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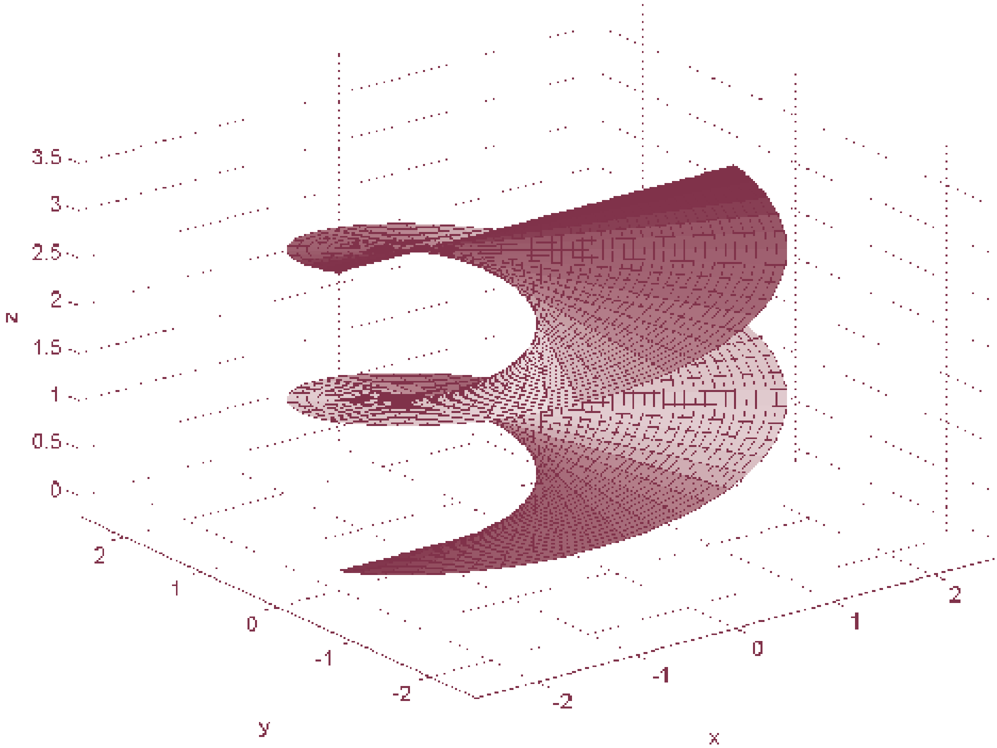

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Holomorphic vector field with one zero on the Grassmannian and cohomology

Zsolt Szilágyi

Abstract. We consider a holomorphic vector field on the complex Grassmannian constructed from a nilpotent matrix. We show that this vector field vanishes only at a single point. Using the Baum-Bott localization theorem we give a Grothendieck residue formula for the intersection numbers of the Grassmannian. Knowing that Chern classes of the tautological bundle generate the cohomology ring of the Grassmannian we can compute the ideal of relations explicitly from the residue formula. This shows that the cohomology ring of the Grassmannian is determined by holomorphic vector field around its only zero.

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Keywords: Cohomology ring, residues, localization, holomorphic vector field.

1. Introduction

In this article we show that the Baum-Bott localization formula [2, Theorem 1] using holomorphic vector fields can be used to compute the cohomology ring and the intersection numbers of the complex Grassmannian. Moreover, the holomorphic vector field can be chosen such that it has only a zero point, hence the cohomology ring of the Grassmannian is determined by this vector field near its single zero. From the residue formula of Baum-Bott for Chern numbers we are able to deduce the relations between the generators of the cohomology ring of the Grassmannian, hence to compute its cohomology ring.

The structure of the article is as follows. In Section 2 we recall the definition and properties of the Grothendieck residue which we use in the sequel. In Section 3 we recall the Baum-Bott localization theorem for holomorphic vector fields (Theorem

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3.1). In Section 4 we construct the holomorphic vector field on the Grassmannian from the action of a nilpotent matrix of form (4.1). We show that this vector field has only a zero point (Proposition 4.1). The vector field of same type of matrix on $\mathbb{CP}^{n-1}$ was considered in [5] and [4, §7] to demonstrate how vector fields with isolated zeroes can be used in computing cohomology rings. In [1] this construction of holomorphic vector fields with single zero is generalized to G/P, where G is a connected linear group algebraic group defined over a an algebraically closed field of characteristic zero and P is a parabolic subgroup. We note that the Grassmannian $Gr_k(n, \mathbb{C})$ can be viewed as homogeneous space. Then we express this vector field in local coordinates around the zero point (cf. (4.8)). In Section 5 we write up a residue formula for the Chern numbers of the tangent bundle of the Grassmannian $Gr_k(n, \mathbb{C})$ by the Baum-Bott theorem and we simplify it by eliminating all but n variables (Theorem 5.6). In Section 6 we recall the properties of Chern classes and the cohomology ring of the complex Grassmannian in terms of Chern classes of the tautological and the quotient bundle. In Lemma 6.3 we show that when $n \neq 2k$ then the Chern classes of the tangent bundle also generate the cohomology ring. In Theorem 6.4 and Corollary 6.5 we reinterpret the results of Theorem 5.6 to give the final version of the residue formula in terms of Chern classes of the tautological and quotient bundle when $n \neq 2k$. Using the similarities between Local Duality property (P2) of the Grothendieck residue and Poincaré duality we can easily calculate the relations between the generators of the cohomology ring (see Subsection 6.2.1).

Finally, in Section 7 we show the connection between the Grothendieck residue formula (6.5) and the Jeffrey-Kirwan residue formula for the Grassmannian constructed as symplectic quotient (cf. (7.1) and [7, Proposition 7.2]).

2. The Grothendieck residue

In this section we recall the definition of the Grothendieck residue and its properties used in the sequel. Let $h, f_1, \dots, f_r$ be holomorphic functions in an open neighborhood $\mathcal{U} \subset \mathbb{C}^r$ of a point p. Suppose that $p \in \mathcal{U}$ is the only common zero of $f_1, \dots, f_r$ in $\mathcal{U}$.

Definition 2.1. The *Grothendieck residue* is defined as

$$\operatorname{Res}_p \left(\frac{h\, dz_1 \dots dz_r}{f_1| \dots |f_r} \right) = \frac{1}{(2\pi\sqrt{-1})^r} \int_{\Gamma(\varepsilon)} \frac{h(z) dz_1 \dots dz_r}{f_1(z) \dots f_r(z)}, \tag{2.1}$$

where $\Gamma(\varepsilon) = \{z \in \mathcal{U} : |f_i(z)| = \varepsilon_i,\ i = 1, \dots, r\}$ for a small regular value

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \mathbb{R}^r_{>0}$$

of $(|f_1|, \dots, |f_r|) : \mathcal{U} \to \mathbb{R}^r$ and the torus $\Gamma(\varepsilon)$ is oriented according to the differential form $d(\arg f_1(z)) \dots d(\arg f_r(z))$.

We note that the Grothendieck residue does not depend on the choice of the small regular value ε. We list some properties of the Grothendieck residue, which we will use in the sequel.

(P1) The functions $f_1,\dots,f_r$ in the denominator of the residue anticommute. That is, if σ is a permutation of indices then

$$\mathrm{Res}_p\left(\frac{h\,dz_1\dots dz_r}{f_1|\dots|f_r}\right)=\mathrm{sgn}(\sigma)\cdot\mathrm{Res}_p\left(\frac{h\,dz_1\dots dz_r}{f_{\sigma(1)}|\dots|f_{\sigma(r)}}\right),$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ. This sign is the result of changing the orientation of the cycle $\Gamma(\varepsilon)$ on which we integrate.

(P2) (Local Duality) We have $\mathrm{Res}_p\left(\frac{hg\,dz_1\dots dz_r}{f_1|\dots|f_r}\right)=0$ for any local holomorphic germ $g\in\mathcal{O}_p$ if and only if h belongs to the ideal $\langle f_1,\dots,f_r\rangle_p\subset\mathcal{O}_p$ (cf. [6, p.659]). Assume that $f_1,\dots,f_r\in\mathbb{C}[z_1,\dots,z_r]$ are polynomials and $\{p\}=\{z\in\mathbb{C}^r\,|\,f_1(z)=\dots=f_r(z)=0\}$ is their only common zero in $\mathbb{C}^r$. We note that this is the cases for isolated zero $p=0\in\mathbb{C}^r$ when $f_1,\dots,f_r$ are graded homogeneous polynomials, i.e. there are degrees $d_i,\delta_i\in\mathbb{Z}_{>0}$, for $i=1,\dots,r$ such that $f_i(t^{d_1}z_1,\dots,t^{d_r}z_r)=t^{\delta_i}f_i(z_1,\dots,z_r)$ for all $i=1,\dots,r$. Then we have the following version of the Local Duality. $\mathrm{Res}_p\left(\frac{hg\,dz_1\dots dz_r}{f_1|\dots|f_r}\right)=0$ for any polynomial $g\in\mathbb{C}[z_1,\dots,z_r]$ if and only if h belongs to the ideal $\langle f_1,\dots,f_r\rangle\subset\mathbb{C}[z_1,\dots,z_r]$ (cf. [9, p.44]).

(P3) When $f=(f_1,\dots,f_r)$ is nondegenerate at p, i.e. the Jacobian determinant $J_f(p)=\det\left(\frac{\partial f_i}{\partial z_j}(p)\right)_{i,j=1}^{r}\neq 0$ then $\mathrm{Res}_p\left(\frac{h\,dz_1\dots dz_r}{f_1|\dots|f_r}\right)=\frac{h(p)}{J_f(p)}$, (cf. [6, p.650]).

(P4) (Transformation Law) Let $A=(A_{ij})_{i,j=1}^r$ be an r-by-r matrix with holomorphic coefficients $A_{ij}\in\mathcal{O}(\mathcal{U})$ such that p is the only locally common zero of $g_i=\sum_{j=1}^r A_{ij}f_j$, $i=1,\dots,r$. Then

$$\mathrm{Res}_p\left(\frac{h\,dz_1\dots dz_r}{f_1|\dots|f_r}\right)=\mathrm{Res}_p\left(\frac{h\det(A)\,dz_1\dots dz_r}{g_1|\dots|g_r}\right),$$

(cf. [6, p.657]). In the case of $p=0\in\mathbb{C}^r$ and $g_i=z_i^{\mu_i}$, $i=1,\dots,r$ with $\mu_i\geq 1$ the above residue becomes an iterated residue and can be evaluated by expanding $h\det(A)$ into power series in variables $z_1,\dots,z_r$ around $p=0\in\mathbb{C}^r$ and taking the coefficient of the term $z_1^{\mu_1-1}\dots z_r^{\mu_r-1}$.

(P5) Assume that $0\in\mathcal{U}$ is the only zero of polynomials $f_1,\dots,f_{r-1},f_r=z_r\in\mathbb{C}[z_1,\dots,z_r]$ in the open set $\mathcal{U}\subseteq\mathbb{C}^r$ and consider the inclusion $\iota_r:\mathbb{C}^{r-1}\to\mathbb{C}^r$, $\iota_r(z_1,\dots z_{r-1})=(z_1,\dots,z_{r-1},0)$. Then by division with remainder with respect to z_r we have $f_j=\varphi_j z_r+\rho_j$, where $\rho_j=\iota_r^*f_j=f_j\circ\iota_r\in\mathbb{C}[z_1,\dots z_{r-1}]$ non-zero for all $j=1,\dots,r-1$. If we set $\rho_r=z_r$, then $\rho_i=\sum_{j=1}^r A_{ij}f_j$ with $A_{ii}=1$, $1\leq i\leq r$, $A_{ir}=-\varphi_i$, $1\leq i\leq r-1$ and $A_{ij}=0$ otherwise. Thus, $\det(A)=1$ and by (P4), (P2) we have

$$\mathrm{Res}_0\left(\frac{h\,dz_1\dots dz_r}{f_1|\dots|f_{r-1}|z_r}\right)=\mathrm{Res}_0\left(\frac{h\,dz_1\dots dz_r}{\iota_r^*f_1|\dots|\iota_r^*f_{r-1}|z_r}\right)$$
$$=\mathrm{Res}_0\left(\frac{\iota_r^*h\,dz_1\dots dz_r}{\iota_r^*f_1|\dots|\iota_r^*f_{r-1}|z_r}\right)=\mathrm{Res}_0\left(\frac{\iota_r^*h\,dz_1\dots dz_{r-1}}{\iota_r^*f_1|\dots|\iota_r^*f_{r-1}}\right).$$

(P6) (Pull-back) If $\varphi = (\varphi_1, \ldots, \varphi_r) : \mathbb{C}^r \to \mathbb{C}^r$ is a finite map, generically m-fold cover then

$$\operatorname{Res}_p \left(\frac{h(z)\, dz_1 \ldots dz_r}{f_1(z)| \ldots |f_r(z)} \right) = \frac{1}{m} \sum_{\zeta \in \varphi^{-1}(p)} \operatorname{Res}_\zeta \left(\frac{h(\varphi(w)) J_\varphi(w)\, dw_1 \ldots dw_r}{f_1(\varphi(w))| \ldots |f_r(\varphi(w))} \right).$$

(P7) Assume that z_i has degree δ_i and $f_i \in \mathbb{C}[z_1, \ldots, z_r]$ are graded homogeneous polynomials of degree d_i, for $i = 1, \ldots, r$. Let h be also homogeneous polynomial of degree d. Rescaling $z \mapsto \lambda z = (\lambda^{\delta_1} z_1, \ldots, \lambda^{\delta_r} z_r)$ yields

$$\begin{aligned}\operatorname{Res}_0 \left(\frac{h(z)\, dz_1 \ldots dz_r}{f_1(z)| \ldots |f_r(z)} \right) &= \operatorname{Res}_0 \left(\frac{h(\lambda z) d(\lambda^{\delta_1} z_1) \ldots d(\lambda^{\delta_r} z_r)}{f_1(\lambda z)| \ldots |f_r(\lambda z)} \right) \\ &= \lambda^D \operatorname{Res}_0 \left(\frac{h(z)\, dz_1 \ldots dz_r}{f_1(z)| \ldots |f_r(z)} \right),\end{aligned}$$

where $D = d + \delta_1 + \ldots + \delta_r - d_1 - \ldots - d_r$, hence the residue vanishes if $D \neq 0$, i.e. when $d \neq d_1 + \ldots + d_r - \delta_1 - \ldots - \delta_r$.

3. The Baum-Bott localization theorem

We recall the Baum-Bott theorem from [2], which states that Chern numbers of the tangent bundle can be computed by Grothendieck residues at zeroes of holomorphic vector fields. Let M be a compact complex analytic manifold of dimension $\dim_{\mathbb{C}} M = r$ and let ϑ be a holomorphic vector field on M. We assume that ϑ vanishes only at isolated points. We consider holomorphic local coordinates $z_1, \ldots, z_r$ on M centered at a zero p, i.e. $z_1(p) = \ldots = z_r(p) = 0$. In these local coordinates we can write

$$\vartheta = \sum_{i=1}^{r} \vartheta_i \frac{\partial}{\partial z_i},$$

where $\vartheta_1, \ldots, \vartheta_r$ are holomorphic functions in a neighborhood $\mathcal{U}_p$ of p and this point p is their only common zero in $\mathcal{U}_p$.

Let $V_p = \left(\frac{\partial \vartheta_i}{\partial z_j} \right)_{i,j=1}^{r}$ be the Jacobian. The *Chern classes* $c_i(V_p)$ of the matrix V_p are defined by the formula

$$c(V_p; t) = \sum_{i=0}^{r} c_i(V_p) t^i = \det(I + tV_p). \tag{3.1}$$

Moreover, for any multidegree $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r_{\geq 0}$ such that $|\alpha| = \alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r$ the *Chern numbers* of V_p are given by $c^\alpha(V_p) = c_1(V_p)^{\alpha_1} \cdots c_r(V_p)^{\alpha_r}$.

Denote $c_i(TM) \in H^{2i}(M, \mathbb{C})$, $i = 1, \ldots, r$ the Chern classes of the holomorphic tangent bundle TM. The theorem of Baum and Bott [2, Theorem 1] states that for any multidegree $\alpha = (\alpha_1, \ldots, \alpha_r)$ the *Chern numbers*

$$c^\alpha(TM) = \int_M c_1(TM)^{\alpha_1} \cdots c_r(TM)^{\alpha_r} \tag{3.2}$$

of the holomorphic tangent bundle TM can be computed by Grothendieck residues at the zeroes of the holomorphic vector field ϑ as follows.

Theorem 3.1 ([2, Theorem 1]). *Let ϑ be a holomorphic vector field on a compact complex analytic manifold M of dimension $\dim_{\mathbb{C}} M = r$. Assume that ϑ vanishes at only isolated points. Then the Chern numbers of the tangent bundle of M can be computed as*

$$c^{\alpha}(TM) = \sum_{p \in \text{zeroes of } \vartheta} \operatorname{Res}_p \left(\frac{c^{\alpha}(V_p)\, dz_1 \dots dz_r}{\vartheta_1 | \dots | \vartheta_r} \right). \tag{3.3}$$

4. Nilpotent vector field on the Grassmannian

From the action of a nilpotent matrix of form (4.1) we construct a holomorphic vector field ϑ on the Grassmannian $Gr_k(n, \mathbb{C})$, which vanishes at a single point (Proposition 4.1). Moreover, in (4.8) we express this vector field in local coordinates around its zero and we use it to compute the Chern numbers of the Grassmannian (cf. (5.1)). The result of the computation is given in Theorem 5.6.

4.1. Definition of the vector field ϑ

On the complex vector space $\mathcal{M}_{k,n}(\mathbb{C})$ of k-by-n complex matrices we consider the natural left action of the group $GL_k(\mathbb{C})$ by matrix multiplication. We assume that $k < n$. On the subset of rank k matrices of $\mathcal{M}_{k,n}(\mathbb{C})$ the group $GL_k(\mathbb{C})$ acts properly and freely. The quotient $Gr_k(n, \mathbb{C})$, called the *Grassmannian*, is the set of k-dimensional complex linear subspaces of $\mathbb{C}^n$ and it has the structure of a compact complex analytic manifold. In particular, a full rank matrix $P \in \mathcal{M}_{k,n}(\mathbb{C})$ represents a point in $Gr_k(n, \mathbb{C})$, namely the k-dimensional complex linear subspace spanned by the rows of P.

On $\mathcal{M}_{k,n}(\mathbb{C})$ the multiplication by n-by-n matrices $GL_n(\mathbb{C})$ on the right commutes with the action of $GL_k(n)$ on the left, hence this right action descends to the Grassmannian $Gr_k(n, \mathbb{C})$.

For any $\ell \geq 2$ we consider nilpotent matrices of the following form

$$N_\ell = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{M}_\ell(\mathbb{C}). \tag{4.1}$$

We consider the vector field ϑ on the Grassmannian induced by the nilpotent matrix $N_n \in \mathcal{M}_n(\mathbb{C})$ as follows. Let ϑ be the vector field associated with the holomorphic flow $Fl(t, P) = P \cdot \exp(tN_n)$, $t \in \mathbb{C}$ on the Grassmannian $Gr_k(n, \mathbb{C})$, where $P \in \mathcal{M}_{k,n}(\mathbb{C})$ is a rank k matrix representing a point on the Grassmannian.

4.2. Zeroes of the vector field ϑ

The following proposition gives a description of matrices representing the zeroes of the vector field ϑ and shows that they represent the same point on the Grassmannian.

Proposition 4.1. *A matrix $P \in \mathcal{M}_{k,n}(\mathbb{C})$ of rank k represents a zero of the vector field ϑ on $Gr_k(n,\mathbb{C})$ if and only if there exists matrix $S \in \mathcal{M}_k(\mathbb{C})$ such that $SP = PN_n$. Moreover, any solution $(S,P) \in \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_{k,n}(\mathbb{C})$ of $SP = PN_n$ is of form $S = UN_kU^{-1}$ and $P = \begin{pmatrix} O_{k,n-k} & U \end{pmatrix}$, where $U \in GL_k(\mathbb{C})$ and $O_{k,n-k} \in \mathcal{M}_{k,n-k}(\mathbb{C})$ is the zero matrix. The point in $Gr_k(n,\mathbb{C})$ represented by $P = \begin{pmatrix} O_{k,n-k} & I_k \end{pmatrix}$ is the only zero of the vector field ϑ.*

Proof. The zeroes of the vector field ϑ correspond to the fixed points of the flow Fl, hence P represents a zero of ϑ if and only if there exists $A(t) \in GL_k(\mathbb{C})$ for all $t \in \mathbb{C}$ such that $A(t)P = P\exp(tN_n)$. Moreover, $A(t)$ is differentiable and denote $S = A'(0)$. Therefore, from the previous relation we get $SP = PN_n$ by taking the differential at $t = 0$. On the other hand, if there is a pair (S,P) with P of rank k such that $SP = PN_n$, then $S^jP = PN_n^j$ for all $j \in \mathbb{Z}_{\geq 0}$, hence $\exp(St)P = P\exp(tN_n)$. This implies that P represents a zero of ϑ.

Consider a pair $(S,P) \in \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_{k,n}(\mathbb{C})$ satisfying $SP = PN_n$. In particular, $S^nP = PN_n^n = O_{k,n-k}$ and P has rank k, thus $S^n = O_k$, i.e. S is nilpotent. The Jordan form of $S \in \mathcal{M}_k(\mathbb{C})$ implies that already $S^k = O_k$. Since $O_k = S^kP = PN_n^k$, thus we must have $P = \begin{pmatrix} O_{k,n-k} & U \end{pmatrix}$ for some matrix $U \in GL_k(\mathbb{C})$ and $S = UN_kU^{-1}$.

The rank k matrices $\begin{pmatrix} O_{k,n-k} & U \end{pmatrix}$ and $\begin{pmatrix} O_{k,n-k} & I_k \end{pmatrix}$ represent the same point in $Gr_k(n,\mathbb{C})$. □

4.3. Expression of the vector field ϑ in local coordinates at the zero point

The local parametrization on the Grassmannian around the point represented by P of Proposition 4.1 is given by the matrix $Z = \begin{pmatrix} z & I_k \end{pmatrix}$ with block $z = (z_{i,j})_{i,j}$ of $k \times (n-k)$ complex coordinates $z_{i,j}$, $i = 1,\dots,k$, $j = 1,\dots,n-k$. The ordering of coordinates will be lexicographical.

First, we compute the flow Fl of ϑ as follows. We set matrices $B(t)$ and $D(t)$ by the relation $\begin{pmatrix} B(t) & D(t) \end{pmatrix} = \begin{pmatrix} z & I_k \end{pmatrix}\exp(tN_n)$, hence $z(t) = D(t)^{-1}B(t)$ gives the local coordinates of the flow at the zero point. We note that $B(0) = z$, $D(0) = I_k$ and $D(t)$ is a matrix of polynomials in t of degree at most n and has of form $D(t) = I_k + D_1t + \ldots + D_nt^n$ with

$$D_1 = \begin{pmatrix} z_{1,n-k} & 1 & 0 & \dots & 0 \\ z_{2,n-k} & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ z_{k-1,n-k} & 0 & 0 & \dots & 1 \\ z_{k,n-k} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.2}$$

Finally, in local coordinates z the vector field ϑ can be computed as

$$\vartheta = \sum_{i=1}^{k}\sum_{j=1}^{n-k} \frac{d}{dt}\left(D^{-1}(t)B(t)\right)_{i,j}(0)\frac{\partial}{\partial z_{i,j}} = \sum_{i=1}^{k}\sum_{j=1}^{n-k}(-D_1z + zN_{n-k})_{i,j}\,\frac{\partial}{\partial z_{i,j}}.$$

Note that zN_{n-k} is the matrix got from z by shifting its columns to the right and inserting zeroes in the first column. In more details, if $\vartheta = \sum_{i=1}^{k}\sum_{j=1}^{n-k}\vartheta_{i,j}\frac{\partial}{\partial z_{i,j}}$ then

$$\vartheta_{i,1} = -z_{i,n-k}z_{1,1} - z_{i+1,1}, \qquad (i = 1,\dots,k-1), \tag{4.3}$$

$$\vartheta_{k,1} = -z_{k,n-k}z_{1,1}, \tag{4.4}$$

$$\vartheta_{i,j} = -z_{i,n-k}z_{1,j} - z_{i+1,j} + z_{i,j-1}, \qquad (i = 1,\dots,k-1,\ j = 2,\dots,n-k), \tag{4.5}$$

$$\vartheta_{k,j} = -z_{k,n-k}z_{1,j} + z_{k,j-1}, \qquad (j = 2,\dots,n-k). \tag{4.6}$$

If we introduce notations $z_{k+1,j} = 0$ for $j = 1,\dots,n-k$ and $z_{i,0} = 0$ for $i = 1,\dots,k$ then we can give a uniform description for these functions. For any $i = 1,\dots,k$ and $j = 1,\dots,n-k$ we have

$$\vartheta_{i,j} = -z_{i,n-k}z_{1,j} - z_{i+1,j} + z_{i,j-1}. \tag{4.7}$$

With these notations the nilpotent vector field around the zero equals to

$$\vartheta = \sum_{i=1}^{k}\sum_{j=1}^{n-k}\left(-z_{i,n-k}z_{1,j} - z_{i+1,j} + z_{i,j-1}\right)\frac{\partial}{\partial z_{i,j}}. \tag{4.8}$$

5. Residue formula for the Grassmannian

We start by writing up the Baum-Bott residue formula (3.3) for the vector field (4.8) on the Grassmannian $Gr_k(n,\mathbb{C})$ and we prove a series of lemmas about the building blocks of this residue formula to reduce the number of variables from $k(n-k)$ to n. At the end of this section in Theorem 5.6 we get a simplified residue formula for Chern numbers of the Grassmannian. In Section 6 this formula will be reinterpreted in terms of Chern classes of the tautological and quotient bundle on the Grassmannian (Theorem 6.4 and Corollary 6.5) to get an even simpler formula.

By Theorem 3.1 we get the following formula for Chern numbers (3.2) of the tangent bundle of the Grassmannian:

$$c^{\alpha}(TGr_k(n,\mathbb{C})) = \mathrm{Res}_0\left(\frac{\prod_{i=1}^{k}\prod_{j=1}^{n-k} dz_{i,j}}{\prod_{i=1}^{k}\prod_{j=1}^{n-k}\vartheta_{i,j}}c^{\alpha}(V)\right), \tag{5.1}$$

where $V = \left(\dfrac{\partial\vartheta_{i,j}}{\partial z_{h,l}}\right)_{(i,j),(h,l)}$ with lexicographical ordering on pairs (i,j) and (h,l). This ordering is compatible with the ordering of functions $\vartheta_{i,j}$ in the denominator of (5.1). We will drop the vertical line notation from the denominator in the Grothendieck residue, but we will keep track of the order of functions.

We make the following change of variables. We express the variables $z_{a,b}$, $a = 1,\dots,k$, $b = 1,\dots,n-k$ in terms of $z_{1,j}$, $j = 1,\dots,n-k-1$, $z_{i,n-k}$, $i = 1,\dots,k$, and $\vartheta_{i,j}$, $i = 1,\dots,k-1$, $j = 1,\dots,n-k-1$. We keep the common variables, while the others can expressed in new variables as follows.

Lemma 5.1. *For any* $i = 1, \dots, k-1$ *and* $j = 1, \dots, n-k-1$ *we have*

$$z_{i+1,j} = -\sum_{\ell=0}^{\min\{i,\,j-1\}} z_{i-\ell,n-k}\, z_{1,j-\ell} - \sum_{\ell=0}^{\min\{i-1,\,j-1\}} \vartheta_{i-\ell,j-\ell} \tag{5.2}$$

with notation $z_{0,n-k} = -1$.

Proof. We fix a pair (i, j) with $1 \le i \le k-1$ and $1 \le j \le n-k-1$. From relations (4.7) we construct the following recursion between variables on the same diagonal as $z_{i+1,j}$

$$z_{i+1-\ell,j-\ell} = -z_{i-\ell,n-k} z_{1,j-\ell} + z_{i-\ell,j-1-\ell} - \vartheta_{i-\ell,j-\ell}, \tag{5.3}$$

for $\ell = 0, \dots, \min\{i-1, j-1\}$. We specify the following edge cases.
For $\ell = i-1 = \min\{i-1, j-1\}$ we have

$$\begin{aligned} z_{2,j-i+1} &= -z_{1,n-k} z_{1,j-i+1} + z_{1,j-i} - \vartheta_{1,j-i+1} \\ &= -z_{1,n-k} z_{1,j-i+1} - z_{0,n-k} z_{1,j-i} - \vartheta_{1,j-i+1}, \end{aligned}$$

with notation $z_{0,n-k} = -1$. Moreover, for $\ell = j-1 = \min\{i-1, j-1\}$ we have

$$\begin{aligned} z_{i-j+2,1} &= -z_{i-j+1,n-k} z_{1,1} + z_{i-j+1,0} - \vartheta_{i-j+1,1} \\ &= -z_{i-j+1,n-k} z_{1,1} - \vartheta_{i-j+1,1} \end{aligned}$$

by earlier introduced notation $z_{i-j+1,0} = 0$.

Finally, substituting the relations (5.3) into each other yields the statement of the lemma. □

The differential form in the numerator of (5.1) can be expressed in terms of new variables as follows.

Lemma 5.2.

$$\prod_{i=1}^{k} \prod_{j=1}^{n-k} dz_{i,j} = \prod_{i=1}^{k-1} \Big(\prod_{j=1}^{n-k-1} d\vartheta_{i,j} \Big) dz_{i,n-k} \Big(\prod_{j=1}^{n-k-1} dz_{1,j} \Big) dz_{k,n-k}.$$

Proof. Taking the differential of (5.2) for every $i = 1, \dots, k-1$ and $j = 1, \dots, n-k-1$, then substituting into the product of differentials basically replaces $dz_{i+1,j}$ with $-d\vartheta_{i,j}$, thus we get

$$\begin{aligned} \prod_{i=1}^{k} \prod_{j=1}^{n-k} dz_{i,j} = {} & (dz_{1,1} \dots dz_{1,n-k-1} dz_{1,n-k}) \, ((-d\vartheta_{1,1}) \dots (-d\vartheta_{1,n-k-1}) dz_{2,n-k}) \\ & \dots ((-d\vartheta_{k-1,1}) \dots (-d\vartheta_{k-1,n-k-1}) dz_{k,n-k}) \\ = {} & (d\vartheta_{1,1} \dots d\vartheta_{1,n-k-1} dz_{1,n-k}) \dots \\ & (d\vartheta_{k-1,1} \dots d\vartheta_{k-1,n-k-1} dz_{k-1,n-k}) \, (dz_{1,1} \dots dz_{1,n-k-1} dz_{k,n-k}) \\ = {} & \Big(\prod_{i=1}^{k-1} \Big(\prod_{j=1}^{n-k-1} d\vartheta_{i,j} \Big) dz_{i,n-k} \Big) \Big(\prod_{j=1}^{n-k-1} dz_{1,j} \Big) dz_{k,n-k}. \end{aligned}$$

□

The functions $\vartheta_{k,j}$, $j = 1, \dots, n-k$ and $\vartheta_{i,n-k}$, $i = 1, \dots, k-1$ in the denominator of the residue (5.1) can be expressed in terms of new variables as follows.

Lemma 5.3. *For $i = k$ or $j = n-k$ we have*

$$\vartheta_{i,j} = -\sum_{h=-1}^{\min\{i,j-1\}} z_{i-h,n-k}\, z_{1,j-h} - \sum_{h=1}^{\min\{i-1,j-1\}} \vartheta_{i-h,j-h} \tag{5.4}$$

with notations $z_{1,n-k+1} = 1$, $z_{0,n-k} = -1$ and $z_{k+1,j} = 0$.

Proof. In the first case when $i = k$ we substitute (5.2) into (4.6) and we get

$$\begin{aligned}
\vartheta_{k,j} &= -z_{k,n-k}\, z_{1,j} - \sum_{\ell=0}^{\min\{k-1,j-2\}} z_{k-1-\ell,n-k}\, z_{1,j-1-\ell} - \sum_{\ell=0}^{\min\{k-2,j-2\}} \vartheta_{k-1-\ell,j-1-\ell} \\
&= -z_{k,n-k}\, z_{1,j} - \sum_{h=1}^{\min\{k,j-1\}} z_{k-h,n-k}\, z_{1,j-h} - \sum_{h=1}^{\min\{k-1,j-1\}} \vartheta_{k-h,j-h} \\
&= -\sum_{h=-1}^{\min\{k,j-1\}} z_{k-h,n-k}\, z_{1,j-h} - \sum_{h=1}^{\min\{k-1,j-1\}} \vartheta_{k-h,j-h}
\end{aligned}$$

by earlier introduced notation $z_{k+1,j} = 0$.

In the second case when $j = n-k$, we substitute (5.2) into (4.5) in place of $z_{i,n-k-1}$ and we get

$$\begin{aligned}
\vartheta_{i,n-k} = &- z_{i,n-k}\, z_{1,n-k} - z_{i+1,n-k} - \sum_{\ell=0}^{\min\{i-1,n-k-2\}} z_{i-1-\ell,n-k}\, z_{1,n-k-1-\ell} - \\
&- \sum_{\ell=0}^{\min\{i-2,n-k-2\}} \vartheta_{i-1-\ell,n-k-1-\ell} \\
= &- z_{i,n-k}\, z_{1,n-k} - z_{i+1,n-k} z_{1,n-k+1} - \sum_{h=1}^{\min\{i,n-k-1\}} z_{i-h,n-k}\, z_{1,n-k-h} \\
&- \sum_{h=1}^{\min\{i-1,n-k-1\}} \vartheta_{i-h,n-k-h} \\
= &- \sum_{h=-1}^{\min\{i,n-k-1\}} z_{i-h,n-k}\, z_{1,n-k-h} - \sum_{h=1}^{\min\{i-1,n-k-1\}} \vartheta_{i-h,n-k-h}
\end{aligned}$$

by setting $z_{1,n-k+1} = 1$. □

We introduce notation for the first sum of (5.4)

$$\zeta_{i,j} = -\sum_{h=-1}^{\min\{i,j-1\}} z_{i-h,n-k} z_{1,j-h} \quad \text{for} \quad i = k \text{ or } j = n-k, \tag{5.5}$$

and $\zeta_{i,j} = \vartheta_{i,j}$ otherwise. Hence by (5.4) we have $\zeta_{i,j} = \vartheta_{i,j} + \sum_{h=1}^{\min\{i-1,j-1\}} \vartheta_{i-h,j-h}$ if $i = k$ or $j = n-k$ and $\vartheta_{i,j} = \zeta_{i,j}$ otherwise.

Remark 5.4. The transformation matrix between $(\vartheta_{i,j})_{i=\overline{1,k},\, j=\overline{1,n-k}}$ and $(\zeta_{i,j})_{i=\overline{1,k},\, j=\overline{1,n-k}}$ (lexicographically ordered) is lower triangular with 1 on the diagonal.

Next, we compute the final component of the residue formula (5.1), namely $c(V;t)$ (cf. (3.1)). We introduce the following notations:

$$\widetilde{u}_i = -z_{i,n-k},\ i = 0,\dots,k+1 \quad \text{and} \quad w_j = z_{1,n-k+1-j} = z_{1,j^*},\ j = 0,\dots,n-k, \tag{5.6}$$

where $j^* = n-k+1-j$. By earlier notations $\widetilde{u}_0 = -z_{0,n-k} = 1$, $w_0 = z_{1,n-k+1} = 1$, $\widetilde{u}_{k+1} = -z_{k+1,n-k} = 0$ and $\widetilde{u}_1 = -w_1 = -z_{1,n-k}$.

We associate with $w = (w_1,\dots,w_{n-k})$ the following matrix

$$\Lambda_w = \begin{pmatrix} w_1 & -1 & 0 & \dots & 0 \\ w_2 & 0 & -1 & \dots & 0 \\ \vdots & & & & \\ w_{n-k-1} & 0 & 0 & \dots & -1 \\ w_{n-k} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{5.7}$$

Lemma 5.5. *Using notations of (5.6) we have*

$$c(V;t) = \det\Big(\sum_{i=0}^{k} \widetilde{u}_i t^i (I - t\Lambda_w)^{k-i}\Big).$$

Proof. From (4.7) we can compute

$$\frac{\partial\vartheta_{1,n-k}}{\partial z_{1,n-k}} = -2z_{1,n-k},\ \frac{\partial\vartheta_{i,j}}{\partial z_{i,n-k}} = -z_{1,j}$$

and $\dfrac{\partial\vartheta_{i,j}}{\partial z_{1,j}} = -z_{i,n-k}$ if $i \neq 1$ or $j \neq n-k$, $\dfrac{\partial\vartheta_{i,j}}{\partial z_{i+1,j}} = -1$, $\dfrac{\partial\vartheta_{i,j}}{\partial z_{i,j-1}} = 1$ and $\dfrac{\partial\vartheta_{i,j}}{\partial z_{h,\ell}} = 0$ otherwise.

Let $W = [W_{i,j}]_{i,j=1}^{n-k}$ be the matrix with $W_{i+1,i} = 1$, $W_{i,n-k} = -z_{1,i} = -w_{n-k+1-i}$ and $W_{i,j} = 0$ otherwise (centrally symmetric image of $-\Lambda_w$). Denote $I = I_{n-k}$ and let $U_i = -z_{i,n-k}I = \widetilde{u}_i I$ for $i = 0,\dots,k$. Then $V = \left(\dfrac{\partial\vartheta_{i,j}}{\partial z_{h,\ell}}\right)_{(i,j),(h,\ell)}$ is the $k(n-k)$-by-$k(n-k)$ matrix composed of $(n-k)$-by-$(n-k)$ blocks $V^{i,j}$, $i,j = 1,\dots,k$ as follows. Let $V^{1,1} = U_1 + W$ and for $i > 1$ let $V^{i,1} = U_i$, $V^{i,i} = W$ and $V^{i,i+1} = -I$, while other blocks are zeroes. Then we compute

$$c(V;t) = \det(I_{k(n-k)} + tV)$$
$$= \begin{vmatrix} tU_1 + I + tW & -tI & 0 & \dots & 0 & 0 \\ tU_2 & I + tW & -tI & \dots & 0 & 0 \\ \vdots & & & & & \\ tU_{k-1} & 0 & 0 & \dots & I + tW & -tI \\ tU_k & 0 & 0 & \dots & 0 & I + tW \end{vmatrix}$$

as follows. To get rid of $I+tW$ from the first $(k-1)$ diagonal blocks, we add successively to the $(k-1)^{\text{th}}, \dots, 1^{\text{st}}$ columns of blocks the subsequent column of blocks multiplied by $t^{-1}(I+tW)$. After this step there will be blocks

$$tU_k + t^{1-k}(I+tW)^k,\ t^{2-k}(I+tW)^{k-1}, \dots, t^{-1}(I+tW)^2,\ I+tW$$

in the last row. Next, we multiply the first $k-1$ rows of blocks by

$$t^{1-k}(I+tW)^{k-1}, \dots, t^{-2}(I+tW)^2,\ t^{-1}(I+tW)$$

respectively and we add them to the last row to get blocks of zeroes in the last row except the first column, where there will be $\sum_{i=0}^{k}(I+tW)^{k-i}t^{1-k+i}U_i$.

Summing up, we get a determinant with blocks $-tI$ above the diagonal, $tU_1, \dots, tU_{k-1}$ in the first $(k-1)$ rows of the first column, respectively, and $\sum_{i=0}^{k}(I+tW)^{k-i}t^{1-k+i}U_i$ in the last row of the first columns, while other blocks are zeroes. Finally, we move the first column of blocks to the end to get an block-upper triangular determinant in exchange of a $(-1)^{(k-1)(n-k)^2}$-sign. Hence,

$$\det(I_{k(n-k)} + tV) = (-1)^{(k-1)(n-k)^2+(k-1)(n-k)} \cdot \det\Big(\sum_{i=0}^{k}(I+tW)^{k-i}t^iU_i\Big)$$
$$= \det\Big(\sum_{i=0}^{k}\widetilde{u}_i t^i (I+tW)^{k-i}\Big) = \det\Big(\sum_{i=0}^{k}\widetilde{u}_i t^i (I-t\Lambda_w)^{k-i}\Big),$$

where the last equality is by reflecting the determinant first with respect to the diagonal and then to the anti-diagonal, which sends W to $-\Lambda_w$, and it is compatible with matrix addition and multiplication. □

Consider the polynomial ring $\mathbb{C}[u,w] = \mathbb{C}[u_1, \dots, u_k, w_1, \dots, w_{n-k}]$ and polynomials

$$P_\ell(u,v) = \sum_{s=0}^{\ell} u_s w_{\ell-s}, \qquad \forall \ell = 1, \dots, n, \tag{5.8}$$

with notations $u_0 = w_0 = 1$. We state the reshaped Baum-Bott residue formula for the Grassmannian $Gr_k(n,\mathbb{C})$ with only n variables.

Theorem 5.6. *For any multidegree $\alpha = (\alpha_1, \dots, \alpha_{k(n-k)}) \in \mathbb{Z}_{\geq 0}^{k(n-k)}$ with*

$$|\alpha| = \alpha_1 + \cdots + \alpha_{k(n-k)} = k(n-k)$$

the Chern numbers can be computed as

$$c^\alpha(TGr_k(n,\mathbb{C})) = \int_{Gr_k(n,\mathbb{C})} \prod_{s=1}^{k(n-k)} c_s(TGr_k(n,\mathbb{C}))^{\alpha_s}$$
$$= \operatorname{Res}_0\left(\frac{\prod_{i=1}^{k} du_i \prod_{j=1}^{n-k} dw_j}{\prod_{\ell=1}^{n} P_\ell(u,w)} \Delta^\alpha(u,w)\right),$$

where

$$\Delta(u,w;t) = \sum_{\ell=0}^{k(n-k)} \Delta_\ell t^\ell = \det\Big(\sum_{i=0}^{k} u_i t^i (I-t\Lambda_w)^{k-i}\Big)$$

and

$$\Delta^{\alpha}(u,w)=\Delta^{\alpha_1}\cdot\ldots\cdot\Delta^{\alpha_{k(n-k)}}.$$

Proof. By the Baum-Bott Theorem we have the residue formula (5.1). In this formula we replace the differential form by the one in Lemma 5.2. We also reorder the functions in the denominator and by (P1) the functions in the denominator are anti-commuting just like the differential forms in the numerator. Thus, the residue (5.1) becomes

$$\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k-1}\left(\prod_{j=1}^{n-k-1}d\vartheta_{i,j}\right)dz_{i,n-k}\left(\prod_{j=1}^{n-k-1}dz_{1,j}\right)dz_{k,n-k}}{\prod_{i=1}^{k}\prod_{j=1}^{n-k}\vartheta_{i,j}}c^{\alpha}(V)\right)= \tag{5.9}$$

$$=\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k-1}\prod_{j=1}^{n-k-1}d\vartheta_{i,j}\prod_{i=1}^{k}dz_{i,n-k}\prod_{j=1}^{n-k-1}dz_{1,j}}{\prod_{i=1}^{k-1}\prod_{j=1}^{n-k-1}\vartheta_{i,j}\prod_{i=1}^{k}\vartheta_{i,n-k}\prod_{j=1}^{n-k-1}\vartheta_{k,j}}c^{\alpha}(V)\right). \tag{5.10}$$

Recall that we have relations $\vartheta_{i,j}=\zeta_{i,j}-\sum_{h=1}^{\min\{i-1,j-1\}}\vartheta_{i-h,j-h}$ for $i=k$ or $j=n-k$ by (5.5) and Lemma 5.3. Thus, when $i=k$ or $j=n-k$ we can replace $\vartheta_{i,j}$ with $\zeta_{i,j}$ in (5.10) by Transformation Law (P4) and Remark 5.4 to get

$$\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k-1}\prod_{j=1}^{n-k-1}d\vartheta_{i,j}\prod_{i=1}^{k}dz_{i,n-k}\prod_{j=1}^{n-k-1}dz_{1,j}}{\prod_{i=1}^{k-1}\prod_{j=1}^{n-k-1}\vartheta_{i,j}\prod_{i=1}^{k}\zeta_{i,n-k}\prod_{j=1}^{n-k-1}\zeta_{k,j}}c^{\alpha}(V)\right). \tag{5.11}$$

We note that $c^{\alpha}(V)$ depends only on $z_{1,1},\ldots,z_{1,n-k},\ldots,z_{k,n-k}$ (see Lemma 5.5). Thus, applying property (P5) to (5.11) yields

$$\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k}dz_{i,n-k}\prod_{j=1}^{n-k-1}dz_{1,j}}{\prod_{i=1}^{k}\zeta_{i,n-k}\prod_{j=1}^{n-k-1}\zeta_{k,j}}c^{\alpha}(V)\right)=$$

$$=\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k}dz_{i,n-k}\prod_{j=2}^{n-k}dz_{1,j^*}}{\prod_{i=1}^{k}\zeta_{i,n-k}\prod_{j=2}^{n-k}\zeta_{k,j^*}}c^{\alpha}(V)\right), \tag{5.12}$$

where $j^*=n-k+1-j$.

By Lemma 5.5 we have $c(V;t)=\Delta(\widetilde{u},w;t)$, hence $c^{\alpha}(V)=\Delta^{\alpha}(\widetilde{u},w)$. Moreover, from (5.5) and (5.6) for $i=k$ and $j=1,\ldots,n-k-1$ we have

$$\zeta_{k,j}=\sum_{h=-1}^{\min\{k,j-1\}}\widetilde{u}_{k-h}w_{n-k+1-j+h}=P_{n+1-j}(\widetilde{u},w),$$

hence $\zeta_{k,j^*}=P_{k+j}(\widetilde{u},w)$. Similarly, from (5.5) and (5.6) for $j=n-k$ and $i=1,\ldots,k$ we have

$$\zeta_{i,n-k}=\sum_{h=-1}^{\min\{i,n-k-1\}}\widetilde{u}_{i-h}w_{h+1}=P_{i+1}(\widetilde{u},w).$$

We recall that $\widetilde{u}_1=-w_1$. Thus, (5.12) becomes

$$\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k}dz_{i,n-k}\prod_{j=2}^{n-k}dz_{1,j^*}}{\prod_{i=1}^{k}\zeta_{i,n-k}\prod_{j=2}^{n-k}\zeta_{k,j^*}}c^{\alpha}(V)\right)=\mathrm{Res}_0\left(\frac{\prod_{i=2}^{k}d\widetilde{u}_i\prod_{j=1}^{n-k}dw_j}{\prod_{i=2}^{n}P_i(\widetilde{u},w)}\Delta^{\alpha}(\widetilde{u},w)\right). \tag{5.13}$$

Finally, to get a more symmetric formula we separate $\widetilde{u}_1$ from w_1. Therefore, let $P_1 = P_1(u, w) = u_1 + w_1$ and $u_i = \widetilde{u}_i$ for $i = 2, \ldots, k$, hence $u_1 = P_1 - w_1 = P_1 + \widetilde{u}_1$. Thus,

$$\operatorname{Res}_0\left(\frac{\prod_{i=1}^k du_i \prod_{j=1}^{n-k} dw_j}{\prod_{i=1}^n P_i(u,w)}\Delta^\alpha(u,w)\right) \overset{(\dagger)}{=}$$

$$\overset{(\dagger)}{=} \operatorname{Res}_0\left(\frac{dP_1 \prod_{i=2}^k du_i \prod_{j=1}^{n-k} dw_j}{P_1 \prod_{i=2}^n P_i(u,w)}\Delta^\alpha(u,w)\right) \overset{(\ddagger)}{=}$$

$$\overset{(\ddagger)}{=} \operatorname{Res}_0\left(\frac{dP_1 \prod_{i=2}^k d\widetilde{u}_i \prod_{j=1}^{n-k} dw_j}{P_1 \prod_{i=2}^n P_i(\widetilde{u},w)}\Delta^\alpha(\widetilde{u},w)\right) \overset{(\dagger\dagger)}{=}$$

$$\overset{(\dagger\dagger)}{=} \operatorname{Res}_0\left(\frac{\prod_{i=2}^k du_i \prod_{j=1}^{n-k} d\widetilde{w}_j}{\prod_{i=2}^n P_i(\widetilde{u},w)}\Delta^\alpha(\widetilde{u},w)\right).$$

In (†) we replaced du_1 with dP_1. In (‡) we made substitution $u_1 = P_1 + \widetilde{u}_1$ and, moreover, we applied Local Duality (P2) to get rid of P_1 from $\Delta^\alpha(u, w)$, appeared after the substitution and to get $\Delta^\alpha(\widetilde{u}, w)$. Furthermore, we used Transformation Law (P4) to remove P_1 from $P_2(u, w), \ldots, P_n(u, w)$ after the aforementioned substitution and to get $P_2(\widetilde{u}, w), \ldots, P_n(\widetilde{u}, w)$ in the denominator. Last, in (††) we used property (P5) to eliminate P_1 from the residue and we got back the right hand side of (5.13). □

6. The residue formula and cohomological relations

We will give an interpretation of variables u_i's and w_j's of Theorem 5.6 in terms of Chern classes of the tautological and quotient bundle on the Grassmannian. Thus, in Theorem 6.4 we can give an even simpler version of Theorem 5.6.

6.1. Cohomology ring of the complex Grassmannian

First, we recall the properties of Chern classes from [3, Ch. IV], then we recall the generators and relations of the cohomology ring of the complex Grassmannian in Theorem 6.1 (cf. [3, Proposition 23.2]).

6.1.1. Chern classes. To a complex vector bundle $\mathcal{E}$ (of rank p) over a manifold M one can associate cohomological classes $c_i(\mathcal{E}) \in H^{2i}(M, \mathbb{C})$, $i = 1, \ldots, p$, called the i^{th} *Chern class* ($c_0(\mathcal{E}) = 1$ and $c_i(\mathcal{E}) = 0$ when $i > p$). One can arrange them into a sequence $c(\mathcal{E}; t) = 1 + c_1(\mathcal{E})t + \ldots + c_p(\mathcal{E})t^p$ and $c(\mathcal{E}) = c(\mathcal{E}; 1)$ is called the *total Chern class.*

Usually, one uses Chern roots to calculate with Chern classes. This is based on the *Splitting Principle* ([3, Ch. IV, §21]): one can pretend that the bundle $\mathcal{E}$ is a direct sum of complex line bundles. The Chern classes $\eta_1, \ldots, \eta_p \in H^2(M, \mathbb{C})$ of these hypothetical line bundles are the *Chern roots* of $\mathcal{E}$, hence $c_i(\mathcal{E}) = e_i(\eta) = \sum_{1 \le \ell_1 < \cdots < \ell_i \le p} \eta_{\ell_1} \cdots \eta_{\ell_i}$ is the i^{th} elementary symmetric polynomial of the Chern roots ($e_0(\eta) = 1$) and $c(\mathcal{E}; t) = \prod_{i=1}^p (1 + t\eta_i)$. For example, the dual bundle $\mathcal{E}^*$ has Chern roots $-\eta_1, \ldots, -\eta_p$, hence $c(\mathcal{E}^*; t) = c(\mathcal{E}; -t)$. Or, if $\mathcal{F}$ is another bundle over M

with Chern roots $\phi_1, \dots, \phi_q$ then $\mathcal{E} \oplus \mathcal{F}$ has Chern roots $\eta_1, \dots, \eta_p, \phi_1, \dots, \phi_q$, hence $c(\mathcal{E} \oplus \mathcal{F}; t) = c(\mathcal{E}; t)c(\mathcal{F}; t)$ (*Whitney product formula*, cf. [3, (20.10.3)]). Similarly, $\mathcal{E} \otimes \mathcal{F}$ has Chern roots $\eta_i + \phi_j$, $i = 1, \dots, p$, $j = 1, \dots, q$, hence $c(\mathcal{E} \otimes \mathcal{F}; t) = \prod_{i=1}^p \prod_{j=1}^q (1+t\eta_i+t\phi_j)$, which can be expressed in terms of Chern classes of $c_i(\mathcal{E})$ and $c_j(\mathcal{F})$ (see Lemma 6.2). In particular, when $\mathcal{E}$ and $\mathcal{F}$ are line bundles then $c_1(\mathcal{E}\otimes\mathcal{F}) = c_1(\mathcal{E}) + c_1(\mathcal{F})$ (cf. [3, (21.9)]). The Chern roots of the trivial bundle are zero, thus $c(M \times \mathbb{C}^s; t) = 1$.

6.1.2. Generators and relations of the cohomology ring of the complex Grassmannians. There is a tautological exact sequence of complex vector bundles over the complex Grassmannian $Gr_k(n, \mathbb{C})$

$$0 \to \mathcal{L} \to Gr_k(n, \mathbb{C}) \times \mathbb{C}^n \to \mathcal{Q} \to 0, \tag{6.1}$$

where $\mathcal{L} = \{(U, u) \in Gr_k(n, \mathbb{C}) \times \mathbb{C}^n \mid u \in U\}$ is the tautological (rank k) complex vector bundle and $\mathcal{Q}$ is the quotient vector bundle (of rank $n - k$). The tautological exact sequence (6.1) induces the relation of total Chern classes $c(\mathcal{L})c(\mathcal{Q}) = c(\mathcal{L}\oplus\mathcal{Q}) = c(Gr_k(n, \mathbb{C})\times\mathbb{C}^n) = 1$, hence we get the following relations between the Chern classes of $\mathcal{L}$ and $\mathcal{Q}$:

$$\sum_{i+j=\ell} c_i(\mathcal{L})c_j(\mathcal{Q}) = 0, \qquad \forall \ell = 1, \dots, n, \tag{6.2}$$

($0 \le i \le k$, $0 \le j \le n - k$). From the first $n - k$ relation one can recursively express each Chern class of the quotient bundle $\mathcal{Q}$ in terms of Chern classes of the tautological bundle $\mathcal{L}$. Substituting them into the remaining k relations we get relations between Chern classes of $\mathcal{L}$.

The Chern classes $c_1(\mathcal{L}), \dots, c_k(\mathcal{L})$ of $\mathcal{L}$ generate the cohomology ring $H^*(Gr_k(n, \mathbb{C}), \mathbb{C})$ with real coefficients, i.e. the ring homomorphism $\mathbb{C}[x_1, \dots, x_k] \to H^*(Gr_k(n, \mathbb{C}), \mathbb{C})$, $x_i \mapsto c_i(\mathcal{L})$ is surjective. Moreover, the above mentioned relation are the only relations between them in the cohomological ring. Nevertheless, one includes the Chern classes of the quotient bundle $\mathcal{Q}$ among the generators for easier description of relations. In this latter case we have the following description of the cohomology ring of the complex Grassmannian $Gr_k(n, \mathbb{C})$.

Theorem 6.1 (cf. [3, Proposition 23.2]**).** *The graded ring morphism induced by $x_i \mapsto c_i(\mathcal{L})$ and $y_j \mapsto c_j(\mathcal{Q})$ induces an isomorphism of graded rings*

$$H(Gr_k(n, \mathbb{C}), \mathbb{C}) \simeq \mathbb{C}[x_1, \dots, x_k, y_1, \dots, y_{n-k}]/\langle P_1(x, y), \dots, P_n(x, y)\rangle,$$

where $P_\ell(x, y) = \sum_{i+j=\ell} x_i y_j$, $\ell = 1, \dots, n$ with convention $x_0 = y_0 = 1$ and $\deg x_i = 2i$ for $i = 1, \dots, k$, $\deg y_j = 2j$ for $j = 1, \dots, n - k$.

6.2. Reinterpretation of the residue formula in terms of tautological and quotient bundle

Since Theorem 5.6 is in terms of the Chern classes of the tangent bundle we have to show that they also generate the cohomology ring.

The tangent bundle of $Gr_k(n, \mathbb{C})$ can be given as $TGr_k(n, \mathbb{C}) \simeq Hom(\mathcal{L}, \mathcal{Q}) = \mathcal{L}^* \otimes \mathcal{Q}$. Thus, if $\sigma_1, \dots, \sigma_k$ and $\tau_1, \dots, \tau_{n-k}$ are Chern roots of $\mathcal{L}$ and $\mathcal{Q}$, respectively, then

$$c(TGr_k(n,\mathbb{C});t) = c(\mathcal{L}^* \otimes \mathcal{Q};t) = \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 - t\sigma_i + t\tau_j). \tag{6.3}$$

Similarly to [8, Lemma 1] we have the following formula.

Lemma 6.2. *If* $e_i(\sigma) = e_i(\sigma_1,\dots,\sigma_k)$, $i = 1,\dots,k$ *and* $e_j(\tau) = e_j(\tau_1,\dots,\tau_{n-k})$, $j = 1,\dots,n-k$ *are elementary symmetric polynomials in formal variables* σ_ℓ *and* τ_ℓ, *respectively, then*

$$\prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 + t\sigma_i + t\tau_j) = \det\Big(\sum_{i=0}^{k} e_i(\sigma)t^i(I + t\Lambda_{e(\tau)})^{k-i}\Big),$$

where $\Lambda_{e(\tau)}$ *is defined in (5.7). Thus,*

$$c(\mathcal{L}^* \otimes \mathcal{Q};t) = \det\Big(\sum_{i=0}^{k} c_i(\mathcal{L}^*)t^i(I + t\Lambda_{c(\mathcal{Q})})^{k-i}\Big).$$

Proof. In the proof of [8, Lemma 1] it was shown that the matrix $\Lambda_{e(\tau)}$ is diagonalizable, $\Lambda_{e(\tau)} = E\mathrm{diag}\,(\tau_1,\dots\,\tau_{n-k})E^{-1}$, hence

$$\det\Big(\sum_{i=0}^{k} e_i(\sigma)t^i(I + t\Lambda_{e(\tau)})^{k-i}\Big) =$$

$$= \det\Big(\sum_{i=0}^{k} e_i(\sigma)t^i E\mathrm{diag}\,(1 + t\tau_1,\dots,1 + t\tau_{n-k})^{k-i}E^{-1}\Big) =$$

$$= \det\Big(E\mathrm{diag}\,\Big(\sum_{i=0}^{k} e_i(\sigma)t^i(1 + t\tau_1),\dots,\sum_{i=0}^{k} e_i(\sigma)t^i(1 + t\tau_{n-k})\Big)^{k-i}E^{-1}\Big) =$$

$$= \prod_{j=1}^{n-k}\sum_{i=0}^{k} e_i(\sigma)t^i(1 + t\tau_j)^{k-i} = \prod_{j=1}^{n-k}\prod_{i=1}^{k}(1 + t\tau_j + t\sigma_i).$$

□

Lemma 6.3. *The Chern classes of the tangent bundle of the Grassmannian* $Gr_k(n,\mathbb{C})$ *also generate the cohomology ring when* $n \neq 2k$.

Proof. The relation (6.2) reads as $c_\ell(\mathcal{Q}) + c_{\ell-1}(\mathcal{Q})c_1(\mathcal{L}) + c_{\ell-2}(\mathcal{Q})c_2(\mathcal{L}) + \dots + c_\ell(\mathcal{L}) = 0$ for $\ell \le n-k$, hence the Chern classes $c_j(Q)$ of the quotient bundle can be expressed recursively in terms of Chern classes of the tautological bundle $\mathcal{L}$,

$$c_j(\mathcal{Q}) = -c_j(\mathcal{L}) + \textit{polynomial of lower order classes of } \mathcal{L}. \tag{6.4}$$

Then by (6.3) and Lemma 6.2 we have

$$c(TGr_k(n,\mathbb{C});t) = \det\Big(\sum_{i=0}^{k} c_i(\mathcal{L}^*)t^i\big(I + t\Lambda_{c(\mathcal{Q})}\big)^{k-i}\Big)$$
$$= \det\Big(\sum_{j=0}^{n-k} c_j(\mathcal{Q})t^j\big(I + t\Lambda_{c(\mathcal{L}^*)}\big)^{n-k-j}\Big),$$

thus $c_\ell(TGr_k(n,\mathbb{C})) = (n-k)c_\ell(\mathcal{L}^*) + kc_\ell(\mathcal{Q})+$ *polynomial of lower order classes*. Hence, by (6.4) we get $c_\ell(TGr_k(n,\mathbb{C})) = [(-1)^\ell(n-k) - k]c_\ell(\mathcal{L}) +$ *polynomial of lower order classes of* $\mathcal{L}$. If $n \neq 2k$ then the coefficient of $c_\ell(\mathcal{L})$ does not vanish, hence it can be expressed recursively in terms of Chern classes of the tangent bundle. □

Theorem 5.6 can be reformulated using Chern classes of the tautological and the quotient bundle.

Theorem 6.4. *Assume that* $n \neq 2k$. *For any polynomial*

$$R(x,y) \in \mathbb{C}[x_1,\dots,x_k,y_1,\dots,y_{n-k}]$$

we have

$$\int_{Gr_k(n,\mathbb{C})} R(c(\mathcal{L}^*),c(\mathcal{Q}^*)) = \operatorname{Res}_0\left(\frac{\prod_{i=1}^k dx_i \prod_{j=1}^{n-k} dy_j}{\prod_{\ell=1}^n P_\ell(x,y)} R(x,y)\right), \tag{6.5}$$

where $P_\ell(x,y) = \sum_{i+j=\ell} x_i y_j$, *with convention* $x_0 = y_0 = 1$.

Proof. On the polynomial ring $\mathbb{C}[x_1,\dots,x_k,y_1,\dots,y_{n-k}]$ we consider the grading induced by $\deg x_i = 2i$, $i = 1,\dots,k$ and $\deg y_j = 2j$, $j = 1,\dots,n-k$. First, assume that $R(x,y)$ is graded homogeneous polynomial of degree $2k(n-k) = \dim_{\mathbb{R}} Gr_k(n,\mathbb{C})$. By (6.3) and Lemma 6.2 we have

$$c(TGr_k(n,\mathbb{C});t) = c(\mathcal{L}^*\otimes\mathcal{Q};t) = \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 - t\sigma_i + t\tau_j) =$$
$$= \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 - t\sigma_i - t(-\tau_j)) = \det\Big(\sum_{i=0}^{k} e_i(\sigma)(-t)^i(I - t\Lambda_{e(-\tau)})^{k-i}\Big) =$$
$$= \det\Big(\sum_{i=0}^{k} e_i(-\sigma)t^i(I - t\Lambda_{e(-\tau)})^{k-i}\Big) =$$
$$= \det\Big(\sum_{i=0}^{k} c_i(\mathcal{L}^*)t^i(I - t\Lambda_{c(\mathcal{Q}^*)})^{k-i}\Big) = \Delta(c(\mathcal{L}^*),c(\mathcal{Q}^*);t),$$

hence $c_\ell(TGr_k(n,\mathbb{C})) = \Delta_\ell(c(\mathcal{L}^*),c(\mathcal{Q}^*))$ (cf. Theorem 5.6). Thus, we can write Theorem 5.6 in the form

$$\int_{Gr_k(n,\mathbb{C})} \Delta^\alpha(c(\mathcal{L}^*),c(\mathcal{Q}^*)) = \operatorname{Res}_0\left(\frac{\prod_{i=1}^k dx_i \prod_{j=1}^{n-k} dy_j}{\prod_{\ell=1}^n P_\ell(x,y)} \Delta^\alpha(x,y)\right),$$

for any multidegree $\alpha = (\alpha_1, \dots, \alpha_{k(n-k)})$ with $|\alpha| = k(n-k)$. Since the Chern classes of the tangent bundle generate the cohomology ring, any polynomial $R(c(\mathcal{L}^*), c(\mathcal{Q}^*))$ can be expressed as a linear combination of $\Delta^\alpha(c(\mathcal{L}^*), c(\mathcal{Q}^*))$'s, thus (6.5) follows.

Finally, when R is homogeneous of degree $\deg R \neq 2k(n-k)$ then the left hand side of (6.5) vanishes by definition. Moreover, the right hand side also vanishes by (P7), since $\deg R \neq \sum_{\ell=1}^n \deg P_\ell - \sum_{i=1}^k 2i - \sum_{j=1}^{n-k} 2j$. □

Corollary 6.5. *Under the assumptions of Theorem 6.4 we have*

$$\int_{Gr_k(n,\mathbb{C})} R(c(\mathcal{L}), c(\mathcal{Q})) = (-1)^{k(n-k)} \operatorname{Res}_0 \left(\frac{\prod_{i=1}^k dx_i \prod_{j=1}^{n-k} dy_j}{\prod_{\ell=1}^n P_\ell(x,y)} R(x,y) \right). \tag{6.6}$$

6.2.1. Cohomology relations from the residue formula. One benefit of the formula (6.5) or (6.6) is that we can easily deduce the relations of the cohomology ring using Poincaré duality.

Proof of Theorem 6.1. We prove the theorem for $n \neq 2k$. By Poincaré duality a (homogeneous) cohomology class $\alpha = A(c(\mathcal{L}), c(\mathcal{Q}))$ vanishes exactly when

$$\int_{Gr_k(n,\mathbb{C})} \alpha\beta = 0$$

for every class $\beta = B(c(\mathcal{L}), c(\mathcal{Q}))$. By (6.6) and Local Duality (P2) follows that $\alpha = A(c(\mathcal{L}), c(\mathcal{Q}))$ vanishes exactly when $A(x,y) \in \langle P_1(x,y), \dots, P_n(x,y) \rangle$, hence this latter being the ideal of relations. □

7. Iterated residues

Since the Chern classes $c_i(\mathcal{L}^*)$, $i = 1, \dots, k$ generates the cohomology ring, hence we will give an iterated residue formula for $\int_{Gr_k(n,\mathbb{C})} \Phi(c(\mathcal{L}^*))$, where Φ is a polynomial in k variables.

We introduce shorter notations $t' = (t_1, \dots, t_k)$ and $t'' = (t_{k+1}, \dots, t_n)$ and we consider the finite map $F : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$F(t_1, \dots, t_n) = (e_1(t'), \dots, e_k(t'), e_1(t''), \dots, e_{n-k}(t'')),$$

where e_i denotes the i^{th} elementary symmetric polynomial.

Below, we use property (P6) to pull back the residue (6.5) along F. We note that F generically is a $k!(n-k)!$-fold cover and the pull-back of polynomials

$$F^* P_\ell(x,y) = \sum_{i=0}^{\ell} e_i(t') e_{\ell-i}(t'') = e_\ell(t_1, \dots, t_n) = e_\ell(t).$$

Moreover, the Jacobian

$$\begin{aligned} J_F(t) &= \det[e_{i-1}(t_1, \dots, \widehat{t_j}, \dots, t_k)]_{i,j=1}^k \cdot \det[e_{i-1}(t_{k+1}, \dots, \widehat{t_{k+j}}, \dots, t_n)]_{i,j=1}^{n-k} \\ &= \prod_{1 \le i < j \le k} (t_i - t_j) \prod_{k+1 \le h < l \le n} (t_h - t_l), \end{aligned}$$

where $\widehat{t_j}$ means that t_j is omitted.

$$\operatorname{Res}_0\left[\frac{\prod_{i=1}^k dx_i \prod_{j=1}^{n-k} dy_j}{\prod_{i=1}^n P_i(x,y)}\Phi(x)\right] =$$

$$= \frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{i=1}^k de_i(t')\prod_{j=1}^{n-k} de_j(t'')}{\prod_{\ell=1}^n e_\ell(t)}\Phi(e(t'))\right]$$

$$= \frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{\ell=1}^n dt_\ell}{\prod_{\ell=1}^n e_\ell(t)}\prod_{1\le i<j\le k}(t_i-t_j)\prod_{k+1\le h<l\le n}(t_h-t_l)\Phi(e(t'))\right].$$

Next, we use the Transformation Law (P4) for the transformation $\left[t_i^n\right]_{i=1}^n = \left[(-1)^{j-1}t_i^{n-j}\right]_{i,j=1}^n\left[e_j(t)\right]_{j=1}^n$. Finally, the coefficient of $t_{k+1}^{n-1}\dots t_n^{n-1}$ in $\prod_{k+1\le h\ne l\le n}(t_h-t_l)\prod_{i=1}^k\prod_{h=k+1}^n(t_h-t_i)$ is $(n-k)!$. Thus,

$$\frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{\ell=1}^n dt_\ell}{\prod_{\ell=1}^n e_\ell(t)}\prod_{1\le i<j\le k}(t_i-t_j)\prod_{k+1\le h<l\le n}(t_h-t_l)\Phi(e(t'))\right] =$$

$$= \frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{\ell=1}^n dt_\ell}{\prod_{\ell=1}^n t_\ell^n}\prod_{1\le i<j\le k}(t_i-t_j)\prod_{k+1\le h<l\le n}(t_h-t_l)\right.$$

$$\left.\prod_{1\le a<b\le n}(t_b-t_a)\Phi(e(t'))\right] =$$

$$= \frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{\ell=1}^n dt_\ell}{\prod_{\ell=1}^n t_\ell^n}\prod_{1\le i\ne j\le k}(t_i-t_j)\prod_{k+1\le h\ne l\le n}(t_h-t_l)\right.$$

$$\left.\prod_{i=1}^k\prod_{h=k+1}^n(t_h-t_i)\Phi(e(t'))\right] =$$

$$= \frac{1}{k!}\operatorname{Res}_0\left[\frac{\prod_{i=1}^k dt_i}{\prod_{i=1}^k t_i^n}\prod_{1\le i\ne j\le k}(t_i-t_j)\Phi(e(t'))\right] =$$

$$= \frac{1}{k!}\operatorname*{Res}_{t_1=0}\dots\operatorname*{Res}_{t_k=0}\left[\frac{\prod_{i=1}^k dt_i}{\prod_{i=1}^k t_i^n}\prod_{1\le i\ne j\le k}(t_i-t_j)\Phi(e(t'))\right],$$

hence

$$\int_{Gr_k(n,\mathbb{C})}\Phi(c(\mathcal{L}^*)) = \frac{1}{k!}\operatorname*{Res}_{t_1=0}\dots\operatorname*{Res}_{t_k=0}\left[\frac{\prod\limits_{i=1}^k dt_i}{\prod\limits_{i=1}^k t_i^n}\prod_{1\le i\ne j\le k}(t_i-t_j)\Phi(e(t_1,\dots,t_k))\right]. \tag{7.1}$$

This iterated residue formula agrees with the Jeffrey-Kirwan formula for the Grassmannian constructed as symplectic quotient (cf. [7, Proposition 7.2]).

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A class of harmonic univalent functions associated with modified $q-$Catas operator

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Abstract. Using the modified $q-$Catas operator, we define a class of harmonic univalent functions and obtain various properties for functions in this class.

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1. Introduction

A function $f = u + iv$, continuous and defined in a simply connected complex domain $\mathcal{D}$ is called harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. If h, g are analytic in $\mathcal{D}$, then f can be written in the form

$$f = h + \overline{g}, \tag{1.1}$$

where, h and g are the analytic and co-analytic parts, respectively. The necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'| > |g'|$ in $\mathcal{D}$ (see [17]).

The class of harmonic, univalent, and orientation preserving functions, of the form (1.1) defined in $\mathcal{E} = \{z : |z| < 1\}$ is denoted by $\mathcal{H}$, for which $f(0) = f'(0)-1 = 0$.

Thus, for $f = h + \overline{g} \in \mathcal{H}$, h and g can be expressed in the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g = \sum_{k=1}^{\infty} b_k z^k, \ |b_1| < 1,$$

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then f is of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=1}^{\infty}\overline{b_kz^k}. \tag{1.2}$$

If the co-analytic part $g\equiv 0$, then $\mathcal{H}$ reduces to class $\mathcal{S}$ of normalized analytic univalent functions.

Let $\overline{\mathcal{H}}$ denotes the subclass of $\mathcal{H}$ consisting of functions $f=h+\overline{g}$ such that h and g given by

$$h(z)=z+\sum_{k=2}^{\infty}a_kz^k,\quad g=(-1)^n\sum_{k=1}^{\infty}b_kz^k,|b_1|<1. \tag{1.3}$$

Recently, several researchers studied classes of harmonic functions (see Aouf [3], Aouf et al. [8, 10, 11], Dixit and Porwal [18], Porwal and Dixit [26, 27]).

For $f\in\mathcal{S}$, and $0<q<1$, the Jackson's $q-$derivative is given by [22] (see also [2, 4, 7, 12, 19, 20, 21, 28, 30, 31, 32]):

$$D_qf(z)=\begin{cases}\frac{f(z)-f(qz)}{(1-q)z} & z\neq 0\\ 0 & z=0\end{cases}\ (z\in\mathcal{E}), \tag{1.4}$$

where

$$[k]_q=\frac{1-q^k}{1-q}\quad (0<q<1).$$

As $q\to 1^-$, $[k]_q\to k$ and, so $D_qf(z)=f'(z)$.

For $f(z)\in\mathcal{S},\delta,l\geq 0$ and $q\in(0,1)$, Aouf and Madian [5, with $p=1$] defined the $q-$Catas operator by:

$$\begin{aligned}
I_q^0(\delta,l)f(z) &= f(z),\\
I_q^1(\delta,l)f(z) &= (1-\delta)f(z)+\frac{\delta}{[l+1]_qz^{l-1}}D_q(z^lf(z))=I_q(\delta,l)f(z)\\
&= z+\sum_{k=2}^{\infty}\frac{[l+1]_q+\delta([k+l]_q-[l+1]_q)}{[l+1]_q}a_kz^k\\
&\ \vdots\\
I_q^n(\delta,l)f(z) &= (1-\delta)I_q^{n-1}(\delta,l)f(z)+\frac{\delta}{[l+1]_qz^{l-1}}D_q(z^lI_q^{n-1}(\delta,l)f(z))\\
(n &\in \mathcal{N},\mathcal{N}=\{1,2,...\}).
\end{aligned}$$

That is

$$I_q^n(\delta,l)f(z)=z+\sum_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)a_kz^k\ .(n\in\mathcal{N}_0=\mathcal{N}\cup\{0\}), \tag{1.5}$$

where

$$\sigma_{q,k}^n(\delta,l)=\left[\frac{[l+1]_q+\delta([k+l]_q-[l+1]_q)}{[l+1]_q}\right]^n. \tag{1.6}$$

From (1.5) we have:

$$z\delta q^lD_q(I_q^n(\delta,l)f(z))=[l+1]_qI_q^{n+1}(\delta,l)f(z)-\{(1-\delta)q^l+[l]_q\}I_q^n(\delta,l)f(z),\delta\neq 0.$$

Note that:

(i) $\lim_{q\to 1-} I_q^n(\delta,l)f(z) = I^n(\delta,l)f(z)$, (see [14]);

(ii) $I_q^n(1,0)f(z) = D_q^n f(z)$ (see Govindaraj and Sivasubramanian [21] and [9]);

(iii) $I_q^n(\delta,0)f(z) = D_{\delta,q}^n f(z)$:

$$\left\{ f \in \mathcal{S} : D_{\delta,q}^n f(z) = z + \sum_{k=2}^{\infty} \left[1+\delta([k]_q - 1)\right]^n a_k z^k \right\},$$

which reduces to Al-Oboudi operator when $q \to 1-$, (see [1]) which is the Salagean operator when $\delta = 1$ (see [29] and [6]);

(iv) $I_q^n(1,l)f(z) = I_q^n(l)f(z)$ which when $q \to 1-$ reduces to $I_l^n f(z)$ (see Cho and Srivastava [15], see also [16]).

Motivated with the definition of modified Salagean operator introduced by Jahangiri et al. [23], Mostafa et al. [24], defined the modified Catas operator by

$$I^n(\delta,l)\, f(z) = I^n(\delta,l)\, h(z) + (-1)^n\, \overline{I^n(\delta,l)\, g(z)},$$

where

$$I^n(\delta,l)\, h(z) = z + \sum_{k=2}^{\infty} \left(\frac{l+1+\delta(k-1)}{l+1}\right)^n a_k z^k$$

and

$$I^n(\delta,l)\, g(z) = (-1)^n \sum_{k=1}^{\infty} \left(\frac{l+1+\delta(k-1)}{l+1}\right)^n b_k z^k.$$

Now, we define the modified $q-$Catas operator by:

$$I_q^n(\delta,l)\, f(z) = I_q^n(\delta,l)\, h(z) + (-1)^n\, \overline{I_q^n(\delta,l)\, g(z)}, \tag{1.7}$$

where

$$I_q^n(\delta,l)\, h(z) = z + \sum_{k=2}^{\infty} \sigma_{q,k}^n(\delta,l) a_k z^k$$

and

$$I_q^n(\delta,l)\, g(z) = \sum_{k=1}^{\infty} \sigma_{q,k}^n(\delta,l) b_k z^k.$$

For $1 \le \beta < \frac{4}{3}, n \in \mathcal{N}_0, \delta, l \ge 0, q \in (0,1)$ and for all $z \in \mathcal{E}$, let $\mathcal{G}_q^n(\delta,l,\beta)$ denote the family of harmonic functions f of the form (1.2) and satisfying:

$$\operatorname{Re}\left\{\frac{I_q^{n+1}(\delta,l)\, f(z)}{I_q^n(\delta,l)\, f(z)}\right\} < \beta. \tag{1.8}$$

Choosing different values of n, l, δ, β when $q \to 1-$, we obtain many subclasses of $\mathcal{G}_q^n(\delta,l,\beta)$ for example:

(1) Putting $\delta = 1$, then it reduces to the class $S_H(n,l,\beta)$ studied by Porwal [25].

(2) Putting $\delta = 1$ and $l = 0$, then it reduces to the class $S_H(n,\beta)$ studied by Porwal and Dixit [27];

(3) Putting $n = 0, l = 0$ and $\delta = 1$, then it reduces to the class $L_H(\beta)$ studied by Porwal and Dixit [26];

(4) Putting $n = 1, l = 0$ and $\delta = 1$, then it reduces to the class $M_H(\beta)$ studied by Porwal and Dixit [26];

(5) Putting $n = 0$ and $n = 1$ with $l = 0, \delta = 1$, $g \equiv 0$, then it reduces to the classes $\mathcal{N}(\beta)$ and $\mathcal{M}(\beta)$ studied by Uralegaddi et al. [33].

Also we can obtain the following subclasses:
i) $\mathcal{G}_q^n(\delta, 0, \beta) = \mathcal{G}_q^n(\delta, \beta)$:

$$\operatorname{Re}\left\{\frac{D_q^{n+1}(\delta)\, f(z)}{D_q^n(\delta)\, f(z)}\right\} < \beta, D_q^n(\delta)\, f(z) = D_q^n(\delta)\, h(z) + (-1)^n\, \overline{D_q^n(\delta)\, g(z)};$$

ii) $\mathcal{G}_q^n(1, 0, \beta) = \mathcal{G}_q^n(\beta)$:

$$\operatorname{Re}\left\{\frac{D_q^{n+1} f(z)}{D_q^n f(z)}\right\} < \beta, D_q^n f(z) = D_q^n h(z) + (-1)^n\, \overline{D_q^n g(z)};$$

iii) $\mathcal{G}_q^n(1, l, \beta) = \mathcal{G}_q^n(l, \beta)$:

$$\operatorname{Re}\left\{\frac{I_q^{n+1}(l) f(z)}{I_q^n(l) f(z)}\right\} < \beta, I_q^n(l) f(z) = I_q^n(l) h(z) + (-1)^n\, \overline{I_q^n(l) g(z)}.$$

Let $\overline{\mathcal{G}_q}^n(\delta, l, \beta)$ be the subclass of $\mathcal{G}_q^n(\delta, l, \beta)$ consisting functions $f = h + \overline{g}$ such that h and g given by (1.3).

2. Main results

Unless otherwise mentioned, we assume in the reminder of this paper that, $1 \leq \beta < \frac{4}{3}, n \in \mathcal{N}_0, \delta, l \geq 0, q \in (0, 1)$, $\sigma_{q,k}^n(\delta, l)$ is given by (1.6) and f is of the form (1.3).

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.2). Furthermore, let*

$$\sum_{k=2}^{\infty} \frac{\sigma_{q,k}^n(\delta, l)\, \{[l+1]_q + \delta([k+l]_q - [l+1]_q) - \beta[l+1]_q\}}{[l+1]_q(\beta - 1)}\, |a_k| +$$

$$+ \sum_{k=1}^{\infty} \frac{\sigma_{q,k}^n(\delta, l)\, \{[l+1]_q + \delta([k+l]_q - [l+1]_q) + \beta[l+1]_q\}}{[l+1]_q(\beta - 1)}\, |b_k| \leq 1. \tag{2.1}$$

Then $f\ (z)$ is sense-preserving, harmonic univalent in $\mathcal{E}$ and $f\,(z) \in \mathcal{G}_q^n(\delta, l, \beta)$.

Proof. If $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum\limits_{k=1}^{\infty} b_k\left(z_1^k-z_2^k\right)}{\left(z_1^k-z_2^k\right)+\sum\limits_{k=2}^{\infty} a_k\left(z_1^k-z_2^k\right)}\right|$$

$$> 1-\frac{\sum\limits_{k=1}^{\infty} k\,|b_k|}{1-\sum\limits_{k=2}^{\infty} k\,|a_k|}$$

$$\geq 1-\frac{\frac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{[l+1]_q(\beta-1)}\,|b_k|}{1-\frac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{[l+1]_q(\beta-1)}\,|a_k|} \geq 0,$$

which proves univalence. $f(z)$ is sense-preserving in $\mathcal{E}$ since

$$\begin{aligned}\left|h'(z)\right| &\geq 1-\sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1}\\ &> 1-\sum_{k=2}^{\infty} k\,|a_k| \geq 1-\sum_{k=2}^{\infty}\frac{\sigma_{q,k}^n(\delta,l)\,\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{[l+1]_q(\beta-1)}\,|a_k|\\ &\geq \sum_{k=1}^{\infty}\frac{\sigma_{q,k}^n(\delta,l)\,\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{[l+1]_q(\beta-1)}\,|b_n| \geq \sum_{k=1}^{\infty} k\,|b_k|\\ &> \sum_{k=1}^{\infty} k\,|b_k|\left|z^{k-1}\right| \geq \left|g'(z)\right|.\end{aligned}$$

Now to show that $f \in \mathcal{G}_q^n(\delta,l;\beta)$, we may show that if (2.1) holds then (1.8) is satisfied. Using the fact that $Re\{w\} < \beta$ if and only if $|w-1| < |w+1-2\beta|$, it suffices to show that

$$\left|\frac{\frac{I_q^{n+1}(\delta,l)\,f(z)}{I_q^n(\delta,l)\,f(z)}-1}{\frac{I_q^{n+1}(\delta,l)\,f(z)}{I_q^n(\delta,l)\,f(z)}+1-2\beta}\right| < 1. \tag{2.2}$$

The L.H.S. of (2.2):

$$=\left|\frac{\left\{\begin{array}{c}\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\,\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-[l+1]_q\}\,a_k z^k\\ +(-1)^{n+1}\sum\limits_{k=1}^{\infty}\sigma_{q,k}^n(\delta,l)\,\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+[l+1]_q\}\,\overline{b_k z^k}\end{array}\right.}{\left\{\begin{array}{c}2(1-\beta)z+\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\,\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+(1-2\beta)[l+1]_q\}\,a_k z^k\\ +(-1)^{n+1}\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\,\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-(1-2\beta)[l+1]_q\}\,\overline{b_k z^k}\end{array}\right.}\right|$$

$$\le \frac{\left\{\begin{array}{c}\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-[l+1]_q\right\}|a_k|\,|z|^k\\+\sum\limits_{k=1}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+[l+1]_q\right\}|b_k|\,|z|^k\end{array}\right.}{\left\{\begin{array}{c}2(\beta-1)-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-(2\beta-1)[l+1]_q\right\}|a_k|\,|z|^k\\-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+(2\beta-1)[l+1]_q\right\}|b_k|\,|z|^k\end{array}\right.}$$

$$< \frac{\left\{\begin{array}{c}\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-[l+1]_q\right\}|a_k|\\+\sum\limits_{k=1}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+[l+1]_q\right\}|b_k|\end{array}\right.}{\left\{\begin{array}{c}2(\beta-1)-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-(2\beta-1)[l+1]_q\right\}|a_k|\\-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+(2\beta-1)[l+1]_q\right\}|b_k|\end{array}\right.},$$

which according to (1.8) is bounded by 1. The harmonic univalent function of the form

$$\begin{aligned} f(z) &= z+\sum_{k=2}^{\infty}\frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\right\}}x_k z^k \\ &\quad+\sum_{k=1}^{\infty}\frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\right\}}\overline{y_k z^k}, \quad (2.3)\end{aligned}$$

where $\sum\limits_{k=2}^{\infty}|x_k|+\sum\limits_{k=1}^{\infty}|y_k|=1$, shows that the coefficient bound given by (2.1) is sharp. It is worthy to note that the function of the form (2.3) belongs to the class $\mathcal{G}_q^n(\delta,l,\beta)$ for all $\sum\limits_{k=2}^{\infty}|x_k|+\sum\limits_{k=1}^{\infty}|y_k|\le 1$ since (2.1) holds. □

Theorem 2.2. *A function $f\in\overline{\mathcal{G}_q}^n(\delta,l,\beta)$ if and only if*

$$\begin{aligned} &\sum_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\right\}|a_k| \quad (2.4)\\ &+\sum_{k=1}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\right\}|b_k| \\ \le\ &(\beta-1)[l+1]_q.\end{aligned}$$

Proof. Since $\overline{\mathcal{G}_q}^n(\delta, l, \beta) \subset \mathcal{G}_q^n(\delta, l, \beta)$, we only need to prove the "only if" part. The condition (1.8) is equivalent to

$$\operatorname{Re}\left\{\frac{\begin{array}{c}(\beta-1)z-\sum_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\right\}|a_k|\,z^k\\ -(-1)^{2n-1}\sum_{k=1}^{\infty}\sigma_{q,k}^n(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\right\}|b_k|\,\overline{z^k}\end{array}}{z+\sum_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\,|a_k|\,z^k+\sum_{k=1}^{\infty}(-1)^{2n-1}\sigma_{q,k}^n(\delta,l)\,|b_k|\,\overline{z^k}}\right\}>0.$$

The above condition must hold for all z, $|z| = r < 1$. Choosing the values of z on the positive real axis where $0 \le r < 1$, we must have

$$\operatorname{Re}\left\{\frac{\begin{array}{c}(\beta-1)-\sum_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\right\}|a_k|\,r^{k-1}\\ -\sum_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\right\}|b_k|\,r^{k-1}\end{array}}{1-\sum_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\,|a_k|\,r^{k-1}-\sum_{k=1}^{\infty}\sigma_{q,k}^n(\delta,l)\,|b_k|\,r^{k-1}}\right\}\ge 0. \tag{2.5}$$

If condition (2.4) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0 \in (0, 1)$ for which the quotient in (2.5) is negative. This contradicts the required condition for $f(z) \in \overline{\mathcal{G}_q}^n(\delta, l, \beta)$. This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let $f(z) \in \overline{\mathcal{G}_q}^n(\delta, l, \beta)$. Then for $|z| = r < 1$, we have*

$$\begin{aligned}&(1+|b_1|)r-\frac{1}{\sigma_{q,2}^n(\delta,l)}\left(\frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}-\frac{(\beta+1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}\,|b_1|\right)r^2\\ \le\ & |f(z)| \le\\ &(1+|b_1|)r+\frac{1}{\sigma_{q,2}^n(\delta,l)}\left(\frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^{m-n}(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}+\frac{(\beta+1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}\,|b_1|\right)r^2.\end{aligned} \tag{2.6}$$

The results are sharp with equality for $f(z)$ defined by

$$f(z)=z\pm b_1\overline{z}\pm\frac{1}{\sigma_{q,2}^n(\delta,l)}\left(\frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^{m-n}(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}-\frac{(\beta+1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}\,|b_1|\right)\overline{z}^2, \tag{2.7}$$

where

$$\sigma_{q,2}^n(\delta,l)=\left[\frac{[l+1]_q+\delta q^{l+1}}{[l+1]_q}\right]^n. \tag{2.8}$$

Proof. We only prove the right-hand inequality and the proof of the left-hand is similar and will be omitted. Let $f(z) \in \overline{\mathcal{G}}^n(\delta, l, \beta)$.Taking the absolute value of f we have:

$$|f(z)| \le (1+|b_1|)r + r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|)$$
$$= (1+|b_1|)r + \frac{(\beta-1)\,[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} \sum_{n=2}^{\infty} \left(\frac{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}}{(\beta-1)\,[l+1]_q} |a_k| + \frac{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}{(\beta-1)[l+1]_q} |b_k| \right) r^2$$
$$\le (1+|b_1|)r + \frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} \sum_{k=2}^{\infty} \left(\frac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q} |a_k| + \frac{\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q} |b_k| \right) r^2$$
$$\le (1+|b_1|)r + \frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} \left(1 - \frac{(1+\beta)}{(\beta-1)} |b_1|\right) r^2$$
$$= (1+|b_1|)r + \left(\frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} - \frac{(1+\beta)\,[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} |b_1| \right) r^2.$$

This completes the proof of the Theorem 2.3. □

Theorem 2.4. *The function* $f(z) \in \overline{\mathcal{G}_q}^n(\delta,l,\beta)$ *if and only if*

$$f(z) = \sum_{k=1}^{\infty} (\gamma_k h_k(z) + \eta_k g_k(z)), \tag{2.9}$$

where $h_1(z) = z$,

$$h_k(z) = z + \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}} z^k, k = 2,3,... \tag{2.10}$$

and

$$g_k(z) = z + (-1)^{n-1} \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}} \overline{z}^k,\ k=1,2,..., \tag{2.11}$$

$\gamma_k \ge 0, \eta_k \ge 0, \sum_{k=1}^{\infty} (\gamma_k + \eta_k) = 1$ *and the extreme points of the class* $\overline{\mathcal{G}_q}^n(\delta,l,\beta)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} (\gamma_k h_k(z) + \eta_k g_k(z)) \\ &= z + \sum_{k=2}^{\infty} \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}} \gamma_k z^k \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}} \eta_k \overline{z^k}. \end{aligned}$$

Then

$$\sum_{k=2}^{\infty} \tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q} \left(\tfrac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}\gamma_k\right)$$
$$+\sum_{k=1}^{\infty} \tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q} \left(\tfrac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}\eta_k\right)$$
$$= \sum_{k=2}^{\infty}\gamma_k+\sum_{k=1}^{\infty}\eta_k = 1-\gamma_1 \le 1$$

and so $f(z) \in \overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$.

Conversely, if $f(z) \in \overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$, then

$$|a_k| \le \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}$$

and

$$|b_k| \le \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}.$$

Setting

$$\gamma_k = \frac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}|a_k| \ \ (k=2,3,...)$$

and

$$\eta_k = \tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}|b_k| \ \ (k=1,2,...),$$

we have $0 \le \gamma_k \le 1$ $(k=2,3,...)$ and $0 \le \eta_k \le 1$ $(k=1,2,...)$,

$$\gamma_1 = 1-\sum_{k=2}^{\infty}\gamma_k-\sum_{k=1}^{\infty}\eta_k \ge 0,$$

then, $f(z)$ can be expressed in the form (2.9).
For harmonic functions of the form:

$$f(z) = z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^{n-1}\sum_{k=1}^{\infty}|b_k|\overline{z^k} \tag{2.12}$$

and

$$G(z) = z+\sum_{k=2}^{\infty}|d_k|z^k+(-1)^{n-1}\sum_{k=1}^{\infty}|c_k|\overline{z^k}, \tag{2.13}$$

the convolution of f and G is given by

$$(f*G)(z) = (G*f)(z) = z+\sum_{k=2}^{\infty}|a_kd_k|z^k+\sum_{k=1}^{\infty}|b_kc_k|\overline{z^k}. \qquad \square$$

The next theorem shows that the class $\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$ is closed under convolution.

Theorem 2.5. *For* $1 \le \beta \le \zeta < \frac{4}{3}$, *let* $f \in \overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$ *and* $G \in \overline{\mathcal{G}_q}^{n}(\delta,l,\zeta)$. *Then* $f*G \in \overline{\mathcal{G}_q}^{n}(\delta,l,\beta) \subset \overline{\mathcal{G}_q}^{n}(\delta,l,\zeta)$.

Proof. Since $G \in \overline{\mathcal{G}_q}^n(\delta, l, \zeta)$ then $|d_k| \leq 1$ and $|c_k| \leq 1$. For $f * G$ we have

$$\begin{aligned}
& \sum_{k=2}^{\infty} \tfrac{\sigma^n_{q,k}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1_q} |a_k d_k| \, z^k \\
& + \sum_{k=1}^{\infty} \tfrac{\sigma^n_{q,k}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1_q} |b_k c_k| \, \overline{z^k} \\
\leq \; & \sum_{k=2}^{\infty} \tfrac{\sigma^n_{q,k}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1_q} |a_k| \, z^k \\
& + \sum_{k=1}^{\infty} \tfrac{\sigma^n_{q,k}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1_q} |b_k| \, \overline{z^k} \\
\leq \; & \sum_{k=2}^{\infty} \tfrac{\sigma^n_{q,k}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1_q} |a_k| \\
& + \sum_{k=1}^{\infty} \tfrac{\sigma^n_{q,k}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1_q} |b_k| \\
\leq \; & 1,
\end{aligned}$$

since $f \in \overline{\mathcal{G}_q}^n(\delta, l, \beta)$. Therefore by Theorem 2.1, $f * G \in \overline{\mathcal{G}}^n_q(\delta, l, \beta) \subset \overline{\mathcal{G}_q}^n(\delta, l, \zeta)$. □

The class $\overline{\mathcal{G}_q}^n(\delta, l, \beta)$ is closed under convex combinations by the following theorem.

Theorem 2.6. *The class $\overline{\mathcal{G}_q}^n(\delta, l, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...,$ let $f_i \in \overline{\mathcal{G}_q}^n(\delta, l, \beta)$, where f_i is given by

$$f_i = z + \sum_{k=2}^{\infty} |a_{k_i}| \, z^k + \sum_{k=1}^{\infty} |b_{k_i}| \, \overline{z^k}.$$ □

Then by using Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \tfrac{\sigma^n_{q,k}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q} |a_{k_i}|$$

$$+ \sum_{k=1}^{\infty} \tfrac{\sigma^n_{q,k}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q} |b_{k_i}| \leq 1. \tag{2.14}$$

For $\sum\limits_{k=1}^{\infty} \mu_i = 1$, $0 \leq \mu_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} \mu_i f_i(z) = z + \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} \mu_i |a_{k_i}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \mu_i |b_{k_i}| \right) \overline{z^k}. \tag{2.15}$$

Then by (2.14), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\left(\sum_{i=1}^{\infty}\mu_i\,|a_{k_i}|\right)\\
&\quad+\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}\mu_i\,|b_{k_i}|\right)\\
=&\sum_{i=1}^{\infty}\mu_i\left(\sum_{k=2}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\,|a_{k_i}|\right.\\
&\quad\left.+\sum_{k=1}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\,|b_{k_i}|\right)\\
\le&\sum_{i=1}^{\infty}\mu_i=1.
\end{aligned}$$

By Theorem 2.2, $\sum\limits_{i=1}^{\infty}\mu_i f_i\,(z)\in\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$.

Let $f(z)=h(z)+\overline{g(z)}$ be defined by (1.2) then $F(z)$ defined by

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt}(c>-1),$$

have the representation

$$F(z)=z+\sum_{k=2}^{\infty}\frac{c+1}{k+c}a_kz^k+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{c+1}{k+c}b_k\overline{z}^k. \tag{2.16}$$

Theorem 2.7. *Let $f(z)=h(z)+\overline{g(z)}\in\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$, then $F(z)$ defined by (2.16) also belongs to $\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$.*

Proof. Since $f(z)\in\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$, then (2.1) is satisfied.

Now,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\frac{c+1}{k+c}\,|a_k|\\
&\quad+\sum_{k=1}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\frac{c+1}{k+c}\,|b_k|\\
\le&\sum_{k=2}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\,|a_k|\\
&\quad+\sum_{k=1}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\,|b_k|\\
\le&\ 1,
\end{aligned}$$

that is, $F(z)\in\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$. □

Remark 2.8. (i) Taking $\delta = 1$ and $q \to 1-$, in the above results, we obtain the results obtained by Porwal [25].

(ii) Specializing the parameters β, l, δ and n in the above results, we obtain the corresponding results for the subclasses $\mathcal{G}_q^n(\delta, \beta), \mathcal{G}_q^n(\beta)$ and $\mathcal{G}_q^n(1, \beta)$.

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Univalence conditions of an integral operator on the exterior unit disk

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Abstract. Into this article, we consider the subclasses $V_j, V_{j,\mu}$ and $\sum_j(p)$, with $j = 2, 3, ...$, and generalize univalence conditions for the integral operator $G_{\alpha_i,\beta}$ of the analytic functions g in the exterior unit disk. We want to see if some univalent conditions for analytic functions obtained on the interior unit disk can be extended on the exterior unit disk, so we make use of the usual transformation

$$g(z) = \frac{1}{f(\frac{1}{z})}.$$

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1. Introduction

Let O be the class of analytical functions g defined in the exterior of the unit disk $W = \{z \in \mathbb{C} | 1 < |z| < \infty\}$.

Let $\sum$ be the subclass of O which contains the univalent functions of W.

Let O_j be the subclass of O which contains the meromorphic, normalized and injective functions $g : W \longrightarrow \mathbb{C}_\infty$, that looks like [4]:

$$g(z) = z + \sum_{k=j+1}^{\infty} \frac{b_k}{z^k}, 1 < |z| < \infty. \tag{1.1}$$
$$(j \in \mathbb{N}_1^* := \mathbb{N} - \{0, 1\} = \{2, 3, ...\})$$

With $g(\infty) = \infty$, $g'(\infty) = 1$.

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Let V be the subclass of univalent functions from O such that:

$$\left|\frac{g'(z)}{z^2}+1\right|>1, z\in W. \tag{1.2}$$

Let V_j be the subclass of V, for which $g^{(k)}(\infty)=0$, $(k=2,3,...,j)$. Let $V_{j,\mu}$ be the subclass of V_j which contains the functions of the form (1.1) and satisfies the condition:

$$\left|\frac{g'(z)}{z^2}+1\right|>\mu, z\in W,$$

for $\mu>1$ and we denote $V_{j,1}\equiv V_j$.

Let $p\in\mathbb{R}$, with $1<p\leq 2$, let $\sum(p)$ be the subclass of O with all the functions $g\in O_j$ such that:

$$\left|\left(\frac{g(z)}{z}\right)''\right|\geq p, z\in W,$$

$$\left|\frac{g'(z)}{z^2}+1\right|\geq\frac{p}{|z|^j}, z\in W, j\in\mathbb{N}_1^*,$$

and we denote $\sum_2(p)\equiv\sum(p)$.

Let A be the class of analytic functions f defined in the open unit disk

$$U:=\{z\in\mathbb{C}:|z|<1\}$$

and normalized by the conditions $f(0)=0=f'(0)-1$.

Let S be the subclass of A consisting of univalent functions in U, of the form [3]:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k.$$

It is known that between the S class and the $\sum$ class there are the following links:

Proposition 1.1. [4]
(i) Let $f\in S$ and $g(\varsigma)=1/f(1/\varsigma)$, $\varsigma\in W$. Then $g\in\sum$ and $g(\varsigma)\neq 0$, $\varsigma\in W$.
(ii) If $g\in\sum$ and $g(\varsigma)\neq 0$, $\varsigma\in W$, then $f\in S$ where $f(z)=1/g(1/z)$, $z\in U$.

Let $F_{\alpha_1,\alpha_2,...,\alpha_n,\beta}$ be the integral operator introduced by Daniel Breaz and Narayanasamy Seenivasagan [10]:

$$F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)=\left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left[\frac{f_i(t)}{t}\right]^{\frac{1}{\alpha_i}}dt\right\}^{\frac{1}{\beta}}\in S,$$

and we take into account that $f_i(t)\in S$.

When $\alpha_i=\alpha$ for all $i=1,2,...,n$, $F_{\alpha_i,\beta}(z)$ becomes the integral operator $F_{\alpha,\beta}(z)$ considered in [1].

We may say that between A and O_1 there is a bijection.

We start from equation (1.2), and we apply the following transformations:

$$
\begin{aligned}
t &\to \frac{1}{t}|()', \\
dt &\to \frac{-1}{t^2}dt, \\
g_i(t) &= \frac{1}{f_i(\frac{1}{t})} \in O_1.
\end{aligned}
\tag{1.3}
$$

With $g_i(t) \neq 0$; $t \in O_1$.

We can form the integral operator from the definition below:

Definition 1.1. (see [9]) Let $g_i \in O_1$ with $i = 1, 2, ..., n$ and $\alpha_1, \alpha_2, ..., \alpha_n, \beta \in \mathbb{C}$ we define the integral operator $G_{\alpha_1,\alpha_2,...,\alpha_n,\beta} : O_1^n \longrightarrow O_1$, considering $|z| > 1$:

$$
G_{\alpha_i,\beta}(z) = \left[\beta \int_1^z t^{-1-\beta} \prod_{i=1}^{n} \left(\frac{t}{g_i(t)}\right)^{\frac{1}{\alpha_i}} dt\right]^{\frac{1}{\beta}}. \tag{1.4}
$$

Theorem 1.1. *(see [7]) Let $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $f \in A$. If the function f satisfies:*

$$
\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \cdot \left|\frac{z \cdot f''(z)}{f'(z)}\right| \leq 1, (z \in U),
$$

then, for any complex number β with $\Re(\beta) \geq \Re(\alpha)$, the integral operator:

$$
F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} \cdot f'(t)dt\right\}^{\frac{1}{\beta}},
$$

is in the class S.

Lemma 1.1. (The Schwarz lemma) (*see* [2], [5], [6]) *Let the analytic function f be regular in the unit disk and let $f(0) = 0$. If $|f(z)| \leq 1$, then:*

$$
|f(z)| \leq |z|,
$$

for all $z \in U$, where the equality can hold only if $|f(z)| = K \cdot z$ and $K = 1$.

Lemma 1.2. (General Schwarz Lemma) (*see* [6]) *Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If has one zero with multiplicity order bigger than m for $z = 0$, then:*

$$
|f(z)| \leq \frac{M}{R^m} \cdot |z|^m, \ (z \in U_R).
$$

The equality can hold only if $f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m$, where θ is constant.

2. Main results

Lemma 2.1. *Let the analytic function g be regular in the exterior of the unit disk and let $g(\infty) = \infty, g'(\infty) = 1$. If $|g(z)| \geq 1$, then:*

$$\left|f\left(\frac{1}{z}\right)\right| \leq \left|\frac{1}{z}\right|,$$

$$\frac{1}{|g(z)|} \leq \frac{1}{|z|} \quad \Bigg|()^{-1},$$

$$|g(z)| \geq |z|,$$

for all $z \in W$, where the equality can hold only if $|g(z)| = K \cdot z$ and $K = 1$.

In Lemma 1.2 we apply the transformation from equation (1.3), and we get the following Lemma:

Lemma 2.2. *Let the function g be regular in the exterior unit disk*

$$W_R = \{z \in \mathbb{C} : |z| > R\},$$

with $|f(z)| > M$ for fixed M. If has one zero with multiplicity order bigger than m for $z = \infty$, then:

$$\left|f\left(\frac{1}{z}\right)\right| \leq \frac{M}{R^m} \cdot \left|\frac{1}{z}\right|^m,$$

$$\left|\frac{1}{g(z)}\right| \leq \frac{M}{R^m} \cdot \frac{1}{|z|^m}\Bigg|()^{-1},$$

$$|g(z)| \geq \frac{R^m}{M} \cdot |z|^m,$$

for all $z \in W$, where the equality can hold only if $f(z) = e^{i\theta} \cdot \frac{R^m}{M} \cdot z^m$, where θ is constant.

Theorem 2.1. *Let $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $k \in O$. If k satisfies:*

$$\frac{|z|^{2\Re(\alpha)} - 1}{\Re(\alpha) \cdot |z|^{2\Re(\alpha)}} \cdot \left|\frac{k''(z)}{z \cdot k'(z)}\right| > 1, (z \in W),$$

and

$$\left|\frac{k''(z)}{zk'(z)}\right| > \Re(\alpha) \cdot |z|, \tag{2.1}$$

then, for any complex number β with $\Re(\beta) \leq \Re(\alpha)$, the integral operator:

$$G_\beta(z) = \left\{\beta \int_1^z t^{-\beta-1} \cdot k'(t)dt\right\}^{\frac{1}{\beta}},$$

is in the class $\sum$.

Proof. We apply in Theorem 1.1, the transformation $z \to \frac{1}{z}|(\ )'$.
We use $k(z) = \frac{1}{h(\frac{1}{z})}$, $\left|\frac{k''(z)}{z \cdot k'(z)}\right| > 1$ (see [9]) and $|k(z)| > 1$.

We multiply (2.1) with $\frac{|z|^{2\Re(\alpha)}-1}{\Re(\alpha)\cdot|z|^{2\Re(\alpha)}}$, and we get:

$$\frac{|z|^{2\Re(\alpha)}-1}{\Re(\alpha)\cdot|z|^{2\Re(\alpha)}}\cdot\left|\frac{k''(z)}{z\cdot k'(z)}\right|\geq\frac{|z|^{2\Re(\alpha)}-1}{|z|^{2\Re(\alpha)-1}}$$
$$\geq\frac{|z|^{2\Re(\alpha)-1}+|z|^{2\Re(\alpha)-2}+...+|z|+1}{|z|^{2\Re(\alpha)-1}}>1.$$

We obtain that, for any complex number β with $\Re(\beta)\leq\Re(\alpha)$, the integral operator:

$$G_\beta(z)=\left\{\beta\int_1^z t^{-\beta-1}\cdot k'(t)dt\right\}^{\frac{1}{\beta}},$$

is in the class $\sum$. ☐

Theorem 2.2. *Let* g_i *defined by:*

$$g_i(z)=z+\sum_{k=j+1}^{\infty}\frac{b_k^i}{z^k},|z|>1, \tag{2.2}$$

be in the class V_j *for* $i\in\{1,2,...,n\},n\in\mathbb{N}^*$, $j\in\mathbb{N}_1^*$.

If $|g_i(z)|\geq M_i(M_i\geq 1,\ z\in W)$.

Then the integral operator $G_{\alpha_i,\beta}$ *defined by (1.4) is in the class* $\sum$, *with* $\alpha,\beta\in\mathbb{C}$,

$$\Re(\alpha)\leq\sum_{i=1}^{n}\frac{1}{M_i|\alpha_i|}, \tag{2.3}$$

and $\Re(\beta)\leq\Re(\alpha)$.

Proof. We define a function:

$$k(z)=\int_0^z\prod_{i=1}^{n}\left(\frac{t}{g_i(t)}\right)^{\frac{1}{\alpha_i}}dt,$$

then we consider that $k(\infty)=\infty,k'(\infty)=1$.
After computation (see [8]) we obtain:

$$\frac{k''(z)}{z\cdot k'(z)}=\sum_{i=1}^{n}\frac{1}{\alpha_i}\cdot\left(\frac{1}{z^2}-\frac{g_i'(z)}{z\cdot g_i(z)}\right)$$
$$\left|\frac{k''(z)}{z\cdot k'(z)}\right|\geq\sum_{i=1}^{n}\frac{1}{M_i|\alpha_i|}.$$

We apply Theorem 2.1 and we consider (2.3), so we get:

$$\frac{|z|^{2\Re(\alpha)}-1}{\Re(\alpha)\cdot|z|^{2\Re(\alpha)}}\cdot\left|\frac{k''(z)}{z\cdot k'(z)}\right|\geq\frac{|z|^{2\Re(\alpha)}-1}{\Re(\alpha)\cdot|z|^{2\Re(\alpha)}}\cdot\sum_{i=1}^{n}\frac{1}{M_i|\alpha_i|}\geq$$
$$\geq\frac{1}{\Re(\alpha)}\cdot\sum_{i=1}^{n}\frac{1}{M_i|\alpha_i|}>1.$$

Applying Theorem 2.1, we obtain that $G_{\alpha_i,\beta}$ is univalent. ☐

Corollary 2.1. *Let g_i defined by (2.2) be in the class V_j for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M$ ($M \geq 1$, $z \in W$).*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Theorem 2.2, we consider $M_1 = M_2 = ... = M_n = M$. □

Corollary 2.2. *Let g_i defined by (2.2) be in the class V_j for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M$ ($M \geq 1, z \in W$).*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.1, we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$. □

Corollary 2.3. *Let g_i defined by (2.2) be in the class V_2, for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, If $|g_i(z)| \geq M$, ($M \geq 1, z \in W$).*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{M|\alpha|}$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.2, we consider $j = 2$. □

Corollary 2.4. *Let g_i defined by (2.2) be in the class V_2 for $i \in \{1, 2, ..., n\}, n \in \mathbb{N}^*$. If $|g_i(z)| \geq 1$ ($z \in W$).*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.2, we consider $j = 2$ and $M = 1$. □

Theorem 2.3. *Let g_i defined by (2.2) be in the class V_{j,μ_i} for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M_i$ ($M_i \geq 1, z \in W$).*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1 + \mu_i)M_i|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. The proof of this theorem is very similar with the proof of Theorem 2.2. □

Corollary 2.5. *Let g_i defined by (2.2) be in the class V_{j,μ_i} for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M$ $(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+\mu_i)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Theorem 2.3, we consider $M_1 = M_2 = ... = M_n = M$. □

Corollary 2.6. *Let g_i defined by (2.2) be in the class V_{j,μ_i} for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+\mu_i)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.5, we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$. □

Corollary 2.7. *Let g_i defined by (2.2) be in the class $V_{j,\mu}$ for $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+\mu)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.5, we consider $\mu_1 = \mu_2 = ... = \mu_n = \mu$. □

Corollary 2.8. *Let g_i defined by (2.2) be in the class V_{2,μ_i} for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+\mu_i)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.5, we set $j = 2$. □

Corollary 2.9. *Let g_i defined by (2.2) be in the class $V_{2,\mu}$ for $n \in \mathbb{N}^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+\mu)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.7, we set $j = 2$. □

Corollary 2.10. *Let g_i defined by (2.2) be in the class $V_{2,\mu}$ for $n \in \mathbb{N}^*$. If $|g_i(z)| \geq$ $(z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+\mu)|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.7, we set $j = 2$ and $M = 1$. □

Theorem 2.4. *Let g_i defined by (2.2) be in the class $\sum_j(p_i)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M_i(M_i \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M_i|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. The proof of this theorem is very similar with the proof of Theorem 2.2. □

Corollary 2.11. *Let g_i defined by (2.2) be in the class $\sum_j(p_i)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Theorem 2.4, we consider $M_1 = M_2 = ... = M_n = M$. □

Corollary 2.12. *Let g_i defined by (2.2) be in the class $\sum_j(p_i)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.11, we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$. □

Corollary 2.13. *Let g_i defined by (2.2) be in the class $\sum_j(p)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.11, we consider $p_1 = p_2 = ... = p_n = p$. □

Corollary 2.14. *Let* g_i *defined by (2.2) be in the class* $\sum_j(p)$ *for* $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If* $|g_i(z)| \geq M(M \geq 1, z \in W)$.
Then the integral operator $G_{\alpha,\beta}$ *defined by (1.4) is in the class* $\sum$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+p)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. If in Corollary 2.12, we consider $p_1 = p_2 = ... = p_n = p$ or in Corollary 2.13 we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$, we get the same result. □

Corollary 2.15. *Let* g_i *defined by (2.2) be in the class* $\sum_2(p)$ *for* $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$. *If* $|g_i(z)| \geq M(M \geq 1, z \in W)$.
Then the integral operator $G_{\alpha_i,\beta}$ *defined by (1.4) is in the class* $\sum$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.13, we set $j = 2$. □

Corollary 2.16. *Let* g_i *defined by (2.2) be in the class* $\sum_2(p)$ *for* $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$. *If* $|g_i(z)| \geq M(M \geq 1, z \in W)$.
Then the integral operator $G_{\alpha,\beta}$ *defined by (1.4) is in the class* $\sum$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+p)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.14, we set $j = 2$. □

Corollary 2.17. *Let* g_i *defined by (2.2) be in the class* $\sum_2(p)$ *for* $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$. *If* $|g_i(z)| \geq 1(z \in W)$.
Then the integral operator $G_{\alpha,\beta}$ *defined by (1.4) is in the class* $\sum$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+p)|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.14, we set $j = 2$ and $M = 1$. □

3. Final remarks

The main issue of the class of analytic functions defined on the exterior unit disk is that there are few studies in this branch. In this article, the authors studied some univalent conditions in the subclasses V_j, $V_{j,\mu}$ and $\sum_j(p)$ for analytic functions of an integral operator defined on the exterior of the unit disk, in order to find out if in the exterior unit disk and in the interior unit disk can be applied the same properties.

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Some applications of a Wright distribution series on subclasses of univalent functions

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Abstract. The purpose of the present paper is to find the sufficient conditions for the subclasses of analytic functions associated with Wright distribution series to be in subclasses of univalent functions and inclusion relations for such subclasses in the open unit disk $\mathbb{D}$. Further, we consider the properties of integral operator related to Wright distribution series.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = \sum_{n=1}^{\infty} a_n\, z^n; (a_1 := 1), \tag{1.1}$$

which are analytic in the open unit disk given by $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathcal{A}$ is said to be starlike of order $\gamma(0 \le \gamma < 1)$, if and only if $\operatorname{Re}(zf'(z)/f(z)) > \gamma$, which is denoted by $S^*(\gamma)$. We also write $S^*(\gamma) \subseteq S^*(0) := S^*$, where S^* denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. Also, a function $f \in \mathcal{A}$ is said to be convex of order $\gamma(0 \le \gamma < 1)$, if and only if $\operatorname{Re}(1 + (zf''(z)/f'(z))) > \gamma$. This function class is denoted by $\mathcal{K}(\gamma)$. We also write $\mathcal{K}(\gamma) \subseteq \mathcal{K}(0) := \mathcal{K}$, the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\gamma) \Leftrightarrow zf' \in S^*(\gamma)$.

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A function $f \in \mathcal{A}$ is said to be starlike of reciprocal order $\gamma (0 \leq \gamma < 1)$, if and only if

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) > \gamma, \quad (z \in \mathbb{U}). \tag{1.2}$$

We denote the class of such functions by $S_r^*(\gamma)$. Also, a function $f \in \mathcal{A}$ is said to be convex of reciprocal order $\gamma (0 \leq \gamma < 1)$, if and only if

$$\operatorname{Re}\left(\frac{f'(z)}{f'z) + zf''(z)}\right) > \gamma, \quad (z \in \mathbb{U}). \tag{1.3}$$

This function class is denoted by $\mathcal{K}_r^*(\gamma)$. We also write $S_r^*(0) := S^*, \mathcal{K}_r^*(0) = \mathcal{K}$ and $f \in \mathcal{K}_r^*(\gamma) \Leftrightarrow zf' \in S_r^*(\gamma)$.

In 2002, Owa and Srivastava [24] studied the classes of p-valent starlike and p-valent convex functions of reciprocal order γ with $\gamma > p$, and further investigated by Polatoglu et al. [25]. Uyanik et al. [36] introduced the classes of p-valently spirallike and p-valently Robertson functions (cf. [31]). Frasin and Sabri [9] derived sufficient condition for starlikeness of reciprocal order. Ravichandran and Kumar [30] investigated the argument estimates for the analytic functions $f \in S_r^*(\gamma)$. Al-Hawar and Frasin [2] determine coefficient bounds and subordination results of analytic functions of reciprocal order by means of Hadamard product. For more related results of some associated classes, see [1, 4, 6, 13, 16, 20, 22, 32, 33, 37].

Frasin et al. [10] introduced the subclasses of analytic functions of reciprocal order as

Definition 1.1. [10] A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{G}^{-1}(\gamma)$ of order γ if and only if it satisfies the condition

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) > \gamma, \quad (z \in \mathbb{U}), \tag{1.4}$$

for some $\gamma > 1$.

Definition 1.2. [10] A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}^{-1}(\gamma)$ of order γ if and only if it satisfies the condition

$$\operatorname{Re}\left(\frac{f'(z)}{f'(z) + zf''(z)}\right) > \gamma, \quad (z \in \mathbb{U}), \tag{1.5}$$

for some $\gamma > 1$.

It can be seen from (1.4) and (1.5) that

$$f(z) \in \mathcal{H}^{-1}(\gamma) \text{ if and only if } zf'(z) \in \mathcal{G}^{-1}(\gamma).$$

Remark 1.3. Silverman [34], consider the condition

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \gamma, \quad (z \in \mathbb{U}), \tag{1.6}$$

for the class $S^*(\gamma)$. This condition shows that the image of $\mathbb{U}$ by $\dfrac{zf'(z)}{f(z)}$ is inside of the circle with the center at 1 and the radius $1 - \gamma$, which is very small circle. If we consider the condition

$$\left|\frac{zf'(z)}{f(z)}-\frac{1}{2\gamma}\right|<\frac{1}{2\gamma},\quad (z\in\mathbb{U}), \tag{1.7}$$

for $0<\gamma<1$ the condition (1.7) shows that

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right)>\gamma,\quad (z\in\mathbb{U}),$$

which means that $f(z)\in S_r^*(\gamma)$. This condition (1.7) shows that the image of $\mathbb{U}$ by $\frac{zf'(z)}{f(z)}$ is inside of the circle with the center at $\frac{1}{2\gamma}$ and the radius $\frac{1}{2\gamma}$. Thus if $0<\gamma<\frac{1}{2}$, the condition (1.7) is better than (1.6). This is the motivation to discuss of the classes $S_r^*(\gamma)$ and $\mathcal{K}_r^*(\gamma)$.

Example 1.4. The function $f(z)=\frac{z}{(1-z)^{2(1-\gamma)}},(0<\gamma<1)$ is a starlike function of reciprocal order 0 in $\mathbb{U}$ ([23], Example 1).

Example 1.5. The function $f(z)=ze^{(1-\gamma)z},(0<\gamma<1)$ is a starlike function of reciprocal order $1/(2-\gamma)$ in $\mathbb{U}$ ([23], Example 2).

In recent years, several interesting subclasses of analytic functions were introduced and investigated from different view points. Several researchers including Altinkaya and Yalçin [3], Eker et al. [8], El-Deeb and Bulboacă [7], Nazeer et al. [21], Porwal and Ahamad [26], Porwal and Kumar [27], Wanas and Khuttar [38], and many more have studied interesting results on certain classes of univalent functions for various distribution series (see also, [17, 32]).

In 1933, Wright [39] introduced a special function known as Wright functions, is given by:

$$W_{\lambda,\kappa}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\lambda n+\kappa)}\frac{z^n}{n!}, \tag{1.8}$$

for $\lambda>-1,\kappa\in\mathbb{C}$ which is convergent for all $z\in\mathbb{C}$, while for $\lambda>-1$ this is absolutely convergent in $\mathbb{U}$. Gorenflo et al. [11] and Mustafa [18] gave insight of some characterizations and basic properties for the Wright functions. Prajapat [29] obtained certain geometric properties including univalency, starlikeness, convexity and close-to-convexity in the open unit disk $\mathbb{U}$ (see also, [15, 14, 19]). It is easy to see that the series (1.8) is not in normalized form so we normalized it as

$$\mathbb{W}_{\lambda,\kappa}=\Gamma(\kappa)zW_{\lambda,\kappa}(z)$$

$$\mathbb{W}_{\lambda,\kappa}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(\kappa)}{\Gamma(\lambda n+\kappa)}\frac{z^{n+1}}{n!}, \tag{1.9}$$

for $\lambda>-1,\kappa>0$ and $z\in\mathbb{U}$. Wright distribution recognized as a vitally important distribution in its own right, first we define the series

$$\mathbb{W}_{\lambda,\kappa}(s)=\sum_{n=0}^{\infty}\frac{\Gamma(\kappa)}{\Gamma(\lambda n+\kappa)}\frac{s^{n+1}}{n!},$$

which is convergent for all $\lambda, \kappa, s > 0$. The probability mass function of Wright distribution is given by

$$p(n) = \frac{\Gamma(\kappa)}{\Gamma(\lambda n + \kappa)\mathbb{W}_{\lambda,\kappa}(s)} \frac{s^{n+1}}{n!}, \quad \lambda, \kappa, s > 0; n = 0, 1, 2, \cdots .$$

The Wright distribution is a particular case of the familiar Poisson distribution which widely used as analysing traffic flow, fault prediction in electric cables, defects occurring in manufactured objects such as castings, email messages arriving at a computer and in the prediction of randomly occurring events or accidents.

Recently in 2022, Porwal et al. [28] invented Wright distribution series and gave a nice application of it on certain classes of univalent functions. Porwal et al. [28] introduce the Wright distribution series as follows

$$K_{\psi}(\lambda, \kappa, s, z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\kappa) s^n}{\Gamma(\lambda(n-1) + \kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)} z^n. \tag{1.10}$$

Porwal et al. [28] introduced the linear operator $\mathcal{I}(\lambda, \kappa, s) : \mathcal{A} \to \mathcal{A}$ defined by using the Hadamard (convolution) product as

$$\mathcal{I}(\lambda, \kappa, s)f(z) = K_{\psi}(\lambda, \kappa, s, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\kappa) s^n}{\Gamma(\lambda(n-1) + \kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)} a_n z^n. \tag{1.11}$$

To establish our main results, we need to recall the following lemmas due to Frasin et al. [10] and Dixit and Pal [5].

Lemma 1.6. [10] *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} (\gamma n - 1)|a_n| \leq \gamma - 1, \tag{1.12}$$

for some $\gamma > 1$, then $f(z) \in \mathcal{G}^{-1}(\gamma)$.

Lemma 1.7. [10] *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(\gamma n - 1)|a_n| \leq \gamma - 1, \tag{1.13}$$

for some $\gamma > 1$, then $f(z) \in \mathcal{H}^{-1}(\gamma)$.

Definition 1.8. A function $f \in \mathcal{A}$ is said to in the class $\mathcal{R}^{\tau}(\vartheta, \delta)$, if it satisfies the inequality

$$\left| \frac{(1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1}{2\tau(1-\delta) + (1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1} \right| < 1, \ \ (z \in \mathbb{D}),$$

where $\tau \in \mathbb{C} \setminus \{0\}, \delta < 1, 0 < \vartheta \leq 1$.The class $\mathcal{R}^{\tau}_{\delta}(\vartheta)$ was introduced by Swaminathan [35].

Lemma 1.9. [5] *If $f \in \mathcal{R}^{\tau}(\vartheta, \delta)$ is of the form* (1.1) *then*

$$|a_n| \leq \frac{|\tau|(\vartheta - \delta)}{n}, \ \ n \in \mathbb{N} \setminus \{1\}. \tag{1.14}$$

The result is sharp.

Motivated by the stated research works, we establish some sufficient conditions for the Wright distribution series $K_\psi(\lambda,\kappa,s,z)$ belonging to the classes $\mathcal{G}^{-1}(\gamma)$ and $\mathcal{H}^{-1}(\gamma)$. We also obtain inclusion relations for aforecited classes with $\mathcal{R}^\tau(C,D)$ by applying certain convolution operator $\mathcal{I}(\lambda,\kappa,s)$ defined by (1.11).

2. Main result

In this section, first we establish a sufficient condition for the function $f \in \mathcal{A}$ to be in the class $\mathcal{G}^{-1}(\lambda)$ and $\mathcal{H}^{-1}(\lambda)$.

Theorem 2.1. *Let $\lambda,\kappa,s>0$ and for some $\gamma(\gamma>1)$. Then $K_\psi(\lambda,\kappa,s,z)\in\mathcal{G}^{-1}(\gamma)$ if*

$$\gamma\Gamma(\kappa)\mathbb{W}_{\lambda,\kappa+\lambda}(s)\le(\gamma-1)\Gamma(\kappa+\lambda). \tag{2.1}$$

Proof. To prove that $K_\psi(\lambda,\kappa,s,z)\in\mathcal{G}^{-1}(\gamma)$, according to Lemma 1.6, we must show that

$$\sum_{n=2}^{\infty}(\gamma n-1)\frac{\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)}\le\gamma-1$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}(\gamma n-1)\frac{\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)}\\
&=\sum_{n=2}^{\infty}\{\gamma(n-1)+\gamma-1\}\frac{\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)}\\
&=\frac{1}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\sum_{n=2}^{\infty}\frac{\gamma\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-2)!}+\sum_{n=2}^{\infty}\frac{(\gamma-1)\Gamma(\kappa)s^n}{(\lambda(n-1)+\kappa)(n-1)!}\right]\\
&=\frac{1}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\gamma s\frac{\Gamma(\kappa)}{\Gamma(\kappa+\lambda)}\mathbb{W}_{\lambda,\kappa+\lambda}(s)+(\gamma-1)\{\mathbb{W}_{\lambda,\kappa}(s)-s\}\right]\\
&\le\gamma-1,\ \text{by the given hypothesis.}
\end{aligned}$$

This concludes the proof of Theorem 2.1. □

Theorem 2.2. *Let $\lambda,\kappa,s>0$ and for some $\gamma(\gamma>1)$. Then $K_\psi(\lambda,\kappa,s,z)\in\mathcal{H}^{-1}(\gamma)$ if*

$$\gamma s\frac{\Gamma(\kappa)}{\Gamma(\kappa+2\lambda)}\mathbb{W}_{\lambda,\kappa+2\lambda}(s)+(3\gamma-1)\frac{\Gamma(\kappa)}{\Gamma(\kappa+\lambda)}\mathbb{W}_{\lambda,\kappa+\lambda}(s)\le\gamma-1. \tag{2.2}$$

Proof. The proof is similar to Theorem 2.1. Therefore, we omit the details involved. □

Theorem 2.3. *Let $\lambda,\kappa,s>0, f\in\mathcal{R}^\tau(\vartheta,\delta)$ and for some $\gamma(\gamma>1)$. Then $\mathcal{I}(\lambda,\kappa,s)f\in\mathcal{H}^{-1}(\gamma)$ if*

$$\frac{(\vartheta-\delta)|\tau|}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\gamma s\frac{\Gamma(\kappa)}{\Gamma(\kappa+\lambda)}\mathbb{W}_{\lambda,\kappa+\lambda}(s)+(\gamma-1)\{\mathbb{W}_{\lambda,\kappa}(s)-s\}\right]\le\gamma-1. \tag{2.3}$$

Proof. To prove that $\mathcal{I}(\lambda,\kappa,s)f \in \mathcal{H}^{-1}(\gamma)$, according to Lemma 1.7, we must show that

$$\sum_{n=2}^{\infty} n(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)} \le \gamma - 1.$$

Since $f \in \mathcal{R}^{\tau}(\vartheta,\delta)$, from Lemma 1.9, we have $|a_n| \le \frac{|\tau|(\vartheta-\delta)}{n}$.
Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)}\\
&= \frac{(\vartheta-\delta)|\tau|}{\mathbb{W}_{\lambda,\kappa}(s)} \sum_{n=2}^{\infty}(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{\Gamma(\lambda(n-1)+\kappa)(n-1)!}\\
&= \frac{(\vartheta-\delta)|\tau|}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\sum_{n=2}^{\infty}\frac{\gamma\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-2)!} + \sum_{n=2}^{\infty}\frac{(\gamma-1)\Gamma(\kappa)s^n}{(\lambda(n-1)+\kappa)(n-1)!}\right]\\
&= \frac{(\vartheta-\delta)|\tau|}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\gamma s\frac{\Gamma(\kappa)}{\Gamma(\kappa+\lambda)}\mathbb{W}_{\lambda,\kappa+\lambda}(s) + (\gamma-1)\left\{\mathbb{W}_{\lambda,\kappa}(s) - s\right\}\right]\\
&\le \gamma - 1, \text{ by the given hypothesis.}
\end{aligned}$$

This concludes the proof of Theorem 2.3. □

3. An integral operator

Theorem 3.1. *If the function $\mathcal{G}(\lambda,\kappa,s,z)$ is given by*

$$\mathcal{G}(\lambda,\kappa,s,z) = \int_0^z \frac{K_\psi(\lambda,\kappa,s,t)}{t}dt, \quad z \in \mathbb{U} \tag{3.1}$$

then $\mathcal{G}(\lambda,\kappa,s,z) \in \mathcal{H}^{-1}(\gamma)$ if

$$\gamma\Gamma(\kappa)\mathbb{W}_{\lambda,\kappa+\lambda}(s) \le (\gamma-1)\Gamma(\kappa+\lambda). \tag{3.2}$$

Proof. Since

$$\mathcal{G}(\lambda,\kappa,s,z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(\kappa)s^n}{n!\Gamma(\lambda(n-1)+\kappa)\mathbb{W}_{\lambda,\kappa}(s)}z^n$$

by Lemma 1.7, we need only to show that

$$\sum_{n=2}^{\infty} n(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{n!\Gamma(\lambda(n-1)+\kappa)\mathbb{W}_{\lambda,\kappa}(s)} \le \gamma - 1.$$

or, consistently

$$\sum_{n=2}^{\infty}(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{(n-1)!\Gamma(\lambda(n-1)+\kappa)\mathbb{W}_{\lambda,\kappa}(s)} \le \gamma - 1.$$

The enduring part of the proof of Theorem 3.1 is parallel to that of Theorem 2.1, and so we omit the details. □

4. Conclusions

In this paper we have considered the subclasses $\mathcal{G}^{-1}(\lambda)$ and $\mathcal{H}^{-1}(\lambda)$ of reciprocal order related with Wright distribution series. We obtained sufficient condition, inclusion relation and properties related to integral operator for functions of these subclasses related to Wright distribution series.

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New subclasses of univalent functions on the unit disc in $\mathbb{C}$

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Abstract. In this paper we consider a differential operator $\mathcal{G}_k$ defined on the family of holomorphic normalized functions $\mathcal{H}_0(\mathbb{U})$ that can be used in the construction of new subclasses of univalent functions on the unit disc $\mathbb{U}$. These new subclasses are closely related to the families of convex, respectively starlike functions on $\mathbb{U}$. We study general results related to these new subclasses, such as growth and distortion theorems, coefficients estimates and duality results. We also present examples of functions that belongs to the subclasses defined.

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1. Preliminaries

Let us denote by $\mathbb{U} = \mathcal{U}(0;1)$ the open unit disc in the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{U})$ the family of all holomorphic functions on the unit disc $\mathbb{U}$. Also, let us denote by $\mathcal{H}_0(\mathbb{U})$ the class of normalized holomorphic functions on $\mathbb{U}$, i.e. $f \in \mathcal{H}(\mathbb{U})$ with $f(0) = 0$ and $f'(0) = 1$. An important class that will be used in our paper is the class of normalized univalent (holomorphic and injective) functions on the unit disc $\mathbb{U}$, denoted by S. For more details about the holomorphic functions and the class of normalized univalent functions, one may consult [2], [3], [5], [10] and [16].

Let us consider $\alpha \in [0,1)$. In [17] Robertson introduced two important subclasses of the class S, namely the family

$$S^*(\alpha) = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha, \quad z \in \mathbb{U} \right\}$$

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of normalized starlike functions of order α, respectively the family

$$K(\alpha) = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, \mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha, \quad z \in \mathbb{U} \right\}$$

of normalized convex functions of order α. In particular, we obtain that $S^* = S^*(0)$ and $K = K(0)$ are the usual families of starlike, respectively convex functions on the unit disc $\mathbb{U}$. For more details about these families of univalent functions, one may consult [2], [3], [5], or [16].

An important result related to the class S is due to Noshiro and Warschawski (see e.g. [2, Theorem 2.16]) and present a sufficient condition of univalence, as follows:

Theorem 1.1. *Let $f \in \mathcal{H}_0(\mathbb{U})$. If $\mathfrak{Re} f'(z) > 0$, for all $z \in \mathbb{U}$, then f is univalent on $\mathbb{U}$.*

Strongly related to the family S is the class of normalized holomorphic functions whose derivative has positive real part (of order α), denoted by

$$\mathcal{R}(\alpha) = \left\{ f \in \mathcal{H}_0(\mathbb{U}) \,:\, \mathfrak{Re} f'(z) > \alpha, \quad z \in \mathbb{U} \right\}, \quad \alpha \in [0, 1).$$

In view of Theorem 1.1 it is clear that $\mathcal{R}(\alpha) \subset S$. For more details about the class $\mathcal{R}(\alpha)$ of univalent functions whose derivatives have positive real part, one may consult [6], [11], [12] and [13] . In particular, the class $\mathcal{R}(0) = \mathcal{R}$ was studied by T.H. MacGregor in [13].

In the following sections of this paper we introduce a differential operator $\mathcal{G}_k$ defined on $\mathcal{H}_0(\mathbb{U})$ that is useful in the construction of new subclasses of univalent functions on $\mathbb{U}$ (denoted by E_k, respectively E_k^*) closely related to the class of convex, respectively starlike functions on the unit disc $\mathbb{U}$. An interesting property of these subclasses is that we can obtain coefficient estimates of the form $|a_n| \leq \frac{1}{(n-k)!}$, for $n \geq k$, where $k \in \mathbb{N}$ and $a_k, ..., a_n$ are the coefficients from the Taylor series expansion of the function $f \in \mathcal{H}_0(\mathbb{U})$.

Remark 1.2. It is important to mention that the operator $\mathcal{G}_k$ can be defined also in the case of several complex variables (see [8]). Although for $n = 1$ we have that $E_0(\mathbb{U}) = E_1^*(\mathbb{U}) = K(\mathbb{U})$, in the case of several complex variables we can prove that $E_1^*(\mathbb{B}^n) \cap K(\mathbb{B}^n) \neq \emptyset$, but $E_1^*(\mathbb{B}^n) \neq K(\mathbb{B}^n)$, where $K(\mathbb{B}^n)$ is the family of convex mappings on the Euclidean unit ball $\mathbb{B}^n$ (for details about univalent mappings in higher dimensions, one may consult [5] and [9]). Another interesting property of E_k^* studied in [8] is related to the Graham-Kohr extension operator (introduced by I. Graham and G. Kohr in [4]).

2. The differential operator $\mathcal{G}_k$

In this section we introduce the differential operator $\mathcal{G}_k$ defined on the family $\mathcal{H}_0(\mathbb{U})$ of normalized holomorphic functions on $\mathbb{U}$. For this operator we present some properties related to the linearity and univalence on the unit disc $\mathbb{U}$ and we discuss about how the convolution product is preserved under the action of the operator $\mathcal{G}_k$.

Definition 2.1. Let $k \in \mathbb{N} = \{0, 1, 2, ...\}$ and let $\mathcal{G}_k : \mathcal{H}_0(\mathbb{U}) \to \mathcal{H}(\mathbb{U})$ be the differential operator defined on the class of normalized holomorphic functions on $\mathbb{U}$, as follows

$$(\mathcal{G}_k f)(z) = \begin{cases} z^k f^{(k)}(z) + a_{k-1}z^{k-1} + ... + a_2 z^2 + a_1 z + a_0, & k \geq 1 \\ f(z) & k = 0, \end{cases} \tag{2.1}$$

for all $f \in \mathcal{H}_0(\mathbb{U})$ and $z \in \mathbb{U}$. Notice that, for $k \geq 1$, $a_0, ..., a_{k-1}$ are the first k coefficients from the Taylor series expansion of the function $f \in \mathcal{H}_0(\mathbb{U})$.

Remark 2.2. In view of the above definition, it is easy to see that the operator $\mathcal{G}_0$ (of order 0) is the identity operator, i.e. $\mathcal{G}_0 f = f$. Another particular form of the operator $\mathcal{G}_k$ is for $k = 1$ (of order 1). In this case, $(\mathcal{G}_1 f)(z) = zf'(z)$, for all $z \in \mathbb{U}$.

Remark 2.3. Let us denote $id : U \to \mathbb{C}$ the identity function on $\mathbb{U}$, given by $id(z) = z$, for all $z \in \mathbb{U}$. Then $\mathcal{G}_k(id) = id$, for all $k \in \mathbb{N}$.

The connection between two differential operators of consecutive orders $k - 1$, respectively k, where $k \in \mathbb{N}$ with $k \geq 1$, is given in the following result:

Proposition 2.4. *Let $f \in \mathcal{H}_0(\mathbb{U})$. Then for any $k \in \mathbb{N}^* = \{1, 2, ...\}$ the following relation holds*

$$(\mathcal{G}_k f)(z) = z(\mathcal{G}_{k-1} f)'(z) - (k-1)(\mathcal{G}_{k-1} f)(z) + \sum_{n=0}^{k-1}(k-n)a_n z^n, \quad z \in \mathbb{U}. \tag{2.2}$$

Proof. We prove relation (2.2) by mathematical induction. Assume that

$$P(k): \quad (\mathcal{G}_k f)(z) = z(\mathcal{G}_{k-1} f)'(z) - (k-1)(\mathcal{G}_{k-1} f)(z) + \sum_{n=0}^{k-1}(k-n)a_n z^n$$

is true for a fixed $k \in \mathbb{N}$ with $k \geq 2$. Then

$$z(\mathcal{G}_k f)'(z) - k(\mathcal{G}_k f)(z) = z^{k+1} f^{(k+1)}(z) + \sum_{n=0}^{k-1}(n-k)a_n z^n, \quad z \in \mathbb{U}.$$

Adding $\sum_{n=0}^{k}(k+1-n)a_n z^n$ at the previous equality, we obtain

$$z^{k+1} f^{(k+1)}(z) + \sum_{n=0}^{k-1}(n-k)a_n z^n + \sum_{n=0}^{k}(k+1-n)a_n z^n = (\mathcal{G}_{k+1} f)(z),$$

for all $z \in \mathbb{U}$ and this completes the proof. □

Proposition 2.5. *Let $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathcal{H}_0(\mathbb{U})$. Then*

$$\mathcal{G}_k(\alpha f + \beta g) = \alpha \mathcal{G}_k f + \beta \mathcal{G}_k g. \tag{2.3}$$

Proof. Let $f, g \in \mathcal{H}_0(\mathbb{U})$ be such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, for all $z \in \mathbb{U}$, with $a_0 = b_0 = 0$ and $a_1 = b_1 = 1$. Then

$$\begin{aligned} \mathcal{G}_k(\alpha f + \beta g)(z) &= z^k(\alpha f + \beta g)^{(k)}(z) + \sum_{n=0}^{k-1}(\alpha a_n + \beta b_n)z^n \\ &= \alpha(\mathcal{G}_k f)(z) + \beta(\mathcal{G}_k g)(z), \end{aligned}$$

for all $z \in \mathbb{U}$ and $\alpha, \beta \in \mathbb{R}$. □

Remark 2.6. For $f \in \mathcal{H}_0(\mathbb{U})$, we can rewrite (2.1) as

$$(\mathcal{G}_k f)(z) = z + a_2 z^2 + ... + a_{k-1} z^{k-1} + k! a_k z^k + (k+1)! a_{k+1} z^{k+1} + ...$$
$$... + \frac{(k+n)!}{n!} a_{k+n} z^{k+n} + ...,$$

for all $z \in \mathbb{U}$. In other words,

$$(\mathcal{G}_k f)(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad \text{where} \quad A_n = \begin{cases} a_n, & n \leq k-1 \\ \frac{n!}{(n-k)!} a_n, & n \geq k, \end{cases} \tag{2.4}$$

for all $z \in \mathbb{U}$.

Another interesting property of the operator $\mathcal{G}_k$ is related to the Hadamard (convolution) product (for details, one may consult [2], [3], [5]). Let $f, g \in \mathcal{H}_0(\mathbb{U})$ be given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. We denote by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U} \tag{2.5}$$

the Hadamard (convolution) product of the functions f and g on $\mathbb{U}$ (see e.g. [2], [3], [5]). There is a nice connection between the convolution product of two different operators and the operator applied on a convolution product, as follows in the next result.

Proposition 2.7. *Let $k \in \mathbb{N}$ and $f, g \in \mathcal{H}_0(\mathbb{U})$. Then*

1. $\mathcal{G}_k(f * g) = (\mathcal{G}_k f) * g = f * (\mathcal{G}_k g)$;
2. $(\mathcal{G}_k f) * (\mathcal{G}_k g) = \mathcal{G}_k(\mathcal{G}_k(f * g))$.

Proof. Let $f, g \in \mathcal{H}_0(\mathbb{U})$ be such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, for all $z \in \mathbb{U}$, with $a_0 = b_0 = 0$ and $a_1 = b_1 = 1$. Then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

for all $z \in \mathbb{U}$. Moreover, taking into account Remark 2.6, we deduce that

$$\mathcal{G}_k(f * g)(z) = z + \sum_{n=2}^{k-1} a_n b_n z^n + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{2.6}$$

1. First, in view of (2.4) and the definition of the convolution product, we obtain

$$\begin{aligned} \big((\mathcal{G}_k f) * g\big)(z) &= \left(z + \sum_{n=2}^{k-1} a_n z^n + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^n \right) * \left(z + \sum_{n=2}^{\infty} b_n z^n \right) \\ &= z + \sum_{n=2}^{k-1} a_n b_n z^n + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n b_n z^n \\ &= \mathcal{G}_k(f * g)(z) \end{aligned}$$

for all $z \in \mathbb{U}$. Similarly, we can prove that $\mathcal{G}_k(f * g) = f * (\mathcal{G}_k g)$.

2. For the second part, it is enough to consider relations (2.4) and (2.6). Then

$$\begin{aligned}\mathcal{G}_k(\mathcal{G}_k(f*g))(z) &= z + \sum_{n=2}^{k-1} a_n b_n z^n + \sum_{n=k}^{\infty}\left[\frac{n!}{(n-k)!}\right]^2 a_n b_n z^n \\ &= (\mathcal{G}_k f)(z) * (\mathcal{G}_k g)(z),\end{aligned}$$

for all $z \in \mathbb{U}$ and this completes the proof.

□

Remark 2.8. Notice that we can obtain the second statement from Proposition 2.7 by replacing f with $\mathcal{G}_k h$ (where $h \in \mathcal{H}_0(\mathbb{U})$) and using only the first part of the result. Then

$$(\mathcal{G}_k h) * (\mathcal{G}_k g) = f * (\mathcal{G}_k g) = \mathcal{G}_k(f * g) = \mathcal{G}_k((\mathcal{G}_k h) * g) = \mathcal{G}_k(\mathcal{G}_k(h * g))$$

and this completes the argument.

It is important that we can prove a sufficient condition of univalence for $\mathcal{G}_k$ (in terms of modulus of coefficients a_n), as follows

Proposition 2.9. *Let $k \in \mathbb{N}$ and $f \in \mathcal{H}_0(\mathbb{U})$. Also, let σ_k be defined by*

$$\sigma_k = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{n \cdot n!}{(n-k)!}|a_n|, & k \le 2 \\ \displaystyle\sum_{n=2}^{k-1} n|a_n| + \sum_{n=k}^{\infty} \frac{n \cdot n!}{(n-k)!}|a_n|, & k \ge 3. \end{cases} \tag{2.7}$$

If $\sigma_k \le 1$, then $\mathcal{G}_k f$ is univalent on the unit disc $\mathbb{U}$. In particular, $\mathcal{G}_k f \in S$.

Proof. It is easy to observe that $(\mathcal{G}_k f)(0) = 0$, $(\mathcal{G}_k f)'(0) = 1$ and $\mathcal{G}_k f$ is a holomorphic function on $\mathbb{U}$. In view of relation (2.7), we consider the following two cases:

- If $k \ge 3$, then

$$\begin{aligned}\left|(\mathcal{G}_k f)'(z) - 1\right| &= \left|1 + \sum_{n=2}^{k-1} n a_n z^{n-1} + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} n a_n z^{n-1} - 1\right| \\ &\le |z|\left(\sum_{n=2}^{k-1} n|a_n| + \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} n|a_n|\right) < 1,\end{aligned}$$

  for all $z \in \mathbb{U}$ and $k \ge 3$. Hence, $(\mathcal{G}_k f)'(z) \in \mathcal{U}(1;1)$, for all $z \in \mathbb{U}$ and this implies that $\mathfrak{Re}(\mathcal{G}_k f)'(z) > 0$, for all $z \in \mathbb{U}$.
- Similarly, for $k \le 2$, we have

$$\begin{aligned}\left|(\mathcal{G}_k f)'(z) - 1\right| &= \left|1 + \sum_{n=2}^{\infty} \frac{n \cdot n!}{(n-k)!} a_n z^{n-1} - 1\right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{n \cdot n!}{(n-k)!}|a_n| < 1,\end{aligned}$$

for all $z \in \mathbb{U}$ and $k \geq 2$. As we seen before, we obtain that $(\mathcal{G}_k f)'(z) \in \mathcal{U}(1;1)$ which implies that $\mathfrak{Re}(\mathcal{G}_k f)'(z) > 0$, for all $z \in \mathbb{U}$.

Finally, according to the univalence criterion given in Theorem 1.1 we deduce that $\mathcal{G}_k f \in S$, for all $k \in \mathbb{N}$ and this completes the proof. □

Remark 2.10. In particular, for $k = 0$, we obtain the well-known univalence condition for a holomorphic function on the unit disc (see for example [5, Exercise 1.1.4]): if $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then f is univalent on $\mathbb{U}$.

3. Subclasses of univalent functions

Using the differential operator $\mathcal{G}_k$ defined above, we can construct some particular subclasses of univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$. These subclasses, denoted here by $E_k^*(\alpha)$, respectively $E_k(\alpha)$, where $\alpha \in [0,1)$, are related to the classes of starlike, respectively convex functions of order α on $\mathbb{U}$.

3.1. The subclass $E_k^*(\alpha)$

First, we present some general results about the subclass $E_k^*(\alpha)$ and connections of this class with another important classes of univalent functions (for example, the class of starlike functions of order α or the class of univalent functions introduced by Sălăgean in [18]).

Definition 3.1. Let $\alpha \in [0,1)$ and $k \in \mathbb{N}$. Let $\mathcal{G}_k$ be the differential operator defined by formula (2.1). Then

$$E_k^*(\alpha) = \{ f \in S \, : \, \mathcal{G}_k f \in S^*(\alpha) \}$$

is the family of normalized univalent functions f on the unit disc such that $\mathcal{G}_k f$ is starlike of order α. In particular, we denote by $E_k^* = E_k^*(0)$.

Remark 3.2. It is clear that $E_0^*(\alpha) = S^*(\alpha)$ is the family of normalized starlike functions of order α on $\mathbb{U}$.

Remark 3.3. Taking into account the definition of starlikeness of order α, we deduce that

$$E_k^*(\alpha) = \left\{ f \in S \, : \, \mathfrak{Re}\left[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)} \right] > \alpha, \quad z \in \mathbb{U} \right\}. \tag{3.1}$$

Indeed, if $f \in S$, then $\mathcal{G}_k f \in \mathcal{H}(\mathbb{U})$, $(\mathcal{G}_k f)(0) = 0$ and $(\mathcal{G}_k f)'(0) = 1$. Together with the condition $\mathfrak{Re}\big[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\big] > \alpha$, for all $z \in \mathbb{U}$, all the assumptions from the definition of starlikeness of order α are satisfied.

Proposition 3.4. *Let $\alpha \in [0,1)$. Then $E_1^*(\alpha) = K(\alpha)$.*

Proof. Indeed, according to the previous definition and Remark 2.2, we have that

$$\begin{aligned} E_1^*(\alpha) &= \left\{ f \in S \, : \, \mathfrak{Re}\left[\frac{z(\mathcal{G}_1 f)'(z)}{(\mathcal{G}_1 f)(z)} \right] > \alpha, \quad z \in \mathbb{U} \right\} \\ &= \left\{ f \in S \, : \, \mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)} \right] > \alpha, \quad z \in \mathbb{U} \right\} = K(\alpha), \end{aligned}$$

for every $\alpha \in [0,1)$ and this completes the proof. □

Remark 3.5. As a consequence of the previous two remarks, we obtain that $E_0^* = S^*$ and $E_1^* = K$. It is important to mention here that the second equality is no longer true in the case of several complex variables (see [8]).

Remark 3.6. It is very important to mention here that

$$E_0^*(\alpha) = S_0(\alpha) \quad \text{and} \quad E_1^*(\alpha) = S_1(\alpha),$$

where $S_0(\alpha)$ and $S_1(\alpha)$ are particular forms of the class $S_n(\alpha)$ introduced by Sălăgean in [18] for $\alpha \in [0,1)$. These equalities holds because

$$D^0 f(z) = f(z) = (\mathcal{G}_0 f)(z) \quad \text{and} \quad D^1 f(z) = zf'(z) = (\mathcal{G}_1 f)(z),$$

for all $z \in \mathbb{U}$, where D^n is the differential operator introduced by Sălăgean. However, for $n = k \geq 2$, we have that

$$E_k^*(\alpha) \neq S_n(\alpha),$$

since the Sălăgean differential operator $D^n f$ (see [18]) is different from the operator $\mathcal{G}_k f$, for every $n = k \geq 2$. For example, if $n = 2$, then

$$D^2 f(z) = D(Df(z)) = z^2 f''(z) + zf'(z) \neq z^2 f''(z) + z = (G_2 f)(z),$$

for all $z \in \mathbb{U}$. Hence, the common results from this thesis and the ones obtained by Sălăgean in [18] are only for the particular cases $k = 0$ and $k = 1$ (which are already well-known, as reduces to the classes $S^*(\alpha)$, respectively $K(\alpha)$).

Using a similar argument as in Proposition 2.9, we can prove the following result. We mention here that this result is a general form of the theorem proved by Merkes, Robertson and Scott in [14].

Theorem 3.7. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in S$. Also, let $\sigma_{k,\alpha}$ be defined by*

$$\sigma_{k,\alpha} = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \leq 2 \\ \\ \displaystyle\sum_{n=2}^{k-1}(n-\alpha)|a_n| + \sum_{n=k}^{\infty} \frac{(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \geq 3. \end{cases} \tag{3.2}$$

If $\sigma_{k,\alpha} \leq 1-\alpha$, then $f \in E_k^*(\alpha)$.

Proof. Let $\alpha \in [0,1)$. Using (3.2) and Proposition 2.9, we obtain that $\mathcal{G}_k f$ is a normalized univalent function on $\mathbb{U}$. Morevoer,

$$\begin{aligned} &\big|z(\mathcal{G}_k f)'(z) - (\mathcal{G}_k f)(z)\big| - (1-\alpha)\big|(\mathcal{G}_k f)(z)\big| = \\ &= \left|z + \sum_{n=2}^{\infty} nA_n z^n - z - \sum_{n=2}^{\infty} A_n z^n\right| - (1-\alpha)\left|z + \sum_{n=2}^{\infty} A_n z^n\right| \\ &\leq \sum_{n=2}^{\infty}(n-1)|A_n||z|^n - (1-\alpha)\left(|z| - \sum_{n=2}^{\infty}|A_n||z|^n\right) \\ &\leq |z|\left(\sum_{n=2}^{\infty}(n-\alpha)|A_n| - (1-\alpha)\right) \leq 0, \end{aligned}$$

where A_n is given by relation (2.4). Since $(\mathcal{G}_k f)(z) \neq 0$ for $z \neq 0$ and in view of relation

$$\left|z(\mathcal{G}_k f)'(z) - (\mathcal{G}_k f)(z)\right| - (1-\alpha)\left|(\mathcal{G}_k f)(z)\right| \leq 0,$$

we deduce that

$$\left|\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)} - 1\right| \leq 1-\alpha, \tag{3.3}$$

for all $z \in \mathbb{U}$. Therefore

$$\mathfrak{Re}\left[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\right] > \alpha, \quad z \in \mathbb{U}$$

which implies $\mathcal{G}_k f \in S^*(\alpha)$. According to Definition 3.1, we conclude that $f \in E_k^*(\alpha)$ and this completes the proof. □

In the next corollary we present two particular cases of the previous theorem (results proved by Merkes, Robertson and Scott in [14]; see also [5]).

Corollary 3.8. *Let $f \in \mathcal{H}_0(\mathbb{U})$ and $k \in \{0,1\}$.*

1. *If $\sigma_{0,\alpha} = \sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha$, then $f \in E_0^*(\alpha) = S^*(\alpha)$.*
2. *If $\sigma_{1,\alpha} = \sum_{n=2}^{\infty}(n-\alpha)n|a_n| \leq 1-\alpha$, then $f \in E_1^*(\alpha) = K(\alpha)$.*

Remark 3.9. It is clear that for $\alpha = 0$, we obtain the classical conditions for starlikeness, respectively convexity on the unit disc (see e.g. [2], [3], [5]).

In this subsection we present some results regarding to coefficient estimates and distortion theorems for the class $E_k^*(\alpha)$. For the proof of our first result, we use the coefficient estimates for the class $S^*(\alpha)$ given by Robertson in [17] (see also [5]).

Theorem 3.10. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in E_k^*(\alpha)$. Then*

$$|a_n| \leq \frac{(n-k)!}{(n-1)! \cdot n!} \prod_{m=2}^{n}(m-2\alpha), \quad n \geq k \geq 2. \tag{3.4}$$

Proof. Let $f \in E_k^*(\alpha)$. Then $f \in S$ and $\mathcal{G}_k f \in S^*(\alpha)$. According to Remark 2.6 and the coefficient bounds for the class $S^*(\alpha)$ given in [17] (see also [5]), we know that

$$|A_n| \leq \frac{1}{(n-1)!} \prod_{m=2}^{n}(m-2\alpha), \tag{3.5}$$

for all $n \geq 2$, where A_n are the coefficients of $\mathcal{G}_k f$ defined by relation (2.4). Since

$$|A_n| = \begin{cases} |a_n|, & n \leq k-1 \\ \frac{n!}{(n-k)!}|a_n|, & n \geq k, \end{cases}$$

we obtain that

$$|a_n| \leq \frac{(n-k)!}{(n-1)! \cdot n!} \prod_{m=2}^{n}(m-2\alpha), \quad n \geq k.$$

Taking into account the product considered in the last relation, we impose the condition $n \geq k \geq 2$ and this completes the proof. □

Corollary 3.11. *Let* $k \in \mathbb{N}$ *and* $f \in E_k^*$. *Then*

$$|a_n| \le \frac{n}{n(n-1)(n-2)\cdot \ldots \cdot (n-k+1)} = \frac{n\cdot(n-k)!}{n!}, \quad n \ge k. \tag{3.6}$$

Proof. In view of Theorem 3.10 for $\alpha = 0$. □

Remark 3.12. As a consequence of the previous Corollary, we obtain the following well-known results (see e.g. [2], [3], [5], [16]):

1. If $k = 0$, then $E_0^* = S^*$ and $|a_n| \le n$, for all $n \ge 0$.
2. If $k = 1$, then $E_1^* = K$ and $|a_n| \le 1$, for all $n \ge 1$.

Following the idea presented by Duren in [2] and treated by Goodman in [3] (also by Grigoriciuc in [7]), we can prove a general distortion result for the class E_k^*. In fact, we obtain upper bounds for the m-th derivative of a function $f \in E_k^*$, where $m \in \mathbb{N}$ such that $m \ge k$.

Remark 3.13. Based on [7, Remark 2.5], we have that

$$\frac{1}{(1-r)^k} = \sum_{n=0}^{\infty} \frac{(k+n-1)!}{n!(k-1)!} r^n, \quad k \in \mathbb{N}, \quad r \in [0,1).$$

Theorem 3.14. *Let* $k \in \mathbb{N}$. *If* $f \in E_k^*$, *then*

$$\left|f^{(m)}(z)\right| \le \frac{\left[m + (1-k)|z|\right]\cdot(m-k)!}{(1-|z|)^{m-k+2}}, \tag{3.7}$$

for all $m \ge k$ *and* $z \in \mathbb{U}$.

Proof. Let $f \in E_k^*$. Then $f \in S$ and $\mathcal{G}_k f \in S^*$. Moreover, for $m \in \mathbb{N}$, we have that

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n, \quad z \in \mathbb{U}. \tag{3.8}$$

Let $r = |z| < 1$. In view of relation (3.6), we obtain

$$\begin{aligned}\left|f^{(m)}(z)\right| = \left|\sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n\right| &\le \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} |a_{m+n}||z|^n \\ &= \sum_{n=0}^{\infty} \frac{(m+n)(m+n-k)!}{n!} r^n\end{aligned}$$

In view of Remark 3.13 and elementary computations we deduce that

$$\left|f^{(m)}(z)\right| \le \sum_{n=0}^{\infty} \frac{(m+n)(m+n-k)!}{n!} r^n = \frac{(m-k)!\left[m + r(1-k)\right]}{(1-r)^{m-k+2}},$$

where $r = |z| < 1$. Finally, we conclude that

$$\left|f^{(m)}(z)\right| \le \frac{\left[m + (1-k)|z|\right]\cdot(m-k)!}{(1-|z|)^{m-k+2}}, \quad z \in \mathbb{U},$$

for all $m \ge k \ge 0$ and this completes the proof. □

Remark 3.15. Obviously, for $k \in \{0,1\}$ we obtain the classical results proved by Goodman in [3].

Based on the previous theorem and the result proved in [7], we propose the following conjecture (already proved for the particular cases $k=0$, $\alpha=0$ and $\alpha=\frac{1}{2}$):

Conjecture 3.16. *Let $\alpha \in [0,1)$ and $m,k \in \mathbb{N}$. If $f \in E_k^*(\alpha)$, then*

$$\left|f^{(m)}(z)\right| \le \frac{\left[m+(1-k)(1-2\alpha)|z|\right] \cdot B(m-k,\alpha)}{(1-|z|)^{m-k+2-2\alpha}}, \tag{3.9}$$

for all $m \ge k+1$ and $z \in \mathbb{U}$, where

$$B(m-k,\alpha) = \begin{cases} \frac{1}{m}(m-k)!, & \alpha = \frac{1}{2} \\ \dfrac{1}{1-2\alpha} \displaystyle\prod_{j=1}^{m-k} (j-2\alpha), & \alpha \ne \frac{1}{2}. \end{cases} \tag{3.10}$$

Remark 3.17. It is clear that for $k=0$, Conjecture 3.16 reduces to [7, Theorem 3.4]. Moreover, for $\alpha=0$, the previous Conjecture reduces to Theorem 3.14 proved in this section.

Remark 3.18. If $\alpha=\frac{1}{2}$, then (3.9) can be written as

$$\left|f^{(m)}(z)\right| \le \frac{(m-k)!}{(1-|z|)^{m-k+1}},$$

for all $m \ge k+1$ and $z \in \mathbb{U}$. Following a similar proof as in Theorem 3.14, we obtain that

$$\begin{aligned} \left|f^{(m)}(z)\right| = \left|\sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n\right| &\le \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} |a_{m+n}||z|^n \\ &\le \sum_{n=0}^{\infty} \frac{(m+n)!(m+n-k)!r^n}{n!(m+n-1)!(m+n)!} \prod_{j=2}^{m+n} (j-1) \\ &= \frac{(m-k)!}{(1-r)^{m-k+1}}, \end{aligned}$$

where $r=|z|<1$. Hence, Conjecture 3.16 is true for $\alpha=\frac{1}{2}$ as we proposed above.

Remark 3.19. The main idea of the results presented in this section is that starting from an index $n \ge k$ we can obtain better estimations for the coefficients a_n of $f \in E_k^*(\alpha)$, respectively upper bounds for the modulus of the m-th derivative of the function $f \in E_k^*(\alpha)$.

3.2. The subclass $E_k(\alpha)$

Similarly as in the previous section, we can use the operator $\mathcal{G}_k$ to define the class $E_k(\alpha)$ of holomorphic functions on the unit disc for which $\mathcal{G}_k f$ is a convex function of order α on $\mathbb{U}$. In the first part, we present some general results for the class $E_k(\alpha)$ related to coefficient estimates and general distortion results. The final part of this section is dedicated to the particular case $k=1$.

In this subsection we introduce the subclass $E_k(\alpha)$ together with some general properties of it.

Definition 3.20. Let $\alpha \in [0,1)$ and $k \in \mathbb{N}$. Let $\mathcal{G}_k$ be the differential operator defined by formula (2.1). Then

$$E_k(\alpha) = \{f \in S \,:\, \mathcal{G}_k f \in K(\alpha)\}$$

is the family of normalized univalent functions f on the unit disc such that $\mathcal{G}_k f$ is convex of order α. In particular, we denote by $E_k = E_k(0)$.

Remark 3.21. Taking into account the definition of convexity of order α (see [5], [17], [16]), we deduce that

$$E_k(\alpha) = \left\{ f \in S \,:\, \mathfrak{Re}\left[1 + \frac{z(\mathcal{G}_k f)''(z)}{(\mathcal{G}_k f)'(z)}\right] > \alpha, \quad z \in \mathbb{U} \right\}. \tag{3.11}$$

It is clear that $E_0(\alpha) = K(\alpha)$ is the family of normalized convex functions of order α on $\mathbb{U}$.

Taking into account Theorem 3.7, we can prove a similar criteria for the family $E_k(\alpha)$, as follows

Theorem 3.22. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in S$. Also, let $\sigma_{k,\alpha}$ be defined by*

$$\sigma_{k,\alpha} = \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{n(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \le 2 \\[2ex] \displaystyle\sum_{n=2}^{k-1} n(n-\alpha)|a_n| + \sum_{n=k}^{\infty} \frac{n(n-\alpha)\cdot n!}{(n-k)!}|a_n|, & k \ge 3. \end{cases} \tag{3.12}$$

If $\sigma_{k,\alpha} \le 1-\alpha$, then $f \in E_k(\alpha)$.

Proof. Similar to the proof of Theorem 3.7. □

Remark 3.23. If $k = 0$, then $E_0(\alpha) = K(\alpha)$ and we obtain the sufficient condition for convexity of order α (one may consult [5] or [14]).

Similar with Theorem 3.10, we can obtain some bounds for the coefficients of a function $f \in E_k(\alpha)$, as follows

Theorem 3.24. *Let $\alpha \in [0,1)$, $k \in \mathbb{N}$ and $f \in E_k(\alpha)$. Then*

$$|a_n| \le \frac{(n-k)!}{n!\cdot n!}\prod_{j=2}^{n}(j-2\alpha), \quad n \ge k \ge 2. \tag{3.13}$$

Proof. Let $f \in E_k(\alpha)$. Then $f \in S$ and $\mathcal{G}_k f \in K(\alpha)$. According to Remark 2.6 and the estimations proved by Robertson in [17] (see also [5]), we deduce that

$$|A_n| \le \frac{1}{n!}\prod_{j=2}^{n}(j-2\alpha), \tag{3.14}$$

for all $n \geq 2$, where A_n are the coefficients of $\mathcal{G}_k f$ defined by relation (2.4). Since

$$|A_n| = \begin{cases} |a_n|, & n \leq k-1 \\ \frac{n!}{(n-k)!}|a_n|, & n \geq k, \end{cases}$$

we obtain that

$$|a_n| \leq \frac{(n-k)!}{n! \cdot n!} \prod_{j=2}^{n} (j - 2\alpha), \quad n \geq k.$$

Taking into account the product considered in the last relation, we impose the condition $n \geq k \geq 2$ and this completes the proof. □

Corollary 3.25. *Let $k \in \mathbb{N}$ and $f \in E_k$. Then*

$$|a_n| \leq \frac{1}{n(n-1)(n-2) \cdot \ldots \cdot (n-k+1)} = \frac{(n-k)!}{n!}, \quad n \geq k. \tag{3.15}$$

Proof. In view of Theorem 3.24 for $\alpha = 0$. □

Remark 3.26. If $k = 0$, then $E_0 = K$ and we obtain the classical result related to the coefficient estimates for convex functions (see e.g. [2]).

Following the remarks presented before Theorem 3.14, we can prove the following general distortion result:

Theorem 3.27. *Let $k \in \mathbb{N}$. If $f \in E_k$, then*

$$\left|f^{(m)}(z)\right| \leq \frac{(m-k)!}{(1-|z|)^{m-k+1}}, \tag{3.16}$$

for all $m \geq k$ and $z \in \mathbb{U}$.

Proof. Let $f \in E_k$. Then $f \in S$ and $\mathcal{G}_k f \in K$. Moreover, for $m \in \mathbb{N}$, we have that

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n, \quad z \in \mathbb{U}.$$

Let $r = |z| < 1$. In view of relation (3.15), we obtain

$$\begin{aligned} \left|f^{(m)}(z)\right| = \left| \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n \right| &\leq \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} |a_{m+n}| |z|^n \\ &\leq \sum_{n=0}^{\infty} \frac{(m+n)!(m+n-k)! r^n}{n!(m+n)!} r^n \\ &= (m-k)! \sum_{n=0}^{\infty} \frac{(m+n-k)!}{n!(m-k)!} r^n \\ &= (m-k)! \cdot \frac{1}{(1-r)^{m-k+1}}, \end{aligned}$$

according to Remark 3.13. Finally, we conclude that

$$\left|f^{(m)}(z)\right| \leq \frac{(m-k)!}{(1-|z|)^{m-k+1}}, \quad z \in \mathbb{U},$$

for all $m \geq k \geq 0$ and this completes the proof. □

Remark 3.28. It is clear that for $k = 0$ we obtain the result proved by Goodman in [3, Theorem 9, Chapter 8].

3.2.1. The particular case $k = 1$ and $\alpha = 0$. The next section is dedicated to the study of a special form ($k = 1$ and $\alpha = 0$) of the class $E_k(\alpha)$. Because we consider such a particular case, we obtain some nice results and examples related to classical properties of univalent functions on the unit disc. According to Definition 3.20, we have that E_1 is defined by

$$E_1 = \{f \in S : \mathcal{G}_1 f \in K\},$$

where $\mathcal{G}_1 f(z) = zf'(z)$, for all $z \in \mathbb{U}$.

Remark 3.29. In view of the analytical characterization of convexity, we have the following equivalent definition

$$E_1 = \left\{ f \in S : \mathfrak{Re}\left[1 + \frac{z^2 f'''(z) + 2zf''(z)}{f'(z) + zf''(z)}\right] > 0, \quad z \in \mathbb{U} \right\}. \tag{3.17}$$

Indeed, $f \in S$ implies that $\mathcal{G}_1 f \in H(U)$. Moreover, according to the analytical characterization of convexity (see for example [5], [16]), it follows that $(\mathcal{G}_1 f)'(0) \neq 0$ (in fact, $(\mathcal{G}_1 f)'(0) = 1$) and $\mathfrak{Re}\left[1 + \frac{z(\mathcal{G}_1 f)''(z)}{(\mathcal{G}_1 f)'(z)}\right] > 0$, for all $z \in \mathbb{U}$. In view of Definition 2.1, we have that

$$\mathfrak{Re}\left[1 + \frac{z(\mathcal{G}_1 f)''(z)}{(\mathcal{G}_1 f)'(z)}\right] = \mathfrak{Re}\left[1 + \frac{z^2 f'''(z) + 2zf''(z)}{f'(z) + zf''(z)}\right] > 0,$$

for all $z \in \mathbb{U}$, which leads to the definition of E_1 given by (3.17).

Example 3.30. Let $f : \mathbb{U} \to \mathbb{C}$ be given by $f(z) = -\log(1 - z)$, for all $z \in \mathbb{U}$, where log is the principal branch of the complex logarithm. Then $f \in E_1$.

Proof. Indeed, $f \in S$ and $f'(z) = \frac{1}{1-z}$, for all $z \in \mathbb{U}$. Moreover,

$$\mathcal{G}_1 f(z) = zf'(z) = \frac{z}{1 - z}, \quad z \in \mathbb{U}.$$

Then $\mathcal{G}_1 f \in S$ and $\mathcal{G}_1 f(\mathbb{U}) = \{w \in \mathbb{C} : \mathfrak{Re} w > -\frac{1}{2}\}$ is a convex domain in $\mathbb{C}$. Hence, $\mathcal{G}_1 f \in K$ and this completes the proof. □

Next, we present an important result that establishes the connection between classes E_1 and $K(1/2)$. In particular, we obtain that every function from E_1 is also convex. This proof of this result was given by the author and is based on the proof of [5, Theorem 2.3.2] given by Suffridge.

Proposition 3.31. *If $f \in E_1$, then $f \in K(1/2)$. This result is sharp.*

Proof. Let $f \in E_1$. Then $f \in S$ and $\mathcal{G}_1 f \in K$. Taking into account a classical result given by Sheil-Small (see [19]) and Suffridge (see [20]; also, one may consult [5]), we know that

$$\mathcal{G}_1 f \in K \quad \Leftrightarrow \quad \mathfrak{Re}\left[\frac{2z(\mathcal{G}_1 f)'(z)}{(\mathcal{G}_1 f)(z) - (\mathcal{G}_1 f)(\zeta)} - \frac{z + \zeta}{z - \zeta}\right] \geq 0,$$

for all $z, \zeta \in \mathbb{U}$. In particular, for $\zeta = 0$, we obtain that $\mathcal{G}_1 f \in K$ is equivalent to

$$\mathfrak{Re}\left[\frac{z(\mathcal{G}_1 f)'(z)}{(\mathcal{G}_1 f)(z)}\right] \geq 0, \quad z \in \mathbb{U}.$$

In view of (2.1) and the minimum principle for harmonic functions, we deduce that

$$\mathfrak{Re}\left[\frac{z^2 f''(z) + zf'(z)}{zf'(z)}\right] = \mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \tag{3.18}$$

for all $z \in \mathbb{U}$. Hence, according to the definition of the convex functions of order α, we conclude that $f \in K(1/2)$. In order to prove that the result is sharp, it suffices to consider the function $f : \mathbb{U} \to \mathbb{C}$, given by $f(z) = -\log(1-z)$, for all $z \in \mathbb{U}$, where log is the principal branch of the complex logarithm and this completes the proof. □

Remark 3.32. In order to prove that the inclusion $E_1 \subsetneq K(1/2)$ is strict, we can use the example given in the proof of Theorem 3 from [8]. If we consider $f : \mathbb{U} \to \mathbb{C}$ given by $f(z) = z + \frac{1}{6}z^2$, for all $z \in \mathbb{U}$, then $f \in K(1/2) \setminus E_1$.

Proof. Indeed, the main idea of the proof (cf. [8]) is the following: according to Corollary 3.8 we have that $f \in K(1/2)$. However, it is easy to prove that $\mathcal{G}_1 f \notin K$, where $(\mathcal{G}_1 f)(z) = zf'(z)$, for all $z \in \mathbb{U}$. Hence, $f \notin E_1$ and this completes the proof. □

Proposition 3.33. *If $f \in E_1$, then $f \in \mathcal{R}(1/2)$, i.e. $\mathfrak{Re} f'(z) > 1/2$, for all $z \in \mathbb{U}$.*

Proof. Let $f \in E_1$. Then $f \in S$ and $\mathcal{G}_1 f \in K$, where $(\mathcal{G}_1 f)(z) = zf'(z)$, for all $z \in \mathbb{U}$. In view of a result due to Marx and Strohhäcker (see for example [5]), we have that

$$\frac{1}{2} < \mathfrak{Re}\left[\frac{(\mathcal{G}_1 f)(z)}{z}\right] = \mathfrak{Re}\left[\frac{zf'(z)}{z}\right] = \mathfrak{Re} f'(z),$$

for all $z \in \mathbb{U}$. Hence, $\mathfrak{Re} f'(z) > \frac{1}{2}$, for all $z \in \mathbb{U}$ and this completes the proof. □

Theorem 3.34. *Let $f \in E_1$. Then*

$$\log(1 + |z|) \leq |f(z)| \leq -\log(1 - |z|) \tag{3.19}$$

and

$$\frac{1}{1+|z|} \leq |f'(z)| \leq \frac{1}{1-|z|}, \tag{3.20}$$

for all $z \in \mathbb{U}$. All of these estimates are sharp.

Proof. Since $f \in E_1$, we have that $f \in S$ and $\mathcal{G}_1 f \in K$, where $(\mathcal{G}_1 f)(z) = zf'(z)$, for all $z \in \mathbb{U}$. According to distortion theorem for the class K (see e.g. [5], [16]), we know that

$$\frac{r}{1+r} \leq \left|zf'(z)\right| \leq \frac{r}{1-r},$$

where $|z| = r$. Then

$$\frac{1}{1+r} \leq \left|f'(z)\right| \leq \frac{1}{1-r}, \tag{3.21}$$

where $|z| = r < 1$ and we obtain the distortion result for the class E_1. The upper bound in (3.19) follows easily by integrating the upper bound in (3.20) and the lower bound in (3.19) can be obtained using the arguments presented in the proof of Theorem 2.2.8 from [5]. Hence,

$$\log(1+r) \leq |f(z)| \leq -\log(1-r),$$

where $|z| = r < 1$. The sharpness of all of these estimates is ensured by the function defined in Example 3.30. □

Corollary 3.35. *If $f \in E_1$, then $f(\mathbb{U})$ contains the open disc $\mathcal{U}_{\ln 2}$.*

Proof. The result follows from the lower estimate in relation (3.19) on letting $r \to 1$. □

3.3. Connections between E_k^* and E_k

Based on the Alexander's duality theorem between convex and starlike functions on $\mathbb{U}$ (see [1], [2], [16]), we prove in this section similar duality results for the subclasses E_k^* and E_k.

Lemma 3.36. *Let $k \in \mathbb{N}$ and $f, g \in S$ be such that $g(z) = zf'(z)$, for all $z \in \mathbb{U}$. Then*

$$z(\mathcal{G}_k f)'(z) = (\mathcal{G}_k g)(z), \quad z \in \mathbb{U}. \tag{3.22}$$

Proof. It is clear that for $k = 0$, relation (3.22) reduces to the definition of g. Let us consider $k \geq 1$ and $f, g \in S$ such that $g(z) = zf'(z)$, for all $z \in \mathbb{U}$. By (2.1) we have

$$(\mathcal{G}_k f)(z) = z^k f^{(k)}(z) + a_{k-1} z^{k-1} + \dots + a_1 z + a_0,$$

for all $z \in \mathbb{U}$, where $a_1 = 1$ and $a_0 = 0$. Then

$$z(\mathcal{G}_k f)'(z) = z^{k+1} f^{(k+1)}(z) + k z^k f^{(k)}(z) + \sum_{n=1}^{k-1} n a_n z^n, \tag{3.23}$$

for all $z \in \mathbb{U}$. According to Leibniz's formula, we deduce that

$$\begin{aligned}(\mathcal{G}_k g)(z) &= z^k g^{(k)}(z) + b_{k-1} z^{k-1} + \dots + b_2 z^2 + b_1 z + b_0 \\ &= z^{k+1} f^{(k+1)}(z) + k z^k f^{(k)}(z) + b_{k-1} z^{k-1} + \dots + b_2 z^2 + b_1 z + b_0 \\ &= z^{k+1} f^{(k+1)}(z) + k z^k f^{(k)}(z) + \sum_{n=1}^{k-1} n a_n z^n, \end{aligned} \tag{3.24}$$

for all $z \in \mathbb{U}$. Finally, in view of (3.23) and (3.24) we obtain that

$$z(\mathcal{G}_k f)'(z) = (\mathcal{G}_k g)(z), \quad z \in \mathbb{U}$$

and this completes the proof. □

Based on the previous lemma, we can obtain an Alexander type theorem for the families E_k^* and E_k.

Theorem 3.37. *Let $k \in \mathbb{N}$ and $f, g \in S$. Then $f \in E_k$ if and only if $g \in E_k^*$, where $g(z) = zf'(z)$, for all $z \in \mathbb{U}$.*

Proof. Let $f \in E_k$. According to the definition of the class E_k, we have that

$$f \in E_k \quad \Leftrightarrow \quad f \in S \quad \text{and} \quad \mathcal{G}_k f \in K.$$

Moreover,

$$\mathcal{G}_k f \in K \Leftrightarrow z(\mathcal{G}_k f)'(z) \in S^*,$$

for all $z \in \mathbb{U}$, in view of the Alexander's duality theorem. Using Lemma 3.36 we can rewrite the previous equivalence as

$$\mathcal{G}_k f \in K \Leftrightarrow z(\mathcal{G}_k f)'(z) = (\mathcal{G}_k g)(z) \in S^*,$$

for all $z \in \mathbb{U}$. Then

$$f \in E_k \Leftrightarrow \mathcal{G}_k f \in K \Leftrightarrow \mathcal{G}_k g \in S^* \Leftrightarrow g \in E_k^*,$$

where $g(z) = zf'(z)$, for all $z \in \mathbb{U}$, and this completes the proof. □

Remark 3.38. Since Theorem 3.37 is based on the Alexander's duality theorem, it is clear that for $k = 0$, we have that

$$f \in E_0 = K \quad \Leftrightarrow \quad g \in E_0^* = S^*,$$

where $g(z) = zf'(z)$, for all $z \in \mathbb{U}$.

Remark 3.39. Another interesting remark is that, taking into account Definition 2.1, we can rewrite Theorem 3.37 as follows

$$f \in E_k \Leftrightarrow \mathcal{G}_1 f \in E_k^*, \tag{3.25}$$

for all $k \in \mathbb{N}$, where $\mathcal{G}_1 f$ is given by (2.1).

Theorem 3.40. *Let $k \in \mathbb{N}$. If $f \in E_k$, then $f \in E_k^*(1/2)$.*

Proof. Let $f \in E_k$. Then $f \in S$ and $\mathcal{G}_k f \in K$. According to a result given by Sheil-Small and Suffridge (see e.g. [5]), we know that

$$\mathfrak{Re}\left[\frac{z(\mathcal{G}_k f)'(z)}{(\mathcal{G}_k f)(z)}\right] > 0, \quad z \in \mathbb{U}.$$

Hence, since $f \in S$ and $\mathcal{G}_k f \in S^*(1/2)$, it follows that $f \in E_k^*(1/2)$ and this completes the proof. □

Remark 3.41. It is clear that Theorem 3.40 is a generalization of Proposition 3.31 (where $k = 1$). On the other hand, if $k = 0$, then Theorem 3.40 reduces to [5, Theorem 2.3.2] due to Marx and Strohhäcker.

Finally, we end this section with some questions related to the subclasses E_k and E_k^* studied above. First question is a generalization of Proposition 3.31:

Question 3.42. Is it true that $E_{k+1} \subset E_k$, for all $k \in \mathbb{N}$?

Clearly, a similar question can be formulated also for the subclass E_k^*. Another important property of these subclasses is the compactness. Hence, one may ask

Question 3.43. Is it true that the subclasses E_k and E_k^* are compact in $\mathcal{H}(\mathbb{U})$?

Since E_k^* and E_k are subclasses of the class S, it would be interesting to study also other geometric and analytic properties of them.

3.4. The subclass $E_{\mathbb{N}}$

Let $k \in \mathbb{N}$ and $f \in \bigcap\limits_{k\in\mathbb{N}} E_k$. Then, for every $k \in \mathbb{N}$, we have that $f \in E_k$. Moreover, according to Corollary 3.25, it follows that for every $k \in \mathbb{N}$

$$|a_n| \le \frac{(n-k)!}{n!}, \quad n \ge k.$$

In particular, for $n = k$ we obtain that $|a_k| \le \frac{1}{k!}$, for every $k \in \mathbb{N}$. Let us denote by

$$E_{\mathbb{N}} = \left\{ f \in S \,:\, |a_n| \le \frac{1}{n!},\, n \ge 2 \right\}. \tag{3.26}$$

Then, we obtain the following remark

Remark 3.44. Let $E_{\mathbb{N}}$ be the set defined by (3.26). Then $\bigcap_{k\in\mathbb{N}} E_k \subsetneqq E_{\mathbb{N}}$, i.e. the intersection of all subclasses E_k is included in $E_{\mathbb{N}}$, but it is not equal with $E_{\mathbb{N}}$.

Indeed, we can construct an example of a function $f \in S$ that belongs to the family $E_{\mathbb{N}}$, but not to $\bigcap_{k\in\mathbb{N}} E_k$ (in fact, we prove that $f \notin E_1$), as follows

Example 3.45. Let $f : \mathbb{U} \to \mathbb{C}$ be defined by $f(z) = z + az^2$, for all $z \in \mathbb{U}$, where $a = 1/2$. Then $f \in E_{\mathbb{N}}$, but $f \notin \bigcap_{k\in\mathbb{N}} E_k$.

Proof. It is clear that $f \in S$ and $|a_2| = |a| \le \frac{1}{2}$. Hence, in view of relation (3.26) we deduce that $f \in E_{\mathbb{N}}$. On the other hand,

$$(\mathcal{G}_1 f)(z) = zf'(z) = z(1 + 2az) = z + 2az^2, \quad z \in \mathbb{U}, \quad a = 1/2.$$

Let us denote

$$h(z) = 1 + \frac{z(\mathcal{G}_1 f)''(z)}{(\mathcal{G}_1 f)'(z)} = 1 + \frac{4az}{1+4az} = \frac{1+8az}{1+4az},$$

for all $z \in \mathbb{U}$, where $a = 1/2$. Then h is a Möbius function on $\mathbb{U}$ such that $h(0) = 1$, $h(1) = \frac{5}{3}$, $h(i) = \frac{9}{5} + \frac{2}{5}i$ and $h(-i) = \frac{9}{5} - \frac{2}{5}i$. In other words, we obtain that

$$h(\mathbb{U}) = \mathbb{C} \setminus \overline{\mathcal{U}}(7/3, 2/3) = \left\{ x + iy \in \mathbb{C} \,:\, \left(x - \frac{7}{3}\right)^2 + y^2 > \left(\frac{2}{3}\right)^2 \right\},$$

i.e., $h(\mathbb{U})$ is the complementary part of the closed disc $\overline{\mathcal{U}}\left(\frac{7}{3}, \frac{2}{3}\right)$ of center $w_0 = \frac{7}{3}$ and radius $r = \frac{2}{3}$. Moreover, for every point $w \in h(\mathbb{U}) \cap \{x + iy \in \mathbb{C} : x < 0\}$ we have that $\mathfrak{Re}\, w < 0$, i.e. there exists $z_0 \in \mathbb{U}$ such that $\mathfrak{Re}\, h(z_0) < 0$. For example, if $z_0 = -\frac{1}{3}$, then $z_0 \in \mathbb{U}$ and simple computations show that

$$\mathfrak{Re}\, h(z_0) = \mathfrak{Re}\left[1 + \frac{z_0(\mathcal{G}_1 f)''(z_0)}{(\mathcal{G}_1 f)'(z_0)}\right] = -1 < 0.$$

Hence, according to the behavior of the function h on $\mathbb{U}$ and the analytical characterization of convexity (see for example [3] or [5]), we deduce that $\mathcal{G}_1 f \notin K$.

Since $f \in S$, but $\mathcal{G}_1 f \notin K$, we obtain (according to Definition 3.20) that $f \notin E_1$. Now, it is clear that $f \notin \bigcap_{k\in\mathbb{N}} E_k$ and this completes the proof. □

Another interesting example (considered also in [15]) which generates important remarks about the class $E_{\mathbb{N}}$ is the following

Example 3.46. Let $f : \mathbb{U} \to \mathbb{C}$ be given by $f(z) = e^z - 1$, for all $z \in \mathbb{U}$. Then $f \in E_{\mathbb{N}}$.

Proof. Indeed, $f \in S$ and

$$f(z) = e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = z + \frac{z^2}{2} + \frac{z^3}{6} + ... = z + \sum_{n=2}^{\infty} a_n z^n,$$

where $a_n = \frac{1}{n!}$, for all $n \geq 2$. Hence, $f \in E_{\mathbb{N}}$. □

Taking into account relation (3.8), we can prove the following result

Proposition 3.47. *Let $m \in \mathbb{N}$. If $f \in E_{\mathbb{N}}$, then $\left|f^{(m)}(z)\right| \leq e^{|z|}$, for all $z \in \mathbb{U}$.*

Proof. Let $f \in E_{\mathbb{N}}$ and $|z| = r < 1$. Then $f \in S$ and in view of (3.8) we have that

$$\left|f^{(m)}(z)\right| = \left|\sum_{n=0}^{\infty} \frac{(m+n)!}{n!} a_{m+n} z^n\right| \leq \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} |a_{m+n}||z|^n = \sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r,$$

where $|z| = r < 1$. Hence, we obtain that

$$\left|f^{(m)}(z)\right| \leq e^{|z|},$$

for $z \in \mathbb{U}$ and this completes the proof. □

It is clear that Proposition 3.47 has the following consequence (for the particular case $z = 0$):

Corollary 3.48. *Let $m \in \mathbb{N}$. If $f \in E_{\mathbb{N}}$, then $|f^{(m)}(0)| \leq 1$, for all $m \in \mathbb{N}$.*

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Bounds of third and fourth Hankel determinants for a generalized subclass of bounded turning functions subordinated to sine function

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Abstract. The objective of this paper is to investigate the bounds of third and fourth Hankel determinants for a generalized subclass of bounded turning functions associated with sine function, in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. The results are also extended to two-fold and three-fold symmetric functions. This investigation will generalize the resuls of some earlier works.

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1. Introduction

Let the complex plane is expressed by $\mathbb{C}$. By $\mathcal{A}$, let us denote the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, defined in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$, we denote the subclass of $\mathcal{A}$ which consists of univalent functions in E.

Let f and g be two analytic functions in E. We say that f is subordinate to g (denoted as $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then the subordination leads to $f(0) = g(0)$ and $f(E) \subset g(E)$.

In the theory of univalent functions, Bieberbach [5] stated a result that, for $f \in \mathcal{S}$, $|a_n| \leq n$, $n = 2, 3, ...$ This result is known as Bieberbach's conjecture and it remained as a challenge for the mathematicians for a long time. Finally, L. De-Branges [8],

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proved this conjecture in 1985. During the course of proving this conjecture, various results related to the coefficients were come into existence which gave rise to some new subclasses of analytic functions.

For better understanding of the main content, let's have a look on some fundamental subclasses of $\mathcal{A}$:

$\mathcal{S}^* = \left\{f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ or } \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, z \in E\right\}$, the class of starlike functions.

$\mathcal{K} = \left\{f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 \text{ or } \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, z \in E\right\}$, the class of convex functions.

Reade [24] introduced the class $\mathcal{CS}^*$ of close-to-star functions which is defined as $\mathcal{CS}^* = \left\{f : f \in \mathcal{A}, Re\left(\frac{f(z)}{g(z)}\right) > 0 \text{ or } \frac{f(z)}{g(z)} \prec \frac{1+z}{1-z}, g \in \mathcal{S}^*, z \in E\right\}$. Further for $g(z) = z$, MacGregor [17] studied the following subclass of close-to-star functions:

$$\mathcal{R}' = \left\{f : f \in \mathcal{A}, Re\left(\frac{f(z)}{z}\right) > 0 \text{ or } \frac{f(z)}{z} \prec \frac{1+z}{1-z}, z \in E\right\}.$$

MacGregor [16] established a very useful class $\mathcal{R}$ of bounded turning functions which is defined as

$$\mathcal{R} = \left\{f : f \in \mathcal{A}, Re(f'(z)) > 0 \text{ or } f'(z) \prec \frac{1+z}{1-z}, z \in E\right\}.$$

Later on, Murugusundramurthi and Magesh [19] studied the following class:

$$\mathcal{R}(\alpha) = \left\{f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right) > 0, 0 \leq \alpha \leq 1, z \in E\right\}.$$

Particularly, $\mathcal{R}(1) \equiv \mathcal{R}$ and $\mathcal{R}(0) \equiv \mathcal{R}'$.

Various subclasses of $\mathcal{S}$ were investigated by associating to different functions. Recently, Arif et al. [3], Cho et al. [7] and Khan et al. [11] studied the classes $\mathcal{S}^*_{sin}$, $\mathcal{K}_{sin}$ and $\mathcal{R}_{sin}$, which are the subclasses of starlike functions, convex functions and bounded turning functions associated with sine function, respectively. Getting motivated by these works, now we define the following class of analytic functions by subordinating to $1 + sinz$.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\alpha_{sin}$ $(0 \leq \alpha \leq 1)$ if it satisfies the condition

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + sinz.$$

We have the following observations:

(i) $\mathcal{R}^0_{sin} \equiv \mathcal{R}'_{sin}$.

(ii) $\mathcal{R}^1_{sin} \equiv \mathcal{R}_{sin}$.

For $q \geq 1$ and $n \geq 1$, Pommerenke [21] introduced the q^{th} Hankel determinant as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For specific values of q and n, the Hankel determinant $H_q(n)$ reduces to the following functionals:
(i) For $q = 2$ and $n = 1$, it redues to $H_2(1) = a_3 - a_2^2$, which is the well known Fekete-Szegö functional.
(ii) For $q = 2$ and $n = 2$, the Hankel determinant takes the form of $H_2(2) = a_2a_4 - a_3^2$, which is known as Hankel determinant of second order.
(iii) For $q = 3$ and $n = 1$, the Hankel determinant reduces to $H_3(1)$, which is the Hankel determinant of third order.
(iv) For $q = 4$ and $n = 1$, $H_q(n)$ reduces to $H_4(1)$, which is the Hankel determinant of fourth order.

Ma [15] introduced the functional $J_{n,m}(f) = a_na_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, which is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2a_3 - a_4$ is a specific case of the generalized Zalcman functional. The upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions was computed by various authors. It is very useful in establishing the bounds for the third Hankel determinant.

On expanding, the third Hankel determinant can be expressed as

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and after applying the triangle inequality, it yields

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.1}$$

Also the expansion of fourth Hankel determinant can be expressed as

$$\begin{aligned} H_4(1) = a_7H_3(1) &- 2a_4a_6(a_2a_4 - a_3^2) - 2a_5a_6(a_2a_3 - a_4) - a_6^2(a_3 - a_2^2) \\ &+ a_5^2(a_2a_4 - a_3^2) + a_5^2(a_2a_4 + 2a_3^2) - a_5^3 + a_4^4 - 3a_3a_4^2a_5. \end{aligned} \tag{1.2}$$

A lot of work has been done on the estimation of second Hankel determinant by various authors including Noor [20], Ehrenborg [9], Layman [12], Singh [26], Mehrok and Singh [18] and Janteng et al. [10]. The estimation of third Hankel determinant is little bit complicated. Babalola [4] was the first researcher who successfully obtained the upper bound of third Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. Further a few researchers including Shanmugam et al. [25], Bucur et al. [6], Altinkaya and Yalcin [1], Singh and Singh [27] have been actively engaged in the study of third Hankel determinant for various subclasses of analytic functions. Now a days, the study of fourth Hankel determinant for various subclasses of analytic functions, is an active topic of research. A few authors including Arif et al. [2], Singh et al. [28, 29] and Zhang and Tang [30] established the bounds of fourth Hankel determinant for certain subclasses of $\mathcal{A}$.

In this paper, we establish the upper bounds of the third and fourth Hankel determinants for the class $\mathcal{R}_{sin}^{\alpha}$. Also various known results follow as particular cases.

Let $\mathcal{P}$ denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

In order to prove our main results, the following lemmas have been used:

Lemma 1.2. [3] *If $p \in \mathcal{P}$, then*

$$|p_k| \leq 2, k \in \mathbb{N},$$

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2},$$

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1,$$

$$|p_{n+2k} - \lambda p_n p_k^2| \leq 2(1 + 2\lambda), (\lambda \in \mathbb{R}),$$

$$|p_m p_n - p_k p_l| \leq 4, (m + n = k + l; m, n \in \mathbb{N}),$$

and for complex number ρ, we have

$$|p_2 - \rho p_1^2| \leq 2 \max\{1, |2\rho - 1|\}.$$

Lemma 1.3. [3] *Let $p \in \mathcal{P}$, then*

$$|J p_1^3 - K p_1 p_2 + L p_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$$

In particular, it is proved in [22] *that*

$$|p_1^3 - 2p_1 p_2 + p_3| \leq 2.$$

Lemma 1.4. [13, 14] *If $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for $|x| \leq 1$ and $|z| \leq 1$.

Lemma 1.5. [23] *Let m, n, l and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$ and $8r(1-r)\left[(mn - 2l)^2 + (m(r + m) - n)^2\right] + m(1-m)(n-2rm)^2 \leq 4m^2(1-m)^2 r(1-r)$. If $p \in \mathcal{P}$, then*

$$\left| l p_1^4 + r p_2^2 + 2m p_1 p_3 - \frac{3}{2} n p_1^2 p_2 - p_4 \right| \leq 2.$$

2. Coefficient bounds for the class $\mathcal{R}_{sin}^{\alpha}$

Theorem 2.1. *If $f \in \mathcal{R}_{sin}^{\alpha}$, then*

$$|a_2| \leq \frac{1}{1+\alpha}, \tag{2.1}$$

$$|a_3| \leq \frac{1}{1+2\alpha}, \tag{2.2}$$

$$|a_4| \leq \frac{1}{1+3\alpha}, \tag{2.3}$$

and

$$|a_5| \leq \frac{1}{1+4\alpha}. \tag{2.4}$$

Proof. Since $f \in \mathcal{R}_{sin}^{\alpha}$, by the principle of subordination, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + sin(w(z)). \tag{2.5}$$

Define $p(z) = \dfrac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$, which implies $w(z) = \dfrac{p(z)-1}{p(z)+1}$. On expanding, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1+(1+\alpha)a_2 z+(1+2\alpha)a_3 z^2+(1+3\alpha)a_4 z^3+(1+4\alpha)a_5 z^4+... \tag{2.6}$$

Also

$$1 + sin(w(z)) = 1 + \frac{1}{2}p_1 z + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{5p_1^3}{48} - \frac{p_1 p_2}{2} + \frac{p_3}{2}\right) z^3 + \left(-\frac{p_1^4}{32} + \frac{5p_1^2 p_2}{16} - \frac{p_3 p_1}{2} - \frac{p_2^2}{4} + \frac{p_4}{2}\right) z^4 + ... \tag{2.7}$$

Using (2.6) and (2.7), (2.5) yields

$$\begin{aligned} 1 + (1+\alpha)a_2 z + (1+2\alpha)a_3 z^2 + (1+3\alpha)a_4 z^3 + (1+4\alpha)a_5 z^4 + ... \\ = 1 + \frac{1}{2}p_1 z + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{5p_1^3}{48} - \frac{p_1 p_2}{2} + \frac{p_3}{2}\right) z^3 \\ + \left(-\frac{p_1^4}{32} + \frac{5p_1^2 p_2}{16} - \frac{p_3 p_1}{2} - \frac{p_2^2}{4} + \frac{p_4}{2}\right) z^4 + ... \end{aligned} \tag{2.8}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (2.8) and on simplification, we obtain

$$a_2 = \frac{1}{2(1+\alpha)} p_1, \tag{2.9}$$

$$a_3 = \frac{1}{1+2\alpha}\left[\frac{p_2}{2} - \frac{p_1^2}{4}\right], \tag{2.10}$$

$$a_4 = \frac{1}{48(1+3\alpha)}\left[5p_1^3 - 24p_1 p_2 + 24p_3\right], \tag{2.11}$$

and

$$a_5 = \frac{1}{2(1+4\alpha)}\left[\frac{p_1^4}{16} + \frac{p_2^2}{2} + p_3 p_1 - \frac{5p_1^2 p_2}{8} - p_4\right]. \tag{2.12}$$

Using first inequality of Lemma 1.2 in (2.9), the result (2.1) is obvious.
From (2.10), we have

$$|a_3| = \frac{1}{2(1+2\alpha)}\left|p_2 - \frac{1}{2}p_1^2\right|. \tag{2.13}$$

Using sixth inequality of Lemma 1.2 in (2.13), the result (2.2) can be easily obtained.
(2.11) can be expressed as

$$|a_4| = \frac{1}{48(1+3\alpha)}\left|5p_1^3 - 24p_1p_2 + 24p_3\right|. \tag{2.14}$$

On applying Lemma 1.3 in (2.14), the result (2.3) is obvious.
Further, on using Lemma 1.5 in (2.12), the result (2.4) is obvious. □

On putting $\alpha = 0$, Theorem 2.1 yields the following result:

Remark 2.2. If $f \in \mathcal{R}'_{sin}$, then

$$|a_2| \leq 1, |a_3| \leq 1, |a_4| \leq 1, |a_5| \leq 1.$$

For $\alpha = 1$, Theorem 2.1 gives the following result due to Khan et al. [11]:

Remark 2.3. If $f \in \mathcal{R}_{sin}$, then

$$|a_2| \leq \frac{1}{2}, |a_3| \leq \frac{1}{3}, |a_4| \leq \frac{1}{4}, |a_5| \leq \frac{1}{5}.$$

Conjecture. If $f \in \mathcal{R}^{\alpha}_{sin}$, then

$$|a_n| \leq \frac{1}{1+(n-1)\alpha}, n \geq 2.$$

Theorem 2.4. *If $f \in \mathcal{R}^{\alpha}_{sin}$ and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{1+2\alpha}\max\left\{1, \frac{(1+2\alpha)}{(1+\alpha)^2}|\mu|\right\}. \tag{2.15}$$

Proof. From (2.9) and (2.10), we obtain

$$|a_3 - \mu a_2^2| = \frac{1}{2(1+2\alpha)}\left|p_2 - \frac{(1+\alpha)^2 + \mu(1+2\alpha)}{2(1+\alpha)^2}p_1^2\right|. \tag{2.16}$$

Using sixth inequality of Lemma 1.2, (2.16) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{1}{1+2\alpha}\max\left\{1, \frac{(1+2\alpha)}{(1+\alpha)^2}|\mu|\right\}. \tag{2.17}$$

□

Substituting for $\alpha = 0$, Theorem 2.4 yields the following result:

Remark 2.5. If $f \in \mathcal{R}'_{sin}$, then

$$|a_3 - \mu a_2^2| \leq \max\{1, |\mu|\}.$$

Putting $\alpha = 1$, Theorem 2.4 yields the following result due to Khan et al. [11]:

Remark 2.6. If $f \in \mathcal{R}_{sin}$, then

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{3}, \frac{1}{4}|\mu|\right\}.$$

Putting $\mu = 1$, Theorem 2.4 yields the following result:

Remark 2.7. If $f \in \mathcal{R}_{sin}^{\alpha}$, then

$$|a_3 - a_2^2| \leq \frac{1}{1+2\alpha}.$$

Theorem 2.8. *If $f \in \mathcal{R}_{sin}^{\alpha}$, then*

$$|a_2 a_3 - a_4| \leq \frac{1}{1+3\alpha}. \tag{2.18}$$

Proof. Using (2.9), (2.10), (2.11) and after simplification, we have

$$|a_2 a_3 - a_4| = \frac{1}{48(1+\alpha)(1+2\alpha)(1+3\alpha)}$$

$$\cdot \left|(11 + 33\alpha + 10\alpha^2)p_1^3 - (36 + 108\alpha + 48\alpha^2)p_1 p_2 + 24(1+\alpha)(1+2\alpha)p_3\right|. \tag{2.19}$$

On applying Lemma 1.3 in (2.19), it yields (2.18). □

For $\alpha = 0$, the following result is a consequence of Theorem 2.8:

Remark 2.9. If $f \in \mathcal{R}_{sin}^{'}$, then

$$|a_2 a_3 - a_4| \leq 1.$$

On putting $\alpha = 1$ in Theorem 2.8, we can obtain the following result due to Khan et al. [11]:

Remark 2.10. If $f \in \mathcal{R}_{sin}$, then

$$|a_2 a_3 - a_4| \leq \frac{1}{4}.$$

Theorem 2.11. *If $f \in \mathcal{R}_{sin}^{\alpha}$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2}. \tag{2.20}$$

Proof. Using (2.9), (2.10) and (2.11), we have

$$|a_2 a_4 - a_3^2| = \frac{1}{96(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$$

$$\cdot \left|24(1+2\alpha)^2 p_1 p_3 - 24\alpha^2 p_1^2 p_2 + (-1 - 4\alpha + 2\alpha^2)p_1^4 - 24(1+\alpha)(1+3\alpha)p_2^2\right|.$$

Substituting for p_2 and p_3 from Lemma 1.4 and letting $p_1 = p$, we get

$$|a_2 a_4 - a_3^2| = \frac{1}{96(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Bigg| - (4\alpha^2 + 4\alpha + 1)p^4$$

$$-6(1+2\alpha)^2 p^2(4-p^2)x^2 - 6(1+\alpha)(1+3\alpha)(4-p^2)^2 x^2 + 12(1+2\alpha)^2 p(4-p^2)(1-|x|^2)z\Bigg|.$$

Since $|p| = |p_1| \leq 2$, we may assume that $p \in [0, 2]$. By using triangle inequality and $|z| \leq 1$ with $|x| = t \in [0, 1]$, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{1}{96(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Bigg[(4\alpha^2+4\alpha+1)p^4 + 6(1+2\alpha)^2p^2(4-p^2)t^2$$

$$+6(1+\alpha)(1+3\alpha)(4-p^2)^2t^2 + 12(1+2\alpha)^2p(4-p^2) - 12(1+2\alpha)^2p(4-p^2)t^2\Bigg] = F(p,t).$$

$$\frac{\partial F}{\partial t} = \frac{(4-p^2)t}{8(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Bigg[\alpha^2p^2 - 2(1+2\alpha)^2p + 4(1+\alpha)(1+3\alpha)\Bigg] \geq 0,$$

and so $F(p,t)$ is an increasing function of t for $p \leq \frac{3}{2}$.
Therefore,

$$\max\{F(p,t)\} = F(p,1) = \frac{1}{192(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Bigg[(\alpha^2+4\alpha+1)p^4$$

$$+12\alpha^2p^2(4-p^2) + 12(1+2\alpha)^2p^2(4-p^2) + 12(1+\alpha)(1+3\alpha)(4-p^2)^2\Bigg] = H(p).$$

$H'(p) = 0$ gives $p = 0$. Also $H''(p) < 0$ for $p = 0$.
This implies $\max\{H(p)\} = H(0) = \dfrac{1}{(1+2\alpha)^2}$, which proves (2.20). □

Putting $\alpha = 0$, Theorem 2.11 gives the following result:

Remark 2.12. If $f \in \mathcal{R}'_{sin}$, then

$$|a_2a_4 - a_3^2| \leq 1.$$

Substituting for $\alpha = 1$ in Theorem 2.11, the following result due to Khan et al. [11], is obvious:

Remark 2.13. If $f \in \mathcal{R}_{sin}$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{9}.$$

Theorem 2.14. *If $f \in \mathcal{R}^{\alpha}_{sin}$, then*

$$|H_3(1)| \leq \frac{(2+8\alpha+4\alpha^2)(1+3\alpha)^2 + (1+4\alpha)(1+2\alpha)^3}{(1+2\alpha)^3(1+3\alpha)^2(1+4\alpha)}. \tag{2.21}$$

Proof. By using (2.2), (2.3), (2.4), (2.18), (2.20) and Remark 2.7 in (1.1), the result (2.21) can be easily obtained. □

For $\alpha = 0$, Theorem 2.14 yields the following result:

Remark 2.15. If $f \in \mathcal{R}'_{sin}$, then

$$|H_3(1)| \leq 3.$$

For $\alpha = 1$, Theorem 2.14 yields the following result due to Khan et al. [11]:

Remark 2.16. If $f \in \mathcal{R}_{sin}$, then

$$|H_3(1)| \leq \frac{359}{2160}.$$

Theorem 2.17. *If $f \in \mathcal{R}^{\alpha}_{sin}$, then*

$$|H_4(1)| \leq \frac{2}{(1+2\alpha)^2(1+4\alpha)}\left[\frac{1+4\alpha+2\alpha^2}{(1+2\alpha)(1+6\alpha)}+\frac{3+12\alpha+2\alpha^2}{(1+4\alpha)^2}+\frac{2+8\alpha+4\alpha^2}{(1+3\alpha)(1+5\alpha)}\right]$$
$$+\frac{1}{(1+3\alpha)^2}\left[\frac{2+12\alpha+9\alpha^2}{(1+6\alpha)(1+3\alpha)^2}+\frac{3}{(1+2\alpha)(1+4\alpha)}\right].$$

Proof. We have

$$|a_2a_4 + 2a_3^2| \leq |a_2a_4 - a_3^2| + 3|a_3|^2.$$

Applying the triangle inequality in (1.2) and using the above inequality along with Theorem 2.1, Theorem 2.4, Theorem 2.8, Theorem 2.11 and Theorem 2.14, the proof of the Theorem 2.17 is obvious. □

For $\alpha = 0$, Theorem 2.17 yields the following result:

Remark 2.18. If $f \in \mathcal{R}^{'}_{sin}$, then

$$|H_4(1)| \leq 17.$$

For $\alpha = 1$, Theorem 2.17 yields the following result due to Khan et al. [11]:

Remark 2.19. If $f \in \mathcal{R}_{sin}$, then

$$|H_4(1)| \leq 0.10556.$$

3. Bounds of $|H_3(1)|$ for two-fold and three-fold symmetric functions

A function f is said to be n-fold symmetric if is satisfy the following condition:

$$f(\xi z) = \xi f(z)$$

where $\xi = e^{\frac{2\pi i}{n}}$ and $z \in E$.
By $S^{(n)}$, we denote the set of all n-fold symmetric functions which belong to the class S.
The n-fold univalent function have the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1}. \tag{3.1}$$

An analytic function f of the form (3.1) belongs to the family $\mathcal{R}^{\alpha(n)}_{sin}$ if and only if

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + sin\left(\frac{p(z)-1}{p(z)+1}\right), p \in \mathcal{P}^{(n)},$$

where

$$\mathcal{P}^n = \left\{p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} p_{nk} z^{nk}, z \in E\right\}. \tag{3.2}$$

Theorem 3.1. *If $f \in \mathcal{R}^{\alpha(2)}_{sin}$, then*

$$|H_3(1)| \leq \frac{1}{(1+2\alpha)(1+4\alpha)}. \tag{3.3}$$

Proof. If $f \in \mathcal{R}_{sin}^{\alpha(2)}$, so there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + sin\left(\frac{p(z)-1}{p(z)+1}\right). \tag{3.4}$$

Using (3.1) and (3.2) for $n = 2$, (3.4) yields

$$a_3 = \frac{1}{2(1+2\alpha)}p_2, \tag{3.5}$$

$$a_5 = \frac{1}{2(1+4\alpha)}\left(p_4 - \frac{1}{2}p_2^2\right). \tag{3.6}$$

Also

$$H_3(1) = a_3a_5 - a_3^3. \tag{3.7}$$

Using (3.5) and (3.6) in (3.7), it yields

$$H_3(1) = \frac{1}{4(1+2\alpha)(1+4\alpha)}p_2\left[p_4 - \frac{(1+2\alpha)^2 + (1+4\alpha)}{2(1+2\alpha)^2}p_2^2\right]. \tag{3.8}$$

On applying triangle inequality in (3.8) and using fourth inequality of Lemma 1.2, we can easily get the result (3.3). □

Putting $\alpha = 0$, the following result can be easily obtained from Theorem 3.1:

Remark 3.2. If $f \in \mathcal{R}_{sin}^{'(2)}$, then

$$|H_3(1)| \leq 1.$$

For $\alpha = 1$, Theorem 3.1 agrees with the following result:

Remark 3.3. If $f \in \mathcal{R}_{sin}^{(2)}$, then

$$|H_3(1)| \leq \frac{1}{15}.$$

Theorem 3.4. *If $f \in \mathcal{R}_{sin}^{\alpha(3)}$, then*

$$|H_3(1)| \leq \frac{1}{(1+3\alpha)^2}. \tag{3.9}$$

Proof. If $f \in \mathcal{R}_{sin}^{\alpha(3)}$, so there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + sin\left(\frac{p(z)-1}{p(z)+1}\right). \tag{3.10}$$

Using (3.1) and (3.2) for $n = 3$, (3.10) gives

$$a_4 = \frac{1}{2(1+3\alpha)}p_3. \tag{3.11}$$

Also

$$H_3(1) = -a_4^2. \tag{3.12}$$

Using (3.11) in (3.12), it yields

$$H_3(1) = -\frac{1}{4(1+3\alpha)^2}p_3^2. \tag{3.13}$$

On applying triangle inequality and using first inequality of Lemma 1.2, (3.9) can be easily obtained. □

For $\alpha = 0$, Theorem 3.4 yields the following result:

Remark 3.5. If $f \in \mathcal{R}_{sin}^{'(3)}$, then

$$|H_3(1)| \leq 1.$$

For $\alpha = 1$, Theorem 3.4 yields the following result:

Remark 3.6. If $f \in \mathcal{R}_{sin}^{(3)}$, then

$$|H_3(1)| \leq \frac{1}{16}.$$

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Subclass of analytic functions on q-analogue connected with a new linear extended multiplier operator

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Abstract. Using a new linear extended multiplier q-Choi-Saigo-Srivastava operator $D_{\alpha,\beta}^{m,q}(\mu,\tau)$ we define a subclass $\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$ subordination and the newly defined q-analogue of the Choi-Saigo-Srivastava operator to the class of analytic functions. For this class, conclusions are drawn that include coefficient estimates, integral representation, linear combination, weighted and arithmetic means, and radius of starlikeness.

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1. Introduction and preliminaries

Let A denote the normalized analytical function family f of the form:

$$f(\varsigma)=\varsigma+\sum_{\vartheta=2}^{\infty}a_{\vartheta}\varsigma^{\vartheta},\ \varsigma\in\mathbb{D}, \tag{1.1}$$

in the open unit disc $\mathbb{D}:=\{\varsigma\in\mathbb{C}:|\varsigma|<1\}$. Let $S\subset$ A be a class of functions which are univalent in $\mathbb{D}$. If f and $\hbar$ are analytic in $\mathbb{D}$ we say that f is *subordinate* to $\hbar$, denoted $f(\varsigma)\prec\hbar(\varsigma)$, if there exists an analytic function ϖ, with $\varpi(0)=0$ and $|\varpi(\varsigma)|<1$ for all $\varsigma\in\mathbb{D}$, such that $f(\varsigma)=\hbar(\varpi(\varsigma))$, $\varsigma\in\mathbb{D}$. In addition, if $\hbar$ is univalent in $\mathbb{D}$, then the next equivalent ([8, 22] and [23]) holds:

$$f(\varsigma)\prec\hbar(\varsigma)\Leftrightarrow f(0)=\hbar(0)\quad\text{and}\quad f(\mathbb{D})\subset\hbar(\mathbb{D}).$$

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For f given by (1.1) and $\hbar$ of the form

$$\hbar(\varsigma)=\varsigma+\sum_{\vartheta=2}^{\infty} b_{\vartheta}\varsigma^{\vartheta},\ \varsigma\in\mathbb{D},$$

the well-known *convolution product* is

$$(f*\hbar)(\varsigma):=\varsigma+\sum_{\vartheta=2}^{\infty} a_{\vartheta}b_{\vartheta}\varsigma^{\vartheta},\ \varsigma\in\mathbb{D}.$$

The class $S^*(\delta)$ of *starlike functions of order* δ, is said to include a function $f\in \mathrm{A}$ if

$$\mathrm{Re}\left(\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\right)>\delta,\ (0\leq\delta<1).$$

We observe that the class of *starlike functions*, $S^*(0)=S^*$, holds true. An analytic function $\hbar$ with $\hbar(0)=1$ is definitely in the Ĵanowski class $P[N,M]$, iff

$$\hbar(\varsigma)\prec\frac{1+N\varsigma}{1+M\varsigma}\quad(-1\leq M<N\leq 1).$$

The class $P[N,M]$ of Ĵanowski functions was investigated by Ĵanowski [16].

Scholars have recently been inspired by the study of the q-derivative, it is useful in mathematics and related fields. Jackson [13, 14], presented the q-analogue of the derivative and integral operator and also suggested some of its applications. Kanas and Raducanu [17] provided the q-analogue of the Ruscheweyh differential operator and looked into some of its features by using the concept of convolution. Aldweby and Darus [1], Mahmood and Sokol [20], and others looked into various sorts of analytical functions defined by the q-analogue of the Ruscheweyh differential operator see [2, 3, 7, 12, 15, 18, 21, 24, 29, 30] for further details.

The primary goal of the current study is to express a Choi-Saigo-Srivastava operator q-analogue based on convolutions. It also offers a few intriguing applications for this operator at the outset.We will now discuss the essential concept of the q-calculus, which was created by Ĵackson [14] and is pertinent to our ongoing research.

Ĵackson [13, 14] defined the q-derivative operator D_q of a function f:

$$D_q f(\varsigma):=\partial_q f(\varsigma)=\frac{f(q\varsigma)-f(\varsigma)}{(q-1)\varsigma},\ q\in(0,1),\ \varsigma\neq 0.$$

Remark that if the function f is in the type (1.1), thus, it implies

$$D_q f(\varsigma)=D_q\left(\varsigma+\sum_{\vartheta=2}^{\infty}a_{\vartheta}\varsigma^{\vartheta}\right)=1+\sum_{\vartheta=2}^{\infty}[\vartheta]_q a_{\vartheta}\varsigma^{\vartheta-1},\tag{1.2}$$

where $[\vartheta]_q$ is

$$[\vartheta]_q:=\frac{1-q^{\vartheta}}{1-q}=1+\sum_{\kappa=1}^{\vartheta-1}q^{\kappa},\ [0]_q:=0,$$

and

$$\lim_{q\to 1^-}[\vartheta]_q=\vartheta.$$

The definition of the q-number shift factorial for every non-negative integer ϑ is

$$[\vartheta, q]! := \begin{cases} 1, & \text{if} \quad \vartheta = 0, \\ [1, q]\,[2, q]\,[3, q]\ldots\ldots[\vartheta, q]\,, & \text{if} \quad \vartheta \in \mathbb{N}. \end{cases}$$

By combining the notion of convolution with a definition of the q-derivative,Wang et al. introduced in [30] the *q-analogue Choi-Saigo-Srivastava operator* $I^q_{\alpha,\beta} : \mathrm{A} \to \mathrm{A}$,

$$I^q_{\alpha,\beta} f(\varsigma) := f(\varsigma) * \mathcal{F}_{q,\alpha+1,\beta}(\varsigma),\ \varsigma \in \mathbb{D} \quad (\alpha > -1,\ \beta > 0), \tag{1.3}$$

where

$$\begin{aligned} \mathcal{F}_{q,\alpha+1,\beta}(\varsigma) &= \varsigma + \sum_{\vartheta=2}^{\infty} \frac{\Gamma_q(\beta + \vartheta - 1)\Gamma_q(\alpha + 1)}{\Gamma_q(\beta)\Gamma_q(\alpha + \vartheta)} \varsigma^{\vartheta} \\ &= \varsigma + \sum_{\vartheta=2}^{\infty} \frac{[\beta, q]_{\vartheta-1}}{[\alpha + 1, q]_{\vartheta-1}} \varsigma^{\vartheta},\ \varsigma \in \mathbb{D}, \end{aligned} \tag{1.4}$$

where $[\beta, q]_\vartheta$ is the *q-generalized Pochhammer symbol* for $\beta > 0$ defined by

$$[\beta, q]_\vartheta := \begin{cases} 1, & \text{if} \quad \vartheta = 0, \\ [\beta]_q\ [\beta + 1]_q \ldots [\beta + \vartheta - 1]_q\,, & \text{if} \quad \vartheta \in \mathbb{N}. \end{cases} \tag{1.5}$$

Thus,

$$I^q_{\alpha,\beta} f(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} \frac{[\beta, q]_{\vartheta-1}}{[\alpha + 1, q]_{\vartheta-1}} a_\vartheta \varsigma^{\vartheta},\ \varsigma \in \mathbb{D}, \tag{1.6}$$

while

$$I^q_{0,2} f(\varsigma) = \varsigma D_q f(\varsigma) \quad \text{and} \quad I^q_{1,2} f(\varsigma) = f(\varsigma).$$

Definition 1.1. [4] For $\mu \geq 0$, and $\tau > -1$, with the aid of the operator $I^q_{\alpha,\beta}$ we will define a new linear extended multiplier q-Choi-Saigo-Srivastava operator $D^{m,q}_{\alpha,\beta}(\mu, \tau) :$ $\mathrm{A} \to \mathrm{A}$ as follows:

$$\begin{aligned} D^{0,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) =&: D^q_{\alpha,\beta}(\mu, \tau) f(\varsigma) = f(\varsigma), \\ D^{1,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) =& \left(1 - \frac{\mu}{\tau + 1}\right) I^q_{\alpha,\beta} f(\varsigma) + \frac{\mu}{\tau + 1} \varsigma D_q \left(I^q_{\alpha,\beta} f(\varsigma)\right) \\ =& \varsigma + \sum_{\vartheta=2}^{\infty} \left(\frac{[\beta, q]_{\vartheta-1}}{[\alpha + 1, q]_{\vartheta-1}} \cdot \frac{\tau + 1 + \mu\left([\vartheta]_q - 1\right)}{\tau + 1} \right) a_\vartheta \varsigma^{\vartheta}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ D^{m,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) =& D^q_{\alpha,\beta}(\mu, \tau) \left(D^{m-1,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) \right),\ m \geq 1, \end{aligned}$$

where $\mu \geq 0$, $\tau > -1$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\beta > 0$ and $0 < q < 1$.

If $f \in \mathrm{A}$ given by (1.1), from (1.6) and the above definition Thus, it implies

$$D^{m,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta \varsigma^{\vartheta},\ \varsigma \in \mathbb{D}, \tag{1.7}$$

where

$$\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau) := \left(\frac{[\beta,q]_{\vartheta-1}}{[\alpha+1,q]_{\vartheta-1}}\cdot\frac{\tau+1+\mu\left([\vartheta]_q-1\right)}{\tau+1}\right)^m. \tag{1.8}$$

From (1.3) and (1.8), then

$$D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) = \underbrace{\left[\left(I^q_{\alpha,\beta}f(\varsigma)*\wp^q_{\mu,\tau}(\varsigma)\right)*\ldots*\left(I^q_{\alpha,\beta}f(\varsigma)*\wp^q_{\mu,\tau}(\varsigma)\right)\right]}_{n-\text{times}}*f(\varsigma),$$

where

$$\wp^q_{\mu,\tau}(\varsigma) := \frac{\varsigma-\left(1-\frac{\mu}{\tau+1}\right)q\varsigma^2}{(1-\varsigma)(1-q\varsigma)}.$$

Remark 1.2. The operator $D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)$ should be noticed that the following generalizes a number of other operators previously covered, for instance:

(i) For $q\to 1^-$, $\alpha=1$, $\beta=2$, and $\tau=0$, we obtain the operator D^m_μ investigated by Al-Oboudi [5];

(ii) If $q\to 1^-$, $\alpha=1$, $\beta=2$, $\mu=1$ and $\tau=0$, we obtain the operator D^m introduced by Sălăgean [27];

(iii) Taking $q\to 1^-$, $\alpha=1$ and $\beta=2$, we obtain the operator $I^m(\lambda,\kappa)$ studied Cătaş [9];

(iv) Considering $\alpha=1$, $\beta=2$ and $\tau=0$, we get $D^m_{\mu,q}$ presented and analysed by Aouf et al. [7];

(v) For $\alpha=1$, $\beta=2$, $\mu=1$ and $\tau=0$, we obtain the operator S^m_q investigated by Govindaraj and Sivasubramanian [12];

(vi) If $q\to 1^-$ we obtain $D^{m,\alpha}_{\mu,\tau,\beta}$ presented and investigated by El-Ashwah et al. [11] for $q=2$, $s=1$, $\alpha_1=\beta$, $\alpha_2=1$, $\beta_1=\alpha+1$;

(vii) If $q\to 1^-$, $\alpha=1$, $\beta=2$ and $\mu=1$, we obtain the operator I^m_τ, $\tau\geq 0$, investigated by Cho and Srivastava [10];

(viii) Given $q\to 1^-$, $\mu=\tau=0$ and $m=1$, we get $I^q_{\alpha,\beta}$ presented and analysed by Wang et al. [30];

(ix) Given $q\to 1^-$, $\alpha:=1-\alpha$, $\beta=2$, and $\tau=0$, we obtain the operator $D^{m,\alpha}_\mu$ investigate by Al-Oboudi and Al-Amoudi [6];

(x) If $\alpha:=1-\varrho$ and $\beta=2$, we get $D^{m,\lambda,\kappa}_{q,\varrho}$ investigated by Kota and El-Ashwah [18];

(xi) Given $\beta=2$, $\mu=0$ and $\tau=0$, we obtain the q-analogue integral operator of Noor $I^q_{\alpha,2}$ presented and investigated by [29];

(xii) If $q\to 1^-$, $\beta=2$, $\mu=0$ and $\tau=0$, we get the differential operator I^ϑ studied in [25, 26];

(xiii) For $q\to 1^-$, $\beta=2$, $\alpha:=1-\alpha$, $\mu=0$ and $\tau=0$, we obtain the Owa-Srivastava operator $I_{1-\alpha,2}$ presented and analysed in [28].

Definition 1.3. Let $-1\leq M<N\leq 1$ and $0<q<1$. $f\in\Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$ if it satisfies

$$\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)}\prec\frac{1+N\varsigma}{1+M\varsigma}.$$

Equivalently, $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$ iff

$$\left| \frac{\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)} - 1}{N - M\left(\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)}\right)} \right| < 1. \tag{1.9}$$

We must apply the following lemma in order to validate one of our findings.

Lemma 1.4. [19] *Let* $-1 \le M_2 \le M_1 < N_1 \le N_2 \le 1$. *Then*

$$\frac{1+N_1\varsigma}{1+M_1\varsigma} \prec \frac{1+N_2\varsigma}{1+M_2\varsigma}.$$

Throughout this paper, we suppose that $\mu \ge 0$, $\tau > -1$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\beta > 0$, $0 < q < 1$ and $-1 \le M < N \le 1$, We furthermore assume that all coefficients a_n of f are real positive numbers.

2. Main results

Theorem 2.1. *Suppose that* $f \in \mathrm{A}$ *given by (1.1). Then* $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$ *iff*

$$\sum_{\vartheta=2}^{\infty} \left[([\vartheta] + N) - (M\,[\vartheta] + 1)\right] \aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta < N - M. \tag{2.1}$$

Proof. Let (2.1) holds. then from (1.9) we have

$$\begin{aligned}
\left| \frac{\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)} - 1}{N - M\left(\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)}\right)} \right| &= \left| \frac{\sum\limits_{\vartheta=2}^{\infty}([\vartheta]_q - 1)\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta\varsigma^\vartheta}{(N-M)\varsigma + \sum\limits_{\vartheta=2}^{\infty}(N - M\,[\vartheta]_q)\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta\varsigma^\vartheta} \right| \\
&\le \frac{\sum\limits_{\vartheta=2}^{\infty}([\vartheta]_q - 1)\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta}{(N-M) - \sum\limits_{\vartheta=2}^{\infty}(N - M\,[\vartheta]_q)\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta} < 1,
\end{aligned}$$

then from (1.2), (1.7), and (2.1) this completes the direct part.
Conversely, $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$ then from (1.9) and (1.7), hence

$$\left| \frac{\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)} - 1}{N - M\left(\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)}\right)} \right| = \left| \frac{\sum\limits_{\vartheta=2}^{\infty}([\vartheta]_q - 1)\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta\varsigma^\vartheta}{(N-M)\varsigma + \sum\limits_{\vartheta=2}^{\infty}(N - M\,[\vartheta]_q)\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta\varsigma^\vartheta} \right| < 1.$$

Since $|\Re(\varsigma)| \le |\varsigma|$, we get

$$\Re\left(\frac{\sum\limits_{\vartheta=2}^{\infty}([\vartheta]_q - 1)\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta\varsigma^\vartheta}{(N-M) + \sum\limits_{\vartheta=2}^{\infty}(N - M\,[\vartheta]_q)\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_\vartheta\varsigma^\vartheta} \right) < 1. \tag{2.2}$$

We now select ς values along the real axis such that $\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)}$ is real. Then for letting $\varsigma \to 1^-$, we get (2.1). □

If we set $\alpha = 1$, $\beta = 2$, and $\tau = 0$ in Theorem 2.1 we have:

Corollary 2.2. *$f \in \Theta^{m,q}_{\alpha,\beta}(\mu, N, M)$ iff*

$$\sum_{\vartheta=2}^{\infty} \left[\left([\vartheta]_q + N \right) - \left(M [\vartheta]_q + 1 \right) \right] \left(1 + \mu \left([\vartheta]_q - 1 \right) \right)^m a_\vartheta < N - M.$$

Theorem 2.3. *Suppose that $f \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$. Therefore*

$$D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) = \exp\left(\frac{\ln q}{q-1} \int_0^\varsigma \frac{1}{t} \left(\frac{1+N\varphi(t)}{1+M\varphi(t)} \right) d_q(t) \right),$$

where $|\varphi(t)| < 1$.

Proof. Let $f \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$ and putting

$$\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)} = \omega(\varsigma),$$

with

$$\omega(\varsigma) \prec \frac{1+N\varsigma}{1+M\varsigma},$$

equivalently, we can write

$$\left| \frac{\omega(\varsigma)-1}{N-M\omega(\varsigma)} \right| < 1,$$

hence, there is

$$\frac{\omega(\varsigma)-1}{N-M\omega(\varsigma)} = \varphi(\varsigma),$$

such that $|\varphi(\varsigma)| < 1$. Hence,

$$\frac{\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)} = \frac{1}{\varsigma}\left(\frac{1+N\varphi(t)}{1+M\varphi(t)} \right).$$

Using simple calculation we get the result. □

Theorem 2.4. *Let $f_j \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$ and*

$$f_j(\varsigma) = \varsigma + \sum_{\iota=1}^{\infty} a_{\iota,j}\varsigma^\iota, \quad (j = 1, 2, 3, ..., \kappa).$$

Therefore $F \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$, such that

$$f(\varsigma) = \sum_{j=1}^{\kappa} c_j f_j(\varsigma) \quad \text{with} \quad \sum_{j=1}^{\kappa} c_j = 1.$$

Proof. FromTheorem 2.1, hence

$$\sum_{\vartheta=2}^{\infty}\left\{\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)}{N-M}\right\}a_{\vartheta,j}<1.$$

Therefore, we get

$$f(\varsigma)=\sum_{j=2}^{\kappa}c_j(\varsigma+\sum_{\vartheta=2}^{\infty}a_{\vartheta,j}\varsigma^{\vartheta})=\varsigma+\sum_{j=2}^{\kappa}\sum_{\vartheta=2}^{\infty}c_ja_{\vartheta,j}\varsigma^{\vartheta}=\varsigma+\sum_{\vartheta=2}^{\infty}\left(\sum_{j=2}^{\kappa}c_ja_{\vartheta,j}\right)\varsigma^{\vartheta}.$$

However,

$$\begin{aligned}&\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)}{N-M}\left(\sum_{j=2}^{\kappa}c_ja_{\vartheta,j}\right)\\ =\;&\sum_{j=2}^{\kappa}\left\{\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)}{N-M}a_{\vartheta,j}\right\}c_j\leq 1,\end{aligned}$$

then $F\in\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$ and the proof is complete. □

Theorem 2.5. *If $f,\hbar\in\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$, then $h_j(j\in\mathbb{N})$ is in $\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$, such that h_j denoted by*

$$h_j(\varsigma)=\frac{(1-j)f(\varsigma)+(1+j)\hbar(\varsigma)}{2}. \tag{2.3}$$

Proof. By (2.3), then

$$h_j(\varsigma)=\varsigma+\sum_{\vartheta=2}^{\infty}\left[\frac{(1-j)a_{\vartheta}+(1+j)b_{\vartheta}}{2}\right]\varsigma^{\vartheta}.$$

To prove $h_j(\varsigma)\in\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$, we need to show that

$$\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]}{N-M}\left\{\frac{(1-j)a_{\vartheta}+(1+j)b_{\vartheta}}{2}\right\}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)<1.$$

For this, consider

$$\begin{aligned}&\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]}{N-M}\left\{\frac{(1-j)a_{\vartheta}+(1+j)b_{\vartheta}}{2}\right\}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)\\ =\;&\frac{(1-j)}{2}\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]}{N-M}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)a_{\vartheta}\\ &+\frac{(1+j)}{2}\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]}{N-M}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)b_{\vartheta}\\ <\;&\frac{(1-j)}{2}+\frac{(1+j)}{2}=1,\end{aligned}$$

by using 2.1 we get the result. □

Theorem 2.6. *Let $f_j \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$ with $j = 1,2,.....\alpha(\alpha \in \mathbb{N})$. Then*

$$h(\varsigma) = \frac{1}{\alpha}\sum_{j=1}^{\alpha} f_j(\varsigma), \tag{2.4}$$

also is in the class $\Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$.

Proof. By (2.4), therefore

$$h(\varsigma) = \frac{1}{\alpha}\sum_{j=1}^{\alpha}\left(\varsigma + \sum_{\vartheta=2}^{\infty} a_{\vartheta,j}\varsigma^{\vartheta}\right) = \varsigma + \sum_{\vartheta=2}^{\infty}\left(\frac{1}{\alpha}\sum_{j=1}^{\alpha} a_{\vartheta,j}\right)\varsigma^{\vartheta}. \tag{2.5}$$

Since $f_j \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$, Through (2.5) and (2.1), there will be

$$\begin{aligned} &\sum_{\vartheta=2}^{\infty}\left[\left([\vartheta]_q + N\right) - \left(M[\vartheta]_q + 1\right)\right]\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)\left(\frac{1}{\alpha}\sum_{j=1}^{\alpha} a_{\vartheta,j}\right) \\ = \;& \frac{1}{\alpha}\sum_{j=1}^{\alpha}\left(\sum_{\vartheta=2}^{\infty}\left[\left([\vartheta]_q + N\right) - \left(M[\vartheta]_q + 1\right)\right]\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_{\vartheta,j}\right) \\ \le \;& \frac{1}{\alpha}\sum_{j=1}^{\alpha}(N-M) = N-M, \end{aligned}$$

the proof is completed. □

Theorem 2.7. *Suppose that $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$. Then $f \in S^*(\delta)$, for $|\varsigma| < r_1$, where*

$$r_1 = \left(\frac{(1-\delta)\left[\left([\vartheta]_q + N\right) - \left(M[\vartheta]_q + 1\right)\right]}{(\vartheta-\delta)(N-M)}\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)\right)^{\frac{1}{\vartheta-1}}.$$

Proof. Let $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$. To show that $f \in S^*(\delta)$, we need

$$\left|\frac{\frac{\varsigma f'(\varsigma)}{f(\varsigma)} - 1}{\frac{\varsigma f'(\varsigma)}{f(\varsigma)} + 1 - 2\delta}\right| < 1.$$

By using (1.1) along with some simple computations we have

$$\sum_{\vartheta=2}^{\infty}\left(\frac{\vartheta-\delta}{1-\delta}\right)|a_{\vartheta}|\,|\varsigma|^{\vartheta-1} < 1. \tag{2.6}$$

Since $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$, from (2.1), there are

$$\sum_{\vartheta=2}^{\infty}\frac{\left[\left([\vartheta]_q + N\right) - \left(M[\vartheta]_q + 1\right)\right]}{N-M}\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)\,|a_{\vartheta}| < 1. \tag{2.7}$$

And then, (2.6) is true, if

$$\sum_{\vartheta=2}^{\infty}\left(\frac{\vartheta-\delta}{1-\delta}\right)|a_\vartheta|\,|\varsigma|^{\vartheta-1}<\sum_{\vartheta=2}^{\infty}\frac{\left[\left([\vartheta]_q+N\right)-\left(M\,[\vartheta]_q+1\right)\right]}{N-M}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)\,|a_\vartheta|\,,$$

holds, which implies that

$$|\varsigma|^{\vartheta-1}<\frac{(1-\delta)\left[\left([\vartheta]_q+N\right)-\left(M\,[\vartheta]_q+1\right)\right]}{(\vartheta-\delta)\,(N-M)}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau),$$

and thus we get the required result. □

If we set $\alpha=1$, $\beta=2$, and $\tau=0$ in Theorem 2.7 we get:

Corollary 2.8. *Suppose that* $f\in\Theta_{\alpha,\beta}^{m,q}(\mu,N,M)$. *Then* $f\in S^*(\delta)$, *for* $|\varsigma|<r_1$, *where*

$$r_1=\left(\frac{(1-\delta)\left[\left([\vartheta]_q+N\right)-\left(M\,[\vartheta]_q+1\right)\right]}{(\vartheta-\delta)\,(N-M)}\left(1+\mu\left([\vartheta]_q-1\right)\right)^m\right)^{\frac{1}{\vartheta-1}}.$$

Remark 2.9. For $q\to1^-$, $\mu=\tau=0$ and $m=1$, in the above results we get the results investigated by Wang et al. [30].

3. Conclusion

This study introduces a subclass $\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$ of analytic functions on q-analogue associated with a new linear extended multiplier q-Choi-Saigo-Srivastava operator $D_{\alpha,\beta}^{m,q}(\mu,\tau)$ in the open unit disk $\mathbb{D}$. We have obtain coefficient estimates, integral representation, linear combination, weighted and arithmetic means, and radius of starlikeness belonging to the class $\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$. Some of the earlier efforts of numerous writers are generalized by our findings.

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Nonlocal conditions for fractional differential equations

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Abstract. In this work we use the method of lower and upper solutions to develop an iterative technique, which is not necessarily monotone, and combined with a fixed point theorem to prove the existence of at least one solution of nonlinear fractional differential equations with nonlocal boundary conditions of integral type.

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1. Introduction

Numerous phenomena in the applied sciences can be described by fractional differential equations. In fact, several monographs and research papers have been devoted to the study of fractional differential equations and related boundary value problems. We can mention the following books [12], [16], [17], [23], [25], [32], the research papers [1, 4, 11, 13, 14] and the references therein. Following [21], Picone, in 1908, was the first to introduce nonlocal boundary conditions for linear systems of ordinary differential equations. The following survey [19] contains a great number of references on nonlocal boundary problems for second order differential equations. The analysis of elliptic boundary value problems with nonlocal conditions were initiated in the paper [5]. The nonlocal condition has been proven more appropriate and more precise in many physical problems than the classical initial condition. We refer the reader to [3, 6, 9] and the references therein for a motivation regarding nonlocal conditions. The lower and upper solutions method has been proven instrumental for

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proving the existence and location of solutions of boundary value problems for ordinary differential equation and partial differential equation problems of integer orders. See for example [7, 10, 15, 26]. Many people have been interested in the study of the existence of solutions to boundary value problems for fractional differential equations with nonlocal conditions, see [8], [29], [28] and the references therein.To our modest knowledge only few research articles using the lower and upper solutions method for fractional differential equations are available. See [13, 18, 24].

In this paper, we consider the following class of fractional differential equations

$$D^{\alpha}u(t) + f(t,u) = 0, \quad t \in (0,1)\,,\ 1 < \alpha \leq 2, \tag{1.1}$$

with a Neumann condition at the initial point and a nonlocal boundary condition of integral type at the terminal point

$$u'(0) = 0, \quad u(1) = \int_0^1 g\left(u(t)\right) dt. \tag{1.2}$$

Here D^{α} is the Caputo fractional derivative of order $\alpha \in (1,2]$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy conditions that will be specified later. We use the lower and upper solutions method to develop an iterative method, which is not necessarily monotone (see [22], [30]) and combined with the Schauder fixed theorem to prove the existence of at least one solution for problem (1.1) - (1.2).

The rest of this paper is organized as follows. In Section 2 we recall some basic definitions and results that are needed in the rest of the paper. In Section 3, we develop the iterative technique in order to prove our main result concerning the existence of a solution of the problem (1.1) - (1.2). Finally, we give an example to illustrate our main result.

2. Preliminaries

In this section, we recall some basic definitions, notations and few results from fractional calculus that we shall use in the remainder of the paper. Let I denote the compact real interval $[0,1]$ and let $C(I)$ denote the space of continuous functions $\omega : I \to \mathbb{R}$, equipped with the norm

$$\|\omega\|_0 = \max_{t\in I} |\omega(t)|\,.$$

$C^n(I)$, $n \in \mathbb{N}$, is the space of continuous functions $\omega : I \to \mathbb{R}$, such that $\omega^{(k)} \in C(I)$ $k = 0, 1, 2, ..., n$, equipped with the norm

$$\|\omega\|_{C^n} = \sum_{k=0}^{n} \max_{0\leq t\leq 1} \left|\omega^{(k)}(t)\right|.$$

Definition 2.1. (see [16]) The Riemann-Liouville fractional primitive of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$, and where Γ is the gamma function.

For instance, $I^{\alpha} f$ exists for all $\alpha > 0$, when $f \in L^1(I)$. Notice, also, that when $f \in C(I)$, then $I^{\alpha} f \in C(I)$ and moreover $I^{\alpha} f(0) = 0$. The law of composition $I^{\alpha} I^{\beta} = I^{\alpha+\beta}$ holds for all $\alpha, \beta > 0$.

Definition 2.2. (see [16]) The Caputo fractional derivative of order $\alpha > 0$ of a C^n function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha} f(t) = I^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Notice that $D^{\alpha} c = 0$, where c is a real constant.

Remark 2.3. It is well known (see for instance [16, Lemma 2.22 page 96], [31, Lemma 3.6 page 6]) that for $\alpha > 0$

$$I^{\alpha} D^{\alpha} u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1}, \text{ for all } t \in I,$$

where $n = [\alpha] + 1$, and $c_0, c_1, ..., c_{n-1}$ are real constants.

Lemma 2.4. *Let $\alpha > 0$. Then the differential equation on I*

$$D^{\alpha} u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1}$, $t \in I$, $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Lemma 2.5. *Let $\alpha \in (1, 2)$. Then the homogeneous problem*

$$\begin{cases} D^{\alpha} u(t) = 0, & t \in I \\ u'(0) = 0, \ u(1) = 0 \end{cases}$$

has only the trivial solution $u(t) = 0$ for all $t \in I$.

Lemma 2.6. *Let $f \in C^2(I)$. Then for any $\alpha \in (1, 2)$ $D^{\alpha} f$ exists and is continuous on I.*

Proof. It follows from (2.2) with $\alpha \in (1, 2)$ that

$$D^{\alpha} f(t) = I^{2-\alpha} f''(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{f''(\tau)}{(t-\tau)^{\alpha-1}} d\tau,$$

so that

$$|D^{\alpha}f(t)| \le \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{|f''(\tau)|}{(t-\tau)^{\alpha-1}}d\tau \le \frac{\|f''\|_0}{\Gamma(2-\alpha)}\int_0^t \frac{1}{(t-\tau)^{\alpha-1}}d\tau.$$

Since

$$\int_0^t \frac{1}{(t-\tau)^{\alpha-1}}d\tau = \frac{t^{2-\alpha}}{2-\alpha} \le \frac{1}{2-\alpha}$$

we obtain

$$\|D^{\alpha}f\|_0 \le \frac{\|f''\|_0}{(2-\alpha)\,\Gamma(2-\alpha)}.$$

To prove the continuity of $D^{\alpha}f$ on I, let $t \ge t_0 \in I$. Then

$$D^{\alpha}f(t) - D^{\alpha}f(t_0) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{f''(\tau)}{(t-\tau)^{\alpha-1}}d\tau - \frac{1}{\Gamma(2-\alpha)}\int_0^{t_0} \frac{f''(\tau)}{(t_0-\tau)^{\alpha-1}}d\tau$$

$$= \frac{1}{\Gamma(2-\alpha)}\int_0^{t_0} \frac{f''(\tau)}{(t-\tau)^{\alpha-1}}d\tau + \frac{1}{\Gamma(2-\alpha)}\int_{t_0}^t \frac{f''(\tau)}{(t-\tau)^{\alpha-1}}d\tau - \frac{1}{\Gamma(2-\alpha)}\int_0^{t_0} \frac{f''(\tau)}{(t_0-\tau)^{\alpha-1}}d\tau$$

$$= \frac{1}{\Gamma(2-\alpha)}\int_0^{t_0} f''(\tau)\left(\frac{1}{(t-\tau)^{\alpha-1}} - \frac{1}{(t_0-\tau)^{\alpha-1}}\right)d\tau + \frac{1}{\Gamma(2-\alpha)}\int_{t_0}^t \frac{f''(\tau)}{(t-\tau)^{\alpha-1}}d\tau.$$

Notice that $(t_0-\tau)^{\alpha-1} \le (t-\tau)^{\alpha-1}$ so that

$$\left|\frac{1}{(t-\tau)^{\alpha-1}} - \frac{1}{(t_0-\tau)^{\alpha-1}}\right| = \frac{1}{(t_0-\tau)^{\alpha-1}} - \frac{1}{(t-\tau)^{\alpha-1}}.$$

Hence

$$|D^{\alpha}f(t) - D^{\alpha}f(t_0)| \le \frac{\|f''\|_0}{\Gamma(2-\alpha)}\int_0^{t_0}\left(\frac{1}{(t_0-\tau)^{\alpha-1}} - \frac{1}{(t-\tau)^{\alpha-1}}\right)d\tau$$

$$+\frac{\|f''\|_0}{\Gamma(2-\alpha)}\int_{t_0}^t \frac{1}{(t-\tau)^{\alpha-1}}d\tau.$$

Simple integrations give

$$\int_0^{t_0}\left(\frac{1}{(t_0-\tau)^{\alpha-1}} - \frac{1}{(t-\tau)^{\alpha-1}}\right)d\tau = \frac{1}{2-\alpha}\left((t-t_0)^{2-\alpha} + t_0^{2-\alpha} - t^{2-\alpha}\right), \qquad (2.3)$$

and

$$\int_{t_0}^t \frac{1}{(t-\tau)^{\alpha-1}}d\tau = \frac{(t-t_0)^{2-\alpha}}{2-\alpha}.$$

Using lemma 2 in [14] we have for $t \geq t_0$ and $1 > 2-\alpha \geq 0$

$$\left|t_0^{2-\alpha} - t^{2-\alpha}\right| \leq (t-t_0)^{2-\alpha}.$$

Combining the above computations we see that

$$|D^\alpha f(t) - D^\alpha f(t_0)| \leq \frac{3\,(t-t_0)^{2-\alpha}}{(2-\alpha)\,\Gamma(2-\alpha)}.$$

If $t_0 > t$, we interchange the role of t and t_0 in the preceding computations and we arrive at the same result. Therefore

$$\lim_{t\to t_0} |D^\alpha f(t) - D^\alpha f(t_0)| = 0.$$

□

The following results play an important role in the proof of our main result.

Theorem 2.7. [2, Corollary 2.1 page 3] *Let $f \in C^2(I)$ attains its minimum over the interval I at the point $t_0 \in (0,1)$ and $f'(0) \leq 0$. Then $D^\alpha f(t_0) \geq 0$ for any $\alpha \in (1,2)$.*

Changing f to $-f$ we obtain

Theorem 2.8. *Let $f \in C^2(I)$ attains its maximum over the interval I at the point $t_0 \in (0,1)$ and $f'(0) \geq 0$. Then $D^\alpha f(t_0) \leq 0$ for any $\alpha \in (1,2)$.*

Definition 2.9. [20]. Let E and F be Banach spaces. An operator $T : E \to F$ is called a completely continuous operator if T is continuous and maps any bounded subset of E into relatively compact subset of F.

Theorem 2.10. *(Schauder fixed point theorem,* [27]*) If Ω is a closed bounded convex subset of a Banach space E and $T : \Omega \to \Omega$ is completely continuous, then T has at least one fixed point in Ω.*

We shall use the following notation. For $U, V \in C^2(I)$, $U \leq V$ means $U(t) \leq V(t)$ for all $t \in I$. Also, $[U, V] := \{v \in C^2(I);\ U \leq v \leq V\}$.

3. Main result

In this section, we shall apply the lower and upper solutions method to develop an iterative technique to prove the existence of solutions to problem (1.1) - (1.2).

Definition 3.1. We call a function $\underline{u}$ a lower solution for problem (1.1) - (1.2), if $\underline{u} \in C^2(I)$ and

$$\begin{cases} D^\alpha \underline{u}(t) + f(t, \underline{u}(t)) \geq 0, & t \in (0,1) \\ \underline{u}'(0) = 0, \quad \underline{u}(1) \leq \int\limits_0^1 g\left(\underline{u}(t)\right) dt. \end{cases}$$

Definition 3.2. We call a function $\overline{u}$ an upper solution for problem (1.1) - (1.2), if $\overline{u} \in C^2(I)$ and

$$\begin{cases} D^\alpha \overline{u}(t) + f(t, \overline{u}(t)) \leq 0, & t \in (0,1) \\ \overline{u}'(0) = 0, \quad \overline{u}(1) \geq \int\limits_0^1 g\left(\overline{u}(t)\right) dt. \end{cases}$$

Definition 3.3. A solution of (1.1) - (1.2) is a function $u \in C^2(I)$ that is both a lower solution and an upper solution of the problem.

Define a truncation operator $\tau : C^2(I) \to [\underline{u}, \overline{u}]$ by

$$\tau(y) = \max\{\underline{u}, \min(y, \overline{u})\}.$$

Then $\tau(y) = \underline{u}$ if $y \le \underline{u}$, $\tau(y) = y$ if $y \in [\underline{u}, \overline{u}]$ and $\tau(y) = \overline{u}$ if $y \ge \overline{u}$. Moreover τ is a continuous and bounded operator. In fact, we have

$$\|\tau(u)\|_0 \le \max\left(\|\underline{u}\|_0, \|\overline{u}\|_0\right).$$

We now provide sufficient conditions on the nonlinearities f, g that will allow us to investigate problem (1.1) - (1.2).
(**H1**) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $(f(t, v_1) - f(t, v_2))(v_1 - v_2) < 0$, for all $t \in I$, $v_1 > v_2$.
(**H2**) $g : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing.

Theorem 3.4. *Assume that Problem (1.1) - (1.2) has a lower solution $\underline{u}$, an upper solution $\overline{u}$ such that $\underline{u}(t) \le \overline{u}(t)$, for all $t \in I$, and (H1), (H2) hold. Then Problem (1.1) - (1.2) has at least one solution $u^* \in C^2(I)$ such that $\underline{u}(t) \le u^*(t) \le \overline{u}(t), t \in I$.*

Proof. The proof will be given in several steps.
Step1: Modification of the problem. Let $\phi : I \times [\underline{u}, \overline{u}] \to \mathbb{R}$ and $\psi : [\underline{u}, \overline{u}] \to \mathbb{R}$ be defined, respectively, by

$$\phi(t, u) = f(t, \tau(u)), \ \psi(u) = g(\tau(u)).$$

It is clear that ϕ, ψ are continuous and bounded. Moreover, for $v_1 > v_2$ in $[\underline{u}, \overline{u}]$, we have $\tau(v_1) = v_1$ and $\tau(v_2) = v_2$, so that $\phi(t, v_1) = f(t, v_1)$ and $\phi(t, v_2) = f(t, v_2)$ hence $(\phi(t, v_1) - \phi(t, v_2))(v_1 - v_2) < 0$ for all $t \in I$ and $\underline{u} \le v_2 < v_1 \le \overline{u}$. Similarly, $\psi(u) = g(u)$, for all $u \in [\underline{u}, \overline{u}]$, so that ψ is nondecreasing in $[\underline{u}, \overline{u}]$.

We consider the following modified boundary value problem

$$\begin{cases} D^\alpha u(t) + \phi(t, u(t)) = 0, & t \in (0,1), 1 < \alpha \le 2 \\ u'(0) = 0, \quad u(1) = \int_0^1 \psi(u(t))\, dt \end{cases} \tag{3.1}$$

We will show that the modified problem (3.1) has at least one solution $u^* \in [\underline{u}, \overline{u}]$. It follows that $\tau(u^*) = u^*$ so that $\phi(t, u^*) = f(t, u^*)$, $\psi(u^*) = g(u^*)$. This implies that u^* is a solution of our original problem (1.1) - (1.2).
Step2. Let $b \in \mathbb{R}$. Consider the auxiliary problem

$$\begin{cases} D^\alpha u(t) + \phi(t, u(t)) = 0, & t \in (0,1), 1 < \alpha \le 2 \\ u'(0) = 0, \quad u(1) = b \end{cases} \tag{3.2}$$

Claim. If (H1) is satisfied then (3.2) has a unique solution.
Proof. Uniqueness. Assume (3.2) has two solutions y, z. We show that $y(t) = z(t)$ for all $t \in I$. Suppose, on the contrary, that there is $\xi \in I$ such that $y(\xi) \ne z(\xi)$. Assume, for definiteness that $y(\xi) > z(\xi)$. Let $w(t) = y(t) - z(t)$ for all $t \in I$. Then $w(\xi) > 0$. By the continuity of w on I it follows that there exists $\xi_0 \in I$ such that

$w(\xi_0) := \max_{t\in I} w(t) > 0$. Then $\xi_0 \in [0,1)$ because $w'(0) = w(1) = 0$. Theorem 2.8 implies that $D^\alpha w(\xi_0) \leq 0$. Then

$$0 \geq D^\alpha w(\xi_0) w(\xi_0) = (D^\alpha y(\xi_0) - D^\alpha z(\xi_0))(y(\xi_0) - z(\xi_0)).$$

It follows from the first equation in (3.2) and (H1) that

$$0 \geq -(\phi(\xi_0, y(\xi_0) - \phi(\xi_0, z(\xi_0))(y(\xi_0) - z(\xi_0)) > 0.$$

This contradiction shows that $y(t) \leq z(t)$ for all $t \in I$. Similarly we show that $z(t) \leq y(t)$ for all $t \in I$. Therefore $y(t) = z(t)$ for all $t \in I$.
Existence. It follows from Remark 2.3 that for $u \in C^2(I)$ and any $\alpha \in (1,2)$ we have

$$I^\alpha D^\alpha u(t) = u(t) - c_0 - c_1 t, \text{ for all } t \in I,$$

where c_0, c_1 are real constants. The first equation in (3.2) implies that

$$u(t) = -I^\alpha \phi(t, u(t)) + c_0 + c_1 t, \text{ for all } t \in I.$$

Simple computations lead to

$$u(t) = \int_0^1 G(t,s)\phi(s,u(s))ds + b, \tag{3.3}$$

where $G(t,s)$ is Green's function corresponding to the linear homogeneous problem. This function exists because the homogeneous problem has only the trivial solution, see Lemma 2.5. It is given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1}, & 0 \leq t < s \leq 1 \\ (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s < t \leq 1 \end{cases}.$$

Conversely, if $u \in C(I)$ is a solution of (3.3) then $u \in C^2(I)$ and is a solution of (3.2). Indeed, let $v(t) = \int_0^1 G(t,s)\phi(s,u(s))ds + b$. Then

$$v(t) = b - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s,u(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi(s,u(s))ds.$$

$$v(t) = b - I^\alpha \phi(\cdot, u(\cdot))(t) + I^\alpha \phi(\cdot, u(\cdot))(1)$$

Obviously $v(1) = b$. Also

$$v'(t) = -\frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \phi(s,u(s))ds,$$

so that $v'(0) = 0$. Moreover, it ias clear that $v' \in C^1(I)$, i.e. $v \in C^2(I)$. By Lemma 2.6 $D^\alpha v(t)$ exists and

$$\begin{aligned} D^\alpha v(t) &= D^\alpha (b - I^\alpha \phi(\cdot, u(\cdot))(t) + I^\alpha \phi(\cdot, u(\cdot))(1)) \\ &= -D^\alpha I^\alpha \phi(\cdot, u(\cdot))(t) = -\phi(t, u(t)). \end{aligned}$$

Hence

$$D^\alpha u(t) = -\phi(t, u(t)),$$

i.e.

$$D^\alpha u(t) + \phi(t, u(t)) = 0.$$

Now, define an operator $T : C(I) \to C(I)$ by the right-hand side of (3.3), i.e.

$$(Tu)(t) = \int_0^1 G(t,s)\phi(s,u(s))ds + b, \text{ for all } t \in I. \tag{3.4}$$

We show that T is continuous and uniformly bounded. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C(I)$ which converges uniformly to u. Then $u \in C(I)$. It follows from the uniform continuity of $G(.,.)$ on the compact rectangle $I\times I$ there is $G_0 > 0$ such that $|G(t,s)| \leq G_0$ for all $(t,s) \in I \times I$. Also, $\phi : I \times [\underline{u}, \overline{u}] \to \mathbb{R}$ is continuous and bounded. It follows that the exists $M_\phi > 0$ such that $|\phi(t,u(t))| \leq M_\phi$ for all $t \in I$. The equation (3.4) implies that

$$\|Tu_n - Tu\|_0 \leq G_0 \int_0^1 |\phi(s,u_n(s)) - \phi(s,u(s))|\, ds \to 0 \text{ as } n \to \infty.$$

Moreover

$$\|T(u)\|_0 \leq G_0 M_\phi + |b| := \rho.$$

Let $\Omega := \{u \in C(I);\ \|u\|_0 \leq \rho\}$. Then Ω is a closed, bounded and convex subset of $C(I)$. Also, $T(\Omega) \subset \Omega$. Now, we show that $\overline{T(\Omega)}$ is a compact subset of $C(I)$. First, $T(\Omega)$ is equicontinuous. Let $(t,s) \in I \times I$ with $s \leq t$. Then for all $u \in \Omega$

$$(Tu)(t) - (Tu)(s) = \int_0^1 (G(t,\sigma) - G(s,\sigma))\, \phi(\sigma, u(\sigma))d\sigma.$$

It follows that

$$|(Tu)(t) - (Tu)(s)| \leq \int_0^1 |G(t,\sigma) - G(s,\sigma)|\, |\phi(\sigma,u(\sigma))|\, d\sigma.$$

$$|(Tu)(t) - (Tu)(s)| \leq M_\phi \int_0^1 |G(t,\sigma) - G(s,\sigma)|\, d\sigma. \tag{3.5}$$

The uniform continuity of Green's function implies that for every $\epsilon > 0$ there is $\delta_1 > 0$ such that for all $(t,s) \in I \times I$ with $|t-s| < \delta_1$ we have

$$|G(t,\sigma) - G(s,\sigma)| \leq \frac{\epsilon}{M_\phi}.$$

It follows from (3.5) that for all $(t,s) \in I \times I$ with $|t-s| < \delta_1$ we have for all $u \in \Omega$

$$|(Tu)(t) - (Tu)(s)| < \epsilon.$$

Next, we show that $\overline{T(\Omega)}$ is equicontinuous. Given $\epsilon > 0$ there is $\delta > 0$ such that for any $u \in \Omega$ and $|t-s| < \delta$ we have

$$|(Tu)(t) - (Tu)(s)| < \epsilon/3. \tag{3.6}$$

Now, let $v \in \overline{T(\Omega)}$. Then there is $u \in \Omega$ such that

$$\|v - T(u)\|_0 \leq \epsilon/3, \tag{3.7}$$

i.e.

$$|(Tu)(t) - v(t)| < \epsilon/3.$$

Hence for $|t-s|<\delta$ we have for all $v\in\overline{T(\Omega)}$

$$|v(t)-v(s)|\le|v(t)-(Tu)(t)|+|(Tu)(t)-(Tu)(s)|+|(Tu)(s)-v(s)|<\epsilon.$$

By Ascoli-Arzela Theorem we conclude that $\overline{T(\Omega)}$ is a compact subset of $C(I)$. Hence the operator T is completely continuous. Schauder fixed point theorem (see Theorem 2.10) implies that T has a fixed point v^* in Ω, which is unique as shown earlier. So that

$$v^*(t)=T(v^*(t))=\int_0^1 G(t,s)\phi(s,v^*(s))ds+b,\ \text{for all } t\in I. \tag{3.8}$$

It follows from (3.8) that v^* is the (unique) solution of the auxiliary problem (3.2). **Step3**. We develop an iterative method to show that the modified problem has at least one solution. Define a sequence $(u_k)_{k\in\mathbb{N}}$ in the following way. Let $u_0=\underline{u}$ and for $k\ge 1$

$$\begin{cases} D^\alpha u_k(t)+\phi(t,u_k(t))=0, & t\in(0,1),1<\alpha\le 2 \\ u_k'(0)=0,\quad u_k(1)=\int\limits_0^1\psi(u_{k-1}(t))\,dt \end{cases} \tag{3.9}$$

Notice that $u_k'(0)$ and $u_k(1)$ do not depend on the unknown function u_k. We see that problem (3.9) is similar to the previous auxiliary problem (3.2). Therefore, for each $k\in\mathbb{N}$, (3.9) has a unique solution $u_k\in\Omega$. This implies that the sequence $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded. Hence it has convergent subsequence $(u_{kj})_{j\in\mathbb{N}}$. Observe that the subsequence $(u_{k_j-1})_{j\in\mathbb{N}}$ may not converge to the same limit as the subsequence $(u_{kj})_{j\in\mathbb{N}}$. We use a diagonalization process to have $\lim\limits_{kj\to\infty}u_{kj}=\lim\limits_{k_j-1}u_{k_j-1}=u^*$. It follows from (3.3) that (3.9) is equivalent to

$$u_k(t)=\int_0^1 G(t,s)\phi(s,u_k(s))ds+\int\limits_0^1\psi(u_{k-1}(t))\,dt,\ \text{for all } t\in I.$$

Take limit as $kj\to\infty$, using the continuity of ϕ and ψ, we obtain

$$u^*(t)=\int_0^1 G(t,s)\phi(s,u^*(s))ds+\int\limits_0^1\psi(u^*(t))\,dt,\ \text{for all } t\in I.$$

Therefore

$$\begin{cases} D^\alpha u^*(t)+\phi(t,u^*(t))=0, & t\in(0,1),1<\alpha\le 2 \\ u^{*'}(0)=0,\quad u^*(1)=\int\limits_0^1\psi(u^*(t))\,dt \end{cases} \tag{3.10}$$

Step4. To complete the proof of our main result we need to prove that $\underline{u}\le u^*\le\overline{u}$, i.e. for all $t\in I$

$$\underline{u}(t)\le u^*(t)\le\overline{u}(t).$$

We only show that $\underline{u}(t)\le u^*(t)$ for all $t\in I$. We proceed by contradiction. Assume, on the contrary that there is $t_1\in(0,1)$ such that $\underline{u}(t_1)>u^*(t_1)$. Let $w(t)=\underline{u}(t)-u^*(t)$ for all $t\in I$. Then $w\in C(I)\cap C^2(0,1)$ and $w(t_1)>0$. It follows that there is $t_0 \in I$ such that $w(t_0) = \max_{t \in I} w(t) > 0$. It follows from Theorem 2.8 that $D^\alpha w(t_0) \leq 0$. The first equation in (3.1) and assumption (H1) imply that

$$0 \geq D^\alpha w(t_0)\ \ w(t_0) = (D^\alpha \underline{u}(t_0) - D^\alpha u^*(t_0))(\underline{u}(t_0) - u^*(t_0))$$
$$= -(\phi(t_0, \underline{u}(t_0)) - \phi(t_0, u^*(t_0)))(\underline{u}(t_0) - u^*(t_0)) > 0.$$

We arrive at a contradiction. Thus, $w(t) = \underline{u}(t) - u^*(t) \leq 0$ for all $t \in (0,1)$. Now, if $t_0 = 0$ we have $w'(0) = 0$. If $w(0) > 0$, it follows from the continuity of w that there exists a small interval $[0,a] \subset I$ such that $w(t) > 0$ for all $t \in [0,a]$. This is not possible from the previous argument. Hence $w(0) = \underline{u}(0) - u^*(0) \leq 0$. Also, if $t_0 = 1$ we have from the definition of $\underline{u}$ and u^* and the monotonicity of ψ

$$\underline{u}(1) - u^*(1) \leq \int_0^1 \psi(\underline{u}(t))\,dt - \int_0^1 \psi(u^*(t))\,dt = \int_0^1 (\psi(\underline{u}(t)) - \psi(u^*(t)))\,dt \leq 0.$$

Therefore $\underline{u}(t) \leq u^*(t)$ for all $t \in I$. Similarly we show that $u^*(t) \leq \overline{u}(t)$ for all $t \in I$. We infer that $\underline{u}(t) \leq u^*(t) \leq \overline{u}(t)$ for all $t \in I$, i.e. $u^* \in [\underline{u}, \overline{u}]$. We deduce that for all $t \in I$

$$\phi(t, u^*(t)) = f(t, u^*(t)), \text{ and } \psi(u^*(t)) = g(u^*(t)).$$

Consequently,

$$\begin{cases} D^\alpha u^*(t) + f(t, u^*(t)) = 0, \quad t \in (0,1), 1 < \alpha \leq 2 \\ u'^*(0) = 0, \quad u^*(1) = \int_0^1 g(u^*(t))\,dt \end{cases}.$$

Finally, we see that u^* is the desired solution to our original problem. This completes the proof of our main result. □

Example 3.5. We consider the following boundary value problem

$$\begin{cases} D^{\frac{3}{2}} u(t) - 1 + e^{-u(t)} = 0, \ t \in (0,1) \\ u'(0) = 0, \quad u(1) = \int_0^1 \left(1 - e^{-u(t)}\right) dt. \end{cases} \tag{3.11}$$

We have $\alpha = 3/2$, $f(t,u) = -1 + e^{-u}$, and $g(u) = 1 - e^{-u}$. We see that f, g are continuous. For $u > v$ the mean value theorem implies

$$f(t,u) - f(t,v) = \left(-1 + e^{-u}\right) - \left(-1 + e^{-v}\right) = \left(e^{-u} - e^{-v}\right) = -e^{-z}(u - v),$$

where z is in the segment $[v,u]$. Hence $(f(t,u) - f(t,v))(u - v) = -e^{-z}(u - v)^2 < 0$. Hence f satisfies (H1). Also, $g'(u) = e^{-u} > 0$, so that g satisfies (H2). We see that $\underline{u}(t) = 0$ is a lower solution for problem (3.11) and $\overline{u}(t) = 1$ is an upper solution for problem (3.11). Applying Theorem 3.4, we see that the problem (3.11) has at least one solution $u^* \in C^2(I)$ with $0 \leq u^*(t) \leq 1$, for all $t \in I$. Notice that we have obtained the existence of a nonnegative solution.

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Coupled system of sequential partial $\sigma(.,.)-$Hilfer fractional differential equations with weighted double phase operator: Existence, Hyers-Ulam stability and controllability

Nadir Benkaci-Ali

Abstract. In this paper, we are concerned by a sequential partial Hilfer fractional differential system with weighted double phase operator. First, we introduce the concept of Hyers-Ulam stability with respect to an operator L for an abstract equation of the form $u = LFu$ in Banach lattice by using the fixed point arguments and spectral theory. Then, we prove the controllability and apply the previous results obtained for abstract equation to prove existence and Hyers-Ulam stability of a coupled system of sequential fractional partial differential equations involving a weighted double phase operator. Finally, example illustrating the main results is constructed. This work contains several new ideas, and gives a unified approach applicable to many types of differential equations.

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1. Introduction

Fractional order and Hilfer fractional order differential equations involving a p-Laplacian operator are of great importance and are interesting class of problems. Such kinds of problems have been studied by many authors, see [3, 4, 5, 17]. At the same time, the studies of Hyers-Ulam stability have attracted a great deal of attention in the last ten years, (see [1, 2, 9, 10, 11, 12, 15, 13, 16]), and the references therein.

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In [19], the authors discussed the existence of positive solutions for the double phase differential equation

$$-D_{p,q}(u)(x) = f(x,u), \ x \in \Omega \subset \mathbb{R}^n,$$

with double phase differential operator $D_{p,q}(u) = \Delta_p u + a.\Delta_q u$.

In [14], existence and uniquness of solutions to sequential fractional differential equation

$$\lambda D^\alpha u(t) + D^\beta u(t) = f(t, u(t))$$

was investigated.

In [8], the authors worked on the existence and Hyers–Ulam stability for the following sequential fractional differential system:

$$\left[{}^cD_q^\nu + r.{}^cD_q^\sigma\right] u(t) = f\left(t, u(t), u(\alpha t), {}^cD_q^\sigma(\alpha t)\right), \ t \in (0,T)$$

where D^ν, D^σ are the Caputo fractional derivatives of orders $\nu \in (1,2]$ and $\sigma \in (0,1]$ respectively.

Motivated by the works mentioned above, in this paper, we give the existence, Hyers-Ulam stability and controllability results for the abstract equation $LFu = u$ and their application to the following coupled sequential partial Hilfer fractional differential system with weighted double phase partial differential operator:

$$\begin{cases} \left(\zeta_1(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_1(x)\frac{\partial u_1}{\partial x}\right)\right)\right)(t,x) + f_1(t,x,u_1,u_2) = 0, \\ t,x > 0, \\ \left(\zeta_2(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_2(x)\frac{\partial u_2}{\partial x}\right)\right)\right)(t,x) + f_2(t,x,u_1,u_2) = 0, \\ t,x > 0, \\ u_j(0,x) = u_j(t,0) = \lim\limits_{x\to+\infty}\frac{\partial u_j}{\partial x}(t,x) = 0, \ j \in \{1,2\}, \end{cases} \tag{1.1}$$

where $D_{0^+,t}^{\alpha,\omega,\sigma}$is the partial $\sigma(.,.)-$Hilfer fractional derivative with respect to the variable t of order α and type $0 \le \omega \le 1$ with $0 < \alpha < 1$,

$$\phi = \phi_{p^-} + \phi_{p^+}, \ 1 < p^- < p^+$$

with

$$\phi_{p^\nu}(x) = |x|^{p^\nu - 2}.x, \ \text{for } \nu \in \{-,+\},$$

and for $j \in \{1,2\}$,

$$\zeta_j(t) = a_j + t, \ a_j > 0,$$

The function $\sigma(t,x)$ is bounded and positive on $\mathbb{R}^+ \times \mathbb{R}^+$ having a continuous and positive derivative $\frac{\partial \sigma}{\partial t}(t,x) > 0$ with respect to the variable t on $(0,+\infty)$ with $\sigma(0,x) = 0$ for all $x \ge 0$ and such that

$$\left(\sigma^+\right)^\alpha \in L^1\left(\mathbb{R}^+\right) \ \text{and } \sigma^+(x) = \lim_{t\to+\infty}\sigma(t,x).$$

2. Abstract background

Let $(E, \|.\|)$ be a real Banach space. A nonempty subset P of E is said to be a cone if P is closed and convex, $P \cap (-P) = 0$ and for all $t \geq 0$, $tP \subset P$. In this situation, P induces a partial order in the Banach space E defined by $x \leq y$ if $y - x \in P$.

The mapping $L : E \to E$ is said to be bounded if it maps bounded subsets in E into bounded subsets in E. L is said to be compact if it is continuous and maps bounded subsets in E into relatively compact subsets in E.

Definition 2.1. A normed lattice E is a vector space with a norm $\|.\|$ and a partial ordering $(\leq)$ under which it is a Riesz space and the following condition holds:
if $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where

$$|u| = \sup\{u, -u\}.$$

If $(E, \|\|)$ is complete, it is called a Banach lattice.

Let us recall the definition and some properties of the resolvent:

Definition 2.2. [7, 18]Let $L : E \to E$ be a bounded and linear operator. The resolvent set of L is the set

$$\rho(L) = \{\lambda \in \mathbb{C} : \lambda I - L \text{ is invertible in } Q(E)\},$$

where $Q(E)$ is the unital Banach algebra defined by

$$Q(E) = \{f : E \to E : f \text{ is linear and bounded}\}$$

and $I : E \to E$ is the identity.

The resolvent of L is $r_L : \rho(L) \to Q(E)$ defined by

$$r_L(\lambda) = (\lambda I - L)^{-1} \in Q(E).$$

The spectrum of L, $\sigma(L) = \mathbb{C} \backslash \rho(L)$ is non-empty, compact and

$$r(L) = \max_{\lambda \in \sigma(L)} |\lambda| = \lim_{n \to \infty} \|L^n\|^{\frac{1}{n}},$$

called the spectral radius of L.

The serie's representation of the resolvent: If $|\lambda| > r(L)$, then $\lambda \in \rho(L)$ and $r_L(\lambda)$ is given by

$$r_L(\lambda) = \sum_{k=0}^{+\infty} \lambda^{-k-1} L^k.$$

Let $E^+ = \{u \in E, u \geq 0\}$ be the positive cone of a real Banach lattice $(E, \|.\|, \leq)$.

We consider an operator $T : E \to E$ defined by

$$Tu = LFu, \; u \in E$$

where $L : E \to E$ is a completely continuous operator and $F : E \to E$ is a continuous and bounded map.

Remark 2.3. T is completely continuous, because it is the composition of the completely continuous operator L and the bounded continuous map F.

We consider the equation

$$u = Tu. \tag{2.1}$$

Definition 2.4. Equation (2.1) is said to be Hyers-Ulam stable in E with respect to L (or L-Hyers-Ulam stable), if $T = LF$ and there exists $N > 0$, such that the following (p_N) property is satisfied:

$$\begin{cases} \text{For all } \epsilon > 0 \text{ and all } (v,w) \in E \times \bar{B}(0,\epsilon) \setminus \{0\}, \\ \text{if } v = L(F(v) + w) \text{ then } T \text{ admits a fixed point } u \in G \text{ such that} \\ \|u - v\| \leq N.\epsilon. \end{cases} \tag{p_N}$$

The main tools of this work are the following Theorems:

Theorem 2.5. [6] *Let E be a Banach space, C be a nonempty bounded convex and closed subset of E, and $T : C \to C$ be a compact and continuous map. Then T has at least one fixed point in C.*

3. Main results

3.1. Existence and Hyers-Ulam stability of abstract equation

Throughout this paper, we assume that the following hypothesis hold:

$$\begin{cases} \text{There exists an operator } L^{(k)} : E^+ \to E^+ \text{ such that, for all } u \in E \\ |L(u)| \leq L^{(k)}(|u|), \end{cases} \tag{3.1}$$

where $L^{(k)}$ is bounded, increasing, $k-$positively homogeneous and sub-additive on E, $k \in (0,1]$, with $L^{(k)}(E^+ \setminus \{0\}) \subset E^+ \setminus \{0\}$.

$F : E \to E$ is a continuous mapping such that

$$\begin{cases} \text{There exist } (g,h) \in E^+ \setminus \{0\} \times E^+ \text{ such that } \left\|L^{(k)}(g)\right\| < 1 \text{ and} \\ |F(u)| \leq g \|u\|^{\frac{1}{k}} + h, \text{ for all } u \in E. \end{cases} \tag{3.2}$$

Lemma 3.1. *Assume that If the hypothesis (3.1) and (3.2) hold true, and let Then T admits a fixed point u in $\bar{B}(0,r)$, $r > r_0$, where*

$$r_0 = \frac{\left\|L^{(k)}(h)\right\|}{1 - \left\|L^{(k)}(g)\right\|} \geq 0.$$

Proof. Let $u \in \bar{B}(0,r)$, $r > r_0$. So,

$$\begin{aligned} |Tu| &= |LFu| \leq L^{(k)}(|Fu|) \leq L^{(k)}\left(\|u\|^{\frac{1}{k}}.g + h\right) \\ &\leq \|u\|.L^{(k)}(g) + L^{(k)}(h) \end{aligned}$$

this implies that

$$\|Tu\| \leq r.\left\|L^{(k)}(g)\right\| + \left\|L^{(k)}(h)\right\| = (r - r_0).\left\|L^{(k)}(g)\right\| + r_0 \leq r,$$

then $T\left(\bar{B}(0,r)\right) \subset \bar{B}(0,r)$. From Schauder fixed point theorem, we deduce that T has at least one fixed point $u \in \bar{B}(0,r)$. □

Lemma 3.2. *Assume that hypothesis (3.1) and (3.2) hold true.*
If $(v, w) \in E \times \bar{B}(0, \epsilon) \setminus \{0\}$, $\epsilon > 0$ *such that*

$$v = L(F(v) + w),$$

then $v \in \bar{B}(0, r_\epsilon)$, *with*

$$r_\epsilon = \frac{\left\|L^{(k)}(h)\right\| + \epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|} \text{ and } M = \sup\left\{\left\|L^{(k)}(x)\right\|, x \in \bar{B}(0,1)\right\}.$$

Proof. Indeed, if $v = L(Fv + w)$, then

$$\begin{aligned} |v| &= |L(Fv + w)| \leq L^{(k)}(|Fv| + |w|) \leq L^{(k)}\left(\|v\|^{\frac{1}{k}}.g + h + |w|\right) \\ &\leq \|v\|.L^{(k)}(g) + L^{(k)}(h) + L^{(k)}(|w|). \end{aligned}$$

This leads

$$\|v\| \leq \|v\|.\left\|L^{(k)}(g)\right\| + \left\|L^{(k)}(h)\right\| + \left\|L^{(k)}(|w|)\right\|.$$

Thus

$$\|v\| \leq \frac{\left\|L^{(k)}(h)\right\| + \left\|L^{(k)}(|w|)\right\|}{1 - \left\|L^{(k)}(g)\right\|} \leq \frac{\left\|L^{(k)}(h)\right\| + \epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|}.$$ □

Let $r_* = \max\left\{r_0, (r_0)^{\frac{1}{k}}\|g\| + \|h\|\right\} \geq 0$, where r_0 is the constant given in Lemma (3.1). We consider the following hypothesis:

There exist $\rho \in E^+ \setminus \{0\}$, $\lambda > 0$ and $r > r_*$ such that, for all $u, v \in \bar{B}(0, r)$,

$$|Fu - Fv| \leq \rho \|u - v\|, \tag{3.3}$$

and

$$|L(u) - L(v)| \leq \lambda L_+ |u - v|. \tag{3.4}$$

where L_+ is a linear, bounded and strictly positive operator on E.

Theorem 3.3. *Assume that hypothesis (3.1), (3.2), (3.3) and (3.4) hold true, and*

$$\lambda \in \left(0, \|L_+(\rho)\|^{-1}\right). \tag{3.5}$$

Then, equation (2.1) is L-Hyers-Ulam stable in E.

Proof. Suppose that

$$v = L(F(v) + w),$$

where $(v, w) \in E \times \bar{B}(0, \epsilon) \setminus \{0\}$, $\epsilon > 0$.
Let $r > r_* = \max\left\{r_0, (r_0)^{\frac{1}{k}}\|g\| + \|h\|\right\}$ be the constant given in the hypothesis (3.3).

We deduce from lemmas (3.1) and (3.2) that T admits a fixed point $u \in \bar{B}(0, r)$ and $v \in \bar{B}(0, r_\epsilon)$, with

$$r_\epsilon = \frac{\left\|L^{(k)}(h)\right\| + \epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|} \text{ and } M = \sup\left\{\left\|L^{(k)}(x)\right\|, x \in \bar{B}(0,1)\right\}.$$

Now, let $x_0 > 0$ be the unique positive solution of the algebraic equation

$$\left(r_0 + \frac{M}{1 - \left\|L^{(k)}(g)\right\|}.x^k\right)^{\dfrac{1}{k}} \|g\| + \|h\| + x - r = 0.$$

We distinguish the following three cases:

Case 1. If $r < (r_\epsilon)^{\frac{1}{k}} \|g\| + \|h\| + \epsilon$, then $\epsilon > x_0$. This leads

$$\|u - v\| \leq r + r_\epsilon \leq x_0^{-1}\left(2r + \frac{M.x_0^k}{1 - \left\|L^{(k)}(g)\right\|}\right).\epsilon.$$

Case 2. If $r < r_\epsilon$, then $\epsilon > \mu$, with

$$\mu = \left[\frac{(r - r_0)\left(1 - \left\|L^{(k)}(g)\right\|\right)}{M}\right]^{\dfrac{1}{k}},$$

and so,

$$\|u - v\| \leq 2r + \frac{\epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|} \leq \mu^{-1}\left(2r + \frac{M.\mu^k}{1 - \left\|L^{(k)}(g)\right\|}\right).\epsilon.$$

Case 3. If $\max\left\{r_\epsilon, (r_\epsilon)^{\frac{1}{k}} \|g\| + \|h\| + \epsilon\right\} \leq r$, then $(Fu, (Fv) + w) \in \bar{B}(0, r) \times \bar{B}(0, r)$, and from hypothesis (3.4), it follows that

$$|L(Fu) - L(Fv + w)| \leq \lambda L_+ |Fu - Fv - w|.$$

And by using (3.3), we obtain

$$\begin{aligned} |u - v| &\leq \lambda L_+ |Fu - Fv - w| \\ &\leq \lambda L_+ |Fu - Fv| + \lambda L_+ (|w|) \\ &\leq \lambda. \|u - v\| L_+(\rho) + \lambda \epsilon L_+\left(\frac{|w|}{\|w\|}\right) \end{aligned}$$

thus

$$\|u - v\| \leq \left(\frac{\lambda \|L_+\|}{1 - \lambda. \|L_+(\rho)\|}\right).\epsilon.$$

Consequently,

$$\|u - v\| \leq N.\epsilon$$

where

$$N = \max\left\{\gamma_1'\left(2r + \frac{M.\gamma_2'}{1 - \left\|L^{(k)}(g)\right\|}\right), \left(\frac{\lambda \|L_+\|}{1 - \lambda. \|L_+(\rho)\|}\right)\right\},$$

with

$$\gamma_1' = \max\left\{x_0^{-1}, \mu^{-1}\right\} \text{ and } \gamma_2' = \max\left(x_0^k, \mu^k\right).$$

Proving our claim. □

Now, we replace the hypothesis (3.3) and (3.4) by the following conditions:
There exists $\lambda_0 > 0$ and $r > r_*$ such that, for all $u, v \in \bar{B}(0, r)$,

$$|F(u) - F(v)| \leq \lambda_0 |u - v|, \tag{3.6}$$

and

$$|L(u) - L(v)| \leq L_0 |u - v|, \tag{3.7}$$

where $L_0 : E \to E$ is a linear, compact and strictly positive operator.

Theorem 3.4. *Assume that hypothesis (3.1), (3.2), (3.6) and (3.7) hold, and*

$$r(L_0) < \lambda_0^{-1}. \tag{3.8}$$

Then equation (2.1) is L-Hyers-Ulam stable in E.

Proof. Suppose that $v = L(F(v) + w)$, $(v, w) \in E \times \bar{B}(0, \epsilon) \setminus \{0\}$, $\epsilon > 0$.
Let $r > r_* = \max\left\{r_0, (r_0)^{\frac{1}{k}} \|g\| + \|h\|\right\}$ is the constant given in the hypothesis (3.6).
It follows from lemmas (3.1) and (3.2), that $v \in \bar{B}(0, r_\epsilon)$ and T admits a fixed point $u \in \bar{B}(0, r)$, with

$$r_\epsilon = \frac{\left\|L^{(k)}(h)\right\| + \epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|} \text{ and } M = \sup\left\{\left\|L^{(k)}(x)\right\|, x \in \bar{B}(0,1)\right\}.$$

We have seen in the proof of theorem (3.3) that, if

$$r \leq \max\left\{r_\epsilon, (r_\epsilon)^{\frac{1}{k}} \|g\| + \|h\| + \epsilon\right\},$$

then $\epsilon \geq \max\{\mu, x_0\}$, where $x_0 > 0$ is the positive solution of the algebraic equation

$$\left(r_0 + \frac{M}{1 - \left\|L^{(k)}(g)\right\|}.x^k\right)^{\frac{1}{k}} \|g\| + \|h\| + x - r = 0$$

In this case, we have

$$\|u - v\| \leq \gamma_1' \left(2r + \frac{M.\gamma_2'}{1 - \left\|L^{(k)}(g)\right\|}\right).\epsilon,$$

where

$$\gamma_1' = \max\left\{x_0^{-1}, \mu^{-1}\right\} \text{ and } \gamma_2' = \max\left(x_0^k, \mu^k\right).$$

Now, we assume that $\max\left\{r_\epsilon, (r_\epsilon)^{\frac{1}{k}} \|g\| + \|h\| + \epsilon\right\} \leq r$. Then $(Fu, (Fv) + w) \in \bar{B}(0, r) \times \bar{B}(0, r)$, and by using hypothesis (3.4), it follows that

$$|L(Fu) - L(Fv + w)| \leq L_0 |Fu - Fv - w|. \tag{3.9}$$

By using (3.6), inequality (3.9) leads

$$\begin{aligned} |u - v| &\leq L_0 |Fu - Fv - w| \\ &\leq L_0 |Fu - Fv| + L_0 (|w|) \\ &\leq \lambda_0 . L_0 (|u - v|) + \epsilon.\pi_w, \end{aligned}$$

where

$$\pi_w = L_0\left(\frac{|w|}{\|w\|}\right) \in E^+\backslash\{0\}.$$

Then

$$\begin{aligned} z &= |u-v| \leq \lambda_0.L_0(z) + \epsilon L_0\left(\frac{|w|}{\|w\|}\right) \\ &\leq \lambda_0.L_0(z) + \epsilon\pi_w \\ &\leq \lambda_0.L_0(\lambda_0.L_0(z) + \epsilon\pi_w) + \epsilon\pi_w \\ &\leq \lambda_0^3.L_0^3(z) + \epsilon.\left(\lambda_0^2.L_0^2(\pi_w) + \lambda_0.L_0(\pi_w) + \pi_w\right) \\ &\leq \lambda_0^n.L_0^n(z) + \epsilon.\sum_{k=0}^{n-1}\lambda_0^k L_0^k(\pi_w) \in E^+\backslash\{0\}, \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

As $\lambda_0.r(L_0) = \lambda_0.\lim_{n\to\infty}\sqrt[n]{\|L_0^n\|} < 1$ then $\lim_{n\to\infty}\lambda_0^n.L_0^n(z) = 0$, $\lambda_0^{-1} \in \rho(L_0)$ and $(I - \lambda_0.L_0)$ is invertible. The serie's representation of the resolvent r_{L_0} at λ_0^{-1} is given by

$$r_{L_0}\left(\lambda_0^{-1}\right) = \left(\lambda_0^{-1}I - L_0\right)^{-1} = \sum_{k=0}^{+\infty}(\lambda_0)^{k+1}L_0^k.$$

Then

$$\sum_{k=0}^{+\infty}\lambda_0^k L_0^k(\pi_w) = (I - \lambda_0.L_0)^{-1}(\pi_w) \in E^+\backslash\{0\}.$$

Thus,

$$\|u-v\| \leq \left\|(I-\lambda_0.L_0)^{-1}(\pi_w)\right\|.\epsilon \leq \left\|(I-\lambda_0.L_0)^{-1}\right\|\|L_0\|.\epsilon.$$

Consequently,

$$\|u-v\| \leq N.\epsilon$$

where

$$N = \max\left\{\gamma_1'\left(2r_0 + \frac{M.\gamma_2'}{1-\left\|L^{(k)}(g)\right\|}\right), \left\|(I-\lambda_0.L_0)^{-1}\right\|\|L_0\|\right\}.$$

Proving our claim. □

3.2. Existence and Hyers-Ulam stability of coupled system IVS

In this section, we use the results obtained in the previous section to prove existence and Hyers-Ulam stability of the coupled system of sequential time $\sigma-$Hilfer fractional differential equations (1.1), where $D_{0^+,t}^{\alpha,\omega,\sigma}$ is the $\sigma-$Hilfer fractional derivative with respect to the variable t of order α and type $0 \leq \omega \leq 1$ with $0 < \alpha < 1$,

$$\phi = \phi_{p^-} + \phi_{p^+}\ ,\ 1 < p^- < p^+$$

with

$$\phi_{p^\nu}(x) = |x|^{p^\nu - 2}.x, \text{ for } \nu \in \{-,+\},$$

and for $j \in \{1,2\}$,

$$\zeta_j(t) = a_j + t,\ a_j > 0.$$

We suppose that the following conditions hold,

$$\begin{cases} f_j \in C\left(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}\right), \ \frac{1}{\theta_j} \in L^1\left(\mathbb{R}^+, \mathbb{R}^+\right) \\ \text{and} \\ \sigma^+ \in L^\alpha\ \left(\mathbb{R}^+\right), \end{cases} \tag{3.10}$$

with

$$0 < \sigma^+(x) = \sup\{\sigma(t,x), \ t \geq 0\} < \infty, \ \forall x \geq 0.$$

Next, we recall the definitions of $\sigma-$Hilfer fractional orders integrals and derivatives of order α and type $0 \leq \omega \leq 1$, where $J \subset \mathbb{R}^n$ and $\sigma : I \times J \to \mathbb{R}^+$ is the positive function on $I \times J \subset \mathbb{R}^+ \times \mathbb{R}^+$ having a continuous and positive derivative $\dfrac{\partial \sigma}{\partial t}(t,x) > 0$ with respect to the variable t on $(0, +\infty)$ with $\sigma(0,x) = 0$ for all $x \geq 0$.

Definition 3.5. [17] Let $a \in \mathbb{R}^+$, $\alpha > 0$ and $J \subset \mathbb{R}^n$. Then the $\sigma-$left-sided fractional integral of a function u with respect to t on $\mathbb{R}^+$ is defined by

$$I_{a^+,t}^{\alpha,\sigma} u(t,x) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\partial \sigma}{\partial t}(t,x)\left(\sigma(t,x) - \sigma(\tau,x)\right)^{\alpha-1} u(\tau,x) d\tau.$$

In the case $\alpha = 0$, this integral is interpreted as the identity operator $I_{a^+}^{0,\sigma} u = u$.

Definition 3.6. [17] Let $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$, u and σ two functions such that $t \mapsto u(t,.) \in C^n(\mathbb{R}^+, \mathbb{R})$ and $t \mapsto \sigma(t,.) \in C^n(\mathbb{R}^+, \mathbb{R})$. The σ-Hilfer fractional derivative $D_{a^+,t}^{\alpha,\omega,\sigma}$ of u with respect to t of order $n-1 < \alpha < n$ and type $0 \leq \omega \leq 1$ is defined by

$$D_{a^+,t}^{\alpha,\omega,\sigma} u(t,x) = I_{a^+,t}^{\omega(n-\alpha),\sigma} \left(\frac{1}{\sigma_t'(t,x)} \frac{\partial}{\partial t}\right)^n I_{a^+,t}^{(1-\omega)(n-\alpha),\sigma} u(t,x),$$

where $\sigma_t'(t,x) = \dfrac{\partial \sigma}{\partial t}(t,x)$.

Let's also recall the following important result ([17]):

Theorem 3.7. *If* $t \mapsto u(t,x) \in C^n(\mathbb{R}^+)$, $n-1 < \beta < \alpha < n$, $0 \leq \omega \leq 1$ *and* $\xi = \alpha + \omega(n-\alpha)$, *then*

$$I_{a^+,t}^{\alpha,\sigma} . D_{a^+,t}^{\alpha,\omega,\sigma} u(t,x)$$

$$= u(t,x) - \sum_{k=1}^{n} \frac{\left(\sigma(t,x) - \sigma(a,x)\right)^{\xi-k}}{\Gamma(\xi - k + 1)} \left(\frac{1}{\sigma_t'(t,x)} \frac{\partial}{\partial t}\right)^{n-k} I_{a^+,t}^{(1-\omega)(n-\alpha),\sigma} u(a,x).$$

Moreover,

$$I_{a^+,t}^{\alpha,\sigma} I_{a^+,t}^{\beta,\sigma}(u) = I_{a^+,t}^{\alpha+\beta,\sigma}, \quad D_{a^+,t}^{\alpha,\omega,\sigma}\left(D_{a^+,t}^{\beta,\omega,\sigma} u\right) = D_{a^+,t}^{\alpha+\beta,\omega,\sigma} u,$$

$$D_{a^+,t}^{1,\omega,\sigma} u = D_t^1 u = \frac{\partial u}{\partial t} \ \textit{and} \ D_{a^+,t}^{\alpha,\omega,\sigma} I_{a^+,t}^{\alpha,\sigma}(u) = u.$$

Remark 3.8. In this paper, we assume that $\sigma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous having a positive and continuous derivative $\frac{\partial \sigma}{\partial t}(t,x)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $\sigma(0,x) = 0$, for all $x \in \mathbb{R}^+$. If $\alpha \in (0,1)$, then $n = 1$ and for $t, x > 0$

$$I_{0^+,t}^{\alpha,\sigma}.D_{0^+,t}^{\alpha,\omega,\sigma} u(t,x) = u(t,x) - \frac{(\sigma(t,x))^{\xi-1}}{\Gamma(\xi)} \left(I_{0^+,t}^{(1-\omega)(1-\alpha),\sigma} u \right) (0^+, x).$$

Moreover, if u is continuous, then

$$\lim_{t \to 0^+} \left(I_{0^+,t}^{(1-\omega)(1-\alpha),\sigma} u \right) (t,x) = 0, \ \forall x \geq 0$$

and so $I_{0^+,t}^{\alpha,\sigma}.D_{0^+,t}^{\alpha,\omega,\sigma} u(t,x) = u(t,x)$.

Definition 3.9. We say that IVS (1.1) has the Hyers-Ulam stability in a Banach space $E = G \times G$ if there exits a constant $N > 0$ such that for every $\epsilon > 0$, $v = (v_1, v_2) \in E$, if

$$\begin{cases} \left| \left(\zeta_1(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma} \right) \left(\frac{\partial}{\partial x} \left(\phi \left(\theta_1(x) \frac{\partial v_1}{\partial x} \right) \right) \right) (t,x) + f_1(t,x,v_1,v_2) \right| \leq \epsilon, \\ t, x > 0, \\ \left| \left(\zeta_2(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma} \right) \left(\frac{\partial}{\partial x} \left(\phi \left(\theta_2(x) \frac{\partial v_2}{\partial x} \right) \right) \right) (t,x) + f_2(t,x,v_1,v_2) \right| \leq \epsilon, \\ t, x > 0, \\ v_j(0,x) = v_j(t,0) = \lim_{x \to +\infty} \frac{\partial v_j}{\partial x}(t,x) = 0, \ j \in \{1,2\}, \end{cases} \tag{3.11}$$

then there exists a solution $u \in E$ of IVS (1.1), such that

$$\|u - v\| \leq N.\epsilon. \tag{3.12}$$

We call such N a Hyers-Ulam stability constant.

Let $E = G \times G$ be a real Banach space with

$$G = \left\{ u \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}) : \sup_{t,x \geq 0} |u(t,x)| < \infty \right\}$$

equipped with the norm $\|(u,v)\| = \max(\|u\|_0, \|v\|_0)$ where

$$\|u\|_0 = \sup_{t,x \in \mathbb{R}^+} (|u(t,x)|).$$

Remark 3.10. E is a Banach lattice under the partial ordering $(\leq)$ defined by

$$(u_1, u_2) \leq (v_1, v_2) \Leftrightarrow u_1(x) \leq v_1(x) \text{ and } u_2(x) \leq v_2(x) \text{ for all } x \geq 0.$$

under which it is a Riesz space and $|(u,v)| = (|u|, |v|)$.
Moreover, $E^+ = \{(u,v) \in E, (u,v) \geq 0\}$ is the positive cone of $(E, \|.\|, \leq)$.

We consider the operator $T : E \to E$ defined by

$$T(u_1, u_2) = LF(u_1, u_2), \ \ (u_1, u_2) \in E$$

where

$$L(u_1, u_2) = (L_1(u_1, u_2), L_2(u_1, u_2)) \text{ and } F(u_1, u_2) = (F_1(u_1, u_2), F_2(u_1, u_2)),$$

such that for $j \in \{1, 2\}$

$$\begin{aligned} L_j(u_1, u_2)(t, x) &= \int_0^x \frac{1}{\theta_j(z)} \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t (u_j)(\tau, s)\, d\tau\right)(t, s)\, ds\right) dz, \\ F_j(u_1, u_2)(t, x) &= f_j(t, x, u_1(t, x), u_2(t, x)), \end{aligned}$$

where $\psi = \phi^{-1} : \mathbb{R} \to \mathbb{R}$ is the inverse function of sum of p_i-Laplacian operators

$$\phi = \sum_{i=1}^{i=N} \phi_{p_i},$$

with $\phi_{p_i}(x) = |x|^{p_i - 2}.x$ and ψ_{p_i} is the inverse function of ϕ_{p_i}.

We denote

$$T = (T_1, T_2)$$

with

$$T_j = L_j F, \ j \in \{1, 2\}.$$

Remark 3.11. Let $p^- = \min\{p_1, p_2 ... p_N\}$ and $p^+ = \max\{p_1, p_2 ... p_N\}$. For all $x \geq 0$, $i \in \{1, 2 ... N\}$

$$\phi_{p_i}(x) \leq \phi(x) \leq N.\phi^+(x)$$

where

$$\phi^+(x) = \begin{cases} \phi_{p^+}(x) & if\ x \geq 1 \\ \phi_{p^-}(x) & if\ x \leq 1 \end{cases}$$

and so, we conclude that

$$\psi^+\left(\frac{x}{N}\right) \leq \psi(x) \leq \psi_{p_i}(x) \tag{3.13}$$

where

$$\psi^+\left(\frac{x}{N}\right) = \begin{cases} \psi_{p^+}\left(\frac{x}{N}\right) & if\ x \geq 1 \\ \psi_{p^-}\left(\frac{x}{N}\right) & if\ x \leq 1. \end{cases}$$

Moreover, for $x \geq y \geq 0$,

$$\begin{cases} \psi_p(x+y) \leq \psi_p(x) + \psi_p(y), & \text{if } p \geq 2, \\ \psi_p(x+y) \leq (2)^{\frac{2-p}{p-1}}.\left[\psi_p(x) + \psi_p(y)\right], & \text{if } p < 2. \end{cases} \tag{3.14}$$

Remark 3.12. The condition (3.10) makes that the operator L_j is completely continuous and F_j is bounded for each $j \in \{1, 2\}$, and so, T is completely continuous.

Lemma 3.13. *Let $h_1, h_2 \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ be continuous and bounded functions. $(u_1, u_2) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+) \times C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ is solution of IVS (3.15)*

$$\begin{cases} \left(\zeta_1(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_1(x)\frac{\partial u_1}{\partial x}\right)\right)\right)(t,x) + h_1(t,x) = 0, \\ \quad t, x > 0, \\ \left(\zeta_2(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_2(x)\frac{\partial u_2}{\partial x}\right)\right)\right)(t,x) + h_2(t,x) = 0, \\ \quad t, x > 0, \\ u_j(0,x) = u_j(t,0) = \lim_{x\to+\infty}\frac{\partial u_j}{\partial x}(t,x) = 0, \ j \in \{1,2\}, \end{cases} \tag{3.15}$$

if and only if

$$u_j(t,x) = \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,s)\,d\tau\right)(t,s)\,ds\right)dz, \ \textit{for } j \in \{1,2\}.$$

(u_1, u_2) is fixed point of T (i.e $T(u_1,u_2) = (u_1,u_2)$).

Proof. First, assume that $(u_1, u_2) \in E$ is a solution of IVS (3.15), then for each $j \in \{1,2\}$, The function u_j satisfies equation

$$D_t^1\left((a_j + t).D_{0^+,t}^{\alpha,\omega,\sigma}\left[\frac{\partial}{\partial x}\left(\phi\left(\theta_j(x)\frac{\partial u_j}{\partial x}\right)\right)\right]\right)(t,x) = -h_j(t,x),$$

where $\phi = \phi_{p^-} + \phi_{p^+}$. Integrating, we have

$$D_{0^+,t}^{\alpha,\omega,\sigma}\left[\frac{\partial}{\partial x}\left(\phi\left(\theta_j(x)\frac{\partial u_j}{\partial x}\right)\right)\right](t,x) = \frac{-1}{a_j + t}\int_0^t h_j(\tau,x)\,d\tau, \ t > 0. \tag{3.16}$$

Applying $I_{0^+,t}^{\alpha,\sigma}$ on both sides of equation (3.16) and using Lemma (3.7) and initial condition $\frac{\partial u_j}{\partial x}(0,x) = 0$, we obtain

$$\frac{\partial}{\partial x}\left(\phi\left(\theta_j(x)\frac{\partial u_j}{\partial x}\right)\right)(t,x) = -I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,x)\,d\tau\right)(t,x)$$

By integrating on $[x, +\infty[$ and using the boundary conditions

$$u_j(t,0) = \lim_{x\to+\infty}\frac{\partial u_j}{\partial x}(t,x) = 0,$$

we have

$$\phi\left(\theta_j(x)\frac{\partial u_j}{\partial x}\right) = \int_x^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,s)\,d\tau\right)(t,s)\,ds$$

and so

$$u_j(t,x) = \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,s)\,d\tau\right)(t,s)\,ds\right)dz.$$

Conversely, assume that $(u_1, u_2) \in E$ such that for $j \in \{1, 2\}$,

$$u_j(t,x) = \int_0^x \frac{1}{\theta_j(z)} \psi \left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t h_j(\tau, s)\, d\tau \right)(t,s)\, ds \right) dz.$$

Then $u_j \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ and verifies

$$u_j(x,0) = u_j(0,x) = 0.$$

Moreover, by derivating with respect to the variable x, we obtain

$$\frac{\partial u_j}{\partial x}(t,x) = \frac{1}{\theta_j(x)} \psi \left(\int_x^{+\infty} I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t h_j(\tau, s)\, d\tau \right)(t,s)\, ds \right), \tag{3.17}$$

and so

$$\frac{\partial}{\partial x} \phi \left(\theta_j(x) \frac{\partial u_j}{\partial x} \right) = -I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_i(t)} \int_0^t h_j(\tau, x)\, d\tau \right)(t,x). \tag{3.18}$$

Applying $D_{0^+,t}^{\alpha,\omega,\sigma}$ on both sides of equation (3.18) and using Lemma (3.7) we have

$$\zeta_j(t) . D_{0^+,t}^{\alpha,\omega,\sigma} \left[\frac{\partial}{\partial x} \left(\phi \left(\theta_j(x) \frac{\partial u_j}{\partial x} \right) \right) \right](t,x) = -\int_0^t h_j(\tau, x)\, d\tau,$$

so, u_j is solution of the equation

$$D_t^1 \left(\zeta_j(t) . D_{0^+,t}^{\alpha,\omega,\sigma} \left[\frac{\partial}{\partial x} \left(\phi \left(\theta_j(x) \frac{\partial u_j}{\partial x} \right) \right) \right] \right)(t,x) = -h_j(t,x).$$

Now, we show that $\lim_{x \to +\infty} \frac{\partial u_j}{\partial x}(t,x) = 0$. Let $H_j = \sup\{h_j(t,x), t, x \geq 0\}$. We have

$$I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t h_j(\tau, s)\, d\tau \right)(t,s) \leq H_j . I_{0^+,t}^{\alpha,\sigma}(1)(t,s) = \frac{H_j}{\Gamma(\alpha+1)} \sigma^\alpha(t,s),$$

then

$$\int_x^{+\infty} I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t h_j(\tau, s)\, d\tau \right)(t,s)\, ds \leq \frac{H_j}{\Gamma(\alpha+1)} \int_x^{+\infty} \sigma^\alpha(t,s)\, ds$$

so, it follows from equation (3.17) that

$$\begin{aligned} \frac{\partial u_j}{\partial x}(t,x) &\leq \frac{1}{\theta_j(x)} \psi \left(\frac{H_j}{\Gamma(\alpha+1)} \int_x^{+\infty} \sigma^\alpha(t,s)\, ds \right) \\ &\leq \frac{1}{\theta_j(x)} \psi \left(\frac{H_j}{\Gamma(\alpha+1)} \int_0^{+\infty} (\sigma^+)^\alpha(s)\, ds \right) \end{aligned}$$

Since $\frac{1}{\theta_i(x).} \in L^1(\mathbb{R}^+, \mathbb{R}^+)$ then

$$\lim_{x \to +\infty} \frac{\partial u_j}{\partial x}(t,x) = 0.$$

Thus, (u_1, u_2) is solution of IVS (3.15). This completes the proof. □

Remark 3.14. We deduce from Lemma (3.13) that, $(u_1, u_2) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ is solution of IVS (1.1) if and only if (u_1, u_2) is a fixed point of T.

Lemma 3.15. *If equation (2.1) is L-Hyers-Ulam stable in E then IVS (1.1) has the Hyers-Ulam stability in E.*

Proof. Assume that equation (2.1) is L-Hyers-Ulam stable in E. Let $\epsilon > 0$ and $v = (v_1, v_2) \in E$ verifying inequalities (3.11). Let $w = (w_1, w_2) \in \bar{B}_E(0, \epsilon)$ such that

$$w_j(t) = -\left(\zeta_j(t) . D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_j(x)\frac{\partial v_j}{\partial x}\right)\right)\right)(t,x) - f_j(t, v_1(t), v_2(t)),$$

$$j \in \{1, 2\}.$$

We have from Lemma (3.13) that

$$v_j(x) = T_j(v_1, v_2)(x) == L_j(F(v_1, v_2) + w),$$

then

$$v = L(F(v) + w).$$

If $w = (0, 0)$ then v is a fixed point of T, and so, $u = v$ is solution of IVS (1.1) and we have

$$\|u - v\| = 0 \le N.\epsilon.$$

Now, if $w \in \bar{B}_E(0, \epsilon) \setminus \{0\}$, as (2.1) is L-Hyers-Ulam stable then there exists a fixed point u of T which is solution of IVS (1.1) such that

$$\|u - v\| \le N.\epsilon.$$

Thus, IVS (1.1) has the Hyers-Ulam stability in E. □

Lemma 3.16. *Assume that*

$$p^+ \ge 2. \tag{3.19}$$

Then L verifies the condition (3.1), with $L^{(k)} = \left(L_1^{(k)}, L_2^{(k)}\right)$ *such that*

$$k = \frac{1}{p^+ - 1} \le 1,$$

where for $j \in \{1, 2\}$

$$L_j^{(k)}(u_1, u_2)(t, x) = \int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t u_j(\tau, s)\, d\tau\right)(t, s)\, ds\right) dz.$$

Proof. Let $u = (u_1, u_2) \in E$. For $j \in \{1, 2\}$

$$\begin{aligned} |L_j(u_1, u_2)(t, x)| &= \left|\int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t u_j(\tau, s)\, d\tau\right)(t, s)\, ds\right) dz\right| \\ &\le \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t |u_j(\tau, s)|\, d\tau\right)(t, s)\, ds\right) dz. \end{aligned}$$

By using the inequality (3.13) we find that for all $t, x \ge 0$,

$$\begin{aligned} |L_j(u_1, u_2)(tx)| &\le \int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t |u_j(\tau, s)|\, d\tau\right)(t, s)\, ds\right) dz \\ &= L_j^{(k)}(|u_1|, |u_2|)(x) \end{aligned}$$

and then $|L(u)| \leq L^{(k)}(|u|)$. Moreover, $L^{(k)}$ is bounded, increasing, $k-$positively homogeneous and verifies

$$L^{(k)}\left(E^{+}\backslash\{0\}\right) \subset E^{+}\backslash\{0\}.$$

And the condition (3.14) leads that $L^{(k)}$ is sub-additive. □

Lemma 3.17. *Assume that*

$$1 < p^{-} \leq 2. \tag{3.20}$$

Then For all $r > 0$ and for all $u, v \in \bar{B}(0, r)$,

$$|L(u) - L(v)| \leq \lambda L_{+}|u - v|.$$

where

$$L_{+} = (L_{+,1}, L_{+,2})$$

with

$$L_{+,j}(u_1, u_2) = \int_0^x \frac{1}{\theta_j(z)} \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)} \int_0^t u_j(\tau, s)\, d\tau\right)(t, s)\, ds dz,\ j \in \{1, 2\},$$

$$\lambda = \lambda(r) = \frac{1}{p^{-} - 1}\left(\frac{r.\left\|\left(\sigma^{+}\right)^{\alpha}\right\|_{L^1}}{\Gamma(\alpha + 1)}\right)^{\frac{2 - p^{-}}{p^{-} - 1}} > 0,$$

and

$$\sigma^{+}(x) = \lim_{t \to \infty} \sigma(t, x).$$

Proof. Let $r > 0$ and $u, v \in \bar{B}(0, r)$, for each $j \in \{1, 2\}$,we have

$$\begin{aligned}
&|L_j(u) - L_j(v)| \\
&= \left|\int_0^x \frac{1}{\theta_j(z)}\left[\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds\right) - \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right)\right] dz\right| \\
&\leq \int_0^x \frac{1}{\theta_j(z)}\left|\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds\right) - \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right)\right| dz,
\end{aligned}$$

where

$$B_j u_j(t, s) = \frac{1}{\zeta_j(t)} \int_0^t u_j(\tau, s)\, d\tau \leq \|u\|, \text{ for all } u \in E.$$

Let $t, x > 0$ such that $u_j \neq v_j$ on $[0, t] \times [x, +\infty[$, and let $\chi_{t,x} \in [b_{t,x}, c_{t,x}] \setminus \{0\}$ where

$$\begin{aligned}
b_{t,x} &= \min\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds, \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right) \text{ and} \\
c_{t,x} &= \max\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds, \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right),
\end{aligned}$$

such that

$$\begin{aligned}
&\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds\right) - \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right) \\
&\qquad = A(\chi_{t,x}) \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j(u_j - v_j))(t, s)\, ds
\end{aligned}$$

where

$$A(\chi_t) = \frac{1}{(p^+ - 1)\left|\psi(\chi_{t,x})\right|^{p^+-2} + (p^- - 1)\left|\psi(\chi_{t,x})\right|^{p^--2}}.$$

We have

$$\begin{aligned} A(\chi_t) &= \frac{1}{(p^+ - 1)\left(\psi(|\chi_{t,x}|)\right)^{p^+-2} + (p^- - 1)\left(\psi(|\chi_{t,x}|)\right)^{p^--2}} \\ &\leq \frac{\left(\psi(|\chi_{t,x}|)\right)^{2-p^-}}{p^- - 1} \\ &\leq \frac{\left(\psi_{p^-}(|\chi_{t,x}|)\right)^{2-p^-}}{p^- - 1}. \end{aligned}$$

Moreover,

$$\begin{aligned} |\chi_{t,x}| &\leq |c_{t,x}| \\ &\leq \max\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(B_j(|u_j|)(t,s)\right)ds, \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(B_j(|v_j|)(t,s)\right)ds\right) \\ &\leq r.\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(B_j(1)\right)ds \\ &\leq r.\int_0^{+\infty} I_{0^+,t}^{\alpha,\sigma}(1)\,ds = r.\int_0^{+\infty} \frac{\sigma^\alpha(s,t)}{\Gamma(\alpha+1)}ds \\ &\leq r.\frac{\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)}, \end{aligned}$$

this leads

$$\left|\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(B_j u_j(t,s)\right)ds\right) - \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(B_j v_j(t,s)\right)ds\right)\right|$$

$$\leq \lambda\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(B_j(|u_j - v_j|)\right)(t,s)\,ds$$

and so,

$$|L_j(u) - L_j(v)| \leq \lambda\int_0^x \frac{1}{\theta_j(z)}\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(B_j(|u_j - v_j|)\right)(t,s)\,ds.$$

Thus

$$|L(u) - L(v)| \leq \lambda L_+ |u - v|.$$

□

Remark 3.18. Since L_+ is linear, bounded and strictly positive on E, then Lemma (3.17) implies that the condition (3.4) holds for all $r_* > 0$. Moreover, the operator

$$L_0 = \lambda L_+ = (\lambda L_{+,1}, \lambda L_{+,2})$$

is linear, compact and strictly positive operator, so, the condition (3.7) is also satisfied.

Lemma 3.19. *Let* $\theta_0 = \min\{\theta_1, \theta_2\}$. *Then*

$$r(L_0) \le \beta = \frac{\lambda \left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)} \int_0^\infty \frac{dt}{\theta_0(t)}, \tag{3.21}$$

where $r(L_0)$ *is the spectral raidus of* L_0.

Proof. Assume that (3.21) holds. Let $u = (u_1, u_2) \in \partial B_E(0,1)$. For $j \in \{1,2\}$

$$\begin{aligned} L_{0,j}(u) &= \lambda \int_0^x \frac{1}{\theta_j(z)} \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)} \int_0^t u_j(\tau,s)\,d\tau\right)(t,s)\,dsdz \\ &\le \lambda \int_0^x \frac{1}{\theta_0(z)} \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(1)(t,s)\,dsdz \\ &\le \frac{\lambda\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)} \int_0^\infty \frac{dz}{\theta_0(z)}, \end{aligned}$$

then for all $n \in \mathbb{N}^*$,

$$L_0^n(\mu) \le (\beta^n, \beta^n).$$

Thus,

$$r(L_0) = \lim_{n\to+\infty} \sqrt[n]{\|L_0^n\|} \le \beta. \qquad \square$$

We consider the following hypothesis:

$$\left\{\begin{array}{c} \text{There exist } (g_1,g_2) \in E^+\backslash\{0\} \text{ and } (h_1,h_2) \in E^+ \text{ such that} \\ \left\|L^{(k)}(g_1,g_2)\right\| < 1, \text{ and for all } (t,x,y_1,y_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^2 \\ |f_j(t,x,y_1,y_2)| \le g_j(t,x).(\max(|y_1|,|y_2|))^{\frac{1}{k}} + h_j(t,x),\ \forall j \in \{1,2\}. \end{array}\right. \tag{3.22}$$

Let $r_* = \max\left\{r_0, (r_0)^{\frac{1}{k}}\|(g_1,g_2)\| + \|(h_1,h_2)\|\right\}$ and

$$r_0 = \frac{\left\|L^{(k)}(h_1,h_2)\right\|}{1 - \left\|L^{(k)}(g_1,g_2)\right\|}.$$

Theorem 3.20. *Assume that the condition (3.22) holds and*

$$1 < p^- \le 2 \le p^+.$$

If there exist $r > r_*$, $\rho^* > 0$ *and* $\rho_0 \in G\backslash\{0\}$ *such that for all* $j \in \{1,2\}$, f_j *verifies one of the following conditions for all* $t, x \in \mathbb{R}^+$ *and all* $(x_1,x_2), (y_1,y_2) \in [-r,r]^2$;

$$\left\{\begin{array}{c} |f_j(t,x,x_1,x_2) - f_j(t,x,y_1,y_2)| \le \rho_0(t).\max(|x_1-y_1|,|x_2-y_2|) \\ \text{and} \\ \lambda < \|L_+(\rho_0,\rho_0)\|^{-1} \end{array}\right. \tag{3.23}$$

or

$$\left\{\begin{array}{c} |f_j(t,x,x_1,x_2) - f_j(t,x,y_1,y_2)| \le \rho^*.|x_j-y_j|, \\ \text{and} \\ \dfrac{\lambda.\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)} \int_0^\infty \dfrac{dt}{\theta_j(t)} < (\rho^*)^{-1}, \end{array}\right. \tag{3.24}$$

then IVS (1.1) is Hyers-Ulam stable in E.

Proof. We have from hypothesis (3.22) and remark 3.18 that the conditions (3.1), (3.2), (3.4) and (3.7) hold.
1. Assume that the condition (3.23), this means that the hypothesis (3.3) and (3.5) hold with

$$\rho = (\rho_1, \rho_2) = (\rho_0, \rho_0),$$

so, it follows from theorem 3.3) that equation (2.1) is L-Hyers-Ulam stable, and from Lemma (3.15) that IVS (1.1) is Hyers-Ulam stable in E.
2. Now, assume that f verifies (3.24). It follows from Lemma (3.19) and (3.24) that

$$r(L_0) \leq \beta = \frac{\lambda. \left\| (\sigma^+)^\alpha \right\|_{L^1}}{\Gamma(\alpha+1)} \int_0^\infty \frac{dt}{\theta_0(t)} < (\rho^*)^{-1}$$

and so, the conditions (3.6) and (3.8) of theorem (3.4) hold with

$$\lambda_0 = \rho^*.$$

Consequently, IVS (1.1) is Hyers-Ulam stable in E. □

3.3. Existence and controllability

In this section, we assume that for all $(t, x, u_1, u_2) \in (\mathbb{R}^+)^2 \times \mathbb{R}^2$:

$$f(t, x, u_1, u_2) = G(t, x, u_1, u_2) + h(t, x),$$

where $h \in E$ is the control function of IVS (1.1) and $G \in E^+$ such that, for each $j \in \{1, 2\}$,

$$G_j(u_1, u_2) \leq \bar{\lambda} \max\left(|u_1|^{p^+-1}, |u_2|^{p^+-1}\right), \tag{3.25}$$

with

$$\bar{\lambda} \left\| \frac{1}{\theta_j} \right\|_{L^1}^{p^+-1} \left(\frac{\left\| (\sigma^+)^\alpha \right\|_{L^1}}{\Gamma(\alpha+1)} \right) < 1. \tag{3.26}$$

We denote by $C^1_{0,\phi}(\mathbb{R}^+)$ the set

$$C^1_{0,\phi}(\mathbb{R}^+) = \left\{ u \in C^1(\mathbb{R}^+) : \phi(u) \in AC(\mathbb{R}^+),\ u(0) = \lim_{x\to+\infty} u'(x) = 0 \right\}.$$

Definition 3.21. IVS (1.1) is said to be controllable in E at ∞, if given any $x^\infty \in C^1_{0,\phi}(\mathbb{R}^+) \times C^1_{0,\phi}(\mathbb{R}^+)$, there exists a control function $h \in E$, such that the solution u of IVS (1.1) satisfies $\lim_{x\to+\infty} u(t, x) = x^\infty$.

Lemma 3.22. *We have* $\lim_{t\to\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{t}{\zeta_j(t)}\right)(t, x) > 0$, $\forall x \geq 0$.

Proof. Let $x \geq 0$. Since $\dfrac{\partial \sigma}{\partial t}(t,x) > 0$;

$$\begin{aligned}
&\lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{t}{\zeta_j(t)}\right) \\
&= \frac{1}{\Gamma(\alpha)}\lim_{t\to\infty}\int_0^{\sigma(t,x)} \frac{T\sigma_t'(T,x)}{\zeta_j(T)}\left(\sigma(t,x)-\sigma(T,x)\right)^{\alpha-1} dT \\
&\geq \frac{1}{\Gamma(\alpha)}\lim_{t\to\infty}\int_{\sigma(1,x)}^{\sigma(t,x)} \frac{T\sigma_t'(T,x)}{a_j+T}\left(\sigma(t,x)-\sigma(T,x)\right)^{\alpha-1} dT \\
&\geq \lim_{t\to\infty}\frac{\sigma(1,x)}{\Gamma(\alpha)(a_j+\sigma(t,x))}\int_{\sigma(1,x)}^{\sigma(t,x)} \sigma_t'(T,x)\left(\sigma(t,x)-\sigma(T,x)\right)^{\alpha-1} dT \\
&\geq \lim_{t\to\infty}\frac{\sigma(1,x)}{\Gamma(\alpha)(a_j+\sigma(t,x))}\int_{\sigma(1,x)}^{\sigma(t,x)} \left(\sigma(t,x)-\sigma\right)^{\alpha-1} d\sigma \\
&\geq \frac{\sigma(1,x)}{\Gamma(\alpha+1)(a_j+\sigma^+(x))}\left(\sigma^+(x)-\sigma(1,x)\right)^{\alpha} > 0.
\end{aligned}$$

□

Theorem 3.23. *Assume that (3.25) and (3.26) hold true. Then for all $h \in E$, IVS (1.1) admits a solution.*

Proof. Let $h \in E$. We show that there exists $R > 0$ such that $T\left(\bar{B}(0,R)\right) \subset \bar{B}(0,R)$ and then we deduce from Schauder's theorem that the compactness of T guarantees the existence of at least one fixed point of T which is, from Lemma (3.13), a solution of IVS (1.1).

Assume on the contrary that for all $n \in \mathbb{N}^*$, there is $u^{(n)} = \left(u_1^{(n)}, u_2^{(n)}\right) \in \bar{B}(0,n)$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $j \in \{1,2\}$, such that

$$n \leq \left|T_j\left(u^{(n)}\right)(t,x)\right|$$

$$= \left|\int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t \left(G_j\left(u_1^{(n)},u_2^{(n)}\right)+h_j\right)(\tau,s)\,d\tau\right) ds\right) dz\right|.$$

By using the inequality (3.13) of Remark (3.11), it follows:

$$\begin{aligned}
1 &\leq \frac{1}{n}\int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t \left(G_j\left(u_1^{(n)},u_2^{(n)}\right)+|h_j|\right)(\tau,s)\,d\tau\right) ds\right) dz \\
&\leq \int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t \left(\frac{G_j\left(u_1^{(n)},u_2^{(n)}\right)+|h_j|}{n^{p^+-1}}\right)(\tau,s)\,d\tau\right) ds\right) dz \\
&\leq \psi_{p^+}\left(\bar{\lambda}+\frac{\|h_j\|_0}{n^{p^+-1}}\right)\int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{t}{\zeta_j(t)}\right) ds\right) dz \\
&\leq \psi_{p^+}\left(\bar{\lambda}+\frac{\|h_j\|_0}{n^{p^+-1}}\right)\int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_0^{+\infty} I_{0^+,t}^{\alpha,\sigma}(1)\,ds\right) dz
\end{aligned}$$

$$\leq \left(\bar{\lambda} + \frac{\|h_j\|_0}{n^{p^+-1}}\right)^{\frac{1}{p^+-1}} \left\|\frac{1}{\theta_j}\right\|_{L^1} \left(\frac{\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)}\right)^{\frac{1}{p^+-1}}.$$

Letting $n \to \infty$, we have

$$\bar{\lambda} \left\|\frac{1}{\theta_j}\right\|_{L^1}^{p^+-1} \left(\frac{\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)}\right) \geq 1.$$

This contradicts hypothesis (3.26) and the proof is finished. $\square$

Theorem 3.24. *Assume that (3.25) and (3.26) hold true. Then IVS (1.1) is controllable.*

Proof. For each $u^\infty = (u_1^\infty, u_2^\infty) \in C_0^2(\mathbb{R}^+) \times C_0^2(\mathbb{R}^+ \times \mathbb{R}^+)$, let

$$h(t,x) = -\frac{1}{\lim\limits_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{t}{\zeta_j(t)}\right)} \left(\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial u_j^\infty}{\partial x}\right)(x) + \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t G_j(u_1,u_2)(\tau,x)\,d\tau\right)\right). \tag{3.27}$$

Let $u = (u_1, u_2) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+) \times C^2(\mathbb{R}^+ \times \mathbb{R}^+)$ be solution of IVS (1.1). We have from Lemma (3.13) that for each $j \in \{1,2\}$;

$$u_j(t,x) = \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t (G_j(u_1,u_2) + h_j)(\tau,s)\,d\tau\right)ds\right)dz.$$

This means that for every $x \geq 0$,

$$y_j(x) = \lim_{t\to\infty} u_j(t,x)$$

$$= \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t (G_j(u_1,u_2) + h_j)(\tau,s)\,d\tau\right)ds\right)dz$$

$$\Rightarrow -\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial y_j}{\partial x}\right)(x) = \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t (G_j(u_1,u_2) + h_j)(\tau,x)\,d\tau\right)$$

$$\Rightarrow -\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial y_j}{\partial x}\right)(x) - \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t G_j(u_1,u_2)(\tau,x)\,d\tau\right) = \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,x)\,d\tau\right).$$

then

$$-\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial y_j}{\partial x}\right)(x) - \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t G_j(u_1,u_2)(\tau,x)\,d\tau\right) = \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,x)\,d\tau\right). \tag{3.28}$$

Substituting (3.27) into (3.28), we find that

$$\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial u_j^\infty}{\partial x}\right)(x) = \frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial y_j}{\partial x}\right)(x),$$

and using $\lim_{x\to\infty} \frac{\partial u_j^\infty}{\partial x}(x) = \lim_{x\to\infty} \frac{\partial y_j}{\partial x}(x) = 0$ and the fact that ϕ is invertible, we can get

$$\frac{\partial u_j^\infty}{\partial x}(x) = \frac{\partial y_j}{\partial x}(x),$$

and also, from $u_j^\infty(0) = y_j(0)$, it follows that

$$\lim_{t\to\infty} u_j(t,x) = y_j(x) = u_j^\infty(x).$$

Thus, at the stat ∞, $u(\infty,.) = u_j^\infty$. So, IVS (1.1) is controllable. □

Example 3.25. Let $\alpha = \frac{1}{2}$, $\sigma(t,x) = \frac{\pi}{4}\left(1-e^{-t}\right)^2 e^{-2x}$ and $\phi(x) = |x|^{-\frac{1}{2}}.x + |x|.x$. For $j \in \{0,1\}$, we have

$$\begin{aligned} f_j(t,x,x_1,x_2) &= G_j(t,x,x_1,x_2) + h_j(t,x), \\ \theta_j(x) &= 1+x^2, \end{aligned}$$

where $h_j(t,x) \in E$ is a control function.
1. If $G_j(t,x,x_1,x_2) = g_j(t,x).x_j$, with

$$g_j(t,x) = \frac{1}{\pi^2} = \bar{\lambda}.$$

Then $p^- = \frac{3}{2} < 2 < p^+ = 3$, $\left\|(\sigma^+)^\alpha\right\|_{L^1} = \left\|\sqrt{\sigma^+}\right\|_{L^1} = \frac{\sqrt{\pi}}{2}$

$$\sigma^+(x) = \frac{\pi}{4}e^{-2x}.$$

We have $\bar{\lambda} = \frac{1}{\pi^2}$ and

$$\bar{\lambda}\left\|\frac{1}{\theta_j}\right\|_{L^1}^{p^+-1}\left(\frac{\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)}\right) = \bar{\lambda}\left(\frac{\pi}{2}\right)^2\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)}\sqrt{\frac{\pi}{4}}\right) = \frac{1}{4} < 1.$$

So, the conditions (3.25) and (3.26) of theorems (3.23) and (3.24) hold true. Then IVS (1.1) is controllable.
2. Now, we assume that $G_j(t,x,x_1,x_2) = g_j(t,x).x_j^2$ and $h_j(t,x) = \eta \in \mathbb{R}^+$ with

$$g_j(t,x) = \frac{1}{\pi^2} = g^+$$

and η verifies

$$\eta < \min\left\{\frac{\sqrt{\pi}}{4\pi}, \frac{\sqrt{\pi\sqrt{\pi}}}{2(2\pi+1)}\right\}. \tag{3.29}$$

We have

$$\begin{aligned} L_j^{(k)}(g_1,g_2)(t,x) &= \int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t g_j(\tau,s)\,d\tau\right)(t,s)\,ds\right)dz \\ &= \int_0^x \frac{1}{1+z^2}\sqrt{g^+\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(1)(t,s)\,ds\right)}dz \\ &\le \int_0^x \frac{dz}{1+z^2}\sqrt{\frac{g^+\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma\left(\frac{3}{2}\right)}} \\ &\le \int_0^x \frac{dz}{1+z^2}\sqrt{\frac{g^+\frac{\sqrt{\pi}}{2}}{\Gamma\left(\frac{3}{2}\right)}} = \sqrt{g^+}.\arctan(x), \end{aligned}$$

then

$$\left\|L^{(k)}(g_1,g_2)\right\| \le \frac{1}{2} < 1.$$

This means that 3.22 holds.
Moreover,

$$\begin{aligned} L_j^{(k)}(h_1,h_2) &\le \int_0^x \frac{1}{\theta_j(z)}\sqrt{\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t \eta.d\tau\right)(t,s)\,ds\right)}dz \\ &< \frac{\pi}{2}\sqrt{\eta} \end{aligned}$$

then

$$r_0 = \frac{\left\|L^{(k)}(h_1,h_2)\right\|}{1-\left\|L^{(k)}(g_1,g_2)\right\|} \le 2\left\|L^{(k)}(h_1,h_2)\right\| < \pi\sqrt{\eta}.$$

Then, from (3.29), we have

$$\begin{aligned} r_* &= \max\left\{r_0, \frac{2}{\pi}(r_0)^2 + \|(h_1,h_2)\|\right\} \le \max\left\{\pi\sqrt{\eta}, (2\pi+1).\eta\right\} \\ &< \frac{\sqrt{\pi\sqrt{\pi}}}{2}. \end{aligned}$$

Now, let $r > 0$ such that

$$r_* < r < \frac{\sqrt{\pi\sqrt{\pi}}}{2}.$$

For all $t, x \ge 0$, $(x_1,x_2)\,[-r,r]^2, (y_1,y_2) \in [-r,r]^2$ we have

$$\begin{aligned} |f_j(t,x,x_1,x_2) - f_j(t,x,y_1,y_2)| &= g_j(t,x).\left|x_j^2 - y_j^2\right| \\ &\le 2.r.g^+.\left|x_j - y_j\right| = \rho^*.\left|x_j - y_j\right|, \end{aligned}$$

where

$$\rho^* = \frac{2.r}{\pi^2},$$

and

$$\begin{aligned} \lambda &= \frac{1}{p^- - 1}\left(\frac{r.\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)}\right)^{\frac{2-p^-}{p^- -1}} \\ &= \frac{4}{\sqrt{\pi}} r. \end{aligned}$$

As $r < \dfrac{\sqrt{\pi\sqrt{\pi}}}{2}$, we have

$$\begin{aligned} \frac{\rho^*}{\Gamma(\alpha+1)} \int_0^\infty \frac{\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\theta_j(t)} dt &\leq \frac{2.r}{\pi^2} \int_0^\infty \frac{1}{1+t^2} dt \\ &\leq \frac{r}{\pi} < \frac{\sqrt{\pi}}{4r} = \lambda^{-1}. \end{aligned}$$

Then, hypothesis (3.24) is also satisfied. Thus, we deduce from theorem (3.20) that IVS (1.1) is Hyers-Ulam stable in E.

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Existence results for some anisotropic possible singular problems via the sub-supersolution method

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Abstract. Using the sub-super solution method, we prove the existence of the solutions for the following anisotropic problem with singularity:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and a given singular nonlinearity $f : \Omega \times (0,\infty) \longrightarrow [0,\infty)$.

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1. Introduction

Partial differential equations with anisotropic operators appear in several scientific domains, in physics for example, such kind of operators models the dynamics of liquids with different conductivities in different directions. Furthermore, in biology for example, such type of operators are related to model describing the spread of epidemics in heterogeneous environments. Regarding the mentioned examples, we point

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out the references [14, 18, 23, 24].
Problems involving anisotropic operators $\vec{p}$-Laplacian

$$-\Delta_{\vec{p}}\, u = -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2}\, \partial_i u \right), \tag{1.1}$$

are extensively studied in the literature and we cite them as examples [1, 3, 6, 7, 11]. We note that the operator (1.1) becomes the Laplacian operator in the case of $p_i = 2$ and the p-Laplacian operator that is $\Delta_p u = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$ in the case of $p_i = p$ for all i. There are many studies on Laplacian and p-Laplacian problems with singularity in the second member, we refer to [19, 4, 22, 16, 25]. There is now a substantial body of work and growing interest in singular problems involving anisotropic operators, some recent results can be found in [2, 20, 17, 14].

In this paper, we study the following anisotropic problem with singularity:

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary and $f : \Omega \times (0,\infty) \to [0,\infty)$ is a continuous function such that $f(.,t)$ is in $C^{\theta}(\Omega)$ with $0 < \theta < 1$. Without loss of generality, we assume that $p_1 \leq ... \leq p_N$.

Against several works that used the approximation methods, we focuse in this work on singular problems which have applications in anisotropic operator using the sub and supersolution method. More precisely, we generalize the existence results existing in [21] through replacing the p-Laplacian operator by the anisotropic one. Moreover, we have weakened conditions given on f. In other part, this work generalise the second member existing in [20, 17] with keeping the same anisotropic operator.

The natural functional space relevant to the problem (1.2) is the anisotropic Sobolev spaces

$$W^{1,\vec{p}}(\Omega) = \left\{ v \in W^{1,1}(\Omega); \partial_i v \in L^{p_i}(\Omega) \right\},$$

and

$$W_0^{1,\vec{p}}(\Omega) = W^{1,\vec{p}}(\Omega) \cap W_0^{1,1}(\Omega),$$

endowed by the usual norm

$$\|v\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}.$$

Where ∂u_i denotes the $i-$ th weak partial derivative of u.
In the following, we assume that $\overline{p} < N$, with

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i} \quad , \qquad \sum_{i=1}^{N}\frac{1}{p_i} > 1,$$

$$\overline{p}^* = \frac{\overline{p}N}{N-\overline{p}} \quad \text{and} \quad p_\infty = \max\{\overline{p}^*, p_N\}.$$

Then for every $r \in [1, p_\infty]$ the embedding

$$W_0^{1,\vec{p}}(\Omega) \subset L^r(\Omega),$$

is continuous, and compact if $r < p_\infty$. We refer to see [13].
Owing to the absence of a strong maximum principle, we will usually assume that $p_i \geq 2$ for all i.

Definition 1.1. We will say that $u \in W_0^{1,\vec{p}}(\Omega)$ is a solution to (1.2) if and only if, the following equality holds:

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi \, dx \; = \; \int_\Omega f(x,u)\varphi \, dx \, , \tag{1.3}$$

for all $\varphi \in W_0^{1,\vec{p}}(\Omega)$.

Now, we are in a position to present our first results. For this, let g be a continuous positive function on $(0,\infty)$. Assume that f and g satisfy the following conditions

(G) $g(0^+) = \lim_{t\to 0^+} g(t) = +\infty$.

(H_0) $\varsigma_\mu(x) = \sup_{t\geq\mu} f(x,t) \in L^r(\Omega)$ for each $\mu > 0$ with $r > \dfrac{N}{\overline{p}}$.

(H_1) There exist two measurable nontrivial functions β, γ and a positive constant λ such that

$$\beta(x) \leq f(x,s) \leqslant \gamma(x)g(s) \text{ for every } \; 0 < s < \lambda, \;\; \text{a.e. } \; x \in \Omega,$$

with $0 \leq \beta(x) \leq \gamma(x)$ a.e. $x \in \Omega$, $\gamma \in L^r(\Omega)$, $r > \dfrac{N}{\overline{p}}$.

Theorem 1.2. *If* $(H_0) - (H_1)$, (G) *hold and* g *is non-increasing, then problem* (1.2) *has a solution in* $W_0^{1,\vec{p}}(\Omega)$.

Theorem 1.3. *If* $(H_0) - (H_1)$, (G) *hold and* g *satisfies the following condition*

$$\limsup_{t\longrightarrow 0^+} tg(t) < +\infty,$$

then problem (1.2) *has a solution in* $W_0^{1,\vec{p}}(\Omega)$.

Remark 1.4. Consider $g(s) = \frac{1}{s^\alpha ln^\beta(s+1)}$, with $0 < \alpha < 1$ and $\beta \geq 1-\alpha$. The function g satisfies the conditions of Theorem 1.2, however g doesn't verify the condition (3) of $(G2)$ of Theorem3.1 in [21].
Also, the function g given by $g(t) = \frac{1}{t^\theta}$ satisfies the conditions of Theorem 1.2 for each $\theta > 0$, but the same function g verifies the condition (3) of $(G2)$ of Theorem [21] for only $\theta > 1$.

This paper is organized as follows: in section 2, we recall some necessary definitions of the classical anisotropic operator, also we mention a technical Lemma and we prove it. In section 3, by using comparison principle and sub-supersolution method, we give the proofs of our results.

2. Preliminaries

Consider the following anisotropic problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = f(x,u) & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where τ in $W^{1,\vec{p}}(\Omega)$.

Definition 2.1. Let $u \in W^{1,\vec{p}}(\Omega)$ such that $u - \tau \in W_0^{1,\vec{p}}(\Omega)$, u is a solution of (2.1) if and only if for every $\varphi \in W_0^{1,\vec{p}}(\Omega)$

$$\int_\Omega \left(\sum_{i=1}^{N} |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi \; - \; f(x,u)\varphi \right) dx \; = \; 0 \, . \tag{2.2}$$

Definition 2.2. Let $(\underline{u}, \overline{u}) \in W^{1,\vec{p}}(\Omega) \times W^{1,\vec{p}}(\Omega)$,
$\underline{u}$ is called a subsolution of the problem (2.1), if

$$\int_\Omega \sum_{i=1}^{N} |\partial_i \underline{u}|^{p_i-2} \partial_i \underline{u} \partial_i \varphi \, dx \leq \int_\Omega f(x,\underline{u})\varphi \, dx \quad \text{and} \quad (\underline{u} - \tau)^+ \in W_0^{1,\vec{p}}(\Omega),$$

$\overline{u}$ is said a supersolution of the problem (2.1), if

$$\int_\Omega \sum_{i=1}^{N} |\partial_i \overline{u}|^{p_i-2} \partial_i \overline{u} \partial_i \varphi \, dx \geq \int_\Omega f(x,\overline{u})\varphi \, dx \quad \text{and} \quad (\overline{u} - \tau)^- \in W_0^{1,\vec{p}}(\Omega),$$

for all functions $0 \leq \varphi \in W_0^{1,\vec{p}}(\Omega)$.

Now, we need to proved the following lemma.

Lemma 2.3. *Let f satisfies (H_0) and $\tau \in W^{1,\overrightarrow{p}}(\Omega)$ with $\tau > 0$ in Ω. Let ϕ_{sub} and ϕ_{super} be sub-solution and super-solution of* (2.1) *respectively with $\phi_{super} > \phi_{sub}$ a.e. in Ω.*
If $0 < \mu < \phi_{sub}$ a.e. in Ω, where μ is a constant, then the problem (2.1) *has at least one positive solution $u \in W^{1,\overrightarrow{p}}(\Omega)$ such that $\phi_{sub} < u < \phi_{super}$ a.e. in Ω.*

Proof. Let $T : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$T(x,t) := \begin{cases} f(x,\mu) & \text{if } t < \mu, \\ f(x,t) & \text{if } t \geqslant \mu. \end{cases}$$

We will consider the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = T(x,u) & \text{in } \quad \Omega, \\ u = \tau & \text{on } \quad \partial\Omega. \end{cases} \tag{2.3}$$

It is easy to see that ϕ_{sub} and ϕ_{super} are sub and super-solution respectively of this problem. Since $T(x,.)$ is Hölder continuous in $\mathbb{R}$ for each $x \in \Omega$, $|T(x,t)| \leq \varsigma_\mu(x)$ in $\Omega \times \mathbb{R}$ and $\varsigma_\mu \in L^r(\Omega)$ with $r > \frac{N}{\overline{p}}$, then by [[5], Theorem 4.14] the problem (2.3) has a solution $u \in W^{1,\vec{p}}(\Omega)$ such that $\phi_{sub} \leq u \leq \phi_{super}$, a.e. in Ω. Since $\mu < \phi_{sub}$ a.e. in Ω, then $T(x,u) = f(x,u)$ a.e. in Ω. Finally, we note that u is a solution of (2.1) as claimed. □

3. Proof of the main results

Proof of Theorem 1.2. Let ϕ be a solution of the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = \gamma(x) & \text{in} \quad \Omega, \\ u = 1 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

As $\gamma \in L^r(\Omega)$ with $r \geq \frac{N}{\overline{p}}$, then according to [[6], Theorem 2.1], we have $\phi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Using comparison lemma in [[10], Lemma 2.5], we get $\phi \geq 1$ a.e. in Ω. We can assume without loss of generality that $\phi < \lambda$ a.e. in Ω. If not, we replace λ by $\lambda^* = \max\{\lambda\ ,\ \|\phi\|_{L^\infty(\Omega)} + 1\}$.
From (H_1) and as $\phi \geq 1$ a.e. in Ω, then

$$\begin{aligned} \int_\Omega f(x,\phi)\varphi &\leq \int_\Omega \gamma(x) g(\phi)\varphi \\ &= \int_{\{\phi\geq 1\}} \gamma(x) g(\phi)\varphi \\ &\leq \int_{\{\phi\geq 1\}} \gamma(x) g(1)\varphi. \end{aligned}$$

Without lost of generality, by replacing γ by $g(1)\gamma$ and g by $\frac{g}{g(1)}$, we deduce that

$$\int_\Omega f(x,\phi)\varphi \leq \int_\Omega \gamma(x)\varphi. \tag{3.2}$$

Let $k \in \mathbb{N}^*$, we consider the following problem

$$(P_k) \qquad \begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = f(x,u) & \text{in} \quad \Omega, \\ u = \frac{1}{k} & \text{on} \quad \partial\Omega. \end{cases}$$

From the inequality (3.2) and the condition (H_0), we obtain

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{N} |\partial_i \phi|^{p_i-2}\, \partial_i \phi \partial_i \varphi\, dx - \int_\Omega f(x,\phi)\varphi dx \\ &\qquad \geq \int_\Omega \sum_{i=1}^{N} |\partial_i \phi|^{p_i-2}\, \partial_i \phi \partial_i \varphi\, dx - \int_\Omega \gamma\varphi dx = 0, \end{aligned}$$

for all positive function $\varphi \in W_0^{1,\vec{p}}(\Omega)$ and $(\phi - \frac{1}{k})^- \in W_0^{1,\vec{p}}(\Omega)$. Thus, ϕ is a super-solution of the problem (P_k) in Ω for all $k = 1, 2, ...$

Take ϕ_k be the solution of

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = \beta_k(x) & \text{in} \quad \Omega, \\ u = 1/k & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3}$$

for $k = 1, 2, ...$, where $\beta_k(x) = \min\{\beta(x)\frac{k+1}{k}\}$, for $x \in \Omega$.
Let ϕ_∞ the solution of (3.3) when $k = \infty$ and $\beta_\infty(x) = \min\{\beta(x)\}$. As $\beta_k \in L^r(\Omega)$ with $r > \frac{N}{P}$, it follows that $\phi_k \in L^\infty(\Omega)$ (see [[6], Theorem 2.1]). By the comparison lemma in [[10], Lemma 2.5], we have

$$0 \leq \phi_\infty \leq \phi_k \leq \phi_1 \;\text{ a.e. in }\; \Omega, \;\text{ for all }\; k = 1, 2, ...$$

Moreover $\phi_k \geq k^{-1}$ a.e. in Ω for all $k = 1, 2, ...$
Since $\beta_\infty \in L^\infty(\Omega)$, $\beta_\infty \neq 0$ in Ω and $p_1 \geq 2$, using the Strong Maximum Principle see ([8], Corollary 4.4.) and ([7], Theorem 1.1), we easily see that $\phi_\infty > 0$ for all compact K in Ω.
By comparison lemma in [[10], Lemma 2.5], since $0 \leq \beta \leq \gamma$ a.e. x in Ω, we deduce that $\phi_k \leq \phi$ for a.e. x in Ω and every $k = 1, 2, ...$
Then from the condition (H_0) and since $\phi_k \leq \phi < \lambda$ a.e. in Ω for all $k = 1, 2, ...$, we get

$$\int_\Omega \sum_{i=1}^{N} |\partial_i \phi_k|^{p_i-2} \partial_i \phi_k \partial_i \varphi \, dx - \int_\Omega f(x, \phi_k) \varphi dx$$
$$\leq \int_\Omega \sum_{i=1}^{N} |\partial_i \phi_k|^{p_i-2} \partial_i \phi_k \partial_i \varphi \, dx - \int_\Omega \gamma \varphi dx = 0,$$

for all positive function φ in $W_0^{1,\vec{p}}(\Omega)$ and $(\phi_k - \frac{1}{k})^+ \in W_0^{1,\vec{p}}(\Omega)$. Hence ϕ_k is a sub-solution of (P_k) for all $k = 1, 2, ...$
Now let $j \in \mathbb{N}^*$, by Lemma 2.3 there exist a solution u_j of the problem (P_j) such that $\phi_j \leq u_j \leq \phi$ a.e. in Ω. Moreover u_j is a super-solution of (P_{j+1}), using again Lemma 2.3, there is a solution u_{j+1} of the problem (P_{j+1}) where $\phi_{j+1} \leq u_{j+1} \leq u_j$ a.e. in Ω. By continuing to do so, we build a sequence (u_k) of solutions of the problem (P_k) such that for every $k \geq j$ we have

$$\phi_\infty \leq u_{k+1} \leq u_k \leq ... \leq u_j \leq \phi \qquad \text{a.e. in} \quad \Omega.$$

We should also note that $u_k \geq k^{-1}$ a.e. in Ω. We define $u(x) = \lim_{k \to \infty} u_k(x)$ a.e in Ω.
Now, as ϕ_∞ is locally Hölder continuous in Ω (see [7]) and $\phi_\infty > 0$ for all compact K in Ω, hence $\inf_{supp(\phi)} \phi_\infty > 0$. Take

$$\zeta_k = \frac{u_k - k^{-1}}{g\left(\inf_{supp(\phi)} \phi_\infty \right)}$$

as a test function, then in view of (H_0) and [[12], Theorem 1.3.], we distinguish two cases:

If $g\left(\inf_{supp(\phi)} \phi_\infty\right) \geq 1$, we get the following inequality

$$\begin{aligned}
\frac{\|\zeta_k\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)}}{N^{p_N-1}} - N &\leq \sum_{i=1}^{N} \int_\Omega |\partial_i \zeta_k|^{p_i} dx \\
&\leq \frac{1}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} \sum_{i=1}^{N} \int_\Omega |\partial_i u_k|^{p_i} dx \\
&= \int_\Omega f(x,u_k) \frac{u_k - k^{-1}}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} dx \\
&\leq \int_\Omega f(x,u_k) \frac{u_k}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} dx \ ,
\end{aligned}$$

where $p_0 = p_1$ if $\|\zeta_k\|_{W_0^{1,\vec{p}}(\Omega)} \geq 1$ and $p_0 = p_N$ if $\|\zeta_k\|_{W_0^{1,\vec{p}}(\Omega)} < 1$.
From (H_1) and since $u_k \leq \phi < \lambda$ for all $k = 1, 2, ...$, a.e. in Ω, we obtain

$$\begin{aligned}
\frac{\|\zeta_k\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)}}{N^{p_N-1}} - N &\leq \int_\Omega \gamma(x) g(u_k) \frac{\phi}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} dx \\
&= \int_{supp(\phi)} \gamma(x) g(u_k) \frac{\phi}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} dx.
\end{aligned}$$

On the other hand as g is non-increasing, $g(u_k) \leq g(\phi_\infty)$ a.e. in Ω and $g(\phi_\infty) \leq g\left(\inf_{supp(\phi)} \phi_\infty\right)$ a.e. in $supp(\phi)$. Then according to the above equality, we find

$$\|\zeta_k\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)} \leq \lambda N^{p_N-1} \|\gamma\|_{L^1(\Omega)} + N^{p_N}.$$

If $g\left(\inf_{supp(\phi)} \phi_\infty\right) < 1$, we have

$$\begin{aligned}
\frac{\|u_k - k^{-1}\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)}}{N^{p_N-1}} - N &\leq \sum_{i=1}^{N} \int_\Omega \left|\partial_i \left(u_k - k^{-1}\right)\right|^{p_i} dx \\
&= \int_\Omega f(x,u_k) \left(u_k - k^{-1}\right) dx \\
&\leq \int_{supp(\phi)} \gamma(x) g(u_k) \phi dx \ ,
\end{aligned}$$

where $p_0 = p_1$ if $\|u_k - k^{-1}\|_{W_0^{1,\vec{p}}(\Omega)} \geq 1$ and $p_0 = p_N$ if $\|u_k - k^{-1}\|_{W_0^{1,\vec{p}}(\Omega)} < 1$.
Since $g(u_k) \leq g\left(\inf_{supp(\phi)} \phi_\infty\right) < 1$ a.e. in $supp(\phi)$ and $\phi < \lambda$ for a.e. in Ω, then we

obtain

$$\|u_k - k^{-1}\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} \leq \lambda N^{p_N-1}\|\gamma\|_{L^1(\Omega)} + N^{p_N},$$

which implies the inequality

$$\begin{aligned}\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} &= \frac{1}{g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)^{p_0}}\|u_k - k^{-1}\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} \\ &\leq \frac{1}{g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)^{p_0}}\left(\lambda N^{p_N-1}\|\gamma\|_{L^1(\Omega)} + N^{p_N}\right)\end{aligned}$$

and thus

$$\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} \leq \frac{\lambda N^{p_N-1}\|\gamma\|_{L^1(\Omega)} + N^{p_N}}{g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)^{p_0}}.$$

Finally, we conclude that $\zeta_k \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ for every k.
Since (ζ_k) is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$, it follows that $\zeta_k \rightharpoonup v$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and (ζ_k) converge weakly to the same limit in $W^{1,\overrightarrow{p}}(\Omega)$. As (u_k) is bounded in $W^{1,\overrightarrow{p}}(\Omega)$, we have $u_k \rightharpoonup u$ in $W^{1,\overrightarrow{p}}(\Omega)$, strongly in $L^p(\Omega)$ and almost everywhere in Ω.
In other part, we have $u_k = g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)\zeta_k + k^{-1} \rightharpoonup g\left(\inf\limits_{supp(\phi)} \phi_\infty\right) v$ in $W^{1,\overrightarrow{p}}(\Omega)$, strongly in $L^p(\Omega)$ and almost everywhere in Ω. Therefore, we can conclude that $u = g\left(\inf\limits_{supp(\phi)} \phi_\infty\right) v$ almost everywhere in Ω, we easily see that $v \in W_0^{1,\overrightarrow{p}}(\Omega)$ which implies that $u \in W_0^{1,\overrightarrow{p}}(\Omega)$.

Let Ω_0 be a compact domain in Ω. We define $\mu = \min\limits_{\Omega_0} \phi_\infty$, from ([7], Theorem 1.1), $\phi_\infty > 0$ a.e. in Ω, we have $\mu > 0$. Hence

$$|(f(x,u_k) - f(x,u_j))(u_k - u_j)| \leqslant 4\varsigma_\mu(x)\phi,$$

which implies that

$$\sum_{i=1}^{N}\int_{\Omega_0}\left(|\partial_i u_k|^{p_i-2}\partial_i u_k - |\partial_i u_j|^{p_i-2}\partial_i u_j\right)\partial_i(u_k - u_j)\,dx \to 0 \tag{3.4}$$

as $k, j \to \infty$. From ([15], Proposition 1.) and (3.4), we get

$$\sum_{i=1}^{N}\int_{\Omega_0}|\partial_i u_k - \partial_i u_j|^{p_i}dx \to 0, \quad k, j \to \infty. \tag{3.5}$$

We observe that

$$u_k \longrightarrow u \quad \text{in} \quad L^{p_i}(\Omega_0). \tag{3.6}$$

From (3.5), (3.6), we obtain that (u_k) is Cauchy sequence in $W^{1,\overrightarrow{p}}(\Omega_0)$ which is a Banach space, therefore $u_k \longrightarrow u$ in $W^{1,\overrightarrow{p}}(\Omega_0)$. We conclude that for any compact

set Ω_0 in Ω, there exist a subsequence (u_k) such that $u_k \longrightarrow u$ in $W^{1,\overrightarrow{p}}(\Omega_0)$.
We mention the following estimates. We have for all $p_i \geq 2$ with $i \in \{1, 2, ..., N\}$

$$\begin{aligned} \| (|\partial_i u_k| + |\partial_i u|)^{\frac{(p_i-2)p_i}{p_i-1}} \|_{L^{p_i-1/(p_i-2)}(\Omega_0)} &= \left(\int_{\Omega_0} (|\partial_i u_k| + |\partial_i u|)^{p_i} \, dx \right)^{p_i-2/(p_i-1)} \\ &\leq 2^{p_i-2} \left(\int_{\Omega_0} |\partial_i u_k|^{p_i} + |\partial_i u|^{p_i} dx \right)^{p_i-2/(p_i-1)} \\ &\leq 2^{p_i-2} M, \end{aligned} \tag{3.7}$$

where M is a positive constant independent of x. Using Hölders inequality, we get

$$\int_{\Omega_0} (|\partial_i u_k| + |\partial_i u|)^{(p_i-2)p_i'} \, dx \leq \| (|\partial_i u_k| + |\partial_i u|)^{\frac{(p_i-2)p_i}{p_i-1}} \|_{L^{p_i-1/(p_i-2)}(\Omega_0)} (|\Omega_0|^{p_i-1}). \tag{3.8}$$

By the inequality (3.7), we have

$$\int_{\Omega_0} (|\partial_i u_k| + |\partial_i u|)^{(p_i-2)p_i'} \, dx \leq 2^{p_i-2} M |\Omega_0|^{p_i-1}. \tag{3.9}$$

Using again Hölders inequality, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega_0} |\partial_i u_k - \partial_i u| \, (|\partial_i u_k| + |\partial_i u|)^{p_i-2} \, dx \\ &\qquad \leq \sum_{i=1}^{N} \|\partial_i u_k - \partial_i u\|_{L^{p_i}(\Omega_0)} \| \, (|\partial_i u_k| + |\partial_i u|)^{p_i-2} \, \|_{L^{p_i'}(\Omega_0)}, \end{aligned}$$

from the inequality (3.9), we deduce that

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega_0} |\partial_i u_k - \partial_i u| \, (|\partial_i u_k| + |\partial_i u|)^{p_i-2} \, dx \\ &\qquad \leq \; M 2^{p_N-2} \, (|\Omega_0| + 1)^{p_N-1} \sum_{i=1}^{N} \|\partial_i u_k - \partial_i u\|_{L^{p_i}(\Omega_0)} \\ &\qquad \leq \; M 2^{p_N-2} \, (|\Omega_0| + 1)^{p_N-1} \, \|u_k - u\|_{W^{1,\overrightarrow{p}}(\Omega_0)}. \end{aligned} \tag{3.10}$$

Now, we recall the fallowing useful inequality (see [9]) that hold for all a, b in $\mathbb{R}^N$ and $p_i \geq 2$ for all $i = 1, 2, ..., N$

$$||a|^{p_i-2}a - |b|^{p_i-2}b| \leq c(|a| + |b|)^{p_i-2}|a - b|, \tag{3.11}$$

where c is a positive constant independent of a and b. By estimation (3.10) and inequality (3.11), it follows that

$$\lim_{k\to+\infty} \sum_{i=1}^{N} \int_{\Omega_0} ||\partial_i u_k|^{p_i-2}\partial_i u_k - |\partial_i u|^{p_i-2}\partial_i u| dx = 0 \, . \tag{3.12}$$

Let $\xi \in C_0^\infty(\Omega)$ such that supp $(\xi) \subseteq \Omega_0 \subset \Omega$. From the limite (3.12), we conclude that

$$\sum_{i=1}^N \int_\Omega |\partial_i u_k|^{p_i-2}\partial_i u_k \partial_i \xi \, dx \longrightarrow \sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \xi \, dx \qquad \text{as } k \longrightarrow +\infty. \quad (3.13)$$

On the other hand, since $|f(x,u_k)\xi| \leq C\varsigma_\mu(x)$ a.e. in Ω_0, where C is a positive constant independent of x and $\varsigma_\mu \in L^1(\Omega)$, we obtain

$$\int_\Omega f(x,u_k)\, \xi \, dx \to \int_\Omega f(x,u)\xi \, dx. \quad (3.14)$$

Hence by (3.13) and (3.14), we conclude that for all $\xi \in C_0^\infty(\Omega)$

$$\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \xi \, dx \; = \; \int_\Omega f(x,u)\xi \, dx.$$

Consequently, the identity (1.3) holds for every ξ in $C_0^\infty(\Omega)$. Now it remains to shows that identity (1.3) is satisfied for every $\xi \in W_0^{1,\overrightarrow{p}}(\Omega)$. Let $\nu \in W_0^{1,\overrightarrow{p}}(\Omega)$, choose a sequence (η_k) of non-negative functions in $C_0^\infty(\Omega)$ such that

$$\eta_k \to |\nu| \text{ in } W_0^{1,\overrightarrow{p}}(\Omega).$$

For subsequence if necessary, we can suppose that $\eta_k \to |\nu|$ a.e. in Ω, then through the Fatou's lemma and Hölder's inequality, we have

$$\begin{aligned}
\left|\int_\Omega f(x,u)\nu\right| \leq \int_\Omega f(x,u)|\nu| &\leq \liminf_{k\to\infty} \int_\Omega f(x,u)\eta_k \\
&= \liminf_{k\to\infty} \sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \eta_k \\
&\leq \liminf_{k\to\infty} \sum_{i=1}^N \left\| |\partial_i u|^{p_i-2}\partial_i u \right\|_{L^{p_i'}(\Omega)} \|\partial_i \eta_k\|_{L^{p_i}(\Omega)} \\
&\leq \liminf_{k\to\infty} \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}(\Omega)}^{p_i-1} \|\partial_i \eta_k\|_{L^{p_i}(\Omega)} \\
&\leq \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}^{q-1} \liminf_{k\to\infty} \sum_{i=1}^N \|\partial_i \eta_k\|_{L^{p_i}(\Omega)} \\
&\leq \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}^{q-1} \liminf_{k\to\infty} \|\eta_k\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \\
&\leq \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}^{q-1} \|\nu\|_{W_0^{1,\overrightarrow{p}}(\Omega)},
\end{aligned}$$

with $q = p_1$ if $\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} < 1$ and $q = p_N$ if $\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \geq 1$. Now for $\xi \in W_0^{1,\overrightarrow{p}}(\Omega)$, choosing again a sequence (ξ_k) of function in $C_0^\infty(\Omega)$ such that $\xi_k \to \xi$. By taking $\nu = \xi_k - \xi$ in the previous inequality, we get

$$\lim_{k\to\infty} \int_\Omega f(x,u)\xi_k \, dx = \int_\Omega f(x,u)\xi \, dx \, .$$

Furthermore

$$\lim_{k\to+\infty}\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i-2}\partial_i u\partial_i\xi_k\,dx \;=\; \sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i-2}\partial_i u\partial_i\xi\,dx.$$

Hence (1.3) holds for every ξ in $W_0^{1,\overrightarrow{p}}(\Omega)$. Consequently $u\in W_0^{1,\overrightarrow{p}}(\Omega)$ is a solution of (1.2) such that $\phi_\infty\le u\le\phi$ a.e. in Ω. □

Proof of Theorem 1.3. From Lemma 2.3 and comparison lemma in [[10], Lemma 2.5], and by following the same steps of the proof of Theorem 1.2, we can build a sequence (u_k) of solutions of the problem (P_k) such that

$$\phi_\infty\le u_{k+1}\le u_k\le ...\le u_j\le\phi \qquad \text{a.e. in}\quad \Omega,\ \text{for } k\ge j,$$

where (P_k) is defined in the proof of Theorem 1.2. We also note that $u_k\ge k^{-1}$ a.e. in Ω. We define $u(x)=\lim\limits_{k\to\infty}u_k(x)$ a.e in Ω.
We take $\zeta_k=u_k-k^{-1}$ as a test function. From the condition (H_0) and [[12], Theorem 1.3.], we have

$$\begin{aligned}
\frac{\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)}}{N^{p_N-1}}-N &\le \sum_{i=1}^{N}\int_\Omega|\partial_i u_k|^{p_i}dx\\
&=\int_\Omega f(x,u_k)\left(u_k-k^{-1}\right)dx\\
&\le\int_\Omega f(x,u_k)u_k\,dx\\
&\le\int_{supp(u_k)}\gamma(x)g(u_k)u_k\,dx\,,
\end{aligned}\tag{3.15}$$

where $p_0=p_1$ if $\|\zeta_k\|_{W_0^{1,\overrightarrow{p}}(\Omega)}\ge1$ and $p_0=p_N$ if $\|\zeta_k\|_{W_0^{1,\overrightarrow{p}}(\Omega)}<1$.
Since $\limsup\limits_{t\longrightarrow0^+}tg(t)<+\infty$, then there exist tow positive constants C and ϵ such that

$$tg(t)\leqslant C\ \text{ for all }\ 0<t<\epsilon.$$

If $0<u_k<\epsilon$, we obtain

$$\gamma(x)g(u_k)u_k\;\le\;C\gamma(x)\qquad\text{a.e. in}\quad supp(u_k).\tag{3.16}$$

If $\epsilon\le u_k\le\lambda$, as g is continuous on $(0,\infty)$, we get

$$\gamma(x)g(u_k)u_k\;\le\;\lambda M\gamma(x)\qquad\text{a.e. in}\quad supp(u_k),\tag{3.17}$$

with M is a constant positive such that $g(s)<M$ for all $\epsilon\le s\le\lambda$. By the inequality (3.16) and (3.17), we deduce

$$\gamma(x)g(u_k)u_k\;\le\;\max\{\lambda M,C\}\gamma(x)\qquad\text{a.e. in}\quad supp(u_k).\tag{3.18}$$

From the inequality (3.15), (3.18) and as $\gamma\in L^r(\Omega)$ with $r>\frac{N}{\overline{p}}$, we obtain

$$\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)}<\max\{\lambda M,C\}N^{p_N-1}\|\gamma\|_{L^1(\Omega)}+N^{p_N}.$$

Thus the sequence (ζ_k) is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$.
Following the same techniques of the proof of Theorem 1.2. We prove the existence of solution
$u \in W_0^{1,\overrightarrow{p}}(\Omega)$ of the problem (1.2) such that $\phi_\infty \le u \le \phi$ a.e. in Ω. □

Remark 3.1. Note that if the conditions $(H_0)-(H_1)$, (G) are satisfied and we replace the condition of g in the Theorem 1.2 by $h(s) = sg(s)$ where $s > 0$ is nondecreasing. Then the problem (1.2) has a solution.
It suffices to show that

$$\int_\Omega f\,(x, u_k)\, u_k\, dx \ < \infty.$$

In fact

$$\int_\Omega f\,(x, u_k)\, u_k\, dx \le \int_\Omega \gamma(x) g(u_k) u_k\, dx.$$

As h is nondecreasing for all $s > 0$, it follows that

$$\begin{aligned}\int_\Omega f\,(x, u_k)\, u_k\, dx &\le \int_{supp(\phi)} \gamma(x) g(\phi)\phi\, dx \\ &\le \int_{supp(\phi)} \gamma(x) g(\|\phi\|_{L^\infty(\Omega)})\|\phi\|_{L^\infty(\Omega)}\, dx \\ &\le g(\|\phi\|_{L^\infty(\Omega)})\|\phi\|_{L^\infty(\Omega)}\|\gamma\|_{L^1(\Omega)} < \infty\,.\end{aligned}$$

Corollary 3.2. *Let g be a nonincreasing function from $(0,\infty)$ to $(0,\infty)$, satisfies (G). Suppose that*

$$\int_0^\lambda g(x)\, dx \ < +\infty$$

for same $\lambda > 0$. If $f(x,t) = \gamma(x)g(t)$ for some non-trivial and non-negative $\gamma \in L^r(\Omega)$ with $r > \frac{N}{\overline{p}}$, then (1.2) has a weak solution in $W_0^{1,\overrightarrow{p}}(\Omega)$.

Proof. Using the fact that $f(x,t) = \gamma(x)g(t)$ and $\gamma \in L^r(\Omega)$ with $r > \frac{N}{\overline{p}}$, then conditions $(H_0) - (H_1)$ are satisfied. Hence, similar to the proof of Theorem 1.3, we can build a sequence (u_k) of solutions of the problem (P_k) such that

$$\phi_\infty \le u_{k+1} \le u_k \le ... \le u_j \le \phi \qquad \text{a.e. in} \quad \Omega,\ \text{for } k \ge j.$$

In addition, since $\int_0^\lambda g(x)\, dx \ < +\infty$, then $tg(t) \le M$ for all $0 < t < \lambda$ and some positive constant M, thus

$$\gamma(x) g(u_k) u_k \ \le\ M\gamma(x) \qquad \text{a.e. in} \quad supp(u_k).$$

As in the proof of Theorem 1.3, we combine the above inequality with (3.15), we get

$$\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} < MN^{p_N - 1}\|\gamma\|_{L^1(\Omega)} + N^{p_N},$$

where $\zeta_k = u_k - k^{-1}$. Thus ζ_k is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$. The proof is completed. □

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Multiplicity of weak solutions for a class of non-homogeneous anisotropic elliptic systems

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Abstract. We study the existence of infinitely many weak solutions for a new class of nonhomogeneous Neumann elliptic systems involving operators that extend both generalized Laplace operators and generalized mean curvature operators in the framework of anisotropic variable spaces.

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1. Introduction

In the recent years, the anisotropic variable exponent Sobolev space $W^{1,\vec{z}(\cdot)}(\Omega)$ have captured the attention of many mathematicians, physicists and engineers. The impulse for this mainly comes from their important applications in modelling real world problems in electrorheological, magneto-rheological fluids, elastic materials and image restoration (see for example [11, 20, 21]). Predominantly, the focus lies on boundary value problems featuring generalized Laplace operators or generalized mean curvature operators. An attractive proposal is to employ operators of greater generality, capable of producing both Laplace-style and mean curvature-style operators. This includes equations structured as follows:

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\Big(\partial_3 A_i\Big(\cdot, u, \partial_{x_i} u\Big)\Big) = f(x,u) \quad \text{in } \Omega, \\ \sum_{i=1}^{N} \partial_3 A_i\Big(\cdot, u, \partial_{x_i} u\Big)\gamma_i = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $A_i : \Omega \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1, \ldots, N$ are Carathéodory functions satisfy suitable conditions.

Moreover, on one hand the operator introduced in the previous equation has the potential to be transformed into the $\vec{z}(\cdot)$-Laplace anisotropic operator given by

$$\Delta_{\vec{z}(\cdot)} u = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{z_i(x)-2} \partial_{x_i} u \right), \tag{1.2}$$

when

$$A_i(x, u, t) = \frac{1}{z_i(x)} |t|^{z_i(x)}$$

which fulfills the assumptions (H_1)-(H_4) in section 3. It is clear that by selecting $z_1(\cdot) = \cdots = z_N(\cdot) = z(\cdot)$, we find an operator known as the $z(\cdot)$-orthotropic operator, which possesses analogous characteristics to the variable exponent $z(\cdot)$-Laplace operator, the relation between the $\vec{z}(\cdot)$-Laplace anisotropic operator, the $z(\cdot)$-Laplacian, and the $z(\cdot)$-orthotropic operator is noteworthy. When $z_1, \ldots, z_N$ are constant functions, we find the $\vec{z}$-Laplacian operator. Noting that $\vec{z}(\cdot)$-Laplacian operator acts as a versatile bridge between these different operational modes, facilitating the analysis of diverse situations. For some existing results for strongly nonlinear elliptic equations in the anisotropic variable exponent Sobolev spaces, see [4, 13, 26]. Notice that the general operator given by (1.2) can admit degenerate and singular points. It is no surprise to find that there are already papers treating problems with this kind of operator. To give some examples, we refer the reader to [8, 17, 18], where the authors were concerned with Dirichlet problems. We, on the other hand, are interested in a Neumann problem. We refer the reader to [1, 12].

On the other hand, the operator in (1.1) generalized the operator corresponding to the anisotropic variable mean curvature given by

$$\sum_{i=1}^{N} \partial_{x_i} \left((1 + |\partial_{x_i} u|^2)^{\frac{z_i(x)-2}{2}} \partial_{x_i} u \right), \tag{1.3}$$

when

$$A_i(x, u, t) = \frac{1}{z_i(x)} (1 + |t|^2)^{\frac{z_i(x)}{2}}$$

which satisfies the assumptions (H_1)-(H_4) in section 3.

Despite the fact that a specialized form of the operator described in (1.1) with $A_i(x, u, t) = a_i(x, t)$ was initially addressed by Boureanu in [9], it is essential to emphasize that the assumptions we have employed in our research are entirely unique. As a result, our outcomes are distinct, stemming from the utilization of a variational principle presented by Ricceri in [24].

In this paper, we are interested in the following problem:

$$(\mathcal{P})\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}\Big(\partial_4 A_i\Big(\cdot,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\Big)+a_0(x,u,v)=\eta(x)f(u,v) \quad \text{in } \Omega,\\ -\sum_{i=1}^{N}\partial_{x_i}\Big(\partial_5 B_i\Big(\cdot,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\Big)+b_0(x,u,v)=\eta(x)g(u,v) \quad \text{in } \Omega,\\ \sum_{i=1}^{N}\partial_4 A_i\Big(\cdot,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\gamma_i=0 \quad \text{on } \partial\Omega,\\ \sum_{i=1}^{N}\partial_5 B_i\Big(\cdot,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\gamma_i=0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ be a rectangular-like domain, $\partial_4 A_i$ (resp $\partial_5 B_i$) stands for the partial derivative with respect to the fourth variable of A_i (resp the fifth variable of B_i), satisfying some conditions in Section 3.

Boureanu in [10] revolves around exploring the concept of weak solvability in the context of two distinct anisotropic systems characterized by variable exponents. The first system is situated within a rectangular like domain and is governed by no-flux boundary conditions, while the second system is located within a general bounded domain and is subject to zero Dirichlet boundary conditions. Both systems incorporate Leray-Lions type operators which is a particular case of the operator introduced in (1.1) and involve a function F exhibiting sublinear behavior at both zero and infinity. The operators we consider encompass a wide range of possibilities, including generalized Laplace operators, generalized orthotropic Laplace operators, Laplace-type operators stemming from capillary phenomena, and generalized mean curvature operators. The operators considered encompass a wide spectrum of possibilities, including generalized Laplace operators, generalized orthotropic Laplace operators, Laplace-type operators arising from capillary phenomena, and generalized mean curvature operators. The problem under consideration is characterized by carefully crafted hypotheses tailored to capture its unique intricacies, rendering it challenging to encapsulate within a single equation. The provided examples of function F illustrate the diversity inherent in our approach, and the multiplicity results are established through the application of critical point theory.

A large number of papers was devoted to the study the existence of solutions of elliptic systems under various assumptions and in different contexts for a review on classical results, see [2, 3, 6, 7, 20, 21, 25].

The main difficulties in this kind of problem are the framework of anisotropic Sobolev spaces and the fact that we have new class of non-homogeneous Neumann elliptic systems that make some difficulties in the application of Theorem 1.1.

We introduce the following theorem, which will be essential to establish the existence of weak solutions for our main problem.

Theorem 1.1. (*See [24], Theorem 2.5*). *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two sequentially weakly lower semi-continuous and Gâteaux differentiable functionnals. Assume also that Ψ is (strongly) continuous and satisfies*

$\lim\limits_{\|u\|\to+\infty} \Psi(u) = +\infty$. *For each* $\rho > \inf_X \Psi$, *put*

$$\varphi(\rho) = \inf_{u\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf\limits_{v\in\overline{\Psi^{-1}(]-\infty,\rho[)}^w} \Phi(v)}{\rho - \Psi(u)}, \tag{1.4}$$

where $\overline{\Psi^{-1}(]-\infty,\rho[)}^w$ *is the closure of* $\Psi^{-1}(]-\infty,\rho[)$ *in the weak topology. Furthermore, set*

$$\gamma = \liminf_{\rho\to+\infty} \varphi(\rho), \tag{1.5}$$

and

$$\delta = \liminf_{\rho\to(\inf_X \Psi)^+} \varphi(\rho). \tag{1.6}$$

Then, the following conclusions hold:
(a) *For each* $\rho > \inf_X \Psi$ *and each* $t > \varphi(\rho)$, *the functional* $\Phi + t\Psi$ *has a critical point which lies in* $\Psi^{-1}(]-\infty,\rho[)$.
(b) *If* $\gamma < +\infty$, *then, for each* $t > \gamma$, *the following alternative holds: either* $\Phi + t\Psi$ *has a global minimum, or there exists a sequence* $(u_n)_n$ *of critical points of* $\Phi + t\Psi$ *such that* $\lim\limits_{n\to\infty} \Psi(u_n) = +\infty$.
(c) *If* $\delta < +\infty$, *then, for each* $t > \delta$, *the following alternative holds: either there exists a global minimum of* Ψ *which is a local minimum of* $\Phi + t\Psi$, *or there exists a sequence of pairwise distinct critical points of* $\Phi + t\Psi$ *which weakly converges to a global minimum of* Ψ.

This paper is organized as follows: In Section 2, we present some necessary preliminary knowledge on the anisotropic Sobolev spaces with variable exponents. We introduce in the Section 3, some assumptions for which our problem has a solutions and we prove the existence of infinitely many weak solutions for our Neumann elliptic problem.

2. Preliminaries results

In this section we summarize notation, definitions and properties of our framework. For more details we refer to [14]. Let Ω be a bounded domain in $\mathbb{R}^N$, we define:

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{measurable function } p(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R} \text{ such that } 1 < p^- \le p^+ < \infty\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x) \,/\, x \in \overline{\Omega}\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\{p(x) \,/\, x \in \overline{\Omega}\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \longmapsto \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx,$$

is finite, then

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \le 1\},$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(\cdot)} + \dfrac{1}{p'(\cdot)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_{\Omega} uv\, dx\right| \leq \Big(\frac{1}{p^-} + \frac{1}{(p'^-)}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}, \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.
An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1. (*See [14, 17].*) *If $u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:*

(i). $\|u\|_{p(\cdot)} < 1$(*respectively*, $= 1, > 1$) $\Leftrightarrow \rho_{p(\cdot)}(u) < 1$(*respectively*, $= 1, > 1$),

(ii). $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+}$,

(iii). $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^-}$.

We define the Sobolev space with variable exponent by:

$$W^{1,p(\cdot)}(\Omega) = \Big\{u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\},$$

equipped with the following norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We refer to [14] for the elementary properties of these spaces.

Remark 2.2. Recall that the definition of these spaces requires only the measurability of $p(\cdot)$. In this work, we do not need to use Sobolev and Poincaré inequalities. Note that the sharp Sobolev inequality is proved for $p(\cdot)$-log-Hölder continuous, while the Poincaré inequality requires only the continuity of $p(\cdot)$ (see [14]).

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of our main problem.
Let $p_1(\cdot), \dots, p_N(\cdot)$ be N variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We denote

$$\vec{p}(\cdot) = \big\{p_1(\cdot), \dots, p_N(\cdot)\big\}, \quad \text{and } D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \dots, N,$$

and for all $x \in \overline{\Omega}$ we put

$$p_M(\cdot) = \max\Big\{p_1(\cdot), ..., p_N(\cdot)\Big\} \;\; \textit{and} \;\; p_m(\cdot) = \min\Big\{p_1(\cdot), ..., p_N(\cdot)\Big\}.$$

We define

$$\underline{p} = \min\{p_1^-, p_2^-, \dots, p_N^-\} \qquad \text{then} \qquad \underline{p} > 1, \tag{2.2}$$

and

$$\overline{p} = \max\{p_1^+, p_2^+, \dots, p_N^+\}. \tag{2.3}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows

$$W^{1,\vec{p}(\cdot)}(\Omega)=\Big\{u\in L^{p_M(\cdot)}(\Omega) \quad \text{and} \quad D^i u\in L^{p_i(\cdot)}(\Omega), \quad i=1,2,\dots,N\Big\},$$

endowed with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}=\|u\|_{1,\vec{p}(\cdot)}=\|u\|_{L^{p_M(\cdot)}(\Omega)}+\sum_{i=1}^{N}\|D^i u\|_{L^{p_i(\cdot)}(\Omega)}. \tag{2.4}$$

(Cf. [5, 22, 23] for the constant exponent case). The space $\big(W^{1,\vec{p}(\cdot)}(\Omega),\|\cdot\|_{1,\vec{p}(\cdot)}\big)$ is a reflexive Banach space (cf [15]). The theory of such spaces was developed in [15, 16, 17, 19].

3. Essential assumptions and main results

Here and in the sequel, we assume that Ω is a rectangular like domain and let $p_1(\cdot),\dots,p_N(\cdot)$ and $q_1(\cdot),\dots,q_N(\cdot)$ be $2N$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$ satisfying that so called log-Hölder continuity, there exists a positive constant $L>0$ such that

$$|p_i(x)-p_i(y)|\le -\frac{L}{\log(x-y)}, \text{ for all } x,y\in\overline{\Omega} \text{ with } |x-y|\le\frac{1}{2}, \tag{3.1}$$

$$|q_i(x)-q_i(y)|\le -\frac{L}{\log(x-y)}, \text{ for all } x,y\in\overline{\Omega} \text{ with } |x-y|\le\frac{1}{2}, \tag{3.2}$$

and we suppose also

$$\underline{p}>N \text{ and } \underline{q}>N. \tag{3.3}$$

The previous assumption gives the following result.

Proposition 3.1. *Since $W^{1,\vec{p}(\cdot)}(\Omega)$ (respectively $W^{1,\vec{q}(\cdot)}(\Omega)$) is continuously embedded in $W^{1,\underline{p}}(\Omega)$(respectively $W^{1,\underline{q}}(\Omega)$), and since $W^{1,\underline{p}}(\Omega)$ and $W^{1,\underline{q}}(\Omega)$ are compactly embedded in $C^0(\overline{\Omega})$ (the space of continuous functions), thus the spaces $W^{1,\vec{p}(\cdot)}(\Omega)$ and $W^{1,\vec{q}(\cdot)}(\Omega)$ are compactly embedded in $C^0(\overline{\Omega})$.*

Then we can set

$$C_1=\sup_{u\in W^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}}\frac{\|u\|_\infty}{\|u\|_{1,\vec{p}(\cdot)}}. \tag{3.4}$$

$$C_2=\sup_{u\in W^{1,\vec{q}(\cdot)}(\Omega)\setminus\{0\}}\frac{\|u\|_\infty}{\|u\|_{1,\vec{q}(\cdot)}}. \tag{3.5}$$

We would like to highlight the relevance of the upcoming density result, as it is instrumental in assuring the sound definition of weak solutions pertaining to system $(\mathcal{P})$.

Theorem 3.2. (*See [10, 15].*) *Let Ω be a rectangular-like domain of $\mathbb{R}^N$. Under the assumptions (3.1) and (3.2), it can be affirmed that $\mathcal{C}^\infty(\overline{\Omega})$ serves as a dense subset within both $W^{1,\vec{p}(\cdot)}(\Omega)$ and $W^{1,\vec{q}(\cdot)}(\Omega)$.*

We present now the characteristics of the functions $A_i,B_i:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $i=1,\dots,N$, and $A_0,B_0:\Omega\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$.

(H_1). For all $i = 0, \dots, N$, A_i and B_i are continuous in x and of class $\mathcal{C}^1$ in (s,t), with $A_i(x,0,0,0,0) = B_i(x,0,0,0,0) = 0$ for a.e. $x \in \overline{\Omega}$.

(H_2). For all $i = 1, \dots, N$, there are positive constants α_i and non-negative functions $c_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$\begin{aligned}&\Big|\partial_r A_i(x,s,t,r,\sigma)\Big| + \Big|\partial_\sigma B_i(x,s,t,r,\sigma)\Big| \\ &\le \alpha_i\Big(c_i(x) + |s|^{p_i(x)} + |t|^{q_i(x)} + |r|^{p_i(x)-1} + |\sigma|^{q_i(x)-1}\Big), \qquad \text{for a.e. } x \in \overline{\Omega},\end{aligned} \tag{3.6}$$

and all $s,t,r,\sigma \in \mathbb{R}$, and, there are non-negative functions $\lambda_1, \lambda_2 \in L^1(\Omega)$ such that

$$\Big|A_0(x,s,t)\Big| \le \lambda_1(x)\Big(|s|^{p_M(x)} + |t|^{q_M(x)}\Big), \qquad \text{for a.e. } x \in \overline{\Omega} \text{ and all } s,t \in \mathbb{R}.$$

$$\Big|B_0(x,s,t)\Big| \le \lambda_2(x)\Big(|s|^{p_M(x)} + |t|^{q_M(x)}\Big), \qquad \text{for a.e. } x \in \overline{\Omega} \text{ and all } s,t \in \mathbb{R},$$

where $A_0(x,s,t) = \int_0^s a_0(x,\sigma,t)d\sigma$ and $B_0(x,s,t) = \int_0^t b_0(x,s,\sigma)d\sigma$.

(H_3). For all $i = 1, \dots, N$ and for all $s,t,\sigma, r \neq r' \in \mathbb{R}$ and all $x \in \Omega$, one has

$$\sum_{i=1}^N \Big(\partial_s A_i\Big(x,s,t,r,\sigma\Big) - \partial_s A_i\Big(x,s,t,r',\sigma\Big)\Big)(r - r') > 0,$$

$$\sum_{i=1}^N \Big(\partial_t B_i\Big(x,s,t,r,\sigma\Big) - \partial_t B_i\Big(x,s,t,r,\sigma'\Big)\Big)(\sigma - \sigma') > 0$$

and,

$$\Big(\partial_s A_0(x,s,t) - \partial_s A_0(x,s,t')\Big)(t - t') > 0, \text{ for all } s, t \neq t' \in \mathbb{R}, \text{ and all } x \in \Omega.$$

$$\Big(\partial_s B_0(x,s,t) - \partial_s B_0(x,s,t')\Big)(t - t') > 0, \text{ for all } s, t \neq t' \in \mathbb{R}, \text{ and all } x \in \Omega.$$

(H_4). There are constants $\delta_0, \delta_1, \theta_0, \theta_1 > 0$ such that, for all $i = 1, \dots, N$ we have

$$A_i(x,s,t,r,\sigma) \ge \delta_0 |r|^{p_i(x)}, \qquad \text{for all } x \in \Omega \text{ and } s,t,r,\sigma \in \mathbb{R},$$

$$B_i(x,s,t,r,\sigma) \ge \delta_1 |\sigma|^{q_i(x)}, \qquad \text{for all } x \in \Omega \text{ and } s,t,r,\sigma \in \mathbb{R},$$

$$A_0(x,s,t) \ge \theta_0 |s|^{p_M(x)}, \qquad \text{for all } x \in \Omega \text{ and } s,t \in \mathbb{R},$$

and,

$$B_0(x,s,t) \ge \theta_1 |t|^{q_M(x)}, \qquad \text{for all } x \in \Omega \text{ and } s,t \in \mathbb{R}.$$

(H_5). $\eta \in \mathcal{C}(\overline{\Omega})$ and $f, g \in \mathcal{C}(\mathbb{R}^2)$ such that the differential form $f(u,v)du + g(u,v)dv$ is exact.

Remark 3.3. (H_5) implies that exists $H : \mathbb{R}^2 \longmapsto \mathbb{R}$ is the integral of the differential form $f(u,v)du + g(u,v)dv$ such that $H(0,0) = 0$.

Let X be the Cartesian product between Sobolev spaces $W^{1,\vec{p}(\cdot)}(\Omega)$ and $W^{1,\vec{q}(\cdot)}(\Omega)$ with the norm $\|(u,v)\|_X = \sqrt{\|u\|^2_{1,\vec{p}(\cdot)} + \|v\|^2_{1,\vec{q}(\cdot)}}$ or another equivalent to it.
We introduce the functionals $\Psi(\cdot,\cdot)$, $\Phi(\cdot,\cdot) : W^{1,\vec{p}(\cdot)}(\Omega) \times W^{1,\vec{q}(\cdot)}(\Omega) \longmapsto \mathbb{R}$ by

$$\begin{aligned}\Psi(u,v) &= \sum_{i=1}^{N}\int_\Omega A_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\,dx + \sum_{i=1}^{N}\int_\Omega B_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\,dx \\ &+ \int_\Omega A_0(x,u,v)dx + \int_\Omega B_0(x,u,v)dx, \end{aligned} \tag{3.7}$$

and

$$\Phi(u,v) = -\int_\Omega F(x,u(x),v(x))\,dx, \tag{3.8}$$

where $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $F(x,u,v) = \eta(x)H(u,v)$.

Lemma 3.4. (*See [9]*). *The functionals $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are well defined on X. In addition, $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are of class $\mathcal{C}^1(X,\mathbb{R})$ and*

$$\begin{aligned}\Psi'(u,v)(w,\phi) &= \sum_{i=1}^{N}\int_\Omega \partial_4 A_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\partial_{x_i}w dx \\ &+ \sum_{i=1}^{N}\int_\Omega \partial_5 B_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\partial_{x_i}u\phi dx \\ &+ \int_\Omega \partial_2 A_0(x,u,v)w dx + \int_\Omega \partial_3 B_0(x,u,v)\phi dx, \end{aligned} \tag{3.9}$$

and

$$\Phi'(u,v)(w,\phi) = -\int_\Omega \eta(x)\Big(\partial_1 H(u(x),v(x))w(x) + \partial_2 H(u(x),v(x))\phi(x)\Big)\,dx, \tag{3.10}$$

for all $(u,v)(w,\phi) \in X$.

Lemma 3.5. (*See* [*9*]). *Under the hypothesis (H_1)-(H_5) and (3.3) the functionals $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are weakly lower semi-continuous.*

Lemma 3.6. *Under the hypothesis (H_1)-(H_5) the functional $\Psi(\cdot,\cdot)$ is coercive, that is,*

$$\Psi(u,v) \longrightarrow +\infty \quad \text{as } \|(u,v)\|_X \longrightarrow +\infty \quad \text{for } (u,v) \in X.$$

Proof. Let $(u,v) \in X$. One has

$$\begin{aligned}\Psi(u,v) &= \sum_{i=1}^{N}\int_\Omega A_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)dx + \sum_{i=1}^{N}\int_\Omega B_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\,dx \\ &+ \int_\Omega A_0(x,u,v)dx + \int_\Omega B_0(x,u,v)dx,\end{aligned}$$

then by using (H_4), we get

$$\begin{aligned}
\Psi(u,v) &\geq \sum_{i=1}^{N}\int_\Omega \delta_0\Big|\partial_{x_i}u\Big|^{p_i(x)}\,dx+\sum_{i=1}^{N}\int_\Omega \delta_1\Big|\partial_{x_i}v\Big|^{q_i(x)}\,dx+\theta_0\int_\Omega |u|^{p_M(x)}\,dx\\
&\quad +\theta_1\int_\Omega |u|^{q_M(x)}\,dx\\
&\geq \min\Big(\delta_0,\theta_0\Big)\Big[\sum_{i=1}^{N}\int_\Omega \Big|\partial_{x_i}u\Big|^{p_i(x)}\,dx+\int_\Omega |u|^{p_M(x)}\,dx\Big]\\
&\quad +\min\Big(\delta_1,\theta_1\Big)\Big[\sum_{i=1}^{N}\int_\Omega \Big|\partial_{x_i}v\Big|^{q_i(x)}\,dx+\int_\Omega |v|^{q_M(x)}\Big]dx\\
&\geq \min\Big(\delta_0,\theta_0\Big)\Big[\frac{1}{N^{\underline{p}-1}}\Big(\sum_{i=1}^{N}\Big\|\partial_{x_i}u\Big\|_{p_i(\cdot)}\Big)^{\underline{p}}+\|u\|_{p_M(\cdot)}^{\underline{p}}-N-1\Big]\\
&+\min\Big(\delta_1,\theta_1\Big)\Big[\frac{1}{N^{\underline{q}-1}}\Big(\sum_{i=1}^{N}\Big\|\partial_{x_i}v\Big\|_{q_i(\cdot)}\Big)^{\underline{q}}+\|u\|_{q_M(\cdot)}^{\underline{q}}-N-1\Big]\\
&\geq \min\Big(\delta_0,\theta_0\Big)\Big[\frac{1}{(2N)^{\underline{p}-1}}\Big(\sum_{i=1}^{N}\Big\|\partial_{x_i}u\Big\|_{p_i(\cdot)}+\|u\|_{p_M(\cdot)}\Big)^{\underline{p}}-N-1\Big]\\
&+\min\Big(\delta_1,\theta_1\Big)\Big[\frac{1}{(2N)^{\underline{q}-1}}\Big(\sum_{i=1}^{N}\Big\|\partial_{x_i}v\Big\|_{q_i(\cdot)}+\|u\|_{q_M(\cdot)}\Big)^{\underline{q}}-N-1\Big]\\
&= \frac{\min\Big(\delta_0,\theta_0\Big)}{(2N)^{\underline{p}-1}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}}+\frac{\min\Big(\delta_1,\theta_1\Big)}{(2N)^{\underline{q}-1}}\|u\|_{1,\vec{q}(\cdot)}^{\underline{q}}-K_2\\
&\geq K_1\Big(\|u\|_{1,\vec{q}(\cdot)}^{\underline{p}}+\|v\|_{1,\vec{q}(\cdot)}^{\underline{q}}\Big)-K_2\\
&\geq K_1\|(u,v)\|_X-K_2,
\end{aligned}$$

where $K_1, K_2 > 0$ constants.
Thus, if $\|(u,v)\|_X \longrightarrow +\infty$ then $\Psi(u,v) \longrightarrow +\infty$. □

Now, we set $\eta_1 = \left(\frac{C_1}{\theta_0 \mathrm{meas}(\Omega)}\right)^{\overline{p}}$ and $\eta_2 = \left(\frac{C_2}{\theta_1 \mathrm{meas}(\Omega)}\right)^{\overline{q}}$,

$$\mu = \min\left\{\frac{1}{\eta_1^{\overline{p}}}, \frac{1}{\eta_1^{\underline{p}}}\right\}, \quad \text{and} \quad \nu = \min\left\{\frac{1}{\eta_2^{\overline{q}}}, \frac{1}{\eta_2^{\underline{q}}}\right\}.$$

The sets $A(r), B(r),\ r > 0$, below satisfied, play an important role in our exposition

$$A(r) = \big\{(\xi,\eta)\in\mathbb{R}^2 \text{ such that } \mu F_{\vec{p}(\cdot)}(\xi)+\nu F_{\vec{q}(\cdot)}(\eta)\leq r\big\}$$

and

$$B(r) = \left\{(\xi,\zeta)\in\mathbb{R}^2 : \int_\Omega A_0(x,\xi,\zeta)dxD_{\vec{p}(\cdot)}(\xi)+\int_\Omega B_0(x,\xi,\zeta)dxD_{\vec{q}(\cdot)}(\zeta)\leq r\right\},$$

where $D_{\vec{r}(\cdot)}(t) = \max(|t|^{\overline{r}}, |t|^{\underline{r}})$ and $F_{\vec{r}(\cdot)}(t) = \min(|t|^{\overline{r}}, |t|^{\underline{r}})$ with $\vec{r}(\cdot) \in \{\vec{p}(\cdot), \vec{q}(\cdot)\}$ and $t \in \{\xi, \eta\}$.

Lemma 3.7. *For all $r > 0$, we have*

$$B(r) \subset A(r).$$

Proof. We observe that, by the definition of constants C_1 and C_2, we have

$$\|u\|_\infty \leq C_1\|u\|_{1,\vec{p}(\cdot)}, \forall u \in W^{1,\vec{p}(\cdot)}(\Omega),$$

and

$$\|v\|_\infty \leq C_2\|v\|_{1,\vec{q}(\cdot)}, \forall v \in W^{1,\vec{q}(\cdot)}(\Omega).$$

For $u \equiv v \equiv 1$, we get

$$1 \leq \eta_1^{\overline{p}}\theta_0 \text{meas}(\Omega) \leq \eta_1^{\overline{p}} \int_\Omega A_0(x, \xi, \eta)dx,$$

and,

$$1 \leq \eta_2^{\overline{q}}\theta_1 \text{meas}(\Omega) \leq \eta_2^{\overline{q}} \int_\Omega B_0(x, \xi, \eta)dx.$$

Thus, we obtain

$$\mu \leq \frac{1}{\eta_1^{\overline{p}}} \leq \int_\Omega A_0(x, \xi, \eta)dx, \quad \text{ and } \nu \leq \frac{1}{\eta_2^{\overline{q}}} \leq \int_\Omega B_0(x, \xi, \eta)dx.$$

Since

$$F_{\vec{p}(\cdot)}(t) \leq D_{\vec{p}(\cdot)}(t), \quad \text{ and } F_{\vec{q}(\cdot)}(t) \leq D_{\vec{q}(\cdot)}(t), \forall t \in \mathbb{R}.$$

Thus, the inequality

$$\mu F_{\vec{p}(\cdot)}(\xi) + \nu F_{\vec{q}(\cdot)}(\zeta) \leq \int_\Omega A_0(x, \xi, \eta)dx D_{\vec{p}(\cdot)}(\xi) + \int_\Omega B_0(x, \xi, \eta)dx D_{\vec{q}(\cdot)}(\zeta),$$

holds for every $(\xi, \zeta) \in \mathbb{R}^2$ and therefore the inclusion

$$B(r) \subset A(r), \forall r > 0,$$

holds. □

Definition 3.8. We say that $(u, v) \in X$ a weak solution to the problem $(\mathcal{P})$ if for all $(w, \phi) \in X$, we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \partial_4 A_i\Big(x, u, v, \partial_{x_i}u, \partial_{x_i}v\Big)\partial_{x_i}w dx \\
&+ \sum_{i=1}^N \int_\Omega \partial_5 B_i\Big(x, u, v, \partial_{x_i}u, \partial_{x_i}v\Big)\partial_{x_i}\phi dx \\
&+ \int_\Omega \partial_u A_0(x, u, v)w\, dx + \int_\Omega \partial_v B_0(x, u, v)\phi dx \\
&= \int_\Omega \eta(x)\Big(\partial_u H(u(x), v(x))w(x) + \partial_v H(u(x), v(x))\phi(x)\Big)\, dx.
\end{aligned}$$

Remark 3.9. Note that the weak solutions of $(\mathcal{P})$ are precisely critical points of $\Psi + \Phi$.

Our first main result is the following theorem.

Theorem 3.10. *Suppose that* $\Psi(\cdot,\cdot)$ *and* $\Phi(\cdot,\cdot)$ *are as in (3.7) and (3.8) and* (H_1)*-*(H_5) *and (3.3) hold true. If there exist* $\rho_0 > 0$, $(\xi_0, \eta_0) \in \mathbb{R}^2$ *with* $(\xi_0, \zeta_0) \in Int(B(\rho_0))$ $\Big(Int(B)$ *is the interior of* $B\Big)$ *and* $\max_{A(\rho_0)} H(\xi,\zeta) = H(\xi_0,\zeta_0)$*. Then, problem* $(\mathcal{P})$ *admits a weak solution* $(u,v) \in X$ *such that* $\Psi(u,v) < \rho_0$*.*

Proof. We apply the part (a) of Theorem 1.1 for showing that $\varphi(\rho_0) = 0$ (here φ is the function defined in the Theorem 1.1 and $t = 1$ is assumed).
First, we observe that, for all $(u,v) \in \Psi^{-1}(]-\infty,\rho_0[)$, one has

$$\begin{aligned} 0 \le \varphi(\rho_0) &= \inf_{\Psi^{-1}(]-\infty,\rho_0[)} \frac{\Phi(u,v) - \inf_{\overline{(\Psi^{-1}(]-\infty,\rho_0[))}^w} \Phi(u,v)}{\rho_0 - \Psi(u,v)} \\ &\le \frac{\Phi(u,v) - \inf_{\overline{\Psi^{-1}(]-\infty,\rho_0[)}^w} \Phi(u,v)}{\rho_0 - \Psi(u,v)}. \end{aligned} \tag{3.11}$$

Let $u_0(x) = \xi_0$, $v_0(x) = \zeta_0$, $\forall x \in \Omega$. Then $\nabla u_n = \nabla v_0 = 0$, and since $(\xi_0,\zeta_0) \in Int(B(\rho_0))$, one has

$$\Psi(u_0,v_0) = \int_\Omega \Big[A_0(x,\xi_0,\zeta_0) + B_0(x,\xi,\zeta_0)\Big]dx < \rho_0.$$

Then, for almost evrey $x \in \overline{\Omega}$ and $\forall (u,v) \in \overline{\Psi^{-1}(]-\infty,\rho_0[)}^w$, one has

$$\mu F_{\vec{p}(\cdot)}(u(x)) + \nu F_{\vec{q}(\cdot)}(v(x)) \le \Psi(u,v) \le \rho_0. \tag{3.12}$$

The first inequality in (3.12) is obtained by the Proposition 2.1, while the second inequality in (3.12) follows from the fact that $\overline{\Psi^{-1}(]-\infty,\rho_0[)}^w = \Psi^{-1}(]-\infty,\rho_0])$.
Thus, since $(u(x),v(x)) \in A(\rho_0)$ and $H(u(x),v(x)) \le H(\xi_0,\zeta_0)$, $\forall x \in \overline{\Omega}$.
Hence $-\Phi(u,v) \le -\Phi(u_0,v_0)$ $\forall (u,v) \in \overline{\Psi^{-1}(]-\infty,\rho_0[)}^w$. Because,

$$-\Phi(u_0,v_0) = \sup_{\overline{\Psi^{-1}(]-\infty,\rho_0[)}^w} (-\Phi(u,v)) = -\inf_{\overline{\Psi^{-1}(]-\infty,\rho_0[)}_w} \Phi(u,v),$$

and since $\Phi(u_0,v_0) < \rho_0$, it follows that

$$\Phi(u_0,v_0) - \inf_{\overline{\Psi^{-1}(]-\infty,\rho_0[)}^w} \Phi(u,v) = \Phi(u_0,v_0) - \Phi(u_0,v_0) = 0.$$

Then, by choosing $(u,v) = (u_0,v_0)$ in the inequality (3.11), one has $\varphi(\rho_0) = 0$.
The conclusion (a) of the Theorem 1.1 assures that there is a critical point of $\Psi+\Phi$. □

Now, we announce our second main result.

Theorem 3.11. *Suppose that* $\Psi(\cdot,\cdot)$ *and* $\Phi(\cdot,\cdot)$ *are as in (3.7) and (3.8) and* (H_1)*-*(H_5) *and (3.3) hold true. If there exist a sequences,* $(\rho_n)_n \subset \mathbb{R}^+$ *with* $\rho_n \to \infty$ *as* $n \to +\infty$ *and* $(\xi_n)_n, (\zeta_n)_n \subset \mathbb{R}$ *such that* $(\xi_n,\zeta_n) \in Int(B(\rho_n))$ *and*

$$\max_{A(\rho_n)} H(\xi,\eta) = H(\xi_n,\zeta_n),\ \forall n > 0$$

and if

$$\limsup_{(\xi,\zeta)\to+\infty} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$
$$> \big(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\big)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

Then, the problem $(\mathcal{P})$ admits an unbounded sequence of a weak solutions in X.

Proof. From the part (a). we know that $\varphi(\rho_n)=0,\ \forall n\in\mathbb{N}$.
Then, since

$$\lim_{n\to\infty}\rho_n=+\infty,$$

one has

$$\liminf_{\rho\to\infty}\varphi(\rho)\le\liminf_{n\to\infty}\varphi(\rho_n)=0<1=t.$$

Now, we fix h satisfying that

$$\limsup_{(\xi,\zeta)\to+\infty} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$
$$> h > \big(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\big)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

and we choose a sequence $(\varsigma_n,\tau_n)_n$ in $\mathbb{R}^2$ such that $\sqrt{\varsigma_n^2+\tau_n^2}\ge n$ and $\forall n\in\mathbb{N}$ one has

$$\begin{aligned}&H(\varsigma_n,\tau_n)\int_\Omega \eta(x)dx\\&>h\Big[D_{\vec{p}(\cdot)}(\varsigma_n)+D_{\vec{p}(\cdot)}(\tau_n)\Big]\int_\Omega A_0(x,\varsigma_n,\tau_n)dx\\&+\Big[D_{\vec{q}(\cdot)}(\varsigma_n)+D_{\vec{q}(\cdot)}(\tau_n)\Big]\int_\Omega B_0(x,\varsigma_n,\tau_n)dx.\end{aligned}$$

If we denote by u_n and v_n the constant functions on Ω which take the ς_n and τ_n values respectively, by using assumptions (H_2) we have

$$\begin{aligned}
&\Phi(u_n,v_n)+\Psi(u_n,v_n)\\
&=\Phi(\varsigma_n,\tau_n)+\Psi(\varsigma_n,\tau_n)\\
&=-H(\varsigma_n,\tau_n)\int_\Omega \eta(x)dx+\int_\Omega A_0(x,\varsigma_n,\tau_n)dx+\int_\Omega B_0(x,\varsigma_n,\tau_n)dx\\
&\le -hD_{\vec{p}(\cdot)}(\varsigma_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx-hD_{\vec{p}(\cdot)}(\tau_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx\\
&-hD_{\vec{q}(\cdot)}(\varsigma_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx-hD_{\vec{q}(\cdot)}(\tau_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx\\
&+\|\lambda_1\|_{L^1(\mathbb{R})}\Big[D_{\vec{p}(\cdot)}(\varsigma_n)+D_{\vec{q}(\cdot)}(\tau_n)\Big]+\|\lambda_2\|_{L^1(\mathbb{R})}\Big[D_{\vec{p}(\cdot)}(\varsigma_n)+D_{\vec{q}(\cdot)}(\tau_n)\Big]\\
&=\left(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}-h\int_\Omega A_0(x,\varsigma_n,\tau_n)dx\right)D_{\vec{p}(\cdot)}(\varsigma_n)\\
&+\left(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}-h\int_\Omega B_0(x,\varsigma_n,\tau_n)dx\right)D_{\vec{q}(\cdot)}(\tau_n)\\
&-hD_{\vec{p}(\cdot)}(\tau_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx-hD_{\vec{q}(\cdot)}(\varsigma_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx\\
&<0,\quad \forall n\in\mathbb{N}.
\end{aligned}$$

Since $(\sqrt{\varsigma_n^2+\tau_n^2})_n$ is unbounded, at least one of the two sequences $(\varsigma_n)_n$ or $(\tau_n)_n$ admits one divergent subsequence.
Hence $(D_{\vec{p}(\cdot)}(\tau_n))_n$ and $(D_{\vec{q}(\cdot)}(\tau_n))_n$ admit one divergent subsequence, thus, the functional $\Phi+\Psi$ is unbounded from below.
The conclusion (b) of the Theorem 1.1 assures that there is a sequence $(x_n,y_n)_n$ of critical points of $\Phi+\Psi$ such that $\lim_{n\to+\infty}\Psi(x_n,y_n)=+\infty$.
Moreover, since Ψ is bounded on each bounded subset of X, the sequence $(x_n,y_n)_n$ must be unbounded in X. □

The following result is a practicable form of Theorem 3.11.

Corollary 3.12. *Let $(a_n)_n$ and $(b_n)_n$ be two sequences in $\mathbb{R}^+$ satisfying*

$$b_n<a_n\quad \forall n\in\mathbb{N},\ \lim_{n\to+\infty}b_n=+\infty,\qquad \lim_{n\to+\infty}\frac{a_n}{b_n}=+\infty,$$

and let

$$A_n=\Big\{(\xi,\zeta)\in\mathbb{R}^2:\mu F_{\vec{p}(\cdot)}(\xi)+\nu F_{\vec{q}(\cdot)}(\eta)\le a_n\Big\},$$

$$B_n=\Big\{(\xi,\zeta)\in\mathbb{R}^2:\int_\Omega A_0(x,\xi,\zeta)dxD_{\vec{p}(\cdot)}(\xi)+\int_\Omega B_0(x,\xi,\zeta)dxD_{\vec{q}(\cdot)}(\zeta)\le b_n\Big\},$$

be such that $\sup_{A_n\setminus IntB_n} H \leq 0$ *for all* $n \in \mathbb{N}$.
Finally, let us assume that

$$\limsup_{(\xi,\zeta)\to+\infty} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$

$$> \big(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\big)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

Then, Problem $(\mathcal{P})$ *admits an unbounded sequence of weak solutions in* X.

Proof. Since $b_n < a_n$ it follows that $B_n \subseteq A_n$.
Let

$$\gamma = \min\{\mu,\nu\} > 0$$

$$\delta = \big(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\big)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right) > 0.$$

Then $\frac{\delta}{\gamma} > 0$ and in virtue of $\lim_{n\to+\infty}\frac{a_n}{b_n} = +\infty$, then we get $\frac{\delta}{\gamma} < \frac{a_n}{b_n}$ for $n \in \mathbb{N}$ large enough.
Let $\rho_n = \gamma a_n$. Then $\{\rho_n\}_n \subset \mathbb{R}^+$ is a divergent sequence and for n large enough, the following inclusions hold

$$IntB_n \subseteq B_n \subseteq B(\rho_n) \subseteq A(\rho_n) \subseteq A_n.$$

Then, since H is negative in the set $A_n \setminus \mathrm{Int}B_n$ for all $n \in \mathbb{N}$, we have

$$\max_{\mathrm{Int}B_n} H = \max_{A_n} H,$$

in particular, we have $\max_{\mathrm{Int}B_n} H = \max_{A(\rho_n)} H$ for $n \in \mathbb{N}$ large enough, i.e. there exist at least one sequence $(\xi_n,\zeta_n)_n \subset \mathrm{Int}B_n$ such that for n large enough, we have

$$\max_{A(\rho_n)} H(\xi,\zeta) = H(\xi_n,\zeta_n).$$

Thus, the sequences $(\xi_n)_n$, $(\zeta_n)_n$ and $(\rho_n)_n$ have got the properties required in Theorem 3.10(b).
This completes the proof. □

Our third main result reads as follows.

Theorem 3.13. *Suppose that* $\Psi(\cdot,\cdot)$ *and* $\Phi(\cdot,\cdot)$ *are as in (3.7) and (3.8) and* (H_1)-(H_5) *and (3.3) hold true. If there exist sequence,* $(\rho_n)_n \subset \mathbb{R}^+$ *with* $\rho_n \longrightarrow 0$ *as* $n \longrightarrow +\infty$ *and* $(\xi_n)_n,(\zeta_n)_n \subset \mathbb{R}$ *such that* $(\xi_n,\zeta_n) \in Int(B(\rho_n))$ *and* $\max_{A(\rho_n)} H(\xi,\zeta) =$

$H(\xi_n, \zeta_n)$, $\forall n > 0$ and if

$$\limsup_{(\xi,\zeta)\to(0,0)} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$

$$> \left(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\right)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

Then the problem $(\mathcal{P})$ admits a sequence of non trivial weak solutions which strongly converges to (u,v) in X.

Proof. We apply the part (c) of Theorem 1.1. As before, from the (a). we know that $\varphi(\rho_n) = 0$, $\forall n \in \mathbb{N}$.
Therefore after observing that $\inf_X \Psi = \Psi(u,v) = 0$, since $\lim_{n\to\infty}\rho_n = 0$, we have

$$\delta = \liminf_{\rho\to 0^+}\varphi(\rho) \le \liminf_{n\to+\infty}\varphi(\rho_n) = 0 < 1 = t.$$

Now, we fix h satisfying

$$\limsup_{(\xi,\zeta)\to(0,0)} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$

$$> h > \left(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\right)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

and choose a sequence $((\varsigma_n,\tau_n))_n$ in $\mathbb{R}^2\setminus\{(0,0)\}$ such that $\sqrt{\varsigma_n^2+\tau_n^2}\le\frac{1}{n}$ and for all $n\in\mathbb{N}$, one has

$$H(\varsigma_n,\tau_n)\int_\Omega \eta(x)dx > h\Bigg(\Big[D_{\vec{p}(\cdot)}(\varsigma_n)+D_{\vec{p}(\cdot)}(\tau_n)\Big]\int_\Omega A_0(x,\varsigma_n,\tau_n)dx$$
$$+\Big[D_{\vec{q}(\cdot)}(\varsigma_n)+D_{\vec{q}(\cdot)}(\tau_n)\Big]\int_\Omega B_0(x,\varsigma_n,\tau_n)dx\Bigg).$$

Once more if we denote by u_n and v_n the constant functions on Ω which equal ς_n and ς_n respectively.
Then, from Proposition 2.1 the sequence $((u_n,v_n))_n$ strongly converges to (u,v) in X and one has

$$\Phi(u_n,v_n)+\Psi(u_n,v_n)=\Phi(\varsigma_n,\tau_n)+\Psi(\varsigma_n,\tau_n)$$

$$
\begin{aligned}
&= -H(\varsigma_n, \tau_n)\int_\Omega \eta(x)dx + \int_\Omega A_0(x,\varsigma_n,\tau_n)dx + \int_\Omega B_0(x,\varsigma_n,\tau_n)dx \\
&\leq -hD_{\vec{p}(\cdot)}(\varsigma_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx - hD_{\vec{p}(\cdot)}(\tau_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx \\
&-hD_{\vec{q}(\cdot)}(\varsigma_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx - hD_{\vec{q}(\cdot)}(\tau_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx \\
&+\|\lambda_1\|_{L^1(\mathbb{R})}\Big[D_{\vec{p}(\cdot)}(\varsigma_n) + D_{\vec{q}(\cdot)}(\tau_n)\Big] + \|\lambda_2\|_{L^1(\mathbb{R})}\Big[D_{\vec{p}(\cdot)}(\varsigma_n) + D_{\vec{q}(\cdot)}(\tau_n)\Big] \\
&= \left(\|\lambda_1\|_{L^1(\mathbb{R})} + \|\lambda_2\|_{L^1(\mathbb{R})} - h\int_\Omega A_0(x,\varsigma_n,\tau_n)dx\right)D_{\vec{p}(\cdot)}(\varsigma_n) \\
&+ \left(\|\lambda_1\|_{L^1(\mathbb{R})} + \|\lambda_2\|_{L^1(\mathbb{R})} - h\int_\Omega B_0(x,\varsigma_n,\tau_n)dx\right)D_{\vec{q}(\cdot)}(\tau_n) \\
&-hD_{\vec{p}(\cdot)}(\tau_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx - hD_{\vec{q}(\cdot)}(\varsigma_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx \\
&< 0, \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

Since $\Phi(u,v) + \Psi(u,v) = 0$ in virtue of the last inequality (u,v) can't be a local minimum of $\Phi + \Psi$.
Then, since (u,v) is the only global minimum of Ψ, the conclusion (c) of the Theorem 1.1 assures that there is a sequence of pairwise distinct critical points of $\Phi+\Psi$ such that $\lim_{n\to\infty}\Psi(x_n,y_n) = 0$ with $x_n, y_n \rightharpoonup 0$, thus $(x_n,y_n)_n$ must be in norm infinitesimal. □

As an immediate consequence of Theorem 3.13 we get the following corollary.

Corollary 3.14. *Let $(a_n)_n$ and $(b_n)_n$ be two sequences in $\mathbb{R}^+$ satisfying*

$$
b_n < a_n \quad \forall n \in \mathbb{N}, \lim_{n\to+\infty} a_n = 0, \quad \lim_{n\to+\infty}\frac{a_n}{b_n} = +\infty,
$$

and let

$$
A_n = \Big\{(\xi,\zeta) \in \mathbb{R}^2 : \mu F_{\vec{p}(\cdot)}(\xi) + \nu F_{\vec{q}(\cdot)}(\eta) \leq a_n\Big\},
$$

$$
B_n = \Big\{(\xi,\zeta) \in \mathbb{R}^2 : \int_\Omega A_0(x,\xi,\zeta)dx D_{\vec{p}(\cdot)}(\xi) + \int_\Omega B_0(x,\xi,\zeta)dx D_{\vec{q}(\cdot)}(\zeta) \leq b_n\Big\},
$$

be such that $\sup_{A_n\setminus IntB_n} H \leq 0$ for all $n \in \mathbb{N}$.
Finally, let us assume that

$$
\limsup_{(\xi,\zeta)\to(0,0)} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi) + D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx + \Big[D_{\vec{q}(\cdot)}(\xi) + D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}
$$

$$
> \left(\|\lambda_1\|_{L^1(\mathbb{R})} + \|\lambda_2\|_{L^1(\mathbb{R})}\right)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx}, \frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).
$$

Then, problem $(\mathcal{P})$ admits a sequence of non-zero weak solutions which strongly converges to (u,v) in X.

Proof. Likewise, by applying Theorem 1.1 part (c), we get the Corollary 3.14, whose proof will be omitted. □

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Topological degree methods for a nonlinear elliptic systems with variable exponents

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Abstract. In this paper, we consider the existence of a distributional solution for nonlinear elliptic system governed by $(p(x),q(x))$-Laplacian operators. We show that the system has at least one solution by using the topological degree theory. Our results improve and generalize existing results with another technical approach.

Mathematics Subject Classification (2010): 35J60, 35J40, 47H11.

Keywords: $p(x)$-Laplacian, operator of (S_+) type, variable exponent, topological degree.

1. Introduction

The main purpose of this paper is to obtain existence of distributional solution for the following nolinear elliptic system

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, w, \nabla w) \quad \text{in } \Omega, \\ -\operatorname{div}(|\nabla w|^{q(x)-2}\nabla w) = h(x, u, \nabla u) \quad \text{in } \Omega, \\ u = w = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p(\cdot), q(\cdot) \in C_+(\bar{\Omega})$. We assume also that $p(\cdot), q(\cdot)$ are log-Hölder continuous functions (see Lemma 2.10).

For it's various applications in various fields, the study of elliptic equations or systems with variable exponents became the most interesting and fascinating area of research (see [1, 11, 28, 29, 34] and so on).

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In the previous decades, the existence of the nontrivial solutions for elliptic equation involving p and $p(x)$-Laplacian have been a large investigation. We refer the interested readers to [4, 9, 10, 14, 15, 16, 17, 18, 13, 20, 2, 25, 26, 27, 30, 23, 31, 24] and the references therein. Now let us briefly comment certain known results of them.

In [10], Chabrowski and Fu studied the $p(x)$-Laplacian problem

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = f(x,u), \quad x \in \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

wheere Ω is a bounded domain in $\mathbb{R}^n$, $0 < a_0 \leq a(x) \in L^\infty(\Omega)$, $0 \leq b_0 \leq b(x) \in L^\infty(\Omega), p$ is Lipschitz continuous on $\bar{\Omega}$ and satisfies $1 < p_1 \leq p(x) \leq p_2 < n$. When $f(x,u)$ is assumed to satisfy their prototype cases, they obtained the existence of nontrivial and nonnegative solutions for problem (1.2).

Fan and Zhang [18] presents several sufficient conditions for the existence of solutions for the problem (1.2) with $a(x) \equiv 1$ and $b(x) = 0$. Especially, an existence criterion for infinite many pairs of solutions for the problem was obtained by them. By using the degree theory for $p(x)$ is a constant function with values in $(2, N)$, Kim and Hong [20] studied the problem

$$\begin{cases} -\Delta_p u = u + f(x,u,\nabla u), \quad x \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. When $p(x)$ is a variable function, Ait Hammou et al [2] studied the problem on bounded domains. Under certain conditions, they established some results on the existence of solutions by the topological degree theory for a class of demicontinuous operators of generalized (S_+) type.

Inspired by the works mentioned above, especially by [20, 2], we try to extend the results in [2] to the system (1.1). More precisely, the aim of this paper is to show the existence of solutions for (1.1) in the variational frame work by using the topological degree constructed by Kim and Hong [20]. This method may be one of the most effective tools in the study of nonlinear equations. For more details about the important stages in the history of this method, the reader can see [3, 6, 7, 8, 22].

The rest of this paper is organized as follows. In Section 2, we introduce some classes of mappings of generalized (S_+) type, topological degree, some basic properties for variable exponent Sobolev spaces and we present several important properties of $p(x)-$Laplacian which will be later needed. In Section 3, we give our basic assumptions and we prove the main results of this paper. Finally, in Section 4, we present a discussion about our research results.

Notation. Throughout this paper, we shall denoted by "$\rightarrow$" and "$\rightharpoonup$" the strong and weak convergence. We use $B_R(a)$ to denote the open ball in the Banach space X of radius $R > 0$ centered at a. The symbol "$\hookrightarrow$" means the continuous embedding.

2. Mathematical preliminaries

2.1. Classes of mappings and topological degree

For the reader's convenience, we bring in some necessary properties and definitions of the classes of mappings mentioned in the introduction which will be the key to proving the existence solution of system (1.1).

Definition 2.1. Let X and Y be two real separable, reflexive Banach spaces and Ω a nonempty subset of X. A mapping $F : \Omega \subset X \to Y$ is

1. demicontinuous, if for each $u \in \Omega$ and any sequence (u_n) in Ω, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.
2. bounded, if it takes any bounded set into a bounded set.
3. compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 2.2. Let X be a real separable reflexive Banach space with dual space X^*. An operator $F : \Omega \subset X \to X^*$ is said to be

1. of class (S_+), if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$ and $\limsup\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.
2. quasimonotone, if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$, we have $\limsup\langle Fu_n, u_n - u\rangle \geq 0$.

Definition 2.3. Let $T : \Omega_1 \subset X \to X^*$ be a bounded mapping such that $\Omega \subset \Omega_1$. For any mapping $F : \Omega \subset X \to X$, we say that

1. F satisfies condition $(S_+)_T$, if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$.
2. F has the property $(QM)_T$, if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup\langle Fu_n, y_n - y\rangle \geq 0$.

Now, let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* | F \text{ is bounded, demicontinuous and of class } (S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X | F \text{ is bounded, demicontinuous and of class } (S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X | F \text{ is demicontinuous and of class } (S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\overline{G}) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\},\\
\mathcal{F}(X) &:= \{F \in \mathcal{F}_T(\overline{G}) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\}.
\end{aligned}$$

Here, $T \in \mathcal{F}_1(\overline{G})$ is called an essential inner map to F.

Lemma 2.4 ([5], **Lemmas** 2.2 **and** 2.4)**.** *Let $T \in \mathcal{F}_1(\overline{G}), G \in \mathcal{O}$, be continuous and $S : D_S \subset X^* \to X$ a bounded demicontinuous mapping such that $T(\overline{G}) \subset D_S$. Then the following statements are true:*

1. *If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\overline{G})$, where I denote the identity operator.*
2. *If S of class (S_+), then $SoT \in \mathcal{F}_T(\overline{G})$.*

Definition 2.5. Let $F, S \in \mathcal{F}_T(\overline{G})$ and let G be a bounded open subset of a real reflexive Banach space X. The affine homotopy $H : [0,1] \times \overline{G} \to X$ given by

$$H(\lambda, u) := (1-\lambda)Fu + \lambda Su, \text{ for } (\lambda, u) \in [0,1] \times \overline{G}$$

is called an admissible affine homotopy with the continuous essential inner map T.

Remark 2.6. [5] The above affine homotopy satisfies condition (S_+).

Now, we introduce the Berkovits topological degree for the class $\mathcal{F}_B(X)$. For more details see [5].

Theorem 2.7. *There exists a unique degree function*

$$\deg_B : \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G}), F \in \mathcal{F}_{T,B}(\overline{G}), h \notin F(\partial G)\} \to \mathbb{Z}$$

that satisfies the following properties:

1. *(Existence) If* $\deg_B(F, G, h) \neq 0$*, then the equation* $Fu = h$ *has a solution in* G*.*
2. *(Normalization) For any* $h \in G$*, we have* $\deg_B(I, G, h) = 1$*.*
3. *(Additivity) Let* $F \in \mathcal{F}_{T,B}(\overline{G})$*. If* G_1 *and* G_2 *are two disjoint open subsets of* G *such that* $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$*, then we have*
$$\deg_B(F, G, h) = \deg_B(F, G_1, h) + \deg_B(F, G2, h).$$
4. *(Homotopy invariance) If* $H : [0,1] \times \overline{G} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* $h :: [0,1] \times X$ *is a continuous path in* X *such that* $h(\lambda) \notin H(\lambda, \partial G)$ *for all* $\lambda \in [0,1]$*, then the value of* $\deg_B(H(\lambda, \cdot), G, h(\lambda))$ *is constant for all* $\lambda \in [0,1]$

2.2. Notation and preliminary results

In order to solve the problem (1.1), we need some necessary properties on variable exponent spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For a deeper treatment on these spaces, we refer to [12, 14, 15, 17, 19, 21], and the references therein.

In the sequel, we consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ with a Lipschitz boundary $\partial\Omega$ and the set

$$C_+(\overline{\Omega}) = \{g \in C(\overline{\Omega}) \mid \inf_{x \in \overline{\Omega}} g(x) > 1\},$$

$$g^- = \min_{x \in \overline{\Omega}} g(x), \quad g^+ = \max_{x \in \overline{\Omega}} g(x), \quad \text{for any } g \in C_+(\overline{\Omega}).$$

For any $p \in C_+(\overline{\Omega})$, we define the generalized Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \Big\{u \mid u : \Omega \to \mathbb{R} \text{ is a measurable function, } \rho_{p(x)}(u) < \infty\Big\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx,$$

this space endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf\{\lambda > 0 \mid \rho_{p(x)}(\frac{u}{\lambda}) \leq 1\},$$

and $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ becomes a Banach space.

Lemma 2.8. [21]

1. *The space* $L^{p(x)}(\Omega)$ *is a separable and reflexive Banach space.*
2. *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{p'(x)}(\Omega)$, *where* $1/p(x) + 1/p'(x) = 1$. *Then for any* $u \in L^{p(x)}(\Omega)$ *and* $w \in L^{p'(x)}(\Omega)$, *we have the following Hölder inequality*
$$\Big|\int_\Omega uwdx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(x)}\|w\|_{p'(x)} \le 2\|u\|_{p(x)}\|w\|_{p'(x)}.$$
3. *If* $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ *for any* $x \in \overline{\Omega}$, *then there exists the continuous embedding* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$

Lemma 2.9. [19, 33] *If* $u, u_n \in L^{p(x)}(\Omega)$, *then the following assertions hold true:*

1. $\|u\|_{p(x)} < 1\ (= 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (= 1, > 1)$.
2. $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^-}$.
3. $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^+}$.
4. $\lim\limits_{n\to\infty} \|u_n - u\|_{p(x)} = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.
5. $\|u\|_{p(x)} \le \rho_{p(x)}(u) + 1$.
6. $\rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}$.

Now, we define the usual Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$ as
$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$
whose norm is defined as
$$\|u\|_{W^{1,p(x)}} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \tag{2.1}$$
Let $W_0^{1,p(x)}(\Omega)$ denote the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1).

Lemma 2.10. [12, 19, 21]

1. *The two spaces* $W_0^{1,p(x)}(\Omega)$ *and* $W^{1,p(x)}(\Omega)$ *are a Banach spaces separable and reflexive.*
2. *If* $p(x)$ *satisfies the log-Hölder continuity condition, i.e., there is a constant* $\alpha > 0$ *such that for every* $x, y \in \Omega, x \notin y$ *with* $|x - y| \le \frac{1}{2}$ *one has*
$$|p(x) - p(y)| \le \frac{\alpha}{-\log|x - y|},$$
then there exists a constant $C > 0$, *such that*
$$\|u\|_{p(x)} \le C\|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$
3. *If* $p \in C_+(\overline{\Omega})$ *for any* $x \in \overline{\Omega}$, *then the imbedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ *is compact.*

Remark 2.11. By (2) of Lemma 2.10, we know that $\|\nabla u\|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

The dual space of $W_0^{1,p(x)}(\Omega)$ is $W^{-1,p'(x)}(\Omega)$, which endowed with the norm

$$\|w\|_{-1,p'(x)} = \inf\Big\{\|w_0\|_{p'(x)} + \sum_{i=1}^{N}\|w_i\|_{p'(x)}\Big\},$$

where the infinimum is taken on all possible decompositions $w = w_0 - divF$ with $w_0 \in L^{p'(x)}(\Omega)$ and $F = (w_1, \cdots, w_N) \in (L^{p'(x)}(\Omega))^N$.

Let us define $V = W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ endowed with the norm $\|(u,w)\|_V = \max(\|u\|_{1,p(x)}, \|w\|_{1,q(x)})$ where $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$ and $(V, \|\cdot\|)$ is a Banach space, separable and reflexive.

2.3. Properties of $(p(x), q(x))$-Laplacian operators

In the present subsection, we discuss the properties of $(p(x), q(x))$-Laplacian operators

$$-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u),$$

and

$$-\Delta_{q(x)}w = -\text{div}(|\nabla w|^{q(x)-2}\nabla w).$$

We consider the following functional:

$$\mathcal{J}(u,w) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|\nabla w|^{q(x)}}{q(x)}dx.$$

It is well known that $\mathcal{J} \in C^1(V, \mathbb{R})$ and for any $(\varphi, \phi) \in V$

$$\langle \mathcal{J}'(u,w), (\varphi, \phi)\rangle$$
$$= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi dx + \int_\Omega |\nabla w|^{q(x)-2}\nabla w\nabla\phi dx, \quad \forall u, w \in V.$$

Denote $M = \mathcal{J}' : V \to V^*$.

Theorem 2.12. [18]

1. *$M : V \to V^*$ is a mapping of type (S_+).*
2. *$M : V \to V^*$ is a continuous, bounded and strictly monotone operator.*
3. *$M : V \to V^*$ is a homeomorphism.*

The proof of the above theorem can be found in [18].

3. Hypotheses and the main results

3.1. Hypotheses

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary $\partial\Omega$. Let $p, q \in C_+(\bar{\Omega})$, $1 < p^- \leq p(x) \leq p^+ < \infty$, $1 < q^- \leq q(x) \leq q^+ < \infty$ and $f, h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are a real-valued functions such that

(A_1). (Continuity) f, h are the Carathéodory functions (i.e., $f(x, \cdot, \cdot)$ is continuous in (s_1, s_2) for almost every $x \in \Omega$ and $f(\cdot, s_1, s_2)$ is measurable in x for each $(s_1, s_2) \in \mathbb{R} \times \mathbb{R}^N$)

(A_2). (Growth) There exist a positive constants c_1, c_2, $b \in L^{p'(x)}(\Omega), d \in L^{q'(x)}(\Omega)$ and $1 < \alpha^- \leq \alpha(x) \leq \alpha^+ < p^-$, $1 < \beta^- \leq \beta(x) \leq \beta^+ < q^-$, such that

$$|f(x, s_1, s_2)| \leq c_1(b(x) + |s_1|^{\alpha(x)-1} + |s_2|^{\alpha(x)-1}),$$
$$|h(x, \xi_1, \xi_2)| \leq c_2(d(x) + |\xi_1|^{\beta(x)-1} + |\xi_2|^{\beta(x)-1}).$$

3.2. Main results

The main tool that we shall use to prove the existence of weak solutions of the problem (1.1) is the degree theory introduced in section 2.

Definition 3.1. We say that $(u, w) \in V$ is a distributional solution of the system (1.1) if for any $(\varphi, \phi) \in V$ we have

$$\begin{aligned} &\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx + \int_\Omega |\nabla w|^{q(x)-2} \nabla w \nabla \phi dx \\ &= \int_\Omega f(x, w, \nabla w) \varphi dx + \int_\Omega h(x, u, \nabla u) \phi dx \end{aligned} \tag{3.1}$$

Lemma 3.2. *Assume that (A_1) and (A_2) hold. Then the operator $T : V \to V^*$ given by*

$$\begin{cases} (u, w) \in V, \\ \langle T(u, w), (\varphi, \phi) \rangle = -\int_\Omega f(x, w, \nabla w) \varphi dx - \int_\Omega h(x, u, \nabla u) \phi dx, \ \forall (\varphi, \phi) \in V \end{cases}$$

is compact.

Proof. First, let $\chi : W_0^{1,p(x)} \to L^{p'(x)}(\Omega)$, $\pi : W_0^{1,q(x)} \to L^{q'(x)}(\Omega)$ be two operators defined by

$$\chi u(x) = -h(x, u, \nabla u) \ \text{ for } u \in W_0^{1,p(x)} \ \text{ and } x \in \Omega,$$

and

$$\pi w(x) = -f(x, w, \nabla w) \ \text{ for } w \in W_0^{1,q(x)} \ \text{ and } x \in \Omega.$$

We divide the proof into three steps.

Step 1. We show that χ and π are bounded.

For each $u \in W_0^{1,p(x)}(\Omega)$, we have by (5), (6) of Lemma 2.9 and the assumption (A_2) that

$$\begin{aligned} \|\chi u\|_{p'(x)} &\leq \rho_{p'(x)}(\chi u) + 1 \\ &= \int_\Omega |h(x, u(x), \nabla u(x))|^{p'(x)} + 1 \\ &\leq \text{const}\Big(\int_\Omega \Big(|d| + |u|^{\beta(x)-1} + |\nabla u|^{\beta(x)-1} \Big)^{p'(x)} dx \Big) \\ &\leq \text{const}\Big(\rho_{p'(x)}(d) + \rho_{\gamma(x)}(u) + \rho_{\gamma(x)}(\nabla u) \Big) + 1 \\ &\leq \text{const}\Big(\|d\|_{p'(x)}^{p'^-} + \|d\|_{p'(x)}^{p'^+} + \|u\|_{\gamma(x)}^{\gamma^-} + \|u\|_{\gamma(x)}^{\gamma^+} + \|\nabla u\|_{\gamma(x)}^{\gamma^-} \\ &\quad + \|\nabla u\|_{\gamma(x)}^{\gamma^+} \Big) + 1, \end{aligned}$$

where

$$\gamma(x) = (\beta(x) - 1)p'(x) < p(x).$$

By (2) of Lemma 2.10 and the continuous embedding $L^{p(x)} \hookrightarrow L^{\gamma(x)}$, we get

$$\|\chi u\|_{p'(x)} \leq \text{ const}\Big(\|d\|_{p'(x)}^{p'^-} + \|d\|_{p'(x)}^{p'^+} + \|u\|_{1,p(x)}^{\gamma^-} + \|u\|_{1,p(x)}^{\gamma^+}\Big) + 1,$$

which implies that χ is bounded on $W_0^{1,p(x)}$.
Similarly, we can show that π is bounded on $W_0^{1,q(x)}$.

Step 2. We show that χ and π are continuous.

Let (u_n, w_n) converge to (u, w) in V. Then

$$u_n \to u \text{ and } \nabla u_n \to \nabla u \text{ in } W_0^{1,p(x)},$$

$$w_n \to w \text{ and } \nabla w_n \to \nabla w \text{ in } W_0^{1,q(x)}.$$

Hence there exist two subsequences denote again by (u_n), (w_n) and measurable functions g_1 (resp. g_2) in $L^{p(x)}(\Omega)$ (resp. in $L^{q(x)}(\Omega)$) and g_1^* (resp.g_2^*) in $(L^{p(x)}(\Omega))^N$ (resp. in $(L^{q(x)}(\Omega))^N$), such that

$$u_n(x) \to u(x) \text{ and } \nabla u_n(x) \to \nabla u(x),$$

$$w_n(x) \to w(x) \text{ and } \nabla w_n(x) \to \nabla w(x),$$

$$|u_n(x)| \leq g_1(x),\ |\nabla u_n(x)| \leq |g_1^*(x)|$$

and

$$|w_n(x)| \leq g_2(x),\ |\nabla w_n(x)| \leq |g_2^*(x)|,$$

for almost all $x \in \Omega$ and all $n \in N$. From (A_1) and (A_2), we have

$$h(x, u_n(x), \nabla u_n(x)) \to h(x, u(x), \nabla u(x)) \text{ for almost all } x \in \Omega,$$

and

$$|h(x, u_n(x), \nabla u_n(x))| \leq \text{ const}\Big(d(x) + |g_1(x)|^{\beta(x)-1} + |g_1^*(x)|^{\beta(x)-1}\Big),$$

for almost all $x \in \Omega$ and all $n \in N$ and

$$d + |g_1|^{\beta(x)-1} + |g_1^*|^{\beta(x)-1} \in L^{p'(x)}(\Omega).$$

Taking into account the equality

$$\rho_{p'(x)}(\chi u_n - \chi u) = \int_\Omega |h(x, u_n(x), \nabla u_n(x)) - h(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

the equivalence (4) of Lemma 2.9 and the Lebesgue dominated convergence theorem imply that

$$\chi u_n \to \chi u \text{ in } L^{p'(x)}(\Omega),$$

which shows that the entire sequence (χu_n) is continuous.
Similarly, we obtain that the entire sequence (πw_n) is continuous.

Step 3. As the embedding $\mathcal{I} : V \to U$ is compact, it is known that the adjoint operator $\mathcal{I}^* : U^* \to V^*$is also compact. So the compositions $\mathcal{I}^* o \chi$ and $\mathcal{I}^* o \pi : V \to V^*$ are compact, which completes the proof. $\square$

Let us now mention our main result in this paper:

Theorem 3.3. *Under conditions* (A_1) *and* (A_2), *problem* (1.1) *has a distributional solution* (u, w) *in* V.

Proof. Let T be an operator from V into its dual V^* as defined in Lemma 3.2, ant let $M : V \to V^*$, as in subsection 2.3, given by

$$\begin{cases} (u, w) \in V, \\ \langle M(u, w), (\varphi, \phi)\rangle = \displaystyle\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \varphi dx + \int_\Omega |\nabla w|^{q(x)-2}\nabla w \nabla \phi dx, \end{cases}$$

for all $(\varphi, \phi) \in V$. Then $(u, w) \in V$ is a distributional solution of (1.1) if and only if

$$M(u, w) = -T(u, w). \tag{3.2}$$

Thanks to Lemma 3.2, the operator T is bounded, continuous and quasimonotone. On the other hand, according to the properties of the operator M seen in Theorem 2.12 and by using the Minty-Browder Theorem (see [32], Theorem 26A), the inverse operator $N = M^{-1} : V^* \to V$ is bounded, continuous and satisfies condition (S_+).

Therefore, equation (3.2) is equivalent to

$$(u, w) = N(\varphi, \phi) \text{ and } (\varphi, \phi) + ToN(\varphi, \phi) = 0. \tag{3.3}$$

To solve (3.3), we shall using the degree theory introduced in subsection 2.1. For this, we first show that the set

$$\Sigma = \{(\varphi, \phi) \in V^* | (\varphi, \phi) + \lambda ToN(\varphi, \phi) = 0 \text{ for some } \lambda \in [0, 1]\}$$

is bounded. Indeed, let $(\varphi, \phi) \in \Sigma$ and take $(u, w) = N(\varphi, \phi)$, then

$$\|N(\varphi, \phi)\|_V = \|(u, w)\|_V = \max(\|\nabla u\|_{p(x)}, \|\nabla w\|_{q(x)}).$$

If $\|\nabla u\|_{p(x)} \leq 1$ and $\|\nabla w\|_{q(x)} \leq 1$, then $\|N(\varphi, \phi)\|_V$ is bounded.

If $\|\nabla u\|_{p(x)} > 1$ and $\|\nabla w\|_{q(x)} > 1$, then by using the assumption (A_2), (3), (6) of Lemma 2.9, (2) of Lemma 2.8 and the Young inequality, we obtain the estimate

$$\begin{aligned}
\|N(\varphi,\phi)\|_V^{\min(p^-,q^-)} &= \|(u,w)\|_V^{\min(p^-,q^-)} \\
&\leq \rho_{p(x)}(\nabla u)+\rho_{q(x)}(\nabla w) \\
&= \langle M(u,w),(u,w)\rangle \\
&= \langle(\varphi,\phi),N(\varphi,\phi)\rangle \\
&= -\lambda\langle ToN(\varphi,\phi),N(\varphi,\phi)\rangle \\
&= \lambda\Big(\int_\Omega f(x,w,\nabla w)udx+\int_\Omega h(x,u,\nabla u)wdx\Big) \\
&\leq \text{const}\ \Big(\|b\|_{p'(x)}\|u\|_{p(x)}+\frac{1}{\alpha'^-}\rho_{\alpha(x)}(w)+\frac{1}{\alpha^-}\rho_{\alpha(x)}(u) \\
&\quad +\frac{1}{\alpha'^-}\rho_{\alpha(x)}(\nabla w)+\frac{1}{\alpha^-}\rho_{\alpha(x)}(u)+\|d\|_{q'(x)}\|w\|_{q(x)} \\
&\quad +\frac{1}{\beta'^-}\rho_{\beta(x)}(u)+\frac{1}{\beta^-}\rho_{\beta(x)}(w)+\frac{1}{\beta'^-}\rho_{\beta(x)}(\nabla u) \\
&\quad +\frac{1}{\beta^-}\rho_{\beta(x)}(w)\Big) \\
&\leq \text{const}\ \Big(\|u\|_{p(x)}+\|w\|_{\alpha(x)}^{\alpha^+}+\|u\|_{\alpha(x)}^{\alpha^+}+\|\nabla w\|_{\alpha(x)}^{\alpha^+} \\
&\quad +\|w\|_{q(x)}+\|u\|_{\beta(x)}^{\beta^+}+\|w\|_{\beta(x)}^{\beta^+}+\|\nabla u\|_{\beta(x)}^{\beta^+}\Big).
\end{aligned}$$

By (2) of Lemma 2.10 and the continuous embedding $L^{p(x)} \hookrightarrow L^{\alpha(x)}$ and $L^{q(x)} \hookrightarrow L^{\beta(x)}$, we get

$$\|N(\varphi,\phi)\|_V^{\min(p^-,q^-)} \leq \text{const}\ (\|N(\varphi,\phi)\|_V+\|N(\varphi,\phi)\|_V^{\max(\alpha^+,\beta^+)}).$$

If $\|\nabla u\|_{p(x)} > 1$ and $\|\nabla w\|_{q(x)} \leq 1$ (resp. if $\|\nabla u\|_{p(x)} \leq 1$ and $\|\nabla w\|_{q(x)} > 1$), we can also get that $\|N(\varphi,\phi)\|_V$ is bounded.
Consequently $\{N(\varphi,\phi)|(\varphi,\phi)\in\Sigma\}$ is bounded.

Since the operator T is bounded, it is obvious from (3.3) that the set Σ is bounded in V^*. Hence, we can choose a positive constant R such that

$$\|(\varphi,\phi)\|_{V^*} < R \ \text{ for all } (\varphi,\phi)\in\Sigma.$$

It follows that

$$(\varphi,\phi)+\lambda ToN(\varphi,\phi)\neq 0 \ \text{ for all } (\varphi,\phi)\in\partial B_R(0) \ \text{ and all } \lambda\in[0,1],$$

where $B_R(0)$ is the ball of radius R and center 0 in V^*.
By Lemma 2.4, we have

$$I+ToN\in\mathcal{F}_T(\overline{B_R(0)}) \ \text{ and } I=MoN\in\mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I,T and N are bounded, $I+ToN$ is also bounded. We conclude that

$$I+ToN\in\mathcal{F}_{T,B}(\overline{B_R(0)}) \ \text{ and } I\in\mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Now, we consider an affine homotopy $H:[0,1]\times\overline{B_R(0)}\to V^*$ given by

$$H(\lambda,\varphi,\phi):=(\varphi,\phi)+\lambda ToN(\varphi,\phi) \ \text{ for } (\lambda,\varphi,\phi)\in[0,1]\times\overline{B_R(0)}.$$

All those properties allow us to apply the homotopy invariance and normalization property of the degree $\deg_B$ stated in Theorem 2.7 and obtain

$$\deg_B(I + ToN, B_R(0), 0) = \deg_B(I, B_R(0), 0) = 1,$$

consequently, there exists a point $(\varphi, \phi) \in B_R(0)$ such that

$$(\varphi, \phi) + ToN(\varphi, \phi) = 0.$$

This implies that $(u, w) = N(\varphi, \phi)$ is a distributional solution of (1.1). The proof is complete. □

4. Conclusion

In this paper, we have studied the existence of distributional solutions for a nonlinear elliptic systems with variable exponents. By using the topological degree theory, we showed that system (1.1) has at least one solutions when the functions f and h satisfying some suitable conditions. This study can be extend in the futur works to more general boundary value problems involving fractional derivatives models.

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Overiteration of d-variate tensor product Bernstein operators: A quantitative result

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Dedicated to the memory of Professor Sorin Gal.

Abstract. Extending an earlier estimate for the degree of approximation of overiterated univariate Bernstein operators towards the same operator of degree one, it is shown that an analogous result holds in the d-variate case. The method employed can be carried over to many other cases and is not restricted to Bernstein-type or similar methods.

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1. Introduction and historical remarks

The question behind this note is well-known. What is a classical Bernstein operator doing if its powers are raised to infinity?

For the univariate version of this operator the answer is known. Already in 1966 the Dutch mathematician P.C. Sikkema proved in the Romanian journal Mathematica (Cluj) that for each function $f \in \mathbb{R}^{[0,1]}$ the powers $B_n^k f$, n fixed, $k \to \infty$ converge to the linear function interpolating f at 0 and 1 (see [15]). Later on his result become known as the Kelisky-Rivlin (1967) or Karlin-Ziegler (1970) theorem (cf. [10, 9]).

However, even earlier T. Popoviciu [12] posed this problem in an (informal) problem book of 1955. We learned this from the note [3] by Albu cited by Precup [13]. The latter author also deals with multivariate operators but from a different point of view.

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Some notation is needed here. For $x \in [0,1]$, $n \geq 1$, and $f \in \mathbb{R}^{[0,1]}$ the Bernstein operator is given by

$$\begin{aligned} B_n(f,x) &:= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \\ &:= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}. \end{aligned}$$

Thus B_n is a polynomial operator, is linear and positive, reproduces all affine linear functions $l(x) = ax + b$, and for each f the polynomial $B_n f$ is of degree $\leq n$.

Moreover, for any $k, n \in \mathbb{N}$, Gonska et al. [6] proved in 2006, extending earlier work of Nagel [11] and Gonska [4],

$$|B_n^k(f,x) - B_1(f,x) \leq \frac{9}{2}\omega_2\left(f; \sqrt{x(1-x)\left(1-\frac{1}{n}\right)^k}\right), x \in [0,1]. \tag{1.1}$$

Here $\omega_2(f;\cdot)$ is the classical second order modulus of f. Hence the right hand side converges to 0 as n is fixed and $k \to \infty$ (some more general situations are possible). It also shows that the powers are interpolatory at 0 and 1 and keep reproducing linear functions. Moreover, the convergence is uniform with respect to $\|\cdot\|_\infty$.

When it comes to multivariate Bernstein operators, all the time operators on generalized simplices or hypercubes are meant. While for simplices the convergence of powers was investigated by, e.g., Wenz [16] and many others, the hypercube case remained allegedly open until a 2009 article of Jachymski [8] appeared. However, for the bivariate case a paper by Agratini and Rus was published already in 2003, cf. [2].

In this note we will use the term tensor product although in other publications one might find 'product of parametric extensions' meaning exactly the same (see, e.g., [5]).

Using functional-analytic methods Jachymski showed the following. For $l, m \geq 1$ let the bivariate tensor product operator

$$((B_l \otimes B_m)f)(x,y) := ({}_sB_l \circ {}_tB_m)(f(s,t); x, y)$$

be given by

$$\sum_{i=0}^{l}\sum_{j=0}^{m} f\left(\frac{i}{l}, \frac{j}{m}\right) p_{l,i}(x) \cdot p_{m,j}(y),\ f \in C([0,1]^2),\ x, y \in [0,1].$$

Theorem A. *For any $l, m \in \mathbb{N}$ fixed, the sequence $((B_l \otimes B_m)^n)_{n\in\mathbb{N}}$ uniformly converges to the operator L (independent of l and m) given by the following formula for $f \in C([0,1]^2)$ and $x, y \in [0,1]$:*

$$\begin{aligned} (Lf)(x,y) &= f(0,0) + (f(1,0) - f(0,0))x + (f(0,1) - f(0,0))y \\ &\quad + (f(0,0) + f(1,1) - f(1,0) - f(0,1))xy \\ &= (1-x, x)\begin{pmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{pmatrix}\binom{1-y}{y}. \end{aligned}$$

In other words, $Lf = (B_1 \otimes B_1)f$.

Jachymski [8] also gave the limit of n-powers of d-variate Bernstein operators, i.e., of

$$((B_{l_1} \otimes \cdots \otimes B_{l_d})f)(x_1, \ldots, x_d)$$
$$= ({}_{s_1}B_{l_1} \circ \cdots \circ {}_{s_d}B_{l_d})(f(s_1, \ldots, s_d); x_1, \ldots, x_d).$$

They map $C([0,1]^d)$ into $\Pi_{l_1,\ldots,l_d}$, the space of d-variate polynomials of total degree $\leq \sum_{\delta=1}^{d} l_\delta$.

The limiting operator in this case is

$$(Lf)(x_1, \ldots, x_d) = \sum_{(\epsilon_1,\ldots,\epsilon_d)\in V} f(\epsilon_1, \ldots, \epsilon_d)p_{\epsilon_1}(x_1) \cdot \cdots \cdot p_{\epsilon_d}(x_d),$$

where $V = \{0,1\}^{\{1,\ldots,d\}}$, and for $s \in [0,1]$, $p_0(s) := 1 - s$ and $p_1(s) := s$. Thus L equals $B_1 \otimes ... \otimes B_1$.

In the present note we will show first that the fixpoint approach of (Agratini and) Rus also works in the d-variate case. Our main emphasis is on the quantitative situation where we will demonstrate how the pointwise ω_2-result may be carried over to d dimensions.

2. The non-quantitative approach of Agratini and Rus revisited

As mentioned above, Jachymski used a functional-analytic framework to derive his result. Here we show that a more elementary approach does the job as well. We recall the three papers by Rus and Agratini & Rus and present their approach for d dimensions.

Some reminders concerning d-variate hypercubes are in order. More details are available in the German Wikipedia, keyword "Hyperwrfel" [17]. Such a hypercube in d dimensions possesses $\binom{d}{0}2^{d-0} = 2^d$ 0-dimensional boundary elements (vertices), in the bivariate case these are the 4 corners of $[0,1]^2$. Adopting the above notation these are all d-tuples

$$(\epsilon_1, \ldots, \epsilon_d) \in V, \; V = \{0,1\}^{\{1,\ldots,d\}}.$$

We will now follow Rus' proof of his Theorem 1. First introduce the sets

$$X_{\alpha_1,\ldots,\alpha_d} = \{f \in C([0,1]^d) \, : \, f(\epsilon_1) = \alpha_1, \ldots, f(\epsilon_d) = \alpha_d\},$$

$(\epsilon_1, \ldots, \epsilon_d) \in V$, $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$. Note that

(a) $X_{\alpha_1,\ldots,\alpha_d}$ is a closed subset of $C([0,1]^d)$;
(b) $X_{\alpha_1,\ldots,\alpha_d}$ is an invariant subset of $B_{l_1} \otimes \cdots \otimes B_{l_d}$, for all $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$ and $l_1, \ldots, l_d \in \mathbb{N}$;
(c) $C([0,1]^d) = \bigcup_{\alpha_1,\ldots,\alpha_d\in\mathbb{R}} X_{\alpha_1,\ldots,\alpha_d}$ is a partition of $C([0,1]^d)$.

Next it is shown that

$$(B_{l_1} \otimes \cdots \otimes B_{l_d})|_{X_{\alpha_1,\ldots,\alpha_d}}$$

maps $X_{\alpha_1,\ldots,\alpha_d}$ onto itself and is a contraction.

For $f, g \in X_{\alpha_1,\ldots,\alpha_d}$ we have

$$\begin{aligned}
&|((B_{l_1} \otimes \cdots \otimes B_{l_d})f)(x_1,\ldots,x_d) - ((B_{l_1} \otimes \cdots \otimes B_{l_d})g)(x_1,\ldots,x_d)| \\
&= \left| \sum_{\lambda_1=0}^{l_1} \cdots \sum_{\lambda_d=0}^{l_d} (f-g)\left(\frac{\lambda_1}{l_1},\ldots,\frac{\lambda_d}{l_d}\right) p_{l_1,\lambda_1}(x_1)\cdot\cdots\cdot p_{l_d,\lambda_d}(x_d)\right| \leq \\
&\sum_{(\lambda_1,\ldots,\lambda_d)\in\{0,\ldots,l_1\}\times\cdots\times\{0,\ldots,l_d\}\setminus V} \left|(f-g)\left(\frac{\lambda_1}{l_1},\ldots,\frac{\lambda_d}{l_d}\right) p_{l_1,\lambda_1}(x_1)\cdot\cdots\cdot p_{l_d,\lambda_d}(x_d)\right| \\
&\leq \|f-g\|_\infty \sum_{(\lambda_1,\ldots,\lambda_d)\in\{0,\ldots,l_1\}\times\cdots\times\{0,\ldots,l_d\}\setminus V} p_{l_1,\lambda_1}(x_1)\cdot\cdots\cdot p_{l_d,\lambda_d}(x_d) \\
&\leq \|f-g\|_\infty \left(1 - \min \sum_{(\lambda_1,\ldots,\lambda_d)\in V} p_{l_1,\lambda_1}(x_1)\cdot\cdots\cdot p_{l_d,\lambda_d}(x_d)\right) \\
&= \|f-g\|_\infty \cdot \left(1 - \min\left\{\left[(1-x_1)^{l_1} + x_1^{l_1}\right]\cdot\cdots\cdot\left[(1-x_d)^{l_d} + x_d^{l_d}\right]\right\}\right) \\
&\leq \|f-g\|_\infty \cdot \left(1 - \frac{1}{\prod_{\delta=1}^d 2^{l_\delta - 1}}\right) < 1.
\end{aligned}$$

Thus $B_{l_1} \otimes \cdots \otimes B_{l_d}$ on $X_{\alpha_1,\ldots,\alpha_d}$ is a contraction for all $\alpha_1,\ldots,\alpha_d \in \mathbb{R}$. On the other hand, $(Lf)(x_1,\ldots,x_d)$ is a fixed point of $B_{l_1} \otimes \cdots \otimes B_{l_d}$.

So $f \in C([0,1]^d)$ is in $X_{f(\epsilon_1),\ldots,f(\epsilon_d)}$ and from the contraction principle we have

$$\lim_{n\to\infty} (B_{l_1} \otimes \cdots \otimes B_{l_d})^n f = Lf.$$

We summarize our observation in

Theorem 2.1. *(Jachymski [8]) For fixed $l_1,\ldots,l_d \in \mathbb{N} = \{1,2,\ldots\}$ one has*

$$\lim_{n\to\infty} (B_{l_1} \otimes \cdots \otimes B_{l_d})^n f = Lf \text{ uniformly.}$$

Here $B_{l_1} \otimes \cdots \otimes B_{l_d}$ is the d-variate tensor product operator on $C([0,1]^d)$ and

$$(Lf)(x_1,\ldots,x_d) = \sum_{(\epsilon_1,\ldots,\epsilon_d)} f(\epsilon_1,\ldots,\epsilon_d) p_{\epsilon_1}(x_1)\cdot\cdots\cdot p_{\epsilon_d}(x_d),\ V = \{0,1\}^{\{1,\ldots,d\}}.$$

In particular, for $d = 2$ we have the representation of Theorem A.

3. The Zhuk extension in the bi- and d-variate cases

Since the articles of Zhuk [18] and Gonska & Kovacheva [7] are hard to obtain, we briefly describe the extension in the univariate situation, then carry it over to the bivariate case and finally show what has to be done in d variables.

3.1. Zhuk construction-univariate case

For $f \in C[0,1]$ and $0 < h \leq \frac{1}{2}(b-a)$ define $f_h : [a-h, b+h] \to \mathbb{R}$ by

$$f_h(x) := \begin{cases} P_-(x), \ a-h \leq x < a, \\ f(x), \ a \leq x \leq b, \\ P_+(x), \ b < x \leq a+h. \end{cases}$$

$$\|f - P_-\|_{C[a,a+2h]} = E_1(f; a, a+2h),$$
$$\|f - P_+\|_{C[b-2h,b]} = E_1(f; b-2h, b).$$

Here P_- and P_+ denote the best approximations in Π_1 on the intervals indicated and with respect to the uniform norm.

Zhuk put

$$S_h(f; x) := \frac{1}{h} \int_{-h}^{h} \left(1 - \frac{|t|}{h}\right) f_h(x+t) dt, \ x \in [a,b].$$

He showed [18, Lemma 1]: For $f \in C[a,b], 0 < h \leq \frac{1}{2}(b-a)$,

$$\|f - S_h f\|_\infty \leq \frac{3}{4}\omega_2(f; h),$$
$$\|(S_h f)''\|_{L_\infty[a,b]} \leq \frac{3}{2} h^{-2}\omega_2(f; h).$$

3.2. Construction of the bivariate Zhuk extension

Let $f \in C([0,1]^2)$. On a fixed y-level we extend the partial function $f_y(x) = f(\cdot, y)$ from $[0,1] \times \{y\}$ to $[-h, 1+h] \times \{y\}$ in complete analogy to the univariate case. After integration, for each $y \in [0,1]$, we obtain

$$S_h(f_y; x) := \frac{1}{h} \int_{-h}^{h} \left(1 - \frac{|t|}{h}\right) (f_y)_h(x+t) dt, \ \ x \in [0,1],$$

satisfying for $0 < h \leq \frac{1}{2}$:

$$\|f_y - S_h f_y\|_\infty \leq \frac{3}{4}\omega_2(f_y; h),$$
$$\|(S_h f_y)''\|_{L_\infty[0,1]} \leq \frac{3}{2} h^{-2}\omega_2(f_y; h).$$

(On each y-level we could have even chosen h_y with $0 < h_y \leq \frac{1}{2}$).

The same procedure we carry out for $f_x(y), y \in [0,1]$, producing functions $S_h f_x$ such that

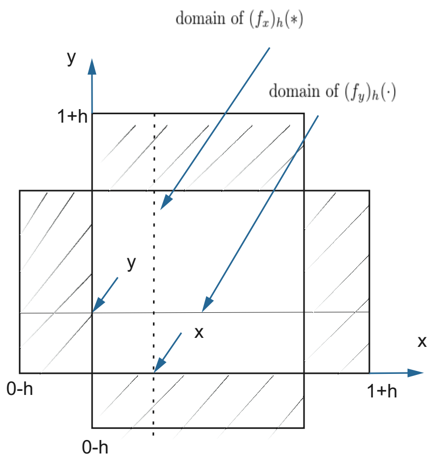
$$\|f_x - S_h f_x\|_C \leq \frac{3}{4}\omega_2(f_x; h),$$
$$\|(S_h f_x)''\|_{L_\infty[0,1]} \leq \frac{3}{2} h^{-2}\omega_2(f_x; h).$$

This can be done for all $x \in [0,1]$.
More explicitly,

$$\begin{aligned}\omega_2(f_y;h) &= \sup\{|f_y(x-\delta)-2f_y(x)+f_y(x+\delta)| : |\delta|\le h, x\pm\delta\in[0,1]\}\\ &= \sup\{|f(x-\delta,y)-2f(x,y)+f(x+\delta,y)| : |\delta|\le h, x\pm\delta\in[0,1]\}\\ &\le \sup_{y\in[0,1]}\sup\{|f(x-\delta,y)-2f(x,y)+f(x+\delta,y)| : |\delta|\le h, x\pm\delta\in[0,1]\}\\ &= \omega_2(f;h,0).\end{aligned}$$

Also, $\omega_2(f_x;h)\le\omega_2(f;0,h)$.

The quantities $\omega_2(f;h,0)$ and $\omega_2(f;0,h)$ are called "partial moduli of smoothness". We have thus constructed auxiliary extensions of $f_y(\cdot)$, $y\in[0,1]$, and $f_x(*)$, $x\in[0,1]$, on the domain shown below



$S_h(f_y;\cdot)$ and $S_h(f_x;*)$ are given on the inner (white) square only.

3.3. Zhuk extension, d-variate case

The construction described for the bivariate case can be easily generalized for $d\ge 3$ dimensions. To this end fix $d-1\ge 2$ variables, say $s_2,\dots,s_d$. Then extend the partial function $f_{s_2,\dots,s_d}(s_1)$, $0\le s_1\le 1$, to $-h\le s_1\le 1+h$, $0<h\le\frac{1}{2}$, and define

$$S_h(f_{s_2,\dots,s_d})(s_1) := \frac{1}{h}\int_{-h}^{h}\left(1-\frac{|t|}{h}\right)\cdot\left(f_{s_2,\dots,s_d}\right)_h(s_1+t)dt.$$

This gives

$$\|f_{s_2,\dots,s_d}-S_hf_{s_2,\dots,s_d}\|_\infty \le \frac{3}{4}\omega_2(f_{s_2,\dots,s_d};h),$$
$$\|(S_hf_{s_2,\dots,s_d})''\|_{L_\infty[0,1]} \le \frac{3}{2}h^{-2}\omega_2(f_{s_2,\dots,s_d};h),$$

for each fixed $s_2,\dots,s_d\in[0,1]$. Moreover, a common upper bound is

$$\omega_2(f_{s_2},\dots,s_d;h)\le\omega_2(f;h,0,\dots,0),\ \text{for all } s_2,\dots,s_d\in[0,1],$$

and a corresponding inequality holds for any other choice of s_δ, $2\le\delta\le d$.

4. An estimate for d-variate tensor product Bernstein operators

We first recall our 2006 estimate for the univariate case:

$$|B_l^n(f;x) - B_1(f;x)| \le \frac{9}{4}\omega_2\left(f;\sqrt{x(1-x)\left(1-\frac{1}{l}\right)^n}\right).$$

In two dimensions, it can be easily derived that

$$\begin{gathered}|(B_l \otimes B_m)^n(f;x,y) - (B_1 \otimes B_1)(f;x,y)|\\ |[(B_l^n - B_1)\otimes(B_m^n - B_1)](f;x,y)|\\ \le \frac{9}{4}\left[\omega_2\left(f;\sqrt{x(1-x)\left(1-\frac{1}{l}\right)^n},0\right) + \omega_2\left(f;0,\sqrt{y(1-y)\left(1-\frac{1}{m}\right)^n}\right)\right].\end{gathered}$$

This extends to d dimensions. Here we have

$$\begin{gathered}|({}_{s_1}B_{l_1} \circ \cdots \circ {}_{s_d}B_{l_d})^n (f(s_1,\dots,s_d);x_1,\dots,x_d)\\ - ({}_{s_1}B_1 \circ \cdots \circ {}_{s_d}B_1)(f(s_1,\dots,s_d);x_1,\dots,x_d)|\\ \le \frac{9}{4}\sum_{\delta=1}^{d}\omega_2\left(f;0,\dots,0,\sqrt{x_\delta(1-x_\delta)\left(1-\frac{1}{l_\delta}\right)^n},0,\dots,0\right).\end{gathered}$$

For d dimensions it is, without additional effort, possible to show

$$\begin{gathered}\left|\left({}_{s_1}B_{l_1}^{n_1} \circ \cdots \circ {}_{s_d}B_{l_d}^{n_d}\right)(f(s_1,\dots,s_d);x_1,\dots,x_d)\right.\\ \left. - ({}_{s_1}B_1^{n_1} \circ \cdots \circ {}_{s_d}B_1^{n_d})(f(s_1,\dots,s_d);x_1,\dots,x_d)\right|\\ = \left[({}_{s_1}B_{l_1}^{n_1} -{}_{s_1} B_1) \circ \cdots \circ ({}_{s_d}B_{l_d}^{n_d} -{}_{s_d} B_1)\right](f(s_1,\dots,s_d);x_1,\dots,x_d)\\ \le \frac{9}{4}\sum_{\delta=1}^{d}\omega_2\left(f;0,\dots 0,\sqrt{x_\delta(1-x_\delta)\left(1-\frac{1}{l_\delta}\right)^{n_\delta}},0,\dots,0\right).\end{gathered}$$

Note that for $n = n_1 = \cdots = n_\delta$ the difference from above becomes

$$({}_{s_1}B_{l_1} \otimes \cdots \otimes {}_{s_d}B_{l_d})^n - ({}_{s_1}B_1 \otimes \cdots \otimes {}_{s_d}B_1)$$

and is this the multivariate quantity considered by Jachymski. However, there is no need to restrict oneself to this case.

5. Optimality

Questions are in order in how far our estimates are "optimal".

1. The constant $\frac{9}{4}$ appearing repeatedly in this note most likely is not. There is need for work in this direction.

2. If the function f is d-linear, then the sum of d ω_2 -terms equals zero. If the sum is zero, then each of its terms does so. This may occur if

(i) $(x_1,\dots,x_d)$ is at a 'corner' of the hypercube, and/or

(ii) l_δ, the degree of B_{l_δ}, is equal to 1 for $1 \le \delta \le d$.

In any other case f must be d-linear to fulfill the condition $\omega_2(f;\dots) = 0$ for all d terms and for an interior point of the hypercube while $l_\delta \ge 2$, $1 \le \delta \le d$.

From (i) and (ii) it is evident that the sum of $d\,\omega_2$-terms is the correct expression for tensor product Bernstein approximation over a (generalized) hypercube.

6. Concluding remark

It should have become clear that our, or a similar approach, may be used to prove analogous results for many other operator sequences (which different authors may consider). We feel that sums of partial moduli of smoothness are among the right tools for tensor product approximation since they show the mutual independence of the variables. Nonetheless, even better pointwise results are available but do not really contribute to a better understanding.

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Nonnegative solutions for a class of fourth order singular eigenvalue problems

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Abstract. In this paper, we discuss the existence of nonnegative solutions to a fourth order singular boundary value problem at two points. Our result is based on a recent Birkhoff-Kellogg type fixed point theorem developed on translates of a cone on a Banach space.

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Keywords: Fixed point, fourth-order boundary value problem, positive solution, cone.

1. Introduction

In the present paper, we investigate the following fourth order singular differential equation with parameter

$$v^{(4)} = \lambda g(t) f(v(t)), \quad 0 < t < 1, \tag{1.1}$$

subject to the boundary conditions

$$v(0) = a_1, \quad v(1) = a_2, \quad v''(0) = a_3, \quad v''(1) = a_4, \tag{1.2}$$

where $a_j \geq 0$, $j \in \{1, 2, 3, 4\}$, are given constants,

(H1). $f \in \mathcal{C}([0, \infty))$,

$$0 < A_1 \leq f(x) \leq A_2 + \sum_{j=0}^{k} B_j x^j, \quad x \in [0, \infty),$$

$A_2 \geq A_1 > 0$ and $B_j \geq 0$, $j \in \{0, \ldots, k\}$, $k \in \mathbb{N}_0$, are given constants.

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(H2). $g : (0, 1) \to \mathbb{R}^+$ is continuous and may be singular at $t = 0$ or/and $t = 1$, $g \not\equiv 0$ on $(0, 1)$ and $\int_0^1 s(1-s)g(s)ds < \infty$.

Fourth order two-point boundary value problems (BVPs for short) have been received much attention by many authors due to their importance in physics. Usually, they are essential in describing a vast class of elastic deflections with several types of boundary conditions such as whose ends are simply-supported at 0 and 1 ($v(0) = v(1) = v''(0) = v''(1) = 0$). A great number of research has been devoted to investigate the existence of positive solutions to this class of problems, see [2, 1, 3, 4, 8, 9, 10, 11, 12, 13] and the references therein. The authors in [2] discussed the existence, uniqueness and multiplicity of positive solutions to the following eigenvalue BVP by means of fixed point theorem and degree theory

$$v^{(4)} = \lambda f(t, (v(t)), \quad 0 < t < 1, \tag{1.3}$$

$$v(0) = v(1) = v''(0) = v''(1) = 0, \tag{1.4}$$

where $\lambda > 0$ is a constant and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. In [12] by applying a Krasnosel'skii fixed point theorem of cone expansion and compression the author obtained the existence and multiplicity results of equation (1.3) with boundary conditions $v(0) = v(1) = v'(0) = v'(1) = 0$. In the literature, there are few papers devoted to study fourth order singular eigenvalue problems. In the case when $a_j = 0$, $j \in \{1, 2, 3, 4\}$, the BVP (1.1)-(1.2) is investigated in [7] when $f \in \mathcal{C}([0, \infty))$, $f > 0$ on $[0, \infty)$, f is nondecreasing on $[0, \infty)$ and there exist $\delta > 0$, $m \geq 2$ such that $f(u) > \delta u^m$, $u \in [0, \infty)$, and $g \in \mathcal{C}(0, 1)$, $g > 0$ on $(0, 1)$ and $0 < \int_0^1 s(1-s)g(s)ds < \infty$. In [7], Feng and Ge used the method of upper and lower solutions and the fixed point index to discuss the existence of positive solutions.

Our main result is as follows where we do not require any monotonicity assumptions on f, and we do not assume that f is either superlinear or sublinear.

Theorem 1.1. *Suppose that* $(H1)$ *and* $(H2)$ *hold. Then there is a* $\lambda^* > 0$ *such that the BVP* (1.1)-(1.2) *has at least one nonnegative solution for* $\lambda = \lambda^*$.

Note that our main result, in the particular case $a_j = 0$, $j \in \{1, 2, 3, 4\}$, is valid in the case when f is decreasing on $[0, \infty)$, while the corresponding result in [7] is not valid. For instance, $f(x) = 1 + \frac{1}{1+x^2}$, $x \in [0, \infty)$, satisfies $(H1)$ for $A_1 = 1$, $A_2 = 2$, $B_j = 0$, $j \in \{0, \ldots, k\}$, and f is decreasing on $[0, \infty)$, whereupon it does not satisfy the conditions in [7]. Also, the conditions for g in [7] are more restrictive than $(H2)$. For instance, $g(t) = \frac{\left(\frac{1}{2}-t\right)^2}{t(1-t)}$, $t \in (0, 1)$, satisfies $(H2)$ and does not satisfy the conditions in [7] because $g\left(\frac{1}{2}\right) = 0$. Thus, we can consider the particular case of our main result, $a_j = 0$, $j \in \{1, 2, 3, 4\}$, as a complementary result to the result in [7]. The approach used in this paper is to rewrite the (BVP) (1.1)-(1.2) into a perturbed integral equation of which we search for solutions in a suitable subset of a Banach space by means of recent fixed point theorem of Birkhoff-Kellogg type developed by Calamai and Infante in [5]. Note that this fixed point theorem has been applied very recently to discuss the solvability of fourth order retarded equations in [6].

The paper is organized as follows. In Section 2, we give some auxiliary results needed for the proof of our main result. In Section 3, we prove our main result. In Section 4, we give an example.

2. Auxiliary results

Let X be a real Banach space.

Definition 2.1. A mapping $F : \Omega \subset X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.2. A closed, convex set $\mathcal{K}$ of X is said to be cone if

1. $\alpha x \in \mathcal{K}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{K}$,
2. $x, -x \in \mathcal{K}$ implies $x = 0$.

For a given $y \in X$, we consider the translate of a cone $\mathcal{K}$, namely

$$\mathcal{K}_y = \mathcal{K} + y = \{x + y \ : x \in \mathcal{K}\}.$$

Given an open bounded subset D of X we denote $D_{\mathcal{K}_y} = D \cap \mathcal{K}_y$, an open subset of $\mathcal{K}_y$.

Theorem 2.3. [5, Corollary 2.4] *Let $(X, \| \ \|)$ be a real Banach space, $\mathcal{K} \subset X$ be a cone, and $D \subset X$ be an open bounded set with $y \in D_{\mathcal{K}_y}$ and $\overline{D}_{\mathcal{K}_y} \neq \mathcal{K}_y$. Assume that $F : \overline{D}_{\mathcal{K}_y} \to \mathcal{K}$ is a completely continuous map and assume that*

$$\inf_{x \in \partial D_{\mathcal{K}_y}} \|Fx\| > 0.$$

Then there exists $x^* \in \partial D_{\mathcal{K}_y}$ and $\lambda^* \in (0, \infty)$ such that

$$x^* = y + \lambda^* F(x^*).$$

Let

$$y_1(t) = \left(a_1 + \frac{a_4}{6}\right)(1-t) + a_2 t + \frac{a_3}{6}(1-t)^3 + \frac{a_4}{6}(t^3 - 1) + \frac{a_3}{6}(t-1), \quad t \in [0,1].$$

We have

$$0 \leq y_1(t) \leq a_1 + a_2 + a_3 + a_4, \quad t \in [0,1],$$

and

$$\begin{aligned} y_1'(t) &= -a_1 - \frac{a_4}{6} + a_2 - \frac{1}{2}a_3(1-t)^2 + \frac{1}{2}a_4 t^2 + \frac{a_3}{6}, \quad t \in [0,1]. \\ y_1''(t) &= a_3(1-t) + a_4 t, \quad t \in [0,1]. \end{aligned}$$

Hence,

$$y_1(0) = a_1, \quad y_1(1) = a_2, \quad y_1''(0) = a_3, \quad y''(1) = a_4.$$

Set

$$y(t) = -y_1(t), \quad t \in [0,1].$$

Now, consider the BVP

$$\begin{aligned} u^{(4)} &= \lambda g(t) f(u(t) - y(t)), \quad 0 < t < 1, \\ u(0) &= u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{2.1}$$

where f and g satisfy $(H1)$ and $(H2)$, respectively.

Let $X = \mathcal{C}([0,1])$ be endowed with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define

$$\mathcal{K} = \{u \in X : u(t) \geq 0, \quad t \in [0,1]\}.$$

Since $0 \leq \int_0^1 s(1-s)g(s)ds < \infty$, there exists a nonnegative constant C_0 such that

$$\int_0^1 s(1-s)g(s)ds = C_0.$$

Because $g \not\equiv 0$ on $(0,1)$, there are $C_1 > 0$, $s_0 \in (0,1)$ and $\epsilon > 0$ such that $s_0-\epsilon, s_0+\epsilon \in (0,1)$ and

$$g(s) \geq C_1, \ \ s \in (s_0-\epsilon, s_0+\epsilon).$$

Define

$$G(t,s) = \begin{cases} t(1-s)\frac{2s-s^2-t^2}{6}, & 0 \leq t \leq s \leq 1, \\ s(1-t)\frac{2t-t^2-s^2}{6}, & 0 \leq s \leq t \leq 1. \end{cases}$$

We have

$$0 \leq G(t,s) \leq \frac{1}{6}s(1-s) \leq \frac{1}{6}, \quad 0 \leq t,s \leq 1,$$

Note that

$$\begin{aligned} \int_0^1 G(s_0+\epsilon,s)g(s)ds &\geq \int_{s_0-\epsilon}^{s_0+\epsilon} G(s_0+\epsilon,s)g(s)ds \\ &\geq C_1\int_{s_0-\epsilon}^{s_0+\epsilon} G(s_0+\epsilon,s)ds \\ &= C_1\int_{s_0-\epsilon}^{s_0+\epsilon} s(1-s_0-\epsilon)\frac{2(s_0+\epsilon)-(s_0+\epsilon)^2-s^2}{6}ds \\ &\geq \frac{2}{3}C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2 \\ &> 0. \end{aligned}$$

For $u \in X$, define the operator

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s)-y(s))ds, \quad t \in [0,1].$$

In [7], it is proved that any fixed point $u \in X$ of the operator λT is a solution to the BVP (2.1). Fix $C_2 > a_1+a_2+a_3+a_4$ arbitrarily. Define

$$D = \{u \in X : \|u\| < C_2\}.$$

We have that D is an open bounded set in X, $y \in D$ and $D_{\mathcal{K}_y} = D \cap \mathcal{K}_y \neq \mathcal{K}_y$. Note that for any $u \in \overline{D}_{\mathcal{K}_y}$, we have

$$u(t) = y(t) + z(t), \ \ t \in [0,1],$$

for some $z \in \mathcal{K}$, and so $u(t)-y(t)=z(t)\geq 0$, $t\in[0,1]$, and

$$\begin{aligned} f(u(t)-y(t)) &\leq \left(A_2+\sum_{j=0}^{k} B_j (u(t)-y(t))^j\right) \\ &\leq \left(A_2+\sum_{j=0}^{k} B_j 2^j \left(|u(t)|^j+|y_1(t)|^j\right)\right) \\ &\leq \left(A_2+\sum_{j=0}^{k} B_j 2^j \left(C_2^j+(a_1+a_2+a_3+a_4)^j\right)\right), \quad t\in[0,1]. \end{aligned}$$

2.1. Proof of the main result

Since $f\in\mathcal{C}([0,\infty))$ and $g\in\mathcal{C}(0,1)$, we have that $T: D_{\mathcal{K}_y}\to\mathcal{K}$ is a continuous operator. Next, for $u\in\overline{D}_{\mathcal{K}_y}$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)g(s)f(u(s)-y(s))ds \\ &\leq \frac{1}{6}\left(A_2+\sum_{j=0}^{k} B_j 2^j \left(C_2^j+(a_1+a_2+a_3+a_4)^j\right)\right)\int_0^1 s(1-s)g(s)\,ds \\ &= \frac{1}{6}C_0\left(A_2+\sum_{j=0}^{k} B_j 2^j \left(C_2^j+(a_1+a_2+a_3+a_4)^j\right)\right), \quad t\in[0,1], \end{aligned}$$

whereupon

$$\|Tu\|\leq C_0\left(A_2+\sum_{j=0}^{k} B_j 2^j \left(C_2^j+(a_1+a_2+a_3+a_4)^j\right)\right).$$

Then, $T(\overline{D}_{\mathcal{K}_y})$ is uniformly bounded. Moreover, for $u\in\overline{D}_{\mathcal{K}_y}$ and $t_1,t_2\in[0,1], t_1<t_2$, the Lebesgue dominated convergence theorem guarantees that

$$\begin{aligned} &|Tu(t_1)-Tu(t_2)| \\ \leq\ & \int_0^1 |G(t_1,s)-G(t_2,s)|g(s)f(u(s)-y(s))ds\,ds \\ \leq\ & \left(A_2+\sum_{j=0}^{k} B_j 2^j \left(C_2^j+(a_1+a_2+a_3+a_4)^j\right)\right)\int_0^1 g(s)|G(t_1,s)-G(t_2,s)|ds \\ &\to 0, \quad t_1\to t_2, \end{aligned}$$

Therefore, $T(\overline{D}_{\mathcal{K}_y})$ is equicontinuous. According to the Arzelà-Ascoli compactness criterion, we conclude that the operator $T:\overline{D}_{\mathcal{K}_y}\to\mathcal{K}$ is completely continuous.

Observe that, for $u \in \partial D_{\mathcal{K}_y}$,

$$\begin{aligned}\max_{t\in[0,1]} |Tu(t)| \geq Tu(s_0+\epsilon) &= \int_0^1 G(s_0+\epsilon, s)g(s)f(u(s)-y(s))ds \\ &\geq A_1 \int_0^1 G(s_0+\epsilon, s)g(s)ds \\ &\geq \frac{2}{3}A_1C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2 \\ &> 0.\end{aligned}$$

Consequently

$$\inf_{u\in\partial D_{\mathcal{K}_y}} \|Tu\| \geq \frac{2}{3}A_1C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2 > 0.$$

Now, applying Theorem 2.3, we conclude that there are $\lambda^* \in (0,\infty)$ and $u^* \in \partial D_{\mathcal{K}_y}$ such that

$$u^*(t) = y(t) + \lambda^* \int_0^1 G(t,s)g(s)f(u^*(s)-y(s))ds, \quad t \in [0,1].$$

Let

$$v^*(t) = u^*(t) - y(t), \quad t \in [0,1].$$

Then

$$\begin{aligned} v^*(0) &= u^*(0) - y(0) = a_1, \\ v^*(1) &= u^*(1) - y(1) = a_2, \\ v^{*\prime\prime}(0) &= u^{*\prime\prime}(0) - y''(0) = a_3, \\ v^{*\prime\prime}(1) &= u^{*\prime\prime}(1) - y''(1) = a_4 \end{aligned}$$

and

$$v^*(t) = \lambda \int_0^1 G(t,s)g(s)f(v^*(s))ds, \quad t \in [0,1],$$

whereupon

$$v^{*(4)}(t) = \lambda g(t) f(v^*(t)), \quad 0 < t < 1.$$

Since $u^* \in \partial D_{\mathcal{K}_y}$, we have that $u^*(t) = y(t) + z^*(t)$, $t \in [0,1]$, for some $z^* \in \mathcal{K}$, and then

$$v^*(t) = u^*(t) - y(t) = z^*(t) + y(t) - y(t) = z^*(t) \geq 0, \quad t \in [0,1].$$

3. An example

Consider the BVP

$$\begin{aligned} u^{(4)} &= \lambda \frac{\left(\frac{1}{2}-t\right)^2}{t(1-t)} \left(1 + \frac{1}{1+(u(t))^2}\right), \quad t \in (0,1), \\ u(0) &= 0, \quad u(1) = 1, \quad u''(0) = \tfrac{1}{2}, \quad u''(1) = 1. \end{aligned} \tag{3.1}$$

Here

$$f(x) = 1 + \frac{1}{1+x^2}, \quad x \in [0,\infty), \quad g(t) = \frac{\left(\frac{1}{2}-t\right)^2}{t\,(1-t)}, \quad t \in (0,1),$$

and

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = 1.$$

By our main result, it follows that the BVP (3.1) has at least one nonnegative solution.

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Optimal control of a frictional contact problem with unilateral constraints

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Abstract. We consider a mathematical model that describes a static contact with a nonlinear elastic body and a foundation. The contact boundary is composed of two measurable parts. In one part, the contact is frictionless with Signorini's conditions. In the other part, the normal stress is given and associated with Coulomb's friction law. We state an optimal control problem that consists of leading the stress tensor as close as possible to a given target by acting with a control on the boundary. Then, we study the penalized and regularized control problem for which we establish a convergence result.

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1. Introduction

Contact problems involving deformable bodies are very common in industry and everyday life and play a large role in structural and mechanical systems. Given the significance of these processes, considerable effort has been devoted to modelling and numerical simulation of these problems. The first of frictional contact problems in the context of variational inequalities was carried out in [9]. To get a background in contact mechanics from the mathematical or engineering point of view, the reader can consult for instance [2,12,14,18,21,22,26,23,24,25]. In addition to the numerical study of contact problems at present, we are also interested in studying the optimal control of such problems. Recall that the theory of optimal control of variational inequalities is very elaborate, see for instance [10,18]. In [19], we find the study of the optimal control of linear or nonlinear elliptic problems and variational inequalities. However,

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the optimal control issues for contact models are very significant, but they are not overly developed, see [1,3,4,5,6,7,8,10,13,15,16,18,19,20,26] and the references therein. Recently, in [16,17] two optimal control problems for elastic frictional contact models were studied. In particular, in [17], the authors investigated the optimal control of a frictional contact problem with normal compliance.

In this paper, we consider a nonlinear elastic body which is in static contact with a foundation. The boundary contact is divided into two measurable parts such that their measures must not equal zero at the same time. In one part, the contact is frictionless with unilateral constraints. In the other part, the normal stress is given and the contact is described by Coulomb's friction law. This model of contact was used in [27] to study a viscoelastic contact problem with a long memory. Thus, we contribute by proposing the model from which we derive a variational formulation (Problem P_2) of the mechanical problem and prove the existence and uniqueness of a weak solution. Next, the optimal control problem concerning this model is denoted by C1. It consists of minimizing a cost functional which is convex and continuous. Indeed, we are interested to led the stress tensor field as close as possible to a given target when we act with control on the boundary of the body. We prove that Problem C1 admits at least one solution, and then we introduce a penalized and regularized problem (Problem P_δ) such that the solution converges to the solution of Problem P_2. Also, we introduce a regularized and penalized optimal control problem C2 and obtain a convergence result.

The paper is structured as follows. In section 2, we describe the mechanical model, introduce some notations, establish a variational formulation and prove its weak solvability, Theorem 2.1. In section 3, we state the optimal control problem C1 and prove that it has at least one solution, Theorem 3.2. In section 4, we state and analyze a penalized and regularized optimal control problem, Theorem 4.4.

2. The model and its weak solvability

We denote by $\mathbf{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d (d = 2, 3)$, while '.' and $|.|$ represent the inner product and the norm on $\mathbf{S}_d$. Thus, for every σ, $\tau \in \mathbf{S}_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{\frac{1}{2}}$. Here and below, the indices i and j lie between 1 and d and the summation convention over repeated indices is adopted. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $v_\nu = v.\nu = v_i\nu_i$, $v_\tau = v - v_\nu\nu$, $\sigma_\nu = \sigma\nu.\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$.

We consider the following physical setting. Let an elastic body occupy a bounded Lipschitzian domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$. The boundary Γ of Ω is partitioned into three measurable parts such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where Γ_i, $i = 1, 2, 3$, are disjoint and $meas\,(\Gamma_1) > 0$. The body is subjected to volume forces of density φ_0 and tractions φ on Γ_2. On Γ_1, the displacement vanishes and the body is clamped here. Γ_3 is divided into $\Gamma_{3,1}$ and $\Gamma_{3,2}$ such that their measures must not equal zero at the same time. This latter hypothesis allows that where one of the two subsets $\Gamma_{3,1}$ and $\Gamma_{3,2}$ is empty, then the corresponding contact condition below is suppressed from the problem. We assume a frictionless contact with Signorini's conditions on $\Gamma_{3,1}$, and Coulomb's law of dry friction on $\Gamma_{3,2}$.

Under these conditions, the classic formulation for the contact problem is as follows.

Problem P_1. Find a displacement field $u : \Omega \to \mathbb{R}^d$ such that

$$div\sigma(u) = -\varphi_0 \text{ in } \Omega, \tag{2.1}$$

$$\sigma(u) = \mathcal{F}\varepsilon(u) \quad \text{in } \Omega\ , \tag{2.2}$$

$$u = 0 \qquad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma\nu = \varphi \qquad \text{on } \Gamma_2, \tag{2.4}$$

$$u_\nu \leq 0,\ \sigma_\nu \leq 0,\ \sigma_\nu u_\nu = 0,\ \sigma_\tau = 0 \text{ on } \Gamma_{3,1}, \tag{2.5}$$

$$\left.\begin{array}{l} -\sigma_\nu = S,\ |\sigma_\tau| \leq \mu|\sigma_\nu|, \\ -\sigma_\tau = \mu|\sigma_\nu| \dfrac{u_\tau}{|u_\tau|} \quad \text{if } u_\tau \neq 0 \end{array}\right\} \text{ on } \Gamma_{3,2}. \tag{2.6}$$

Here (2.1) represents the equilibrium equation where $\sigma = \sigma(u)$ denotes the stress tensor and $div\sigma = \sigma_{ij,j}$ is the divergence of σ. Next, equation (2.2) is the elastic constitutive law in which $\varepsilon(u)$ is the strain tensor defined by $\varepsilon(u) = (\varepsilon_{ij}(u))$, $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ and $\mathcal{F}$ is a given nonlinear function. Equations (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. Over $\Gamma_{3,1}$, (2.5) describes the frictionless contact with Signorini's conditions. On $\Gamma_{3,2}$, Coulomb's law of dry friction with the hypothesis that the normal stress is given. In (2.6) S is a nonnegative function, μ is a coefficient of friction and μS a friction bound.

To proceed with the variational formulation, Problem P_1, we need additional notations and need to recall some assumptions in the sequel.

$$H = L^2(\Omega)^d\ ,\ Q = \left\{\tau = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\}$$
$$H_1 = \left\{u = (u_i)\,|u_i \in H^1(\Omega),\ i = \overline{1,d}\right\},\ Q_1 = \{\sigma \in Q\,|\text{div } \sigma \in H\}$$

H, Q, H_1, H_d are real Hilbert spaces endowed with the respective inner products:

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad \langle\sigma,\tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij}dx,$$
$$(u,v)_{H_1} = \langle u,v\rangle_H + (\varepsilon(u),\varepsilon(v))_Q\ ,\ (\sigma,\tau)_{H_d} = \langle\sigma,\tau\rangle_Q + (div\ \sigma, div\tau)_H\ .$$

We denote respectively the norms associated with $\|.\|_H$, $\|.\|_Q$, $\|.\|_{H_1}$ and $\|.\|_{H_d}$.
Recall that the following *Green's* formula holds:

For every element $v \in H_1$, we also write v for the trace of v on Γ. Recall that if σ is a regular function, then the following Green's formula holds:

$$(\sigma,\varepsilon(v))_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1,$$

where da is the measure surface element.
Next, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1\}\ .$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds [9],

$$\|\varepsilon\,(v)\|_Q \geq c_\Omega\,\|v\|_{H_1} \quad \forall v \in V, \tag{2.7}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product given by

$$(u, v)_V = (\varepsilon\,(u)\,, \varepsilon(v))_Q,$$

and let $\|.\|_V$ be the associated norm. It follows from (2.7) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent and $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by Sobolev's trace theorem, there exists a constant $d_\Omega > 0$ depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega\,\|v\|_V \quad \forall v \in V. \tag{2.8}$$

We introduce the closed convex set of admissible displacements defined as

$$K = \{v \in V;\, v_\nu \leq 0 \; a.e. \text{ on } \Gamma_{3,1}\}\,.$$

For the study of Problem (P) we adopt the following assumptions on the data:

The operator of elasticity $\mathcal{F}$ satisfies

$$\begin{cases} (a) \;\; \mathcal{F} : \Omega \times S_d \to S_d; \\ (b) \;\; \text{there exists } M > 0 \text{ such that} \\ \qquad |\mathcal{F}\,(x, \varepsilon_1) - \mathcal{F}\,(x, \varepsilon_2)| \leq M\,|\varepsilon_1 - \varepsilon_2| \\ \qquad \forall \varepsilon_1, \varepsilon_2 \in S_d, \; a.e. \; x \in \Omega; \\ (c) \;\; \text{there exists } m > 0 \text{ such that} \\ \qquad (\mathcal{F}\,(x, \varepsilon_1) - \mathcal{F}\,(x, \varepsilon_2))\,.\,(\varepsilon_1 - \varepsilon_2) \geq m\,|\varepsilon_1 - \varepsilon_2|^2 \\ \qquad \forall \varepsilon_1, \varepsilon_2 \in S_d, \; a.e. \; x \in \Omega; \\ (d) \;\; \text{the mapping } x \to \mathcal{F}\,(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \varepsilon \in S_d; \\ (e) \;\; \mathcal{F}\,(x, 0_{S_d}) = 0 \text{ for } a.e. \; x \in \Omega. \end{cases} \tag{2.9}$$

Examples of nonlinear elasticity operators can be found in $[11, 28]$.

We assume that the densities of the body force and the surface traction satisfies

$$\varphi_0 \in H, \qquad \varphi \in \left(L^2\,(\Gamma_2)\right)^d. \tag{2.10}$$

Finally, the coefficient of friction μ and the normal stress S are assumed to satisfy

$$\mu \in L^\infty\,(\Gamma_{3,2}) \text{ and } \mu \geq 0 \; a.e. \text{ on } \Gamma_{3,2}, \tag{2.11}$$

$$S \in L^2\,(\Gamma_{3,2}) \text{ and } S \geq 0 \; a.e. \text{ on } \Gamma_{3,2}. \tag{2.12}$$

Next, we define the functional $j : V \to \mathbb{R}$ by

$$j\,(v) = \int_{\Gamma_{3,2}} (S v_\nu + \mu S\,|v_\tau|) da, \; \forall v \in V.$$

Using Riesz representation theorem, there exists $f \in V$ such that

$$(f, v)_V = (\varphi_0, v)_H + (\varphi, v)_{(L^2(\Gamma_2))^d} \; \forall v \in V.$$

A standard procedure allows us to derive the following variational formulation from the mechanical P_1.

Problem P_2. *Find* $u \in K$ *such that*

$$(Au, v-u)_V + j(v) - j(u) \geq (f, v-u)_V, \ \forall v \in K. \tag{2.13}$$

Here, the operator A is defined by

$$(Au, v)_V = (\mathcal{F}\varepsilon(u), \varepsilon(v))_Q, \ \forall u, v \in V.$$

The main result of this section is on the existence and uniqueness of the weak formulation P_2. One has the following theorem.

Theorem 2.1. *Let* (2.9)*,* (2.10)*,* (2.11) *and* (2.12) *hold. Then, there exists a unique solution of Problem* P_2*.*

Proof. We use (2.9) (b), (2.9) (c) to show that the operator A is Lipschitz continuous and strongly monotone. Using (2.11) and (2.12), we see that the functional $j: V \to \mathbb{R}$ is proper, convex and lower semicontinuous; K is a non empty closed convex of V. Then, it follows from the theory of elliptic variational inequalities (see [24]) that the inequality (2.13) has a unique solution. □

3. The optimal control problem

For a fixed $\varphi_0 \in H$, we consider the state problem below.

Problem Q1. For a given $\varphi \in (L^2(\Gamma_2))^d$ (called control), find $u \in K$ such that

$$\begin{cases} (Au, v-u)_V + \ j(v) - j(u) \\ \geq (\varphi_0, v-u)_H + (\varphi, v-u)_{(L^2(\Gamma_2))^d}, \quad \forall v \in K. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let* (2.9)*,* (2.10)*,* (2.11) *and* (2.12) *hold. Then Problem Q1 has a unique solution.*

By the same arguments used in the proof of Theorem 2.1, this problem has a unique solution $u = u(\varphi)$.

Now, by acting the control on the boundary Γ_2, we focus that the resulting stress be as close to a given target σ_d. We assume that $\sigma_d = \mathcal{F}\varepsilon(u_d)$ where $u_d \in V$ and recall that $\sigma = \mathcal{F}\varepsilon(u)$. Then we have $\|\sigma - \sigma_d\|_Q \leq M \|u - u_d\|_V$ and we see that if $\|u - u_d\|_V$ is sufficiently small, it follows that σ approach σ_d in the sense of $Q-$norm. Thus, we consider the cost functional $\mathcal{L}: V \times (L^2(\Gamma_2))^d \to \mathbb{R}_+$ defined as

$$\mathcal{L}(u, \varphi) = \alpha \|u - u_d\|_V^2 + \beta \|\varphi\|^2_{(L^2(\Gamma_2))^d}, \tag{3.2}$$

where $\alpha, \beta > 0$. We define the set U_{ad} of admissible pairs by

$$U_{ad} = \left\{(u, \varphi) \in (K \times (L^2(\Gamma_2))^d), \text{ such that } (3.1) \text{ is satisfied}\right\}.$$

Then we consider the following optimal control problem.

Problem C1. Find $(u^*, \varphi^*) \in U_{ad}$ such that

$$\mathcal{L}(u^*, \varphi^*) = \min_{(u,\varphi) \in U_{ad}} \mathcal{L}(u, \varphi).$$

Theorem 3.2. *Assume* (2.9)*,* (2.10)*,* (2.11) *and* (2.12)*. Then Problem C1 has at least one solution.*

Proof. We put $v = 0_V$ in (3.1), then, using (2.7), (2.8) and (2.9) (c), we deduce that the solution u of Problem Q1 is bounded in V as

$$\|u\|_V \leq \frac{c_0}{m}\left(\|\varphi_0\|_H + d_\Omega \|\varphi\|_{(L^2(\Gamma_2))^d} + d_\Omega \|S\|_{L^2(\Gamma_{3,2})}\right),$$

where $c_0 > 0$. This estimate below implies that

$$\inf_{(u,\varphi)\in U_{ad}} \{\mathcal{L}(u,\varphi)\} < \infty.$$

Now, let us denote

$$\inf_{(u,\varphi)\in U_{ad}} \{\mathcal{L}(u,\varphi)\} = \theta. \tag{3.3}$$

Then, there exists a minimizing sequence $(u^n,\varphi^n) \subset U_{ad}$ such that

$$\lim_{n\to\infty} \mathcal{L}(u^n,\varphi^n) = \theta. \tag{3.4}$$

The sequence (u^n,φ^n) is bounded in $V \times (L^2(\Gamma_2))^d$, so there exists an element

$$(u^*,\varphi^*) \in V \times (L^2(\Gamma_2))^d$$

such that passing to a subsequence still denoted by (u^n,φ^n), we deduce that as $n \to \infty$,

$$u^n \to u^* \text{ weakly in } V. \tag{3.5}$$

We note that K is a closed convex subset of the space V and $(u^n) \subset K$. Then the convergence (3.5) implies that $u^* \in K$.

$$\varphi^n \to \varphi^* \text{ weakly in } (L^2(\Gamma_2))^d. \tag{3.6}$$

Now, we need to prove that

$$u^n \to u^* \text{ strongly in } V \text{ as } n \to \infty. \tag{3.7}$$

Indeed, as $(u^n,\varphi^n) \in U_{ad}$, then u^n is the solution of the inequality below.

$$\begin{cases} (Au^n, v - u^n)_V + j(v) - j(u^n) \\ \geq (\varphi_0, v - u^n)_H + (\varphi^n, v - u^n)_{(L^2(\Gamma_2))^d}, \quad \forall v \in K. \end{cases} \tag{3.8}$$

Using (2.9) (c) and (3.8), we deduce that

$$\begin{aligned} m\|u^n - u^*\|_V^2 &\leq (Au^n - Au^*, u^n - u^*)_V \\ &\leq -(Au^*, u^n - u^*)_V + j(u^*) - j(u^n) \\ &\quad + (\varphi_0, u^n - u^*)_H + (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d}. \end{aligned} \tag{3.9}$$

Using (3.5), we have that

$$\lim_{n\to\infty} (Au^*, u^n - u^*)_V = 0.$$

On the other hand, since $u^n \to u^*$ weakly in V implies $u^n \to u^*$ strongly in H, then $\lim\limits_{n\to+\infty} (\varphi_0, u^n - u^*)_H = 0$. Also, as (φ^n) is bounded in $(L^2(\Gamma_2))^d$, then using that (3.5) implies $u^n \to u^*$ strongly in $(L^2(\Gamma_2))^d$. It follows that

$$\lim_{n\to\infty} (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d} = 0 \text{ and } \lim_{n\to\infty} j(u^n) = j(u^*).$$

Thus, the right hand side of inequality (3.9) tends to zero as $n \to +\infty$ and then we get (3.7). Moreover, using (3.6) and (3.7), we pass to the limit as $n \to +\infty$ in (3.8) to obtain that u^* satisfies the inequality (3.1) with $\varphi = \varphi^*$. Hence, from Theorem 3.1, we deduce that

$$(u^*, \varphi^*) \in U_{ad}. \tag{3.10}$$

On the other hand, the functional $\mathcal{L}$ is convex and lower semicontinuous, then it is weakly lower semicontinuous. So we deduce that

$$\liminf_{n\to+\infty} \mathcal{L}(u^n, \varphi^n) \geq \mathcal{L}(u^*, \varphi^*). \tag{3.11}$$

It follows now from (3.4) and (3.11) that

$$\theta \geq \mathcal{L}(u^*, \varphi^*). \tag{3.12}$$

In addition, (3.3) yields

$$\mathcal{L}(u^*, \varphi^*) \geq \theta. \tag{3.13}$$

Then, to end the proof, it suffices to combine the inequalities (3.12) and (3.13) .□ □

4. The penalized and regularized optimal control problem

Let $\delta > 0$, we replace the contact condition (2.5) by the condition

$$\sigma_{\delta\nu}(u) = -\frac{1}{\delta}(u_\nu)_+$$

where we recall that for $r \in \mathbb{R}$, $r_+ = \max(r, 0)$, and consider the smooth function

$$\psi(x) = \sqrt{x^2 + \delta^2}.$$

Now, we introduce the following penalized and regularized problem.

Problem P_δ. Find $u^\delta \in V$ such that

$$\begin{aligned}(Au^\delta, v - u^\delta)_V &+ \frac{1}{\delta}\left((u^\delta_\nu)_+, v_\nu - u^\delta_\nu\right)_{L^2(\Gamma_{3,1})} + \int_{\Gamma_{3,2}} \mu S\left(\psi(v_\tau) - \psi\left(u^\delta_\tau\right)\right) da \\ &+ \int_{\Gamma_{3,2}} S\left(v_\nu - u^\delta_\nu\right) da \geq (f, v - u^\delta)_V \quad \forall v \in V.\end{aligned} \tag{4.1}$$

Theorem 4.1. *Assume that* (2.9), (2.10), (2.11) *and* (2.12) *hold.Then, there exists a unique solution of Problem* P_δ.

Proof. We define the operator $B : V \to V$ by

$(Bu, v)_V = (Au, v)_V + \frac{1}{\delta}\left((u_\nu)_+, v_\nu\right)_{L^2(\Gamma_{3,1})} \ \forall u, v \in V.$

Using that for $a, b \in \mathbb{R}$, $(a - b)(a_+ - b_+) \geq (a_+ - b_+)^2$ and $|a_+ - b_+| \leq |a - b|$, we deduce by (2.8) and (2.9) that the operator B is Lipschitz continuous and strongly monotone as for all $u, v \in V$:

$$\begin{aligned}&\|Bu - Bv\|_V \leq (M + \frac{d_\Omega^2}{\delta}) \|u - v\|_V, \\ &(Bu - Bv, u - v)_V \geq m \|u - v\|_V^2.\end{aligned}$$

So, there exists a unique solution u^δ of (4.1). In addition, take $v=0$ in (4.1) and use (2.7), (2.8) and (2.9) (c) implies that

$$\left\|u^\delta\right\|_V \leq \frac{c_0}{m}(\|\varphi_0\|_H + d_\Omega \|\varphi\|_{(L^2(\Gamma_2))^d} + d_\Omega \|S\|_{L^2(\Gamma_{3,2})}). \tag{4.2}$$

□

Now for a fixed $\varphi_0 \in H$, we define the penalized and regularized state problem as follows.

Problem Q2. For a given $\varphi \in (L^2(\Gamma_2))^d$ (called control), find $u^\delta \in V$ such that

$$(Au^\delta, v-u^\delta)_V + \frac{1}{\delta}\left((u_\nu^\delta)_+, v_\nu - u_\nu^\delta\right)_{L^2(\Gamma_{3,1})} + \int_{\Gamma_{3,2}} \mu S\left(\psi(v_\tau) - \psi\left(u_\tau^\delta\right)\right) da$$

$$+\int_{\Gamma_{3,2}} S(v_\nu - u_\nu)\, da \geq (\varphi_0, v-u^\delta)_H + (\varphi, v-u^\delta)_{(L^2(\Gamma_2))^d} \quad \forall v \in V.$$

With the same arguments used in Theorem 4.1 this problem has a unique solution. Moreover, we define the set of admissible pairs as

$$U_{ad}^\delta = \left\{(u,\varphi) \in V \times (L^2(\Gamma_2))^d, \text{ such that } (4.1) \text{ is satisfied}\right\}.$$

Then using the functional $\mathcal{L}$, given by (3.2), we formulate below the regularized and penalized optimal control problem.

Problem C2. Find $(\bar{u}^\delta, \bar{\varphi}^\delta) \in U_{ad}^\delta$ such that

$$\mathcal{L}\left(\bar{u}^\delta, \bar{\varphi}^\delta\right) = \min_{(u,\varphi)\in U_{ad}^\delta} \left\{\mathcal{L}(u,\varphi)\right\}.$$

With arguments similar to those used in Theorem 3.1, the following result can be proved.

Theorem 4.2. *Assume* (2.9), (2.10), (2.11) *and* (2.12) *hold. Then, Problem C2 has at least one solution.*

In the first part of this section, we prove that the unique solution of the penalized and regularized state problem Q2 converges to the unique solution of the state problem Q1. More precisely, the following theorem takes place.

Theorem 4.3. *Assume that* (2.9), (2.10), (2.11) *and* (2.12) *hold. Then, the following strong convergence holds:*

$$u^\delta \to u \textit{ strongly in } V \quad \textit{as } \delta \to 0. \tag{4.3}$$

Proof. Taking into account (4.2), it follows that there exists an element $\tilde{u} \in V$ such that passing to a subsequence still denoted in the same way, we have the convergence:

$$u^\delta \to \tilde{u} \text{ weakly in } V \quad \text{as } \delta \to 0. \tag{4.4}$$

Now take $v \in K$ in (4.1) and taking account that for $a, b \in \mathbb{R}$,

$$(a_+ - b_+)(a-b) \geq (a_+ - b_+)^2,$$

we deduce that

$$\begin{aligned}&(Au^\delta, v-u^\delta)_V+\int_{\Gamma_{3,2}}\mu S\left(\psi\left(v_\tau\right)-\psi\left(u_\tau^\delta\right)\right)da+\int_{\Gamma_{3,2}}S(v_\nu-u_\nu^\delta)da\\&\geq(\varphi_0, v-u^\delta)_H+(\varphi, v-u^\delta)_{(L^2(\Gamma_2))^d}\quad\forall v\in K.\end{aligned}\tag{4.5}$$

Using (2.11) and (2.12), we have that $\int_{\Gamma_{3,2}}\mu S(\psi\left(v_\tau\right)-|v_\tau|)da=O\left(\delta\right)$, then

$$\int_{\Gamma_{3,2}}\mu S(\psi\left(v_\tau\right))\to\int_{\Gamma_{3,2}}\mu S\left|v_\tau\right|da \text{ as } \delta\to 0.\tag{4.6}$$

On the other hand, we have

$$\int_{\Gamma_{3,2}}\mu S\psi\left(u_\tau^\delta\right)da=\int_{\Gamma_{3,2}}\mu S(\psi\left(u_\tau^\delta\right)-\left|u_\tau^\delta\right|)da+\int_{\Gamma_{3,2}}\mu S\left|u_\tau^\delta\right|da.$$

By (2.11) and (2.12), we have that

$$\int_{\Gamma_{3,2}}\mu S(\psi\left(u_\tau^\delta\right)-\left|u_\tau^\delta\right|)da=O\left(\delta\right).$$

Then,

$$\int_{\Gamma_{3,2}}\mu S\psi\left(u_\tau^\delta\right)da=O\left(\delta\right)+\int_{\Gamma_{3,2}}\mu S\left|u_\tau^\delta\right|da.\tag{4.7}$$

With compactness arguments, as $u_\tau^\delta\to\tilde{u}_\tau$ strongly in $\left(L^2\left(\Gamma_2\right)\right)^d$, we have that

$$\int_{\Gamma_{3,2}}\mu S\left|u_\tau^\delta\right|da\to\int_{\Gamma_{3,2}}\mu S\left|\tilde{u}_\tau\right|da \text{ as } \delta\to 0.$$

Then from (4.7) we deduce that

$$\int_{\Gamma_{3,2}}\mu S\psi\left(u_\tau^\delta\right)da\to\int_{\Gamma_{3,2}}\mu S\left|\tilde{u}_\tau\right|da \text{ as } \delta\to 0.\tag{4.8}$$

Then using (2.11), (2.12), (4.4), (4.5), (4.6), (4.8) and the compact imbedding $H^{\frac{1}{2}}\left(\Gamma\right)\hookrightarrow L^2\left(\Gamma\right)$, yields

$$\begin{aligned}\limsup_{\delta\to 0}\left(Au^\delta, u^\delta-v\right)_V&\leq(\varphi_0,\tilde{u}-v)_H+(\varphi,\tilde{u}-v)_{(L^2(\Gamma_2))^d}\\&\quad+j(\tilde{u})-j\left(v\right)\quad\forall v\in K.\end{aligned}\tag{4.9}$$

Using now the pseudo-monotonicity of A, we deduce that

$$\liminf_{\delta\to 0}\left(Au^\delta, u^\delta-v\right)_V\geq(A\tilde{u},\tilde{u}-v)_V\quad\forall v\in V.\tag{4.10}$$

Then, we combine (4.9) and (4.10) to get that

$$\left\{\begin{array}{l}(A\tilde{u}\ , v-\tilde{u})_V+j\left(v\right)-j(\tilde{u})\\\geq(\varphi_0, v-\tilde{u})_H+(\varphi, v-\tilde{u})_{(L^2(\Gamma_2))^d}\quad\forall v\in K.\end{array}\right.\tag{4.11}$$

On the other hand, take $v=0$ in (4.1) implies that

$$\left((\tilde{u}_\nu^\delta)_+,\tilde{u}_\nu^\delta\right)_{L^2(\Gamma_{3,1})}\leq\delta\left((\varphi_0, u^\delta)_H+(\varphi, u^\delta)_{(L^2(\Gamma_2))^d}-\left(S, u_\nu^\delta\right)_{L^2(\Gamma_{3,2})}\right).$$

Then, from this inequality, we deduce that

$$\|(\tilde{u}_\nu)_+\|_{L^2(\Gamma_{3,1})} \leq \liminf_{\delta\to 0} \left\|(u^\delta_\nu)_+\right\|_{L^2(\Gamma_{3,1})} \leq$$
$$\lim_{\delta\to 0}\sqrt{\frac{c_0\delta}{m}}\left(\|\varphi_0\|_H + d_\Omega\,\|\varphi\|_{(L^2(\Gamma_2))^d} + d_\Omega\,\|S\|_{L^2(\Gamma_{3,2})}\right).$$

This inequality above implies that $\|(\tilde{u}_\nu)_+\|_{L^2(\Gamma_{3,1})} = 0$, then $(\tilde{u}_\nu)_+ = 0$ *a.e.* on $\Gamma_{3,1}$. Hence, it follows that $\tilde{u} \in K$. Then, we deduce that $\tilde{u}$ is a solution of Problem P_1, so that $u = \tilde{u}$ from the uniqueness part of Theorem 2.1. Now, we have all ingredients to end the proof of Theorem 4.2. Indeed, by the arguments used above, it follows that any weakly convergent subsequence of the sequence $(u_\delta) \subset V$ converges weakly to the unique solution u of Problem P_2. Estimate (4.2) implies that the sequence (u_δ) is bounded in V. Thus, by a standard compactness argument, we conclude that the whole sequence (u_δ) converges weakly to u. Then we use (2.9) (c) to have

$$\begin{aligned} m\left\|u^\delta - u\right\|_V^2 &\leq (Au^\delta - Au, u^\delta - u)_V \\ &= (Au^\delta, u^\delta - u)_V - (Au, u^\delta - u)_V \end{aligned} \tag{4.12}$$

Now take $v = u$ in (4.5) and (4.9), then as $u = \tilde{u}$, we get

$$0 \leq \liminf_{\delta\to 0}\left(Au^\delta, u^\delta - u\right)_V \leq \limsup_{\delta\to 0}\left(Au^\delta, u^\delta - u\right)_V \leq 0.$$

Hence,

$$\lim_{\delta\to 0}\left(Au^\delta, u^\delta - u\right)_V = 0.$$

Moreover, from (4.12) since $\lim_{\delta\to 0}\left(Au, u^\delta - u\right)_V = 0$, we deduce that

$$\lim_{\delta\to 0}\left\|u^\delta - u\right\|_V = 0.$$

Then, we obtain (4.3). □

Next, we prove the convergence result below.

Theorem 4.4. *Assume that* (2.9), (2.10), (2.11), (2.12) *hold and let* $(\bar{u}^\delta, \bar{\varphi}^\delta)$ *be a solution of Problem C2. Then, there exists a solution* $(\bar{u}, \bar{\varphi})$ *of Problem C1 such that after passing to a subsequence still denoted in the same way, the following convergences as* $\delta \to 0$ *hold :*

$$\begin{aligned} &(a)\ \bar{u}^\delta \to \bar{u} \textit{ strongly in } V, \\ &(b)\ \ \bar{\varphi}^\delta \to \bar{\varphi} \ \textit{ weakly in } (L^2(\Gamma_2))^d. \end{aligned} \tag{4.13}$$

Proof. Let $u_0^\delta \in V$ be the unique solution of Problem Q2 with $\varphi = 0_{(L^2(\Gamma_2))^d}$. We have

$$\mathcal{L}\left(u_0^\delta, 0_{(L^2(\Gamma_2))^d}\right) = \alpha\left\|u_0^\delta - u_d\right\|_V^2 \leq 2\alpha\left(\left\|u_0^\delta\right\|_V^2 + \|u_d\|_V^2\right).$$

On the other hand, by (2.7), (2.8) and (2.9) (c), we have

$$\left\|u_0^\delta\right\|_V \leq \frac{c_1}{m}\left(\|\varphi_0\|_H + d_\Omega\,\|S\|_{L^2(\Gamma_{3,2})}\right),$$

where $c_1 > 0$. Then, denote $\frac{c_1}{m}\left(\|\varphi_0\|_H + d_\Omega \|S\|_{L^2(\Gamma_{3,2})}\right) = C$, we deduce that

$$\mathcal{L}\left(\bar{u}^\delta, \bar{\varphi}^\delta\right) \leq \mathcal{L}\left(u_0^\delta, 0_{(L^2(\Gamma_2))^d}\right) \leq 2\alpha(C^2 + \|u_d\|_V^2).$$

Therefore, $\left(\bar{u}^\delta, \bar{\varphi}^\delta\right)$ is a bounded sequence in $V \times (L^2(\Gamma_2))^d$. Consequently, there exists $(\bar{u}, \bar{\varphi}) \in V \times (L^2(\Gamma_2))^d$ such that passing to a subsequence still denoted in the same way, we have the convergences as $\delta \to 0$:

$$\begin{array}{rcl} \bar{u}^\delta & \rightharpoonup & \bar{u} \text{ weakly in } V, \\ \bar{\varphi}^\delta & \rightharpoonup & \bar{\varphi} \text{ weakly in } (L^2(\Gamma_2))^d. \end{array}$$

Moreover, denote $j_\delta(v) = \int_{\Gamma_{3,2}} (S\mu\psi(v_\tau) + Sv_\nu)da$, we see that

$$\begin{array}{rcl} m\left\|\bar{u}^\delta - \bar{u}\right\|_V^2 & \leq & \left(A\bar{u} - A\bar{u}^\delta, \bar{u} - \bar{u}^\delta\right)_V \\ & \leq & \left(A\bar{u}, \bar{u} - \bar{u}^\delta\right)_V + j_\delta(\bar{u}) - j_\delta\left(\bar{u}^\delta\right) \\ & & +(\varphi_0, \bar{u} - u^\delta)_H + (\varphi, \bar{u} - u^\delta)_{(L^2(\Gamma_2))^d}. \end{array}$$

Then, taking in mind that $\bar{u}^\delta \rightharpoonup \bar{u}$ weakly in V implies that $\bar{u}^\delta \to \bar{u}$ strongly in $(L^2(\Gamma_2))^d$, it follows that $j_\delta(\bar{u}) - j_\delta\left(\bar{u}^\delta\right) \to 0$ as $\delta \to 0$. Hence we deduce that the right hand side of the above inequality tends to zero, thus we obtain (4.13) (a). Also, we must prove that $(\bar{u}, \bar{\varphi}) \in U_{ad}$. Indeed, using (4.3), it follows that as $\delta \to 0$, the following convergences hold:

$$(A\bar{u}^\delta, v - \bar{u}^\delta)_V \to (A\bar{u}, v - \bar{u})_V,$$

$$\lim_{\delta\to 0}(j_\delta(v) - j_\delta(\bar{u})) = j(v) - j(\bar{u}),$$

$$(\varphi_0, v - u^\delta)_H + (\varphi, v - u^\delta)_{(L^2(\Gamma_2))^d} \to (\varphi_0, v - \bar{u})_H + (\varphi, v - \bar{u})_{(L^2(\Gamma_2))^d}.$$

Therefore, passing to the limit as $\delta \to 0$ in (4.5), we deduce that $(\bar{u}, \bar{\varphi})$ satisfies (3.1) and $(\bar{u}, \bar{\varphi}) \in U_{ad}$.Let now (u^*, φ^*) be a solution of Problem C1 and let us consider the sequence $\left(u^\delta\right)_\delta$ such that, for each $\delta > 0$, u^δ is the unique solution of Problem Q2 with $\varphi^* \in (L^2(\Gamma_2))^d$. Obviously, for every $\delta > 0$, $\left(u^\delta, \varphi^*\right) \in U_{ad}^\delta$. Using Theorem 4.3 we deduce that

$$\left(u^\delta, \varphi^*\right) \to \left(u^*, \varphi^*\right) \text{ in } V \times (L^2(\Gamma_2))^d \text{ as } \delta \to 0. \tag{4.14}$$

Since the functional $\mathcal{L}$ is convex and continuous, we have

$$\mathcal{L}(u^*, \varphi^*) \leq \liminf_{\delta\to 0} \mathcal{L}\left(\bar{\varphi}^\delta, \bar{u}^\delta\right). \tag{4.15}$$

Also, as $\left(\bar{u}^\delta, \bar{\varphi}^\delta\right)$ is a solution of Problem C2, we have

$$\limsup_{\delta\to 0} \mathcal{L}\left(\bar{u}^\delta, \bar{\varphi}^\delta\right) \leq \limsup_{\delta\to 0} \mathcal{L}\left(u^\delta, \bar{\varphi}\right). \tag{4.16}$$

Using (4.13), we have

$$\limsup_{\delta\to 0} \mathcal{L}\left(u^\delta, \bar{\varphi}\right) = \mathcal{L}(\bar{u}, \bar{\varphi}), \tag{4.17}$$

and as $(\bar{u}, \bar{\varphi})$ is a solution of Problem C1, then

$$\mathcal{L}(\bar{u}, \bar{\varphi}) \leq \mathcal{L}(u^*, \varphi^*). \tag{4.18}$$

Thus, from (4.15)-(4.18) , we deduce that $\mathcal{L}(\bar{u}, \bar{\varphi}) = \mathcal{L}(u^*, \varphi^*)$. □

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Some aspects of a coupled system of nonlinear integral equations

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Abstract. In the present work we take a system of two integral equations and prove the existence and uniqueness of their solution. We investigate four aspects of the problem, namely, error estimation and rate of convergence of the iteration leading to the solution, Ulam-Hyers stability, well-posedness and data dependence of the solution sets. We give some new definitions pertaining to the system we analyze here. In order to establish our results we utilize the coupled contraction mapping principle due to Bhaskar and Lakshmikantham (Nonlinear Anal. TMA 65(2006), 1379-1393) and several related results which we deduce here.

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1. Introduction

In this paper, we consider a system of two coupled nonlinear Fredholm type integral equations. Coupled integral equations are of great practical value. Some examples of works are [3], [11], [12] and [20] where they have been applied to contact problems, magnetostatic problems, solidification problems and scattering of nucleons.

Problem I. The problem is to solve the coupled system of nonlinear equations

$$\left.\begin{array}{rcll} u(t) & = & g(t) + \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ u(s),\ v(s))\ ds & \text{and} \\ v(t) & = & g(t) + \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ v(s),\ u(s))\ ds, & \lambda \geq 0, \end{array}\right\} \tag{1.1}$$

for all $t \in [a,\ b]$ under some appropriate conditions on $g, \mathfrak{h}$ and K.

The organization of our work is following.

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• First we describe the coupled contraction mapping theorem of Bhaskar et al. [13]. This result is pivotal to our study here.

• In section 3, we solve Problem I under certain conditions. We also establish that this solution is unique if we take some extra assumptions.

• In section 4, we study the rate of convergence and error estimation for the iteration obtained in section 3.

• In section 5, we discuss the Ulam-Hyers stability of the problem. It is a stability concept of general character which is applicable to diverse domains of mathematics. The essence of the stability is to see whether a mathematical object having approximate behaviour of a given class of objects can actually be approximated by a member of that class.

• In section 6, we investigate the well-posedness aspect of the problem.

• In section 7, we obtain a data dependence result for the solution of the problem.

• In both sections 6 and 7, we offer new definitions pertaining to the problem.

In our analysis we consider the structure of partial order on a metric space.

2. Review of coupled fixed point result of Bhaskar et al.[13]

Here we review a coupled fixed point result due to Bhaskar et al. [13]. This result is instrumental to establishing our results in the following sections of the paper.

Although coupled fixed point was introduced by Guo et al. [14] some time back in 1987, it was only after Bhaskar et al. [13] produced their result in 2006, there have been wide spread interest in this subject. Some prominent references on this topic, amongst others, are [2, 7, 8, 16]. Fixed point method is well known in several areas of mathematics. Coupled fixed point theorems have also been used to solve several problems of mathematics like these discussed in [14, 15]. In the present paper we derive results by use of such methodologies.

In the paper, the notation X^2 stands for $X \times X$ and the notation $(X, d, \preceq)$ stands for a partially ordered metric space.

A coupled fixed point of a mapping $\mathfrak{F} : X^2 \to X$ is an element $(s, t) \in X^2$ satisfying $s = \mathfrak{F}(s, t)$ and $t = \mathfrak{F}(t, s)$.

Problem P. Let $(X, d, \preceq)$ be a metric space with a partial order. The problem is to find a coupled fixed point of a mapping $\mathfrak{F} : X^2 \to X$ under suitable conditions.

Definition 2.1 ([13]). A mapping $\mathfrak{F} : X^2 \to X$, where $(X, \preceq)$ is a partially order set, is called mixed monotonic if for any $u, v \in X$,

$$t_1, t_2 \in X,\ t_1 \preceq t_2 \text{ implies } \mathfrak{F}(t_1, v) \preceq \mathfrak{F}(t_2, v)$$

and

$$s_1, s_2 \in X,\ s_1 \preceq s_2 \text{ implies } \mathfrak{F}(u, s_2) \preceq \mathfrak{F}(u, s_1).$$

Starting with $(X, \preceq)$ we define a partial order " $\leq$ " on the product space X^2 as follows: for $(s, t), (u, v) \in X^2, (u, v) \leq (s, t) \Leftrightarrow u \preceq s$ and $t \preceq v$.

Definition 2.2 ([13]). A partially ordered metric space $(X, d, \preceq)$ is regular if

(i) $x_n \preceq t$, for all n, whenever $\{x_n\}$ is any nondecreasing sequence converging to t;

(ii) $t \preceq x_n$, for all n, whenever $\{x_n\}$ is any nonincreasing sequence converging to t.

Theorem 2.3 ([13]). *Let $(X, d, \preceq)$ be a complete metric space with a partial ordered having regular property. Let $\mathfrak{F} : X^2 \to X$ be a mixed monotonic function such that for all $(s, t),\ (u, v) \in X^2$ with $u \preceq s,\ t \preceq v$,*

$$d(\mathfrak{F}(s,t),\ \mathfrak{F}(u,v)) \leq \frac{\xi}{2}\ [d(s,\ u) + d(t,\ v)],\ \textit{where}\ \xi \in [0,1). \tag{2.1}$$

If there exist $x_0, y_0 \in X$ satisfying $x_0 \preceq \mathfrak{F}(x_0, y_0)$ and $\mathfrak{F}(y_0, x_0) \preceq y_0$, then the sequence $\{(x_n, y_n)\}$ obtained for all $n \geq 1$ as

$$x_n = \mathfrak{F}(x_{n-1},\ y_{n-1}) = \mathfrak{F}^n(x_0,\ y_0) \quad \textit{and} \quad y_n = \mathfrak{F}(y_{n-1}, x_{n-1}) = \mathfrak{F}^n(y_0,\ x_0) \tag{2.2}$$

converges to a coupled fixed point $(x,\ y)$ of $\mathfrak{F}$, that is, $x_n \to x$ and $y_n \to y$ with $x = \mathfrak{F}(x, y)$ and $y = \mathfrak{F}(y, x)$.

Theorem 2.4 ([13]). *The coupled fixed point is unique in Theorem 2.3 if it is further assumed that for every $(x_1, y_1),\ (x_2, y_2) \in X^2$ there exists an element $(x_3, y_3) \in X^2$ which is comparable to both (x_1, y_1) and (x_2, y_2).*

3. Existence and uniqueness of solution of Problem I

In this section we deal with system of nonlinear integral equations and we apply Theorem 2.4 ([13]) to establish the existence and uniqueness of solution of the system in a complete metric space. The system (1.1) will be considered under some suitable conditions.

In this section, we present our main finding, we take help of the coupled results discussed in previous section to prove existence of the unique solution of (1.1).

We take the coupled system of nonlinear integral equations

$$\left.\begin{array}{rcl} x(t) & = & g(t) + \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ x(s),\ y(s))\ ds \qquad \text{and} \\ y(t) & = & g(t) + \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ y(s),\ x(s))\ ds, \quad \lambda \geq 0, \end{array}\right\}$$

where the unknown functions $x(t)$ and $y(t)$ are real valued and continuous on $[a,\ b]$. That is, we investigate the possibility of continuous solution of (1.1).

Consider the metric space $X = C[a,\ b]$, the space of all real valued continuous functions defined on $[a, b]$, endowed with the metric

$$d(x,\ y) = \sup_{t \in [a,\ b]} \mid x(t) - y(t) \mid . \tag{3.1}$$

Assume that this metric space is endowed with the following partial ordered relation $\preceq$. Let in X, the relation $x \preceq y$ holds if $x(t) \leq y(t)$, whenever $a \leq t \leq b$.

We designate the following assumptions by $I_1,\ I_2,\ I_3,\ I_4$ and I_5.

I_1: $g \in X$ and $\mathfrak{h} : [a, b] \times \mathbb{R} \times \mathbb{R} \to [0, \infty),\ K : [a, b] \times [a, b] \to [0, \infty)$ are continuous mappings.

I_2: For $x, y, u, v \in X$ and $s \in [a, b]$, $x \preceq u$ implies $\mathfrak{h}(s, x(s), y(s)) \leq \mathfrak{h}(s, u(s),\ y(s))$ and $y \preceq v$ implies $\mathfrak{h}(s, x(s), v(s)) \leq \mathfrak{h}(s, x(s), y(s))$.

I_3: $\mid \mathfrak{h}(s, x(s), y(s)) - \mathfrak{h}(s, u(s), v(s)) \mid \leq \mathcal{M}(x, y, u, v)$, for $(x, y), (u, v) \in X^2$ with $u \preceq x$ and $y \preceq v$, where

$$\mathcal{M}(x,\ y,\ u,\ v) = \sup_{s\in[a,\ b]} \frac{\mid x(s) - u(s) \mid + \mid y(s) - v(s) \mid}{2}.$$

I_4: $\mid K(t,\ s) \mid \leq m$ and $\xi = \lambda\ (b - a)\ m$ with $0 \leq \xi < 1$.

I_5: There exist $x_0, y_0 \in X$ satisfying the following two inequalities:

$$x_0(t) \leq g(t) + \lambda \int_a^b K(t, s)\ \mathfrak{h}(s, x_0(s), y_0(s))\ ds, \text{ for all } t \in [a, b]$$

and

$$g(t) + \lambda \int_a^b K(t, s)\ \mathfrak{h}(s, y_0(s), x_0(s))\ ds \leq y_0(t), \text{ for all } t \in [a, b].$$

Theorem 3.1. *Let* $(X,\ d) = (C[a,\ b],\ d)$, $\mathfrak{h}$, g, $K(t, s)$ *satisfy all the assumptions* I_1, I_2, I_3, I_4 *and* I_5. *Then the system of equations (1.1) has a unique solution* $(x(t),\ y(t))$ *in* X^2 *and there exist sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *converging respectively to* x *and* y *uniformly in* $[a,\ b]$.

Proof. Define $\mathfrak{F} : X^2 \to X$ as

$$\mathfrak{F}(x,\ y)(t) = g(t) + \lambda \int_a^b K(t,\ s)\ \ \mathfrak{h}(s,\ x(s),\ y(s))\ ds, \quad \text{for all } \ a \leq t \leq b. \tag{3.2}$$

Take $x, y, u, v \in X$ with $x \preceq u$ and $y \preceq v$. By I_1, I_2, we obtain

$$\begin{aligned} \mathfrak{F}(x, y)(t) &= g(t) + \lambda \int_a^b K(t, s)\mathfrak{h}(s, x(s), y(s))ds \\ &\leq g(t) + \lambda \int_a^b K(t, s)\mathfrak{h}(s, u(s), y(s))ds = \mathfrak{F}(u, y)(t), \end{aligned}$$

$$\begin{aligned} \mathfrak{F}(x, y)(t) &= g(t) + \lambda \int_a^b K(t, s)\mathfrak{h}(s, x(s), y(s))ds \\ &\geq g(t) + \lambda \int_a^b K(t, s)\mathfrak{h}(s, x(s), v(s))ds = \mathfrak{F}(x, v)(t), \end{aligned}$$

that is, $\mathfrak{F}(x, y) \preceq \mathfrak{F}(u, y)$ and $\mathfrak{F}(x, v) \preceq \mathfrak{F}(x, y)$. Hence $\mathfrak{F}$ is a mixed monotonic mapping.

By assumptions I_1, I_3 and I_4, for all $(x, y),\ (u, v) \in X^2$ with $u \preceq x,\ y \preceq v$ and for all $a \leq t \leq b$, we get

$$|\mathfrak{F}(x, y)(t) - \mathfrak{F}(u, v)(t)| = \lambda \mid \int_a^b K(t, s)[\mathfrak{h}(s, x(s), y(s)) - \mathfrak{h}(s, u(s), v(s))]ds|$$

$$\leq \lambda \int_a^b m \ \ \mid [\mathfrak{h}(s, x(s), y(s)) - \mathfrak{h}(s, u(s), v(s))]ds \mid$$

$$\leq \lambda\ m \int_a^b \mathcal{M}(x, y, u, v)ds$$

$$= \lambda \ m \int_a^b \sup_{s\in[a,b]} \frac{| \ x(s) - u(s) \ | + | \ y(s) - v(s) \ |}{2} ds$$

$$\leq \lambda \ m \ \frac{[d(x,u) + d(y,v)]}{2} \int_a^b ds$$

$$= \lambda \ m \ (b-a) \ \frac{[d(x,u) + d(y,v)]}{2} = \frac{\xi}{2} \ [d(x,u) + d(y,v)],$$

that is,

$$d(\mathfrak{F}(x,y), \ \mathfrak{F}(u,v)) \leq \frac{\xi}{2} \ [d(x,u) + d(y,v)],$$

where $\xi = \lambda \ m \ (b-a)$ and $\xi \in [0, \ 1)$. From the definition of $\mathfrak{F}$ and the assumption I_5, we have $x_0, y_0 \in X$ satisfying $x_0 \preceq \mathfrak{F}(x_0, y_0)$ and $\mathfrak{F}(y_0, x_0) \preceq y_0$.

Let $\{x_n\}$ be a sequence in X such that $x_n \to x \in X$ as $n \to \infty$. If $\{x_n\}$ is nondecreasing then $x_n \preceq x_{n+1}$, for $n > 0$, that is, $x_n(s) \leq x_{n+1}(s)$, for all n and $s \in [a, \ b]$. Then $x_n(s) \leq x(s)$, for $n > 0$ and $s \in [a,b]$, that is, $x_n \preceq x$, for $n > 0$. If $\{x_n\}$ is nonincreasing then $x_{n+1} \preceq x_n$, for $n > 0$, that is, $x_{n+1}(s) \leq x_n(s)$, for $n > 0$ and $s \in [a,b]$. Then $x(s) \leq x_n(s)$, for $n > 0$ and $s \in [a,b]$, that is, $x \preceq x_n$, for $n > 0$. Therefore, X has regular property.

By application of Theorem 2.3, we get $x, y \in X$ satisfying

$$x(t) = \mathfrak{F}(x, \ y)(t) = g(t) + \lambda \int_a^b K(t, \ s) \ \mathfrak{h}(s, \ x(s), \ y(s)) \ ds$$

and

$$y(t) = \mathfrak{F}(y, \ x)(t) = g(t) + \lambda \int_a^b K(t, \ s) \ \mathfrak{h}(s, \ y(s), \ x(s)) ds,$$

for all $t \in [a,b]$, and corresponding to (2.2) there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\left.\begin{aligned} x_{n+1}(t) = \mathfrak{F}(x_n, \ y_n)(t) = g(t) + \lambda \ \int_a^b K(t, \ s) \ \mathfrak{h}(s, \ x_n(s), \ y_n(s)) \ ds, \\ y_{n+1}(t) = \mathfrak{F}(y_n, \ x_n)(t) = g(t) + \lambda \ \int_a^b K(t, \ s) \ \mathfrak{h}(s, \ y_n(s), \ x_n(s)) \ ds, \end{aligned}\right\} \tag{3.3}$$

and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in X. Then

$$\sup_{t\in[a, \ b]} | \ x_n(t) - x(t) \ | \to 0 \text{ and } \sup_{t\in[a, \ b]} | \ y_n(t) - y(t) \ | \to 0, \text{ as } n \to \infty,$$

that is, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ uniformly on $[a, \ b]$, as $n \to \infty$.

Let $x, y \in X$. Define $z(t) = \max \{x(t), \ y(t)\}$ and $w(t) = \min \{x(t), \ y(t)\}$, for $t \in [a,b]$. Then $x \preceq z$, $y \preceq z$ and $w \preceq x$, $w \preceq y$. Therefore, for any $x, y \in X$, there exist z and $w \in X$ such that z is upper bound of x, y and w is lower bound of x, y.

By application of Theorem 2.4, we have that $(x(t), \ y(t))$ is the unique coupled fixed point of $\mathfrak{F}$, that is, $(x(t), \ y(t))$ is the unique solution of the system (1.1).

Example 3.2. Consider the metric space $X = C[0, \ 1]$ with the metric

$$d(x, \ y) = \sup_{t\in[0, \ 1]} | \ x(t) - y(t) \ |$$

and with a partial ordered relation $\preceq$ defined as $x \preceq y$ if and only if $x(t) \leq y(t)$, whenever $x, y \in X$ and $0 \leq t \leq 1$. Let $h : [0,1] \times \mathbb{R} \times \mathbb{R} \to [0, \infty),\ K : [0,1] \times [1,0] \to [0, \infty),$ and $g \in X$ be defined respectively as follows:

$$h(s,\ u,\ v) = \begin{cases} \frac{u-v}{3}, \text{ if } \ u \geq v \\ 0, \ \text{ otherwise,} \end{cases}$$

$$K(x,\ y) = y, \text{ for } x, y \in [0,\ 1] \text{ and } \ g(t) = 0, \text{ for } t \in [0,1].$$

Take $m = 1$ and $\lambda = \frac{1}{2}$. Let $x_0 = 0$ and $y_0 = c(> 0)$ be two points in X. Then all the conditions of Theorem 3.1 are satisfied and here $(x(t),\ y(t)) = (0,\ 0)$ is the unique solution of the system of equations (1.1). Consider the sequence $\{x_n\}$ and $\{y_n\}$, where $x_n = 0$ for all $n \geq 0$ and $y_0 = c$, $y_n = \frac{c}{3\ 4^n}$ for all $n \geq 1$. Here the sequences $\{x_n\}$ and $\{y_n\}$ in X converge respectively to $x = 0$ and $y = 0$ uniformly in $[0,\ 1]$.

4. Error estimation and rate of convergence

We investigate some aspects of the coupled fixed point problem considered by Bhaskar et al. [13] in this section. We make an error estimation of the coupled fixed point iteration which we construct in this paper. We also investigate the rate of convergence of the iteration process. Such considerations have appeared in the fixed point theory through works like [4].

We now study the rate at which the iteration method of finding the coupled fixed point of Problem P converges if the initial approximation of the coupled fixed point is sufficiently close to the desired coupled fixed point. For this purpose we first define the order of convergence of the Problem P.

Definition 4.1. Problem P is said to be of order r or has the rate of convergence r with respect to $\{(x_n, y_n)\}$ given by equation (2.2) if (i) $\mathfrak{F}$ admits a unique coupled fixed point (x, y), (ii) r is a positive real number for which there exists a finite fixed $C > 0$ for which $R_{n+1} \leq C\ (R_n)^r$, where $R_n = d(x, x_n) + d(y, y_n)$ is the error in n-th iterate and (x_n, y_n) is the n-th approximation of the coupled fixed point (x, y). The constant C is called the asymptotic error.

We study here the rate at which the iteration method of finding the solution of system of integral equations converges if the initial approximation of the solution of the system is sufficiently close to the desired solution of the system. For this purpose we define the order of convergence of the solution of system of integral equations.

Definition 4.2. Problem I is said to be of order r or has the rate of convergence r with respect to $\{(x_n, y_n)\}$ given by equation (3.3) if (i) the system of integral equations (1.1) has a unique solution (x, y), (ii) r is a positive real number for which there exists a finite fixed $C > 0$ for which $R_{n+1} \leq C\ (R_n)^r$, where $R_n = \sup_{s \in [a,\ b]}[\ |\ x(s) - x_n(s)\ |\ +\ |\ y(s) - y_n(s)\ |\ ]$ is the error in n-th iterate and (x_n, y_n) is the n-th approximation of the solution (x, y) of the system of integral equations (1.1). The constant C is called the asymptotic error.

Theorem 4.3. *Let* $(x_0, y_0) \in X^2$ *be the initial approximation of the unique coupled fixed point* (x, y) *of* $\mathfrak{F}$ *in Theorem 2.4. Then* $R_{n+1} \leq \dfrac{\xi^{n+1}}{(1-\xi)}\ [d(x_1, x_0) + d(y_1, y_0)]$, *where* $R_n = d(x, x_n) + d(y, y_n)$ *is the error in n-th iterate and* (x_n, y_n) *is the n-th approximation of the coupled fixed point* (x, y).

Proof. Following the same techniques used in establishing Theorem 2.4 (see [13]), we have the sequence $\{(x_n, y_n)\}$ in X^2 given by equation (2.2). Also,

- $x_n \preceq x_{n+1}$ and $y_{n+1} \preceq y_n$, for all $n \geq 0$,
- both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and $\{(x_n,\ y_n)\}$ converges to a coupled fixed point of $\mathfrak{F}$ in X^2.

As, we consider that $(x,\ y)$ is the unique coupled fixed point of $\mathfrak{F}$, we have $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. By equation (2.2) and the regularity assumption, $x_n \preceq x$ and $y \preceq y_n$, for $n \geq 0$. Using (2.1), we have

$$d(x,\ x_{n+1}) = d(\mathfrak{F}(x,\ y),\ \mathfrak{F}(x_n,\ y_n)) \leq \frac{\xi}{2}\ [d(x, x_n) + d(y, y_n)].$$

Similarly,

$$d(y, y_{n+1}) = d(\mathfrak{F}(y, x), \mathfrak{F}(y_n, x_n)) \leq \frac{\xi}{2}\ [d(x, x_n) + d(y, y_n)].$$

Therefore,

$$R_{n+1} = d(x,\ x_{n+1}) + d(y,\ y_{n+1}) \leq \xi\ [d(x_n, x) + d(y_n, y)] = \xi\ R_n. \tag{4.1}$$

Let,

$$r_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}).$$

It follows from (4.1) that

$$\begin{aligned} R_{n+1} &= d(x,\ x_{n+1}) + d(y,\ y_{n+1}) \leq \xi\ [d(x_n, x) + d(y_n, y)] \\ &\leq \xi\ [d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n+1}, x) + d(y_{n+1}, y)] = \xi\ [R_{n+1} + r_n], \end{aligned} \tag{4.2}$$

which implies that

$$R_{n+1} \leq \frac{\xi}{(1-\xi)}\ r_n. \tag{4.3}$$

Using (2.2), we obtain

$$\begin{aligned} r_{n+1} &= d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \\ &= d(\mathfrak{F}(x_n, y_n),\ \mathfrak{F}(x_{n+1}, y_{n+1})) + d(\mathfrak{F}(y_n, x_n),\ \mathfrak{F}(y_{n+1}, x_{n+1})) \\ &= d(\mathfrak{F}(x_{n+1}, y_{n+1}),\ \mathfrak{F}(x_n, y_n)) + d(\mathfrak{F}(y_n, x_n),\ \mathfrak{F}(y_{n+1}, x_{n+1})) \\ &\leq \frac{\xi}{2}\ [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] + \frac{\xi}{2}\ [d(y_n, y_{n+1}) + d(x_n, x_{n+1})] \\ &= \xi\ [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = \xi\ r_n. \end{aligned}$$

Applying (4.3) repeatedly and using the above inequality, we get

$$R_{n+1} \leq \frac{\xi^{n+1}}{(1-\xi)}\ r_0 = \frac{\xi^{n+1}}{(1-\xi)}\ [d(x_1,\ x_0) + d(y_1,\ y_0)]. \tag{4.4}$$

Remark 4.4. In general, the speed of the iteration depends on the value of ξ; the smaller is the value of ξ, the faster would be the convergence.

Remark 4.5. Above theorem shows that if $0 < \xi < 1$, the error in n-th iterate does not exceed $\frac{\xi^n}{1-\xi}\ [d(x_1,\ x_0) + d(y_1,\ y_0)]$. This error can be made less than a preassigned real number $\varepsilon > 0$ by taking $n \geq \max\left\{\left[\frac{\log(\frac{d(x_1,x_0)+d(y_1,y_0)}{\varepsilon\ (1-\xi)})}{\log\ (\frac{1}{\xi})}\right],\ \ 0\right\} + 1$, where $[y]$ denotes the greatest integer function. This gives the number of iterations n needed to bring the point $(x_n,\ y_n)$ within ε distance of the actual coupled fixed point.

Theorem 4.6. *Assume that the conditions of Theorem 3.1 are satisfied and R_n is the error at n^{th} stage of approximation of solution of the system (1.1). Then*

$$R_{n+1} \leq \frac{[\lambda\ (b-a)m]^{n+1}}{1-\lambda\ (b-a)m} \sup_{s\in[a,b]} [|\ x_1(s) - x_0(s)\ | + |\ y_1(s) - y_0(s)\ |],$$

where (x_0, y_0) is the initial approximation of the solution of (1.1),

$$x_1(t) = g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, x_0(s), y_0(s))ds,$$

$$y_1(t) = g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, y_0(s), x_0(s))ds$$

and m is given in I_4.

Proof. Define $\mathfrak{F} : C[a,\ b] \times C[a,\ b] \to C[a,\ b]$ as in (3.2). Applying Theorems 3.1 and 4.3, we have

$$\begin{aligned} R_{n+1} &\leq \frac{\xi^{n+1}}{(1-\xi)}\ \ [d(x_1,\ x_0) + d(y_1,\ y_0)] \\ &= \frac{[\lambda\ (b-a)\ m]^{n+1}}{[1-\lambda\ (b-a)\ m]} \sup_{s\in[a,\ b]} [\ |\ x_1(s) - x_0(s)\ | + |\ y_1(s) - y_0(s)\ |\ ]\ \ [\text{ by } I_4]. \end{aligned}$$

5. Ulam-Hyers stability

In present section, we investigate Ulam-Hyers stability of the above fixed point problem of coupled mapping. It is a type of stability which was initiated by a mathematical question by Ulam [27] and subsequent partial answer by Hyers [17] and Rassias [23]. The investigation of such stability has been of profound interest in various contexts of mathematics like functional equations, isometries [24], etc.

We consider the issue of stability of the afore-mentioned coupled fixed points. The kind of stability we consider is known as Hyers - Ulam stability which is also known as Hyers - Ulam - Rassias stability or H-U-S stability in contemporary literatures. It has its origin in the work of Ulam [27] and was extended by Hyers [17], Rassias [23] and many others. Its generality makes it applicable to a wide variety of domains like functional equations [9], isometries [24], group homomorphisms [18] and the like. In fixed point theory this kind of stability was considered in recent works like [5, 6, 19]. In [25] one can find the following definition as well as some related notions concerning the Ulam-Hyers stability which is relevant to the present consideration.

Let $S : M \to M$, where (M, d) be a metric space. We say that the fixed point problem $x = Sx$ is Ulam-Hyers stable if for each $\epsilon > 0$ and $y \in X$ satisfying $d(y, Sy) \leq \epsilon$ there exists $x_0 \in X$ for which $x_0 = Sx_0$ and $d(y, x_0) \leq \epsilon$. The essence of the problem of stability is to investigate the fact whether approximate fixed points are approximations of actual fixed points at the same level of accuracy as is evident from the above statement.

Definition 5.1 ([6]). Problem P is Ulam-Hyers stable if for each $\epsilon > 0$ and for each solution $(u^*,\ v^*) \in X^2$ of the inequalities $d(x,\ \mathfrak{F}(x,\ y)) \leq \epsilon$ and $d(y,\ \mathfrak{F}(y,\ x)) \leq \epsilon$ there exists a solution $(x^*,\ y^*) \in X^2$ of Problem P satisfying $\max\{d(u^*,\ x^*),\ d(v^*,\ y^*)\} \leq \phi(\epsilon)$, where $\phi : [0,\ \infty) \to [0,\ \infty)$ is monotone increasing and continuous at 0 with $\phi(0) = 0$.

Being inspired by above definition we give the following definition in case of the system of equations (1.1).

Definition 5.2. Coupled system of nonlinear equations (1.1) is called Ulam-Hyers stable if for each $\epsilon > 0$ and for each solution $(u^*,\ v^*)$ of the inequations

$$\left.\begin{array}{l} \sup_{t\in[a,b]} \mid x(t) - g(t) - \lambda \int_a^b K(t,\ s)\, \mathfrak{h}(s,\ x(s),\ y(s))\, ds \mid < \epsilon \ \text{ and} \\ \sup_{t\in[a,b]} \mid y(t) - g(t) - \lambda \int_a^b K(t,\ s)\, \mathfrak{h}(s,\ y(s),\ x(s))\, ds \mid < \epsilon,\ \lambda \geq 0, \end{array}\right\}$$

there exists a solution $(x^*,\ y^*)$ of (1.1) satisfying

$$\sup_{s\in[a,\ b]} \max\{\mid u^*(s) - x^*(s) \mid, \mid v^*(s) - y^*(s) \mid\} \leq \phi(\epsilon),$$

where $\phi : [0,\ \infty) \to [0,\ \infty)$ is monotone increasing and continuous at 0 with $\phi(0) = 0$.

We use the following assumption to assure the Ulam-Hyers stability of fixed point problem of mixed monotone mapping:

(A1): If $(x^*,\ y^*)$ be any solution of Problem P, then $x \preceq x^*,\ \ y^* \preceq y$, for any $(x,\ y) \in X^2$.

Theorem 5.3. *Problem P is Ulam-Hyers stable if the assumption (A1) is included in Theorem 2.4.*

Proof. By Theorem 2.4, $\mathfrak{F}$ has a unique coupled fixed point (x^*, y^*) (say). Therefore, (x^*, y^*) is a solution of Problem P. Let $\epsilon > 0$ and $(u^*, v^*) \in X^2$ be a solution of the inequalities $d(x, \mathfrak{F}(x,\ y)) \leq \epsilon$ and $d(y, \mathfrak{F}(y,\ x)) \leq \epsilon$. Then $d(u^*,\ \mathfrak{F}(u^*,\ v^*)) \leq \epsilon$ and $d(v^*,\ \mathfrak{F}(v^*,\ u^*)) \leq \epsilon$. By the assumption (A1), we have $u^* \preceq x^*,\ \ y^* \preceq v^*$. Using (2.1), we have

$$\begin{aligned} d(x^*, u^*) &= d(\mathfrak{F}(x^*, y^*), u^*) \leq d(\mathfrak{F}(x^*, y^*),\ \mathfrak{F}(u^*, v^*)) + d(\mathfrak{F}(u^*, v^*), u^*) \\ &\leq \frac{\xi}{2}\, [d(x^*,\ u^*) + d(y^*,\ v^*)] + \epsilon. \end{aligned}$$

Similarly, we have

$$d(y^*,\ v^*) \leq \frac{\xi}{2}\ [d(x^*,\ u^*) + d(y^*,\ v^*)] + \epsilon.$$

Therefore,

$$\begin{aligned} \max\ \{d(x^*,\ u^*),\ d(y^*,\ v^*)\} &\leq \frac{\xi}{2}\ [d(x^*,\ u^*) + d(y^*,\ v^*)] + \epsilon \\ &\leq \xi \max\ \{d(x^*,\ u^*),\ d(y^*,\ v^*)\} + \epsilon, \end{aligned}$$

which implies that

$$\max\ \{d(x^*, u^*),\ d(y^*, v^*)\} \leq \frac{\epsilon}{(1-\xi)}. \tag{5.1}$$

Define $\phi : [0,\ \infty) \to [0,\ \infty)$ as $\phi(t) = \dfrac{t}{(1-\xi)}$. Then

$$\max\ \{d(x^*, u^*),\ d(y^*, v^*)\} \leq \frac{\epsilon}{(1-\xi)} = \phi(\epsilon). \tag{5.2}$$

Since ϕ is monotone increasing, continuous at 0 with $\phi(0) = 0$. Therefore, Problem P is Ulam-Hyers stable.

Now we establish Ulam-Hyers stability of the system (1.1).
Take the following system of integral inequations

$$\left.\begin{array}{c} \sup_{t\in[a,\ b]} \mid x(t) - g(t) - \lambda\ \int_a^b K(t,s)\ \mathfrak{h}(s, x(s), y(s))\ ds \mid\leq \epsilon, \\ \sup_{t\in[a,\ b]} \mid y(t) - g(t) - \lambda\ \int_a^b K(t,s)\ \mathfrak{h}(s, y(s), x(s))\ ds \mid\leq \epsilon, \\ \text{where}\ \ \lambda \geq 0, \quad t \in [a,b]\ \text{ and } \epsilon > 0. \end{array}\right\} \tag{5.3}$$

In the next theorem, we take an extra assumption for assuring the Ulam-Hyers stability of (1.1).

(I_6) If $(x^*,\ y^*)$ is any solution of (1.1), then $u \preceq x^*$ and $y^* \preceq v$ for any $(u, v) \in X \times X$.

Theorem 5.4. *The solution of (1.1) is Ulam-Hyers stable if the assumption* (I_6) *is included in Theorem 3.1.*

Proof. With the help of Theorem 3.1 we get a unique point $(x^*,\ y^*) \in X^2$ which satisfies (1.1). Hence it is the unique coupled fixed point of $\mathfrak{F}$ defined in (3.2). Let $(u^*,\ v^*)$ be a solution of the system of integral inequation (5.3). Hence $(u^*,\ v^*)$ is a solution of $d(x,\ \mathfrak{F}(x,y)) \leq \epsilon$ and $d(y,\ \mathfrak{F}(y,x)) \leq \epsilon$. Also by (I_6), we have $u^* \preceq x^*$ and $y^* \preceq v^*$. By (5.2) of Theorem 5.3, we obtain

$$\sup_{s\in[a,b]} \max\{\mid u^*(s) - x^*(s) \mid, \mid v^*(s) - y^*(s) \mid\} = \max\ \{d(x^*, u^*), d(y^*, v^*)\}$$

$$\leq \frac{\epsilon}{(1-\xi)} = \frac{\epsilon}{1 - \lambda\ (b-a)\ m} = \phi(\epsilon).$$

Therefore, the solution of (1.1) is Ulam-Hyers stable.

6. Well-Posedness

In the current section, we also investigate the well-posedness of the fixed point problem considered here. The study of well-posedness has appeared in several recent works related to fixed point theory as, for instances, in [21, 26].

The notion of well-posedness of a fixed point problem has evoked interests of several mathematicians (see for example [1, 19, 22]). Let $S : M \to M$, where (M, d) is a metric space. The fixed point problem $x = Sx$ is well-posed if S admits a unique fixed point $x \in X$ and $d(x_n, x) \to 0$ as $n \to \infty$ for any sequence $\{x_n\}$ in X with $d(x_n,\ Sx_n) \to 0$ as $n \to \infty$.

Incorporating the ideas and technicalities described above, the followings are the corresponding concepts for coupled mapping and also for system of equations (1.1).

Definition 6.1 ([6]). Problem P is well-posed if (i) $\mathfrak{F}$ has a unique coupled fixed point $(x^*,\ y^*)$, (ii) $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$, whenever $\{(x_n,\ y_n)\}$ is any sequence in X^2 satisfying $d(x_n,\ \mathfrak{F}(x_n,\ y_n)) \to 0$ and $d(y_n,\ \mathfrak{F}(y_n,\ x_n)) \to 0$, as $n \to \infty$.

Definition 6.2. The coupled system of nonlinear integral equations (1.1) is well-posed if

(i) the system has a unique unique solution $(x^*,\ y^*)$,

(ii) $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$, whenever $\{(x_n,\ y_n)\}$ is any sequence of functions satisfying

$$\sup_{t\in[a,\ b]} \mid x_n(t) - g(t) - \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ x_n(s),\ y_n(s))\ ds \mid \to 0$$

and

$$\sup_{t\in[a,\ b]} \mid y_n(t) - g(t) - \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ y_n(s),\ x_n(s))\ ds \mid \to 0,$$

as $n \to \infty$, where $\lambda \geq 0$.

We consider the following condition for the well-posedness of mixed monotone mapping.

(A2): If $(x^*,\ y^*)$ is any solution of Problem P and $\{(x_n,\ y_n)\}$ is any sequence in X^2 with $\lim_{n\to\infty} d(x_n, \mathfrak{F}(x_n,\ y_n)) = 0$ and $\lim_{n\to\infty} d(y_n, \mathfrak{F}(y_n,\ x_n)) = 0$, then $x^* \preceq x_n,\ \ y_n \preceq y^*$, for $n > 0$.

Theorem 6.3. *Problem P is well-posed, if (A2) is taken as the additional assumption in Theorem 2.4.*

Proof. By Theorem 2.4, $\mathfrak{F}$ has a unique coupled fixed point $(x^*,\ y^*)$ (say). Then $(x^*,\ y^*)$ is a solution of Problem P. Let $\{(x_n,\ y_n)\} \in X^2$ be a sequence for which $d(x_n,\ \mathfrak{F}(x_n,\ y_n)) \to 0$ and $d(y_n,\ \mathfrak{F}(y_n,\ x_n)) \to 0$ as $n \to \infty$. By the assumption (A2), we have $x^* \preceq x_n,\ \ y_n \preceq y^*$, for all n. Using (2.1), we have

$$\begin{aligned} d(x_n, x^*) &= d(x_n,\ \mathfrak{F}(x^*, y^*)) \leq d(x_n,\ \mathfrak{F}(x_n, y_n)) + d(\mathfrak{F}(x_n, y_n),\ \mathfrak{F}(x^*, y^*)) \\ &\leq d(x_n,\ \mathfrak{F}(x_n,\ y_n)) + \frac{\xi}{2}\ [d(x_n, x^*) + d(y_n, y^*)]. \end{aligned}$$

Similarly,

$$d(y_n, y^*) \leq d(y_n,\ \mathfrak{F}(y_n, x_n)) + \frac{\xi}{2}\ [d(x_n, x^*) + d(y_n, y^*)].$$

Therefore,

$$\begin{aligned} d(x_n, x^*) + d(y_n, y^*) &\leq d(x_n,\ \mathfrak{F}(x_n,\ y_n)) + d(y_n,\ \mathfrak{F}(y_n, x_n)) \\ &\quad + \xi\ [d(x_n, x^*) + d(y_n, y^*)], \end{aligned}$$

which implies that

$$d(x_n, x^*) + d(y_n, y^*) \leq \frac{d(x_n,\ \mathfrak{F}(x_n,\ y_n)) + d(y_n,\ \mathfrak{F}(y_n, x_n))}{(1-\xi)}.$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} [d(x_n, x^*) + d(y_n, y^*)] &\leq \lim_{n\to\infty} \frac{d(x_n,\ \mathfrak{F}(x_n, y_n)) + d(y_n, \mathfrak{F}(y_n, x_n))}{(1-\xi)} \\ &= 0, \end{aligned}$$

which implies that $\lim_{n\to\infty}[d(x_n, x^*) + d(y_n, y^*)] = 0$, that is, $\lim_{n\to\infty} d(x_n, x^*) = 0$ and $\lim_{n\to\infty} d(y_n, y^*) = 0$, that is, $x_n \to x^*$ and $y_n \to y^*$, as $n \to \infty$. Hence Problem P is well-posed.

In the next theorem, we take an assumption for assurance the well-posedness for the system (1.1).

(I_7) For any sequence $\{(x_n,\ y_n)\}$,

$$\sup_{t\in[a,\ b]} \mid x_n(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,\ x_n(s),\ y_n(s))\ ds \mid \to 0 \quad \text{and}$$

$$\sup_{t\in[a,\ b]} \mid y_n(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,\ y_n(s),\ x_n(s))\ ds \mid \to 0,$$

as $n \to \infty$ imply $x^* \preceq x_n$ and $y_n \preceq y^*$, for all n, where (x^*, y^*) is a solution of (1.1).

Theorem 6.4. *The system (1.1) is well-posed if (I_7) holds in Theorem 3.1.*

Proof. Applying Theorem 3.1, we get a unique point (x^*, y^*) in X^2 which satisfies (1.1). Hence it is a unique coupled fixed point of $\mathfrak{F}$ defined in (3.2). Let $\{(x_n,\ y_n)\}$ be a sequence such that

$$\sup_{t\in[a,\ b]} \mid x_n(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,\ x_n(s),\ y_n(s))\ ds \mid \to 0 \qquad \text{and}$$

$$\sup_{t\in[a,\ b]} \mid y_n(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,\ y_n(s),\ x_n(s))\ ds \mid \to 0,$$

as $n \to \infty$. By the assumption (I_7), we have $x^* \preceq x_n$ and $y_n \preceq y^*$, for all n. Hence we have $d(x_n,\ \mathfrak{F}(x_n,\ y_n)) \to 0$ and $d(y_n,\ \mathfrak{F}(y_n,\ x_n)) \to 0$, as $n \to \infty$ with $x^* \preceq x_n$ and $y_n \preceq y^*$, for all n, where $\mathfrak{F}$ defined in (3.2). As (by application of Theorem 6.3) Problem P is well-posed, the coupled system of nonlinear equations (1.1) is also so.

7. Data dependence result

Let $S_1,\ S_2 : M \to M$ be two mappings, where (M, d) is a metric space such that $d(S_1x,\ S_2x) \leq \eta$ for all $x \in X$, where η is some positive number. Then the problem of data dependence is to estimate the distance between the fixed points of these two mappings. Several research papers on data dependence have been published in recent literatures of which we mention a few in references [6, 10, 25].

Our problem of data dependence is with coupled mappings and their coupled fixed point sets. Such problems for coupled fixed point sets have already appeared in work of Chifu et al. [6]. We formulate a version of the problem suitable to our needs.

Being inspired of the aforesaid ideas we give definitions of data dependence for the case of aforementioned system of integral equations.

Definition 7.1. Let $(x^*,\ y^*)$ be a solution of (1.1) and $(u^*,\ v^*)$ be a solution of the following system

$$\left.\begin{array}{lll} x(t) & = & f(t) + \lambda\ \int_a^b K_1(t,\ s)\ \mathfrak{h}_1(s,\ x(s),\ y(s))\ ds \qquad \text{and} \\ y(t) & = & f(t) + \lambda\ \int_a^b K_1(t,\ s)\ \mathfrak{h}_1(s,\ y(s),\ x(s))\ ds, \quad \lambda \geq 0, \end{array}\right\}$$

for all $t \in [a, b]$. The problem of data dependence is to find

$$\sup_{t\in[a,b]} [\mid x^*(t) - u^*(t) \mid + \mid y^*(t) - v^*(t) \mid].$$

Theorem 7.2. *Let $(X, d, \preceq)$ be a complete and partially ordered metric space having regular property and $\mathfrak{F} : X^2 \to X$. Suppose that all the assumptions of Theorem 2.4 are satisfied. Then $\mathfrak{F}$ has a unique coupled fixed point (x^*, y^*). Moreover, let $T : X^2 \to X$ has nonempty coupled fixed point set. Assume that there exists $M > 0$ for which $d(\mathfrak{F}(x, y), T(x, y)) \leq M$, whenever $(x, y) \in X^2$ and for any coupled fixed point (x, y) of the mapping T, $x \preceq \mathfrak{F}(x, y)$ and $\mathfrak{F}(y, x) \preceq y$ hold. Then*

$$d(x, x^*) + d(y, y^*) \leq \frac{4M}{(1-\xi)},$$

whenever (x, y) is any coupled fixed point of T.

Proof. From Theorem 2.4, $\mathfrak{F}$ has a unique coupled fixed point (x^*, y^*). Suppose that (x, y) is a coupled fixed point of T. Take $x_0 = x$ and $y_0 = y$. Then

$$x_0 = T(x_0,\ y_0) \ \text{ and } \ \ y_0 = T(y_0,\ x_0). \tag{7.1}$$

Let $x_1 = \mathfrak{F}(x_0, y_0)$ and $y_1 = \mathfrak{F}(y_0, x_0)$. Then

$$\left.\begin{array}{c} d(x_0, x_1) = d(T(x_0, y_0),\ \mathfrak{F}(x_0, y_0)) \leq M \\ \text{and} \\ d(y_0, y_1) = d(T(y_0, x_0),\ \mathfrak{F}(y_0, x_0)) \leq M. \end{array}\right\} \tag{7.2}$$

Applying the assumption of the theorem, we get $x_0 \preceq x_1$ and $y_1 \preceq y_0$. Let $x_2 = \mathfrak{F}(x_1, y_1)$ and $y_2 = \mathfrak{F}(y_1, x_1)$. Then by a property of $\mathfrak{F}$, it follows that $x_1 \preceq x_2$ and $y_2 \preceq y_1$. Then taking the technicalities as in establishing of Theorem 2.4 (see [13]), we have a sequence $\{(x_n, y_n)\}$ in X^2 given by equation (2.2) and

- $x_n \preceq x_{n+1}$ and $y_{n+1} \preceq y_n$, for all $n \geq 0$;
- both $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in (X, d) and there exist $u, v \in X$

such that $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = v$;
• (u, v) is a coupled fixed point of $\mathfrak{F}$. As coupled fixed point of $\mathfrak{F}$ unique, we have $u = x^*$, $v = y^*$.

Using (2.1), we obtain

$$\begin{aligned} r_{n+1} &= d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \\ &= d(\mathfrak{F}(x_n, y_n),\ \mathfrak{F}(x_{n+1}, y_{n+1})) + d(\mathfrak{F}(y_n, x_n),\ \mathfrak{F}(y_{n+1}, x_{n+1})) \\ &= d(\mathfrak{F}(x_{n+1}, y_{n+1}),\ \mathfrak{F}(x_n, y_n)) + d(\mathfrak{F}(y_n, x_n),\ \mathfrak{F}(y_{n+1}, x_{n+1})) \\ &\le \frac{\xi}{2}\ [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] + \frac{\xi}{2}\ [d(y_n, y_{n+1}) + d(x_n, x_{n+1})] \\ &= \xi\ [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = \xi\ r_n, \end{aligned}$$

where $r_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$.

Applying the above inequality repeatedly, we get

$$r_{n+1} \le \xi\ r_n \le \xi^2\ r_{n-1} \le ...\xi^n\ r_1 \le \xi^{n+1}\ r_0.$$

Using (7.2) and the above inequality, we have

$$\begin{aligned} d(x_0, x^*) = d(x_0, u) &\le \sum_{i=0}^{n} d(x_i, x_{i+1}) + d(x_{n+1},\ u) \\ &\le \sum_{i=0}^{n} r_i + d(x_{n+1}, u) \le \sum_{i=0}^{n} \xi^i\ r_0 + d(x_{n+1}, u) \end{aligned}$$

and

$$\begin{aligned} d(y_0, y^*) = d(y_0, v) &\le \sum_{i=0}^{n} d(y_i, y_{i+1}) + d(y_{n+1},\ v) \\ &\le \sum_{i=0}^{n} r_i + d(y_{n+1}, v) \le \sum_{i=0}^{n} \xi^i\ r_0 + d(y_{n+1}, u). \end{aligned}$$

Using (7.2), we obtain

$$d(x_0,\ u) \le \sum_{i=0}^{\infty} \xi^i r_0 = \frac{r_0}{(1-\xi)} = \frac{d(x_0, x_1) + d(y_0, y_1)}{(1-\xi)} \le \frac{2\ M}{(1-\xi)} \quad \text{and}$$

$$d(y_0,\ v) \le \sum_{i=0}^{\infty} \xi^i r_0 = \frac{r_0}{(1-\xi)} = \frac{d(x_0, x_1) + d(y_0, y_1)}{(1-\xi)} \le \frac{2\ M}{(1-\xi)}.$$

Hence, $d(x_0, u) + d(y_0, v) \le \dfrac{4\ M}{(1-\xi)}$, that is, $d(x, x^*) + d(y, y^*) \le \dfrac{4\ M}{(1-\xi)}$.

Theorem 7.3. *In Theorem 3.1, we also assume that if* $(x,\ y)$ *is any solution of the following system*

$$\left.\begin{aligned} x(t) = f(t) + \lambda\ \int_a^b K_1(t,\ s)\ \mathfrak{h}_1(s,\ x(s),\ y(s))\ ds \qquad\qquad & \textit{and} \\ y(t) = f(t) + \lambda\ \int_a^b K_1(t, s)\ \mathfrak{h}_1(s, y(s), x(s))\ ds,\ \ \lambda \ge 0 \textit{ for all } t \in [a, b], & \end{aligned}\right\} \quad (7.3)$$

then for $a \leq t \in b$,

$$x(t) \leq g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, x(s), y(s))\ ds$$

and

$$g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, y(s), x(s))\ ds \leq y(t).$$

Further suppose that there exist $\nu, \eta > 0$ *such that*

$$\sup_{t\in[a,b]} |K_1(t,s)\ \mathfrak{h}_1(s, x(s), y(s)) - K(t,s)\ \mathfrak{h}(s, x(s), y(s))| \leq \eta$$

and

$$\sup_{t\in[a,b]} |f(t) - g(t)| \leq \nu.$$

If (x, y) *is any solution of the system (7.3) and* $(x^*,\ y^*)$ *is any solution of the system (1.1), then*

$$\sup_{t\in[a,\ b]} [\ |\ x(t) - x^*(t)\ |\ \ + |\ y(t) - y^*(t)\ |\ ]\ \leq \frac{4\ [\ \nu + \lambda\ \eta\ (b-a)\ ]}{(1-\xi)},$$

where ξ *is given in* I_4.

Proof. Applying Theorem 3.1, we get that the system (1.1) has a unique solution (x^*, y^*) (say). Define $T : X^2 \to X$, where $X = C[a,\ b]$, by

$$T(x,\ y)(t) = g(t) + \lambda \int_a^b K_1(t,\ s)\ \mathfrak{h}_1(s,\ x(s),\ y(s))\ ds, \qquad \text{for all}\ \ a \leq t \leq b. \quad (7.4)$$

Since $(x,\ y)$ is a solution of (7.3), it is a coupled fixed point of T. By the assumptions of the theorem, we have $x(t) \leq \mathfrak{F}(x,\ y)(t)$ and $\mathfrak{F}(y,\ x)(t) \leq y(t)$, for all $t \in [a, b]$, which imply that $x \preceq \mathfrak{F}(x,\ y)$ and $\mathfrak{F}(y,\ x) \preceq y$. Also,

$$\begin{aligned}
&|\ \mathfrak{F}(x,y)(t) - T(x,y)(t)\ |\\
&= |\ f(t) - g(t) + \lambda \int_a^b [K(t,s)\ \mathfrak{h}(s,x(s),y(s)) - K_1(t,s)\ \mathfrak{h}_1(s,x(s),y(s))]ds\ |\\
&\leq |\ f(t) - g(t)\ | + |\ \lambda \int_a^b [K(t,s)\ \mathfrak{h}(s,x(s),y(s)) - K_1(t,s)\ \mathfrak{h}_1(s,x(s),y(s))]ds\ |\\
&\leq \nu + \lambda \int_a^b |\ [K(t,s)\ \mathfrak{h}(s,x(s),y(s)) - K_1(t,\ s)\ \mathfrak{h}_1(s,\ x(s),y(s))]\ |\ \ ds\\
&\leq \nu + \lambda \int_a^b \eta\ ds = \nu + \lambda\ \eta\ (b-a) = M \quad (\text{say }),\ \text{for all}\ \ t \in [a,\ b],
\end{aligned}$$

which means that $\sup_{t\in[a,\ b]} |\ \mathfrak{F}(x,\ y)(t) - T(x,\ y)(t)\ | \leq M$, whenever $(x, y) \in X^2$, that is, $d(\mathfrak{F}(x,y), T(x,y)) \leq M$, whenever $(x,y) \in X^2$. By application of Theorem 7.2, we have

$$\sup_{t\in[a,b]} [\ |\ x(t) - x^*(t)\ | + |\ y(t) - y^*(t)\ |\ ] = d(x, x^*) + d(y, y^*) \leq \frac{4M}{(1-\xi)} = \frac{4\ [\ \nu + \lambda\ \eta\ (b-a)\ ]}{(1-\xi)}.$$

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