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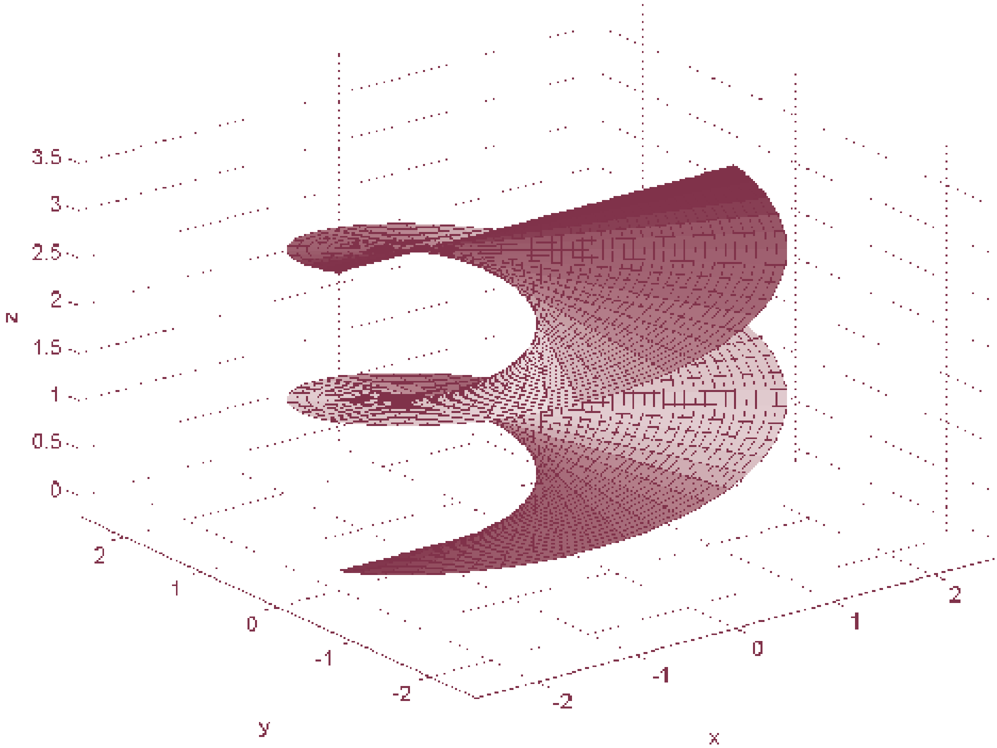

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CONTENTS

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Some operators of fractional calculus and their applications regarding various complex functions analytic in certain domains

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Abstract. In this academic research note, some familiar operators prearranged by fractional-order calculus will first be introduced and various characteristic properties of those operators will next be propounded. Through the instrumentality of various earlier results associating with both those operators and some complex-exponential forms, and also in the light of certain special information in [1], [20], [17] and [38], an extensive result together with a variety of its implications consisting of several exponential type inequalities will then be determined. A number of its possible implications will extra be pointed out.

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1. Introduction and rudiments

In the literature consisted of mathematically academic studies, particularly, fractional-order calculations have been continually encountering either as fractional-order integral(s) *or* as fractional-order derivative(s) in metamathematics. The mentioned-specially calculations, *which* are closely related to each other, are extensive calculations that are frequently applied for both the functions with real variable and the functions with complex variable. There are a wide range of both theoretical and applied research in relation with those. In this respect, in particular, a great variety of

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scientific articles are also presented as extensive works in the section of the references of this research. For instance, one may refer to certain main works in [4], [6], [7], [11], [13], [24], [34], [39] and [40].

As indicated in the abstract, in this scientific note, various fundamental operators associating with fractional calculus, which are specially fractional type derivative operators, will be firstly considered for certain complex functions which are regular in certain domains of the complex plane. In special, in consideration of the fractional derivative(s) operator, a fractional type operator, *which* is encountered as the Tremblay operator in the academic literature, will be then introduced here. Especially, comprehensive studies are still ongoing regarding both this operator and the other specified operators. Nevertheless, for related researchers, we can also offer the results in the earlier papers given in [2], [9], [14], [20], [19] and [31] as a variety of examples.

Furthermore, as several applications of fractional (order) calculus used in various different fields of sciences, numerous papers are also presented in the papers in [3], [10], [12], [15], [16], [22], [25]-[27], [30], [32], [33] and [37]-[40] as examples.

We have given some literature information above. We can now begin to introduce various special information, definitions and several important relationships between those operators that will be necessary for our investigations.

Firstly, let the familiar notations:

$$\mathbb{U} \quad , \quad \mathbb{C} \quad , \quad \mathbb{R} \quad \text{and} \quad \mathbb{N}$$

represent, respectively, the *open unit* disk, the *complex* numbers' set, the *real* numbers' set and the *natural* numbers' set.

Next, for the following numbers:

$$\mathrm{s} \in \mathbb{N} \quad , \quad \tilde{\alpha} \in \mathbb{C} - \{0\} \quad \text{and} \quad \tilde{\alpha}_{\mathrm{S}} \in \mathbb{C}\,,$$

the notation $\mathbf{H}_{\tilde{\alpha}}(\mathrm{s})$ represents the family of the functions $\varrho := \varrho(z)$ being of the forms given by the complex-series expansion:

$$\varrho(z) = \tilde{\alpha} z^{\mathrm{S}} + \tilde{\alpha}_{\mathrm{S}+1} z^{\mathrm{S}+1} + \tilde{\alpha}_{\mathrm{S}+2} z^{\mathrm{S}+2} + \tilde{\alpha}_{\mathrm{S}+3} z^{\mathrm{S}+3} + \cdots \,, \tag{1.1}$$

which are also regular in $\mathbb{U}$.

Most especially, we indicate here that, as simpler expression and more convenient, the following special classes of the regular-functions in the class $\mathbf{H}_{\tilde{\alpha}}(\mathrm{s})$:

$$\mathbf{H}(\mathrm{s}) := \mathbf{H}_1(\mathrm{s}) \quad \text{and} \quad \mathbf{H} := \mathbf{H}(1)$$

can be pointed out as examples and they will also be played important roles for investigations. For this reason, those (more) special classes will taken consideration as revealing various applications of our basic result for researchers. We specially note that, in the mathematical literature, the functions in the class $\mathbf{H}_{\tilde{\alpha}}(\mathrm{s})$ are called as multivalently (*or* s-valently) regular functions (in $\mathbb{U}$) and the functions in the class $\mathbf{H}$ are also called the normalized regular functions in the open set $\mathbb{U}$. For their details and some examples, see [3], [5], [8], [9], [16], [21] and [30].

Secondly, for a function $\varrho := \varrho(z) \in \mathbf{H}_{\tilde{\alpha}}(\mathrm{s})$, we also denote the notation of the Tremblay operator, *which* is specified by fractional derivative (of order λ ($\lambda :=$

$\alpha - \beta; 0 \leq \lambda < 1)$), by any one of the equivalent notations:

$$\mathbf{T}_z^{\alpha,\beta}[\varrho] \quad \Big(or, \quad \mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\Big).$$

At that time, it is generally defined by

$$\mathbf{T}_z^{\alpha,\beta}[\varrho] = \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta}\, \mathbf{D}_z^{\alpha-\beta}\left[z^{\alpha-1}\varrho(z)\right], \tag{1.2}$$

where

$$\beta \in \big(0,1\big] \quad , \quad \alpha \in \big(0,1\big] \quad , \quad \alpha - \beta \in \big[0,1\big) \quad \text{and} \quad z \in \mathbf{U}\,, \tag{1.3}$$

and, for a function $\zeta := \zeta(z)$, any one of the equivalent notations:

$$\mathbf{D}_z^{\delta}[\zeta] \quad \Big(or, \quad \mathbf{D}_z^{\delta}\big[\zeta(z)\big]\Big)$$

denotes the Fractional Derivative Operator (of order δ) and it also identified as in the form given by

$$\mathbf{D}_z^{\delta}[\zeta] = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{\zeta(q)}{(z-q)^{\delta}}\, dq \quad \big(0 \leq \delta < 1\big)\,, \tag{1.4}$$

where ζ is a regular function in a simply connected region of the complex plane comprising its origin, and the multiplicity of $(z-q)^{-\delta}$ is raised by behoving $log(z-q)$ to be real *when* $z - q > 0$.

By taking notice of the restricted conditions in (1.3), as a fairly simple implementation of the respective operators designated by (1.2) and (1.4), for a simple-complex power function (just below), the following-special calculations can be easily propounded:

$$\mathbf{D}_z^{\delta}\left[z^{\mathrm{s}}\right] = \frac{\Gamma(\mathrm{s}+1)}{\Gamma(\mathrm{s}-\delta+1)} z^{\mathrm{s}-\delta} \tag{1.5}$$

and

$$\mathbf{T}_z^{\alpha,\beta}\left[z^{\mathrm{s}}\right] = \frac{\Gamma(\beta)\Gamma(\mathrm{s}+\alpha)}{\Gamma(\alpha)\Gamma(\mathrm{s}+\beta)} z^{\mathrm{s}}\,, \tag{1.6}$$

where

$$\delta \in \big[0,1\big) \quad , \quad \alpha \in \big(0,1\big] \quad , \quad \beta \in \big(0,1\big] \quad , \quad \alpha - \beta \in \big[0,1\big) \quad \text{and} \quad \mathrm{s} \in \mathbb{N}. \tag{1.7}$$

In the same time, in the light of the conditions created by (1.7) and also with the help of the results (1.5) and (1.6), respectively, the following-extra-special results can be also given by

$$z\frac{d}{dz}\Big(\mathbf{D}_z^{\delta}\left[z^{\mathrm{s}}\right]\Big) \equiv z\mathbf{D}_z^{1+\delta}\left[z^{\mathrm{s}}\right] = \frac{\Gamma(\mathrm{s}+1)}{\Gamma(\mathrm{s}-\delta)} z^{\mathrm{s}-\delta} \tag{1.8}$$

and

$$z\frac{d}{dz}\Big(\mathbf{T}_z^{\alpha,\beta}\left[z^{\mathrm{s}}\right]\Big) = \frac{\mathrm{s}\Gamma(\beta)\Gamma(\mathrm{s}+\alpha)}{\Gamma(\alpha)\Gamma(\mathrm{s}+\beta)} z^{\mathrm{s}} \tag{1.9}$$

for all $\mathrm{s} \in \mathbb{N}$. For these determinations in (1.5)-(1.6) and (1.8)-(1.9) and also some of their applications, one can see the recent works in [21] and [20].

In terms of this academic study, we specially note here that, for convenience, both the indicated functions belonging to the general class $\mathbf{H}_{\tilde{\alpha}}(n)$ (defined by any forms of the complex-series expansions like (1.1)) will be considered for our major results and the fundamental definitions (given by (1.2) and (1.4) together with the related special conclusions (determined by (1.5)-(1.9)) will be a quite basic-necessary information for main results of this scientific investigations.

As both a final reminder of this introductory section and one of various properties of the special operator advertised by (1.2), especially, for any regular function $\varrho(z)$ having the form (1.1), the following two-important relationships consisting of a form of the Srivastava-Owa operator and an identity transformation of the Tremblay operator:

$$\mathbf{T}_z^{1,\beta}[\varrho(z)] \equiv \Gamma(\beta) z^{1-\beta}\, \mathbf{D}_z^{1-\beta}\big[\varrho(z)\big] \quad \big(0 < \beta \leq 1\big) \tag{1.10}$$

and

$$\mathbf{T}_z^{\gamma,\gamma}[\varrho(z)] \equiv \varrho(z) \quad \big(0 \leq \gamma < 1\big) \tag{1.11}$$

can be easily ascertained in terms of the character of that operator as its implications *when* the concerned parameters are then selected by letting

$$\alpha := 1$$

and

$$\alpha := \gamma \quad \text{and} \quad \gamma =: \beta\,,$$

respectively.

Specially, for pertinent researchers, recently, by taking advantage of the mentioned fractional derivative operator, the main works (in relation with the Tremblay operator) can be firstly presented and certain relations and also several elementary results for normalized analytic functions (with negative coefficient) are also determined. (cf., e.g., [35]; and, see also [14].)

For various operators specified by fractional-order calculus, it can be looked over the results in the papers in [8], [16] and [17]. By considering certain different methods (*or* ideas), numerous interesting applications of related operators to certain functions analytic in $\mathbb{U}$ can be given in [17], as examples.

Additionally, we also indicate that the fractional derivative(s) operator, identified by (1.4), has comprehensive implications of the well-recognized operator for the literature, *which* also is the Srivastava-Owa fractional derivative operator being of similar form like (1.10). For those and their special forms, one check the work in [1]. See also the results in [15], [16] and [17].

2. Lemmas and results

In this section, in order to get a line on our essential objective, we need some fundamental lemmas with some of applications of fractional calculus (derivatives). Those are only three lemmas, *which* will be taken advantage of starting and then proving for principal results of this investigations.

Firstly, in the light of the conditions given in (1.7), the first assertion, *which* is Lemma 2.1 just below, can be easily demonstrated by applying the elementary results stated in (1.9) and (1.8) (a long with the results in (1.5) and (1.6)) to any function belonging to the class $\mathbf{H}_{\tilde{\alpha}}(n)$. Accordingly, its detail is omitted here. For similar results, one may also center upon the recent papers in [8, 21, 17, 19].

Lemma 2.1. *Let a regular function $\varrho(z)$ have the series form given as in the family $\mathbf{H}_{\tilde{\alpha}}(n)$. For $z \in \mathbb{U}$ and any function $\varrho := \varrho(z)$, the following basic result then holds:*

$$\begin{aligned}\mathbf{T}_z^{\alpha,\beta}[\varrho] &\equiv \mathbf{T}_z^{\alpha,\beta}[\varrho(z)] \\ &= \tilde{\alpha}\,\frac{\Gamma(\beta)\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+\beta)}z^n + \sum_{\ell=n+1}^{\infty}\tilde{\alpha}_n\frac{\Gamma(\beta)\Gamma(\ell+\alpha)}{\Gamma(\alpha)\Gamma(\ell+\beta)}z^\ell. \end{aligned} \tag{2.1}$$

The second assertion is just below and it is a special form of the well-known-elementary results of complex exponential. For both it and some of its implications, one may check the paper given in [18].

Lemma 2.2. *Let $\omega \in \mathbb{R}$ and also let $z \in \mathbb{C} - \{0\}$. Then, the following-complex exponentiation is true.*

$$z^\omega = |z|^\omega\big[Cos\big(\omega\, arg(z)\big) + i\, Sin\big(\omega\, arg(z)\big)\big]. \tag{2.2}$$

The last assertion, *which* is Lemma 2.3 just below, is a well-known important tool and very useful auxiliary theorem proven in [28]. For some of its applications, one can easily arrive at various works in the literature. For its detail, one may also refer to the paper given by [23].

Lemma 2.3. *Let $\xi := \xi(z)$ be a regular function in the domain $\mathbb{U}$ and also be of the form given as in (1.1). For $z \in \mathbb{U}$ and for any $z_0 \in \mathbb{U}$, if*

$$\big|\xi(z_0)\big| = \max\Big\{\,\big|\xi(z)\big| \;:\; |z| \le |z_0|\,\Big\}, \tag{2.3}$$

then there exists any positive number λ such that

$$z_0\xi'(z_0) = \lambda\xi(z_0), \tag{2.4}$$

where $\lambda \in \mathbb{R}$ with $\lambda \geq n$ $(n \in \mathbb{N})$.

In accordance with principal assertions, namely, Lemmas 2.1-2.3 just above, we can then compose our comprehensive result appertaining to the functions belonging to in the class $\mathbf{H}_{\tilde{\alpha}}(n)$, *which* will be specified by the special operator (1.2).

Theorem 2.4. *Under the mentioned conditions of both the parameters in (1.3) and the definitions in (1.2) and (1.4), let the parameters Υ, Λ, ∇ and Θ have the conditions determined as follows:*

$$\Upsilon \in \mathbb{R} - \{0\} \quad , \quad \Lambda \geq m \quad , \quad \nabla \in \mathbb{C} \quad \textit{and} \quad 0 \leq \Theta < 2\pi\,, \tag{2.5}$$

where $m \in \mathbb{N}$ and $0 < |\nabla| < 1$. Then, for some $z \in \mathbb{U}$ and for any function $\varrho(z) \in \mathbf{H}_{\tilde{\alpha}}(n)$, if any one of the statements given by

$$\Re e\left\{\left[z\Big(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\Big)^{(n+1)}\right]^{\Upsilon}\right\} \neq \Lambda^{\Upsilon}|\nabla|^{\Upsilon}\, Cos\Big(\Upsilon\big[\Theta + Arg(\nabla)\big]\Big) \tag{2.6}$$

and

$$\Im m\left\{\left[z\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n+1)}\right]^{\Upsilon}\right\}\neq \lambda^{\Upsilon}|\nabla|^{\Upsilon}\,Sin\Big(\Upsilon\left[\Theta+Arg(\nabla)\right]\Big) \tag{2.7}$$

is satisfied, then the statement given by

$$\left|\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)}-\tilde{\alpha}\,\mathbb{I}_n^n(\alpha,\beta)\right|<|\nabla| \tag{2.8}$$

is satisfied, which also is quite clear that

$$\left|\Re e\left[\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)}\right]-\Re e\big(\tilde{\alpha}\big)\,\mathbb{I}_n^n(\alpha,\beta)\right|\leq|\nabla| \tag{2.9}$$

and

$$\left|\Im m\left[\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)}\right]-\Im m\big(\tilde{\alpha}\big)\,\mathbb{I}_n^n(\alpha,\beta)\right|\leq|\nabla| \tag{2.10}$$

where

$$\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)}:=\frac{d^n}{dz^n}\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)\quad\big(n\in\mathbb{N}\cup\{0\}\big) \tag{2.11}$$

and

$$\mathbb{I}_v^u(\alpha,\beta)=\frac{u!}{(u-v)!}\,\frac{\Gamma(\beta)\Gamma(u+\alpha)}{\Gamma(\alpha)\Gamma(u+\beta)}\quad\big(v<u;u\in\mathbb{N};v\in\mathbb{N}\big), \tag{2.12}$$

and, also, here and throughout this research note, the values of the complex powers in (2.6) and (2.7) are considered as their principal values.

Proof. Let the interested function $\varrho:=\varrho(z)$ be of the form in the class $\mathbf{H}_{\tilde{\alpha}}(n)$. When taking into account the equivalent relation in (2.11) and the determined result in (1.9) (of Lemma 2.1), its nth derivative:

$$\begin{aligned}\left(\mathbf{T}_z^{\alpha,\beta}[\varrho]\right)^{(n)}&=\frac{d^n}{dz^n}\left(\tilde{\alpha}\,\frac{\Gamma(\beta)\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+\beta)}\,z^{\mathrm{n}}\right.\\&\qquad\left.+\,\tilde{\alpha}_{\mathrm{n+1}}\frac{\Gamma(\beta)\Gamma(n+1+\alpha)}{\Gamma(\alpha)\Gamma(n+1+\beta)}\,z^{\mathrm{n+1}}+\;\cdots\right)\\&=\tilde{\alpha}\,\mathbb{I}_n^n(\alpha,\beta)+\tilde{\alpha}_{\mathrm{n+1}}\,\mathbb{I}_n^{n+1}(\alpha,\beta)z^1+\tilde{\alpha}_{\mathrm{n+2}}\,\mathbb{I}_n^{n+2}(\alpha,\beta)z^2+\;\cdots\end{aligned} \tag{2.13}$$

can be easily calculated, *where* the notation $\mathbb{I}_r^s(\alpha,\beta)$ above is defined by (2.12).

For the proof of Theorem 1, in the light of such information (2.12) and (2), for a n-valently regular function like any form $\varrho:=\varrho(z)$ in $\mathbf{H}_{\tilde{\alpha}}(n)$, there is a need to consider a function $\Omega(z)$ in the form given by

$$\left(\mathbf{T}_z^{\alpha,\beta}[\varrho]\right)^{(n)}=\tilde{\alpha}\,\mathbb{I}_n^n(\alpha,\beta)+\Phi\,\Omega(z)\quad\big(0<|\Phi|<1;z\in\mathbb{U}\big). \tag{2.14}$$

In that case, as a result of simple elementary operations, one can easily distinguish that the described function $\Omega(z)$ belongs to the class $\mathbf{H}_{\tilde{\alpha}}(m)$ $(m\in\mathbb{N})$. Thereby, both the related function $\Omega(z)$ both is regular in the set $\mathbb{U}$ and it can be considered for the proof of Theorem 2.4. By differentiating of both sides of (2.14) with respect to the complex variable z, we then get that

$$\frac{d}{dz}\left\{\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)}\right\}\equiv\left(\mathbf{T}_z^{\alpha,\beta}[\varrho]\right)^{(n+1)}=\Phi\,\Omega'(z)\quad\big(0<|\Phi|<1;z\in\mathbb{U}\big). \tag{2.15}$$

We now assert that $|\Omega(z)| < 1$ in $\mathbb{U}$. In fact, if not, then, according to (2.3) (of Lemma 2.3), there exists a point z_0 belonging to $\mathbb{U}$ such that

$$\max\Big\{ \big|\Omega(z)\big| \,:\, |z| \leq |z_0| \quad \big(z, z_0 \in \mathbb{U}\big) \Big\} = \big|\Omega(z_0)\big| = 1\,,$$

which readily yields that

$$\Omega(z_0) = e^{i\Delta} \quad \big(0 \leq \Delta < 2\pi; z_0 \in \mathbb{U}\big).$$

In the present case, the expression (2.4) (of Lemma 2.3) also gives rise to

$$z_0\,\Omega'(z_0) = \lambda\Omega(z_0) = \lambda e^{i\Delta} \quad \big(\lambda \geq m; m \in \mathbb{N}\big).$$

Therefore, of course, for all $\lambda \geq m \geq n$ $(n, m \in \mathbb{N})$, by setting $z := z_0$ and also by means of the main relations (2.2) (of Lemma 2.2) and (2.4) (of Lemma 2.3), the expression (2.15) lightly follows that

$$\begin{aligned}
\Re e&\left\{ \left(z\Big(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\Big)^{(n+1)} \right)^{\mathrm{r}} \Bigg|_{z:=z_0} \right\} \\
&= \Re e\left\{ \Big(\Phi\, z_0\,\Omega'\big(z_0\big)\Big)^{\mathrm{r}} \right\} \\
&= \Re e\left\{ \Big(\Phi\lambda\Omega\big(z_0\big)\Big)^{\mathrm{r}} \right\} \\
&= \Re e\left\{ \Big[\lambda\Phi e^{i\Delta}\Big]^{\mathrm{r}} \right\} \qquad (2.16)\\
&= \Re e\left\{ \big|\lambda\Phi e^{i\Delta}\big|^{\mathrm{r}} e^{i\mathrm{r}\mathrm{Arg}\left(\lambda\Phi e^{i\Delta}\right)} \right\} \\
&= \Re e\left\{ \big|\lambda\Phi\big|^{\mathrm{r}} e^{i\mathrm{r}\mathrm{Arg}\left(\Phi e^{i\Delta}\right)} \right\} \quad (\textit{since}\ \lambda \geq m \geq 1) \\
&= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Cos}\big[\mathrm{r}\mathrm{Arg}\big(\Phi e^{i\Delta}\big)\big] \\
&= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Cos}\big[\mathrm{r}\big(\mathrm{Arg}(\Phi) + \mathrm{Arg}\big(e^{i\Delta}\big)\big] \\
&= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Cos}\big[\mathrm{r}\big(\Delta + \mathrm{Arg}(\Phi)\big)\big]
\end{aligned}$$

and

$$\begin{aligned}
\Im m&\left\{ \left(z\Big(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\Big)^{(n+1)} \right)^{\mathrm{r}} \Bigg|_{z:=z_0} \right\} \\
&= \Im m\left\{ \Big(\Phi\, z_0\,\Omega'\big(z_0\big)\Big)^{\mathrm{r}} \right\} \\
&= \Im m\left\{ \Big(\Phi\lambda\Omega\big(z_0\big)\Big)^{\mathrm{r}} \right\} \\
&= \Im m\left\{ \Big[\lambda\Phi e^{i\Delta}\Big]^{\mathrm{r}} \right\} \qquad (2.17)\\
&= \Im m\left\{ \big|\lambda\Phi e^{i\Delta}\big|^{\mathrm{r}} e^{i\mathrm{r}\mathrm{Arg}\left(\lambda\Phi e^{i\Delta}\right)} \right\}
\end{aligned}$$

$$= \Im m\left\{\left|\lambda\Phi\right|^{\mathrm{r}} e^{i\mathrm{r}\mathrm{Arg}\left(\Phi e^{i\Delta}\right)}\right\} \quad (since\ \lambda \geq m \geq 1)$$
$$= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Sin}\left[\mathrm{r}\mathrm{Arg}\left(\Phi e^{i\Delta}\right)\right]$$
$$= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Sin}\left[\mathrm{r}\left(\Delta + \mathrm{Arg}(\Phi)\right)\right],$$

where $\lambda \geq m$ ($m \in \mathbb{N}$), $0 \leq \Delta < 2\pi$, $\mathrm{r} \in \mathbb{R}$ and $\Phi \in \mathbb{C}$ ($0 < |\Phi| < 1$). But, unfortunately, the results determined as in (2.16) and (2.17) are, respectively, contradictions with the hypotheses of Theorem 2.4, *which* are the mentioned results presented by (2.16) and (2.17) *when* setting

$$\lambda := \Lambda \quad , \quad \Phi := \nabla \quad , \quad \mathrm{r} := \Upsilon \quad \text{and} \quad \Delta := \Theta.$$

This means that there is no $z_0 \in \mathbb{U}$ satisfying the condition $\left|\Omega(z_0)\right| = 1$. Therefore, we decide upon that it has to be in the form $\left|\Omega(z)\right| < 1$ for all $z \in \mathbb{U}$. Consequently, for all functions $\varrho(z) \in \mathbf{H}_{\tilde{\alpha}}(n)$, the expression (2.14) follows that

$$\left|\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)} - \tilde{\alpha}\,\mathbb{I}_n^n(\alpha,\beta)\right| = \left|\Phi\,\Omega(z)\right| < \left|\Phi\right| \quad \left(0 < |\Phi| < 1; z \in \mathbb{U}\right),$$

which is equivalent to the provision of Theorem 2.4, namely, the statement in (2.8) *when* $\Phi := \nabla$. Finally, the basic relationships, *which* are both between (2.8) and(2.9) and between (2.8) and (2.10), can be easily seen propositions. Thus, this ends the desired proof. □

3. Conclusion and recommendations

In this part, *which* is the last part of this comprehensive-research note, we would like to mention various special implications and suggestions relating to our investigations for our readers.

Here we want to bring forward certain conclusions and also to give implicit recommendations concerning our main results. As emphasized in the abstract of this study, the main purpose of this comprehensive study was to present both the basic concepts about some operators of fractional derivatives and to introduce a special operator defined with the help of those, *which* is expressed as various works relating with the Tremblay operator (cf., e.g., [1], [38], [15] and [8]), as indicated before. In any case, these were also carried out in the first chapter. In fact, some important relationships between the respective operators as in (1.2) and (1.4), and also, for certain regular functions like (1.1), a number of their basic applications were presented as in (1.5), (1.6) and (1.8)-(1.11). Clearly, those relevant relations and special implications play a big role both for our essential result given in this section above and for all of its possible special consequences. By focusing especially on our main comprehensive result, namely Theorem 2.4, and its proof, that is, with the help of the relevant theorem and its proof, different analytical and geometrical new results specified by the mentioned operators and naturally the mentioned functions (in the classes $\mathbf{H}_{\tilde{\alpha}}(n)$ *or* its special subclasses $\mathbf{H}(s)$ and $\mathbf{H}$) can also be determined (*or* calculated) (cf., e.g., [5], [10], [13] and [28]). Lastly, most particularly, the parameter Υ, considered in Theorem 2.4, can be chosen as complex number. For possible details of both this and other suggestions, one can check the works given in [8], [15], [21] and [17]-[22] as some of different investigations.

We choose to leave the special details of the relationships between those operators of fractional calculus (that is, that derivatives) and a large number of possible-logic implications of our principal result as an exercise for the interested researchers (*or* readers). Nevertheless, we also want to find out only one private result together with one of its special forms, *which* both associates with Theorem 2.4 and has wide range of (more) special results according to the facts of suitable values of the related parameters.

Inside of the extra information in the first section, through the instrumentality of the special relation (1.11) together with taking $\tilde{\alpha} := 1$ in Theorem 2.4, the following-extensive result can be easily designated for all nth derivative of any regular function $\varrho(z)$ in the special class $\mathbf{H}(n)$ (of the general class $\mathbf{H}_{\tilde{\alpha}}(n)$), *which* also includes numerous geometric properties of n-valently regular functions in $\mathbb{U}$. Specifically, for more detailed information in relation to those analytic-geometric properties, one may center on the main works given by [3], [10] and [27].) Shortly, the desired-special result can be easily constituted as in the following Proposition (just below).

Proposition 3.1. *Under the mentioned conditions of both the parameters* Υ, Λ, ∇ *and* Θ *designated as in (2.5) and for any* n*-valently regular function* $\varrho(z)$ *in the class* $\mathbf{H}(n)$, *if any one of the statements given by*

$$\Re e\left\{\left[z\Big(\varrho(z)\Big)^{(n+1)}\right]^{\Upsilon}\right\} \equiv \Re e\left\{\Big(z\varrho^{(n+1)}(z)\Big)^{\Upsilon}\right\}$$

$$\neq \Lambda^{\Upsilon}|\nabla|^{\Upsilon}\,Cos\Big(\Upsilon\left[\Theta + Arg(\nabla)\right]\Big)$$

and

$$\Im m\left\{\left[z\Big(\varrho(z)\Big)^{(n+1)}\right]^{\Upsilon}\right\} \equiv \Im m\left\{\Big(z\varrho^{(n+1)}(z)\Big)^{\Upsilon}\right\}$$

$$\neq \Lambda^{\Upsilon}|\nabla|^{\Upsilon}\,Sin\Big(\Upsilon\left[\Theta + Arg(\nabla)\right]\Big)$$

is true, then the statement given by

$$\left|\varrho^{(n)}(z) - n!\right| < |\nabla|$$

is also true, which also requires to more simple inequalities given by

$$\left|\Re e\Big(\varrho^{(n)}(z)\Big) - n!\right| \leq |\nabla|$$

and

$$\left|\Im m\Big(\varrho^{(n)}(z)\Big) - n!\right| \leq |\nabla|$$

where

$$\varrho^{(n)}(z) := \frac{d^n}{dz^n}\Big(\varrho(z)\Big)$$

for all $n \in \mathbb{N} \cup \{0\}$ *and for some* $z \in \mathbb{U}$.

By putting $n := 1$, and $\Lambda := 1$ in Proposition 3.1 (*or,* equivalently, by taking $\alpha := \gamma$, $\beta := \gamma$, $\Lambda := 1$ and $n := 1$ in the concerned theorem, *i.e.,* in Theorem 2.4), one of the exclusive-special results of any normalized-regular function $\varrho(z)$ in the more special class $\mathbf{H}$ (of the general class $\mathbf{H}_{\tilde{\alpha}}(n)$) can be easily determined as in the following Proposition (below).

Proposition 3.2. *Under the mentioned conditions of both the parameters* Λ, ∇ *and* Θ *designated as in (16) and for any normalized regular function* $\varrho(z)$ *in the class* $\mathbf{H}$, *if any one of*

$$\Re e\Big(\varrho''(z)\Big) \neq \Lambda|\nabla|Cos\Big(\Theta + Arg(\nabla)\Big)$$

and

$$\Im m\Big(\varrho''(z)\Big) \neq \Lambda|\nabla|Sin\Big([\Theta + Arg(\nabla)\Big)$$

holds true, then

$$|\varrho'(z) - 1| < \big|\nabla\big|$$

also holds true, which also requires to

$$\Big|\Re e\Big(\varrho'(z)\Big) - 1\Big| \leq \big|\nabla\big| \quad and \quad \Big|\Im m\Big(\varrho'(z)\Big) - 1\Big| \leq \big|\nabla\big|$$

where $z \in \mathbb{U}$.

As a final note of this research, in the light of the two-special propositions of our extensive result above or/and by considering certain extra conditions *when* there needs any necessity, we want to present to the attention of the related researchers to describe (*or* redescribe) each one of those possible-special results can be designated by making use of various types of the normalized-regular functions (*or* the multivalently-regular functions) in certain domains of the complex plane.

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Existence and Ulam stability of initial value problem for fractional perturbed functional q-difference equations

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Abstract. In this work, we discuss the existence and uniqueness of solutions to the initial value problem for perturbed functional fractional q-difference equations involving q-derivative of the Caputo sense. By applying Banach contraction principle and Burton and Kirk's fixed point theorems. Further, we present the Ulam-Hyers and Ulam-Hyers-Rassias stabilities results by using direct analysis methods. Finally, we give two examples illustrating of the results.

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1. Introduction

Fractional calculus is a significant branch in mathematical analysis. Indeed, Leibniz and Newton developed differential calculus, it has numerous applications in various sciences, for example, mechanics, electricity, biology. Also, Fractional differential equations play a fundamental role in the modeling of a considerable number of phenomena in many areas. Currently being addressed by many researchers of various fields of science and engineering such as physics, chemistry, biology, economics, control theory, and biophysics, etc. For more details, see the books of Hilfer [18] and Tarasov *et al.* [35], Kilbas *et al.* [24] and Samko *et al.* [33], Podlubny [26] and Miller *et al.* [25].

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The q-difference calculus is an interesting and old subject. In 1910, Jackson [21, 20] introduced and developed q-difference calculus or quantum calculus in a systematic way, basic definitions and properties of q-difference calculus can be found in [16, 23]. Then, Al-Salam [11] and Agarwal [5] proposed the fractional q-difference calculus. Due to it applicability in mathematical modeling in different branches like technical sciences, engineering, physics and biomathematics, it has drawn wide attention to many researchers.

Fractional q-difference equations initiated at the beginning of the nineteenth century [4, 15] and received significant attention in recent years. While some interesting details about initial and boundary value problems of fractional q-difference equations can be found in books of Ahmad *et al.* [7] and Annaby *et al.* [12]; see the papers of Ahmed *et al.* [6], Abbas *et al.* [2, 3], Allouch *et al.* [9, 8, 10] and Samei *et al.* [32].

The stability of functional equations was originally emerged Ulam [36, 37] and Hyers [19]. Thereafter, the stability of this type is called Ulam-Hyers Stability. In 1978, Rassias [29] provided a generalization of the Hyers theorem which allows the Cauchy difference to be unbounded. Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of fractional differential equations. See the papers of Rassias [29], Rus [30, 31], Abbes *et al.* [1, 3], Jung [22], Taieb *et al.* [34] and Wang *et al.* [38].

In [13], Belarbi *et al.* studied the initial value problem (IVP for short) for perturbed fractional order functional differential equations of the form:

$$D^{\alpha}y(t) = f(t, y_t) + g(t, y_t), \text{ for a.e. } t \in J = [0, b], \quad 0 < \alpha < 1,$$

$$y(t) = \phi(t), \ t \in [-r, 0],$$

where ${}^{C}D^{\alpha}$ is the Riemman-Liouville fractional derivative, $f, g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are given functions and $\phi \in C([-r, 0], \mathbb{R})$ with $\phi(0) = 0$. For any continuous function y defined on $[-r, b]$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ defined by:

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(.)$ represents the history of the state from time $t - r$ up to the present time t.

Motivated by aforementioned work, in this paper, we concentrate on the existence, uniqueness and Ulam stability of solutions of the initial value problem (IVP for short) for perturbed functional fractional q-difference equations of the form:

$${}^{C}D_q^{\alpha}y(t) = f(t, y_t) + g(t, y_t), \text{ for a.e. } t \in J = [0, T], \quad 0 < \alpha < 1, \tag{1.1}$$

$$y(t) = \varphi(t), \ t \in \overline{J} = [-d, 0], \tag{1.2}$$

where $T > 0, \ d > 0, \ q \in (0, 1), \ {}^{C}D_q^{\alpha}$ is the Caputo fractional q-derivative of order α, $f, g : J \times C([-d, 0], \mathbb{R}) \to \mathbb{R}$ are given functions and $\varphi \in C([-d, 0], \mathbb{R})$ with $\varphi(0) = 0$. For any continuous function y defined on $[-d, T]$ and any $t \in J$, we denote by y_t the element of $C([-d, 0], \mathbb{R})$ defined by:

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-d, 0].$$

Here $y_t(.)$ represents the history of the state from time $t-d$ up to the present time t.

The work is arranged as follows : In Section 2, we introduce some preliminary, including basic definitions and properties of fractional q-calculus. In Section 3, we prove the existence and uniqueness results for the problem (1.1)-(1.2), we give two results, the first one is based on Banach contraction principle (Theorem 3.3), the second is based on Burton and Kirk's fixed point theorem (Theorem 3.4). In Section 4, Ulam-Hyers and Ulam-Hyers-Rassias stabilities theorems are presented. In Section 5, we give two examples to illustrate the obtained results. Finally, we end with the conclusion.

2. Preliminaries

In this section, we present some basic definitions, lemmas and notations which will be used in this paper.

Let $T > 0$, $d > 0$ and define $J := [0, T]$, $\overline{J} := [-d, 0]$. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the usual supremum norm:

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\}.$$

Also, $C(\overline{J}, \mathbb{R})$ is endowed with the norm $\|.\|_*$ defined by:

$$\|y\|_* = \sup\{|y(t)| : -d \le t \le 0\}.$$

Let $\mathcal{C} = \{y : [-d, T] \to \mathbb{R} : y|_{[-d,0]} \in C(\overline{J}, \mathbb{R}) \text{ and } y|_{[0,T]} \in C(J, \mathbb{R})\}$ is a Banach space with the norm:

$$\|y\|_{\mathcal{C}} = \sup\{|y(t)| : -d \le t \le T\}.$$

Now, we introduce some definitions and properties of fractional q-calculus [16, 23].

For $0 < q < 1$, we set:

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power $(a-b)^{(n)}$ is expressed by:

$$(a-b)^{(0)} = 1, \; (a-b)^{(n)} = \prod_{k=0}^{n-1}(a - bq^k), \; a, b \in \mathbb{R}, \; n \in \mathbb{N}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}} \right), \; a, b \in \mathbb{R}.$$

Note that if $b = 0$, then $a^{(\alpha)} = a^\alpha$.

Definition 2.1. [23] The q-gamma function is defined by:

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \; \alpha \in \mathbb{R} - \{0, -1, -2, \ldots\}.$$

Notice that the q-gamma function satisfies $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$.

Definition 2.2. [23] The q-derivative of order $n \in \mathbb{N}$ of a function $f : J \to \mathbb{R}$, is defined by $(D_q^0 f)(t) = f(t)$,

$$(D_q f)(t) = (D_q^1 f)(t) = \frac{f(t) - f(qt)}{(1-q)t},\ t \neq 0,\ (D_q f)(0) = \lim_{t\to 0}(D_q f)(t),$$

and

$$(D_q^n f)(t) = (D_q^1 D_q^{n-1} f)(t),\ t \in J,\ n \in \{1, 2, \ldots\}.$$

Set $J_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3. [23] The q-integral of a function $f : J_t \to \mathbb{R}$, is given by:

$$(I_q f)(t) = \int_0^t f(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q f)(t) = f(t)$, while if f is continuous at 0, then

$$(I_q D_q f)(t) = f(t) - f(0).$$

Definition 2.4. [5] Let $\alpha \in \mathbb{R}_+$ and function $f : J \to \mathbb{R}$. The fractional q-integral of the Riemann-Liouville type of order α is defined by $(I_q^0 f)(t) = f(t)$, and

$$(I_q^\alpha f)(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s,\ t \in J.$$

Note that for $\alpha = 1$, we have $(I_q^1 f)(t) = (I_q f)(t)$.

Lemma 2.5. [27] *For $\alpha \in \mathbb{R}_+$ and $\beta \in (-1, +\infty)$, we have:*

$$(I_q^\alpha (t-a)^{(\beta)})(t) = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(t-a)^{(\alpha+\beta)},\ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)}.$$

Definition 2.6. [28] The fractional q-derivative of the Riemann-Liouville type of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$, is defined by $(D_q^0 f)(t) = f(t)$, and

$$(D_q^\alpha f)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(t),\ t \in J,$$

where $[\alpha]$ is the integer part of α.

Definition 2.7. [28] The fractional q-derivative of the Caputo type of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$, is defined by $(D_q^0 f)(t) = f(t)$, and

$$({}^C D_q^\alpha f)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(t),\ t \in J.$$

where $[\alpha]$ is the integer part of α.

Lemma 2.8. [28] *Let $\alpha, \beta \in \mathbb{R}_+$ and let f be a function defined on J. Then, the next identities hold:*

(i) $(I_q^\alpha I_q^\beta f)(t) = (I_q^{\alpha+\beta} f)(t)$.
(ii) $(D_q^\alpha I_q^\alpha f)(t) = f(t)$.

Lemma 2.9. [28] *Let $\alpha \in \mathbb{R}_+$ and let f be a function defined on J. Then, the following equality holds:*

$$(I_q^\alpha \, {}^C D_q^\alpha f)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha \, {}^C D_q^\alpha f)(t) = f(t) - f(0).$$

Next, we introduce the main fixed point theorems.

Theorem 2.10. *(Banach Contraction Principle)* [17]
Let $\mathcal{M}$ be a non-empty closed subset of a Banach space X, then any contraction mapping F of $\mathcal{M}$ into itself has a unique fixed point.

Theorem 2.11. *(Burton and Kirk)* [14]
Let X be a Banach space, and $A, B : X \to X$ two operators satisfying:

(i) *A is a contraction, and*
(ii) *B is completely continuous.*

Then either

(a) *the operator equation $y = A(y) + B(y)$ has a solution, or*
(b) *the set $\Omega = \left\{y \in X : \lambda A\left(\frac{y}{\lambda}\right) + \lambda B(y) = y\right\}$ is unbounded for $\lambda \in (0,1)$.*

Finally, we state the following generalization of **Gronwall**'s lemma.

Lemma 2.12. *(Gronwall lemma)* [39]
Let $u : J \to [0, +\infty)$ be a real function and $v(.)$ is a nonnegative, locally integrable function on J. Assume that there is a constant $c > 0$ and $0 < \alpha < 1$ such that

$$u(t) \leq v(t) + c \int_0^t (t-s)^{-\alpha} u(s) ds.$$

Then, there exists a constant $\delta = \delta(\alpha)$ such that

$$u(t) \leq v(t) + \delta c \int_0^t (t-s)^{-\alpha} v(s) ds, \text{ for every } t \in J.$$

3. Existence and uniqueness results

In this section, we present the existence and uniqueness of solutions for the problem (1.1)-(1.2).

Let us start by defining what we mean by a solution of the problem (1.1)-(1.2).

Definition 3.1. A function $y \in \mathcal{C}$ is said to be a solution of the problem (1.1)-(1.2) if y satisfies the equation $({}^C D_q^\alpha y)(t) = f(t, y_t) + g(t, y_t)$ on J, and satisfies the condition $y(t) = \varphi(t)$ on $\overline{J}$.

For the existence of solutions for the problem (1.1)-(1.2), we need the following auxiliary lemma.

Lemma 3.2. *Let $h : J \to \mathbb{R}$ be continuous, the solution of the initial value problem:*

$$({}^C D_q^\alpha y)(t) = h(t),\ t \in J = [0,T], \quad 0 < \alpha < 1, \tag{3.1}$$

$$y(t) = \varphi(t),\ t \in \overline{J} = [-d, 0], \tag{3.2}$$

is given by:

$$y(t) = \begin{cases} \phi(t), & t \in \overline{J} = [-d,0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s, & t \in J = [0,T]. \end{cases} \tag{3.3}$$

Proof. Applying the Riemann-Liouville fractional q-integral of order α to both sides of equation (3.1), and by using Lemma 2.9, we have:

$$y(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + c_0.$$

Using the initial condition of the problem (3.1)-(3.2) and $y(0) = \phi(0) = 0$, we obtain:

$$c_0 = 0.$$

So,

$$y(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s,\ t \in J = [0,T],$$

and

$$y(t) = \varphi(t),\ t \in \overline{J} = [-d,0].$$

The proof is completed. □

In the following subsection, we prove uniqueness and existence results for the problem (1.1)-(1.2) by means fixed point theorems.

The first result is based on Banach contraction principle (Theorem 2.10).

Theorem 3.3. *Assume that the following hypotheses hold:*

($H1$) *The functions $f, g : J \times C(\overline{J}, \mathbb{R}) \to \mathbb{R}$ are continuous.*

($H2$) *There exist $L_f > 0$, such that for each $t \in J$ and each $y, x \in \mathbb{R}$, we have:*

$$|f(t,y) - f(t,x)| \le L_f |y - x|.$$

($H3$) *There exist $L_g > 0$, such that for each $t \in J$ and each $y, x \in \mathbb{R}$, we have:*

$$|g(t,y) - g(t,x)| \le L_g |y - x|.$$

If

$$\frac{(L_f + L_g) T^{(\alpha)}}{\Gamma_q(\alpha+1)} < 1. \tag{3.4}$$

Then, the problem (1.1)-(1.2) has a unique solution.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem, we consider the operator

$$F : \mathcal{C} \longrightarrow \mathcal{C}$$

Defined by:

$$(Fy)(t) = \begin{cases} \phi(t), & t \in \overline{J} = [-d, 0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, y_s) d_q s + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, y_s) d_q s, & t \in J = [0, T]. \end{cases}$$

Clearly, the fixed points of operator F are solution of the problem (1.1)-(1.2).

Now, we shall prove that F is a contraction mapping on $\mathcal{C}$.

Let $y, x \in \mathcal{C}$, if $t \in \overline{J}$, then we have:

$$|(Fy)(t) - (Fx)(t)| = |\phi(t) - \phi(t)| = 0.$$

Hence,

$$\|(Fy)(t) - (Fx)(t)\|_* = 0. \tag{3.5}$$

For $t \in J$, we have:

$$\begin{aligned} |(Fy)(t) - (Fx)(t)| &= \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s, y_s) - f(s, x_s)\right) d_q s \right. \\ &\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s, y_s) - g(s, x_s)\right) d_q s \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} |(Fy)(t) - (Fx)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \Big(|f(s, y_s) - f(s, x_s)| \\ &\quad + |g(s, y_s) - g(s, x_s)| \Big) d_q s. \end{aligned}$$

By hypothesis $(H2)$-$(H3)$, we get:

$$\begin{aligned} |(Fy)(t) - (Fx)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \Big(Ł_f \, |y_s - x_s| + L_g \, |y_s - x_s| \Big) d_q s, \\ &\leq (L_f + L_g) \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|y_s - x_s\|_* \, d_q s. \end{aligned}$$

Thus,

$$\|(Fy)(t) - (Fx)(t)\|_\infty \leq \frac{(L_f + L_g) \, T^{(\alpha)}}{\Gamma_q(\alpha + 1)} \|y - x\|_{\mathcal{C}}. \tag{3.6}$$

From equations (3.5) and (3.6), we conclude that:

$$\|(Fy)(t) - (Fx)(t)\|_{\mathcal{C}} \leq \frac{(L_f + L_g) \, T^{(\alpha)}}{\Gamma_q(\alpha + 1)} \|y - x\|_{\mathcal{C}}.$$

By condition (3.4), F is a contraction operator, and by Banach contraction mapping principle, we deduce that the operator F has a unique fixed point, which is the unique solution of the problem (1.1)-(1.2). □

The second result is based on Burton and Kirk's fixed point theorem (Theorem 2.11).

Theorem 3.4. *Assume that the hypotheses* $(H1)$*-*$(H2)$*-*$(H3)$ *are satisfied and*

$(H4)$ *There exists constant* $M_g > 0$*, such that for each* $t \in J$ *and each* $y \in \mathbb{R}$*, we have:*

$$|g(t,y)| \leq M_g.$$

If

$$\frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)} < 1. \tag{3.7}$$

Then, the problem (1.1)-(1.2) has at least one solution.

Proof. Consider the operators

$$F_1, F_2 : \mathcal{C} \longrightarrow \mathcal{C}$$

Defined by:

$$(F_1 y)(t) = \begin{cases} 0, & t \in \overline{J} = [-d,0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y_s) d_q s, & t \in J = [0,T]. \end{cases}$$

And

$$(F_2 y)(t) = \begin{cases} \phi(t), & t \in \overline{J} = [-d,0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s) d_q s, & t \in J = [0,T]. \end{cases}$$

Then, the problem of finding the solution of the initial problem (1.1)-(1.2) is reduced to finding the solution of the operator equation $(F_1 y)(t) + (F_2 y)(t) = y(t)$, $t \in [-d,T]$.

Next, we shall show that the operators F_1 and F_2 satisfy all the conditions of Theorem 2.11. For better readability, we break the proof into a sequence of steps.

Step 1: F_1 *is contraction operator.*

Let $y, x \in \mathcal{C}$, if $t \in \overline{J}$, then we have:

$$|(F_1 y)(t) - (F_1 x)(t)| = 0.$$

Hence,

$$\|(F_1 y)(t) - (F_1 x)(t)\|_* = 0. \tag{3.8}$$

For $t \in J$, we have:

$$|(F_1 y)(t) - (F_1 x)(t)| = \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s,y_s) - f(s,x_s)\right) d_q s \right|.$$

Therefore,

$$|(F_1 y)(t) - (F_1 x)(t)| \leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,y_s) - f(s,x_s)| \, d_q s.$$

By hypothesis $(H2)$, we get:

$$\begin{aligned} |(F_1y)(t)-(F_1x)(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_f\,|y_s-x_s|\,d_qs, \\ &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_f\,\|y_s-x_s\|_*\,d_qs. \end{aligned}$$

Thus,

$$\|(F_1y)(t)-(F_1x)(t)\|_\infty \le \frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)}\|y-x\|_{\mathcal{C}}. \tag{3.9}$$

From equations (3.8) and (3.9), we conclude that:

$$\|(F_1y)(t)-(F_1x)(t)\|_{\mathcal{C}} \le \frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)}\|y-x\|_{\mathcal{C}}.$$

Consequently, the operator F_1 is contraction.

Step 2: *F_2 is continuous.*

Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y$ in $\mathcal{C}$. If $t \in \overline{J}$, then we have:

$$|(F_2y_n)(t)-(F_2y)(t)| = |\varphi(t)-\varphi(t)| = 0.$$

Hence,

$$\|(F_2y_n)(t)-(F_2y)(t)\|_* = 0. \tag{3.10}$$

For each $t \in J$, we have:

$$|(F_2y_n)(t)-(F_2y)(t)| \le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\,|g(s,y_{ns})-g(s,y_s)|\,d_qs.$$

Thus,

$$\|(F_2y_n)(t)-(F_2y)(t)\|_\infty \le \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}\,\|g(.,y_{n}.)-g(.,y.)\|_\infty\,.$$

Since g is a continuous function, we get:

$$\|F_2(y_n)-F_2(y)\|_\infty \to 0 \;\; as \;\; n\to\infty. \tag{3.11}$$

From equations (3.10) and (3.11), we conclude that:

$$\|F_2(y_n)-F_2(y)\|_{\mathcal{C}} \to 0 \;\; as \;\; n\to\infty.$$

Consequently, F_2 is continuous in $\mathcal{C}$.

Step 3: *F_2 maps bounded sets into bounded sets in $\mathcal{C}$.*

Indeed, it is enough to show that for any $r>0$, there exists a positive constant R such that for each $y \in B_r = \{y \in \mathcal{C} : \|y\|_{\mathcal{C}} \le r\}$ we have $\|F_2(y)\|_{\mathcal{C}} \le R$.

Let $y \in B_r$. If $t \in \overline{J}$, then we have:

$$|(F_2y)(t)| = |\varphi(t)|.$$

Hence,

$$\|(F_2y)\|_* \le \|\varphi\|_*. \tag{3.12}$$

For each $t \in J$, we have:

$$\begin{aligned} |(F_2y)(t)| &= \left|\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s)d_qs\right|, \\ &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s,y_s)|d_qs. \end{aligned}$$

By hypothesis $(H4)$, we get:

$$|(F_2y)(t)| \leq M_g \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs.$$

Thus,

$$\|(F_2y)\|_\infty \leq \frac{M_gT^{(\alpha)}}{\Gamma_q(\alpha+1)} := l. \tag{3.13}$$

From equations (3.12) and (3.13), we conclude that:

$$\|(F_2y)\|_{\mathcal{C}} \leq \max\{\|\varphi\|_*, l\} := R. \tag{3.14}$$

Consequently, the operator F_2 is uniformly bounded in B_r.
Step 4: *F_2 maps bounded sets into equicontinuous sets in $\mathcal{C}$.*
Let $t_1, t_2 \in J,\ t_1 < t_2$ and let B_r be a bounded set of $\mathcal{C}$ as in Step 2. Let $y \in B_r$, then we have:

$$\begin{aligned} |(F_2y)(t_2) - (F_2y)(t_1)| &= \left|\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s)d_qs \right. \\ &\quad \left. - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s)d_qs\right|, \\ &= \left|\int_0^{t_1} \frac{\left((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right)}{\Gamma_q(\alpha)} g(s,y_s)d_qs \right. \\ &\quad \left. + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s)d_qs\right|, \\ &\leq \int_0^{t_1} \frac{\left((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right)}{\Gamma_q(\alpha)} |g(s,y_s)|d_qs \\ &\quad + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s,y_s)|d_qs. \end{aligned}$$

By hypothesis $(H4)$, we get:

$$\begin{aligned} |(F_2y)(t_2) - (F_2y)(t_1)| &\leq \frac{M_g}{\Gamma_q(\alpha)} \int_0^{t_1} \left((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right) d_qs \\ &\quad + \frac{M_g}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2-qs)^{(\alpha-1)} d_qs, \\ &\leq \frac{M_g}{\Gamma_q(\alpha+1)} \left(t_2^{(\alpha)} - t_1^{(\alpha)}\right). \end{aligned} \tag{3.15}$$

And if $t_1, t_2 \in \overline{J}$, then we have:

$$|(F_2y)(t_2) - (F_2y)(t_1)| = |\varphi(t_2) - \varphi(t_1)|. \tag{3.16}$$

The right hand sides of equations (3.15) and (3.16) tend to zero independently of $y \in B_r$ as $t_1 \to t_2$.
As a consequence of Steps 2 to 4, together with the Arzela-Ascoli theorem, we can conclude that the operator F_2 is completely continuous.
Step 5: *A priori bound.*
the set $\Omega = \{y \in \mathcal{C} : y = \lambda F_1\left(\frac{y}{\lambda}\right) + \lambda F_2(y)\}$ is bounded.
Let $y \in \Omega$, then $y = \lambda F_1\left(\frac{y}{\lambda}\right) + \lambda F_2(y)$ for some $0 < \lambda < 1$. If $t \in \overline{J}$, then

$$y(t) = \lambda\varphi(t).$$

For each $t \in J$, we have:

$$y(t) = \lambda \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, \frac{y_s}{\lambda}) d_q s + \lambda \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, y_s) d_q s.$$

Thus,

$$\begin{aligned}
|y(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, \frac{y_s}{\lambda})| d_q s + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s, y_s)| d_q s, \\
&\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, \frac{y_s}{\lambda}) - f(s,0)| d_q s + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,0)| d_q s \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s, y_s)| d_q s.
\end{aligned}$$

This implies by hypothesis $(H2)$ and $(H4)$ that for each $t \in J$, we get:

$$\begin{aligned}
|y(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_f |y_s| d_q s + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,0)| d_q s \\
&\quad + M_g \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s, \\
&\leq L_f \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|y_s\|_* d_q s + f^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&\quad + M_g \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s, \\
&\leq \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} + L_f \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|y_s\|_* d_q s,
\end{aligned}$$

where $f^* = \sup_{s \in J} |f(s,0)|$.
Now, we consider the function ρ defined by:

$$\rho(t) = \sup\{|y(s)| : 0 \leq s \leq t\},\ t \in J.$$

Then, there exists $t^* \in [-d, t]$ be such that $\rho(t) = |y(t^*)|$.
If $t^* \in \overline{J}$, then

$$\rho(t) = \|\varphi\|_*. \tag{3.17}$$

If $t^* \in J$, then by the previous inequality we have for $t \in J$:

$$\rho(t) \leq \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} + L_f \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \rho(s) d_q s.$$

Then, from Lemma 2.12, there exists $\delta = \delta(\alpha)$ such that we get:

$$\rho(t) \leq \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} + L_f \delta \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} d_q s.$$

Thus,

$$\rho(t) \leq \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} \left(1 + \frac{L_f \delta T^{(\alpha)}}{\Gamma_q(\alpha+1)}\right) =: k. \tag{3.18}$$

Thus for any $t \in J$, $\|y\|_{\mathcal{C}} \leq \rho(t)$, from (3.17) and (3.18), we conclude that:

$$\|y\|_{\mathcal{C}} \leq \max(\|\varphi\|_{\mathcal{C}}, k).$$

This shows that the set Ω is bounded.
As a consequence of Theorem 3.4, we deduce that $F_1(y) + F_2(y)$ has a fixed point which is an integral solution of the problem (1.1)-(1.2). □

4. Ulam stability results

In this section, we will define and study some types of Ulam stability for problem (1.1)-(1.2). The following definitions were adopted from [31].

Definition 4.1. The problem (1.1)-(1.2) is Ulam-Hyers stable if there exists a real number $C > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathcal{C}$ of the following inequality:

$$|({}^C D_q^\alpha x)(t) - f(t, x_t) - g(t, x_t)| \leq \varepsilon, \ t \in J = [0, T], \tag{4.1}$$

there exists a solution $y \in \mathcal{C}$ of the problem (1.1)-(1.2) with the norm:

$$\|x - y\|_{\mathcal{C}} \leq C\varepsilon.$$

Definition 4.2. The problem (1.1)-(1.2) is generalized Ulam-Hyers stable if there exists $\vartheta \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\vartheta(0) = 0$, such that for each $\varepsilon > 0$, and for each solution $x \in \mathcal{C}$ of the inequality (4.1), there exists a solution $y \in \mathcal{C}$ of the problem (1.1)-(1.2) with the norm:

$$\|x - y\|_{\mathcal{C}} \leq \vartheta(\varepsilon).$$

Definition 4.3. The problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to ϕ if there exists $C_\phi > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathcal{C}$ of the following inequality:

$$|({}^C D_q^\alpha x)(t) - f(t, x_t) - g(t, x_t)| \leq \varepsilon\phi(t), \ t \in J = [0, T], \tag{4.2}$$

there exists a solution $y \in \mathcal{C}$ of the problem (1.1)-(1.2) with the norm:

$$\|x - y\|_{\mathcal{C}} \leq C_\phi \varepsilon \phi(t), \ t \in J = [0, T].$$

Remark 4.4. A function $x \in \mathcal{C}$ is a solution of the inequality

$$|({}^C D_q^\alpha x)(t) - f(t, x_t) - g(t, x_t)| \leq \varepsilon, \ t \in J = [0, T],$$

if and only if there exists a function $k \in C([0, T], \mathbb{R})$ (which depend on y) such that:

(i) $|k(t)| \leq \varepsilon, \ t \in J = [0, T]$.
(ii) $({}^C D_q^\alpha x)(t) = f(t, x_t) + g(t, x_t) + k(t), \ t \in J = [0, T]$.

Theorem 4.5. *Assume that the hypotheses* $(H1)$-$(H2)$-$(H3)$ *and condition (3.4) are satisfied. Then, the problem (1.1)-(1.2) is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and let $x \in \mathcal{C}$ be a solution of the inequality (4.1) and let $y \in \mathcal{C}$ be the solution of the problem (1.1)- (1.2). Then, we have:

$$y(t) = \begin{cases} \phi(t), & t \in \overline{J} = [-d, 0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, y_s) d_q s + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, y_s) d_q s, & t \in J = [0, T]. \end{cases}$$

From the inequality (4.1) for each $t \in J$, we obtain:

$$\begin{aligned} \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, x_s) d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, x_s) d_q s \right| &\leq I_q^\alpha \varepsilon, \\ &\leq \frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)} \varepsilon. \end{aligned}$$

Using the hypotheses $(H1)$-$(H2)$ and $(H3)$, for each $t \in J$, we can write:

$$\begin{aligned} |x(t) - y(t)| &\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, y_s) d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, y_s) d_q s \right|, \\ &\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, x_s) d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, x_s) d_q s \right. \\ &\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s, x_s) - f(s, y_s) \right) d_q s \\ &\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s, x_s) - g(s, y_s) \right) d_q s \right|, \\ &\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, x_s) d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, x_s) d_q s \right| \\ &\quad + \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s, x_s) - f(s, y_s) \right) d_q s \right. \\ &\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s, x_s) - g(s, y_s) \right) d_q s \right|, \end{aligned}$$

Thus,

$$\begin{aligned} |x(t)-y(t)| &\leq \frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)}\varepsilon+\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\,|f(s,x_s)-f(s,y_s)|\,d_qs \\ &\quad +\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\,|g(s,x_s)-g(s,y_s)|\,d_qs, \\ &\leq \frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)}\varepsilon+L_f\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\,\|x_s-y_s\|_*\,d_qs \\ &\quad +L_g\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\,\|x_s-y_s\|_*\,d_qs. \end{aligned}$$

Hence,

$$\|x-y\|_{\mathcal{C}} \leq \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}\varepsilon+\frac{(L_f+L_g)\,T^{(\alpha)}}{\Gamma_q(\alpha+1)}\|x-y\|_{\mathcal{C}}.$$

By condition (3.4), we get:

$$\begin{aligned} \|x-y\|_{\mathcal{C}} &\leq \frac{\frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}}{1-\frac{(L_f+L_g)T^{(\alpha)}}{\Gamma_q(\alpha+1)}}\varepsilon, \\ &:= C\varepsilon. \end{aligned}$$

Consequently, the problem (1.1)-(1.2) is Ulam-Hyers stable.
Taking $\vartheta(\varepsilon)=C\varepsilon$; $\vartheta(0)=0$, we can state that the problem (1.1)-(1.2) is generalized Ulam-Hyers stable. □

Theorem 4.6. *Assume that the hypotheses* $(H1)$*-*$(H2)$*-*$(H3)$ *and condition (3.4) are satisfied and*

$(H5)$ *Let* $\phi \in C(J, R_+)$ *be an increasing function. There exists* $\lambda_\phi > 0$ *such that for each* $t \in J$*, we have:*

$$I_q^\alpha \phi(t) \leq \lambda_\phi \phi(t).$$

Then, problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable.

Proof. Let $\varepsilon > 0$ and let $x \in \mathcal{C}$ be a solution of the inequality (4.2) and let $y \in \mathcal{C}$ be the solution of the problem (1.1)-(1.2). Then, we have:

$$y(t)=\begin{cases} \phi(t), & t \in \overline{J}=[-d,0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y_s)d_qs+\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s)d_qs, & t \in J=[0,T]. \end{cases}$$

From the inequality (4.2) and $(H5)$, for each $t \in J$, we obtain:

$$\begin{aligned} \left|x(t)-\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,x_s)d_qs-\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,x_s)d_qs\right| &\leq \varepsilon I_q^\alpha \phi(t), \\ &\leq \varepsilon \lambda_\phi \phi(t). \end{aligned}$$

Using the hypotheses $(H1)$-$(H2)$ and $(H3)$, for each $t \in J$, we can write:

$$\begin{aligned}
|x(t)-y(t)| &\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y_s)d_qs - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s)d_qs \right|, \\
&\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,x_s)d_qs - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,x_s)d_qs \right. \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s,x_s)-f(s,y_s)\right) d_qs \\
&\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s,x_s)-g(s,y_s)\right) d_qs \right|, \\
&\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,x_s)d_qs - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,x_s)d_qs \right| \\
&\quad + \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s,x_s)-f(s,y_s)\right) d_qs \right. \\
&\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s,x_s)-g(s,y_s)\right) d_qs \right|, \\
&\leq \varepsilon\lambda_\phi\phi(t) + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f(s,x_s)-f(s,y_s)\right| d_qs \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|g(s,x_s)-g(s,y_s)\right| d_qs, \\
&\leq \varepsilon\lambda_\phi\phi(t) + L_f \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|x_s-y_s\|_* \, d_qs \\
&\quad + L_g \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|x_s-y_s\|_* \, d_qs.
\end{aligned}$$

Hence,

$$\|x-y\|_{\mathcal{C}} \leq \varepsilon\lambda_\phi\phi(t) + \frac{(L_f+L_g)\,T^{(\alpha)}}{\Gamma_q(\alpha+1)}\|x-y\|_{\mathcal{C}}.$$

By condition (3.4), we get:

$$\begin{aligned}
\|x-y\|_{\mathcal{C}} &\leq \frac{\varepsilon\lambda_\phi\phi(t)}{1-\frac{(L_f+L_g)T^{(\alpha)}}{\Gamma_q(\alpha+1)}} \\
&:= C_\phi\varepsilon\phi(t).
\end{aligned}$$

Consequently, the problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable. □

5. Examples

In this section, we give two examples illustrating our main results.

Example 5.1. Consider the following initial value problem of perturbed functional fractional q-difference equations:

$$({}^{C}D^{\frac{2}{3}}_{\frac{1}{3}}y)(t) = 1 + 2t + \frac{\sin(y_t)}{e^{-t}+1} + \frac{e^{-t}\sin(y_t)}{t+6}, \quad t \in J = [0,1],\ 0 < \alpha \leq 1, \tag{5.1}$$

$$y(t) = t^2,\ t \in \overline{J} = [-1,0], \tag{5.2}$$

where $\varphi(t) = t^2$ and $\alpha = \frac{2}{3}$, $q = \frac{1}{3}$, $d = 1$, $T = 1$, and

$$f(t,y) = 1 + 2t + \frac{\sin(y)}{e^{-t}+1},\ (t,y) \in J \times \mathbb{R},$$

$$g(t,y) = \frac{e^{-t}\sin(y)}{t+6},\ (t,y) \in J \times \mathbb{R}.$$

Clearly, the function f, g are continuous.
Let $y, x \in \mathbb{R}$ and $t \in J$. Then, we have:

$$\begin{aligned} |f(t,y) - f(t,x)| &= \left|\frac{\sin(y)-\sin(x)}{e^{-t}+1}\right|, \\ &\leq \frac{1}{2}|y-x|, \\ |g(t,y)-g(t,x)| &\leq \frac{1}{6}|y-x|. \end{aligned}$$

Hence, the hypothesis $(H2)$-$(H3)$ are satisfied with $L_f = \frac{1}{2}$ and $L_g = \frac{1}{6}$. Now, we shall check that the condition (3.4) is satisfied with $T = 1$. Indeed,

$$\begin{aligned} \frac{(L_f + L_g)\,T^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \frac{\left(\frac{1}{2}+\frac{1}{6}\right)}{\Gamma_{\frac{1}{3}}(\frac{5}{3})}, \\ &= 0.7028 < 1. \end{aligned}$$

Then, by Theorem 3.3, the problem (5.1)-(5.2) has a unique solution on $[-1,1]$, and from Theorem 4.5, the problem (5.1)-(5.2) is Ulam-Hyers stable on $[0,1]$.
On the other hand, we have:

$$|g(t,y)| \leq \frac{1}{6},\ (t,y) \in J \times \mathbb{R}.$$

Thus, the condition $(H4)$ holds. Next, we shall check that the condition (3.7) is satisfied with $T = 1$. Indeed,

$$\begin{aligned} \frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \frac{1}{2\Gamma_{\frac{1}{3}}(\frac{5}{3})}, \\ &= 0.5271 < 1. \end{aligned}$$

Then, by Theorem 3.4, the problem (5.1)-(5.2) has at least one solution on $[-1,1]$.

Example 5.2. Consider the following initial value problem of perturbed functional fractional q-difference equations:

$$({}^C D_{\frac{1}{4}}^{\frac{1}{2}} y)(t) = \frac{t^2}{e^t+5}\left(1+\frac{|y_t|}{1+|y_t|}\right)+\frac{t^2 sin(y_t)}{3}, \quad t\in J=[0,1],\ 0<\alpha\le 1, \quad (5.3)$$

$$y(t) = \frac{t}{6+t},\ t\in \overline{J}=[-2,0], \quad (5.4)$$

where $\varphi(t)=\frac{t}{6+t}$ and $\alpha=\frac{1}{2}$, $q=\frac{1}{2}$, $d=2$, $T=1$, and

$$f(t,y)=\frac{t^2}{e^t+5}\left(1+\frac{|y|}{1+|y|}\right),\ (t,y)\in J\times\mathbb{R},$$

$$g(t,y)=\frac{t^2 sin(y)}{3}\in J\times\mathbb{R}.$$

Clearly, the function f,g are continuous.
Let $y,x\in\mathbb{R}$ and $t\in J$. Then, we have:

$$\begin{aligned} |f(t,y)-f(t,x)| &= \left|\frac{1}{e^t+5}\left(\frac{|y|}{1+|y|}-\frac{|x|}{1+|x|}\right)\right|, \\ &\le \frac{1}{6}|y-x|, \\ |g(t,y)-g(t,x)| &\le \frac{1}{3}|y-x|. \end{aligned}$$

Hence, the hypothesis $(H2)$-$(H3)$ are satisfied with $L_f=\frac{1}{6}$ and $L_g=\frac{1}{3}$. Now, we shall check that the condition (3.4) is satisfied with $T=1$. Indeed,

$$\begin{aligned} \frac{(L_f+L_g)\,T^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \frac{\left(\frac{1}{6}+\frac{1}{3}\right)}{\Gamma_{\frac{1}{4}}(\frac{3}{2})}, \\ &= 0.5275<1. \end{aligned}$$

Then, by Theorem 3.3, the problem (5.3)- (5.4) has a unique solution on $[-2,1]$, and from Theorem 4.5, the problem (5.3) is Ulam-Hyers stable on $[0,1]$.
On the other hand, we have:

$$|g(t,y)|\le\frac{1}{3},\ (t,y)\in J\times\mathbb{R}.$$

Thus, the condition $(H4)$ holds. Next, we shall check that the condition (3.7) is satisfied with $T=1$. Indeed,

$$\begin{aligned} \frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \frac{1}{6\Gamma_{\frac{1}{4}}(\frac{3}{2})}, \\ &= 0.1758<1. \end{aligned}$$

Then, by Theorem 3.4, the problem (5.3)-(5.4) has at least one solution on $[-2,1]$.
Now, let $\phi(t)=t^2$ for each $t\in J$, we have:

$$I_{\frac{1}{4}}^{\frac{1}{2}}\phi(t)=\frac{\Gamma_{\frac{1}{4}}(3)}{\Gamma_{\frac{1}{4}}(\frac{7}{2})}t^{2+\frac{1}{2}}\le\frac{5}{4\Gamma_{\frac{1}{4}}(\frac{7}{2})}t^2=\lambda_\phi\phi(t). \quad (5.5)$$

Thus, the condition $(H5)$ is satisfied with $\phi(t) = t^2$ and $\lambda_\phi = \frac{5}{4\Gamma_{\frac{1}{4}}(\frac{7}{2})}$. Then, it follows from Theorem 4.6 that the problem (5.3)- (5.4) is Ulam-Hyers-Rassias stable on $[0, 1]$.

6. Conclusions

In this work, we have provided sufficient conditions for the existence of solutions for the initial value problem (IVP for short) for perturbed functional fractional q-difference equations involving the Caputo's fractional q-derivative. The uniqueness result is obtained by applying the Banach contraction mapping principle, while the existence result is obtained by using Burton and Kirk's fixed point theorem. In addition, we presented some results for Ulam-Hyers stability and Ulam-Hyers-Rassias stability. For the justification, examples are we given to illustrate the main results.

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Existence results for some fractional order coupled systems with impulses and nonlocal conditions on the half line

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Abstract. In this paper, we deal with initial value problems for coupled systems of nonlinear fractional differential equations, subject to coupled nonlocal initial and impulsive conditions on the half line. Global existence-uniqueness results are obtained under weak conditions allowing the reaction part of the problem to increase indefinitely with time. Our approach relies mainly to some fixed point theorem of Perov's type in generalized gauge spaces. The obtained results improve, generalize and complement many existing results in the literature. An example illustrating our main finding is also given.

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Keywords: Fractional differential equation, coupled systems, generalized spaces in Perov's sens, nonlocal initial conditions, impulses.

1. Introduction

Recently, an intensive interest has been given to the investigation of differential equations of fractional order. This is motivated by the natural introduction of fractional operators in the modeling of several phenomena whose nonlocal dynamics involving long-term effects are taken into account. These models have been applied successfully in many fields such as in mechanics, bio-chemistry, electrical engineering, control, porous media, medicine, etc. (see [6, 11]).

On the other hand, differential equations involving impulse effects appear as an appropriate model for some evolutionary problems. It is the case of many real-world processes that are subject of abrupt of changes in certain moments of times

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and arising in a variety of disciplines, including biology, population dynamics, electric technology, control theory, engineering, etc. For more details on this subject, we reefer to the monographs [3, 12].

Banach's contractive principle is one of the most useful tools in nonlinear functional analysis that ensures the existence and uniqueness of a fixed point on complete metric spaces. One of the extensions of this principle for contractive mappings on spaces endowed with vector valued metrics, was done by Perov in [16] and Perov and Kibento in [17]. Many other generalizations in this direction have been investigated. In [15], Precup established the extension in Perov's sens of some fixed point theorem in spaces endowed with a family of pseudo-metrics. Many authors applied the vector version's fixed point theorems in the study of the existence of solutions for systems of differential and integral equations, see for example [4, 5, 9, 10, 20] and the references therein. In this line of research, we consider in this work, the following nonlinear coupled system of fractional differential equations:

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = f(t,u(t),v(t)), & t \in \mathcal{I}_i = \left]t_i, t_{i+1}\right], \, i \in \mathbb{N} \\ {}^{C}D^{\beta}_{0^+}v(t) = g(t,u(t),v(t)), & t \in \mathcal{I}_i = \left]t_i, t_{i+1}\right], \, i \in \mathbb{N} \end{cases} \tag{1.1}$$

with coupled nonlocal initial conditions:

$$\begin{cases} u(0) = \varphi(u,v), \\ v(0) = \psi(u,v), \end{cases} \tag{1.2}$$

and subject to coupled impulsive conditions:

$$\begin{cases} \Delta u(t_i) = I_i(u(t_i),v(t_i)), & i \in \mathbb{N}^* = \mathbb{N}\backslash\{0\} \\ \Delta v(t_i) = J_i(u(t_i),v(t_i)), & i \in \mathbb{N}^* = \mathbb{N}\backslash\{0\} \end{cases} \tag{1.3}$$

where ${}^{C}D^{\alpha}_{0^+}$ and ${}^{C}D^{\beta}_{0^+}$ denote the Caputo fractional derivative operators with the fixed lower limit equals zero, of order α and β in $]0,1[$ respectively, $f,g : \mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are nonlinear continuous functions, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, where $u(t_i^+)$ and $u(t_i^-)$ represent the right and left limits of u at $t = t_i$ and $\{t_i\}_{i\in\mathbb{N}^*}$ is a sequence of points in $\mathbb{R}_+$ such that $t_i < t_{i+1}$ for $i \in \mathbb{N}^*$, $I_i, J_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are nonlinear continuous functions, $\phi, \psi : X \longrightarrow \mathbb{R}$ are nonlinear continuous functional where X is a generalized complete gauge space, which will be defined later.

It should be noted that the coupled nonlocal initial conditions (1.2) generalizes many other types of initial conditions considered in the literature, such as: classical initial conditions, multi-point conditions and integral conditions.

After converting (1.1)- (1.3) into an equivalent fixed point problem in generalized gauge space, we apply some fixed point theorem of Perov's type, established in [15]. Using this approach, we obtain a global existence-uniqueness results for (1.1)- (1.3) under weak conditions allowing the nonlinearity to increase indefinitely with time, which is not the case in many earlier results in the literature (see Remark 3.1). This study allows us also, to improve and generalize some other existence results in the literature for systems of fractional differential equations without impulses (see Remark 3.6).

The rest of the paper is organized as follows. In Section 2 we recall some definitions from fractional calculus. We introduce also the fixed point theorem in generalized gauge spaces, on which our result is based, as well as some related concepts. The main result concerning the global existence-uniqueness result for (1.1)- (1.3) is established in Section 3. Finally, in Section 4, we provide an illustrative example.

2. Preliminaries

Let us recall the notion of the fractional derivatives. For further details on some essential related properties, we refer to [6, 11].
Let n be a positive integer, α the positive real such that $n-1<\alpha\leq n$ and d^n/dt^n the classical derivative operator of order n.

Definition 2.1. The Riemann-Liouville fractional integral, and the Riemann-Liouville fractional derivative, of a real function u defined on $\mathbb{R}_+$ of order α, are defined respectively by

$$I_{0^+}^{\alpha}u\left(t\right):=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}u\left(s\right)ds,\qquad t>0,$$

$$D_{0^+}^{\alpha}u\left(t\right):=\frac{d^n}{dt^n}I_{0^+}^{n-\alpha}u\left(t\right):=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^n}{dt^n}\int_0^t\left(t-s\right)^{n-\alpha-1}u\left(s\right)ds,\qquad t>0,$$

where $\Gamma\left(.\right)$ is the Gamma function, provided that the right hand sides exist point wise.

Definition 2.2. The Caputo fractional derivative of a real function u defined on $\mathbb{R}_+$ of order α, noted by ${}^{C}D_{0^+}^{\alpha}$, is defined by

$${}^{C}D_{0^+}^{\alpha}u\left(t\right):=\left(D_{0^+}^{\alpha}\left[u-\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}(.)^k\right]\right)\left(t\right),\qquad t>0,$$

provided that the right hand side exists point wise.

We denote by $\mathcal{M}_n(\mathbb{R}_+)$, the set of all square matrices of order n with positive real elements, I the identity matrix of order n and by O the zero matrix of order n.

Definition 2.3. [18] A square real matrix M of order n, is said to be convergent to zero, if $M^k\longrightarrow O,$ as $k\longrightarrow\infty$.

Definition 2.4. [18] Let $M\in\mathcal{M}_n(\mathbb{R}_+)$ with eigenvalues λ_i, $1\leq i\leq n$, that is $\lambda_i\in\mathbb{R}$ such that $det(M-\lambda_iI)=O$. Then

$$\rho\left(M\right)=\max_{1\leq i\leq n}\left|\lambda_i\right|$$

is called the spectral radius of M.

Lemma 2.5. [18] Let $M\in\mathcal{M}_n(\mathbb{R}_+)$. The following assumptions are equivalent.

(i) M is convergent to zero.
(ii) The matrix $I-M$ is non singular, and

$$(I-M)^{-1}=I+M+M^2+...+M^n+...,$$

(iii) $\rho(M) < 1$.

As it is pointed out in [13], the following lemma follows immediately from the characterization (iii) in Lemma 2.5.

Lemma 2.6. [13] If A is a square matrix that converges to zero and the elements of another matrix B are small enough, then $A + B$ also converges to zero.

We state now the extension of Gheorghiu's theorem for generalized contractions on complete generalized gauge spaces established in [15].

Let X be a generalized gauge space endowed with a complete gauge structure $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathcal{N}}$, where $\mathcal{N}$ is an index set. For further details on gauge spaces and generalized gauge spaces we reefer to [7, 15].

Definition 2.7. [15] (Generalized contraction) Let $(X, \mathfrak{D})$ be a generalized gauge space with $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathcal{N}}$. A map $T : D(T) \subset X \longrightarrow X$ is called a generalized contraction, if there exists a function $w : \mathcal{N} \longrightarrow \mathcal{N}$ and $M \in \mathcal{M}_n(\mathbb{R}_+)^{\mathcal{N}}$, $M = \{M_\nu\}_{\nu\in\mathcal{N}}$ such that

$$D_\nu(T(u), T(v)) \leq M_\nu D_{w(\nu)}(u, v), \;\; \forall u, v \in D(T), \forall \nu \in \mathcal{N} \tag{2.1}$$

and

$$\sum_{k=1}^{\infty} M_\nu M_{w(\nu)} ... M_{w^{k-1}(\nu)} D_{w^k(\nu)}(u, v) < \infty, \;\; \forall u, v \in D(T), \forall \nu \in \mathcal{N} \tag{2.2}$$

Theorem 2.8. [15, Theorem 2.1] Let $(X, \mathfrak{D})$ be a complete generalized gauge space and let $T : X \longrightarrow X$ be a generalized contraction. Then, T has a unique fixed point in X, which can be obtained by successive approximations starting from any element of X.

2.1. Equivalent system of integral equations

In the fractional case, there are two different approaches defining the concept of solutions for impulsive differential equations, which can be briefly described as follows (see [1, 2]):

Fractional derivatives with a fixed lower limit at the initial time. This approach (denoted respectively by V_2 in [1] and by A_1 in [2]) considers that the lower limit of the fractional derivative is kept equal to the initial time on any interval between two consecutive impulses, with only modified initial conditions.

Fractional derivatives with varying lower limits. This approach (denoted respectively by V_1 in [1] and by A_2 in [2]) neglects the lower limit of the fractional derivative at the initial time and moves it to each impulsive time.

In this work, we will adopt the case of fixed lower limit.

For any interval $\mathcal{I}$ of $\mathbb{R}_+$ (which may be unbounded), we denote by $\mathcal{C}(\mathcal{I})$ the set of all real continuous functions on $\mathcal{I}$ and by u_i the restriction of $u \in \mathcal{C}(\mathbb{R}_+)$ to $\mathcal{I}_i =]t_i, t_{i+1}]$, $(i \in \mathbb{N})$.

Let $\mathcal{PC}(\mathbb{R}_+)$ be the set of all real valued piece-wise continuous functions on $\mathbb{R}_+$:

$$\mathcal{PC}(\mathbb{R}_+) = \{u : \mathbb{R}_+ \to \mathbb{R} : u_i \in \mathcal{C}\,(\mathcal{I}_i) \text{ and } u(t_i^+) \text{ exist for every } i \in \mathbb{N}\} \tag{2.3}$$

endowed with the saturated family $\{d_\nu : \nu \in \mathcal{N}\}$ of pseudo-metrics, generating its topology, defined by

$$d_\nu(u,v) = \max_{t\in\nu}\left\{e^{-\lambda t}\left|u(t)-v(t)\right|\right\}, \forall u,v \in \mathcal{PC}(\mathbb{R}_+), \tag{2.4}$$

where ν runs over the set of all compact subsets of $\mathbb{R}_+$ denoted by $\mathcal{N}$, and λ is a positive real number to be specified later.

In what follows, we consider $X = \mathcal{PC}(\mathbb{R}_+) \times \mathcal{PC}(\mathbb{R}_+)$, endowed with the generalized complete gauge structure $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathcal{N}}$ defined for $W_1 = (u_1,v_1), W_2 = (u_2,v_2) \in X$ by:

$$D_\nu(W_1,W_2) = \begin{pmatrix} d_\nu(u_1,u_2) \\ \\ d_\nu(v_1,v_2) \end{pmatrix}, \tag{2.5}$$

where d_ν is the pseudo-metric on $\mathcal{PC}(\mathbb{R}_+)$ given in (2.4).

Reproducing the proof of [14, Lemma 1], in addition of [8, Lemma 2.6] with a slight adaptation, we get the system of integral equations equivalent to (1.1)-(1.2) given by the following lemma.

Lemma 2.9. Let f, g, I_i, J_i $(i \in \mathbb{N}^*)$ be continuous functions and φ, ψ continuous functionals such that:

$$\begin{gathered} \forall (u,v),(\tilde{u},\tilde{v}) \in X \text{ if } u = \tilde{u} \text{ and } v = \tilde{v} \text{ on } [0,t_1[, \text{ then} \\ \varphi(u,v) = \varphi(\tilde{u},\tilde{v}) \text{ and } \psi(u,v) = \psi(\tilde{u},\tilde{v}) \end{gathered} \tag{2.6}$$

Then, $(u,v) \in X$ is a solution of (1.1)-(1.3) if and only if (u,v) is a solution of the following system of integral equations

$$\begin{cases} u(t) = \begin{cases} \varphi(u,v) + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds, & t \in \mathcal{I}_0 \\ \varphi(u,v) + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds + \sum_{j=1}^{i} I_j(u(t_j),v(t_j)), & t \in \mathcal{I}_i \end{cases} \\ v(t) = \begin{cases} \psi(u,v) + \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds, & t \in \mathcal{I}_0 \\ \psi(u,v) + \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds + \sum_{j=1}^{i} J_j(u(t_j),v(t_j)), & t \in \mathcal{I}_i \end{cases} \end{cases} \tag{2.7}$$

For $i = 1,2$, let $T_i : X \to \mathcal{PC}(\mathbb{R}_+)$ be the operators defined for every $W := (u,v) \in X$ by

$$T_1(W)(t) = \varphi(u,v) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds + \sum_{t_j<t} I_j(u(t_j),v(t_j)) \tag{2.8}$$

$$T_2(W)(t) = \psi(u,v) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds + \sum_{t_j<t} J_j(u(t_j),v(t_j)) \tag{2.9}$$

Let us consider the operator: $T : X \to X$ defined by

$$T(u,v) = (T_1(u,v), T_2(u,v)), \quad \forall (u,v) \in X, \tag{2.10}$$

where T_1 and T_2 are given respectively by (2.8) and (2.9).

Thus, according to Lemma 2.9, the solutions of (1.1)-(1.3) can be regarded as fixed points of T.

3. Main results

In this section, we will prove a global existence-uniqueness result for (1.1)-(1.3), to this end, we consider the following assumptions:

(H_1) There exist continuous positive real valued functions $A_i, B_i : i = 1, 2$ defined on $\mathbb{R}_+$, and satisfying

(i) $|f(t, \xi_1, \eta_1) - f(t, \xi_2, \eta_2)| \leq A_1(t) |\xi_1 - \xi_2| + A_2(t) |\eta_1 - \eta_2|$
$|g(t, \xi_1, \eta_1) - g(t, \xi_2, \eta_2)| \leq B_1(t) |\xi_1 - \xi_2| + B_2(t) |\eta_1 - \eta_2|$,
whenever the left hand sides are defined.

(ii) For $\lambda > 0, \mu > 1$, $q := 1 + 1/\alpha$ and $\tilde{q} := 1 + 1/\beta$, we have

$$S_{\lambda,\mu} := \int_0^{+\infty} A_1^q(s) e^{\frac{-q\lambda s}{\mu}} ds < \infty \text{ and } \tilde{S}_{\lambda,\mu} := \int_0^{+\infty} A_2^q(s) e^{\frac{-q\lambda s}{\mu}} ds < \infty$$

$$R_{\lambda,\mu} := \int_0^{+\infty} B_1^{\tilde{q}}(s) e^{\frac{-\tilde{q}\lambda s}{\mu}} ds < \infty \text{ and } \tilde{R}_{\lambda,\mu} := \int_0^{+\infty} B_2^{\tilde{q}}(s) e^{\frac{-\tilde{q}\lambda s}{\mu}} ds < \infty$$

(H_2) There exist fixed compacts $K_i, \tilde{K}_j$ and non negative real numbers $L_i, \tilde{L}_i, M_j, \tilde{M}_j, (1 \leq i \leq l, 1 \leq j \leq m)$, satisfying what follows for every $(u_1, v_1), (u_2, v_2) \in X$:

$$|\varphi(u_1, v_1) - \varphi(u_2, v_2)| \leq \sum_{i=1}^{l} \Big(L_i d_{K_i}(u_1 - u_2) + \tilde{L}_i d_{K_i}(v_1 - v_2) \Big)$$
$$|\psi(u_1, v_1) - \psi(u_2, v_2)| \leq \sum_{j=1}^{m} \Big(M_j d_{\tilde{K}_j}(u_1 - u_2) + \tilde{M}_j d_{\tilde{K}_j}(v_1 - v_2) \Big)$$

(H_3) There exist positive real sequences $\{h_i\}$,$\{\tilde{h}_i\}$,$\{k_i\}$ and $\{\tilde{k}_i\}$ that converge to $H, \tilde{H}, K$ and $\tilde{K}$ respectively and satisfying for every $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}$ and $i \in \mathbb{N}^*$, the following estimations:

$$|I_i(\xi_1, \eta_1) - I_i(\xi_2, \eta_2)| \leq h_i |\xi_1 - \xi_2| + \tilde{h}_i |\eta_1 - \eta_2|$$
$$|j_i(\xi_1, \eta_1) - j_i(\xi_2, \eta_2)| \leq k_i |\xi_1 - \xi_2| + \tilde{k}_i |\eta_1 - \eta_2|$$

Remark 3.1. It is not hard to see that hypothesis (H_1) includes as special cases the Lipschitz condition with constant or integrable arguments, widely used in the literature (see for example [9, 20, 19, 5]). This being said, we emphasize here that hypothesis $(H_1.(ii))$ allows the nonlinearity to increase indefinitely with time, which can not be covered by the previous special cases (that is when A_i, B_i are constants or $A_i, B_i \in L^1(\mathbb{R}_+)$). Therefore, our work generalizes and complements many existing results in the literature.

For $\lambda > 0$ and $\mu > 1$, let $M_{\alpha,\beta}(\lambda,\mu)$ be the square matrix defined by:

$$M_{\alpha,\beta}(\lambda,\mu) := \begin{pmatrix} \sum_{i=1}^{l} L_i + \Lambda_{\lambda,\mu}^{\alpha} + H & \sum_{i=1}^{l} \tilde{L}_i + \tilde{\Lambda}_{\lambda,\mu}^{\alpha} + \tilde{H} \\ \sum_{i=1}^{m} M_i + \Lambda_{\lambda,\mu}^{\beta} + K & \sum_{i=1}^{m} \tilde{M}_i + \tilde{\Lambda}_{\lambda,\mu}^{\beta} + \tilde{K} \end{pmatrix} \tag{3.1}$$

Where

$$\begin{aligned} \Lambda_{\lambda,\mu}^{\alpha} &= \tfrac{1}{\Gamma(\alpha)\lambda^{\alpha}} \left(\tfrac{1}{(\alpha+1)^{\alpha^2}}\Gamma(\alpha^2)\right)^{\frac{1}{1+\alpha}} \left(S_{\lambda,\mu}\right)^{\frac{\alpha}{1+\alpha}} \\ \tilde{\Lambda}_{\lambda,\mu}^{\alpha} &= \tfrac{1}{\Gamma(\alpha)\lambda^{\alpha}} \left(\tfrac{1}{(\alpha+1)^{\alpha^2}}\Gamma(\alpha^2)\right)^{\frac{1}{1+\alpha}} \left(\tilde{S}_{\lambda,\mu}\right)^{\frac{\alpha}{1+\alpha}} \\ \Lambda_{\lambda,\mu}^{\beta} &= \tfrac{1}{\Gamma(\beta)\lambda^{\beta}} \left(\tfrac{1}{(\beta+1)^{\beta^2}}\Gamma(\beta^2)\right)^{\frac{1}{1+\beta}} \left(R_{\lambda,\mu}\right)^{\frac{\beta}{1+\beta}} \\ \tilde{\Lambda}_{\lambda,\mu}^{\beta} &= \tfrac{1}{\Gamma(\beta)\lambda^{\beta}} \left(\tfrac{1}{(\beta+1)^{\beta^2}}\Gamma(\beta^2)\right)^{\frac{1}{1+\beta}} \left(\tilde{R}_{\lambda,\mu}\right)^{\frac{\beta}{1+\beta}} \end{aligned} \tag{3.2}$$

Theorem 3.2. Let $(H_1)-(H_3)$ and (2.6) hold true. Then, the system (1.1)-(1.3) admits a unique global solution in X provided that: there exist $\lambda > 0$ and $\mu > 1$ such that

$$\text{The matrix } M_{\alpha,\beta}(\lambda,\mu) \text{ given in (3.1), converges to zero.} \tag{3.3}$$

Proof. Recall that the solutions of (1.1)-(1.3) are the fixed points of the operator T defined in (2.10). We shall prove that T is a generalized contraction in the sens of Definition 2.7, to deduce the result from Theorem 2.8. To this end, let us define a mapping $w : \mathcal{N} \longrightarrow \mathcal{N}$ as follows:

$$w(\nu) = \left[0, \max_{1\le i\le l, 1\le j\le n} \{\nu^m, K_i^m, \tilde{K}_j^m\}\right], \tag{3.4}$$

where ν^m denotes $\max \nu$ and K_i, $\tilde{K}_j$ are the compacts given by (H_2).

Note that according to (3.4), it follows that

$$\text{For every } \nu \in \mathcal{N} : w^n(\nu) = w(\nu), \quad \forall n \ge 2 \tag{3.5}$$

Let $\nu \in \mathcal{N}$ and $t \in \nu$. Using $(H_1(i)), (H_2), (H_3)$, we get:

$$|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \le$$

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{A_1(s)|u_1(s)-u_2(s)| + A_2(s)|v_1(s)-v_2(s)|\} ds$$

$$+\sum_{t_i<t} \left\{h_i |u_1(t_i)-u_2(t_i)| + \tilde{h}_i |v_1(t_i)-v_2(t_i)|\right\}$$

$$+\sum_{i=1}^{l} \left\{L_i d_{K_i}(u_1-u_2) + \tilde{L}_i d_{K_i}(v_1-v_2)\right\}$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ A_1(s) e^{\lambda s \frac{\mu-1}{\mu}} \max_{\sigma\in[0,t]} e^{-\lambda\sigma} |u_1(\sigma)-u_2(\sigma)| \right.$$

$$\left. + A_2(s)\, e^{\lambda s \frac{\mu-1}{\mu}} \max_{\sigma\in[0,t]} e^{-\lambda\sigma} |v_1(\sigma)-v_2(\sigma)| \right\} ds$$

$$+ \sum_{i=1}^{l} \left\{ L_i d_{K_i}(u_1-u_2) + \tilde{L}_i d_{K_i}(v_1-v_2) \right\}$$

$$+ H e^{\lambda t} \max_{\sigma\in[0,t]} e^{-\lambda\sigma} |u_1(\sigma)-u_2(\sigma)| + \tilde{H} e^{\lambda t} \max_{\sigma\in[0,t]} e^{-\lambda\sigma} |v_1(\sigma)-v_2(\sigma)|,$$

where λ is the positive parameter introduced in (2.4) and $\mu > 1$. Note that according to (3.4), the compacts $[0,t]$ and K_i $(1 \leq i \leq l)$ are included in $w(\nu)$. Hence

$$|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \leq \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A_1(s) e^{\lambda s \frac{\mu-1}{\mu}} ds \right\} d_{w(\nu)}(u_1-u_2)$$

$$+ \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A_2(s) e^{\lambda s \frac{\mu-1}{\mu}} ds \right\} d_{w(\nu)}(v_1-v_2)$$

$$+ \sum_{i=1}^{l} \left\{ L_i d_{w(\nu)}(u_1-u_2) + \tilde{L}_i d_{w(\nu)}(v_1-v_2) \right\}$$

$$+ H e^{\lambda t} d_{w(\nu)}(u_1-u_2) + \tilde{H} e^{\lambda t} d_{w(\nu)}(v_1-v_2)$$

Now, multiplying the above inequality by $e^{-\lambda t}$, we get:

$$e^{-\lambda t} |T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \leq$$

$$\left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t - s\frac{\mu-1}{\mu})} A_1(s) ds \right\} d_{w(\nu)}(u_1-u_2)$$

$$+ \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t - s\frac{\mu-1}{\mu})} A_2(s) ds \right\} d_{w(\nu)}(v_1-v_2) \tag{3.6}$$

$$+ \sum_{i=1}^{l} \left\{ L_i d_{w(\nu)}(u_1-u_2) + \tilde{L}_i d_{w(\nu)}(v_1-v_2) \right\}$$

$$+ H d_{w(\nu)}(u_1-u_2) + \tilde{H} d_{w(\nu)}(v_1-v_2)$$

Let us find estimates for the integrals in (3.6):

$$I := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t - s\frac{\mu-1}{\mu})} A_1(s) ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t-s)} A_1(s) e^{\frac{-\lambda s}{\mu}} ds$$

Performing the change of variable $X = \lambda(t-s)$, we get:

$$I = \frac{1}{\Gamma(\alpha)\lambda^\alpha} \int_0^{\lambda t} X^{\alpha-1} e^{-X} A_1\left(t - \frac{X}{\lambda}\right) e^{-\frac{\lambda}{\mu}(t - \frac{X}{\lambda})} dX$$

In view of $(H_1.(ii))$, Hölder's inequality gives:

$$I \leq \frac{1}{\Gamma(\alpha)\lambda^{\alpha}}\left\{\int_0^{\lambda t}\left(X^{\alpha-1}e^{-X}\right)^{1+\alpha}dX\right\}^{\frac{1}{1+\alpha}} \times$$

$$\left\{\int_0^{\lambda t}\left(A_1\left(t-\frac{X}{\lambda}\right)e^{-\frac{\lambda}{\mu}(t-\frac{X}{\lambda})}\right)^{1+\frac{1}{\alpha}}dX\right\}^{\frac{\alpha}{1+\alpha}}$$

Consequently:

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda(t-s\frac{\mu-1}{\mu})}A_1(s)ds \leq \Lambda^{\alpha}_{\lambda,\mu} \tag{3.7}$$

In the same way, we can prove that

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda(t-s\frac{\mu-1}{\mu})}A_2(s)ds \leq \tilde{\Lambda}^{\alpha}_{\lambda,\mu} \tag{3.8}$$

In view of (3.7) and (3.8), and after taking the maximum on ν, the estimation (3.6) can be rewritten as:

$$\begin{aligned} d_{\nu}\left(T_1(u_1,v_1),T_1(u_2,v_2)\right) \leq{}& \Lambda^{\alpha}_{\lambda,\mu}\, d_{w(\nu)}(u_1-u_2)+\tilde{\Lambda}^{\alpha}_{\lambda,\mu}\, d_{w(\nu)}(v_1-v_2) \\ &+\sum_{i=1}^{l}\left\{L_i d_{w(\nu)}(u_1-u_2)+\tilde{L}_i d_{w(\nu)}(v_1-v_2)\right\} \\ &+Hd_{w(\nu)}(u_1-u_2)+\tilde{H}d_{w(\nu)}(v_1-v_2) \end{aligned} \tag{3.9}$$

Similarly, we prove that the following inequality holds true for every $(u_1,v_1),(u_2,v_2) \in X$ and every $\nu \in \mathcal{N}$:

$$\begin{aligned} d_{\nu}\left(T_2(u_1,v_1),T_2(u_2,v_2)\right) \leq{}& \Lambda^{\beta}_{\lambda,\mu}\, d_{w(\nu)}(u_1-u_2)+\tilde{\Lambda}^{\beta}_{\lambda,\mu}\, d_{w(\nu)}(v_1-v_2) \\ &+\sum_{i=1}^{m}\left\{M_i d_{w(\nu)}(u_1-u_2)+\tilde{M}_i d_{w(\nu)}(v_1-v_2)\right\} \\ &+Hd_{w(\nu)}(u_1-u_2)+\tilde{K}d_{w(\nu)}(v_1-v_2) \end{aligned} \tag{3.10}$$

Now, (3.9) together with (3.10) lead to what follows for every $(u_1,v_1),(u_2,v_2) \in X$ and every $\nu \in \mathcal{N}$:

$$D_{\nu}\left(T(u_1,v_1),T(u_2,v_2)\right) \leq M_{\alpha,\beta}\left(\lambda,\mu\right)D_{w(\nu)}\left(\left(u_1,u_2\right),\left(v_1,v_2\right)\right) \tag{3.11}$$

That is (2.1) holds true with $M_{\nu} = M_{\alpha,\beta}\left(\lambda,\mu\right)$, which is independent of ν. Consequently the series (2.2) turns in our case into

$$\sum_{n=0}^{\infty} M^{n+1}_{\alpha,\beta}\left(\lambda,\mu\right)D_{w^n(\nu)}\left(u,v\right) \tag{3.12}$$

According to (3.5), we have:

$$sup\left\{D_{w^n(\nu)}\left(u,v\right): n=0,1,2,\dots\right\} = sup\left\{D_{\nu}\left(u,v\right),D_{w(\nu)}\left(u,v\right)\right\} < \infty.$$

Since, moreover $M_{\alpha,\beta}(\lambda,\mu)$ is convergent to zero, then the series in (3.12) converges too. That is T is a generalized contraction and the result follows so, from Theorem 2.8 □

Remark 3.3. In view of Lemme 2.5, the following condition is equivalent to (3.3)

$$\sqrt{\left(\sum_{i=1}^{l} L_i+\Lambda^{\alpha}_{\lambda,\mu}+H-\sum_{i=1}^{m}\tilde{M}_i-\tilde{\Lambda}^{\beta}_{\lambda,\mu}-\tilde{K}\right)^2+4\left(\sum_{i=1}^{l}\tilde{L}_i+\tilde{\Lambda}^{\alpha}_{\lambda,\mu}+\tilde{H}\right)\left(\sum_{i=1}^{m} M_i+\Lambda^{\beta}_{\lambda,\mu}+K\right)} + \sum_{i=1}^{l} L_i+\Lambda^{\alpha}_{\lambda,\mu}+H+\sum_{i=1}^{m}\tilde{M}_i+\tilde{\Lambda}^{\beta}_{\lambda,\mu}+\tilde{K}<2 \tag{3.13}$$

The following Corollary provides a global existence-uniqueness result for a particular class of (1.1)-(1.3).

Corollary 3.4. Assume that in addition of $(H_1)-(H_3)$ and (2.6), the following hypothesis holds true:

$$\forall\epsilon>0,\ \exists\lambda>0,\ \mu>1,\ \text{such that: } S_{\lambda,\mu},\ \tilde{S}_{\lambda,\mu},\ R_{\lambda,\mu},\ \tilde{R}_{\lambda,\mu}<\epsilon. \tag{3.14}$$

Then, (1.1)-(1.3) admits a unique global solution in X provided that:

$$Q:=\begin{pmatrix} \sum_{i=1}^{l} L_i+H & \sum_{i=1}^{l}\tilde{L}_i+\tilde{H} \\ \sum_{i=1}^{m} M_i+K & \sum_{i=1}^{m}\tilde{M}_i+\tilde{K}\end{pmatrix},\ \text{converges to zero} \tag{3.15}$$

Proof. Note first that $M_{\alpha,\beta}(\lambda,\mu)=P_{\alpha,\beta}(\lambda,\mu)+Q$, where

$$P_{\alpha,\beta}(\lambda,\mu):=\begin{pmatrix}\Lambda^{\alpha}_{\lambda,\mu} & \tilde{\Lambda}^{\alpha}_{\lambda,\mu} \\ \Lambda^{\beta}_{\lambda,\mu} & \tilde{\Lambda}^{\beta}_{\lambda,\mu}\end{pmatrix}$$

It is not hard to see that under hypothesis (3.14), the elements of $P_{\alpha,\beta}(\lambda,\mu)$ are small enough.

Hence, in view of (3.15) together with Lemma 2.6, $M_{\alpha,\beta}(\lambda,\mu)$ is convergent to zero and the result follows so from Theorem 3.2. □

When $I_i=J_i=0$ for every $i\in\mathbb{N}^*$, that is by omitting the impulsive condition (1.3), then the problem (1.1)-(1.3) is reduced to:

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t)=f(t,u(t),v(t)), & t>0 \\ {}^{C}D^{\beta}_{0^+}v(t)=g(t,u(t),v(t)), & t>0 \\ u(0)=\varphi(u,v), \\ v(0)=\psi(u,v), \end{cases} \tag{3.16}$$

In this particular case, we have:

$$\Delta u(t_i):=u(t_i^+)-u(t_i^-)=I_i(u(t_i),v(t_i))=0$$

and

$$\Delta v(t_i):=v(t_i^+)-v(t_i^-)=J_i(u(t_i),v(t_i))=0$$

Which means that the space $X = \mathcal{PC}(\mathbb{R}_+) \times \mathcal{PC}(\mathbb{R}_+)$, where $\mathcal{PC}(\mathbb{R}_+)$ is defined by (2.3), becomes $\mathcal{C}(\mathbb{R}_+) \times \mathcal{C}(\mathbb{R}_+)$. So, as particular cases of Theorem 3.2 and Corollary 3.4, we have the following Corollary.

Corollary 3.5. Under hypotheses $(H_1) - (H_2)$, the system (3.16) admits a unique global solution in $\mathcal{C}(\mathbb{R}_+) \times \mathcal{C}(\mathbb{R}_+)$ provided that: there exist $\lambda > 0$ and $\mu > 1$ such that:

$$\tilde{M}_{\alpha,\beta}(\lambda,\mu) = \begin{pmatrix} \sum\limits_{i=1}^{l} L_i + \Lambda^{\alpha}_{\lambda,\mu} & \sum\limits_{i=1}^{l} \tilde{L}_i + \tilde{\Lambda}^{\alpha}_{\lambda,\mu} \\ \sum\limits_{i=1}^{m} M_i + \Lambda^{\beta}_{\lambda,\mu} & \sum\limits_{i=1}^{m} \tilde{M}_i + \tilde{\Lambda}^{\beta}_{\lambda,\mu} \end{pmatrix} \text{ converges to zero} \tag{3.17}$$

where $\Lambda^{\alpha}_{\lambda,\mu}, \tilde{\Lambda}^{\alpha}_{\lambda,\mu}, \Lambda^{\beta}_{\lambda,\mu}, \tilde{\Lambda}^{\beta}_{\lambda,\mu}$ are given by (3.2). If in addition, (3.14) holds true, then (3.17) is weakened to

$$\tilde{Q} := \begin{pmatrix} \sum\limits_{i=1}^{l} L_i & \sum\limits_{i=1}^{l} \tilde{L}_i \\ \sum\limits_{i=1}^{m} M_i & \sum\limits_{i=1}^{m} \tilde{M}_i \end{pmatrix} \text{ converges to zero.} \tag{3.18}$$

Remark 3.6. Note that (3.14) includes the Lipschitz condition with constant and integrable arguments. In this case, the matrices Q in (3.15) and $\tilde{Q}$ in (3.18) are independent of A_i, B_i $(i = 1, 2)$. Moreover, with the classical initial conditions, $\sum\limits_{i=1}^{l} L_i$, $\sum\limits_{i=1}^{l} \tilde{L}_i$, $\sum\limits_{i=1}^{m} M_i$ and $\sum\limits_{i=1}^{m} \tilde{M}_i$ vanish. All this, allows us to see clearly that Corollary 3.4 and Corollary 3.5 provide significant improvements and generalizations of many recent results in the literature, such as [9, Theorem 15], [19, Theorem 3.3], [20, Theorem 3.1], [20, Theorem 3.2] and [5, Theorem 3.2].

4. Example

Let us consider the following system:

$$\begin{cases} {}^{C}D^{\frac{15}{20}}_{0^+} u(t) = \dfrac{1}{10} e^{\frac{t}{80}} \left(2u(t) + v(t)\right), & t > 0,\, t \neq t_i = 10^i,\, i \in \mathbb{N}^* \\ {}^{C}D^{\frac{13}{20}}_{0^+} v(t) = \dfrac{1}{10} e^{\frac{t}{80}} \left(u(t) + v(t)\right), & t > 0,\, t \neq t_i = 10^i,\, i \in \mathbb{N}^* \\ \Delta u(t_i) = \frac{7}{25i(i+1)(1+|u(t_i)|)} + \frac{5}{25i(i+1)(1+|v(t_i)|)}, & i \in \mathbb{N}^* \\ \Delta v(t_i) = \frac{6}{25\times 2^i(1+|u(t_i)|)} + \frac{9}{25\times 2^i(1+|v(t_i)|)}, & i \in \mathbb{N}^* \\ u(0) = \frac{1}{10} \sup\limits_{t\in[0,\,1]} u(t) + \frac{1}{5} \sup\limits_{t\in[0,\,\frac{1}{2}]} v(t) & \\ v(0) = \frac{1}{5} sin\left(u(\frac{1}{6}) + v(\frac{1}{3})\right) & \end{cases} \tag{4.1}$$

The problem (4.1) is identified to (1.1)-(1.3), with:

$$\alpha = \tfrac{15}{20}, f\left(t,\xi,\eta\right) = \frac{1}{10}e^{\frac{t}{80}}\left(2\xi+\eta\right),$$

$$I_i\left(\xi,\eta\right) = \frac{7}{25i(i+1)\left(1+|\xi|\right)} + \frac{5}{25i(i+1)\left(1+|\eta|\right)}$$

$$\beta = \tfrac{13}{20}, g\left(t,\xi,\eta\right) = \frac{1}{10}e^{\frac{t}{80}}\left(\xi+\eta\right),$$

$$J_i\left(\xi,\eta\right) = \frac{6}{25\times 2^i\left(1+|\xi|\right)} + \frac{9}{25\times 2^i\left(1+|\eta|\right)}$$

$$\varphi(u,v) = \tfrac{1}{10}\sup_{t\in[0,\,1]} u\left(t\right) + \tfrac{1}{5}\sup_{t\in[0,\,\frac{1}{2}]} v\left(t\right),\ \ \psi(u,v) = \tfrac{1}{5}sin\left(u(\tfrac{1}{6}) + v(\tfrac{1}{3})\right)$$

It is not hard to see that $(H_1.(i))$ is satisfied with:

$$A_1(t) = \frac{1}{5}e^{\frac{t}{80}},\ \ A_2(t) = B_1(t) = B_2(t) = \frac{1}{10}e^{\frac{t}{80}}$$

A straightforward computation leads to:

$$S_{\lambda,\mu} = \frac{7\lambda}{30\mu},\ \tilde{S}_{\lambda,\mu} = \frac{\lambda}{10\mu},\ R_{\lambda,\mu} = \tilde{R}_{\lambda,\mu} = \frac{23\lambda}{260\mu}$$

Which means that $(H_1.(ii))$ is satisfied too.
It can be easily seen that (H_2) is satisfied with:

$$l = 2, L_1 = \frac{1}{10}e^{\lambda}, L_2 = 0,\ \tilde{L}_1 = 0,\ \tilde{L}_2 = \frac{1}{5}e^{\frac{\lambda}{2}},\ K_1 = [0,1],\ K_2 = [0,\tfrac{1}{2}]$$

$$m = 2, M_1 = \frac{1}{10}e^{\frac{\lambda}{6}}, M_2 = 0,\ \tilde{M}_1 = 0,\ \tilde{M}_2 = \frac{1}{5}e^{\frac{\lambda}{3}},\ \tilde{K}_1 = [0,\tfrac{1}{6}],\ \ \tilde{K}_2 = [0,\tfrac{1}{3}]$$

For all $i \in \mathbb{N}^*$ we have:

$$\left|I_i\left(\xi_1,\eta_1\right) - I_i\left(\xi_2,\eta_2\right)\right| \le \frac{7}{25i(i+1)}|\xi_1-\xi_2| + \frac{5}{25i(i+1)}|\eta_1-\eta_2|$$

$$\left|J_i\left(\xi_1,\eta_1\right) - J_i\left(\xi_2,\eta_2\right)\right| \le \frac{6}{25\times 2^i}|\xi_1-\xi_2| + \frac{9}{25\times 2^i}|\eta_1-\eta_2|$$

That is, (H_3) is satisfied with:

$$\{h_i\} = \left\{\tfrac{7}{25i(i+1)}\right\}, \{\tilde{h}_i\} = \left\{\tfrac{5}{25i(i+1)}\right\},\ \{k_i\} = \left\{\tfrac{6}{25\times 2^i}\right\},\ \{\tilde{k}_i\} = \left\{\tfrac{9}{25\times 2^i}\right\}$$

$$H = \tfrac{7}{25},\ \ \tilde{H} = \tfrac{5}{25},\ \ K = \tfrac{6}{25},\ \tilde{K} = \tfrac{9}{25}$$

If we choose $\lambda = \frac{1}{2}$ and $\mu = 20$, the matrix $M_{\alpha,\beta}\left(\lambda,\mu\right)$ given in (3.1), becomes in this case:

$$M_{\alpha,\beta}\left(\lambda,\mu\right) := \begin{pmatrix} 0.486788 & 0.498721 \\ \\ 0.474757 & 0.495513 \end{pmatrix},$$

which admits the following eigenvalues: $\lambda_1 = 0.977761 < 1$ and $\lambda_2 = 0.00453945 < 1$ and consequently $M_{\alpha,\beta}(\lambda, \mu)$ converges to zero.

Hence, all conditions of Theorem 3.2 are fulfilled, and therefore the system (4.1) admits a unique global solution in $\mathcal{PC}(\mathbb{R}_+) \times \mathcal{PC}(\mathbb{R}_+)$.

Note that f and g in (4.1) increase indefinitely with time, and therefore many existing results in the literature fail to be applicable.

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Hankel and symmetric Toeplitz determinants for Sakaguchi starlike functions

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Abstract. In this paper, we consider the class of starlike functions with respect to symmetric points which are also known as Sakaguchi starlike functions. We determine best possible bounds on Zalcman conjecture $|a_n^2 - a_{2n-1}|$ and generalized Zalcman conjecture $|a_m a_n - a_{m+n-1}|$ for $n = 2$ and $n = 4$, $m = 2$, respectively for such functions. Further, we compute estimate on third order and fourth order Hankel determinants. As well, we also obtain estimates on third and fourth symmetric Toeplitz determinants.

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1. Introductory text

Let $\mathcal{A}$ be the family of all normalized analytic functions f defined on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with series expansion $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. The subfamily $\mathcal{S} \subset \mathcal{A}$ contains univalent functions. Let $\mathcal{S}^*$ and $\mathcal{K}$ represent the subfamily of $\mathcal{S}$ containing starlike and convex functions, respectively. Analytically, $\mathcal{S}^* = \{f \in \mathcal{S} : \operatorname{Re}(zf'(z)/f(z)) > 0, z \in \mathbb{D}\}$ and $\mathcal{K} = \{f \in \mathcal{S} : 1 + \operatorname{Re}(zf''(z)/f'(z)) > 0, z \in \mathbb{D}\}$ [11]. The class $\mathcal{P}$ consists of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ satisfying conditions $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. Recent results for a more general class of $\mathcal{P}$ can be found in [3]. In 1959, Sakaguchi [33] studied the subclass $\mathcal{S}^*_{\mathcal{S}}$ of $\mathcal{S}$ consisting of starlike functions

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with respect to the symmetric points. The analytical description of these functions is

$$\mathcal{S}_{\mathcal{S}}^{*} = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z)-f(-z)} \in \mathcal{P},\, z \in \mathbb{D} \right\}.$$

The functions $f \in \mathcal{S}_{\mathcal{S}}^{*}$ are also called Sakaguchi starlike functions. The coefficient estimates related literature gives the geometric properties of univalent functions. The bound on the initial coefficient a_2 contribute in growth, distortion and covering theorems. Zalcman conjecture and Hankel determinants are two of the coefficient problems that have been discussed by several authors. In recent years, many authors have studied the Toeplitz determinant $T_q(n)$ for various values of q and n for several subclasses of analytic functions. A significant problem concerning the coefficients in the series expansion of the the function $f \in \mathcal{A}$ is the Zalcman conjecture which is defined as

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2,\, n \geq 2.$$

From [7], we observe that the Zalcman conjecture implies the Bieberbach conjecture. Ma [24] verified Zalcman conjecture ($n \geq 4$) for close-to-convex functions. Further, Ma [25] explored the generalized Zalcman conjecture which is defined as

$$|a_m a_n - a_{m+n-1}| \leq (m-1)(n-1); \qquad m \geq 2,\, n \geq 2$$

for the starlike functions and the univalent functions with real coefficients. In [32], Ravichandran and Verma established the generalized Zalcman conjecture for certain starlike and convex functions. In [34], the Zalcman conjecture and the generalized Zalcman conjecture for the locally univalent functions were discussed using extreme point theory. Recently, in [26] the Zalcman conjecture and the generalized Zalcman conjecture were shown for the class $\mathcal{U}$ defined as $\mathcal{U} = \{z \in \mathcal{A} : \left|(z/f(z))^2 f'(z) - 1\right| < 1,\, z \in \mathbb{D}\}$. For $q \geq 1$ and $n \geq 1$, the q^{th} Hankel determinant $H_q(n)(f)$ for a function $f \in \mathcal{S}$ is given by $H_q(n)(f) := \det\{a_{n+i+j-2}\}_{i,j}^{q}$, $1 \leq i,\, j \leq q$, where $a_1 = 1$. For $q = 2$ and $n = 1$, the Hankel determinant $H_2(1) = a_3 - a_2^2$ is the Fekete Szegö functional. The study of Hankel determinant was initiated by Pommerenke [27, 28] for the starlike functions. Since then the growth of $H_q(n)(f)$ has been studied for different subclasses of univalent functions. One of the notable results in this direction is by Hayman [12] giving the best possible upper bound as $Mn^{1/2}$ on $H_2(n)(f)$, where M is an absolute constant. For $q = 2$ and $n = 2$, Janteng *et al.*[13] obtained the sharp estimates on second order Hankel determinant $H_2(2)(f) = a_2a_4 - a_3^2$ for the classes of starlike and convex functions. However, the sharp bound for the whole class $\mathcal{S}$ is not known till now. For the class of Bazilevic functions, Krishna and RamReddy [16] determined $H_2(2)(f)$. Recently, Anand *et al.* [4] studied the second order Hankel determinant for a class of normalized analytic functions.

For $q = 3$ and $n = 1, 2, 3$, the third Hankel determinants are given as

$$H_3(1)(f) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2) \tag{1.1}$$

$$H_3(2)(f) = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2) \tag{1.2}$$

$$H_3(3)(f) = a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2). \tag{1.3}$$

The study of the third order Hankel determinant $H_3(1)(f)$ for the classes $\mathcal{S}^*$ and $\mathcal{K}$ was initiated by Babalola (2010) [6] which was later improved by Zaprawa [36]. However, the bounds obtained in [36] were not sharp. The best possible bound on third order Hankel determinant $H_3(1)(f)$ for the class of convex functions was computed by Kowalczyk *et al.* [15]. Also, Lecko *et al.* [23] computed the best possible upper bound on $H_3(1)(f)$ for the starlike functions of order 1/2. Krishna *et al.* [17] obtained the bound on $H_3(1)(f)$ for the class $\mathcal{S}^*_S$. Recently, Kumar *et al.* [20] improved the existing bound for the class $\mathcal{S}^*_S$. For more recent developments on coefficient estimates and third order Hankel determinant, see [14, 17, 29, 37, 22, 21, 35]. For $q = 4$ and $n = 1$, the fourth order Hankel determinant is given by

$$H_4(1)(f) = a_7 H_3(1)(f) - a_6\Delta_1 + a_5\Delta_2 - a_4\Delta_3 \tag{1.4}$$

where

$$\Delta_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2),$$
$$\Delta_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2)$$

and

$$\Delta_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2).$$

Arif *et al.* [5] obtained the bound on $H_4(1)(f)$ for the functions with bounded turning. Cho and Kumar [9] computed the bound on $H_4(1)(f)$ for starlike functions associated with a lune-shaped region. For recent results on fourth order Hankel determinant, see [19, 10]. For $q \geq 1$ and $n \geq 1$, the symmetric Toeplitz determinant $T_q(n)$ for a function $f \in \mathcal{S}$ is defined as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}$$

where $a_1 = 1$. In particular, for $q = 2$ and $n = 2, 3$ the second Toeplitz deteminants are given by $T_2(2) = a_3^2 - a_2^2$ and $T_2(3) = a_4^2 - a_3^2$.

For $q = 3$ and $n = 1, 2$ the third Toeplitz determinants are as follows

$$T_3(1) = 1 + 2a_2^2(a_3 - 1) - a_3^2 \text{ and } T_3(2) = (a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4). \tag{1.5}$$

For $q = 4$ and $n = 2$ the fourth Toeplitz determinant is given by

$$\begin{aligned} T_4(2) = (a_2^2 - a_3^2)^2 + 2(a_3^2 - a_2a_4)(a_2a_4 - a_3a_5) - (a_2a_3 - a_3a_4)^2 \\ + (a_4^2 - a_3a_5)^2 - (a_3a_4 - a_2a_5)^2. \end{aligned} \tag{1.6}$$

In 2019, Zhang *et al.* [38] computed the upper bound on the Toeplitz determinant $T_3(2)$ for the starlike functions associated with the sine function. Ahuja *et al.* [1] studied the Toeplitz determinants $T_2(2)$ and $T_3(1)$ for unified class of starlike and convex functions. Recently, in [39], Zhang and Tang obtained the upper bound on fourth Toeplitz determinant $T_4(2)$ for the starlike functions associated with the sine function. For more recent details, see [2, 18]

In this manuscript, we prove Zalcman Conjecture $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ for $n = 2$ and generalised Zalcman Conjecture $|a_m a_n - a_{m+n-1}| \leq (m-1)(n-1)$ for

$m = 2$, $n = 4$. Further, we obtain the estimates on the third order Hankel determinant $H_3(1)(f)$ for such functions which is an improvement to the existing estimate computed in [20]. In addition, we compute the bounds on third order Hankel determinants $H_3(2)(f)$, $H_3(3)(f)$ and the fourth order Hankel determinant $H_4(1)(f)$. Moreover, bounds on the symmetric Toeplitz determinants $T_2(2), T_2(3), T_3(1), T_3(2)$ and $T_4(2)$ are also determined.

2. Inductive lemmas

In order to establish the main results, we need following lemmas related to coefficient estimates.

Lemma 2.1. [30] *Let $w(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then*

$$|c_3 + \mu c_1 c_2 + \nu c_1^3| \leq 1,$$

where $1/2 \leq |\mu| \leq 2,\ 4(|\mu|+1)^3/27 - (|\mu|+1) \leq \nu \leq 1$.

Let $\mathcal{B}$ be the class of functions $f \in \mathcal{A}$ satisfying $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Lemma 2.2. [8] *Let $f(z) = a_0 + \sum\limits_{n=1}^{\infty} a_n z^n$ be in $\mathcal{B}$. Then*

$$|a_{2n+1}| \leq 1 - |a_0|^2 - |a_1|^2 - \cdots - |a_n|^2,\ n = 0, 1, \cdots \tag{2.1}$$

and

$$|a_{2n}| \leq 1 - |a_0|^2 - |a_1|^2 - \cdots - |a_{n-1}|^2 - \frac{|a_n|^2}{1+|a_0|},\ n = 1, 2, \cdots . \tag{2.2}$$

Equality in (2.1) holds for

$$f(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n + \varepsilon z^{2n+1}}{1 + (\overline{a_n} z^{n+1} + \overline{a_{n-1}} z^{n+2} + \cdots + \overline{a_0} z^{2n+1})\varepsilon},\ |\varepsilon| = 1$$

and in (2.2) for

$$f(z) = \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + \frac{a_n}{1+|a_0|} + \varepsilon z^{2n}}{1 + (\frac{\overline{a_n}}{1+|a_0|} z^n + \overline{a_{n-1}} z^{n+1} + \cdots + \overline{a_0} z^{2n})\varepsilon},\ |\varepsilon| = 1$$

where $a_0 \overline{a_n}^2 \varepsilon$ is non-positive real.

In view of Lemma 2.2, for a Schwarz function $w(z) = c_1 z + c_2 z^2 + \cdots$, we have

$$|c_2| \leq 1 - |c_1|^2,\ |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|} \text{ and } |c_4| \leq 1 - |c_1|^2 - |c_2|^2. \tag{2.3}$$

Lemma 2.3. [33] *Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be univalent and starlike with respect to symmetric points in $\mathbb{D}$. Then*

$$|a_n| \leq 1,\ n \geq 2$$

equality being attained by the function $z/(1+\varepsilon z)$, $|\varepsilon| < 1$.

Lemma 2.4. [31] *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ *then for all* $n, m \in \mathbb{N}$

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1 \\ 2|2\mu - 1|, & \textit{elsewhere.} \end{cases}$$

If $0 < \mu < 1$, *the inequality is sharp for the function* $p(z) = (1 + z^{n+m})/(1 - z^{n+m})$. *In other cases, the inequality is sharp for the function* $p(z) = (1 + z)/(1 - z)$.

3. Zalcman conjecture

In this section, we first prove Zalcman conjecture ($n = 2$) for starlike functions with respect to the symmetric space.

Theorem 3.1. *If the function* $f \in \mathcal{S}_{\mathcal{S}}^*$ *is of the form* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. *Then*

$$|a_2^2 - a_3| \leq 1.$$

The inequality is sharp.

Proof. Let $f \in \mathcal{S}_{\mathcal{S}}^*$. Then we have

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + z}{1 - z}$$

for all $z \in \mathbb{D}$ so that

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z)$$

where $\prec$ denotes subordination and $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$. On comparing the coefficients of like power terms on both sides, we get

$$a_2 = \frac{p_1}{2}; \tag{3.1}$$

$$a_3 = \frac{p_2}{2}; \tag{3.2}$$

$$a_4 = \frac{1}{8}(p_1 p_2 + 2p_3); \tag{3.3}$$

$$a_5 = \frac{1}{8}(p_2^2 + 2p_4); \tag{3.4}$$

$$a_6 = \frac{1}{48}(4p_2 p_3 + p_1(p_2^2 + 2p_4) + 8p_5); \tag{3.5}$$

$$a_7 = \frac{1}{48}(p_2^3 + 6p_2 p_4 + 8p_6); \tag{3.6}$$

It follows from (3.1) and (3.2) that

$$a_2^2 - a_3 = \frac{p_1^2}{4} - \frac{p_2}{2}.$$

By using Lemma 2.4, we get

$$|a_2^2 - a_3| = \frac{1}{2}\left|\frac{1}{2}p_1^2 - p_2\right| \leq 1.$$

The inequality is sharp for the function

$$\frac{2zf'(z)}{f(z)-f(-z)} = \frac{1+z^2}{1-z^2} = 1+2z^2+2z^4+\cdots \tag{3.7}$$

by noting the fact $a_2 = 0, a_3 = 1$ implies $|a_2^2 - a_3| = 1$. □

Next we prove the generalized Zalcman conjecture for $m = 2$ and $n = 4$.

Theorem 3.2. *Let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}^*_{\mathcal{S}}$. Then*

$$|a_2a_4 - a_5| \leq 1.$$

The inequality is sharp.

Proof. If the function $f \in \mathcal{S}^*_{\mathcal{S}}$, then using (3.1),(3.3) and (3.4), we get

$$\begin{aligned} a_2a_4 - a_5 &= \frac{1}{16}p_1(p_1p_2+2p_3) - \frac{1}{8}(p_2^2+2p_4) \\ &= \frac{1}{8}p_2\left(\frac{1}{2}p_1^2 - p_2\right) + \frac{1}{4}\left(\frac{1}{2}p_1p_3 - p_4\right). \end{aligned}$$

Using triangle inequality

$$|a_2a_4 - a_5| \leq \frac{1}{8}|p_2|\left|\frac{1}{2}p_1^2 - p_2\right| + \frac{1}{4}\left|\frac{1}{2}p_1p_3 - p_4\right|.$$

Applying Lemma 2.4 and the fact $|p_n| \leq 2$, we get

$$|a_2a_4 - a_5| \leq 1.$$

The inequality is sharp for the function f defined by (3.7). □

4. Hankel determinants

Using the technique discussed in [37], the following theorem gives an improved estimate on $H_3(1)$ for the functions f in the class $\mathcal{S}^*_{\mathcal{S}}$.

Theorem 4.1. *Let the function $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|H_3(1)(f)| \leq \frac{329}{400} \simeq 0.8225.$$

Proof. Let $f \in \mathcal{S}^*_{\mathcal{S}}$. Then we have

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+z}{1-z}$$

so that

$$\frac{2zf'(z)}{f(z)-f(-z)} = \frac{1+w(z)}{1-w(z)}, \qquad z \in \mathbb{D}$$

where $\prec$ denotes subordination and $w(z) = c_1 z + c_2 z^2 + \cdots$ is a Schwarz function. On comparing the coefficients of like powers of z, we get

$$a_2 = c_1,\ a_3 = c_1^2 + c_2,\ a_4 = \frac{1}{2}(c_3 + 3c_1c_2 + 2c_1^3) \tag{4.1}$$

$$a_5 = \frac{1}{2}(c_4 + 2c_1c_2 + 5c_1^2c_2 + 2c_1^4 + 2c_2^2). \tag{4.2}$$

Therefore, in view of (1.1), (4.1) and (4.2) the third order Hankel determinant $H_3(1)$ becomes

$$\begin{aligned} H_3(1)(f) &= \frac{1}{4}(c_1^2c_2^2 + 2c_1c_2c_3 - c_3^2 + 2c_2c_4) \\ &= \frac{1}{4}(-2c_3(c_3 - c_1c_2) + c_3^2 + c_1^2c_2^2 + 2c_2c_4). \end{aligned}$$

Hence, applying Lemma 2.1 ($\mu = -1, \nu = 0$) and inequalities given in (2.3), we get

$$\begin{aligned} |H_3(1)(f)| &\le \frac{1}{4}(2\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + \left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right)^2 \\ &\qquad + |c_1|^2|c_2|^2 + 2|c_2|\left(1 - |c_1|^2 - |c_2|^2\right)) \\ &= \frac{1}{4}G(|c_1|, |c_2|). \end{aligned}$$

The function $G(x, y)$ is given by

$$G(x, y) = g_1(x, y) + g_2(x, y) + g_3(x, y),$$

where

$$\begin{aligned} g_1(x, y) &= \left(\frac{y}{(1+x)^2} - 1\right) y^3 \\ g_2(x, y) &= 2(1 - x^2)y - \frac{4 - 3x^2 - x^3}{1+x} y^2 \\ g_3(x, y) &= -y^3 + x^4 - 4x^2 + 3 \end{aligned}$$

where $x = |c_1|$ and $y = |c_2|$. In view of $|c_2| \le 1 - |c_1|^2$, we maximize the function $G(x, y)$ in the region

$$\Omega = \{(x, y) : x \ge 0, y \ge 0, y \le 1 - x^2\}.$$

It is noted that

$$g_1(x, y) \le 0. \tag{4.3}$$

Since $g_2(x, y)$ is a quadratic expression in y, so it attains its maximum value at

$$y_0 = \frac{(1 - x^2)(1 + x)}{4 - 3x^2 - x^3}.$$

Also $y_0 < 1 - x^2$ for all $x \in [0, 1]$ and thus we have

$$g_2(x, y) \le g_2(x, y_0) = \frac{(1 - x)(1 + x)^3}{(2 + x)^2} =: f(x).$$

A simple calcultaion shows that $x_2 = 0.3$ is a critical point of the function f in $(0,1)$. Hence,

$$g_2(x,y) \leq f(x_2) = \frac{29}{100}. \tag{4.4}$$

For the function $g_3(x,y)$, it is evident that

$$g_3(x,y) \leq g_3(x,0) = x^4 - 4x^2 + 3 =: h(x).$$

Now $h'(x) = 4x(x^2-2)$, so $h(x) \leq h(0)$. This gives

$$g_3(x,y) \leq h(0) = 3. \tag{4.5}$$

Using (4.3), (4.4) and (4.5), we get $G(x,y) \leq 329/100$.
Therefore, we have $|H_3(1)(f)| \leq 329/400 \simeq 0.8225$. □

Remark 4.2. The obtained upper bound $\frac{329}{400} \simeq 0.8225$ on $H_3(1)(f)$ (4.1) improves the existing bound $\frac{5}{4} \simeq 1.25$ [20, Theorem 2.3, p.227] for the functions $f \in \mathcal{S}^*_{\mathcal{S}}$.

Next theorem gives bound on $H_3(2)$ for the functions $f \in \mathcal{S}^*_{\mathcal{S}}$.

Theorem 4.3. *If $f \in \mathcal{S}^*_{\mathcal{S}}$ is of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|H_3(2)(f)| < \frac{83}{24} \simeq 3.45.$$

Proof. On substituting the values of a_4, a_5 and a_6 from (3.3), (3.4) and (3.5), respectively in the expression $a_4a_6 - a_5^2$, we have

$$\begin{aligned} a_4a_6 - a_5^2 &= \frac{1}{384}(p_1p_2 + 2p_3)(4p_2p_3 + p_1p_2^2 + 2p_1p_4 + 8p_5) - \frac{1}{64}(p_2^4 + 4p_4^2 + 4p_2^2p_4) \\ &= \frac{1}{384}(4p_1p_2^2p_3 + p_1^2p_2^3 + 2p_1^2p_2p_4 + 8p_1p_2p_5 + 8p_2p_3^2 + 2p_1p_2^2p_3 + 4p_1p_3p_4 \\ &\qquad + 16p_3p_5) - \frac{1}{64}p_2^4 - \frac{1}{16}p_4^2 - \frac{1}{16}p_2^2p_4 \\ &= \frac{1}{384}p_1^2p_2^3 + \frac{1}{64}p_1p_2^2p_3 + \frac{1}{48}p_2p_3^2 + \frac{1}{192}p_1^2p_2p_4 + \frac{1}{96}p_1p_4p_3 + \frac{1}{48}p_1p_2p_5 \\ &\qquad + \frac{1}{24}p_3p_5 - \frac{1}{64}p_2^4 - \frac{1}{16}p_4^2 - \frac{1}{16}p_2^2p_4 \\ &= \frac{1}{64}p_2^3\left(\frac{1}{6}p_1^2 - p_2\right) + \frac{1}{16}p_2^2\left(\frac{1}{4}p_1p_3 - p_4\right) + \frac{1}{24}p_3\left(\frac{1}{2}p_2p_3 + p_5\right) \\ &\qquad + \frac{1}{16}p_4\left(\frac{1}{6}p_1p_3 - p_4\right) + \frac{1}{48}p_1p_2\left(\frac{1}{4}p_1p_4 + p_5\right). \end{aligned}$$

By triangle inequality, we get

$$\begin{aligned} |a_4a_6 - a_5^2| \leq \frac{1}{64}|p_2^3|\left|\frac{1}{6}p_1^2 - p_2\right| + \frac{1}{16}|p_2^2|\left|\frac{1}{4}p_1p_3 - p_4\right| + \frac{1}{24}|p_3|\left|\frac{1}{2}p_2p_3 + p_5\right| \\ + \frac{1}{16}|p_4|\left|\frac{1}{6}p_1p_3 - p_4\right| + \frac{1}{48}|p_1||p_2|\left|\frac{1}{4}p_1p_4 + p_5\right|. \end{aligned}$$

Using Lemma 2.4 and the inequality $|p_n| \le 2$, we get

$$|a_4a_6 - a_5^2| \le \frac{19}{12}. \tag{4.6}$$

Again, on substituting the values of a_3, a_4, a_5 and a_6 from (3.2), (3.3), (3.4) and (3.5), respectively in the expression $a_3a_6 - a_4a_5$, we have

$$\begin{aligned} a_3a_6 - a_4a_5 &= \frac{1}{96}(4p_2^2p_3 + p_1p_2^3 + 2p_1p_2p_4 + 8p_2p_5) \\ &\quad - \frac{1}{64}(p_1p_2^3 + 2p_1p_2p_4 + 2p_2^2p_4 + 2p_2^2p_3 + 4p_3p_4) \\ &= \frac{1}{96}p_2^2p_3 - \frac{1}{192}p_1p_2^3 - \frac{1}{96}p_1p_2p_4 + \frac{1}{12}p_2p_5 - \frac{1}{16}p_3p_4. \end{aligned}$$

So that

$$|a_3a_6 - a_4a_5| \le \frac{1}{16}|p_3|\left|\frac{1}{12}p_2^2 - p_4\right| + \frac{1}{192}|p_2^2||p_1p_2 - p_3| + \frac{1}{12}|p_2|\left|\frac{1}{8}p_1p_4 - p_5\right|.$$

By Lemma 2.4 and the fact $|p_n| \le 2$, we have

$$|a_3a_6 - a_4a_5| \le \frac{5}{8}. \tag{4.7}$$

On substituting the values of a_3, a_4 and a_5 from (3.2), (3.3) and (3.4), respectively in the expression $a_3a_5 - a_4^2$, we have

$$a_3a_5 - a_4^2 = -\frac{1}{16}p_2^2\left(\frac{1}{4}p_1^2 - p_2\right) - \frac{1}{8}p_2\left(\frac{1}{2}p_1p_3 - p_4\right) - \frac{1}{16}p_3^2$$

so that

$$|a_3a_5 - a_4^2| \le \frac{1}{16}|p_2|^2\left|\frac{1}{4}p_1^2 - p_2\right| + \frac{1}{8}|p_2|\left|\frac{1}{2}p_1p_3 - p_4\right| + \frac{1}{16}|p_3|^2.$$

Using Lemma 2.4 and the inequality $|p_n| \le 2$, we have

$$|a_3a_5 - a_4^2| \le \frac{5}{4}. \tag{4.8}$$

It follows from (1.2) that

$$|H_3(2)(f)| \le |a_2||a_4a_6 - a_5^2| + |a_3||a_3a_6 - a_4a_5| + |a_4||a_3a_5 - a_4^2|.$$

Using inequality (4.6), (4.7) and (4.8) and Lemma 2.3, we have $|H_3(2)(f)| \le 83/24 \simeq 3.45$. □

In the next theorem we estimate third order Hankel determinant $H_3(3)$ for $f \in \mathcal{S}_{\mathcal{S}}^*$.

Theorem 4.4. *If* $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in \mathcal{S}_{\mathcal{S}}^*$, *then*

$$|H_3(3)(f)| \le \frac{89}{24} \simeq 3.7.$$

Proof. On substituting (3.4),(3.5) and (3.6), we have

$$\begin{aligned}
a_5a_7-a_6^2 &= \frac{1}{384}(p_2^5+6p_2^3p_4+8p_2^2p_6+2p_2^3p_4+12p_2p_4^2+16p_4p_6)-\frac{1}{2304}(16p_2^2p_3^2\\
&\quad +p_1^2p_2^4+4p_1^2p_4^2+64p_5^2+8p_1p_2^3p_3+16p_1p_2p_3p_4+64p_2p_3p_5+4p_1^2p_2^2p_4\\
&\quad +16p_1p_2^2p_5+32p_1p_4p_5)\\
&= \frac{1}{96}p_2^2\left(p_6-\frac{2}{3}p_3^2\right)+\frac{1}{384}p_2^4\left(p_2-\frac{1}{6}p_1^2\right)+\frac{1}{192}p_4^2\left(p_2-\frac{1}{3}p_1^2\right)\\
&\quad +\frac{1}{64}p_2^3\left(p_4-\frac{2}{9}p_1p_3\right)+\frac{5}{192}p_2p_4\left(p_4-\frac{4}{15}p_1p_3\right)+\frac{1}{192}p_2^2p_4\\
&\quad \left(p_2-\frac{1}{3}p_1^2\right)+\frac{1}{24}p_4\left(p_6-\frac{1}{3}p_1p_5\right)+\frac{1}{96}p_2^2\left(p_6-\frac{2}{3}p_1p_5\right)-\frac{1}{36}p_5^2\\
&\quad -\frac{1}{36}p_2p_3p_5.
\end{aligned}$$

Using triangle inequality, Lemma 2.4 and the fact $|p_n|\le 2$, we get

$$|a_5a_7-a_6^2|\le\frac{4}{3}. \tag{4.9}$$

Again in view of (3.3),(3.4),(3.5) and (3.6), we have

$$\begin{aligned}
a_4a_7-a_6a_5 &= \frac{1}{384}(p_1p_2^4+6p_1p_2^2p_4+8p_1p_2p_6+2p_2^3p_3+12p_2p_3p_4+16p_3p_6)\\
&\quad -\frac{1}{384}(4p_2^3p_3+8p_2p_3p_4+p_1p_2^4+4p_1p_2^2p_4+4p_1p_4^2+8p_2^2p_5+16p_4p_5)\\
&= \frac{1}{48}p_2^2\left(\frac{1}{4}p_1p_4-p_5\right)+\frac{1}{24}p_4\left(\frac{1}{8}p_2p_3-p_5\right)+\frac{1}{192}p_2p_3(p_4-p_2^2)\\
&\quad +\frac{1}{24}p_6\left(\frac{1}{2}p_1p_2+p_3\right)+\frac{1}{96}p_1p_4^2.
\end{aligned}$$

Using triangle inequality, by Lemma 2.4 and $|p_n|\le 2$, we have

$$|a_4a_7-a_6a_5|\le\frac{19}{24}. \tag{4.10}$$

It follows from (1.3) that

$$|H_3(3)|\le|a_3||a_5a_7-a_6^2|+|a_4||a_4a_7-a_6a_5|+|a_5||a_4a_6-a_5^2|.$$

Using (4.6),(4.9), and (4.10) and Lemma 2.3, we have $|H_3(3)(f)|\le 89/24\simeq 3.7$. □

Next we compute an estimate on the fourth Hankel determinant $H_4(1)$.

Theorem 4.5. *Let $f\in\mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$. Then*

$$|H_4(1)(f)|\le 1.84.$$

Proof. Since $f \in \mathcal{S}^*_{\mathcal{S}}$, then in view of (1.4), (3.1), (3.2), (3.3), (3.4),(3.5) and (3.6), we get

$$
\begin{aligned}
36864H_4(1)(f) &= p_1^4(p_2^2-4p_4)^2+8p_1^3(p_2^2-4p_4)(p_2p_3-4p_5)\\
&-32p_1(2p_2^4p_3-2p_2^2p_3p_4-2p_2^3p_5+12p_3(-2p_4^2+p_3p_5)\\
&+p_2(-3p_3^3+20p_4p_5-12p_3p_6))+8(3p_2^6-6p_2^4p_4+4p_2^2(9p_4^2+20p_3p_5)\\
&-4p_2^3(p_3^2+12p_6)+6(3p_3^4-12p_4^3+16p_3p_4p_5-8p_3^2p_6)\\
&-32p_2(3p_3^2p_4+2p_5^2-3p_4p_6))-8p_1^2(p_2^5-8p_2^3p_4\\
&+16p_2(p_4^2+p_3p_5)-p_2^2(5p_3^2+12p_6)+4(3p_3^2p_4-8p_5^2+12p_4p_6)).
\end{aligned}
$$

$$
\begin{aligned}
36864H_4(1)(f) &= p_1^4p_2^4-8p_1^2p_2^5+24p_2^6+8p_1^3p_2^3p_3-64p_1p_2^4p_3+40p_1^2p_2^2p_3^2\\
&-32p_2^3p_3^2+96p_1p_2p_3^3+144p_3^4-8p_1^4p_2^2p_4+64p_1^2p_2^3p_4-48p_2^4p_4\\
&-32p_1^3p_2p_3p_4+64p_1p_2^2p_3p_4-96p_1^2p_3^2p_4-768p_2p_3^2p_4+16p_1^4p_4^2\\
&-128p_1^2p_2p_4^2+288p_2^2p_4^2+768p_1p_3p_4^2-576p_4^3-32p_1^3p_2^2p_5+64p_1p_2^3p_5\\
&-128p_1^2p_2p_3p_5+640p_2^2p_3p_5-384p_1p_3^2p_5+128p_1^3p_4p_5\\
&-640p_1p_2p_4p_5+768p_3p_4p_5+256p_1^2p_5^2-512p_2p_5^2+96p_1^2p_2^2p_6\\
&-384p_2^3p_6+384p_1p_2p_3p_6-384p_3^2p_6-384p_1^2p_4p_6+768p_2p_4p_6.
\end{aligned}
$$

A simple calculation gives

$$
\begin{aligned}
36864H_4(1)(f) &= 8p_1^4p_2^2\left(\frac{1}{8}p_2^2-p_4\right)+\frac{1}{2}p_1^3\left(\frac{1}{4}p_2^2-p_4\right)\left(\frac{1}{4}p_2p_3-p_5\right)\\
&-64p_1^2p_2^3\left(\frac{1}{8}p_2^2-p_4\right)-64p_1p_2^2p_3(p_2^2-p_4)+32p_2^2p_3^2\left(\frac{5}{4}p_1^2-p_2\right)\\
&+48p_2^4\left(\frac{1}{2}p_2^2-p_4\right)-96p_1p_3p_4(p_1p_3-p_4)+576p_4^2(p_1p_3-p_4)\\
&-768p_3p_4(p_2p_3-p_5)-96p_1p_4^2(p_1p_2-p_3)-640p_2p_3p_5\left(\frac{1}{5}p_1^2-p_2\right)\\
&-640p_2p_4(p_1p_5-p_6)+384p_3p_6(p_1p_2-p_3)-128p_4p_6(3p_1^2-p_2)\\
&+512p_5^2\left(\frac{1}{2}p_1^2-p_2\right)+192p_2^2p_6\left(\frac{1}{2}p_1^2-p_2\right)+192p_2^3\left(\frac{1}{3}p_1p_5-p_6\right)\\
&-384p_1p_3^2\left(\frac{1}{8}p_2p_3-p_5\right)-288p_2p_4^2\left(\frac{1}{9}p_1^2-p_2\right)\\
&-144p_3^3\left(-\frac{1}{3}p_1p_2-p_3\right)
\end{aligned}
$$

which implies

$$\begin{aligned}
36864|H_4(1)(f)| &\le 8p_1^4p_2^2\left|\frac{1}{8}p_2^2-p_4\right|+\frac{1}{2}p_1^3\left|\frac{1}{4}p_2^2-p_4\right|\left|\frac{1}{4}p_2p_3-p_5\right|\\
&+64p_1^2p_2^3\left|\frac{1}{8}p_2^2-p_4\right|+64p_1p_2^2p_3|p_2^2-p_4|+32p_2^2p_3^2\left|\frac{5}{4}p_1^2-p_2\right|\\
&+48p_2^4\left|\frac{1}{2}p_2^2-p_4\right|+96p_1p_3p_4|p_1p_3-p_4|+576p_4^2|p_1p_3-p_4|\\
&+768p_3p_4|p_2p_3-p_5|+96p_1p_4^2|p_1p_2-p_3|+640p_2p_3p_5\left|\frac{1}{5}p_1^2-p_2\right|\\
&+640p_2p_4|p_1p_5-p_6|+384p_3p_6|p_1p_2-p_3|+128p_4p_6|3p_1^2-p_2|\\
&+512p_5^2\left|\frac{1}{2}p_1^2-p_2\right|+192p_2^2p_6\left|\frac{1}{2}p_1^2-p_2\right|+192p_2^3\left|\frac{1}{3}p_1p_5-p_6\right|\\
&+384p_1p_3^2\left|\frac{1}{8}p_2p_3-p_5\right|+288p_2p_4^2\left|\frac{1}{9}p_1^2-p_2\right|+144p_3^3\left|-\frac{1}{3}p_1p_2-p_3\right|.
\end{aligned}$$

Using Lemma 2.4 and the fact $|p_n|\le 2$, we get

$$|H_4(1)(f)|\le\frac{4241}{2304}\simeq 1.84.$$

Thus, we have the required bound for $|H_4(1)(f)|$. □

5. Toeplitz determinants

In this section, we first compute the bound on second Toeplitz determinant $T_2(2)$.

Theorem 5.1. *If $f\in\mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$. Then*

$$|T_2(2)(f)|\le 2.$$

The inequality is sharp.

Proof. Since $f\in\mathcal{S}^*_{\mathcal{S}}$, then on putting the values of a_2 and a_3 from (3.1) and (3.2) in expression $T_2(2)=a_3^2-a_2^2$, we get

$$|T_2(2)|=|a_3^2-a_2^2|=\left|\frac{p_2^2}{4}-\frac{p_1^2}{4}\right|.$$

Applying triangle inequality and using the fact $|p_n|\le 2$, we get

$$|T_2(2)|\le 2.$$

The inequality is sharp for the function $f:\mathbb{D}\to\mathbb{C}$ defined as

$$f(z)=\frac{z}{1-iz}.$$

It is noted that $a_2=i, a_3=-1$ and thus $|a_3^2-a_2^2|=2$. □

Next, we obtain an estimate for second Toeplitz determinant $T_2(3)$.

Theorem 5.2. *Let $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. Then*

$$|T_2(3)(f)| \leq 2.$$

The inequality is sharp.

Proof. For $f \in \mathcal{S}^*_{\mathcal{S}}$, then on putting the values of a_3 and a_4 from (3.2) and (3.3) in expression $T_2(3) = a_4^2 - a_3^2$, we get

$$\begin{aligned} |T_2(3)| = |a_4^2 - a_3^2| &= \left| \frac{1}{64}(p_1 p_2 + 2p_3)^2 - \frac{p_2^2}{4} \right| \\ &= \left| \frac{1}{16}\left(-\frac{1}{2}p_1 p_2 - p_3 \right)^2 - \frac{p_2^2}{4} \right|. \end{aligned}$$

Applying triangle inequality, Lemma 2.4 and the fact $|p_n| \leq 2$, we get

$$|T_2(3)| \leq 2.$$

To prove the sharpness, consider the function $f : \mathbb{D} \to \mathbb{C}$ defined as

$$f(z) = \frac{z}{1 - iz}.$$

Here $a_3 = -1$ and $a_4 = -i$ and thus $|a_4^2 - a_3^2| = 2$. □

In the next theorem we obtain an estimate for the bound on third Toeplitz determinant $T_3(1)$.

Theorem 5.3. *If $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. Then*

$$|T_3(1)(f)| \leq 4.$$

The inequality is sharp.

Proof. Let $f \in \mathcal{S}^*_{\mathcal{S}}$. Then in view of (1.5), (3.1) and (3.2), we get

$$\begin{aligned} |T_3(1)| = |1 + 2a_2^2(a_3 - 1) - a_3^2| &= \left| 1 + 2\frac{p_1^2}{4}\left(\frac{p_2}{2} - 1\right) - \left(\frac{p_2^2}{4}\right) \right| \\ &= \frac{1}{4}\left|4 + p_1^2 p_2 - 2p_1^2 - p_2^2\right| \\ &= \frac{1}{4}\left|4 + p_2(p_1^2 - p_2) - 2p_1^2\right|. \end{aligned}$$

Using triangle inequality, we obtain

$$|T_3(1)| \leq \frac{1}{4}(4 + |p_2||p_1^2 - p_2| + 2|p_1^2|).$$

Applying Lemma 2.4 and using the fact that $|p_n| \leq 2$, we get

$$|T_3(1)| \leq 4.$$

For the function $f(z) = \dfrac{z}{1 - iz}$, we have $a_2 = i$ and $a_3 = -1$. Thus, we get

$$|1 + 2a_2^2(a_3 - 1) - a_3^2| = 4.$$

This proves the sharpness of the result. □

Next we compute the bound on third Toeplitz determinant $T_3(2)$.

Theorem 5.4. *Let $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|T_3(2)(f)| = |(a_2 - a_4)(a_2^2 - 2a_2^3 + a_2a_4)| \leq 6.$$

Proof. In view of (3.1) and (3.3), we get

$$\begin{aligned}
&|T_3(2)(f)| \\
&= |(a_2 - a_4)(a_2^2 - 2a_2^3 + a_2a_4)| \\
&= \left| \left(\frac{p_1}{2} - \frac{1}{8}(p_1p_2 + 2p_3) \right) \left(\frac{p_1^2}{4} - \frac{p_1^3}{4} + \frac{p_1}{16}(p_1p_2 + 2p_3) \right) \right| \\
&= \left| \frac{p_1^3}{8} - \frac{p_1^4}{8} + \frac{1}{32}p_1^4p_2 - \frac{1}{128}p_1^3p_2^2 + \frac{1}{16}p_1^3p_3 - \frac{1}{32}p_1^2p_2p_3 - \frac{1}{32}p_1p_3^2 \right| \\
&= \left| \frac{p_1^3}{8} - \frac{p_1^4}{8} - \frac{1}{16}p_1^3 \left(-\frac{1}{2}p_1p_2 - p_3 \right) + \frac{1}{32}p_1^2p_2 \left(-\frac{1}{4}p_1p_2 - p_3 \right) - \frac{1}{32}p_1p_3^2 \right|.
\end{aligned}$$

Using triangle inequality, we obtain

$$\begin{aligned}
|T_3(2)(f)| \leq \frac{1}{8}|p_1|^3 + \frac{1}{8}|p_1|^4 + \frac{1}{16}|p_1|^3 \left| -\frac{1}{2}p_1p_2 - p_3 \right| + \frac{1}{32}|p_1||p_3|^2 \\
+ \frac{1}{32}|p_1|^2|p_2| \left| -\frac{1}{4}p_1p_2 - p_3 \right|.
\end{aligned}$$

By using Lemma 2.4 and the inequality $|p_n| \leq 2$, we get $|T_3(2)(f)| \leq 6$. □

The following theorem gives an estimate on fourth Toeplitz deteminant $T_4(2)$.

Theorem 5.5. *Let $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$\begin{aligned}
|T_4(2)(f)| =& |(a_2^2 - a_3^2)^2 + 2(a_3^2 - a_2a_4)(a_2a_4 - a_3a_5) - (a_2a_3 - a_3a_4)^2 \\
&+ (a_4^2 - a_3a_5)^2 - (a_3a_4 - a_2a_5)^2| \leq 15.12.
\end{aligned}$$

Proof. In view of (1.6), (3.1), (3.2), (3.3) and (3.4) and on rearranging the terms, we have

$$
\begin{aligned}
&4096T_4(2)(f)\\
&= 256(p_1^2-p_2^2)^2 - 16p_2^2(p_1(-4+p_2)+2p_3)^2 - 64(p_2p_3-p_1p_4)^2\\
&\quad + ((p_1p_2+2p_3)^2 - 4p_2(p_2^2+2p_4))^2 - 32(p_1^2p_2 - 4p_2^2 + 2p_1p_3)\\
&\quad (p_1^2p_2 + 2p_1p_3 - p_2(p_2^2+2p_4))\\
&= 256p_1^4 - 768p_1^2p_2^2 - 32p_1^4p_2^2 + 256p_1^2p_2^3 + 256p_2^4 + 16p_1^2p_2^4 + p_1^4p_2^4\\
&\quad - 128p_2^5 - 8p_1^2p_2^5 + 16p_2^6 - 128p_1^3p_2p_3 + 512p_1p_2^2p_3 + 8p_1^3p_2^3p_3\\
&\quad - 32p_1p_2^4p_3 - 128p_1^2p_3^2 - 128p_2^2p_3^2 + 24p_1^2p_2^2p_3^2 - 32p_2^3p_3^2 + 32p_1p_2p_3^3\\
&\quad + 16p_3^4 + 64p_1^2p_2^2p_4 - 256p_2^3p_4 - 16p_1^2p_2^3p_4 + 64p_2^4p_4 + 256p_1p_2p_3p_4\\
&\quad - 64p_1p_2^2p_3p_4 - 64p_2p_3^2p_4 - 64p_1^2p_4^2 + 64p_2^2p_4^2\\
&= -256p_1^2p_2^2\left(\frac{1}{8}p_1^2 - p_2\right) + 128p_2^4\left(\frac{1}{8}p_1^2 - p_2\right) - 16p_2^5\left(\frac{1}{2}p_1^2 - p_2\right)\\
&\quad - 512p_1p_2p_3\left(\frac{1}{4}p_1^2 - p_2\right) - 64p_2^4\left(\frac{1}{2}p_1p_3 - p_4\right) + 32p_2^2p_3^2\left(\frac{3}{4}p_1^2 - p_2\right)\\
&\quad + 16p_1^2p_2^3\left(\frac{1}{2}p_1p_3 - p_4\right) + 64p_2p_3^2\left(\frac{1}{2}p_1p_3 - p_4\right) + 64p_1^2p_4(p_2^2 - p_4)\\
&\quad - 64p_2^2p_4(p_1p_3 - p_4) + 256p_1^4 - 768p_1^2p_2^2 + 256p_2^4 + p_1^4p_2^4 - 128p_1^2p_3^2\\
&\quad - 128p_2^2p_3^2 + 16p_3^4 - 256p_2^3p_4 + 256p_1p_2p_3p_4.
\end{aligned}
$$

Using triangle inequality, we get

$$
\begin{aligned}
4096|T_4(2)(f)| &\le 256|p_1|^2|p_2|^2\left|\frac{1}{8}p_1^2 - p_2\right| + 128|p_2|^4\left|\frac{1}{8}p_1^2 - p_2\right|\\
&\quad + 16|p_2|^5\left|\frac{1}{2}p_1^2 - p_2\right| + 512|p_1||p_2||p_3|\left|\frac{1}{4}p_1^2 - p_2\right| + 64|p_2|^4\left|\frac{1}{2}p_1p_3 - p_4\right|\\
&\quad + 32|p_2|^2|p_3|^2\left|\frac{3}{4}p_1^2 - p_2\right| + 16|p_1|^2|p_2|^3\left|\frac{1}{2}p_1p_3 - p_4\right|\\
&\quad + 64|p_2||p_3|^2\left|\frac{1}{2}p_1p_3 - p_4\right| + 64|p_1|^2|p_4||p_2^2 - p_4|\\
&\quad + 64|p_2|^2|p_4||p_1p_3 - p_4| + 256|p_1|^4 + 768|p_1|^2|p_2|^2 + 256|p_2|^4\\
&\quad + |p_1|^4|p_2|^4 + 128|p_1|^2|p_3|^2 + 128|p_2|^2|p_3|^2 + 16|p_3|^4 + 256|p_2|^3|p_4|\\
&\quad + 256|p_1||p_2||p_3||p_4|.
\end{aligned}
$$

Applying Lemma 2.4 and the fact that $|p_n| \le 2$, we get $|T_4(2)(f)| \le 15.12$. □

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Certain theorems involving differential superordination and sandwich-type results

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Abstract. To obtain the main result of the present paper, we use the technique of differential superordination. As special cases of our main result, we obtain sufficient conditions for $f \in \mathcal{A}$ to be $\phi-$like, parabolic $\phi-$like, starlike, parabolic starlike, close-to-convex and uniform close-to-convex. We also obtain sandwich-type results regarding these functions. For demonstration of the results, we have plotted the images of open unit disk under certain functions using Mathematica 7.0.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots.$$

Let $\mathcal{A}$ be the class of functions f, analytic in the unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. The Taylor series expansion of any function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots.$$

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Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1$, $\phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to the rotation of $\mathbb{E}$.

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic and univalent function in domain $\mathbb{C}^2 \times \mathbb{E}$, h be analytic function in $\mathbb{E}$, p be analytic and univalent in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called the solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), h(0) = \Psi(p(0), 0; 0). \tag{1.2}$$

An analytic function q is called a subordinant of the differential superordination (1.2) if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2), is said to be the best subordinant of (1.2). The best subordinant is unique up to the rotation of $\mathbb{E}$.

A function $f \in \mathcal{A}$ is said to be starlike in the open unit disk $\mathbb{E}$, if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. The well known condition for the members of class $\mathcal{A}$ to be starlike is that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E}.$$

Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of all univalent starlike functions with respect to the origin.
A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$, if there exists a starlike function g (not necessarily normalized) such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E}.$$

In addition, if g is normalized by the conditions $g(0) = 0 = g'(0) - 1$, then the class of close-to-convex functions is denoted by $\mathcal{C}$.
A function $f \in \mathcal{A}$ is called parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathbb{E}, \tag{1.3}$$

and the class of such functions is denoted by S_P.
A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|, \ z \in \mathbb{E}, \tag{1.4}$$

for some $g \in \mathcal{S}_P$. Let UCC denote the class of all such functions. Note that the function $g(z) \equiv z \in \mathcal{S}_P$. Therefore, for $g(z) \equiv z$, condition (1.4) becomes:

$$\Re\left(f'(z)\right) > \left|f'(z) - 1\right|, \ z \in \mathbb{E}. \tag{1.5}$$

Ronning [11] and Ma and Minda [6] studied the domain Ω and the function $q(z)$ defined below:

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

Clearly the function

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the conditions (1.3) and (1.5) are, respectively, equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z), \ z \in \mathbb{E},$$

and

$$f'(z) \prec q(z).$$

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0, \ z \in \mathbb{E}.$$

This concept was introduced by Brickman [2]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is $\phi-$ like for some analytic function ϕ. Later, Ruscheweyh [12] investigated the following general class of $\phi-$like functions:
Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E})\backslash\{0\}$, then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to a univalent function q, $q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|, \ z \in \mathbb{E}. \tag{1.6}$$

Equivalently, condition (1.6) can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

In 2005, Ravichandran et al. [10] proved the following result for ϕ-like functions: Let $\alpha \neq 0$ be a complex number and $q(z)$ be a convex univalent function in $\mathbb{E}$. Suppose $h(z) = \alpha q^2(z) + (1-\alpha)q(z) + \alpha z q'(z)$ and

$$\Re\left\{\frac{1-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0, \ z \in \mathbb{E}.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))}\left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha(f'(z) - (\phi(f(z)))'}{\phi(f(z))}\right) \prec h(z),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E},$$

and $q(z)$ is best dominant. Later on, Shanmugam et al. [13] and Ibrahim [9] also obtained the results for ϕ-like functions similar to the above mentioned results of Ravichandran [10].

In 2017, Kaur and Billing [4] investigated the following operator

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

to obtain $\phi-$likeness, starlikeness and close-to-convexity of normalized analytic functions.

Later, in 2019, Adegani et al. [1] studied the operator

$$\frac{\lambda z f'(z)}{g(z)}\left(1 + \frac{1}{\lambda} + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right)$$

and derived criteria for close-to-convexity of normalized analytic functions.

Recently, Mohammed et al. [8] studied the geometric properties of some subfamilies of holomorphic functions in this direction.

In this paper, we obtain the superordination theorem for the differential operator

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

where $f, \ g \in \mathcal{A}$ and $\beta, \ \gamma$ be complex numbers such that $\beta \neq 0$. Also ϕ is an analytic function in a domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$, for real numbers $a, \ b(\neq 0)$. Further, we derive sandwich-type theorem. As consequences of our main results, we obtain sufficient conditions for ϕ-like, parabolic $\phi-$like, starlike, parabolic starlike, close-to-convex, and uniform close-to-convex functions.

2. Preliminaries

We shall need the following definition and lemma to prove our main result.

Definition 2.1. ([7], Definition 2, p.817) Denote by $\mathbb{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\bar{\mathbb{E}} \setminus \mathbb{E}(f)$, where

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{E} \ : \ \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{E}(f)$.

Lemma 2.2. *([3]). Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*
(i) Q_1 is starlike and
(ii) $\Re\left(\dfrac{\theta' q(z)}{\varphi(q(z)})\right) > 0$ for all $z \in \mathbb{E}$.
If $p \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\varphi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\varphi[q(z)] \prec \theta[p(z)] + zp'(z)\varphi[p(z)],\ z \in \mathbb{E},$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3. A superordination theorem

Theorem 3.1. *Let β and γ be complex numbers such that $\beta \neq 0$ and $a,\ b(\neq 0)$ are real numbers. Let $q(z) \neq 0$ with $q(0) = 1$ be a univalent function in $\mathbb{E}$, such that*
(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\gamma}{\beta} - 1\right)\dfrac{zq'(z)}{q(z)}\right] > 0$ and
(ii) $\Re\left[\dfrac{a}{b}\left(1 + \dfrac{\gamma}{\beta}\right)q(z)\right] > 0.$
Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\setminus\{0\}$. If $f,\ g \in \mathcal{A},\ \dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ and

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

is univalent in $\mathbb{E}$, satisfy

$$(q(z))^{\gamma}\left[aq(z) + b\frac{zq'(z)}{q(z)}\right]^{\beta} \prec \left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}$$
$$\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta} \tag{3.1}$$

then

$$q(z) \prec \frac{zf'(z)}{\phi(g(z))},\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

Proof. On writing $p(z) = \dfrac{zf'(z)}{\phi(g(z))}$, the superordination (3.1) can be rewritten as:

$$(q(z))^{\gamma}\left(aq(z)+b\frac{zq'(z)}{q(z)}\right)^{\beta} \prec (p(z))^{\gamma}\left(ap(z)+b\frac{zp'(z)}{p(z)}\right)^{\beta}$$

or

$$a(q(z))^{\frac{\gamma}{\beta}+1}+b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z) \prec a(p(z))^{\frac{\gamma}{\beta}+1}+b(p(z))^{\frac{\gamma}{\beta}-1}zp'(z)$$

Let us define the functions θ and ϕ as follows:

$$\theta(w) = aw^{\frac{\gamma}{\beta}+1} \text{ and } \phi(w) = bw^{\frac{\gamma}{\beta}-1}$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$.
Therefore,

$$Q(z) = \phi(q(z))zq'(z) = b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = a(q(z))^{\frac{\gamma}{\beta}+1}+b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we obtain

$$\frac{zQ'(z)}{Q(z)} = 1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{\theta'(q(z))}{\phi(q(z))} = \frac{zh'(z)}{Q(z)}-\frac{zQ'(z)}{Q(z)} = \frac{a}{b}\left(1+\frac{\gamma}{\beta}\right)q(z).$$

In view of the given condition (i) and (ii), we see that Q is starlike and

$$\Re\left(\frac{\theta'(q(z))}{\phi(q(z))}\right) > 0.$$

Therefore, the proof, now follows from the Lemma [2.2]. □

Remark 3.2. Together with the corresponding result for differential subordination (see Kaur et al. [5]), we get the following "sandwich result".

4. Sandwich-type result and its applications

Theorem 4.1. *Let β and γ be complex numbers such that $\beta \neq 0$ and a, $b(\neq 0)$ are real numbers. Let q_1, q_2 $(q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$, be univalent functions in $\mathbb{E}$, such that*
(i) $\Re\left[1+\dfrac{zq_i''(z)}{q_i'(z)}+\left(\dfrac{\gamma}{\beta}-1\right)\dfrac{zq_i'(z)}{q_i(z)}\right] > 0$ *and*
(ii) $\Re\left[\dfrac{a}{b}\left(1+\dfrac{\gamma}{\beta}\right)q_i(z)\right] > 0;\ i = 1,\ 2.$

Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0)-1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ and

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma} \left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

is univalent in $\mathbb{E}$, satisfy

$$(q_1(z))^{\gamma}\left[aq_1(z) + b\frac{zq_1'(z)}{q_1(z)}\right]^{\beta}$$
$$\prec \left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma} \left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$
$$\prec (q_2(z))^{\gamma}\left[aq_2(z) + b\frac{zq_2'(z)}{q_2(z)}\right]^{\beta} \tag{4.1}$$

then

$$q_1(z) \prec \frac{zf'(z)}{\phi(g(z))} \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.

Remark 4.2. When we select $q_1(z) = 1 + m_1 z$, $q_2(z) = 1 + m_2 z$; $0 < m_1 < m_2 \leq 1$, $\beta = 1$, $\gamma = 0$ in Theorem 4.1, we obtain:

Corollary 4.3. *Let a, $b(\neq 0)$ are real numbers such that $\dfrac{a}{b} > 0$. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

is univalent in $\mathbb{E}$ and satisfy

$$a(1+m_1 z) + \frac{bm_1 z}{1+m_1 z} \prec \left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]$$
$$\prec a(1+m_2 z) + \frac{bm_2 z}{1+m_2 z}$$

then

$$1 + m_1 z \prec \frac{zf'(z)}{\phi(g(z))} \prec 1 + m_2 z,\ \textit{where}\ 0 < m_1 < m_2 \leq 1,\ z \in \mathbb{E}.$$

By selecting $a = 1,\ b = 1,\ m_1 = \frac{1}{3},\ m_2 = 1$ in Corollary 4.3, we get

Example 4.4. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}$$

is univalent in $\mathbb{E}$ and satisfy

$$\frac{z^2 + 9z + 9}{3z + 9} \prec 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \prec \frac{z^2 + 3z + 1}{z + 1}$$

then

$$1 + \frac{z}{3} \prec \frac{zf'(z)}{\phi(g(z))} \prec 1 + z,\ z \in \mathbb{E}.$$

By selecting $g(z) = f(z)$ in Example 4.4, we have

Example 4.5. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(f(z))} - \frac{z(\phi(f(z)))'}{\phi(f(z))}$$

is univalent in $\mathbb{E}$ and satisfy

$$\frac{z^2 + 9z + 9}{3z + 9} \prec 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(f(z))} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \prec \frac{z^2 + 3z + 1}{z + 1}$$

then

$$1 + \frac{z}{3} \prec \frac{zf'(z)}{\phi(f(z))} \prec 1 + z,\ z \in \mathbb{E}.$$

i.e. f is $\phi-$like.

By selecting $\phi(z) = z$ and $g(z) = f(z)$ in Example 4.4, we get

Example 4.6. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies

$$\frac{z^2 + 9z + 9}{3z + 9} \prec 1 + \frac{zf''(z)}{f'(z)} \prec \frac{z^2 + 3z + 1}{z + 1}$$

then

$$1 + \frac{z}{3} \prec \frac{zf'(z)}{f(z)} \prec 1 + z,\ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

By selecting $\phi(z) = g(z) = z$ in Example 4.4, we have

Example 4.7. If $f \in \mathcal{A}$, $f'(z) \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$, with $f'(z) + \dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfy

$$\frac{z^2+9z+9}{3z+9} \prec f'(z) + \frac{zf''(z)}{f'(z)} \prec \frac{z^2+3z+1}{z+1}$$

then

$$1+\frac{z}{3} \prec f'(z) \prec 1+z,\ z \in \mathbb{E},$$

and hence $f(z)$ is close-to-convex.

For illustration, in Figure 4.1, we plot the images of unit disk $\mathbb{E}$ under the functions

$$w_1(z) = \frac{z^2+9z+9}{3z+9} \text{ and } w_2(z) = \frac{z^2+3z+1}{z+1}.$$

In Figure 4.2, the images of unit disk $\mathbb{E}$ under the functions

$$q_1(z) = 1+\frac{z}{3} \text{ and } q_2(z) = 1+z$$

are given. In the light of Example 4.4, when the differential operator

$$1+\frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}$$

takes values in the light shaded portion as shown in Figure 4.1, then $\dfrac{zf'(z)}{\phi(g(z))}$ takes values in the light shaded region as given in Figure 4.2. Consequently, in view of Example 4.5, Example 4.6, Example 4.7, $f(z)$ is $\phi-like$, starlike and close-to-convex respectively.

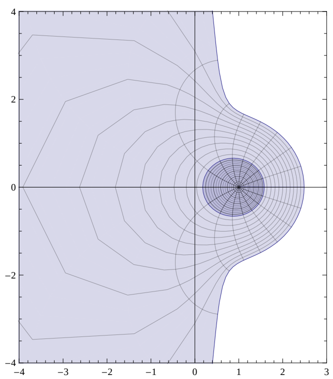

Figure 4.1

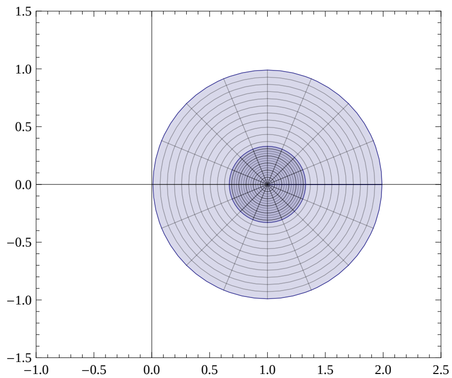

Figure 4.2

Remark 4.8. When we select

$$q_1(z) = \left(\frac{1+z}{1-z}\right)^{\delta_1},\ q_2(z) = \left(\frac{1+z}{1-z}\right)^{\delta_2},\ 0 < \delta_1 < \delta_2 \leq 1,\ \beta = 1,\ \gamma = 0$$

in Theorem 4.1, we obtain the following result:

Corollary 4.9. *For real numbers $a,\ b(\neq 0)$ with same sign. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

is univalent in $\mathbb{E}$ and satisfy

$$a\left(\frac{1+z}{1-z}\right)^{\delta_1} + \left(\frac{2b\delta_1 z}{1-z^2}\right) \prec a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$
$$\prec a\left(\frac{1+z}{1-z}\right)^{\delta_2} + \left(\frac{2b\delta_2 z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{\delta_1} \prec \frac{zf'(z)}{\phi(g(z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta_2}; 0 < \delta_1 < \delta_2 \leq 1,\ z \in \mathbb{E}.$$

Selecting $\delta_1 = 0.3,\ \delta_2 = 1$ and $a = 1,\ b = 1$ in Corollary 4.9, we have:

Example 4.10. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}$$

is univalent in $\mathbb{E}$ and satisfy

$$\left(\frac{1+z}{1-z}\right)^{0.3} + \left(\frac{0.6z}{1-z^2}\right) \prec \frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$
$$\prec \left(\frac{1+z}{1-z}\right) + \left(\frac{2z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{0.3} \prec \frac{zf'(z)}{\phi(g(z))} \prec \left(\frac{1+z}{1-z}\right);\ z \in \mathbb{E}.$$

By selecting $g(z) = f(z)$ in Example 4.10, we get

Example 4.11. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(f(z))} - \frac{z(\phi(f(z)))'}{\phi(f(z))}$$

is univalent in $\mathbb{E}$ and satisfy

$$\left(\frac{1+z}{1-z}\right)^{0.3} + \left(\frac{0.6z}{1-z^2}\right) \prec \frac{zf'(z)}{\phi(f(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$

$$\prec \left(\frac{1+z}{1-z}\right) + \left(\frac{2z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{0.3} \prec \frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z}\right);\ z \in \mathbb{E}.$$

i.e. f is $\phi-$like.

By selecting $\phi(z) = z$ and $g(z) = f(z)$ in Example 4.10, we obtain

Example 4.12. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \frac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies

$$\left(\frac{1+z}{1-z}\right)^{0.3} + \left(\frac{0.6z}{1-z^2}\right) \prec \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right) + \left(\frac{2z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{0.3} \prec \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right);\ z \in \mathbb{E}.$$

i.e. f is starlike.

By selecting $\phi(z) = g(z) = z$ in Example 4.10, we have

Example 4.13. If $f \in \mathcal{A}$, $f'(z) \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$, with $f'(z) + \frac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfy

$$\left(\frac{1+z}{1-z}\right)^{0.3} + \left(\frac{0.6z}{1-z^2}\right) \prec f'(z) + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right) + \left(\frac{2z}{1-z^2}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{0.3} \prec f'(z) \prec \left(\frac{1+z}{1-z}\right);\ z \in \mathbb{E}.$$

i.e. f is close-to-convex.

Using Mathematica 7.0, we plot the images of unit disk $\mathbb{E}$ under the functions

$$w_3(z) = \left(\frac{1+z}{1-z}\right)^{0.3} + \frac{0.6z}{1-z^2} \text{ and } w_4(z) = \frac{1+z}{1-z} + \frac{2z}{1-z^2},$$

which are given by Figure 4.3 and the images of unit disk $\mathbb{E}$ under the functions

$$q_1(z) = \left(\frac{1+z}{1-z}\right)^{0.3} \text{ and } q_2(z) = \frac{1+z}{1-z},$$

which are shown in Figure 4.4. It follows from Example 4.10 that the differential operator $\frac{zf'(z)}{\phi(g(z))}$ takes values in the light shaded region of Figure 4.4 when the differential operator

$$\frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

takes values in the light shaded region of Figure 4.3. Therefore, from Example 4.11, Example 4.12, Example 4.13, we can say that $f(z)$ is $\phi - like$, starlike and close-to-convex respectively.

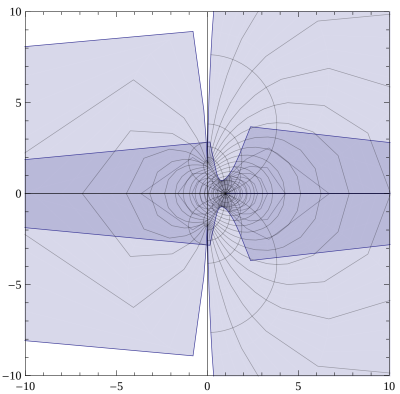

Figure 4.3

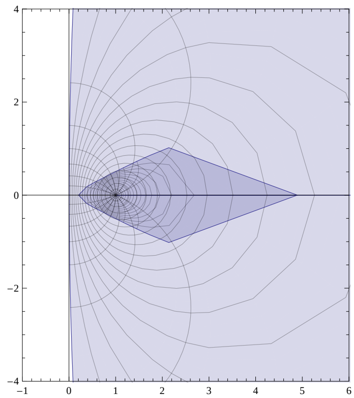

Figure 4.4

Remark 4.14. When we select $q_1(z) = e^{z/2},\ q_2(z) = \frac{1+z}{1-z},\ \beta = 1,\ \gamma = 0$ in Theorem 4.1, we get the following result:

Corollary 4.15. *For real numbers* $a,\ b(\neq 0)$ *of same sign. Let* ϕ *be analytic function in the domain containing* $g(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in g(\mathbb{E})\backslash\{0\}$. *If* $f,\ g \in \mathcal{A},\ \frac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ *with*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

is univalent in $\mathbb{E}$ *and satisfy*

$$ae^{z/2} + \frac{bz}{2} \prec a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

$$\prec a\left(\frac{1+z}{1-z}\right) + \left(\frac{2bz}{1-z^2}\right),$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(g(z))} \prec \frac{1+z}{1-z},\ 0 \le \delta < 1,\ z \in \mathbb{E}.$$

Selecting $a = 1$ and $b = 1$ in Corollary 4.15, we obtain:

Example 4.16. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{zf'(z)}{\phi(g(z))} - \dfrac{z(\phi(g(z)))'}{\phi(g(z))}$ is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2} + \frac{z}{2} \prec \frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right) \prec \frac{z^2+4z+1}{1-z^2},$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(g(z))} \prec \frac{1+z}{1-z},\ 0 \leq \delta < 1,\ z \in \mathbb{E}.$$

By selecting $g(z) = f(z)$ in Example 4.16, we get

Example 4.17. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{zf'(z)}{\phi(f(z))} - \dfrac{z(\phi(f(z)))'}{\phi(f(z))}$ is univalent in $\mathbb{E}$ and satisfy

$$e^{z/2} + \frac{z}{2} \prec \frac{zf'(z)}{\phi(f(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec \frac{z^2+4z+1}{1-z^2},$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z},\ 0 \leq \delta < 1,\ z \in \mathbb{E}.$$

i.e. f is $\phi-$like.

By selecting $\phi(z) = z$ and $g(z) = f(z)$ in Example 4.16, we have

Example 4.18. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2} + \frac{z}{2} \prec \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{z^2+4z+1}{1-z^2},$$

then

$$e^{z/2} \prec \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E}.$$

i.e. f is starlike.

By selecting $\phi(z) = g(z) = z$ in Example 4.10, we obtain

Example 4.19. If $f\ \in \mathcal{A}$, $f'(z) \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$, with $f'(z) + \dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfy

$$e^{z/2} + \frac{z}{2} \prec f'(z) + \frac{zf''(z)}{f'(z)} \prec \frac{z^2+4z+1}{1-z^2},$$

then

$$e^{z/2} \prec f'(z) \prec \frac{1+z}{1-z}, \ z \in \mathbb{E}.$$

i.e. f is close-to-convex.

For demonstration, we plot the images of unit disk $\mathbb{E}$ under the functions

$$w_5(z) = e^{z/2} + \frac{z}{2} \text{ and } w_6(z) = \frac{z^2+4z+1}{1-z^2},$$

which are shown by Figure 4.5. In Figure 4.6, the images of unit disk $\mathbb{E}$ under the functions

$$q_1(z) = e^{z/2} \text{ and } q_2(z) = \frac{1+z}{1-z}$$

are given. It follows from Example 4.16 that the differential operator $\dfrac{zf'(z)}{\phi(g(z))}$ takes values in the light shaded region of Figure 4.6 when the differential operator

$$\frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

takes values in the light shaded portion of Figure 4.5. Thus in view of Example 4.17, Example 4.18, Example 4.19, $f(z)$ is $\phi-like$, starlike and close-to-convex respectively.

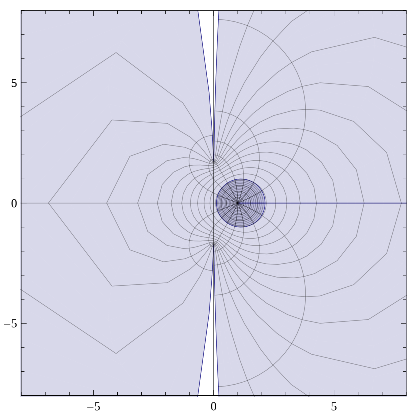

Figure 4.5

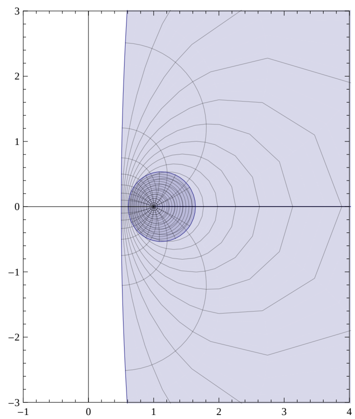

Figure 4.6

Remark 4.20. When we select

$$q_1(z) = e^{z/2}, \ q_2(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \ \beta = 1, \ \gamma = 0$$

in Theorem 4.1, we derive the following result:

Corollary 4.21. *For real numbers a, $b(\neq 0)$ of same sign. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for*

$w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

is univalent in $\mathbb{E}$ and satisfy

$$ae^{z/2} + \frac{bz}{2} \prec a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

$$\prec \left\{a + \frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(g(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E}.$$

Selecting $a = 1$ and $b = 1$ in Corollary 4.21, we obtain:

Example 4.22. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}$$

is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2} + \frac{z}{2} \prec \frac{zf'(z)}{\phi(g(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)$$

$$\prec \left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(g(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E}.$$

By selecting $g(z) = f(z)$ in Example 4.22, we get

Example 4.23. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that

$$\phi(0) = 0 = \phi'(0) - 1 \text{ and } \phi(w) \neq 0 \text{ for } w \in f(\mathbb{E})\backslash\{0\}.$$

If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $1 + \dfrac{zf''(z)}{f'(z)} + \dfrac{zf'(z)}{\phi(f(z))} - \dfrac{z(\phi(f(z)))'}{\phi(f(z))}$ is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2} + \frac{z}{2} \prec \frac{zf'(z)}{\phi(f(z))} + \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)$$

$$\prec \left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2} \prec \frac{zf'(z)}{\phi(f(z))} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. f is parabolic $\phi-$like.

By selecting $\phi(z)=z$ and $g(z)=f(z)$ in Example 4.22, we have

Example 4.24. If $f\in\mathcal{A}$, $\dfrac{zf'(z)}{f(z)}\in\mathcal{H}[1,\ 1]\cap\mathbb{Q}$ with $1+\dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfy

$$e^{z/2}+\frac{z}{2}\prec\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2} \prec \frac{zf'(z)}{f(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. f is parabolic starlike.

By selecting $\phi(z)=g(z)=z$ in Example 4.22, we obtain

Example 4.25. If $f\ \in\mathcal{A}$, $f'(z)\in\mathcal{H}[1,\ 1]\cap\mathbb{Q}$, with $f'(z)+\dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies

$$e^{z/2}+\frac{z}{2}\prec f'(z)+\frac{zf''(z)}{f'(z)}\prec\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

then

$$e^{z/2}\prec f'(z)\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. f is uniform close-to-convex.

Using Mathematica 7.0, we draw the images of unit disk $\mathbb{E}$ under the functions

$$w_7(z)=e^{z/2}+\frac{z}{2}\text{ and }w_8(z)=\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\},$$

which are shown by Figure 4.7 and the images of unit disk $\mathbb{E}$ under the functions

$$q_1(z)=e^{z/2}\text{ and }q_2(z)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

are given by Figure 4.8. Hence from Example 4.22, we can say that the differential operator $\dfrac{zf'(z)}{\phi(g(z))}$ takes values in the light shaded portion of Figure 4.8 when the differential operator $\dfrac{zf'(z)}{\phi(g(z))} + \left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{z(\phi(g(z)))'}{\phi(g(z))}\right)$ takes values in the light shaded region of Figure 4.7. Therefore, in light of Example 4.23, Example 4.24, Example 4.25, $f(z)$ is parabolic $\phi-$like, parabolic starlike and uniform close-to-convex respectively.

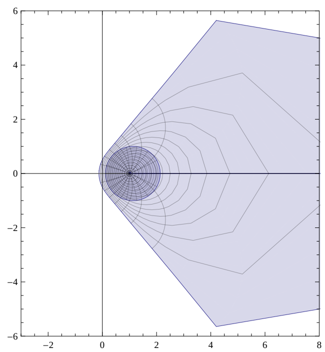

Figure 4.7

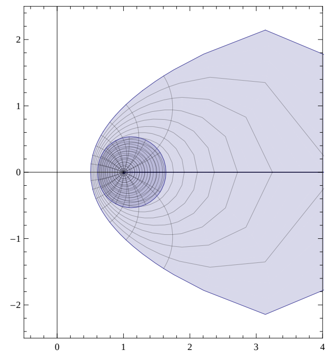

Figure 4.8

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Monotone iterative technique for a sequential δ-Caputo fractional differential equations with nonlinear boundary conditions

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Abstract. In this article, we discuss the existence of extremal solutions for a class of nonlinear sequential δ–Caputo fractional differential equations involving nonlinear boundary conditions. Our results are founded on advanced functional analysis methods. To be more specific, we use the monotone iterative approach in conjunction with the upper and lower solution method to create adequate requirements for the existence of extremal solutions. As an application, we give an example to illustrate our results.

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Keywords: Sequential δ–Caputo derivative, nonlinear boundary conditions, monotone iterative technique, upper and lower solutions.

1. Introduction

Nowadays, fractional differential equations appear in diverse fields such as physics, fluid mechanics, viscoelasticity, biology, control theory, chemistry, and so on, (see, for example, [19, 28, 31, 36]). For some fundamental results in the theory of fractional calculus and fractional differential equations, we suggest the monographs of several scientists [2, 3, 4, 20, 26, 29, 41, 42].

There are various techniques to defining fractional integrals and derivatives in the literature, such as Riemann–Liouville, Caputo, Caputo–Hadamard, Hilfer, δ–Caputo and δ–Hilfer. For more details, we refer the readers to [1, 5, 6, 7, 10, 11, 12, 14, 15, 17, 23, 32, 33, 34, 35, 37].

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On the other hand, much attention has been focused on the study of the existence and uniqueness of solutions for initial and boundary value problems involving sequential fractional differential equations, we refer the reader to [8, 9, 25] and the references cited therein. Additionally, it is well known that the monotone iterative technique [22] combined with the method of upper and lower solutions is used as a fundamental tool to prove the existence and approximation of solutions to many applied problems of nonlinear differential equations and integral equations. Moreover, this technique has more advantages, such as it not only proves the existence of solutions but also can provide computable monotone sequences that converge to the extremal solutions in a sector generated by upper and lower solutions. Recent results by means of the monotone iterative method are obtained in [13, 16, 18, 24, 27, 38, 39, 40] and the references therein. However, to the best of the author's knowledge, no results yet exist for the sequential fractional differential equations involving the δ–Caputo fractional derivative by using the monotone iterative technique.

Motivated by this fact together with recent works [5, 10, 18, 21, 24, 39], we investigate the existence of extremal solutions for the following boundary value problem of δ–Caputo sequential fractional differential equations involving nonlinear boundary conditions:

$$\begin{cases} \left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1+} + \lambda \; {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\right)\xi(\vartheta) = \Psi(\vartheta, \xi(\vartheta)), \;\; \vartheta \in \Theta := [\kappa_1, \kappa_2], \\ \Phi(\xi(\kappa_1), \xi(\kappa_2)) = 0, \quad \xi'(\kappa_1) = 0, \end{cases} \tag{1.1}$$

where ${}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}$ is the δ–Caputo fractional derivative of order $\zeta \in (0, 1], \Psi\colon [\kappa_1, \kappa_2] \times \mathbb{R} \longrightarrow \mathbb{R}$, $\Phi\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are both continuous functions and λ is a positive real number.

The following is how the paper is structured. We provide some essential definitions and lemmas in section 2. The major findings are discussed in section 3. Lastly, an illustration is provided to demonstrate the applicability of the generated results.

2. Preliminaries

In this part, we provide certain fractional calculus notations and concepts, as well as definitions and lemmas that will be used later in our proofs.

Definition 2.1 ([10, 20]**).** For $\zeta > 0$, the left-sided δ–Riemann-Liouville fractional integral of order ζ for an integrable function $\xi\colon \Theta \longrightarrow \mathbb{R}$ with respect to another function $\delta\colon \Theta \longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\delta'(\vartheta) \neq 0$, for all $\vartheta \in \Theta$ is defined as follows

$$\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \frac{1}{\Gamma(\zeta)} \int_{\kappa_1}^{\vartheta} \delta'(\varrho)(\delta(\vartheta) - \delta(\varrho))^{\zeta-1}\xi(\varrho)d\varrho. \tag{2.1}$$

Definition 2.2 ([10]**).** Let $\beta \in \mathbb{N}$ and let $\delta, \xi \in C^{\beta}(\Theta, \mathbb{R})$ be two functions such that δ is increasing and $\delta'(\vartheta) \neq 0$, for all $\vartheta \in \Theta$. The left-sided δ–Riemann–Liouville fractional derivative of a function ξ of order ζ is defined by

$$\begin{aligned}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) &= \left(\frac{1}{\delta'(\vartheta)}\frac{d}{d\vartheta}\right)^{\beta} \mathcal{I}^{\beta-\zeta;\delta}_{\kappa_1+}\xi(\vartheta)\\ &= \frac{1}{\Gamma(\beta-\zeta)}\left(\frac{1}{\delta'(\vartheta)}\frac{d}{d\vartheta}\right)^{\beta}\int_{\kappa_1}^{\vartheta}\delta'(\varrho)(\delta(\vartheta)-\delta(\varrho))^{\beta-\zeta-1}\xi(\varrho)d\varrho,\end{aligned}$$

where $\beta = [\zeta] + 1$.

Definition 2.3 (**[10]**)**.** Let $\beta \in \mathbb{N}$ and let $\delta, \xi \in C^{\beta}(\Theta, \mathbb{R})$ be two functions such that δ is increasing and $\delta'(\vartheta) \neq 0$, for all $\vartheta \in \Theta$. The left-sided δ-Caputo fractional derivative of ξ of order ζ is defined by

$$^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \mathcal{I}^{\beta-\zeta;\delta}_{\kappa_1+}\left(\frac{1}{\delta'(\vartheta)}\frac{d}{d\vartheta}\right)^{\beta}\xi(\vartheta),$$

where $\beta = [\zeta] + 1$ for $\zeta \notin \mathbb{N}$, $\beta = \zeta$ for $\zeta \in \mathbb{N}$.

In the sequel, we will employ the following:

$$\xi^{[\beta]}_{\delta}(\vartheta) = \left(\frac{1}{\delta'(\vartheta)}\frac{d}{d\vartheta}\right)^{\beta}\xi(\vartheta). \tag{2.2}$$

From the previous definition, it is clear that

$$^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \begin{cases}\int_{\kappa_1}^{\vartheta}\frac{\delta'(\varrho)(\delta(\vartheta)-\delta(\varrho))^{\beta-\zeta-1}}{\Gamma(\beta-\zeta)}\xi^{[\beta]}_{\delta}(\varrho)d\varrho & , \quad \text{if } \zeta \notin \mathbb{N},\\ \xi^{[\beta]}_{\delta}(\vartheta) & , \quad \text{if } \zeta \in \mathbb{N}.\end{cases} \tag{2.3}$$

This generalization (2.3) yields the Caputo fractional derivative operator when $\delta(\vartheta) = \vartheta$. Moreover, for $\delta(\vartheta) = \ln\vartheta$, it gives the Caputo–Hadamard fractional derivative.

We note that if $\xi \in C^{\beta}(\Theta, \mathbb{R})$ the δ–Caputo fractional derivative of order ζ of ξ is determined as

$$^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \mathcal{D}^{\zeta;\delta}_{\kappa_1+}\left[\xi(\vartheta) - \sum_{\jmath=0}^{\beta-1}\frac{\xi^{[\jmath]}_{\delta}(\kappa_1)}{\jmath!}(\delta(\vartheta)-\delta(\kappa_1))^{\jmath}\right].$$

(For more details, see [10, Theorem 3]).

Lemma 2.4 (**[12]**)**.** *Let $\zeta, \varkappa > 0$, and $\xi \in L^1(\Theta, \mathbb{R})$. Then,*

$$\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\mathcal{I}^{\varkappa;\delta}_{\kappa_1+}\xi(\vartheta) = \mathcal{I}^{\zeta+\varkappa;\delta}_{\kappa_1+}\xi(\vartheta), \ \textit{a.e. } \vartheta \in \Theta.$$

If $\xi \in C(\Theta, \mathbb{R})$, then $\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\mathcal{I}^{\varkappa;\delta}_{\kappa_1+}\xi(\vartheta) = \mathcal{I}^{\zeta+\varkappa;\delta}_{\kappa_1+}\xi(\vartheta)$, $\vartheta \in \Theta$.

Next, we recall the property describing the composition rules for fractional δ-integrals and δ-derivatives.

Lemma 2.5 (**[12]**)**.** *Let $\zeta > 0$, The following holds:*

- *If $\xi \in C(\Theta, \mathbb{R})$, then*

$$^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \xi(\vartheta), \ \vartheta \in \Theta.$$

- *If* $\xi \in C^{\beta}(\Theta, \mathbb{R})$, $\beta - 1 < \zeta < \beta$, *then*

$$\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\ {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \xi(\vartheta) - \sum_{\jmath=0}^{\beta-1} \frac{\xi^{[\jmath]}_{\delta}(\kappa_1)}{\jmath!} \left[\delta(\vartheta) - \delta(\kappa_1)\right]^{\jmath},\ \vartheta \in \Theta.$$

Lemma 2.6 ([12, 20])**.** *Let* $\vartheta > \kappa_1$, $\zeta \geq 0$, *and* $\varkappa > 0$. *Then*

- $\mathcal{I}^{\zeta;\delta}_{\kappa_1+}(\delta(\vartheta) - \delta(\kappa_1))^{\varkappa-1} = \frac{\Gamma(\varkappa)}{\Gamma(\varkappa+\zeta)}(\delta(\vartheta) - \delta(\kappa_1))^{\varkappa+\zeta-1}$,
- ${}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}(\delta(\vartheta) - \delta(\kappa_1))^{\varkappa-1} = \frac{\Gamma(\varkappa)}{\Gamma(\varkappa-\zeta)}(\delta(\vartheta) - \delta(\kappa_1))^{\varkappa-\zeta-1}$,
- ${}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}(\delta(\vartheta) - \delta(\kappa_1))^{\jmath} = 0$, *for all* $\jmath \in \{0, \ldots, \beta - 1\}, \beta \in \mathbb{N}$.

Now, we give the definitions of lower and upper solutions of problem (1.1).

Definition 2.7. A function $\xi_0 \in C(\Theta, \mathbb{R})$ is called a lower solution of problem (1.1), if it satisfies

$$\begin{cases} \left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1+} + \lambda\ {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\right)\xi_0(\vartheta) \leq \Psi(\vartheta, \xi_0(\vartheta)), \quad \vartheta \in (\kappa_1, \kappa_2], \\ \Phi(\xi_0(\kappa_1), \xi_0(\kappa_2)) \leq 0, \quad \xi_0'(\kappa_1) = 0. \end{cases} \tag{2.4}$$

Definition 2.8. $\sigma_0 \in C(\Theta, \mathbb{R})$ is an upper solution of problem (1.1), if it satisfies

$$\begin{cases} \left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1+} + \lambda\ {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\right)\sigma_0(\vartheta) \geq \Psi(\vartheta, \sigma_0(\vartheta)), \quad \vartheta \in (\kappa_1, \kappa_2], \\ \Phi(\sigma_0(\kappa_1), \sigma_0(\kappa_2)) \geq 0, \quad \sigma_0'(\kappa_1) = 0. \end{cases} \tag{2.5}$$

Lemma 2.9. *For any* $h \in C(\Theta, \mathbb{R})$, *the unique solution of the following sequential fractional differential equation,*

$$\left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1+} + \lambda\ {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\right)\xi(\vartheta) = h(\vartheta), \quad \vartheta \in \Theta = [\kappa_1, \kappa_2], \tag{2.6}$$

supplemented with the initial conditions

$$\xi(\kappa_1) = \xi_{\kappa_1}, \quad \xi'(\kappa_1) = 0, \tag{2.7}$$

is given by

$$\xi(\vartheta) = \xi_{\kappa_1} + \int_{\kappa_1}^{\vartheta} \delta'(\varrho)e^{-\lambda(\delta(\vartheta)-\delta(\varrho))}\left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho) - \delta(r))^{\zeta-1}}{\Gamma(\zeta)} h(r)dr\right) d\varrho. \tag{2.8}$$

Proof. Applying the δ–Riemann–Liouville fractional integral of order ζ to both sides of (2.6) and using Lemma 2.5, we get

$$\xi^{[1]}_{\delta}(\vartheta) + \lambda\xi(\vartheta) = \mathcal{I}^{\zeta;\delta}_{\kappa_1+}h(\vartheta) + c_0, \quad c_0 \in \mathbb{R}.$$

Using the notation of $\xi^{[1]}_{\delta}$ given by Eq (2.2) we obtain

$$\xi'(\vartheta) + \delta'(\vartheta)\lambda\xi(\vartheta) = \delta'(\vartheta)\left(\mathcal{I}^{\zeta;\delta}_{\kappa_1+}h(\vartheta) + c_0\right). \tag{2.9}$$

By multiplying $e^{\lambda(\delta(\vartheta)-\delta(\kappa_1))}$ to both sides of (2.9), we can write

$$\left(\xi(\vartheta)e^{\lambda(\delta(\vartheta)-\delta(\kappa_1))}\right)' = \delta'(\vartheta)e^{\lambda(\delta(\vartheta)-\delta(\kappa_1))}\mathcal{I}^{\zeta;\delta}_{\kappa_1+}h(\vartheta) + c_0\delta'(\vartheta)e^{\lambda(\delta(\vartheta)-\delta(\kappa_1))}.$$

Integrating from κ_1 to ϑ, we have

$$\xi(\vartheta) = c_1 e^{-\lambda(\delta(\vartheta)-\delta(\kappa_1))} + \frac{c_0}{\lambda} + \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \mathcal{I}_{\kappa_1+}^{\zeta;\delta} h(\varrho) d\varrho, \tag{2.10}$$

where c_1 is an arbitrary constant. Differentiating (2.10), we obtain

$$\begin{aligned}\xi'(\vartheta) = &-\lambda c_1 \delta'(\vartheta) e^{-\lambda(\delta(\vartheta)-\delta(\kappa_1))} + \delta'(\vartheta)\mathcal{I}_{\kappa_1+}^{\zeta;\delta} h(\vartheta) \\ &- \lambda\delta'(\vartheta) \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \mathcal{I}_{\kappa_1+}^{\zeta;\delta} h(\varrho) d\varrho.\end{aligned} \tag{2.11}$$

Using the initial conditions given by equation (2.7) together with equations (2.10) and (2.11), we obtain

$$c_0 = \lambda \xi_{\kappa_1}, \quad c_1 = 0.$$

Substituting the value of c_0, c_1 in (2.10) we get (2.8). The converse of the lemma follows by direct computation. This completes the proof. □

Now consider the following linear fractional initial value problem.

Lemma 2.10. *Let $0 < \zeta \le 1$ and $p, q \in C(\Theta, \mathbb{R})$. Then the following linear fractional initial value problem*

$$\begin{cases} \left({}^c\mathcal{D}_{\kappa_1+}^{\zeta+1;\delta} + \lambda\, {}^c\mathcal{D}_{\kappa_1+}^{\zeta;\delta} \right)\xi(\vartheta) - p(\vartheta)\xi(\vartheta) = q(\vartheta), \quad \vartheta \in \Theta := [\kappa_1, \kappa_2], \\ \xi(\kappa_1) = \xi_{\kappa_1}, \quad \xi'(\kappa_1) = 0, \end{cases} \tag{2.12}$$

has a unique solution $\xi \in C(\Theta, \mathbb{R})$, provided that

$$\|p\| < \frac{\lambda\Gamma(\zeta+1)}{(\delta(\kappa_2) - \delta(\kappa_1))^{\zeta}}. \tag{2.13}$$

Proof. It follows from Lemma 2.9 that problem (2.12) is equivalent to the following integral equation:

$$\begin{aligned}\xi(\vartheta) = \xi_{\kappa_1} + &\int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\ &\times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} \big(p(r)\xi(r) + q(r)\big) dr \right) d\varrho.\end{aligned}$$

Define the operator $\aleph\colon C(\Theta, \mathbb{R}) \longrightarrow C(\Theta, \mathbb{R})$ as follows

$$\begin{aligned}\aleph x(\vartheta) = \xi_{\kappa_1} + &\int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\ &\times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} \big(p(r)\xi(r) + q(r)\big) dr \right) d\varrho, \ \vartheta \in \Theta.\end{aligned}$$

Now, we have to show that the operator $\aleph$ has a unique fixed point. To do this, we will prove that $\aleph$ is a contraction map.

Let $\xi, \sigma \in C(\Theta, \mathbb{R})$ and $\vartheta \in [\kappa_1, \kappa_2]$. Then, we have

$$\begin{aligned}|\aleph x(\vartheta) - \aleph y(\vartheta)| &\leq \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\ &\quad \times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} |p(r)||\xi(r) - \sigma(r)| dr \right) d\varrho \\ &\leq \frac{(\delta(\kappa_2)-\delta(\kappa_1))^{\zeta}}{\Gamma(\zeta+1)} \|p\| \|\xi - \sigma\| \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} d\varrho \\ &\leq \frac{(\delta(\kappa_2)-\delta(\kappa_1))^{\zeta}}{\lambda\Gamma(\zeta+1)} \|p\| \|\xi - \sigma\|.\end{aligned}$$

By (2.13) it follows that the operator $\aleph$ is a contraction. Consequently, by Banach's fixed point theorem, the operator $\aleph$ has a unique fixed point. That is, problem (2.12) has a unique solution. This completes the proof. □

The following result will play a very important role in this paper.

Lemma 2.11 (Comparison result). *Assume that* $p \in C(\Theta, \mathbb{R}_+^*)$ *and satisfies* (2.13). *If* $\theta \in C(\Theta, \mathbb{R})$ *satisfies the following inequalities*

$$\begin{cases} \left({}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta+1;\delta} + \lambda\, {}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta;\delta} \right) \theta(\vartheta) \geq p(\vartheta)\theta(\vartheta), \quad \vartheta \in \Theta := [\kappa_1, \kappa_2], \\ \theta(\kappa_1) \geq 0, \quad \theta'(\kappa_1) = 0, \end{cases} \tag{2.14}$$

then $\theta(\vartheta) \geq 0$ *on* $[\kappa_1, \kappa_2]$.

Proof. Let

$$\left({}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta+1;\delta} + \lambda\, {}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta;\delta} \right) \theta(\vartheta) - p(\vartheta)\theta(\vartheta) = q(\vartheta),$$
$$\theta(\kappa_1) = \xi_{\kappa_1} \quad \text{and } \theta'(\kappa_1) = 0.$$

We know that

$$q(\vartheta) \geq 0, \ \xi_{\kappa_1} \geq 0.$$

Suppose that the inequality $\theta(\vartheta) \geq 0, \vartheta \in [\kappa_1, \kappa_2]$ is not true. It means that there exists at least a $\vartheta_0 \in [\kappa_1, \kappa_2]$ such that $\theta(\vartheta_0) < 0$. Without loss of generality, we assume $\theta(\vartheta_0) = \min_{\vartheta \in [\kappa_1, \kappa_2]} \theta(\vartheta)$. Then by Lemma 2.10 we have

$$\begin{aligned}\theta(\vartheta) &= \xi_{\kappa_1} + \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\ &\quad \times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} \big(p(r)\theta(r) + q(r)\big) dr \right) d\varrho \\ &\geq \theta(\vartheta_0) \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\ &\quad \times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} p(r) dr \right) d\varrho.\end{aligned}$$

For $\vartheta = \vartheta_0$, we can get

$$\theta(\vartheta_0) \geq \theta(\vartheta_0) \int_{\kappa_1}^{\vartheta_0} \delta'(\varrho) e^{-\lambda(\delta(\vartheta_0)-\delta(\varrho))} \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} p(r) dr \right) d\varrho.$$

Therefore, keeping in mind that $\theta(\vartheta_0) < 0$, we have

$$1 \leq \int_{\kappa_1}^{\vartheta_0} \delta'(\varrho) e^{-\lambda(\delta(\vartheta_0)-\delta(\varrho))} \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} p(r) dr \right) d\varrho.$$

Hence,

$$\|p\| \geq \frac{\lambda\Gamma(\zeta+1)}{(\delta(\kappa_2)-\delta(\kappa_1))^{\zeta}},$$

which is in contradiction to (2.13). Hence, $\theta(\vartheta) \geq 0$ for all $\vartheta \in [\kappa_1, \kappa_2]$. □

3. Main results

In this paper, we will apply the monotone iterative method to present a result on the existence of the solution of problem (1.1).

Theorem 3.1. *Let the function $\Psi \in C(\Theta \times \mathbb{R}, \mathbb{R})$. In addition assume that:*

(H_1) *There exist ξ_0, $\sigma_0 \in C(\Theta, \mathbb{R})$ such that ξ_0 and σ_0 are lower and upper solutions of problem (1.1), respectively, with $\xi_0(\vartheta) \leq \sigma_0(\vartheta), \vartheta \in \Theta$.*

(H_2) *There exists $p \in C(\Theta, \mathbb{R}^+)$ satisfies (2.13) such that*

$$\Psi(\vartheta, \varpi_2) - \Psi(\vartheta, \varpi_1) \geq p(\vartheta)(\varpi_2 - \varpi_1) \quad \textit{for} \quad \xi_0 \leq \varpi_1 \leq \varpi_2 \leq \sigma_0.$$

(H_3) *There exist $\mathbb{k}_1 > 0$ and $\mathbb{k}_2 \geq 0$, where for $\xi_0(\kappa_1) \leq u_1 \leq u_2 \leq \sigma_0(\kappa_1)$, $\xi_0(\kappa_2) \leq v_1 \leq v_2 \leq \sigma_0(\kappa_2)$,*

$$\Phi(u_2, v_2) - \Phi(u_1, v_1) \leq \mathbb{k}_1(u_2 - u_1) - \mathbb{k}_2(v_2 - v_1).$$

Consequently, there exist monotone iterative sequences $\{\xi_\beta\}$ and $\{\sigma_\beta\}$, which converge uniformly on Θ to the extremal solutions of (1.1) in $[\xi_0, \sigma_0]$, where

$$[\xi_0, \sigma_0] = \big\{\varpi \in C(\Theta, \mathbb{R}) : \xi_0(\vartheta) \leq \varpi(\vartheta) \leq \sigma_0(\vartheta), \quad \vartheta \in \Theta\big\}.$$

Proof. First, for any $\xi_0, \sigma_0 \in C(\Theta, \mathbb{R})$, consider:

$$\begin{cases} \big({}^c\mathcal{D}_{\kappa_1^+}^{\zeta+1;\delta} + \lambda \, {}^c\mathcal{D}_{\kappa_1^+}^{\zeta;\delta} \big)\xi_{\beta+1}(\vartheta) = \Psi\big(\vartheta, \xi_n(\vartheta)\big) + p(\vartheta)\big(\xi_{\beta+1}(\vartheta) - \xi_\beta(\vartheta)\big), \\ \xi_{\beta+1}(\kappa_1) = \xi_\beta(\kappa_1) - \frac{1}{\mathbb{k}_1}\Phi\big(\xi_\beta(\kappa_1), \xi_\beta(\kappa_2)\big), \quad \xi'_{\beta+1}(\kappa_1) = 0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \big({}^c\mathcal{D}_{\kappa_1^+}^{\zeta+1;\delta} + \lambda \, {}^c\mathcal{D}_{\kappa_1^+}^{\zeta;\delta} \big)\sigma_{\beta+1}(\vartheta) = \Psi\big(\vartheta, \sigma_n(\vartheta)\big) + p(\vartheta)\big(\sigma_{\beta+1}(\vartheta) - \sigma_n(\vartheta)\big), \\ \sigma_{\beta+1}(\kappa_1) = \sigma_\beta(\kappa_1) - \frac{1}{\mathbb{k}_1}\Phi\big(\sigma_\beta(\kappa_1), \sigma_\beta(\kappa_2)\big), \quad \sigma'_{\beta+1}(\kappa_1) = 0. \end{cases} \tag{3.2}$$

By Lemma 2.10, we know that (3.1) and (3.2) have a unique solutions in $C(\Theta, \mathbb{R})$. We will divide the proof in the following steps.

Step 1: We prove that $\xi_\beta, \sigma_\beta (\beta \geq 1)$ are lower and upper solutions of problem (1.1), respectively and

$$\xi_0(\vartheta) \leq \xi_1(\vartheta) \leq \cdots \leq \xi_\beta(\vartheta) \leq \cdots \leq \sigma_\beta(\vartheta) \leq \cdots \leq \sigma_1(\vartheta) \leq \sigma_0(\vartheta), \quad \vartheta \in \Theta. \tag{3.3}$$

First, we prove that

$$\xi_0(\vartheta) \leq \xi_1(\vartheta) \leq \sigma_1(\vartheta) \leq \sigma_0(\vartheta), \quad \vartheta \in \Theta. \tag{3.4}$$

Set $\theta(\vartheta)=\xi_1(\vartheta)-\xi_0(\vartheta)$. From (3.1) and Definition 2.7, we obtain

$$\begin{aligned}\big({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+}+\lambda\,{}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\big)\theta(\vartheta)&=\Psi\big(\vartheta,\xi_0(\vartheta)\big)-\big({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+}+\lambda\,{}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\big)\xi_0(\vartheta)\\&\quad+p(\vartheta)\theta(\vartheta)\\&\geq p(\vartheta)\theta(\vartheta).\end{aligned}$$

Again, since $\theta'(\kappa_1)=0$ and

$$\theta(\kappa_1)=-\frac{1}{\Bbbk_1}\Phi\big(\xi_0(\kappa_1),\xi_0(\kappa_2)\big)\geq 0.$$

By Lemma 2.11, $\theta(\vartheta)\geq 0$, for $\vartheta\in\Theta$. That is, $\xi_0(\vartheta)\leq\xi_1(\vartheta)$. Also, we have

$$\sigma_1(\vartheta)\leq\sigma_0(\vartheta),\ \ \vartheta\in\Theta.$$

Now, let $\theta(\vartheta)=\sigma_1(\vartheta)-\xi_1(\vartheta)$. From (3.1), (3.2) and $(H2)$, we get

$$\begin{aligned}\big({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+}+\lambda\,{}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\big)\theta(\vartheta)&=\Psi\big(\vartheta,\sigma_0(\vartheta)\big)-\Psi\big(\vartheta,\xi_0(\vartheta)\big)\\&\quad+p(\vartheta)\big(\sigma_1(\vartheta)-\sigma_0(\vartheta)\big)-p(\vartheta)\big(\xi_1(\vartheta)-\xi_0(\vartheta)\big)\\&\geq p(\vartheta)\big(\sigma_0(\vartheta)-\xi_0(\vartheta)\big)+p(\vartheta)\big(\sigma_1(\vartheta)-\sigma_0(\vartheta)\big)\\&\quad-p(\vartheta)\big(\xi_1(\vartheta)-\xi_0(\vartheta)\big)\\&=p(\vartheta)\theta(\vartheta).\end{aligned}$$

Since, $\theta'(\kappa_1)=0$ and

$$\begin{aligned}\theta(\kappa_1)&=\big(\sigma_0(\kappa_1)-\xi_0(\kappa_1)\big)-\frac{1}{\Bbbk_1}\left(\Phi\big(\sigma_0(\kappa_1),\sigma_0(\kappa_2)\big)-\Phi\big(\xi_0(\kappa_1),\xi_0(\kappa_2)\big)\right)\\&\geq\frac{\Bbbk_2}{\Bbbk_1}\big(\sigma_0(\kappa_2)-\xi_0(\kappa_2)\big)\geq 0.\end{aligned}$$

By Lemma 2.11, we get $\xi_1(\vartheta)\leq\sigma_1(\vartheta),\ \vartheta\in\Theta$.

Next, we prove that $\xi_1(\vartheta),\sigma_1(\vartheta)$ are lower and upper solutions of (1.1), respectively. Since ξ_0 and σ_0 are lower and upper solutions of (1.1), by (H_2), it follows that

$$\big({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+}+\lambda\,{}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\big)\xi_1(\vartheta)=\Psi\big(\vartheta,\xi_0(\vartheta)\big)+p(\vartheta)\big(\xi_1(\vartheta)-\xi_0(\vartheta)\big)\leq\Psi\big(\vartheta,\xi_1(\vartheta)\big),$$

also $\xi_1'(\kappa_1)=0$ and

$$\begin{aligned}0&=\Bbbk_1\big(\xi_1(\kappa_1)-\xi_0(\kappa_1)\big)+\Phi\big(\xi_0(\kappa_1),\xi_0(\kappa_2)\big)\\&\geq\Phi\big(\xi_1(\kappa_1),\xi_1(\kappa_2)\big)+\Bbbk_2\big(\xi_1(\kappa_2)-\xi_0(\kappa_2)\big).\end{aligned}$$

Thus,

$$\Phi(\xi_1(\kappa_1),\xi_1(\kappa_2))\leq 0.$$

Therefore, $\xi_1(\vartheta)$ is a lower solution of (1.1). Also, we get that $\sigma_1(\vartheta)$ is an upper solution of (1.1).

By induction, we demonstrate that $\xi_\beta(\vartheta),\sigma_\beta(\vartheta),(\beta\geq 1)$ are lower and upper solutions of problem (1.1), respectively and the following relation holds

$$\xi_0(\vartheta)\leq\xi_1(\vartheta)\leq\cdots\leq\xi_\beta(\vartheta)\leq\cdots\leq\sigma_\beta(\vartheta)\leq\cdots\leq\sigma_1(\vartheta)\leq\sigma_0(\vartheta),\quad\vartheta\in\Theta.$$

Step 2: The sequences $\{\xi_\beta(\vartheta)\}$, $\{\sigma_\beta(\vartheta)\}$ uniformly converge to their limit functions $\xi^*(\vartheta), \sigma^*(\vartheta)$.
Note that $\{\xi_\beta(\vartheta)\}$ is monotone nondecreasing and is bounded from above by $\sigma_0(\vartheta)$. Also, since the sequence $\{\sigma_\beta(\vartheta)\}$ is monotone nonincreasing and is bounded from below by $\xi_0(\vartheta)$, thus the pointwise limits ξ^* and σ^* exist. And, since $\{\xi_\beta(\vartheta)\}$, $\{\sigma_\beta(\vartheta)\}$ are sequences of continuous functions defined on $[\kappa_1, \kappa_2]$, hence by Dini's theorem [30], the convergence is uniform. This is

$$\lim_{\beta\to\infty} \xi_\beta(\vartheta) = \xi^*(\vartheta) \quad \text{and} \quad \lim_{\beta\to\infty} \sigma_\beta(\vartheta) = \sigma^*(\vartheta),$$

uniformly on $\vartheta \in \Theta$ and the limit functions ξ^*, σ^* satisfy problem (1.1). Furthermore, ξ^* and σ^* satisfy the relation

$$\xi_0 \le \xi_1 \le \cdots \le \xi_\beta \le \xi^* \le \sigma^* \le \cdots \le \sigma_\beta \le \cdots \le \sigma_1 \le \sigma_0.$$

Step 3: ξ^* and σ^* are extremal solutions of problem (1.1) in $[\xi_0, \sigma_0]$.
Let $\varpi \in [\xi_0, \sigma_0]$ be any solution of (1.1). We assume that the following relation holds for some $\beta \in \mathbb{N}$:

$$\xi_\beta(\vartheta) \le \varpi(\vartheta) \le \sigma_\beta(\vartheta), \quad \vartheta \in \Theta. \tag{3.5}$$

Let $\theta(\vartheta) = \varpi(\vartheta) - \xi_{\beta+1}(\vartheta)$. We have

$$\begin{aligned}\left({}^c\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+} + \lambda\, {}^c\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\right)\theta(\vartheta) &= \Psi\big(\vartheta, \varpi(\vartheta)\big) - \Psi\big(\vartheta, \xi_\beta(\vartheta)\big) \\ &\quad - p(\vartheta)\big(\xi_{\beta+1}(\vartheta) - \xi_\beta(\vartheta)\big) \\ &\ge p(\vartheta)\big(\varpi(\vartheta) - \xi_\beta(\vartheta)\big) - p(\vartheta)\big(\xi_{\beta+1}(\vartheta) - \xi_\beta(\vartheta)\big) \\ &= p(\vartheta)\theta(\vartheta).\end{aligned} \tag{3.6}$$

Furthermore, $\theta'(\kappa_1) = 0$ and

$$\begin{aligned}0 &= \Phi\big(\varpi(\kappa_1), \varpi(\kappa_2)\big) - \Phi\big(\xi_\beta(\kappa_1), \xi_\beta(\kappa_2)\big) + \mathbb{k}_1\big(\xi_{\beta+1}(\kappa_1) - \xi_\beta(\kappa_1)\big) \\ &\ge \mathbb{k}_1\big(\varpi(\kappa_1) - \xi_\beta(\kappa_1)\big) - \mathbb{k}_2\big(\varpi(\kappa_2) - \xi_\beta(\kappa_2)\big) + \mathbb{k}_1\big(\xi_{\beta+1}(\kappa_1) - \xi_\beta(\kappa_1)\big) \\ &= \mathbb{k}_1\theta(\kappa_1) - \mathbb{k}_2\big(\varpi(\kappa_2) - \xi_\beta(\kappa_2)\big).\end{aligned}$$

That is,

$$\theta(\kappa_1) \ge \frac{\mathbb{k}_2}{\mathbb{k}_1}\big(\varpi(\kappa_2) - \xi_\beta(\kappa_2)\big) \ge 0.$$

By Lemma 2.11, we obtain $\theta(\vartheta) \ge 0,\ \vartheta \in \Theta$, which means

$$\xi_{\beta+1}(\vartheta) \le \varpi(\vartheta),\ \vartheta \in \Theta.$$

Using the same method, we can show that

$$\varpi(\vartheta) \le \sigma_{\beta+1}(\vartheta),\ \vartheta \in \Theta.$$

Hence, we have

$$\xi_{\beta+1}(\vartheta) \le \varpi(\vartheta) \le \sigma_{\beta+1}(\vartheta),\ \vartheta \in \Theta.$$

Therefore, (3.5) holds on Θ for all $\beta \in \mathbb{N}$. Taking the limit as $\beta \to \infty$ on (3.5), we obtain

$$\xi^* \le \varpi \le \sigma^*.$$

Consequently, ξ^* and σ^* are the extremal solutions of (1.1) in $[\xi^*, \sigma^*]$. □

Example 3.2. Consider the following boundary value problem:

$$\begin{cases} \left({}^{c}\mathcal{D}^{\frac{3}{2}}_{\kappa_1^+} + \frac{2}{\sqrt{\pi}}\, {}^{c}\mathcal{D}^{\frac{1}{2}}_{\kappa_1^+} \right)\xi(\vartheta) = \sin(\vartheta)(\xi - 1) + e^{-\vartheta}, \quad \vartheta \in \Theta := [0,1], \\ \xi(0) = 1, \quad \xi'(0) = 0. \end{cases} \tag{3.7}$$

Note that, this problem is a particular case of BVP (1.1), where

$$\zeta = \frac{1}{2}, \quad \lambda = \frac{2}{\sqrt{\pi}}, \quad \delta(\vartheta) = \vartheta,$$
$$\Psi(\vartheta, \xi) = \sin(\vartheta)(\xi - 1) + e^{-\vartheta}, \quad \Phi(\xi, \sigma) = \xi - 1.$$

Obviously, $\Psi \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\Phi \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. On the other hand, taking $\xi_0(\vartheta) = 1$ and $\sigma_0(\vartheta) = 1 + \vartheta\sqrt{\vartheta}$, it is not difficult to verify that ξ_0, σ_0 are lower and upper solutions of (3.7), respectively, and $\xi_0 \leq \sigma_0$. So condition (H_1) holds.

Moreover, for $\xi_0 \leq \xi \leq \sigma \leq \sigma_0$ we have

$$\Psi(\vartheta, \sigma) - \Psi(\vartheta, \xi) \geq \sin\vartheta(\sigma - \xi). \tag{3.8}$$

And if $\xi_0(\kappa_1) \leq u_1 \leq u_2 \leq \sigma_0(\kappa_1)$, $\xi_0(\kappa_2) \leq v_1 \leq v_2 \leq \sigma_0(\kappa_2)$, we have

$$\Phi(u_2, v_2) - \Phi(u_1, v_1) \leq (u_2 - u_1). \tag{3.9}$$

In view of (3.8) and (3.9), we can choose $p(\vartheta) = \sin\vartheta, \Bbbk_1 = 1$ and $\Bbbk_2 = 0$ in Theorem 3.1. At last, by a simple computation, we have

$$\frac{(\delta(\kappa_2) - \delta(\kappa_1))^{\zeta}}{\lambda\Gamma(\zeta + 1)}\|p\| < 1.$$

Hence, all conditions of Theorem 3.1 are satisfied and consequently the problem (3.7) has extremal solutions on $[\xi_0, \sigma_0]$.

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New results on asymptotic stability of time-varying nonlinear systems with applications

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Abstract. In this paper, we present a converse Lyapunov theorem for the new notion of global generalized practical uniform h-stability of nonlinear systems of differential equations. We derive some sufficient conditions which guarantee the global generalized practical uniform h-stability of time-varying perturbed systems. In addition, these results are used to study the practical h-stability of models of infectious diseases and vaccination.

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Keywords: Epidemic models, generalized practical uniform h-stability, Gronwall's inequalities, Lyapunov functions.

1. Introduction

The most important stability concept used in the qualitative theory of differential equations is the uniform exponential stability. In some situations, particularly, in the non-autonomous setting, the notion of uniform exponential stability is to restrictive and it is important to look for a more general behavior. In the last century, Manual Pinto (see [21, 20]) introduced a new notion of stability called h-stability for nonlinear differential equations on the Euclidean space $\mathbb{R}^n$, with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. Some important properties about h-stability for various differential systems and nonlinear differential systems are given. In [4], the authors investigated the h-stability properties for nonlinear differential systems using the notion of t_∞-similarity and Lyapunov functions. Goo and al. studied h-stability for the nonlinear Volterra integro-differential system

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(see [10]) and for the linear perturbed Volterra integro-differential systems (see [9]). An interesting and fruitful technique that gained increasing significance and has given decisive impetus for modern development of stability theory of differential equations is Lyapunov's method. The strength of this technique is that it is possible to ascertain stability without solving the underlying differential equation. This method states that if one can find an appropriate Lyapunov function, then the system has some stability property (see [6, 18]). However, a system might be stable or asymptotically stable in theory, nevertheless, it is actually unstable in practice because the stable domain or the domain of the desired attractor is not large enough. Thus, from an engineering point of view we need a notion of stability that more suitable in several situations than Lyapunov stability. Such a concept is called practical stability (see [2, 5]). The novelty of this paper is to present a new notion of stability called generalized practical uniform h-stability as an extension of the generalized exponential asymptotic stability in [17] and practical uniform h-stability in [6, 7, 8]. In recent years, mathematical models of infectious diseases have been studied by a numbers of authors, see [11, 12, 13, 15, 19, 22] and many others. For instance, Ito in [12] considered a variety of models of infectious diseases and vaccination through the language of iISS and ISS. The method of Lyapunov functions is widely used to establish global stability results for biological models (see [12, 13, 15]).

The remainder of this work is organized as follows. In Section 2, we recall a new concept of stability and some tools used in the proofs. In Section 3, under growth conditions on the perturbed term, we investigate the global practical uniform h-stability of a nonlinear perturbed system using the Nonlinear Gronwall Inequality. In addition, we propose sufficient conditions with the extended of a Lyapunov function to indicate the global generalized practical uniform h-stability of the nonlinear system. The main result is provided in Section 4 in which the generalization of converse Lyapunov theorem is established by requiring the existence of a continuously non-differentiable Lyapunov function that satisfying certain properties. Moreover, a practical approach is obtained of time-varying dynamical perturbed system using the indirect Lyapunov's method, the comparison principle and the Generalized Gronwall-Bellman Inequality. However, Section 5 employs the notion of practical h-stability to evaluate robustness of infectious diseases with respect to integrable perturbation of the newborn/immigration rate and time-varying death rate. Finally, our conclusion is proposed in Section 6.

2. Preliminaries

The notation used throughout this note is standard. $\mathbb{R}_+$ indicates the set of non-negative real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\|\cdot\|$ stands for its Euclidean vector norm. Also, we denote by:

- $I, J \subset \mathbb{R}$ are two intervals that are not empty and not reduce to a singleton.
- $\mathcal{BC}(I, J)$ is the space of continuous bounded functions on I to J endowed with the norm $\|f\|_\infty = \sup_{t\in I} |f(t)|$.
- $\mathcal{C}(I, J)$ is the space of continuous functions on I to J.

- $\mathcal{C}^1(I, J)$ is the space of continuous differentiable functions on I to J.

We consider the nonlinear non-autonomous differential system:

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{2.1}$$

where $t \in \mathbb{R}_+$ is the time, $x \in \mathbb{R}^n$ is the state and $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t.
Let $x(t) = x(t, t_0, x_0)$ be denoted by the unique solution of (2.1) through $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.
We assume that the Jacobian matrix $f_x = \left[\dfrac{\partial f}{\partial x}\right]$ exists and continuous on $\mathbb{R}_+ \times \mathbb{R}^n$.
We consider also the associated variational system:

$$\dot{z}(t) = f_x(t, x(t, t_0, x_0))z(t), \qquad z(t_0) = z_0, \qquad t \geq t_0 \geq 0. \tag{2.2}$$

Theorem 2.1. (See [1]) *If f is differentiable in $\mathbb{R}^n$ for $t \in \mathbb{R}_+$ and $x(t, t_0, x_0)$ is in $\mathbb{R}^n$ for $t \in \mathbb{R}_+$, then $x(t, t_0, x_0)$ is differentiable with respect to x_0 and*

$$R(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$$

is the fundamental matrix of solutions of the variational system (2.2)*, such that $R(t_0, t_0, x_0) = I$ is the identity matrix which is independent of x_0.*

A precise definition of the global generalized practical uniform h-stability will be given as follows.

Definition 2.2. Let $h \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$.

- System (2.1) is called generalized practically uniformly h-stable if there exist $\eta \geq 0$ and a function $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$, such that for any initial state x_0, with $\|x_0\| \leq r$ and for all $t \geq 0$, we have

$$\|x(t)\| \leq \eta + K(t_0)\|x_0\|h(t)h(t_0)^{-1}, \qquad \forall\, t \geq t_0. \tag{2.3}$$

- System (2.1) is said to be globally generalized practically uniformly h-stable if the previous definition is satisfied for any initial state $x_0 \in \mathbb{R}^n$.

Here, $h(t)^{-1} = \dfrac{1}{h(t)}\cdot$

Remark 2.3. Definition 2.2 generalizes the notions of h-stability (see [20]). More precisely, when $\eta = 0$ we obtained the definition of global generalized uniform h-stability. Moreover, for $\eta > 0$ and for some special cases of h, the generalized practical uniform h-stability coincides with known practical types of stability:

- If $K(t) = c > 0$, we say that the system (2.1) is globally practically uniformly h-stable (see [6]).
- The practical uniform exponential stability is a particular case of generalized practical h-stability by taking $K(t) = c > 0$ and $h(t) = e^{-\beta t}$ with $\beta > 0$ (see [2]).
- If $h(t) = \dfrac{1}{(1+t)^\gamma}$ with $\gamma > 0$, we say that the system (2.1) is generalized practically uniformly polynomially stable (see [6]).

We are now in position to present the following lemmas which are important tools in the subsequent discussion.

Lemma 2.4. (See [16]) *Assume that $x(t,t_0,x_0)$ and $x(t,t_0,y_0)$ be any two solutions of system* (2.1) *through $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}^n$ and $(t_0,y_0)\in\mathbb{R}_+\times\mathbb{R}^n$, respectively, existing for $t\geq t_0$, such that x_0 and y_0 belong to a convex subset D of $\mathbb{R}^n$. Then,*

$$x(t,t_0,x_0)-x(t,t_0,y_0)=\int_0^1 R\big(t,t_0,x_0+s(y_0-x_0)\big)ds(y_0-x_0),\qquad t\geq t_0\geq 0,\ (2.4)$$

where $R(t,t_0,x_0)$ is the fundamental matrix solution of system (2.2).

Lemma 2.5. *The variational system* (2.2) *is globally generalized uniformly h-stable if and only if there exist functions $K\in\mathcal{BC}(\mathbb{R}_+,\mathbb{R}_+^\star)$ and $h\in\mathcal{BC}(\mathbb{R}_+,\mathbb{R}_+^\star)$, such that for all $x_0\in\mathbb{R}^n$ and all $t_0\in\mathbb{R}_+$, we have*

$$\|R(t,t_0,x_0)\|\leq K(t_0)h(t)h(t_0)^{-1},\qquad \forall\ t\geq t_0.$$

Definition 2.6. (Lyapunov Functions)
We define the upper-right hand derivative Lyapunov functions of (2.1) as follows:

$$D^+V_{(2.1)}(t,x)=\limsup_{T\to 0^+}\frac{1}{T}\Big(V\big(t+T,x+Tf(t,x)\big)-V(t,x)\Big),$$

for $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$ and for the solution $x(t)=x(t,t_0,x_0)$ of (2.1),

$$D^+V(t,x(t))=\limsup_{T\to 0^+}\frac{1}{T}\Big(V\big(t+T,x(t+T)\big)-V(t,x)\Big).$$

Lemma 2.7. *Assume that the continuous function $V:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$ is Lipschitz in x for a function $K\in\mathcal{BC}(\mathbb{R}_+,\mathbb{R}_+^\star)$, that is,*

$$\Big|V(t,x)-V(t,y)\Big|\leq K(t)|x-y|,\qquad \forall\ t\geq 0,\quad \forall\ x,y\in\mathbb{R}^n,$$

Then,

$$D^+V_{(2.1)}(t,x)=D^+V(t,x(t)).$$

Proof. We have,

$$\begin{aligned}V(t+T,x(t+T))-V(t,x) &= V(t+T,x+Tf(t,x)+\circ(T))-V(t,x)\\ &= \Big(V(t+T,x+Tf(t,x)+\circ(T))-\\ &\quad V(t+T,x+Tf(t,x))\Big)\\ &+ \Big(V(t+T,x+Tf(t,x))-V(t,x)\Big).\end{aligned}$$

Since $V(t,x)$ is Lipschitz in x for a continuous function $K(t)>0$ for all $t\in\mathbb{R}_+$, one easily sees that

$$V(t+T,x+Tf(t,x)+\circ(T))-V(t+T,x+Tf(t,x))=\circ(T).$$

Therefore, by Definition 2.6 we immediately deduce that

$$\begin{aligned} D^+V(t,x(t)) &= \limsup_{T\to 0^+} \frac{1}{T}\Big(V\big(t+T,x(t+T)\big)-V(t,x)\Big) \\ &= \limsup_{T\to 0^+} \frac{1}{T}\Big(V\big(t+T,x+Tf(t,x)\big)-V(t,x)\Big) \\ &= D^+V_{(2.1)}(t,x). \end{aligned}$$

□

Remark 2.8. If $V(t,x)\in\mathcal{C}^1(\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}^n)$, then

$$D^+V_{(2.1)}(t,x)=D^+V(t,x(t))=\dot{V}_{(2.1)}(t,x)=\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)f(t,x).$$

We use also the following lemmas to prove our results.

Lemma 2.9. (Nonlinear Gronwall Inequality)
Let $\mu:\mathbb{R}_+\to\mathbb{R}$ be a continuous function that satisfies the integral inequality

$$\mu(t)\le d+\int_{t_0}^t b(s)\mu^\alpha(s)ds,\quad d\ge 0,\qquad 0\le\alpha<1,\qquad t\ge t_0,$$

where b is a non-negative continuous function on $\mathbb{R}_+$. Then, we have

$$\mu(t)\le\left(d^{1-\alpha}+(1-\alpha)\int_{t_0}^t b(s)ds\right)^{\frac{1}{1-\alpha}},\qquad\forall\ t\ge t_0.$$

Proof. Let,

$$\varpi(t)=d+\int_{t_0}^t b(s)\varpi^\alpha(s)ds,\qquad 0\le\alpha<1,\qquad t\ge t_0.$$

Then,

$$\dot{\varpi}(t)=b(t)\varpi(t))^\alpha,\qquad \varpi(t_0)=d,\qquad\forall\ t\ge t_0.$$

It is follows that,

$$\varpi^{1-\alpha}(t_0)=d^{1-\alpha}+(1-\alpha)\int_{t_0}^t b(s)ds.$$

Therefore,

$$\varpi(t)\le\left(d^{1-\alpha}+(1-\alpha)\int_{t_0}^t b(s)ds\right)^{\frac{1}{1-\alpha}}.$$

□

Lemma 2.10. (Generalized Gronwall-Bellman Inequality) (See [23])
Let $\rho,\ \varphi:\mathbb{R}_+\to\mathbb{R}$ be continuous functions and $\mu:\mathbb{R}_+\to\mathbb{R}_+$ is a function, such that

$$\dot{\mu}(t)\le\rho(t)\mu(t)+\varphi(t),\quad\forall\ t\ge t_0.$$

Then, for all $t_0\ge 0$, we have

$$\mu(t)\le\mu(t_0)\exp\Big(\int_{t_0}^t\rho(\tau)d\tau\Big)+\int_{t_0}^t\exp\Big(\int_s^t\rho(\tau)d\tau\Big)\varphi(s)ds,\quad\forall\ t\ge t_0.$$

3. Sufficient conditions for practical h-stability results

We start this section by studying the global practical uniform h-stability of a perturbed system under sufficient conditions on the perturbed term using the Nonlinear Gronwall Inequality. We need Alekseev formula to compare between the solutions of system (2.1) and the solutions of the following perturbed nonlinear system:

$$\dot{y} = f(t, y) + p(t, y), \qquad y(t_0) = y_0, \qquad t \geq t_0 \geq 0, \tag{3.1}$$

where $p \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$. Let $y(t) = y(t, t_0, y_0)$ represent the solution of the perturbed system passing through the point $(t_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.
The next lemma is a generalization to nonlinear system of the variation of constants formula on account of Alekseev (see [3]).

Lemma 3.1. *If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$, $x(t, t_0, y_0) \in \mathbb{R}^n$ and $y(t, t_0, y_0) \in \mathbb{R}^n$, we have*

$$y(t, t_0, y_0) - x(t, t_0, y_0) = \int_{t_0}^{t} R(t, s, y(s))p(s, y(s, t_0, y_0))ds.$$

Let consider the following theorem.

Theorem 3.2. *We consider the perturbed system* (3.1) *with the perturbation $p \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x. Let the origin be globally uniformly h-stable of system* (2.1) *and $z = 0$ of system* (2.2) *is globally uniformly h-stable. Assume that $p(t, x)$ satisfies the following condition:*

$$\|p(t, y)\| \leq \vartheta(t)\|y\|^{\alpha} + \nu(t), \qquad 0 \leq \alpha < 1, \qquad \forall\, y \in \mathbb{R}^n, \qquad \forall\, t \geq 0, \tag{3.2}$$

where ϑ, ν are non-negative continuous functions on $\mathbb{R}_+$ and there exist positive constants M_1 and M_2, such that

$$\int_0^t \vartheta(s)h(s)^{-1}ds \leq M_1, \qquad \int_0^t \nu(s)h(s)^{-1}ds \leq M_2, \qquad \forall\, t \geq 0. \tag{3.3}$$

Then, the system (3.1) *is globally practically uniformly h-stable.*

Proof. Let $y(t) = y(t, t_0, y_0)$ and $x(t) = x(t, t_0, y_0)$ be solutions of systems (3.1) and (2.1), respectively, then by Lemma 3.1, we have

$$y(t) = x(t) + \int_{t_0}^{t} R(t, s, y(s))p(t, y(s))ds.$$

Thus, from the global uniform h-stability of system (2.1), there exists $c > 0$, such that

$$\|y(t)\| = c\|y_0\|h(t)h(t_0)^{-1} + ch(t)\int_{t_0}^{t} \vartheta(s)h(s)^{-1}\|y(s)\|^{\alpha}ds + ch(t)\int_{t_0}^{t} \nu(s)h(s)^{-1}ds.$$

Hence,

$$h(t)^{-1}\|y(t)\| \leq \Big(c\|y_0\|h(t_0)^{-1} + cM_2\Big) + c\int_{t_0}^{t} \vartheta(s)h(s)^{\alpha-1}\big(h(s)^{-1}\|y(s)\|\big)^{\alpha}ds.$$

Let, $\rho(t) = h(t)^{-1}\|y(t)\|$, then

$$\rho(t) \leq \Big(c\rho(t_0) + cM_2\Big) + c\int_{t_0}^{t} \vartheta(s)h(s)^{\alpha-1}\rho^{\alpha}(s)ds.$$

Applying the Nonlinear Gronwall Inequality and the fact that

$$(\lambda_1 + \lambda_2)^r \leq 2^{r-1}(\lambda_1^r + \lambda_2^r),$$

for all $\lambda_1,\ \lambda_2 \geq 0$ and $r \geq 1$, we get

$$\rho(t) \leq 2^{\frac{\alpha}{1-\alpha}}\big(c\rho(t_0) + cM_2\big) + 2^{\frac{\alpha}{1-\alpha}}\Big(cM_1(1-\alpha)\|h\|_{\infty}^{\alpha}\Big)^{\frac{1}{1-\alpha}},$$

with $\|h\|_\infty = \sup\limits_{t\geq 0}\{h(t)\}$. This yields, for all $y_0 \in \mathbb{R}^n$ and all $t \geq t_0$ the solution of system (3.1) satisfies:

$$\|y(t)\| \leq \eta + c_1\|y_0\|h(t)h(t_0)^{-1},$$

with $c_1 = 2^{\frac{\alpha}{1-\alpha}}c$ and $\eta = 2^{\frac{\alpha}{1-\alpha}}cM_2\|h\|_\infty + 2^{\frac{\alpha}{1-\alpha}}\Big(cM_1(1-\alpha)\|h\|_\infty\Big)^{\frac{1}{1-\alpha}}$.
Consequently, system (3.1) is globally practically uniformly h-stable. This completes the proof. □

The stability properties of the solutions of nonlinear differential equations can be studied using the Lyapunov functions and the theory of differential and integral inequalities. This interesting and useful technique is called Lyapunov's second method. The following theorem proves the global generalized practical uniform h-stability of solutions of system (2.1) by requiring the existence of a continuously non-differentiable Lyapunov function that satisfying sufficient conditions.

Theorem 3.3. *Suppose that h is a positive bounded continuously differentiable function on $\mathbb{R}_+$. Furthermore, assume that there exist $a > 0$, $b \geq 1$, $\varrho \geq 0$, a function $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$ and a continuously non-differentiable Lyapunov function $V(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n$, such that the following conditions are hold.*

1. *$V(t,x)$ is Lipschitzian in x for a function $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$,*
2. $a\|x\|^b \leq V(t,x) \leq K(t)\|x\|^b, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$
3. $D^+V_{(2.1)}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - \varrho h'(t)h(t)^{-1}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$

Then, the system (2.1) *is globally generalized practically uniformly $h^{\frac{1}{b}}$-stable.*

Proof. One has,

$$D^+V_{(2.1)}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - \varrho h'(t)h(t)^{-1}.$$

We apply the comparison principle (see [14]), where

$$\dot{u}(t) = h'(t)h(t)^{-1}u(t) - \varrho h'(t)h(t)^{-1}, \qquad u(t_0) = u_0, \qquad t \geq t_0 \geq 0, \qquad (3.4)$$

with $V(t_0, x_0) \leq u_0 \leq K(t_0)\|x_0\|^b$. Then, by using the Generalized Gronwall-Bellman Inequality, the maximal solution of the scalar equation (3.4) is as follows:

$$\begin{aligned} u(t) &\leq u_0 h(t)h(t_0)^{-1} - \varrho \int_{t_0}^{t} \exp\left(\int_s^t h'(\tau)h(\tau)^{-1}d\tau\right) h'(s)h(s)^{-1}ds \\ &= u_0 h(t)h(t_0)^{-1} - \varrho h(t) \int_{t_0}^{t} h'(s)(h(s)^{-1})^2 ds \\ &\leq \varrho + K(t_0)\|x_0\|^b h(t)h(t_0)^{-1}. \end{aligned}$$

Hence, for all $x_0 \in \mathbb{R}^n$ and all $t \geq t_0$, we have

$$\|x(t)\| \leq \left(\frac{\varrho}{a}\right)^{\frac{1}{b}} + \left(\frac{K(t_0)}{a}\right)^{\frac{1}{b}} \|x_0\| h(t)^{\frac{1}{b}} h(t_0)^{-\frac{1}{b}}.$$

Consequently, the system (2.1) is globally generalized practically uniformly $h^{\frac{1}{b}}$-stable. □

4. Converse theorem

The purpose of this section is to represent a converse Lyapunov result for nonlinear time-varying systems that are globally generalized practically uniformly h-stable.

Theorem 4.1. *Assume that the system* (2.1) *is globally generalized practically uniformly h-stable and the solution $z = 0$ of system* (2.2) *is globally generalized uniformly h-stable. Suppose further that $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function. Then, there exist $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$, a positive constant η and a continuously non-differentiable Lyapunov function $V(t, x)$, such that the following properties are hold.*

1. *$V(t, x)$ is Lipschitzian in x for a function $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$,*
2. $\|x\| \leq V(t, x) \leq K(t)\|x\| + \eta, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$
3. $D^+V_{(2.1)}(t, x) \leq h'(t)h(t)V(t, x) - \eta h'(t)h(t)^{-1}, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n.$

Proof. Since system (2.1) is globally generalized practically uniformly h-stable, then there exist $\eta \geq 0$, functions $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$ and $h \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$, such that for all $t, \ \tau \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$, we have

$$\|x(t + \tau, t, x)\| \leq \eta + K(t)\|x\|h(t + \tau)h(t)^{-1},$$

We define the Lyapunov function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ as

$$V(t, x) = \sup_{\tau \geq 0} \Big(h(t + \tau)^{-1}h(t)(\|x(t + \tau, t, x\| - \eta)\Big) + \eta,$$

where $x(t+\tau,t,x)$ is the solution of system (2.1) through $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$.
To prove the Lipschitzian of $V(t,x)$, let $(t,x),\ (t,y)\in\mathbb{R}_+\times\mathbb{R}^n$, then one has

$$\begin{aligned}\Big|V(t,x)-V(t,y)\Big| &= \Big|\sup_{\tau\geq 0}\Big(\|x(t+\tau,t,x)\|-\eta\Big)h(t+\tau)^{-1}h(t)\\ &- \sup_{\tau\geq 0}\Big(\|x(t+\tau,t,y)\|-\eta\Big)h(t+\tau)^{-1}h(t)\Big|\\ &\leq \sup_{\tau\geq 0}\Big\|x(t+\tau,t,x)-x(t+\tau,t,y)\Big\|h(t+\tau)^{-1}h(t).\end{aligned}$$

Since for each x and y in a convex subset $D\subset\mathbb{R}^n$, thus by Lemma 2.4, we obtain the following inequalities

$$\begin{aligned}\Big|V(t,x)-V(t,y)\Big| &\leq \|x-y\|h(t+\tau)^{-1}h(t)\sup_{\rho\in D}\|\phi(t+\tau,t,\rho)\|\\ &\leq K(t)h(t+\tau)^{-1}h(t)h(t)^{-1}h(t+\tau)\|x-y\|\\ &= K(t)\|x-y\|,\end{aligned}$$

where $K\in\mathcal{BC}(\mathbb{R}_+,\mathbb{R}_+^\star)$ and D is a convex subset of $\mathbb{R}^n$ containing x and y. Then, the first inequality holds.
We show next the continuity of $V(t,x)$. For that, let $T\geq 0$, then

$$\begin{aligned}\Big|V(t+T,\hat{x})-V(t,x)\Big| &\leq \Big|V(t+T,\hat{x})-V(t+T,x)\Big|\\ &+ \Big|V(t+T,x)-V\big(t+T,x(t+T,x(t+T,t,x))\big)\Big|\\ &+ \Big|V\big(t+T,x(t+T,x(t+T,t,x))\big)-V(t,x)\Big|.\end{aligned}$$

Since $V(t,x)$ is Lipschitzian in x and $x(t+T,t,x)$ is continuous in T, the first two terms on the right-hand side of the proceeding inequality are small when $\|x-\hat{x}\|$ and T are small.
Let us consider the third term. We have,

$$\begin{aligned}\Big|V\big(t+T,x(t+T,t,x)\big)-V(t,x)\Big| &= \Big|\sup_{\tau\geq 0}\Big(\|x(t+\tau+T,t+T,x(t+T,t,x))\|\\ &\quad -\eta\Big)h(t+\tau+T)^{-1}h(t+T)\\ &- \sup_{\tau\geq 0}\Big(\|x(t+\tau,t,x)\|-\eta\Big)h(t+\tau)^{-1}h(t)\Big|\\ &= \Big|\sup_{\tau\geq T}\Big(\|x(t+\tau,t,x)\|-\eta\Big)h(t+\tau)^{-1}\\ &\quad h(t+T)-\sup_{\tau\geq 0}\Big(\|x(t+\tau,t,x)\|-\eta\Big)\\ &\quad h(t+\tau)^{-1}h(t)\Big|.\end{aligned}$$

Put,

$$\alpha(T)=\sup_{\tau\geq T}\Big(\|x(t+\tau,t,x)\|-\eta\Big)h(t+\tau)^{-1}h(t+T).$$

We notice that, the function $\alpha(T)$ is non-decreasing and since $\Big(\|x(t+\tau,t,x)\| - \eta\Big)h(t+\tau)^{-1}h(t)$ is a bounded continuous function for all $\tau \geq 0$, then $\alpha(T) \to \alpha(0)$ as $T \to 0$. Hence,

$$\Big|V\big(t+T, x(t+T,t,x)\big) - V(t,x)\Big| = \Big|\alpha(T) - \alpha(0)\Big|$$

implies that the third term tends to zero as $T \to 0^+$. Consequently, the continuity of $V(t,x)$ is satisfied. On the other hand, we have

$$V(t,x) = \sup_{\tau\geq 0}\Big(h(t+\tau)^{-1}h(t)(\|x(t+\tau,t,x\| - \eta)\Big) + \eta \geq (\|x(t,t,x)\| - \eta) + \eta = \|x\|.$$

In addition,

$$V(t,x) \leq \Big(h(t+\tau)^{-1}h(t)\big(K(t)\|x\|h(t+\tau)h(t)^{-1} + \eta - \eta\big)\Big) + \eta = K(t)\|x\| + \eta.$$

Hence, the second property of the theorem is satisfied.
The last property can be proved using the uniqueness of solutions and the definition of generalized practical h-stability.

$$\begin{aligned}
D^+V(t,x(t)) &= \limsup_{T\to 0^+}\frac{1}{T}\Big[V(t+T,x(t+T,t,x)) - V(t,x)\Big]\\
&= \limsup_{T\to 0^+}\frac{1}{T}\Big[\sup_{\tau\geq 0}\Big(h(t+\tau+T)^{-1}h(t+T)\\
&\quad \big(\|x(t+\tau+T,t+T,x(t+T,t,x))\| - \eta\big)\Big)\\
&- \sup_{\tau\geq 0}\Big(h(t+\tau)^{-1}h(t)\big(\|x(t+\tau,t,x)\| - \eta\Big)\Big]\\
&= \limsup_{T\to 0^+}\frac{1}{T}\Big[\sup_{\tau\geq T}\Big(h(t+\tau)^{-1}h(t+T)\big(\|x(t+\tau,t,x)\| - \eta\big)\Big)\\
&- \sup_{\tau\geq 0}\Big(h(t+\tau)^{-1}h(t)\big(\|x(t+\tau,t,x)\| - \eta\Big)\Big]\\
&\leq \limsup_{T\to 0^+}\frac{1}{T}\Big[\Big(h(t+T)h(t)^{-1} - 1\Big)\sup_{\tau\geq 0}\Big(h(t+\tau)^{-1}h(t)\\
&\quad \big(\|x(t+\tau,t,x)\| - \eta\big)\Big)\\
&+ \eta\big(h(t+T)h(t)^{-1} - 1\big) - \eta\big(h(t+T)h(t)^{-1} - 1\big)\Big]\\
&\leq h'(t)h(t)^{-1}V(t,x) - \eta h'(t)h(t)^{-1}.
\end{aligned}$$

Since, for small $T > 0$

$$\begin{aligned}
V(t+T,x+Tf(t,x)) - V(t,x) &\leq \big|V\big(t+T,x+Tf(t,x)\big)\\
&- V\big(t+T,x(t+T,t,x)\big)\big|\\
&+ \big|V\big(t+T,x(t+T,t,x)\big) - V(t,x)\big|\\
&\leq K(t)\|x+Tf(t,x) - x(t+T,t,x)\|\\
&+ \big|V\big(t+T,x(t+T,t,x)\big) - V(t,x)\big|,
\end{aligned}$$

therefore

$$D^+V_{(2.1)}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - \eta h'(t)h(t)^{-1}.$$

This ends the proof. □

Next, we use Lyapunov's indirect method and the Generalized Gronwall-Bellman Inequality to show the global generalized practical uniform h-stability of perturbed systems.

Theorem 4.2. *Consider the perturbed system:*

$$\dot{x} = f(t,x) + p(t,x), \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{4.1}$$

where $p \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x and satisfies the following condition:

$$\|p(t,x)\| \leq \vartheta(t)\|x\| + \nu(t), \quad \forall\, x \in \mathbb{R}^n, \qquad \forall\, t \geq 0, \tag{4.2}$$

where ϑ and ν are non-negative continuous and integrable functions on $\mathbb{R}_+$. Let the origin be globally generalized practically uniformly h-stable of system (2.1) with $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function and the solution $z = 0$ of system (2.2) is globally generalized uniformly h-stable. Then, the perturbed system (4.1) is globally generalized practically uniformly h-stable.

Proof. From Theorem 4.1, there exists a Lyapunov function $V(t,x)$ satisfies the properties in that theorem. Then, we have

$$\begin{aligned} D^+V_{(4.1)}(t,x) &\leq D^+V_{(2.1)}(t,x) + K(t)\|p(t,x)\| \\ &\leq h'(t)h(t)^{-1}V(t,x) - \eta h'(t)h(t)^{-1} + K(t)\vartheta(t)\|x\| + K(t)\nu(t) \\ &= \Big(h'(t)h(t)^{-1} + K(t)\vartheta(t)\Big)V(t,x) + K(t)\nu(t) - \eta h'(t)h(t)^{-1}. \end{aligned}$$

By applying the comparison principle, where

$$\dot{u}(t) = \Big(h'(t)h(t)^{-1} + K(t)\vartheta(t)\Big)u(t) + K(t)\nu(t) - \eta h'(t)h(t)^{-1}, \quad u(t_0) = u_0, \tag{4.3}$$

for $t \geq t_0 \geq 0$, such that $V(t_0, x_0) \leq u_0 \leq K(t_0)\|x_0\| + \eta$ and using the Generalized Gronwall-Bellman Inequality, the maximal solution of (4.3) is given by:

$$\begin{aligned} u(t) &\leq u_0 h(t)h(t_0)^{-1}\exp\left(\int_{t_0}^t K(s)\vartheta(s)ds\right) + h(t)\int_{t_0}^t h(s)^{-1} \\ &\quad \exp\left(\int_s^t K(\tau)\vartheta(\tau)d\tau\right)\Big(K(s)\nu(s) - \eta h'(s)h(s)^{-1}\Big)ds \\ &\leq K(t_0)e^{\|K\|_\infty M_\vartheta}\|x_0\|h(t)h(t_0)^{-1} + \eta e^{\|K\|_\infty M_\vartheta}h(t)h(t_0)^{-1} + e^{\|K\|_\infty M_\vartheta} \\ &\quad \big(\|K\|_\infty M_\nu + \eta\big) - \eta e^{\|K\|_\infty M_\vartheta}h(t)h(t_0)^{-1} \\ &= e^{\|K\|_\infty M_\vartheta}\big(\|K\|_\infty M_\nu + \eta\big) + K(t_0)e^{\|K\|_\infty M_\vartheta}\|x_0\|h(t)h(t_0)^{-1}, \end{aligned}$$

where $\|K\|_\infty = \sup_{t \in \mathbb{R}_+}\{K(t)\}$, $M_\vartheta = \int_0^\infty \vartheta(t)dt$ and $M_\nu = \int_0^\infty \nu(t)dt$.

Therefore, for all $x_0 \in \mathbb{R}^n$ and all $t \geq t_0$, the solution $x(t)$ of system (4.1) satisfies

$$\|x(t)\| \leq \eta_1 + K_1(t_0)\|x_0\|h(t)h(t_0)^{-1},$$

with $K_1(t_0) = K(t_0)e^{\|K\|_\infty M_\vartheta}$ and $\eta_1 = e^{\|K\|_\infty M_\vartheta}\big(\|K\|_\infty M_\nu + \eta\big)$. Consequently, the system (4.1) is globally generalized practically uniformly h-stable. □

Proposition 4.3. *Consider the perturbed system* (4.1). *If the nonlinear system* (2.1) *is globally practically uniformly h-stable with $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function, the solution $z = 0$ of system* (2.2) *is globally uniformly h-stable and $p(t,x)$ satisfies the condition* (4.2) *where ϑ and ν are non-negative continuous and integrable functions on $\mathbb{R}_+$. Then, the system* (4.1) *is globally practically uniformly h-stable.*

A particular case of Theorem 4.2 is given in the following corollary.

Corollary 4.4. *Consider the perturbed system* (4.1). *Assume that the system* (2.1) *is globally generalized practically uniformly h-stable with $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function, the solution $z = 0$ of system* (2.2) *is globally generalized h-stable. Suppose that the perturbed term $p(t,x)$ satisfies the condition:*

$$\|p(t,x)\| \leq \gamma(t), \qquad \forall\, t \geq 0, \tag{4.4}$$

where γ is a non-negative continuous and integrable function on $\mathbb{R}_+$. Hence, the system (4.1) *is globally generalized practically uniformly h-stable.*

5. Applications

In this section, the systematic method developed and applied to various diseases models to illustrate several aspects of these methods.

5.1. SIR model

We consider the solution $x(t) = (S(t), I(t), R(t))^T \in \mathbb{R}_+^3$ of the ordinary differential equation:

$$\begin{cases} \dot{S} = B(t) - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \dot{I} = \beta IS - \nu I - \mu(t)I, \\ \dot{R} = \nu I - \mu(t)R, \end{cases} \tag{5.1}$$

defined for any $x(t_0) = (S(t_0), I(t_0), R(t_0))^T \in \mathbb{R}_+^3$, any continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ and any continuous and integrable function $B : \mathbb{R}_+ \to \mathbb{R}_+$. Here, the variable $S(t)$ describes the number of susceptible population and $I(t)$ is the number of infected individuals, while $R(t)$ is of individuals recovered with immunity. $B(t)$ is the newborn/ immigration rate. $\mu(t)$ is the death rate. The positive number β and ν are parameters describing the contact rate and the recovery rate, respectively.
Select the appropriate state variable as $x_1 = S$, $x_2 = I$ and $x_3 = R$. Thus, the equations describing a SIR Model can be written as

$$\begin{cases} \dot{x}_1 = B(t) - \mu(t)x_1 - \beta x_2 x_1, \\ \dot{x}_2 = \beta x_2 x_1 - \nu x_2 - \mu(t)x_2, \\ \dot{x}_3 = \nu x_2 - \mu(t)x_3. \end{cases} \tag{5.2}$$

The state model (5.1) is equivalent to system (4.1), where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3_+$ and

$$f(t,x) = \begin{pmatrix} -\mu(t)x_1 - \beta x_2 x_1 \\ \beta x_2 x_1 - \nu x_2 - \mu(t)x_2 \\ \nu x_2 - \mu(t)x_3 \end{pmatrix}.$$

and $p(t,x) = (B(t), 0, 0)^T$. We consider the following Lyaunov function

$$V(t,x) = x_1(t) + x_2(t) + x_3(t).$$

The derivative of V in t along the solution of the system $\dot{x} = f(t,x)$ leads to

$$\begin{aligned} D^+V(t,x) &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3 \\ &= -\mu(t)V(t,x). \end{aligned}$$

Then, the nominal system $\dot{x} = f(t,x)$ is uniformly h-stable with $K(t) = 1$ and

$$h(t) = \exp\left(-\int_0^t \mu(s)ds\right).$$

On the other hand, the perturbed term $p(t,x)$ satisfies the condition (4.4) with $\gamma(t) = B(t)$, which is non-negative, continuous and integrable function on $\mathbb{R}_+$. Thus, all assumptions of Corollary 4.4 are satisfied. We conclude that the SIR Model (5.1) is practically uniformly h-stable.

From Figure 1, we can see that the SIR Model (5.1) is practically uniformly h-stable with $h(t) = \dfrac{1}{1+t}$. In this case, for integrable newborn/immigration rate $B(t)$ the convergence of $I(t)$, $S(t)$ and $R(t)$ to a neighborhood of the origin are guaranteed where the initial values $S(0) = 600$, $I(0) = 100$ and $R(0) = 60$. The parameters of SIR Model (5.1) are $\beta = 0.0002$ and $\nu = 0.020$.

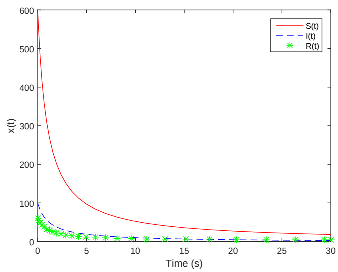

FIGURE 1. *Populations of the SIR model with* $B(t) = \frac{1}{1+t^2}$ *and* $\mu(t) = \frac{1}{1+t}$.

Remark 5.1. If we suppose that $\mu(t) = \tilde{\mu}$ is constant and $B(t)$ is a measurable and locally essentially bounded, then by using Theorem 3.3 the SIR Model (5.1) is practically uniformly h-stable with $h(t) = \exp(-\tilde{\mu}t)$.

5.2. SEIS model

Let $x(t) = (S(t), E(t), I(t))^T \in \mathbb{R}^3_+$ for

$$\begin{cases} \dot{S} = B(t) - \mu(t)S - \beta IS + \nu I, \qquad t \geq t_0 \geq 0, \\ \dot{E} = \beta IS - \varepsilon E - \mu(t)E, \\ \dot{I} = \varepsilon E - \nu I - \mu(t)I, \end{cases} \tag{5.3}$$

with $x(t_0) = (S(t_0), E(t_0), I(t_0))^T \in \mathbb{R}^3_+$, any continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ and any continuous and integrable function $B : \mathbb{R}_+ \to \mathbb{R}_+$. The variable E individuals move to the class I at the rate ε. Model (5.3) is refereed to as the SEIS model [15] when $\mu(t)$ is constant. The SEIS model is known to be useful for describing diseases which have non-negligible incubation periods and also consider infections which do not give long lasting immunity and recovered individuals become susceptible again. Select the appropriate state variable as $x_1 = S$, $x_2 = E$ and $x_3 = I$. Thus, the equations describing a SEIS model can be written as:

$$\begin{cases} \dot{x}_1 = B(t) - \mu(t)x_1 - \beta x_3 x_1 + \nu x_3, \\ \dot{x}_2 = \beta x_3 x_1 - \varepsilon x_2 - \mu(t)x_2, \\ \dot{x}_3 = \varepsilon x_2 - \nu x_3 - \mu(t)x_3. \end{cases} \tag{5.4}$$

The state model (5.3) is equivalent to system (4.1), where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3_+$,

$$f(t, x) = \begin{pmatrix} -\mu(t)x_1 - \beta x_3 x_1 + \nu x_3 \\ \beta x_3 x_1 - \varepsilon x_2 - \mu(t)x_2 \\ \varepsilon x_2 - \nu x_3 - \mu(t)x_3 \end{pmatrix}$$

and $p(t, x) = (B(t), 0, 0)^T$. Let

$$V(t, x) = x_1(t) + x_2(t) + x_3(t).$$

The derivative of V in t along the solution of the nominal system $\dot{x} = f(t, x)$ leads to

$$\begin{aligned} D^+V(t, x) &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3 \\ &= -\mu(t)V(t, x). \end{aligned}$$

Then, the nominal system $\dot{x} = f(t, x)$ is uniformly h-stable with $K(t) = 1$ and $h(t) = \exp\left(-\int_0^t \mu(s)ds\right)$. On the other hand, the perturbed term $p(t, x)$ satisfies the condition (4.4) with $\gamma(t) = B(t)$ which is non-negative, continuous and integrable function on $\mathbb{R}_+$. We deduce that all hypothesis of Corollary 4.4 are satisfied. Therefore, the SEIS model (5.3) is practically uniformly h-stable.
From Figure 2, we can see that the SEIS model (5.3) is practically uniformly h-stable with $h(t) = \exp(-t^2)$. The parameters of SEIS model are $\varepsilon = 0.15$, $\beta = 0.002$ and $\nu = 0.032$ with the initial state is $\Big(S(0), E(0), I(0)\Big) = (100, 60, 200)$.

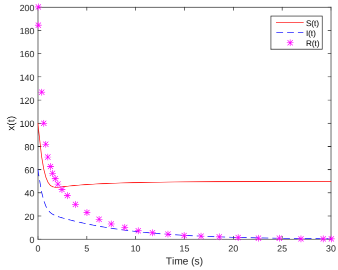

FIGURE 2. *Populations of the SEIS model with* $B(t) = \frac{1}{1+t^2}$ *and* $\mu(t) = t$.

5.3. SEIR model

Let $x(t) = (S(t), E(t), I(t), R(t)^T \in \mathbb{R}^4_+$ for

$$\begin{cases} \dot{S} = B(t) - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \\ \dot{E} = \beta IS - \varepsilon E - \mu(t)E, \\ \\ \dot{I} = \varepsilon E - \nu I - \mu(t)I, \\ \\ \dot{R} = \nu I - \mu(t)R, \end{cases} \tag{5.5}$$

which called the SEIR model. The analysis of SEIR model is almost the same as the SIR Model.

Select the appropriate state variable as $x_1 = S$, $x_2 = E$, $x_3 = I$ and $x_4 = R$. Thus, the equations describing a SEIR Model can be written as:

$$\begin{cases} \dot{x}_1 = B(t) - \mu(t)x_1 - \beta x_3 x_1, \\ \dot{x}_2 = \beta x_3 x_1 - \varepsilon x_2 - \mu(t)x_2, \\ \dot{x}_3 = \varepsilon x_2 - \nu x_3 - \mu(t)x_3, \\ \dot{x}_4 = \nu x_3 - \mu(t)x_4. \end{cases} \tag{5.6}$$

The state model (5.5) is equivalent to system (4.1), where $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4_+$ and

$$f(t, x) = \begin{pmatrix} -\mu(t)x_1 - \beta x_3 x_1 \\ \beta x_3 x_1 - \varepsilon x_2 - \mu(t)x_2 \\ \varepsilon x_2 - \nu x_3 - \mu(t)x_3 \\ \nu x_3 - \mu(t)x_4 \end{pmatrix}.$$

and $p(t, x) = (B(t), 0, 0, 0)^T$. Let

$$V(t, x) = x_1(t) + x_2(t) + x_3(t) + x_4(t).$$

The derivative of V in t along the solution of the system $\dot{x} = f(t, x)$ leads to

$$\begin{aligned} D^+V(t, x) &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4 \\ &= -\mu(t)V(t, x). \end{aligned}$$

Then, the nominal system $\dot{x} = f(t,x)$ is uniformly h-stable with $K(t) = 1$ and $h(t) = \exp\left(-\int_0^t \mu(s)ds\right)$. On the other hand, the perturbed term $p(t,x)$ satisfies the condition (4.4) with $\gamma(t) = B(t)$ which is non-negative, continuous and integrable function on $\mathbb{R}_+$. Thus, all assumptions of Corollary 4.4 are satisfied. We conclude that the SEIR Model (5.5) is practically uniformly h-stable.

5.4. Vaccination models

One way of eradicating infections diseases is to vaccinate newborns and entering individuals. Let $P \in (0,1)$ the vaccination fraction. Considering a vaccine giving lifelong immunity [11], the SIR model can be modified as

$$\begin{cases} \dot{S} = B(t)(1-P) - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \dot{I} = \beta IS - \nu I - \mu(t)I, \\ \dot{R} = \nu I - \mu(t)R, \\ \dot{A} = B(t)P - \mu(t)A, \end{cases} \tag{5.7}$$

where A is the number of vaccinated individuals.
Select the appropriate state variable as $x_1 = S$, and $x_2 = I$, $x_3 = R$ and $x_4 = A$. Thus, the equations describing a SIR Model can be written as:

$$\begin{cases} \dot{x}_1 = B(t)(1-P) - \mu(t)x_1 - \beta x_2 x_1, \\ \dot{x}_2 = \beta x_2 x_1 - \nu x_2 - \mu(t)x_2, \\ \dot{x}_3 = \nu x_2 - \mu(t)x_3. \\ \dot{x}_4 = B(t)P - \mu(t)x_4. \end{cases} \tag{5.8}$$

The state model (5.8) is equivalent to system (4.1), where $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4_+$ and

$$f(t,x) = \begin{pmatrix} B(t)(1-P) - \mu(t)x_1 - \beta x_2 x_1 \\ \beta x_2 x_1 - \nu x_2 - \mu(t)x_2 \\ \nu x_2 - \mu(t)x_3 \\ B(t)P - \mu(t)x_4 \end{pmatrix}.$$

and $p(t,x) = (B(t), 0, 0, 0)^T$. Let

$$V(t,x) = x_1(t) + x_2(t) + x_3(t) + x_4(t).$$

The derivative of V in t along the solution of the system $\dot{x} = f(t,x)$ leads to

$$\begin{aligned} D^+V(t,x) &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4(t) \\ &= -\mu(t)V(t,x). \end{aligned}$$

Then, the nominal system $\dot{x} = f(t,x)$ is uniformly h-stable with $K(t) = 1$ and $h(t) = \exp\left(-\int_0^t \mu(s)ds\right)$. Moreover, the perturbed term $p(t,x)$ satisfies the condition (4.4) with $\gamma(t) = B(t)$ which is non-negative, continuous and integrable function on $\mathbb{R}_+$. Thus, all assumptions of Corollary 4.4 are satisfied. We conclude that the SIR Model (5.8) is practically uniformly h-stable.

Figure 3 is the simulation result of the SIR Model (5.8) with the initial values $S(0) = 600$, $I(0) = 150$, $R(0) = 70$ and $A(0) = 50$. The parameters of SIR Model are $\beta = 0.0002$, $\nu = 0.035$ and $P = 0.5$.
From the simulation, we see that the states trajectories converge eventually to a small neighborhood of the origin.

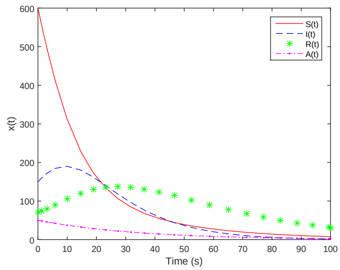

FIGURE 3. *Populations of Vaccination model with* $B(t) = \exp(-t)$ *and* $\mu = 0.0015$.

Another way to model the newborn vaccination within the SIR model is

$$\begin{cases} \dot{S} = B(t)(1-P) - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \dot{I} = \beta IS - \nu I - \mu(t)I, \\ \dot{R} = \nu I - \mu(t)R + B(t)P. \end{cases} \tag{5.9}$$

In the same way as of the model (5.9), we have the newborn vaccination model is practically uniformly h-stable with $h(t) = \exp\left(-\int_0^t \mu(s)ds\right)$.
If non-newborns/non-immigrants are vaccinated [19, 22] with a continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, a way to modifier the SIR model is

$$\begin{cases} \dot{S} = B(t) - \rho S - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \dot{I} = \beta IS - \nu I + \mu(t)I, \\ \dot{R} = \nu I - \mu(t)R, \\ \dot{A} = \rho S - \mu(t)A, \end{cases} \tag{5.10}$$

where $\rho \in \mathbb{R}_+$ is the vaccination rate. The analysis is the same as of the model (5.9), the model (5.10) also is practically uniformly h-stable with

$$h(t) = \exp\left(-\int_0^t \mu(s)ds\right).$$

6. Conclusion

Non-differentiable Lyapunov-like function is proposed for obtaining the global generalized practical uniform h-stability of the nonlinear system. Sufficient conditions are given to study the practical approach of nonlinear time-varying perturbed systems using Lyapunov's indirect method, the comparison principle and some generalizations of Gronwall's inequality. This results can be viewed as an extension of [4] and [17]. The models considered in this paper are practically uniformly h-stable. This conclusion is valid for non-autonomous systems.

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q-Deformed and λ-parametrized A-generalized logistic function induced Banach space valued multivariate multi layer neural network approximations

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Abstract. Here we research the multivariate quantitative approximation of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We investigate also the case of approximation by iterated multilayer neural network operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its partial derivatives. Our multivariate operators are defined by using a multidimensional density function induced by a q-deformed and λ-parametrized A-generalized logistic function, which is a sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network are with one or multi hidden layers.

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1. Introduction

The author in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined

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neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there. The author started with [1].

Motivations for this work are the article [14] of Z. Chen and F. Cao, also by [4]-[13], [15], [16].

Here we perform a q-deformed and λ-parametrized, $q, \lambda > 0$, $A > 1$, A-generalized logistic sigmoid function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$ and also iterated, multi layer approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its partial derivatives and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by the q-deformed and λ-parametrized A-generalized logistic sigmoid function.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is a kind of logistic sigmoid function. About neural networks read [17] - [19].

2. Preliminaries

We consider the q-deformed and λ-parametrized function

$$\varphi_{q,\lambda}(x) = \frac{1}{1 + qA^{-\lambda x}}, \quad x \in \mathbb{R}, \text{ where } q, \lambda > 0, \ A > 1. \tag{2.1}$$

This is an A-generalized logistic type function.

We easily observe that

$$\varphi_{q,\lambda}(+\infty) = 1, \ \ \varphi_{q,\lambda}(-\infty) = 0. \tag{2.2}$$

Furthermore we have

$$1 - \varphi_{\frac{1}{q},\lambda}(-x) = 1 - \frac{1}{1 + \frac{1}{q}A^{\lambda x}} = \frac{1 + \frac{1}{q}A^{\lambda x} - 1}{1 + \frac{1}{q}A^{\lambda x}}$$

$$= \frac{\frac{1}{q}A^{\lambda x}}{1+\frac{1}{q}A^{\lambda x}} = \frac{1}{\frac{1}{\frac{1}{q}A^{\lambda x}}+1} = \frac{1}{1+qA^{-\lambda x}} = \varphi_{q,\lambda}(x),$$

proving

$$\varphi_{q,\lambda}(x) = 1 - \varphi_{\frac{1}{q},\lambda}(-x). \tag{2.3}$$

We also have that

$$\varphi_{q,\lambda}(0) = \frac{1}{1+q}. \tag{2.4}$$

Consider the activation function

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}. \tag{2.5}$$

Then

$$G_{q,\lambda}(-x) = \frac{1}{2}\left(\varphi_{q,\lambda}(-x+1) - \varphi_{q,\lambda}(-x-1)\right)$$
$$= \frac{1}{2}\Big(1 - \varphi_{\frac{1}{q},\lambda}(x-1) - 1 + \varphi_{\frac{1}{q},\lambda}(x+1)\Big)$$
$$= \frac{1}{2}\Big(\varphi_{\frac{1}{q},\lambda}(x+1) - \varphi_{\frac{1}{q},\lambda}(x-1)\Big) = G_{\frac{1}{q},\lambda}(x). \tag{2.6}$$

That is

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}. \tag{2.7}$$

We have

$$\varphi'_{q,\lambda}(x) = \Big(\left(1+qA^{-\lambda x}\right)^{-1}\Big)'$$
$$= -1\left(1+qA^{-\lambda x}\right)^{-2} q(\ln A)A^{-\lambda x}(-\lambda) = q\lambda(\ln A)\left(1+qA^{-\lambda x}\right)^{-2}A^{-\lambda x} > 0. \tag{2.8}$$

So that $\varphi_{q,\lambda}$ is a strictly increasing function over $\mathbb{R}$.
Hence it holds

$$\varphi'_{q,\lambda}(x) = \frac{q\lambda(\ln A)}{\left(1+qA^{-\lambda x}\right)^2 A^{\lambda x}}$$
$$= \frac{q\lambda(\ln A)}{\left(1+q^2A^{-2\lambda x}+2qA^{-\lambda x}\right)A^{\lambda x}} = \frac{q\lambda(\ln A)}{\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)}. \tag{2.9}$$

That is

$$\varphi'_{q,\lambda}(x) = q\lambda(\ln A)\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)^{-1}. \tag{2.10}$$

Therefore it holds

$$\varphi''_{q,\lambda}(x) = q\lambda(\ln A)(-1)\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)^{-2}\left((\ln A)A^{\lambda x}\lambda + q^2(\ln A)A^{-\lambda x}(-\lambda)\right)$$
$$= q\lambda^2(\ln A)^2\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)^{-2}\left(q^2A^{-\lambda x}-A^{\lambda x}\right). \tag{2.11}$$

That is

$$\varphi''_{q,\lambda}(x) = q\lambda^2(\ln A)^2\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)^{-2}\left(q^2A^{-\lambda x}-A^{\lambda x}\right) \in C(\mathbb{R}). \tag{2.12}$$

We have

$$q^2A^{-\lambda x} - A^{\lambda x} > 0, \text{ iff } q^2A^{-\lambda x} > A^{\lambda x}, \text{ iff } q^2 > A^{2\lambda x}, \text{ iff } q > A^{\lambda x},$$
$$\text{iff } \log_A q > \lambda x, \text{ iff } x < \frac{\log_A q}{\lambda}.$$

So, $\varphi''_{q,\lambda}(x) > 0$, for $x < \frac{\log_A q}{\lambda}$ and there $\varphi_{q,\lambda}$ is concave up.

When $x > \frac{\log_A q}{\lambda}$, we have $\varphi''_{q,\lambda}(x) < 0$ and $\varphi_{q,\lambda}$ is concave down.
Of course

$$\varphi''_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = 0.$$

So, $\varphi_{q,\lambda}$ is a sigmoid function, see [12].
We have that

$$G'_{q,\lambda}(x) = \frac{1}{2}\left(\varphi'_{q,\lambda}(x+1) - \varphi'_{q,\lambda}(x-1)\right).$$

We got that $\varphi'_{q,\lambda}$ is strictly increasing for $x < \frac{\log_A q}{\lambda}$. Let $x < \frac{\log_A q}{\lambda} - 1$, then

$$x - 1 < x + 1 < \frac{\log_A q}{\lambda}.$$

Hence $\varphi'_{q,\lambda}(x+1) > \varphi'_{q,\lambda}(x-1)$. Thus $G'_{q,\lambda} > 0$, i.e. $G_{q,\lambda}$ is strictly increasing over $\left(-\infty, \frac{\log_A q}{\lambda} - 1\right)$.

Let now $x > \frac{\log_A q}{\lambda} + 1$, then $x+1 > x-1 > \frac{\log_A q}{\lambda}$, and $\varphi'_{q,\lambda}(x+1) < \varphi'_{q,\lambda}(x-1)$, by $\varphi'_{q,\lambda}$ being strictly decreasing over $\left(\frac{\log_A q}{\lambda}, +\infty\right)$. Hence $G'_{q,\lambda} < 0$, and $G_{q,\lambda}$ is strictly decreasing over $\left(\frac{\log_A q}{\lambda}, +\infty\right)$.

Let now $\frac{\log_A q}{\lambda} - 1 \leq x \leq \frac{\log_A q}{\lambda} + 1$. We have that

$$G''_{q,\lambda}(x) = \frac{1}{2}\left(\varphi''_{q,\lambda}(x+1) - \varphi''_{q,\lambda}(x-1)\right)$$

$$= \frac{q\lambda^2(\ln A)^2}{2}\left[\frac{\left(q^2A^{-\lambda(x+1)} - A^{\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2A^{-\lambda(x+1)} + 2q\right)^2} - \frac{\left(q^2A^{-\lambda(x-1)} - A^{\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2A^{-\lambda(x-1)} + 2q\right)^2}\right] \tag{2.13}$$

$$= \frac{q\lambda^2(\ln A)^2}{2}\left[\frac{\left(q^2 - A^{2\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2A^{-\lambda(x+1)} + 2q\right)^2 A^{\lambda(x+1)}} - \frac{\left(q^2 - A^{2\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2A^{-\lambda(x-1)} + 2q\right)^2 A^{\lambda(x-1)}}\right]$$

$$= \frac{q\lambda^2(\ln A)^2}{2}\left[\frac{\left(q - A^{\lambda(x+1)}\right)\left(q + A^{\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2A^{-\lambda(x+1)} + 2q\right)^2 A^{\lambda(x+1)}} - \frac{\left(q - A^{\lambda(x-1)}\right)\left(q + A^{\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2A^{-\lambda(x-1)} + 2q\right)^2 A^{\lambda(x-1)}}\right].$$

By $\frac{\log_A q}{\lambda} \leq x + 1 \Leftrightarrow \log_A q \leq \lambda(x+1) \Leftrightarrow q \leq A^{\lambda(x+1)} \Leftrightarrow q - A^{\lambda(x+1)} \leq 0$.

By $x \leq \frac{\log_A q}{\lambda} + 1 \Leftrightarrow x - 1 \leq \frac{\log_A q}{\lambda} \Leftrightarrow \lambda(x-1) \leq \log_A q \Leftrightarrow A^{\lambda(x-1)} \leq q \Leftrightarrow q - A^{\lambda(x-1)} \geq 0$.

Clearly, when $\frac{\log_A q}{\lambda} - 1 \leq x \leq \frac{\log_A q}{\lambda} + 1$ by the above we get that $G''_{q,\lambda}(x) \leq 0$, that is $G''_{q,\lambda}$ is concave down there.

Clearly $G_{q,\lambda}$ is strictly concave down over $\left(\frac{\log_A q}{\lambda} - 1, \frac{\log_A q}{\lambda} + 1\right)$.

Overall $G_{q,\lambda}$ is a bell-shaped function over $\mathbb{R}$.

Of course it holds $G''_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) < 0$.

We have that

$$G'_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{1}{2}\left(\varphi'_{q,\lambda}\left(\frac{\log_A q}{\lambda}+1\right) - \varphi'_{q,\lambda}\left(\frac{\log_A q}{\lambda}-1\right)\right)$$

$$= \frac{q\lambda(\ln A)}{2}\left[\frac{1}{A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}+2q} - \frac{1}{A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+2q}\right] \tag{2.14}$$

$$= \frac{q\lambda(\ln A)}{2}\left[\frac{A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}-A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)}-q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}}{\left(A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}+2q\right)\left(A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+2q\right)}\right] \tag{2.15}$$

$$= \frac{q\lambda(\ln A)}{2}\left[\frac{qA^{-\lambda}+q^2q^{-1}A^{\lambda}-qA^{\lambda}-q^2q^{-1}A^{-\lambda}}{\left(qA^{\lambda}+q^2q^{-1}A^{-\lambda}+2q\right)\left(qA^{\lambda}+q^2q^{-1}A^{-\lambda}+2q\right)}\right]$$

$$= \frac{q\lambda(\ln A)}{2}\left[\frac{qA^{-\lambda}+qA^{\lambda}-qA^{\lambda}-qA^{-\lambda}}{\left(qA^{\lambda}+qA^{-\lambda}+2q\right)\left(qA^{\lambda}+qA^{-\lambda}+2q\right)}\right] = 0. \tag{2.16}$$

So $\frac{\log_A q}{\lambda}$ is the only critical number of $G_{q,\lambda}$ over $\mathbb{R}$. Therefore $G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right)$ is the maximum of $G_{q,\lambda}$.

We calculate it:

We have that

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{1}{2}\left(\varphi_{q,\lambda}\left(\frac{\log_A q}{\lambda}+1\right) - \varphi_{q,\lambda}\left(\frac{\log_A q}{\lambda}-1\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{1+qA^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}} - \frac{1}{1+qA^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}}\right) \tag{2.17}$$

$$= \frac{1}{2}\left(\frac{1}{1+qq^{-1}A^{-\lambda}} - \frac{1}{1+qq^{-1}A^{\lambda}}\right) = \frac{1}{2}\left(\frac{1}{1+A^{-\lambda}} - \frac{1}{1+A^{\lambda}}\right)$$

$$= \frac{1}{2}\left(\frac{A^{\lambda}-A^{-\lambda}}{\left(1+A^{-\lambda}\right)\left(1+A^{\lambda}\right)}\right) = \frac{A^{\lambda}-1}{2\left(A^{\lambda}+1\right)}.$$

The global maximum of $G_{q,\lambda}$ is

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^{\lambda}-1}{2\left(A^{\lambda}+1\right)}. \tag{2.18}$$

Finally we have that

$$\lim_{x\to+\infty} G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(+\infty) - \varphi_{q,\lambda}(+\infty)\right) = 0, \tag{2.19}$$

and

$$\lim_{x\to-\infty} G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(-\infty) - \varphi_{q,\lambda}(-\infty)\right) = 0. \tag{2.20}$$

Consequently the x-axis is the horizontal asymptote of $G_{q,\lambda}$. Of course $G_{q,\lambda}(x) > 0$, $\forall\, x \in \mathbb{R}$.

We need

Theorem 2.1. *It holds*

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x-i) = 1, \ \ \forall\ x \in \mathbb{R}, \forall\ q, \lambda > 0,\ A > 1. \tag{2.21}$$

Proof. We observe that

$$\sum_{i=-\infty}^{\infty} \left(\varphi_{q,\lambda}(x-i) - \varphi_{q,\lambda}(x-1-i)\right)$$

$$= \sum_{i=0}^{\infty} \left(\varphi_{q,\lambda}(x-i) - \varphi_{q,\lambda}(x-1-i)\right) + \sum_{i=-\infty}^{-1} \left(\varphi_{q,\lambda}(x-i) - \varphi_{q,\lambda}(x-1-i)\right).$$

Furthermore ($\lambda \in \mathbb{Z}^+$)

$$\sum_{i=0}^{\infty} \left(\varphi_{q,\lambda}(x-i) - \varphi_{q,\lambda}(x-1-i)\right) \tag{2.22}$$

$$= \lim_{\lambda \to \infty} \sum_{i=0}^{\lambda^*} \left(\varphi_{q,\lambda}(x-i) - \varphi_{q,\lambda}(x-1-i)\right) \quad \text{(telescoping sum)}$$

$$= \lim_{\lambda^* \to \infty} \left(\varphi_{q,\lambda}(x) - \varphi_{q,\lambda}(x-(\lambda^*+1))\right) = \varphi_{q,\lambda}(x).$$

Similarly, it holds

$$\sum_{i=-\infty}^{-1} \left(\varphi_{q,\lambda}(x-i) - \varphi_{q,\lambda}(x-1-i)\right) = \lim_{\lambda^* \to \infty} \sum_{i=-\lambda^*}^{-1} \left(\varphi_{q,\lambda}(x-i) - \varphi_{q,\lambda}(x-1-i)\right)$$

$$= \lim_{\lambda * \to \infty} \left(\varphi_{q,\lambda}(x+\lambda^*) - \varphi_{q,\lambda}(x)\right) = 1 - \varphi_{q,\lambda}(x). \tag{2.23}$$

Therefore we derive

$$\sum_{i=-\infty}^{\infty} \left(\varphi_{q,\lambda}(x-i) - \varphi_{q,\lambda}(x-1-i)\right) = 1, \ \ \forall\ x \in \mathbb{R}, \tag{2.24}$$

and

$$\sum_{i=-\infty}^{\infty} \left(\varphi_{q,\lambda}(x+1-i) - \varphi_{q,\lambda}(x-i)\right) = 1, \ \ \forall\ x \in \mathbb{R}. \tag{2.25}$$

Adding the last two equations we get

$$\sum_{i=-\infty}^{\infty} \left(\varphi_{q,\lambda}(x+1-i) - \varphi_{q,\lambda}(x-1-i)\right) = 2, \ \forall x \in \mathbb{R}. \tag{2.26}$$

Since

$$G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right),$$

we have that

$$G_{q,\lambda}(x-i) = \frac{1}{2}\left[\varphi_{q,\lambda}(x+1-i) - \varphi_{q,\lambda}(x-1-i)\right], \tag{2.27}$$

giving

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x-i) = 1.$$

Thus

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(nx-i) = 1, \ \forall\, n \in \mathbb{N}, \ \forall\, x \in \mathbb{R}. \tag{2.28}$$

Similarly, it holds

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x-i) = 1, \ \forall\, x \in \mathbb{R}. \tag{2.29}$$

But $G_{\frac{1}{q},\lambda}(x-i) \overset{(2.7)}{=} G_{q,\lambda}(i-x)$, $\forall\, x \in \mathbb{R}$.
Hence

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(i-x) = 1, \ \forall\, x \in \mathbb{R}, \tag{2.30}$$

and

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(i+x) = 1, \ \forall\, x \in \mathbb{R}. \tag{2.31}$$

□

It follows

Theorem 2.2. *It holds*

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = 1, \quad \lambda, q > 0, \ A > 1. \tag{2.32}$$

Proof. We observe that

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{j}^{j+1} G_{q,\lambda}(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{0}^{1} G_{q,\lambda}(x+j)\, dx \tag{2.33}$$

$$= \int_0^1 \left(\sum_{j=-\infty}^{\infty} G_{q,\lambda}(x+j)\, dx \right) = \int_0^1 1dx = 1.$$

□

So that $G_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0, \ A > 1$.

We need the following result

Theorem 2.3. *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *Then*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} G_{q,\lambda}(nx-k) < \max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda(n^{1-\alpha}-2)}} = \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{2.34}$$

where $q, \lambda > 0, \ A > 1$; $\gamma := \max\left\{q, \frac{1}{q}\right\}$.

Proof. Let $x \geq 1$. That is $0 \leq x-1 < x+1$. Applying the mean value theorem we obtain

$$G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right)$$
$$= \frac{1}{2}\cdot 2\cdot \varphi'_{q,\lambda}(\xi) = q\lambda(\ln A)\frac{A^{-\lambda\xi}}{\left(1+qA^{-\lambda\xi}\right)^2}, \tag{2.35}$$

where $0 \leq x-1 < \xi < x+1$.

Notice that

$$G_{q,\lambda}(x) < q\lambda(\ln A) A^{-\lambda\xi} < q\lambda(\ln A) A^{-\lambda(x-1)}, \quad \forall\, x \geq 1. \tag{2.36}$$

Thus, we observe that

$$\sum_{\substack{k=-\infty \\ :|nx-k|\geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(|nx-k|)$$
$$< q\lambda(\ln A)\sum_{\substack{k=-\infty \\ :|nx-k|\geq n^{1-\alpha}}}^{\infty} A^{-\lambda(|nx-k|-1)} \leq q\lambda(\ln A)\int_{n^{1-\alpha}-1}^{\infty} A^{-\lambda(x-1)}dx$$
$$= q\lambda(\ln A)\int_{n^{1-\alpha}-2}^{\infty} A^{-\lambda z}d(z) \overset{(y=\lambda z)}{=} q(\ln A)\int_{n^{1-\alpha}-2}^{\infty} A^{-y}dy \tag{2.37}$$
$$= (-q)\int_{n^{1-\alpha}-2}^{\infty}\left(-(\ln A)A^{-y}\right)dy = -q\left(\int_{n^{1-\alpha}-2}^{\infty} dA^{-y}\right) = (-q)\left(A^{-y}\Big|_{n^{1-\alpha}-2}^{\infty}\right)$$
$$= q\left(A^{-y}\Big|_{\infty}^{n^{1-\alpha}-2}\right) = q\left(A^{-\lambda z}\Big|_{\infty}^{n^{1-\alpha}-2}\right) = qA^{-\lambda\left(n^{1-\alpha}-2\right)} = \frac{q}{A^{\lambda\left(n^{1-\alpha}-2\right)}}.$$

We have proved that

$$\sum_{\substack{k=-\infty \\ :|nx-k|\geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(|nx-k|) < \frac{q}{A^{\lambda\left(n^{1-\alpha}-2\right)}}, \tag{2.38}$$

for $n^{1-\alpha} > 2$, $n \in \mathbb{N}$; $\lambda, q > 0$, $A > 1$.

If $(nx-k) > 0$, then

$$\sum_{\substack{k=-\infty \\ :|nx-k|\geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(nx-k) < \frac{q}{A^{\lambda\left(n^{1-\alpha}-2\right)}}. \tag{2.39}$$

Similarly, it holds

$$\sum_{\left\{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}\right.}^{\infty} G_{\frac{1}{q},\lambda}\left(|nx-k|\right) < \frac{1}{qA^{\lambda(n^{1-\alpha}-2)}}, \quad \lambda, q > 0, \ A > 1. \tag{2.40}$$

Assume now that $nx - k \leq 0$, then

$$\sum_{\left\{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}\right.}^{\infty} G_{q,\lambda}\left(nx-k\right) \overset{(2.7)}{=} \sum_{\left\{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}\right.}^{\infty} G_{\frac{1}{q},\lambda}\left(-\left(nx-k\right)\right)$$

$$< \frac{1}{qA^{\lambda(n^{1-\alpha}-2)}}, \quad \lambda, q > 0, \ A > 1. \tag{2.41}$$

Therefore, it holds (by (2.39), (2.41))

$$\sum_{\left\{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}\right.}^{\infty} G_{q,\lambda}\left(nx-k\right) < \max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda(n^{1-\alpha}-2)}}, \tag{2.42}$$

where $q, \lambda > 0, \ A > 1$.

The claim is proved. □

Let $\lceil \cdot \rceil$ the ceiling of the number, and $\lfloor \cdot \rfloor$ the integral part of the number.

Theorem 2.4. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0$, $A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}\left(z_0\right) = G_{q,\lambda}\left(0\right)$ and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}\left(nx-k\right)} < \max\left\{\frac{1}{G_{q,\lambda}\left(\lambda_q\right)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: K\left(q\right). \tag{2.43}$$

Proof. By Theorem 2.1 we have

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(x-i\right) = 1, \ \forall \ x \in \mathbb{R}, \ \forall \ \lambda, q > 0; \ A > 1,$$

and by (2.30), we have that

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(i-x\right) = 1, \ \forall \ x \in \mathbb{R}, \ \forall \ \lambda, q > 0; \ A > 1. \tag{2.44}$$

Therefore we get

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(|x-i|\right) = 1, \ \forall \ x \in \mathbb{R}, \ \forall \ \lambda, q > 0; \ A > 1. \tag{2.45}$$

Hence

$$1=\sum_{k=-\infty}^{\infty}G_{q,\lambda}\left(|nx-k|\right)>\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(|nx-k|\right)>G_{q,\lambda}\left(|nx-k_0|\right), \tag{2.46}$$

$\forall\ k_0\in[\lceil na\rceil,\lfloor nb\rfloor]\cap\mathbb{Z}$.

We can choose $k_0\in[\lceil na\rceil,\lfloor nb\rfloor]\cap\mathbb{Z}$, such that $|nx-k_0|<1$.

Notice that $|nx-k_0|$ could be $\lesseqqgtr \frac{\log_A q}{\lambda}$. If $0\leq|nx-k_0|<\frac{\log_A q}{\lambda}$, by down concavity of $G_{q,\lambda}$ over $\mathbb{R}$, we can choose $z\in[\frac{\log_A q}{\lambda},+\infty)$ such that $G_{q,\lambda}\left(|nx-k_0|\right)=G_{q,\lambda}\left(z\right)$. If $|nx-k_0|\geq\frac{\log_A q}{\lambda}$ we just set $z:=|nx-k_0|$. Next, we can choose large enough $\lambda_q>1$, and such that $\lambda_q>z_0>0$ where $G_{q,\lambda}\left(z_0\right)=G_{q,\lambda}\left(0\right)$. Clearly, it is $z\leq z_0<\lambda_q$.

Since $G_{q,\lambda}$ is decreasing over $[\frac{\log_A q}{\lambda},+\infty)$ we get that

$$G_{q,\lambda}\left(|nx-k_0|\right)\geq G_{q,\lambda}\left(\lambda_q\right).$$

Consequently,

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(|nx-k|\right)>G_{q,\lambda}\left(\lambda_q\right),$$

and

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(|nx-k|\right)}<\frac{1}{G_{q,\lambda}\left(\lambda_q\right)}, \tag{2.47}$$

$\forall\ \lambda,q>0;\ A>1$.

If $nx-k>0$, by (2.47), we get

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(nx-k\right)}<\frac{1}{G_{q,\lambda}\left(\lambda_q\right)},\quad\forall\ \lambda,q>0;\ A>1. \tag{2.48}$$

We have also that

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{\frac{1}{q},\lambda}\left(|nx-k|\right)}<\frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)},\quad\forall\ \lambda,q>0;\ A>1. \tag{2.49}$$

Let now $nx-k\leq 0$, then

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(nx-k\right)}\overset{(2.7)}{=}\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{\frac{1}{q},\lambda}\left(-\left(nx-k\right)\right)}\overset{(2.49)}{<}\frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \tag{2.50}$$

$\forall\ \lambda,q>0;\ A>1$.

Consequently, it holds

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)} < \max\left\{\frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\}, \tag{2.51}$$

$\forall\ \lambda, q > 0;\ A > 1.$

The claim is proved. □

We make

Remark 2.5. (i) We also notice for $q \geq 1$ that

$$1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nb-k) = \sum_{k=-\infty}^{\lceil na\rceil-1} G_{q,\lambda}(nb-k) + \sum_{k=\lfloor nb\rfloor+1}^{\infty} G_{q,\lambda}(nb-k)$$

$$> G_{q,\lambda}(nb - \lfloor nb\rfloor - 1) \tag{2.52}$$

(call $\varepsilon := nb - \lfloor nb\rfloor$, $0 \leq \varepsilon < 1$)

$$= G_{q,\lambda}(\varepsilon - 1) = G_{q,\lambda}(-(1-\varepsilon)) = G_{\frac{1}{q},\lambda}(1-\varepsilon)$$

($0 < \frac{1}{q} \leq 1$ and $0 < 1-\varepsilon \leq 1$)

($G_{\frac{1}{q},\lambda}$ is decreasing on $[0,+\infty)$).

$$\geq G_{\frac{1}{q},\lambda}(1) > 0.$$

Therefore

$$\lim_{n\to\infty}\left(1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nb-k)\right) > 0, \quad q \geq 1,\ \lambda > 0;\ A > 1. \tag{2.53}$$

(ii) Let now $0 < q \leq 1$, then we work as in (i), and we have

$$1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nb-k) > G_{\frac{1}{q},\lambda}(1-\varepsilon) \tag{2.54}$$

($\varepsilon := nb - \lfloor nb\rfloor$, $0 \leq \varepsilon < 1$).

That is $\frac{1}{q} \geq 1$, and choose $\overline{\lambda} : 0 < 1-\varepsilon \leq 1 < \overline{\lambda}$, where $\overline{\lambda} > \frac{\log_A \frac{1}{q}}{\lambda} = -\frac{\log_A q}{\lambda}$.

First assume that $1-\varepsilon \in [-\frac{\log_A q}{\lambda}, +\infty)$. Hence

$$G_{\frac{1}{q},\lambda}(1-\varepsilon) > G_{\frac{1}{q},\lambda}\left(\overline{\lambda}\right) > 0, \tag{2.55}$$

by $G_{\frac{1}{q},\lambda}$ being decreasing on $[-\frac{\log_A q}{\lambda}, +\infty)$.

If $0 < 1-\varepsilon < -\frac{\log_A q}{\lambda}$, then we use the concavity-bell shape of $G_{q,\lambda}$.

So, there exists $z_\varepsilon \in \left(-\frac{\log_A q}{\lambda}, +\infty\right)$ such that $G_{\frac{1}{q},\lambda}(1-\varepsilon) = G_{\frac{1}{q},\lambda}(z_\varepsilon)$. We also consider $z_0 \in \left(-\frac{\log_A q}{\lambda}, +\infty\right)$ such that $G_{\frac{1}{q},\lambda}(z_0) = G_{\frac{1}{q},\lambda}(0)$. Clearly it holds $-\frac{\log_A q}{\lambda} < z_\varepsilon \leq z_0$ and we choose $\overline{\lambda} : z_0 < \overline{\lambda}$. Therefore, it holds

$$G_{\frac{1}{q},\lambda}(1-\varepsilon) \geq G_{\frac{1}{q},\lambda}(0) \geq G_{\frac{1}{q},\lambda}\left(\overline{\lambda}\right) > 0,$$

by $G_{\frac{1}{q},\lambda}$ being decreasing on $[-\frac{\log_A q}{\lambda}, +\infty)$.

Again it holds

$$\lim_{n\to\infty}\left(1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nb-k)\right) > 0, \ \ 0 < q \leq 1, \ \lambda > 0, \ A > 1. \tag{2.56}$$

(iii) Similarly, $(q > 0)$

$$\begin{aligned} 1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(na-k) &= \sum_{k=-\infty}^{\lceil na\rceil-1} G_{q,\lambda}(na-k) + \sum_{k=\lfloor nb\rfloor+1}^{\infty} G_{q,\lambda}(na-k) \\ &> G_{q,\lambda}(na - \lceil na\rceil + 1) \\ &\text{(call } \ \eta := \lceil na\rceil - na, \ 0 \leq \eta < 1) \\ &= G_{q,\lambda}(1-\eta), \quad \text{etc.} \end{aligned} \tag{2.57}$$

Acting as in (i), (ii) we derive that

$$\lim_{n\to+\infty}\left(1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(na-k)\right) > 0. \tag{2.58}$$

Conclusion: (i) We have that

$$\lim_{n\to+\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k) \neq 1, \quad \text{for at least some } x \in [a,b], \tag{2.59}$$

where $\lambda, q > 0$.

(ii) Let $[a,b] \subset \mathbb{R}$. For large n we always have $\lceil na\rceil \leq \lfloor nb\rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na\rceil \leq k \leq \lfloor nb\rfloor$. In general it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k) \leq 1. \tag{2.60}$$

We make

Remark 2.6. We introduce

$$Z_{q,\lambda}(x_1,\ldots,x_N) := Z_{q,\lambda}(x) := \prod_{i=1}^{N} G_{q,\lambda}(x_i), \quad x = (x_1,\ldots,x_N) \in \mathbb{R}^N, \tag{2.61}$$

$\lambda, q > 0, \ A > 1, \ N \in \mathbb{N}$.

It has the properties:

(i) $Z_{q,\lambda}(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_{q,\lambda}(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z_{q,\lambda}(x_1-k_1,\ldots,x_N-k_N) = 1, \tag{2.62}$$

where $k := (k_1,\ldots,k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,
hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_{q,\lambda}(nx-k) = 1, \tag{2.63}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,
and

(iv)

$$\int_{\mathbb{R}^N} Z_{q,\lambda}(x)\, dx = 1, \tag{2.64}$$

that is Z_q is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|,\ldots,|x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty,\ldots,\infty)$, $-\infty := (-\infty,\ldots,-\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil,\ldots,\lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor,\ldots,\lfloor nb_N \rfloor), \end{aligned} \tag{2.65}$$

where $a := (a_1,\ldots,a_N)$, $b := (b_1,\ldots,b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_{q,\lambda}(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i) \right)$$

$$= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} G_{q,\lambda}(nx_i-k_i) \right). \tag{2.66}$$

For $0 < \beta^* < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_{q,\lambda}(nx-k)$$

$$= \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^{\beta^*}} \end{cases}}^{\lfloor nb \rfloor} Z_{q,\lambda}(nx-k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\lfloor nb \rfloor} Z_{q,\lambda}(nx-k). \tag{2.67}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^{\beta^*}}$, where $r \in \{1,\ldots,N\}$.

(v) By Theorem 2.3 and as in [10], pp. 379-380, we derive that

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor} Z_{q,\lambda}\left(nx-k\right) < \gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)}, \ \ 0<\beta^*<1, \tag{2.68}$$

with $n\in\mathbb{N}: n^{1-\beta^*}>2,\ x\in\prod_{i=1}^{N}\left[a_i,b_i\right].$

(vi) By Theorem 2.4 we get that

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}\left(nx-k\right)}<\left(K\left(q\right)\right)^{N}, \tag{2.69}$$

$\forall\ x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right),\ \ n\in\mathbb{N}.$

It is also clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}}}^{\infty} Z_{q,\lambda}\left(nx-k\right) < \gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)}, \tag{2.70}$$

$0<\beta^*<1,\ n\in\mathbb{N}: n^{1-\beta^*}>2,\ x\in\mathbb{R}^N.$

Furthermore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}\left(nx-k\right)\neq 1, \tag{2.71}$$

for at least some $x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right).$

Here $\left(X,\left\|\cdot\right\|_{\gamma}\right)$ is a Banach space.

Let $f\in C\left(\prod_{i=1}^{N}\left[a_i,b_i\right],X\right),\ x=\left(x_1,\ldots,x_N\right)\in\prod_{i=1}^{N}\left[a_i,b_i\right],\ n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor,\ i=1,\ldots,N.$

We introduce and define the following multivariate linear normalized neural network operator ($x:=\left(x_1,\ldots,x_N\right)\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$):

$$A_n\left(f,x_1,\ldots,x_N\right):=A_n\left(f,x\right):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z_{q,\lambda}\left(nx-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}\left(nx-k\right)}$$

$$=\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}\cdots\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},\ldots,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N} G_{q,\lambda}\left(nx_i-k_i\right)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} G_{q,\lambda}\left(nx_i-k_i\right)\right)}. \tag{2.72}$$

For large enough $n\in\mathbb{N}$ we always obtain $\lceil na_i\rceil\leq\lfloor nb_i\rfloor,\ i=1,\ldots,N.$
Also $a_i\leq\frac{k_i}{n}\leq b_i$, iff $\lceil na_i\rceil\leq k_i\leq\lfloor nb_i\rfloor,\ i=1,\ldots,N.$

When $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} g\left(\frac{k}{n}\right) Z_{q,\lambda}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)}. \tag{2.73}$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1,x) = 1, \ \ \forall\, x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right).$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Furthermore it holds

$$\|A_n(f,x)\|_\gamma \leq \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z_{q,\lambda}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)} = \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{2.74}$$

$\forall\, x \in \prod_{i=1}^{N}[a_i,b_i]$.

Clearly $\|f\|_\gamma \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

So, we have that

$$\|A_n(f,x)\|_\gamma \leq \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{2.75}$$

$\forall\, x \in \prod_{i=1}^{N}[a_i,b_i]$, $\forall\, n \in \mathbb{N}$, $\forall\, f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$.

Let $c \in X$ and $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, then $cg \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$.

Furthermore it holds

$$A_n(cg,x) = c\widetilde{A}_n(g,x), \ \ \forall\, x \in \prod_{i=1}^{N}[a_i,b_i]. \tag{2.76}$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$A_n(c) = c, \ \ \forall\, c \in X. \tag{2.77}$$

We call $\widetilde{A}_n$ the companion operator of A_n.

For convenience we call

$$A_n^*(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z_{q,\lambda}(nx-k)$$

$$= \sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor} \cdots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},\ldots,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right), \tag{2.78}$$

$\forall\, x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

That is

$$A_n(f,x) := \frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)}, \tag{2.79}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)}. \tag{2.80}$$

Consequently we derive

$$\|A_n(f,x) - f(x)\|_\gamma \overset{(2.69)}{\leq} (K(q))^N \left\| A_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)\right\|_\gamma, \tag{2.81}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

We will estimate the right hand side of (2.81).

For the last and others we need

Definition 2.7. ([11], p. 274) Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M,X)$. We define the first modulus of continuity of f as

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \quad 0 < \delta \leq diam(M). \tag{2.82}$$

If $\delta > diam(M)$, then

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{2.83}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 2.8. *([11], p. 274) We have $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M,X)$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.*

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (2.82). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

Let now $f \in C^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1,\ldots,N$, and $|\alpha| := \sum_{i=1}^{N}\alpha_i = l$, where $l = 0,1,\ldots,m$. We write also $f_\alpha := \frac{\partial^n f}{\partial x^n}$ and we say it is of order l.

We denote

$$\omega_{1,m}^{\max}(f_\alpha, h) := \max_{\alpha:|\alpha|=m} \omega_1(f_\alpha, h). \tag{2.84}$$

Call also

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m}\{\|f_\alpha\|_\infty\}, \tag{2.85}$$

where $\|\cdot\|_{\infty}$ is the supremum norm.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ we define,

$$B_n(f,x) := B_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z_{q,\lambda}(nx-k)$$

$$:= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},\ldots,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right), \quad (2.86)$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f,x) := C_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty}\left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z_{q,\lambda}(nx-k)$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}\cdots\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,\ldots,t_N)\,dt_1\ldots dt_N\right)$$

$$\cdot\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right), \quad (2.87)$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1,\ldots,\theta_N) \in \mathbb{N}^N$, $r = (r_1,\ldots,r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1,r_2,\ldots r_N} \geq 0$, such that $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}\cdots\sum_{r_N=0}^{\theta_N} w_{r_1,r_2,\ldots r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n,k_1,k_2,\ldots,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n}+\frac{r}{n\theta}\right)$$

$$= \sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}\cdots\sum_{r_N=0}^{\theta_N} w_{r_1,r_2,\ldots r_N} f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\frac{k_2}{n}+\frac{r_2}{n\theta_2},\ldots,\frac{k_N}{n}+\frac{r_N}{n\theta_N}\right), \quad (2.88)$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1},\frac{r_2}{\theta_2},\ldots,\frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f,x) := D_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f)\, Z_{q,\lambda}(nx-k) \quad (2.89)$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,k_2,\ldots,k_N}(f)\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right),$$

$\forall\, x \in \mathbb{R}^N$.

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates, that is acting with multilayer neural networks. Thus the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3. Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 3.1. *Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$, $0<\beta^*<1$, $q,\lambda>0$, $A>1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N,n \in \mathbb{N}$ with $n^{1-\beta^*}>2$. Then*

1)

$$\left\|A_n(f,x)-f(x)\right\|_\gamma$$
$$\leq (K(q))^N\left[\omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_\gamma\right\|_\infty\right]=:\lambda_1(n), \tag{3.1}$$

and

2)

$$\left\|\|A_n(f)-f\|_\gamma\right\|_\infty \leq \lambda_1(n). \tag{3.2}$$

We notice that $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, pointwise and uniformly.

Above ω_1 is with respect to $p=\infty$.

Proof. We observe that

$$\overline{\Delta}(x):=A_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{q,\lambda}(nx-k)$$
$$=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)Z_{q,\lambda}(nx-k)-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f(x)Z_{q,\lambda}(nx-k)$$
$$=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)Z_{q,\lambda}(nx-k). \tag{3.3}$$

Thus

$$\left\|\overline{\Delta}(x)\right\|_\gamma \leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z_{q,\lambda}(nx-k)$$
$$=\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z_{q,\lambda}(nx-k)$$

$$+\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)-f(x)\right\|_{\gamma} Z_{q,\lambda}(nx-k)$$

$$\overset{(2.63)}{\leq} \omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\left\|\|f\|_{\gamma}\right\|_{\infty} \sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)$$

$$\overset{(2.68)}{\leq} \omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_{\gamma}\right\|_{\infty}. \tag{3.4}$$

So that

$$\left\|\overline{\Delta}(x)\right\|_{\gamma}\leq \omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_{\gamma}\right\|_{\infty}. \tag{3.5}$$

Now using (2.81) we finish the proof. □

When $X=\mathbb{R}$, next we discuss the high order of approximation.

Theorem 3.2. *Let* $f\in C^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta^*<1$, $n,m,N\in\mathbb{N}$, $n^{1-\beta^*}\geq 3$, $A>1$, $\lambda>0$, $q>0$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$. *Then*

i)

$$\left|\widetilde{A}_n(f,x)-f(x)-\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{=1}^{N}\alpha_i!}\right)\widetilde{A}_n\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right)\right| \tag{3.6}$$

$$\leq (K(q))^N\left\{\frac{N^m}{m!n^{m\beta^*}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^{\beta^*}}\right)+\left(\frac{\|b-a\|_{\infty}^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\right\}.$$

ii)

$$\left|\widetilde{A}_n(f,x)-f(x)\right|\leq (K(q))^N \tag{3.7}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{|f_\alpha(x)|}{\prod_{=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta^*j}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\right]\right)\right.$$

$$\left.+\frac{N^m}{m!n^{m\beta^*}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^{\beta^*}}\right)+\left(\frac{\|b-a\|_{\infty}^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\right\}.$$

iii)

$$\left\|\widetilde{A}_n(f)-f\right\|_{\infty}\leq (K(q))^N \tag{3.8}$$

$$\left\{ \sum_{j=1}^{m} \left(\sum_{|\alpha|=j} \left(\frac{\||f_\alpha|\|_\infty}{\prod_{=1}^{N} \alpha_i!} \right) \left[\frac{1}{n^{\beta^* j}} + \left(\prod_{i=1}^{N} (b_i - a_i)^{\alpha_i} \right) \gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \right] \right) \right.$$

$$\left. + \frac{N^m}{m! n^{m\beta^*}} \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^{\beta^*}} \right) + \left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \right\}.$$

iv) Assume $f_\alpha(x_0) = 0$, *for all* $\alpha : |\alpha| = 1, \ldots, m$; $x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$. *Then*

$$\left| \widetilde{A}_n(f, x_0) - f(x_0) \right| \tag{3.9}$$

$$\leq (K(q))^N \left\{ \frac{N^m}{m! n^{m\beta^*}} \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^{\beta^*}} \right) + \left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta^*(m+1)}$.

Proof. As similar to [10], pp. 389-391, is omitted. □

We continue with

Theorem 3.3. *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta^* < 1$, $x \in \mathbb{R}^N$, $q > 0$, $\lambda > 0$, $A > 1$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta^*} > 2$, ω_1 *is for* $p = \infty$. *Then*

1)

$$\|B_n(f, x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n^{\beta^*}} \right) + 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \left\| \|f\|_\gamma \right\|_\infty =: \lambda_2(n), \tag{3.10}$$

2)

$$\left\| \|B_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_2(n). \tag{3.11}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right) \right)$, *we obtain* $\lim_{n\to\infty} B_n(f) = f$, *uniformly.*

Proof. We have that

$$B_n(f, x) - f(x) \overset{(2.63)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z_{q,\lambda}(nx - k) - f(x) \sum_{k=-\infty}^{\infty} Z_{q,\lambda}(nx - k) \tag{3.12}$$

$$= \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z_{q,\lambda}(nx - k).$$

Hence

$$\|B_n(f, x) - f(x)\|_\gamma \leq \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_{q,\lambda}(nx - k)$$

$$= \sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^{\beta^*}} \end{cases}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_{q,\lambda}(nx - k)$$

$$+\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}} \end{cases}}^{\infty} \left\|f\left(\frac{k}{n}\right)-f(x)\right\|_{\gamma} Z_{q,\lambda}(nx-k)$$

$$\overset{(2.63)}{\leq} \omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\left\|\|f\|_{\gamma}\right\|_{\infty} \sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}} \end{cases}}^{\infty} Z_{q,\lambda}(nx-k)$$

$$\overset{(2.70)}{\leq} \omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_{\gamma}\right\|_{\infty}, \tag{3.13}$$

proving the claim. □

We give

Theorem 3.4. *Let* $f\in C_B\left(\mathbb{R}^N,X\right)$, $0<\beta^*<1$, $x\in\mathbb{R}^N$, $q>0$, $\lambda>0$, $A>1$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta^*}>2$, ω_1 *is for* $p=\infty$. *Then*

1)

$$\|C_n(f,x)-f(x)\|_{\gamma}\leq\omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_{\gamma}\right\|_{\infty}=:\lambda_3(n), \tag{3.14}$$

2)

$$\left\|\|C_n(f)-f\|_{\gamma}\right\|_{\infty}\leq\lambda_3(n). \tag{3.15}$$

Given that $f\in\left(C_U\left(\mathbb{R}^N,X\right)\cap C_B\left(\mathbb{R}^N,X\right)\right)$, *we obtain* $\lim_{n\to\infty}C_n(f)=f$, *uniformly.*

Proof. We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt=\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}\ldots\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,t_2,\ldots,t_N)\,dt_1dt_2\ldots dt_N$$

$$=\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\ldots\int_0^{\frac{1}{n}} f\left(t_1+\frac{k_1}{n},t_2+\frac{k_2}{n},\ldots,t_N+\frac{k_N}{n}\right)dt_1\ldots dt_N=\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt. \tag{3.16}$$

Thus it holds (by (2.87))

$$C_n(f,x)=\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt\right)Z_{q,\lambda}(nx-k). \tag{3.17}$$

We observe that

$$\|C_n(f,x)-f(x)\|_{\gamma}$$

$$=\left\|\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt\right)Z_{q,\lambda}(nx-k)-\sum_{k=-\infty}^{\infty}f(x)\,Z_{q,\lambda}(nx-k)\right\|_{\gamma}$$

$$=\left\|\sum_{k=-\infty}^{\infty}\left(\left(n^N\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt\right)-f(x)\right)Z_{q,\lambda}(nx-k)\right\|_{\gamma}$$

$$= \left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(f\left(t + \frac{k}{n}\right) - f(x) \right) dt \right) Z_{q,\lambda}(nx - k) \right\|_\gamma \tag{3.18}$$

$$\leq \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z_{q,\lambda}(nx - k)$$

$$= \sum_{\begin{cases} k = -\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^{\beta^*}} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z_{q,\lambda}(nx - k)$$

$$+ \sum_{\begin{cases} k = -\infty \\ \left\|\frac{k}{n} - x\right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z_{q,\lambda}(nx - k)$$

$$\leq \sum_{\begin{cases} k = -\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^{\beta^*}} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1 \left(f, \|t\|_\infty + \left\| \frac{k}{n} - x \right\|_\infty \right) dt \right) Z_{q,\lambda}(nx - k)$$

$$+2 \left\| \|f\|_\gamma \right\|_\infty \left(\sum_{\begin{cases} k = -\infty \\ \left\|\frac{k}{n} - x\right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\infty} Z_{q,\lambda}(|nx - k|) \right)$$

$$\leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta^*}} \right) + 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \left\| \|f\|_\gamma \right\|_\infty, \tag{3.19}$$

proving the claim. □

We also present

Theorem 3.5. *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta^* < 1$, $x \in \mathbb{R}^N$, $q > 0$, $\lambda > 0$, $A > 1$, $N, n \in \mathbb{N}$ with $n^{1-\beta^*} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\|D_n(f, x) - f(x)\|_\gamma \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta^*}} \right) + 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \left\| \|f\|_\gamma \right\|_\infty = \lambda_4(n), \tag{3.20}$$

2)

$$\left\| \|D_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_4(n). \tag{3.21}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} D_n(f) = f$, uniformly.

Proof. Similar to the proof of Theorem 3.4, as such is omitted. □

Definition 3.6. Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, $q > 0$, $\lambda > 0$, $A > 1$, where $\left(X, \|\cdot\|_\gamma\right)$ is a Banach space. We define the general neural network operator

$$F_n(f,x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z_{q,\lambda}(nx-k) =$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{3.22}$$

Clearly $l_{nk}(f)$ is an X-valued bounded linear functional such that

$$\|l_{nk}(f)\|_\gamma \le \left\|\|f\|_\gamma\right\|_\infty .$$

Hence $F_n(f)$ is a bounded linear operator with $\left\|\|F_n(f)\|_\gamma\right\|_\infty \le \left\|\|f\|_\gamma\right\|_\infty$.
We need

Theorem 3.7. *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \ge 1$, $\lambda, q > 0$, $A > 1$. Then*

$$F_n(f) \in C_B\left(\mathbb{R}^N, X\right).$$

Proof. It is very lengthy and very similar to [13], pp. 167-171. As such is omitted. □

Remark 3.8. By (2.72) it is obvious that $\left\|\|A_n(f)\|_\gamma\right\|_\infty \le \left\|\|f\|_\gamma\right\|_\infty < \infty$, and $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, given that $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Call L_n any of the operators A_n, B_n, C_n, D_n.

Clearly then

$$\left\|\|L_n^2(f)\|_\gamma\right\|_\infty = \left\|\|L_n(L_n(f))\|_\gamma\right\|_\infty \le \left\|\|L_n(f)\|_\gamma\right\|_\infty \le \left\|\|f\|_\gamma\right\|_\infty, \tag{3.23}$$

etc.

Therefore we get

$$\left\|\|L_n^k(f)\|_\gamma\right\|_\infty \le \left\|\|f\|_\gamma\right\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{3.24}$$

the contraction property.

Also we see that

$$\left\|\|L_n^k(f)\|_\gamma\right\|_\infty \le \left\|\|L_n^{k-1}(f)\|_\gamma\right\|_\infty \le \ldots \le \left\|\|L_n(f)\|_\gamma\right\|_\infty \le \left\|\|f\|_\gamma\right\|_\infty. \tag{3.25}$$

Here L_n^k are bounded linear operators.

Notation 3.9. Here $q > 0$, $\lambda > 0$, $A > 1$, $N \in \mathbb{N}$, $0 < \beta^* < 1$. Denote by

$$c_N := \begin{cases} (K(q))^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{3.26}$$

$$\varphi(n) := \begin{cases} \frac{1}{n^{\beta^*}}, & \text{if } L_n = A_n, B_n, \\ \frac{1}{n} + \frac{1}{n^{\beta^*}}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{3.27}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^{N}[a_i,b_i], X\right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{3.28}$$

and

$$Y := \begin{cases} \prod_{i=1}^{N}[a_i,b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{3.29}$$

We give the following combined result.

Theorem 3.10. *Let $f \in \Omega$, $0 < \beta^* < 1$, $x \in Y$; $q > 0$, $\lambda > 0$, $A > 1$, n, $N \in \mathbb{N}$ with $n^{1-\beta^*} > 2$. Then*

(i)

$$\left\| L_n(f,x) - f(x) \right\|_\gamma \leq c_N \left[\omega_1(f, \varphi(n)) + 2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)} \left\| \|f\|_\gamma \right\|_\infty \right] =: \tau(n), \tag{3.30}$$

where ω_1 is for $p = \infty$,
x and

(ii)

$$\left\| \left\| L_n(f) - f \right\|_\gamma \right\|_\infty \leq \tau(n) \to 0, \text{ as } n \to \infty. \tag{3.31}$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n\to\infty} L_n(f) = f,$$

pointwise and uniformly.

Proof. By Theorems 3.1, 3.3, 3.4, 3.5. □

Next we talk about iterated multilayer neural network approximation (see also [9]).

We give

Theorem 3.11. *All here as in Theorem 3.10 and $r \in \mathbb{N}$, $\tau(n)$ as in (3.30). Then*

$$\left\| \left\| L_n^r f - f \right\|_\gamma \right\|_\infty \leq r\tau(n). \tag{3.32}$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n.

Proof. As similar to [13], pp. 172-173, is omitted. □

We also present the more general

Theorem 3.12. *Let $f \in \Omega$; $q > 0$, $\lambda > 0$, $A > 1$, $N, m_1, m_2, \ldots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \ldots \leq m_r$, $0 < \beta^* < 1$; $m_i^{1-\beta^*} > 2$, $i = 1, \ldots, r$, $x \in Y$, and let $(L_{m_1}, \ldots, L_{m_r})$ as $(A_{m_1}, \ldots, A_{m_r})$ or $(B_{m_1}, \ldots, B_{m_r})$ or $(C_{m_1}, \ldots, C_{m_r})$ or $(D_{m_1}, \ldots, D_{m_r})$, $p = \infty$. Then*

$$\left\| L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1} f\right)\right)\right)(x) - f(x) \right\|_\gamma$$
$$\leq \left\| \left\| L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1} f\right)\right)\right) - f \right\|_\gamma \right\|_\infty$$

$$\leq \sum_{i=1}^{r} \left\| \left\| L_{m_i} f - f \right\|_{\gamma} \right\|_{\infty}$$

$$\leq c_N \sum_{i=1}^{r} \left[\omega_1 \left(f, \varphi \left(m_i \right) \right) + 2\gamma A^{-\lambda \left(n^{1-\beta^* -2} \right)} \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \right]$$

$$\leq r c_N \left[\omega_1 \left(f, \varphi \left(m_1 \right) \right) + 2\gamma A^{-\lambda \left(n^{1-\beta^* -2} \right)} \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \right]. \tag{3.33}$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1}.

Proof. As similar to [13], pp. 173-175, is omitted. □

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Modified inertia Halpern method for split null point problem in Banach spaces

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Abstract. In this paper, we study split null point problem in reflexive Banach spaces. Using the Bregman technique together with a modified inertial Halpern method, we approximate a solution of split null point problem. Also, we establish a strong convergence result for approximating the solution of the aforementioned problems. It is worth mentioning that the iterative algorithm employ in this study is design in such a way that it does not require prior knowledge of operator norm. We display some numerical examples to illustrate the performance of the proposed iterative method. The result discuss in this paper extends and complements many related results in literature.

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Keywords: Monotone variational inclusion problem, split feasibility problem, firmly nonexpansive-type mapping, fixed point problem, inertial method.

1. Introduction

Let E be a reflexive Banach space with E^* its dual and Q be a nonempty closed and convex subset of E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of g denoted as $g^* : E^* \to (-\infty, +\infty]$ is define as

$$g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) : x \in E\},\ x^* \in E^*.$$

Let the domain of g be denoted as $dom(g) = \{x \in E : g(x) < +\infty\}$, hence for any $x \in intdom(g)$ and $y \in E$, we define the right-hand derivative of g at x in the direction

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of y by

$$g^0(x,y) = \lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}.$$

Let $g : E \to (-\infty, +\infty]$ be a function, then g is said to be:

(i) Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case, $g^0(x,y)$ coincides with $\nabla g(x)$ (the value of the gradient ∇g of g at x);

(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in intdomg$;

(iii) Fréchet differentiable at x, if its limit is attained uniformly in $\|y\| = 1$;

(iv) Uniformly Fréchet differentiable on a subset Q of E, if the above limit is attained uniformly for $x \in Q$ and $\|y\| = 1$.

(v) essentially smooth, if the subdifferential of g denoted as ∂g is both locally bounded and single-valued on its domain, where

$$\partial g(x) = \{w \in E : g(x) - g(y) \geq \langle w, y-x\rangle,\ y \in E\};$$

(vi) essentially strictly convex, if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of *dom* ∂g;

(vii) Legendre, if it is both essentially smooth and essentially strictly convex. See [8, 9] for more details on Legendre functions.

Alternatively, a function g is said to be Legendre if it satisfies the following conditions:

(i) The $intdom(g)$ is nonempty, g is Gâteaux differentiable on $intdom(g)$ and $dom\nabla g = intdom(g)$;

(ii) The $intdomg^*$ is nonempty, g^* is Gâteaux differentiable on $intdomg^*$ and $dom\nabla g^* = intdom(g)$.

Let E be a Banach space and $B_s := \{z \in E : \|z\| \leq s\}$ for all $s > 0$. Then, a function $g : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E, [see pp. 203 and 221] [51] if $\rho_s t > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y\in B_s, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha(x) + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, with ρ_s denoting the gauge of uniform convexity of g. The function g is also said to be uniformly smooth on bounded subsets of E, [see pp. 221] [51], if $\lim_{t\downarrow 0} \frac{\sigma_s}{t}$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x\in B, y\in S_E, \alpha\in(0,1)} \frac{\alpha g(x) + (1-\alpha)ty) + (1-\alpha)g(x-\alpha ty) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, and uniformly convex if the function $\delta g : [0, +\infty) \to [0, +\infty)$ defined by

$$\delta g(t) := \sup\{\frac{1}{2}g(x) + \frac{1}{2}g(y) - g(\frac{x+y}{2}) : \|y-x\| = t\},$$

satisfies $\lim_{t\downarrow 0} \frac{\delta g(t)}{t} = 0$.

Definition 1.1. [11] Let E be a Banach space. A function $g : E \to (-\infty, \infty]$ is said to be proper if the interior of its domain $dom(g)$ is nonempty. Let $g : E \to (-\infty, \infty]$

be a convex and Gâteaux differentiable function. Then the Bregman distance corresponding to g is the function $D_g : dom(g) \times intdom(g) \to \mathbb{R}$ defined by

$$D_g(x,y) := g(x) - g(y) - \langle x - y, \nabla_E^g(y)\rangle, \ \forall\ x, y \in E. \tag{1.1}$$

It is clear that $D_g(x,y) \geq 0$ for all $x, y \in E$.

It is well-known that Bregman distance D_g does not satisfy all the properties of a metric function because D_g fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in dom(g)$ and $y, z \in intdom(g)$,

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x - y, \nabla_E^g(y) - \nabla_E^g(z)\rangle. \tag{1.2}$$

In particular,

$$D_g(x,y) = -D_g(y,x) + \langle y - x, \nabla_E^g(y) - \nabla_E^g(x)\rangle, \ \forall\ x, y \in E.$$

The relationship between D_g and $\|.\|$ is guaranteed when g is strongly convex with strong convexity constant $\rho > 0$ i.e.

$$D_g(x,y) \geq \frac{\rho}{2}\|x - y\|^2, \ \forall\ x \in dom(g), \ y \in intdom(g). \tag{1.3}$$

Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function and $T : Q \to intdom(g)$ be a mapping, a point $x \in Q$ is called a fixed point of T, if for all $x \in Q$, $Tx = x$. We denote by $Fix(T)$ the set of all fixed points of T. Furthermore, a point $p \in Q$ is called an asymptotic fixed point of T if Q contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote by $\hat{Fix}(T)$ the set of asymptotic fixed points of T.
Let Q be a nonempty closed and convex subset of int(dom g), then we define an operator $T : Q \to int(domg)$ to be :

(i) Bregman relatively nonexpansive, if $Fix(T) \neq \emptyset$, and

$$D_f(p, Tx) \leq D_f(p, x), \ \forall\ p \in Fix(T), \ x \in Q \text{ and } Fi\hat{x}(T) = Fix(T).$$

(ii) Bregman quasi-nonexpansive mapping if $Fix(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \forall\ x \in Q \text{ and } p \in Fix(T).$$

(iii) Bregman firmly nonexpansive (BFNE), if

$$\langle \nabla_E^g(Tx) - \nabla_E^g(Ty), Tx - Ty\rangle \leq \langle \nabla_E^g(x) - \nabla_E^g(y), Tx - Ty\rangle, \ \forall\ x, y \in E.$$

Definition 1.2. [20] Let Q be a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let β and γ be real numbers with $\beta \in (-\infty, 1)$ and $\gamma \in [0, \infty)$, respectively. Then a mapping $T : Q \to E$ with $Fix(T) \neq \emptyset$ is called Bregman (β, γ)-demigeneralized if for any $x \in Q$ and $p \in Fix(T)$,

$$\langle x - p, \nabla_E^g(x) - \nabla_E^g(Tx)\rangle \geq (1 - \beta)D_g(x, Tx) + \gamma D_g(Tx, x), \ \forall\ x \in E \text{ and } p \in F(T).$$

For modelling inverse problems which arises from phase retrievals and medical image reconstruction, (see [12]), Censor and Elfving [17] introduced the Split Feasibility Problem (SFP) in 1994, which is to find

$$u^* \in C \text{ such that } Ku^* \in Q; \tag{1.4}$$

where C and Q are nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 respectively, and $K : E_1 \to E_2$ is a bounded linear operator. The SFP have been well studied in the framework of real Hilbert spaces, uniformly convex and uniformly smooth Banach spaces, see ([2, 19, 24, 43] and other references contained in). Different optimization problems have been formulated in terms of SFP (1.4), for instance, If $Q = \{b\}$ in SFP (1.4) is a singleton, then we have the following convexly constrained linear inverse problem (in short, CCLIP) defined as follows:

$$\text{Find a point } u^* \in C \text{ such that } Ku^* = b.$$

The Split Null Point Problem (SNPP) introduced by Bryne et al. [13] is formulated as finding a point

$$x \in H_1 \text{ such that } 0 \in B_1(x) \text{ and } 0 \in B_2(Kx), \tag{1.5}$$

where H_1 and H_2 are real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multivalued mappings and $K : H_1 \to H_2$ are real Hilbert spaces.
In 2018, Jailoka and Suantai [23] introduced the following Halpern iterative method for approximating the split null point and fixed point problems for maximal monotone operators and multivalued demicontractive mapping T as follows:

$$\begin{cases} u, x_1 \in H_1, \\ y_n = J_{\lambda_n}^{B_1}(x_n + \gamma K^*(J_{\lambda_n}^{B_2} - I)Kx_n), \\ u_n = (1-\delta)y_n + \delta z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \ n \geq 1, \end{cases}$$

where $z_n \in Ty_n$. Also, Oyewole et al. [33] introduced a new iterative method with self adaptive step-size for approximating solutions of a SFP for sum of two monotone operators and fixed point problem of a demimetric mapping in real Hilbert spaces. Strong convergence result was proved and numerical experiment to illustrate the performance of the algorithm were displayed.
In the framework of uniformly convex and smooth Banach spaces, Takahashi and Takahashi [45] introduced a shrinking projection method to approximate a solution of SNPP. Using their iterative method, they proved a strong convergence theorem.
Question: Can the results of [3, 6, 13, 22, 23, 32, 33, 45] be establish in a more general Banach spaces (reflexive Banach spaces)?

Let $B : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of B by $domB = \{x \in E : Bx \neq \emptyset\}$ and $ranB = \bigcup_{x \in E} Bx$, respectively. The graph of B denoted by $G(B) = \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. The mapping $B \subset E \times E^*$ is said to be monotone [38] if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in B$. It is also said to be maximal monotone [37] if its graph is not contained in the graph of any other monotone operator on E. If $B \subset E \times E^*$ is maximal monotone, then the set $B^{-1}(0) = \{z \in E : 0 \in Bz\}$ is closed and convex. Also, the resolvent associated with B and λ for any $\lambda > 0$ is the mapping $J^g_{\lambda B} : E \to 2^E$ with $Fix(J^g_{\lambda B}) = B^{-1}(0)$ defined by

$$J^g_{\lambda B} := (\nabla^g_E + \lambda B)^{-1} \circ \nabla^g_E.$$

It is worth mentioning that a mapping $B : E \to 2^{E^*}$ is called Bregman inverse strongly monotone (BISM) on the set C if

$$C \cap (domg) \cap (int\ dom\ g) \neq \emptyset,$$

and for any $x, y \in C \cap (int\ dom\ g)$, $\eta \in Ax$ and $\xi \in Ay$, we have

$$\langle \eta - \xi, (\nabla^{g^*}_{E^*}(x) - \eta) - \nabla^{g^*}_{E^*}(\nabla^g_E(y) - \xi)\rangle \geq 0.$$

The anti-resolvent $B^g_\lambda : E \to 2^E$ associated with the mapping $b : E \to 2^{E^*}$ and $\lambda > 0$ is defined by

$$B^g_\lambda := \nabla^g_E \circ (\nabla^g_E - \lambda B). \tag{1.6}$$

Let $A : E \to E^*$ be a single-valued monotone mapping and $B : E \to 2^{E^*}$ be a multivalued monotone mapping. Then, the Monotone Variational Inclusion Problem (MVIP) (also known as the problem of finding a zero of sum of two monotone mappings) is to find $x \in E$ such that

$$0^* \in A(x) + B(x). \tag{1.7}$$

We denote by Ω, the solution set of problem (1.7).
A simple and efficient method for solving (1.7) is the forward-backward splitting method introduced by Lions and Mercier [26] in a Hilbert space H. It is known that this method converges weakly to an element in (1.7) under the assumption that A is α-inverse strongly monotone. Note that the inverse strongly monotonicity of A is a strict assumption. To avoid this assumption, Tseng [48] introduced the following algorithm which is known as Tseng's splitting algorithm for solving (1.7) as follows:

$$\begin{cases} x_1 \in H, \\ y_n = J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n), \ \forall\ n \geq 1, \end{cases} \tag{1.8}$$

where $A : H \to H$ is monotone and L-Lipschitz continuous and $\{\lambda_n\}$ is the sequence of suitable stepsize in $(0, \frac{1}{L})$. He proved that the sequence $\{x_n\}$ generated by (1.8) converges weakly to an element in (1.7). It is well-known that the step size of Tseng's splitting method requires prior knowledge of the Lipschitz constant of the mapping. However, from a practical point of view, the Lipschitz constant is very difficult to approximate.
It is well known that many interesting problems arising from mechanics, economics, finance, nonlinear programming, applied sciences, optimization such as equilibrium and variational inequality problems can be solved using MVIP. Considerable efforts have been devoted to develop efficient iterative method to approximate solutions of MVIP in which the resolvent operator technique is one of the vital technique.
Many authors have considered approximating solutions of (1.7) together with fixed

point problems in real Hilbert and Banach spaces, see [3, 1, 5, 33, 42].
For instance, Okeke and Izuchukwu [32] studied and analysed an iterative method for approximating split feasibility problem and variational inclusion problem in p-uniformly convex Banach spaces which are uniformly smooth, they proved a strong convergence result for approximating the solution of the aforementioned problems. Shehu [40] considered the splitting method for finding zeros of the sum of maximal monotone operator and Lipschitz continuous monotone operator in Banach space. He proved weak and strong convergence results and give some applications of his main result. In the framework of 2-uniformly convex real Banach spaces which are also uniformly smooth, Abass et al. [4] investigated a shrinking algorithm for finding zeros of the sum of maximal monotone operators and Lipschitz continuous monotone operators which is also a common fixed point for finite family of relatively quasi-nonexpansive mappings.
Suppose $A = 0$ in (1.7), then (1.7) reduces to the following Monotone Inclusion Problem (MIP), which is to find $x \in E$ such that

$$0^* \in B(x). \tag{1.9}$$

Many results on MIP have been extended by authors from real Hilbert spaces to more general Banach spaces. For instance, Reich and Sabach [36] introduced some iterative algorithms and proved two strong convergence results for approximating a common solution of a finite family of MIP (1.9) in a reflexive Banach spaces. Recently, Timnak et al. [47] introduced a new Halpern-type iterative scheme for finding a common zero of finitely many maximal monotone mappings in a reflexive Banach spaces and prove the following strong convergence theorem.

Theorem 1.3. *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subset of E. Let $A_i : E \to 2^{E^*}, i = 1, 2, ...,$ be N maximal monotone operators such that $Z := \cap_{i=1}^{N} A_i^{-1}(0^*) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $(0, 1)$ satisfying the following control conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E, \ x_1 \in E \text{ chosen arbitrarily}, \\ y_n = \nabla f^*[\beta_n \nabla f(x_n) + (1 - \beta_n)\nabla f(Res^f_{\lambda_n A_N}) \cdots (Res^f_{r_1 A_1}(x_n))], \\ x_{n+1} = \nabla f^*[\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_n)], \end{cases} \tag{1.10}$$

for $n \in \mathbb{N}$, where ∇f is the gradient of f. If $r_i > 0$, for each $i = 1, 2, ..., N$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (1.10) converges strongly to $proj^f_Z u$ as $n \to \infty$.

Very recently, Ogbuisi and Izuchukwu [30] introduced an iterative algorithm and obtained a strong convergence result for approximating a zero of sum of two maximal monotone operators which is also a fixed point of a Bregman strongly nonexpansive mapping in the framework of a reflexive Banach spaces.

We will also like to emphasize that approximating a common solution of SNPP have some possible applications to mathematical models whose constraints can be expressed as SNPP. In fact, this happens in practical problems like signal processing, network resource allocation, image recovery, to mention a few, (see [21]). It is worth mentioning that the problem considered in this article generalizes the ones in [6, 18, 29].

Inspired by the results discussed above, we introduce an iterative algorithm which does not require the prior knowledge of operator norm as this may give difficulty in computing, to approximate a solution of split null point problem involving single-valued, multi-valued monotone and Lipschitz continuous monotone mappings in reflexive Banach spaces. Using our iterative algorithm, we prove a strong convergence result for approximating solutions of the aforementioned problems. Finally, we illustrate some numerical experiments to show the performance and behavior of our main result. The result discussed in this paper complements and extends many related results in literature.
We state our contributions in this article as follows:

1. The main result in this paper generalizes the results in [10], [**?**] and [32] from p-uniformly Banach spaces which are also uniformly smooth to reflexive Banach spaces and [5, 6, 18, 29, 31, 32, 47] from real Hilbert spaces to a reflexive Banach spaces.

2. The iterative method defined in this article is design in such a way that it does not depend on the operator norm, see [20, 33].

3. We proved a strong convergence result which is more desirable than the weak convergence result obtained in [44].

4. The sequence of stepsizes of our algorithms is chosen without the prior knowledge of the Lipschitz constant and the uniform smoothness constant of the mapping, see [40].

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main result. In the sequel, we denote strong and weak convergence by ”$\rightarrow$” and ”$\rightharpoonup$”, respectively.

Definition 2.1. A function $g : E \to \mathbb{R}$ is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} \frac{g(x)}{\|x\|} = \infty.$$

Definition 2.2. A mapping $T : C \to E$ is said to be demiclosed at p if $\{x_n\}$ is a sequence in C such that $\{x_n\}$ converges weakly to some $x^* \in C$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Lemma 2.3. [47] *Let E be a Banach space, $s > 0$ be a constant, ρ_s be the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then,*
(i) For any $x, y \in B_s$ and $\alpha \in (0,1)$, we have

$$D_g\big(x, \nabla_{E^*}^{g^*}[\alpha\nabla_E^g\nabla_E^g(y) + (1-\alpha)\nabla_E^g(z)]\big) \leq \alpha D_g(x,y) + (1-\alpha)D_g(x,z) - \alpha(1-\alpha)\rho_s(\|\nabla_E^g(y) - \nabla_E^g(z)\|),$$

(ii) For any $x, y \in B_s := \{z \in E : \|z\| \leq s\},\ s > 0$,

$$\rho_s(\|x-y\|) \leq D_g(x,y).$$

Lemma 2.4. [16] *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ be a strongly coercive Bregman function and V be a function defined by*

$$V(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*),\ x \in E,\ x^* \in E^*.$$

The following assertions also hold:

$$D_g(x, \nabla_{E^*}^{g^*}(x^*)) = V(x, x^*),\ \textit{for all } x \in E \textit{ and } x^* \in E^*.$$

$$V(x,x^*) + \langle\nabla_{E^*}^{g^*}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*) \textit{ for all } x \in E \textit{and } x^*, y^* \in E^*.$$

Also, following a similar approach as in Lemma 2.4 and for any $x \in E, y^*,\ z^* \in B_r$ and $\alpha \in (0,1)$, we have

$$V_g(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_g(x,y^*) + (1-\alpha)V_g(x,z^*) - \alpha(1-\alpha)\rho_r^*(\|y^* - x^*\|). \tag{2.1}$$

Lemma 2.5. [20] *Let E_1 and E_2 be two Banach spaces. Let $F : E_1 \to E_2$ be a bounded linear operator and $T : E_2 \to E_2$ be a Bregman (ϕ, σ)-demigeneralized for some $\phi \in (-\infty, 1)$ and $\sigma \in [0, \infty)$. Suppose that $K = ran(A) \cap Fix(T) \neq \emptyset$ (where $ran(A)$ denotes the range of (A). Then for any $(x, q) \in E_1 \times K$,*

$$\begin{aligned}\langle x - q, F^*(\nabla_{E_2}^{g_2}(T(Fx)))\rangle &\geq (1-\phi)D_{g_2}(Fx, T(Fx)) + \sigma D_{g_2}(T(Fx), Fx)\\ &\geq (1-\phi)D_{g_2}(Fx, T(Fx)).\end{aligned} \tag{2.2}$$

So, given any real numbers ξ_1 and ξ_2, the mapping $L_1 : E_1 \to [0, \infty)$ and $L_2 : E_2 \to [0.\infty)$ formulated for $x \in E_1$ as

$$L_1(x) = \begin{cases}\frac{D_{g_2}(Fx,TFx)}{D_{g_1}^*(F^*(\nabla_{E_2}^{g_2}(Fx)),F^*(\nabla_{E_2}^{g_2}(TFx))}, & \textit{if} \quad (I-T)Fx \neq 0,\\ \xi_1, & \textit{otherwise},\end{cases} \tag{2.3}$$

and

$$L_2(x) = \begin{cases}\frac{D_{g_1}^*(\nabla_{E_1}^{g_1}(x) - \gamma F^*(\nabla_{E_2}^{g_2}(Fx) - \nabla_{E_2}^{g_2}(TFx)), \nabla_{E_1}^{g_1}(x))}{D_{g_1}^*(F^*(\nabla_{E_2}^{g_2}(Fx)),F^*(\nabla_{E_2}^{g_2}(TFx))}, & \textit{if}, \quad (I-T)Fx \neq 0,\\ \xi_2, & \textit{otherwise},\end{cases} \tag{2.4}$$

are well-defined, where γ is any nonnegative real number. Moreover, for any $(x,p) \in E_1 \times K$, we have

$$D_{g_1}(q,y) \leq D_{g_1}(q,x) - (\gamma(1-\phi)L_1(x) - L_2(x))D_{g_1^*}(F^*(\nabla_{E_2}^{g_2}(Fx)), F^*(\nabla_{E_2}^{g_2}(TFx)), \tag{2.5}$$

where

$$y = (\nabla_{E_1}^{g_1})^{-1}[\nabla_{E_1}^{g_1}(x) - \gamma F^*(\nabla_{E_2}^{g_2}(Fx) - \nabla_{E_2}^{g_2}(TFx))].$$

Remark 2.6. From Definition 2.2 of [20], It can be seen that $J_{\lambda B}^g$ is $(0,1)-$ demigeneralized. Therefore, we conclude from (2.5) that

$$D_{g_1}(q,y) \le D_{g_1}(q,x) - (\gamma L_1(x) - L_2(x)) D_{g_1^*}(F^*(\nabla_{E_2}^{g_2}(Fx)), F^*(\nabla_{E_2}^{g_2}(J_{\lambda B}^g Fx)), \tag{2.6}$$

where $T = J_{\lambda B}^g$ *and* $B : E \to 2^{E^*}$ *is a maximal monotone operator.*

Lemma 2.7. [16] *Let* E *be a Banach space and* $g : E \to \mathbb{R}$ *a Gâteaux differentiable function which is uniformly convex on bounded subsets of* E*. Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be bounded sequences in* E*. Then,*

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.8. [7] *Let* $A : E \to E^*$ *be a monotone, hemicontinuous and bounded operator, and* $B : E \to 2^{E^*}$ *be a maximal monotone operator. Then* $A+B$ *is maximal monotone.*

Lemma 2.9. [36] *Let* $g : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_0 \in E$ *and the sequence* $\{D_g(x_n, x_0)\}$ *is bounded, then the sequence* $\{x_n\}$ *is also bounded.*

Definition 2.10. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. A Bregman projection of $x \in int(dom(g))$ onto $C \subset int(domg)$ is the unique vector $P_C^g(x) \in C$ satisfying

$$D_g(P_C^g(x), x) = int\{D_g(y, x) : y \in C\}.$$

Lemma 2.11. [34] *Let* C *be a nonempty closed and convex subset of a reflexive Banach space* E *and* $x \in E$*. Let* $g : E \to \mathbb{R}$ *be a strongly coercive Bregman function. Then,*
(i) $z = P_C^g(x)$ *if and only if* $\langle \nabla_E^g(x) - \nabla_E^g(z), y - z\rangle \le 0, \ \forall \ y \in C$*.*
(ii) $D_g(y, P_C^g(x)) + D_g(P_C^g(x), x) \le D_g(y, x), \ \forall \ y \in C$*.*

Lemma 2.12. [50] *Let* $\{a_n\}$*,* $\{\gamma_n\}$*,* $\{\delta_n\}$ *and* $\{t_n\}$ *be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - t_n - \gamma_n)a_n + \gamma_n n a_{n-1} + t_n s_n + \delta_n, \ \forall n \ge 0,$$

where $\sum\limits_{n=n_0}^{\infty} t_n = +\infty$, $\sum\limits_{n=n_0}^{\infty} \delta_n < +\infty$*, for each* $n \ge n_0$ *(where* n_0 *is a positive integer) and* $\{\gamma_n\} \subset [0, \frac{1}{2}]$, $\limsup\limits_{n\to\infty} s_n \le 0$*. Then, the sequence* $\{a_n\}$ *converges weakly to zero.*

Lemma 2.13. [27] *Let* Γ_n *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_k}\}_{k\ge 0}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_k} \le \Gamma_{n_j+1}$ *for all* $j \ge 0$*. Also, consider a sequence of integers* $\{\tau(n)\}_{n\ge n_0}$ *defined by*

$$\tau(n) := \max\{k \le n \mid \Gamma_{n_k} \le \Gamma_{n_k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence satisfying* $\lim_{n\to\infty}\tau(n)=\infty$. *If it holds that* $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ *for all* $n\geq n_0$ *then we have*

$$\Gamma_\tau(n)\leq\Gamma_{\tau(n)+1}.$$

3. Main result

Throughout this section, we assume that

Assumption 3.1.

1. E_1 *and* E_2 *be two reflexive Banach spaces,* $g_1 : E_1 \to (-\infty,+\infty]$ *and* $g_2 : E_2 \to (-\infty,+\infty]$ *be strongly coercive Bregman functions which are bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of* E_1 *and* E_2 *with constant* $\beta>0$, *respectively.*

2. $\nabla^{g_1}_{E_1}$ *and* $\nabla^{g_2}_{E_2}$ *be the gradients of* E_1 *dependent on* g_1 *and* E_2 *dependent on* g_2 *respectively.*

3. $A_1 : E_1 \to E_1^*$ *be a monotone and L-Lipschitz continuous mapping,* $B_1 : E_1 \to 2^{E_1^*}$ *and* $B_2 : E_2 \to 2^{E_2^*}$ *are maximal monotone mappings respectively, and* $J^{g_2}_{\lambda B_2}$ *be the resolvent of* g_2 *on* B_2 *for* $\lambda>0$, *and* $\lambda_n=\rho l^{m_n}$ *where* m_n *is the smallest nonnegative integer such that*
$$\lambda_n\|A_1z_n-A_1y_n\|\leq\mu\|z_n-y_n\|. \tag{3.1}$$

4. *Suppose that* $K : E_1\to E_2$ *is a bounded linear operator such that* $K\neq 0$ *and* $K^* : E_2^*\to E_1^*$ *be the adjoint of* K. *Given that* $\rho>0$, $l\in(0,1)$, $\mu\in(0,\sigma)$, *where* σ *is a constant given by* (1.3).

5. *The control sequence* $\{\alpha_n\},\{\beta_n\}$ *and* $\{\delta_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n+\beta_n+\delta_n=1$, $\{\theta_n\}\subset[0,\frac{1}{2}]$ *and the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty}\alpha_n=0,\ \sum\limits_{n=1}^{\infty}\alpha_n=\infty,$

(ii) $0<\liminf\limits_{n\to\infty}\beta_n\leq\limsup\limits_{n\to\infty}\beta_n<1,$

(iii) $0<a\leq\theta_n<\delta_n\leq\frac{1}{2},\ \forall\ n\geq 1.$

Algorithm 3.2. *Define a sequence* $\{x_n\}_{n=1}^\infty$ *generated arbitrarily by chosen* $x_0,x_1\in E_1$ *and any fixed* $u\in E_1$, *such that*

$$\begin{cases} w_n=(\nabla^{g_1}_{E_1})^{-1}[\nabla^{g_1}_{E_1}(x_n)+\theta_n(\nabla^{g_1}_{E_1}(x_{n-1})-\nabla^{g_1}_{E_1}(x_n))],\\ z_n=(\nabla^{g_1}_{E_1})^{-1}\big[\nabla^{g_1}_{E_1}(w_n)-\gamma K^*(\nabla^{g_2}_{E_2}(Kw_n)-\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda B_2}Kw_n))\big]\\ y_n=J^{g_1}_{\lambda_nB_1}\big[(\nabla^{g_1}_{E_1})(z_n)-\lambda_nA_1z_n\big]\\ u_n=(\nabla^{g_1}_{E_1})^{-1}\big[\nabla^{g_1}_{E_1}(y_n)-\lambda_n(A_1y_n-A_1z_n)\big],\\ x_{n+1}=(\nabla^{g_1}_{E_1})^{-1}\big[\alpha_n\nabla^{g_1}_{E_1}(u)+\beta_n\nabla^{g_1}_{E_1}(x_n)+\delta_n\nabla^{g_1}_{E_1}(u_n)\big]. \end{cases} \tag{3.2}$$

Suppose that $\Omega := \{p \in (A_1 + B_1)^{-1}(0) : Kp \in B_2^{-1}(0)\} \neq \emptyset$, *let* $\gamma > 0$, *let the sequences* $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ *and* $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ *satisfy the following conditions:*

(i) *there exists a positive real number* ϕ_1 *such that*

$$0 < \phi_1 < \liminf_{n\to\infty} \frac{\xi_{2,n}}{\xi_{1,n}} < \gamma,$$

where

$$\xi_{1,n} = \begin{cases} \frac{D_{g_2}(Kw_n, J^{g_2}_{\lambda_n B_2} w_n)}{D^*_{g_1}(K^*(\nabla^{g_2}_{E_2}(Kw_n)), K^*(\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda_n B_2} Kw_n))}, & \text{if} \quad (I - J^{g_2}_{\lambda_n B_2})Kw_n \neq 0, \\ \xi_1, & \text{otherwise}, \end{cases}$$

and

$$\xi_{2,n} = \begin{cases} \frac{D^*_{g_1}(\nabla^{g_1}_{E_1}(w_n) - \gamma K^*(\nabla^{g_2}_{E_2}(Kw_n) - \nabla^{g_2}_{E_2}(J^{g_2}_{\lambda_n B_2} Kw_n)), \nabla^{g_1}_{E_1}(w_n))}{D^*_{g_1}(K^*(\nabla^{g_2}_{E_2}(Kw_n)), K^*(\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda_n B_2} Kw_n))}, \\ \quad \text{if } (I - J^{g_2}_{\lambda_n B_2})Kw_n \neq 0, \\ \xi_2, \quad \text{otherwise}. \end{cases}$$

Then, the sequence $\{x_n\}$ *generated iteratively converges strongly to* $z = P^{g_1}_\Omega u$, *where* $P^{g_1}_\Omega$ *is the Bregman projection of* E_1 *onto* Ω.

Proof. It can be seen in Lemma 3.2 of [44] that the Armijo linesearch rule defined by (3.1) is well-defined and

$$\min\{\rho, \frac{\mu l}{L}\} \leq \lambda_n \leq \rho.$$

Now, let $x^* \in \Omega$ then, using definition of u_n in (3.2) we have from (1.1) that

$$\begin{aligned} D_{g_1}(x^*, u_n) &= D_{g_1}\big(x^*, (\nabla^{g_1}_{E_1})^{-1}[\nabla^{g_1}_{E_1}(y_n) - \lambda_n(A_1 y_n - A_1 z_n)]\big) \\ &= g_1(x^*) - g_1(u_n) - \langle x^* - u_n, \nabla g_1(y_n) - \lambda_n(A_1 y_n - A_1 z_n)\rangle \\ &= g_1(x^*) - g_1(u_n) - \langle x^* - u_n, \nabla g_1(y_n)\rangle + \lambda_n\langle x^* - u_n, A_1 y_n - A_1 z_n\rangle \\ &= g_1(x^*) - g_1(y_n) - \langle x^* - y_n, \nabla g_1(y_n)\rangle + \langle x^* - y_n, \nabla g_1(y_n)\rangle \\ &+ g_1(y_n) - g_1(u_n) - \langle x^* - u_n, \nabla g_1(y_n)\rangle + \lambda_n\langle x^* - u_n, A_1 y_n - A_1 z_n\rangle \\ &= g_1(x^*) - g_1(y_n) - \langle x^* - y_n, \nabla g_1(y_n)\rangle - g_1(u_n) + g_1(y_n) \\ &+ \langle u_n - y_n, \nabla g_1(y_n)\rangle + \lambda_n\langle x^* - u_n, A_1 y_n - A_1 z_n\rangle \\ &= D_{g_1}(x^*, y_n) - D_{g_1}(u_n, y_n) + \lambda_n\langle x^* - u_n, A_1 y_n - A_1 z_n\rangle. \end{aligned} \tag{3.3}$$

Using (1.2), we get

$$D_{g_1}(x^*, u_n) = D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) + \langle x^* - y_n, \nabla g_1(z_n) - \nabla g_1(y_n)\rangle. \tag{3.4}$$

On substituting (3.4) into (3.3), we obtain

$$\begin{aligned}
D_{g_1}(x^*, u_n) &= D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) \\
&\quad + \langle x^* - y_n, \nabla g_1(z_n) - \nabla g_1(y_n)\rangle + \lambda_n\langle x^* - u_n, A_1y_n - A_1z_n\rangle \\
&= D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \langle x^* - y_n, \nabla g_1(z_n) - \nabla g_1(y_n)\rangle \\
&\quad + \lambda_n\langle y_n - u_n, A_1y_n - A_1z_n\rangle - \lambda_n\langle y_n - x^*, A_1y_n - A_1z_n\rangle \\
&= D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \lambda_n\langle y_n - u_n, A_1y_n - A_1z_n\rangle \\
&\quad - \langle y_n - x^*, \nabla g_1(z_n) - \nabla g_1(y_n) - \lambda_n(A_1z_n - A_1y_n)\rangle. \qquad (3.5)
\end{aligned}$$

By applying the definition of y_n, we have $\nabla g_1(z_n) - \lambda_n A_1 z_n \in \nabla g_1(y_n) + \lambda_n B_1$. Since $B_1 : E_1 \to 2^{E_1^*}$ is a maximal monotone mapping, there exists $a_n \in B_1y_n$ such that $\nabla g_1(z_n) - \lambda_n A_1 z_n = \nabla g_1(y_n) + \lambda_n a_n$, it follows that

$$a_n = \frac{1}{\lambda_n}(\nabla g_1(z_n) - \nabla g_1(y_n) - \lambda_n A_1 z_n). \qquad (3.6)$$

Since $0 \in (A_1 + B_1)x^*$ and $A_1y_n + a_n \in (A_1 + B_1)y_n$, it follows from Lemma 2.8 that $A_1 + B_1$ is maximal monotone, hence

$$\langle y_n - x^*, A_1y_n + a_n\rangle \geq 0. \qquad (3.7)$$

On substituting (3.6) into (3.7), we get

$$\frac{1}{\lambda_n}\langle y_n - x^*, \nabla g_1(z_n) - \nabla g_1(y_n) - \lambda_n A_1z_n + \lambda_n A_1y_n\rangle \geq 0.$$

That is

$$\langle y_n - x^*, \nabla g_1(z_n) - \nabla g_1(y_n) - \lambda_n(A_1z_n - A_1y_n)\rangle \geq 0. \qquad (3.8)$$

Combining (3.5) and (3.8), and using (1.3), we have

$$\begin{aligned}
D_{g_1}(x^*, u_n) &\leq D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \lambda_n\langle y_n - u_n, A_1y_n - A_1z_n\rangle \\
&\leq D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \lambda_n||y_n - u_n||\ ||A_1y_n - A_1z_n|| \\
&\leq D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \mu\|y_n - u_n\|\ \|y_n - z_n\| \\
&\leq D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \mu\big(\|y_n - u_n\|^2 + \|y_n - z_n\|^2\big) \\
&\leq D_{g_1}(x^*, z_n) - \big(1 - \frac{\mu}{\sigma}\big)D_{g_1}(y_n, z_n) - \big(1 - \frac{\mu}{\sigma}\big)D_{g_1}(y_n, u_n) \qquad (3.9) \\
&\leq D_{g_1}(x^*, z_n). \qquad (3.10)
\end{aligned}$$

Also, from (2.6) and (3.2), we get

$$\begin{aligned} D_{g_1}(x^*, z_n) &= D_{g_1}\Big((\nabla^{g_1}_{E_1})^{-1}\big(\nabla^{g_1}_{E_1}(w_n) - \gamma K^*(\nabla^{g_2}_{E_2}(Kw_n - \nabla^{g_2}_{E_2}(J^{g_2}_{\lambda B_2}Kw_n))))\Big) \\ &\le D_{g_1}(x^*, w_n) - (\gamma\xi_{1,n} - \xi_{2,n}) D_{g_1^*}\big(K^*(\nabla^{g_2}_{E_2}(Kw_n)), K^*(\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda B_2}Kw_n))\big) \quad (3.11) \\ &\le D_{g_1}(x^*, w_n) \quad (3.12) \\ &= D_{g_1}\Big(x^*, (\nabla^{g_1}_{E_1})^{-1}\big(\nabla^{g_1}_{E_1}(x_n) + \theta_n(\nabla^{g_1}_{E_1}(x_{n-1}) - \nabla^{g_1}_{E_1}(x_n))\big)\Big) \\ &\le (1-\theta_n) D_{g_1}(x^*, x_n) + \theta_n D_{g_1}(x^*, x_{n-1}). \quad (3.13) \end{aligned}$$

From (2.1), (3.2), (3.9) and (3.10), we get

$$\begin{aligned} D_{g_1}(x^*, x_{n+1}) &\le D_{g_1}\Big(x^*, (\nabla^{g_1}_{E_1})^{-1}\big(\alpha_n \nabla^{g_1}_{E_1}(u) + \beta_n \nabla^{g_1}_{E_1}(x_n) + \delta_n \nabla^{g_1}_{E_1}(u_n)\big)\Big) \\ &\le V_{g_1}\Big(x^*, \alpha_n \nabla^{g_1}_{E_1}(u) + \beta_n \nabla^{g_1}_{E_1}(x_n) + \delta_n \nabla^{g_1}_{E_1}(u_n)\Big) \\ &= g_1(x^*) - \langle x^*, \alpha_n \nabla^{g_1}_{E_1}(u) + \beta_n \nabla^{g_1}_{E_1}(x_n) + \delta_n \nabla^{g_1}_{E_1}(u_n)\rangle \\ &+ g_1^*\big(\alpha_n \nabla^{g_1}_{E_1}(u) + \beta_n \nabla^{g_1}_{E_1}(x_n) + \delta_n \nabla^{g_1}_{E_1}(u_n)\big) \\ &\le \alpha_n g_1(x^*) + \beta_n g_1(x^*) + \delta_n g_1(x^*) - \beta_n \langle x^*, \nabla^{g_1}_{E_1}(x_n)\rangle \\ &- \delta_n \langle x^*, \nabla^{g_1}_{E_1}(u_n)\rangle - \alpha_n \langle x^*, \nabla^{g_1}_{E_1}(u)\rangle + \beta_n g_1^*(\nabla^{g_1}_{E_1}(x_n)) \\ &+ \delta_n g_1^*(\nabla^{g_1}_{E_1}(u_n)) + \alpha_n g_1^*(\nabla^{g_1}_{E_1}(u)) - \beta_n\delta_n \rho_r^*(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u_n)\|) \\ &- \beta_n\delta_n \rho_r^*(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u)\|) \\ &\le \beta_n\Big(g_1(x^*) - \langle x^*, \nabla^{g_1}_{E_1}(x_n)\rangle + g_1^*(\nabla^{g_1}_{E_1}(x_n))\Big) \\ &+ \delta_n\Big(g_1(x^*) - \langle x^*, \nabla^{g_1}_{E_1}(u_n)\rangle + g_1^*(\nabla^{g_1}_{E_1}(u_n))\Big) \\ &+ \alpha_n\Big(g_1(x^*) - \langle x^*, \nabla^{g_1}_{E_1}(u)\rangle + g_1^*\Big(\nabla^{g_1}_{E_1}(u)\Big)\Big) \\ &- \beta_n\delta_n \rho_r^*\big(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u_n)\|\big) \\ &= \beta_n V_{g_1}(x^*, \nabla^{g_1(x_n)}_{E_1}) + \delta_n V_{g_1}(x^*, \nabla^{g_1}_{E_1}(u_n)) + \alpha_n V_{g_1}(x^*, \nabla^{g_1}_{E_1}(u)) \\ &- \beta_n\delta_n \rho_r^*\big(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u_n)\|\big) \\ &\le \beta_n D_{g_1}(x^*, x_n) + \delta_n D_{g_1}(x^*, u_n) + \alpha_n D_{g_1}(x^*, u) \\ &- \beta_n\delta_n \rho_r^*\big(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u_n)\|\big) \\ &\le \beta_n D_{g_1}(x^*, x_n) \\ &+ \delta_n\Big(D_{g_1}(x^*, w_n) - \big(1 - \frac{\mu}{\sigma}\big) D_{g_1}(y_n, z_n) - \big(1 - \frac{\mu}{\sigma}\big) D_{g_1}(y_n, u_n) \\ &- (\gamma\xi_{1,n} - \xi_{2,n}) D_{g_1^*}\big(K^*(\nabla^{g_2}_{E_2}(Kw_n)), K^*(\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda B_2}Kw_n))\big) \end{aligned}$$

$$
\begin{aligned}
&+\alpha_n D_{g_1}(x^*,u)-\beta_n\delta_n\rho_r^*\big(\|\nabla_{E_1}^{g_1}(x_n)-\nabla_{E_1}^{g_1}(u_n)\|\big)\\
&\le \beta_n D_{g_1}(x^*,x_n)+\delta_n(1-\theta_n)\big(D_{g_1}(x^*,x_n)+\delta_n\theta_n D_{g_1}(x^*,x_{n-1})\\
&-\delta_n\big(1-\frac{\mu}{\sigma}\big)\big(D_{g_1}(y_n,z_n)-D_{g_1}(y_n,u_n)\big)\\
&-\delta_n(\gamma\xi_{1,n}-\xi_{2,n})D_{g_1^*}\big(K^*(\nabla_{E_2}^{g_2}(Kw_n)),K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2}Kw_n))\big)\\
&+\alpha_n D_{g_1}(x^*,u)-\beta_n\delta_n\rho_r^*\big(\|\nabla_{E_1}^{g_1}(x_n)-\nabla_{E_1}^{g_1}(u_n)\|\big)\\
&\le (1-\alpha_n-\delta_n\theta_n)D_{g_1}(x^*,x_n)+\delta_n\theta_n D_{g_1}(x^*,x_{n-1})+\alpha_n D_{g_1}(x^*,u)\\
&-\delta_n\big(1-\frac{\mu}{\sigma}\big)\big(D_{g_1}(y_n,z_n)-D_{g_1}(y_n,u_n)\big)\\
&-\delta_n(\gamma\xi_{1,n}-\xi_{2,n})D_{g_1^*}\big(K^*(\nabla_{E_2}^{g_2}(Kw_n)),K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2}Kw_n)\big)\\
&-\beta_n\delta_n\rho_r^*\big(\|\nabla_{E_1}^{g_1}(x_n)-\nabla_{E_1}^{g_1}(u_n)\|\big) \qquad (3.14)\\
&\le (1-\alpha_n-\delta_n\theta_n)D_{g_1}(x^*,x_n)+\delta_n\theta_n D_{g_1}(x^*,x_{n-1})+\alpha_n D_{g_1}(x^*,u)\\
&\le \max\{D_{g_1}(x^*,x_n),D_{g_1}(x^*,x_{n-1}),D_{g_1}(x^*,u)\},\ \forall\, n\ge 1. \qquad (3.15)
\end{aligned}
$$

By induction, we obtain that

$$D_{g_1}(x^*,x_n)\le \max\{D_{g_1}(x^*,x_1),D_{g_1}(x^*,x_0),D_{g_1}(x^*,u)\}.$$

Hence, $\{D_{g_1}(x^*,x_n)\}$ is bounded and therefore we conclude that from Lemma 2.9 that $\{x_n\}$ is bounded. More so, $\{w_n\},\{z_n\},\{y_n\}$ and $\{u_n\}$ are bounded. The remaining proof is divided into two cases.

Case A: If there exists $n_0\in\mathbb{N}$ such that $\{D_{g_1}(x^*,x_n)\}_{n=n_0}^{N}$ is decreasing, then $\{D_{g_1}(x^*,x_n)\}_{n\in\mathbb{N}}$ is convergent. Thus, we have that $D_{g_1}(x^*,x_n)-D_{g_1}(x^*,x_{n+1})\to 0$, as $n\to\infty$. Hence, from (3.14), we have that

$$
\begin{aligned}
&\delta_n\big(1-\frac{\mu}{\sigma}\big)\Big(D_{g_1}(y_n,z_n)-D_{g_1}(y_n,u_n)\Big)\\
&-\delta_n(\gamma\xi_{1,n}-\xi_{2,n})D_{g_1^*}\big(K^*(\nabla_{E_2}^{g_2}(Kw_n)),K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2}Kw_n))\big)\\
&\le (1-\alpha_n)D_{g_1}^*(x^*,x_n)-D_{g_1}(x^*,x_{n+1})+\delta_n\theta_n\big(D_{g_1}(x^*,x_{n-1})-D_{g_1}(x^*,x_n)\big)\\
&+\alpha_n D_{g_1}(x^*,u). \qquad (3.16)
\end{aligned}
$$

On applying condition (i) and (ii), we obtain that

$$\lim_{n\to\infty} D_{g_1}(y_n,z_n)=0=\lim_{n\to\infty} D_{g_1}(y_n,u_n). \tag{3.17}$$

From Lemma 2.7, we get that

$$\lim_{n\to\infty}\|y_n-z_n\|=0=\lim_{n\to\infty}\|y_n-u_n\|. \tag{3.18}$$

Since g_1 is bounded and uniformly smooth on bounded sets of E_1, it follows that $\nabla_{E_1}^{g_1}$ is uniformly continuous on bounded subsets of E_1. Thus, we conclude from (3.18) that

$$\lim_{n\to\infty}\|\nabla_{E_1}^{g_1}(y_n)-\nabla_{E_1}^{g_1}(z_n)\|=0. \tag{3.19}$$

From (3.18), we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.20}$$

Also, from (3.16), we have

$$\lim_{n\to\infty} \beta_n\delta_n\rho_r^*\left(\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(u_n)\| \right) = 0 \tag{3.21}$$

$$= \lim_{n\to\infty} \delta_n(\gamma\xi_{1,n} - \xi_{2,n})D_{g_1^*}\big(K^*(\nabla_{E_2}^{g_2}(Kw_n)), K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2}Kw_n))\big). \tag{3.22}$$

By Lemma 2.7 and from properties of the functions $\rho_r,\ D_{g_1}^*$ and K, we have

$$\lim_{n\to\infty} \|K^*(\nabla_{E_2}^{g_2}(Kw_n)) - K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2}Kw_n))\| = 0, \tag{3.23}$$

and

$$\lim_{n\to\infty} (\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(u_n)\|) = 0. \tag{3.24}$$

Employing Lemma 2.7, we arrive at

$$\lim_{n\to\infty} \|Kw_n - J_{\lambda B_2}^{g_2}Kw_n) = 0. \tag{3.25}$$

and

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.26}$$

In view of (3.2), we obtain that

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.27}$$

From (3.20) and (3.26), we ge that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.28}$$

From (3.2), it is easy to see that

$$\begin{aligned} \|\nabla_{E_1}^{g_1}(x_{n+1}) - \nabla_{E_1}^{g_1}(x_n)\| &\le \alpha_n\|\nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(x_n)\| + \beta_n\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(x_n)\| \\ &\quad + \delta_n\|\nabla_{E_1}^{g_1}(u_n) - \nabla_{E_1}^{g_1}(x_n)\|. \end{aligned} \tag{3.29}$$

Hence, we have from (3.29) and condition (i) of (3.2) that

$$\lim_{n\to\infty} \|\nabla_{E_1}^{g_1}(x_{n+1}) - \nabla_{E_1}^{g_1}(x_n)\| = 0. \tag{3.30}$$

Since $\nabla_{E_1}^{g_1}$ is norm to norm uniformly continuous on bounded subset of E_1^*, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.31}$$

From (3.18) and (3.26), we get that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.32}$$

From (3.2), we obtain from (3.31)

$$\|\nabla_{E_1}^{g_1}(w_n) - \nabla_{E_1}^{g_1}(x_n)\| = \theta_n\|\nabla_{E_1}^{g_1}(x_{n-1}) - \nabla_{E_1}^{g_1}(x_n)\| \to 0,\ \text{as } n\to\infty.. \tag{3.33}$$

Using the fact that $\nabla^{g_1}_{E_1}$ is norm to norm uniformly continuous on bounded subset of E_1^*, we have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.34}$$

Lastly, with (3.27) and (3.34), we arrive at

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.35}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded and E_1 is reflexive, we deduce that there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to z. Also, from (3.28), (3.32), (3.34) and (3.35), we have that there exist subsequences $\{u_{n_j}\}_{j\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$, $\{y_{n_j}\}_{j\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$, $\{w_{n_j}\}_{j\in\mathbb{N}}$ of $\{w_n\}_{n\in\mathbb{N}}$ and $\{z_{n_j}\}_{j\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ converge weakly to z respectively. Hence, from (3.25) and the demiclosedness principle we have that $J^{g_2}_{\lambda B_2}(Kz) = Kz$, therefore we conclude that $Kz \in B_2^{-1}(0)$. To show that $z \in (A_1 + B_1)^{-1}(0)$. Let $(v, w) \in G(A_1 + B_1)$, we have $w - A_1 v \in B_1 v$. From the definition of y_n, we observe that

$$\nabla^{g_1}_{E_1}(z_n) - \lambda_n A_1 z_n \in \nabla^{g_1}_{E_1}(y_n) + \lambda_n B_1 y_n,$$

or equivalently

$$\frac{1}{\lambda_n}(\nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n) - \lambda_n A_1 z_n) \in B_1 y_n.$$

By the maximal monotonicity of B_1, we get

$$\langle v - y_n, w - A_1 v + \frac{1}{\lambda_n}(\nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n) - \lambda_n A_1 z_n)\rangle \geq 0.$$

Also, from the monotonicity of A_1, we have

$$\begin{aligned}
\big\langle v - y_n, w\big\rangle &\geq \langle v - y_n, A_1 v + \frac{1}{\lambda_n}(\nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n) - \lambda_n A_1 z_n)\big\rangle \\
&= \langle v - y_n, A_1 v - A_1 z_n\rangle + \frac{1}{\lambda_n}\langle v - y_n, \nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n)\rangle \\
&= \langle v - y_n, A_1 v - A_1 y_n\rangle + \langle v - y_n, A_1 y_n - A_1 z_n\rangle \\
&\quad + \frac{1}{\lambda_n}\langle v - y_n, \nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n)\rangle \\
&\geq \langle v - y_n, A_1 y_n - A_1 z_n\rangle + \frac{1}{\lambda_n}\langle v - y_n, \nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n)\rangle.
\end{aligned} \tag{3.36}$$

Since A_1 is Lipschitz continuous and $y_{n_j} \rightharpoonup z$, it follows from (3.18) and (3.19) that

$$\langle v - z, w\rangle \geq 0.$$

By the monotonicity of $A_1 + B_1$, we get $0 \in (A_1 + B_1)z$, that is $z \in (A_1 + B_1)^{-1}(0)$. Hence $z \in \Omega$.

Next, we show that $\{x_n\}$ converges strongly to z, where $z = P^{g_1}_{\Omega} u$.

From Lemma 2.11, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - x^*, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(x^*)\rangle &= \lim_{j\to\infty}\langle x_{n_j} - x^*, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(x^*)\rangle \\ &= \langle z - x^*, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(x^*)\rangle \\ &\le 0, \end{aligned} \tag{3.37}$$

and hence from (3.31), we obtain

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(x^*)\rangle \le 0. \tag{3.38}$$

Using Lemma 2.4, (3.10) and (3.12), we obtain

$$\begin{aligned} D_{g_1}(z, x_{n+1}) &\le D_{g_1}\Big(z, (\nabla_{E_1}^{g_1})^{-1}(\beta_n\nabla_{E_1}^{g_1}(x_n) + \delta_n\nabla_{E_1}^{g_1}(u_n) + \alpha_n\nabla_{E_1}^{g_1}(u))\Big) \\ &= V_{g_1}(z, \beta_n\nabla_{E_1}^{g_1}(x_n) + \delta_n\nabla_{E_1}^{g_1} + \alpha_n\nabla_{E_1}^{g_1}(u)) - \alpha_n(\nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)) \\ &+ \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \\ &= \beta_n D_{g_1}(z, x_n) + \delta_n D_{g_1}(z, u_n) + \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \\ &\le \beta_n D_{g_1}(z, x_n) + \delta_n D_{g_1}(z, w_n) + \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \\ &\le \beta_n D_{g_1}(z, x_n) + \delta_n((1-\theta_n)D_{g_1}(z, x_n) + \theta_n D_{g_1}(z, x_{n-1})) \\ &+ \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \\ &\le (1 - \alpha_n - \delta_n\theta_n)D_{g_1}(z, x_n) + \delta_n\theta_n D_{g_1}(z, x_{n-1}) \\ &+ \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle. \end{aligned} \tag{3.39}$$

By applying (3.39) and Lemma 2.12, we have that $x_n \to z$.
Case B: Assume $\{D_{g_1}(z, x_n)\}$ is non-decreasing. Set Γ_n of Lemma 2.13, as $\Gamma_n := D_{g_1}(z, x_n)$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough), defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Then τ is non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$. Thus

$$0 < \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \ \forall\, n \ge n_0,$$

this implies that

$$D_{g_1}(z, x_{\tau(n)}) \le D_{g_1}(z, x_{\tau(n)+1}), \ n > n_0.$$

Since $\{D_{g_1}(z, x_{\tau(n)})\}$ is bounded, therefore $\lim_{n\to\infty} D_{g_1}(z, x_{\tau(n)})$ exists. Then the following estimates can be obtained, using same argument as in case A above.

$$\begin{cases} \lim_{n\to\infty} \|y_{\tau(n)} - z_{\tau(n)}\| = 0, \\ \lim_{n\to\infty} \|Kw_{\tau(n)} - J_{\lambda B_2}^{g_2} Kw_{\tau(n)}\| = 0, \\ \lim_{n\to\infty} \|z_{\tau(n)} - x_{\tau(n)}\| = 0, \\ \lim_{n\to\infty} \|w_{\tau(n)} - x_{\tau(n)}\| = 0, \\ \limsup_{n\to\infty}\langle x_{\tau(n)} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \le 0. \end{cases} \tag{3.40}$$

From (3.39) and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have

$$D_{g_1}(z, x_{\tau(n)}) \leq (1-\alpha_{\tau(n)})D_{g_1}(z, x_{\tau(n)}) + \delta_{\tau(n)}\theta_{\tau(n)}\big(D_{g_1}(z, x_{\tau(n)-1} - D_{g_1}(z, x_{\tau(n)})\big) \\ + \alpha_{\tau(n)}\langle x_{\tau(n)+1} - z, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)\rangle.$$

and hence

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0,$$

for all $n \geq n_0$, we have $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_{k+1} \leq \Gamma_k$, for $\tau(n) \leq k \leq n$. This gives for all $n \geq n_0$

$$0 < \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

This implies that $\lim_{n\to\infty} \Gamma_n = 0$ which yields that $\lim_{n\to\infty} D_{g_1}(z, x_n) = 0$. Hence, $x_n \to z = P^{g_1}_{\Omega} u$ as $n \to \infty$. □

Remark 3.3. Our main result improve and generalize the main results of [22, 23, 33, 40, 45] in the following ways:

- (i) We extend Theorem 3.1 of [40] from 2-uniformly Banach spaces which are uniformly smooth to a reflexive Banach space and also extend the results of [22, 23, 45] from real Hilbert spaces to reflexive Banach spaces.
- (ii) We relax the strict assumption of the mapping A in [22, 23, 33] with the weaker assumption that A is a monotone and L-Lipschitz continuous mapping.

4. Numerical examples

In this section, we give a couple of examples to implement our main result.

Example 4.1. This is an implementation of our result in infinite dimensional Hilbert space with our application to split feasibility problem. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $K : H_1 \to H_2$ be a bounded linear operator with its adjoint K^* and Θ denote the solution set of (1.4). Let $H_1 = H_2 = L_2([0,1])$ with norm

$$\|x\|_2 = \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}},$$

and inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt,$$

for all $x, y \in L_2([0,1])$.
Now, let

$$C = \{x \in L_2([0,1]) : \|x\| \leq 1\},$$

and

$$Q = \{x \in L_2([0,1]) : \langle \frac{t}{2}, x\rangle = 0\}.$$

Let $K : L_2([0,1]) \to L_2([0,1])$ be a mapping defined by $(Kx)(t) = \frac{x(t)}{3}$ for all $x \in L_2([0,1])$. Then, we have $(K^*x)(t) = \frac{x(t)}{3}$ and $\|K\| = \frac{1}{3}$. We see that the $\Theta \neq \emptyset$ because $x^*(t) = 0$ is a solution. We define

$$A_1(x) = \nabla\left(\frac{1}{2}\|Kx - P_QKx\|^2\right) = K^*(I - P_Q)Kx,\ B_1(x) = N_C(x)$$

and

$$B_2(x) = N_Q(x) \text{ for all } x \in L_2([0,1]).$$

For our algorithm, we take

$$\alpha_n = \frac{1}{12n+3},\ \beta_n = \frac{8n+1}{12n+3},\ \delta_n = \frac{4n+1}{12n+3},$$

$$\gamma = 0.002,\ l = 0.0001,\ \mu = 0.03 \text{ and } \theta_n = \frac{1}{4}.$$

We present the result of this experiment in Figure 1 with $\|x_{n+1} - x_n\|_2 = 10^{-4}$ and varying initial values of x_0 and x_1 as follows:

(I) $x_0 = t^{\frac{2}{3}} + 11t$ and $x_1 = t$;
(II) $x_0 = 2t$ and $x_1 = \cos t$;
(III) $x_0 = -2t + 5$ and $x_1 = t + 1$;
(IV) $x_0 = 2t$ and $x_1 = \frac{7t^2}{11}$;

Example 4.2. Let $E_1 = E_2 = E = \mathbb{R}^2$ be the two-dimensional Euclidean space of the real number with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\langle x, y \rangle = x \cdot y = x_1y_1 + x_2y_2$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|x\| = (x_1^2 + x_1^2)^{\frac{1}{2}}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Let $B_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $B_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined respectively by

$$B_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Since B_1 and B_2 are positive definite, they are maximal monotone operators. Also, let $A_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A_1(x) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Now, define $h_i : \mathbb{R} \to (-\infty, +\infty]$ by $h_i(x) = \frac{x^2}{2}$ for $i = 1, 2$, then $\nabla h_i(x) = x$. We also define $g_1 = g_2 = g$ by

$$g : \mathbb{R}^2 \to (-\infty, +\infty], \quad g(x) = h_1(x_1) + h_2(x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2}, \quad x = (x_1, x_2).$$

Therefore, we have

$$\nabla g(x) = (\nabla h_1(x_1), \nabla h_2(x_2)) = (x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

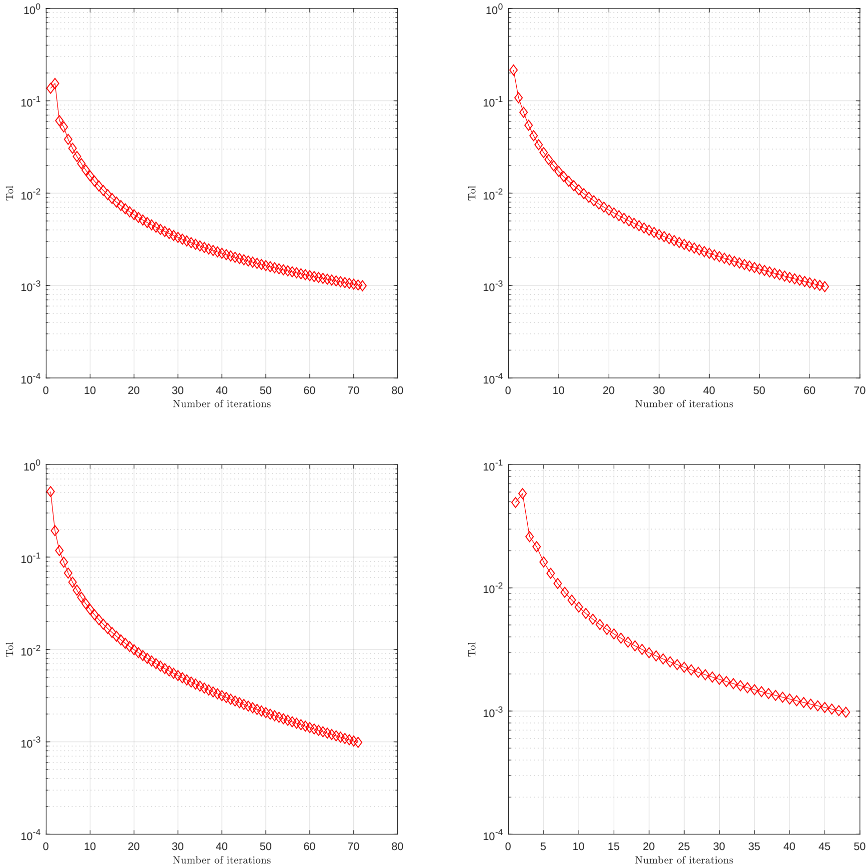

FIGURE 1. Example 4.1. Top left: Case I, Top right: II, Bottom left: III, Bottom right: IV.

For $\lambda > 0$, we compute the resolvents of B_1 and B_2 as follows:

$$J^{g_1}_{\lambda B_1} = \nabla g_1 + rB_1 = \begin{pmatrix} 1+\lambda & 2\lambda \\ 0 & 1+\lambda \end{pmatrix}, \quad (\nabla g_1 + rB_1)^{-1} = \frac{1}{(1+\lambda)^2}\begin{pmatrix} 1+\lambda & -2\lambda \\ 0 & 1+\lambda \end{pmatrix}$$

and

$$J^{g_2}_{\lambda B_2} = \nabla g_1 + rB_2 = \begin{pmatrix} 1+\lambda & 2\lambda \\ 2\lambda & 1+\lambda \end{pmatrix},$$

$$(\nabla g_1 + rB_2)^{-1} = \frac{1}{1+6\lambda+\lambda^2}\begin{pmatrix} 1+5\lambda & -2\lambda \\ -2\lambda & 1+\lambda \end{pmatrix}.$$

Let the operator $K : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$K(x) = (2x_1 - x_2, x_1 + 2x_2) \text{ for all } x = (x_1, x_2) \in \mathbb{R}^2$$

and $K^*: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$K^*(y) = (2y_1 - y_2, y_1 + 2y_2) \text{ for all } y = (y_1, y_2) \in \mathbb{R}^2.$$

For this experiment, we choose the parameters

$$\alpha_n = \frac{3n}{4n^2+5n+3}, \ \beta_n = \frac{n^2+3}{4n^2+5n+3}, \ \delta_n = \frac{3n^2+2n}{4n^2+5n+3},$$

$$\gamma = 0.002, \ l = 0.0001, \ \mu = 0.03 \text{ and } \theta_n = \frac{1}{4}.$$

For $u = 0.1$ and initial values of x_0 and x_1, we report our test for the following cases in Figure 2 with $\|x_{n+1} - x_n\| = 10^{-5}$.

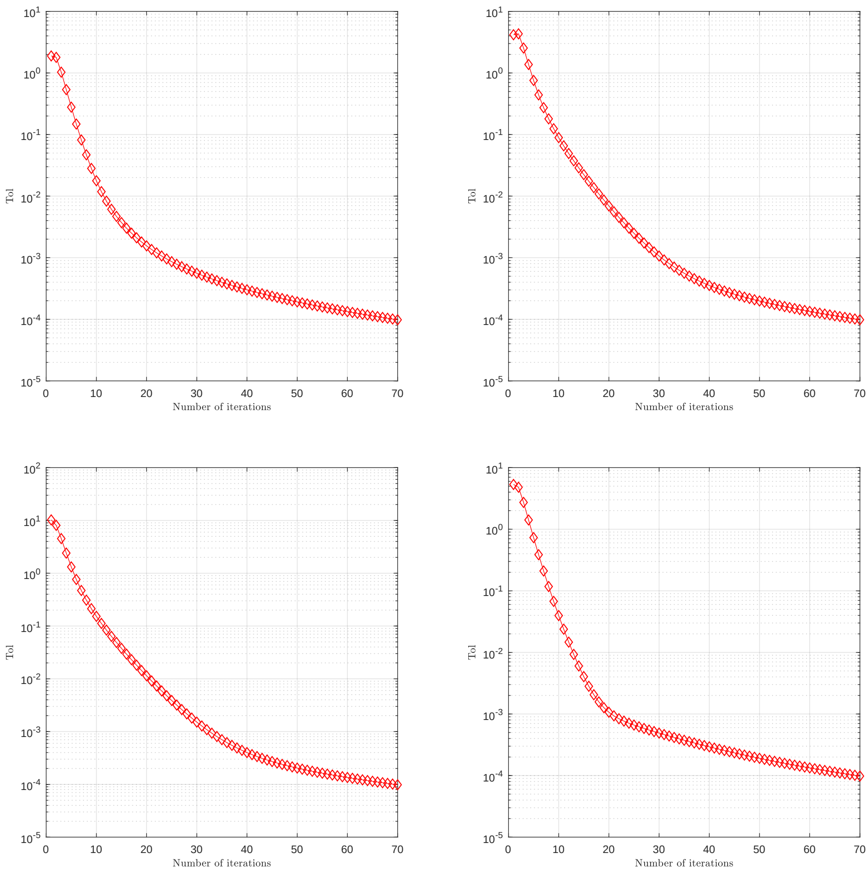

FIGURE 2. Example 4.2. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

Case 1. $x_0 = [5, -5]$ and $x_1 = [3, 5]$;
Case 2. $x_0 = [-5, -5]$ and $x_1 = [10, 10]$;
Case 3. $x_0 = [10, 10]$ and $x_1 = [20, 20]$;
Case 4. $x_0 = [10, -5]$ and $x_1 = [5, 15]$.

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Existence of positive solutions to impulsive nonlinear differential systems of second order with two point boundary conditions

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Abstract. In this paper the authors consider the existence of positive solutions to a two point boundary value problem for nonlinear second-order impulsive systems. They use a vector version of Krasnosel'skii's fixed point theorem in cones in their proofs. Examples are provided to illustrate the results.

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1. Introduction

The existence of positive solutions to second order impulsive differential equations and systems has been studied by many authors such as in [7, 9, 10, 11, 12].

Liu *et al.* [10, 11, 12] studied the existence of one and multiple positive solutions to two point boundary value problems for systems of nonlinear second-order singular impulsive differential equations by using fixed point index theory. In [7], He investigated the existence of positive solutions to second order periodic boundary value problems with impulse actions by applying fixed point index theory.

The existence and location of positive solutions for ordinary differential systems has been studied in [4, 8, 13, 14] by using a technique based on a vector version of

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Krasnosel'skii's fixed point theorem in cones. In [8], Herlea considered the system of first order equations with integral boundary conditions

$$\begin{cases} u_1'(t) = f_1(t, u_1, u_2), \\ u_2'(t) = f_2(t, u_1, u_2), \\ u_1(0) - a_1 u_1(1) = g_1[u_1], \\ u_2(0) - a_2 u_2(1) = g_2[u_2], \end{cases}$$

where $f_i,\ f_2 \in C([0,1] \times \mathbb{R}_+^2, \mathbb{R}^+)$ and $g_i : C[0,1] \to \mathbb{R}$, $i = 1, 2$, are linear functionals given by

$$g_i[u] = \int_0^1 u(s) d\gamma_i, \quad i = 1, 2$$

with $g_i[1] < 1$ and $\gamma_i \in C^1[0,1]$ is increasing and satisfies $0 < a_i < 1 - g_i[1]$ for $i = 1, 2$. Herlea obtained the existence and the location of positive solutions by using a vector version of Krasnosel'skii's fixed point theorem in cones.

Precup [14] also used the vector version of Krasnosel'skii's fixed point theorem to study the existence and localization of positive solutions of the nonlinear differential system

$$\begin{cases} u_1''(t) + f_1(t, u_1, u_2) = 0, \\ u_2'(t) + f_2(t, u_1, u_2) = 0, \\ u_1(0) = u_1(1) = 0, \\ u_2(0) = u_2(1) = 0. \end{cases}$$

Other authors have recently studied the existence of solution for system of impulsive differential equations using vector versions of fixed point theorems, such as in [1, 3, 2, 5, 6].

With this background in mind, in this paper we examine the existence and location of positive solutions of the two point boundary value problem for the system of nonlinear second-order impulsive differential equations

$$\begin{cases} -u_1''(t) = f_1(t, u_1(t), u_2(t)), & t \in J', \\ -u_2''(t) = f_2(t, u_1(t), u_2(t)), & t \in J', \\ -\Delta u_1' \mid_{t=t_k} = I_{1,k} u_1(t_k), & k = 1, 2, \cdots, m, \\ -\Delta u_2' \mid_{t=t_k} = I_{2,k} u_2(t_k), & k = 1, 2, \cdots, m, \\ \alpha u_1(0) - \beta u_1'(0) = 0,\ \alpha u_2(0) - \beta u_2'(0) = 0, & \\ \gamma u_1(1) + \delta u_1'(1) = 0,\ \gamma u_2(1) + \delta u_2'(1) = 0, & \end{cases} \tag{1.1}$$

where $\alpha,\ \beta,\ \gamma,\ \delta \geq 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$, $J = [0,1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $f_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_{i,k} \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, $k \in \{1, 2, \cdots, m\}$. Here, $\Delta u' \mid_{t=t_k} = u_1(t_k^+) - u_1(t_k^-)$ and $\Delta u_2' \mid_{t=t_k} = u_2(t_k^+) - u_2(t_k^-)$, where $u_1'(t_k^+)$ and $u_2'(t_k^+)$, ($u_1'(t_k^-)$ and $u_2'(t_k^-)$) denote the right (left) hand limits of $u_1'(t)$ and $u_2'(t)$ at $t = t_k$, respectively.

Motivated by the work mentioned above, here we study the existence and location of positive solution of the system (1.1) using the vector version of Krasnosel'skii's fixed point theorem in cones given in [13]. As we will see, this approach allows the nonlinear terms and impulses in the system to have different types of behaviors in their variables.

2. Preliminaries

In this paper we need the following concepts. For a normed linear space $(X, \|\cdot\|)$, a cone $K \subset X$ is a closed and convex set with $K \setminus \{0\} \neq \emptyset$, $\lambda K \subset K$ for all $\lambda \in R^+$, and $K \cap (-K) = \{0\}$. A cone K in X induces a partial order relation in X that we will denote by $\preceq$; we write $u \preceq v$ if and only if $v - u \in K$. We say that $u \prec v$ if $v - u \in K \setminus \{0\}$ and $u \not\prec v$ if $v - u \notin K \setminus \{0\}$. Finally, $u \succeq v$ means $v \preceq u$.

Consider two cones K_1 and K_2 in X and the corresponding cone $K := K_1 \times K_2$ in X^2. We use the same symbol $\preceq$ to denote the partial order relation induced by K in X^2 as we do for K_1 or K_2 in X. In X^2, $u = (u_1, u_2) \prec v = (v_1, v_2)$ means $u_i \prec v_i$ for $i = 1, 2$. For r, $R \in \mathbb{R}^2_+$ with $r = (r_1, r_2)$ and $R = (R_1, R_2)$, we will write $0 < r < R$ to mean $0 < r_1 < R_1$ and $0 < r_2 < R_2$. Also, we set

$$(K_i)_{r_i,R_i} := \{u \in K_i : r_i \leq \|u\| \leq R_i\},\ i = 1, 2,$$

$$K_{r,R} := \{u \in K : r_i \leq \|u_i\| \leq R_i \ for\ i = 1, 2\},$$

and we see that $K_{r,R} = (K_1)_{r_1,R_1} \times (K_2)_{r_2,R_2}$.

The following vector version of Krasnosel'skii's fixed point theorem in a cone [13, Theorem 2.1] will be used to obtain our main existence result.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space, K_1, $K_2 \subset X$ be two cones in X, $K := K_1 \times K_2$, r, $R \in R^+$ with $0 < r < R$, and $N : K_{r,R} \to K$ given by $N = (N_1, N_2)$ be a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) $N_i(u) \not\prec u_i$ if $\|u_i\| = r_i$ and $N_i(u) \not\succ u_i$ if $\|u_i\| = R_i$;
(b) $N_i(u) \not\succ u_i$ if $\|u_i\| = r_i$ and $N_i(u) \not\prec u_i$ if $\|u_i\| = R_i$.

Then N has a fixed point u in K with $r_i \leq \|u_i\| \leq R_i$ for $i \in \{1, 2\}$.

3. Main Result

We first formulate problem (1.1) as a fixed point problem for a vector-valued mapping $N = (N_1, N_2)$. Then, $u := (u_1, u_2)$ will satisfy an operator system

$$\begin{cases} u_1 = N_1(u_1, u_2), \\ u_2 = N_2(u_1, u_2), \end{cases} \tag{3.1}$$

in the vector conical shell $K_{r,R}$ with $u \in K$ and

$$r_1 \leq \|u_1\| \leq R_1,\ r_2 \leq \|u_2\| \leq R_2.$$

We denote by $G(t, s)$ the Green's function for the boundary value problem

$$\begin{cases} -x''(t) = 0, \\ \alpha x(0) - \beta x'(0) = 0, \\ \gamma x(1) + \delta x'(1) = 0. \end{cases} \tag{3.2}$$

It is given explicitly by

$$G(t, s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1 \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1. \end{cases}$$

The function $G(t,s)$ is positive and satisfies the properties (see [10, p. 552], [11, p. 3775]):

$$\begin{aligned} G(t,s) &\leq G(s,s), \text{ for all } t,\ s \in [0,1], & (3.3)\\ 0<\sigma G(s,s) &\leq G(t,s),\ t\in[a,b],\ s\in[0,1], & (3.4)\end{aligned}$$

where $a \in [0,t_1],\ b\in[t_m,1]$ and $0\leq \sigma = \min\{\frac{(1-b)\gamma+\delta}{\gamma+\delta}, \frac{a\alpha+\beta}{\alpha+\beta}\} \leq 1$.

In this paper, we consider the space

$$PC(J,\mathbb{R}^+) = \{x : [0,1] \longrightarrow \mathbb{R}^+ \mid x_k \in C(J',\mathbb{R}), k=1,\dots,m,$$
$$x(t_k^-) \text{ and } x(t_k^+) \text{ exist},\ k=1,\dots,m, \text{ and } x(t_k^-)=x(t)\}.$$

We see that $PC(J,\mathbb{R}^+)$ is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t\in J}|x(t)|.$$

Let P be the cone of all nonnegative functions in $PC([0,1],\mathbb{R}^+)$.

Definition 3.1. A pair $(u_1,u_2) \in PC(J,\mathbb{R}^+)\times PC(J,\mathbb{R}^+)$ is called a solution of system (1.1) if it satisfies system (1.1).

The following lemma is obvious.

Lemma 3.2. *The vector* $(u_1,u_2) \in PC(J,\mathbb{R}^+)\times PC(J,\mathbb{R}^+)$ *is a solution of the differential system* (1.1) *if and only if* $(u_1,u_2) \in PC^1(J,\mathbb{R}^+)\times PC^1(J,\mathbb{R}^+)$ *is a solution of the integral system*

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 G(t,s)f_1(s,u_1(s),u_2(s))ds + \sum_{k=1}^m G(t,t_k)I_{1,k}(u_1(t_k)),\\ \\ u_2(t) = \displaystyle\int_0^1 G(t,s)f_2(s,u_1(s),u_2(s))ds + \sum_{k=1}^m G(t,t_k)I_{2,k}(u_2(t_k)). \end{cases} \tag{3.5}$$

Let $N : P^2 \to P^2$ be the completely continuous map $N=(N_1,N_2)$ given by

$$N_i(u(t)) = \int_0^1 G(t,s)f_i(s,u(s),v(s))ds + \sum_{k=1}^m G(t,t_k)I_{i,k}(u_i(t_k)) \quad i=1,2.$$

Then (3.5) is equivalent to the fixed point problem

$$u = N(u), \quad u\in P^2.$$

If $v\in P$,

$$u_i(t) := \int_0^1 G(t,s)v(s)ds + \sum_{k=1}^m G(t,t_k)I_{i,k}(u_i(t_k)),$$

and if $u_i(t') = \|u_i\|_{PC}$, then in view of (3.4), for every $t\in[0,1]$, we have

$$u_i(t) \geq \sigma\int_0^1 G(s,s)v(s)ds + \sigma\sum_{k=1}^m G(t_k,t_k)I_{i,k}(u_i(t_k)).$$

If $t' \neq t_k$ for $k = 1, 2, \cdots, m$, then

$$\begin{aligned} u_i(t) &\geq \sigma \int_0^1 G(t', s)v(s)ds + \sigma \sum_{k=1}^{m} G(t_k, t_k)I_{i,k}(u_i(t_k)) \\ &\geq \sigma \int_0^1 G(t', s)v(s)ds + \sigma \sum_{k=1}^{m} G(t', t_k)I_{i,k}(u_i(t_k)) = \sigma u_i(t') = \sigma \|u\|_{PC}. \end{aligned}$$

If $t' = t_k$ for $k = 1, 2, \cdots, m$, then

$$\begin{aligned} u_i(t) &\geq \sigma \int_0^1 G(s, s)v(s)ds + \sigma \sum_{k=1}^{m} G(t', t_k)I_{i,k}(u_i(t_k)) \\ &\geq \sigma \int_0^1 G(t', s)v(s)ds + \sigma \sum_{k=1}^{m} G(t', t_k)I_{i,k}(u_i(t_k)) = \sigma u_i(t') = \sigma \|u\|_{PC}. \end{aligned}$$

Define the cones K_i in P by

$$K_i = \{u_i \in P : u_i(t) \geq \sigma \|u_i\|_{PC} \text{ for all } t \in [a, b]\},\ i = 1, 2,$$

and the product cone $K = K_1 \times K_2$ in X^2. Then $N(K) \subset K$. Before we state our main result we introduce the following notations. For any α_i, $\beta_i > 0$ with $\alpha_i \neq \beta_i$, let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$, and

$$\gamma_1 = \min\{f_1(t, u_1, u_2) :\ a \leq t \leq b,\ \sigma\beta_1 \leq u_1 \leq \beta_1,\ \sigma r_2 \leq u_2 \leq R_2\},$$

$$\gamma_2 = \min\{f_2(t, u_1, u_2) :\ a \leq t \leq b,\ \sigma r_1 \leq u_1 \leq R_1,\ \sigma\beta_2 \leq u_2 \leq \beta_2\},$$

$$\Gamma_1 = \max\{f_1(t, u_1, u_2) :\ 0 \leq t \leq 1,\ \sigma\alpha_1 \leq u_1 \leq \alpha_1,\ \sigma r_2 \leq u_2 \leq R_2\},$$

$$\Gamma_2 = \max\{f_2(t, u_1, u_2) :\ 0 \leq t \leq 1,\ \sigma r_1 \leq u_1 \leq R_1,\ \sigma\alpha_2 \leq u_2 \leq \alpha_2\}.$$

Also, let

$$B = \max\{G(t, s) : 0 \leq t \leq 1,\ 0 \leq s \leq 1\},$$

$$A = \min\{G(t, s) : a \leq t \leq b,\ a \leq s \leq b\},$$

$$\lambda_1 = \min_{1 \leq k \leq m} \{\min\{I_{1,k}(u_1) : \sigma\beta_1 \leq u_1 \leq \beta_1\}\},$$

$$\lambda_2 = \min_{1 \leq k \leq m} \{\min\{I_{2,k}(u_2) : \sigma\beta_2 \leq u_2 \leq \beta_2\}\},$$

$$\Lambda_1 = \max_{1 \leq k \leq m} \{\max\{I_{1,k}(u_1) : \sigma\alpha_1 \leq u_1 \leq \alpha_1\}\},$$

$$\Lambda_2 = \max_{1 \leq k \leq m} \{\max\{I_{2,k}(u_2) : \sigma\alpha_2 \leq u_2 \leq \alpha_2\}\}.$$

Theorem 3.3. *Assume that there exist α_i, $\beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, such that*

$$\begin{array}{ll} B(\Gamma_1 + \Lambda_1 m) \leq \alpha_1, & A(\gamma_1(b - a) + \lambda_1 m) \geq \beta_1, \\ B(\Gamma_2 + \Lambda_2 m) \leq \alpha_2, & A(\gamma_2(b - a) + \lambda_2 m) \geq \beta_2. \end{array} \tag{3.6}$$

Then (1.1) has a positive solution $u = (u_1, u_2)$ with $r_i \leq \|u_i\|_{PC} \leq R_i$, $i = 1, 2$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$. Moreover, the corresponding orbit of u is included in the rectangle $[\sigma r_1, R_1] \times [\sigma r_2, R_2]$.

Proof. If $u \in K_{r,R}$, then $r_1 \le \|u_1\|_{PC} \le R_1$ and $r_2 \le \|u_2\|_{PC} \le R_2$, so from the definition of K,

$$\sigma r_1 \le \|u_1\|_{PC} \le R_1 \quad \text{and} \quad \sigma r_2 \le \|u_2\|_{PC} \le R_2,$$

for $t \in [a,b]$, that is, for $t \in [a,b]$, $u(t) \in [\sigma r_1, R_1] \times [\sigma r_2, R_2]$. Also, if $\|u_1\|_{PC} = \alpha_1$, then $u_1(t) \le \alpha_1$ for $t \in [0,1]$, and

$$\sigma\alpha_1 \le u_1(t) \le \alpha_1 \text{ for all } t \in [a,b].$$

We wish to show that for every $u \in K_{r,R}$ and each $i \in \{1,2\}$, we have

$$\begin{aligned} \|u_i\|_{PC} = \alpha_i \text{ implies } u_i \not\prec N_i(u), \\ \|u_i\|_{PC} = \beta_i \text{ implies } u_i \not\succ N_i(u). \end{aligned} \tag{3.7}$$

If $\|u_1\|_{PC} = \alpha_1$ and $u_1 \prec N_1(u)$, then

$$u_1(t) < N_1(u)(t) \le B(\Gamma_1 + \Lambda_1 m) \le \alpha_1$$

for $t \in [0,1]$, which leads to the contradiction $\alpha_1 < \alpha_1$.

If $\|u_1\|_{PC} = \beta_1$ and $u_2 \succ N_2(u)$, then for $t \in [a,b]$, we obtain

$$\begin{aligned} u_1(t) > N_1(u)(t) &\ge \int_a^b G(t,s) f_1(s, u_1(s), u_2(s))ds + \sum_{k=1}^m G(t,t_k) I_{1,k}(u_1(t_k)) \\ &\ge A(\gamma_1(b-a) + \lambda_1 m) \ge \beta_1, \end{aligned}$$

yielding the contradiction $\beta_1 > \beta_1$. Hence, (3.7) holds for $i = 1$. In a similar way we can show that (3.7) holds for $i = 2$. By Theorem 2.1, we see that N has at least one nonzero fixed point in K. Therefore, system (1.1) has at least one positive solution. This completes the proof of the theorem. □

Analogous to the discussion by Precup in [13] and [14], we examine the situation where f_1 and f_2 are independent of t, i.e., suppose $f_1 = f_1(u_1, u_2)$ and $f_2 = f_2(u_1, u_2)$. If f_1, f_2, $I_{1,k}$, and $I_{2,k}$, $k = 1, 2, \ldots\ldots, m$, satisfy various monotonicity conditions, then we can obtain specific estimates for γ_1, γ_2, Γ_1, Γ_2, λ_1, λ_2, Λ_1, Λ_2. As examples, we have the following cases.

Case 1. If f_1 and f_2 are nondecreasing in u_1 and u_2, and $I_{1,k}$ and $I_{2,k}$ are nondecreasing respectively in u_1 and u_2 for $k = 1, 2, \ldots, m$, then

$$\begin{aligned} \Gamma_1 &= f_1(\alpha_1, R_2), & \gamma_1 &= f_1(\sigma\beta_1, \sigma r_2), \\ \Gamma_2 &= f_2(R_1, \alpha_2), & \gamma_2 &= f_2(\sigma r_1, \sigma\beta_2), \\ \Lambda_1 &= \max_{1\le k\le m}\{I_{1,k}(\alpha_1)\}, & \lambda_1 &= \min_{1\le k\le m}\{I_{1,k}(\sigma\beta_1)\}, \\ \Lambda_2 &= \max_{1\le k\le m}\{I_{2,k}(\alpha_2)\}, & \lambda_2 &= \min_{1\le k\le m}\{I_{2,k}(\sigma\beta_2)\}. \end{aligned}$$

Case 2. If f_1 is nondecreasing in u_1 and u_2, f_2 is nondecreasing in u_1 and non increasing in u_2, and on the other hand $I_{1,k}$ are nondecreasing in u_1 and $I_{2,k}$ are non increasing in u_2 for $k = 1, 2, \ldots, m$, then

$$\begin{array}{ll} \Gamma_1 = f_1(\alpha_1, R_2), & \gamma_1 = f_1(\sigma\beta_1, \sigma r_2), \\ \Gamma_2 = f_2(R_1, \sigma\alpha_2), & \gamma_2 = f_2(\sigma r_1, \beta_2), \\ \Lambda_1 = \max\limits_{1\le k\le m} \{I_{1,k}(\alpha_1)\}, & \lambda_1 = \min\limits_{1\le k\le m} \{I_{1,k}(\sigma\beta_1)\}, \\ \Lambda_2 = \max\limits_{1\le k\le m} \{I_{2,k}(\sigma\alpha_2)\}, & \lambda_2 = \min\limits_{1\le k\le m} \{I_{2,k}(\beta_2)\}. \end{array}$$

Case 3. If f_1 is nondecreasing in u_1 and non increasing in u_2, f_2 is non increasing in u_1 and nondecreasing in u_2, and on the other hand $I_{1,k}$ are non increasing in u_1 and $I_{2,k}$ are nondecreasing in u_2 for $k = 1, 2, \ldots\ldots, m$, then

$$\begin{array}{ll} \Gamma_1 = f_1(\alpha_1, \sigma r_2), & \gamma_1 = f_1(\sigma\beta_1, R_2), \\ \Gamma_2 = f_2(\sigma r_1, \alpha_2), & \gamma_2 = f_2(R_1, \sigma\beta_2), \\ \Lambda_1 = \max\limits_{1\le k\le m} \{I_{1,k}(\sigma\alpha_1)\}, & \lambda_1 = \min\limits_{1\le k\le m} \{I_{1,k}(\beta_1)\}, \\ \Lambda_2 = \max\limits_{1\le k\le m} \{I_{2,k}(\alpha_2)\}, & \lambda_2 = \min\limits_{1\le k\le m} \{I_{2,k}(\sigma\beta_2)\}. \end{array}$$

Case 4. If f_1 and f_2 are nondecreasing in u_1 and nonincreasing in u_2, and $I_{1,k}$ are nondecreasing in u_1 and $I_{2,k}$ are nonincreasing in u_2 for $k = 1, 2, \ldots, m$, then

$$\begin{array}{ll} \Gamma_1 = f_1(\alpha_1, \sigma r_2), & \gamma_1 = f_1(\sigma\beta_1, R_2), \\ \Gamma_2 = f_2(R_1, \sigma\alpha_2), & \gamma_2 = f_2(\sigma r_1, \beta_2), \\ \Lambda_1 = \max\limits_{1\le k\le m} \{I_{1,k}(\alpha_1)\}, & \lambda_1 = \min\limits_{1\le k\le m} \{I_{1,k}(\sigma\beta_1)\}, \\ \Lambda_2 = \max\limits_{1\le k\le m} \{I_{2,k}(\sigma\alpha_2)\}, & \lambda_2 = \min\limits_{1\le k\le m} \{I_{2,k}(\beta_2)\}. \end{array}$$

4. Examples

We conclude this paper with two examples to illustrate Theorem 3.3 in the Cases 1 and 4 above.

Example 4.1. Consider the second-order impulsive system

$$\begin{cases} u_1''(t) + u_1^\theta + u_2^\varepsilon = 0, & 0 < \theta < \varepsilon < 1, \quad t \neq \frac{1}{4}, \quad 0 \le t \le 1, \\ u_2''(t) + u_1^\varepsilon + u_2^\theta = 0, & 0 < \theta < \varepsilon < 1, \quad t \neq \frac{1}{4}, \quad 0 \le t \le 1, \\ -\Delta u_1' \mid_{t=\frac{1}{4}} = c\sqrt{u_1\left(\frac{1}{4}\right)}, & c > 0, \\ -\Delta u_2' \mid_{t=\frac{1}{4}} = d\sqrt{u_2(\frac{1}{4})}, & d > 0, \\ u_1(0) - u_1'(0) = 0, \quad u_1(1) - u_1'(1) = 0, \\ u_2(0) + u_2'(0) = 0, \quad u_2(1) + u_2'(1) = 0. \end{cases} \tag{4.1}$$

We can establish that system (4.1) has at least one positive solution $u = (u_1, u_2)$. Here,

$$f_1(u_1, u_2) = u_1^\theta + u_2^\varepsilon, \quad f_2(u_1, u_2) = u_1^\varepsilon + u_2^\theta,$$

$$I_{1,1}\left(u_1\left(\frac{1}{4}\right)\right) = c\sqrt{u_1\left(\frac{1}{4}\right)}, \quad I_{2,1}\left(u_2\left(\frac{1}{4}\right)\right) = d\sqrt{u_2\left(\frac{1}{4}\right)}.$$

System (4.1) is equivalent to the integral system

$$\begin{cases} u_1(t) = \int_0^1 G(t,s)[u_1(s)^\theta + u_2(s)^\varepsilon]ds + cG\left(t, \frac{1}{4}\right)\sqrt{u_1\left(\frac{1}{4}\right)}, \\ u_2(t) = \int_0^1 G(t,s)[u_1(s)^\varepsilon + u_2(s)^\theta]ds + dG\left(t, \frac{1}{4}\right)\sqrt{u_2\left(\frac{1}{4}\right)}. \end{cases}$$

where $G(t,s)$ is the Green's function

$$G(t,s) = \frac{1}{3}\begin{cases} (2-t)(1+s), & 0 \le s \le t \le 1 \\ (2-s)(1+t), & 0 \le t \le s \le 1 \end{cases}$$

Clearly $B = \frac{9}{4}$ and $A = \sigma$. In this case $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ are both nondecreasing in u_1 and u_2, while $I_{1,1}$ and $I_{2,1}$ are nondecreasing respectively in u_1 and u_2 for u_1, $u_2 \in R^+$, so we are in Case 1. We choose $\alpha_1 = \alpha_2 =: \alpha^*$ and $\beta_1 = \beta_2 =: \beta^*$, with $\beta^* < \alpha^*$, and so $r_1 = r_2 = \beta^*$, $R_1 = R_2 = \alpha^*$, and $\gamma_i = f_i(\sigma\beta^*, \sigma\beta^*)$, $\Gamma_i = f_i(\alpha^*, \alpha^*)$, $\Lambda_i = I_{i,1}(\alpha^*)$, $\lambda_i = I_{i,2}(\sigma\beta^*)$ for $i = 1, 2$. The values of α^* and β^* will be made precise in what follows. Since

$$\lim_{x\to\infty} \frac{f_i(x,x)}{x} = 0, \ \lim_{x\to 0} \frac{f_i(x,x)}{x} = \infty,$$

$$\lim_{x\to\infty} \frac{I_{1,1}(x)}{x} = 0 \ \text{ and } \ \lim_{x\to 0} \frac{I_{2,1}(x)}{x} = \infty,$$

we may find β^* small enough and α^* large enough that the conditions

$$\frac{f_i(\alpha^*, \alpha^*)}{\alpha^*} \le \frac{1}{2B}, \qquad \frac{f_i(\sigma\beta^*, \sigma\beta^*)}{\sigma\beta^*} \ge \frac{1}{2\sigma A(b-a)},$$
$$\frac{I_{i,1}(\alpha^*)}{\alpha^*} \le \frac{1}{2Bm}, \qquad \frac{I_{i,1}(\sigma\beta^*)}{\sigma\beta^*} \ge \frac{1}{2\sigma Am},$$

$i \in \{1, 2\}$, are satisfied. Thus, condition (3.6) holds. Hence, system (4.1) has at least one positive solution (u_1, u_2) with $\beta^* \le \|u_i\|_{PC} \le \alpha^*$ for $i \in \{1, 2\}$.

Example 4.2. Consider the second-order impulsive system

$$\begin{cases} u_1''(t) + \dfrac{u_1^{\frac{1}{4}}}{u_2+1} = 0, & t \ne \frac{1}{2}, \quad 0 \le t \le 1, \\ u_1''(t) + \dfrac{u_1}{u_2+1} = 0, & t \ne \frac{1}{2}, \quad 0 \le t \le 1, \\ -\Delta u_1' \mid_{t=\frac{1}{2}} = u_1^{\frac{1}{3}}(\frac{1}{2}), & \\ -\Delta u_2' \mid_{t=\frac{1}{2}} = e^{-u_2(\frac{1}{2})}, & \\ u_1(0) - u_1'(0) = 0, \quad u_1(1) - u_1'(1) = 0, & \\ u_2(0) + u_2'(0) = 0, \quad u_2(1) + u_2'(1) = 0. & \end{cases} \tag{4.2}$$

Here we have

$$f_1(u_1, u_2) = \frac{u_1^{\frac{1}{4}}}{u_2+1}, \ f_2(u_1, u_2) = \frac{u_1}{u_2+1},$$

$$I_{1,1}\left(u_1\left(\frac{1}{2}\right)\right) = u_1^{\frac{1}{3}}(\frac{1}{2}) \text{ and } I_{2,1}\left(u_2\left(\frac{1}{2}\right)\right) = e^{-u_2(\frac{1}{2})}.$$

System (4.2) is equivalent to the integral system

$$\begin{cases} u_1(t) = \int_0^1 G(t,s)\dfrac{u_1(s)^{\frac{1}{4}}}{u_2(s)+1}ds + G\left(t, \dfrac{1}{2}\right) u_1^{\frac{1}{3}}\left(\dfrac{1}{2}\right), \\ u_2(t) = \int_0^1 G(t,s)\dfrac{u_1(s)}{u_2(s)+1}ds + G\left(t, \dfrac{1}{2}\right) e^{-u_2(\frac{1}{2})}. \end{cases}$$

The Green function $G(t,s)$ is the same as in Example 4.1. In this case $f_1(u_1,u_2)$ and $f_2(u_1,u_2)$ are nondecreasing in u_1 and nonincreasing in u_2. Also, $I_{1,1}$ is nondecreasing in u_1 and $I_{2,1}$ is nonincreasing in u_2 for u_1, $u_2 \in R^+$, so we are in Case 4. We choose $\alpha_1 = \alpha_2 =: \alpha^*$, $\beta_1 = \beta_2 =: \beta^*$, with $\beta^* < \alpha^*$. Then $r_1 = r_2 = \beta^*$, $R_1 = R_2 = \alpha^*$ and $\Gamma_1 = f_1(\alpha^*, \sigma\beta^*)$, $\Gamma_2 = f_2(\alpha^*, \sigma\alpha^*)$, $\gamma_1 = f_1(\sigma\beta^*, \alpha^*)$, $\gamma_2 = f_2(\sigma\beta^*, \beta^*)$, $\Lambda_1 = I_{1,1}(\alpha^*)$, $\lambda_1 = I_{1,1}(\sigma\beta^*)$, $\Lambda_2 = I_{2,1}(\sigma\alpha^*)$, $\lambda_2 = I_{2,1}(\beta^*)$, where α^* and β^* will be made precise below. Since

$$\lim_{x\to\infty} \frac{f_1(x,0)}{x} = 0, \ \lim_{y\to\infty} \frac{f_2(x,\sigma y)}{y} = 0,$$

$$\lim_{x\to\infty} \frac{I_{1,1}(x)}{x} = 0, \quad \text{and} \quad \lim_{y\to\infty} \frac{I_{2,1}(\sigma y)}{y} = 0,$$

we may find $\alpha^* > 0$ large enough so that

$$\frac{f_1(\alpha^*,0)}{\alpha^*} \leq \frac{1}{2B}, \quad \frac{f_2(\alpha^*,\sigma\alpha^*)}{\alpha^*} \leq \frac{1}{2B},$$
$$\frac{I_{1,1}(\alpha^*)}{\alpha^*} \leq \frac{1}{2Bm}, \quad \frac{I_{1,2}(\sigma\alpha^*)}{\alpha^*} \leq \frac{1}{2Bm}.$$

Since

$$\frac{f_1(\alpha^*,\sigma\beta^*)}{\alpha^*} \leq \frac{f_1(\alpha^*,0)}{\alpha^*},$$

we have

$$\frac{f_1(\alpha^*,\sigma\beta^*)}{\alpha^*} \leq \frac{1}{2B}.$$

And since

$$\lim_{x\to 0} \frac{f_1(\sigma x, y)}{x} = \infty, \quad \lim_{y\to 0} \frac{f_2(x,y)}{y} = \infty,$$
$$\lim_{x\to 0} \frac{I_{1,1}(\sigma x)}{x} = \infty, \quad \lim_{y\to 0} \frac{I_{2,1}(y)}{y} = \infty,$$

with α fixed as above, we can choose β small enough that

$$\frac{f_1(\sigma\beta^*,\alpha^*)}{\beta^*} \geq \frac{1}{2A(b-a)}, \quad \frac{f_2(\sigma\beta^*,\beta^*)}{\beta^*} \geq \frac{1}{2A(b-a)},$$
$$\frac{I_{1,1}(\sigma\beta^*)}{\beta^*} \geq \frac{1}{2Am}, \quad \frac{I_{1,2}(\beta^*)}{\beta^*} \geq \frac{1}{2Am}.$$

Conditions (3.6) are satisfied, hence system (4.2) has at least one positive solution $u = (u_1, u_2)$.

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An optimal quadrature formula exact to the exponential function by the phi function method

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Abstract. The numerical integration of definite integrals is essential in fundamental and applied sciences. The accuracy of approximate integral calculations is contingent upon the initial data and specific requirements, leading to the imposition of diverse conditions on the resultant computations. Classical methods for the numerical analysis of definite integrals are known, such as the quadrature formulas of Gregory, Newton-Cotes, Euler, Gauss, Markov, etc. Since the middle of the last century, the theory of constructing optimal formulas for numerical integration based on variational methods began to develop. It should be noted that there are optimal quadrature formulas in the sense of Nikolsky and Sard.
In this paper, we study the problem of constructing an optimal quadrature formula in the sense of Sard. When constructing a quadrature formula, the method of φ-functions is used. The error of the formula is estimated from above using the integral of the square of the function φ from a specific Hilbert space. Next, such a φ function is selected, and the integral of the square in this interval takes the smallest value. The coefficients of the optimal quadrature formula are calculated using the resulting φ function. The optimal quadrature formula in this work is exact on the functions $e^{\sigma x}$ and $e^{-\sigma x}$, where σ is a nonzero real parameter.

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1. Introduction

Many problems of science and technology lead to integral and differential equations or their systems. Solutions to such equations are often expressed in terms of definite integrals. In most cases, these integrals cannot be calculated accurately. Therefore, it is necessary to calculate the approximate value of such integrals with the highest possible accuracy and at a low cost.

Given the known geometric value, the problem of finding the numerical value of integrals is often called quadrature and cubature, respectively. Various quadrature and cubature methods allow the calculation of the integral using a finite number of values of the integrated function. These methods are universal and can be used where other calculation methods fail.

Many researchers have constructed various quadrature formulas based on certain ideas and taking into account the properties of the integrand. Thus, the well-known quadrature formulas of Gregory, Newton-Cotes, Simpson, Euler, Gauss, Chebyshev, Markov and others appeared, still used in practice.

Currently, in the theory of constructing quadrature and cubature formulas, there are the following main approaches: *algebraic, probabilistic, numerical theoretic and functional.*

1. *In the algebraic approach*, it is necessary to choose the nodes and coefficients of quadrature and cubature formulas so that these formulas are accurate for all functions of a particular set F. Taking into account the properties of the integrand. Usually, the set F is taken to be algebraic and trigonometric polynomials whose degrees do not exceed a certain number of m or bounded rational functions.
2. *The probabilistic approach* to constructing cubature formulas is based on the Monte Carlo method.
3. *Number-theoretic approach* to constructing cubature formulas is based on methods of number theory.
4. *For functional approach* to constructing quadrature and cubature formulas, in the functional system, it is considered that the integrands belong to some Banach space, and the difference between the integral and the combination of values of the integrand that approximates it is considered some linear continuous functional. This functional is called the error functional of the cubature formula, and the error of the formula is estimated through the norms of the error functional. By minimizing the norms of the error functional according to the parameters of quadrature and cubature formulas, optimal formulas for numerical integration of various meanings are obtained.

Since the research in this work relates to the latter approach, we will provide an overview of the results in this area.

The construction of quadrature formulas and the study of their error estimates, based on the methods of functional analysis, were first given in the scientific works of A. Sard [20, 21] (minimizing the norm of the error functional can be achieved by adjusting the coefficients at fixed nodes) and S. M. Nikolsky [18] (minimization of the error functional by coefficients and nodes), and the emergence of the theory of cubature formulas is associated with the scientific research of S.L. Sobolev [29].

The works of S.M. Nikolsky, N.P. Korneichuk, N.E. Lushpay, T.N. Busarova, B. Boyanov, V.P. Motorny, A.A. Ligun, A.A. Zhensykbaev, K.I. Oskolkov, M.A. Chakhkiev, T.A. Grankina are devoted to the problems of minimizing the norm of the error functional over coefficients and over nodes in various spaces in the one-dimensional case. For example, detailed results and a complete bibliography are given in the creation of S.M. Nikolsky [19].

Note that there is a spline method, a method of φ-functions and a Sobolev method for constructing optimal formulas obtained by minimizing the norm of the error functional over coefficients at fixed nodes. A.Sard [20, 21], L.F.Meyers [17], G.Coman [6, 7], I.J.Schoenberg [22, 23, 24, 25], S.D.Silliman [25], P.Köhler [15], based on the spline method, and A.Ghizzetti and A.Ossicini [9], F.Lanzara [16], T.Catinaş and G.Coman [5] , using the method of φ-functions, constructed optimal quadrature formulas in the space $L_2^{(m)}$. In constructing optimal cubature formulas using the Sobolev method, the results of S.L. Sobolev [30] on finding the coefficients of optimal quadrature formulas generalized the studies mentioned earlier in which the spline method was applied. The algorithm proposed by S.L. Sobolev in the $L_2^{(m)}$ space was implemented in scientific research by Z.Z.Zhamalov, F.Ya.Zagirova, Kh.M. Shadimetov, A.R. Hayotov and others. Recent results on optimal formulas obtained using the Sobolev method can be found, for example, in the works [1, 27].

Note that the results of this work are closely related to the results of the works [26, 2, 3, 8, 14, 10, 28, 4, 11, 12], which are devoted to the construction of optimal quadrature formulas using the Sobolev method. In particular, our results generalize the results of recent work [13].

2. Statement of the problem

In this work, we study the construction of an optimal quadrature formula using the method of φ-functions. In this regard, consider quadrature formulas of the form

$$\int_a^b f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f), \tag{2.1}$$

where A_k and x_k are the coefficients and the nodes of the quadrature formula. Let the nodes of the formula be located on the segment $[a, b]$ as follows

$$a = x_0 < x_2 < \ldots < x_n = b, \tag{2.2}$$

and $R_n(f)$ is the error of formula (2.1).

Suppose that the integrand function $f(x)$ is from the space $W_{2,\sigma}^{(1,0)}(a, b)$, where $W_{2,\sigma}^{(1,0)}(a, b)$ is the Hilbert space of absolutely continuous functions that are quadratically integrable with the first-order derivative on the interval $[a, b]$. The scalar product of any two functions $f(x)$ and $g(x)$ from this space is defined by the following formula

$$\langle f(x), g(x)\rangle = \int_a^b (f'(x) + \sigma f(x))(g'(x) + \sigma g(x))dx, \tag{2.3}$$

where $\sigma \in \mathbb{R}$ and $\sigma \neq 0$. This space is provided with the corresponding norm

$$\|f(x)\|_{W_{2,\sigma}^{(1,0)}} = \left(\int_a^b (f'(x) + \sigma f(x))^2 dx \right)^{1/2}. \quad (2.4)$$

One of the important problems in the theory of quadrature formulas is the problem of the optimality of quadrature formulas relative to the error of this formula. In this paper, we will consider the problem of optimality of a formula in the sense of Sard. We use the one-to-one correspondence between quadrature formulas and φ - functions in this.

For convenience, we introduce the multi-index notations

$$A = (A_0, A_1, \ldots, A_n) \text{ and } X = (x_0, x_1, \ldots, x_n). \quad (2.5)$$

Definition 2.1. The quadrature formula (2.1) is called *optimal in the sense of Nikolsky* in space $W_{2,\sigma}^{(1,0)}$, if the value

$$F_n(W_{2,\sigma}^{(1,0)}, A, X) = \sup_{f \in W_{2,\sigma}^{(1,0)}} |R_n(f)| \quad (2.6)$$

reaches its smallest value relative to A and X, where A and X are defined in (2.5).

Definition 2.2. The quadrature formula (2.1) is called *optimal in the sense of Sard* in the space $W_{2,\sigma}^{(1,0)}$ if the quantity

$$F_n(W_{2,\sigma}^{(1,0)}, A) = \sup_{f \in W_{2,\sigma}^{(1,0)}} |R_n(f)| \quad (2.7)$$

reaches its smallest value relative to A for fixed X, where A and X are defined in (2.5).

In this work, we solve the problem of constructing an optimal quadrature formula of the form (2.1) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}(a,b)$, i.e., let us find such coefficients of the formula (2.1) that give the smallest value to the quantity (2.7) for fixed X. In this case, we use the method of φ-functions.

Next, the rest of this work is organized as follows. Section 3 describes the method of φ-functions for constructing quadrature formulas of the form (2.1) in the space $W_{2,\sigma}^{(1,0)}$ and provides the relationship between the coefficients and φ-functions. Section 4 is devoted to obtaining φ-functions that give the smallest error value of a quadrature formula of the form (2.1). Using the obtained φ-functions, the coefficients of the optimal quadrature formula of the form (2.1) are calculated.

3. Method of φ - functions for constructing quadrature formulas in the space $W_{2,\sigma}^{(1,0)}$

In this section, we explain the method of φ - functions for constructing optimal quadrature formulas of the form (2.1) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}$. For more details on the φ - function method see, for instance, [5, 16, 9].

Let functions $f(x)$ be from the space $W_{2,\sigma}^{(1,0)}(a,b)$ and for a given positive integer n the nodes of the quadrature formula under consideration are located as in (2.5). Then for each subinterval $[x_{k-1},x_k]$, $k=1,2,\dots,n$, consider the functions φ_k, $k=1,2,\dots,n$ having the following property

$$\varphi_k'(x)-\sigma\varphi_k(x)=1,\quad k=1,2,\dots,n. \tag{3.1}$$

Then the function φ is defined as follows

$$\varphi|_{[x_{k-1},x_k]}=\varphi_k(x),\quad k=1,2,\dots,n. \tag{3.2}$$

That is, the restriction of the function φ to the interval $[x_{k-1},x_k]$ is equal to φ_k.

Let us introduce the following notation

$$I(f):=\int_a^b f(x)dx, \tag{3.3}$$

$$Q_n(f):=\sum_{k=0}^{n}A_kf(x_k). \tag{3.4}$$

Now, using the property of additivity of a definite integral, taking into account the equalities (3.1), from (3.3), we have (see, e.g., [5, 16, 9])

$$\begin{aligned}
I(f):&=\int_a^b f(x)dx=\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}(\varphi_k'(x)-\sigma\varphi_k(x))f(x)dx\\
&=\sum_{k=1}^{n}\Big(\int_{x_{k-1}}^{x_k}\varphi_k'(x)f(x)dx-\sigma\int_{x_{k-1}}^{x_k}\varphi_k(x)f(x)dx\Big)\\
&=\sum_{k=1}^{n}\Big(\varphi_k(x)f(x)\Big|_{x_{k-1}}^{x_k}-\int_{x_{k-1}}^{x_k}\varphi_k(x)f'(x)dx-\sigma\int_{x_{k-1}}^{x_k}\varphi_k(x)f(x)dx\Big)\\
&=\sum_{k=1}^{n}\Big(\varphi_k(x)f(x)\Big|_{x_{k-1}}^{x_k}-\int_{x_{k-1}}^{x_k}\varphi_k(x)(f'(x)+\sigma f(x))\Big)dx\\
&=\sum_{k=1}^{n}\Big(\varphi_k(x_k)f(x_k)-\varphi_k(x_{k-1})f(x_{k-1})\Big)\\
&-\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\varphi_k(x)\Big(f'(x)+\sigma f(x)\Big)dx\\
&=\sum_{k=1}^{n}\varphi_k(x_k)f(x_k)-\sum_{k=1}^{n}\varphi_k(x_{k-1})f(x_{k-1})\\
&-\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\varphi_k(x)\Big(f'(x)+\sigma f(x)\Big)dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \varphi_k(x_k) f(x_k) - \sum_{k=0}^{n-1} \varphi_{k+1}(x_k) f(x_k) \\
&- \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) \Big(f'(x) + \sigma f(x) \Big) dx \\
&= \varphi_n(x_n) f(x_n) + \sum_{k=1}^{n-1} \varphi_k(x_k) f(x_k) - \sum_{k=1}^{n-1} \varphi_{k+1}(x_k) f(x_k) \\
&- \varphi_1(x_0) f(x_0) - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) \Big(f'(x) + \sigma f(x) \Big) dx.
\end{aligned}
$$

From here we have

$$
\begin{aligned}
I(f) : \quad &= \quad -\varphi_1(x_0) f(x_0) + \sum_{k=1}^{n-1} \Big(\varphi_k(x_k) - \varphi_{k+1}(x_k) \Big) f(x_k) \\
& \quad\quad + \varphi_n(x_n) f(x_n) - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) \Big(f'(x) + \sigma f(x) \Big) dx \\
&= \quad A_0 f(x_0) + \sum_{k=1}^{n-1} A_k f(x_k) + A_n f(x_n) + R_n[f]. \qquad (3.5)
\end{aligned}
$$

From (3.5) we get

$$
\begin{aligned}
A_0 &= -\varphi_1(x_0), \\
A_k &= \varphi_k(x_k) - \varphi_{k+1}(x_k), \quad k = 1, 2, \ldots, n-1, \qquad (3.6) \\
A_n &= \varphi_n(x_n)
\end{aligned}
$$

and the error of the formula has the form

$$
\begin{aligned}
R_n[f] &= -\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) \Big(f'(x) + \sigma f(x) \Big) dx \\
&= -\int_a^b \varphi(x) \Big(f'(x) + \sigma f(x) \Big). \qquad (3.7)
\end{aligned}
$$

Remark 3.1. Knowing the function φ from (3.6) we can find the coefficients A_k, $k = 0, 1, \ldots, n$. This method of constructing a quadrature formula is called the method of φ - functions (see, [5, 16, 9]).

Remark 3.2. From the expression (3.7) it is clear that the quadrature formula (2.1) is exact on functions that are a solution to the equation

$$
f'(x) + \sigma f(x) = 0. \qquad (3.8)
$$

Further, in the next section, we are engaged in calculating the coefficients of the optimal quadrature formula of the form (2.1) in the space $W_{2,\sigma}^{(1,0)}(a, b)$.

4. The optimality problem for a quadrature formula

In this section, we will discuss the problem of optimality of a quadrature formula of the form (2.1) in the space $W_{2,\sigma}^{(1,0)}(a,b)$.

Using the Cauchy-Schwartz inequality for the absolute value of the error (3.7) of the quadrature formula (2.1) we have the following

$$\begin{aligned} |R_n(f)| &\leq \|f'(x)+\sigma f(x)\|_{L_2(a,b)} \left(\int_a^b \varphi^2(x)dx \right)^{1/2} \\ &= \|f(x)\|_{W_{2,\sigma}^{(1,0)}} \|\varphi(x)\|_{L_2(a,b)}. \end{aligned} \tag{4.1}$$

It should be noted that here the task of constructing an optimal quadrature formula of the form (2.1) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}(a,b)$ is the task of finding the coefficients $A=(A_0,A_1,\ldots,A_n)$ (for fixed nodes $X=(x_0,x_1,\ldots,x_n)$ satisfying the condition (2.2)) giving the smallest value to the quantity

$$F_n(A) = \int_a^b \varphi^2(x)dx. \tag{4.2}$$

In turn, this problem is equivalent to finding functions $\varphi_k(x)$, $k=1,2,\ldots,n$, satisfying the equation (3.1) and giving the smallest value to the quantity (4.2) on each interval $[x_{k-1},x_k]$, $k=1,2,\ldots,n$.

Next, for the beginning we will find the functions $\varphi_k(x)$, $k=1,2,\ldots,n$ that give the smallest value to the quantity (4.2) and then using the formulas (3.6) we will calculate coefficients A_k, $k=0,1,\ldots,n$ of the optimal quadrature formula (2.1).

4.1. Finding functions φ_k

Now we are engaged in finding the functions φ_k on each interval $[x_{k-1},x_k]$ for $k=1,2,\ldots,n$, which are the solution to the equation

$$y' - \sigma y = 1. \tag{4.3}$$

We will seek a solution to this equation in the form of the product $y=uy_1$ of the functions $u(x)$ and $y_1(x)$, where y_1 is the solution to the corresponding homogeneous equation

$$y' - \sigma y = 0. \tag{4.4}$$

It is easy to check that one of the solutions to the equation (4.4) has the form

$$y_1(x) = e^{\sigma x}. \tag{4.5}$$

Now we can solve equation (4.3). By assumption, the solution to equation (4.3) has the form

$$y(x) = uy_1, \tag{4.6}$$

where $y_1(x)$ is defined by equality (4.5). Then we just need to find the function $u(x)$. To find it, we first calculate the first-order derivative of the unknown function $y(x)$ in (4.6). Then we have

$$y' = u'y_1 + uy_1'. \tag{4.7}$$

Substituting (4.6) into (4.3), taking into account (4.5), we get

$$u'e^{\sigma x} + u\sigma e^{\sigma x} - u\sigma e^{\sigma x} = 1.$$

From here

$$u' = e^{-\sigma x}.$$

Integrating both sides of the last equality we have

$$u(x) = -\frac{1}{\sigma}e^{-\sigma x} + C.$$

This means, taking into account the last equality and (4.5), for the solution of equation (4.3) we obtain

$$y = -\frac{1}{\sigma} + Ce^{\sigma x}. \tag{4.8}$$

Next, on each interval $[x_{k-1}, x_k]$, $k = 1, 2, \dots, n$ we take functions $\varphi_k(x)$ in the form (4.8), i.e.

$$\varphi_k(x) = -\frac{1}{\sigma} + C_1^{(k)}e^{\sigma x}, \quad x \in [x_{k-1}, x_k],\ k = 1, 2, \dots, n. \tag{4.9}$$

From here we conclude that to find the functions $\varphi_k(x)$ we need to find such coefficients $C_1^{(k)}$, $k = 1, 2, \dots, n$, which give the smallest values to the quantity (4.2) on each of the intervals $[x_{k-1}, x_k]$ for $k = 1, 2, \dots, n$. Next, we find $C_1^{(k)}$ such that the integral of the square of the function $\varphi_k(x)$ defined by equality (4.9) on the interval $[x_{k-1}, x_k]$ takes the smallest value. In this regard, consider the following functions

$$\mathcal{F}_k(C_1^{(k)}) = \int\limits_{x_{k-1}}^{x_k} \varphi_k^2(x)dx, \quad k = 1, 2, \dots, n.$$

Then from here, taking into account (4.9), we have

$$\begin{aligned} \mathcal{F}_k(C_1^{(k)}) &= \int\limits_{x_{k-1}}^{x_k} \left(-\frac{1}{\sigma} + C_1^{(k)}e^{\sigma x}\right)^2 dx \\ &= \int\limits_{x_{k-1}}^{x_k} \frac{1}{\sigma^2}dx - 2C_1^{(k)}\frac{1}{\sigma}\int\limits_{x_{k-1}}^{x_k} e^{\sigma x}dx \\ &\quad +(C_1^{(k)})^2\int\limits_{x_{k-1}}^{x_k} e^{2\sigma x}dx,\ k = 1, 2, \dots, n. \end{aligned}$$

Then calculating the first order derivatives of the functions $\mathcal{F}_k(C_1^{(k)})$ with respect to $C_1^{(k)}$ and equating them to zero, we have

$$2C_1^{(k)}\int\limits_{x_{k-1}}^{x_k} e^{2\sigma x}dx - \frac{2}{\sigma}\int\limits_{x_{k-1}}^{x_k} e^{\sigma x}dx = 0,\ k = 1, 2, \dots, n.$$

From the last equalities we get the following

$$C_1^{(k)} = \frac{\frac{1}{\sigma}\int\limits_{x_{k-1}}^{x_k} e^{\sigma x}dx}{\int\limits_{x_{k-1}}^{x_k} e^{2\sigma x}dx} = \frac{2}{\sigma\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)}, \quad k=1,2,\ldots,n. \tag{4.10}$$

It is easy to check that this value of $C_1^{(k)}$ gives the smallest value to the function $\mathcal{F}_k(C_1^{(k)})$ on the interval $[x_{k-1},x_k]$. Then, taking into account (4.10), from (4.9) we have

$$\varphi_k(x) = -\frac{1}{\sigma} + \frac{2e^{\sigma x}}{\sigma\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)}, \quad x\in[x_{k-1},x_k],\ k=1,2,\ldots,n. \tag{4.11}$$

4.2. Calculation of coefficients of the optimal quadrature formula

Now, using (4.11), from the formulas (3.6) we calculate the coefficients A_k, $k=0,1,\ldots,n$ of the optimal quadrature formula of the form (2.1).

First, let's calculate A_0. From (3.6), taking into account $\varphi_1(x)$, we have

$$\begin{aligned} A_0 &= -\varphi_1(x_0) = -\left(-\frac{1}{\sigma}+\frac{2e^{\sigma x_0}}{\sigma\left(e^{\sigma x_1}+e^{\sigma x_0}\right)}\right) \\ &= \frac{e^{\sigma x_1}-e^{\sigma x_0}}{\sigma(e^{\sigma x_1}+e^{\sigma x_0})}. \end{aligned} \tag{4.12}$$

Now let's calculate the coefficients A_k, $k=1,2,\ldots,n-1$. From (3.6), using $\varphi_k(x)$ for $k=1,2,\ldots,n-1$, we have

$$\begin{aligned} A_k &= \varphi_k(x_k)-\varphi_{k+1}(x_k) \\ &= \left(-\frac{1}{\sigma}+\frac{2e^{\sigma x_k}}{\sigma\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)}\right)-\left(-\frac{1}{\sigma}+\frac{2e^{\sigma x_k}}{\sigma\left(e^{\sigma x_{k+1}}+e^{\sigma x_k}\right)}\right) \\ &= \frac{2e^{\sigma x_k}\left(e^{\sigma x_{k+1}}-e^{\sigma x_{k-1}}\right)}{\sigma(e^{\sigma x_{k+1}}+e^{\sigma x_k})(e^{\sigma x_k}+e^{\sigma x_{k-1}})}. \end{aligned} \tag{4.13}$$

Finally, let's calculate the last coefficient A_n. Then, from (3.6), taking into account (4.11), we obtain

$$\begin{aligned} A_n &= -\varphi_n(x_n) = -\left(-\frac{1}{\sigma}+\frac{2e^{\sigma x_n}}{\sigma\left(e^{\sigma x_n}+e^{\sigma x_{n-1}}\right)}\right) \\ &= \frac{e^{\sigma x_n}-e^{\sigma x_{n-1}}}{\sigma(e^{\sigma x_n}+e^{\sigma x_{n-1}})}. \end{aligned} \tag{4.14}$$

Thus, summing up the results of (4.12), (4.13) and (4.14), we obtain the following main theorem of this work.

Theorem 4.1. *In the space $W_{2,\sigma}^{(1,0)}(a,b)$ for each fixed positive integer n, there is a unique quadrature formula that is optimal in the sense of Sard of the form*

$$\int\limits_a^b f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f)$$

with coefficients

$$
\begin{aligned}
A_0 &= \frac{e^{\sigma x_1}-e^{\sigma x_0}}{\sigma(e^{\sigma x_1}+e^{\sigma x_0})},\\
A_k &= \frac{2e^{\sigma x_k}(e^{\sigma x_{k+1}}-e^{\sigma x_{k-1}})}{\sigma(e^{\sigma x_{k+1}}+e^{\sigma x_k})(e^{\sigma x_k}+e^{\sigma x_{k-1}})}, \quad k=1,2,\dots,n-1,\\
A_n &= \frac{e^{\sigma x_n}-e^{\sigma x_{n-1}}}{\sigma(e^{\sigma x_n}+e^{\sigma x_{n-1}})},
\end{aligned}
$$

for fixed nodes x_k, $k=0,1,\dots,n$ *satisfying the inequality* $a=x_0<x_1<\ldots<x_k=b$.

Remark 4.2. It should be noted that for $[a,b]=[0,1]$ and $x_k=kh$, where $k=0,1,\dots,N$, $h=1/N$ from Theorem 1 we obtain the result of the work [13].

5. The norm of φ–function

According to inequality (4.1), we need to calculate the norm of the function φ to get an upper bound of the absolute value of the error (3.7)

$$
\|\varphi\|^2_{L_2(a,b)} = \int_0^1 \varphi^2(x)\mathrm{d}x = \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varphi_k^2(x)\mathrm{d}x. \tag{5.1}
$$

From the expression of the function φ_k in equation (4.11), we can calculate the following

$$
\begin{aligned}
\varphi_k^2(x) &= \left(\frac{-1}{\sigma}+\frac{2e^{\sigma x}}{\sigma(e^{\sigma x_k}+e^{\sigma x_{k-1}})}\right)^2\\
&= \frac{1}{\sigma^2}-\frac{4e^{\sigma x}}{\sigma^2(e^{\sigma x_k}+e^{\sigma x_{k-1}})}+\frac{4e^{2\sigma x}}{\sigma^2(e^{\sigma x_k}+e^{\sigma x_{k-1}})^2}.
\end{aligned}
$$

Substituting the last expression into equation (5.1), we get the following

$$
\begin{aligned}
\int_{x_{k-1}}^{x_k}\varphi_k^2(x)dx &= \int_{x_{k-1}}^{x_k}\left(\frac{1}{\sigma^2}-\frac{4e^{\sigma x}}{\sigma^2(e^{\sigma x_k}+e^{\sigma x_{k-1}})}+\frac{4e^{2\sigma x}}{\sigma^2(e^{\sigma x_k}+e^{\sigma x_{k-1}})^2}\right)dx\\
&= \int_{x_{k-1}}^{x_k}\frac{1}{\sigma^2}dx-\int_{x_{k-1}}^{x_k}\frac{4e^{\sigma x}}{\sigma^2(e^{\sigma x_k}+e^{\sigma x_{k-1}})}dx\\
&\quad+\int_{x_{k-1}}^{x_k}\frac{4e^{2\sigma x}}{\sigma^2\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)^2}dx\\
&= \frac{x_k-x_{k-1}}{\sigma^2}-\frac{4\left(e^{\sigma x_k}-e^{\sigma x_{k-1}}\right)}{\sigma^3\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)}+\frac{2\left(e^{2\sigma x_k}-e^{2\sigma x_{k-1}}\right)}{\sigma^3\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)^2}\\
&= \frac{x_k-x_{k-1}}{\sigma^2}-\frac{2\left(e^{\sigma x_k}-e^{\sigma x_{k-1}}\right)}{\sigma^3\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)}.
\end{aligned}
$$

Thus, putting the obtained expression into equation (5.1), we get the following result

$$\begin{aligned}\|\varphi\|^2_{L_2(a,b)} &= \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varphi_k^2(x)dx = \sum_{k=1}^{N}\left(\frac{x_k-x_{k-1}}{\sigma^2}-\frac{2\left(e^{\sigma x_k}-e^{\sigma x_{k-1}}\right)}{\sigma^3\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)}\right)\\ &= \frac{x_n-x_0}{\sigma^2}-\frac{2}{\sigma^3}\sum_{k=1}^{N}\frac{e^{\sigma x_k}-e^{\sigma x_{k-1}}}{e^{\sigma x_k}+e^{\sigma x_{k-1}}}.\end{aligned}$$

We have the next result in the case of equally spaced nodes $x_k = hk$, $h=\frac{1}{N}$

$$\|\varphi\|^2_{L_2(a,b)} = \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varphi_k^2(x)dx = \frac{1}{\sigma^2}-\frac{2}{\sigma^3 h}\cdot\frac{e^{\sigma h}-1}{e^{\sigma h}+1}.$$

6. Conclusion

In this work, we constructed an optimal quadrature formula in the space $W_{2,\sigma}^{(1,0)}(a,b)$, where $W_{2,\sigma}^{(1,0)}(a,b)$ is the Hilbert space of absolutely continuous functions whose first-order derivatives are square-integrable on the interval $[a,b]$. Here the quadrature sum consists of a linear combination of the values $f(x_k)$ of the function $f(x)$ at the nodes $x_k \in [a,b]$, where $a = x_0 < x_1 < \ldots < x_n = b$. The error of the quadrature formula under consideration is estimated from above using the product of the norm of the integrand and the L_2 norm of the particular φ function from the space $W_{2,\sigma}^{(1,0)}(a,b)$. Moreover, this φ function is determined by an unknown factor on each subinterval. The optimal quadrature formula is obtained by choosing these factors, which provide the smallest value of the L_2-norm of the φ function. In this work, we found such a φ - function. Explicit coefficients for optimal quadrature are found using φ-function. The resulting quadrature formula is exact for the functions $e^{\sigma x}$ and $e^{-\sigma x}$. In particular, well-known results are obtained from the results of this work.

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A modified inertial shrinking projection algorithm with adaptive step size for solving split generalized equilibrium, monotone inclusion and fixed point problems

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Abstract. In this paper, we study the common solution problem of split generalized equilibrium problem, monotone inclusion problem and common fixed point problem for a countable family of strict pseudo-contractive multivalued mappings. We propose a modified shrinking projection algorithm of inertial form with self-adaptive step sizes for finding a common solution of the aforementioned problem. The self-adaptive step size eliminates the difficulty of computing the operator norm while the inertial term accelerates the rate of convergence of the proposed algorithm. Moreover, unlike several of the existing results in the literature, the monotone inclusion problem considered is a more general problem involving the sum of Lipschitz continuous monotone operators and maximal monotone operators, and knowledge of the Lipschitz constant is not required to implement our algorithm. Under some mild conditions, we establish strong convergence result for the proposed method. Finally, we present some applications and numerical experiments to illustrate the usefulness and applicability of our algorithm as well as comparing it with some related methods. Our results improve and extend corresponding results in the literature.

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1. Introduction

Let H be a real Hilbert space with induced norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let C be a nonempty closed convex subset of a real Hilbert space and let $F : C \times C \to \mathbb{R}$ be A bifunction. The *equilibrium problem* (shortly, (EP)) in the sense of Blum and Oettli [8] is to find $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0, \quad \forall\, y \in C. \tag{1.1}$$

The set of all solutions of EP (1.1) is denoted by $EP(F)$. The EP attracts considerable research efforts and serves as a unifying framework for studying many well-known problems, such as the Nonlinear Complementarity Problems (NCPs), Optimization Problems (OPs), Variational Inequality Problems (VIPs), Saddle Point Problems (SPPs), the Fixed Point Problem (FPP), the Nash equilibria and many others, and has many applications in physics and economics, (see, for example [1, 11, 12, 34, 33, 48] and the references therein).

On the other hand, the *generalized equilibrium problem* (GEP) is defined as finding a point $x \in C$ such that

$$F(x, y) + \phi(x, y) \geq 0, \forall y \in C, \tag{1.2}$$

where $F, \phi : C \times C \to \mathbb{R}$ are bifunctions. We denote the solution set of GEP (1.2) by $GEP(F, \phi)$. If $\phi = 0$, then the GEP (1.2) reduces to the equilibrium problem (1.1).

Let $C \subseteq H_1$ and $Q \subseteq H_2$ where H_1 and H_2 are real Hilbert spaces. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, and $A : H_1 \to H_2$ be a bounded linear operator. The *split generalized equilibrium problem* (SGEP) introduced by Kazmi and Rizvi [23] is defined as follows: Find $\bar{x} \in C$ such that

$$F_1(\bar{x}, x) + \phi_1(\bar{x}, x) \geq 0, \forall x \in C, \tag{1.3}$$

and such that

$$\bar{y} = A\bar{x} \in Q \text{ solves } F_2(\bar{y}, y) + \phi_2(\bar{y}, y) \geq 0,\ \forall y \in Q. \tag{1.4}$$

The solution set of the split generalized equilibrium problem is denoted by

$$SGEP(F_1, \phi_1, F_2, \phi_2) = \{\bar{x} \in C : \bar{x} \in GEP(F_1, \phi_1) \text{ and } A\bar{x} \in GEP(F_2, \phi_2)\}. \tag{1.5}$$

If $\phi_1 = 0$ and $\phi_2 = 0$, we obtain a special case of the split generalized equilibrium problem (1.3)-(1.4) called the *split equilibrium problem* (SEP) which is defined as follows:

$$F_1(\bar{x}, x) \geq 0, \forall x \in C, \tag{1.6}$$

and such that

$$\bar{y} = A\bar{x} \in Q \text{ solves } F_2(\bar{y}, y) \geq 0,\ \forall y \in Q. \tag{1.7}$$

We denote the solution set of the SEP (1.6)-(1.7) by $\Omega := \{\bar{x} \in EP(F_1) : A\bar{x} \in EP(F_2)\}$. The split generalized equilibrium problem has been studied by numerous authors and several iterative algorithms have been proposed by many authors for solving the problem (see, [39, 42]).

Another important problem that we consider is the *monotone inclusion problem* (MIP), which is defined as finding a point $z \in H$ such that

$$0 \in (B + D)z, \tag{1.8}$$

where $B : H \to H$ is a nonlinear operator and $D : H \to 2^H$ is a set-valued operator. We denote the set of solutions of (1.8) by $(B + D)^{-1}(0)$. The MIP (1.8) and related optimization problems have been studied by several authors with various iterative algorithms proposed for approximating their solutions in Hilbert spaces and Banach spaces (see, for instance [3, 31, 32, 47, 50, 49]). One of the most efficient methods for solving the MIP is the forward-backward splitting method (see [6, 9, 14, 17, 18, 26]). Martinez [29] first introduced the Proximal Point Algorithm (PPA) for finding the zero point of a maximal monotone operator B. The sequence generated by PPA is defined as follows:

$$x_{n+1} = J^D_{r_n} x_n,$$

where $0 < r_n < \infty$, $J^D_{r_n} = (I + r_n D)^{-1}$ is the resolvent operator of D and I is the identity mapping. This algorithm was eventually modified by Rockafellar [40] to the following PPA with errors:

$$x_{n+1} = J^D_{r_n} x_n + e_n,$$

where $\{e_n\}$ is an error sequence. It was proved that if $e_n \to 0$ such that

$$\sum_{n=1}^{\infty} \|e_n\| < +\infty,$$

and the solution set $D^{-1}(0) \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges weakly to a zero point of D.

Also, Moudafi and Théra [31] introduced the following iterative algorithm for solving MIP (1.8):

$$\begin{cases} x_n = J^D_r v_n, \\ v_{n+1} = t v_n + (1-t)x_n - \mu(1-t)Bx_n, \end{cases} \tag{1.9}$$

where $t \in (0, 1)$, $r > 0$, B is Lipschitz continuous and strongly monotone and D is maximal monotone. They proved that the sequence $\{x_n\}$ generated by the iterative algorithm converges weakly to an element in $(B + D)^{-1}(0)$.

Alvarez and Attouch [5] proposed the following modified PPA of inertial form:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J^D_{\lambda_n} y_n, \qquad n \geq 1, \end{cases} \tag{1.10}$$

where $\{\mu_n\} \subset [0, 1)$, $\{\lambda_n\}$ is non-decreasing and

$$\sum_{n=1}^{\infty} \mu_n \|x_n - x_{n-1}\|^2 < \infty, \quad \forall \mu_n < \frac{1}{3}. \tag{1.11}$$

It was proved that Algorithm (1.10) converges weakly to a zero of D.

Recently, Moudafi and Oliny [30] introduced the following inertial PPA for approximating the zero point problem of the sum of two monotone operators:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J^D_{\lambda_n}(y_n - \lambda_n B x_n), n \geq 1, \end{cases} \tag{1.12}$$

where $D : H \to 2^H$ is maximal monotone and B is Lipschitz continuous. They proved that the sequence generated by Algorithm (1.12) converges weakly if $\lambda_n < \frac{2}{L}$, where L is the Lipschitz constant of B.

Moreover, the following inertial forward-backward algorithm was introduced by Lorenz and Pock [26]:

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = J^D_{\lambda_n}(y_n - \lambda_n B y_n), n \geq 1, \end{cases} \tag{1.13}$$

where $\{\lambda_n\}$ is a positive real sequence. Algorithm (1.13) differs from Algorithm (1.12) since the operator B is evaluated as the inertial extrapolate y_n. The proposed algorithm was also proved to converge weakly to a solution of the MIP (1.8).

In 2016, Deepho [16] introduced the general Cesáro mean iterative method for approximating a common solution of split generalized equilibrium, fixed point of nonexpansive mappings T_j and variational inequality problems:

$$\begin{cases} z_n = T^{(F_1,\phi_1)}_{r_n}(x_n + \gamma A^*(T^{(F_2,\phi_2)}_{r_n} - I)Ax_n), \\ u_n = P_C(z_n - \lambda_n G z_n), \\ x_{n+1} = \alpha_n \eta f(x_n) + \beta x_n + ((1-\beta_n)I - \alpha_n K)\frac{1}{n+1}\sum_{j=0}^{n} T_j u_n, \ \forall n \geq 0, \end{cases} \tag{1.14}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \in [a,b] \subset (0, 2\beta)$ and $\{r_n\} \subset (0,\alpha)$ and $\gamma \in \left(0, \frac{1}{L}\right)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Under the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $0 \leq \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;

(C3) $\lim\limits_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(C4) $\liminf\limits_{n\to\infty} r_n > 0$, $\lim\limits_{n\to\infty} |r_{n+1} - r_n| = 0$.

the authors proved that the sequence $\{x_n\}$ converges strongly to an element q in the solution set Ω, where $q = P_\Omega(I - K + \gamma f)(q)$ is the unique solution of the variational inequality problem

$$\langle (K - \gamma f)q, x - q\rangle \geq 0, \ \forall x \in \Omega.$$

Also, in 2017, Sitthithakerngkiet [42] proposed and studied the following iterative method for approximating a common solution of split generalized equilibrium, variational inequality for an inverse-strongly monotone mapping and fixed point problems of nonexpansive mappings in Hilbert spaces:

$$\begin{cases} z_n = T^{(F_1,\phi_1)}_{r_n}(x_n + \gamma A^*(T^{(F_2,\phi_2)}_{r_n} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \xi_n T[\sigma_n v + (1-\sigma_n)P_C(z_n - \lambda_n G z_n)], \end{cases} \tag{1.15}$$

where $v \in C$ is a fixed point, $r_n \in (0,\infty), \mu \in \left(0, \frac{1}{L}\right)$, L is the spectral radius of the operator A^*A, A^* is the adjoint of A, sequences $\{\alpha_n\}, \{\beta_n\}, \{\xi_n\}$ and $\{\sigma_n\}$ are in $(0,1)$ and satisfy $\alpha_n + \beta_n + \xi_n = 1, \lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\beta_n$ and $\{\gamma_n\} \subset [c,1]$ for some $c \in (0,1)$. Assume that the following conditions are satisfied:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $\lim\limits_{n\to\infty} \sigma_n = 0$;

(C3) $0 \leq \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;

(C4) $\lim\limits_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(C5) $\liminf\limits_{n\to\infty} r_n > 0$, $\lim\limits_{n\to\infty} |r_{n+1} - r_n| = 0$,

the authors proved that the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, where

$$z = P_\Omega f(z).$$

Recently, Phuengrattana and Lerkchaiyaphum [39] introduced the following shrinking projection method for solving SGEP and FPP for a countable family of nonexpansive multivalued mappings: For $x_1 \in C$ and $C_1 = C$, then

$$\begin{cases} z_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma A^*(I - T_{r_n}^{(F_2,\phi_2)})A)x_n, \\ y_n = \delta_{n,0} x_n + \sum_{j=1}^{n} \delta_{n,j} u_{n,j}, \qquad u_{n,j} \in P_j z_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\|^2 \leq \|x_n - p\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \qquad n \in \mathbb{N}. \end{cases} \tag{1.16}$$

They proved that if

(i) $\liminf\limits_{n\to\infty} r_n > 0$,

(ii) The limits $\lim\limits_{n\to\infty} \delta_{n,j} \in (0,1)$ exist for all $j \geq 0$,

then the sequence $\{x_n\}$ generated by (1.16) converges strongly to $P_\Gamma x_1$, where

$$\Gamma = \bigcap_{j=1}^{\infty} F(P_j) \cap SGEP(F_1, \phi_1, F_2, \phi_2) \neq \emptyset,$$

$F(P_j)$ is the set of fixed points of P_j and P_j is a countable family of nonexpansive multivalued mappings.

In 2021, Olona et al. [37] proposed an inertial shrinking projection defined as follows for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings : for $x_0, x_1 \in C$ with $C_1 = C$, then

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})A)w_n, \\ z_n = \delta_{n,0} u_n + \sum_{i=1}^{n} \delta_{n,j} y_{n,j},\ y_{n,j} \in P_j u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 \\ -2\theta_n \langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2 \|x_{n-1} - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{1.17}$$

$$\gamma_n = \begin{cases} \frac{\tau_n \|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{(F_2,\phi_2)} Aw_n, \\ \gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}), \end{cases}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $0 < a \leq \tau_n \leq b < 1, \{\theta_n\} \subset \mathbb{R}$, $\{\delta_{n,j}\} \subset (0,1)$, such that $\sum_{j=0}^{n} \delta_{n,j} = 1$, and $\{r_n\} \subset (0,\infty)$. $\{P_j\}$ is a countable family of nonexpansive multivalued mappings, $F_1, \phi_1 : C \times C \to \mathbb{R},\ F_2, \phi_2 : Q \times Q \to \mathbb{R}$ are bifunctions. Under some appropriate conditions, it was proved that the sequence $\{x_n\}$ converges strongly to $P_\Omega x_1$, where $\Omega = \bigcap_{j=1}^{\infty} F(P_j) \cap SGEP(F_1, \phi_1, F_2, \phi_2) \neq \emptyset$.
Motivated by the above results and the current research interest in this direction, in this paper, we propose a new iterative algorithm of inertial type with self-adaptive step size for approximating the common solution of SGEP (1.3)-(1.4), MIP (1.8) and FPP of strictly pseudo-contractive multivalued mappings. We prove that the sequence generated by our algorithm converges strongly to a solution of the investigated problem. Finally, we present some applications and numerical examples to illustrate the usefulness and efficiency of the proposed method in comparison with some related methods. Our proposed method uses self-adaptive step size and employs inertial technique to accelerate the rate of convergence of the proposed method. The implementation of our proposed algorithm does not require a prior knowledge of the norm of the bounded linear operator.
Subsequent sections of this paper are organised as follows: In Section 2, we recall some basic definitions and lemmas that are relevant in establishing our main results. In Section 3, we present our proposed algorithm and highlight some of its features. In Section 4, we prove some lemmas that are useful in establishing the strong convergence of our proposed algorithm and also prove the strong convergence theorem for the algorithm. In Section 5, we apply our result to study some optimization problems while in Section 6, we present some numerical experiments to illustrate the performance of our method and compare it with some related methods in the literature. Finally, in Section 7 we give a concluding remark.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote $x_n \to x$ to mean that sequence $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. We write $w_\omega(x_n)$ to denote set of weak limits of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

The nearest point projection of H onto C denoted by P_C is defined for each $x \in H$, as the unique element $P_C x \in C$ such that

$$\|x - P_C x\| \leq \|x - y\|,\ \forall\ y \in C. \tag{2.1}$$

It is well known that P_C is nonexpansive and has the following characteristics (see [4, 21]:

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle,\ \forall\ x, y \in H_1, \tag{2.2}$$

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.3}$$

$$\|x - y\|^2 \leq \|x - P_C x\|^2 + \|y - P_C x\|^2,\ \forall x \in H, y \in C, \tag{2.4}$$

$$\|(x - y) - (P_C x - P_C y)\|^2 \geq \|x - y\|^2 - \|P_C x - P_C y\|^2,\ x, y \in H. \tag{2.5}$$

A mapping $B : C \to H$ is said to be monotone if

$$\langle Bu - Bv, u - v\rangle \geq 0,\ \forall u, v \in C. \tag{2.6}$$

Moreover, if B satisfies

$$\langle Bu - Bv, u - v\rangle \geq \alpha\|Bu - Bv\|^2,\ \forall u, v \in C, \tag{2.7}$$

for some positive real number α. Then, B is called an α-inverse-strongly monotone mapping. It is clear that every inverse-strongly monotone mapping is monotone.

Lemma 2.1. [37, 28] *Let H be a real Hilbert space, $\lambda \in \mathbb{R}$, then $\forall x, y \in H$, we have*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
(ii) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$;
(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(iv) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$.

Lemma 2.2. [32] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $P_C : H \to C$ be the metric projection. Then*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2,\ \forall x \in H, y \in C.$$

Lemma 2.3. [?] *Let $x_i \in H, (1 \leq i \leq m), \sum_{i=1}^{m} \alpha_i = 1$, where $\{\alpha_i\} \subseteq (0, 1)$. Then*

$$\left\|\sum_{i=1}^{m} \alpha_i x_i\right\|^2 = \sum_{i=1}^{m} \alpha_i\|x_i\|^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j\|x_i - x_j\|^2.$$

Lemma 2.4. [24] *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and $a \in (R)$, the set $D = \{v \in C : \|y-v\|^2 \leq \|x-v\|^2+\langle z, v\rangle+a\}$ is convex and closed.*

Assumption 2.5. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions that satisfy the following conditions:*

(A1) $F_1(x, x) = 0$ *for all* $x \in C$,
(A2) F *is monotone, that is,* $F_1(x, y) + F_1(y, x) \leq 0$ *for all* $x, y \in C$,
(A3) F *is upper hemicontinuous, that is, for all* $x, y, z \in C$, $\lim_{t\downarrow 0} F\big(tz + (1 - t)x, y\big) \leq F(x, y)$,
(A4) *for each* $x \in C, y \mapsto F_1(x, y)$ *is convex and lower semicontinuous,*
(A5) $\phi_1(x, x) \geq 0$, *for all* $x \in C$,
(A6) *for each* $y \in C, x \mapsto \phi_1(x, y)$ *is upper semicontinuous,*
(A7) *for each* $x \in C, y \mapsto \phi_1(x, y)$ *is convex and lower semicontinuous,*

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F_1(y, x) + \phi_1(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0,\ \forall y \in C \setminus K.$$

Lemma 2.6. [27] *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions that satisfy Assumption 2.5. Assume that ϕ is monotone. For $r > 0$ and and $x \in H$. Define mapping $T_r^{(F,\phi)} : H \to C$ as follows:*

$$T_r^{(F,\phi)}(x) = \big\{z \in C : F(z,y) + \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\big\}$$

for all $x \in H_1$. Then

(1) *for each $x \in H_1$, $T_r^{(F,\phi)} \neq \emptyset$,*
(2) *$T_r^{(F,\phi)}$ is single-valued,*
(3) *$T_r^{(F,\phi)}$ is firmly nonexpansive, that is, for any $x, y \in H_1$,*

$$\|T_r^{(F,\phi)}x - T_r^{(F,\phi)}y\|^2 \leq \langle T_r^{(F,\phi)}x - T_r^{(F,\phi)}y, x - y\rangle,$$

(4) *$F(T_r^{(F,\phi)}) = GEP(F,\phi)$,*
(5) *$GEP(F,\phi)$ is closed and convex.*

Lemma 2.7. [44] *Let X be a Banach space space satisfying Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that*

$$\lim_{n\to\infty} \|x_n - u\| \text{ and } \lim_{n\to\infty} \|x_n - v\| \text{ exist.}$$

If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.

Lemma 2.8. [10] *Let $B : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous and monotone mapping. Then, the mapping $A + B$ is a maximal monotone mapping.*

Lemma 2.9. [20] *Let $B : H \to 2^H$ be a maximal monotone operator and $A : H \to H$ be a mapping on H. Define $T_\lambda := (I + \lambda B)^{-1}(I - \lambda A)$, $\lambda > 0$. Then, we have the following*

$$Fix(T_\lambda) = (A + B)^{-1}(0), \ \forall \lambda > 0. \tag{2.8}$$

Let D be a nonempty subset of H. D is said to be proximal if there exists $y \in D$ such that

$$\|x - y\| = d(x, D), \ x \in H.$$

Let $CC(C), CB(C)$ and $P(C)$ be the family of nonempty closed convex subset of H, nonempty closed bounded subsets of H and nonempty proximal bounded subsets of H respectively. The Hausdorff metric on $CB(C)$ is defined as follows:

$$H(A,B) := \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A), \ \forall A, B \in CB(C)\Big\}.$$

Let $S : C \to 2^C$ be a multivalued mapping. An element $x \in H$ is said to be a fixed point of S if $x \in Sx$. We say that S satisfies the endpoint condition if $Sp = \{p\}$ for all $p \in F(S)$. For multivalued mappings $S_i : H \to 2^H$ $(i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, we say S_i satisfies the common endpoint condition if $S_i(p) = \{p\}$ for all $i \in \mathbb{N}$, $p \in \cap_{i=1}^{\infty} F(S_i)$.

Definition 2.10. Let $A : H \to H$ be a nonlinear operator. Then A is called

(i) Lipschitz continuous if for all $L > 0$

$$\|Ax - Ay\| \leq L\|x - y\|, \ \forall x, y \in H;$$

if $0 \leq L < 1$, then A is a contraction mapping,

(ii) $\beta-$strongly monotone if for all $\beta > 0$

$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \ \forall x, y \in H.$$

Definition 2.11. Let $S : C \to CB(C)$ be a multivalued mapping. S is said to be

(i) nonexpansive if

$$H(Sx, Sy) \leq \|x - y\|, \ \forall x, y \in C,$$

(ii) quasi-nonexpansive if $F(S) \neq \emptyset$ such that

$$H(Sx, Sp) \leq \|x - p\|, \ \forall x \in C, p \in F(S),$$

(iii) k- strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$(H(Sx, Sy))^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \ \forall u \in Sx, v \in Sy \tag{2.9}$$

If $k = 1$ in (2.9), then the mapping S is said to be pseudo-contractive.

Clearly, the class of k-strict pseudo-contractive mappings properly contains the class of nonexpansive mappings. That is, S is nonexpansive if and only if S is 0-strict pseudo-contractive. It is known that if S is a k-strict pseudo-contraction and $F(S) \neq \emptyset$, then $F(S)$ is a closed convex subset of H (see [51]). Strict pseudo-contractions have many applications, due to their ties with inverse strongly monotone operators. It is known that, if B is a strongly monotone operator, then $S = I - B$ is a strict pseudo-contraction, and so we can recast a problem of zeros for B as a fixed point problem for S, and vice versa (see e.g. [13, 41]).

Let $S : H \to CB(H)$ be a multivalued mapping. The multivalued mapping $I - S$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset H$ which converges weakly to p and the sequence $\{\|x_n - u_n\|\}$ converges strongly to 0, where $u_n \in Sx_n$, then $p \in F(S)$.

3. Proposed method

In this section, we present our proposed algorithm.

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}_{i=1}^m$ be a countable family of k_i-strictly pseudo-contractive multivalued mappings of C into $CB(C)$ such that $I - S_i$ is demiclosed at zero for each $i = 1, 2, \ldots, m$, $S_ip = \{p\}$ for each $p \in \cap_{i=1}^m F(S_i)$ and $k = \max\{k_i\}$. Let $F_1, \phi_1 : C \times C \to \mathbb{R}, F_2, \phi_2 : Q \times Q \to$ be bifunctions satisfying Assumptions 2.5. Let ϕ_1, ϕ_2 be monotone, ϕ_1 be upper hemicontinuous, and F_2 and ϕ_2 be upper semicontinuous in the first argument. Let $B : H_1 \to H_1$ be L-Lipschitz continuous and monotone and $D : H_1 \to 2^{H_1}$ be a maximal monotone operator such that $\Gamma = SGEP(F_1, \phi_1, F_2, \phi_2) \cap \cap_{i=1}^m F(S_i) \cap (B + D)^{-1}(0) \neq \emptyset$. We establish the convergence of our algorithm under the following conditions on the control parameters:

(C1) $0 < a \leq \tau_n \leq b < 2, \{r_n\} \subset (0, \infty), \liminf_{n\to\infty} r_n > 0,$

(C2) $\liminf_{n} \alpha_{n,i}(\alpha_{n,0} - k) > 0$ and $\lim_{n\to\infty} \alpha_{n,i} \in (0,1)$ exists for all $i \geq 0$.

Now, we present our proposed algorithm as follows:

Algorithm 3.1.

Initialization: *Select* $x_0, x_1 \in H_1$, $s_1 > 0, \mu \in (0,1), \theta_n \in [-\theta, \theta]$ *for some* $\theta > 0$ *and* $C_1 = C$.

Iterative Step: *Given the current iterate* x_n, *calculate the next iterate as follows:*

Step 1 : *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2 : *Compute*

$$z_n = T_{r_n}^{(F_1,\phi_1)}(I - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})A)w_n.$$

Step 3 : *Compute*

$$y_n = \alpha_{n,0} z_n + \sum_{i=1}^{m} \alpha_{n,i} u_{n,i}, \quad u_{n,i} \in S_i z_n.$$

Step 4 : *Compute*

$$\begin{cases} v_n = (I + s_n D)^{-1}(I - s_n B)y_n = J_{s_n}^D (I - s_n B)y_n \\ t_n = v_n - s_n(Bv_n - By_n) \\ C_{n+1} = \{p \in C_n : \|t_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2\} \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases}$$

Step 5 : *Compute*

$$s_{n+1} = \begin{cases} \min\left\{\frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|}, s_n\right\} & \text{if } By_n - Bv_n \neq 0. \\ s_n & \text{otherwise}, \end{cases} \tag{3.1}$$

Set $n := n+1$ *and return to* **Step 1**,

where

$$\gamma_n = \begin{cases} \tau_n \frac{||(I - T_{r_n}^{(F_2,\phi_2)})Aw_n||^2}{||A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n||^2} & \text{If } Aw_n \neq T_{r_n}^{(F_2,\phi_2)} Aw_n \\ \gamma & \text{otherwise } (\gamma \text{ being any non-negative real number}). \end{cases}$$

Remark 3.2. We observe that

(i) The implementation of our proposed algorithm does not require prior knowledge of the operator norm. Hence, this makes our method easily implementable.

(ii) We employ the inertial technique to accelerate the rate of convergence.

(iii) The underlying single-valued operator $B : H_1 \to H_1$ for most of the results on monotone inclusion problem in the literature are either strongly monotone or inverse strongly monotone while the single-valued operator in our proposed algorithm is only required to be monotone and Lipschitz continuous. Moreover, knowledge of the Lipschitz constant of the operator is not required to implement our proposed algorithm. Thus, our method is more applicable than several of the existing methods in the literature.

(iv) Our result extends and improves on the results of Deepho et al. [16], Sitthithakerngkiet et al. [42], Phuengrattana and Lerkchaiyaphum [39], Olona et al. [37] and several other results in the current literature in this direction.

4. Convergence analysis

In this section, we analyze the convergence of our proposed algorithm.

Lemma 4.1. *Let* $\{s_n\}$ *be a sequence generated by* (3.1)*. Then,* $\{s_n\}$ *is a nonincreasing sequence and*

$$\lim_{n\to\infty} s_n = s \geq \min\left\{s_1, \frac{\mu}{L}\right\}. \tag{4.1}$$

Proof.
From (3.1), it is clear that $\{s_n\}$ is a nonincreasing sequence. Moreover, observe that if $By_n - Bv_n \neq 0$, then

$$\frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|} \geq \frac{\mu}{L}. \tag{4.2}$$

Hence, the sequence $\{s_n\}$ has the lower bound $\min\left\{s_1, \frac{\mu}{L}\right\}$. □

Lemma 4.2. [20] *Let* $\{x_n\}$ *be a sequence generated by Algorithm 3.1. Then the following inequality holds for all* $p \in \Gamma$:

$$\|t_n - p\|^2 \leq \|y_n - p\|^2 - \left(1 - \mu^2 \frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2, \ p \in \Gamma, \tag{4.3}$$

and

$$\|t_n - v_n\| \leq \mu \frac{s_n}{s_{n+1}}\|y_n - v_n\|. \tag{4.4}$$

Proof. By the definition of s_n, we have

$$\|By_n - Bv_n\| \leq \frac{\mu}{s_{n+1}}\|y_n - v_n\| \quad \forall\, n \in \mathbb{N}. \tag{4.5}$$

Clearly, if $By_n = Bv_n$, then (4.5) holds. Otherwise , we have

$$s_{n+1} = \min\left\{\frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|}, s_n\right\} \leq \frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|}.$$

This implies that

$$\|By_n - Bv_n\| \leq \frac{\mu}{s_{n+1}}\|y_n - v_n\|.$$

Thus, (4.5) holds when $By_n = Bv_n$ and $By_n \neq Bv_n$. Let $p \in \Gamma$, then by Lemma 2.1, we have

$$\begin{aligned}\|t_n - p\|^2 &= \|v_n - s_n(Bv_n - By_n) - p\|^2\\ &= \|v_n - p\|^2 + s_n^2\|Bv_n - By_n\|^2 - 2s_n\langle v_n - p, Bv_n - By_n\rangle\\ &= \|y_n - p\|^2 + \|y_n - v_n\|^2 + 2\langle v_n - y_n, y_n - p\rangle\\ &+ s_n^2\|Bv_n - By_n\|^2 - 2s_n\langle v_n - p, Bv_n - By_n\rangle\\ &= \|y_n - p\|^2 + \|y_n - v_n\|^2 - 2\langle v_n - y_n, v_n - y_n\rangle + 2\langle v_n - y_n, v_n - p\rangle\\ &+ s_n^2\|Bv_n - By_n\|^2 - 2s_n\langle v_n - p, Bv_n - By_n\rangle\\ &= \|y_n - p\|^2 - \|y_n - v_n\|^2 + 2\langle v_n - y_n, y_n - p\rangle\\ &+ s_n^2\|Bv_n - By_n\|^2 - 2s_n\langle v_n - p, Bv_n - By_n\rangle\\ &= \|y_n - p\|^2 - \|y_n - v_n\|^2 - 2\langle y_n - v_n - s_n(By_n - Bv_n), v_n - p\rangle\\ &+ s_n^2\|Bv_n - By_n\|^2. \end{aligned} \tag{4.6}$$

By applying (4.5) in (4.6), we obtain

$$\|t_n - p\|^2 \leq \|y_n - p\|^2 - \left(1 - \mu^2\frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2 - 2\langle y_n - v_n - s_n(By_n - Bv_n), v_n - p\rangle. \tag{4.7}$$

We now prove that $\langle y_n - v_n - s_n(By_n - Bv_n), v_n - p\rangle \geq 0$. Since

$$v_n = (I + s_nD)^{-1}(I - s_nB)y_n,$$

then we have $(I - s_nB)y_n \in (I + s_nD)v_n$. Recall that D is maximal monotone. Then there exists $u_n \in Dy_n$ such that

$$(I - s_nB)y_n = v_n + s_nu_n,$$

from which we obtain

$$u_n = \frac{1}{s_n}(y_n - v_n - s_nBy_n). \tag{4.8}$$

Moreover, we have $0 \in (B + D)p$ and $Bv_n + u_n \in (B + D)v_n$. Since $B + D$ is maximal monotone, we get

$$\langle Bv_n + u_n, v_n - p\rangle \geq 0. \tag{4.9}$$

By substituting (4.8) into (4.9), we obtain

$$\frac{1}{s_n}\langle y_n - v_n - s_nBy_n + s_nBv_n, v_n - p\rangle \geq 0.$$

This implies that

$$\langle y_n - v_n - s_n(By_n - Bv_n), v_n - p\rangle \geq 0. \tag{4.10}$$

By applying (4.10) in (4.7), we have

$$\|t_n - p\|^2 \leq \|y_n - p\|^2 - \left(1 - \mu^2\frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2. \tag{4.11}$$

On the other hand, one can see that (4.4) follows from (4.5). □

Remark 4.3. By Lemma 4.1 and $\mu \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that

$$1-\mu^2\frac{s_n^2}{s_{n+1}^2} > \epsilon > 0$$

for all $n \geq n_0$. Consequently, it follows from (4.3) that for all $p \in \Gamma$ and $n \geq n_0$

$$\|t_n - p\|^2 \leq \|y_n - p\|^2 - \epsilon\|y_n - v_n\|^2.$$

Theorem 4.4. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of k_i-strictly pseudo-contractive multivalued mappings of C into $CB(C)$. Let $F_1, \phi_1 : C \times C \to \mathbb{R}, F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.5. Suppose ϕ_1, ϕ_2 are monotone, ϕ_1 is upper hemicontinuous, and F_2 and ϕ_2 are upper semicontinuous in the first argument. Let $B : H_1 \to H_1$ be an $L-$Lipschitz continuous monotone mapping and $D : H_1 \to 2^{H_1}$ be a maximal monotone operator such that $\Gamma = SGEP(F_1, \phi_1, F_2, \phi_2) \cap \bigcap_{i=1}^m F(S_i) \cap \Omega \neq \emptyset$, where $\Omega = (B + D)^{-1}(0)$ and $S_i p = \{p\}$ for each $p \in \cap_{i=1}^m F(S_i)$. Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that conditions (C1) and (C2) hold. Then, the sequence $\{x_n\}$ converges strongly to $q = P_\Gamma x_0$.*

Proof. We divide the proof of the strong convergence Theorem 4.4 into various steps as follows:

Step 1: We show that sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded and well defined.

Let $p \in \Gamma$, then we have $p = T_{r_n}^{(F_1,\phi_1)}p$ and $Ap = T_{r_n}^{(F_1,\phi_1)}Ap, S_i p = p$, for all $i = 1, 2, ..., m$.

Since $T_{r_n}^{(F_1,\phi_1)}$ is nonexpansive, then by Lemma 2.1 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|T_{r_n}^{(F_1,\phi_1)}(w_n - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n) - p\|^2 \\
&\leq \|w_n - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n - p\|^2 \\
&= \|w_n - p\|^2 + \gamma_n^2\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \\
&\quad - 2\gamma_n\langle w_n - p, A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle.
\end{aligned} \tag{4.12}$$

By the firmly nonexpansivity of $I - T_{r_n}^{(F_2,\phi_2)}$, we get

$$\begin{aligned}
\langle w_n - p, A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle &= \langle Aw_n - Ap, (I - T_{r_n}^{(F_2,\phi_2)})Aw_n\rangle \\
&= \langle Aw_n - Ap, (I - T_{r_n}^{(F_2,\phi_2)})Aw_n \\
&\quad - (I - T_{r_n}^{(F_2,\phi_2)})Ap\rangle \\
&\geq \|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2.
\end{aligned} \tag{4.13}$$

By substituting (4.13) in (4.12) and applying the condition on τ_n, we have

$$\begin{aligned}\|z_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma_n^2 \|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \\ &\quad - 2\gamma_n \|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \\ &= \|w_n - p\|^2 - \gamma_n \big[2\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \\ &\quad - \gamma_n \|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2\big] \\ &= \|w_n - p\|^2 - \gamma_n(2 - \tau_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 \qquad (4.14) \\ &\leq \|w_n - p\|^2. \qquad (4.15)\end{aligned}$$

By Lemma 2.3 and applying the fact that S_i, $i = 1, 2, \ldots, m$ is strictly pseudo-contractive together with condition (C2), we get

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_{n,0} z_n + \sum_{i=1}^{m} \alpha_{n,i} u_{n,i} - p\|^2 \\ &= \alpha_{n,0}\|z_n - p\|^2 + \sum_{i=1}^{m} \alpha_{n,i}\|u_{n,i} - p\|^2 \\ &\quad - \sum_{i=1}^{m} \alpha_{n,0}\alpha_{n,i}\|u_{n,i} - z_n\|^2 - \sum_{1\leq i<j\leq m} \alpha_{n,i}\alpha_{n,j}\|u_{n,i} - u_{n,j}\|^2 \\ &\leq \alpha_{n,0}\|z_n - p\|^2 + \sum_{i=1}^{m} \alpha_{n,i}\big(H(S_i z_n, S_i p)\big)^2 \\ &\quad - \sum_{i=1}^{m} \alpha_{n,0}\alpha_{n,i}\|u_{n,i} - z_n\|^2 - \sum_{1\leq i<j\leq m} \alpha_{n,i}\alpha_{n,j}\|u_{n,i} - u_{n,j}\|^2 \\ &\leq \alpha_{n,0}\|z_n - p\|^2 + \sum_{i=1}^{m} \alpha_{n,i}\big(\|z_n - p\|^2 + k_i\|u_{n,i} - z_n\|^2\big) \\ &\quad - \sum_{i=1}^{m} \alpha_{n,0}\alpha_{n,i}\|u_{n,i} - z_n\|^2 \\ &\quad - \sum_{1\leq i<j\leq m} \alpha_{n,i}\alpha_{n,j}\|u_{n,i} - u_{n,j}\|^2 \\ &\leq \|z_n - p\|^2 - \sum_{i=1}^{m} \alpha_{n,i}(\alpha_{n,0} - k_i)\|u_{n,i} - z_n\|^2 \qquad (4.16) \\ &\leq \|z_n - p\|^2, \qquad (4.17)\end{aligned}$$

which implies that

$$\|y_n - p\| \leq \|z_n - p\|. \qquad (4.18)$$

By applying (4.17) and (4.15) into (4.11), we get

$$\|t_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{s_n^2}{s_{n+1}^2}\right)\|y_n - v_n\|^2, \forall p \in \Gamma. \qquad (4.19)$$

By Lemma 2.4, we have that C_{n+1} is closed and convex. Furthermore, from (4.19) it follows that $p \in C_{n+1}$. Hence, we have $\Gamma \subset C_{n+1} \subset C_n$ for all n and thus $x_{n+1} = P_{C_{n+1}} x_0$ is well defined. Therefore, $\{x_n\}$ is well defined.
We now show that $\{x_n\}$ is bounded. It is known that Γ is a nonempty closed convex subset of H_1, then there exists a unique $q \in \Gamma$ such that $q = P_\Gamma x_0$. From $x_n = P_{C_n} x_0$ and $x_{n+1} \in C_{n+1}$ for all $n \in \mathbb{N}$, we obtain

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \ \forall\ n \in \mathbb{N}.$$

On the other hand, since $\Gamma \subset C_n$, we get

$$\|x_n - x_0\| \leq \|q - x_0\|, \ \forall\ n \in \mathbb{N}.$$

This implies that $\{\|x_n - x_0\|\}$ is bounded. Hence, $\{x_n\}$ is bounded. Consequently $\{w_n\}, \{t_n\}, \{z_n\}$ and $\{y_n\}$ are bounded. Thus, $\lim_{n\to\infty} \|x_n - x_0\|$ exists.
Step 2: We claim that $\lim_{n\to\infty} x_n = q$, for some $q \in C$.
It is clear from the definition of C_n that $x_m = P_{C_m} x_0 \in C_m \subset C_n$, $m > n \geq 1$. By Lemma 2.2, we obtain

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{4.20}$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, then it follows from (4.20) that $\|x_m - x_n\| \to 0$ as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. Since H_1 is complete and C is closed, there exists $q \in C$ such that $x_n \to q$ as $n \to \infty$.
Step 3: We now show that $q \in \Gamma$.
From (4.20), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.21}$$

From the definition of w_n and by applying (4.21), we get

$$\|w_n - x_n\| = |\theta_n| \|x_n - x_{n-1}\| \leq |\theta| \|x_n - x_{n-1}\| \to 0, \quad n \to \infty. \tag{4.22}$$

From (4.21) and (4.22), we obtain

$$\|w_n - x_{n+1}\| \to 0, \quad n \to \infty. \tag{4.23}$$

We known that $x_{n+1} \in C_{n+1}$. Then, from the definition of C_{n+1} we obtain

$$\|t_n - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2.$$

Combining this with (4.23) gives

$$\lim_{n\to\infty} \|t_n - x_{n+1}\| = 0. \tag{4.24}$$

From (4.21) and (4.24), we obtain

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{4.25}$$

From (4.22) and (4.25), we obtain

$$\lim_{n\to\infty} \|t_n - w_n\| = 0. \tag{4.26}$$

By applying (4.17) and (4.15) into Remark 4.3, we have

$$\|t_n - p\|^2 \leq \|w_n - p\|^2 - \epsilon \|y_n - v_n\|^2.$$

From which we get

$$\begin{aligned}\epsilon\|y_n - v_n\|^2 &\leq \|w_n - p\|^2 - \|t_n - p\|^2\\ &\leq \|w_n - t_n\|(\|w_n - p\| + \|t_n - p\|),\end{aligned}$$

which together with (4.26) implies that

$$\|y_n - v_n\| \to 0, \quad n \to \infty. \tag{4.27}$$

Applying Lemma 4.1 together with (4.27) to (4.4), we have

$$\|t_n - v_n\| \to 0, \quad n \to \infty. \tag{4.28}$$

From (4.26)-(4.28), we obtain

$$\|y_n - w_n\| \to 0, \quad n \to \infty. \tag{4.29}$$

From (4.15) and (4.16), we obtain

$$\|y_n - p\|^2 \leq \|w_n - p\|^2 - \sum_{i=1}^{m} \alpha_{n,i}(\alpha_{n,0} - k_i)\|u_{n,i} - z_n\|^2.$$

From this we have

$$\begin{aligned}\alpha_{n,i}(\alpha_{n,0} - k_i)\|u_{n,i} - z_n\|^2 &\leq \sum_{i=1}^{m} \alpha_{n,i}(\alpha_{n,0} - k_i)\|u_{n,i} - z_n\|^2\\ &\leq \|w_n - p\|^2 - \|y_n - p\|^2\\ &\leq (\|w_n - y_n\|)(\|w_n - p\| + \|y_n - p\|).\end{aligned}$$

By applying Condition (C2) and (4.29), we get

$$\|u_{n,i} - z_n\| \to 0, \quad n \to \infty. \tag{4.30}$$

From the definition of y_n and by applying (4.30), we get

$$\|y_n - z_n\| \leq \alpha_{n,0}\|z_n - z_n\| + \sum_{i=1}^{m} \alpha_{n,i}\|u_{n,i} - z_n\| \to 0, \quad n \to \infty. \tag{4.31}$$

Also, by applying (4.22), (4.29) and (4.31), we obtain

$$\lim_{n\to\infty} \|w_n - z_n\| = 0; \quad \lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{4.32}$$

From (4.14), we have

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \gamma_n(2 - \gamma_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2,$$

which implies that

$$\begin{aligned}\gamma_n(2 - \gamma_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2 &\leq \|w_n - p\|^2 - \|z_n - p\|^2\\ &\leq \|w_n - z_n\|(\|w_n - p\| + \|z_n - p\|).\end{aligned}$$

Using the definition of γ_n, the condition on τ_n and applying (4.32), it follows that

$$\frac{\tau_n(2 - \tau_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^4}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)Aw_n})Aw_n\|^2} \to 0 \quad \text{as } n \to \infty.$$

From which we get

$$\frac{\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|} \to 0 \quad \text{as } n \to \infty.$$

Since $\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|$ is bounded, then it follows that

$$\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\| \to 0, \ n \to \infty. \tag{4.33}$$

Consequently, we have

$$\begin{aligned}\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n)\| &\leq \|A^*\|\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n)\| \\ &= \|A\|\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n)\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.34}$$

Since $t_n = v_n - s_n(Bv_n - By_n)$ and B is Lipschitz continuous, then by applying (4.27) we have

$$\|t_n - v_n\| = \|v_n - s_n(Bv_n - By_n) - v_n\| = s_n\|By_n - Bv_n\| \to 0, \quad n \to \infty.$$

Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $q \in w_\omega(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ as $k \to \infty$. Let $z \in w_\omega(x_n)$ and $\{x_{n_j}\} \subset \{x_n\}$ be such that $x_{n_j} \rightharpoonup z$ as $j \to \infty$. From (4.32), we get $z_{n_k} \rightharpoonup q$ and $z_{n_j} \rightharpoonup z$. Since $I - S_i$ is demiclosed at zero for each $i = 1, 2, \ldots, m$, then it follows from (4.30) that $q, z \in F(S_i)$ for all $i = 1, 2, \ldots, m$, which implies that $q, z \in \cap_{i=1}^m F(S_i)$.

Next, let $(g, h) \in \text{Graph}(B + D)$, that is $h - Bg \in Dg$. Since

$$v_{n_k} = (I + s_{n_k}D)^{-1}(I - s_{n_k}B)y_{n_k},$$

we have

$$(I - s_{n_k}B)y_{n_k} \in (I + s_{n_k}D)v_{n_k},$$

which implies that

$$\frac{1}{s_{n_k}}(y_{n_k} - v_{n_k} - s_{n_k}By_{n_k}) \in Dv_{n_k}.$$

Since D is maximal monotone, we get

$$\left\langle g - v_{n_k}, h - Bg - \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k} - s_{n_k}By_{n_k}) \right\rangle \geq 0.$$

From this we obtain

$$\begin{aligned}\langle g - v_{n_k}, h\rangle &\geq \left\langle g - v_{n_k}, Bg + \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k} - s_{n_k}By_{n_k}) \right\rangle \\ &= \langle g - v_{n_k}, Bg - By_{n_k}\rangle + \left\langle g - v_{n_k}, \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k}) \right\rangle \\ &= \langle g - v_{n_k}, Bg - Bv_{n_k}\rangle + \langle g - v_{n_k}, Bv_{n_k} - By_{n_k}\rangle \\ &\quad + \left\langle g - v_{n_k}, \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k}) \right\rangle \\ &\geq \langle g - v_{n_k}, Bv_{n_k} - By_{n_k}\rangle + \left\langle g - v_{n_k}, \frac{1}{s_{n_k}}(y_{n_k} - v_{n_k}) \right\rangle. \end{aligned}$$

Since B is Lipschitz continuous and $\lim_{n\to\infty}\|v_n - y_n\| = 0$, we have

$$\lim_{n\to\infty}\|Bv_{n_k} - By_{n_k}\| = 0.$$

Applying this together with $\lim_{n\to\infty} s_n = s \geq \min\left\{s_1, \frac{\mu}{L}\right\}$, we get

$$\langle g - q, h\rangle = \lim_{k\to\infty}\langle g - v_{n_k}, h\rangle \geq 0. \tag{4.35}$$

Following similar argument, we obtain

$$\langle g - z, h\rangle = \lim_{j\to\infty}\langle g - v_{n_j}, h\rangle \geq 0. \tag{4.36}$$

By the maximal monotonicity of $(B + D)$, it follows from (4.35) and (4.36) that $q, z \in (B + D)^{-1}(0)$.
Next, since $z_{n_k} = T_{r_{n_k}}^{(F_1,\phi_1)}(I - \gamma_{n_k} A^*(I - T_{r_{n_k}}^{F_2,\phi_2})A)w_{n_k}$, then by applying Lemma 2.6, we get

$$\begin{aligned}F_1(z_{n_k}, y) &+ \phi_1(z_{n_k}, y)\\ &+ \frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k} - \gamma_{n_k} A^*(I - T_{r_{n_k}}^{(F_2,\phi_2)})Aw_{n_k}\rangle\\ &\geq 0,\ \forall y \in C,\end{aligned}$$

which implies that

$$\begin{aligned}F_1(z_{n_k}, y) &+ \phi_1(z_{n_k}, y)\\ &+ \frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k}\rangle\\ &- \frac{1}{r_{n_k}}\langle y - z_{n_k}, \gamma_{n_k} A^*(I - T_{r_{n_k}}^{(F_2,\phi_2)})Aw_{n_k}\rangle\\ &\geq 0,\ \forall y \in C.\end{aligned}$$

From the monotonicity of F_1 and ϕ_1, it follows that

$$\begin{aligned}\frac{1}{r_{n_k}}\langle y - z_{n_k}, z_{n_k} - w_{n_k}\rangle&\\ &- \frac{1}{r_{n_k}}\langle y - z_{n_k}, \gamma_{n_k} A^*(I - T_{r_{n_k}}^{(F_2,\phi_2)})Aw_{n_k}\rangle\\ &\geq F_1(y, z_{n_k}) + \phi_1(y, z_{n_k}),\ \forall y \in C.\end{aligned}$$

By (4.32) and $x_{n_k} \rightharpoonup q$, we obtain $z_{n_k} \rightharpoonup q$. Applying condition $(C1)$, (4.32), (4.34) and Assumption 2.5 (A1)-(A7), we obtain

$$0 \geq F_1(y, q) + \phi_1(y, q),\ \forall y \in C.$$

Suppose $y_t = ty + (1 - t)q, \forall t \in (0, 1]$ and $y \in C$.
Then, $y_t \in C$ and $F_1(y_t, q) + \phi_1(y_t, q) \leq 0$. Therefore, by Assumption 2.5 (A1)-(A7),

we get

$$\begin{aligned}0 &\le F_1(y_t, y_t) + \phi_1(y_t, y_t)\\ &\le t\big(F_1(y_t, y) + \phi_1(y_t, y)\big) + (1-t)\big(F_1(y_t, q) + \phi_1(y_t, q)\big)\\ &\le t\big(F_1(y_t, y) + \phi_1(y_t, y)\big).\end{aligned}$$

Thus, we have

$$F_1(y_t, y) + \phi_1(y_t, y) \ge 0, \ \forall y \in C.$$

Letting $t \to 0$, and applying condition (A3) together with the upper hemicontinuity of ϕ_1, we have

$$F_1(q, y) + \phi_1(q, y) \ge 0, \ \forall y \in C. \tag{4.37}$$

By similar argument, we have

$$F_1(z, y) + \phi_1(z, y) \ge 0, \ \forall y \in C. \tag{4.38}$$

It follows from (4.37) and (4.38) that $q, z \in GEP(F_1, \phi_1)$.
Next, we show that $Aq, Az \in GEP(F_2, \phi_2)$. Since A is a bounded linear operator, then by (4.22) we have $Aw_{n_k} \rightharpoonup Aq$. Hence, from (4.33), we obtain

$$T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k} \rightharpoonup Aq, \quad k \to \infty. \tag{4.39}$$

By the definition of $T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k}$, we have

$$\begin{aligned}F_2(T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k}, y)&\\ &+ \phi_2(T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k}, y)\\ &+ \frac{1}{r_{n_k}} \langle y - T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k}, T_{r_{n_k}}^{(F_2,\phi_2)} Aw_{n_k} - Aw_{n_k} \rangle\\ &\ge 0, \ \forall y \in Q.\end{aligned}$$

Since F_2 and ϕ_2 are upper semicontinuous in the first argument, then by (4.33), (4.39) and $\liminf_{k\to\infty} r_{n_k} > 0$, we have

$$F_2(Aq, y) + \phi_2(Aq, y) \ge 0, \ \forall y \in Q. \tag{4.40}$$

Following similar argument, we have

$$F_2(Az, y) + \phi_2(Az, y) \ge 0, \ \forall y \in Q. \tag{4.41}$$

From (4.40) and (4.41), it follows that $Aq, Az \in GEP(F_2, \phi_2)$.
Therefore $q, z \in SGEP(F_1, \phi_1, F_2, \phi_2)$. By Invoking Lemma 2.7, we get $q = z$. Hence, we have that $q \in \Gamma$.
Step 4. Lastly, we show that $q = P_\Gamma x_0$.
Since $x_n = P_{C_n} x_0$ and $\Gamma \subset C_n$, we have $\langle x_0 - x_n, x_n - p\rangle \ge 0$ for all $p \in \Gamma$. By taking limit as $n \to \infty$, we have $\langle x_0 - q, q - p\rangle \ge 0$ for all $p \in \Gamma$. This shows that $q = P_\Gamma x_0$. Therefore, we can conclude by the steps above that $\{x_n\}$ converges strongly to $q = P_\Gamma x_0$. This completes the proof. □

If $\phi_1 = \phi_2 = 0$ in (1.3)-(1.4), then the split generalized equilibrium problem reduces to split equilibrium problem. Hence from Theorem 3.1 , we obtain the following consequent result.

Corollary 4.5. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}_{i=1}^m$ be a countable family of k_i-strictly pseudo-contractive multivalued mappings of C into $CB(C)$ such that $I - S_i$ is demiclosed at zero for each $i = 1, 2, \ldots, m$, $S_i p = \{p\}$ for each $p \in \cap_{i=1}^m F(S_i)$ and $k = \max\{k_i\}$. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.5 such that F_2 is upper semicontinuous in the first argument. Let $B : H_1 \to H_1$ be L-Lipschitz continuous and monotone and $D : H_1 \to 2^{H_1}$ be a maximal monotone operator such that $\Gamma = SGEP(F_1, F_2) \cap_{i=1}^m F(S_i) \bigcap (B + D)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 4.6.

Initialization: *Select $x_0, x_1 \in H_1$, $\mu \in (0, 1), \theta_n \in [-\theta, \theta]$ for some $\theta > 0$ and $C_1 = C$.*
Iterative Step: *Given the current iterate x_n, calculate the next iterate as follows:*
Step 1 : *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2 : *Compute*

$$z_n = T_{r_n}^{F_1}(I - \gamma_n A^*(I - T_{r_n}^{F_2})A)w_n.$$

Step 3 : *Compute*

$$y_n = \alpha_{n,0} z_n + \sum_{i=1}^m \alpha_{n,i} u_{n,i}, \quad u_{n,i} \in S_i z_n.$$

Step 4 : *Compute*

$$\begin{cases} v_n = (I + s_n D)^{-1}(I - s_n B)y_n \\ t_n = v_n - s_n(Bv_n - By_n) \\ C_{n+1} = \{p \in C_n : \|t_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle \\ +\theta_n^2\|x_{n-1} - x_n\|^2\} \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases}$$

Step 5 : *Compute*

$$s_{n+1} = \begin{cases} \min\left\{\frac{\mu\|y_n - v_n\|}{\|By_n - Bv_n\|}, s_n\right\} & \text{if } By_n - Bv_n \neq 0. \\ s_n & \text{otherwise,} \end{cases} \tag{4.42}$$

Set $n := n + 1$ and return to **Step 1**. *where*

$$\gamma_n = \begin{cases} \tau_n \frac{||(I - T_{r_n}^{(F_2)})Aw_n||^2}{||A^*(I - T_{r_n}^{(F_2)})Aw_n||^2} & \text{If } Aw_n \neq T_{r_n}^{(F_2}Aw_n \\ \gamma & \text{otherwise } (\gamma \text{ being any non-negative real number.}) \end{cases}$$

Suppose other conditions of Theorem 3.1 hold. Then, the sequence $\{x_n\}$ converges strongly to $q = P_\Gamma x_0$.

5. Applications

5.1. Split minimization problem

Let $H_1,\ H_2$ be two real Hilbert spaces, and let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed, and convex subsets. Let $f : C \to \mathbb{R},\ g : Q \to \mathbb{R}$ be two operators and $A : H_1 \to H_2$ be a bounded linear operator. The split minimization problem (SMP) is formulated as finding

$$x^* \in C \quad \text{such that} \quad f(x^*) \leq f(x),\ \forall x \in C, \tag{5.1}$$

and

$$y^* = Ax^* \quad \text{such that} \quad g(y^*) \leq g(y),\ y \in Q. \tag{5.2}$$

Let Ω denote the set of solution of SMP (5.1)-(5.2), and we assume $\Omega \neq \emptyset$. Let $\phi_1 = \phi_2 = 0$, and

$$F_1(x, y) := f(y) - f(x) \quad \text{for all } x, y \in C;$$

and

$$F_2(u, v) := g(v) - g(u) \quad \text{for all } u, v \in Q.$$

Suppose f and g are convex and lower semi-continuous on C and Q, respectively. Then, F_1, F_2, ϕ_1 and ϕ_2 satisfy all the conditions of Assumption 2.5. Consequently, from Theorem 3.1 we obtain a strong convergence theorem for approximating a common solution of split minimization problem, monotone variational inclusion problem and fixed point problem for a countable family of strict pseudo-contractive multivalued mappings in Hilbert spaces.

5.2. Split variational inequality problem

Let C be a nonempty closed convex subset of a real Hilbert space H, and $f : H \to H$ be a single-valued mapping. The variational inequality problem (VIP) introduced independently by Fichera [19] and Stampacchia [43] is formulated as follows:

$$\text{find } x^* \in C \text{ such that } \langle y - x^*, fx^* \rangle \geq 0, \qquad \forall\, y \in C. \tag{5.3}$$

The VIP can be modelled to solve several optimization problems and has vast applications in different fields, such as in physics, engineering, economics, etc, (see [3, 8, 12, 16, 34, 36, 42]).
The split variational inequality problem (SVIP), which was first introduced by Censor et al. [12] is defined as finding a point:

$$x^* \in C \text{ such that } \langle x - x^*, f(x^*) \rangle \geq 0 \qquad \forall\, x \in C, \tag{5.4}$$

and

$$y^* = Ax^* \in Q \text{ solves } \langle y - y^*, g(y^*) \rangle \geq 0 \qquad \forall\, y \in Q, \tag{5.5}$$

where C and Q are nonempty, closed, convex subsets of real Hilbert spaces H_1 and H_2, respectively, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are monotone mappings, and

$A : H_1 \to H_2$ is a bounded linear operator, see [25]. Let $\Omega \neq \emptyset$ denote the set of solution of SVIP (5.4)-(5.5). By setting $\phi_1 = \phi_2 = 0$, and

$$F_1(x, y) := \langle y - x, f(x)\rangle \quad \text{for all } x, y \in C;$$

and

$$F_2(u, v) := \langle v - u, g(u)\rangle \quad \text{for all } u, v \in Q.$$

Then, F_1, F_2, ϕ_1 and ϕ_2 satisfy all the conditions of Assumption 2.5. Hence, from Theorem 3.1, we obtain a strong convergence theorem for approximating a common solution of split variational inequality problem, monotone variational inclusion problem and fixed point problem for a countable family of strict pseudo-contractive multivalued mappings in Hilbert spaces.

6. Numerical examples

In this section, we present a numerical experiments to illustrate the performance of our Algorithm 3.1 as well as comparing it with Algorithm (1.14), Algorithm (1.15), Algorithm (1.16) and Algorithm (1.17) in the literature.
In our computation, we choose $\alpha_{n,0} = \frac{n}{2n+1}, \alpha_{n,i} = \frac{n+1}{5(2n+1)},\ i = 1, 2, \ldots, 5, \tau_n = 1.5, \theta_n = 1.9, r_n = 2.0, s_0 = 0.1$ and $\mu = 0.7$ in our Algorithm 3.1. $Gx = \frac{1}{3}x, fx = \frac{2}{3}x, Kx = \frac{2}{5}x, \lambda_n = \frac{2n}{5n+1}, \alpha_n = \frac{2}{2n+3}, \beta_n = \frac{n+1}{2n+3}, \eta = \frac{2}{5}, \gamma = 0.2, T_j x = \frac{2}{(3+j)}x$ in Algorithm (1.14),$\beta_n = \xi_n = \frac{1}{2}(1 - \alpha_n), \sigma_n = \frac{2}{2n+1}$ in Algorithm (1.15) while in Algorithm (1.16) and Algorithm (1.17). Let the sequences $\{\delta_{n,j}\}$ be defined as follows for each $j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$:

$$\delta_{n,j} = \begin{cases} \frac{1}{b^{j+1}}\left(\frac{n}{n+1}\right), & n > j, \\ 1 - \frac{n}{n+1}\left(\sum_{k=1}^{n} \frac{1}{b^k}\right), & n = j, \\ 0, & n < j, \end{cases} \tag{6.1}$$

where $b > 1$.

Example 6.1. Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = [0, 10]$. Let $A : H_1 \to H_2$ be defined by $Ax = \frac{x}{5}$ for all $x \in H_1$. Then, we have that $A^*y = \frac{y}{5}$ for all $y \in H_2$. For $x \in C, j \in \mathbb{N}$ and $i = 1, 2, \ldots, 5$, let $P_j, S_i : C \to CB(C)$ be multivalued mappings defined as follows:

$$P_j(x) = \left[0, \frac{x}{10j}\right], \quad S_i(x) = \left[0, \frac{x}{10i}\right]. \tag{6.2}$$

One can easily verify that P_j and S_i are nonexpansive and strictly pseudo-contractive, respectively. Define mappings $B : H_1 \to H_1$ by $Bx = 2x$, $D : H_1 \to H_1$ by $Dx = 3x$, and let the bifunctions $F_1, \phi_1 : C\times C \to \mathbb{R}$ be defined by $F_1(x, y) = y^2 + 3xy - 4x^2$ and $\phi_1(x, y) = y^2 - x^2$ for $x, y \in C$, and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ by $F_2(w, v) = 2v^2 + wv - 3w^2$ and $\phi_2(w, v) = w - v$ for $w, v \in Q$. It is easy to verify that all the conditions of Theorem 4.4 are satisfied. Next, we compute $T_r^{(F_1,\phi_1)}(x)$. We find $u \in C$ such that for all $z \in C$

$$\begin{aligned}
0 &\leq F_1(u,z) + \phi_1(u,z) + \frac{1}{r}\langle z-u, u-x\rangle \\
&= 2z^2 + 3uz - 5u^2 + \frac{1}{r}\langle z-u, u-x\rangle \\
&\Leftrightarrow \\
0 &\leq 2rz^2 + 3ruz - 5ru^2 + (z-u)(u-x) \\
&= 2rz^2 + 3ruz - 5ru^2 + uz - xz - u^2 + ux \\
&= 2rz^2 + (3ru + u - x)z + (-5ru^2 - u^2 + ux).
\end{aligned}$$

Suppose $h(z) = 2rz^2 + (3ru+u-x)z + (-5ru^2 - u^2 + ux)$. Then, $h(z)$ is a quadratic function of z with coefficients $a = 2r, b = 3ru + u - x$, and $c = -5ru^2 - u^2 + ux$. We determine the discriminant $\triangle$ of $h(z)$ as follows:

$$\begin{aligned}
\triangle &= (3ru + u - x)^2 - 4(2r)(-5ru^2 - u^2 + ux) \\
&= 49r^2u^2 + 14ru^2 - 14rux + u^2 - 2ux + x^2 \\
&= ((7r+1)u - x)^2. \qquad (6.3)
\end{aligned}$$

By Lemma 2.6, $T_r^{(F_1,\phi_1)}$ is single-valued. Thus, it follows that $h(z)$ has at most one solution in $\mathbb{R}$. Hence, from (6.3), we have that $u = \frac{y}{7r+1}$. This implies that $T_r^{(F_1,\phi_1)}(y) = \frac{y}{7r+1}$. Similarly, we compute $T_r^{(F_2,\phi_2)}(y)$. Find $w \in Q$ such that for all $d \in Q$

$$T_r^{(F_2,\phi_2)}(y) = \left\{ w \in Q : F_2(w,d) + \phi_2(w,d) + \frac{1}{r}\langle d-w, w-y\rangle \geq 0, \quad \forall\, d \in Q \right\}.$$

By following similar procedure as above, we obtain $w = \frac{y+r}{5r+1}$. This implies that $T_r^{(F_2,\phi_2)}(y) = \frac{y+r}{5r+1}$.

In this example, we set the parameter b on $\{\delta_{n,i}\}$ in (6.1) to be $b = 40, v = 3.5$ and we choose different initial values as follows:

Case I: $x_0 = 7, x_1 = 3$;
Case II: $x_0 = 6, x_1 = 2$;
Case III: $x_0 = 8, x_1 = 4$;
Case IV: $x_0 = 9, x_1 = 5$.

We compare the performance of our Algorithm 3.1 with Algorithms (1.14), (1.15), (1.16) and (1.17). The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 1 and Table 1.

Example 6.2. Let $H_1 = H_2 = L_2([0,1])$ with the inner product defined as

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall x, y \in L_2([0,1]).$$

Let

$$C := \{x \in H_1 : \langle a, x\rangle \geq d\},$$

TABLE 1. **Numerical results for Example 6.1**

		Alg. (1.14)	Alg. (1.15)	Alg. (1.16)	Alg. (1.17)	Alg. 3.1
Case I	No. of Iter.	9	20	4	9	2
	CPU time (sec)	0.0057	0.0078	1.6693	0.3383	0.0032
Case II	No. of Iter.	8	20	4	8	2
	CPU time (sec)	0.0051	0.0059	1.6884	0.3124	0.0039
Case III	No. of Iter.	9	20	4	9	2
	CPU time (sec)	0.0053	0.0057	1.6625	0.3566	0.0041
Case IV	No. of Iter.	9	20	4	9	2
	CPU time (sec)	0.0054	0.0067	1.6623	0.3449	0.0039

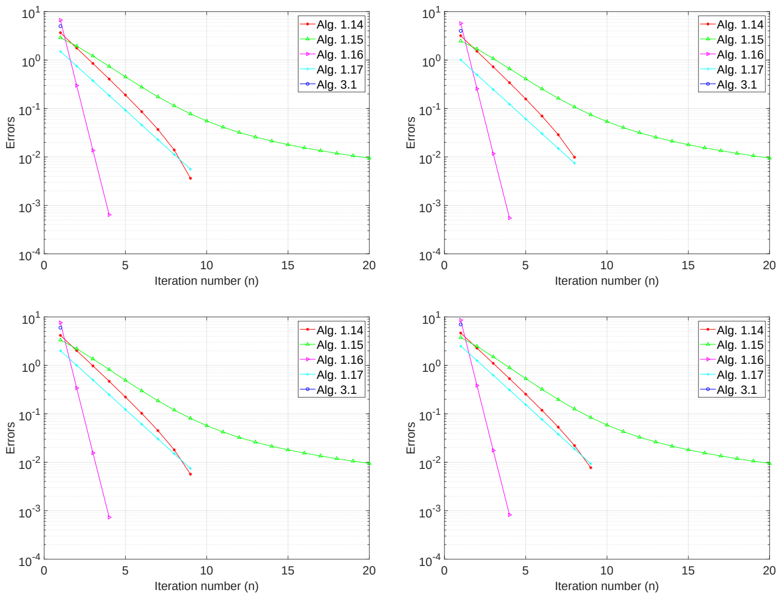

FIGURE 1. Top left: Case I ; Top right: Case II; Bottom left: Case III ; Bottom right: Case IV.

where $a = 2t^2$ and $d = 0$. Here, we have

$$P_C(x) = x + \frac{d - \langle a, x\rangle}{||a||^2} a.$$

Also, let

$$Q := \{x \in H_2 : \langle c, x\rangle \le e\},$$

where $c = \frac{t}{3}$, $e = 1$ and we have

$$P_Q(x) = x + \max\left\{0, \frac{e - \langle c, x\rangle}{||c||^2}c\right\}.$$

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be defined as $F_1(x, y) = \langle L_1 x, y - x\rangle$ and $F_2(x, y) = \langle L_2 x, y - x\rangle$, where $L_1 x(t) = \frac{x(t)}{3}$ and $L_2 x(t) = \frac{x(t)}{4}$. It can easily be verified that F_1 and F_2 satisfy conditions (A1)-(A4). Also, let $\phi_1 = \phi_2 = 0$. Furthermore, define $B : H_1 \to H_1$ by $Bx = 3x$, $D : H_1 \to H_1$ by $Dx = 7x$, and let $A : L_2([0, 1]) \to L_2([0, 1])$ be defined by $Ax(t) = \frac{x(t)}{3}$ and $A^*y(t) = \frac{y(t)}{3}$. Then, A is a bounded linear operator. We consider the case for which the multivalued mappings $\{S_j\}$ and $\{S_i\}$ are single-valued. Let $S_j, S_i : L^2([0, 1]) \to L^2([0, 1])$ be defined by

$$(S_j x)(t) = \int_0^1 t^j x(s)ds \qquad \text{and} \qquad (S_i x)(t) = \int_0^1 t^i x(s)ds \quad \text{for all } t \in [0, 1].$$

Note that S_i and S_j are nonexpansive for each i, j. Select $r_n = \frac{2n}{2n+1}, \theta_n = 0.8, \tau_n = 0.7$. It can easily be checked that all the conditions of Theorem 4.4 are satisfied. Now, we compute $T_r^{(F_1,\phi_1)}(x)$. We find $z \in C$ such that for all $y \in C$

$$\begin{aligned}
&F_1(z, y) + \phi_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\\
&\Leftrightarrow \langle \frac{z}{2}, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\\
&\Leftrightarrow \frac{z}{3}(y - z) + \frac{1}{r}(y - z)(z - x) \geq 0\\
&\Leftrightarrow (y - z)[rz + 3(z - x)] \geq 0\\
&\Leftrightarrow (y - z)[(r + 3)z - 3x] \geq 0. \qquad (6.4)
\end{aligned}$$

By Lemma 2.6, we obtain

$$T_r^{(F_1,\phi_1)}(x) = \left\{z \in C : F_1(z, y) + \phi_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\, y \in C\right\},$$

($\forall\ x \in H_1$), is single-valued. Thus, from (6.4) we obtain $z = \frac{3x}{r+3}$. This implies that $T_r^{(F_1,\phi_1)}(x) = \frac{3x}{r+3}$. Similarly, we compute $T_r^{(F_2,\phi_2)}(v)$. We find $w \in Q$ such that for all $d \in Q$

$$T_s^{(F_2,\phi_2)}(v) = \left\{w \in Q : F_2(w, d) + \phi_2(w, d) + \frac{1}{s}\langle d - w, w - v\rangle \geq 0, \quad \forall\ d \in Q\right\}.$$

By using similar approach as above, we obtain $w = \frac{4v}{s+4}$. This implies that $T_s^{(F_2,\phi_2)}(v) = \frac{4v}{s+4}$.

Here, we set the parameter b on $\{\delta_{n,i}\}$ in (6.1) to be $b = 3, v = t^2$ and we choose different initial values as follows:

Case I: $x_0 = t^4, x_1 = t^2 + t^4 + t^6 + 3$;
Case II: $x_0 = t^5, x_1 = t^2 + t^5 + 2$;
Case III: $x_0 = t^4, x_1 = t^3 + t^5 + t^7 + 2$;
Case IV: $x_0 = t^5, x_1 = t + t^2 + 1$.

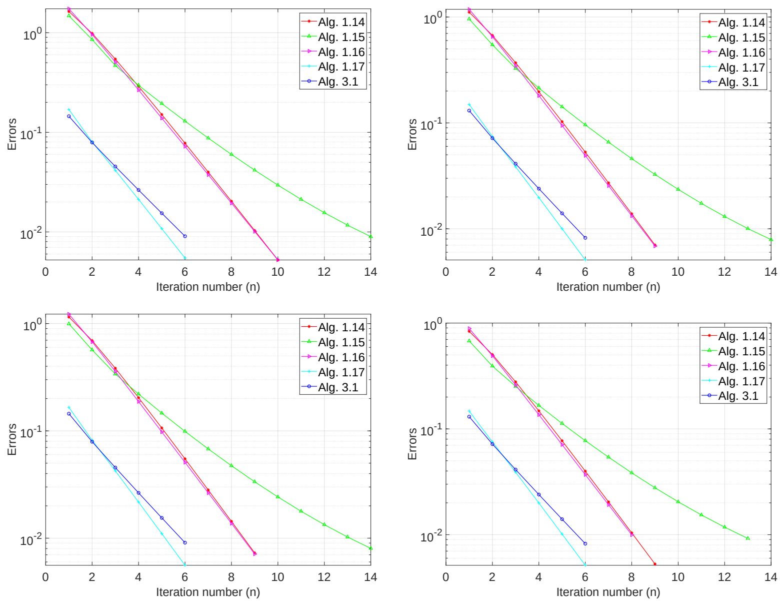

FIGURE 2. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

We compare the performance of our Algorithm 3.1 with Algorithms (1.14), (1.15), (1.16) and (1.17). The stopping criterion used for our computation is $||x_{n+1} - x_n|| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 2 and Table 2.

TABLE 2. **Numerical results for Example 6.2**

		Alg. (1.14)	Alg. (1.15)	Alg. (1.16)	App. (1.17)	Alg. 3.1
Case I	No. of Iter.	10	14	10	6	6
	CPU time (sec)	0.7297	0.7237	1.2541	0.2548	0.3256
Case II	No. of Iter.	9	14	9	6	6
	CPU time (sec)	0.6743	0.7004	1.1791	0.2628	0.3091
Case III	No. of Iter.	9	14	9	6	6
	CPU time (sec)	0.6507	0.6825	1.1474	0.2599	0.3087
Case IV	No. of Iter.	9	13	8	6	6
	CPU time (sec)	0.6353	0.6458	1.1130	0.2631	0.3166

7. Conclusion

In this article, we proposed a new modified inertial shrinking projection algorithm for finding common solution of split generalized equilibrium problem, monotone inclusion problem and fixed point problems for a countable family of strictly pseudocontractive multivalued mappings. We established strong convergence result for the proposed method. We applied our results to study related optimization problems and presented some numerical examples to demonstrate the efficiency of our proposed method in comparison with other existing methods. Our results extend and improve several existing results in this direction in the current literature.

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Bernstein polynomials iterative method for weakly singular and fractional Fredholm integral equations

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Abstract. A novel iterative method based on Picard iterations and Berstein polynomials is proposed for solving weakly singular and fractional Fredholm integral equations. On a uniform mesh, at each iterative step a Bernstein type spline is constructed by using the values computed at the previous step. The error estimates are obtained in terms of the Lipschitz constants and the convergence of the method is proved. Some numerical examples are presented in order to illustrate the accuracy of this iterative method.

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Introduction

The interest for fractional order differential and integral equations is motivated by the multiple applications of fractional calculus in fluid dynamics, viscoelasticity (see [7] and [38] for the Bagley-Torvik fractional differential model), heat transfer, diffusive transport, signal processing and various areas of engineering, economy, plasma physics, hematopoiesis, epidemiology, and in modeling of memory and hereditary properties of materials (see [12], [14], [15], [20], [30], [37]). According to the Scot Blair model the fractional order of a derivative is an index of memory (see [14]). A significant development in the field of fractional calculus, including fractional differential and integral equations, was realized in recent years and the results are presented in the monographs of Baleanu et al. (see [8]), Diethelm (see [12]), Kilbas et al. (see [20]),

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Lakshmikantham et al. (see [21]), Miller and Ross (see [29]), Muskhelishvili and Radok (see [31]), and Podlubny (see [33]). The numerical integration of fractional type integrals is usually realized by product integration and adapted quadrature rules (see [6]). Fractional integral equations are suitable models for several phenomena from physics and electro-chemistry such as crystal growth and heat transfer (see [17] and [39]). The corresponding fractional integral equations equivalent with various types of boundary value problems associated to nonlinear fractional differential equations with Caputo fractional derivative and existence results can be found in [1]. Usually, the existence and uniqueness of the solution for fractional integral equations is investigated by using the Banach fixed point theorem (see [1], [12], and [27]). Regularity properties of the solution of weakly singular and fractional Fredholm integral equations were obtained in [19] and [35].

In order to solve Volterra fractional integral equations, various numerical methods were proposed based on the following techniques: product integration and quadrature rules (see [6], [5], [27], [28]), collocation (see [9], [10], [13], and [44]), Runge-Kutta techniques (see [23]), Adams-Bashforth procedures (see [12]), Bernstein's approximation (see [39]), Haar, Legendre and Riesz wavelets (see [30] and [43]), variational iteration (see [40]). In the case of weakly singular and fractional Fredholm integral equations, the numerical solution is obtained by applying sinc, spectral and Haar wavelet collocation (see [3], [24], [32] and [41]), B-spline wavelets Galerkin technique (see [25]), product integration (see [2] and [36]), Taylor-series expansion (see [34]), hybrid collocation (see [11]), Galerkin and iterated Galerkin methods (see [18] and [26]).

In this paper, we approximate the solution of the following type Fredholm integral equation with singularities

$$x(t) = g(t) + \lambda \int_0^T b(t) \left|t - s\right|^{\alpha - 1} f(s, x(s))\, ds,\ t \in [0, T] \tag{0.1}$$

where $\lambda > 0$, $\alpha \in (0,1)$ and $g, b : [0,T] \to \mathbb{R}$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous with $b(t) \geq 0$, $\forall t \in [0,T]$. The choice $\lambda = \frac{1}{\Gamma(\alpha)}$ corresponds to the case of fractional integral equations, while $\lambda = 1$ usually describes weakly singular integral equations.

In the case $\lambda = \frac{1}{\Gamma(\alpha)}$ of fractional integral equations, we use the left-sided and right-sided Riemann-Liouville fractional integrals which are defined as follows.

Definition 0.1. (see [39]) Let $f : [0,T] \to \mathbb{R}$.The left-sided fractional integral of f of order $\alpha \in (0,1)$ is defined as

$$I_+^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \text{for } t > 0$$

where $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx$, for $x > 0$. The right-sided fractional integral of f of order $\alpha \in (0,1)$ is

$$I_-^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1} f(s)ds, \quad \text{for } t < T.$$

Our method comes from the product integration technique and an iterative procedure is obtained based on piecewise Bernstein polynomials involved at each iterative step. More precisely, at each iterative step we construct a Bernstein spline based on the values computed in the previous step and the integral is approximated by using the Bernstein type quadrature formula. This method differs by the technique developed in [39] where the solution was directly approximated by Bernstein polynomials inserted in the two sides of the integral equation and the convergence analysis was based on Voronovskaia's type theorem. The product integration method firstly appears in 1954, in the work of Young (see [42]), and as it is specified in [16] the most used procedures are rectangular and trapezoidal schemes with the order of convergence $O\left(h^{\min(1+\alpha,2)}\right)$. For integral equations such as (0.1), our Bernstein splines method has the order of convergence $O\left(h^\alpha\right)$ as it is specified in Theorem 2.1.

The paper is organized as follows: in Section 1 we present some uniform boundedness and uniform Hölder type Lipschitz properties of the Picard iterations, including the description of the iterative algorithm for solving the integral equation (0.1). Section 2 is devoted to the convergence analysis of this iterative method. In order to confirm the obtained theoretical result and to illustrate the accuracy of the method, in Section 3 we present some numerical experimets. Finally, we point out some concluding remarks.

1. The properties of Picard's iterations and the iterative method

We see that in (0.1) the singularity appears inside the open interval $(0,T)$ which can be moved at extremeness by writing (0.1) as

$$x(t) = g(t) + \lambda \int_0^t b\left(t\right)\left(t-s\right)^{\alpha-1} f\left(s, x(s)\right) ds + \lambda \int_t^T b\left(t\right)\left(s-t\right)^{\alpha-1} f\left(s, x(s)\right) ds$$

and we consider the corresponding integral operator $A : C[0,T] \to C[0,T]$ that is well-defined according to [4],

$$A\left(x\right)\left(t\right) := g(t) + \lambda \int_0^t b\left(t\right)\left(t-s\right)^{\alpha-1} f\left(s, x(s)\right) ds +$$

$$+ \lambda \int_t^T b\left(t\right)\left(s-t\right)^{\alpha-1} f\left(s, x(s)\right) ds. \tag{1.1}$$

Concerning the existence and uniqueness of the solution and the properties of Picard's iterations we obtain the following result.

Theorem 1.1. *If* $g, b : [0, T] \to \mathbb{R}$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *are continuous,* $b(t) \geq 0$, $\forall t \in [0, T]$, $L \geq 0$ *is such that*

$$|f(s, u) - f(s, v)| \leq L |u - v|, \quad \forall s \in [0, T],\ u, v \in \mathbb{R} \tag{1.2}$$

and if $\theta = \frac{\lambda L M_b T^{\alpha}}{\alpha} < 1$, *then the integral equation (0.1) has unique solution* $x^* \in C[0, T]$ *where* $M_b \geq 0$, *with* $|b(t)| \leq M_b$, $\forall t \in [0, T]$, *and the sequence of Picard iterations given by* $x_0 = g$, $x_m = A(x_{m-1})$, $m \in \mathbb{N}^*$, *is uniformly bounded having* $\lim_{m \to \infty} x_m = x^*$ *in* $(C[0, T], \|\cdot\|_{\infty})$ *and*

$$|x^*(t) - x_m(t)| \leq \frac{\theta^m \lambda M_b M_0 T^{\alpha}}{\alpha(1 - \theta)}, \quad \forall t \in [0, T],\ m \in \mathbb{N}^*, \tag{1.3}$$

$$|x^*(t) - x_m(t)| \leq \frac{\theta}{1 - \theta} |x_m(t) - x_{m-1}(t)|, \quad \forall t \in [0, T],\ m \in \mathbb{N}^*, \tag{1.4}$$

where $\|x\|_{\infty} = \max_{t \in [0, T]} |x(t)|$. *If in addition, there exist* $\beta, \gamma, \eta \geq 0$ *such that*

$$|g(t) - g(t')| \leq \eta |t - t'|, \quad |b(t) - b(t')| \leq \beta |t - t'|,\ \forall t, t' \in [0, T] \tag{1.5}$$

$$|f(s, u) - f(s', u)| \leq \gamma |s - s'|, \quad \forall s, s' \in [0, T],\ u \in \mathbb{R} \tag{1.6}$$

then the sequence $(x_m)_{m \in \mathbb{N}^*}$ *of Picard iterations is uniform Hölder type Lipschitz.*

Proof. Elementary calculus lead to

$$|A(x)(t) - A(y)(t)| \leq \frac{\lambda L M_b T^{\alpha}}{\alpha} \|x - y\|_{\infty}$$

for all $x, y \in C[0, T]$, $t \in [0, T]$ and according to Banach's fixed point principle the integral operator A has unique fixed point that is the unique solution $x^* \in C[0, T]$ of (0.1) with $\lim_{m \to \infty} x_m(t) = x^*(t)$ uniformly for $t \in [0, T]$ and the apriori and a posteriori error estimates (1.3) and (1.4) follows. For the Picard iterations

$$x_{m+1}(t) = g(t) + \lambda \int_0^t b(t)(t - s)^{\alpha - 1} f(s, x_m(s))\, ds$$

$$+ \lambda \int_t^T b(t)(s - t)^{\alpha - 1} f(s, x_m(s))\, ds \tag{1.7}$$

in inductive manner we get

$$|x_m(t) - x_{m-1}(t)| \leq \theta \|x_{m-1} - x_{m-2}\|_{\infty} \leq \ldots \leq \theta^{m-1} \|x_1 - x_0\|_{\infty}$$

and thus,

$$|x_m(t)| \leq |x_m(t) - x_0(t)| + |x_0(t)| \leq \left(1 + \theta + \ldots + \theta^{m-1}\right) \frac{\lambda M_b M_0 T^{\alpha}}{\alpha} + M_g$$

for all $t \in [0,T]$, $m \in \mathbb{N}^*$, where $M_0, M_g \geq 0$ are such that $|f(t, g(t))| \leq M_0$, $|g(t)| \leq M_g$, $\forall t \in [0,T]$. By denoting

$$R = M_g + \frac{\lambda M_b M_0 T^\alpha}{\alpha (1-\theta)}$$

we have $|x_m(t)| \leq R$ for all $t \in [0,T]$, $m \in \mathbb{N}^*$, that is the uniform boundedness of the sequence $(x_m)_{m\in\mathbb{N}^*}$ of Picard iterations. If we denote $F_m(t) = f(t, x_m(t))$ for $t \in [0,T]$ and $m \in \mathbb{N}$, and use the Lipschitz property it obtains,

$$\begin{aligned} |F_m(t)| &\leq |f(t, x_m(t)) - f(t, x_0(t))| + |f(t, x_0(t))| \\ &\leq \frac{\lambda L M_b M_0 T^\alpha}{\alpha (1-\theta)} + M_0 = M \end{aligned} \tag{1.8}$$

for all $t \in [0,T]$ and $m \in \mathbb{N}^*$, and thus the sequence $(F_m)_{m\in\mathbb{N}}$ is uniformly bounded, too.

Now, by considering arbitrary $t, t' \in [0,T]$, if $t \leq t'$ (the case $t' \leq t$ being approached similarly) we have $(t'-s)^{\alpha-1} \leq (t-s)^{\alpha-1}$ and consequently,

$$\begin{aligned} &\int_t^{t'} \left| b(t')(t'-s)^{\alpha-1} - b(t)(s-t)^{\alpha-1} \right| ds \\ &\leq \int_t^{t'} |b(t') - b(t)| (t'-s)^{\alpha-1} ds + |b(t)| \int_t^{t'} \left(\left|(t'-s)^{\alpha-1}\right| + \left|(s-t)^{\alpha-1}\right| \right) ds \\ &\leq \beta |t-t'| \cdot \frac{|t-t'|^\alpha}{\alpha} + \frac{2M_b |t-t'|^\alpha}{\alpha} \end{aligned}$$

obtaining,

$$\begin{aligned} &|x_m(t') - x_m(t)| \leq \eta |t-t'| \\ &+\lambda \int_0^t \left| b(t')(t'-s)^{\alpha-1} - b(t)(t-s)^{\alpha-1} \right| \cdot |f(s, x_{m-1}(s))| ds \\ &+\lambda \int_t^{t'} (\left| b(t') |t'-s|^{\alpha-1} - b(t)(s-t)^{\alpha-1} \right| \cdot |f(s, x_{m-1}(s))| ds \\ &+\lambda \int_{t'}^T \left| b(t')(s-t')^{\alpha-1} - b(t)(s-t)^{\alpha-1} \right| \cdot |f(s, x_{m-1}(s))| ds \\ &\leq (\eta + \frac{2\lambda M \beta T^\alpha}{\alpha}) |t-t'| + \frac{\lambda M \beta}{\alpha} |t-t'|^{\alpha+1} + \frac{4\lambda M M_b}{\alpha} |t-t'|^\alpha, \quad \forall m \in \mathbb{N}^*, \end{aligned}$$

that is the uniform Hölder type Lipschitz property of the sequence $(x_m)_{m\in\mathbb{N}^*}$ of Picard iterations. Under the Lipschitz conditions (1.5) and (1.6) we have

$$|F_m(t) - F_m(t')| \leq \gamma |t-t'| + L |x_m(t') - x_m(t)| \leq \frac{\lambda L M \beta}{\alpha} |t-t'|^{\alpha+1}$$

$$+\left[\gamma+L\left(\eta+\frac{2\lambda\beta MT^{\alpha}}{\alpha}\right)\right]\left|t-t'\right|+\frac{4\lambda LMM_b}{\alpha}\left|t-t'\right|^{\alpha}, \quad \forall m\in\mathbb{N}^{*} \tag{1.9}$$

for all $t,t'\in[0,T]$, that is the uniform Hölder type Lipschitz property of the sequence $(F_m)_{m\in\mathbb{N}}$. In that follows, we will denote

$$L_0=\gamma+L\left(\eta+\frac{2\lambda\beta MT^{\alpha}}{\alpha}\right),\ L''=\frac{\lambda LM\beta}{\alpha},\ \text{and}\ L'=\frac{4\lambda LMM_b}{\alpha}. \qquad \square$$

For the case of fractional integral equations with $\lambda=\frac{1}{\Gamma(\alpha)}$ the contraction condition becomes

$$\theta=\frac{LM_bT^{\alpha}}{\Gamma(\alpha+1)}<1.$$

Our iterative method is based on approximating the Picard iterations (1.7) and for this purpose we consider a uniform mesh of $[0,T]$ with the knots $t_i=i\cdot h$, $i=\overline{0,n}$, where $h=\frac{T}{n}$ is the stepsize. On these knots the Picard iterations become

$$x_{m+1}(t_i)=g(t_i)+\lambda\int_0^{t_i}b(t_i)(t_i-s)^{\alpha-1}f(s,x_m(s))\,ds$$

$$+\lambda\int_{t_i}^{T}b(t_i)(s-t_i)^{\alpha-1}f(s,x_m(s))\,ds, \quad i=\overline{0,n},\ m\in\mathbb{N}^{*} \tag{1.10}$$

and on each subinterval $[t_{i-1},t_i]$, $i=\overline{1,n}$, we approximate the continuous function F_m by the Bernstein polynomial of a given degree $q\geq 1$:

$$B_{m,i}(t)=\frac{1}{h^q}\sum_{k=0}^{q}C_q^k(t-t_{i-1})^k(t_i-t)^{q-k}\cdot F_m\left(t_{i-1}+\frac{kh}{q}\right),\ t\in[t_{i-1},t_i] \tag{1.11}$$

where $C_q^k=\frac{q!}{k!\cdot(q-k)!}$, and in this way F_m will be approximated on $[0,T]$ by a Bernstein spline B_m for all $m\in\mathbb{N}^{*}$. For estimating the remainder in the Bernstein approximation formula $F_m(t)=B_{m,i}(t)+R_{m,i}(t)$, we use the inequality of Lorentz (see [22]) described in terms of the modulus of continuity,

$$|R_{m,i}(t)|\leq\frac{5}{4}\cdot\omega\left(F_m,\frac{h}{\sqrt{q}}\right),\ \forall t\in[t_{i-1},t_i],\ \forall i=\overline{1,n},\ m\in\mathbb{N}.$$

According to the uniform Hölder type Lipschitz property of the sequence $(F_m)_{m\in\mathbb{N}^{*}}$, this inequality becomes

$$|R_{m,i}(t)|\leq\frac{5}{4}\left(\frac{L_0h}{\sqrt{q}}+\frac{L'h^{\alpha}}{(\sqrt{q})^{\alpha}}+\frac{L''h^{\alpha+1}}{(\sqrt{q})^{\alpha+1}}\right),\ \forall t\in[t_{i-1},t_i],\ \forall i=\overline{1,n} \tag{1.12}$$

for all $m\in\mathbb{N}$.

Based on (1.10) and (1.11) we obtain the following iterative algorithm:

Step 1: $x_0(t)=g(t)$, $\forall t\in[0,T]$ and for $k=\overline{0,n-1}$, $l=\overline{0,q-1}$ let

$$x_1\left(t_k+\frac{lh}{q}\right)=g\left(t_k+\frac{lh}{q}\right)+\lambda b\left(t_k+\frac{lh}{q}\right)\int_0^T\left|t_k+\frac{lh}{q}-s\right|^{\alpha-1}f(s,g(s))\,ds$$

$$= g\left(t_k + \frac{lh}{q}\right) + \lambda b\left(t_k + \frac{lh}{q}\right)\sum_{i=1}^{n}\int\limits_{t_{i-1}}^{t_i}\left|t_k + \frac{lh}{q} - s\right|^{\alpha-1}\left(B_{0,i}(s) + R_{0,i}(s)\right)ds$$

$$= g\left(t_k + \frac{lh}{q}\right) + \frac{\lambda}{h^q}b\left(t_k + \frac{lh}{q}\right)\sum_{i=1}^{n}\sum_{j=0}^{q}C_q^j\varphi_{k,l,j}\cdot$$

$$\cdot f\left(t_{i-1} + \frac{jh}{q}, g\left(t_{i-1} + \frac{jh}{q}\right)\right) + \overline{R_{1,(k,l)}} = \overline{x_1}\left(t_k + \frac{lh}{q}\right) + \overline{R_{1,(k,l)}} \qquad (1.13)$$

and

$$x_1(t_n) = g(t_n) + \lambda b(t_n)\int\limits_0^T (t_n - s)^{\alpha-1} f(s, g(s))\,ds = g(t_n) + \lambda b(t_n)\cdot$$

$$\cdot\sum_{i=1}^{n}\int\limits_{t_{i-1}}^{t_i}(t_n - s)^{\alpha-1}\left(B_{0,i}(s) + R_{0,i}(s)\right)ds = g(t_n) + \frac{\lambda}{h^q}b(t_n)\sum_{i=1}^{n}\sum_{j=0}^{q}C_q^j\varphi_{n,j}\cdot$$

$$\cdot f\left(t_{i-1} + \frac{jh}{q}, g\left(t_{i-1} + \frac{jh}{q}\right)\right) + \overline{R_{1,n}} = \overline{x_1}(t_n) + \overline{R_{1,n}} \qquad (1.14)$$

where

$$\varphi_{k,l,j} = \int\limits_{t_{i-1}}^{t_i}\left|t_k + \frac{lh}{q} - s\right|^{\alpha-1}(s - t_{i-1})^j (t_i - s)^{q-j}\,ds,$$

$$k = \overline{0, n-1},\ l = \overline{0, q-1} \qquad (1.15)$$

$$\varphi_{n,j} = \int\limits_{t_{i-1}}^{t_i}(t_n - s)^{\alpha-1}(s - t_{i-1})^j (t_i - s)^{q-j}\,ds, \quad j = \overline{0, q}. \qquad (1.16)$$

In the computation of the integrals (1.15)-(1.16) we use the change of variable

$$s = t_{i-1} + uh$$

obtaining $\varphi_{k,l,j} = h^{q+\alpha}\varphi_{k,l,j}(i)$ and $\varphi_{n,j} = h^{q+\alpha}\varphi_{n,j}(i)$ with

$$\varphi_{k,l,j}(i) = \int\limits_0^1 u^j (1-u)^{q-j}\left|k + \frac{l}{q} - (i-1) - u\right|^{\alpha-1} du$$

and $\varphi_{n,j}(i) = \int\limits_0^1 u^j (1-u)^{q-j}(n - i - u + 1)^{\alpha-1}\,du.$

Step 2: Construct the Bernstein splines B_1 and $\overline{B_1}$ given for $i = \overline{1, n}$ by

$$B_{1,i}(t) = \frac{1}{h^q}\sum_{k=0}^{q}C_q^k (t - t_{i-1})^k (t_i - t)^{q-k}\cdot F_1\left(t_{i-1} + \frac{kh}{q}\right),\ t \in [t_{i-1}, t_i] \qquad (1.17)$$

and

$$\overline{B_{1,i}}(t)=\frac{1}{h^q}\sum_{j=0}^{q}C_q^j(t-t_{i-1})^j(t_i-t)^{q-j}\cdot f\left(t_{i-1}+\frac{jh}{q},\overline{x_1}\left(t_{i-1}+\frac{jh}{q}\right)\right), \quad (1.18)$$

$t\in[t_{i-1},t_i]$.

Step 3: In inductive way, for $m\geq 2$, with $k=\overline{0,n-1}$, $l=\overline{0,q-1}$ let

$$\begin{aligned}x_m\left(t_k+\frac{lh}{q}\right)&=g\left(t_k+\frac{lh}{q}\right)\\&+\lambda b\left(t_k+\frac{lh}{q}\right)\int_0^T\left|t_k+\frac{lh}{q}-s\right|^{\alpha-1}f\left(s,x_{m-1}(s)\right)ds=g\left(t_k+\frac{lh}{q}\right)\\&+\lambda b\left(t_k+\frac{lh}{q}\right)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left|t_k+\frac{lh}{q}-s\right|^{\alpha-1}\left(B_{m-1,i}(s)+R_{m-1,i}(s)\right)ds\\&=g\left(t_k+\frac{lh}{q}\right)+\frac{\lambda}{h^q}b\left(t_k+\frac{lh}{q}\right)\sum_{i=1}^{n}\sum_{j=0}^{q}C_q^j\varphi_{k,l,j}\\&\cdot f\left(t_{i-1}+\frac{jh}{q},\overline{x_{m-1}}\left(t_{i-1}+\frac{jh}{q}\right)\right)+\overline{R_{m,(k,l)}}\\&=\overline{x_m}\left(t_k+\frac{lh}{q}\right)+\overline{R_{m,(k,l)}} \qquad (1.19)\end{aligned}$$

and

$$x_m(t_n)=g(t_n)+\lambda b(t_n)\int_0^T(t_n-s)^{\alpha-1}f\left(s,x_{m-1}(s)\right)ds=g(t_n)$$

$$+\lambda b(t_n)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_n-s)^{\alpha-1}\left(B_{m-1,i}(s)+R_{m-1,i}(s)\right)ds=g(t_n)+\frac{\lambda}{h^q}b(t_n)$$

$$\cdot\sum_{i=1}^{n}\sum_{j=0}^{q}C_q^j\varphi_{n,j}\cdot f\left(t_{i-1}+\frac{jh}{q},\overline{x_{m-1}}\left(t_{i-1}+\frac{jh}{q}\right)\right)+\overline{R_{m,n}}=\overline{x_m}(t_n)+\overline{R_{m,n}} \quad (1.20)$$

where

$$B_{m,i}(t)=\frac{1}{h^q}\sum_{k=0}^{q}C_q^k(t-t_{i-1})^k(t_i-t)^{q-k}\cdot F_m\left(t_{i-1}+\frac{kh}{q}\right), t\in[t_{i-1},t_i] \quad (1.21)$$

and

$$\overline{B_{m,i}}(t)=\frac{1}{h^q}\sum_{k=0}^{q}C_q^k(t-t_{i-1})^k(t_i-t)^{q-k}\cdot f\left(t_{i-1}+\frac{kh}{q},\overline{x_m}\left(t_{i-1}+\frac{kh}{q}\right)\right) \quad (1.22)$$

for $t\in[t_{i-1},t_i]$ and $i=\overline{1,n}$ are the Bernstein splines B_{m-1} and $\overline{B_{m-1}}$. The algorithm is stopped to a previously chosen iteration m and at this iterative step we construct the Bernstein spline $\widetilde{B_m}$ given on the subintervals $[t_{i-1}, t_i]$, $i = \overline{1,n}$, by

$$\widetilde{B_{m,i}}(t) = \frac{1}{h^q} \sum_{k=0}^{q} C_q^k (t - t_{i-1})^k (t_i - t)^{q-k} \cdot \overline{x_m}\left(t_{i-1} + \frac{kh}{q}\right), t \in [t_{i-1}, t_i]. \quad (1.23)$$

This spline $\widetilde{B_m}$ will be the continuous approximation of the solution.

2. Convergence analysis

Concerning the convergence of the above presented iterative method we obtain the following main result.

Theorem 2.1. *Under the conditions of Theorem 1.1, including (1.5) and (1.6), the sequence* $\left(\overline{x_m}\left(t_k + \frac{lh}{q}\right)\right)_{m\in\mathbb{N}^*}$ *with* $k = \overline{0, n-1}$, $l = \overline{0, q}$, *approximates the solution of the integral equation (0.1) on the knots of a uniform mesh and the sequence of Bernstein splines* $\left(\widetilde{B_m}\right)_{m\in\mathbb{N}^*}$ *approximates the same solution on* $[0, T]$. *The error estimates in the discrete and continuous approximation is*

$$\left|x^*\left(t_k + \frac{lh}{q}\right) - \overline{x_m}\left(t_k + \frac{lh}{q}\right)\right| \le \frac{\theta^m \lambda M_b M_0 T^\alpha}{\alpha(1-\theta)}$$

$$+ \frac{5\lambda M_b T^\alpha}{4\alpha\left(1 - \frac{\lambda L M_b T^\alpha}{\alpha}\right)} \left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^\alpha}{(\sqrt{q})^\alpha} + \frac{L'' h^{\alpha+1}}{(\sqrt{q})^{\alpha+1}}\right), \quad \forall m \in \mathbb{N}^* \quad (2.1)$$

for $k = \overline{0, n-1}$, $l = \overline{0, q}$, *and*

$$\left|x^*(t) - \widetilde{B_m}(t)\right|$$

$$\le \frac{\theta^m \lambda M_b M_0 T^\alpha}{\alpha(1-\theta)} + \frac{5\lambda M_b T^\alpha}{4\alpha\left(1 - \frac{\lambda L M_b T^\alpha}{\alpha}\right)} \left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^\alpha}{(\sqrt{q})^\alpha} + \frac{L'' h^{\alpha+1}}{(\sqrt{q})^{\alpha+1}}\right)$$

$$+ \frac{5}{4}\left(\left(\eta + \frac{2\lambda\beta M T^\alpha}{\alpha}\right)\frac{h}{\sqrt{q}} + \frac{L' h^\alpha}{(\sqrt{q})^\alpha} + \frac{L'' h^{\alpha+1}}{(\sqrt{q})^{\alpha+1}}\right), \quad \forall t \in [0, T] \quad (2.2)$$

where $\theta = \frac{\lambda L M_b T^\alpha}{\alpha}$.

Proof. Since

$$\left|x^*\left(t_k + \frac{lh}{q}\right) - \overline{x_m}\left(t_k + \frac{lh}{q}\right)\right| \le \left|x^*\left(t_k + \frac{lh}{q}\right) - x_m\left(t_k + \frac{lh}{q}\right)\right|$$

$$+ \left|x_m\left(t_k + \frac{lh}{q}\right) - \overline{x_m}\left(t_k + \frac{lh}{q}\right)\right|$$

by (1.3) we have to estimate $\left|\overline{R_{m,(k,l)}}\right| = \left|x_m\left(t_k + \frac{lh}{q}\right) - \overline{x_m}\left(t_k + \frac{lh}{q}\right)\right|$ for $m \in \mathbb{N}^*$, $k = \overline{0, n-1}$, $l = \overline{0, q}$. Based on (1.12) and (1.13) we have

$$\begin{aligned}\left|\overline{R_{1,(k,l)}}\right| &= \left|x_1\left(t_k + \frac{lh}{q}\right) - \overline{x_1}\left(t_k + \frac{lh}{q}\right)\right| \\ &\leq \lambda M_b \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left|R_{0,i}(s)\right| \left|t_k + \frac{lh}{q} - s\right|^{\alpha-1} ds \\ &\leq \frac{5\lambda M_b}{4}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{\left(\sqrt{q}\right)^{\alpha}} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right) \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left|t_k + \frac{lh}{q} - s\right|^{\alpha-1} ds \\ &\leq \frac{5\lambda M_b T^{\alpha}}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{\left(\sqrt{q}\right)^{\alpha}} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right), \; k = \overline{0, n-1}, \; l = \overline{0, q-1}\end{aligned}$$

and by (1.14) we get

$$\left|\overline{R_{1,n}}\right| \leq \frac{5\lambda M_b T^{\alpha}}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{\left(\sqrt{q}\right)^{\alpha}} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right).$$

Now, let us consider $\left|\overline{R_{m-1}}\right| = \max\{\left|\overline{R_{m-1,n}}\right|, \max_{k=\overline{0,n-1}, l=\overline{0,q-1}} \left|\overline{R_{m-1,(k,l)}}\right|\}$, and since

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\sum_{j=0}^{q} C_q^j (s - t_{i-1})^j (t_i - s)^{q-j}\right) \left|t_k + \frac{lh}{q} - s\right|^{\alpha-1} ds \leq \frac{h^q T^{\alpha}}{\alpha}$$

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left[\sum_{j=0}^{q} C_q^j (s - t_{i-1})^j (t_i - s)^{q-j}\right] (t_n - s)^{\alpha-1} ds \leq \frac{h^q T^{\alpha}}{\alpha}$$

by induction for $m \geq 2$, and by (1.2) and (1.17)-(1.22) it obtains

$$\begin{aligned}\left|\overline{R_{m,(k,l)}}\right| &\leq \lambda M_b \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left|R_{m-1,i}(s)\right| \left|t_k + \frac{lh}{q} - s\right|^{\alpha-1} ds \\ &+ \frac{\lambda}{h^q} M_b \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big[\sum_{j=0}^{q} C_q^j (s - t_{i-1})^j (t_i - s)^{q-j} \\ &\cdot L\left|x_{m-1}\left(t_{i-1} + \frac{jh}{q}\right) - \overline{x_{m-1}}\left(t_{i-1} + \frac{jh}{q}\right)\right|\Big] (t_k + \frac{lh}{q} - s)^{\alpha-1} ds \\ &\leq \frac{5\lambda M_b T^{\alpha}}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{\left(\sqrt{q}\right)^{\alpha}} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right) + \frac{\lambda L M_b T^{\alpha}}{\alpha} \left|\overline{R_{m-1}}\right|\end{aligned}$$

with $k=\overline{0,n-1}$, $l=\overline{0,q-1}$. Similarly, we get

$$\left|\overline{R_{m,n}}\right| \le \frac{5\lambda M_b T^\alpha}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)+\frac{\lambda L M_b T^\alpha}{\alpha}\left|\overline{R_{m-1}}\right|.$$

For estimating $\left|\overline{R_{m-1}}\right|$ we have

$$\left|\overline{R_{2,(k,l)}}\right| \le \left[1+\frac{\lambda L M_b T^\alpha}{\alpha}\right]\frac{5\lambda M_b T^\alpha}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right),$$
$$k=\overline{0,n-1}, l=\overline{0,q-1}$$

and

$$\left|\overline{R_{2,n}}\right| \le \frac{5\lambda M_b T^\alpha}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)\left(1+\frac{\lambda L M_b T^\alpha}{\alpha}\right),$$

obtaining in inductive manner for $m \ge 3$, the estimate

$$\begin{aligned}\left|\overline{R_{m,(k,l)}}\right| &\le \left[1+\frac{\lambda L M_b T^\alpha}{\alpha}+\ldots+\left(\frac{\lambda L M_b T^\alpha}{\alpha}\right)^{m-1}\right]\frac{5\lambda M_b T^\alpha}{4\alpha} \\ &\quad\cdot\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right) \\ &\le \frac{5\lambda M_b T^\alpha}{4\alpha\left(1-\frac{\lambda L M_b T^\alpha}{\alpha}\right)}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)\end{aligned}$$

with $k=\overline{0,n-1}$, $l=\overline{0,q}$. Now, the inequality (2.1) follows.
The estimate (2.2) will be obtained by using the scheme

$$x^* \to x_m \to \widehat{B_m} \to \widetilde{B_m}$$

where $\widehat{B_{m,i}}(t) = \frac{1}{h^q}\sum_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i-t)^{q-k}\cdot x_m\left(t_{i-1}+\frac{kh}{q}\right)$ for $t\in[t_{i-1},t_i]$, $i=\overline{1,n}$. According to the proof of Theorem 1.1 we have

$$\left|x_m(t')-x_m(t)\right| \le \left(\eta+\frac{2\lambda\beta M T^\alpha}{\alpha}\right)\left|t-t'\right|+L'\left|t-t'\right|^\alpha+L''\left|t-t'\right|^{\alpha+1}$$

for $t\in[0,T]$, $m\in\mathbb{N}^*$, and with the inequality of Lorentz we get

$$\left|x_m(t)-\widehat{B_m}(t)\right| \le \frac{5}{4}\left(\left(\eta+\frac{2\lambda\beta M T^\alpha}{\alpha}\right)\frac{h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)$$

for all $t\in[0,T]$. By (1.23) and (2.1) it follows,

$$\left|\widehat{B_m}(t)-\widetilde{B_m}(t)\right|$$

$$\le \frac{1}{h^q}\sum_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i-t)^{q-k}\cdot\left|x_m\left(t_{i-1}+\frac{kh}{q}\right)-\overline{x_m}\left(t_{i-1}+\frac{kh}{q}\right)\right|$$

$$\leq \frac{5\lambda M_b T^\alpha}{4\alpha\left(1-\frac{\lambda L M_b T^\alpha}{\alpha}\right)}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)$$

for $t \in [t_{i-1}, t_i]$, $i = \overline{1,n}$, and with (1.3) we obtain the error estimate (2.2). □

By the error estimate (2.2) we observe that the order of convergence of this iterative method is $\left\| x_m - \widetilde{B_m} \right\|_\infty = O(h^\alpha)$.

3. Numerical experiments

In order to test the obtained theoretical result and to illustrate the accuracy of the Bernstein spline iterative method, in that follows we present some numerical examples.

Example 3.1. The weakly singular linear integral equation (Example 6.1. in [32])

$$x(t) = g(t) + \frac{1}{4}\int_0^1 \sqrt{ts}\,|t-s|^{-\frac{1}{2}}\, x(s)\, ds, \quad t \in [0,1] \tag{3.1}$$

with $\lambda = \frac{1}{4}$, $b(t) = \sqrt{t}$, $\alpha = \frac{1}{2}$, and

$$g(t) = \frac{1}{5}\sqrt{t}(1-t)\left[15 - \sqrt{1-t}(1+4t)\right] + \frac{1}{5}t^2(4t-5)$$

has the exact solution $x^*(t) = 3\sqrt{t}(1-t)$. By considering separately the degree of Bernstein polynomials $q = 1$ and $q = 4$, we apply the algorithm (1.13)-(1.22) with $m = 30$ iterations, and take $n = 10$, $n = 50$, and $n = 100$ for the test of convergence. The results are presented in Tables 1 and 2, where $e_i = |\overline{x_m}(t_i) - x^*(t_i)|$, $i = \overline{0,n}$, are the pointwise errors. Investigating Tables 1 and 2, the convergence is confirmed and improved results can be observed when the degree of Bernstein polynomials increases by $q = 1$ to $q = 4$. It is interesting to see that the case $q = 1$ corresponds to the trapezoidal product integration and as was expected, the Bernstein splines iterative method provides better results.

Table 1. Numerical results for (3.1) with $q = 1$

	$m = 30$ $q = 1$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
0,0	0,00	0,00	0,00
0,2	$1,86E-03$	$7,88E-05$	$2,00E-05$
0,4	$2,98E-03$	$1,26E-04$	$3,20E-05$
0,6	$3,96E-03$	$1,68E-04$	$4,25E-05$
0,8	$4,63E-03$	$1,97E-04$	$5,00E-05$
1,0	$3,88E-03$	$1,64E-04$	$4,15E-05$

Table 2. Numerical results for (3.1) with $q = 4$

	$m = 30\ q = 4$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$0,00$	$0,00$	$0,00$
$0,2$	$4,75E-04$	$1,99E-05$	$5,02E-06$
$0,4$	$7,66E-04$	$3,20E-05$	$8,06E-06$
$0,6$	$1,02E-03$	$4,25E-05$	$1,07E-05$
$0,8$	$1,20E-03$	$5,00E-05$	$1,26E-05$
$1,0$	$1,01E-03$	$4,16E-05$	$1,05E-05$

Example 3.2. We test the Bernstein spline iterative method (1.17)-(1.23) on fractional integral equations too, and for the nonlinear integral equation

$$x(t) = g(t) + \frac{1}{4\Gamma\left(\frac{1}{2}\right)} \int_0^1 |t-s|^{-\frac{1}{2}} [x(s)]^2 ds, \quad t \in [0,1] \tag{3.2}$$

with

$$\lambda = 1,\ b(t) = \frac{1}{4},\ \alpha = \frac{1}{2},$$

and

$$g(t) = \sqrt{t(1-t)} + \frac{1}{15\sqrt{\pi}} \left[t^{\frac{3}{2}} (4t-5) - (1-t)^{\frac{3}{2}} (4t+1) \right]$$

the exact solution is $x^*(t) = \sqrt{t(1-t)}$. The contraction condition

$$\frac{LM_bT^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\sqrt{\pi}} < 1$$

is fulfilled and the iterative method (1.17)-(1.23) applied with $m = 30$, $n = 10$, $n = 50$, $n = 100$, $q = 1$ and $q = 4$, respectively, provides the results presented in Tables 3 and 4. The convergence is confirmed and we observed better results when we pass by $q = 1$ to $q = 4$. So, the Bernstein splines iterative method is better than the trapezoidal product integration method for fractional integral equations, too.

Table 3. Numerical results for (3.2) with $q = 1$

	$m = 30\ q = 1$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$5,87E-04$	$2,48E-05$	$6,27E-06$
$0,2$	$8,15E-04$	$3,48E-05$	$8,81E-06$
$0,4$	$8,93E-04$	$3,80E-05$	$9,62E-06$
$0,6$	$8,93E-04$	$3,80E-05$	$9,62E-06$
$0,8$	$8,15E-04$	$3,48E-05$	$8,81E-06$
$0,7$	$8,64E-04$	$3,68E-05$	$9,32E-06$
$1,0$	$5,87E-04$	$2,48E-05$	$6,27E-06$

Table 4. Numerical results for (3.2) with $q = 4$

	$m = 30\ q = 4$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$1,51E-04$	$6,26E-06$	$1,58E-06$
$0,2$	$2,10E-04$	$8,78E-06$	$2,22E-06$
$0,4$	$2,30E-04$	$9,60E-06$	$2,42E-06$
$0,6$	$2,30E-04$	$9,60E-06$	$2,42E-06$
$0,8$	$2,10E-04$	$8,78E-06$	$2,22E-06$
$1,0$	$1,51E-04$	$6,26E-06$	$1,58E-06$

Example 3.3. In order to make a comparison with other methods from the existing literature we present the results obtained on the following example. The linear weakly singular integral equation (Example 1. in [25], Example 6.2. in [32], Example 4. in [34])

$$x(t) = g(t) + \frac{1}{10}\int_0^1 |t-s|^{-\frac{1}{3}} x(s)\,ds, \quad t \in [0,1] \tag{3.3}$$

with $\lambda = \frac{1}{10}$, $b(t) = 1$, $\alpha = \frac{2}{3}$, and

$$g(t) = t^2(1-t)^2 - \frac{27}{30800}\left[t^{\frac{8}{3}}\left(54t^2 - 126t + 77\right) + (1-t)^{\frac{8}{3}}\left(54t^2 + 18t + 5\right)\right]$$

has the exact solution $x^*(t) = t^2(1-t)^2$. By applying the iterative method (1.17)-(1.23) with $m = 30$, $n = 10$, $n = 50$, $n = 100$, and taking $q = 1$ and $q = 4$, we obtain the results presented in Tables 5 and 6. Comparing the results between Table 6 ($n = 100$) and Table 1 in [25] (where the accuracy is $O\left(10^{-6}\right)$), we see better accuracy for our method. In Tables 5 and 6 we see that the accuracy is improved by passing from $n = 10$ to $n = 100$, that confirm the convergence of Bernstein splines method stated in Theorem 2.1. Moreover, for $q = 4$ the accuracy is better than those for $q = 1$, which means that again the Bernstein splines method provides better accuracy than the trapezoidal product integration method.

Table 5. Numerical results for (3.3) with $q = 1$

	$m = 30\ q = 1$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$1,70E-05$	$8,94E-07$	$2,32E-07$
$0,1$	$1,74E-05$	$8,48E-07$	$2,19E-07$
$0,2$	$1,59E-06$	$4,51E-08$	$1,06E-08$
$0,3$	$1,99E-05$	$8,73E-07$	$2,22E-07$
$0,4$	$3,23E-05$	$1,43E-06$	$3,63E-07$
$0,5$	$3,67E-05$	$1,62E-06$	$4,13E-07$
$0,6$	$3,23E-05$	$1,43E-06$	$3,63E-07$
$0,7$	$1,99E-05$	$8,73E-07$	$2,22E-07$
$0,8$	$1,59E-06$	$4,51E-08$	$1,06E-08$
$0,9$	$1,74E-05$	$8,48E-07$	$2,19E-07$
$1,0$	$1,70E-05$	$8,94E-07$	$2,32E-07$

Table 6. Numerical results for (3.3) with $q = 4$

	$m = 30$ $q = 4$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$4,48E-06$	$2,25E-07$	$5,81E-08$
$0,1$	$4,40E-06$	$2,13E-07$	$5,48E-08$
$0,2$	$4,13E-07$	$1,13E-08$	$2,66E-09$
$0,3$	$5,03E-06$	$2,19E-07$	$5,56E-08$
$0,4$	$8,16E-06$	$3,58E-07$	$9,10E-08$
$0,5$	$9,26E-06$	$4,06E-07$	$1,03E-07$
$0,6$	$8,16E-06$	$3,58E-07$	$9,10E-08$
$0,7$	$5,03E-06$	$2,19E-07$	$5,56E-08$
$0,8$	$4,13E-07$	$1,13E-08$	$2,66E-09$
$0,9$	$4,40E-06$	$2,13E-07$	$5,48E-08$
$1,0$	$4,48E-06$	$2,25E-07$	$5,81E-08$

4. Conclusions

The iterated Bernstein splines method was applied to second kind weakly singular and fractional Fredholm integral equations and in Theorem 2.1 the convergence of this method was proved providing the order of convergence $O(h^{\alpha})$. The condition that ensures the convergence is the same as the contraction condition and therefore, the applicability of this method is limited by the contraction condition. On the other hand the accuracy of this method is better than those provided by the trapezoidal product integration, as was observed in the previously presented numerical examples and, on some cases, provides better accuracy than the existing methods from literature.

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