p,k}}{\binom{n+p}{k}}. \end{aligned} \tag{3.2}$$

Now, let $q(x)=T_n(2x-1),\ x\in[0,1]$ be Chebyshev poynomials. Then $\|q\|=1$ and from Theorem 2.4 $|c_k(q)|=d_{n+p,k}$. So,

$$\|\tilde{S}^{-1}_{n,p}\|\ge\max_{0\le k\le n+p}\frac{|c_k(q)|}{\binom{n+p}{k}}=\max_{0\le k\le n+p}\frac{d_{n+p,k}}{\binom{n+p}{k}}. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\|\tilde{S}^{-1}_{n,p}\|=\max_{0\le k\le n+p}\frac{d_{n+p,k}}{\binom{n+p}{k}}=\max_{0\le k\le n+p}\frac{\binom{2n+2p}{2k}}{\binom{n+p}{k}} \qquad \text{[By (2.4)]}$$

Let $a_k=\dfrac{\dbinom{2n+2p}{2k}}{\dbinom{n+p}{k}}$, $0\le k\le n+p$. Then, $\dfrac{a_{k+1}}{a_k}=\dfrac{2n+2p-2k-1}{2k+1}$, $0\le k\le n+p-1$.
The inequality $\dfrac{a_{k+1}}{a_k}\ge 1$ is satisfied if and only if $k\le\left[\frac{n+p-1}{2}\right]$, where $[x]$ denotes the greatest integer function. So, maximum value of $a_k,\ 0\le k\le n+p$ will be at $\left[\frac{n+p-1}{2}\right]+1$.

i.e. $\max\limits_{0\le k\le n+p} a_k=a_{\left[\frac{n+p-1}{2}\right]+1}=\begin{cases} a_{\left[\frac{n+p}{2}\right]}, & \text{if } n+p \text{ is even} \\ a_{\left[\frac{n+p}{2}\right]+1}=a_{\left[\frac{n+p}{2}\right]}, & \text{if } n+p \text{ is odd.}\end{cases}$

Hence, $\max\limits_{0\le k\le n+p} a_k=a_{\left[\frac{n+p}{2}\right]}$.

Finally, using (3.3), $\|\tilde{S}_{n,p}^{-1}\| = \dfrac{\binom{2(n+p)}{2\left[\frac{n+p}{2}\right]}}{\binom{n+p}{\left[\frac{n+p}{2}\right]}}$.

When $p = 0$, it will reduce to HUS constant for Bernstein-Stancu operators. Also, when $p = a = b = 0$, it will reduce the HUS constant for Bernstein operators.

3.2. Bernstein-Durrmeyer operators

Durrmeyer [5] in 1967 defined Bernstein-Durrmeyer operators $D_n : C[0,1] \to C[0,1]$ as:

$$D_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t)\, f(t)\, dt, \quad x \in [0,1],\, n \geq 1. \tag{3.4}$$

As the range of the operators (3.4) is Π_n, which is finite dimensional. Hence, the operators are HU-stable. Now, we will find the HUS constant for these operators. Therefore, we will check that boundedness of its inverse operators. The kernel of $D_n(.;x)$ is:

$$N(D_n) = \left\{ f \in C[0,1];\, \int_0^1 p_{n,k}(t)\, f(t)\, dt = 0 \right\}.$$

$N(D_n)$ is closed subspace of $C[0,1]$ and $R(D_n) = \Pi_n$.

Hence, $\tilde{D}_n : \dfrac{C[0,1]}{N(D_n)} \to \Pi_n$ is bijective. So, $\tilde{D}_n^{-1}$ exists and bijective, where

$$\tilde{D}_n^{-1} : \Pi_n \to \frac{C[0,1]}{N(D_n)}.$$

Let Lorentz representation of $g(x) = \sum_{k=0}^{n} x^k (1-x)^{n-k} c_k(g)$ such that $g \in \Pi_n$ and $\|g\| \leq 1$.

Define a function $f_g \in C[0,1]$ as: $f_g(x) = \dfrac{c_k(g)}{\binom{n}{k}},\ 0 \leq k \leq n.$

Clearly, $D_n(f_g;x) = g(x)$, therefore $\tilde{D}_n^{-1}(g(x)) = f_g + N(D_n)$.

$$\begin{aligned}
\|\tilde{D}_n^{-1}\| &= \sup_{\|g\|\leq 1} \|\tilde{D}_n^{-1}(g)\| = \sup_{\|g\|\leq 1} \inf_{h\in N(D_n)} \|f_g + h\| \\
&= \sup_{\|g\|\leq 1} \|f_g\| \leq \sup_{\|g\|\leq 1} \max_{0\leq k\leq n} \frac{|\, c_k(g)\, |}{\binom{n}{k}} \\
&\leq \sup_{\|g\|\leq 1} \max_{0\leq k\leq n} \frac{d_{n,k}\|g\|}{\binom{n}{k}} \leq \max_{0\leq k\leq n} \frac{d_{n,k}}{\binom{n}{k}} \quad \text{[Using Theorem 2.4]}.
\end{aligned} \tag{3.5}$$

Now, choose $q(x) = T_n(2x-1),\ x \in [0,1]$. Clearly, $\|q\| = 1$ and $|c_k(q)| = d_{n,k}$.

$$\|\tilde{D}_n^{-1}\| \geq \max_{0\leq k\leq n} \frac{|\, c_k(q)\, |}{\binom{n}{k}} = \max_{0\leq k\leq n} \frac{d_{n,k}}{\binom{n}{k}}. \tag{3.6}$$

Using (3.5) and (3.6), we get:

$$\|\tilde{D}_n^{-1}\| = \max_{0\le k\le n} \frac{d_{n,k}}{\binom{n}{k}} = \max_{0\le k\le n} \frac{\binom{2n}{2k}}{\binom{n}{k}} \quad \text{[By (2.4)]}. \tag{3.7}$$

Consider $a_k = \frac{\binom{2n}{2k}}{\binom{n}{k}}$ and $a_{k+1} = \frac{\binom{2n}{2k+2}}{\binom{n}{k+1}}$. By simple calculations, we get

$$\frac{a_{k+1}}{a_k} = \frac{2n-2k-1}{2k+1}.$$

For $k \le \left[\frac{n-1}{2}\right]$, we have $a_{k+1} \ge a_k$.
Therefore,

$$\begin{aligned} \max_{0\le k\le n} a_k &= a_{\left[\frac{n-1}{2}\right]+1} \\ &= \begin{cases} a_{\left[\frac{n}{2}\right]}, & \text{if } n \text{ is even} \\ a_{\left[\frac{n}{2}\right]+1} = a_{\left[\frac{n}{2}\right]}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{3.8}$$

Thus, $\max\limits_{0\le k\le n} a_k = a_{\left[\frac{n}{2}\right]}$, and by (3.7)

$$\|\tilde{D}_n^{-1}\| = \frac{\binom{2n}{2\left[\frac{n}{2}\right]}}{\binom{n}{\left[\frac{n}{2}\right]}},$$

which is the HUS constant for Bernstein-Durrmeyer operators.

4. Unstability of Beta Operators with Jacobi Weights

For any $\alpha, \beta \ge -1$, the operators are defined as:

$$B_n^{\alpha,\beta}(f;x) = \frac{\displaystyle\int_0^1 t^{nx+\alpha}(1-t)^{n-nx+\beta} f(t)\,dt}{B(nx+\alpha+1, n-nx+\beta+1)}, \tag{4.1}$$

where $B(m,n)$ is the beta function. For $\alpha = \beta = 0$, these operators reduce to beta operators by Lupaş.

Theorem 4.1. *For each $n \ge 1$, the beta operators with Jacobi weights are HU-unstable.*

Proof. To define the inverse of the operators (4.1), firstly, we prove that the operators $B_n^{\alpha,\beta}(.;x)$ are injective.
Consider $B_n^{\alpha,\beta} f = 0$, for some $f \in C[0,1]$.
Thus,

$$\int_0^1 t^{nx+\alpha}(1-t)^{n-nx+\beta} f(t)\,dt = 0.$$

Now, by changing the variable $\dfrac{t}{1-t}=u$, we get

$$\int_0^\infty \frac{u^{nx+\alpha}}{(1+u)^{n+\alpha+\beta+2}}\, f\left(\frac{u}{1+u}\right) du = 0.$$

As $f \in C[0,1]$, therefore g defined as:

$$g(u) = \frac{1}{(1+u)^{n+\alpha+\beta+2}} f\left(\frac{u}{1+u}\right), \quad u \in [0,\infty).$$

is also continuous function on $[0,\infty)$.
Now, we have $\displaystyle\int_0^\infty u^{nx+\alpha}\, g(u)\, du = 0, \quad x \in [0,1],$
Using Mellin transformation, we get:

$$M[g](nx+\alpha+1) = 0, \quad x \in [0,1].$$

Put $nx+\alpha+1 = s$, we have: $M[g](s) = 0 \quad \forall s \in [\alpha+1, n+\alpha+1]$, which gives $g(u) = 0$ a.e. on $[0,\infty)$.
But $g \in C[0,\infty)$, which implies $g(u) = 0$ on $[0,\infty)$. Therefore, $f(t) = 0$ on $[0,1]$. Hence, $B_n^{\alpha,\beta}(.;x)$ are injective.
Now, consider the inverse operators

$$(B_n^{\alpha,\beta})^{-1} : R(B_n^{\alpha,\beta}) \to C[0,1].$$

Denote $e_j(x) = x^j, j = 0,1,\cdots, \quad x \in [0,1]$.
Clearly, $B_n^{\alpha,\beta}(e_0;x) = 1$ and

$$B_n^{\alpha,\beta}(e_j;x) = \frac{(nx+\alpha+1)(nx+\alpha+2)\cdots(nx+\alpha+j)}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)\cdots(n+\alpha+\beta+j+1)}.$$

The eigenvalues of

$$B_n^{\alpha,\beta}(f;x) = \frac{n^j}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)\cdots(n+\alpha+\beta+j+1)}.$$

Thus, eigenvalues of $(B_n^{\alpha,\beta})^{-1}$ are

$$\frac{(n+\alpha+\beta+2)(n+\alpha+\beta+3)\cdots(n+\alpha+\beta+j+1)}{n^j}.$$

Since, $\displaystyle\lim_{j\to\infty} \frac{(n+\alpha+\beta+2)(n+\alpha+\beta+3)\cdots(n+\alpha+\beta+j+1)}{n^j} = \infty.$
We can say that $(B_n^{\alpha,\beta})^{-1}$ is unbounded, so the operators $B_n^{\alpha,\beta}(.;x)$ are HU-unstable.

□

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On reachability and controllability for a Volterra integro-dynamic system on time scales

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Abstract. The paper studies and relates the notions of reachability and controllability for the Volterra integro-dynamic system on time scales. More specifically, we obtain necessary and sufficient conditions for reachability and controllability. In addition, we obtain an equivalence between the concepts of reachability and controllability studied.

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1. Introduction

The Volterra integro-dynamic systems on time scales have been considered in several articles in the literature, which can be witnessed by [1], [2], [8], [10], and [11]. In [1], [2] and [10], the authors have studied the linear Volterra integro-dynamic system on time scales of the type

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + B(t)u(t), \ t \in [0,\infty)_{\mathbb{T}^\kappa} \\ x(t_0) = x_0. \end{cases} \tag{1.1}$$

Adıvar [1] introduced the variation of parameters for Eq. (1.1) and then Adıvar and Raffoul [2] used it to obtain the necessary and sufficient conditions for the uniform stability of the zero solutions of Eq. (1.1) employing the resolvent equation. Lupulescu et al. [10] studied asymptotic behaviour of solutions for (1.1). Karpuz and Koyuncuoğlu [8] obtained the necessary and sufficient conditions for the positivity and uniform exponential stability for the Volterra integro-dynamical systems means of Metzler

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matrices. Further, Younus and ur Rahaman [11] studied controllability, observability, and asymptotic behaviour for the Volterra integro-dynamic system on time scales.

Inspired by [3], [9], and [11], the present study investigates the reachability and controllability for system (1.1). Here $\mathbb{T}$ is a time scale $\mathbb{T}_0 = [0, \infty)_{\mathbb{T}^\kappa}$ and $t_0 \in \mathbb{T}_0$ is fixed, $u\colon \mathbb{T}_0 \to \mathbb{R}^m$ is control function, the functions $A\colon \mathbb{T}_0 \to \mathbb{R}^{n\times n}$ and $B\colon \mathbb{T}_0 \to \mathbb{R}^{n\times m}$ are continuous on $\mathbb{T}_0$, and $K\colon \mathbb{T}_0 \times \mathbb{T}_0 \to \mathbb{R}^{n\times n}$ is continuous on $\Omega := \{(t,s) \in \mathbb{T}^\kappa \times \mathbb{T}^\kappa \colon 0 \le s \le t < \infty\}$. Also, the control functions u can admit a finite number of discontinuities at $t_{u_1}, \ldots, t_{u_p}$ in $\mathbb{T}_0 \setminus \{0, \sup \mathbb{T}\}$ with $p \in I(u) \subset \mathbb{N}$ and $t_{u_i} > t_0$ for every $i \in \{1, \ldots, p\}$, such way that for $1 \le i \le p$, there exist the left and right limit of $u(t)$ at $t = t_{u_i}$ in time scale context, i.e., $u(t_{u_i}^-) = \lim_{h\to 0^+} u(t_{u_i} - h)$ and $u(t_{u_i}^+) = \lim_{h\to 0^+} u(t_{u_i} + h)$, respectively. Further, we have $u(t_{u_i}^-) \neq u(t_{u_i}^+) = u(t_{u_i})$. We emphasize that throughout the work $\mathbb{R}^n$ denotes the space of n-dimensional column vectors, equivalently, $\mathbb{R}^n$ also denotes the space of real matrices $n \times 1$.

For system (1.1), we use the notions of reachability and controllability analogous to those given in [3] and establish necessary and sufficient conditions similar to [3, Theorem 1] and [3, Proposition 5]. We also establish an equivalence between the reachability and controllability of (1.1) analogous to [3, Proposition 6]. To do this, we first state and prove the existence result to system (1.1). The novelty of the results obtained here on controllability in relation to [11] are the new necessary and sufficient conditions to controllability. On the other hand, to the best of our knowledge, there is no studies in the time scales literature related to the reachability for Volterra integro-dynamic system on time scales.

The paper is organized as follows. The next section provides useful background concepts of time scales theory, such as the Δ-derivative in addition to the Δ-integral for reading the paper. In Section 3, we define and obtain the existence of solution to system (1.1). Section 4 contains the results concerning the reachability and controllability to system (1.1). Finally, Section 5 brings the conclusions of the work.

2. Preliminaries

In this section, we include basic concepts of time scales theory that will be used throughout the work.

2.1. Time Scales

Given a time scale $\mathbb{T}$, i.e., a nonempty closed subset of the real numbers, here we assume that there exist $a, b \in \mathbb{T}_0$ such that $a < b$. The forward jump operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} \colon s > t\}$$

and the backward jump operator $\rho\colon \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} \colon s < t\}.$$

In this case, we assume that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. Also, the graininess function $\mu\colon \mathbb{T} \to [0, +\infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

We say that $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, and right-scattered whenever $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, and $\sigma(t) > t$, respectively. For $A \subset \mathbb{R}$, we write $A_{\mathbb{T}} = A \cap \mathbb{T}$. In case $\sup \mathbb{T} < +\infty$, we set $\mathbb{T}^{\kappa} = \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}]_{\mathbb{T}}$, otherwise, if $\sup \mathbb{T} = +\infty$ we set $\mathbb{T}^{\kappa} = \mathbb{T}$.
Take a function $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. If $\xi \in \mathbb{R}$ is such that, for all $\varepsilon > 0$ there exists $\delta > 0$ satisfying

$$|f(\sigma(t)) - f(s) - \xi(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in (t-\delta, t+\delta)_{\mathbb{T}}$, it is said that ξ is the delta derivative of f at t and we denote it by $f^{\Delta}(t)$.
Now, consider a function $f\colon \mathbb{T} \to \mathbb{R}^n$, $f = (f_1, f_2, \cdots, f_n)$, and $t \in \mathbb{T}^{\kappa}$. We say that f is Δ-differentiable at t if each component $f_i\colon \mathbb{T} \to \mathbb{R}$ of f is Δ-differentiable at t. In this case $f^{\Delta}(t) = (f_1^{\Delta}(t), \ldots, f_n^{\Delta}(t))$.

2.2. Δ-Integrability

For fixed $a_1, b_1 \in \mathbb{T}_0$ with $a_1 < b_1$, without loss of generality, we consider the time scale $\mathbb{T}_1 = [a_1, b_1]_{\mathbb{T}}$. We denote the family of Δ-measurable sets of $\mathbb{T}_1$ by Δ. We recall that Δ is a σ-algebra of $\mathbb{T}_1$ (see, for instance, [7]).
Suppose that $f\colon \mathbb{T}_1 \to \mathbb{R}$ is a Δ-measurable function, that is, for any $r \in \mathbb{R}$ the set $\{t \in \mathbb{T}_1 \colon f(t) < r\}$ is Δ-measurable. If $E \in \Delta$, we indicate by

$$\int_E f(s)\Delta s$$

the Lebesgue Δ-integral of f over E. Now, if $f\colon \mathbb{T}_1 \to \mathbb{R}^m$ and $E \in \Delta$, then f is Lebesgue Δ-integrable over E if each component $f_i\colon \mathbb{T}_1 \to \mathbb{R}$ of f is Lebesgue Δ-integrable over E. In this case, we have

$$\int_E f(s)\Delta s = \Big(\int_E f_1(s)\Delta s, \ldots, \int_E f_m(s)\Delta s \Big).$$

Also, if $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$, we will indicate by $\mathrm{L}^2(E;\mathbb{R}^m)$ the set of functions $f : \mathbb{T}_1 \to \mathbb{R}^m$ such that the function $\|f\|^2$ is Lebesgue Δ-integrable over E.
In the vector space $\mathrm{L}^2([a_1, b_1]_{\mathbb{T}};\mathbb{R}^m)$, we can define the inner product

$$\langle f, g\rangle_{L^2} := \int_{[a_1,b_1)_{\mathbb{T}}} g^T(s) f(s)\Delta s,$$

where $f, g \in \mathrm{L}^2([a_1, b_1]_{\mathbb{T}};\mathbb{R}^m)$ and $g^T(s)$ denotes the transpose of the column vector $g(s) \in \mathbb{R}^m$.

Similarly to [5, Théorème IV.8.], we have the following remark.

Remark 2.1. The vector space $\mathrm{L}^2([a_1, b_1]_{\mathbb{T}};\mathbb{R}^m)$ is a Banach Space when equipped with the norm induced by the inner product $\langle\cdot,\cdot\rangle_{L^2}$.

We recall that a function $f\colon \mathbb{T}_1 \to \mathbb{R}^n$ is said to be right-dense continuous (rd-continuous) if f is continuous at each right-dense point $t \in \mathbb{T}_1$, and if $\lim_{h\to 0^+} f(t-h)$ exists and finite at each left-dense point $t \in \mathbb{T}_1$.

The Cauchy integral and the Lebesgue Δ-integral of an rd-continuous function can be related as follows. For this, let $f\colon \mathbb{T}_1 \to \mathbb{R}$ be an rd-continuous function. From [4], we can see that function f has an antiderivative $F : \mathbb{T} \to \mathbb{R}$, if $F^{\Delta}(t) = f(t)$ for each $t \in \mathbb{T}_1^{\kappa}$. Thus, the Cauchy integral of f is defined as

$$\int_c^d f(s)\Delta s = F(d) - F(c)$$

for all $c, d \in \mathbb{T}_1$. Hence, the Cauchy integral and the Lebesgue Δ-integral of f relate as

$$\int_c^d f(s)\Delta s = \int_{[c,d)_{\mathbb{T}_1}} f(s)\Delta s$$

with $c, d \in \mathbb{T}_1$ and $c \leq d$.
More about the integration on time scales, can be found in [4], [6], and [7].

3. Existence to Eq. (1.1)

Here we define the solution to system (1.1) and then establish the existence of solution in Theorem 3.1. For this, we first consider the principal matrix solution $Z(t, s)$ of the integro-dynamic equation

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + \displaystyle\int_s^t K(t,\tau)x(\tau)\Delta\tau, \ t \in [s,\infty)_{\mathbb{T}^{\kappa}} \\ x(s) = x_0, \end{cases} \tag{3.1}$$

where $s \in \mathbb{T}^{\kappa}$. The principal matrix solution of Eq. (3.1) is the $n \times n$ matrix function $Z(t, s)$ defined as

$$Z(t, s) = [x^1(t, s), \ldots, x^n(t, s)]$$

where $x^i(t, s)$, $i = 1, \ldots, n$, are the linearly independent solutions of Eq. (3.1). Given the control function $u(t) \in \mathbb{R}^m$, we define the solution x of system (1.1) as follows. If u is continuous on $\mathbb{T}_0$, as can be seen in [1, Theorem 19], the solution x of Eq. (1.1) on $\mathbb{T}_0$ is given by

$$x(t) = Z(t, t_0)x_0 + \int_{t_0}^t Z(t, \sigma(s))B(s)u(s)\Delta s, \tag{3.2}$$

where $Z(t, s)$ is the principal matrix solution of Eq. (3.1).
Now, if the control function $u(t) \in \mathbb{R}^m$ admits the discontinuities $t_{u_1}, \ldots, t_{u_p}$ in $\mathbb{T}_0 \setminus \{0, \sup \mathbb{T}\}$, with $p \in I(u) \subset \mathbb{N}$, for $i = 1$, we consider the continuous function $w_1 : [0,\infty)_{\mathbb{T}^{\kappa}} \to \mathbb{R}^m$ defined by

$$w_i(t) = \begin{cases} u(t) & \text{if } t \in [0, t_{u_i})_{\mathbb{T}^{\kappa}}, \\ u(t_{u_i}^-) & \text{if } t \in [t_{u_i}, \infty)_{\mathbb{T}^{\kappa}}, \end{cases} \tag{3.3}$$

and for $1 < i \leq p$, we take the continuous function $w_i : [0,\infty)_{\mathbb{T}^{\kappa}} \to \mathbb{R}^m$ given by

$$w_i(t) = \begin{cases} u(t_{u_{i-1}}^+) & \text{if } t \in [0, t_{u_{i-1}})_{\mathbb{T}^{\kappa}}, \\ u(t) & \text{if } t \in [t_{u_{i-1}}, t_{u_i})_{\mathbb{T}^{\kappa}}, \\ u(t_{u_i}^-) & \text{if } t \in [t_{u_i}, \infty)_{\mathbb{T}^{\kappa}}. \end{cases} \tag{3.4}$$

We also consider the continuous function $w_{p+1} : [0,\infty)_{\mathbb{T}^\kappa} \to \mathbb{R}^m$ given by

$$w_{p+1}(t) = \begin{cases} u(t_{u_p}^+) & \text{if } t \in [0, t_{u_p})_{\mathbb{T}^\kappa}, \\ u(t) & \text{if } t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}. \end{cases} \tag{3.5}$$

Hence for $i = 1$, let $x_i = x_1$ be the solution of integro-dynamic equation

$$\begin{cases} x_1^\Delta(t) = A(t)x_1(t) + \displaystyle\int_{t_0}^t K(t,s)x_1(s)\Delta s + B(t)w_1(t), \\ x_1(t_0) = x_0 \end{cases} \tag{3.6}$$

on $[0,\infty)_{\mathbb{T}^\kappa}$ and for $1 < i \le p$, let x_i be the solution of integro-dynamic equation

$$\begin{cases} x_i^\Delta(t) = A(t)x_i(t) + \displaystyle\int_{t_{u_{i-1}}}^t K(t,s)x_i(s)\Delta s + B(t)w_i(t), \\ x_i(t_{u_{i-1}}) = x_{i-1}(t_{u_{i-1}}) \end{cases} \tag{3.7}$$

on $[t_{u_{i-1}}, \infty)_{\mathbb{T}^\kappa}$. Futhermore, let x_{p+1} be the solution of integro-dynamic equation

$$\begin{cases} x_{p+1}^\Delta(t) = A(t)x_{p+1}(t) + \displaystyle\int_{t_{u_p}}^t K(t,s)x_{p+1}(s)\Delta s + B(t)w_{p+1}(t), \\ x_{p+1}(t_{u_p}) = x_p(t_{u_p}) \end{cases} \tag{3.8}$$

on $[t_{u_p}, \infty)_{\mathbb{T}^\kappa}$. Thus, we define the solution x of system (1.1) as

$$x(t) = \begin{cases} x_1(t) & \text{if } t \in [0, t_{u_1})_{\mathbb{T}^\kappa}, \\ x_i(t) & \text{if } t \in [t_{u_{i-1}}, t_{u_i})_{\mathbb{T}^\kappa},\ 1 < i \le p, \\ x_{p+1}(t) & \text{if } t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}. \end{cases}$$

The principal matrix $Z(t,s)$ is said to be transition matrix if $Z(s,s) = \mathrm{Id}$. According to [11, Lemma 2.2], the transition matrix $Z(t,s)$ of Eq. (3.1) admits, among others, the following properties:

(i) $Z(t,s) = Z(t,\tau)Z^{-1}(s,\tau)$;
(ii) $Z(t,s) = Z^{-1}(s,t)$;
(iii) $Z(t,r)Z(r,s) = Z(s,t)$.

Theorem 3.1. *Suppose that the control function $u\colon \mathbb{T}_0 \to \mathbb{R}^m$ in Eq.* (1.1) *admits the discontinuities $t_{u_1}, \ldots, t_{u_p}$ in $\mathbb{T}_0 \setminus \{0, \sup \mathbb{T}\}$. Then the solution x of system* (1.1) *is given by*

$$x(t) = Z(t,t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

on $[0,\infty)_{\mathbb{T}^\kappa}$.

Proof. For the control function $u(t) \in \mathbb{R}^m$ we consider continuous functions w_i $(1 \le i \le p)$ and w_{p+1} as defined in Eqs. (3.3), (3.4), and (3.5). Let x_1 be the solution of Eq. (3.6) on $[0,\infty)_{\mathbb{T}^\kappa}$, and for $1 < i \le p$, x_i be the solution of Eq. (3.7) on $[t_{u_{i-1}}, \infty)_{\mathbb{T}^\kappa}$. Also, let x_{p+1} be the solution of Eq. (3.8) on $[t_{u_p}, \infty)_{\mathbb{T}^\kappa}$. Hence, the solution x of system (1.1) is given by

$$x(t) = \begin{cases} x_1(t) & \text{if } t \in [0, t_{u_1})_{\mathbb{T}^\kappa}, \\ x_i(t) & \text{if } t \in [t_{u_{i-1}}, t_{u_i})_{\mathbb{T}^\kappa},\ 1 < i \le p, \\ x_{p+1}(t) & \text{if } t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}. \end{cases}$$

Using [1, Theorem 19] repeatedly, we have

$$x_1(t) = Z(t, t_0)x_0 + \int_{t_0}^{t} Z(t, \sigma(s))B(s)w_1(s)\Delta s$$

for $t \in [0, t_{u_1}]_{\mathbb{T}^\kappa}$,

$$x_i(t) = Z(t, t_{u_{i-1}})x_{i-1}(t_{u_{i-1}}) + \int_{t_{u_{i-1}}}^{t} Z(t, \sigma(s))B(s)w_i(s)\Delta s$$

for $t \in [t_{u_{i-1}}, t_{u_i}]_{\mathbb{T}^\kappa}$, $1 < i \leq p$, and

$$x_{p+1}(t) = Z(t, t_{u_p})x_p(t_{u_p}) + \int_{t_{u_p}}^{t} Z(t, \sigma(s))B(s)w_{p+1}(s)\Delta s$$

for $t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}$. Thus the solution x is given by

$$x(t) = Z(t, t_0)x_0 + \int_{[t_0, t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

for $t \in [0, t_{u_1}]_{\mathbb{T}^\kappa}$,

$$x(t) = Z(t, t_{u_{i-1}})x(t_{u_{i-1}}) + \int_{[t_{u_{i-1}}, t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

for $t \in [t_{u_{i-1}}, t_{u_i}]_{\mathbb{T}^\kappa}$, $1 < i \leq p$, and for $t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}$, it is expressed by

$$x(t) = Z(t, t_{u_p})x(t_{u_p}) + \int_{[t_{u_p}, t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s.$$

Note that for $p = 1$ and $t \in [t_{u_1}, \infty)_{\mathbb{T}^\kappa}$, the solution is given by

$$x(t) = Z(t, t_{u_1})x(t_{u_1}) + \int_{[t_{u_1}, t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

with

$$x(t_{u_1}) = Z(t_{u_1}, t_0)x_0 + \int_{[t_0, t_{u_1})_{\mathbb{T}}} Z(t_{u_1}, \sigma(s))B(s)u(s)\Delta s,$$

and by using properties of transition matrices and integrals, we can conclude that

$$x(t) = Z(t, t_0)x_0 + \int_{[t_0, t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

for each $t \in [t_{u_1}, \infty)_{\mathbb{T}^\kappa}$. Thus,

$$x(t) = Z(t, t_0)x_0 + \int_{[t_0, t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

for every $t \in [0, \infty)_{\mathbb{T}^\kappa}$. Now, for $p > 1$ and $t \in [t_{u_1}, t_{u_2}]_{\mathbb{T}^\kappa}$, the solution is

$$x(t) = Z(t, t_{u_1})x(t_{u_1}) + \int_{[t_{u_1}, t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s,$$

where

$$x(t_{u_1}) = Z(t_{u_1}, t_0)x_0 + \int_{[t_0, t_{u_1})_{\mathbb{T}}} Z(t_{u_1}, \sigma(s))B(s)u(s)\Delta s.$$

Again, we can use properties of transition matrices and integrals to conclude that

$$x(t) = Z(t, t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

for every $t \in [t_{u_1}, t_{u_2}]_{\mathbb{T}^\kappa}$. Similarly, recursively we get

$$x(t) = Z(t, t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

for $t \in [t_{u_{i-1}}, t_{u_i}]_{\mathbb{T}^\kappa}$ and $1 \leq i \leq p$. In this case, for $t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}$, we have

$$x(t) = Z(t, t_{u_p})x(t_{u_p}) + \int_{[t_{u_p},t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

with

$$x(t_{u_p}) = Z(t_{u_p}, t_0)x_0 + \int_{[t_0,t_{u_p})_{\mathbb{T}}} Z(t_{u_p}, \sigma(s))B(s)u(s)\Delta s.$$

Hence, we can also deduce that

$$x(t) = Z(t, t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

for each $t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}$. Therefore

$$x(t) = Z(t, t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

for each $t \in [0, \infty)_{\mathbb{T}^\kappa}$. This completes the proof. □

4. Reachability and controllability

Now, we consider the notions of reachability and controllability for system (1.1) analogous to those given in [3]. Then we establish the results on necessary and sufficient conditions for reachability and controllability (theorems 4.3, 4.4, 4.6, and 4.7). Thus, in theorems 4.3 and 4.4, we establish the results on reachability and in theorems 4.6 and 4.7, we establish results on controllability. Besides, we also establish an equivalence that relates reachability and controllability (Theorem 4.8).

Suppose that $\mathcal{U}$ denotes the set of control functions to system (1.1). Hence, if $\tau \in (0, \infty)_{\mathbb{T}^\kappa}$ and $t_0 \in [0, \tau)_{\mathbb{T}}$, $\mathcal{U}(t_0, \tau)$ will denote the functions from the set $\mathcal{U}$ restricted to $[t_0, \tau]_{\mathbb{T}}$. We point out that $\mathcal{U}(t_0, \tau)$ is a subspace of $\mathrm{L}^2([t_0, \tau]_{\mathbb{T}}; \mathbb{R}^m)$.

Definition 4.1.

1. A state $x_1 \in \mathbb{R}^n$ is said to be reachable at time $\tau \in [0, \infty)_{\mathbb{T}^\kappa}$ if for some $t_0 \in [0, \tau)_{\mathbb{T}}$ and for every initial state $x_0 \in \mathbb{R}^n$, there is an input $u \in \mathcal{U}(t_0, \tau)$ such that the state x of system (1.1) with $x(t_0) = x_0$ satisfies $x(\tau) = x_1$.
2. The system (1.1) is called reachable at time $\tau \in [0, \infty)_{\mathbb{T}^\kappa}$ if each state $x_1 \in \mathbb{R}^n$ is reachable at time τ.

3. The reachability map $\mathcal{B}_r(t_0,\tau)\colon \mathcal{U}(t_0,\tau)\to\mathbb{R}^n$ is defined as
$$\mathcal{B}_r(t_0,\tau)[u]=\int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau,\sigma(s))B(s)u(s)\Delta s.$$
4. The adjoint of $\mathcal{B}_r(t_0,\tau)$, indicated by $\mathcal{B}_r^*(t_0,\tau)\colon \mathbb{R}^n\to\mathcal{U}(t_0,\tau)$, is defined by
$$\mathcal{B}_r^*(t_0,\tau)[w](t)=B^T(t)Z^T(\tau,\sigma(t))w,$$
for all $w\in\mathbb{R}^n$ and for any $t\in[t_0,\tau]_{\mathbb{T}}$.
5. The Gramian reachability map $\mathcal{G}_B^r(t_0,\tau)\colon\mathbb{R}^n\to\mathbb{R}^n$ is given by
$$\begin{aligned}\mathcal{G}_B^r(t_0,\tau)[w]&=\mathcal{B}_r(t_0,\tau)\circ\mathcal{B}_r^*(t_0,\tau)[w]\\&=\int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau,\sigma(s))B(s)B^T(s)Z^T(\tau,\sigma(s))w\Delta s.\end{aligned}$$

Definition 4.2.

1. A state $x_0\in\mathbb{R}^n$ is said to be controllable at time $t_0\in[0,\infty)_{\mathbb{T}^\kappa}$ if for some $\tau\in(t_0,\infty)_{\mathbb{T}^\kappa}$ there exists an input $u\in\mathcal{U}(t_0,\tau)$ such that the state x of system (1.1) with $x(t_0)=x_0$ satisfies $x(\tau)=0$.
2. The system (1.1) is called controllable at time $t_0\in[0,\infty)_{\mathbb{T}^\kappa}$ if each state $x_0\in\mathbb{R}^n$ is controllable at time t_0.
3. The controllability map $\mathcal{B}_c(t_0,\tau)\colon\mathcal{U}(t_0,\tau)\to\mathbb{R}^n$ is defined as
$$\mathcal{B}_c(t_0,\tau)[u]=\int_{[t_0,\tau)_{\mathbb{T}}} Z(t_0,\sigma(s))B(s)u(s)\Delta s.$$
4. The adjoint of $\mathcal{B}_c(t_0,\tau)$, indicated by $\mathcal{B}_c^*(t_0,\tau)\colon\mathbb{R}^n\to\mathcal{U}(t_0,\tau)$, is given by
$$\mathcal{B}_c^*(t_0,\tau)[w](t)=B^T(t)Z^T(t_0,\sigma(t))w,$$
for all $w\in\mathbb{R}^n$ and all $t\in[t_0,\tau]_{\mathbb{T}}$.
5. The Gramian controllability map $\mathcal{G}_B^c(t_0,\tau)\colon\mathbb{R}^n\to\mathbb{R}^n$ is defined by
$$\begin{aligned}\mathcal{G}_B^c(t_0,\tau)[w]&=\mathcal{B}_c(t_0,\tau)\circ\mathcal{B}_c^*(t_0,\tau)[w]\\&=\int_{[t_0,\tau)_{\mathbb{T}}} Z(t_0,\sigma(s))B(s)B^T(s)Z^T(t_0,\sigma(s))w\Delta s.\end{aligned}$$

The necessary and sufficient condition for the reachability of (1.1) in terms of the Gramian reachability matrix is proved in Theorem 4.3.

Theorem 4.3. *For a fixed $\tau\in[0,\infty)_{\mathbb{T}^\kappa}$, the system* (1.1) *is reachable at time τ if and only if there exists $t_0\in[0,\tau)_{\mathbb{T}}$ such that the $n\times n$ Gramian reachability matrix defined by*
$$\mathcal{G}_r(t_0,\tau):=\int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau,\sigma(s))B(s)B^T(s)Z^T(\tau,\sigma(s))\Delta s$$
is invertible, where $Z(t,s)$ is the transition matrix of Eq. (3.1).

Proof. Suppose the Gramian matrix $\mathcal{G}_r(t_0,\tau)$ is invertible. Define the input $u\in\mathcal{U}(t_0,\tau)$ by
$$u(t)=-B^T(t)Z^T(\tau,\sigma(t))\mathcal{G}_r^{-1}(t_0,\tau)(Z(\tau,t_0)x_0-x_1),\tag{4.1}$$

where $x_0, x_1 \in \mathbb{R}^n$. Then, the state x of system (1.1) with $x(t_0) = x_0$ is such that

$$x(\tau) = Z(\tau, t_0)x_0 + \int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)u(s)\Delta s.$$

Now, substituting the value of $u(s)$ from (4.1), we get

$$\begin{aligned} x(\tau) &= Z(\tau, t_0)x_0 \\ &\quad - \mathcal{G}_r^{-1}(t_0, \tau)(Z(\tau, t_0)x_0 - x_1) \int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)B^T(s) \times Z^T(\tau, \sigma(s))\Delta s \\ &= x_1. \end{aligned}$$

Therefore the system (1.1) is reachable at time τ. On the other hand, assume $\mathcal{G}_r(t_0, \tau)$ not invertible and the system (1.1) reachable at time τ. Hence there exists a nonzero vector $x_a \in \mathbb{R}^n$ such that

$$\begin{aligned} 0 &= x_a^T \mathcal{G}_r(t_0, \tau)x_a \\ &= \int_{[t_0,\tau)_{\mathbb{T}}} x_a^T Z(\tau, \sigma(s))B(s)B^T(s)Z^T(\tau, \sigma(s))x_a \Delta s \\ &= \int_{[t_0,\tau)_{\mathbb{T}}} \left\|B^T(s)Z^T(\tau, \sigma(s))x_a\right\|^2 \Delta s. \end{aligned}$$

This gives

$$B^T(s)Z^T(\tau, \sigma(s))x_a = 0, s \in [t_0, \tau]_{\mathbb{T}},$$

i.e.,

$$x_a^T Z(\tau, \sigma(s))B(s) = 0, s \in [t_0, \tau]_{\mathbb{T}}. \tag{4.2}$$

Since the system (1.1) is reachable at time τ, for $x_0 = Z(t_0, \tau)x_a + Z(t_0, \tau)x_1$, there exists an input $u \in \mathcal{U}(t_0, \tau)$ such that the state x of system (1.1) with $x(t_0) = x_0$ obeys

$$x(\tau) = x_1 = Z(\tau, t_0)x_0 + \int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)u(s)\Delta s.$$

Hence,

$$x_1 = x_a + x_1 + \int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)u(s)\Delta s$$

and we deduce that

$$x_a = -\int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)u(s)\Delta s.$$

Now, multiplying the last equation by x_a^T and using Eq. (4.2) we get

$$\begin{aligned} x_a^T x_a &= -\int_{[t_0,\tau)_{\mathbb{T}}} x_a^T Z(\tau, \sigma(s))B(s)u(s)\Delta s \\ &= 0. \end{aligned}$$

That is, $\|x_a\| = 0$ and hence $x_a = 0$, a contradiction. Thus, the Gramian matrix $\mathcal{G}_r(t_0, \tau)$ is invertible. □

Theorem 4.4 given below establishes necessary and sufficient conditions to reachability for system (1.1).

Theorem 4.4. *Suppose $\tau \in [0,\infty)_{\mathbb{T}^\kappa}$. The system* (1.1) *is reachable at time τ if and only if there exists $t_0 \in [0,\tau)_{\mathbb{T}}$ such that one of the following statements is satisfied.*

(i). *The operator $\mathcal{B}_r(t_0,\tau)$ is onto (surjective).*
(ii). *The operator $\mathcal{B}_r^*(t_0,\tau)$ is one to one (injective).*
(iii). *The Gramian reachability operator $\mathcal{G}_B^r(t_0,\tau)$ is invertible.*
(iv). *There is a positive constant γ such that*

$$\|w\|^2 \le \gamma \int_{[t_0,\tau)_{\mathbb{T}}} \|B^T(s)Z^T(\tau,\sigma(s))w\|^2 \Delta s$$

for all $w \in \mathbb{R}^n$.

Remark 4.5. In Theorem 4.4, the equivalence between statement (iii) and reachability of system (1.1) at time τ can be obtained from Theorem 4.3, since the $n \times n$ Gramian reachability matrix $\mathcal{G}_r(t_0,\tau)$ is the matrix representation of operator $\mathcal{G}_B^r(t_0,\tau)$ relative to the canonical basis.

From [11, Theorem 2.4] we have the following result.

Theorem 4.6. *Assume $t_0 \in [0,\infty)_{\mathbb{T}^\kappa}$. Hence, the system* (1.1) *is controllable at time t_0 if, and only if, there exists $\tau \in (t_0,\infty)_{\mathbb{T}^\kappa}$ such that the $n \times n$ controllability Gramian matrix defined by*

$$\mathcal{G}_c(t_0,\tau) = \int_{[t_0,\tau)_{\mathbb{T}}} Z(t_0,\sigma(s))B(s)B^T(s)Z^T(t_0,\sigma(s))\Delta s$$

is invertible, where $Z(t,s)$ is the transition matrix of Eq. (3.1).

The necessary and sufficient conditions to controllability for system (1.1) is given below.

Theorem 4.7. *Assume $t_0 \in [0,\infty)_{\mathbb{T}^\kappa}$. The system* (1.1) *is controllable at time t_0 if and only if there exists $\tau \in (t_0,\infty)_{\mathbb{T}^\kappa}$ such that one of the following statements is satisfied.*

(i). *The operator $\mathcal{B}_c(t_0,\tau)$ is onto (surjective).*
(ii). *The operator $\mathcal{B}_c^*(t_0,\tau)$ is one to one (injective).*
(iii). *The Gramian controllability operator $\mathcal{G}_B^c(t_0,\tau)$ is invertible.*
(iv). *There is a positive constant γ such that*

$$\|w\|^2 \le \gamma \int_{[t_0,\tau)_{\mathbb{T}}} \|B^T(s)Z^T(t_0,\sigma(s))w\|^2 \Delta s$$

for every $w \in \mathbb{R}^n$.

Proof. The equivalence between controllability of system (1.1) at time τ and statement (iii) follows from Theorem 4.6, since the $n \times n$ Gramian controllability matrix $\mathcal{G}_c(t_0,\tau)$ is the matrix representation of operator $\mathcal{G}_B^c(t_0,\tau)$ relative to the canonical basis. The remaining equivalences can be obtained as in proof of [3, Proposition 5]. □

Finally, we have the following equivalence that relates reachability and controllability.

Theorem 4.8. *If system* (1.1) *is reachable at time* τ *in* $[0,\infty)_{\mathbb{T}^\kappa}$ *if and only if the system* (1.1) *is controllable at some time* $t_0 < \tau$ *in* $[0,\infty)_{\mathbb{T}^\kappa}$.

5. Conclusions

The paper studies the notions of reachability and controllability for linear Volterra integro-dynamic system on time scales. In such a way that the study carried out here on reachability is a pioneer in the time scales literature. In Theorems 4.3, 4.4, 4.6, and 4.7, we establish results on the necessary and sufficient conditions to reachability and controllability. Also, we relate the notions of reachability and controllability in Theorem 4.8. In Theorem 4.7, new necessary and sufficient conditions to controllability are obtained. On the other hand, Theorems 4.3 and 4.4 establish new necessary and sufficient conditions to reachability.

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PC-Asymptotically almost automorphic mild solutions for impulsive integro-differential equations with nonlocal conditions

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Abstract. In this article, we study the existence of PC-asymptotically almost automorphic mild solutions of integro-differential equations with nonlocal conditions via resolvent operators in Banach space. Further, we give sufficient conditions for the solutions to depend continuously on the initial condition. Finally, an example is given to validate the theory part.

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1. Introduction

In one of his most influential papers in 1964, S. Bochner introduced almost automorphic functions [14]. In comparison to almost periodic functions, almost automorphic functions are more general. Many authors had established the almost automorphic solution of differential equations in abstract spaces, totically almost automorphic coefficients. For more on asymptotically almost automorphic functions and related issues, we refer the reader to [25] and the references therein.

N'Guérékata [31] is credited with introducing the concept of asymptotically almost automorphy, which serves as the main topic of discussion in this paper. The study of the existence of almost automorphic and asymptotically almost automorphic

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solutions to differential equations is a highly intriguing subject within mathematical analysis. This topic holds significant appeal due to its potential applications in various fields, including physics, economics, mathematical biology, engineering, etc. We recommend reading [16, 15, 22, 32, 33] and its references for more details on the fundamental theory of almost automorphic functions and its applications.

Grimmer's work on utilizing resolvent operators to demonstrate the existence of integro-differential systems. If one is interested in learning more about resolvent operators and integro-differential systems, one can refer to the following sources: [12, 13, 20, 23, 26, 27]. By consulting these references, one can gain a deeper understanding of the subject and explore further studies cited within them for more in-depth information.

Shocks, harvesting, and natural disasters are a few examples of abrupt changes that frequently affect the dynamics of evolution processes. These brief perturbations are frequently treated as having occurred instantly or as impulses. It is crucial to investigate dynamical systems with impulsive effects. Impulsive differential equations can be used to define a variety of mathematical models in the study of population dynamics, biology, ecology, and epidemics, among other topics. For the theory of impulsive differential equations, and impulsive delay differential equations we refer to [5, 6, 28, 10, 11, 7], and the references therein.

On the other hand, evolution equations with nonlocal initial conditions generalize evolution equations with classical initial conditions. Because more information is considered, this notion is more thorough in explaining natural occurrences than the classical one. See [9, 17, 35, 1, 7], and the references therein for further information on the significance of nonlocal conditions in several branches of applied sciences.

Benchohra *et al.* in [8] have established the existence of asymptotically almost automorphic mild solution to some classes of second order semilinear evolution equation. Moreover, in [18] Cao *et al.* discussed the existence of asymptotically almost automorphic mild solutions for a class of nonautonomous semilinear evolution equations.

Motivated by the last two recent works, we will investigate the existence of PC-asymptotically almost automorphic mild solutions for the following impulsive integro-differential equation with nonlocal conditions:

$$\begin{cases} \phi'(\vartheta) = \mathfrak{Z}\phi(\vartheta) + \int_0^\vartheta \Lambda(\vartheta - \upsilon)\phi(\upsilon)d\upsilon + \Psi\left(\vartheta, \phi(\vartheta)\right); \text{ if } \vartheta \in \widetilde{J}, \\ \phi(\vartheta_\imath^+) - \phi(\vartheta_\imath^-) = I_\imath(\phi(\vartheta_\imath^-)), \ \imath \in \mathbb{N}, \\ \phi(0) = \phi_0 + \Xi(\phi), \end{cases} \tag{1.1}$$

where $J = [0; +\infty)$, $\widetilde{J} = J \setminus \{\vartheta_\imath, \ \imath \in \mathbb{N}\}$, $0 = \vartheta_0 < \vartheta_1 < \vartheta_2 < \ldots < \vartheta_\imath \to +\infty$, and $\mathfrak{Z} : D(\mathfrak{Z}) \subset \mho \to \mho$ is the infinitesimal generator of a strongly continuous semigroup $\{T(\vartheta)\}_{\vartheta \geq 0}$, $\Lambda(\vartheta)$ is a closed linear operator with domain $D(\mathfrak{Z}) \subset D(\Lambda(\vartheta))$, $\phi_0 \in \mho$. The nonlinear term Ψ, Ξ and $I_\imath$ are a given functions. $\phi(\vartheta_\imath^+)$ and $\phi(\vartheta_\imath^-)$ denote the left and right limit of ϕ at $\vartheta = \vartheta_\imath$, respectively. $(\mho, \|\cdot\|)$ is a Banach space.

This paper is organized in five sections. Section 2 is reserved for some preliminary results and definitions which will be utilized throughout this manuscript. In Section 3 we study the existence of PC-asymptotically almost automorphic solutions to the system (1.1). And in section 4 we study the continuous dependence of the mild solutions. In section 5, An example is presented to illustrate the efficiency of the result obtained.

2. Preliminaries

In this section, we will go over some of the notations, definitions, and theorems that will be used throughout the work.

Let $J_0 = [0, \vartheta_1], J_\imath = (\vartheta_\imath, \vartheta_{\imath+1}]$, for $\imath \in \mathbb{N}$, $\xi(\vartheta^+) = \lim_{\vartheta \to \vartheta^+} \xi(\vartheta)$, and define the space of piecewise continuous functions:

$$PC(J, \mho) = \Big\{ \xi : J \to \mho : \xi|_{J_\imath} \text{ is continuous for } \imath \in \mathbb{N} \text{ , such that } \xi(\vartheta_\imath^-) \text{ and } \xi(\vartheta_\imath^+) \text{ exist and satisfy } \xi(\vartheta_\imath^-) = \xi(\vartheta_\imath), \text{ for } \imath \in \mathbb{N} \Big\}.$$

Let

$$BPC(J, \mho) = \{ \xi \in PC(J, \mho) \ : \ \xi \ \text{ is bounded on } \mathbb{R}^+ \},$$

be a Banach space with

$$\|\xi\|_{BPC} = \sup_{\vartheta \in J} \{ \|\xi(\vartheta)\| \}.$$

Let $L^1(J, \mho)$ be the Banach space of measurable functions $\aleph : J \to \mho$ which are Bochner integrable, with the norm

$$\|\aleph\|_{L^1} = \int_0^{+\infty} \|\aleph(\vartheta)\| d\vartheta,$$

We consider the following Cauchy problem

$$\begin{cases} \phi'(\vartheta) = \mathfrak{Z}\phi(\vartheta) + \int_0^\vartheta \Lambda(\vartheta - \upsilon)\phi(\upsilon)d\upsilon; & \text{for } \vartheta \geq 0, \\ \phi(0) = \phi_0 \in \mho. \end{cases} \tag{2.1}$$

The existence and properties of a resolvent operator has been discussed in [26]. In what follows, we suppose the following assumptions:

($R1$) $\mathfrak{Z}$ is the infinitesimal generator of a uniformly continuous semigroup $\{T(\vartheta)\}_{\vartheta>0}$,

($R2$) For all $\vartheta \geq 0, \Lambda(\vartheta)$ is closed linear operator from $D(\mathfrak{Z})$ to $\mho$ and $\Lambda(\vartheta) \in \Lambda(D(\mathfrak{Z}), \mho)$. For any $\phi \in D(\mathfrak{Z})$, the map $\vartheta \to \Lambda(\vartheta)\phi$ is bounded, differentiable and the derivative $\vartheta \to \Lambda'(\vartheta)\phi$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 2.1. [26] *Assume that* $(R1) - (R2)$ *hold, then there exists a unique resolvent operator for the Cauchy problem (2.1).*

The concept of "PC-almost automorphic operator" was defined by G.M. N'Guérékata and A. Pankov in [34]. So now, we recall some basic definitions and results on almost automorphic functions and asymptotically almost automorphic functions.

Definition 2.2. A function $\aleph \in PC(\mathbb{R}, \mho)$ is said to be $PC-$almost automorphic if

1. The sequence of impulsive moments $\{\vartheta_\imath\}_{\imath\in\mathbb{N}}$ is a almost automorphic sequence
2. For every sequence of real numbers $\{\tau'_n\}$, there exists a subsequence $\{\tau_{n_\imath}\}$ such that
$$\widehat{\aleph}(\vartheta) = \lim_{\imath\to\infty} \aleph\left(\vartheta + \tau_{n_\imath}\right),$$
is well defined for each $\vartheta \in \mathbb{R}$ and
$$\lim_{\imath\to\infty} \widehat{\aleph}\left(\vartheta - \tau_{n_\imath}\right) = \aleph(\vartheta) \quad \text{for each } \vartheta \in \mathbb{R}.$$

Denote by $AA_{PC}(\mathbb{R}, \mho)$ the set of all such functions.

Lemma 2.3. [32] $AA_{PC}(\mathbb{R}, \mho)$ *is a Banach space with*
$$\|\aleph\|_{PC^*} = \sup_{\vartheta\in\mathbb{R}} \|\aleph(\vartheta)\|.$$

Definition 2.4. A function $\aleph \in PC(\mathbb{R} \times \mho, \mho)$ is said to be $PC-$almost automorphic if

1. The sequence of impulsive moments $\{\vartheta_\imath\}_{\imath\in\mathbb{N}}$ is a almost automorphic sequence.
2. For every sequence of real numbers $\{\tau'_n\}$, there exists a subsequence $\{\tau_{n_\imath}\}$ such that
$$\lim_{\imath\to\infty} \aleph\left(\vartheta + \tau_{n_\imath}, \phi\right) = \widehat{\aleph}(\vartheta, \phi),$$
is well defined for each $\vartheta \in \mathbb{R}$ and
$$\lim_{\imath\to\infty} \widehat{\aleph}\left(\vartheta - \tau_{n_\imath}, \phi\right) = \aleph(\vartheta, \phi),$$
for each $\vartheta \in \mathbb{R}$ and each $\phi \in \mho$.

The collection of those functions is denoted by $AA_{PC}(\mathbb{R} \times \mho, \mho)$.

The space of all piecewise continuous functions $\widetilde{\aleph} : \mathbb{R}^+ \to \mho$ such that $\lim_{\vartheta\to\infty} \widetilde{\aleph}(\vartheta) = 0$ is denoted by $PC_0\left(\mathbb{R}^+, \mho\right)$. Moreover, we denote $PC_0\left(\mathbb{R}^+ \times \mho, \mho\right)$; the space of all piecewise continuous functions from $\mathbb{R}^+ \times \mho$ to $\mho$ satisfying $\lim_{\vartheta\to\infty} \widetilde{\aleph}(\vartheta, \phi) = 0$ in ϑ and uniformly in $\phi \in \mho$.

Definition 2.5. A function $\aleph : \mathbb{R}^+ \to \mho$ is said to be $PC-$asymptotically almost automorphic if it can be decomposed as
$$\aleph(\vartheta) = \widehat{\aleph}(\vartheta) + \widetilde{\aleph}(\vartheta),$$
where
$$\widehat{\aleph} \in AA_{PC}(\mathbb{R}, \mho),\ \widetilde{\aleph} \in PC_0\left(\mathbb{R}^+, \mho\right).$$

Denote by $\mathfrak{G} = AAA_{PC}\left(\mathbb{R}^+, \mho\right)$ the set of all such functions with the norm
$$\|\phi\|_{\mathfrak{G}} = \sup_{\vartheta\in J} \left\{\|\phi(\vartheta)\|\right\}.$$

Definition 2.6. A function $\aleph : \mathbb{R}^+ \times \mho \to \mho$ is said to be $PC-$asymptotically almost automorphic if it can be decomposed as
$$\aleph(\vartheta, \phi) = \widehat{\aleph}(\vartheta, \phi) + \widetilde{\aleph}(\vartheta, \phi),$$
where
$$\widehat{\aleph} \in AA_{PC}(\mathbb{R} \times \mho, \mho), \quad \widetilde{\aleph} \in PC_0\left(\mathbb{R}^+ \times \mho, \mho\right).$$

Denote by $AAA_{PC}(\mathbb{R}^+ \times \mho, \mho)$ the set of all such functions.

Lemma 2.7. [21] *Let $\aleph \in AA_{PC}(\mathbb{R}^+ \times \mho, \mho)$ and write $\aleph = \widehat{\aleph} + \widetilde{\aleph}$ where $\widehat{\aleph} \in AA_{PC}(\mathbb{R} \times \mho, \mho), \widetilde{\aleph} \in PC_0(\mathbb{R}^+ \times \mho, \mho)$. Suppose that both $\widehat{\aleph}$ and $\widetilde{\aleph}$ are Lipschitz in $x \in \mho$ uniformly in ϑ, i.e., there exists $L_1, L_2 > 0$ such that*

$$\|\widehat{\aleph}(\vartheta,\phi) - \widehat{\aleph}(\vartheta,\widehat{\phi})\| \leq L_1\|\phi - \widehat{\phi}\| \text{ for a.e } \vartheta \in \mathbb{R} \text{and each } \phi, \widehat{\phi} \in \mho.$$

and

$$\|\widetilde{\aleph}(\vartheta,\phi) - \widetilde{\aleph}(\vartheta,\widehat{\phi})\| \leq L_2\|\phi - \widehat{\phi}\| \text{ for a.e } \vartheta \in \mathbb{R}^+ \text{and each } \phi, \widehat{\phi} \in \mho.$$

Then $\phi \in AAA_{PC}(\mathbb{R}^+, \mho)$ implies that $\aleph(\cdot, \phi(\cdot)) \in AAA_{PC}(\mathbb{R}^+, \mho)$.

Lemma 2.8. [29] *$\aleph : \mathbb{R} \times \mho \to \mho$ is PC-almost automorphic, and assume that $\aleph(\vartheta, \cdot)$ is uniformly continuous on each bounded subset $\daleth \subset \mho$ uniformly for $\vartheta \in \mathbb{R}$, that is for any $\varepsilon > 0$, there exists $\varrho > 0$ such that $\phi, \widehat{\phi} \in \daleth$ and $\|\phi(\vartheta) - \widehat{\phi}(\vartheta)\| < \varrho$ imply that $\|\aleph(\vartheta,\phi) - \aleph(\vartheta,\widehat{\phi})\| < \varepsilon$ for all $\vartheta \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mho$ be PC-almost automorphic. Then the function $\widehat{\daleth} : \mathbb{R} \to \mho$ defined by $\widehat{\daleth}(\vartheta) = \aleph(\vartheta, \varphi(\vartheta))$ is PC-almost automorphic.*

Lemma 2.9. [29] *Suppose that $\aleph(\vartheta,\phi) = \widehat{\aleph}(\vartheta,\phi) + \widetilde{\aleph}(\vartheta,\phi)$ is an asymptotically almost automorphic function with $\widehat{\aleph} \in AA_{PC}(\mathbb{R} \times \mho, \mho), \widetilde{\aleph} \in PC_0(\mathbb{R}^+ \times \mho, \mho)$, and $\widehat{\aleph}$ is uniformly continuous on any bounded subset $\daleth \subset X$ uniformly for $\vartheta \in \mathbb{R}$. Then $\phi \in AAA_{PC}(\mathbb{R}, \mho)$ implies $\aleph \in AAA_{PC}(\mathbb{R}, \mho)$*

Now, for $\vartheta \in \mathbb{R}^+$ we define the following functions:

$$\Phi_1(\vartheta) = \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon)Y(\upsilon)d\upsilon, \text{ and } \Phi_2(\vartheta) = \int_0^{\vartheta} \Re(\vartheta - \upsilon)Z(\upsilon)d\upsilon.$$

Lemma 2.10. *We assume that*

($R3$) *The resolvent $\Re(\vartheta)$ is exponentially stable i.e, there exist $\mathfrak{X}_{\Re} \geq 1$ and $b \geq 0$, such that*

$$\|\Re(\vartheta)\|_{B(\mho)} \leq \mathfrak{X}_{\Re} e^{-b\vartheta}, \quad \text{for all } \vartheta \in J.$$

Then

(i) *If $Y \in AA_{PC}(\mathbb{R}, \mho)$, then $\Phi_1 \in AA_{PC}(\mathbb{R}, \mho)$.*
(ii) *If $Z \in PC_0(\mathbb{R}^+, \mho)$, then $\Phi_2 \in PC_0(\mathbb{R}^+, \mho)$.*

Proof. For (i), choose a bounded subset $\daleth$ of $\mho$ such that $Y(\vartheta) \in \daleth$ for all $\vartheta \in \mathbb{R}$. Since $Y \in AA_{PC}(\mathbb{R}, \mho)$ and the resolvent $\Re(\vartheta)$ is exponentially stable it follows that for every sequence of real numbers τ'_n, we can extract a subsequence $\tau_{n_\imath}$ such that

(i_1) $\lim_{\imath \to +\infty} Y(\vartheta + \tau_{n_\imath}) = \widetilde{Y}(\vartheta)$,
(i_2) $\lim_{\imath \to +\infty} \widetilde{Y}(\vartheta - \tau_{n_\imath}) = Y(\vartheta)$.

Write

$$\widetilde{\Phi}_1(\vartheta) := \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon)\widetilde{Y}(\upsilon)d\upsilon, \quad \vartheta \in \mathbb{R}^+.$$

Then

$$\|\Phi_1(\vartheta + \tau_{n_\imath}) - \widetilde{\Phi}_1(\vartheta)\| = \left\| \int_{-\infty}^{\vartheta + \tau_{n_\imath}} \Re(\vartheta + \tau_{n_\imath} - \upsilon)Y(\upsilon)d\upsilon - \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon)\widetilde{Y}(\upsilon) \right\|$$

$$\begin{aligned}
&= \left\| \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon) Y(\upsilon + \tau_{n_\imath}) d\upsilon - \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon) \widetilde{Y}(\upsilon) \right\| \\
&\leq \int_{-\infty}^{\vartheta} \|\Re(\vartheta - \upsilon)\| \left\| Y(\upsilon + \tau_{n_\imath}) - \widetilde{Y}(\upsilon) \right\| d\upsilon \\
&\leq \frac{\mathfrak{X}_{\Re}}{b} \sup_{\vartheta \in \mathbb{R}} \left\| Y(\vartheta + \tau_{n_\imath}) - \widetilde{Y}(\vartheta) \right\|.
\end{aligned}$$

Since the resolvent $\Re(\vartheta)$ is exponentially stable together with the Lebesgue dominated convergence theorem and (i_1) it follows that

$$\lim_{\imath \to +\infty} \Phi_1(\vartheta + \tau_{n_\imath}) = \widetilde{\Phi}_1(\vartheta), \quad \vartheta \in \mathbb{R}.$$

Similarly by (i_2) we can prove that

$$\lim_{\imath \to +\infty} \widetilde{\Phi}_1(\vartheta - \tau_{n_\imath}) = \Phi_1(\vartheta), \quad \vartheta \in \mathbb{R}.$$

Hence, $\Phi_1 \in AA_{PC}(\mathbb{R}, \mho)$.
Now for (ii), one can choose $\varkappa > 0$ such that

$$\|Z(\vartheta)\| < \varepsilon, \qquad \forall \vartheta > \varkappa.$$

This enables us to conclude that for all $\vartheta > \varkappa$,

$$\begin{aligned}
\|\Phi_2(\vartheta)\| &= \left\| \int_0^{\vartheta} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon \right\| \\
&= \left\| \int_0^{\varkappa} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon + \int_{\varkappa}^{\vartheta} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon \right\| \\
&\leq \left\| \int_0^{\varkappa} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon \right\| + \left\| \int_{\varkappa}^{\vartheta} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon \right\| \\
&\leq \frac{\mathfrak{X}_{\Re} e^{-b(\vartheta - \varkappa)}}{b} \|Z\| + \frac{\mathfrak{X}_{\Re} \varepsilon}{b}.
\end{aligned}$$

Consequently $\lim_{\vartheta \to +\infty} \|\Phi_2(\vartheta)\| = 0$.
Now, we define the Kuratowski measure of noncompactness.

Definition 2.11. [4] Let $\Bbbk$ be a Banach space and $\nabla_{\Bbbk}$ the bounded subsets of $\Bbbk$. The Kuratowski measure of noncompactness is the map $\alpha : \nabla_{\Bbbk} \to [0, \infty)$ defined by

$$\alpha(\Im) = \inf\{\epsilon > 0 : \Im \subseteq \cup_{i=1}^{n} \Im_i \text{ and } \operatorname{diam}(\Im_i) \leq \epsilon\}; \text{ here } \Im \in \nabla_{\Bbbk},$$

where

$$\operatorname{diam}(\Im_i) = sup\{\|\xi - \widehat{\xi}\| : \xi, \widehat{\xi} \in \Im_i\}.$$

Lemma 2.12. [24] *If Y is a bounded subset of a Banach space $\Bbbk$, then for each $\epsilon > 0$, there is a sequence $\{\phi_\imath\}_{\imath=1}^{\infty} \subset Y$ such that*

$$\alpha(Y) \leq 2\alpha\left(\{\phi_\imath\}_{\imath=1}^{\infty}\right) + \epsilon.$$

Lemma 2.13. [30] *If* $\{\phi_\imath\}_{\imath=0}^{\infty} \subset L^1$ *is uniformly integrable, then the function* $\vartheta \to \alpha(\{\phi_\imath(\vartheta)\}_{\imath=0}^{\infty})$ *is measurable and*

$$\alpha\left(\left\{\int_0^{\vartheta} \phi_\imath(\upsilon)d\upsilon\right\}_{\imath=0}^{\infty}\right) \leq 2\int_0^{\vartheta} \alpha\left(\{\phi_\imath(\upsilon)\}_{\imath=0}^{\infty}\right)d\upsilon.$$

Theorem 2.14. (*Darbo's fixed point theorem,* [19]). *Let* $\Im$ *be a nonempty, bounded, closed and convex subset of a Banach space* $\Bbbk$ *and let* $T : \Im \to \Im$ *be a continuous mapping. Assume that there exists a constant* $\imath \in [0,1)$*, such that*

$$\alpha(TM) \leq \imath\alpha(M),$$

for any nonempty subset M *of* $\Im$*. Then* T *has a fixed point in set* $\Im$*.*

3. The main result

In this section we discuss existence of PC-asymptotically almost automorphic mild solutions via resolvent operators for problem (1.1). In order to establish a measure of noncompactness in the space $\mathfrak{G}$, let us first recall the specific measure of noncompactness that results from [8]. This measure will be used in our main results. Let us fix a nonempty bounded subset $\Im$ in the space $\mathfrak{G}$, for $\widehat{\xi} \in \Im$, $\varkappa > 0$, $\epsilon > 0$ *and* $\kappa, \tau \in [0,\varkappa]$, such that $|\kappa - \tau| \leq \epsilon$. We denote $\omega^{\varkappa}(\widehat{\xi},\epsilon)$ the modulus of continuity of the function $\widehat{\xi}$ on the interval $[0,\varkappa]$, namely,

$$\begin{aligned}
\omega^{\varkappa}(\widehat{\xi},\epsilon) &= \sup\{\|\widehat{\xi}(\kappa) - \widehat{\xi}(\tau)\| \ ; \ \kappa,\tau \in [0,\varkappa] \cap \widetilde{J}\},\\
\omega_0(\Im) &= \lim_{\varkappa\to+\infty} \lim_{\epsilon\to 0} \sup\{\omega^{\varkappa}(\widehat{\xi},\epsilon) \ ; \ \widehat{\xi} \in \Im\}.
\end{aligned}$$

Finally, consider the function χ_* defined on the family of subset of $\mathfrak{G}$ by the formula

$$\chi_*(\Im) = \omega_0(\Im) + \sup_{\vartheta\in J} \alpha(\Im(\vartheta)),$$

and notice that if the set $\Im$ is equicontinuous and equiconvergent, then $\omega_0(\Im) = 0$.

Definition 3.1. A function $\phi \in \mathfrak{G}$ is called a PC-asymptotically almost automorphic mild solution of problem (1.1), if it satisfies the following integral equation

$$\begin{aligned}
\phi(\vartheta) = \Re(\vartheta)(\phi_0 + \Xi(\phi)) &+ \int_0^{\vartheta} \Re(\vartheta-\upsilon)\Psi(\upsilon,\phi(\upsilon))d\upsilon\\
&+ \sum_{0<\vartheta_\imath<\vartheta} \Re(\vartheta-\vartheta_\imath)I_\imath(\phi(\vartheta_\imath^-)), \quad \vartheta \in J.
\end{aligned}$$

The following hypotheses will be used in the sequel.

(*A*1) Assume that $(R1) - (R3)$ hold.

(*A*2) *i*) The sequence of impulsive moments $\vartheta_\imath$ is asymptotically almost automorphic.

ii) $\Psi : J \times \mho \longrightarrow \mho$ is a Carathéodory function and PC-asymptotically almost automorphic ie., $\Psi(\vartheta,\phi) = \widehat{\aleph}(\vartheta,\phi) + \widetilde{\aleph}(\vartheta,\phi)$ with

$$\widehat{\aleph} \in AA_{PC}(\mathbb{R}\times\mho,\mho), \quad \widetilde{\aleph} \in PC_0(\mathbb{R}^+\times\mho,\mho).$$

iii) There exists a function $\beta \in L^1(J, \mathbb{R}^+)$, such that :

$$\|\Psi(\vartheta, x) - \Psi(\vartheta, \phi)\| \leq \beta(\vartheta)\|x - \phi\|, \text{ for all } x, \phi \in \mho.$$

Also we assume that $\Psi(\vartheta, 0) = 0$.

$(A3)$ $\Xi : \mathfrak{G} \to \mho$ is continuous and there exists $L_\Xi > 0$, such that,

$$\|\Xi(\xi) - \Xi(\widehat{\xi})\| \leq L_\Xi \|\xi - \widehat{\xi}\|_{\mathfrak{G}}, \text{ for all } \xi, \widehat{\xi} \in \mathfrak{G}.$$

Also we assume that $\Xi(0) = 0$.

$(A4)$ $I_\imath : \mho \to \mho$ is Lipschitz continuous with Lipschitz constants $m_\imath$, $\imath \in \mathbb{N}$, such that

$$\|I_\imath(\kappa_3) - I_\imath(\kappa_4)\| \leq m_\imath \|\kappa_3 - \kappa_4\|, \text{for all } \kappa_3, \ \kappa_4 \in \mho, \ \imath \in \mathbb{N}.$$

And $I_\imath(0) = 0$.

Theorem 3.2. *Assume that the conditions* $(A1) - (A4)$ *are satisfied. If*

$$\mathfrak{X}_{\Re}\left(L_\Xi + 4\|\beta\|_{L^1} + \sum_{\imath=0}^{\infty} m_\imath\right) < 1,$$

then the problem (1.1) has a PC-asymptotically almost automorphic mild solution.

Proof. Consider the operator $\Theta : \mathfrak{G} \longrightarrow \mathfrak{G}$ defined by

$$(\Theta\phi)(\vartheta) = \Re(\vartheta)(\phi_0 + \Xi(\phi)) + \int_0^\vartheta \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi(\upsilon))d\upsilon$$
$$+ \sum_{0<\vartheta_\imath<\vartheta} \Re(\vartheta - \vartheta_\imath) I_\imath(\phi(\vartheta_\imath^-)), \quad \vartheta \in J,$$

where $\phi \in \mathfrak{G}$ with $\phi = \phi_1 + \phi_2$, ϕ_1 is the principal term and ϕ_2 the corrective term of ϕ_1.

Step 1 : Θ is well-defined, i.e $\Theta(\mathfrak{G}) \subset \mathfrak{G}$.

We have $\Theta(\mathfrak{G}) \subset PC(J, \mho)$. Now, let

$$\zeta(\vartheta) = \Re(\vartheta)\left(\phi_0 + \Xi(\phi)\right),$$

then

$$\|\zeta(\vartheta)\| \leq \mathfrak{X}_{\Re} e^{-bt}\left(\|\phi_0\| + L_\Xi\|\phi\|\right).$$

Since $b > 0$, we get $\lim_{\vartheta \longrightarrow +\infty} |\zeta(\vartheta)| = 0$. Thus $\zeta \in PC_0(\mathbb{R}^+, \mho)$.

From assumption $(A2)$, we can write

$$\begin{aligned}\Psi(\vartheta, \phi(\vartheta)) &= \widehat{\aleph}(\vartheta, \phi_2(\vartheta)) + \Psi(\vartheta, \phi(\vartheta)) - \Psi(\vartheta, \phi_2(\vartheta)) + \widetilde{\aleph}(\vartheta, \phi_2(\vartheta)) \\ &= \widehat{\aleph}(\vartheta, \phi_2(\vartheta)) + \mathfrak{U}(\vartheta, \phi(\vartheta)).\end{aligned}$$

Then, we get

$$\int_0^\vartheta \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi(\upsilon))d\upsilon$$
$$= \int_0^\vartheta \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon + \int_0^\vartheta \Re(\vartheta - \upsilon)\mathfrak{U}(\upsilon, \phi(\upsilon))d\upsilon$$

$$= \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon + \int_{-\infty}^{0} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi(\upsilon))d\upsilon$$
$$+ \int_{0}^{\vartheta} \Re(\vartheta - \upsilon)\mathfrak{U}(\upsilon, \phi(\upsilon))d\upsilon$$
$$= \Upsilon_1\phi(\vartheta) + \Upsilon_2\phi(\vartheta),$$

where

$$(\Upsilon_1\phi)(\vartheta) = \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon,$$
$$(\Upsilon_2\phi)(\vartheta) = \int_{-\infty}^{0} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi(\upsilon))d\upsilon + \int_{0}^{\vartheta} \Re(\vartheta - \upsilon)\mathfrak{U}(\upsilon, \phi(\upsilon))d\upsilon,$$

and

$$(\Delta_1\phi)(\vartheta) = \int_{0}^{\vartheta} \Re(\vartheta - \upsilon)\mathfrak{U}(\upsilon, \phi(\upsilon))d\upsilon,$$
$$(\Delta_2\phi)(\vartheta) = \int_{-\infty}^{0} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon.$$

Using $(A2)$ and Lemma 2.8, We deduce that $\upsilon \longrightarrow \widehat{\aleph}(\upsilon, \phi_2(\upsilon))$ is in $AA_{PC}(\mathbb{R} \times \mho, \mho)$. Thus, by Lemma 2.8, we obtain

$$\Upsilon_1\phi \in AA_{PC}(\mathbb{R}^+ \times \mho, \mho).$$

Let us prove that $\Delta_1\phi \in PC_0(\mathbb{R}^+ \times \mho, \mho)$, indeed by definition $\mathfrak{U} \in PC_0(\mathbb{R}^+ \times \mho, \mho)$, that means given $\varepsilon > 0$, there exists $\varkappa > 0$ such that for $\vartheta \geq \varkappa$, we have $\|\mathfrak{U}(\vartheta, \phi)\| \leq \varepsilon$. Therefore if $\vartheta \geq \varkappa$, we get

$$\begin{aligned} \int_{\varkappa}^{\vartheta} \|\Re(\vartheta - \upsilon)\|\|\mathfrak{U}(\upsilon, \phi(\upsilon))\|d\upsilon &\leq \mathfrak{X}_{\Re}\varepsilon \int_{\varkappa}^{\vartheta} e^{-b(\vartheta - \upsilon)}d\upsilon \\ &\leq \frac{\mathfrak{X}_{\Re}}{b}\varepsilon, \end{aligned}$$

then

$$\|(\Delta_1\phi)(\vartheta)\| \leq \frac{\mathfrak{X}_{\Re}}{b}\varepsilon.$$

Thus

$$\Delta_1 \in PC_0(\mathbb{R}^+ \times \mho, \mho).$$

Next, let us show that $\Delta_2\phi \in PC_0(\mathbb{R}^+ \times \mho, \mho)$, we have

$$\begin{aligned} \|(\Delta_2\phi)(\vartheta)\| &= \left\| \int_{-\infty}^{0} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon \right\| \\ &\leq \mathfrak{X}_{\Re} \sup_{\vartheta \in \mathbb{R}} \|\widehat{\aleph}(\vartheta, \phi_2(\vartheta))\| \int_{0}^{\varkappa} e^{-b(\vartheta - \upsilon)}d\upsilon \\ &\quad + \mathfrak{X}_{\Re}\|\widehat{\aleph}\|_{PC^*} \frac{e^{-b(\vartheta - \varkappa)}}{b} \longrightarrow 0, \ as\ \vartheta \longrightarrow \infty. \end{aligned}$$

Therefore, $\Delta_2\phi \in PC_0(\mathbb{R}^+ \times \mho, \mho)$.
Also we have

$$\| \sum_{0 < \vartheta_\imath < \vartheta} \Re(\vartheta - \vartheta_\imath)I_\imath(\phi(\vartheta_\imath)\| \leq \mathfrak{X}_{\Re}\|\phi\|_{\mathfrak{S}} \sum_{\imath=0}^{+\infty} e^{-b(\vartheta - \vartheta_\imath)}m_\imath \longrightarrow 0, \ as\ \vartheta \longrightarrow \infty.$$

Consequently, from the previous estimates we deduce that $\Theta(\mathfrak{G}) \subset \mathfrak{G}$.
Next, we shall check that operator Θ satisfies all conditions of Darbo's theorem.
Let $\Im_\theta = \{\phi \in \mathfrak{G}\ ;\ \|\phi\|_{\mathfrak{G}} \leq\ \theta\}$, the set $\Im_\theta$ is bounded, closed and convex.
Step 2 : $\Theta(\Im_\theta) \subset \Im_\theta$.
For each $\phi \in \Im_\theta$ and by $(A1)$, $(A2)$ and $(A3)$, we have

$$\|\Xi(\phi)\| \leq L_{\Xi}\|\phi\|_{\mathfrak{G}}.$$

Then,

$$\begin{aligned}\|\Theta\phi(\vartheta)\| &\leq \mathfrak{X}_{\Re}(\|\phi_0\| + \|\Xi(\phi)\|) + \mathfrak{X}_{\Re}\int_0^{\vartheta}\|\Psi(\upsilon,\phi(\upsilon))\|d\upsilon \\ &\quad + \mathfrak{X}_{\Re}\sum_{0<\vartheta_\imath<\vartheta}\|I_\imath(\phi(\vartheta_\imath^-))\| \\ &\leq \mathfrak{X}_{\Re}(\|\phi_0\| + L_{\Xi}\theta) + \mathfrak{X}_{\Re}\theta\|\beta\|_{L^1} + \mathfrak{X}_{\Re}\theta\sum_{\imath=0}^{\infty} m_\imath.\end{aligned}$$

Hence $\Theta(\Im_\theta) \subset \Im_\theta$, provided that

$$\theta > \frac{\mathfrak{X}_{\Re}\|\phi_0\|}{1 - \mathfrak{X}_{\Re}(L_{\Xi} + \|\beta\|_{L^1} + \sum_{\imath=0}^{\infty} m_\imath)}.$$

Step 3: Θ is continuous.
Let x_m be a sequence such that $\phi_m \to \phi_*$ in $\mathfrak{G}$, then we have,

$$\begin{aligned}&\|(\Theta\phi_m)(\vartheta) - (\Theta\phi_*)(\vartheta)\| \\ &\quad\leq \mathfrak{X}_{\Re}\|\Xi(\phi_m) - \Xi(\phi_*)\| + \mathfrak{X}_{\Re}\int_0^{\vartheta}\|\Psi(\upsilon,\phi_m(\upsilon)) - \Psi(\upsilon,\chi_*(\upsilon))\|d\upsilon \\ &\quad\quad + \mathfrak{X}_{\Re}\sum_{0<\vartheta_\imath<\vartheta} m_\imath\|\phi_m(\vartheta_\imath^-) - \phi_*(\vartheta_\imath)\|.\end{aligned}$$

Since the function Ψ is Carathédory and Ξ is continuous, the Lebesgue dominated converge theorem implies that :

$$\|(\Theta\phi_m) - (\Theta\phi_*)\|_{\mathfrak{G}} \to 0,\ as\ m \to +\infty.$$

Thus, Θ is continuous.
Step 4: $\Theta(\Im_\theta)$ is equicontinuous. Let $\vartheta_1, \vartheta_2 \in J$ with $\vartheta_2 > \vartheta_1$. For all $\phi \in \Im_\theta$, we have

$$\begin{aligned}&\|(\Theta\phi)(\vartheta_2) - (\Theta\phi)(\vartheta_1)\| \\ &\quad= \|\int_0^{\vartheta_2}\Re(\vartheta_2-\upsilon)\Psi(\upsilon,\phi(\upsilon))d\upsilon - \int_0^{\vartheta_1}\Re(\vartheta_1-\upsilon)\Psi(\upsilon,\phi(\upsilon))d\upsilon \\ &\quad\quad + \sum_{0<\vartheta_\imath<\vartheta_2}\Re(\vartheta_2-\vartheta_\imath)I_\imath(\phi(\vartheta_\imath)) - \sum_{0<\vartheta_\imath<\vartheta_1}\Re(\vartheta_1-\vartheta_\imath)I_\imath(\phi(\vartheta_\imath))\| \\ &\quad\leq \int_0^{\vartheta_1}\|\Re(\vartheta_2-\upsilon) - \Re(\vartheta_1-\upsilon)\|\,\|\Psi(\upsilon,\phi(\upsilon))\|\,d\upsilon\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\vartheta_1}^{\vartheta_2} \|\Re(\vartheta_2 - \upsilon)\| \, \|\Psi(\upsilon, \phi(\upsilon))d\upsilon\| \\
&\quad + \sum_{\vartheta_1 < \vartheta_\imath < \vartheta_2} \|\Re(\vartheta_2 - \vartheta_\imath)\| \|I_\imath(\phi(\vartheta_\imath))\| \\
&\quad + \sum_{0 < \vartheta_\imath < \vartheta_1} \|(\Re(\vartheta_2 - \vartheta_\imath) - \Re(\vartheta_1 - \vartheta_\imath)\| \|I_\imath(\phi(\vartheta_\imath))\| \\
&\leq \theta \int_0^{\vartheta_1} \|(\Re(\vartheta_2 - \upsilon) - \Re(\vartheta_1 - \upsilon))\| \beta(\upsilon)d\upsilon + \mathfrak{X}_\Re \theta \int_{\vartheta_1}^{\vartheta_2} \beta(\upsilon)d\upsilon \\
&\quad + \theta \sum_{0 < \vartheta_\imath < \vartheta_1} m_\imath \|\Re(\vartheta_2 - \vartheta_\imath) - \Re(\vartheta_1 - \vartheta_\imath)\| + \mathfrak{X}_\Re \theta \sum_{\vartheta_1 < \vartheta_\imath < \vartheta_2} m_\imath e^{-b(\vartheta_2 - \vartheta_\imath)}.
\end{aligned}
$$

Since $\Re(\vartheta)$ is strongly continuous and $\beta \in L^1$, we get

$$
\|(\Theta\phi)(\vartheta_2) - (\Theta\phi)(\vartheta_1)\| \longrightarrow 0 \quad as \quad \vartheta_2 \longrightarrow \vartheta_1,
$$

which implies that $\Theta(\Im_\theta)$ is equicontinuous.

Step 5: $\Theta(\Im_\theta)$ is equiconvergent.

For $\phi \in \Im_\theta$ and $\vartheta \in J$, we have

$$
\begin{aligned}
\|(\Theta\phi)(\vartheta)\| &\leq \|\Re(\vartheta)\|_{B(\mho)}[\|\phi_0\| + \|\Xi(\phi)\|] + \int_0^{\vartheta} \|\Re(\vartheta - \upsilon)\| \beta(\upsilon) \|\phi(\upsilon)\| d\upsilon \\
&\quad + \sum_{0 < \vartheta_\imath < \vartheta} \|\Re(\vartheta - \vartheta_\imath)\| \|I_\imath(\phi(\vartheta_\imath))\| \\
&\leq \mathfrak{X}_\Re e^{-b\vartheta} (\|\phi_0\| + L_\Xi \theta) + \mathfrak{X}_\Re \theta \int_0^{\vartheta} e^{-b(\vartheta - \upsilon)} \beta(\upsilon) d\upsilon \\
&\quad + \mathfrak{X}_\Re \theta \sum_{\imath = 0}^{p} e^{-b(\vartheta - \vartheta_\imath)} m_\imath \\
&\quad \longrightarrow \mathfrak{X}_\Re (\|\phi_0\| + L_\Xi \theta) + \mathfrak{X}_\Re \theta \|\beta\|_{L^1} \; as \quad \vartheta \longrightarrow +\infty.
\end{aligned}
$$

Then

$$
\|(\Theta\phi)(\vartheta) - (\Theta\phi)(+\infty)\| \longrightarrow 0 \quad as \quad \vartheta \longrightarrow +\infty.
$$

Step 6: Let ∇ be a bounded equicontinuous subset of $\Im_\theta$, we have $\{\Theta(\nabla)\}$ is equicontinuous and in addition to the estimate given in step 1 and step 5 we have, $\omega_0(\Theta(\nabla)) = 0$.

From Lemma 2.12 and 2.13 it follow that for any $\varrho > 0$, there exists a sequence $\{\phi_m\}_{\imath=0}^{\infty} \subset \nabla$ such that

$$
\begin{aligned}
&\alpha\left(\int_0^{\vartheta} \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi(\upsilon))d\upsilon \; ; \phi \in \nabla\right) \\
&\leq 2\alpha\left(\int_0^{\vartheta} \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi_m(\upsilon))d\upsilon \; ; \phi \in \nabla\right) + \varrho \\
&\leq 4 \int_0^{\vartheta} \alpha\Big(\Re(\vartheta - \upsilon)\Psi(\upsilon, \phi_m(\upsilon))d\upsilon \quad ; \phi \in \nabla\Big) + \varrho.
\end{aligned}
$$

For any bounded set $\nabla \subset \mho$ and $\vartheta \in J$, and by [2] the Lipschitz conditions on the functions Ψ, Ξ and $I_\imath$, we get

$$\begin{aligned} \alpha(\Psi(\vartheta, \nabla(\vartheta))) &\leq \beta(\vartheta)\alpha(\nabla(\vartheta)), \\ \alpha(I_\imath(\vartheta, \nabla(\vartheta))) &\leq m_\imath \alpha(\nabla(\vartheta)), \\ \alpha(\Xi(\nabla(\vartheta))) &\leq L_\Xi \chi_*(\nabla). \end{aligned}$$

Then

$$\begin{aligned} \alpha(\Theta\nabla(\vartheta)) &\leq \mathfrak{X}_\Re L_\Xi \chi_*(\nabla) + 4\int_0^\vartheta \mathfrak{X}_\Re \beta(\upsilon)\alpha(\nabla(\upsilon))d\upsilon \\ &\quad + \mathfrak{X}_\Re \sum_{0<\vartheta_\imath<\vartheta} m_\imath \alpha(\nabla(\vartheta_\imath)) + \varrho \\ &\leq \mathfrak{X}_\Re L_\Xi \chi_*(\nabla) + 4\mathfrak{X}_\Re \|\beta\|_{L^1} \sup_{\vartheta\in J} \alpha(\nabla(\vartheta)) \\ &\quad + \mathfrak{X}_\Re \sum_{0<\vartheta_\imath<\vartheta} m_\imath \alpha(\nabla(\vartheta_\imath)) + \varrho. \end{aligned}$$

Since ϱ is arbitrary, we obtain

$$\alpha(\Theta\nabla(\vartheta)) \leq \mathfrak{X}_\Re L_\Xi \chi_*(\nabla) + \mathfrak{X}_\Re \left(4\|\beta\|_{L^1} + \sum_{0<\vartheta_\imath<\vartheta} m_\imath \right) \sup_{\vartheta\in J} \alpha(\nabla(\vartheta)).$$

Therefore

$$\chi_*(\Theta\nabla) \leq \mathfrak{X}_\Re \left(L_\Xi + 4\|\beta\|_{L^1} + \sum_{0<\vartheta_\imath<\vartheta} m_\imath \right) \chi_*(\nabla).$$

Thus Θ is χ_*-contraction. By Theorem 2.13 we conclude that Θ has at least one fixed point $\phi \in \Im_\theta$, which is a PC-asymptotically almost automorphic mild solution of problem (1.1) .

4. Continuous dependence on the initial condition

In this section we need the following lemma:

Lemma 4.1. [3] *Let the following inequality holds:*

$$\xi(\vartheta) \leq a(\vartheta) + \int_0^\vartheta b(\upsilon)d\upsilon + \sum_{0\leq\vartheta_\imath<\vartheta} \varsigma_\imath \xi\left(\vartheta_\imath^-\right), \quad \vartheta \geq 0,$$

where $\xi, a, b \in PC(\mathbb{R}^+, \mathbb{R}^+)$, *and* a *is nondecreasing,* $b(\vartheta) > 0$, $\varsigma_\imath > 0$, $\imath \in \mathbb{N}$. *Then, for* $\vartheta \in \mathbb{R}^+$, *the following inequality is valid:*

$$\xi(\vartheta) \leq a(\vartheta)(1+\varsigma)^\imath \exp\left(\int_0^\vartheta b(\upsilon)d\upsilon\right) \quad \vartheta \in [\vartheta_\imath, \vartheta_{\imath+1}], \imath \in \mathbb{N},$$

where $\varsigma = \max\{\varsigma_\imath : \imath \in \mathbb{N}\}$.

Theorem 4.2. *If the assumption of Theorems 3.2 are fulfilled, then the solution of the problem (1.1) depends continuously on the initial condition.*

Proof. Let $\phi_0, \phi_0^* \in \mho$. From Theorem 3.2, there exist $\phi(\cdot,\phi_0)$, $\phi^*(\cdot,\phi_0^*) \in \mathfrak{G}$ such that

$$\phi(\vartheta) = \Re(\vartheta)(\phi_0+\Xi(\phi))+\int_0^\vartheta \Re(\vartheta-\upsilon)\Psi(\upsilon,\phi(\upsilon))d\upsilon+\sum_{0<\vartheta_\imath<\vartheta}\Re(\vartheta-\vartheta_\imath)I_\imath(\phi(\vartheta_\imath^-)),\ \vartheta\in J,$$

and

$$\phi^*(\vartheta) = \Re(\vartheta)[\phi_0^*+\Xi(\phi)]+\int_0^\vartheta \Re(\vartheta-\upsilon)\Psi(\upsilon,\phi^*(\upsilon))d\upsilon+\sum_{0<\vartheta_\imath<\vartheta}\Re(\vartheta-\vartheta_\imath)I_\imath((\vartheta_\imath^-)),\ \vartheta\in J.$$

Then for $\varpi(\vartheta) = \|\phi(\vartheta)-\phi^*(\vartheta)\|$, we have

$$\begin{aligned}\sup_{\vartheta\in J}\varpi(\vartheta) &\leq \mathfrak{X}_\Re\|\phi_0-\phi_0^*\| + \mathfrak{X}_\Re\|\Xi(\phi)-\Xi(\phi^*)\| + \mathfrak{X}_\Re\int_0^\vartheta \beta(\upsilon)\varpi(\upsilon)d\upsilon \\ &\quad +\mathfrak{X}_\Re\sum_{0<\vartheta_\imath<\vartheta} m_\imath\varpi(\vartheta_\imath^-) \\ &\leq \mathfrak{X}^*\|\phi_0-\phi^*\| + \mathfrak{X}^*\int_0^\vartheta \beta(\upsilon)\varpi(\upsilon)d\upsilon + \sum_{0<\vartheta_\imath<\vartheta}\mathfrak{X}^* m_\imath\varpi(\vartheta_\imath^-),\end{aligned}$$

where $\mathfrak{X}^* = \frac{\mathfrak{X}_\Re}{1-\mathfrak{X}_\Re L_\Xi}$.
Now, applying Lemma 4.1, we get

$$\|\phi-\phi^*\|_{\mathfrak{G}} \leq \mathfrak{X}^*\delta(1+m^*)^\imath\exp(\mathfrak{X}^*\|\beta\|_{L^1}),$$

where $m^* = \mathfrak{X}^*\max_{\imath\in N} m_\imath$.
Therefore if δ is small enough, we obtain

$$\|\phi-\phi^*\|_{\mathfrak{G}} \leq \epsilon.$$

It follows that the PC-Asymptotically almost automorphic mild solutions of the problem (1.1) depends continuously on the initial condition.

5. An example

Consider the following partial differential equation :

$$\begin{cases}\frac{\partial}{\partial\vartheta}(\phi(\vartheta,x)) = \frac{\partial^2\phi(\vartheta,x)}{\partial x^2} + \int_0^\vartheta\Gamma(\vartheta-\upsilon)\frac{\partial^2\phi(\upsilon,x)}{\partial x^2}d\upsilon + \frac{\cos^2(\vartheta)\sin(\pi\phi(\vartheta,x))}{30\sqrt{1+\vartheta^2}(1+|\phi(\vartheta,x)|)e^\vartheta} \\ \qquad +\frac{e^{-\vartheta}\cos^2(\vartheta)}{6\sqrt{1+\vartheta^2}}\sin\left(\frac{1}{\cos(\vartheta)+\cos\sqrt{2}\vartheta+2}\right)|(\phi(\vartheta,x))|,\ \vartheta\in\widehat{J},\quad x\in[0,\pi], \\ I_\imath\phi(\vartheta_\imath,x) = \frac{7^{-\imath}\phi(\vartheta_\imath^-,x)}{9\sqrt{1+|\phi(\vartheta_\imath^-,x)|}},\ \text{for } \imath\in\mathbb{N},\ \textit{and } x\in(0,\pi). \\ \phi(\vartheta,0) = \phi(\vartheta,\pi) = 0,\quad \vartheta\in\mathbb{R}^+, \\ \phi(0,x) + \frac{9}{28}\sum_{i=1}^2\frac{1}{3^i}\phi\left(\frac{1}{i},x\right) = e^x,\quad x\in[0,\pi],\end{cases} \tag{5.1}$$

where $\widehat{J} = \mathbb{R}^+ - \{\vartheta_\imath\}_{\imath\in\mathbb{N}}$, and $\{\vartheta_\imath\}$ is an almost automorphic sequence of positive real numbers.
Let $\mho = L^2(0,\pi)$ be the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}^+$.
Define

$$\phi(\vartheta)(x) = \phi(\vartheta, x),\ \ \Xi(\phi) = \frac{9}{28}\sum_{i=1}^{2}\frac{1}{3^i}\phi\left(\frac{1}{i}, x\right),$$

and

$$\Psi(\vartheta, \phi(\vartheta)) = \frac{e^{-\vartheta}\cos^2(\vartheta)}{6\sqrt{1+\vartheta^2}}\sin\left(\frac{1}{\cos\vartheta + \cos\sqrt{2}\vartheta + 2}\right)\|\phi(\vartheta)\| + \frac{e^{-\vartheta}\cos^2(\vartheta)\sin\pi\phi(\vartheta)}{30\sqrt{1+\vartheta^2}(1+|\phi(\vartheta)|)},$$

$$I_\imath(\phi(\vartheta_\imath^-)) = \frac{7^{-\imath}\phi(\vartheta_\imath^-)}{9\sqrt{1+|\phi(\vartheta_\imath^-)|}}.$$

Consider the operator $\Lambda(\vartheta) : \mho \mapsto \mho$ as follows:

$$\Lambda(\vartheta)z = \Gamma(\vartheta)\mathfrak{Z}z,\ \ for\ \vartheta \geq 0,\ \ z \in D(\mathfrak{Z}),$$

where $\mathfrak{Z}$ is defined by

$$\begin{cases} D(\mathfrak{Z}) = \{\varphi \in \mho \ /\ \varphi,\ \varphi' \text{ are AC},\ \varphi'' \in L^2(0,\pi)\ ,\ \varphi(0) = \varphi(\pi) = 0\}, \\ (\mathfrak{Z}\varphi)(x) = \frac{\partial^2\varphi(\vartheta,x)}{\partial x^2}. \end{cases}$$

It is well known that $\mathfrak{Z}$ generates a strongly continuous semigroup $(T(\vartheta))_{\vartheta\geq 0}$, which is dissipative and compact with $\|T(\vartheta)\| \leq e^{-\phi^2\vartheta}$, and for some $\sigma > \frac{1}{\phi^2}$. We assume that

$$\|\Gamma(\vartheta)\| \leq \frac{e^{-\phi^2\vartheta}}{\sigma},\ \ \text{and}\ \|\Gamma'(\vartheta)\| \leq \frac{e^{-\phi^2\vartheta}}{\sigma^2}.$$

It follows from [26], that $\|\Re(\vartheta)\| \leq e^{-\jmath\vartheta}$, where $\jmath = 1 - \sigma^{-1}$.
Then $(A1)$ hold with $\mathfrak{X}_\Re = 1$ and $b = 1 - \sigma^{-1}$.
Consequently, the problem can be written in the abstract form (1.1) with $\mathfrak{Z}$, Λ, Ξ and Ψ as defined above.
Now, let

$$\Psi(\vartheta, \phi(\vartheta)) = \widehat{\aleph}(\vartheta, \phi(\vartheta)) + \widetilde{\aleph}(\vartheta, \phi(\vartheta)),$$

where

$$\widehat{\aleph}(\vartheta, \phi(\vartheta)) = \frac{e^{-\vartheta}\cos^2(\vartheta)}{6\sqrt{1+\vartheta^2}}\sin\left(\frac{1}{\cos\vartheta + \cos\sqrt{2}\vartheta + 2}\right)|\phi(\vartheta)|,$$

$$\widetilde{\aleph}(\vartheta, \phi(\vartheta)) = \frac{e^{-\vartheta}\cos^2(\vartheta)\sin\pi\phi(\vartheta)}{30\sqrt{1+\vartheta^2}(1+|\phi(\vartheta)|)}.$$

Then it is easy to verify that the function $\widehat{\aleph}, \widetilde{\aleph} : \mathbb{R}^+ \times \mho \longrightarrow \mho$ are continuous and $\widehat{\aleph} \in AA(\mathbb{R}^+ \times \mho; \mho)$, with

$$\|\widehat{\aleph}(\vartheta, z_1(\vartheta)) - \widehat{\aleph}(\vartheta, z_2(\vartheta))\| \leq \frac{1}{6}\|z_1(\vartheta) - z_2(\vartheta)\|,\ \text{for all}\ \vartheta \in J, z_1, z_2 \in \mho,$$

$$\|\widetilde{\aleph}(\vartheta, z(\vartheta))\| \leq \frac{1}{30\sqrt{1+\vartheta^2}},\ \text{for all}\ \vartheta \in J,\ \ z \in \mho,$$

which implies that $\widetilde{\aleph} \in PC_0(\mathbb{R}^+ \times \mho, \mho)$.
For the function Ψ, we can make the following estimates:

$$\|\Psi(\vartheta, \phi_1(\vartheta)) - \Psi(\vartheta, \phi_2(\vartheta))\| \leq \frac{e^{-\vartheta}\cos^2\vartheta}{6\sqrt{1+\vartheta^2}}\|\phi_1(\vartheta) - \phi_2(\vartheta)\|.$$

For every $\vartheta \in J$ and $\Im \subset \mho$, we have

$$\alpha(\Psi(\vartheta, \Im(\vartheta))) \leq \frac{e^{-\vartheta}\cos^2\vartheta}{6\sqrt{1+\vartheta^2}}\alpha(\Im(\vartheta)),$$

Then, $\beta(\vartheta) = \frac{e^{-\vartheta}\cos^2\vartheta}{6\sqrt{1+\vartheta^2}}$, which belongs to $L^1(J, \mathbb{R}^+)$. We have also the following estimates,

$$\|\Xi(\phi_1) - \Xi(\phi_1)\| \leq \frac{1}{7}\|\phi_1 - \phi_2\|_{\mathfrak{G}},$$

$$\|I_\imath\phi_1(\vartheta_\imath) - I_\imath\phi_2(\vartheta_\imath)\| \leq \frac{7^{-\imath}}{9}\|\phi_1(\vartheta_\imath) - \phi_2(\vartheta_\imath)\|,$$

and

$$\mathfrak{X}_{\Re}\left(L_\Xi + 4\|\beta\|_{L^1} + \sum_{\imath=0}^{\infty} m_\imath\right) \simeq 0,6 < 1.$$

Thus, Theorem 3.2 yields, then the problem (5.1) has a PC-asymptotically almost automorphic mild solution.

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Coincidence theory and KKM type maps

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Abstract. In this paper we present a variety of coincidence results for classes of maps defined on Hausdorff topological vector spaces. Our theory is based on fixed point theory in the literature.

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1. Introduction

In this paper we present a coincidence theory for classes of maps defined on Hausdorff topological spaces. These classes include some of the most general type of maps in the literature, namely KKM type maps, PK type maps, DKT type maps and $HLPY$ type maps. We establish coincidence results in both situations, namely when the classes are the same and when the classes are different. Our theory is based on fixed point theory in the literature (some due to the author [1], [12], [13]) and on selection theorems in the literature. Our results generalize and extend many results in the literature; see [1], [3], [5], [4], [6], [7], [11], [14], [15] and the references therein.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to$

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$H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [10]. A upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

We also discuss the following classes of maps in this paper. Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ [11] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\,int\, S^{-1}(w) : \ w \in W\}$; here $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. These maps are related to the DKT maps in the literature and $F \in DKT(Z, W)$ [7] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the $\Phi^\star$ maps. We say $G \in \Phi^\star(Z, W)$ [3] if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$.

Now we consider a general class of maps, namely the PK maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : \ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathbf{X}(Z, Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:

(i). $\mathbf{U}$ contains the class C of single valued continuous functions;

(ii). each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leq 1\}$.

We say $F \in PK(X,Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions [12].

For a subset K of a topological space X, we denote by $Cov_X\,(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov\,(K) = Cov_X\,(K)$). Given two maps F, $G : X \to 2^Y$ and $\alpha \in Cov\,(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov\,(Y)$ and any pair (X,K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V\,(K)$ there exists a continuous function $\pi_\alpha : K \to V$ such that

(i). π_α and $i : K \to V$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES$(compact).

The next results can be found in [1] and [12] respectively.

Theorem 1.1. *Let $X \in ES($compact$)$ and $\Psi \in PK(X,X)$ a compact map. Then there exists a $x \in X$ with $x \in \Psi(x)$.*

Theorem 1.2. *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $\Psi \in PK(X,X)$ a compact upper semicontinuous map with closed values. Then there exists a $x \in X$ with $x \in \Psi(x)$.*

Remark 1.3. (i). Other variations of Theorem 1.2 can be found in [13].

(ii). It is of interest to note that Theorem 1.1 is based on the following fixed point result. If T is the Tychonoff cube and $\Phi \in PK(T,T)$ then Φ has a fixed point and to see the proof of this note since T is compact then there exists a $G \in \mathbf{U}_c(K,Y)$ with $G(x) \subseteq \Phi(x)$ for $x \in T$, so a standard result (see [4, Theorem 3.1]) guarantees a $x \in T$ with $x \in G(x)$, so $x \in \Phi(x)$.

Next we describe a class of maps more general than the PK maps in our setting. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. If S, $T : X \to 2^Y$ are two set valued maps such that $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X then we call S a generalized KKM mappping w.r.t. T. Now the set valued map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM map $S : X \to 2^Y$ w.r.t. T the family $\{\,\overline{S(x)} : \ x \in X\}$ has the finite intersection property (the intersection of each finite subfamily is nonempty).

We let

$$KKM(X,Y) = \{T : X \to 2^Y \,|\, T \text{ has the } KKM \text{ property}\}.$$

Note $PK(X,Y) \subset KKM(X,Y)$ (see [6]). Next we recall the following result [6].

Theorem 1.4. *Let X be a convex subset of a Hausdorff topological vector space and Y, Z be Hausdorff topological spaces.*
(i). $T \in KKM(X,Y)$ iff $T|_\triangle \in KKM(\triangle,Y)$ for each polytope $\triangle$ in X;
(ii). if $T \in KKM(X,Y)$ and $f \in C(Y,Z)$ then $f\,T \in KKM(X,Z)$;
(iii). if Y is a normal space, $\triangle$ a polytope of X and if $T : \triangle \to 2^Y$ is a set valued map such that for each $f \in C(Y,\triangle)$ we have that $f\,T$ has a fixed point in $\triangle$, then $T \in KKM(\triangle,Y)$.

Next we recall the following fixed point result for KKM maps. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

The next result can be found in [4].

Theorem 1.5. *Let X be an admissible convex set in a Hausdorff topological vector space E and $T \in KKM(X,X)$ be a closed compact map. Then T has a fixed point in X.*

Remark 1.6. One could also consider $s - KKM$ maps [5], [4], [15] in this paper and we could obtain similar results to those in section 2.

Next we will present an analogue of Theorem 1.4 (ii) for $T\,f$ and this composition will be needed in a few results in section 2.

Theorem 1.7. *Let X be an admissible convex set in a Hausdorff topological vector space and Z a subset of a Hausdorff topological space. If $T \in KKM(X,Z)$ is a upper semicontinuous compact map with closed (in fact compact) values and $f \in C(Z,X)$ then $T\,f$ has a fixed point in Z.*

Proof. Now $T \in KKM(X,Z)$, $f \in C(Z,X)$ and Theorem 1.4 (ii) implies $f\,T \in KKM(X,X)$. Also $f\,T$ is a compact upper semicontinuous map with compact values (so $f\,F$ is a closed map [2]). Now Theorem 1.5 guarantees that $f\,T$ has a fixed point in X and consequently $T\,f$ has a fixed point in Z. □

Theorem 1.8. *Let X be an admissible convex set in a Hausdorff topological vector space, Y a convex set in a Hausdorff topological vector space and Y a normal space. If $T \in KKM(X,Y)$ is a upper semicontinuous map with compact values and $f \in C(Y,X)$ then $T\,f \in KKM(Y,Y)$.*

Proof. Note $T\,f : Y \to 2^Y$. From Theorem 1.4 (i), (iii) we need to show that for each polytope $\triangle$ in Y that $g(T\,f)$ has a fixed point in $\triangle$ for any $g \in C(Y,\triangle)$. Note from Theorem 1.4 (ii) since $T \in KKM(X,Y)$ and $g \in C(Y,\triangle)$ that $g\,T \in KKM(X,\triangle)$.

Now from Theorem 1.7 (note $Z = \triangle$ is compact and $gT : X \to 2^{\triangle}$ is a upper semicontinuous compact map with compact values) guarantees that $(gT)\,f$ has a fixed point in $\triangle$. $\square$

In section 2 we will make use of the following two properties. Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i). If $T \in KKM(X,Y)$ then $G \equiv T|_C \in KKM(C,Y)$.

This can be seen from Theorem 1.4 (i). Note $T \in KKM(X,Y)$ so $T|_{\triangle} \in KKM(\triangle,Y)$ for each polytope $\triangle$ in X from Theorem 1.4 (i). Thus in particular for any polytope $\triangle$ in C we have $T|_{\triangle} \in KKM(\triangle,Y)$ so from Theorem 1.4 (i) we have $T|_C \in KKM(C,Y)$.

Alternatively we can prove it directly as follows. Let $S : C \to 2^Y$ be a generalized KKM map w.r.t. G i.e. $G(co\,(A)) \subseteq S(A)$ for each finite subset A of C. We must show $\{\,\overline{S(x)} :\ x \in C\}$ has the finite intersection property. To see this let $S^\star : X \to 2^Y$ be given by

$$S^\star(x) = \begin{cases} S(x), & x \in C \\ Y, & x \in X \setminus C. \end{cases}$$

We claim $T(co\,(D)) \subseteq S^\star(D)$ for each finite subset D of X. Now either (a). $x \in C$ for all $x \in D$ or (b). there exists a $y \in D$ with $y \notin C$. Suppose first case (b) occurs. Then since $S^\star(y) = Y$ we have

$$T(co\,(D)) \subseteq Y = S^\star(y) = S^\star(D).$$

It remains to consider case (a). Then since C is convex we have $co\,(D) \subseteq C$ and since $S^\star(z) = S(z)$ for $z \in C$ we have

$$T(co\,(D)) = G(co\,(D)) \subseteq S(D) = S^\star(D).$$

Thus $S^\star : X \to 2^Y$ is a generalized KKM map w.r.t. T. Since $T \in KKM(X,Y)$ then $\{\,\overline{S^\star(x)} :\ x \in X\}$ has the finite intersection property. Now for any finite subset Ω of C (note $S^\star(z) = S(z)$ for $z \in C$) we have

$$\cap_{x\in\Omega}\ \overline{S(x)} = \cap_{x\in\Omega}\ \overline{S^\star(x)} \neq \emptyset,$$

so $G = T|_C \in KKM(C,Y)$.

(ii). If $T \in KKM(X,Y)$, $T(X) \subseteq Z \subseteq Y$ and Z is closed in Y then $T \in KKM(X,Z)$.

Let $S : X \to 2^Z$ be a generalized KKM map w.r.t. T i.e. $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X. We must show $\{\,\overline{S(x)}^Z :\ x \in X\}$ has the finite intersection property. Note since $S : X \to 2^Y$ is a generalized KKM map w.r.t. T then since $T \in KKM(X,Y)$ we have that $\{\,\overline{S(x)}\,(= \overline{S(x)}^Y) :\ x \in X\}$ has the finite intersection property. However note for $x \in X$ that

$$\overline{S(x)}^Z = \overline{S(x)}^Y \cap Z = \overline{S(x)}^Y\,(= \overline{S(x)})$$

since Z is closed in Y (note $S(X) \subseteq Z$ so $\overline{S(x)}^Y \subseteq Z$). Thus $\{\,\overline{S(x)}^Z :\ x \in X\} = \{\,\overline{S(x)}\,(= \overline{S(x)}^Y) :\ x \in X\}$ has the finite intersection property.

2. Coincidence results

In this section we present coincidence results between two different classes of set–valued maps. Results on coincidence points for similar classes of maps is also discussed at the end of section 2.

Theorem 2.1. *Let X be a convex subset of a Hausdorff topological vector space E and Y a subset of a Hausdorff topological space. Suppose $F \in KKM(X,Y)$ is a upper semicontinuous compact map with compact values and $G \in DKT(Y,X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Proof. By assumption there exists a compact set K of Y with $F(X) \subseteq K$. Also since $G \in DKT(Y,X)$ we have $G \in DKT(K,X)$. To see this note there exists a map $S : Y \to X$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$, $S(y) \neq \emptyset$ for each $y \in Y$ and $S^{-1}(x)$ is open (in Y) for each $x \in X$. Let $S^\star$ denote the restriction of S to K and note $co\,(S^\star(y)) = co\,(S(y)) \subseteq G(y)$ for $y \in K$ and for $x \in X$ then (note $K \subseteq Y$)

$$\begin{aligned}(S^\star)^{-1}(x) &= \{z \in K : x \in S^\star(z)\} = \{z \in K : x \in S(z)\} \\ &= K \cap \{z \in Y : x \in S(z)\} = K \cap S^{-1}(x)\end{aligned}$$

which is open in $K \cap Y = K$. Thus $G \in DKT(K,X)$ with K compact so from [7] there exists a continuous single valued selection $g : K \to X$ (i.e. $g \in C(K,X)$) of G and there exists a finite subset A of X with $g(K) \subseteq co\,(A)$, so $g \in C(K, co\,(A)$. Note from Section 1 (see (i) and (ii)) that $F \in KKM(co\,(A), K)$ so $g\,F \in KKM(co\,(A), co\,(A))$ (see Theorem 1.4 (ii)) is a upper semicontinuous map with compact values, so a closed map [2]. Also note $co\,(A)$ is a compact convex subset in a finite dimensional subspace of E. Then Theorem 1.5 guarantees a $x \in co\,(A) \subseteq X$ with $x \in g\,(F(x))$. As a result $G^{-1}(x) \cap F(x) \neq \emptyset$. □

Remark 2.2. Since $PK(X,Y) \subseteq KKM(X,Y)$ then one could replace KKM with PK in Theorem 2.1.

Our next result replaces $G \in DKT(Y,X)$ with $G \in HLPY(Y,X)$ in Theorem 2.1.

Theorem 2.3. *Let X be a convex subset of a Hausdorff topological vector space E and Y a subset of a Hausdorff topological space. Suppose $F \in KKM(X,Y)$ is a upper semicontinuous compact map with compact values and $G \in HLPY(Y,X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Proof. There exists a compact set K of Y with $F(X) \subseteq K$. We claim $G \in HLPY(K,X)$. To see this note there exists a map $S : Y \to X$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$ and $Y = \bigcup\{\,int\,S^{-1}(w) : w \in X\}$. Let S also denote the restriction of S to K. We now show $K = \bigcup\{\,int_K\,S^{-1}(w) : w \in X\}$. To see this first notice that

$$\begin{aligned}K = K \cap Y &= K \cap \left(\bigcup\{\,int\,S^{-1}(w) : w \in X\}\right) \\ &= \bigcup\{K \cap int\,S^{-1}(w) : w \in X\},\end{aligned}$$

so $K \subseteq \bigcup \{ int_K\, S^{-1}(w) : \ w \in X\}$ since for each $w \in X$ we have that $K \cap int\, S^{-1}(w)$ is open in K. On the other hand clearly $\bigcup \{ int_K\, S^{-1}(w) : \ w \in X\} \subseteq K$ so as a result

$$K = \bigcup \{ int_K\, S^{-1}(w) : \ w \in X\}.$$

Thus $G \in HLPY(K,X)$ so from [11] there exists a selection $g \in C(K,X)$ of G and a finite subset A of X with $g(K) \subseteq co\,(A)$. As in Theorem 2.1 we have that $g\,F \in KKM(co\,(A), co\,(A))$ is a upper semicontinuous map with compact values. Now apply Theorem 1.5. □

Remark 2.4. Note $F \in KKM(X,Y)$ can be replaced by $F \in PK(X,Y)$ in Theorem 2.3.

Next we will consider the case where F is a compact map is replaced by G is a compact map in Theorem 2.1.

Theorem 2.5. *Let X be an admissible convex set in a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological space. Suppose $F \in KKM(X,Y)$ is a upper semicontinuous map with compact values and $G \in DKT(Y,X)$ is a compact map. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Proof. Since Y is paracompact then from [7] there exists a selection $g \in C(Y,X)$ of G. Now from Theorem 1.4 (ii) we have that $g\,F \in KKM(X,X)$ is a compact (since G is compact) upper semicontinuous map with compact values, so a closed map [2]. Theorem 1.5 guarantees a $x \in X$ with $x \in g\,(F(x))$. □

Remark 2.6. (i). Note we could replace $G \in DKY(Y,X)$ with $G \in HLPY(Y,X)$ in Theorem 2.5 since [11] guarantees a selection $g \in C(Y,X)$ of G.

(ii). In fact one could replace X an admissible set with $D = co\,(\overline{G(Y)})$ an admissible set in Theorem 2.5. To see this note $G \in DKT(Y,D)$ [there exists a map $S : Y \to X$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$, $S(y) \neq \emptyset$ for each $y \in Y$ and $S^{-1}(x)$ is open (in Y) for each $x \in X$ so since $G(Y) \subseteq D$ then $S : Y \to D$ since $S(y) \subseteq co\,(S(y)) \subseteq G(y) \subseteq D$ for $y \in Y$ and trivially $S^{-1}(x)$ is open (in Y) for each $x \in D$ since $D \subseteq X$]. Now from [7] there exists a selection $g \in C(Y,D)$ of G. Also note since $D \subseteq X$ and $F \in KKM(X,Y)$ then $F \in KKM(D,Y)$ (see (i) in Section 1). Then from Theorem 1.4 (ii) we have that $g\,F \in KKM(D,D)$ and now apply Theorem 1.5. A similar comment applies if $G \in DKY(Y,X)$ is replaced by $G \in HLPY(Y,X)$.

Remark 2.7. Note $F \in KKM(X,Y)$ can be replaced by $F \in PK(X,Y)$ in Theorem 2.5. However it is possible to obtain other results for PK maps as the next theorem shows.

Theorem 2.8. *Let X be a convex set in a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological space. Suppose $F \in PK(X,Y)$ and $G \in DKT(Y,X)$ is a compact map. Also assume one of the following hold:*

(a). $X \in ES(compact)$,

(b). X is Schauder admissible and F is upper semicontinuous with closed values.

Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.

Proof. Let $g \in C(Y, X)$ be a selection of G (guaranteed since Y is paracompact) and note $g\, F \in PK(X, X)$ since the composition of PK maps is PK. Finally note $g\, F$ is a compact map since G is a compact map. If (a) holds then apply Theoren 1.1 to guarantee a $x \in X$ with $x \in g\,(F(x))$. If (b) holds note $g\, F$ is a upper semicontinuous map with closed (in fact compact) values. Now apply Theorem 1.2. □

Remark 2.9. (i). Note we could replace $G \in DKY(Y, X)$ with $G \in HLPY(Y, X)$ in Theorem 2.8.

(ii). In (a) in Theorem 2.8 we could replace $X \in ES(\text{compact})$ with $D = co\,(\overline{G(Y)}) \in ES(\text{compact})$ (respectively, in (b) in Theorem 2.8 we could replace X is Schauder admissible with D is Schauder admissible). To see this note $G \in DKT(Y, D)$ so there exists a selection $g \in C(Y, D)$ of G. Also note since $D \subseteq X$ and $F \in PK(X, Y)$ then $F \in PK(D, Y)$. Thus $g\, F \in PK(D, D)$.

In Theorem 2.5 and Theorem 2.8 we considered $g\, F$. It is also possible to consider $F\, g$ to obtain other results. We first consider the analogue of Theorem 2.8.

Theorem 2.10. *Let X be a convex set in a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological space E. Suppose $F \in PK(X, Y)$ is a upper semicontinuous map with compact values and $G \in DKT(Y, X)$ is a compact map. Also assume one of the following hold:*

(a). $Y \in ES(compact)$,

(b). E is a uniform space and Y is a Schauder admissible subset of E.

Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.

Proof. Let $g \in C(Y, X)$ be a selection of G and note $F\, g \in PK(Y, Y)$. Also note $F\, g$ is a upper semicontinuous, compact map (since G is a compact map and F is upper semicontinuous map with compact values) with closed (in fact compact) values. If (a) holds apply Theorem 1.1 whereas if (b) holds apply Theorem 1.2. □

Remark 2.11. Note in Theorem 2.10 (a), F is upper semicontinuous map with compact values is only needed to guarantee that $F\, g$ is a compact map (this is all that is needed to apply Theorem 1.1). As a result in Theorem 2.10 (a), F is upper semicontinuous map with compact values and G is a compact map can be replaced by the condition that $F\, G$ is a compact map.

Remark 2.12. Note we could replace $G \in DKY(Y, X)$ with $G \in HLPY(Y, X)$ in Theorem 2.10.

We now consider the analogue of Theorem 2.5 when we use $F\, g$ instead of $g\, F$.

Theorem 2.13. *Let X be a convex set in a Hausdorff topological vector space E and Y an admissible convex paracompact subset in a Hausdorff topological vector space. Suppose $F \in KKM(X, Y)$ is a upper semicontinuous map with compact values and $G \in DKT(Y, X)$ is a compact map. Let $D = co\,(\overline{G(Y)})$ and assume D is an admissible subset of E. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Proof. We claim $G \in DKT(Y, D)$. To see this note there exists a map $S : Y \to X$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$, $S(y) \neq \emptyset$ for each $y \in Y$ and $S^{-1}(x)$ is open (in Y) for each $x \in X$. Since $G(Y) \subseteq D$ then $S : Y \to D$ since $S(y) \subseteq co\,(S(y)) \subseteq G(y) \subseteq D$ for $y \in Y$. Trivially $S^{-1}(x)$ is open (in Y) for each $x \in D$ since $D \subseteq X$. Thus $G \in DKT(Y, D)$ so from [7] there exists a selection $g \in C(Y, D)$ of G (note Y is paracompact). Also note since $D \subseteq X$ and $F \in KKM(X, Y)$ then $F \in KKM(D, Y)$ (see (i) in Section 1). Now $g \in C(Y, D)$, $F \in KKM(D, Y)$ and Theorem 1.8 guarantee that $F\,g \in KKM(Y, Y)$ (note D is an admissible convex set in E and Y is normal since Hausdorff paracompact spaces are normal [9]). Also note $F\,g$ is a upper semicontinuous compact map with compact values, so it is a closed map [2]. Now apply Theorem 1.5 to guarantee a $y \in Y$ with $y \in F\,(g(y))$. □

Remark 2.14. Note we could replace $G \in DKY(Y, X)$ with $G \in HLPY(Y, X)$ in Theorem 2.13. To see this we only need to show that $G \in HLPY(Y, D)$. To see this note there exists a map $S : Y \to X$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$ and $Y = \bigcup\{\, int\, S^{-1}(w) :\ w \in X\}$. Now note for any $y \in Y$ there exists a $w \in X$ with $y \in int\, S^{-1}(w)$ so $w \in S(y) \subseteq co\,(S(y)) \subseteq G(y) \subseteq D$. Thus $Y = \bigcup\{\, int\, S^{-1}(w) : w \in D\}$, so $G \in HLPY(Y, D)$.

Remark 2.15. In Theorem 2.13 we used Theorem 1.8 to get $F\,g \in KKM(Y, Y)$ and then we applied Theorem 1.5. If D is an admissible subset of E was replaced by X is an admissible subset of E then since one could obtain a selection $g \in C(Y, X)$ of G and since $F \in KKM(X, Y)$, then from Theorem 1.8 we have that $F\,g \in KKM(Y, Y)$.

In our next set of results we will remove the condition that Y is paracompact in Theorem 2.5, Theorem 2.8, Theorem 2.10 and Theorem 2.13. We begin with the analogue of Theorem 2.8.

Theorem 2.16. *Let X be a convex set and Y a closed set in a Hausdorff topological vector space E. Suppose $F \in PK(X, Y)$ and $G \in DKT(Y, X)$ is a compact map. Let K be a compact subset of X with $G(Y) \subseteq K$, let $L(K)$ be the linear span of K (i.e. the smallest linear subspace of E that contains K) and assume $F(X) \subseteq L(K) \cap Y$. Also suppose one of the following hold:*

(a). $X \in ES(compact)$,

(b). X is Schauder admissible and F is upper semicontinuous with closed values.

Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.

Proof. Let $L(K)$ be as described above. We claim $G \in DKT(Y \cap L(K), X)$. To see this note there exists a map $S : Y \to X$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$, $S(y) \neq \emptyset$ for each $y \in Y$ and $S^{-1}(x)$ is open (in Y) for each $x \in X$. Let $S^\star$ denote the restriction of S to $Y \cap L(K)$ and note if $x \in X$ then

$$\begin{aligned} (S^\star)^{-1}(x) &= \{z \in Y \cap L(K) : x \in S^\star(z)\} = \{z \in Y \cap L(K) : x \in S(z)\} \\ &= L(K) \cap \{z \in Y : x \in S(z)\} = L(K) \cap S^{-1}(x) \end{aligned}$$

which is open in $L(K) \cap Y$. Thus $G \in DKT(Y \cap L(K), X)$. Now recall $L(K)$ is Lindelöf so paracompact [9] and since $Y \cap L(K)$ is closed in $L(K)$ then $Y \cap L(K)$ is paracompact. Now from [7] there exists a selection $g \in C(Y \cap L(K), X)$ of G. Also since $F(X) \subseteq L(K) \cap Y$ then $F \in PK(X, Y \cap L(K))$. Thus $g\,F \in PK(X,X)$ is a compact maps (since G is a compact map). If (a) holds then apply Theoren 1.1 to guarantee a $x \in X$ with $x \in g\,(F(x))$. If (b) holds note $g\,F$ is a upper semicontinuous map with closed (in fact compact) values. Now apply Theorem 1.2. $\square$

Remark 2.17. In Theorem 2.16 we could replace $F(X) \subseteq L(K) \cap Y$ with $F(L(K)) \subseteq L(K) \cap Y$ provided we assume $F \in PK(E,Y)$ and $G \in DKT(Y,E)$ is a compact map (here K is a compact subset of E with $G(Y) \subseteq K$). Also we need to replace (a) and (b) in Theorem 2.16 with one of the following:

(a). $L(K) \in ES(\text{compact})$,

(b). $L(K)$ is Schauder admissible and F is upper semicontinuous with closed values.

To see this we claim $G \in DKT(Y \cap L(K), L(K))$. Note there exists a map $S : Y \to E$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$, $S(y) \neq \emptyset$ for each $y \in Y$ and $S^{-1}(x)$ is open (in Y) for each $x \in E$. Let $S^\star$ denote the restriction of S to $Y \cap L(K)$ and note if $x \in L(K)$ then

$$\begin{aligned}(S^\star)^{-1}(x) &= \{z \in Y \cap L(K) : x \in S^\star(z)\} = \{z \in Y \cap L(K) : x \in S(z)\} \\ &= L(K) \cap \{z \in Y : x \in S(z)\} = L(K) \cap S^{-1}(x)\end{aligned}$$

which is open in $L(K) \cap Y$. Thus $G \in DKT(Y \cap L(K), L(K))$ and so there exists a selection $g \in C(Y \cap L(K), L(K))$ of G. Also since $F(L(K)) \subseteq L(K) \cap Y$ then $F \in PK(L(K), L(K) \cap Y)$ and so $g\,F \in PK(L(K), L(K))$ is a compact map. Now apply Theorem 1.1 or Theorem 1.2.

Next we will replace DKT with $HLPY$ in Theorem 2.16.

Theorem 2.18. *Let X be a convex set and Y a closed set in a Hausdorff topological vector space E. Suppose $F \in PK(X,Y)$ and $G \in HLPY(Y,X)$ is a compact map. Let K be a compact subset of X with $G(Y) \subseteq K$, let $L(K)$ be the linear span of K and assume $F(X) \subseteq L(K) \cap Y$. Also suppose one of the following hold:*

(a). $X \in ES(\text{compact})$,

(b). X is Schauder admissible and F is upper semicontinuous with closed values.

Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.

Proof. Let $L(K)$ be as described above. We claim $G \in HLPY(Y \cap L(K), X)$. To see this note there exists a map $S : Y \to X$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$ and $Y = \bigcup\{\,int\,S^{-1}(w) :\ w \in X\}$. Let S also denote the restriction of S to $L(K) \cap Y$. We now show $Y \cap L(K) = \bigcup\{\,int_{Y \cap L(K)}\,S^{-1}(w) :\ w \in X\}$. To see this first notice that

$$\begin{aligned}L(K) \cap Y &= L(K) \cap \left(\bigcup\{\,int\,S^{-1}(w) :\ w \in X\}\right) \\ &= \bigcup\{L(K) \cap int\,S^{-1}(w) :\ w \in X\},\end{aligned}$$

so $L(K) \cap Y \subseteq \bigcup\{\,int_{Y \cap L(K)}\,S^{-1}(w) :\ w \in X\}$ since for each $w \in X$ we have that $Y \cap int\,S^{-1}(w) = int\,S^{-1}(w)$ so $L(K) \cap int\,S^{-1}(w) = L(K) \cap Y \cap$

$int\, S^{-1}(w) = (L(K) \cap Y) \cap int\, S^{-1}(w)$ with is open in $L(K) \cap Y$. On the other hand clearly $\bigcup \{ int_{Y \cap L(K)}\, S^{-1}(w) : \; w \in X\} \subseteq L(K) \cap Y$. Thus $L(K) \cap Y = \bigcup \{ int_{Y \cap L(K)}\, S^{-1}(w) : \; w \in X\}$ so $G \in HLPY(Y \cap L(K), X)$ and since $Y \cap L(K)$ is paracompact there exists [11] a selection $g \in C(Y \cap L(K), X)$ of G and also note $g\, F \in PK(X, X)$ is a compact map. Now apply Theorem 1.1 or Theorem 1.2. □

Remark 2.19. In Theorem 2.18 we could replace $F(X) \subseteq L(K) \cap Y$ with $F(L(K)) \subseteq L(K) \cap Y$ provided we assume $F \in PK(E, Y)$ and $G \in HLPY(Y, E)$ is a compact map (here K is a compact subset of E with $G(Y) \subseteq K$). Also we need to replace (a) and (b) in Theorem 2.18 with one of the following:

(a). $L(K) \in ES(\text{compact})$,

(b). $L(K)$ is Schauder admissible and F is upper semicontinuous with closed values.

To see this we claim $G \in HLPY(Y \cap L(K), L(K))$. First note there exists a map $S : Y \to E$ with $co\,(S(y)) \subseteq G(y)$ for $y \in Y$ and $Y = \bigcup \{ int\, S^{-1}(w) : \; w \in E\}$. Let S also denote the restriction of S to $L(K) \cap Y$. Notice

$$\begin{aligned} L(K) \cap Y &= L(K) \cap \left(\bigcup \{ int\, S^{-1}(w) : \; w \in E\} \right) \\ &= \bigcup \{ L(K) \cap \, int\, S^{-1}(w) : \; w \in E\}, \end{aligned}$$

so $L(K) \cap Y \subseteq \bigcup \{ int_{Y \cap L(K)}\, S^{-1}(w) : \; w \in X\}$ since for each $w \in E$ we have that $Y \cap int\, S^{-1}(w) = int\, S^{-1}(w)$ so $L(K) \cap int\, S^{-1}(w) = (L(K) \cap Y) \cap int\, S^{-1}(w)$ with is open in $L(K) \cap Y$. On the other hand clearly $\bigcup \{ int_{Y \cap L(K)}\, S^{-1}(w) : \; w \in E\} \subseteq L(K) \cap Y$. Thus

$$L(K) \cap Y = \bigcup \{ int_{Y \cap L(K)}\, S^{-1}(w) : \; w \in E\}.$$

Now for any $y \in L(K) \cap Y$ there exists a $w \in E$ with $y \in int_{Y \cap L(K)}\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(y) \subseteq co\,(S(y)) \subseteq G(y) \subseteq K \subseteq L(K)$. Thus

$$L(K) \cap Y = \bigcup \{ int_{Y \cap L(K)}\, S^{-1}(w) : \; w \in L(K)\},$$

so $G \in HLPY(Y \cap L(K), L(K))$ and since $Y \cap L(K)$ is paracompact there exists a selection $g \in C(Y \cap L(K), L(K))$ of G and also note $g\, F \in PK(L(K), L(K))$ is a compact map. Now apply Theorem 1.1 or Theorem 1.2.

Now we consider the analogue of Theorem 2.5.

Theorem 2.20. *Let X be an admissible convex set and Y a set in a Hausdorff topological vector space E. Suppose $F \in KKM(X, Y)$ is a upper semicontinuous map with compact values and $G \in DKT(Y, X)$ is a compact map. Let K be a compact subset of X with $G(Y) \subseteq K$, let $L(K)$ be the linear span of K and assume $F(X) \subseteq L(K) \cap Y$ and $L(K) \cap Y$ is closed in both Y and $L(K)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Proof. Let $L(K)$ be as described above and as in Theorem 2.16 we have $G \in DKT(Y \cap L(K), X)$. Note $L(K)$ is paracompact and since $Y \cap L(K)$ is closed in $L(K)$ then $Y \cap L(K)$ is paracompact. Now from [7] there exists a selection $g \in C(Y \cap L(K), X)$

of G. Now $F \in KKM(X, Y)$, $F(X) \subseteq L(K) \cap Y$ so from Section 1 (see (ii), note $Y \cap L(K)$ is closed in Y) we have $F \in KKM(X, Y \cap L(K))$. Now Theorem 1.4 (ii) guarantees that $g\,F \in KKM(X, X)$ and note $g\,F$ is a compact map. Now apply Theorem 1.5. □

Remark 2.21. Note we could replace $G \in DKY(Y, X)$ with $G \in HLPY(Y, X)$ in Theorem 2.20 (since $G \in HLPY(Y \cap L(K), X)$ as in Theorem 2.18).

Remark 2.22. In Theorem 2.20 (respectively, Remark 2.21) we could replace $F(X) \subseteq L(K) \cap Y$ with $F(L(K)) \subseteq L(K) \cap Y$ provided we assume $F \in KKM(E, Y)$ and $G \in DKT(Y, E)$ ((respectively, $G \in HLPY(Y, E)$) is a compact map (here K is a compact subset of E with $G(Y) \subseteq K$). Also we need to replace the assumption that X is an admissible subset of E with $L(K)$ is an admissible subset of E.

To see this note (see Remark 2.17, respectively, Remark 2.19) $G \in DKT(Y \cap L(K), L(K))$ (respectively, $G \in HLPY(Y \cap L(K), L(K))$) so since $Y \cap L(K)$ is paracompact there exists a selection $g \in C(Y \cap L(K), L(K))$ of G. Also since $F(L(K)) \subseteq L(K) \cap Y$ from Section 1 (see (i) and (ii)) we have $F \in KKM(L(K), Y \cap L(K))$. Now Theorem 1.4 (ii) guarantees that $g\,F \in KKM(L(K), L(K))$ and note $g\,F$ is a compact map. Now apply Theorem 1.5.

Next we present the analogue of Theorem 2.10.

Theorem 2.23. *Let X be a convex set and Y a closed subset in a Hausdorff topological vector space E. Suppose $F \in PK(X, Y)$ is a upper semicontinuous map with compact values and $G \in DKT(Y, X)$ is a compact map. Let K be a compact subset of X with $G(Y) \subseteq K$, let $L(K)$ be the linear span of K and assume $F(X) \subseteq L(K) \cap Y$. Also suppose one of the following hold:*

(a). $Y \cap L(K) \in ES(compact)$,

(b). $Y \cap L(K)$ is a Schauder admissible subset of E.

Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.

Proof. As in Theorem 2.16 note $G \in DKT(Y \cap L(K), X)$ and $Y \cap L((K)$ is paracompact, so there exists a selection $g \in C(Y \cap L(K), X)$ of G. Also $F \in PK(X, Y \cap L(K))$ so $F\,g \in PK(Y \cap L(K), Y \cap L(K))$ is a upper semicontinuous compact map with compact values. Now apply Theorem 1.1 or Theorem 1.2. □

Remark 2.24. Note we could replace $G \in DKY(Y, X)$ with $G \in HLPY(Y, X)$ in Theorem 2.23 (since $G \in HLPY(Y \cap L(K), X)$ as in Theorem 2.18).

Remark 2.25. In Theorem 2.23 (respectively, Remark 2.24) we could replace $F(X) \subseteq L(K) \cap Y$ with $F(L(K)) \subseteq L(K) \cap Y$ provided we assume $F \in PK(E, Y)$ and $G \in DKT(Y, E)$ (respectively, $G \in HLPY(Y, E)$) is a compact map (here K is a compact subset of E with $G(Y) \subseteq K$). To see this notice from Remark 2.17 (respectively, Remark 2.19) that $G \in DKT(Y \cap L(K), L(K))$ (respectively, $G \in HLPY(Y \cap L(K), L(K))$), so there exists a selection $g \in C(Y \cap L(K), L(K))$ of G. Also since $F(L(K)) \subseteq L(K) \cap Y$ then $F \in PK(L(K), L(K) \cap Y)$ and so $F\,g \in PK(Y \cap L(K), Y \cap L(K))$ is a compact map. Now apply Theorem 1.1 or Theorem 1.2.

We note in [14, Theorem 2.5] that inadventently we assumed $F \in Ad(X,Y)$ and $G \in DKT(Y,X)$ instead of $F \in Ad(E,Y)$ and $G \in DKT(Y,E)$.

Next we present the analogue of Theorem 2.13.

Theorem 2.26. *Let X be an admissible convex set and Y a convex set in a Hausdorff topological vector space E. Suppose $F \in KKM(X,Y)$ is a upper semicontinuous map with compact values and $G \in DKT(Y,X)$ is a compact map. Let K be a compact subset of X with $G(Y) \subseteq K$, let $L(K)$ be the linear span of K and assume $F(X) \subseteq L(K) \cap Y$. Also assume $L(K) \cap Y$ is closed in both Y and $L(K)$ and $Y \cap L(K)$ is an admissible subset of E. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Proof. As in Theorem 2.16 note $G \in DKT(Y \cap L(K), X)$ and $Y \cap L(K)$ is paracompact, so there exists a selection $g \in C(Y \cap L(K), X)$ of G. Also $F \in KKM(X,Y)$, $F(X) \subseteq L(K) \cap Y$ so from Section 1 (see (ii), note $Y \cap L(K)$ is closed in Y) we have $F \in KKM(X, Y \cap L(K))$. Now Theorem 1.8 (note $Y \cap L(K)$ is normal since Hausdorff paracompact spaces are normal [8] and also note X is an admissible convex subset of E) guarantees that $F\,g \in KKM(Y \cap L(K), Y \cap L(K))$ and $F\,g$ is a upper semicontinuous compact map with compact values. Now apply Theorem 1.5. □

Remark 2.27. Now $G \in DKY(Y,X)$ can be replaced by $G \in HLPY(Y,X)$ in Theorem 2.26 (since $G \in HLPY(Y \cap L(K), X)$ as in Theorem 2.18).

Remark 2.28. In Theorem 2.26 (respectively, Remark 2.27) we could replace $F(X) \subseteq L(K) \cap Y$ with $F(L(K)) \subseteq L(K) \cap Y$ provided we assume $F \in KKM(E,Y)$ and $G \in DKT(Y,E)$ (respectively, $G \in HLPY(Y,E)$) is a compact map (here K is a compact subset of E with $G(Y) \subseteq K$). Also we need to replace X is an admissible subset of E with $L(K)$ is an admissible subset of E.

To see this notice from Remark 2.17 (respectively, Remark 2.19) that $G \in DKT(Y \cap L(K), L(K))$ (respectively, $G \in HLPY(Y \cap L(K), L(K))$), so there exists a selection $g \in C(Y \cap L(K), L(K))$ of G. Also since $F(L(K)) \subseteq L(K) \cap Y$ then as in Section 1 (see (i) and (ii)) we have $F \in KKM(L(K), L(K) \cap Y)$. Now Theorem 1.8 (note $Y \cap L(K)$ is normal and $L(K)$ is an admissible convex subset of E) guarantees that $F\,g \in KKM(Y \cap L(K), Y \cap L(K))$ and $F\,g$ is a upper semicontinuous compact map with compact values. Now apply Theorem 1.5.

In all the results so far we choose to obtain coincidences between different classes of maps. When the classes of maps are the same we present the following two results.

Theorem 2.29. *Let X be a subset in a Hausdorff topological space E and Y a subset in a Hausdorff topological space. Suppose $F \in PK(X,Y)$ and $G \in PK(Y,X)$ is a compact map. Also assume one of the following hold:*

(a). $X \in ES(compact)$,

(b). E is a topological vector space, X is a Schauder admissible subset of E and $G\,F : X \to 2^X$ is upper semicontinuous with closed values.

Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.

Proof. Note $G\,F \in PK(X,X)$ is a compact map. Now apply Theorem 1.1 or Theorem 1.2. $\square$

Remark 2.30. (i). If E is a Hausdorff topological vector space and X is convex then in Theorem 2.29 (a) (respectively, Theorem 2.29 (b)) we could replace $X \in ES(\text{compact})$ (respectively, X is a Schauder admissible subset of E and $G\,F : X \to 2^X$ is upper semicontinuous with closed values) with $D = co\,(\overline{G(Y)}) \in ES(\text{compact})$ (respectively, D is a Schauder admissible subset of E and $G\,F : D \to 2^D$ is upper semicontinuous with closed values). To see this we just need to note that $F \in PK(D,Y)$ and $G \in PK(Y,D)$.

(ii). Of course one could have other variations of (a) and (b) in Theorem 2.29 if one uses other results in [12], [13].

(iii). Note in Theorem 2.29 we could replace "$G \in PK(Y,X)$ is a compact map" with "$F \in PK(Y,X)$ is a compact map and $G \in PK(Y,X)$ is a upper semicontinuous map with compact values".

Theorem 2.31. *Let X be a convex set in a Hausdorff topological space E and Y a convex set in a Hausdorff topological space. Suppose $F \in HLPY(X,Y)$ is a compact map and $G \in HLPY(Y,X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Proof. There exists a compact set K of Y with $F(X) \subseteq K$. Now since $G \in HLPY(Y,X)$ then as in Theorem 2.3 we have $G \in HLPY(K,X)$, so there exists a selection $g \in C(K,X)$ of G and a finite subset A of X with $g(K) \subseteq co\,(A)$. Note $F \in HLPY(X,Y)$ so a similar argument as in Theorem 2.3 guarantees that $F \in HLPY(co\,(A),Y)$. Now since $co\,(A)$ is compact then there exists a selection $f \in C(co\,(A),Y)$ of F. Note $f(co\,(A)) \subseteq F(co\,(A)) \subseteq F(X) \subseteq K$ so $f \in C(co\,(A),K)$ and as a result we have that $g\,f \in C(co\,(A), co\,(A))$. Now since $co\,(A)$ is a compact convex subset in a finite dimensional subspace of E then Brouwer's fixed point theorem guarantees a $x \in co\,(A)$ with $x = g\,(f(x))$. $\square$

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Metric conditions, graphic contractions and weakly Picard operators

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Abstract. In the paper of S. Park (*Almost all about Rus-Hicks-Rhoades maps in quasi-metric spaces*, Adv. Theory Nonlinear Anal. Appl., **7**(2023), No. 2, 455-472), the author solves the following problem: *Which metric conditions imposed on f imply that f is a graphic contraction?* In this paper we study the following problem: *Which metric conditions imposed on f imply that f satisfies the conditions of Rus saturated principle of graphic contractions?*

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Keywords: Metric space, generalized metric space, contraction type mapping, metric condition, graphic contraction, successive approximation, Picard mapping, pre-weakly Picard mapping, weakly Picard mapping, interpolative Hardy-Rogers mapping, well-posedness of fixed point problem, Ulam-Hyers stability, Ostrowski property.

1. Introduction and preliminaries

Let (X, d) be an L-space and $f : X \to X$ be a mapping. By following [17], [16] and [15], we present the following notions and notations, which will be used in the sequel of this paper.

By definition, f is a pre-weakly Picard mapping ($PWPM$) if the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$. If f is $PWPM$, then we consider the mapping, $f^\infty : X \to X$ defined by, $f^\infty(x) := \lim_{n\to\infty} f^n(x)$, for all $x \in X$.

If f is a $PWPM$ and $f^\infty(x) \in F_f$, for any $x \in X$, then by definition, f is a weakly Picard mapping (WPM). Each WPM generates a partition of X. Let $x^* \in F_f$

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and $X_{x^*} := \{x \in X \mid f^n(x) \to x^* \text{ as } n \to \infty\}$. Then, $X = \bigcup_{x^* \in F_f} X_{x^*}$ is a partition of X. In this case, we have that: $f(X_{x^*}) \subset X_{x^*}$ and $X_{x^*} \cap F_f = \{x^*\}$, for all $x^* \in F_f$.

If f is WPM and $F_f = \{x^*\}$, then by definition, f is Picard mapping (PM).

The following result was given by I.A. Rus in [15].

Theorem 1.1 (Saturated principle of graphic contractions). *Let (X,d) be a complete metric space and $f : X \to X$ be a graphic l-contraction, i.e., $0 < l < 1$ and $d(f^2(x), f(x)) \leq l d(x, f(x))$, for all $x \in X$. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for all $x \in X$, then:

(ii) $F_f = F_{f^n} \neq \varnothing$, for all $n \in \mathbb{N}^*$.
(iii) f is a WPM.
(iv) $d(x, f^\infty(x)) \leq \frac{1}{1-l} d(x, f(x))$, for all $x \in X$.
(v) The fixed point equation corresponding to f is Ulam-Hyers stable.
(vi) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.
(vii) If in addition, $l < \frac{1}{3}$, then, $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$ $\Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.

Notice that if $cardF_f \leq 1$, then Theorem 1.1 takes the following form:

Theorem 1.2. *Let (X,d) be a complete metric space and $f : X \to X$ be a graphic l-contraction. We assume that $cardF_f \leq 1$. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for all $x \in X$, then:

(ii) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$.
(iii) f is a PM.
(iv) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, for all $x \in X$.
(v) The fixed point equation corresponding to f is Ulam-Hyers stable.
(vi) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.
(vii) If in addition, $l < \frac{1}{3}$, then $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.

On the other hand, in the metric fixed point theory, there is a large number of metric conditions (see [12], [9], [2], [18], [14], [13], [1], [4], [20], ...).

In the paper [10], S. Park solves the following problem: *Which metric conditions imposed on f imply that f is a graphic contraction?*

In this paper we study the following problem: *Which metric conditions imposed on f imply that f satisfies the conditions of Rus saturated principle of graphic contractions?*

Throughout this paper we follow the notation and terminology used in [15], [17], [3] and [19].

2. Conditions with respect to a standard metric

Let (X, d) be a metric space and $f : X \to X$ be a mapping. In many fixed point results, there are several standard metric conditions imposed on f with respect to d, which imply that f is a graphic contraction. For example, we have the Banach, Kannan, Ćirić-Reich-Rus, Ćirić, Berinde, Zamfirescu's metric conditions. More of them can be found in the paper of S. Park [10].

In this section we will focus on some other interesting metric conditions, implying the graphic contraction property of the mapping f.

- *Hardy-Rogers' metric condition* (see [6] and also [13]).
  f is called *HR mapping* if there exist three constants $a, b, c \in \mathbb{R}_+$, with $a + 2b + 2c \in (0, 1)$, such that

$$d(f(x), f(y)) \leq ad(x, y) + b[d(x, f(x)) + d(y, f(y))]+ \\ + c[d(x, f(y)) + d(y, f(x))], \text{ for all } x, y \in X.$$

- *Khojasteh, Abbas and Costache's metric condition* (see [8]).
  f is called *KAC mapping* if

$$d(f(x), f(y)) \leq \frac{d(y, f(x)) + d(x, f(y))}{d(x, f(x)) + d(y, f(y)) + 1} d(x, y), \text{ for all } x, y \in X.$$

- *Interpolative Kannan's metric condition* (see [7]).
  f is called *IK mapping* if there exist two constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda[d(x, f(x))]^{\alpha} \cdot [d(y, f(y))]^{1-\alpha}, \text{ for all } x, y \in X \setminus F_f.$$

2.1. The case of HR mappings

Lemma 2.1. *Let (X, d) be a metric space. Let $f : X \to X$ be a HR mapping. Then f is a graphic l_{HR}-contraction, i.e.,*

$$d(f(x), f^2(x)) \leq l_{HR} \cdot d(x, f(x)), \text{ for all } x \in X,$$

where the constant $l_{HR} = \frac{a+b+c}{1-b-c}$, with $a, b, c \in \mathbb{R}_+$ and $a + 2b + 2c \in (0, 1)$.

Proof. The conclusion follows by replacing y with $f(x)$ in the Hardy-Rogers' metric condition. □

Lemma 2.2. *Let (X, d) be a complete metric space. Let $f : X \to X$ be a HR mapping. Then $f(f^n(x)) \to f(f^{\infty}(x))$ as $n \to \infty$, for all $x \in X$.*

Proof. By replacing x with $f^n(x)$ and y with $f^\infty(x)$ in the Hardy-Rogers' metric condition, we get that

$$\begin{aligned} d(f(f^n(x)), f(f^\infty(x))) \leq ad(f^n(x), f^\infty(x)) +& \\ + b[d(f^n(x), f(f^n(x))) + d(f^\infty(x), f(f^\infty(x)))] +& \\ + c[d(f^n(x), f(f^\infty(x))) + d(f^\infty(x), f(f^n(x)))].& \end{aligned}$$

Next, by using the triangle inequality satisfied by the metric d we get

$$\begin{aligned} d(f(f^n(x)), f(f^\infty(x))) \leq ad(f^n(x), f^\infty(x)) +& \\ + b[d(f^n(x), f(f^n(x))) + d(f^\infty(x), f(f^n(x))) + d(f(f^n(x)), f(f^\infty(x)))] +& \\ + c[d(f^n(x), f(f^n(x))) + d(f(f^n(x)), f(f^\infty(x))) + d(f^\infty(x), f(f^n(x)))].& \end{aligned}$$

By letting $n \to \infty$ in the above inequality and taking into account the continuity of the metric d and the fact that the operator f is a graphic l_{HR}-contraction via Lemma 2.1, we get that $d(f^n(x), f^\infty(x)) \to 0$ as $n \to \infty$ and $d(f^n(x), f(f^n(x))) \to 0$ as $n \to \infty$. It follows that $f(f^n(x)) \to f(f^\infty(x))$ as $n \to \infty$, for all $x \in X$. □

In the paper [6], G.E. Hardy and T.D. Rogers showed that any HR mapping is a PM. In the following theorem, we give a simple proof of this result and several other conclusions concerning HR mappings.

Theorem 2.3 (Saturated principle of HR mappings). *Let (X, d) be a complete metric space and $f : X \to X$ be a HR mapping. Then we have that:*

(i) $F_f = F_{f^n} = \{x^*\}$, *for all* $n \in \mathbb{N}^*$.
(ii) f *is a* PM.
(iii) $d(x, x^*) \leq \frac{1}{1-l_{HR}} d(x, f(x))$, *for all* $x \in X$.
(iv) The fixed point equation corresponding to f *is Ulam-Hyers stable.*
(v) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well-posed.*
(vi) If in addition, $l_{HR} < \frac{1}{3}$, *then* $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e.,* f *has the Ostrowski property.*

Proof. From Lemma 2.1, f is a graphic l_{HR}-contraction.

From Lemma 2.2, it follows that $f^{n+1}(x) \to f(f^\infty(x))$ as $n \to \infty$. But $f^{n+1}(x) \to f^\infty(x)$ as $n \to \infty$. So, $f^\infty(x) \in F_f$. Hence, $F_f \neq \varnothing$.

Let $x^*, y^* \in F_f$ with $x^* \neq y^*$. By replacing x with x^* and y with y^* in the Hardy-Rogers' metric condition, we get $(1 - a - 2c)d(x^*, y^*) \leq 0$, which implies that $x^* = y^*$. So, $cardF_f = 1$. We apply next Theorem 1.2. □

2.2. The case of KAC mappings

Lemma 2.4. *Let (X, d) be a bounded metric space. Let $f : X \to X$ be a KAC mapping. Then f is a graphic l_{KAC}-contraction, i.e.,*

$$d(f(x), f^2(x)) \leq l_{KAC} \cdot d(x, f(x)), \text{ for all } x \in X,$$

where the constant $l_{KAC} = \frac{2\delta(X)}{2\delta(X)+1}$ *and* $\delta(X)$ *is the diameter functional of the space* X.

Proof. By taking $y = f(x)$ in the Khojasteh, Abbas and Costache's metric condition, we obtain the following estimation:

$$d(f(x), f^2(x)) \leq \frac{d(x, f^2(x))}{d(x, f(x)) + d(f(x), f^2(x)) + 1} d(x, f(x)), \text{ for all } x \in X.$$

Let us notice that the number $\frac{d(x,f^2(x))}{d(x,f(x))+d(f(x),f^2(x))+1}$ is not a constant, since it depends on $x \in X$. However, we can find an upper bound for it, by considering the diameter functional of the space X,

$$\delta(X) := \sup\{d(x, y) \mid x, y \in X\}.$$

Let us consider the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $\psi(x) := \frac{x}{x+1}$, for all $x \in \mathbb{R}_+$. By calculating its first derivative, we conclude that the function ψ is increasing on $\mathbb{R}_+$. We have the following estimations:

$$\frac{d(x, f^2(x))}{d(x, f(x)) + d(f(x), f^2(x)) + 1} \leq \frac{d(x, f(x)) + d(f(x), f^2(x))}{d(x, f(x)) + d(f(x), f^2(x)) + 1} =$$

$$= \psi(d(x, f(x)) + d(f(x), f^2(x))) \leq \psi(2\delta(X)) = \frac{2\delta(X)}{2\delta(X) + 1}, \text{ for all } x \in X.$$

Hence, for $l_{KAC} = \frac{2\delta(X)}{2\delta(X)+1}$ we have that $d(f(x), f^2(x)) \leq l_{KAC} \cdot d(x, f(x))$, for all $x \in X$. □

Lemma 2.5. *Let (X, d) be a bounded complete metric space and $f : X \to X$ be a KAC mapping. Then, $f(f^n(x)) \to f(f^\infty(x))$ as $n \to \infty$, for all $x \in X$ and $f^\infty(x) \in F_f$, for all $x \in X$.*

Proof. We have that $d(f(f^n(x)), f(f^\infty(x))) = d(f^{n+1}(x), f(f^\infty(x))) \leq$

$$\leq \frac{d(f^\infty(x), f^{n+1}(x)) + d(f^n(x), f(f^\infty(x)))}{d(f^n(x), f^{n+1}(x)) + d(f^\infty(x), f(f^\infty(x))) + 1} d(f^n(x), f^\infty(x)).$$

By letting $n \to \infty$, it follows that,

$$d(f^\infty(x), f(f^\infty(x))) \leq \frac{d(f^\infty(x), f(f^\infty(x)))}{d(f^\infty(x), f(f^\infty(x))) + 1} \cdot 0 = 0.$$

So, $f(f^n(x)) \to f(f^\infty(x))$ as $n \to \infty$ and $f^\infty(x) \in F_f$, for all $x \in X$. □

Theorem 2.6 (Saturated principle of KAC mappings). *Let (X, d) be a complete bounded metric space and $f : X \to X$ be a KAC mapping. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all* $x \in X$ *and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, *for all* $x \in X$.

(ii) $F_f = F_{f^n} \neq \varnothing$, *for all* $n \in \mathbb{N}^*$.

(iii) f *is a* WPM.

(iv) $d(x, f^\infty(x)) \leq \frac{1}{1-l_{KAC}} d(x, f(x))$, *for all* $x \in X$.

(v) The fixed point equation corresponding to f *is Ulam-Hyers stable.*

(vi) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well-posed.*

(*vii*) *If in addition,* $l_{KAC} < \frac{1}{3}$*, then,* $x^* \in F_f$*,* $y_n \in X_{x^*}$*,* $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$*, i.e.,* f *has the Ostrowski property.*

Proof. The conclusions follow from the saturated principle of graphic contractions. □

2.3. The case of IK mappings

Lemma 2.7. *Let* (X, d) *be a metric space. Let* $f : X \to X$ *be an* IK *mapping. Then* f *is a graphic* l_{IK}*-contraction, i.e.,*

$$d(f(x), f^2(x)) \leq l_{IK} \cdot d(x, f(x)), \text{ for all } x \in X,$$

where $l_{IK} = \lambda^{\frac{1}{\alpha}}$*, with* $\lambda \in [0, 1)$ *and* $\alpha \in (0, 1)$*.*

Proof. By replacing y with $f(x)$ in the interpolative Kannan's metric condition, we get the conclusion. □

Lemma 2.8. *Let* (X, d) *be a metric space. Let* $f : X \to X$ *be an* IK *mapping. Then* $f(f^n(x)) \to f(f^\infty(x))$ *as* $n \to \infty$*, for all* $x \in X$*.*

Proof. If $f^\infty(x) \notin F_f$ then, by replacing x with $f^n(x)$ and y with $f^\infty(x)$ in the interpolative Kannan's metric condition, we have

$$d(f(f^n(x)), f(f^\infty(x))) \leq \lambda[d(f^n(x), f^{n+1}(x))]^\alpha [d(f^\infty(x), f(f^\infty(x)))]^{1-\alpha},$$

for all $x \in X$. By letting $n \to \infty$ and taking into account the Lemma 2.7, $d(f^n(x), f^{n+1}(x)) \to 0$ as $n \to \infty$, for all $x \in X$. The conclusion follows. □

Theorem 2.9 (Saturated principle of IK mappings). *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an* IK *mapping. Then we have that:*

(*i*) $\{f^n(x)\}_{n \in \mathbb{N}}$ *converges for all* $x \in X$ *and* $\sum_{n \in \mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$*, for all* $x \in X$*.*
(*ii*) $F_f = F_{f^n} \neq \varnothing$*, for all* $n \in \mathbb{N}^*$*.*
(*iii*) f *is a* WPM*.*
(*iv*) $d(x, f^\infty(x)) \leq \frac{1}{1 - l_{IK}} d(x, f(x))$*, for all* $x \in X$*.*
(*v*) *The fixed point equation corresponding to* f *is Ulam-Hyers stable.*
(*vi*) $x^* \in F_f$*,* $y_n \in X_{x^*}$*,* $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$*, i.e., the fixed point problem for* f *is well-posed.*
(*vii*) *If in addition,* $l_{IK} < \frac{1}{3}$*, then,* $x^* \in F_f$*,* $y_n \in X_{x^*}$*,* $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$*, i.e.,* f *has the Ostrowski property.*

Proof. The conclusions follow from Lemmas 2.7, 2.8 and Theorem 1.1. □

In the case when $cardF_f \leq 1$, Theorem 2.9 takes the following form:

Theorem 2.10. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an* IK *mapping. We assume that* $cardF_f \leq 1$*. Then we have that:*

(*i*) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all* $x \in X$ *and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, *for all* $x \in X$.
(*ii*) $F_f = F_{f^n} = \{x^*\}$, *for all* $n \in \mathbb{N}^*$.
(*iii*) f *is a* PM.
(*iv*) $d(x, x^*) \leq \frac{1}{1-l_{IK}} d(x, f(x))$, *for all* $x \in X$.
(*v*) *The fixed point equation corresponding to* f *is Ulam-Hyers stable.*
(*vi*) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well-posed.*
(*vii*) *If in addition,* $l_{IK} < \frac{1}{3}$, *then,* $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e.,* f *has the Ostrowski property.*

Proof. The conclusions follow from Lemmas 2.7, 2.8 and Theorem 1.2. □

Remark 2.11. Not any metric condition yields a graphic contraction. For instance, if we consider a metric space (X, d) and the mapping $f : X \to X$ with the property that there exist two constants $\theta \in [0, 1)$ and $L \geq 0$ such that

$$d(f(x), f(y)) \leq \theta d(x, y) + L[d(x, f(x)) + d(y, f(y))],$$

for all $x, y \in X$, then f is not a graphic contraction.

Indeed, by choosing $y := f(x)$ we get $d(f(x), f^2(x)) \leq \frac{\theta+L}{1-L} d(x, f(x))$, for all $x \in X$. By taking $L = \frac{1}{2}$, the condition $\frac{\theta+L}{1-L} < 1$ implies $\theta < 0$, which is a contradiction with $\theta \in [0, 1)$.

3. Conditions with respect to a dislocated metric

Let us recall first the notion of dislocated metric.

Definition 3.1. Let X be a nonempty set. A functional $d : X \times X \to \mathbb{R}_+$ is called dislocated metric on X if the following conditions hold:
(*i*) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$.
(*ii*) $d(x, y) = d(y, x)$, for all $x, y \in X$.
(*iii*) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

If X is a nonempty set and $d : X \times X \to \mathbb{R}_+$ is a dislocated metric on X, then the couple (X, d) is called dislocated metric space.

In the above setting, we have the following results.

Theorem 3.2 (Saturated principle of graphic contraction). *Let* (X, d) *be a complete dislocated metric space and* $f : X \to X$ *be a graphic* l*-contraction, i.e.,* $0 < l < 1$ *and* $d(f^2(x), f(x)) \leq l d(x, f(x))$, *for all* $x \in X$. *Then we have that:*
(*i*) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all* $x \in X$ *and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, *for all* $x \in X$.

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, *for all* $x \in X$, *then:*
(*ii*) $F_f = F_{f^n} \neq \varnothing$, *for all* $n \in \mathbb{N}^*$.

(*iii*) *f is a WPM.*
(*iv*) *$d(x, f^\infty(x)) \leq \frac{1}{1-l} d(x, f(x))$, for all $x \in X$.*
(*v*) *The fixed point equation corresponding to f is Ulam-Hyers stable.*
(*vi*) *$x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.*
(*vii*) *If in addition, $l < \frac{1}{3}$, then, $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$ $\Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.*

Proof. (*i*). Let $x \in X$. We construct the sequence of successive approximations, $\{f^n(x)\}_{n\in\mathbb{N}}$, for f starting from x.

Since f is a graphic l-contraction, we have the following estimations:

$$\begin{aligned} d(f(x), f^2(x)) &\leq l d(x, f(x)), \\ d(f^2(x), f^3(x)) &\leq l d(f(x), f^2(x)) \leq l^2 d(x, f(x)), \\ &\vdots \\ d(f^n(x), f^{n+1}(x)) &\leq l d(f^{n-1}(x), f^n(x)) \leq \ldots \leq l^n d(x, f(x)), \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. By summing up the left hand side of the above inequalities, we have

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) \leq \sum_{n\in\mathbb{N}} l^n d(x, f(x)) = \frac{1}{1-l} d(x, f(x)) < +\infty.$$

It follows that $\{f^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d) and, since (X, d) is complete, we get that $\{f^n(x)\}_{n\in\mathbb{N}}$ is convergent in (X, d).

(*ii*) + (*iii*). Since $\{f^n(x)\}_{n\in\mathbb{N}}$ is convergent in (X, d), there exists $f^\infty(x) = \lim\limits_{n\to\infty} f^n(x) \in X$. By using the assumption $\lim\limits_{n\to\infty} f(f^n(x)) = f(\lim\limits_{n\to\infty} f^n(x))$, for all $x \in X$, it follows that $f^\infty(x) \in F_f$ and also that $f^\infty(x) \in F_{f^n}$. So (*ii*) holds. By definition, f is a WPM. So, (*iii*) also holds.

(*iv*). Let $x \in X$. Since f is a graphic l-contraction, we have

$$\begin{aligned} d(x, f^\infty(x)) &\leq \sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) + d(f^{n+1}(x), f^\infty(x)) \\ &\leq \sum_{k=0}^{n} l^k d(x, f(x)) + d(f^{n+1}(x), f^\infty(x)). \end{aligned}$$

By letting $n \to \infty$, the conclusion follows.

(*v*). We recall that the fixed point equation $x = f(x)$, $x \in X$ is Ulam-Hyers stable if there exists a constant $c > 0$ such that for any $\varepsilon > 0$ and any ε-solution z of the fixed point equation, i.e., $d(z, f(z)) \leq \varepsilon$, there exists $x^* \in F_f$ such that $d(z, x^*) \leq c\varepsilon$.

Let $\varepsilon > 0$ and let z be the ε-solution of the fixed point equation $x = f(x)$, for all $x \in X$. Since $f^\infty(x) \in F_f$, by using the inequality (*iv*), we have

$$d(z, f^\infty(x)) \leq \frac{1}{1-l} d(z, f(z)) \leq \frac{1}{1-l}\varepsilon$$

So, there exists $c := \frac{1}{1-l} > 0$ such that $d(z, f^\infty(x)) \le c\varepsilon$.

(vi). Let $x^* \in F_f$ and $y_n \in X_x^* := \{x \in X \mid f^n(x) \to x^*, \text{ as } n \to \infty\}$, such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. We have the following estimations

$$\begin{aligned} d(y_n, x^*) &\le \sum_{k=0}^{n-1} d(f^k(y_n), f^{k+1}(y_n)) + d(f^n(y_n), x^*) \\ &\le \sum_{k=0}^{n-1} l^k d(y_n, f(y_n)) + d(f^n(y_n), x^*). \end{aligned}$$

By letting $n \to \infty$, it follows that $d(y_n, x^*) \to 0$. So, (vi) holds.

(vii). First, we show that

$$d(f(x), f^\infty(x)) \le \frac{l}{1-2l} d(x, f^\infty(x)), \text{ for all } x \in X. \tag{3.1}$$

Indeed, for any $x \in X$, we have the following estimations

$$\begin{aligned} d(f(x), f^\infty(x)) &\le \sum_{k=0}^{\infty} d(f^k(x), f^{k+1}(x)) - d(x, f(x)) \\ &\le \sum_{k=0}^{\infty} l^k d(x, f(x)) - d(x, f(x)) = \left(\frac{1}{1-l} - 1\right) d(x, f(x)) \\ &\le \frac{l}{1-l} d(x, f^\infty(x)) + \frac{l}{1-l} d(f^\infty(x), f(x)). \end{aligned}$$

It follows that $\frac{1-2l}{1-l} d(f(x), f^\infty(x)) \le \frac{l}{1-l} d(x, f^\infty(x))$. Hence (3.1) holds. Notice also that the constant $\frac{l}{1-2l} < 1$ if and only if $l < \frac{1}{3}$.

Now, let $x^* \in F_f$ and $y_n \in X_x^* := \{x \in X \mid f^n(x) \to x^*, \text{ as } n \to \infty\}$, such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. By using (3.1), we have

$$\begin{aligned} d(y_{n+1}, x^*) &\le d(y_{n+1}, f(y_n)) + d(f(y_n), x^*) \\ &\le d(y_{n+1}, f(y_n)) + \frac{l}{1-2l} d(y_n, x^*) \\ &\le d(y_{n+1}, f(y_n)) + \frac{l}{1-2l} d(y_n, f(y_{n-1})) + \frac{l}{1-2l} d(f(y_{n-1}), x^*) \\ &\le d(y_{n+1}, f(y_n)) + \frac{l}{1-2l} d(y_n, f(y_{n-1})) + \left(\frac{l}{1-2l}\right)^2 d(y_{n-1}, x^*) \\ &\vdots \\ &\le d(y_{n+1}, f(y_n)) + \frac{l}{1-2l} d(y_n, f(y_{n-1})) + \\ &\quad + \ldots + \left(\frac{l}{1-2l}\right)^n d(y_1, f(y_0)) + \left(\frac{l}{1-2l}\right)^n d(f(y_0), x^*). \end{aligned}$$

By letting $n \to \infty$ and applying a Cauchy (or Toeplitz) lemma, we obtain $d(y_{n+1}, x^*) \to 0$. The conclusion follows. □

In the case when $cardF_f \leq 1$, Theorem 3.2 takes the following form:

Theorem 3.3. *Let (X, d) be a complete dislocated metric space and $f : X \to X$ be a graphic l-contraction. We assume that $cardF_f \leq 1$. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for all $x \in X$, then:

(ii) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$.

(iii) f is a PM.

(iv) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, for all $x \in X$.

(v) The fixed point equation corresponding to f is Ulam-Hyers stable.

(vi) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.

(vii) If in addition, $l < \frac{1}{3}$, then, $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.

Remark 3.4. Notice that the Theorem 2.3 for HR mappings and the Theorems 2.9 and 2.10 for IK mappings also hold in the context of a complete dislocated metric space. Theorem 2.6 for KAC mappings also holds in the context of a complete bounded dislocated metric space.

4. Conditions with respect to an $\mathbb{R}^m_+$-metric

In this section we follow the terminology and notations given in [5], concerning vector-valued metric ($\mathbb{R}^m_+$-metric) and matrices convergent to zero. Regarding the properties of these matrices, we recall the following result (see [5]).

Theorem 4.1. *Let $S \in \mathcal{M}_m(\mathbb{R}_+)$. The following assertions are equivalent:*

(1) *S is convergent to zero;*

(2) *$S^n \to O_m$ as $n \to \infty$;*

(3) *the spectral radius $\rho(S)$ is strictly less than 1;*

(4) *the matrix $(I_m - S)$ is nonsingular and*

$$(I_m - S)^{-1} = I_m + S + S^2 + \ldots + S^n + \ldots;$$

(5) *the matrix $(I_m - S)$ is nonsingular and $(I_m - S)^{-1}$ has nonnegative elements;*

(6) *$S^n x \to 0 \in \mathbb{R}^m$ as $n \to \infty$, for all $x \in \mathbb{R}^m$.*

The main result of this section is the following one.

Theorem 4.2 (Saturated principle of graphic contraction). *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space and $f : X \to X$ be a graphic S-contraction, i.e., there exists a matrix convergent to zero, $S \in \mathcal{M}_m(\mathbb{R}_+)$, such that $d(f^2(x), f(x)) \leq Sd(x, f(x))$, for all $x \in X$. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, *for all* $x \in X$, *then:*

(*ii*) $F_f = F_{f^n} \neq \varnothing$, *for all* $n \in \mathbb{N}^*$.
(*iii*) f *is a* WPM.
(*iv*) $d(x, f^\infty(x)) \leq (I_m - S)^{-1} d(x, f(x))$, *for all* $x \in X$.
(*v*) *The fixed point equation corresponding to* f *is Ulam-Hyers stable.*
(*vi*) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well-posed.*
(*vii*) *If in addition, the matrix* $[2I_m - (I_m - S)^{-1}]^{-1}[(I_m - S)^{-1} - I_m]$ *converges to zero, then,* $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e.,* f *has the Ostrowski property.*

Proof. $(i)+(ii)+(iii)+(iv)$. We follow the proof given for Theorem 3.2, by replacing the constant l with the matrix S. We also take into account the assertions (4) and (5) of Theorem 4.1.

(v). We say that the fixed point equation $x = f(x)$, $x \in X$ is Ulam-Hyers stable if there exists a matrix $C \in \mathcal{M}_m(\mathbb{R}_+)$ such that for any $\varepsilon \in \mathbb{R}^m_+$ and any ε-solution z of the fixed point equation, i.e., $d(z, f(z)) \leq \varepsilon$, there exists $x^* \in F_f$ such that $d(z, x^*) \leq C\varepsilon$.

Let $\varepsilon \in \mathbb{R}^m_+$ and let z be the ε-solution of the fixed point equation $x = f(x)$, for all $x \in X$. Since $f^\infty(x) \in F_f$, by using the inequality (iv), we have

$$d(z, f^\infty(x)) \leq (I_m - S)^{-1} d(z, f(z)) \leq (I_m - S)^{-1}\varepsilon$$

So, there exists $C := (I_m - S)^{-1} \in \mathcal{M}_m(\mathbb{R}_+)$ such that $d(z, f^\infty(x)) \leq C\varepsilon$.

(vi). Let $x^* \in F_f$ and $y_n \in X^*_x := \{x \in X \mid f^n(x) \to x^*, \text{ as } n \to \infty\}$, such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. We have the following estimations

$$\begin{aligned} d(y_n, x^*) &\leq \sum_{k=0}^{n-1} d(f^k(y_n), f^{k+1}(y_n)) + d(f^n(y_n), x^*) \\ &\leq \sum_{k=0}^{n-1} S^k d(y_n, f(y_n)) + d(f^n(y_n), x^*) \\ &\leq (I_m - S)^{-1} d(y_n, f(y_n)) + d(f^n(y_n), x^*). \end{aligned}$$

By letting $n \to \infty$, it follows that $d(y_n, x^*) \to 0$. So, (vi) holds.

(vii). First, we show that

$$d(f(x), f^\infty(x)) \leq \Lambda d(x, f^\infty(x)), \text{ for all } x \in X, \tag{4.1}$$

where $\Lambda := [2I_m - (I_m - S)^{-1}]^{-1}[(I_m - S)^{-1} - I_m]$.

Indeed, for any $x \in X$, we have the following estimations

$$\begin{aligned} d(f(x), f^\infty(x)) &\leq \sum_{k=0}^{\infty} d(f^k(x), f^{k+1}(x)) - d(x, f(x)) \\ &\leq \sum_{k=0}^{\infty} S^k d(x, f(x)) - d(x, f(x)) = [(I_m - S)^{-1} - I_m] d(x, f(x)) \\ &\leq [(I_m - S)^{-1} - I_m] d(x, f^\infty(x)) + [(I_m - S)^{-1} - I_m] d(f^\infty(x), f(x)). \end{aligned}$$

It follows that

$$[2I_m - (I_m - S)^{-1}]d(f(x), f^\infty(x)) \leq [(I_m - S)^{-1} - I_m]d(x, f^\infty(x)).$$

and hence, (4.1) holds.

Now, let $x^* \in F_f$ and $y_n \in X_x^* := \{x \in X \mid f^n(x) \to x^*, \text{ as } n \to \infty\}$, such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. By using (4.1) we have

$$\begin{aligned} d(y_{n+1}, x^*) &\leq d(y_{n+1}, f(y_n)) + d(f(y_n), x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \Lambda d(y_n, x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \Lambda d(y_n, f(y_{n-1})) + \Lambda d(f(y_{n-1}), x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \Lambda d(y_n, f(y_{n-1})) + \Lambda^2 d(y_{n-1}, x^*) \\ &\vdots \\ &\leq d(y_{n+1}, f(y_n)) + \Lambda d(y_n, f(y_{n-1})) + \\ &\quad + \ldots + \Lambda^n d(y_1, f(y_0)) + \Lambda^n d(f(y_0), x^*). \end{aligned}$$

By letting $n \to \infty$ and applying a Cauchy (or Toeplitz) lemma, we get that $d(y_{n+1}, x^*) \to 0$. The conclusion follows. □

In the case when $cardF_f \leq 1$, Theorem 4.2 takes the following form:

Theorem 4.3. *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space and $f : X \to X$ be a graphic S-contraction, i.e., there exists a matrix convergent to zero, $S \in \mathcal{M}_m(\mathbb{R}_+)$, such that $d(f^2(x), f(x)) \leq Sd(x, f(x))$, for all $x \in X$. We assume that $cardF_f \leq 1$. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for all $x \in X$, then:

(ii) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$.
(iii) f is a PM.
(iv) $d(x, x^*) \leq (I_m - S)^{-1}d(x, f(x))$, for all $x \in X$.
(v) The fixed point equation corresponding to f is Ulam-Hyers stable.
(vi) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.
(vii) If in addition, the matrix $[2I_m - (I_m - S)^{-1}]^{-1}[(I_m - S)^{-1} - I_m]$ converges to zero, then, $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.

We introduce next the notion of interpolative Kannan mapping defined on a $\mathbb{R}^m_+$-metric space.

Definition 4.4. Let (X, d) be a $\mathbb{R}^m_+$-metric space. A mapping $f : X \to X$ is called interpolative Kannan mapping (IK mapping) on X, if there exists a convergent to zero matrix, $\Lambda \in \mathcal{M}_m(\mathbb{R}_+)$, and a real constant $\alpha \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \Lambda[d(x, f(x))]^\alpha \cdot [d(y, f(y))]^{1-\alpha}, \text{ for all } x, y \in X \setminus F_f.$$

Lemma 4.5. *Let (X, d) be a $\mathbb{R}^m_+$-metric space. Let $f : X \to X$ be an IK mapping. Then f is a graphic L_{IK}-contraction, i.e.,*

$$d(f(x), f^2(x)) \leq L_{IK} \cdot d(x, f(x)),\ \textit{for all } x \in X \setminus F_f,$$

where $L_{IK} = \Lambda^{\frac{1}{\alpha}}$ is a matrix that converges to zero, having positive real values.

Proof. Let $x \in X \setminus F_f$. By replacing y with $f(x)$ in the interpolative Kannan's metric condition, the conclusion follows. □

Lemma 4.6. *Let (X, d) be a $\mathbb{R}^m_+$-metric space. Let $f : X \to X$ be an IK mapping. Then $f(f^n(x)) \to f(f^\infty(x))$ as $n \to \infty$, for all $x \in X$.*

Proof. If $f^\infty(x) \notin F_f$ then, by replacing x with $f^n(x)$ and y with $f^\infty(x)$ in the interpolative Kannan's metric condition, we have

$$d(f(f^n(x)), f(f^\infty(x))) \leq \Lambda[d(f^n(x), f^{n+1}(x))]^\alpha [d(f^\infty(x), f(f^\infty(x)))]^{1-\alpha},$$

for all $x \in X$. By letting $n \to \infty$ and taking into account the Lemma 4.5, $d(f^n(x), f^{n+1}(x)) \to 0 \in \mathbb{R}^m$ as $n \to \infty$, for all $x \in X$. The conclusion follows. □

Theorem 4.7 (Saturated principle of IK mappings). *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space and $f : X \to X$ be an IK mapping. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.
(ii) $F_f = F_{f^n} \neq \varnothing$, for all $n \in \mathbb{N}^*$.
(iii) f is a WPM.
(iv) $d(x, f^\infty(x)) \leq (I_m - L_{IK})^{-1} d(x, f(x))$, for all $x \in X$.
(v) The fixed point equation corresponding to f is Ulam-Hyers stable.
(vi) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.
(vii) If in addition, the matrix $[2I_m - (I_m - L_{IK})^{-1}]^{-1}[(I_m - L_{IK})^{-1} - I_m]$ converges to zero, then, $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.

Proof. The conclusions follow from the Lemmas 4.5, 4.6 and Theorem 4.2. □

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