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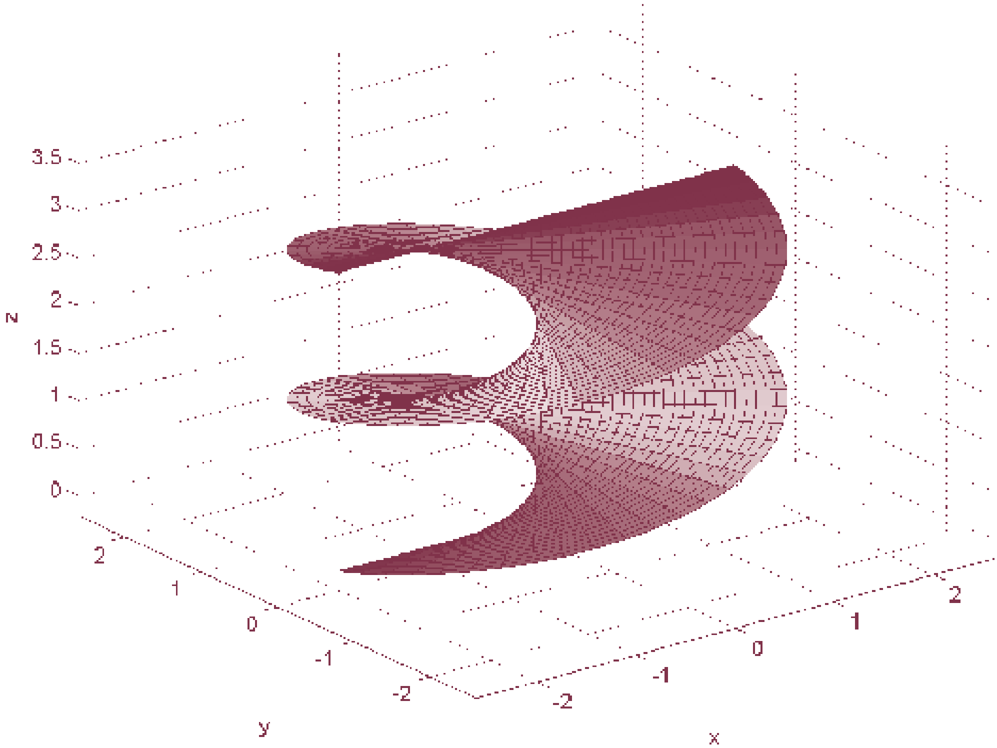

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CONTENTS

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Idempotent and nilpotent elements in octonion rings over $\mathbb{Z}_p$

Michael Aristidou, Philip R. Brown and George Chailos

Abstract. In this paper, we show that the set $\mathbb{O}/\mathbb{Z}_p$, where p is a prime number, does not form a skew field and discuss idempotent and nilpotent elements in the (finite) ring $\mathbb{O}/\mathbb{Z}_p$. We provide examples and establish conditions for idempotency and nilpotency.

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Keywords: Quaternion, octonion, ring, skew field, idempotent, nilpotent.

1. Introduction

Quaternions, denoted by $\mathbb{H}$, were first discovered by William. R. Hamilton in 1843 as an extension of complex numbers into four dimensions [10]. Namely, a quaternion is of the form

$$x = a_0 + a_1 i + a_2 j + a_3 k,$$

where a_i are reals and i, j, k are such that $i^2 = j^2 = k^2 = ijk = -1$. Algebraically speaking, $\mathbb{H}$ forms a division algebra (skew field) over $\mathbb{R}$ of dimension 4 ([10], p.195-196).About the same time, John T. Graves discovered the octonions, denoted by $\mathbb{O}$, which are 8-dimensional numbers of the form

$$x = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7$$

where a_i are reals and e_i's are mutually anti-commuting roots of unity. (i.e. $e_i^2 = -1$ and $e_i e_j = e_k$, $e_j e_i = -e_k$, $i \neq j$) [6]. Algebraically speaking, $\mathbb{O}$ forms a normed division algebra (skew field) over $\mathbb{R}$ of dimension 8 [6]. It is the largest of the (only) four normed division algebras and it is nonassociative.

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A study of the structure and some of its properties of the finite ring[2] $\mathbb{H}/\mathbb{Z}_p$, where p is a prime number, was done in [2]. A more detailed description of the structure $\mathbb{H}/\mathbb{Z}_p$ was given by Miguel and Serodio in [20]. Among others, they found the number of zero-divisors, the number of idempotent elements, and provided an interesting description of the zero-divisor graph. In particularly, they showed that the number of idempotent elements in $\mathbb{H}/\mathbb{Z}_p$ is p^2+p+2, for p odd prime. As discussed in [3], the only scalar idempotents in $\mathbb{H}/\mathbb{Z}_p$ are $a_0 = 0, 1$. Furthermore, there are no purely imaginary idempotents in $\mathbb{H}/\mathbb{Z}_p$. On the other hand, in [4], it was shown that nilpotents x in $\mathbb{H}/\mathbb{Z}_p$ are purely imaginary with norm $N(x) = 0$ and $x^2 = 0$.

In the sections that follow, we look at the structure of the finite ring $\mathbb{O}/\mathbb{Z}_p$. The multiplication of octonions followed the Fano Plane and it was programmed in Maple[3]. We give examples of idempotent and nilpotent elements in $\mathbb{O}/\mathbb{Z}_p$ and provide conditions for idempotency and nilpotency in $\mathbb{O}/\mathbb{Z}_p$.

2. Is $\mathbb{O}/\mathbb{Z}_p$ a finite skew field? A counterexample

In [2] we saw that since $\mathbb{Z}_\mathbf{p}$ is a field, then $\mathbb{H}/\mathbb{Z}_p$ is a quaternion algebra. The theory of quaternion algebras over a field $\mathbb{K}$ (char$\mathbb{K} \neq 2$) tells us that a quaternion algebra Q is either a division ring or $Q = \mathbb{M}_{2\times 2}(\mathbb{K})$ ([16], p.16, 19). Since $\mathbb{H}/\mathbb{Z}_p$ is not a division ring (see [2]), then $\mathbb{H}/\mathbb{Z}_p \cong \mathbb{M}_{2\times 2}(\mathbb{Z}_p)$ if $p \neq 2$.

The real matrix representation of $\mathbb{H}/\mathbb{Z}_p$, where $x = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}/\mathbb{Z}_p$, is achieved by the 4×4 left or right Hamilton Operators as follows:

$$H_x^L = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & -a_1 & -a_0 \end{bmatrix} \quad H_x^R = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}$$

But is the finite ring $\mathbb{O}/\mathbb{Z}_p$ a skew field? Consider the elements

$$x_1 = 2e_2 - e_3, x_2 = e_4 + 3e_5$$

in $\mathbb{O}/\mathbb{Z}_5$. Multiplying the two, we get:

$$x_1 \cdot x_2 = (2e_2 - e_3)(e_4 + 3e_5) = 0(\mathrm{mod}\, 5).$$

This shows that $\mathbb{O}/\mathbb{Z}_5$ has zero-divisors, and hence $\mathbb{O}/\mathbb{Z}_5$ is not a skew field. This was also anticipated by some well-known theorem in algebra, by Wedderburn in 1905 ([11], p.361), which says that: "Every finite skew field is a field". Since $\mathbb{O}/\mathbb{Z}_p$ is not commutative, then it is not a field, and so it is not a skew-field.

So, what is the structure of $\mathbb{O}/\mathbb{Z}_p$? Since $\mathbb{Z}_p$ is a field, then $\mathbb{O}/\mathbb{Z}_p$ is a non-associative octonion algebra. As a matter of fact, is it an alternative, flexible and power associative algebra[4]. It is well known that $\mathbb{O}$ is a skew field, yet it has no "proper" matrix representation due to the non-associativity. Nevertheless, as $\mathbb{O}$ is an extension of $\mathbb{H}$, by the Cayley-Dickson process, some non-proper 8×8 real matrix representations were introduced, by Tian in [26], through the left and right Hamilton

Operators of quaternions analogous to the one above. Namely:

$$H_x^L = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}$$

$$H_x^R = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & a_3 & -a_2 & a_5 & -a_4 & -a_7 & a_6 \\ a_2 & -a_3 & a_0 & a_1 & a_6 & a_7 & -a_4 & -a_5 \\ a_3 & a_2 & -a_1 & a_0 & a_7 & -a_6 & a_5 & -a_4 \\ a_4 & -a_5 & -a_6 & -a_7 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & -a_7 & a_6 & -a_1 & a_0 & -a_3 & a_2 \\ a_6 & a_7 & a_4 & -a_5 & -a_2 & a_3 & a_0 & -a_1 \\ a_7 & -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$

Modifying the above over $\mathbb{Z}_p$, one could easily get the left and right 8×8 real representations of $\mathbb{O}/\mathbb{Z}_p$ as follows[5]:

$$H_x^L = \begin{bmatrix} a_0 & p-a_1 & p-a_2 & p-a_3 & p-a_4 & p-a_5 & p-a_6 & p-a_7 \\ a_1 & a_0 & p-a_3 & a_2 & p-a_5 & a_4 & a_7 & p-a_6 \\ a_2 & a_3 & a_0 & p-a_1 & p-a_6 & p-a_7 & a_4 & a_5 \\ a_3 & p-a_2 & a_1 & a_0 & p-a_7 & a_6 & p-a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & p-a_1 & p-a_2 & p-a_3 \\ a_5 & p-a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & p-a_7 & p-a_4 & a_5 & a_2 & p-a_3 & a_0 & a_1 \\ a_7 & a_6 & p-a_5 & -a_4 & a_3 & a_2 & p-a_1 & a_0 \end{bmatrix}$$

$$H_x^R = \begin{bmatrix} a_0 & p-a_1 & p-a_2 & p-a_3 & p-a_4 & p-a_5 & p-a_6 & p-a_7 \\ a_1 & a_0 & a_3 & p-a_2 & a_5 & p-a_4 & p-a_7 & a_6 \\ a_2 & p-a_3 & a_0 & a_1 & a_6 & a_7 & p-a_4 & p-a_5 \\ a_3 & a_2 & p-a_1 & a_0 & a_7 & p-a_6 & a_5 & p-a_4 \\ a_4 & p-a_5 & p-a_6 & p-a_7 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & p-a_7 & a_6 & p-a_1 & a_0 & p-a_3 & a_2 \\ a_6 & a_7 & a_4 & p-a_5 & p-a_2 & a_3 & a_0 & p-a_1 \\ a_7 & p-a_6 & a_5 & a_4 & p-a_3 & p-a_2 & a_1 & a_0 \end{bmatrix}$$

Notice that for the octonionic cases $\mathbb{O}$ and $\mathbb{O}/\mathbb{Z}_p$, we have that $H_{xy}^L \neq H_x^L H_y^L$ because of the non-associativity.

3. Idempotent and nilpotents elements in $\mathbb{O}/\mathbb{Z}_p$

Recall that an element x in a ring R is called idempotent if $x^2 = x$. In the ring $\mathbb{H}/\mathbb{Z}_p$, p prime, in the special case where $x = a_0, a_0 \neq 0$ (i.e., x is a non-zero scalar in $\mathbb{H}/\mathbb{Z}_p$) one quickly observes that if x is idempotent then $x = 1$, for x in 1, 2, ..., p-1, since $(x, p) = 1$. Therefore, the only scalar idempotent in $\mathbb{H}/\mathbb{Z}_p$ is 1 (we omit the case $x = 0$ as trivial). Another simple case is the case where $x = ai, aj$ or $ak, a \neq 0$ (i.e., a non-zero scalar multiple of the imaginary units). Then, $x^2 = (ai)^2 = -a^2 i^2 = -a^2 \neq ai = x$, which shows that there are no idempotents of the form ai, aj or ak. (Again, we omitted the case $x = 0$ as trivial). Examples of proper idempotents[6] and conditions for idempotency in $\mathbb{H}/\mathbb{Z}_p$ were given in [3]. Due to the isomorphism $\mathbb{H}/\mathbb{Z}_p \cong \mathbb{O}[e_i, e_j, e_i e_j]$ (where $e_i \neq e_j$) idempotents in $\mathbb{H}/\mathbb{Z}_p$ will transfer in some subalgebras[7] of $\mathbb{O}/\mathbb{Z}_p$. For example, $x = 4 + i + 3j + 4k$ is idempotent in $\mathbb{H}/\mathbb{Z}_7$ and therefore $x = 4 + e_1 + 3e_2 + 4e_3$ is idempotent in $\mathbb{O}/\mathbb{Z}_7$. Nevertheless, $x = 4 + e_1 + 3e_3 + 4e_5$ is a non-"quaternionic" idempotent in $\mathbb{O}/\mathbb{Z}_7$. Notice that $x = 7i + 4j$ is nilpotent in $\mathbb{H}/\mathbb{Z}_{13}$ and so $x = 7e_1 + 4e_2$ is also nilpotent in $\mathbb{O}/\mathbb{Z}_{13}$. Nevertheless, $x = 4e_1 + e_2 + 3e_3 + 4e_5$ is a non-"quaternionic" nilpotent in $\mathbb{O}/\mathbb{Z}_7$. As we will show below, purely imaginary octonions in $\mathbb{O}/\mathbb{Z}_p$ cannot be idempotents, just as in $\mathbb{H}/\mathbb{Z}_p$ [3]. And nilpotents in $\mathbb{O}/\mathbb{Z}_p$ are purely imaginary, just as in $\mathbb{H}/\mathbb{Z}_p$ [4].

Theorem 3.1. *Let $x \in \mathbb{O}/\mathbb{Z}_p$ be an octonion of the form $x = a_0 + \sum_{i=1}^{7} a_i e_i$. Then x is idempotent if and only if $a_0 = \frac{1+p}{2}$ and $\sum_{i=1}^{7} a_i^2 = \frac{p^2-1}{4}$.*

Proof. We follow the steps given in the proof for the quaternion case in [3]. Since x is idempotent, we have:

$$x^2 = x \Rightarrow \left(a_0 + \sum_{i=1}^{7} a_i e_i\right)\left(a_0 + \sum_{i=1}^{7} a_i e_i\right) = a_0 + \sum_{i=1}^{7} a_i e_i$$

$$\Rightarrow a_0^2 + 2a_0 \sum_{i=1}^{7} a_i e_i + \left(\sum_{i=1}^{7} a_i e_i\right)\left(\sum_{i=1}^{7} a_i e_i\right) = a_0 + \sum_{i=1}^{7} a_i e_i$$

$$\xrightarrow[\text{Fano}]{\text{distr.}} a_0^2 - \sum_{i=1}^{7} a_i^2 = a_0 \quad \text{and} \quad 2a_0 a_i = a_i$$

From the 2^{nd} equation, we have that either $a_i = 0$ or $2a_0 = 1$. That is $a_0 = \frac{1+p}{2}$, as $p = 0 (\bmod p)$. Substituting the latter in the 1^{st} equation, we get $\sum_{i=1}^{7} a_i^2 = \frac{p^2-1}{4}$. ☐

Corollary 3.2. *Let $x \in \mathbb{O}/\mathbb{Z}_p$ be a purely imaginary octonion of the form*

$$x = \sum_{i=1}^{7} a_i e_i.$$

Then x is not idempotent.

Proof. If x is purely imaginary then $a_0 = 0$. Then from Theorem 3.1, $0 = \frac{1+p}{2}$ which is a contradiction. □

Example 3.3. Consider $x = 4 + e_1 + 3e_3 + 4e_5$ in $\mathbb{O}/\mathbb{Z}_7$. Then x is idempotent. Notice that $4 = \frac{1+7}{2}$ and $1^2 + 3^2 + 4^2 = 26 = \frac{49-1}{4} \bmod(7)$.

Remark 3.4. To find the number of idempotents in $\mathbb{O}/\mathbb{Z}_p$, one could naturally find how many ways $\frac{P^2-1}{4}$ can be written as a sum of seven or fewer squares. The equation $\sum_{i=1}^{7} a_i^2 = \frac{p^2-1}{4}$ in Theorem 3.1 brings to mind the "sum of seven squares problem", which is to find the different values $r_7(n)$ for which $n = \sum_{i=1}^{7} x_i^2$, $n \in \mathbb{N}$. A formula for square-free values of n were stated without proof by Eisenstein in 1847, and those were extended to all positive integers n by Smith in 1864, also without a proof. Hardy in 1920 developed a method in deriving the proof for $r_k(n)$, where k is odd, but he explicitly showed only the $r_5(n)$ case in [13, 12]. More general results for $\mathrm{r}_7(n)$ were given by Cooper in 2001 [8] and Cooper and Hirschhorn in 2007 [9].

Recall that an element x in a ring R is called nilpotent if $x^k = 0$ for some $k \in \mathbb{N}$. In [4], it was shown that if x in $\mathbb{H}/\mathbb{Z}_p$ is nilpotent then the norm $N(x) = 0$ (where $N(x) = xx^* = \sum_{i=0}^{3} a_i^2$) and, furthermore, that x is purely imaginary and $x^2 = 0$. If $x \in \mathbb{O}/\mathbb{Z}_p$, we have similar results. First, consider the following Lemmas:

Lemma 3.5. *For any $x \in \mathbb{O}/\mathbb{Z}_p$, we have that $x^2 - 2a_0x + N(x) = 0$.*

Proof. Let $x = a_0 + \sum_{i=1}^{7} a_ie_i$. Then the left-hand side of the equation becomes:

$$\begin{aligned}
x^2 - 2a_0x + N(x) &= (a_0 + \sum_{i=1}^{7} a_ie_i)(a_0 + \sum_{i=1}^{7} a_ie_i) - 2a_0x + N(x) \\
&= a_0^2 + 2a_0\sum_{i=1}^{7} a_ie_i + (\sum_{i=1}^{7} a_ie_i)(\sum_{i=1}^{7} a_ie_i) - 2a_0(a_0 + \sum_{i=1}^{7} a_ie_i) + \sum_{i=0}^{7} a_i^2 \\
&= a_0^2 + \sum_{i=1}^{7} 2a_0a_ie_i - \sum_{i=1}^{7} a_i^2 - 2a_0(a_0 + \sum_{i=1}^{7} a_ie_i) + \sum_{i=0}^{7} a_i^2 \\
&= a_0^2 + 2a_0\sum_{i=1}^{7} a_ie_i - \sum_{i=1}^{7} a_i^2 - 2a_0^2 - 2a_0\sum_{i=1}^{7} a_ie_i + a_0^2 + \sum_{i=1}^{7} a_i^2 \\
&= 0
\end{aligned}$$

□

Lemma 3.6. *Let $x \in \mathbb{O}/\mathbb{Z}_p$. If x is nilpotent, then $N(x) = 0$.*

Proof. We follow the steps given in the proof for the quaternion case in [4]. If x is nilpotent, then $x^k = 0$ for some k. From Lemma 3.5 above, we have:

$$\begin{aligned} x^2 - 2a_0x + N(x) = 0 &\Rightarrow x(x - 2a_0) = -N(x) \\ &\Rightarrow (x(x - 2a_0))^k = (-N(x))^k \\ &\Rightarrow x^k(x - 2a_0)^k = (-N(x))^k \text{ (see Remark 3.7 below)} \\ &\Rightarrow 0 = (N(x))^k \\ &\Rightarrow N(x) = 0, \text{ because } \mathbb{Z}_\mathbf{p} \text{ is a field.} \end{aligned}$$

□

Remark 3.7. We discuss the statement $(x(x - 2a_0))^k = x^k(x - 2a_0)^k$ in the proof in the Lemma 3.6 above: The statement is taken as obvious, without a proof, in [4] (in Lemma 2.1) for the quaternionic case $\mathbb{H}/\mathbb{Z}_p$, but it deserves a bit more explanation in our case here considering the non-commutativity and non-associativity of $\mathbb{O}/\mathbb{Z}_p$. As we mentioned in Sec.2, $\mathbb{O}/\mathbb{Z}_p$ is an alternative algebra (and flexible). Therefore, it also satisfies the *Moufang Identities*, in particularly the identity $(xy)(zx) = (x(yz))x$. Given this, it is not hard to show the following:

Proposition 3.8. *If A is an alternative algebra such that $xy = yx$, x, $y \in A$, then $(xy)^k = x^k y^k$.*

Proof. We show this for $k = 2$ (the general case follows by iteration). Indeed:

$$\begin{aligned} (xy)^2 = (xy)(xy) \overset{comm.}{=} (yx)(xy) &\overset{Mouf.}{=} (y(xx))y \\ &\overset{altern.}{=} ((yx)x)y \\ &\overset{comm.}{=} ((xy)x)y \\ &\overset{flex.}{=} (x(yx))y \\ &\overset{comm.}{=} (x(xy))y \\ &\overset{altern.}{=} ((xx)y)y \\ &\overset{Mouf.}{=} (xx)(yy) \end{aligned}$$

□

Hence, the statement $(x(x - 2a_0))^k = x^k(x - 2a_0)^k$ is also true in our particular case here, because $\mathbb{O}/\mathbb{Z}_p$ is alternative (and flexible) and $x(x - 2a_0) = (x - 2a_0)x$. It is also clear now why the statement is easy to prove in $\mathbb{H}/\mathbb{Z}_p$, considering that $\mathbb{H}/\mathbb{Z}_p$ is actually associative. Finally, given the above result, one could also obtain the binomial formula $(x + y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$, which could also be used to prove the statement in question. That is:

$$\begin{aligned}(x(x-2a_0))^k = (x^2-2a_0x)^k &= \sum_{j=0}^{k}\binom{k}{j}(x^2)^j(-2a_0x)^{k-j}\\ &= x^k\sum_{j=0}^{k}\binom{k}{j}x^j(-2a_0)^{k-j}\\ &= x^k(x-2a_0)^k\end{aligned}$$

Theorem 3.9. *Let $x \in \mathbb{O}/\mathbb{Z}_p$. Then x is nilpotent if and only if x is purely imaginary and $N(x) = 0$. Furthermore, if x is nilpotent, then $x^2 = 0$.*

Proof. If x is nilpotent, then $x^k = 0$ for some $k > 1$ (where k is the least such natural number). From Lemma 3.6 above, we have that $N(x) = 0$. Combining Lemmas 3.5 and 3.6, we get $x^2 = 2a_0x$. Following the steps given in the proof for the quaternion case in [4], we have:

$$\begin{aligned}\text{If } k \text{ is even}: \quad x^2 = 2a_0x &\Rightarrow (x^2)^{k/2} = (2a_0)^{k/2}x^{k/2}\\ &\Rightarrow x^k = (2a_0)^{k/2}x^{k/2}\\ &\Rightarrow 0 = (2a_0)^{k/2}x^{k/2}\\ &\Rightarrow a_0 = 0\end{aligned}$$

$$\begin{aligned}\text{If } k \text{ is odd}: \quad x^2 = 2a_0x &\Rightarrow (x^2)^{(k+1)/2} = (2a_0)^{(k+1)/2}x^{(k+1)/2}\\ &\Rightarrow (x)^{(k+1)/2} = (2a_0)^{(k+1)/2}x^{(k+1)/2}\\ &\Rightarrow 0 = (2a_0)^{(k+1)/2}x^{k/2}\\ &\Rightarrow a_0 = 0\end{aligned}$$

Hence, $a_0 = 0$ and therefore x is imaginary. Furthermore, since $a_0 = 0$, from $x^2 = 2a_0x$ we have that $x^2 = 0$. For the converse, since $N(x) = 0$, Lemma 3.5 gives $x^2 = 2a_0x$. Since also x is imaginary ($a_0 = 0$) the equation $x^2 = 2a_0x$ gives $x^2 = 0$. Then for any $k > 1$ we have: $x^k = x^{k-2}x^2 = x^{k-2} \cdot 0 = 0$, so x is nilpotent. □

Example 3.10. Consider $x = 4e_1 + e_2 + 3e_3 + 4e_5$ in $\mathbb{O}/\mathbb{Z}_7$. Then x is nilpotent. Notice that $N(x) = 0^2 + 4^2 + 1^2 + 3^2 + 0^2 + 4^2 + 0^2 + 0^2 = 0(\bmod 7)$.

4. Connection to general rings and applications

There is a lot in the literature on idempotents, nilpotents and k-potents in general, in more general rings R. It would be interesting to see if and how some of these results relate to the 'special', in a sense, ring $\mathbb{O}/\mathbb{Z}_p$.

In [16], Hirano and Tominaga proved that in a ring R the following are equivalent: (i) Every element of R is a sum of two commuting idempotents; (ii) R is commutative and every element of R is a sum of two idempotents; (iii) $x^3 = x$, for all x in R.[8]

As $\mathbb{O}/\mathbb{Z}_p$ is not commutative, the above fails. For example, consider the idempotents $a = 3 + e_1$ and $b = 3 + e_2$ in $\mathbb{O}/\mathbb{Z}_5$. Then,

$$x = a + b = (3 + e_1) + (3 + e_2) = 6 + e_1 + e_2 = 1 + e_1 + e_2,$$

but x is not tripotent (indeed, $(1 + e_1 + e_2)^3 = e_1 + e_2 \neq 1 + e_1 + e_2$). The above fails even when the idempotents commute. Take, for example, $a = b = 3 + e_1$ in $\mathbb{O}/\mathbb{Z}_5$.

Also, Mosic in [21] gives the relation between idempotent and tripotent elements in any associative ring R, generalizing the result on matrices by Trenkler and Baksalary [27]. Namely, for any $x \in R$, where 2, 3 are invertible, x is idempotent if and only if x is tripotent and $1 - x$ is tripotent or $1 + x$ is invertible. Notice that even though $\mathbb{O}/\mathbb{Z}_p$ is not associative, the result does hold in some cases. Take for example the tripotent $x = 4 + 3e_1 + e_2 + 4e_3$ in $\mathbb{O}/\mathbb{Z}_7$, which is also an idempotent. It is not hard to check that directly or using the conditions for idempotency in Theorem 3.1 above. Notice also that $1 - x$ is tripotent and $1 + x$ is invertible as $N(x) = 2 \neq 0$. So, we conjecture that Mosic's result may extend to (some) non-associative rings.

Finally, it is interesting to note any possible applications of rings related to the ring $\mathbb{O}/\mathbb{Z}_p$. Malekian and Zakerolhosseini in [19] use octonionic algebras to construct a high speed public key cryptosystem. More specifically, they consider the convolution polynomial rings $R = \mathbb{Z}[x]/(x^N - 1)$, $R_p = \mathbb{Z}_p[x]/(x^N - 1)$ and $R_q = \mathbb{Z}_q[x]/(x^N - 1)$, where p, q are primes such as $q \gg p$. From these they construct the octonionic algebras:

$$\mathbb{A} = \{\, a_0(x) + \sum_{i=1}^{7} a_i(x)e_i \mid a_i(x) \in R \,\},$$

$$\mathbb{A}_p = \{\, a_0(x) + \sum_{i=1}^{7} a_i(x)e_i \mid a_i(x) \ \in R_p \,\},$$

$$\mathbb{A}_q = \{\, a_0(x) + \sum_{i=1}^{7} a_i(x)e_i \mid a_i(x) \in R_q \,\},$$

respectively. Then, the public (and private) key is generated as follows: initially two small octonions $F \in \mathbb{L}_f$ and $G \in \mathbb{L}_g$, where $\mathbb{L}_f, \mathbb{L}_g$ are some specifically constructed subspaces of $\mathbb{A}$, are randomly generated. Namely,

$$F = f_0 + \sum_{i=1}^{7} f_i e_i \mid f_i \in \mathbb{L}_f\,,$$

$$G = g_0 + \sum_{i=1}^{7} g_i e_i \mid g_i \in \mathbb{L}_g\,.$$

The octonion F must be invertible in $\mathbb{A}_p$ and $\mathbb{A}_q$, otherwise a new octionion F is generated. The inverses of F in $\mathbb{A}_p$ and $\mathbb{A}_q$ are denoted by in F_p^{-1} and F_q^{-1}, respectively. The public key, which is an octonion, is then given by $H = F_p^{-1} o\, G \in \mathbb{A}_q$, where o is a multiplication defined on $\mathbb{A}_q$, in terms of the convolution product. Encryption and decryption are done with similar calculations.

Notes

1. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only normed division algebras. This was proved by Hurwitz in 1898 [17].

2. "+" and "." on $\mathbb{H}$ are defined in [14, p. 124]. As $p = 0 (\bmod p)$ on $\mathbb{H}/\mathbb{Z}_p$ they are defined as follows:

$$
\begin{aligned}
x + y &= (a_0 + a_1 i + a_2 j + a_3 k) + (b_0 + b_1 i + b_2 j + b_3 k) \\
&= (a_0 + b_0) + (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) k \\
x \cdot y &= (a_0 + a_1 i + a_2 j + a_3 k) \cdot (b_0 + b_1 i + b_2 j + b_3 k) \\
&= a_0 b_0 + (p-1) a_1 b_1 + (p-1) a_2 b_2 + (p-1) a_3 b_3 + \\
&\quad (a_0 b_0 + a_1 b_0 + a_2 b_3 + (p-1) a_3 b_2) j + \\
&\quad (a_0 b_2 + (p-1) a_1 b_3 + a_2 b_0 + a_3 b_1) j + \\
&\quad (a_0 b_3 + a_1 b_2 + (p-1) a_2 b_1 + a_3 b_0) k
\end{aligned}
$$

3. Fano Plane (Figure 1); Multiplication table (Figure 2); Program in Maple (Figure 3):

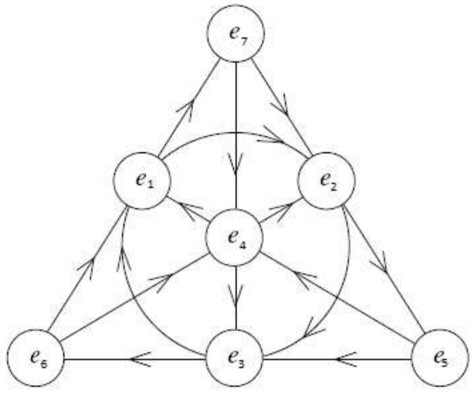

Figure 1. Fano Plane

e_ie_j	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Figure 2. Multiplication table

```
> HypercomplexLib :=`C:\\Hypercomplex\\Hypercomplex.mla`;
> libname:=HypercomplexLib,libname; ### now Maple will find the lib
                HypercomplexLib := C:\Hypercomplex\Hypercomplex.mla
  libname := "C:\Hypercomplex\Hypercomplex.mla", "C:\Program Files\Maple 18\lib", "."      (1)
> with(Hypercomplex) :
> setHypercomplex(octonion) :
> i2·i7
                                   -i5                                                     (2)
> (2 i2 − i3)·(i4 + 3 i5) mod 5;
                                   0                                                       (3)
> (4 + i1 + 3 i3 + 4 i5)·(4 + i1 + 3 i3 + 4 i5) mod 7;
                          4 + i1 + 3 i3 + 4 i5                                             (4)
> (4 i1 + i2 + 3 i3 + 4 i5)·(4 i1 + i2 + 3 i3 + 4 i5) mod 7;
                                   0                                                       (5)
```

Figure 3. Maple program

4. Accordingly, the following hold: $(xx)y = x(xy)$ (alternative), $x(yx) = (xy)x$ (flexible), $\langle x\rangle$ is power associative for all x.

5. These representations are given in [11] without a proof. The proof for $\mathbb{O}/\mathbb{Z}_p$ is actually straightforward, following the exact steps in the proofs of Theorems 2.1 and 2.3 in [26] for the case of $\mathbb{O}$.

6. In Herstein [14, p. 130], we have as an exercise that: In a ring R, if $x^2 = x$, for all x in R, then R is commutative. It is not hard to show that the converse is not true. (e.g. $\mathbb{F} = \mathbb{Z}_3$, 2 is not idempotent). Actually, a field $\mathbb{F}$ has only trivial idempotents. Hence, in $\mathbb{H}/\mathbb{Z}_p$ some elements are non-trivial idempotents and they were described in [3].

7. Namely, the seven quaternionic subalgebras of $\mathbb{O}$ each generated by the seven "line" (including the circle) in the Fano Plane.

8. A ring R is called a *tripotent ring* if $x^3 = x$, for all x in R. The fact that a tripotent ring is commutative is found as an exercise in Hernstein [14, p. 136]. Several proofs of this fact have been given since the 60's [5]. In Bourbaki, we find it also as an exercise with guided steps/hints for the proof [7, p. 176]. See also [23]. Interestingly, a more general result by Jacobson was already known in the 40's [18]. Namely, if in a ring R there exists an integer $n > 1$ such that $x^n{=}x$, for every x in R, then R is commutative. For a proof of Jacobson's Theorem see [5], [15].

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Popoviciu type inequalities for $n-$convex functions via extension of weighted Montgomery identity

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Abstract. In this article, we derive the Popoviciu-type inequalities by using the weighted version of the extension of Montgomery's identity and Green functions. By considering the n-convex function, we prove some identities and related inequalities involving sums $\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i)$ and integrals $\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))\, d\chi$. Some results for n-convex functions at a point are also obtained. Besides that, some Ostrowski-type inequalities are also presented, which are interrelated with the obtained inequalities.

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1. Introduction

Pečarić [15] established the following result (see also [18, p.262]):

Proposition 1.1. *The inequality*

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) \geq 0 \tag{1.1}$$

holds for all convex functions ζ if and only if the $\gamma-$tuples

$$\chi = (\chi_1, \ldots, \chi_\gamma), \quad \varrho = (\varrho_1, \ldots, \varrho_\gamma) \in \mathbb{R}^\gamma$$

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satisfy

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \quad \textit{and} \quad \sum_{i=1}^{\gamma} \varrho_i |\chi_i - \chi_\kappa| \geq 0 \ \forall \ \kappa \in \{1, \dots, \gamma\}. \tag{1.2}$$

Since

$$\sum_{i=1}^{\gamma} \varrho_i |\chi_i - \chi_\kappa| = 2 \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \chi_\kappa)_+ - \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \chi_\kappa),$$

where $y_+ = \max(y, 0)$, it is easy to see that condition (1.2) is equivalent to

$$\sum_{i=1}^{\gamma} \varrho_i = 0, \quad \sum_{i=1}^{\gamma} \varrho_i \chi_i = 0 \quad \text{and} \quad \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \chi_\kappa)_+ \geq 0 \text{ for } \kappa \in \{1, \dots, \gamma-1\}. \tag{1.3}$$

Let $\chi_{(1)} \leq \chi_{(2)} \leq \ldots \leq \chi_{(\gamma)}$ be the sequence χ in ascending order, $w(\chi, \tau) = (\chi - \tau)_+$ and Λ_0 denote the linear operator

$$\Lambda_0(\zeta) = \sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i).$$

Notice that

$$\Lambda(w(\cdot, \chi_\kappa)) = \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \chi_\kappa)_+.$$

For $\tau \in [\chi_{(\kappa)}, \chi_{(\kappa+1)}]$ we have

$$\Lambda(w(\cdot, \tau)) = \Lambda(w(\cdot, \chi_{(\kappa)})) + (\chi_{(\kappa)} - \tau) \sum_{\{i : \chi_i > \chi_{(\kappa)}\}} \varrho_i,$$

so the mapping $\tau \mapsto \Lambda(w(\cdot, \tau))$ is linear on $[\chi_{(\kappa)}, \chi_{(\kappa+1)}]$.
Additionally, $\Lambda(w(\cdot, \chi_{(\gamma)}) = 0$, so condition (1.3) is equivalent to

$$\sum_{i=1}^{\gamma} \varrho_i = 0, \quad \sum_{i=1}^{\gamma} \varrho_i \chi_i = 0 \quad \text{and} \quad \sum_{i=1}^{\gamma} \varrho_i (\chi_i - \tau)_+ \geq 0 \ \forall \ \tau \in [\chi_{(1)}, \chi_{(\gamma-1)}]. \tag{1.4}$$

It comes out that condition (1.4) is suitable for extension of Proposition 1.1 to the integral version and the more general class of n-convex functions (see e.g. [18]).

Definition 1.2. The *nth order divided difference* of a function $\zeta : I \to \mathbb{R}$ at distinct points $\chi_i, \chi_{i+1}, \dots, \chi_{i+n} \in I = [a_1, b_1] \subset \mathbb{R}$ for some $i \in \mathbb{N}$ is defined recursively by:

$$\begin{aligned} [\chi_j; \zeta] &= \zeta(\chi_j), \quad j \in \{i, \dots, i+n\} \\ [\chi_i, \dots, \chi_{i+n}; \zeta] &= \frac{[\chi_{i+1}, \dots, \chi_{i+n}; \zeta] - [\chi_i, \dots, \chi_{i+n-1}; \zeta]}{\chi_{i+n} - \chi_i}. \end{aligned}$$

It may easily be verified that

$$[\chi_i, \dots, \chi_{i+n}; \zeta] = \sum_{\kappa=0}^{n} \frac{\zeta(x_{i+\kappa})}{\prod_{j=i, j \neq i+\kappa}^{i+n} (\chi_{i+\kappa} - \chi_j)}.$$

Remark 1.3. Let us denote $[\chi_i, \ldots, \chi_{i+n}; \zeta]$ by $\Delta_{(n)}\zeta(\chi_i)$. The value $[\chi_i, \ldots, \chi_{i+n}; \zeta]$ is independent of the order of the points $\chi_i, \chi_{i+1}, \ldots, \chi_{i+n}$. This definition can be extended by involving the cases in which two or more points coincide by taking respective limits. □

Definition 1.4. If for all choices of $(n+1)$ distinct points $\chi_i, \ldots, \chi_{i+n}$ we have $\Delta_{(n)}\zeta(\chi_i) \geq 0$ then the function $\zeta : I \to \mathbb{R}$ is called *convex of order* n or $n-$*convex.*

If the function is nth order differentiable such that $\zeta^{(n)} \geq 0$ then ζ is n-convex. A function ζ is n-convex for $1 \leq \kappa \leq n-2$, if and only if $\zeta^{(\kappa)}$ exists and is $(n-\kappa)$-convex.

Popoviciu [19], [20] obtained the following result (see also [17, 18, 22]).

Proposition 1.5. *Let $n \geq 2$. Inequality (1.1) is valid for all n-convex functions $\zeta : [a_1, b_1] \to \mathbb{R}$ if and only if the $\gamma-$tuples $\chi \in [a_1, b_1]^\gamma$, $\varrho \in \mathbb{R}^\gamma$ satisfy*

$$\sum_{i=1}^{\gamma} \varrho_i \chi_i^\kappa = 0, \quad \forall \kappa = 0, 1, \ldots, n-1 \tag{1.5}$$

$$\sum_{i=1}^{\gamma} \varrho_i (\chi_i - \tau)_+^{n-1} \geq 0, \quad \forall \tau \in [a_1, b_1]. \tag{1.6}$$

Definitely Popoviciu established a significant result - it is adequate to postulate that (1.6) holds $\forall\ \tau \in [\chi_{(1)}, \chi_{(\gamma-n+1)}]$ and then, because of (1.5), it is automatically stated $\forall\ \tau \in [a_1, b_1]$. The integral version is given in the following proposition (see [17, 18, 21]).

Proposition 1.6. *Let $n \geq 2$, $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ and $g : [\alpha_1, \beta_1] \to [a_1, b_1]$. Then, the inequality*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))\, d\chi \geq 0 \tag{1.7}$$

holds for all n-convex functions $\zeta : [a_1, b_1] \to \mathbb{R}$ if and only if

$$\begin{aligned} \int_{\alpha_1}^{\beta_1} \varrho(\chi) g(\chi)^\kappa\, d\chi = 0, &\qquad \forall\ \kappa = 0, 1, \ldots, n-1 \\ \int_{\alpha_1}^{\beta_1} \varrho(\chi)\,(g(\chi) - \tau)_+^{n-1}\, d\chi \geq 0, &\qquad \forall\ \tau \in [a_1, b_1]. \end{aligned} \tag{1.8}$$

In this article, we would like to establish some inequalities of type (1.1) and (1.7) by using the following extension of Montgomery's identity via Taylor's formula for n-convex functions obtained in [1].

Proposition 1.7. *Let $\zeta : I \to \mathbb{R}$ be such that $\zeta^{(n-1)}$ is absolutely continuous, $n \in \mathbb{N}$, $a_1, b_1 \in I$, $a_1 < b_1$, $I \subset \mathbb{R}$ an open interval, $w : [a_1, b_1] \to [0, \infty)$ is some probability* *density function. Then the following identity holds*

$$\zeta(\chi) = \int_{a_1}^{b_1} w(\tau)\zeta(\tau)d\tau$$
$$+\sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{a_1}^{\chi} w(\varsigma)\left((\chi-a_1)^{\kappa+1}-(\varsigma-a_1)^{\kappa+1}\right)d\varsigma$$
$$+\sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\chi}^{b_1} w(\varsigma)\left((\chi-b_1)^{\kappa+1}-(\varsigma-b_1)^{\kappa+1}\right)d\varsigma$$
$$+\frac{1}{(n-1)!}\int_{a_1}^{b_1} T_{w,n}(\chi,\varsigma)\zeta^{(n)}(\varsigma)d\varsigma, \tag{1.9}$$

where

$$T_{w,n}(\chi,\varsigma) = \begin{cases} \int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du + W(\chi)(\chi-\varsigma)^{n-1}, \\ \qquad a_1 \le \varsigma \le \chi \\ \int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du + (W(\chi)-1)(\chi-\varsigma)^{n-1}, \\ \qquad \chi < \varsigma \le b_1 \end{cases} \tag{1.10}$$

If we put $w(\tau) = \frac{1}{b_1-a_1}$, $\tau \in [a_1, b_1]$, *the above identity reduces*

$$\begin{aligned} \zeta(\chi) &= \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\tau)\,d\tau + \sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!(\kappa+2)}\frac{(\chi-a_1)^{\kappa+2}}{b_1-a_1} \\ &- \sum_{\kappa=0}^{n-2}\frac{\zeta^{(n-1)}(b_1)}{\kappa!(\kappa+2)}\frac{(\chi-b_1)^{\kappa+2}}{b_1-a_1} + \frac{1}{(n-1)!}\int_{a_1}^{b_1} T_n(\chi,\varsigma)\,\zeta^{(n)}(\varsigma)\,d\varsigma, \end{aligned} \tag{1.11}$$

where

$$T_n(\chi,\varsigma) = \begin{cases} -\frac{(\chi-\varsigma)^n}{n(b_1-a_1)} + \frac{\chi-a_1}{b_1-a_1}(\chi-\varsigma)^{n-1}, & a_1 \le \varsigma \le \chi, \\ -\frac{(\chi-\varsigma)^n}{n(b_1-a_1)} + \frac{\chi-b_1}{b_1-a_1}(\chi-\varsigma)^{n-1}, & \chi < \varsigma \le b_1. \end{cases} \tag{1.12}$$

In case $n = 1$ the sum $\sum_{\kappa=0}^{n-2}\cdots$ is empty, so identity (1.11) encounters to the renowned Montgomery identity (see for instance [13])

$$\zeta(\chi) = \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\tau)\,d\tau + \int_{a_1}^{b_1} P(\chi,\varsigma)\,\zeta'(\varsigma)\,d\varsigma$$

where $P(\chi, \varsigma)$ is the Peano kernel, given by

$$P(\chi, \varsigma) = \begin{cases} \dfrac{s - a_1}{b_1 - a_1}, & a_1 \leq \varsigma \leq \chi, \\ \dfrac{\varsigma - b_1}{b_1 - a_1}, & \chi < \varsigma \leq b_1. \end{cases}$$

The weighted version of Montgomery identity can be found in [14]:

Proposition 1.8. *Let $\zeta \in AC[a_1, b_1]$. Suppose that $w : [a_1, b_1] \to [0, \infty)$ is satisfying*

$$\int_{a_1}^{b_1} w(\tau) d\tau = 1,$$

some probability density function, i.e., it is a positive integrable function and

$$W(\tau) = \begin{cases} 0, & \tau < a_1, \\ \int_{a_1}^{\tau} w(\chi) d\chi, & \tau \in [a_1, b_1], \\ 1, & \tau > b_1. \end{cases}$$

Then

$$\zeta(\chi) = \int_{a_1}^{b_1} w(\tau)\zeta(\tau)\, d\tau + \int_{a_1}^{b_1} P_w(\chi, \tau)\zeta(\tau)\, d\tau,$$

where the weighted Peano kernel is given by

$$P_w(\chi, \tau) = \begin{cases} W(\tau), & a_1 \leq \tau \leq \chi, \\ W(\tau) - 1, & \chi < \tau \leq b_1. \end{cases}$$

Let us denote the Green's function by $G_l : [a_1, b_1] \times [a_1, b_1] \to \mathbb{R}$ with the boundary value problem

$$z''(\lambda) = 0, \ z(a_l) = z(b_l) = 0.$$

The function G_0 is defined as

$$G_0(\tau, \varsigma) = \begin{cases} \dfrac{(\tau - b_1)(\varsigma - a_1)}{b_1 - a_1} & \text{for } a_1 \leq \varsigma \leq \tau, \\ \dfrac{(\varsigma - b_1)(\tau - a_1)}{b_1 - a_1} & \text{for } \tau \leq \varsigma \leq b_1 \end{cases} \tag{1.13}$$

and for any function $\zeta \in C^2[a_1, b_1]$,the following identity induces using integration by parts

$$\zeta(\chi) = \frac{b_1 - \chi}{b_1 - a_1}\zeta(a_1) + \frac{\chi - a_1}{b_1 - a_1}\zeta(b_1) + \int_{b_1}^{a_1} G_0(\chi, \varsigma)\zeta''(\varsigma) d\varsigma. \tag{1.14}$$

The function G_0 is continuous, symmetric and convex with respect to both variables τ and ς.

As a special choice Abel-Gontscharoff polynomial for 'two-point right focal' interpolating polynomial for $n = 2$ can be given as (see [16]):

$$\zeta(\chi) = \zeta(a_1) + (\chi - a_1)\zeta'(b_1) + \int_{a_1}^{b_1} G_1(\chi, \tau)\zeta''(\tau) d\tau. \tag{1.15}$$

where $G_1(\varsigma,\tau)$ is Green's function for two-point right focal problem defined as

$$G_1(\varsigma,\tau) = \begin{cases} a_1 - \tau & \text{for } a_1 \leq \tau \leq \varsigma, \\ a_1 - \varsigma & \text{for } \varsigma \leq \tau \leq b_1 \end{cases} \tag{1.16}$$

Motivated by Abel-Gontscharoff identity (1.15) and related Green's function (1.16), we would recall here some new types of Green functions $G_l : [a_1, b_1] \times [a_1, b_1] \longrightarrow \mathbb{R}$ for $l \in 2, 3, 4$ defined as in [3]:

$$G_2(\varsigma,\tau) = \begin{cases} \varsigma - b_1 & \text{for } a_1 \leq \tau \leq \varsigma, \\ \tau - b_1 & \text{for } \varsigma \leq \tau \leq b_1 \end{cases} \tag{1.17}$$

$$G_3(\varsigma,\tau) = \begin{cases} \varsigma - a_1 & \text{for } a_1 \leq \tau \leq \varsigma, \\ \tau - a_1 & \text{for } \varsigma \leq \tau \leq b_1 \end{cases} \tag{1.18}$$

$$G_4(\varsigma,\tau) = \begin{cases} b_1 - \tau & \text{for } a_1 \leq \tau \leq \varsigma, \\ b_1 - \varsigma & \text{for } \varsigma \leq \tau \leq b_1 \end{cases} \tag{1.19}$$

In [3] (see also [4], [12]), it is also shown that all four Green functions are symmetric and continuous. These new Green functions enable us to present some new identities, stated as follow

$$\zeta(\chi) = \zeta(b_1) + (b_1 - \chi)\zeta'(a_1) + \int_{a_1}^{b_1} G_2(\chi,\tau)\zeta''(\tau)d\tau. \tag{1.20}$$

$$\zeta(\chi) = \zeta(b_1) - (b_1 - a_1)\zeta'(b_1) + (\chi - a_1)\zeta'(a_1) + \int_{a_1}^{b_1} G_3(\chi,\tau)\zeta''(\tau)d\tau. \tag{1.21}$$

$$\zeta(\chi) = \zeta(a_1) + (b_1 - a_1)\zeta'(a_1) - (b_1 - \chi)\zeta'(b_1) + \int_{a_1}^{b_1} G_4(\chi,\varsigma)\zeta''(\tau)d\tau. \tag{1.22}$$

To recall definitions of a generalized convex function and related concepts and results we refer to interested readers following references [11], [6] and [18]. This article is arranged in the following manner. In Section 2 we will obtain inequalities of type (1.1), (1.7) for n-convex functions by using the extension of Montgomery's identity (1.11). In Section 3 we will give some discrete and integral nature identities and corresponding linear inequalities using Green functions and applying extension of weighted Montgomery identity. In both sections, we will discuss a generalization of the class of n-convex functions introduced in [17]. On the basis of this discussion, we will give related inequalities for n-convex functions at a point. we will also provide some Ostrowski-type inequalities by obtaining bounds for the remainders of the identities from obtained results.

We will first prove some results that will have a crucial role in each Section of the paper. Then we will propose some Related Popoviciu type inequalities.

2. Popoviciu type identities and inequalities via extension of weighted Montgomery identity

Theorem 2.1. *Under the assumptions of Proposition* 1.7 *and let* $T_{w,n}$ *be defined by (1.10). Additionally, let* $\chi_i \in [a_1, b_1]$, $\varrho_i \in \mathbb{R}$ *for* $i \in \{1, 2, \dots, \gamma\}$ *and* $\gamma \in \mathbb{N}$ *be s.t.*

$$\sum_{i=1}^{\gamma} \varrho_i = 0.$$

Then

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) = & \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma \\ & + \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma \\ & + \frac{1}{(n-1)!} \int_{a_1}^{b_1} \left(\sum_{i=1}^{\gamma} \varrho_i T_{w,n}(\chi_i, \varsigma) \right) \zeta^{(n)}(\varsigma) d\varsigma. \end{aligned} \tag{2.1}$$

Proof. Putting in the extension of Montgomery identity (1.9) $\chi_i, i = 1, \dots, m$, multiplying with ϱ_i and summing all the identities we obtain

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) = & \int_{a_1}^{b_1} w(\varsigma)\zeta(\tau) d\tau \sum_{i=1}^{\gamma} \varrho_i \\ & + \sum_{i=1}^{\gamma} \varrho_i \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{a_1}^{\chi} w(\varsigma) \left((\chi - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma \\ & + \sum_{i=1}^{\gamma} \varrho_i \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\chi}^{b_1} w(\varsigma) \left((\chi - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma \\ & + \frac{1}{(n-1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{b_1} T_{w,n}(\chi, \varsigma) \zeta^{(n)}(\varsigma) d\varsigma, \end{aligned}$$

By simplifying this expressions we obtain (2.1). □

Remark 2.2. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 1 of [8].

Its integral version is as follows.

Theorem 2.3. *Let* $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ *and* $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ *be integrable functions s.t.*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0.$$

Let $\zeta : I \to \mathbb{R}$ be such that $\zeta^{(n-1)}$ is absolutely continuous, $a_1 < b_1$, $a_1, b_1 \in I$, $I \subset \mathbb{R}$ be an open interval, $n \in \mathbb{N}$, $T_{w,n}$ be given by (1.10). Then

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\,\zeta(g(\chi))\,d\chi$$
$$= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{a_1}^{g(\chi)} w(\varsigma)\left((g(\chi)-a_1)^{\kappa+1} - (\varsigma-a_1)^{\kappa+1}\right) d\varsigma d\chi$$
$$+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{g(\chi)}^{b_1} w(\varsigma)\left((g(\chi)-b_1)^{\kappa+1} - (\varsigma-b_1)^{\kappa+1}\right) d\varsigma d\chi$$
$$+ \frac{1}{(n-1)!} \int_{a_1}^{b_1} \left(\int_{\alpha_1}^{\beta_1} \varrho(\chi) T_{w,n}(g(\chi),\varsigma) d\chi \right) \zeta^{(n)}(\varsigma) d\varsigma, \tag{2.2}$$

Proof. We obtain the required result by putting $\chi = g(\chi)$, multiplying with $\varrho(\chi)$, integrating over $[\alpha_1, \beta_1]$, and using some transformations and then using Fubini's theorem in the extension of Montgomery identity (1.9), □

Remark 2.4. If we put $w(\varsigma) = \frac{1}{b_1-a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 2 of [8].

Now we present some inequalities which can be derived from the previous identities.

Theorem 2.5. *Under the assumptions of Theorem* 2.1 *with the additional condition*

$$\sum_{i=1}^{\gamma} \varrho_i T_{w,n}(\chi_i, \varsigma) \geq 0, \qquad \forall\, \varsigma \in [a_1, b_1]. \tag{2.3}$$

Then, for every $n-$convex function $\zeta : I \to \mathbb{R}$ the following inequality holds

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) \geq \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma)\left((\chi_i-a_1)^{\kappa+1} - (\varsigma-a_1)^{\kappa+1}\right) d\varsigma$$
$$+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma)\left((\chi_i-b_1)^{\kappa+1} - (\varsigma-b_1)^{\kappa+1}\right) d\varsigma. \tag{2.4}$$

If the inequality in (2.3) is reversed, then (2.4) holds with the reversed sign of inequality.

Proof. By using the fact that function ζ is n-convex, so $\zeta^{(n)} \geq 0$ and (2.3) in (2.1), we can easily derive our required result. □

Remark 2.6. If reverse inequality holds in (2.3) then reverse inequality holds in (2.4).

Remark 2.7. If we put $w(\varsigma) = \frac{1}{b_1-a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 3 of [8].

Now we discuss a major consequence.

Theorem 2.8. *Under the assumptions of Theorem 2.1 and let* $w(\chi) \in C^n[a_1, b_1]$*,* $\chi = (x_1, \ldots, x_m) \in [a_1, b_1]^\gamma$*,* $\varrho = (\varrho_1, \ldots, \varrho_m) \in \mathbb{R}^\gamma$ *satisfy (1.5) and (1.6) with* n *replaced by* j *where* $j \in \mathbb{N}$*,* $2 \le j \le n$*. If* ζ *is* n*-convex and* $n - j$ *is even, then*

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) \ge \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma$$
$$+ \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma. \quad (2.5)$$

Proof. Let $\varsigma \in [a_1, b_1]$ be fixed. Notice that

$$T_{w,n}(x, \varsigma) = L_{w,n}(\chi) + (\chi - \varsigma)_+^{n-1}, \quad (2.6)$$

where

$$L_{w,n}(\chi) = \int_{\chi}^{\varsigma} w(u)(u - \varsigma)^{n-1} du + (W(\chi) - 1)(\chi - \varsigma)^{n-1}$$

Using the Leibnitz theorem we have

$$L_{w,n}^{(j)}(\chi) = (n-1) \sum_{i=0}^{j-1} \binom{j-1}{i} \left[\frac{d^{j-1-i}}{d\chi^{j-1-i}} (\chi - \varsigma)^{n-2} \right] \left[\frac{d^i}{d\chi^i} \int_{b_1}^{\chi} w(u) du \right]. \quad (2.7)$$

Therefore, (2.6) and (2.7) for $\varsigma < x \le b_1$ yield

$$\frac{d^j}{d\chi^j} T_{w,n}(\chi, \varsigma) = L_{wn}^{(j)}(\chi) + (n-1)_j (\chi - \varsigma)^{n-j-1}$$
$$= (n-1) \sum_{i=0}^{j-1} \binom{j-1}{i} \left[\frac{d^{j-1-i}}{d\chi^{j-1-i}} (\chi - \varsigma)^{n-2} \right] \left[\frac{d^i}{d\chi^i} \int_{b_1}^{\chi} w(u) du \right]$$
$$+ (n-1)_j (\chi - \varsigma)^{n-j-1} \ge 0, \quad (2.8)$$

while for $a_1 \le \chi < \varsigma$ we have

$$\frac{d^j}{d\chi^j} T_{w,n}(\chi, \varsigma)$$
$$= (-1)^{n-2} (n-1) \sum_{i=0}^{j-1} \binom{j-1}{i} \left[\frac{d^{j-1-i}}{d\chi^{j-1-i}} (\varsigma - \chi)^{n-2} \right] \left[\frac{d^i}{d\chi^i} \int_{b_1}^{\chi} w(u) du \right] \ge 0. \quad (2.9)$$

From (2.6) it is clear that for $j \le n-2$, $\chi \mapsto \frac{d^j}{d\chi^j} T_{w,n}(\chi, \varsigma)$ is continuous. Hence, if $n - j$ is even and $j \le n - 2$, from (2.8) and (2.9) we can conclude that the function $\chi \mapsto T_{w,n}(\chi, \varsigma)$ is j-convex. Furthermore, the conclusion extends towards the case $j = n$, *i. e.* the mapping $\chi \mapsto T_{w,n}(\chi, \varsigma)$ is n-convex, since the mapping $x \mapsto \frac{d^{n-2}}{d\chi^{n-2}} T_n(\chi, \varsigma)$ is 2-convex.

Now, by Proposition 1.5, we see that assumption (2.3) is satisfied, so inequality (2.4) holds. Moreover, due to the assumption (1.5), $\sum_{i=1}^{\gamma} \varrho_i(\chi_i) = 0$ for every polynomial P of degree $\le j - 1$, so the first $j - 2$ terms in the inner sum in (2.4) vanish, *i. e.*, the right hand side of (2.4) under the assumptions of this theorem is equal to the right hand side of (2.5). □

Remark 2.9. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 4 of [8].

Corollary 2.10. *Under the assumptions of Theorem 2.5 we denote*

$$H(\chi) = \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma$$
$$+ \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma. \tag{2.10}$$

If H is j-convex on $[a_1, b_1]$ and $n - j$ is even, then

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) \geq 0.$$

Proof. Applying Proposition 1.5 we conclude that the right hand side of (2.5) is nonnegative for the j-convex function H. □

Remark 2.11. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Corollary 1 of [8].

The rest of this section will present integral versions of the previous results. We will skip the details because the proofs are identical to the discrete case.

Theorem 2.12. *Under the assumptions of Theorem* 2.3 *with the additional condition*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi) T_{w,n}(g(\chi), \varsigma) \, d\chi \geq 0, \qquad \forall \varsigma \in [a_1, b_1].$$

Then, for every $n-$convex function $\zeta : I \to \mathbb{R}$ the following inequality holds

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi) \zeta(g(\chi)) \, d\chi \geq \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \times$$
$$\int_{a_1}^{g(\chi)} w(\varsigma) \left((g(\chi) - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma d\chi$$
$$+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \times$$
$$\int_{g(\chi)}^{b_1} w(\varsigma) \left((g(\chi) - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma d\chi. \tag{2.11}$$

Remark 2.13. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 5 of [8].

Theorem 2.14. *Let all the assumptions from Theorem 2.3 be valid. Moreover, let $w(\chi) \in C^n[a_1, b_1]$, let $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ and $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ satisfy (1.8) with n*

replaced by j where $j \in \mathbb{N}$, $2 \leq j \leq n$. If ζ is n-convex and $n-j$ is even, then

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\,\zeta(g(\chi))\,d\chi \geq \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \times$$
$$\int_{a_1}^{g(\chi)} w(\varsigma)\left((g(\chi)-a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma d\chi$$
$$+ \sum_{\kappa=j-2}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \times$$
$$\int_{g(\chi)}^{b_1} w(\varsigma)\left((g(\chi)-b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma d\chi. \tag{2.12}$$

Remark 2.15. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$, $\varsigma \in [a_1, b_1]$ above identity reduces to Theorem 6 of [8].

Corollary 2.16. *Let n, ϱ, ζ, j and g be as in Theorem 2.14 and let H be given by (2.10). If $n-j$ is even and H is j-convex, then*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))\,d\chi \geq 0.$$

2.1. Inequalities related to n-convex functions at a point

Throughout this section, we will discuss related results obtained in [17] for the class of n-convex functions at a point.

Definition 2.17. Let $n \in \mathbb{N}$, c_1 a point in the interior of I and I be an interval in $\mathbb{R}$. If there exists a constant K such that

$$F_1(\chi) = \zeta(\chi) - \frac{K}{(n-1)!}\chi^{n-1} \tag{2.13}$$

where the function $\zeta : I \to \mathbb{R}$ is said to be n-convex at point c_1 and $(n-1)$-concave on $I \cap (-\infty, c_1]$ and $(n-1)$-convex on $I \cap [c_1, \infty)$. If the function $-\zeta$ is n-convex at point $c_1]$ then ζ is called n-concave at point c_1. For more details, we refer the readers to see [2, 17].

In [17], authors discussed sufficient conditions on two linear functionals $\Lambda : C([a_1, c_1]) \to \mathbb{R}$ and $\Xi : C([c_1, b_1]) \to \mathbb{R}$ so that the inequality $\Lambda(\zeta) \leq \Xi(\zeta)$ holds for every function ζ that is n-convex at c_1.

This section will provide inequalities of this type for specific linear functionals that connect to the inequalities derived in the preceding section. Let e_i denote the monomials $e_i(\chi) = \chi^i$, $i \in \mathbb{N}_0$. More specifically, let $T_{w,n}^{[a_1,c_1]}$ and $T_{w,n}^{[c_1,b_1]}$ represent the same as (1.10) on these intervals, *i. e.*,

$$T_{w,n}^{[a_1,c_1]}(\chi,\varsigma)=\begin{cases}\displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, \\ a_1\leq\varsigma\leq\chi \\ \displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+\left(W(\chi)-1\right)(\chi-\varsigma)^{n-1}, \\ \chi<\varsigma\leq c_1,\end{cases} \tag{2.14}$$

$$T_{w,n}^{[c_1,b_1]}(\chi,\varsigma)=\begin{cases}\displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, \\ c_1\leq\varsigma\leq\chi \\ \displaystyle\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+\left(W(\chi)-1\right)(\chi-\varsigma)^{n-1} \\ \chi<\varsigma\leq b_1,\end{cases} \tag{2.15}$$

Let $\chi\in[a_1,c_1]^{\gamma}$, $\varrho\in\mathbb{R}^{\gamma}$, $\mathbf{y}\in[c_1,b_1]^{\ell}$ and $\mathbf{q}\in\mathbb{R}^{\ell}$ and denote

$$\begin{aligned}\Lambda(\zeta)=&\sum_{i=1}^{\gamma}\varrho_i\zeta\left(\chi_i\right)\\ &-\sum_{\kappa=0}^{n-2}\frac{\zeta^{\kappa+1}(a_1)}{(\kappa+1)!}\sum_{i=1}^{\gamma}\varrho_i\int_{a_1}^{\chi_i} w(\varsigma)\left((\chi_i-a_1)^{\kappa+1}-(\varsigma-a_1)^{\kappa+1}\right)d\varsigma\\ &-\sum_{\kappa=0}^{n-2}\frac{\zeta^{\kappa+1}(c_1)}{(\kappa+1)!}\sum_{i=1}^{\gamma}\varrho_i\int_{\chi_i}^{c_1} w(\varsigma)\left((\chi_i-c_1)^{\kappa+1}-(\varsigma-c_1)^{\kappa+1}\right)d\varsigma,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\Xi(\zeta)=&\sum_{i=1}^{\ell}q_i\zeta\left(y_i\right)\\ &-\sum_{\kappa=0}^{n-2}\frac{\zeta^{\kappa+1}(c_1)}{(\kappa+1)!}\sum_{i=1}^{\ell}q_i\int_{c_1}^{y_i} w(\varsigma)\left((y_i-c_1)^{\kappa+1}-(\varsigma-c_1)^{\kappa+1}\right)d\varsigma\\ &-\sum_{\kappa=0}^{n-2}\frac{\zeta^{\kappa+1}(b_1)}{(\kappa+1)!}\sum_{i=1}^{\ell}q_i\int_{y_i}^{b_1} w(\varsigma)\left((y_i-b_1)^{\kappa+1}-(\varsigma-b_1)^{\kappa+1}\right)d\varsigma.\end{aligned} \tag{2.17}$$

Identity (2.1) applied to the intervals $[a_1,c_1]$ and $[c_1,b_1]$ and by using the functionals Λ and Ξ can be written as

$$\Lambda(\zeta)=\frac{1}{(n-1)!}\int_{a_1}^{c_1}\left(\sum_{i=1}^{\gamma}\varrho_i T_{w,n}^{[a_1,c_1]}\left(\chi_i,\varsigma\right)\right)\zeta^{(n)}\left(\varsigma\right)\,d\varsigma, \tag{2.18}$$

$$\Xi(\zeta)=\frac{1}{(n-1)!}\int_{c_1}^{b}\left(\sum_{i=1}^{\ell}q_i T_{w,n}^{[c_1,b_1]}\left(y_i,\varsigma\right)\right)\zeta^{(n)}\left(\varsigma\right)\,d\varsigma. \tag{2.19}$$

Theorem 2.18. *Let* $\chi \in [a_1, c_1]^\gamma$, $\varrho \in \mathbb{R}^\gamma$, $\mathbf{y} \in [c_1, b_1]^\ell$ *and* $\mathbf{q} \in \mathbb{R}^\ell$ *be such that*

$$\sum_{i=1}^{\gamma} \varrho_i T_{w,n}^{[a_1,c_1]}(\chi_i, \varsigma) \geq 0, \quad \text{for every } \varsigma \in [a_1, c_1], \tag{2.20}$$

$$\sum_{i=1}^{\ell} q_i T_{w,n}^{[c_1,b_1]}(y_i, \varsigma) \geq 0, \quad \text{for every } \varsigma \in [c_1, b_1], \tag{2.21}$$

$$\int_{a_1}^{c_1} \left(\sum_{i=1}^{\gamma} \varrho_i T_{w,n}^{[a_1,c_1]}(\chi_i, \varsigma) \right) d\varsigma = \int_{c_1}^{b_1} \left(\sum_{i=1}^{\ell} q_i T_{w,n}^{[c_1,b_1]}(y_i, \varsigma) \right) d\varsigma, \tag{2.22}$$

where $T_{w,n}^{[a_1,c_1]}$, $T_{w,n}^{[c_1,b_1]}$, Λ *and* Ξ *are given by (2.14), (2.15), (2.16) and (2.17) respectively. If* $\zeta : [a_1, b_1] \to \mathbb{R}$ *is* $(n+1)$*-convex at point* c_1*, then*

$$\Lambda(\zeta) \leq \Xi(\zeta). \tag{2.23}$$

If the inequalities in (2.20) *and* (2.21) *are reversed, then* (2.23) *holds with the reversed sign of inequality.*

Proof. Let the function $F_1 = \zeta - \frac{K}{n!} e_n$ is n-concave on $[a_1, c_1]$ and n-convex on $[c_1, b_1]$(see Definition 2.17). Applying Theorem 2.5 to F_1 on the intervals $[a_1, c_1]$ and $[c_1, b_1]$ respectively we have

$$0 \geq \Lambda(F_1) = \Lambda(\zeta) - \frac{K}{n!} \Lambda(e_n) \tag{2.24}$$

$$0 \leq \Xi(F_1) = \Xi(\zeta) - \frac{K}{n!} \Xi(e_n). \tag{2.25}$$

Identities (2.18) and (2.19) applied to the function e_n yield

$$\Lambda(e_n) = n \int_{a_1}^{c_1} \left(\sum_{i=1}^{\gamma} \varrho_i T_{w,n}^{[a_1,c_1]}(\chi_i, \varsigma) \right) d\varsigma,$$

$$\Xi(e_n) = n \int_{c_1}^{b} \left(\sum_{i=1}^{\ell} q_i T_{w,n}^{[c_1,b_1]}(y_i, \varsigma) \right) d\varsigma.$$

Therefore, assumption (2.22) is equivalent to $\Lambda(e_n) = \Xi(e_n)$. Now, from (2.24) and (2.25) we obtain the stated inequality. □

Remark 2.19. If we put $w(u) = \frac{1}{b_1 - a_1}$, $u \in [a_1, b_1]$ above identity reduces to Theorem 7 of [8].

Remark 2.20. In the Theorem 2.18 we have proved that

$$\Lambda(\zeta) \leq \frac{K}{n!} \Lambda(e_n) = \frac{K}{n!} \Xi(e_n) \leq \Xi(\zeta).$$

Inequality (2.23) still holds if we substitute assumption (2.22) with the weaker assumption that $K(\Xi(e_n) - \Lambda(e_n)) \geq 0$.

Corollary 2.21. *Let $n, j_1, j_2 \in \mathbb{N}$, $\leq j_1, j_2 \leq n$, let $\zeta : [a_1, b_1] \to \mathbb{R}$ be $(n+1)$-convex at point c_1, let $\varrho \in \mathbb{R}^\gamma$ and γ-tuples $\chi \in [a_1, c_1]^\gamma$ satisfy (1.5) and (1.6) with n replaced by j_1, let $\mathbf{q} \in \mathbb{R}^\ell$ and ℓ-tuples $\mathbf{y} \in [c_1, b_1]^\ell$ satisfy*

$$\sum_{i=1}^{\ell} q_i y_i^\kappa = 0, \qquad \forall\ \kappa = 0, 1, \ldots, j_2 - 1$$

$$\sum_{i=1}^{\ell} q_i (y_i - \tau)_+^{j_2-1} \geq 0, \qquad \forall\ \tau \in [y_{(1)}, y_{(\ell-n+1)}]$$

and let (2.22) holds. If $n - j_1$ and $n - j_2$ are even, then

$$\Lambda(\zeta) \leq \Xi(\zeta).$$

Proof. Same as the proof of Theorem 2.8. □

Remark 2.22. Similar results can also be stated for integral versions as well by defining new functionals using identity (2.2).

2.2. Bounds for identities related to the Popoviciu-type inequalities

Let $\zeta_1, \zeta_2 : [a_1, b_1] \to \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional

$$\begin{aligned} T(\zeta_1, \zeta_2) &= \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta_1(\chi)\zeta_2(\chi) d\chi \\ &- \left(\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta_1(\chi) d\chi\right) \left(\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta_2(\chi) d\chi\right). \end{aligned} \tag{2.26}$$

The symbol $L_p[a_1, b_1]$ $(1 \leq p < \infty)$ denotes the space of p-power integrable functions on the interval $[a_1, b_1]$ equipped with the norm

$$\|\zeta_1\|_p = \left(\int_{a_1}^{b_1} |\zeta_1(\tau)|^p d\tau\right)^{\frac{1}{p}}$$

and $L_\infty[a_1, b_1]$ denotes the space of essentially bounded functions on $[a_1, b_1]$ with the norm

$$\|\zeta_1\|_\infty = \operatorname{ess} \sup_{\tau \in [a_1, b_1]} |\zeta_1(\tau)|.$$

The following results can be found in [5].

Proposition 2.23. *Let $\zeta 1 : [a_1, b_1] \to \mathbb{R}$ be a Lebesgue integrable function and $\zeta_2 : [a_1, b_1] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a_1)(b_1 - \cdot)[\zeta_2']^2 \in L[a_1, b_1]$. Then we have the inequality*

$$|T(\zeta_1, \zeta_2)| \leq \frac{1}{\sqrt{2}} \left(\frac{1}{b_1 - a_1} |T(\zeta_1, \zeta_1)| \int_{a_1}^{b_1} (\chi - a_1)(b_1 - \chi)[\zeta_2'(\chi)]^2 d\chi\right)^{1/2}. \tag{2.27}$$

The constant $\frac{1}{\sqrt{2}}$ in (2.27) is the best possible.

Proposition 2.24. *Let $\zeta_2 : [a_1, b_1] \to \mathbb{R}$ be a monotonic nondecreasing function and let $\zeta_1 : [a_1, b_1] \to \mathbb{R}$ be an absolutely continuous function such that $\zeta_1' \in L_\infty[a_1, b_1]$. Then we have the inequality*

$$|T(\zeta_1, \zeta_2)| \leq \frac{1}{2(b_1 - a_1)} \|\zeta_1'\|_\infty \int_{a_1}^{b_1} (\chi - a_1)(b_1 - \chi) d\zeta_2(\chi). \tag{2.28}$$

The constant $\frac{1}{2}$ in (2.28) is the best possible.

Under the assumptions of Theorems 2.1 and 2.3 we denote the following functions. For $\gamma-$tuples $\varrho = (\varrho_1, \ldots, \varrho_\gamma)$, $\chi = (\chi_1, \ldots, \chi_\gamma)$ with $\chi_i \in [a_1, b_1]$, $\varrho_i \in \mathbb{R}$ $(i = 1, \ldots, \gamma)$ such that $\sum_{i=0}^{\gamma} \varrho_i = 0$ and the function $T_{w,n}$ defined as in (1.10), denote

$$\Psi_1(\varsigma) = \sum_{i=1}^{\gamma} \varrho_i T_{w,n}(\chi_i, \varsigma), \qquad \text{for } \varsigma \in [a_1, b_1]. \tag{2.29}$$

Similarly for functions $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ and $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ such that $\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0$, denote

$$\Psi_2(\varsigma) = \int_{\alpha_1}^{\beta_1} \varrho(\chi) T_{w,n}(g(\chi), \varsigma)\, d\chi, \qquad \text{for } \varsigma \in [a_1, b_1]. \tag{2.30}$$

Now, we are ready to state bounds for the integral remainders of identities obtained in Section 2.

Theorem 2.25. *Let $n \in \mathbb{N}$, $\zeta : [a_1, b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $(\cdot - a_1)(b_1 - \cdot)[\zeta^{(n+1)}]^2 \in L[a_1, b_1]$, $\chi_i \in [a_1, b_1]$ and $\varrho_i \in \mathbb{R}$ $(i \in \{1, \ldots, \gamma\})$ such that $\sum_{i=0}^{\gamma} \varrho_i = 0$ and let the functions $T_{w,n}$, T and Ψ_1 be defined in* (1.10), (2.26) *and* (2.29) *respectively. Then*

$$\begin{aligned} \sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) = & \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{a_1}^{\chi_i} w(\varsigma) \left((\chi_i - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma \\ & + \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \sum_{i=1}^{\gamma} \varrho_i \int_{\chi_i}^{b_1} w(\varsigma) \left((\chi_i - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma \\ & + \frac{\left[\zeta^{(n-1)}(b_1) - \zeta^{(n-1)}(a_1)\right]}{(n-1)!(b_1 - a_1)} \int_{a_1}^{b_1} \Psi_1(\varsigma) d\varsigma + R_n^1(\zeta; a_1, b_1), \end{aligned} \tag{2.31}$$

where the remainder $R_n^1(\zeta; a_1, b_1)$ satisfies the estimation

$$\begin{aligned} & |R_n^1(\zeta; a_1, b_1)| \\ & \leq \frac{1}{(n-1)!} \left(\frac{b_1 - a_1}{2} \left| T(\Psi_1, \Psi_1) \int_{a_1}^{b_1} (\varsigma - a_1)(b_1 - \varsigma)[\zeta^{(n+1)}(\varsigma)]^2 d\varsigma \right| \right)^{1/2} \end{aligned} \tag{2.32}$$

Proof. If we apply Proposition 2.23 for $\zeta_1 \to \Psi_1$ and $\zeta_2 \to \zeta^{(n)}$, then we obtain

$$\left| \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \Psi_1(\varsigma)\zeta^{(n)}(\varsigma)d\varsigma - \left(\frac{1}{b_1 - a_1}\int_{a_1}^{b_1} \Psi_1(\varsigma)d\varsigma\right)\left(\frac{1}{b_1 - a_1}\int_{a_1}^{b_1} \zeta^{(n)}(\varsigma)d\varsigma\right)\right| \leq \frac{1}{\sqrt{2}}\left(\frac{1}{b_1 - a_1}|T(\Psi_1, \Psi_1)| \int_{a_1}^{b_1} (\varsigma - a_1)(b_1 - \varsigma)[\zeta^{(n+1)}(\varsigma)]^2 d\varsigma\right)^{1/2}.$$

Furthermore, we have

$$\frac{1}{(n-1)!}\int_{a_1}^{b_1} \Psi_1(\varsigma)\zeta^{(n)}(\varsigma)d\varsigma = \frac{\left[\zeta^{(n-1)}(b_1) - \zeta^{(n-1)}(a_1)\right]}{(n-1)!(b_1 - a_1)}\int_{a_1}^{b_1} \Psi_1(\varsigma)d\varsigma + R_n^1(\zeta; a_1, b_1).$$

where $R_n^1(\zeta; a_1, b_1)$ satisfies inequality (2.32). Now from identity (2.1) we obtain (2.31). □

Remark 2.26. If we put $w(u) = \frac{1}{b_1 - a_1}$, $u \in [a_1, b_1]$ above identity reduces to Theorem 8 of [8].

Here we state the integral version of the previous theorem.

Theorem 2.27. *Let $n \in \mathbb{N}$, $\zeta : [a_1, b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $(\cdot - a_1)(b_1 - \cdot)[\zeta^{(n+1)}]^2 \in L[a_1, b_1]$, let $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ and $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ be functions such that $\int_{\alpha_1}^{\beta_1} \varrho(\chi)d\chi = 0$ and let the functions $T_{w,n}$, T and Ψ_2 be defined in* (1.10), (2.26) *and* (2.30) *respectively. Then*

$$\begin{aligned}
&\int_{\alpha_1}^{\beta_1} \varrho(\chi)\, \zeta(g(\chi))\, d\chi \\
&= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{a_1}^{g(\chi)} w(\varsigma)\left((g(\chi) - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1}\right) d\varsigma d\chi \\
&+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{g(\chi)}^{b_1} w(\varsigma)\left((g(\chi) - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1}\right) d\varsigma d\chi \\
&+ \frac{\left[\zeta^{(n-1)}(b_1) - \zeta^{(n-1)}(a_1)\right]}{(n-1)!(b_1 - a_1)} \int_{a_1}^{b_1} \Psi_2(\varsigma)d\varsigma + R_n^2(\zeta; a_1, b_1), \qquad (2.33)
\end{aligned}$$

where the remainder $R_n^2(\zeta; a_1, b_2)$ satisfies the estimation

$$\left|R_n^2(\zeta; a_1, b_1)\right| \leq \frac{1}{(n-1)!}\left(\frac{b_1 - a_1}{2}\,|T(\Psi_2, \Psi_2)| \int_{a_1}^{b_2} (\varsigma - a_1)(b_1 - \varsigma)[\zeta^{(n+1)}(\varsigma)]^2 d\varsigma\right)^{1/2}. \qquad (2.34)$$

Proof. This result easily follows by proceeding as in the proof of the previous theorem and replacing (2.1) with (2.2). □

Remark 2.28. If we put $w(u) = \frac{1}{b_1 - a_1}$, $u \in [a_1, b_1]$ above identity reduces to Theorem 9 of [8].

By using Proposition 2.24, we obtain the following Grüss type inequality.

Theorem 2.29. *Let $n \in \mathbb{N}$, $\zeta : [a_1, b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $\zeta^{(n+1)} \geq 0$ on $[a_1, b_1]$, $\chi_i \in [a_1, b_1]$ and $\varrho_i \in \mathbb{R}$ ($i \in \{1, \ldots, \gamma\}$) such that*

$$\sum_{i=0}^{\gamma} \varrho_i = 0.$$

Also, let the functions T and Ψ_1 be defined in (2.26) *and* (2.29) *respectively. Then we have representation* (2.31) *and the remainder $R_n^1(\zeta; a_1, b_1)$ satisfies the following estimation*

$$|R_n^1(\zeta; a_1, b_1)| \leq \frac{1}{(n-1)!} \|\Psi_1'\|_\infty \left[\frac{b_1 - a_1}{2} \left[\zeta^{(n-1)}(b_1) + \zeta^{(n-1)}(a_1) \right] - \left[\zeta^{(n-2)}(b_1) - \zeta^{(n-2)}(a_1) \right] \right]. \tag{2.35}$$

Proof. If we apply Proposition 2.24 for $\zeta_1 \to \Psi_1$ and $\zeta_2 \to \zeta^{(n)}$, then we obtain

$$\left| \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \Psi_1(\varsigma) \zeta^{(n)}(\varsigma) d\varsigma - \left(\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \Psi_1(\varsigma) d\varsigma \right) \left(\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta^{(n)}(\varsigma) d\varsigma \right) \right| \leq \frac{1}{2(b_1 - a_1)} \|\Psi_1'\|_\infty \int_{a_1}^{b_1} (\varsigma - a_1)(b_1 - \varsigma) \zeta^{(n+1)}(\varsigma) d\varsigma.$$

Since

$$\begin{aligned} &\int_{a_1}^{b_1} (\varsigma - a_1)(b_1 - \varsigma) \zeta^{(n+1)}(\varsigma) d\varsigma \\ &= \int_{a_1}^{b_1} (2\varsigma - a_1 - b_1) \zeta^{(n)}(\varsigma) d\varsigma \\ &= (b_1 - a_1) \left[\zeta^{(n-1)}(b_1) + \zeta^{(n-1)}(a_1) \right] - 2 \left[\zeta^{(n-2)}(b_1) - \zeta^{(n-2)}(a_1) \right], \end{aligned} \tag{2.36}$$

by using the identities (2.1) and (2.36) we deduce (2.35). □

Remark 2.30. If we put $w(u) = \frac{1}{b_1 - a_1}$, $u \in [a_1, b_1]$ above identity reduces to Theorem 10 of [8].

Here we give the integral version of the above theorem.

Theorem 2.31. *Let $n \in \mathbb{N}$, $\zeta : [a_1, b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $\zeta^{(n+1)} \geq 0$ on $[a_1, b_1]$, let $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ and $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ be functions such that $\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0$. Also, let the functions T and Ψ_2 be defined in*

(2.26) *and* (2.30) *respectively. Then we have representation* (2.33) *and the remainder* $R_n^2(\zeta;a_1,b_1)$ *satisfies the following estimation*

$$|R_n^2(\zeta;a_1,b_1)| \leq \frac{1}{(n-1)!}\|\Psi_2'\|_\infty \left[\frac{b_1-a_1}{2}\left[\zeta^{(n-1)}(b_1)+\zeta^{(n-1)}(a_1)\right] - \left[\zeta^{(n-2)}(b_1)-\zeta^{(n-2)}(a_1)\right]\right]. \tag{2.37}$$

Remark 2.32. If we put $w(u)=\frac{1}{b_1-a_1}$, $u\in[a_1,b_1]$ above identity reduces to Theorem 11 of [8].

2.3. Ostrowski type inequalities via extension of Montgomery identity

Here we present some Ostrowski-type inequalities related to the generalized linear inequalities. Throughout the section, we use the following functions Ψ_1 and Ψ_2 defined as in (2.29) and (2.30).

Theorem 2.33. *Let all the assumptions of Theorem* 2.1 *hold. Additionally, let* $\zeta^{(n)}\in L_q[a_1,b_1]$, $1\leq q,r\leq\infty$, $\frac{1}{q}+\frac{1}{r}=1$, $n\geq 2$, $n\in\mathbb{N}$ *and let* $\chi\in[a_1,b_1]^\gamma$ *and* $\varrho\in\mathbb{R}^\gamma$ *satisfy*

$$\sum_{i=1}^{\gamma}\varrho_i=0 \text{ and } \sum_{i=1}^{\gamma}\varrho_i\chi_i=0.$$

Then

$$\Bigg|\sum_{i=1}^{\gamma}\varrho_i\zeta(\chi_i) - \sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!}\sum_{i=1}^{\gamma}\varrho_i\int_{a_1}^{\chi_i}w(\varsigma)\left((\chi_i-a_1)^{\kappa+1}-(\varsigma-a_1)^{\kappa+1}\right)d\varsigma - \sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!}\sum_{i=1}^{\gamma}\varrho_i\int_{\chi_i}^{b_1}w(\varsigma)\left((\chi_i-b_1)^{\kappa+1}-(\varsigma-b_1)^{\kappa+1}\right)d\varsigma\Bigg| \leq \frac{1}{(n-1)!}\|\zeta^{(n)}\|_q\,\|\Psi_1\|_r. \tag{2.38}$$

The constant on the right hand sides of (2.38) *is the best possible for* $q=1$ *and sharp for* $1<q\leq\infty$.

Proof. Let us denote

$$\mu(\varsigma)=\frac{1}{(n-1)!}\Psi_1(\varsigma).$$

By using Hölder's inequality on identity (2.1) we obtain inequality (2.38), i. e.

$$\text{L.H.S.} \leq \|\zeta^{(n)}\|_q\,\|\mu\|_r. \tag{2.39}$$

Let us find a function ζ for the proof of the sharpness of the constant

$$\left(\int_{a_1}^{b_1}|\mu(\varsigma)|^r\,dt\right)^{1/r},$$

for which the equality in (2.39) is obtained.

For $1 < q < \infty$ take ζ to be s.t.

$$\zeta^{(n)}(\varsigma) = sgn\mu(\varsigma) \cdot |\mu(\varsigma)|^{1/(q-1)}.$$

For $q = \infty$, take ζ s.t.

$$\zeta^{(n)}(\varsigma) = sgn\mu(\varsigma).$$

Finally, for $q = 1$, we prove that

$$\left| \int_{a_1}^{b_1} \mu(\varsigma)\zeta^{(n)}(\varsigma)d\varsigma \right| \leq \max_{\varsigma\in[a_1,b_1]} |\mu(\varsigma)| \int_{a_1}^{b_1} \zeta^{(n)}(\varsigma)d\varsigma \tag{2.40}$$

is the best possible inequality.

Suppose that $|\mu(\varsigma)|$ attains its maximum at $\varsigma_0 \in [a_1, b_1]$. First we consider the case $\mu(\varsigma_0) > 0$. For δ small enough we define $\zeta_{1\delta}(\varsigma)$ by

$$\zeta_{1\delta}(\varsigma) = \begin{cases} 0 \quad , & a_1 \leq \varsigma \leq \varsigma_0, \\ \dfrac{1}{\delta n!}(\varsigma - \varsigma_0)^n \quad , & \varsigma_0 \leq \varsigma \leq \varsigma_0 + \delta, \\ \dfrac{1}{(n-1)!}(\varsigma - \varsigma_0)^{n-1} \quad , & \varsigma_0 + \delta_1 \leq \varsigma \leq b_1. \end{cases}$$

So, we have

$$\left| \int_{a_1}^{b_1} \mu(\varsigma)\zeta_{1\delta}^{(n)}(\varsigma)d\varsigma \right| = \left| \int_{\varsigma_0}^{\varsigma_0+\delta} \mu(\varsigma)\frac{1}{\delta}d\varsigma \right| = \frac{1}{\delta}\int_{\varsigma_0}^{\varsigma_0+\delta} \mu(\varsigma)d\varsigma$$

Now from inequality (2.40) we have

$$\frac{1}{\delta}\int_{\varsigma_0}^{\varsigma_0+\delta} \mu(\varsigma)d\varsigma \leq \mu(\varsigma_0)\frac{1}{\delta}\int_{\varsigma_0}^{\varsigma_0+\delta} d\varsigma = \mu(\varsigma_0)$$

Since

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{\varsigma_0}^{\varsigma_0+\delta} \mu(\varsigma)d\varsigma = \mu(\varsigma_0)$$

the statement follows.

In the case $\mu(\varsigma_0) < 0$, we define $\zeta_{1\delta}(\varsigma)$ by

$$\zeta_{1\delta}(\varsigma) = \begin{cases} \dfrac{1}{(n-1)!}(\varsigma - \varsigma_0 - \delta)^{n-1} \quad , & a \leq \varsigma \leq \varsigma_0, \\ -\dfrac{1}{\delta n!}(\varsigma - \varsigma_0 - \delta)^n \quad , & \varsigma_0 \leq \varsigma \leq \varsigma_0 + \delta, \\ 0 \quad , & \varsigma_0 + \delta \leq \varsigma \leq b_1. \end{cases}$$

and the rest of the proof is the same as above. □

Remark 2.34. If we put $w(\varsigma) = \frac{1}{b_1-a_1}$ in Theorem 2.33, we capture Theorem 12 of [8].

At the end of this section, we will present the integral version of the above Theorem. We will skip the details because the proof is identical.

Theorem 2.35. *Let all the assumptions of Theorem 2.3 hold. Additionally,* $\zeta^{(n)} \in L_q[a_1, b_1]$, $1 \leq q, r \leq \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $n \geq 2$, $n \in \mathbb{N}$ *and let* $g : [\alpha_1, \beta_1] \to [a_1, b_1]$ *and* $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ *satisfy*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0 \text{ and } \int_{\alpha_1}^{\beta_1} \varrho(\chi) g(\chi) d\chi = 0.$$

Then

$$\left| \int_{\alpha_1}^{\beta_1} \varrho(\chi) \zeta\left(g(\chi)\right) \right. - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{a_1}^{g(\chi)} w(\varsigma) \left((g(\chi) - a_1)^{\kappa+1} - (\varsigma - a_1)^{\kappa+1} \right) d\varsigma d\chi$$
$$\left. - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{(\kappa+1)!} \int_{\alpha_1}^{\beta_1} \varrho(\chi) \int_{g(\chi)}^{b_1} w(\varsigma) \left((g(\chi) - b_1)^{\kappa+1} - (\varsigma - b_1)^{\kappa+1} \right) d\varsigma d\chi \right|$$
$$\leq \frac{1}{(n-1)!} \|\zeta^{(n)}\|_q \|\Psi_2\|_r . \quad (2.41)$$

The constant on the right hand side of (2.41) *is the best possible for* $q = 1$ *and sharp for* $1 < q \leq \infty$.

Remark 2.36. If we put $w(\varsigma) = \frac{1}{b_1 - a_1}$ in Theorem 2.35, we capture Theorem 13 of [8].

3. Popoviciu type identities and inequalities via extension of weighted Montgomery identity using Green Functions

In the present section, we obtain some discrete and integral identities and corresponding linear inequalities using Green functions and apply the extension of weighted Montgomery identity. We'll start by proving a few identities that will play a crucial role in the rest of the article.

Theorem 3.1. *Let* $\zeta : I \to \mathbb{R}$ *be such that* $\zeta^{(n-1)}$ *is absolutely continuous,* $n \geq 3$, $n \in \mathbb{N}$, $a_1 < b_1$, $a_1, b_1 \in I$, $I \subset \mathbb{R}$ *an open interval,* $w : [a_1, b_1] \to [0, \infty)$ *is some probability density function. Let* $\varrho \in \mathbb{R}^\gamma$ *satisfy*

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \text{ and } \sum_{i=1}^{\gamma} \varrho_i \chi_i = 0$$

and $\chi \in [a_1, b_1]^\gamma$, G_l *are as given by* (1.13), (1.16) – (1.19). *Then*

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) = \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma)$$
$$\times \left[w(\varsigma)(\varsigma - a_1)^\kappa + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\varsigma$$
$$+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma)$$
$$\times \left[-w(\varsigma)(\varsigma - b_1)^\kappa + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\varsigma$$
$$+ \frac{1}{(n-3)!} \int_{a_1}^{b_1} \left(\int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \tilde{T}_{w,n-2}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du, \quad (3.1)$$

where

$$\tilde{T}_{w,n-2}(\varsigma, u) = \begin{cases} \dfrac{w(\varsigma)(\varsigma - u)^{n-2}}{(n-2)} + W(\varsigma)(\varsigma - u)^{n-3}, & a_1 \le u \le \varsigma \\ \dfrac{w(\varsigma)(\varsigma - u)^{n-2}}{(n-2)} + (W(\varsigma) - 1)(\varsigma - u)^{n-3}, & \varsigma < u \le b_1. \end{cases} \quad (3.2)$$

Moreover, the following identity holds

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) = \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \left(\int_{a_1}^{b_1} w(\tau) \zeta''(\tau) d\tau \right) d\varsigma$$
$$+ \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma)$$
$$\times \left[\zeta^{(\kappa)}(a_1) \int_{a_1}^{\varsigma} w(u) \left((\varsigma - a_1)^{\kappa-2} - (u - a_1)^{\kappa-2} \right) du \right.$$
$$\left. + \zeta^{(\kappa)}(b_1) \int_{\varsigma}^{b_1} w(u) \left((\varsigma - b_1)^{\kappa-2} - (u - b_1)^{\kappa-2} \right) du \right] d\varsigma$$
$$+ \frac{1}{(n-3)!} \int_{a_1}^{b_1} \zeta^{(n)}(u) \left(\int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) T_{w,n-2}(\varsigma, u) d\varsigma \right) du, \quad (3.3)$$

where $T_{w,n}$ *is as defined in* (1.10).

Proof. Using (1.14) in $\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i)$ and the fact that $\sum_{i=1}^{\gamma} \varrho_i = 0$ and $\sum_{i=1}^{\gamma} \varrho_i \chi_i = 0$ we get

$$\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) = \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \zeta''(\varsigma) d\varsigma. \quad (3.4)$$

Differentiating the function f in (1.9) twice gives

$$\begin{aligned}\zeta''(\varsigma) &= \sum_{\kappa=0}^{n-2} \frac{f^{(\kappa+1)}(a_1)}{\kappa!}\left[w(\varsigma)(\varsigma-a_1)^{\kappa} + \kappa\int_{a_1}^{\varsigma} w(u)(\varsigma-a_1)^{\kappa-1}du\right] \\ &+ \sum_{\kappa=0}^{n-2} \frac{f^{(\kappa+1)}(b_1)}{\kappa!}\left[-w(\varsigma)(\varsigma-b_1)^{\kappa} + \kappa\int_{\varsigma}^{b_1} w(u)(\varsigma-b_1)^{\kappa-1}du\right] \\ &+ \frac{1}{(n-3)!}\int_{a_1}^{b_1} \tilde{T}_{w,n-2}(\varsigma,u)\zeta^{(n)}(u)du. \end{aligned} \tag{3.5}$$

Inserting (3.5) in (3.4) yields

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!}\int_{a_1}^{b_1}\sum_{i=1}^{\gamma}\varrho_i G_l(\chi_i,\varsigma) \\ &\times \left[w(\varsigma)(\varsigma-a_1)^{\kappa} + \kappa\int_{a_1}^{\varsigma} w(u)(\varsigma-a_1)^{\kappa-1}du\right]d\varsigma \\ &+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!}\int_{a_1}^{b_1}\sum_{i=1}^{\gamma}\varrho_i G_l(\chi_i,\varsigma) \\ &\times \left[-w(\varsigma)(\varsigma-b_1)^{\kappa} + \kappa\int_{\varsigma}^{b_1} w(u)(\varsigma-b_1)^{\kappa-1}du\right]d\varsigma \\ &+ \frac{1}{(n-3)!}\int_{a_1}^{b_1}\sum_{i=1}^{\gamma}\varrho_i G_l(\chi_i,\varsigma)\left(\int_{a_1}^{b_1} \tilde{T}_{w,n-2}(\varsigma,u)\zeta^{(n)}(u)du\right)d\varsigma. \end{aligned}$$

and in the last term, by applying the Fubini's theorem we get (3.1).

Furthermore, in (1.9) by replacing $\zeta \longrightarrow \zeta''$ and $n \longrightarrow n-2$ respectively, and after some rearrangements we get

$$\begin{aligned}\zeta''(\varsigma) &= \int_{a_1}^{b_1} w(\tau)\zeta''(\tau)d\tau \\ &+ \sum_{\kappa=3}^{n-1}\frac{1}{(\kappa-2)!}\left[\zeta^{(\kappa)}(a_1)\int_{a_1}^{\varsigma} w(u)\left((\varsigma-a_1)^{\kappa-2}-(u-a_1)^{\kappa-2}\right)du\right. \\ &\left.+\zeta^{(\kappa)}(b_1)\int_{\varsigma}^{b_1} w(u)\left((\varsigma-b_1)^{\kappa-2}-(u-b_1)^{\kappa-2}\right)du\right] \\ &+ \frac{1}{(n-3)!}\int_{a_1}^{b_1} T_{w,n-2}(\varsigma,u)\zeta^{(n)}(u)du. \end{aligned} \tag{3.6}$$

Similarly, using (3.6) in (3.4) and applying Fubini's Theorem we get (3.3). □

Remark 3.2. If we put $w(\tau) = \frac{1}{b_1-a_1}$ in Theorem 3.1, we capture Theorem 2.1 of [9].

Now we will discuss some inequalities that can be obtained from the previous identities.

Theorem 3.3. *Under the assumptions of Theorem* 3.1 *with the additional condition*

$$\int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \tilde{T}_{w,n-2}(\varsigma, u) d\varsigma \geq 0, \quad \forall\, u \in [a_1, b_1], \tag{3.7}$$

where G_l *and* $\tilde{T}_{w,n-2}$ *are given in* (1.13), (1.16) – (1.19) *and* (3.2). *If* ζ *is* $n-$*convex, then the following inequality holds*

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &- \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\ &\times \left[w(\varsigma)(\varsigma - a_1)^{\kappa} + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\varsigma \\ &- \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\ &\times \left[-w(\varsigma)(\varsigma - b_1)^{\kappa} + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\varsigma \geq 0. \end{aligned} \tag{3.8}$$

Proof. Using the fact that function ζ is $n-$convex, we have $\zeta^{(n)} \geq 0$ and (3.7) in (3.1) we obtain our required result. □

Remark 3.4. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.3, we capture Theorem 2.2 of [9].

Theorem 3.5. *Under the assumptions of Theorem* 3.1 *with the additional condition*

$$\int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) T_{w,n-2}(\varsigma, u) d\varsigma \geq 0, \quad \forall\, u \in [a_1, b_1], \tag{3.9}$$

where G_l *and* $T_{w,n}$ *are defined in* (1.13), (1.16) – (1.19) *and* (1.10). *If* ζ *is* $n-$*convex, then the following inequality holds*

$$\begin{aligned}\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &- \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \left(\int_{a_1}^{b_1} w(\tau) \zeta''(\tau) d\tau \right) d\varsigma \\ &- \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) \\ &\times \left[\zeta^{(\kappa)}(a_1) \int_{a_1}^{\varsigma} w(u) \left((\varsigma - a_1)^{\kappa-2} - (u - a_1)^{\kappa-2} \right) du \right. \\ &\left. + \zeta^{(\kappa)}(b_1) \int_{\varsigma}^{b_1} w(u) \left((\varsigma - b_1)^{\kappa-2} - (u - b_1)^{\kappa-2} \right) du \right] d\varsigma \geq 0. \end{aligned} \tag{3.10}$$

Proof. Using the fact that the function ζ is $n-$convex, we have $\zeta^{(n)} \geq 0$ and (3.9) in (3.3), we easily arrive at our required result. □

Remark 3.6. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.5, we capture Theorem 2.3 of [9].

Here we discuss a major consequence.

Theorem 3.7. *Under the assumptions of Theorem 3.1 and additionally,*

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \text{ and } \sum_{i=1}^{\gamma} \varrho_i |\chi_i - \chi_k| \geq 0$$

for $\kappa \in \{1, \dots, \gamma\}$. *If* n *is even and* ζ *is* $n-$*convex, then inequalities* (3.8) *and* (3.10) *hold.*

Proof. The Green's function $G_l(\varsigma, \tau)$ is convex w.r.t τ $\forall$ $\varsigma \in [a_1, b_1]$. Therefore, from Proposition 1.5, with conditions (1.5) and (1.6) replaced by (1.4) as in [15], we have

$$\sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma) \geq 0 \quad \forall \quad \varsigma \in [a_1, b_1]. \tag{3.11}$$

Here $\tilde{T}_{w,n-2}(\varsigma, \tau) \geq 0$ and $T_{w,n-2}(\varsigma, \tau) \geq 0$ because n is even. By combining this fact with (3.11) we get inequalities (3.7) and (3.9). As ζ is $n-$convex, the results follow from Theorems 3.3 and 3.5. □

Remark 3.8. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.7, we capture Theorem 2.4 of [9]

Following that, we will present the integral versions of our main findings. We will skip the details because the proofs are identical to discrete version.

Theorem 3.9. *Let* $\zeta : I \to \mathbb{R}$ *be a function such that* $\zeta^{(n-1)}$ *is absolutely continuous,* $n \geq 3$, $n \in \mathbb{N}$, $a_1 < b_1$, $a_1, b_1 \in I$, $I \subset \mathbb{R}$ *an open interval,* $w : [a_1, b_1] \to [0, \infty)$ *is some probability density function. Additionally, let* $\varrho : [\alpha_1, \beta_1] \to \mathbb{R}$ *satisfy* $\int_{\alpha_1}^{\beta_1} \varrho(\chi) d\chi = 0$ *and* $g : [\alpha_1, \beta_1] \to [a_1, b_1]$, $\int_{\alpha_1}^{\beta_1} \varrho(\chi) g(\chi) d\chi = 0$, *and let* G_l, $\tilde{T}_{w,n}$ *and* $T_{w,n}$ *be given by* (1.13), (1.16) – (1.19), (3.2) *and* (1.10). *Then the following two identities hold:*

$$\begin{aligned}
\int_{\alpha_1}^{\beta_1} \varrho(\chi) \zeta(g(\chi)) d\chi &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi) G_l(g(\chi), \varsigma) \\
&\quad \times \left[w(\varsigma)(\varsigma - a_1)^{\kappa} + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\chi d\varsigma \\
&\quad + \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi) G_l(g(\chi), \varsigma) \\
&\quad \times \left[-w(\varsigma)(\varsigma - b_1)^{\kappa} + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\chi d\varsigma \\
&\quad + \frac{1}{(n-3)!} \int_{a_1}^{b_1} \left(\left(\int_{\alpha_1}^{\beta_1} \varrho(\chi) G_l(g(\chi), \varsigma) d\chi \right) \tilde{T}_{w,n-2}(\varsigma, u) d\varsigma \right) \zeta^{(n)}(u) du.
\end{aligned} \tag{3.12}$$

and

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))d\chi = \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi)G_l(g(\chi),\varsigma)d\chi \left(\int_{a_1}^{b_1} w(\tau)\zeta''(\tau)d\tau\right) d\varsigma$$
$$+\sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \left(\int_{\alpha_1}^{\beta_1} \varrho(\chi)G_l(g(\chi),\varsigma)d\chi\right) \times$$
$$\left[\zeta^{(\kappa)}(a_1)\int_{a_1}^{\varsigma} w(u)\left((\varsigma-a_1)^{\kappa-2}-(u-a_1)^{\kappa-2}\right)du\right.$$
$$\left.+\zeta^{(\kappa)}(b_1)\int_{\varsigma}^{b_1} w(u)\left((\varsigma-b_1)^{\kappa-2}-(u-b_1)^{\kappa-2}\right)du\right]d\varsigma$$
$$+\frac{1}{(n-3)!}\int_{a_1}^{b_1} \zeta^{(n)}(u)\left(\int_{a_1}^{b_1}\left(\int_{\alpha_1}^{\beta_1} \varrho(\chi)G_l(g(\chi),\varsigma)d\chi\right)T_{w,n-2}(\varsigma,u)d\varsigma\right)du. \quad (3.13)$$

Remark 3.10. If we put $w(\tau) = \frac{1}{b_1-a_1}$ in Theorem 3.9, we capture Theorem 2.5 of [9].

Theorem 3.11. *Under the assumptions of Theorem* 2.3 *with the additional condition*

$$\int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi)\, G_l(g(\chi),\varsigma)\, \tilde{T}_{w,n-2}(\varsigma,u)\, d\chi\, d\varsigma \geq 0, \quad \forall\, u \in [a_1,b_1] \quad (3.14)$$

where G_l *is defined in* $(1.13),(1.16)-(1.19)$ *and* $\tilde{T}_{w,n}$ *is given in* (3.2). *If* ζ *is* $n-$*convex, then the following inequality holds*

$$\int_{\alpha_1}^{\beta_1} \varrho(\chi)\,\zeta(g(\chi))\,d\chi - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi)G_l(g(\chi),\varsigma)$$
$$\times \left[w(\varsigma)(\varsigma-a_1)^{\kappa} + \kappa\int_{a_1}^{\varsigma} w(u)(\varsigma-a_1)^{\kappa-1}du\right] d\chi d\varsigma$$
$$-\sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi)G_l(g(\chi),\varsigma)$$
$$\times \left[-w(\varsigma)(\varsigma-b_1)^{\kappa} + \kappa\int_{\varsigma}^{b_1} w(u)(\varsigma-b_1)^{\kappa-1}du\right] d\chi d\varsigma \geq 0. \quad (3.15)$$

Remark 3.12. If we put $w(\tau) = \frac{1}{b_1-a_1}$ in Theorem 3.11, we capture Theorem 2.6 of [9].

Theorem 3.13. *Under the assumptions of Theorem* 2.3 *with the additional condition*

$$\int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi)\, G_l(g(\chi),\varsigma)\, T_{w,n-2}(\varsigma,u)d\chi\, d\varsigma \geq 0, \quad \forall\, u \in [a_1,b_1], \quad (3.16)$$

where G_l is defined in (1.13), (1.16) − (1.19) *and* $T_{w,n}$ *is given in* (1.10). *If* ζ *is* $n-$*convex, then the following inequality holds*

$$\begin{aligned}\int_{\alpha_1}^{\beta_1} \varrho(\chi)\zeta(g(\chi))d\chi - \int_{a_1}^{b_1}\int_{\alpha_1}^{\beta_1} \varrho(\chi)G_l(g(\chi),\varsigma)d\chi\left(\int_{a_1}^{b_1} w(\tau)\zeta''(\tau)d\tau\right)d\varsigma \\ -\sum_{\kappa=3}^{n-1}\frac{1}{(\kappa-2)!}\int_{a_1}^{b_1}\left(\int_{\alpha_1}^{\beta_1}\varrho(\chi)G_l(g(\chi),\varsigma)d\chi\right) \\ \times\left[\zeta^{(\kappa)}(a_1)\int_{a_1}^{\varsigma} w(u)\left((\varsigma-a_1)^{\kappa-2}-(u-a_1)^{\kappa-2}\right)du\right. \\ \left.-\zeta^{(\kappa)}(b_1)\int_{\varsigma}^{b_1} w(u)\left((\varsigma-b_1)^{\kappa-2}-(u-b_1)^{\kappa-2}\right)du\right]d\varsigma \geq 0. \quad (3.17)\end{aligned}$$

Remark 3.14. If we put $w(\tau)=\frac{1}{b_1-a_1}$ in Theorem 3.13, we capture Theorem 2.7 of [9].

Theorem 3.15. *Under the assumptions of Theorem* 3.9 *and additionally let* $g:[\alpha_1,\beta_1]\to[a_1,b_1]$ *and* $\varrho:[\alpha_1,\beta_1]\to\mathbb{R}$ *satisfy (1.8). If* n *is even and* ζ *is* $n-$*convex, then inequalities* (3.15) *and* (3.17) *hold.*

Remark 3.16. If we put $w(\tau)=\frac{1}{b_1-a_1}$ in above, we capture Theorem 2.8 of [9]

3.1. Inequalities related to n-convex functions at a point

In the present subsection, we would like to discuss some results related to the Green function following the definition of convexity at a point (Definition 2.17 of subsection 2.1). Here we improve results from previous subsection. More specifically, let $T_{w,n}^{[a_1,c_1]}$ and $T_{w,n}^{[c_1,b_1]}$ represent the same as (1.10) on these intervals, *i.e.*,

$$T_{w,n}^{[a_1,c_1]}(\chi,\varsigma)=\begin{cases}\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, & a_1\leq\varsigma\leq\chi;\\ \int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+\left(W(\chi)-1\right)(\chi-\varsigma)^{n-1}, & \chi<\varsigma\leq c_1;\end{cases} \quad (3.18)$$

$$T_{w,n}^{[c_1,b_1]}(\chi,\varsigma)=\begin{cases}\int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+W(\chi)(\chi-\varsigma)^{n-1}, & c_1\leq\varsigma\leq\chi;\\ \int_{\chi}^{\varsigma} w(u)(u-\varsigma)^{n-1}du+\left(W(\chi)-1\right)(\chi-\varsigma)^{n-1}, & \chi<\varsigma\leq b_1.\end{cases} \quad (3.19)$$

Similarly, $\tilde{T}_{w,n-2}^{[a_1,c_1]}$ and $\tilde{T}_{w,n-2}^{[c_1,b_1]}$ denote equivalent of (3.2) on these intervals, i.e.,

$$\tilde{T}_{w,n-2}^{[a_1,c_1]}(\varsigma,u)=\begin{cases}\dfrac{w(\varsigma)(\varsigma-u)^{n-2}}{(n-2)}+W(\varsigma)(\varsigma-u)^{n-3}, & a_1\le u\le\varsigma;\\ \dfrac{w(\varsigma)(\varsigma-u)^{n-2}}{(n-2)}+\left(W(\varsigma)-1\right)(\varsigma-u)^{n-3}, & \varsigma<u\le b_1;\end{cases} \tag{3.20}$$

$$\tilde{T}_{w,n-2}^{[c_1,b_1]}(\varsigma,u)=\begin{cases}\dfrac{w(\varsigma)(\varsigma-u)^{n-2}}{(n-2)}+W(\varsigma)(\varsigma-u)^{n-3}, & c_1\le u\le\varsigma;\\ \dfrac{w(\varsigma)(\varsigma-u)^{n-2}}{(n-2)}+\left(W(\varsigma)-1\right)(\varsigma-u)^{n-3}, & \varsigma<u\le b_1.\end{cases} \tag{3.21}$$

Let $\chi\in[a_1,c_1]^\gamma$, $\varrho\in\mathbb{R}^\gamma$, $\mathbf{y}\in[c_1,b_1]^\ell$ and $\mathbf{q}\in\mathbb{R}^\ell$ and denote

$$\begin{aligned}\Lambda_1(\zeta)&=\sum_{i=1}^{\gamma}\varrho_i\zeta(\chi_i)-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!}\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_iG(\chi_i,\varsigma)\\&\quad\times\left[w(\varsigma)(\varsigma-a_1)^\kappa+\kappa\int_{a_1}^{\varsigma}w(u)(\varsigma-a_1)^{\kappa-1}du\right]d\varsigma\\&\quad-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(c_1)}{\kappa!}\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_iG(\chi_i,\varsigma)\\&\quad\times\left[-w(\varsigma)(\varsigma-c_1)^\kappa+\kappa\int_{\varsigma}^{c_1}w(u)(\varsigma-c_1)^{\kappa-1}du\right]d\varsigma,\end{aligned} \tag{3.22}$$

$$\begin{aligned}\Xi_1(\zeta)&=\sum_{i=1}^{\ell}q_i\zeta(y)-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(c_1)}{\kappa!}\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_iG(y_i,\varsigma)\\&\quad\times\left[w(\varsigma)(\varsigma-c_1)^\kappa+\kappa\int_{c_1}^{\varsigma}w(u)(\varsigma-c_1)^{\kappa-1}du\right]d\varsigma\\&\quad-\sum_{\kappa=0}^{n-2}\frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!}\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_iG(y_i,\varsigma)\\&\quad\times\left[-w(\varsigma)(\varsigma-b_1)^\kappa+\kappa\int_{\varsigma}^{b_1}w(u)(\varsigma-b_1)^{\kappa-1}du\right]d\varsigma.\end{aligned} \tag{3.23}$$

Identity (3.1) applied to the intervals $[a_1,c_1]$ and $[c_1,b_1]$ and by using the functionals Λ_1 and Ξ_1 can be written as

$$\Lambda_1(\zeta)=\frac{1}{(n-3)!}\int_{a_1}^{c_1}\left(\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_iG_l(\chi_i,\varsigma)\tilde{T}_{w,n-2}^{[a_1,c_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du, \tag{3.24}$$

$$\Xi_1(\zeta)=\frac{1}{(n-3)!}\int_{c_1}^{b_1}\left(\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_iG_l(y_i,\varsigma)\tilde{T}_{w,n-2}^{[c_1,b_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du. \tag{3.25}$$

In the same manner, we can introduce further functionals namely

$$\Lambda_2(\zeta) = \frac{1}{(n-3)!}\int_{a_1}^{c_1}\left(\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_i G_l(\chi_i,\varsigma)T_{w,n-2}^{[a_1,c_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du, \tag{3.26}$$

$$\Xi_2(\zeta) = \frac{1}{(n-3)!}\int_{c_1}^{b_1}\left(\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_i G_l(y_i,\varsigma)T_{w,n-2}^{[c_1,b_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du, \tag{3.27}$$

$$\Lambda_3(\zeta) = \frac{1}{(n-3)!}\int_{a_1}^{c_1}\left(\int_{a_1}^{c_1}\left(\int_{\alpha_1}^{\beta_1}\varrho(\chi)G_l(g(\chi),\varsigma)d\chi\right)\right.$$
$$\left.\times\tilde{T}_{w,n-2}^{[a_1,c_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du, \tag{3.28}$$

$$\Xi_3(\zeta) = \frac{1}{(n-3)!}\int_{c_1}^{b_1}\left(\int_{c_1}^{b_1}\left(\int_{\alpha_1}^{\beta_1}q(y)G_l(g(y),\varsigma)dy\right)\right.$$
$$\left.\times\tilde{T}_{w,n-2}^{[c_1,b_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du, \tag{3.29}$$

$$\Lambda_4(\zeta) = \frac{1}{(n-3)!}\int_{a_1}^{c_1}\left(\int_{a_1}^{c_1}\left(\int_{\alpha_1}^{\beta_1}\varrho(\chi)G_l(g(\chi),\varsigma)d\chi\right)\right.$$
$$\left.\times\, T_{w,n-2}^{[a_1,c_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du, \tag{3.30}$$

$$\Xi_4(\zeta) = \frac{1}{(n-3)!}\int_{c_1}^{b_1}\left(\int_{c_1}^{b_1}\left(\int_{\alpha_1}^{\beta_1}q(y)G_l(g(y),\varsigma)dy\right)\right.$$
$$\left.\times T_{w,n-2}^{[c_1,b_1]}(\varsigma,u)d\varsigma\right)\zeta^{(n)}(u)du. \tag{3.31}$$

Theorem 3.17. *Let $\chi \in [a_1, c_1]^\gamma$, $\varrho \in \mathbb{R}^\gamma$, $y \in [c_1, b_1]^l$ and $q \in \mathbb{R}^l$ be such that*

$$\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_i G_l(\chi_i,\varsigma)\tilde{T}_{n-2}^{[a_1,c_1]}(\varsigma,u)\,d\varsigma \geq 0, \qquad \forall\ u\in[a_1,c_1], \tag{3.32}$$

$$\int_{c_1}^{b_1}\sum_{i=1}^{\gamma}q_i G_l(y_i,\varsigma)\tilde{T}_{n-2}^{[c_1,b_1]}(\varsigma,u)\,d\varsigma \geq 0, \qquad \forall\ u\in[c_1,b_1], \tag{3.33}$$

$$\int_{a_1}^{c_1}\left(\int_{a_1}^{c_1}\sum_{i=1}^{\gamma}\varrho_i G_l(\chi_i,\varsigma)\tilde{T}_{w,n-2}^{[a_1,c_1]}(\varsigma,u)d\varsigma\right)du$$
$$=\int_{c_1}^{b_1}\left(\int_{c_1}^{b_1}\sum_{i=1}^{\ell}q_i G_l(y_i,\varsigma)\tilde{T}_{w,n-2}^{[c_1,b_1]}(\varsigma,u)d\varsigma\right)du, \tag{3.34}$$

where $\tilde{T}_{w,n-2}^{[a_1,c_1]}$, $\tilde{T}_{w,n-2}^{[c_1,b_1]}$, Λ_1 and Ξ_1 are given by (3.20), (3.21), (3.22) *and* (3.23) *respectively. If $\zeta : [a_1, b_1] \to \mathbb{R}$ is $(n+1)-$convex at point c_1, then*

$$\Lambda_1(\zeta) \leq \Xi_1(\zeta). \tag{3.35}$$

If inequalities in (3.32) *and* (3.33) *are reversed, then* (3.35) *is valid with reversed sign of inequality.*

Remark 3.18. From proof of Theorem 3.17 we have

$$\Lambda_1(\zeta) \leq \frac{K}{n!}\Lambda_1(e_n) = \frac{K}{n!}\Xi_1(e_n) \leq \Xi_1(\zeta).$$

In fact, inequality (3.35) still is valid if we replace assumption (3.34) with weaker assumption that

$$K\left(\Xi_1(e_n) - \Lambda_1(e_n)\right) \geq 0.$$

Remark 3.19. If we put $w(u) = \frac{1}{b_1-a_1}$ in above identity, we capture Theorem 2.11 of [7].

Here we have another similar result.

Theorem 3.20. *Let $\chi \in [a_1, c_1]^\gamma$, $\varrho \in \mathbb{R}^\gamma$, $y \in [c_1, b_1]^l$ and $q \in \mathbb{R}^l$ be such that*

$$\int_{a_1}^{c_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) T_{n-2}^{[a_1,c_1]}(\varsigma, u)\, d\varsigma \geq 0, \qquad \forall\ u \in [a_1, c_1], \tag{3.36}$$

$$\int_{c_1}^{b_1} \sum_{i=1}^{\gamma} q_i G_l(y_i, \varsigma) T_{n-2}^{[c_1,b_1]}(\varsigma, u)\, d\varsigma \geq 0, \qquad \forall\ u \in [c_1, b_1], \tag{3.37}$$

$$\begin{aligned} &\int_{a_1}^{c_1} \left(\int_{a_1}^{c_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i, \varsigma) T_{w,n-2}^{[a_1,c_1]}(\varsigma, u) d\varsigma \right) du \\ &= \int_{c_1}^{b_1} \left(\int_{c_1}^{b_1} \sum_{i=1}^{\ell} q_i G_l(y_i, \varsigma) T_{w,n-2}^{[c_1,b_1]}(\varsigma, u) d\varsigma \right) du, \end{aligned} \tag{3.38}$$

where $T_{w,n-2}^{[a_1,c_1]}$, $T_{w,n-2}^{[c_1,b_1]}$, Λ_2 and Ξ_2 are given by (3.18), (3.19), (3.26) *and* (3.27) *respectively. If $\zeta : [a_1, b_1] \to \mathbb{R}$ is $(n+1)-$convex at point c_1, then*

$$\Lambda_2(\zeta) \leq \Xi_2(\zeta). \tag{3.39}$$

If inequalities in (3.36) *and* (3.37) *are reversed, then* (3.39) *is valid with reversed sign of inequality.*

Remark 3.21. If we put $w(u) = \frac{1}{b_1-a_1}$ in above identity, we capture Theorem 2.13 of [7].

Remark 3.22. Similar results can also be stated for integral versions as well by using functionals $\Lambda_3(\zeta)$, $\Xi_3(\zeta)$, $\Lambda_4(\zeta)$ and $\Xi_4(\zeta)$ as defined in (3.28), (3.29) (3.30) and (3.31) respectively.

3.2. Bounds for identities related to the Popoviciu-type inequalities

Under the assumptions of Theorems 3.1 and 3.9, we denote the following functions Ω_j, $j \in \{1,2,3,4\}$, define as

$$\Omega_1(\tau) = \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i,\varsigma) \tilde{T}_{w,n-2}(\varsigma,u) d\varsigma, \quad u \in [a_1,b_1];$$

$$\Omega_2(\tau) = \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i,\varsigma) T_{w,n-2}(\varsigma,u) d\varsigma \geq 0, \quad u \in [a_1,b_1];$$

$$\Omega_3(\tau) = \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi)\, G(g(\chi),\varsigma)\, \tilde{T}_{w,n-2}(\varsigma,u)\, d\chi\, d\varsigma, \quad u \in [a_1,b_1];$$

$$\Omega_4(\tau) = \int_{a_1}^{b_1} \int_{\alpha_1}^{\beta_1} \varrho(\chi)\, G(g(\chi),\varsigma)\, T_{w,n-2}(\varsigma,u) d\chi\, d\varsigma, \quad u \in [a_1,b_1].$$

Theorem 3.23. *Let $n \in \mathbb{N}$, $n \geq 3$, $\zeta : [a_1,b_1] \to \mathbb{R}$ be such that $\zeta^{(n)}$ is an absolutely continuous function with $(\cdot - a_1)(b_1 - \cdot)[\zeta^{(n+1)}]^2 \in L[a_1,b_1]$ and let $\chi \in [a_1,b_1]^\gamma$ and $\varrho \in \mathbb{R}^\gamma$ satisfy*

$$\sum_{i=1}^{\gamma} \varrho_i = 0 \text{ and } \sum_{i=1}^{\gamma} \varrho_i \chi_i = 0.$$

Then

$$\begin{aligned}
\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &= \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i,\varsigma) \\
&\times \left[w(\varsigma)(\varsigma - a_1)^\kappa + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\varsigma \\
&+ \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i,\varsigma) \\
&\times \left[-w(\varsigma)(\varsigma - b_1)^\kappa + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\varsigma \\
&+ \frac{\zeta^{(n-1)}(b_1) - \zeta^{(n-1)}(a_1)}{(n-3)!(b_1 - a_1)} \int_{a_1}^{b_1} \Omega_1(\varsigma) d\varsigma + R_n^1(\zeta; a_1, b_1),
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) &= \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i,\varsigma) \left(\int_{a_1}^{b_1} w(\tau) \zeta''(\tau) d\tau \right) d\varsigma \\
&+ \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G_l(\chi_i,\varsigma)
\end{aligned}$$

$$\times \left[\zeta^{(\kappa)}(a_1)\int_{a_1}^{\varsigma} w(u)\left((\varsigma-a_1)^{\kappa-2}-(u-a_1)^{\kappa-2}\right)du\right.$$
$$\left.+\zeta^{(\kappa)}(b_1)\int_{\varsigma}^{b_1} w(u)\left((\varsigma-b_1)^{\kappa-2}-(u-b_1)^{\kappa-2}\right)du\right]d\varsigma$$
$$+\frac{\zeta^{(n-1)}(b_1)-\zeta^{(n-1)}(a_1)}{(n-3)!(b_1-a_1)}\int_{a_1}^{b_1}\Omega_2(\varsigma)d\varsigma+R_n^2(\zeta;a_1,b_1), \tag{3.41}$$

where the remainders $R_n^j(\zeta;a_1,b_1)$, $j=1,2$, *satisfy the bounds*

$$|R_n^j(\zeta;a_1,b_1)|\leq\frac{1}{(n-3)!}$$
$$\times\left(\frac{(b_1-a_1)}{2}\left|T(\Omega_j,\Omega_j)\int_{a_1}^{b_1}(\varsigma-a_1)(b_1-\varsigma)[\zeta^{(n+1)}(\varsigma)]^2d\varsigma\right|\right)^{1/2}. \tag{3.42}$$

Remark 3.24. If we put $w(u)=\frac{1}{b_1-a_1}$, $u\in[a_1,b_1]$ above identity reduces to Theorem 3.3 of [9].

By using Proposition 2.24, we obtain the following Grüss type inequality.

Theorem 3.25. *Let* $n\in\mathbb{N}$, $n\geq 3$, $\zeta:[a_1,b_1]\to\mathbb{R}$ *be such that* $\zeta^{(n)}$ *is an absolutely continuous function with* $\zeta^{(n+1)}\geq 0$ *and let* $\chi\in[a_1,b_1]^\gamma$ *and* $\varrho\in\mathbb{R}^\gamma$ *satisfy*

$$\sum_{i=1}^{\gamma}\varrho_i=0 \text{ and } \sum_{i=1}^{\gamma}\varrho_i\chi_i=0.$$

Then representations (3.40) *and* (3.41) *hold and the remainders* $R_n^j(\zeta;a_1,b_1)$, $j=1,2$, *satisfy the bounds*

$$|R_n^j(\zeta;a_1,b_1)|\leq\frac{1}{(n-3)!}\|\Omega_j'\|_\infty\left\{\frac{b_1-a_1}{2}\left[\zeta^{(n-1)}(b_1)+\zeta^{(n-1)}(a_1)\right]\right.$$
$$\left.-\left[\zeta^{(n-2)}(b_1)-\zeta^{(n-2)}(a_1)\right]\right\}. \tag{3.43}$$

Remark 3.26. If we put $w(u)=\frac{1}{b_1-a_1}$, $u\in[a_1,b_1]$ above identity reduces to Theorem 3.4 of [9].

Remark 3.27. Similar results can also be stated for the integral version as well by using functional Ψ_j, where $j\in\{3,4\}$.

3.3. Ostrowski type inequalities via extension of Montgomery identity and Green functions

Here we present some Ostrowski-type inequalities related to the generalized linear inequalities. Throughout the section, we use the following functions Ω_j, $j\in\{1,2,3,4\}$ defined as in the previous subsection.

Theorem 3.28. *Let* $\zeta^{(n)} \in L_q[a_1, b_1]$, $1 \le q, r \le \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $n \ge 3$, $n \in \mathbb{N}$, $j \in \{1, 2, 3, 4\}$. *Then*

$$\Bigg|\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) - \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(a_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma)$$
$$\times \left[w(\varsigma)(\varsigma - a_1)^{\kappa} + \kappa \int_{a_1}^{\varsigma} w(u)(\varsigma - a_1)^{\kappa-1} du \right] d\varsigma$$
$$- \sum_{\kappa=0}^{n-2} \frac{\zeta^{(\kappa+1)}(b_1)}{\kappa!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma)$$
$$\times \left[-w(\varsigma)(\varsigma - b_1)^{\kappa} + \kappa \int_{\varsigma}^{b_1} w(u)(\varsigma - b_1)^{\kappa-1} du \right] d\varsigma \Bigg|$$
$$\le \frac{1}{(n-3)!} \|\zeta^{(n)}\|_q \, \|\Omega_j\|_r, \quad (3.44)$$

and

$$\Bigg|\sum_{i=1}^{\gamma} \varrho_i \zeta(\chi_i) - \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma) \left(\int_{a_1}^{b_1} w(\tau) \zeta''(\tau) d\tau \right) d\varsigma$$
$$- \sum_{\kappa=3}^{n-1} \frac{1}{(\kappa-2)!} \int_{a_1}^{b_1} \sum_{i=1}^{\gamma} \varrho_i G(\chi_i, \varsigma)$$
$$\times \Bigg[\zeta^{(\kappa)}(a_1) \int_{a_1}^{\varsigma} w(\tau) \left((\varsigma - a_1)^{\kappa-2} - (\tau - a_1)^{\kappa-2} \right) dt$$
$$+ \zeta^{(\kappa)}(b_1) \int_{\varsigma}^{b_1} w(\tau) \left((\varsigma - b_1)^{\kappa-2} - (\tau - b_1)^{\kappa-2} \right) dt \Bigg] d\varsigma \Bigg|$$
$$\le \frac{1}{(n-3)!} \|\zeta^{(n)}\|_q \, \|\Omega_2\|_r. \quad (3.45)$$

The constant on the right hand sides of (3.44) *and* (3.45) *is the best possible for* $q = 1$ *and sharp for* $1 < q \le \infty$.

Remark 3.29. If we put $w(\tau) = \frac{1}{b_1 - a_1}$ in Theorem 3.28, we capture Theorem 3.5 for $j \in \{1, 2\}$ and Theorem 3.8 for $j \in \{3, 4\}$ of [9].

4. Conclusion and remarks

In this article, we have given a generalization of the results stated in [8] and [9](see also [10]) by introducing weights which are probability density functions. If we put our weights equal to $\frac{1}{b_1 - a_1}$ in our proposed results, we will capture almost all the results of [8], [9] and [10] as our special cases. Due to the general nature of the article, in some places we have used the Leibnitz rule of integration due to the involvement of the variable of integration in the limit of the integral as well. In our subsections we stated results, related to $n-$convexity at a point for Popoviciu-type inequalities involving the weighted version of the extension of Montgomery's identity and similar results for Popoviciu-type inequalities involving the weighted version of the extended Montgomery's identity with Green functions. We have also discussed the bounds of remainders for our proposed results using *Čebyšev* functional and *Grüss* type inequalities. In the end of sections we obtained bounds of Ostrowski type. Such results are also valid in the context of Green functions.

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Certain class of analytic functions defined by $q-$analogue of Ruscheweyh differential operator

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Abstract. In this paper, we obtain coefficient estimates, distortion theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $TB_q^{\lambda}(\alpha, \beta)$ of analytic starlike and convex functions defined by $q-$analogue of Ruscheweyh differential operator. Also we find closure theorems, $N_{k,q,\delta}(e, g)$ neighborhood and partial sums for functions in this class.

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1. Introduction

Let $\mathcal{S}$ be the class of analytic and univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , z \in \mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\} . \tag{1.1}$$

Also let $\mathcal{S}^*(\alpha)$ and $C(\alpha)$ denote the subclasses of $\mathcal{S}$ which are, respectively, starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$, satisfying (see Robertson [30])

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{S} \ \text{ and Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \right\}, \tag{1.2}$$

and

$$C(\alpha) = \left\{ f : f \in \mathcal{S} \ \text{ and Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \right\}. \tag{1.3}$$

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It readily follows from (1.2) and (1.3) that

$$f(z) \in C(\alpha) \Leftrightarrow zf^{'}(z) \in S^{*}(\alpha).$$

For $0 < q < 1$ the Jackson's $q-$derivative of a function $f(z) \in \mathcal{S}$ is given by [22] (see also [2, 3, 8, 13, 17, 20, 24, 34, 35, 39])

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & \text{for} \quad z \neq 0, \\ f^{'}(0) & \text{for} \quad z = 0, \end{cases} \tag{1.4}$$

For $f(z)$ of the form (1.1), we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \tag{1.5}$$

where

$$[n]_q = \frac{1-q^n}{1-q} \quad (0 < q < 1; \; n \in \mathbb{N} = \{1, 2, ...\}). \tag{1.6}$$

Kanas and Raducanu [23] (see also Aldweby and Darus [1]) defined the $q-$analogue of Ruscheweyh operator by

$$R_q^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k \quad (0 < q < 1; \lambda \geq 0), \tag{1.7}$$

where

$$[n]_q! = \begin{cases} [n]_q \, [n-1]_q \ldots [1]_q \, , & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{1.8}$$

From (1.7) we obtain that

$$R_q^0 f(z) = f(z) \quad \text{and } R_q^1 f(z) = zD_q f(z),$$

and

$$\lim_{q \longrightarrow 1^-} R_q^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!}{\lambda! \, (k-1)!} a_k z^k = R^{\lambda} f(z), \tag{1.9}$$

where R^{λ} is the Ruscheweyh differential operator (see [32] and [4, 7, 10, 14, 18]).

Definition 1.1. For $0 < q < 1, 0 \leq \alpha < 1, \beta \geq 0$ and $\lambda \geq 0$, let $B_q^{\lambda}(\alpha, \beta)$ be the class of functions $f \in \mathcal{S}$ satisfying

$$\operatorname{Re}\left\{ \frac{zD_q(R_q^{\lambda} f(z))}{R_q^{\lambda} f(z)} - \alpha \right\} > \beta \left| \frac{zD_q(R_q^{\lambda} f(z))}{R_q^{\lambda} f(z)} - 1 \right|. \tag{1.10}$$

Let $\mathcal{T} \subset \mathcal{S}$ such that:

$$\mathcal{T} = \left\{ f \in \mathcal{S} : f(z) = z - \sum_{k=2}^{\infty} a_k z^k \; , a_k \geq 0 \right\}, \tag{1.11}$$

and

$$TB_q^{\lambda}(\alpha, \beta) = B_q^{\lambda}(\alpha, \beta) \cap \mathcal{T}. \tag{1.12}$$

Note that

(i) $TB_q^0(\alpha,\beta) = S_p^q(\alpha,\beta) = \left\{f \in T : \text{Re}\left\{\frac{zD_qf(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zD_qf(z)}{f(z)} - 1\right|, z \in \mathbb{U}\right\}$;

(ii) $TB_q^0(\alpha,0) = TB_q(\alpha) = \left\{f \in T : \text{Re}\left\{\frac{zD_qf(z)}{f(z)}\right\} > \alpha\right\}$;

(iii) $\lim\limits_{q\to 1^-} TB_q^0(\alpha,\beta) = S_p(\alpha,\beta) = \{f \in T : \text{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\}$
$> \beta\left\{\frac{zf'(z)}{f(z)} - 1\right\}, z \in \mathbb{U}\}$ (see [29] and [36]);

(iv) $TB_q^1(\alpha,\beta) = UCS_p^q(\alpha,\beta) = \{f \in T : \text{Re}\left\{\frac{D_q(zD_qf(z))}{D_qf(z)} - \alpha\right\}$
$> \beta\left|\frac{D_q(zD_qf(z))}{D_qf(z)} - 1\right|, z \in \mathbb{U}\}$;

(v) $TB_q^1(\alpha,0) = C_q(\alpha) = \{f \in T : \text{Re}\left\{\frac{D_q(zD_qf(z))}{D_qf(z)}\right\} > \alpha, z \in \mathbb{U}\}$;

(vi) $\lim\limits_{q\to 1^-} TB_q^1(\alpha,\beta) = UCS_p(\alpha,\beta) = \{f \in T : \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\}$
$> \beta\left|\frac{zf''(z)}{f'(z)}\right|, z \in \mathbb{U}\}$ (see [29]);

(vii) $\lim\limits_{q\to 1^-} TB_q^\lambda(\alpha,\beta) = S_p^\lambda(\alpha,\beta)$ (see Rosy et al. [31]).

2. Coefficient estimates

Unless indicated, we assume that $0 \le \alpha < 1$, $\beta \ge 0, \lambda \ge 0$, $0 < q < 1$ and $f(z) \in \mathcal{T}$.

Theorem 2.1. *A function* $f(z) \in TB_q^\lambda(\alpha,\beta)$ *if and only if*

$$\sum_{k=2}^{\infty}\left[[k]_q(1+\beta) - (\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k \le 1-\alpha. \tag{2.1}$$

Proof. Assume that (2.1) holds. Then it is suffices to show that

$$\beta\left|\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right| - \text{Re}\left\{\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right\} \le 1-\alpha.$$

We have

$$\begin{aligned} &\beta\left|\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right| - \text{Re}\left\{\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right\} \\ \le\ &(1+\beta)\left|\frac{zD_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1\right| \\ \le\ &\frac{(1+\beta)\sum\limits_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\left([k]_q-1\right)a_k}{1-\sum\limits_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k}. \end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ since (2.1) holds.

Conversely if $f(z) \in TB_q^{\lambda}(\alpha,\beta)$ and z is real, then

$$\mathrm{Re}\left\{\frac{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}[k]_q a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^{k-1}}-\alpha\right\} \geq \beta\left|\frac{\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}([k]_q-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^{k-1}}\right|.$$

Letting $z \to 1^-$ along the real axis, we obtain (2.1). Hence the proof is completed. □

Corollary 2.2. *For* $f(z) \in TB_q^{\lambda}(\alpha,\beta)$,

$$a_k \leq \frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}} \quad (k \geq 2) \tag{2.2}$$

and

$$f(z) = z - \frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}z^k \quad (k \geq 2), \tag{2.3}$$

gives the sharpness.

Remark 2.1. Letting $q \to 1^-$ in the results of Section 2, we get the results of Section 2 for the class $S_p^{\lambda}(\alpha,\beta)$ studied by Rosy et al. [31].

3. Growth and distortion theorems

Theorem 3.1. *For* $f(z) \in TB_q^{\lambda}(\alpha,\beta)$ *and* $|z| = r < 1$, *we have*

$$|f(z)| \geq r - \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q}r^2, \tag{3.1}$$

and

$$|f(z)| \leq r + \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q}r^2. \tag{3.2}$$

Equalities hold for

$$f(z) = z - \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q}z^2, \tag{3.3}$$

at $z = r$ *and* $z = re^{i(2k+1)\pi}$ $(k \geq 2)$.

Proof. Since for $k \geq 2$,

$$[[2]_q(1+\beta)-(\alpha+\beta)][1+\lambda]_q\sum_{k=2}^{\infty}a_k \leq \sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k \leq 1-\alpha, \tag{3.4}$$

then

$$\sum_{k=2}^{\infty}a_k \leq \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q}. \tag{3.5}$$

From (1.12) and (3.5), we have

$$|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - \frac{1-\alpha}{\left[[2]_q(1+\beta) - (\alpha+\beta)\right][1+\lambda]_q} r^2 \tag{3.6}$$

and

$$|f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k \leq r + \frac{1-\alpha}{\left[[2]_q(1+\beta) - (\alpha+\beta)\right][1+\lambda]_q} r^2. \tag{3.7}$$

This completes the proof. □

Letting $q \to 1^-$ in Theorem 3.1, we have

Corollary 3.2. *For $f(z) \in S_p^{\lambda}(\alpha, \beta)$, then*

$$|f(z)| \geq r - \frac{1-\alpha}{(2+\beta-\alpha)(1+\lambda)} r^2, \tag{3.8}$$

and

$$|f(z)| \leq r + \frac{1-\alpha}{(2+\beta-\alpha)(1+\lambda)} r^2. \tag{3.9}$$

Equalities hold for

$$f(z) = z - \frac{1-\alpha}{(2+\beta-\alpha)(1+\lambda)} z^2, \tag{3.10}$$

at $z = r$ and $z = re^{i(2k+1)\pi}$ $(k \geq 2)$.

Proof. Letting $q \to 1^-$ in Theorem 3.1, we can show (3.8) and (3.9). □

Theorem 3.3. *Let $f(z) \in TB_q^{\lambda}(\alpha, \beta)$. Then for $|z| = r < 1$,*

$$\left|f'(z)\right| \geq 1 - \frac{2(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r, \tag{3.11}$$

and

$$\left|f'(z)\right| \leq 1 + \frac{2(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r. \tag{3.12}$$

The sharpness are attained for $f(z)$ given by (3.3).

Proof. For $k \geq 2$, we have

$$\left|f'(z)\right| \leq 1 - r \sum_{k=2}^{\infty} k a_k.$$

We find from (2.1) and (3.5) that

$$\begin{aligned} [2]_q(1+\beta)[\lambda+1]_q \sum_{k=2}^{\infty} k a_k &\leq 2(1-\alpha) + 2(\alpha+\beta)[\lambda+1]_q \sum_{k=2}^{\infty} a_k \\ &\leq 2(1-\alpha) + \frac{2(\alpha+\beta)(1-\alpha)}{\left[[2]_q(1+\beta) - (\alpha+\beta)\right]} \\ &\leq \frac{2[2]_q(1+\beta)(1-\alpha)}{\left[[2]_q(1+\beta) - (\alpha+\beta)\right]}, \end{aligned}$$

that is, that

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}. \tag{3.13}$$

From (3.11) and (3.12) that

$$\left|f^{'}(z)\right| \geq 1 - r\sum_{k=2}^{\infty} k a_k \geq 1 - \frac{2(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r \tag{3.14}$$

and

$$\left|f^{'}(z)\right| \leq 1 + r\sum_{k=2}^{\infty} k a_k \leq 1 + \frac{2(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r. \tag{3.15}$$

This completes the proof. □

Theorem 3.4. *For* $f(z) \in TB_q^{\lambda}(\alpha,\beta)$ *and* $|z| = r < 1$,

$$|D_q f(z)| \geq 1 - \frac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r, \tag{3.16}$$

and

$$|D_q f(z)| \leq 1 + \frac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r. \tag{3.17}$$

The sharpness are attained for $f(z)$ given by (3.3).

Proof. For $k \geq 2$, we have

$$|D_q f(z)| \leq 1 - r\sum_{k=2}^{\infty} [k]_q a_k.$$

We find from (2.1) and (3.5) that

$$\begin{aligned}(1+\beta)[\lambda+1]_q \sum_{k=2}^{\infty} [k]_q a_k &\leq (1-\alpha) + (\alpha+\beta)[\lambda+1]_q \sum_{k=2}^{\infty} a_k \\ &\leq (1-\alpha) + \frac{[2]_q(\alpha+\beta)(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]} \\ &\leq \frac{[2]_q(1+\beta)(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right]},\end{aligned}$$

that is, that

$$\sum_{k=2}^{\infty} [k]_q a_k \leq \frac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}, \tag{3.18}$$

From (3.16) and (3.17) that

$$|D_q f(z)| \geq 1 - r\sum_{k=2}^{\infty} [k]_q a_k \geq 1 - \frac{[2]_q(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r \tag{3.19}$$

and

$$|D_q f(z)| \leq 1 + r\sum_{k=2}^{\infty} [k]_q\, a_k \leq 1 + \frac{[2]_q (1-\alpha)}{\left[[2]_q (1+\beta)-(\alpha+\beta)\right][1+\lambda]_q} r. \tag{3.20}$$

This completes the proof. □

Letting $q \to 1^-$ in Theorem 3.4, we have

Corollary 3.5. *For $f(z) \in S_p^{\lambda}(\alpha, \beta)$, then*

$$\left|f^{'}(z)\right| \geq 1 - \frac{2(1-\alpha)}{(2+\beta-\alpha)(1+\lambda)} r, \tag{3.21}$$

and

$$\left|f^{'}(z)\right| \leq 1 + \frac{2(1-\alpha)}{(2+\beta-\alpha)(1+\lambda)} r. \tag{3.22}$$

The sharpness are attained for $f(z)$ given by (3.10).

Proof. Letting $q \to 1^-$ in Theorem 3.4, we can show (3.21) and (3.22). Then Corollary 3.5 corresponds to Theorem 3.3 when $q \to 1^-$. □

4. Closure theorems

Let $f_j(z)$ be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0,\ z \in \mathbb{U}). \tag{4.1}$$

Theorem 4.1. *Let $f_j(z) \in TB_q^{\lambda}(\alpha, \beta)$ for $j = 1, 2, ..., m$. Then*

$$g(z) = \sum_{j=1}^{m} c_j f_j(z), \tag{4.2}$$

is also in the same class, where $c_j \geq 0$, $\sum\limits_{j=1}^{m} c_j = 1$.

Proof. According to (4.2), we can write

$$g(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} c_j a_{k,j}\right) z^k. \tag{4.3}$$

Further, since $f_j(z) \in TB_q^{\lambda}(\alpha, \beta)$, we get

$$\sum_{k=2}^{\infty} \left[[k]_q (1+\beta) - (\alpha+\beta)\right] \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_{k,j} \leq 1 - \alpha. \tag{4.4}$$

Hence

$$\sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}(\sum_{j=1}^{m}c_j a_{k,j})$$
$$=\sum_{j=1}^{m}c_j[\sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_{k,j}]$$
$$\leq\left(\sum_{j=1}^{m}c_j\right)(1-\alpha)=1-\alpha, \tag{4.5}$$

which implies that $g(z)\in TB_q^{\lambda}(\alpha,\beta)$. Thus we have the theorem. □

Corollary 4.2. *The class $TB_q^{\lambda}(\alpha,\beta)$ is closed under convex linear combination.*

Proof. Let $f_j(z)\in TB_q^{\lambda}(\alpha,\beta)$ $(j=1,2)$ and

$$g(z)=\mu f_1(z)+(1-\mu)f_2(z)\quad(0\leq\mu\leq1), \tag{4.6}$$

Then by, taking $m=2$, $c_1=\mu$ and $c_2=1-\mu$ in Theorem 5, we have $g(z)\in TB_q^{\lambda}(\alpha,\beta)$. □

Theorem 4.3. *Let $f_1(z)=z$ and*

$$f_k(z)=z-\frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}z^k\quad(k\geq2). \tag{4.7}$$

Then $f(z)\in TB_q^{\lambda}(\alpha,\beta)$ if and only if

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z), \tag{4.8}$$

where $\mu_k\geq0$ $(k\geq1)$ and $\sum_{k=1}^{\infty}\mu_k=1$.

Proof. Suppose that

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z)=z-\sum_{k=2}^{\infty}\frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}\mu_k z^k. \tag{4.9}$$

Then it follows that

$$\sum_{k=2}^{\infty}\frac{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{1-\alpha}\cdot\frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}\mu_k=\sum_{k=2}^{\infty}\mu_k=1-\mu_1\leq1. \tag{4.10}$$

So by Theorem 2.1, $f(z)\in TB_q^{\lambda}(\alpha,\beta)$. Conversely, assume that $f(z)\in TB_q^{\lambda}(\alpha,\beta)$. Then

$$a_k\leq\frac{1-\alpha}{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}\quad(k\geq2). \tag{4.11}$$

Setting

$$\mu_k = \frac{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{1-\alpha} a_k \quad (k \geq 2), \tag{4.12}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{4.13}$$

we see that $f(z)$ can be expressed in the form (4.8). This completes the proof. □

Corollary 4.4. *The extreme points of* $TB_q^{\lambda}(\alpha,\beta)$ *are* $f_k(z)$ $(k \geq 1)$ *given by Theorem 4.3.*

5. Some radii of the class $TB_q^{\lambda}(\alpha,\beta)$

Theorem 5.1. *Let* $f(z) \in TB_q^{\lambda}(\alpha,\beta)$. *Then for* $0 \leq \rho < 1, k \geq 2, f(z)$ *is*

(i) *close -to- convex of order* ρ *in* $|z| < r_1$, *where*

$$r_1 = r_1(q,\alpha,\beta,\lambda,\rho) := \inf_k \left[\frac{(1-\rho)\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{k(1-\alpha)}\right]^{\frac{1}{(k-1)}}. \tag{5.1}$$

(ii) *starlike of order* ρ *in* $|z| < r_2$, *where*

$$r_2 = r_2(q,\alpha,\beta,\lambda,\rho) := \inf_k \left[\frac{(1-\rho)\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{(k-1)}}. \tag{5.2}$$

(iii) *convex of order* ρ *in* $|z| < r_3$, *where*

$$r_3 = r_3(q,\alpha,\beta,\lambda,\rho) := \inf_k \left[\frac{(1-\rho)\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{k(k-\rho)(1-\alpha)}\right]^{\frac{1}{(k-1)}}. \tag{5.3}$$

The result is sharp for $f(z)$ *is given by (2.3).*

Proof. To prove (i) we must show that

$$\left|f'(z) - 1\right| \leq 1-\rho \quad for \quad |z| < r_1(q,\alpha,\beta,\rho).$$

From (1.12), we have

$$\left|f'(z) - 1\right| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$\left|f'(z) - 1\right| \leq 1-\rho,$$

if

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \tag{5.4}$$

But, by Theorem 2.1, (5.4) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \leq \frac{\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\left[[k]_q(1+\beta)-(\alpha+\beta)\right]\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}}{k(1-\alpha)}\right]^{\frac{1}{(k-1)}} \quad (k \geq 2), \tag{5.5}$$

which gives (5.1).

To prove (ii) and (iii) it is suffices to show

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho \ \text{ for } \ |z| < r_2, \tag{5.6}$$

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\rho \ \text{ for } \ |z| < r_3, \tag{5.7}$$

respectively, by using arguments as in proving (i), we have the results. □

6. Inclusion relations involving $N_{k,q,\delta}(e)$

In this section following the works of Goodman [21] and Ruscheweyh [33] (see also [5], [6], [9], [16], [26] and [28]) defined the k,δ neighborhood of function $f(z) \in T$ by

$$N_{k,\delta}(f;g) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k\,|a_k - b_k| \leq \delta \right\}. \tag{6.1}$$

In particular, for the identity function $e(z) = z$, we have

$$N_{k,\delta}(e;g) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k\,|b_k| \leq \delta \right\}. \tag{6.2}$$

Aouf et al. [12] defined the k,q,δ neighborhood of function $f(z) \in T$ by

$$N_{k,q,\delta}(f;g) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \sum_{k=2}^{\infty} [k]_q\,|a_k - b_k| \leq \delta_q \right\}. \tag{6.3}$$

In particular, for the identity function $e(z) = z$, we have

$$N_{k,q,\delta}(e;g) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} [k]_q\,|b_k| \leq \delta_q \right\}. \tag{6.4}$$

Theorem 6.1. *Let*

$$\delta_q = \frac{(1-\alpha)}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}. \tag{6.5}$$

Then $TB_q^{\lambda}(\alpha,\beta) \subset N_{k,q,\delta}(e)$.

Proof. For $f \in TB_q^{\lambda}(\alpha,\beta)$, Theorem 2.1, (3.5) and (3.18), and in view of the (6.4), Theorem 6.1 follows. □

A function $f \in T$ is in the class $TB_q^{\lambda}(\alpha,\beta,\xi)$ if there exists a function $g \in TB_q^{\lambda}(\alpha,\beta)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\xi_q \quad (z \in \mathbb{U},\ 0 \le \xi_q < 1). \tag{6.6}$$

Now we determine the neighborhood for the class $TB_q^{\lambda}(\alpha,\beta,\xi)$.

Theorem 6.2. *If $g \in TB_q^{\lambda}(\alpha,\beta)$ and*

$$\xi_q = 1 - \frac{\delta_q\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q-(1-\alpha)\right\}}, \tag{6.7}$$

where

$$\delta_q \le \frac{2\left\{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q-(1-\alpha)\right\}}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}.$$

Then $N_{k,q,\delta}(g) \subset TB_q^{\lambda}(\alpha,\beta,\xi)$.

Proof. Suppose that $f \in N_{k,q,\delta}(g)$ then

$$\sum_{k=2}^{\infty}[k]_q|a_k-b_k| \le \delta_q,$$

where δ_q is given by (6.5), which implies that the coefficient inequality

$$\sum_{k=2}^{\infty}|a_k-b_k| \le \frac{\delta_q}{[2]_q}.$$

Next, since $g \in TB_q^{\lambda}(\alpha,\beta)$, we have

$$\sum_{k=2}^{\infty}b_k \le \frac{1-\alpha}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q},$$

so that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=2}^{\infty}|a_k-b_k|}{1-\sum_{k=2}^{\infty}b_k} \le \frac{\delta_q}{[2]_q} \times \frac{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q}{\left[[2]_q(1+\beta)-(\alpha+\beta)\right][\lambda+1]_q-(1-\alpha)} \le 1-\xi_q.$$

Provided that ξ_q is given precisely by (6.7). Thus, by definition, $f \in TB_q^{\lambda}(\alpha,\beta,\xi)$, which completes the proof. □

7. Partial sums

For $f(z)$ of the form (1.1), the sequence of partial sums is given by

$$f_m(z) = z + \sum_{k=2}^{m} a_k z^k \quad (m \in \mathbb{N} \setminus \{1\}).$$

Now following the work of [38] and also the works cited in [11, 15, 19, 25, 27, 31, 37] on partial sums of analytic functions, to obtain our results. Let

$$\Phi_{q,k}^{\lambda} = \Phi_q^{\lambda}(k,\alpha,\beta) = \left[[k]_q (1+\beta) - (\alpha+\beta) \right] \tfrac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}. \tag{7.1}$$

Theorem 7.1. *If $f \in \mathcal{S}$, satisfies the condition (2.1), then*

$$\operatorname{Re}\left(\frac{f(z)}{f_m(z)} \right) \geq \frac{\Phi_{q,m+1}^{\lambda} - 1 + \alpha}{\Phi_{q,m+1}^{\lambda}}, \tag{7.2}$$

where

$$\Phi_{q,k}^{\lambda} \geq \begin{cases} 1-\alpha, & \text{if} \quad k = 2,3,...,m \\ \Phi_{q,m+1}^{\lambda}, & \text{if} \quad k = m+1, m+2, ... \end{cases} \tag{7.3}$$

The result (7.2) is sharp for

$$f(z) = z + \frac{1-\alpha}{\Phi_{q,m+1}^{\lambda}} z^{m+1}. \tag{7.4}$$

Proof. Define $g(z)$ by

$$\tfrac{1+g(z)}{1-g(z)} = \tfrac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} \left[\tfrac{f(z)}{f_m(z)} - \tfrac{\Phi_{q,m+1}^{\lambda}-1+\alpha}{\Phi_{q,m+1}^{\lambda}} \right] = \frac{1+\sum\limits_{k=2}^{m} a_k z^{k-1} + \left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} \right) \sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1+\sum\limits_{k=2}^{m} a_k z^{k-1}}. \tag{7.5}$$

It suffices to show that $|g(z)| \leq 1$. Now from (7.5) we have

$$g(z) = \frac{\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} \right) \sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{2+2\sum\limits_{k=2}^{m} a_k z^{k-1} + \left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} \right) \sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}.$$

Hence we obtain

$$|g(z)| \leq \frac{\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} \right) \sum\limits_{k=m+1}^{\infty} |a_k|}{2-2\sum\limits_{k=2}^{m} |a_k| - \left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} \right) \sum\limits_{k=m+1}^{\infty} |a_k|}.$$

Now $|g(z)| \leq 1$ if and only if

$$2 \left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} \right) \sum_{k=m+1}^{\infty} |a_k| \leq 2 - 2\sum_{k=2}^{m} |a_k|,$$

or, equivalently,

$$\sum_{k=2}^{m} |a_k| + \sum_{k=m+1}^{\infty} \frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha} |a_k| \leq 1.$$

From (2.1), it is sufficient to show that

$$\sum_{k=2}^{m}|a_k|+\sum_{k=m+1}^{\infty}\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}|a_k|\leq\sum_{k=2}^{\infty}\frac{\Phi_{q,k}^{\lambda}}{1-\alpha}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{\Phi_{q,k}^{\lambda}-1+\alpha}{1-\alpha}\right)|a_k|+\sum_{k=m+1}^{\infty}\left(\frac{\Phi_{q,k}^{\lambda}-\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)|a_k|\geq 0. \tag{7.6}$$

For $z=re^{i\pi/m}$ we have

$$\frac{f(z)}{f_m(z)}=1+\frac{1-\alpha}{\Phi_{q,m+1}^{\lambda}}z^k\to 1-\frac{1-\alpha}{\Phi_{q,m+1}^{\lambda}}=\frac{\Phi_{q,m+1}^{\lambda}-1+\alpha}{\Phi_{q,m+1}^{\lambda}}\quad\text{where } r\to 1^-,$$

which shows that $f(z)$ is given by (7.4) gives the sharpness. □

Remark 7.1. (i) Putting $\lambda=0$ and (ii) $\lambda=1$ in Theorem 7.1, we obtain the following results, respectively.

Corollary 7.2. *If $f\in\mathcal{S}$, satisfies the condition (2.1) and $\frac{f(z)}{z}\neq 0(0<|z|<1)$, then*

$$\operatorname{Re}\left(\frac{f(z)}{f_m(z)}\right)\geq\frac{\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]-1+\alpha}{\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]}. \tag{7.7}$$

The result is sharp for

$$f(z)=z+\frac{1-\alpha}{\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]}z^{m+1}. \tag{7.8}$$

Corollary 7.3. *If $f\in\mathcal{S}$, satisfies the condition (2.1) and $\frac{f(z)}{z}\neq 0(0<|z|<1)$, then*

$$\operatorname{Re}\left(\frac{f(z)}{f_m(z)}\right)\geq 1-\frac{1-\alpha}{[m+1]_q\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]}. \tag{7.9}$$

The result is sharp for

$$f(z)=z+\frac{1-\alpha}{[m+1]_q\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]}z^{m+1}. \tag{7.10}$$

Theorem 7.4. *If $f\in\mathcal{S}$, satisfies the condition (2.1), then*

$$\operatorname{Re}\left(\frac{f_m(z)}{f(z)}\right)\geq\frac{\Phi_{q,m+1}^{\lambda}}{\Phi_{q,m+1}^{\lambda}+1-\alpha}, \tag{7.11}$$

where $\Phi_{q,m+1}^{\lambda}$ is defined by (7.1) and satisfies (7.3) and $f(z)$ given by (7.4) gives the sharpness.

Proof. The proof follows by defining

$$\frac{1+g(z)}{1-g(z)}=\frac{\Phi_{q,m+1}^{\lambda}+1-\alpha}{1-\alpha}\left[\frac{f_m(z)}{f(z)}-\frac{\Phi_{q,m+1}^{\lambda}}{\Phi_{q,m+1}^{\lambda}+1-\alpha}\right]$$

and much akin are to similar arguments in Theorem 7.1. So, we omit it. □

Remark 7.2. (i) Putting $\lambda=0$ and (ii) $\lambda=1$ in Theorem 7.4, we obtain the following sharp results, respectively.

Corollary 7.5. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and* $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, *then*

$$\operatorname{Re}\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{[m+1]_q(1+\beta)-(\alpha+\beta)}{[m+1]_q(1+\beta)-(\alpha+\beta)+1-\alpha}. \tag{7.12}$$

Corollary 7.6. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and* $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, *then*

$$\operatorname{Re}\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{[m+1]_q\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]}{[m+1]_q\left[[m+1]_q(1+\beta)-(\alpha+\beta)\right]+1-\alpha}. \tag{7.13}$$

Theorem 7.7. *If $f \in \mathcal{S}$, satisfies the condition (2.1), then*

$$\operatorname{Re}\left(\frac{f'(z)}{f'_m(z)}\right) \geq \frac{\Phi^{\lambda}_{q,m+1}-(m+1)(1-\alpha)}{\Phi^{\lambda}_{q,m+1}}, \tag{7.14}$$

and

$$\operatorname{Re}\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{\Phi^{\lambda}_{q,m+1}}{\Phi^{\lambda}_{q,m+1}+(m+1)(1-\alpha)}, \tag{7.15}$$

where $\Phi^{\lambda}_{q,m+1} \geq (m+1)(1-\alpha)$ *and*

$$\Phi^{\lambda}_{q,k} \geq \begin{cases} k(1-\alpha), & \text{if} \quad k = 2,3,...,m \\ k\left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)}\right), & \text{if} \quad k = m+1, m+2, ... \end{cases} \tag{7.16}$$

$f(z)$ is given by (7.4) gives the sharpness.

Proof. We write

$$\frac{1+g(z)}{1-g(z)} = \frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\left[\frac{f'(z)}{f'_m(z)} - \left(\frac{\Phi^{\lambda}_{q,m+1}-(m+1)(1-\alpha)}{\Phi^{\lambda}_{q,m+1}}\right)\right],$$

where

$$g(z) = \frac{\left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum\limits_{k=m+1}^{\infty} k a_k z^{k-1}}{2+2\sum\limits_{k=2}^{m} k a_k z^{k-1}+\left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum\limits_{k=m+1}^{\infty} k a_k z^{k-1}}.$$

Now $|g(z)| \leq 1$ if and only if

$$\sum_{k=2}^{m} k|a_k| + \left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty} k|a_k| \leq 1.$$

From (2.1), it is sufficient to show that

$$\sum_{k=2}^{m} k|a_k| + \left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty}\frac{\Phi^{\lambda}_{q,k}}{1-\alpha}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{\Phi^{\lambda}_{q,k}-k(1-\alpha)}{1-\alpha}\right)|a_k| + \sum_{k=m+1}^{\infty}\left(\frac{(m+1)\Phi^{\lambda}_{q,k}-k\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\right)|a_k| \geq 0.$$

To prove the result (7.15), define the function $g(z)$ by

$$\frac{1+g(z)}{1-g(z)} = \frac{(m+1)(1-\alpha)+\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)}\left[\frac{f_m^{'}(z)}{f^{'}(z)} - \frac{\Phi^{\lambda}_{q,m+1}}{(m+1)(1-\alpha)+\Phi^{\lambda}_{q,m+1}}\right],$$

and by similar arguments in first part we get desired result. □

Remark 7.3. (i) Putting $\lambda = 0$ and (ii) $\lambda = 1$ in Theorem 7.7, we obtain the following sharp results, respectively.

Corollary 7.8. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and* $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, *then*

$$\operatorname{Re}\left(\frac{f^{'}(z)}{f_m^{'}(z)}\right) \geq 1 - \frac{(m+1)(1-\alpha)}{[m+1]_q(1+\beta)-(\alpha+\beta)}, \tag{7.17}$$

and

$$\operatorname{Re}\left(\frac{f_m^{'}(z)}{f^{'}(z)}\right) \geq \frac{[m+1]_q(1+\beta)-(\alpha+\beta)}{[m+1]_q(1+\beta)-(\alpha+\beta)+(m+1)(1-\alpha)}. \tag{7.18}$$

Corollary 7.9. *If $f \in \mathcal{S}$, satisfies the condition (2.1) and* $\frac{f(z)}{z} \neq 0 (0 < |z| < 1)$, *then*

$$\operatorname{Re}\left(\frac{f^{'}(z)}{f_m^{'}(z)}\right) \geq 1 - \frac{(m+1)(1-\alpha)}{[m+1]_q[(m+1)(1+\beta)-(\alpha+\beta)]}, \tag{7.19}$$

and

$$\operatorname{Re}\left(\frac{f_m^{'}(z)}{f^{'}(z)}\right) \geq \frac{[m+1]_q[(m+1)(1+\beta)-(\alpha+\beta)]}{[m+1]_q[(m+1)(1+\beta)-(\alpha+\beta)]+(m+1)(1-\alpha)}. \tag{7.20}$$

Remark 7.4. Letting $q \to 1^-$ in Theorems 7.1, 7.4 and 7.7, respectively, we get Theorems 4.1 and 4.2, respectively, for the class $S_q^{\lambda}(\alpha, \beta)$ studied by Rosy et al. [31].

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Starlikeness and close-to-convexity involving certain differential inequalities

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Abstract. In the present paper, we study certain differential inequalities involving meromorphic functions in the open unit disk and obtain certain sufficient conditions for starlikeness and close-to-convexity of meromorphic functions. In particular, we obtain:
1. If $f(z) \in \Sigma_p$ satisfies the differential inequality $\left|1 + \frac{zf''(z)}{f'(z)} + p\right| < \frac{1}{2}$, $z \in \mathbb{E}$, then $f(z)$ is meromorphic close-to-convex function.
2. If $f(z) \in \Sigma$ satisfies the differential inequality

$$\left|\frac{zf'(z)}{f(z)} + 1\right|^{1-\gamma} \left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right|^{\gamma} < \frac{1-\alpha}{(1+|1-2\alpha|)^{\gamma}},\ \gamma \geq 0,\ z \in \mathbb{E},$$

then $f(z)$ is meromorphic starlike function of order α.

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1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p}\ (p \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

which are analytic and p-valent in the punctured unit disc $\mathbb{E}_0 = \mathbb{E} \setminus \{0\}$, where $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \Sigma_p$ is said to be meromorphic p-valent starlike

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of order α if $f(z) \neq 0$ for $z \in \mathbb{E}$ and

$$-\Re\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}). \tag{1.1}$$

The class of all such meromorphic p-valent starlike functions is denoted by $\mathcal{MS}_p^*(\alpha)$.
A function $f \in \Sigma_p$ is called meromorphic p-valent close-to-convex of order α if there exists a function $g \in \mathcal{MS}_p^*$ such that and

$$-\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of all such meromorphic p-valent close-to-convex functions defined above is denoted by $\mathcal{MC}_p(\alpha)$.
Since $g(z) = z^{-p} \in \mathcal{MS}_p^*$, it follows that a function $f \in \Sigma_p$ satisfying

$$-\Re(z^{p+1}f'(z)) > 0,\ z \in \mathbb{E},$$

or

$$\left|z^{p+1}f'(z) + p\right| < p,\ z \in \mathbb{E}, \tag{1.2}$$

is a member of the class $\mathcal{MC}_p$.
Let $\Sigma = \Sigma_1$, $\mathcal{MS}^*(\alpha) = \mathcal{MS}_1^*(\alpha)$, $\mathcal{MS}^* = \mathcal{MS}_1^*(0)$, $\mathcal{MC}(\alpha) = \mathcal{MC}_1(\alpha)$ and $\mathcal{MC} = \mathcal{MC}_1(0)$.
In the literature of meromorphic functions, many authors obtained the conditions for meromorphic close-to-convex functions and meromorphic starlike functions. Some of the results from literature are given below:

Goyal and Prajapat [1] proved the following results:

Theorem 1.1. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - z^2f'(z) + 1\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha} \qquad (0 \leq \alpha < 1),$$

then $f \in \mathcal{MC}(\alpha)$.

Theorem 1.2. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - z^2f'(z) + 1\right| < \frac{3}{2},$$

then $f \in \mathcal{MC}$.

Theorem 1.3. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| < \frac{1}{2},$$

then $f \in \mathcal{MC}$.

Theorem 1.4. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)}\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha},\ (0 \leq \alpha < 1),$$

then $f \in \mathcal{MS}^*(\alpha)$.

Theorem 1.5. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right| < \frac{1}{2},$$

then $f \in \mathcal{MS}^*$.

Xu and Yang [4] proved the following results:

Theorem 1.6. *If $f \in \Sigma_n$ satisfies $f'(z) \neq 0$ in $\mathbb{E}_0$ and*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < a,$$

for some a $(0 < a \leq n)$, then $f \in \mathcal{MS}_n^*(e^{-a/n})$ and the order $e^{-a/n}$ is sharp.

Z-G Wang et al. [3] proved the following results:

Theorem 1.7. *If $f(z) \in \Sigma_p$ satisfies the following inequality*

$$\left|\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \mu \ \left(0 < \mu < \frac{1}{p}\right),$$

then $f \in \mathcal{MS}_p^*\left(\dfrac{p}{1+p\mu}\right)$.

Theorem 1.8. *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$\left|\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - 1\right| < \delta \ (0 < \delta < 1),$$

then $f \in \mathcal{MS}_p^*(p(1-\delta))$.

2. Preliminaries

We shall use the following lemma of Jack [2] to prove our result.

Lemma 2.1. *Suppose w is a nonconstant analytic function in $\mathbb{E}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in \mathbb{E}$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = m w(z_0)$, where $m \geq 1$, is some real number.*

Theorem 2.2. *Let $f(z) \in \Sigma_p$ and suppose that it satisfies, for $\gamma \geq 0$, the inequality*

$$\left|z^{p+1}f'(z) + p\right|^{1-\gamma}\left|z^{p+2}f''(z) + (p+1)z^{p+1}f'(z)\right|^{\gamma} < p, \qquad z \in \mathbb{E}. \tag{2.1}$$

Then $|z^{p+1}f'(z) + p| < p$, i.e. $f(z) \in \mathcal{MC}_p$ and is a bounded function in $\mathbb{E}$.

Proof. For a function $f \in \Sigma_p$ satisfying the assumption (2.1), we define a function w by

$$w(z) = \frac{1}{p}\left(z^{p+1}f'(z) + p\right) = b_k z^k + \dots, \qquad z \in \mathbb{E}. \tag{2.2}$$

Then w is analytic in $\mathbb{E}$ with $w(0) = 0$. To prove our conclusion we will show that $|w(z)| < 1, z \in \mathbb{E}$. Differentiating (2.2), we have

$$z^{p+2}f''(z) + (p+1)z^{p+1}f'(z) = pzw'(z) \tag{2.3}$$

From (2.2) and (2.3) we obtain that

$$\begin{aligned}\left|z^{p+1}f'(z)+p\right|^{1-\gamma}\left|z^{p+2}f''(z)+(p+1)z^{p+1}f'(z)\right|^{\gamma}\\ &= |pw(z)|^{1-\gamma}|pzw'(z)|^{\gamma}\\ &= p|w(z)|\left|\frac{zw'(z)}{w(z)}\right|^{\gamma},\ z\in\mathbb{E}.\end{aligned}\tag{2.4}$$

Supposing that there exists a point $z_0 \in \mathbb{E}$ such that $\max\limits_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1$. Then by Lemma 2.1, we have $z_0w'(z_0) = kw(z_0), k \geq 1$.
Hence, from (2.4) we obtain

$$\begin{aligned}\left|z_0^{p+1}f'(z_0)+p\right|^{1-\gamma}\left|z_0^{p+2}f''(z_0)+(p+1)z_0^{p+1}f'(z_0)\right|^{\gamma} &= |pw(z_0)|^{1-\gamma}|pzw'(z_0)|^{\gamma}\\ &= p|k|^{\gamma}\geq p,\end{aligned}$$

which contradicts (2.1).Therefore, $|w(z)| < 1$ for all $z \in \mathbb{E}$, and the conclusion has been proved.
Finally, from (1.2) it follows that $|f'(z)| \leq 2p|z|^{-(p+1)} < 2p,\ z \in \mathbb{E}$, hence

$$|f(z)| = \left|\int_0^z f'(t)dt\right| \leq \int_0^r |f'(\rho e^{\iota\theta})|d\rho \leq 2pr < 2p,$$

$$z = re^{\iota\theta} \in \mathbb{E},\ \theta \in [0, 2\pi).$$

Consequently, f is bounded in $\mathbb{E}$. □

Setting $\gamma = 1$ in Theorem 2.2 reduces to the next result.

Corollary 2.3. *If $f \in \Sigma_p$ satisfies*

$$|z^{p+2}f''(z) + (p+1)z^{p+1}f'(z)| < p,\ z \in \mathbb{E},$$

then the inequality (1.2) holds, i.e., $f \in \mathcal{MC}_p$ and it is bounded function in $\mathbb{E}$.

Theorem 2.4. *Let $f(z) \in \Sigma_p$ and suppose that it satisfies, for $\gamma \geq 0$, the inequality*

$$\left|\frac{z^{p+1}f'(z)}{p}+1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}+p\right|^{\gamma} < \left(\frac{1}{2}\right)^{\gamma},\qquad z\in\mathbb{E}.\tag{2.5}$$

Then $|z^{p+1}f'(z) + p| < p$, i.e. $f(z) \in \mathcal{MC}_p$ and is a bounded function in $\mathbb{E}$.

Proof. For a function $f \in \Sigma_p$ satisfying the assumption (2.5), we define a function w by (2.2). Then w is analytic in $\mathbb{E}$ with $w(0) = 0$ and differentiating (2.2), we have

$$1+\frac{zf''(z)}{f'(z)}+p = \frac{zw'(z)}{w(z)-1}.\tag{2.6}$$

From the assumption (2.5), it follows that the left-hand side of (2.6) is an analytic function in $\mathbb{E}$, hence $w(z) \neq 1$ for all $z \in \mathbb{E}$. From (2.2) and (2.6) we have

$$\left|\frac{z^{p+1}f'(z)}{p}+1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}+p\right|^{\gamma} = |w(z)|^{1-\gamma}\left|\frac{zw'(z)}{w(z)-1}\right|^{\gamma},\ z\in\mathbb{E}.\tag{2.7}$$

If we suppose that there exists a point $z_0 \in \mathbb{E}$ such that $\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$.
Then by Lemma 2.1, we have $z_0 w'(z_0) = kw(z_0), k \geq 1$.
Hence, from (2.7) we have

$$\begin{aligned}\left|\frac{z_0^{p+1} f'(z_0)}{p} + 1\right|^{1-\gamma} \left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} + p\right|^{\gamma} &= |w(z_0)|^{1-\gamma} \left|\frac{z_0 w'(z_0)}{w(z_0) - 1}\right|^{\gamma} \\ &= |w(z_0)| \left|\frac{k}{w(z_0) - 1}\right|^{\gamma} \\ &\geq \left(\frac{1}{2}\right)^{\gamma},\end{aligned}$$

which contradicts (2.5). Therefore, $|w(z)| < 1$ for all $z \in \mathbb{E}$ and our conclusion (1.2) has been proved.
Since under the assumption (2.5) the inequality holds, as in the proof of the previous theorem it follows that f is bounded in $\mathbb{E}$. □

Selecting $\gamma = 1$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.5. *If $f \in \Sigma_p$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)} + p\right| < \frac{1}{2}, \; z \in \mathbb{E},$$

then $|z^{p+1} f'(z) + p| < p$, i.e. $f(z) \in \mathcal{MC}_p$ and is a bounded function in $\mathbb{E}$.

Putting p=1 in the above corollary, we have the following result.

Corollary 2.6. *If $f \in \Sigma$ satisfies*

$$\left|2 + \frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}, \; z \in \mathbb{E},$$

then $|z^2 f'(z) + 1| < 1$, i.e. $f(z) \in \mathcal{MC}$ and is a bounded function in $\mathbb{E}$.

Remark 2.7. From above corollary, we obtained the result of Goyal and Prajapat [1, Corollary 3].

Theorem 2.8. *Let $f(z) \in \Sigma_p$ and suppose that it satisfies, for $\gamma \geq 0$, the inequality*

$$\left|\frac{zf'(z)}{f(z)} + p\right|^{1-\gamma} \left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right|^{\gamma} < \frac{p-\alpha}{(p + |p - 2\alpha|)^{\gamma}}, \qquad z \in \mathbb{E}, \tag{2.8}$$

then assume that for $f(z) \neq 0$, $f(z) \in \mathcal{MS}_p^*(\alpha)$.

Proof. For a function $f \in \Sigma_p$ satisfying the assumption (2.8), we define a function w by

$$\frac{-zf'(z)}{f(z)} = \frac{p + (p - 2\alpha)w(z)}{1 - w(z)}, \; z \in \mathbb{E}, \qquad (0 \leq \alpha < p). \tag{2.9}$$

Since $w(z) = b_k z^k + \ldots$ is analytic in $\mathbb{E}$ with $w(0) = 0$ and from assumption (2.8) it follows that the left hand side of (2.9) is an analytic function in $\mathbb{E}$, hence $w(z) \neq 1$

for all $z \in \mathbb{E}$.
Differentiating (2.9), we have

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{2(p-\alpha)zw'(z)}{(p+(p-2\alpha)w(z))(1-w(z))},\ z\in\mathbb{E}. \tag{2.10}$$

From (2.9) and (2.10), we get

$$\begin{aligned}&\left|\frac{zf'(z)}{f(z)}+p\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|^{\gamma}\\ &=2(p-\alpha)\left|\frac{w(z)}{1-w(z)}\right|\left|\frac{\frac{zw'(z)}{w(z)}}{p+(p-2\alpha)w(z)}\right|^{\gamma},\ z\in\mathbb{E}.\end{aligned} \tag{2.11}$$

If we suppose that there exists a point $z_0 \in \mathbb{E}$ such that $\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1$. Then by Lemma 2.1, we have $z_0w'(z_0) = kw(z_0)$, $k \geq 1$.

$$\begin{aligned}&\left|\frac{z_0f'(z_0)}{f(z_0)}+p\right|^{1-\gamma}\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right|^{\gamma}\\ &=2(p-\alpha)\left|\frac{w(z_0)}{1-w(z_0)}\right|\left|\frac{\frac{z_0w'(z_0)}{w(z_0)}}{p+(p-2\alpha)w(z_0)}\right|^{\gamma},\ z\in\mathbb{E}, \qquad (2.12)\\ &\geq(p-\alpha)\left|\frac{k}{p+(p-2\alpha)w(z_0)}\right|^{\gamma},\end{aligned}$$

$$\left|\frac{z_0f'(z_0)}{f(z_0)}+p\right|^{1-\gamma}\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right|^{\gamma}\geq\frac{p-\alpha}{(p+|p-2\alpha|)^{\gamma}},$$

which contradicts (2.8).
This proves that $|w(z) < 1$ for all $z \in \mathbb{E}$ and hence $f(z) \in \mathcal{MS}_p^*(\alpha)$. □

Putting $p = 1$ in Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let $f \in \Sigma$ and suppose that f satisfies, for $\gamma \geq 0$, the inequality*

$$\left|\frac{zf'(z)}{f(z)}+1\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|^{\gamma}<\frac{1-\alpha}{(1+|1-2\alpha|)^{\gamma}},\ z\in\mathbb{E},$$

then $f \in \mathcal{MS}^*(\alpha)$.

If we take $\alpha = 0$ in Theorem 2.8, then we obtain the next corollary.

Corollary 2.10. *Let $f \in \Sigma_p$ and suppose that f satisfies, for $\gamma \geq 0$, the inequality*

$$\left|\frac{zf'(z)}{f(z)}+p\right|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|^{\gamma}<\frac{p}{(2p)^{\gamma}},\ z\in\mathbb{E},$$

then $f \in \mathcal{MS}_p^*$.

For $p = 1$ and $\gamma = 1$, above corollary reduces to

Corollary 2.11. *Let $f \in \Sigma$ satisfies the inequality*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{1}{2}, \ z \in \mathbb{E},$$

and $f(z) \neq 0$ for all $z \in \mathbb{E}_0$ then $f \in \mathcal{MS}^*$.

Remark 2.12. From above corollary, we obtained another result of Goyal and Prajapat [1, Corollary 7].

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Generalized q-Srivastava-Attiya operator on multivalent functions

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Abstract. In this article, we define a generalized q-integral operator on multivalent functions. It generalizes many known linear operators in Geometric Function Theory (GFT). Inclusions results, convolution properties and q-Bernardi integral preservation of the subclasses of analytic functions are discussed.

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1. Introduction

The study of q-extension of classical calculus has been point of focus for various researchers due to its applications. In Physics, q-calculus is amicably used in theories of quantum fields, Newton quantum gravity, special relativity and many other notable fields. In Mathematics, various branches has been established due to its applications in basic hypergeometric functions, combinatorics, calculus of variations, optimal control problems, q-transform analysis. It dates back to great mathematicians of 17th century L. Euler and C. Jacobi. F.H. Jackson formally defined q-difference operator and q-integral operator in [8, 9]. For comprehensive details of concepts of q-calculus, see [5]. The concepts of GFT has been studied in context of q-calculus and q-analogues of various subclasses of analytic functions are defined by the researchers, see [20, 7, 1, 14, 15, 12, 11, 4, 21, 10]. Using the convolution of normalized analytic functions, several q-operators are defined by the researchers, see details in [19]. In this paper we define a generalized q-Srivastava Attiya operator and study its application on multivalent functions.

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Let $A(p)$ $(p \in \mathbb{N} = \{0, 1, 2, ..\})$ denote the set of multivalent functions say f given as

$$f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \tag{1.1}$$

analytic in the open unit disc $E = \{z : |z| < 1\}$.

Let $f(z)$ be given by (1.1) and $g(z)$ defined as

$$f(z) = z^p + \sum_{k=2}^{\infty} b_{k+p-1} z^{k+p-1}.$$

Then Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} b_{k+p-1} z^{k+p-1}.$$

Let $f, g \in A$. Then f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, $z \in E$, if there exists a Schwartz function $\omega(z)$ analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$ such that $f(z) = g(\omega(z))$.

A subset $D \subset \mathbb{C}$ is called q-geometric if $zq \in D$ whenever $z \in D$ and it contains all the geometric sequences $\left\{zq^k\right\}_0^{\infty}$. In GFT, the q-derivative of $f(z)$ is defined as;

$$d_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \; q \in (0, 1), \quad (z \in D \setminus \{0\}), \tag{1.2}$$

and $d_q f(0) = f'(0)$. For a function $g(z) = z^k$, the q-derivative is

$$d_q g(z) = [k] z^{k-1},$$

where $[k] = \frac{1-q^k}{1-q} = 1 + q + q^2 + ... + q^{k-1}$.

From (1.1) and (1.2) we easily get that

$$d_q f(z) = [p]\, z^{p-1} + \sum_{k=2}^{\infty} [k+p-1]\; a_{k+p-1} z^{k+p-2}.$$

Let $f(z)$ and $g(z)$ be defined on a q-geometric set $D \subset \mathbb{C}$ such that q-derivatives of $f(z)$ and $g(z)$ exist $\forall\, z \in \mathbb{C}$. Then for complex numbers b, c we have:

$$d_q(bf(z) \pm cg(z)) = b d_q f(z) \pm c d_q g(z).$$

$$d_q(f(z)g(z)) = f(qz) d_q g(z) + g(z) d_q f(z).$$

$$d_q \left(\frac{f(z)}{g(z)} \right) = \frac{g(z)\; d_q f(z) - f(z)\; d_q g(z)}{g(z) g(qz)}, \; g(z)g(qz) \neq 0.$$

$$d_q \left(\log f(z)\right) = \frac{\ln q^{-1}}{1-q} \frac{d_q f(z)}{f(z)}.$$

Jackson [8] introduced the q-integral of a function f is given by

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(q^k z),$$

provided that series converges.

Consider a q-analogue of Lerch-Hurwitz function

$$\Phi_q(s,b;z)=\sum_{k=0}^{\infty}\frac{z^k}{[k+b]^s}, z\in E,$$

($b\in\mathbb{C}\setminus\mathbb{Z}_0^-$; $s\in\mathbb{C}$ when $|z|<1$; $\mathrm{Re}(s)>1$ when $|z|=1$), which is a convergent series of radius 1. Now we define the generalized q-Srivastava Attiya operator $J_{q,b}^{s,p}$: $A(p)\to A(p)$ as

$$J_{q,b}^{s,p}f(z)=\Psi_q(s,b;z)*f(z), \tag{1.3}$$

where,

$$\Psi_q(s,b;z)=[1+b]^s\left\{\Phi_q(s,b;z)-[b]^{-s}\right\}. \tag{1.4}$$

From (1.1), (1.3) and (1.4), we have

$$J_{q,b}^{s,p}f(z)=z^p+\sum_{k=2}^{\infty}\left(\frac{[1+b]}{[k+b]}\right)^s a_{k+p-1}z^{k+p-1}. \tag{1.5}$$

We observe that $J_{q,b}^{0,p}f(z)=f(z)$. The operator $J_{q,b}^{s,p}$ reduces to known linear operators for different values of parameters p,b and s as:

(i) For $p=1$, $s=1,b=0$ and $q\to 1^-$, $J_{q,b}^{s,p}$ reduces to Alexander operator [2].

(ii) If $p=1$, it is q-Srivastava Attiya operator discussed in [3].

(iii) For $p=1$, $s=1,b>0$ it reduces to q-Choi-Saigo-Srivastava operator discussed in [22].

(iv) For $s=\alpha$ ($\alpha>0$), $b=p$ and $q\to 1^-$, it is operator discussed in [16].

For any complex number s, the operator $I_{q,s}^{b,p}: A(p)\to A(p)$ is defined as;

$$I_{q,b}^{s,p}f(z)=z^p+\sum_{k=2}^{\infty}\left(\frac{[k+b]}{[1+b]}\right)^s a_{k+p-1}z^{k+p-1}. \tag{1.6}$$

The operator $I_{q,b}^{s,p}$ also reduces to known linear operators as:

(i) For $p=1$, $s\in\mathbb{N}_0$, $b=0$, it is q-Sãlãgean differential operator [6].

(ii) For $p=1$, $s=-1$ and $q\to 1^-$, it reduces to Owa-Srivastava Integral Operator [13].

The following identities holds for the two operators $J_{q,b}^{s,p}(z)$and $I_{q,b}^{s,p}(z)$,

$$zd_q(J_{q,b}^{s+1,p}f(z))=q^{p-1}\left(1+\frac{[b]}{q^b}\right)J_{q,b}^{s,p}f(z)+\left(\frac{[p-1]-[b]}{q^b}\right)J_{q,b}^{s+1,p}f(z). \tag{1.7}$$

$$zd_q(I_{q,s}^{b,p}f(z))=q^{p-1}\left(1+\frac{[b]}{q^b}\right)I_{q,s+1}^{b,p}f(z)+\left(\frac{[p-1]-[b]}{q^b}\right)I_{q,s}^{b,p}f(z). \tag{1.8}$$

Here we prove the identity (1.7) as;

$$zd_q(J_{q,b}^{s+1,p}f(z))=[p]\,z^p+\sum_{k=2}^{\infty}\left(\frac{[1+b]}{[k+b]}\right)^{s+1}a_{k+p-1}\,[k+p-1]\,z^{k+p-1},$$

or equivalently,

$$zd_q(J_{q,b}^{s+1,p}f(z)) = [p]\,z^p + \sum_{k=2}^{\infty}\left(\tfrac{[1+b]}{[k+b]}\right)^{s+1} \cdot a_{k+p-1}\left[(k+b)+(p-b-1)\right]z^{k+p-1}.$$

Using the property $[a+b] = q^b[a]+[b]$ we have:

$$zd_q(J_{q,b}^{s+1,p}f(z)) = [p]z^p + \sum_{k=2}^{\infty}\left(\frac{[1+b]}{[k+b]}\right)^{s+1} a_{k+p-1}\left\{q^{p-b-1}[k+b]+[p-b-1]\right\}z^{k+p-1}.$$

By adding and subtracting the terms $q^{p-b-1}[1+b]z^p$ and $[p-b-1]z^p$, using the property $[a+b] = q^b[a]+[b]$ and rearranging the terms: we get

$$zd_q(J_{q,b}^{s+1,p}f(z)) = q^{p-1}\left(1+\frac{[b]}{q^b}\right)J_{q,b}^{s,p}f(z) + \left(\frac{[p-1]-[b]}{q^b}\right)J_{q,b}^{s+1,p}f(z).$$

On same lines, we can prove the identity (1.8) as well.

Definition 1.1. A function $f \in A(p)$ is said to be in the class $ST_q^p(\varphi)$ if and only if

$$\frac{zd_qf(z)}{[p]\,f(z)} \prec \varphi(z); \tag{1.9}$$

where $\varphi \in \Omega$, the class of analytic and convex multivalent functions in E.

Definition 1.2. A function $f \in A(p)$ is said to be in the class $CV_q^p(\varphi)$ if and only if

$$\frac{d_q(zd_qf(z))}{[p]\,d_qf(z)} \prec \varphi(z);$$

where $\varphi \in \Omega$, the class of analytic and convex multivalent functions in E.

By using operators given by (1.5) and (1.6), we define the classes

$$ST_{q,b}^{s,p}(\varphi) = \left\{f \in A(p) : J_{q,b}^{s,p}f(z) \in ST_q^p(\varphi)\right\}$$

and

$$\widetilde{ST}_{q,s}^{b,p}(\varphi) = \left\{f \in A(p) : I_{q,s}^{b,p}f(z) \in ST_q^p(\varphi)\right\}.$$

Similarly

$$CV_{q,b}^{s,p}(\varphi) = \left\{f \in A(p) : J_{q,b}^{s,p}f(z) \in CV_q^p(\varphi)\right\}$$

and

$$\widetilde{CV}_{q,s}^{b,p}(\varphi) = \left\{f \in A(p) : I_{q,s}^{b,p}f(z) \in CV_q^p(\varphi)\right\}.$$

It is noted

$$f \in CV_{q,b}^{s,p}(\varphi) \Leftrightarrow \frac{zd_qf(z)}{[p]} \in ST_{q,b}^{s,p}(\varphi)$$

and

$$f \in \widetilde{CV}_{q,s}^{b,p}(\varphi) \Leftrightarrow \frac{zd_qf(z)}{[p]} \in \widetilde{ST}_{q,s}^{b,p}(\varphi).$$

We need the following Lemma to obtain our results.

Lemma 1.3. [17] *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z)$ be a regular in E with $h(0) = 1$ and* $\mathrm{Re}[\beta h(z) + \gamma] > 0$. *If $p(z) = 1 + p_1 z + p_2 z^2 + ...$is analytic in E, then $p(z) + \frac{z d_q p(z)}{\beta p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z)$.*

2. Main results

2.1. Inclusions results

In this section, we proved the inclusions results of the classes with respect to parameter s.

Theorem 2.1. *Let $\varphi(z)$ be analytic and convex multivalent function with $\varphi(0) = 1$ and* $\mathrm{Re}(\varphi(z)) > 0$ *for $z \in E$. Then $ST_{q,b}^{s,p}(\varphi) \subset ST_{q,b}^{s+1,p}(\varphi)$ if* $\mathrm{Re}(q^{b-p+1}) > 1$.

Proof. Let $f \in ST_{q,b}^{s,p}(\varphi)$ and we set

$$\frac{z d_q (J_{q,b}^{s+1,p} f(z))}{[p] J_{q,b}^{s+1,p} f(z)} = Q(z), \ z \in E, \tag{2.1}$$

where $Q(z)$ is analytic in E with $Q(0) = 1$.

Using identity (1.7), we get

$$\frac{z d_q (J_{q,b}^{s+1,p} f(z))}{J_{q,b}^{s+1,p} f(z)} = M_q \frac{J_{q,b}^{s,p} f(z)}{J_{q,b}^{s+1,p} f(z)} - \gamma_q, \tag{2.2}$$

where $M_q = q^{p-1}\left(1 + \frac{[b]}{q^b}\right)$ and $\gamma_q = \frac{[p-1]-[b]}{q^b}$.

From (2.1) and (2.2), we have

$$[p]Q(z) + \gamma_q = M_q \frac{J_{q,b}^{s,p} f(z)}{J_{q,b}^{s+1,p} f(z)}.$$

Applying logarithmic q-differentiation,

$$\frac{z d_q (J_{q,b}^{s,p} f(z))}{J_{q,b}^{s,p} f(z)} = [p]Q(z) + \frac{[p] z d_q Q(z)}{Q(z) + \gamma_q}. \tag{2.3}$$

As $f \in ST_{q,b}^{s,p}(\varphi)$ so,

$$\frac{z d_q (J_{q,b}^{s,p} f(z))}{J_{q,b}^{s,p} f(z)} \prec \varphi(z). \tag{2.4}$$

From (2.3) and (2.4), we have

$$[p]Q(z) + \frac{[p] z d_q Q(z)}{Q(z) + \gamma_q} \prec \varphi(z).$$

As $\mathrm{Re}(q^{b-p+1}) > 1$ and $\mathrm{Re}(\varphi) > 0$ then by Lemma 1.3, we have $Q(z) \prec \varphi(z)$ which implies $f \in ST_{q,b}^{s+1,p}(\varphi)$. So $ST_{q,b}^{s,p}(\varphi) \subset ST_{q,b}^{s+1,p}(\varphi)$. □

Theorem 2.2. *Let $\varphi(z)$ be same as in Theorem 2.1. Then $CV_{q,b}^{s,p}(\varphi) \subset CV_{q,b}^{s+1,p}(\varphi)$ if* $\mathrm{Re}(q^{b-p+1}) > 1$.

Proof. It is evident from the fact $f \in CV_{q,b}^{s,p}(\varphi) \Leftrightarrow \frac{zd_q f(z)}{[p]} \in ST_{q,b}^{s,p}(\varphi)$. ☐

We can easily prove the following result by using Lemma 1.3 and identity relation (1.8).

Theorem 2.3. *Let* $\varphi(z)$ *be analytic and convex multivalent function with* $\varphi(0) = 1$ *and* $\operatorname{Re}(\varphi(z)) > 0$ *for* $z \in E$. *Then*

$$\widetilde{ST}_{q,s}^{b,p}(\varphi) \subset \widetilde{ST}_{q,s+1}^{b,p}(\varphi)$$

and

$$\widetilde{CV}_{q,s+1}^{b,p}(\varphi) \subset \widetilde{CV}_{q,s}^{b,p}(\varphi).$$

2.2. Integral preservation under generalized q-Bernardi integral operator

In [18], the q-Bernardi integral operator $L_{c,p}f(z)$ for multivalent functions is defined as:

$$L_{c,p}f(z) = \frac{[p+c]}{z^c} \int_0^z t^{c-1} f(t) d_q t, \tag{2.5}$$

where $f \in A(p)$ given by (1.1) with $c > -p$.

Theorem 2.4. *If* $f \in ST_{q,b}^{s,p}(\varphi)$ *then* $L_{c,p}f(z) \in ST_{q,b}^{s,p}(\varphi)$.

Proof. Let $f(z) \in ST_{q,b}^{s,p}(\varphi)$. Consider

$$\frac{zd_q(J_{q,b}^{s,p}(L_{c,p}f(z)))}{[p]\, J_{q,b}^{s,p}(L_{c,p}f(z))} = Q(z), \tag{2.6}$$

where $Q(z)$ is analytic in E with $Q(0) = 1$. From (2.5), after some calculations, we can write

$$zd_q(L_{b,p}f(z)) = [p+c]\, f(z) - [c]\, L_{c,p}f(z).$$

Now applying the operator $J_{q,b}^{s,p}$ on both sides, we have

$$\frac{zd_q(J_{q,b}^{s,p}(L_{c,p}f(z)))}{J_{q,b}^{s,p}(L_{c,p}f(z))} = [p+c]\, \frac{J_{q,b}^{s,p}f(z)}{J_{q,b}^{s,p}(L_{c,p}f(z))} - [c]. \tag{2.7}$$

Now applying logarithmic q-differentiation on both sides of (2.7), after some calculations and using (2.6), we get

$$\frac{zd_q(J_{q,b}^{s,p}f(z))}{[p]\, J_{q,b}^{s,p}f(z)} = Q(z) + \frac{z[d_q Q(z)]}{[p]\, Q(z) + [c]}. \tag{2.8}$$

As $f \in ST_{q,b}^{s,p}(\varphi)$, so from (2.7) and (2.8), we have

$$Q(z) + \frac{z[d_q Q(z)]}{[p]\, Q(z) + [b]} \prec \varphi(z).$$

As $\operatorname{Re}([p]\varphi(z) + [c]) > 0$ so by Lemma 1.3, we have $Q(z) \prec \varphi(z)$ which implies $L_{c,p}f(z) \in ST_{q,b}^{s,p}(\varphi)$. ☐

Theorem 2.5. *If* $f(z) \in CV_{q,b}^{s,p}(\varphi)$ *then* $L_{c,p}f(z) \in CV_{q,b}^{s,p}(\varphi)$.

Proof. It is immediate consequence of the fact $f \in CV_{q,b}^{s,p}(\varphi) \Leftrightarrow \frac{zd_q f(z)}{[p]} \in ST_{q,b}^{s,p}(\varphi)$. ☐

2.3. Convolution property of $ST_{q,b}^{s,p}(\varphi)$

We now obtain convolution property for the class $ST_{q,b}^{s,p}(\varphi)$.

Theorem 2.6. *Let $f \in ST_{q,b}^{s,p}(\varphi)$. Then*

$$f(z) = z^{[p]}.\exp\left(\frac{\ln q^{-1}}{1-q}[p]\int_0^z \frac{\varphi(\omega(\varsigma))-1}{\varsigma}d_q\varsigma\right) * \left(z^p + \sum_{k=2}^{\infty}\left(\frac{[k+b]}{[1+b]}\right)^s a_{k+p-1}z^{k+p-1}\right), \quad (2.9)$$

where ω is Schwartz function.

Proof. Suppose that $f \in ST_{q,b}^{s,p}(\varphi)$. The subordination condition (1.9) can be written as:

$$\frac{zd_q(J_{q,b}^{s,p}f(z))}{J_{q,b}^{s,p}f(z)} = [p]\,\varphi(\omega(z)), \quad (2.10)$$

where ω is Schwartz function.

From (2.10), after q-integrating we get

$$\log\left(\frac{J_{q,b}^{s,p}f(z)}{z^{[p]}}\right) = \frac{\ln q^{-1}}{1-q}[p]\int_0^z \frac{\varphi(\omega(\varsigma))-1}{\varsigma}d_q\varsigma. \quad (2.11)$$

It follows from (2.11) that

$$J_{q,b}^{s,p}f(z) = z^{[p]}.\exp\left(\frac{\ln q^{-1}}{1-q}[p]\int_0^z \frac{\varphi(\omega(\varsigma))-1}{\varsigma}d_q\varsigma\right). \quad (2.12)$$

The assertion can be obtained easily from (1.5) and (2.12). □

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A dual mapping associated to a closed convex set and some subdifferential properties

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Abstract. In this paper we establish some properties of the multivalued mapping $(x,d) \rightrightarrows D_C(x;d)$ that associates to every element x of a linear normed space X the set of linear continuous functionals of norm $d \geq 0$ and which separates the closed ball $B(x;d)$ from a closed convex set $C \subset X$. Using this mapping we give links with other important concepts in convex analysis (ε-approximation element, ε-subdifferential of distance function, duality mapping, polar cone). Thus, we establish a dual characterization of ε-approximation elements with respect to a nonvoid closed convex set as a generalization of a known result of Garkavi. Also, we give some properties of univocity and monotonicity of mapping D_C.

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1. Introduction and preliminaries

Let C be a nonvoid closed convex set in a real linear normed space X and a closed ball $B(x;d)$, $d > 0$ such that $C \cap int\, B(x,d) = \emptyset$. It is well known that those two sets can be separated by closed hyperplanes (see, for instance, [1],[2]).

We denote by

$$d_C(x) = \inf_{u \in C} \|x - u\|, \; x \in X, \tag{1.1}$$

the *distance function* to a set $C \subset X$. Also, let us denote by X^* the *dual* space of X.

In the special case $d = d_C(x)$, $x \notin C$, using separating hyperplane, Garkavi [4] has obtained a well known dual characterization of best approximation elements of $x \in X$ in C.

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We recall that an element $z \in C$ is a *ε-approximation of* x in C if

$$\|x - z\| \leq \|x - u\| + \varepsilon, \text{ for all } u \in C. \tag{1.2}$$

Therefore using, the distance of x to the set C the property (1.2) is equivalent to

$$\|x - z\| \leq d_C(x) + \varepsilon.$$

Obviously, here it is necessary that $\varepsilon \geq 0$.

If $\varepsilon = 0$, then z is a *best approximation of* x in C, that is $\|x - z\| = d_C(x)$ and $z \in C$.

If $\varepsilon > 0$ then the set of ε-approximations of $x \in C$ is always nonvoid, but the set of the best approximations may be void.

Using separating hyperplanes, Garkavi [4] established a well known dual characterization of best approximation elements as follows.

Theorem 1.1. ([4]) An element $z \in C$ is a best approximation element of $x \in X \in C$ if and only if there exists an element $x_0^* \in X^*$ such that

i) $\|x_0^*\| = \|x - z\|$;

ii) $x_0^*(x - u) \geq \|x - z\|^2$ for all $u \in C$.

Here, the property ii) is equivalent with the following two properties:

i') $x_0^*(x - z) = \|x - z\|^2$;

ii') $x_0^*(z) = \sup\{x_0^*(u); u \in C\}$.

Obviously, if $x \in C$, and z is a best approximation element, then $x = z$, and so we take $x_0^* = 0$. Now, if $x \notin C$, then $d_C(x) > 0$ and we consider a closed separating hyperplane (x_0^*, α) for the sets C and $B(x, d_C(x))$ such that $\|x_0^*\| = \|x - z\|$. Conversely, if $z \in C$ has the property i) and ii) it follows that

$$\|x - u\| \|x_0^*\| \geq x_0^*(x - u) \geq \|x - z\|^2,$$

for all $u \in C$ which prove that z is a best approximation element in C for $x \in X$.

Let us denote by $P_C(x)$ the set of all best approximations of x in C. The (multivalued) mapping $x \rightrightarrows P_C(x)$ $x \in X$ is called the *metric projection* associated to the set C. Clearly, $P_C(x) = x$ for any $x \in C$. Also, we can have $P_C(x) = \emptyset$ for certain elements in X. If $P_C(x) \neq \emptyset$ for any $x \in X$ then the set C is called *proximinal* and if $P_C(x) = \emptyset$ for any $x \in X \backslash C$, the set C is called *antiproximinal.* It is well known that in a reflexive space any closed convex set is proximinal.

Given a convex real extended function $f : X \to \overline{\mathbb{R}}$, its *$\varepsilon$-subdifferential* is defined by

$$\partial_\varepsilon f(x) = \{x^* \in X^*; x^*(x - u) \geq f(x) - f(u) - \varepsilon, \text{ for all } u \in X\}, \ x \in X \tag{1.3}$$

where $\overline{\mathbb{R}} = [-\infty, +\infty]$.

Here, we suppose that f is a *proper function*, that is $f(u) > -\infty$ for all $u \in X$ and there exist elements $\overline{x} \in X$ such that $f(\overline{x}) < \infty$. If $\varepsilon = 0$ we obtain the *subdifferential* of function f in x, denoted by $\partial f(x)$.

The multivalued operator $x \rightrightarrows \partial_\varepsilon f(x)$, $x \in X$, has the following ε-monotonicity property

$$(x_1^* - x_2^*)(x_1 - x_2) \geq -2\varepsilon \text{ for all } x_1^* \in \partial_\varepsilon f(x_1), \ x_2^* \in \partial_\varepsilon f(x_2). \tag{1.4}$$

Generally, a multivalued operator $A : X \rightrightarrows X^*$ which has the property of ε-monotonicity of type (1.4) is called *ε-monotone.* Some properties of those mappings were given in [13]. This type of monotonicity is different of ε-monotonicity defined in [9].

Also, we recall the definition of *duality mapping* $J : X \Rightarrow X^*$,

$$J(x) = \left\{x^* \in X^*; x^*(x) = \|x^*\|^2 = \|x\|^2\right\}, \; x \in X. \tag{1.5}$$

It is well known that J is the subdifferential of the function $x \mapsto \frac{1}{2}\|x\|^2$, $x \in X$ (see, for instance, [1], [2]).

If A is a subset of X, we denote by A^o the *polar set* of $A \subset X$, that is

$$A^o = \{x^* \in X^*; x^*(a) \leq 1 \text{ for all } a \in A\}. \tag{1.6}$$

In this paper we intend to analyze some properties of the (multivalued) mapping $(x, d) \rightrightarrows D_C(x; d)$, $x \in X$, $C \subset X$, $d \geq 0$, where

$$D_C(x; d) = \{x^* \in X^*; x^*(v) \geq x^*(u), \|x^*\| = d, \forall v \in B(x; d), \forall u \in C\}. \tag{1.7}$$

Remark 1.2. Obviously, $D_C(x; 0) = \{0\}$, for all $x \in X$ and $D_C(x; d) = \emptyset$ whenever $d > d_C(x)$.

Geometrically, for each $x \in X$ and $d > 0$, $D_C(x; d)$ coincides with the set of all linear continuous functionals $x^* \in X^*$ such that $\|x^*\| = d$ and for which $x^*(y) = k$, $y \in X$, is a separating hyperplane for the sets C and $B(x; d)$ for a certain $k \in \mathbb{R}$.

Equivalently,

$$D_C(x; d) = \left\{x^* \in X^*; \|x^*\| = d, \; x^*(x - u) \geq d^2, \forall u \in C\right\}. \tag{1.7'}$$

In the special case $d = d_C(x)$ we denote

$$D_C(x) = D_C(x; d_C(x)), \; x \in X. \tag{1.7''}$$

We establish a dual characterization of real number d such that $0 \leq d \leq d_C(x)$ (Theorem 2.1). Consequently, if $x \notin C$, we obtain the basic properties of elements in $D_C(x; d)$. Using this multivalued mapping naturally generated by the geometric problem of separation of a nonvoid closed convex set and a closed ball we give connections with some important concepts and properties of convex analysis (ε-subdifferentials of distance function, ε-approximation elements, duality mapping, polar cone). For example,

$x^* \in D_C(x; d)$ if and only if $\frac{1}{\|x^*\|}x^* \in \partial_\varepsilon d_C(x) \cap Bd\, B^*(0; 1)$ for $\varepsilon = d_C(x) - d$, where $0 < d \leq d_C(x)$. Generally, by BdA we denote the *boundary* of a set $A \subset X$. Also, we denote by $B^*(x_0^*; d)$, $x_0^* \in X^*$, $d \geq 0$, the closed balls in X^*.

Consequently, we give an explicit formula for $\partial_\varepsilon d_C(x)$ in the case $x \notin C$, but $\varepsilon > d_C(x)$ (Theorem 2.5, ii). The special case $d = d_C(x)$ was considered by Ioffe in [8]. A detailed study of subdifferential of distance function was given by Penot, Ratsimahalo in [10] (see also [3] and [6] if $P(x) \neq \emptyset$). In [5] Hiriart-Urruty (see, also, [6]) has obtained formula for the ε-subdifferential of a marginal function. Particularly, one can be obtained formulas for ε-subdifferential of distance function which is considered either as a marginal function, or as the convolution of the norm and the indicator function of the set C. But, by Theorem 2.5, we establish some explicit properties of

$\partial_\varepsilon d_C(x)$. We remark that we have a special situation if $\varepsilon = d_C(x)$. The assertion iii) in Theorem 2.5 is similar to the one shown in [10] for the subdifferential distance function. We also establish a property of univocity of D_C.

Following Jofre, Luc and Thera ([9]), we define a new type of ε-monotonicity by (3.1), in according with D_C (Theorem 3.1). Some monotonicity properties of D_C are given in Section 3.

2. A dual mapping associated to a closed convex set and an arbitrary positive number

Now, we give a dual characterization of the numbers d such that $d_C(x) \geq d \geq 0$.

Theorem 2.1. Let C be a nonvoid closed convex set in a linear normed space X. If $x \in X$ is a fixed element then $d_C(x) \geq d \geq 0$ if and only if there exists $x^* \in X^*$ such that

i) $\|x^*\| = d$;
ii) $x^*(x-u) \geq d^2$, for all $u \in C$.

Proof. If $d = 0$ then i) and ii) are obviously fulfilled taking $x^* = 0$ and conversely. Hence we can suppose that $d > 0$.

Now, if $0 < d \leq d_C(x)$ it follows that $B(x;d)$ has nonvoid interior set and $C \cap int\, B(x;d) = \emptyset$. Thus, using a separation theorem for sets C and $B(x;d)$ (see, for instance, [1] or [2]), there exists a non null element $y^* \in X^*$ such that $y^*(v) \geq y^*(u)$ for all $u \in C$ and $v \in B(x;d)$. Taking $x_0^* = d\|y^*\|^{-1} y^*$ it follows that

$$x_0^*(x - dz) \geq x_0^*(u)$$

for any $z \in B(0;1)$ and $u \in C$, and so $x_0^*(x-u) \geq d\|x_0^*\|$ for all $u \in C$. Obviously, $\|x_0^*\| = d$. Therefore, the properties i) and ii) are fulfilled.

Conversely, if i) and ii) hold, then

$$d^2 \leq x_0^*(x-u) \leq \|x_0^*\|\,\|x-u\| \leq d\,\|x-u\|,$$

for all $u \in C$, and so $d \leq d_C(x)$. □

From the proof of Theorem 2.1 in the case $0 < d \leq d_C(x)$ (and, so, $x \notin C$), we see that every x^* which verifies i) and ii) is in $D_C(x;d)$.

Remark 2.2. Given an element $x \in X$, taking $d = \|x - z\|$, where $z \in C$, by Theorem 2.1 it results that $\|x - z\| \leq d_C(x)$ if and only if the properties i) and ii) in Theorem 2.1 are fulfilled. But it is clear that $\|x - z\| \leq d_C(x)$ and $z \in C$ if and only if z is the best approximation of x in C. Therefore, Theorem 2.1 is a slight extension of a famous characterization established by Garkavi [4] concerning the best approximation elements.

Corollary 2.3. Let X be a linear normed space, C a nonvoid closed convex set of X and $\varepsilon \geq 0$. Then $z_\varepsilon \in C$ is an ε-approximation element for $x \notin C$, $\varepsilon < d_C(x)$, if and only if there exists $x^* \in X^*$ such that the properties i) and ii) in Theorem 2.1 are fulfilled for $d = \|x - z_\varepsilon\| - \varepsilon$.

Proof. According to (1.1) and (1.2) $z_\varepsilon \in C$ is an ε-approximation element for $x \in X$ if and only if $\|x - z_\varepsilon\| - \varepsilon \leq d_C(x)$. Therefore it is sufficient to apply Theorem 2.1 taking $d = \|x - z_\varepsilon\| - \varepsilon$. □

Now, we intent to characterize $x^* \in D_C(x; d)$ using the set $\partial_\varepsilon d_C(x)$, where $\varepsilon = d_C(x) - d \geq 0$, whenever $x \notin C$.

Proposition 2.4. If $x^* \in \partial_\varepsilon d_C(x)$ and $\varepsilon > 0$ then:

$$\|x^*\| \leq 1; \tag{2.1}$$

$$\|x^*\| \geq 1 - \frac{\varepsilon}{d_C(x)}, \text{ for all } x \notin C. \tag{2.2}$$

whenever C is a nonvoid closed convex set in X.

Proof. If $x^* \in \partial_\varepsilon d_C(x)$ then $x^*(x - y) \geq d_C(x) - d_C(y) - \varepsilon$ for any $y \in X$. Taking $y = x + tz$, $t > 0$ and $z \in X$ it follows that

$$tx^*(z) + d_C(x) - \varepsilon \leq d_C(x + tz) \leq \|x + tz - \overline{u}\| \leq t\|z\| + \|x - \overline{u}\|,$$

for a given $\overline{u} \in C$. Therefore, $x^*(z) - \|z\| \leq \frac{1}{t}(\|x - \overline{u}\| - d_C(x))$, for any $t > 0$ and $z \in X$, and so, for $t \to \infty$ we obtain that $x^*(z) \leq \|z\|, z \in X$.

If $x \in C$ and $x^* \in \partial_\varepsilon d_C(x)$ we take $y = x + tz$, $z \in X$, $t < 0$ in inequality

$$x^*(x - y) \geq d_C(x) - d_C(y) - \varepsilon$$

and we obtain $x^*(x - y) \geq d_C(x) - \varepsilon$, so $\|x^*\| \, \|x - y\| \geq d_C(x) - \varepsilon$, equivalently

$$\|x^*\| d_C(x) \geq d_C(x) - \varepsilon.$$

Therefore, if $x \notin C$ then $d_c(x) > 0$. Thus, we obtain the inequality (2.2). □

We recall that if X is a linear normed space, the *conic polar* A^+ of a set $A \subset X$ is defined by

$$A^+ = \{x^* \in X^*; x^*(a) \geq 0 \text{ for all } a \in A\}.$$

If A is a cone, then $A^+ = -A^0$.

In the next result we establish some special properties of ε-subdifferential distance function.

Theorem 2.5. Suppose that X is a real normed space, $x \in X$ and $C \subset X$ is a nonvoid closed convex set.

i) If $x \notin C$, $0 < d \leq d_C(x)$ and $\varepsilon = d_C(x) - d$, then

$$\partial_\varepsilon d_C(x) \cap Bd\ B^*(0; 1) = \frac{1}{d} D_C(x; d);$$

ii) If $x \notin C$ and $\varepsilon > d_C(x)$ then

$$\partial_\varepsilon d_C(x) = (\varepsilon - d_C(x))(C - x)^o \cap B^*(0; 1);$$

iii) If $x \notin C$, and $\varepsilon = d_C(x)$ then $\partial_\varepsilon d_C(x) = (x - C)^+ \cap B^*(0; 1)$.

iv) If $x \in C$ then $\partial_\varepsilon d_C(x) = \varepsilon (C - x)^o \cap B^*(0; 1)$ for every $\varepsilon > 0$.

Proof. i) Using (1.3) it follows that $z^* \in \partial_\varepsilon d_C(x)$ if and only if

$$z^*(x-y) \geq d - d_C(y) \text{ for all } y \in X.$$

If $y \in C$ then $z^*(x-y) \geq d$, which implies $z^* \in \frac{1}{d} D_C(x;d)$ whenever $\|z^*\| = 1$.

Conversely, suppose $z^* \in \frac{1}{d} D_C(x;d)$. Then $\|z^*\| = 1$ and $z^*(x-y) \geq d$ for any $y \in C$, so $z^*(x-y) \geq d - d_C(y)$ for all $y \in C$.

Now, consider $y \in X \backslash C$ and some $u \in C$.

Then $z^*(x-y) = z^*(x-u) - z^*(y-u) \geq d - z^*(y-u)$.

But $z^*(y-u) \leq \|z^*\| \|y-u\| = \|y-u\|$. So, $z^*(x-y) \geq d - \|y-u\|$, for any $u \in C$. Passing to the $\sup_{u \in C}$ in this inequality we obtain $z^* \in \partial_\varepsilon d_C(x) \cap B^*(0;1)$.

ii) If $\varepsilon > d_C(x)$ denote $\eta = \varepsilon - d_C(x) > 0$. Let x^* be an element of $\partial_\varepsilon d_C(x)$. Then $x^*(x-y) \geq -\eta - d_C(y)$, for all $y \in X$. Taking $y \in C$ it results $x^*(y-x) \leq \eta$ for any $y \in C$, that is $x^* \in \left(\frac{C-x}{\eta}\right)^o \cap B^*(0;1) = \eta(C-x)^o \cap B^*(0;1)$ according to (2.1).

Now, if $x^* \in \eta(C-x)^o \cap B^*(0;1)$ then $x^*(u-x) \leq \varepsilon - d_C(x)$ for all $u \in C$. If $y \notin C$ then

$$\begin{aligned} x^*(x-y) &= x^*(x-u) + x^*(u-y) \geq d_C(x) - \varepsilon + x^*(u-y) \\ &\geq d_C(x) - \varepsilon - \|x^*\| \|u-y\| \geq d_C(x) - \varepsilon - \|u-y\| \end{aligned}$$

for all $u \in C$.

Using (1.1) it follows that $x^* \in \partial_\varepsilon d_C(x)$.

iii) Let x^* be an element in $\partial_\varepsilon d_C(x)$. Taking $\varepsilon = d_C(x)$ in the definition of ε-subdifferential of d_C and arbitrary $y \in C$ one obtains $x^*(y-x) \leq 0$, so $x^* \in (x-C)^+$. Now, using (2.1), the conclusion follows.

iv) Let $y \in X$ be arbitrary and $x \in C$. If $\varepsilon > 0$ and $x^* \in \partial_\varepsilon d_C(x)$ then $x^*(x-y) \geq -d_C(y) - \varepsilon$, so $x^*(y-x) \leq \varepsilon$, whenever $y \in C$. Hence $x^* \in \varepsilon(C-x)^o$.

Also, from (2.1) we have $\|x^*\| \leq 1$.

Conversely, for $x^* \in \varepsilon(C-x)^o \cap B(0;1)$ and $y \in X$ we have $x^*(y-u) \leq \|y-u\|$ for all $u \in C$. We deduce

$$x^*(x-y) = x^*(x-u) + x^*(u-y) \geq -\varepsilon - \|y-u\|.$$

Passing to the infimum for $u \in C$ it results $x^*(x-y) \geq -\varepsilon - d_C(y)$ for all $y \in X$ as claimed. □

Corollary 2.6. Let X be a linear normed space. Then:

i) $\frac{1}{d} D_{\{0\}}(x;d) = \partial_\varepsilon \|\cdot\|(x) \cap Bd\, B(0;1)$ where $\varepsilon = \|x\| - d > 0$, $d > 0$;

ii) $D_{\{0\}}(x;\|x\|) = J(x)$.

Proof. i) Observe that $d_C(x) = \|x\|$ if $C = \{0\}$. Now, we apply Theorem 2.5, i).

ii) Consider $x^* \in D_{\{0\}}(x;\|x\|)$, that is $\|x^*\| = \|x\|$ and $x^*(x) \geq \|x\|^2$. But $x^*(x) \leq \|x\|^2$ and so $x^*(x) = \|x\|^2$. According to (1.5) we obtain that $x^* \in J(x)$. □

Remark 2.7. The assertion iii) of Theorem 2.5 has obtained by Hiriart-Urruty in [5] (see also, [6], [7]). The special case $\varepsilon = 0$ was studied by Penot and Ratsimahalo [10].

Remark 2.8. Theorem 2.5, i), can be reformulated as

$$\frac{1}{d} D_C(x;d) = \partial_\lambda (d \cdot d_C(x)) \cap Bd\, B(0;1),$$

where $\lambda = d(d_C(x) - d)$, $0 < d \le d_C(x)$.

We recall that X is a *smooth space* (see [1], [3]) if there is exactly one supporting hyperplane through each boundary point of closed unit ball.

Generally, closed convex set $A \subset X$ is called *smooth at a point* x_0 if there exists only one closed hyperplane which separates x_0 at A. Obviously, it is necessary that $x_0 \in Bd\, A$.

Theorem 2.9. Let C be a nonvoid closed convex set in X and a fixed element $x \in X$. Then, for any $d \in [0, d_C(x)]$ we have:

i) $D_C(x;d) = \{0\}$ if and only if $d = 0$;

ii) $Dom\ D_C = (X \times \{0\}) \cup \{(x,d); x \notin C, d \in (0; d_C(x)]\}$;

iii) If $D_C(x;d)$ is a singleton then $d = 0$ or $d = d_C(x)$.

iv) $D_C(x; d_C(x))$ is a singleton if and only if the set $C - B(x; d_C(x))$ is smooth at origin.

Proof. The properties i), ii) are obvious.

Also, in the sequel we can suppose that $x \notin C$, and so $d_C(x) > 0$.

Now, we prove properties (iii) and (iv): if $d = 0$ then $D_C(x;0) = \{0\}$ is a singleton. Let us consider an arbitrary element $x \notin C$ and $d \in (0, d_c(x)]$. But, if $d < d_C(x)$ then C and $B(x;d)$ are strongly separated, that is there exists many parallel separating hyperplanes (see, for example, [1], Remark 1.46). Therefore, $D_C(x;d)$ is not a singleton. If $d = d_C(x)$ there exists a unique hyperplane which separates C and $B(x; d_C(x))$ if and only if there exists a unique hyperplane which separates the origin and $C - B(x; d_C(x))$, that is $C - B(x; d_C(x))$ is smooth at the origin. □

Remark 2.10. In the spacial case when $P_C(x) \neq \emptyset$, the property iii) was established by Garkavi ([4]).

Now, if $P_C(x) \neq \emptyset$, we have

$$D_C(x) = \left\{x^* \in X^*; \|x^*\| = \|x - z\|,\ x^*(x-u) \ge \|x - z\|^2\ \ \forall u \in C\right\}, \quad z \in P_C(x) \tag{2.3}$$

since $d_C(x) = \|x - z\|$ for any $z \in P_C(x)$.

In the sequel we prove that the mapping D_C can be equivalently defined using a min-max property. Since $B^*(x;d)$ is a convex w*-compact set in X^* and the function $F_x(x^*, u) = x^*(x-u)$, $(u, x^*) \in X \times X^*$ is convex-concave, using a min-max result (see, for instance, [1], [11] and [12]), it implies the following equality:

$$\max_{x^* \in B^*(0;d)} \inf_{u \in C} x^*(x-u) = \inf_{u \in C} \max_{x^* \in B^*(0;d)} x^*(x-u) \ \text{ for all } x \in X,\ d > 0. \tag{2.4}$$

Here, by "max", we mean that "sup" is attained. The elements $x_0^* \in B^*(0; d)$, where "*max*" is attained in the left hand of (2.4) and make valid the equality (2.4) are called *the solutions of the max-inf problem (2.4).*

Proposition 2.11. Given an element $x \in X$ and a nonvoid convex, closed set $C \subset X$, then $x^* \in D_C(x)$ if and only if x^* is a solution of max-inf problem (2.4), where $d = d_C(x)$, that is

$$D_C(x) = \left\{ x^* \in B^*(0; d_C(x)); \inf_{u \in C} x^*(x - u) = d_C^2(x) \right\}. \tag{2.5}$$

Proof. We remark that the saddle value of (2.4) is equal to $d_C(x)\, d$. Consequently, for $d = d_C(x)$, the properties i), ii) in Theorem 2.1 are equivalent to the assertion that x^* is a solution of max-inf problem (2.4). □

Remark 2.12. If in the equality (2.4) "inf" is also attained, these elements of C are even the best approximation elements of x in C. Therefore, if $P_C(x) \neq \emptyset$ and $d = d_C(x)$, then the set of all saddle elements of max-min problem associated to (2.4) is $D_C(x) \times P_C(x)$.

Now, if we return to the dual characterization of the best approximation elements, we observe that in the special case $P_C(x) \neq \emptyset$, we have a conection with the duality map J. Firstly, we remark that if we put in equality (1.7) $d = \|x - z\|$ it results that $D_C(X)$ is exactly the set of all $x^* \in X^*$ with the properties of Garkavi Theorem 1.1. But, the properties i) and i′) in Theorem 1.1 prove that $x_0^* \in J(z - x)$. Also, ii′) say that $x^* \in (x - C)^*$. Consequently we have the following equality

$$D_C(x) = J(x - z) \cap (C - x)^+ \text{ whenever } z \in P_C(x) \text{ and } x \in X.$$

3. Properties of monotonicity

It is well known the relationship between the subdifferentials of convex functions and their property of monotonicity ([9]). Also, the ε-subdifferentials are ε-monotone in the sense of definition (1.4) and they have some good properties (see, for e.g., [13]).

Because the multivalued mapping $x \rightrightarrows D_C(x; d)$ is expressed using the ε-subdifferential of $d_C(\cdot)$ (Theorem 2.5, i)), it is expected to have an ε-monotonicity property.

Now, we establish two special monotonicity properties of D_C.

Theorem 3.1. The mapping $(x, d) \rightrightarrows D_C(x; d)$ is monotone in the following sense:

$\forall x_i \in X \backslash C$, $0 < d_i \leq d_C(x_i)$, $\varepsilon_i = d_C(x_i) - d_i$ and $\forall x_i^* \in D_C(x_i; d_i)$, $i = 1, 2$, then

$$(x_1^* - x_2^*)(x_1 - x_2) \geq -\varepsilon_2 d_1 - \varepsilon_1 d_2. \tag{3.1}$$

Proof. Let us consider $x_i^* \in D_C(x_i, d_i)$, $i = 1, 2$. By property ii) in Theorem 2.1 and the definition of D_C we have $(x_i^*, x_i - u_i) \geq d_i^2$ for any $u_i \in C$, $i = 1, 2$. Therefore it follows that

$$\begin{aligned}(x_1^* - x_2^*)(x_1 - x_2) &= x_1^*(x_1 - u_1) + x_2^*(x_2 - u_2) - x_1^*(x_2 - u_1) - x_2^*(x_1 - u_2) \\ &\geq d_1^2 + d_2^2 - x_1^*(x_2 - u_1) - x_2^*(x_1 - u_2) \\ &\geq d_1^2 + d_2^2 - d_1 \|x_2 - u_1\| - d_2 \|x_1 - u_2\|.\end{aligned}$$

Since u_1, u_2 are arbitrary elements in C we get

$$(x_1^* - x_2^*)(x_1 - x_2) \geq d_1^2 + d_2^2 - d_1 d_C(x_2) - d_2 d_C(x_1) = -d_1\varepsilon_2 - d_2\varepsilon_1,$$

as claimed. □

Also, the mapping D_C has a property of monotonicity with respect to corresponding best approximation elements.

Proposition 3.2. If $x_i^* \in D_C(x_i; d_i)$ and $z_i \in P_C(x_i)$, $i = 1, 2$, then

$$(x_1^* - x_2^*)(z_1 - z_2) \geq 0.$$

Proof. Taking $u_1 = z_2$ and $u_2 = z_1$ in Theorem 2.1, we have

$$\begin{aligned}(x_1^* - x_2^*)(z_1 - z_2) &= x_1^*(x_1 - z_2) + x_2^*(x_2 - z_1) - x_1^*(x_1 - z_1) - x_2^*(x_2 - z_2) \\ &\geq d_1^2 + d_2^2 - x_1^*(x_1 - z_1) - x_2^*(x_2 - z_2).\end{aligned}$$

By properties i) and ii) in Theorem 1.1 it follows that

$$(x_1^* - x_2^*)(z_1 - z_2) \geq d_1^2 + d_2^2 - d_1 \|x_1 - z_1\| - d_2 \|x_2 - z_2\| = 0.$$ □

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On singular $\phi-$Laplacian BVPs of nonlinear fractional differential equation

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Abstract. This paper investigates the existence of multiple positive solutions for a class of $\phi-$Laplacian boundary value problem with a nonlinear fractional differential equation and fractional boundary conditions. Multiple solutions are proved under slight conditions on a possibly degenerating source term. Approximation techniques together with the fixed point index theory a on cone of a Banach space are employed. Some illustrating examples of are also supplied.

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1. Introduction

This paper deals with the existence of multiple positive solutions to the following nonlinear fractional differential equation with a ϕ-Laplacian operator and Riemann-Liouville derivatives:

$$\begin{cases} -D^{\alpha}_{0^+}(\phi(-D^{\beta}_{0^+}x(t)) = q(t)f(t,x(t),D^{\gamma}_{0^+}x(t)), \quad 0<t<1, \\ x(0)=x'(0)=D^{\beta-1}_{0^+}x(1)=D^{\beta}_{0^+}x(0)=[D^{\alpha-1}_{0^+}(\phi(-D^{\beta}_{0^+}x(t))]_{t=1}=0, \end{cases} \tag{1.1}$$

where $\gamma>0$, $\alpha\in(1,2]$, $\beta\in(2,3]$, $\beta-\gamma-2\geq 0$, and $D^{\alpha}_{0^+}, D^{\beta}_{0^+}, D^{\gamma}_{0^+}$ are the standard Riemann-Liouville derivatives. The nonlinear term $f=f(t,x,y)$: $[0,1]\times[0,+\infty)\times[0,+\infty)\longrightarrow\mathbb{R}^+$ is continuous but may be singular at $x=0$ and/or at $y=0$ in a sense to be made precise. The function $\phi:\mathbb{R}\longrightarrow\mathbb{R}$ is an increasing

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homeomorphism such that $\phi(0) = 0$. The sets $\mathbb{R}^+$ and $I = (0, +\infty)$ will stand for the nonnegative real numbers and the positive real numbers, respectively.

In the last couple of years, fractional boundary value problems (BVPs for short) have been the subject of intensive research works, see, e.g., [2, 12, 11, 15] and reference therein. They can thought of as extension of BVPs with ordinary differential equations (see [5, 6]). For the $p-$Laplacian $\varphi_p(s) = |s|^{p-2}s \;\; p > 1$, the authors of [13] discussed the BVP

$$\begin{cases} D^{\beta}_{0+}(\varphi_p(D^{\alpha}_{0+}y(x))) = f(x, y(x)), 0 < x < 1, \\ y(0) = y'(0) = y(1) = D^{\alpha}_{0+}y(0) = 0, D^{\alpha}_{0+}y(1) = \lambda D^{\alpha}_{0+}y(\varepsilon), \end{cases}$$

where $\alpha, \beta \in \mathbb{R}, \;\; \alpha \in (2.3], \;\; \beta \in (1, 2], \;\; \varepsilon \in (0, 1), \;\; \lambda \in [0, +\infty), \; D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the standard Riemann-Liouville derivatives, and $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$. They proved the existence of positive solutions by means of the Guo-Krasnosel'skii fixed point theorem. In [8], Lu *et al.* considered the BVP

$$\begin{cases} D^{\alpha}_{0+}(\varphi_p(D^{\beta}_{0+}u(t))) = f(t, u(t)), 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \; D^{\beta}_{0+}u(0) = D^{\beta}_{0+}u(1) = 0, \end{cases}$$

where $\alpha \in (1, 2], \beta \in (2, 3], \; D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the standard Riemann-Liouville derivatives, and $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$. Existence results are proved by combination of the Guo-Krasnosel'skii fixed point theorem, the Leggett-Williams fixed point theorem, and the method of upper and lower solutions. In [14], the authors considered the BVP

$$\begin{cases} -D^{\alpha_1}_{0+}(\varphi_p(D^{\beta_1}_{0+}u(t)) = f(t, u(t)), 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \; D^{\beta_1}_{0+}u(0) = 0, D^{\beta_1}_{0+}u(1) = bD^{\beta_1}_{0+}u(\eta), \end{cases} \tag{1.2}$$

where $\alpha \in (1, 2], \beta \in (3, 4], \; \eta \in (0, 1), b \in (0, \eta^{\frac{1-\alpha}{p-1}})$, and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$. They established the existence of positive solutions by the upper and lower solutions method combined with the Schauder fixed point theorem. More recently, the existence of positive solutions is proved in [15] by fixed point theory. In [3], A. Boucenna and T. Moussaoui have used the Krasnoselskii fixed point theorem to establish the existence of positive solution on the half-line for the BVP:

$$\begin{cases} -D^{\alpha}_{0+}u(t) = a(t)g(u(t), D^{\beta}_{0+}u(t)), t > 0, \\ u(0) = D^{\alpha-1}_{0+}u(\infty) = 0, \end{cases} \tag{1.3}$$

where $\alpha \in (1, 2], \beta > 0$, and $\alpha - \beta \geq 1$ and the nonlinear function g satisfies some growth assumptions.

This work discusses the existence and the multiplicity of positive solutions to Problem (1.1) where the function f depends on x and on the standard Rieman-Liouville derivative $D^{\gamma}_{0+}x$. The nonlinear term f may be singular point at $x = 0$ and/or $D^{\gamma}_{0+}x = 0$. ϕ is a homeomorphism. We will make use of the fixed point index theory on a suitable cone in some Banach space. Each existence result is illustrated by an example. In this section, we also recall some preliminary results we need in this paper. The first reminders concern the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivation. For more details, we refer to [7, 10, 9].

Definition 1.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,1) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds$$

provided that the right-hand side is pointwise defined on $(0,1)$. $\Gamma(\alpha)$ is the Euler gamma function $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$.

Definition 1.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0,1) \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha} u(t) = \frac{d^n}{dt^n} I_{0^+}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)ds,$$

where n is the smallest integer greater than or equal to α, provided the right-hand side is pointwise defined on $(0,1)$.

Lemma 1.3. *Let $\alpha > 0$. Then for $u \in L(0,1)$ and $D_{0^+}^{\alpha} u(t) \in L(0,1)$, we have*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

where $c_1, c_2, \ldots, c_n \in (-\infty, +\infty), n-1 < \alpha \leq n$.

For the theory and the computation of the fixed point index on cones in Banach spaces, we refer to [1, 2, 4]. An operator $A : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets. A nonempty subset $\mathcal{P}$ of a Banach space E is called a cone if it is convex, closed and satisfies $\alpha x \in \mathcal{P}$ for all $x \in \mathcal{P}$ and $\alpha \geq 0$ and $x, -x \in \mathcal{P}$ implies that $x = 0$.

Lemma 1.4. *Let Ω be a bounded open set in a real Banach space E, $\mathcal{P}$ a cone of E and $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose that $\lambda Ax \neq x, \forall\, x \in \partial\Omega \cap \mathcal{P}, \lambda \in (0,1]$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 1$.*

Lemma 1.5. *Let Ω be a bounded open set in a real Banach space E, $\mathcal{P}$ a cone of E and $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose that $Ax \not\leq x, \forall\, x \in \partial\Omega \cap \mathcal{P}$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 0$.*

The basic space to study Problem (1.1) is

$$E = \{x \in C([0,1], \mathbb{R}) : D_{0^+}^{\gamma} x \in C([0,1], \mathbb{R})\}.$$

E is a Banach space with the norm $\|x\| = \|x\|_1 + \|x\|_2$, where $\|x\|_1 = \sup_{t \in [0,1]} |x(t)|$ and $\|x\|_2 = \sup_{t \in [0,1]} |D_{0^+}^{\gamma} x(t)|$. The following lemma is the fractional version of Ascoli-Arzéla Theorem.

Lemma 1.6. *Let $M \subseteq E$, then M is relatively compact in E if the following conditions hold:*

(*a*) *M is bounded in E,*

(*b*) *the functions belonging to $\{x, x \in M\}$ and $\{z : z(t) = D_{0^+}^{\gamma} x(t), x \in M\}$ are equicotinuous, i.e., $\forall\, \varepsilon > 0, \exists\, \delta > 0, \forall\, t_1, t_2 \in [0,1]$, and for all $x \in M$,*

$$|t_1 - t_2| < \delta \Rightarrow |x(t_1) - x(t_2)| < \varepsilon \text{ and } |D_{0^+}^{\gamma} x(t_1) - D_{0^+}^{\gamma} x(t_2)| < \varepsilon.$$

The cases where f is either regular or singular are discussed separately in Section 3 and Section 4, respectively. Some technical lemmas are collected in the following section.

2. Fixed point setting

Consider the boundary value problem

$$\begin{cases} -D_{0+}^{\alpha} u(t) = v(t), \quad 0 < t < 1, \\ u(0) = D_{0+}^{\alpha-1} u(1) = 0. \end{cases} \tag{2.1}$$

It is easy to verify

Lemma 2.1. *If $v \in C([0,1])$, then Problem (2.1) has the unique solution*

$$u(t) = \int_0^1 H(t,s) v(s) ds,$$

where

$$H(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

If we set $\phi(-D_{0+}^{\beta} x(t)) = u(t)$, then $-D_{0+}^{\beta} x(t) = \phi^{-1}(u)(t)$. Thus the BVP

$$\begin{cases} -D_{0+}^{\alpha} (\phi(-D_{0+}^{\beta} x(t)) = v(t), \quad 0 < t < 1 \\ x(0) = x'(0) = D_{0+}^{\beta-1} x(1) = D_{0+}^{\beta} x(0) = [D^{\alpha-1}(\phi(-D_{0+}^{\beta} x(t)]_{t=1} = 0 \end{cases} \tag{2.2}$$

is equivalent to

$$\begin{cases} -D_{0+}^{\beta} x(t) = \phi^{-1}(\int_0^1 H(t,s) v(s)) ds), \quad t \in (0,1) \\ \\ x(0) = x'(0) = D_{0+}^{\beta-1} x(1) = 0. \end{cases}$$

Lemma 2.2. *Given $v \in C[0,1]$, Problem (2.2) has the unique solution*

$$x(t) = \int_0^1 G(t,s) \phi^{-1} \left(\int_0^1 H(s,\tau) v(\tau) d\tau \right) ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ t^{\beta-1}, & 0 \le t \le s \le 1. \end{cases}$$

A direct computation yields

$$D_{0+}^{\gamma} G(t,s) = \frac{1}{\Gamma(\beta-\gamma)} \begin{cases} t^{\beta-\gamma-1} - (t-s)^{\beta-\gamma-1}, & 0 \le s \le t \le 1, \\ t^{\beta-\gamma-1}, & 0 \le t \le s \le 1. \end{cases}$$

Proof. By Lemma 1.3,

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = -I_{0+}^{\alpha} v(t).$$

Then

$$u(t) = -I_{0+}^{\alpha} v(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions $u(0) = D_{0+}^{\alpha-1}u(1) = 0$ imply $c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 v(s)ds$ and $c_2 = 0$. Hence the solution u of Problem (2.1) is

$$\begin{aligned} u(t) &= - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 v(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1}-(t-s)^{\alpha-1}]v(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}v(s)ds \\ &= \int_0^1 H(t,s)v(s)ds. \end{aligned}$$

Also,

$$I_{0+}^{\beta}D_{0+}^{\beta}x(t) = I_{0+}^{\beta}\phi^{-1}\left(\int_0^1 H(t,s)v(s))ds\right).$$

Then

$$x(t) = -I_{0+}^{\beta}\phi^{-1}\left(\int_0^1 H(t,s)v(s))ds\right) + c_1t^{\beta-1} + c_2t^{\beta-2} + c_3t^{\beta-3},$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. By the boundary conditions $x(0) = x'(0) = D_{0+}^{\beta-1}x(1) = 0$, we have

$$c_1 = \frac{1}{\Gamma(\beta)}\int_0^1 \phi^{-1}\left(\int_0^1 H(s,\tau)v(\tau)d\tau\right)ds, \quad c_2 = c_3 = 0.$$

Finally, the explicit solution x of Problem (2.2) is

$$\begin{aligned} x(t) &= - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi^{-1}(\int_0^1 H(s,\tau)v(\tau)d\tau)ds \\ &\quad + \frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1 \phi^{-1}(\int_0^1 H(s,\tau)v(\tau)d\tau)ds \\ &= \frac{1}{\Gamma(\beta)}\int_0^t [t^{\beta-1}-(t-s)^{\beta-1}]\phi^{-1}(\int_0^1 (H(s,\tau)v(\tau)d\tau)ds \\ &\quad + \frac{1}{\Gamma(\beta)}\int_t^1 t^{\beta-1}\phi^{-1}(\int_0^1 H(s,\tau)v(\tau)d\tau)ds \\ &= \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)v(\tau)d\tau\right)ds. \end{aligned}$$

□

Lemma 2.3. *The function $H(t,s), G(t,s), D_{0+}^{\gamma}G(t,s)$ enjoys the properties*

(a_1) $H, G, D_{0+}^{\gamma}G$ *are continuous on* $[0,1]\times[0,1]$,

(a_2) $H(t,s) \le \frac{t^{\alpha-1}}{\Gamma(\alpha)} \le \frac{1}{\Gamma(\alpha)}$, $\forall\,(t,s)\in[0,1]\times[0,1]$,

(a_3) $G(t,s) \le \frac{t^{\beta-1}}{\Gamma(\beta)} \le \frac{1}{\Gamma(\beta)}$, $\forall\,(t,s)\in[0,1]\times[0,1]$,

(a_4) $D_{0+}^{\gamma}G(t,s) \le \frac{t^{\beta-1}}{\Gamma(\beta-\gamma)} \le \frac{1}{\Gamma(\beta-\gamma)}$, $\forall\,(t,s)\in(0,1)\times(0,1)$,

(a_5) $\frac{(\beta-1)\rho(t)s}{\Gamma(\beta)} \le G(t,s) \le \frac{(\beta-1)s}{\Gamma(\beta)}$, $\forall\,(t,s)\in[0,1]\times[0,1]$,

(a_6) $G(t,s) \ge \rho(t)\sup\limits_{t\in[0,1]} G(t,s)$, $\forall\,(t,s)\in[0,1]\times[0,1]$,

(a_7) $\frac{(\beta-1)\rho(t)s}{\Gamma(\beta-\gamma)} \le D_{0+}^{\gamma}G(t,s) \le \frac{(\beta-1)s}{\Gamma(\beta-\gamma)}$, $\forall\,(t,s)\in(0,1)\times(0,1)$

(a_8) $D_{0+}^{\gamma}G(t,s) \ge \rho(t)\sup\limits_{t\in[0,1]} D_{0+}^{\gamma}G(t,s)$, $\forall\,(t,s)\in(0,1)\times(0,1)$,

(a_9) $G(t,s) \ge \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\sup\limits_{t\in[0,1]} D_{0+}^{\gamma}G(t,s)$, $\forall\,(t,s)\in(0,1)\times(0,1)$,

(a_{10}) $D_{0+}^{\gamma}G(t,s) \ge \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\sup\limits_{t\in[0,1]} G(t,s)$, $\forall\,(t,s)\in(0,1)\times(0,1)$,

where $\rho(t) = \frac{1}{\beta-1}t^{\beta-1}(1-t)^{\beta-1}$.

Proof. Let $(t,s)\in(0,1)\times(0,1)$.

(a_5) If $s\le t$, then

$$\begin{array}{rcl} t^{\beta-1}-(t-s)^{\beta-1} & = & (\beta-1)\int_{t-s}^{t} z^{\beta-2}dz \\ & \le & (\beta-1)(t-t+s)=(\beta-1)s \end{array}$$

and

$$\begin{array}{rcl} t^{\beta-1}-(t-s)^{\beta-1} & \ge & t^{\beta-1}(1-s)^{\beta-1}-(t-s)^{\beta-1} \\ & = & t^{\beta-2}(1-s)^{\beta-2}(t-ts)-(t-s)^{\beta-2}(t-s) \\ & \ge & t^{\beta-2}(1-s)^{\beta-2}(t-ts)-(t-ts)^{\beta-2}(t-s) \\ & \ge & t^{\beta-2}(1-s)^{\beta-2}[(t-ts)-(t-s)] \\ & \ge & t^{\beta-2}(1-s)^{\beta-2}s(1-t) \\ & \ge & t^{\beta-1}(1-t)^{\beta-1}s. \end{array}$$

If $t\le s$, then

$$t^{\beta-1}=(\beta-1)\int_0^t z^{\beta-2}dz\le(\beta-1)s$$

and

$$t^{\beta-1}\ge t^{\beta-1}(1-t)^{\beta-1}s.$$

Hence

$$\frac{t^{\beta-1}(1-t)^{\beta-1}s}{\Gamma(\beta)}\le G(t,s)\le\frac{(\beta-1)s}{\Gamma(\beta)},\quad \forall\,(t,s)\in[0,1]\times[0,1].$$

(a_7) For $s\le t$

$$\begin{array}{rcl} t^{\beta-\gamma-1}-(t-s)^{\beta-\gamma-1} & = & (\beta-\gamma-1)\int_{t-s}^{t} z^{\beta-\gamma-2}dz \\ & \le & (\beta-\gamma-1)(t-t+s) \\ & = & (\beta-\gamma-1)s\le(\beta-1)s \end{array}$$

and

$$\begin{array}{rl} t^{\beta-\gamma-1}-(t-s)^{\beta-\gamma-1} & \ge t^{\beta-\gamma-1}(1-s)^{\beta-\gamma-1}-(t-s)^{\beta-\gamma-1} \\ & = t^{\beta-\gamma-2}(1-s)^{\beta-\gamma-2}(t-ts)-(t-s)^{\beta-\gamma-2}(t-s) \\ & \ge t^{\beta-\gamma-2}(1-s)^{\beta-\gamma-2}(t-ts)-(t-ts)^{\beta-\gamma-2}(t-s) \\ & \ge t^{\beta-\gamma-2}(1-s)^{\beta-\gamma-2}[(t-ts)-(t-s)] \\ & \ge t^{\beta-\gamma-2}(1-s)^{\beta-\gamma-2}s(1-t) \\ & \ge t^{\beta-1}(1-t)^{\beta-1}s. \end{array}$$

If $t\le s$, then

$$t^{\beta-\gamma-1}=(\beta-\gamma-1)\int_0^t z^{\beta-2}dz\le(\beta-1)s$$

and

$$t^{\beta-\gamma-1}\ge t^{\beta-1}(1-t)^{\beta-1}s.$$

Hence

$$\frac{t^{\beta-1}(1-t)^{\beta-1}s}{\Gamma(\beta-\gamma)}\le D_{0+}^{\gamma}G(t,s)\le\frac{(\beta-1)s}{\Gamma(\beta-\gamma)}.$$

(a_9) By (a_5) and (a_7),

$$\begin{aligned} G(t,s) &\geq \tfrac{t^{\beta-1}(1-t)^{\beta-1}s}{\Gamma(\beta)} \\ &= \tfrac{\Gamma(\beta-\gamma)t^{\beta-1}(1-t)^{\beta-1}}{(\beta-1)\Gamma(\beta)} \tfrac{(\beta-1)s}{\Gamma(\beta-\gamma)} \\ &\geq \tfrac{\Gamma(\beta-\gamma)t^{\beta-1}(1-t)^{\beta-1}}{(\beta-1)\Gamma(\beta)} \sup_{t\in[0,1]} D^{\gamma}_{0^+}G(t,s). \end{aligned}$$

(a_{10}) By (a_5) and (a_7)

$$\begin{aligned} D^{\gamma}_{0^+}G(t,s) &\geq \tfrac{t^{\beta-1}(1-t)^{\beta-1}s}{\Gamma(\beta-\gamma)} \\ &= \tfrac{\Gamma(\beta)t^{\beta-1}(1-t)^{\beta-1}}{(\beta-1)\Gamma(\beta)} \tfrac{(\beta-1)s}{\Gamma(\beta)} \\ &\geq \tfrac{\Gamma(\beta)t^{\beta-1}(1-t)^{\beta-1}}{(\beta-1)\Gamma(\beta-\gamma)} \sup_{t\in[0,1]} G(t,s). \end{aligned}$$

□

Define the cone $\mathcal{P}$

$$P = \{x \in E: \ \ x(t) \geq \lambda_1\rho(t)\|x\|, \ D^{\gamma}_{0^+}x(t) \geq \lambda_2\rho(t)\|x\|, \ \ \forall\, t \in [0,1]\},$$

where

$$\lambda_1 = \frac{1}{2(\beta-1)} \max\left\{\frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}, 1\right\}$$

and

$$\lambda_2 = \frac{1}{2(\beta-1)} \max\left\{\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}, 1\right\}.$$

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ with $0 < \alpha_1 < \alpha_2$ be such that

$$t^{\alpha_2}\phi(x) \leq \phi(tx) \leq t^{\alpha_1}\phi(x); \qquad \forall\, t \in [0,1], \ \forall\, x \geq 0. \tag{2.3}$$

Then

$$t^{\frac{1}{\alpha_1}}\phi^{-1}(x) \leq \phi^{-1}(tx) \leq t^{\frac{1}{\alpha_2}}\phi^{-1}(x); \quad \forall\, t \in [0,1], \ \forall\, x \geq 0. \tag{2.4}$$

Let

$$\rho_1(x) = \begin{cases} x^{\frac{1}{\alpha_1}}, & x \leq 1 \\ x^{\frac{1}{\alpha_2}}, & x \geq 1 \end{cases} \tag{2.5}$$

$$\rho_2(x) = \begin{cases} x^{\frac{1}{\alpha_2}}, & x \leq 1 \\ x^{\frac{1}{\alpha_1}}, & x \geq 1. \end{cases} \tag{2.6}$$

From Equation (2.4), we get

$$\rho_1(t)\phi^{-1}(x) \leq \phi^{-1}(tx) \leq \rho_2(t)\phi^{-1}(x); \quad \forall\, t \geq 0, \forall\, x \geq 0. \tag{2.7}$$

By Lemma 2.1 and Lemma 2.2, Problem 1.1 is equivalent to the nonlinear integral equation

$$x(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau, x(\tau), D^{\gamma}_{0^+}x(\tau))d\tau\right) ds. \tag{2.8}$$

Thus the fixed point operator is the operator $A: E \longrightarrow C([0,1])$ given by

$$A(x)(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau, x(\tau), D^{\gamma}_{0^+}x(\tau))d\tau\right) ds,$$

where $x \in \mathcal{P}$. The fractional derivative is

$$D^{\gamma}_{0+}A(x)(t) = \int_0^1 D^{\gamma}_{0+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds.$$

Lemma 2.3 will help in investigating the properties of the fixed point operator. Existence of fixed points will be investigated in the next two sections.

3. Regular nonlinear term

Suppose that $f:[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous such that $f(t_0,0,0)\neq 0$ for some $t_0\in(0,1]$. Let the hypotheses

$(\mathcal{H}_1)$ There exist $m\in C([0,1],\mathbb{R}^+)$ and a nondecreasing function in each argument $g\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ such that

$$f(t,x,y)\le m(t)g(x,y),\quad \forall\, t\in[0,1],\forall\, x,y\in\mathbb{R}^+.$$

$(\mathcal{H}_2)$

$$\sup_{c>0}\frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c,c)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)}>1.$$

$(\mathcal{H}_3)$ There exist a,b $(0<a<b<1)$ such that

$$\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)}=+\infty,\qquad \text{uniformly in } t\in[a,b] \text{ and } y\in\mathbb{R}^+.$$

Proposition 3.1. *Suppose $(\mathcal{H}_1)$. Then the operator A maps $\mathcal{P}$ into $\mathcal{P}$ and it is completely continuous.*

Proof.

(1) $A(\mathcal{P})\subset\mathcal{P}$. $A(x)(t)\ge 0, D^{\gamma}_{0+}Ax(t)\ge 0\ \ \forall\ t\in[0,1]$ and by Lemma 2.3(a_6)

$$\begin{aligned}
A(x)(t) &= \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds\\
&\ge \rho(t)\\
&\quad \sup_{t\in[0,1]}\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds\\
&\ge \rho(t)\sup_{t\in[0,1]}Ax(t)\\
&\ge \rho(t)\|Ax\|_1.
\end{aligned}$$

By Lemma 2.3(a_9),

$$\begin{aligned}
A(x)(t) &= \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds\\
&\ge \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\\
&\quad \sup_{t\in[0,1])}\int_0^1 D^{\gamma}_{0+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^{\gamma}_{0+}x(\tau))d\tau\right)ds\\
&\ge \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\sup_{t\in[0,1]}D^{\gamma}_{0+}Ax(t)\\
&\ge \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\|Ax\|_2.
\end{aligned}$$

Hence

$$\begin{array}{rcl} A(x) & = & \frac{1}{2}(A(x)+A(x)) \\ & \geq & \frac{1}{2}(\rho(t)\|Ax\|_1 + \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)}\rho(t)\|Ax\|_2) \\ & \geq & \lambda_1\rho(t)\|Ax\|. \end{array}$$

Also by Lemma 2.3(a_{10}),

$$\begin{array}{rcl} D^\gamma_{0^+}A(x)(t) & = & \int_0^1 D^\gamma_{0^+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\int_0^1 \sup\limits_{t\in[0,1]} G(t,s) \\ & & \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\sup\limits_{t\in[0,1]}\int_0^1 G(t,s) \\ & & \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\sup\limits_{t\in[0,1]} Ax(t) \\ & \geq & \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\|Ax\|_1 \end{array}$$

and Lemma 2.3(a_8) implies

$$\begin{array}{rcl} D^\gamma_{0^+}A(x)(t) & = & \int_0^1 D^\gamma_{0^+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \rho(t)\sup\limits_{t\in[0,1]}\int_0^1 D^\gamma_{0^+}G(t,s) \\ & & \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right)ds \\ & \geq & \rho(t)\sup\limits_{t\in[0,1]} D^\gamma_{0^+}Ax(t) \\ & \geq & \rho(t)\|Ax\|_2. \end{array}$$

Hence

$$\begin{array}{rcl} D^\gamma_{0^+}A(x) & \geq & \frac{1}{2}(\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\rho(t)\|Ax\|_1 + \rho(t)\|Ax\|_2) \\ & \geq & \lambda_2\rho(t)\|Ax\|, \end{array}$$

proving the claim.

(2) Let $D\subset E$ be a bounded set. Then there exists $r>0$ such that $\forall\, x\in D, \|x\|\leq r$. By $(\mathcal{H}_1)$ and the properties $(a_2),(a_3),(a_4)$ of Lemma 2.3, we have the estimates

$$\begin{array}{rcl} \|A(x)\|_1 & = & \|\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^\gamma_{0^+}(\tau))d\tau\right)ds\|_1 \\ & \leq & \frac{1}{\Gamma(\beta)}\int_0^1 \phi^{-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)f(\tau,x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right)ds \\ & \leq & \frac{1}{\Gamma(\beta)}\phi^{-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)g(x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right) \\ & \leq & \frac{1}{\Gamma(\beta)}\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)(m(\tau)d\tau\right) < +\infty \end{array}$$

and

$$\begin{array}{rcl} \|A(x)\|_2 & = & \|\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right)ds\|_2 \\ & \leq & \frac{1}{\Gamma(\beta-\gamma)}\phi^{-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)g(x(\tau),D^\gamma_{0^+}x(\tau))d\tau\right) \\ & \leq & \frac{1}{\Gamma(\beta-\gamma)}\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right) < +\infty. \end{array}$$

Hence $\|A(x)\| < [\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)(m(\tau)d\tau\right)$, that is $A(D)$ is uniformly bounded.

(3) $A(D)$ is equicontinuous. For $t,t'\in[0,1]$ $(t<t')$, we have

$$\begin{aligned} & |A(x)(t)-A(x)(t')| \\ \le\ & \int_0^1 |G(t,s)-G(t',s)|\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x(\tau),D^\gamma_{0+}x(\tau))d\tau\right)ds \\ \le\ & \int_0^1 |G(t,s)-G(t',s)|\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)ds \end{aligned}$$

and

$$\begin{aligned} & |D^\gamma_{0+}A(x)(t)-D^\gamma_{0+}A(x)(t')| \\ \le\ & \int_0^1 |D^\gamma_{0+}G(t,s)-D^\gamma_{0+}G(t',s)|\phi^{-1}\left(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)ds. \end{aligned}$$

Then for any $\varepsilon>0$, there exists $\delta>0$ such that

$$|Ax(t)-A(x')|<\varepsilon \text{ and } |D^\gamma_{0+}A(x)(t)-D^\gamma_{0+}A(x)(t')|<\varepsilon,$$

for all $t,t'\in[0,1]$ and $|t-t'|<\delta$, proving that $A(D)$ is relatively compact.

(4) A is continuous: Let some sequence $\{x_n\}_{n\ge0}\subset\mathcal{P}$ be such that $\lim\limits_{n\to+\infty}x_n=x_0$. Then there exists $r>0$ such that $\|x_n\|\le r$, $\forall n\ge0$. By $(\mathcal{H}_1)$, for all $t\in[0,1]$, we have

$$\begin{aligned} & |Ax_n(t)-Ax_0(t)| \\ =\ & |\int_0^1 G(t,s)[\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_n(\tau),D^\gamma_{0+}x_n(\tau))d\tau)ds \\ & -\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^\gamma_{0+}x_0(\tau))d\tau)]ds| \\ \le\ & \frac{2}{\Gamma(\beta)}\phi^{-1}(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau) \end{aligned}$$

and

$$|D^\gamma_{0+}Ax_n(t)-D^\gamma_{0+}Ax_0(t)|\le\frac{2}{\Gamma(\beta-\gamma)}\phi^{-1}(\frac{g(r,r)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau).$$

With the Lebegue Dominated convergence theorem, we conclude that

$$\lim_{n\to+\infty}\|Ax_n-Ax_0\|=0,$$

i.e., A is continuous. □

We state and prove our first existence result

Theorem 3.2. *Under Assumptions* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold, BVP* (1.1) *has at least one positive solution.*

Proof. From Condition $(\mathcal{H}_2)$, there exists $R>0$ such that

$$\frac{R}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(R,R)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)}>1. \tag{3.1}$$

Let $\Omega_1=\{x\in E;\|x\|\le R\}$. To prove that $x\ne\lambda Ax$ for all $x\in\partial\Omega_1\cap\mathcal{P}$ and $\lambda\in(0,1]$, suppose by contradiction that there exist $x_0\in\partial\Omega_1\cap\mathcal{P}$ and $\lambda_0\in(0,1]$ such that $x_0 = \lambda_0 Ax_0$. By $(\mathcal{H}_1)$ and the properties $(a_2), (a_3)$ and (a_4) of Lemma 2.3, we have

$$\begin{aligned}
R &= \|x_0\| \\
&= \|\lambda_0 Ax_0\| \\
&\leq \|Ax_0\|_1 + \|Ax_0\|_2 \\
&\leq \sup_{t\in[0,1]} \int_0^1 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^\gamma_{0+}x_0(\tau))d\tau)ds \\
&\quad + \sup_{t\in[0,1]} \int_0^1 D^\gamma_0 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^\gamma_{0+}x_0(\tau))d\tau)ds \\
&\leq \tfrac{1}{\Gamma(\beta)}\phi^{-1}(\tfrac{g(R,R)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau) + \tfrac{1}{\Gamma(\beta-\gamma)}\phi^{-1}(\tfrac{g(R,R)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}(\tfrac{g(R,R)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau),
\end{aligned}$$

which contradicts (3.1). Lemma 1.4 implies that

$$i(A, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1.$$

Then there exists $x_0 \in \Omega_1 \cap \mathcal{P}$ such that $Ax_0 = x_0$. Since

$$f(t_0, 0, 0) \neq 0 \text{ and } x_0(t) \geq \lambda_1 \rho(t)\|x_0\|,$$

x_0 is a positive solution of Problem (1.1). □

Example 3.3. Consider the BVP

$$\begin{cases} -D^{\frac{3}{2}}_{0+}\left(-D^{\frac{5}{2}}_{0+}x(t)\right)^{\frac{5}{3}} = \frac{3}{25}t^{\frac{1}{4}}(1+\cos(\frac{\pi}{4}t^{\frac{5}{4}}))(x + D^{\frac{1}{6}}_{0+}x+1)^{\frac{5}{3}}, t\in(0,1) \\ x(0) = x'(0) = D^{\frac{3}{2}}_{0+}x(1) = D^{\frac{5}{2}}_{0+}x(0) = [D^{\frac{1}{2}}_{0+}(\phi(-D^{\frac{5}{2}}_{0+}x(t))]_{t=1} = 0, \end{cases} \tag{3.2}$$

where

$$f(t,x,y) = (1+\cos(\frac{\pi}{4}t^{\frac{5}{4}}))(x+y+1)^{\frac{5}{3}},\ q(t) = \frac{3}{25}t^{\frac{1}{4}} \text{ and } \phi(t) = t^{\frac{5}{3}}.$$

Then ϕ is an increasing homeomorphism such that $\phi(0) = 0$. For

$$g(x,y) = \frac{3}{25}(x+y+1)^{\frac{5}{3}} \text{ and } m(t) = 1+\cos(\frac{\pi}{4}t^{\frac{5}{4}}),$$

Assumption $(\mathcal{H}_2)$

$$\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c,c)}{\Gamma(\alpha)}, \int_0^1 q(\tau)m(\tau)d\tau\right)} \geq 1.013 > 1$$

is satisfied and then all conditions of Theorem 3.2 hold. Therefore Problem (3.2) has at least one positive solution.

The existence of positive solutions is given by

Theorem 3.4. *Assume that* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold and suppose that there exist* α_1, α_2, $0 < \alpha_1 < \alpha_2$, *such that*

$$t^{\alpha_2}\phi(x) \leq \phi(tx) \leq t^{\alpha_1}\phi(x),\ \forall\, t\in[0,1],\ \forall\, x \geq 0.$$

Then Problem (1.1) *has at least two positive solutions.*

Proof. Choose R as in the proof of Theorem 3.2. Then

$$i(A,\Omega_1\cap\mathcal{P},\mathcal{P})=1 \tag{3.3}$$

and there exist $x_0\in\Omega_1$ solution of Problem (1.1). Let $0<a<b<1$ be as in $(\mathcal{H}_3)$ and

$$a_0=\min_{(t,s)\in[a,b]^2}G(t,s)>0,\ b_0=\min_{(t,s)\in[a,b]^2}H(t,s)>0,$$

$$c=\lambda_1\min_{t\in[a,b]}\rho(t)>0,\ N>1+\frac{1}{c^{\alpha_2}(b-a)^{\alpha_2}a_0^{\alpha_2}b_0\int_a^b q(\tau)d\tau}.$$

By $(\mathcal{H}_3)$, there exists $R'>\lambda_1 R$ such that

$$f(t,x,y)>N\phi(x),\ \forall\, t\in[a,b],\ \ \forall\, x>R',\ \ \forall\, y\in\mathbb{R}^+.$$

Define the open ball $\Omega_2=\left\{x\in E:\|x\|\le\frac{R'}{c}\right\}$.
To show that $Ax\not\leq x$ for all $x\in\partial\Omega_2\cap\mathcal{P}$, suppose on the contrary that there exists $x_0\in\partial\Omega_2\cap\mathcal{P}$ such that $Ax_0\le x_0$. Since $x_0\in\mathcal{P}$, then

$$\begin{array}{rcl}
x_0(t) & \ge & Ax_0(t)\\
 & = & \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds\\
 & \ge & \int_a^b G(t,s)\phi^{-1}\left(\int_a^b H(s,\tau)q(\tau)f(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds\\
 & = & (b-a)a_0\phi^{-1}\left(b_0\int_a^b q(\tau)N\phi(x_0(\tau))d\tau\right)\\
 & = & (b-a)a_0\phi^{-1}\left(b_0N\phi(R')\int_a^b q(\tau)d\tau\right)\\
 & = & (b-a)a_0\phi^{-1}\left([b_0N\int_a^b q(\tau)d\tau]\phi(R')\right)\\
 & \ge & (b-a)a_0\rho_1\left(b_0N\int_a^b q(\tau)d\tau\right)R'\\
 & \ge & (b-a)a_0b_0^{\frac{1}{\alpha_2}}N^{\frac{1}{\alpha_2}}\left(\int_a^b q(\tau)d\tau\right)^{\frac{1}{\alpha_2}}R'\\
 & > & \frac{R'}{c},
\end{array}$$

contradicting $\|x_0\|=\frac{R'}{c}$. By Lemma 1.5, we conclude that

$$i(A,\Omega_2\cap\mathcal{P},\mathcal{P})=0. \tag{3.4}$$

(3.3) and (3.4) imply

$$i(A,(\Omega_2\setminus\Omega_1)\cap\mathcal{P},\mathcal{P})=-1. \tag{3.5}$$

Then A has a second fixed point $y_0\in(\Omega_2\setminus\Omega_1)\cap\mathcal{P}$. Moreover $y_0\ge\lambda_1\rho(t)R$ and $R\le\|y_0\|\ <\frac{R'}{c}$. Then x_0 and y_0 are two positive solutions of Problem (1.1). ☐

Example 3.5. Consider the BVP

$$\left\{\begin{array}{l}
-D^{\frac{3}{2}}_{0+}\left(-D^{\frac{11}{4}}_{0+}x(t)\right)^p=(2\delta t)(x+(D^{\frac{1}{4}}_{0+}x)+1),\ t\in(0,1),\\
x(0)=x'(0)=D^{\frac{9}{4}}_{0+}x(1)=D^{\frac{11}{4}}_{0+}x(0)=[D^{\frac{1}{2}}_{0+}(\phi(-D^{\frac{11}{4}}_{0+}x(t))]_{t=1}=0,
\end{array}\right. \tag{3.6}$$

where $f(t,x,y) = (2\delta\sqrt{t})(x+y+1), q(t) = \sqrt{t},\ \delta > 0$, and $\phi(t) = t^p, (p = \frac{a}{b}$ are such that $0 < a < b$ and $(b-a)$ is an even number. ϕ is an increasing homeomorphism such that $\phi(0) = 0$ and there exist $\alpha_1 = p^2, \alpha_2 = p$

$$t^p\phi(x) \le \phi(tx) \le t^{p^2}\phi(x), \quad \forall\, t \in [0,1], \forall\, x \ge 0.$$

For $g(x,y) = x+y+1$ and $m(t) = 2\delta\sqrt{t}$, Assumption $(\mathcal{H}_2)$

$$\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c,c)}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)d\tau\right)} \ge \sup_{c>0} \frac{0.72(\frac{\sqrt{\pi}}{2})^{\frac{1}{p}}c}{(\delta(2c+1))^{\frac{1}{p}}}$$

and $(\mathcal{H}_3)$

$$\begin{aligned}\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)} &= \lim_{x\to+\infty}\frac{(2\delta\sqrt{t})(x+y+1)}{x^p}\\ &\ge \lim_{x\to+\infty} 2\delta\sqrt{a}x^{1-p} = +\infty, \quad \forall\, t \in [a,b],\ \forall\ y \ge 0\end{aligned}$$

are satisfied for $\delta < \left(\sup_{c>0}\frac{0.72(\frac{\sqrt{\pi}}{2})^{\frac{1}{p}}c}{(\delta(2c+1))^{\frac{1}{p}}}\right)^p$. Finally all hypotheses of Theorem 3.2 are fulfilled. Hence Problem (3.6) has at least two positive solutions.

4. Degenerating nonlinear term

First suppose that f may have a singular point at $x = 0$ only. More precisely $f : [0,1] \times I \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Assume that
$(\mathcal{H}'_1)$ There exist $m \in C([0,1],\mathbb{R}^+), \psi \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $g, h, \in C(I,I)$ such that h is a decreasing function and $\psi, \frac{g}{h}$ are increasing functions with

$$f(t,x,y) \le m(t)g(x)\psi(y), \quad \forall\ t \in [0,1], \forall\, x \in I,\ \forall y \in \mathbb{R}^+$$

and for each $c > 0$,

$$\int_0^1 q(\tau)m(\tau)h(c\rho(\tau))d\tau < +\infty,$$

$(\mathcal{H}'_2)$

$$\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c)\psi(c)}{\Gamma(\alpha)h(c)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)c)d\tau\right)} > 1.$$

$(\mathcal{H}'_3)$ There exist a, b $(0 < a < b < 1)$ such that

$$\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)} = +\infty, \quad \text{uniformly in } t \in [a,b] \text{ and } y \in \mathbb{R}^+.$$

$(\mathcal{H}'_4)$ For any $c > 0$, there exist $\psi_c \in C([0,1],\mathbb{R}^+)$ and an interval $J \subset [0,1]$ such that $\psi_c(t) > 0$ in J and

$$f(t,x,y) \ge \psi_c(t), \quad \forall\ t \in [0,1],\ \forall\, x \in (0,c], \forall\, y \in [0,c].$$

Given $f \in C([0,1] \times I \times \mathbb{R}^+, \mathbb{R}^+)$, define the sequence of functions $\{f_n\}_{n\ge1}$

$$f_n(t,x,y) = f(t, \max\{\tfrac{1}{n}, x\}, y), \quad n \in \{1,2,\ldots\},$$

and for $x \in \mathcal{P}$, define the sequences of operators

$$A_n(x)(t) = \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x(\tau), D^{\gamma}_{0+}x(\tau)\right) d\tau)ds.$$

Then

$$D^{\gamma}_{0+}A_n(x)(t) = \int_0^1 D^{\gamma}_{0+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x(\tau, D^{\gamma}_{0+}x(\tau))d\tau\right) ds.$$

The proof of the following result is the same as that of the operator A in Proposition 3.1. We omit it.

Proposition 4.1. *Suppose* $(\mathcal{H}_1')$ *holds.Then for each* $n \geq 1$*, the operator* A_n *maps* $\mathcal{P}$ *into* $\mathcal{P}$ *and it is completely continuous.*

As in the regular case, we prove two theorems: one of the existence of a single solution and one of a pair of solutions.

Theorem 4.2. *Suppose* $(\mathcal{H}_1'), (\mathcal{H}_2'), (\mathcal{H}_4')$ *hold. Then Problem* (1.1) *has at least one positive solution.*

Proof. **(1) Construction of a sequence $(x_n)_n$ of approximating fixed points.**
By condition $(\mathcal{H}_2')$, there exists $R > 0$ such that

$$\frac{R}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(R)\psi(R)}{\Gamma(\alpha)h(R)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)R)d\tau\right)} > 1. \tag{4.1}$$

Let $\Omega_1 = \{x \in E : \|x\| < R\}$. Then $x \neq \lambda A_n(x)$ for any $x \in \partial\Omega_1 \cap \mathcal{P}, \lambda \in (0,1]$ and $n \geq n_0 \geq \frac{1}{R}$. Otherwise there exist $n_1 \geq n_0$, $x_1 \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_0 \in (0,1]$ such that $x_1 = \lambda_0 A_{n_1}x_1$. Since $x_1 \in \partial\Omega_1 \cap \mathcal{P}$, we have $x_1(t) \geq \lambda_1\rho(t)\|x_1\| = \lambda_1\rho(t)R$, then

$$\begin{aligned}
R &= \|x_1\| \\
&= \|\lambda_0 A_{n_1}x_1\| \\
&\leq \|A_{n_1}x_1\|_1 + \|Ax_1\|_2 \\
&\leq \sup_{t\in[0,1]} \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_{n_1}(\tau, x_1(\tau), D^{\gamma}_{0+}x_1(\tau))d\tau\right) ds \\
&+ \sup_{t\in[0,1]} \int_0^1 D^{\gamma}_{0+}G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_{n_1}(\tau, x_1(\tau), D^{\gamma}_{0+}x_1(\tau))d\tau\right) ds \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\tfrac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)g(\max\{\tfrac{1}{n_1}, x_1(\tau)\})\psi(D^{\gamma}_{0+}x_1(\tau))d\tau\right) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}(\tfrac{1}{\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)h(\max\{\tfrac{1}{n_1}, x_1(\tau)\})\tfrac{g(\max\{\frac{1}{n_1}, x_1(\tau)\})}{h(\max\{\frac{1}{n_1}, x_1(\tau)\})} \\
&\psi(D^{\gamma}_{0+}x_1(\tau))d\tau) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\tfrac{g(R)\psi(R)}{h(R)\Gamma(\alpha)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)R)d\tau\right)
\end{aligned}$$

which is a contradiction to (4.1). By Lemma 1.4, we deduce that

$$i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \text{ for all } n \in \{n_0, n_0+1, \ldots\}. \tag{4.2}$$

Hence there exists an $x_n \in \Omega_1 \cap \mathcal{P}$ such that $A_n x_n = x_n$, $\forall\, n \geq n_0$.

(2) The sequence $(x_n)_n$ is relatively compact.

(a) Since $\|x_n\| < R$, by $(\mathcal{H}_4')$ there exists $\psi_R \in C([0,1], \mathbb{R}^+)$ such that

$$f_n(t, x_n(t), D^{\gamma}_{0+}x_n(t)) \geq \psi_R(t), \quad \forall\, t \in [0,1].$$

Then, by Lemma 2.3(a_5),

$$\begin{array}{rcl} x_n(t) & = & A_n x_n(t) \\ & = & \int_0^1 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x_n(\tau), D^{\gamma}_{0+}x_n(\tau))d\tau)ds \\ & \geq & \int_0^1 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)\psi_R(\tau)d\tau)ds \\ & \geq & \frac{\rho(t)(\beta-1)}{\Gamma(\beta)} \int_0^1 s\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)\psi_R(\tau)d\tau\right)ds. \end{array}$$

Let

$$c^* = \frac{(\beta-1)}{\Gamma(\beta)} \int_0^1 s\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)\psi_R(\tau)d\tau\right)ds > 0.$$

Then

$$x_n(t) \geq c^*\rho(t), \qquad \forall t \in [0,1], \forall\, n \geq n_0.$$

(b) For any $t, t' \in [0,1]$ $(t > t')$,

$$\begin{array}{l} |x_n(t) - x_n(t')| \\ \leq \int_0^1 \left|G(t,s) - G(t',s)\right| \\ \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x_n(\tau), D^{\gamma}_{0+}x_n(\tau))d\tau\right)ds \\ \leq \int_0^1 \left|G(t,s) - G(t',s)\right| \phi^{-1}\left(\frac{g(R)\psi(R)}{\Gamma(\alpha)h(R)} \int_0^1 q(\tau)m(\tau)h(c^*\rho(\tau))d\tau\right). \end{array}$$

Also

$$\begin{array}{l} \left|D^{\gamma}_{0+}x_n(t) - D^{\gamma}_{0+}x_n(t')\right| \\ \leq \int_0^1 \left|D^{\gamma}_{0+}G(t,s) - D^{\gamma}_{0+}G(t',s)\right| \\ \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau, x_n(\tau), D^{\gamma}_{0+}x_n(\tau))d\tau\right)ds \\ \leq \int_0^1 \left|D^{\gamma}_{0+}G(t,s) - D^{\gamma}_{0+}G(t',s)\right| \phi^{-1}\left(\frac{g(R)\psi(R)}{\Gamma(\alpha)h(R)} \int_0^1 q(\tau)m(\tau)h(c^*\rho(\tau))d\tau\right)ds. \end{array}$$

Since G and $D^{\gamma}_{0+}G$ are continuous, by Lemma 1.6 $(x_n)_n$ is relatively compact in E. Then there exists a subsequence $(x_{n_k})_{k\geq 1}$ such that $\lim\limits_{k\to+\infty} x_{n_k} = x_0$. Since $x_{n_k}(t) \geq c^*\rho(t)\ \forall\, k \geq 1, \forall\, t \in [0,1]$, we have $x_0(t) \geq c^*\rho(t), \quad \forall\, t \in [0,1]$. Since f is continuous, by the Lebesgue dominated convergence theorem,

$$\begin{array}{rl} & x_0(t) \\ = & \lim\limits_{k\to+\infty} x_{n_k}(t) \\ = & \lim\limits_{k\to+\infty} \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_{n_k}(\tau, x_{n_k}(\tau), D^{\gamma}_{0+}x_{n_k}(\tau))d\tau\right)ds \\ = & \lim\limits_{k\to+\infty} \int_0^1 G(t,s) \\ & \phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau, \max\{\frac{1}{n_k}, x_{n_k}(\tau)\}, D^{\gamma}_{0+}x_{n_k}(\tau))d\tau\right)ds \\ = & \int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f(\tau, \max\{0, x_0(\tau)\}, D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds \\ = & \int_0^1 G(t,s)\phi^{-1}(\int_0^1 H(s,\tau)q(\tau)f(\tau, x_0(\tau), D^{\gamma}_{0+}x_0(\tau))d\tau)ds. \end{array}$$

Therefore x_0 is a positive solution of Problem (1.1). □

Example 4.3. Consider the BVP

$$\begin{cases} -D_{0^+}^{\frac{7}{4}}\left(-D_{0^+}^{\frac{9}{4}}x(t)\right)^{\frac{1}{3}} = \delta t^{\frac{5}{4}}\frac{e^{2x}}{x}Ch(D_{0^+}^{\frac{1}{6}}x),\ t\in(0,1) \\ x(0)=x'(0)=D_{0^+}^{\frac{5}{4}}x(1)=D_{0^+}^{\frac{9}{4}}x(0)=[D_{0^+}^{\frac{3}{4}}(\phi(-D_{0^+}^{\frac{9}{4}}x(t))]_{t=1}=0, \end{cases} \tag{4.3}$$

where

$$f(t,x,y)=\delta\frac{e^{2x+t}}{x}Ch(y),\ (\delta>0),\ q(t)=t^{\frac{5}{4}}e^{-t} \text{ and } \phi(t)=t^{\frac{1}{3}}.$$

Hence ϕ is an increasing homeomorphism and $\phi(0)=0$ We check the conditions of Theorem 4.2.

($\mathcal{H}_1'$) Let $m(t)=\delta e^t,\ g(x)=\frac{e^x}{x},\ \psi(y)=Ch(y),\ h(y)=\frac{1}{y}$. Then $\frac{g(x)}{h(x)}=e^x$ and ψ are increasing,

$$f(t,x,y)\le m(t)g(x)\psi(y),\ \forall t\in[0,1],\ \forall x\in I,\ \forall y\in\mathbb{R}^+,$$

and for any $c>0$

$$\int_0^1 m(\tau)q(\tau)h(c\rho(\tau))d\tau=\frac{16}{45c}<+\infty.$$

($\mathcal{H}_2'$)

$$\sup_{c>0}\frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c)\psi(c)}{\Gamma(\alpha)h(c)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)c)d\tau\right)}\ge\sup_{c>0}\frac{81c^4}{(\delta e^cCh(c))^3}.$$

($\mathcal{H}_4'$) For every $c>0$, there exists $\psi_c=\frac{\delta e^t}{c}$ such that

$$f(t,x,y)\ge\psi_c(t),\quad \forall\ t\in[0,1],\forall\ \ \forall x\in(0,c],y\in[0,c].$$

Let $0<\delta\le\left(\sup\limits_{c>0}\frac{81c^4}{(e^cCh(c))^3}\right)^{\frac{1}{3}}$. Then Problem (4.3) has at least one positive solution.

The existence of two positive solutions is given by

Theorem 4.4. *Let* ($\mathcal{H}_1'$) $-$ ($\mathcal{H}_4'$) *and suppose that there exist* α_1,α_2 *with* $0<\alpha_1<\alpha_2$ *such that*

$$t^{\alpha_2}\phi(x)\le\phi(tx)\le t^{\alpha_1}\phi(x),\ \forall t\in[0,1],\ \forall x\ge0.$$

Then Problem (1.1) *has at least two positive solutions.*

Proof. With R the same as in the proof of Theorem 4.2, we get

$$i(A_n,\Omega_1\cap\mathcal{P},\mathcal{P})=1,\quad\text{for all } n\in\{n_0,n_1,\ldots\}. \tag{4.4}$$

Then for every $n\in\{n_0,n_1,\ldots\}$, there exists a solution x_n of Problem (1.1) in Ω_1. Let $0<a<b<1$ be as in ($\mathcal{H}_3'$) and a_0,b_0,c as in the proof of Theorem 3.4. Choose

$$N>1+\frac{1}{c^{\alpha_2}(b-a)^{\alpha_2}a_0^{\alpha_2}b_0\int_a^b q(\tau)d\tau}.$$

By ($\mathcal{H}_3'$), there exists a positive constant $R'>\max\{1,\lambda_1R\}$ such that

$$f(t,x,y)>N\phi(x),\ \forall t\in[a,b],\ \ \forall x\ge R',\ \ \forall y\in\mathbb{R}^+.$$

Consider the open ball $\Omega_2=\left\{x\in E:\|x\|\le\frac{R'}{c}\right\}$. Then $A_nx\nleq x$ for all $x\in\partial\Omega_2\cap\mathcal{P}$ and $n\in\{1,2\ldots\}$. Otherwise there exist $n\in\{1,2,\ldots\}$ and $x_0\in\partial\Omega_2\cap\mathcal{P}$ such that $A_nx_0\le x_0$. Since $x_0\in\partial\Omega_2\cap\mathcal{P}$,

$$\begin{aligned}
x_0(t)\ &\ge A_nx_0(t)\\
&=\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau,x_0(\tau),D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds\\
&\ge\int_a^b G(t,s)\phi^{-1}\left(\int_a^b H(s,\tau)q(\tau)f(\tau,\max\{\tfrac{1}{n},x_0(\tau)\},D^{\gamma}_{0+}x_0(\tau))d\tau\right)ds\\
&=(b-a)a_0\phi^{-1}\left(b_0\int_a^b q(\tau)N\phi(x_0(\tau))d\tau\right)\\
&=(b-a)a_0\phi^{-1}\left(b_0N\phi(R')\int_a^b q(\tau)d\tau\right)\\
&=(b-a)a_0\phi^{-1}\left([b_0N\int_a^b q(\tau)d\tau]\phi(R')\right)\\
&\ge(b-a)a_0\rho_1\left(b_0N\int_a^b q(\tau)d\tau\right)R'\\
&\ge(b-a)a_0b_0^{\frac{1}{\alpha_2}}N^{\frac{1}{\alpha_2}}\left(\int_a^b q(\tau)d\tau\right)^{\frac{1}{\alpha_2}}R'\\
&>\tfrac{R'}{c},
\end{aligned}$$

contradicting $\|x_0\|=\frac{R'}{c}$. Finally, Lemma 1.5 entails

$$i(A_n,\Omega_2\cap\mathcal{P},\mathcal{P})=0,\ \forall n\in\mathbb{N}^* \tag{4.5}$$

whereas (4.4) and (4.5) imply

$$i(A_n,(\Omega_2\setminus\Omega_1)\cap\mathcal{P},\mathcal{P})=-1,\ \forall n\ge n_0. \tag{4.6}$$

Then A_n has a second fixed point $y_n\in(\Omega_2\setminus\Omega_1)\cap\mathcal{P}$, $\forall n\ge n_0$.
In addition $y_n(t)\ge\lambda_1\rho(t)R$, $\forall t\in[0,1]$ and $\|y_n\|<\frac{R'}{c}$. As above, we can show that $(y_n)_{n\ge n_0}$ has a subsequence $(y_{n_j})_{j\ge1}$ such that $\lim\limits_{j\to+\infty}y_{n_j}=y_0$ and y_0 is a solution of Problem (1.1). Finally $R\le\|y_0\|<\frac{R'}{c}$, i.e., x_0 and y_0 are two positives solutions of Problem (1.1). □

Example 4.5. Consider the BVP

$$\begin{cases}-D^{\frac{7}{4}}_{0+}\left(-D^{\frac{9}{4}}_{0+}x(t)\right)^{\frac{1}{3}}=\delta t^{\frac{5}{4}}\dfrac{e^{2x}Ch(D^{\frac{1}{6}}_{0+}x)}{x}, & 0<t<1\\ x(0)=x'(0)=D^{\frac{5}{4}}_{0+}x(1)=D^{\frac{9}{4}}_{0+}x(0)=[D^{\frac{3}{4}}_{0+}(\phi(-D^{\frac{9}{4}}_{0+}x(t))]_{t=1}=0,\end{cases} \tag{4.7}$$

where $f(t,x,y)=\delta\frac{e^{2x+t}Ch(y)}{x}$, $(\delta>0)$, $q(t)=t^{\frac{5}{4}}e^{-t}$. $\phi(t)=t^{\frac{1}{3}}$. Hence ϕ is an increasing homeomorphism, $\phi(0)=0$, and there exist $\alpha_1=\frac{1}{4},\alpha_2=2$ such that

$$t^2\phi(x)\le\phi(tx)\le t^{\frac{1}{4}}\phi(x),\forall t\in[0,1],\ \forall x\ge0.$$

$(\mathcal{H}_3')$

$$\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)}\ge\lim_{x\to+\infty}\frac{\delta e^{2x}}{x^{\frac{4}{3}}}=+\infty,\quad \forall t\ge0,\ \forall y\ge0.$$

Choosing $\delta\le\sup\limits_{c>0}\left(\frac{81c^4}{(e^cCh(c))^3}\right)^{\frac{1}{3}}$, all conditions of Theorem 4.4 are fulfilled and Problem (4.7) has at least two positive solutions.

In the last part of this work, the nonlinear function f may be degenerating at both $x=0$ and $y=0$. More precisely $f:[0,1]\times I\times I\longrightarrow\mathbb{R}^+$ satisfies Assumption $(\mathcal{H}_1'')$, i.e., there exist $m\in C([0,1],\mathbb{R}^+)$ and $g,h,\psi,l\in C(I,I)$ such that h,l are decreasing functions and $\frac{\psi}{l},\frac{g}{h}$ are increasing functions and satisfies

$$f(t,x,y)\leq m(t)g(x)\psi(y),\quad \forall\, t\in[0,1],\forall\, x,y\in I,$$

and for any $c,c'>0$,

$$\textstyle\int_0^1 q(\tau)m(\tau)h(c\rho(\tau))l(c'\rho(t)d\tau<+\infty.$$

Assumption $(\mathcal{H}_2'')$ is

$$\sup_{c>0}\frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c)\psi(c)}{\Gamma(\alpha)h(c)l(c)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)c)l(\lambda_2\rho(\tau)c)d\tau\right)}>1.$$

Regarding Assumption $(\mathcal{H}_3'')$, there exist a,b $(0<a<b<1)$ such that

$$\lim_{x\to+\infty}\frac{f(t,x,y)}{\phi(x)}=+\infty,\quad \text{uniformly in } t\in[a,b] \text{ and } y>0.$$

As for Assumption $(\mathcal{H}_4'')$, we have that for any $c>0$, there exist $\psi_c\in C([0,1],\mathbb{R}^+)$ and an interval $J\subset(0,1]$ such that $\psi_c(t)>0$, in J and

$$f(t,x,y)\geq\psi_c(t),\quad \forall\, t\in[0,1],\ \forall\, x,y\in(0,c].$$

For $f\in C([0,1]\times I\times I,\mathbb{R}^+)$, define the sequence $(f_n)_{n\geq1}$ by

$$f_n(t,x,y)=f(t,\max\{\tfrac{1}{n},x\},\max\{\tfrac{1}{n},y\}),\quad n\in\{1,2,\ldots\}$$

and for $x\in\mathcal{P}$, define the sequence of operators

$$A_n(x)(t)=\int_0^1 G(t,s)\phi^{-1}\left(\int_0^1 H(s,\tau)q(\tau)f_n(\tau,x(\tau),D_{0^+}^{\gamma}x(\tau))d\tau\right)ds.$$

Then

$$D_{0^+}^{\gamma}A_n(x)(t)=\int_0^1 D_{0^+}^{\gamma}G(t,s)\phi^{-1}\left(\int_0^1 H_2(s,\tau)q(\tau)f_n(\tau,x(\tau),D_{0^+}^{\gamma}x(\tau))d\tau\right)ds.$$

As for Proposition 3.1, we can prove

Proposition 4.6. *Suppose $(\mathcal{H}_1'')$ holds then, for each $n\geq1$, the operator A_n sends $\mathcal{P}$ into $\mathcal{P}$ and is completely continuous.*

As in the previous cases, we prove the existence of one solution and then two solutions. The first result is

Theorem 4.7. *Assume that $(\mathcal{H}_1''),(\mathcal{H}_2''),(\mathcal{H}_4'')$ hold. Then Problem (1.1) has at least one positive solution.*

Proof. From the condition $(\mathcal{H}_2'')$, there exists $R>0$ such that

$$\frac{R}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(R)\psi(R)}{\Gamma(\alpha)h(R)l(R)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)R)l(\lambda_2\rho(\tau)R)d\tau\right)}>1. \quad (4.8)$$

Let $\Omega_1 = \{x \in E : \|x\| < R\}$. We claim that $x \neq \lambda A_n(x)$, for any $x \in \partial\Omega_1 \cap \mathcal{P}$, $\lambda \in (0,1]$ and $n \geq n_0 \geq \frac{1}{R}$. On the contrary, there exist $n_1 \geq n_0$, $x_1 \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_0 \in (0,1]$ such that $x_1 = \lambda_0 A_{n_1} x_1$. Since $x_1 \in \partial\Omega_1 \cap \mathcal{P}$, then

$$x_1(t) \geq \lambda_1 \rho(t)\|x_1\| = \lambda_1 \rho(t) R, \ \forall\, t \in [0,1]$$

and

$$D^{\gamma}_{0^+} x_1(t) \geq \lambda_2 \rho(t)\|x_1\| = \lambda_2 \rho(t) R, \ \forall\, t \in [0,1].$$

Hence

$$\begin{aligned}
R &= \|x_1\| \\
&= \|\lambda_0 A_{n_1} x_1\| \\
&\leq \|A_{n_1} x_1\|_1 + \|A x_1\|_2 \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}] \\
&\phi^{-1}\left(\int_0^1 H(s,\tau) q(\tau) f(\tau, \max\{\tfrac{1}{n}, x_1(\tau)\}, \max\{\tfrac{1}{n}, D^{\gamma}_{0^+} x_1(\tau)\} d\tau\right) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}] \\
&\phi^{-1}\left(\tfrac{1}{\Gamma(\alpha)} \int_0^1 q(\tau) m(\tau) g(\max\{\tfrac{1}{n_1}, x_1(\tau)\}) \psi(\max\{\tfrac{1}{n}, D^{\gamma}_{0^+} x_1(\tau)\}) d\tau\right) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}] \phi^{-1}(\tfrac{1}{\Gamma(\alpha)} \int_0^1 q(\tau) m(\tau) h(\max\{\tfrac{1}{n_1}, x_1(\tau)\}) \tfrac{g(\max\{\frac{1}{n_1}, x_1(\tau)\})}{h(\max\{\frac{1}{n_1}, x_1(\tau)\})} \\
&\tfrac{\psi(\max\{\frac{1}{n}, D^{\gamma}_{0^+} x_1(\tau)\})}{l(\max\{\frac{1}{n}, D^{\gamma}_{0^+} x_1(\tau)\})} l(\max\{\tfrac{1}{n}, D^{\gamma}_{0^+} x_1(\tau)\}) d\tau) \\
&\leq [\tfrac{1}{\Gamma(\beta)} + \tfrac{1}{\Gamma(\beta-\gamma)}] \phi^{-1}\left(\tfrac{g(R)\psi(R)}{h(R) l(R) \Gamma(\alpha)} \int_0^1 q(\tau) m(\tau) h(\lambda_1 \rho(\tau) R) l(\lambda_2 \rho(\tau) R) d\tau\right)
\end{aligned}$$

which is a contraction to (4.8). By Lemma 1.4, we deduce that

$$i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \ \text{ for all } \ n \in \{n_0, n_0+1, \ldots\}. \tag{4.9}$$

Then there exists $x_n \in \Omega_1 \cap \mathcal{P}$ such that $A_n x_n = x_n$; $\forall\, n \geq n_0$. As in the proof of Theorem 4.2(2), (x_n) is proven to be relatively compact in E and thus there exists a subsequence $(x_{n_k})_{k\geq 1}$ such that $\lim\limits_{k\to+\infty} x_{n_k} = x_0$, where x_0 is a positive solution of Problem (1.1). □

Example 4.8. Consider the BVP

$$\begin{cases} -D^{\frac{11}{6}}_{0^+} \phi\left(-D^{\frac{11}{5}}_{0^+} x(t)\right) = \delta t^{\frac{12}{5}} (1-t)^{\frac{12}{5}} e^{-t} \dfrac{e^{x + D^{\frac{1}{2}}_{0^+} x}}{x D^{\frac{1}{2}}_{0^+} x}, \ t \in (0,1) \\ x(0) = x'(0) = D^{\frac{5}{4}}_{0^+} x(1) = D^{\frac{11}{5}}_{0^+} x(0) = [D^{\frac{5}{6}}_{0^+} (\phi(-D^{\frac{11}{5}}_{0^+} x(t))]_{t=1} = 0, \end{cases} \tag{4.10}$$

where

$$f(t,x,y) = \delta t^{\frac{5}{4}} e^{-t} \frac{e^{x+y}}{xy}, \ (\delta > 0), \ q(t) = t^{\frac{12}{5}} (1-t)^{\frac{12}{5}} \text{ and } \phi(t) = t^3 + t.$$

Hence ϕ is an increasing homeomorphism such that $\phi(0) = 0$.
$(\mathcal{H}_1'')$ Let $m(t) = 1, g(x) = \frac{e^x}{x}, \psi(y) = \frac{e^y}{y}, h(y) = \frac{1}{y}, l(y) = \frac{1}{y}$. Then

$$f(t,x,y) \leq m(t) g(x) \psi(y), \ \ \forall\, t \in [0,1], \forall\, x, y \in I,$$

and for any $c, c' > 0$

$$\int_0^1 m(\tau)q(\tau)h(c\rho(\tau))l(c'\rho(\tau))d\tau = \frac{35}{36cc'} < +\infty.$$

$(\mathcal{H}_2'')$

$$\begin{aligned}&\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\Big(\frac{g(c)\psi(c)}{\Gamma(\alpha)h(c)l(c)}\int_0^1 q(\tau)m(\tau)h(\lambda_1\rho(\tau)c)l(\lambda_2\rho(\tau)c)d\tau\Big)}\\ &\geq \sup_{c>0} \frac{0.94c}{\phi^{-1}(\delta\frac{e^{2c}}{c^2})}\\ &\geq \frac{1}{\delta^{\frac{1}{4}}}\sup_{c>0} \frac{0.94c}{\phi^{-1}(\frac{e^{2c}}{c^2})}.\end{aligned}$$

$(\mathcal{H}_4'')$ For any $c > 0$ there exists $\psi_c = \frac{t^{\frac{12}{5}}(1-t)^{\frac{12}{5}}}{c^2}$ such that

$$f(t, x, y) \geq \psi_c(t), \quad \forall\, t \in [0, 1],\ \forall\, x \in (0, c], y \in (0, c].$$

For $\delta \leq \left(\sup\limits_{c>0} \frac{0.94c}{\phi^{-1}(\frac{e^{2c}}{c^2})}\right)^4$, all conditions of Theorem 4.7 hold. Then Problem (4.10) has at least one positive solution.

The last result of this work concerns the existence of two positive solutions. The proof is similar to the proof of Theorem 4.4 and is omitted.

Theorem 4.9. *Assume that* $(\mathcal{H}_1'') - (\mathcal{H}_4'')$ *hold and there exist* α_1, α_2 *with* $0 < \alpha_1 < \alpha_2$ *such that*

$$t^{\alpha_2}\phi(x) \leq \phi(tx) \leq t^{\alpha_1}\phi(x),\ \forall\, t \in [0, 1],\ \forall\, x \geq 0.$$

Then Problem (1.1) *has at least two positive solutions.*

Example 4.10. Let the BVP

$$\begin{cases} -D_{0^+}^{\frac{11}{6}}\phi\left(-D_{0^+}^{\frac{11}{5}}x(t)\right) = \delta t^{\frac{12}{5}}(1-t)^{\frac{12}{5}}e^{-t}\frac{e^{x+D_{0^+}^{\frac{1}{2}}x}}{xD_{0^+}^{\frac{1}{2}}x},\ t \in (0, 1) \\ x(0) = x'(0) = D_{0^+}^{\frac{5}{4}}x(1) = D_{0^+}^{\frac{11}{5}}x(0) = [D_{0^+}^{\frac{5}{6}}(\phi(-D_{0^+}^{\frac{11}{5}}x(t))]_{t=1} = 0, \end{cases} \tag{4.11}$$

where

$$f(t, x, y) = \delta t^{\frac{5}{4}}e^{-t}\frac{e^{x+y}}{xy},\ (\delta > 0),\ q(t) = t^{\frac{12}{5}}(1-t)^{\frac{12}{5}} \text{ and } \phi(t) = t^3 + t.$$

Hence ϕ is an increasing homeomorphism such that $\phi(0) = 0$. Moreover there exist $\alpha_1 = 1, \alpha_2 = 4$ such that

$$t^4\phi(x) \leq \phi(tx) \leq t\phi(x), \forall\, t \in [0, 1],\ \forall\, x \geq 0.$$

Assumption $(\mathcal{H}_3''')$ reads

$$\lim_{x\to+\infty} \frac{f(t, x, y)}{\phi(x)} \geq \lim_{x\to+\infty} \frac{\delta a^{\frac{5}{4}}e^{-b}e^x}{(x + x^3)x} = +\infty, \quad \forall t \in [a, b], \forall y > 0.$$

If we choose δ such that $\delta \leq \left(\sup\limits_{c>0} \frac{0.94c}{\phi(\frac{e^{2c}}{c^2})^{-1}}\right)^4$, all conditions of Theorem 4.9 hold. Consequently Problem (4.11) has at least two positive solutions.

Remark 4.11. The same results can be obtained in case the nonlinear function f has a singular point at $y = 0$ but not at $x = 0$. The corresponding assumptions are
($\mathcal{H}_1'''$) There exist $m \in C([0,1], \mathbb{R}^+), \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $g, l, \in C(I, I)$ such that l is a decreasing function and $g, \frac{\psi}{l}$ are increasing functions with

$$f(t,x,y) \leq m(t)g(x)\psi(y), \quad \forall\, t \in [0,1], \forall\, x \in \mathbb{R}^+,\ \forall\, y \in I$$

and for each $c > 0$,

$$\int_0^1 q(\tau)m(\tau)l(c\rho(\tau))d\tau < +\infty,$$

($\mathcal{H}_2'''$)

$$\sup_{c>0} \frac{c}{[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-\gamma)}]\phi^{-1}\left(\frac{g(c)\psi(c)}{\Gamma(\alpha)l(c)} \int_0^1 q(\tau)m(\tau)h(\lambda_2\rho(\tau)c)d\tau\right)} > 1.$$

($\mathcal{H}_3'''$) There exist a, b $(0 < a < b < 1)$ such that

$$\lim_{x\to+\infty} \frac{f(t,x,y)}{\phi(x)} = +\infty, \quad \text{uniformly in } \ t \in [a,b] \text{ and } y > 0.$$

($\mathcal{H}_4'''$) For any $c > 0$ there exists $\psi_c \in C([0,1], \mathbb{R}^+)$ and there exists an interval $J \subset (0,1]$ such that $\psi_c(t) > 0$, in J and

$$f(t,x,y) \geq \psi_c(t), \quad \forall\, t \in [0,1], \forall\ \ \forall\, x \in [0,c], y \in (0,c]$$

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Oscillation criteria for third-order semi-canonical differential equations with unbounded neutral coefficients

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Abstract. In this paper, we investigate the oscillatory behavior of solutions to a class of third-order differential equations of the form

$$\mathcal{L}z(t) + f(t)y^{\beta}(\sigma(t)) = 0,$$

where $\mathcal{L}z(t) = (p(t)(q(t)z'(t))')'$ is a semi-canonical operator and $z(t) = y(t) + g(t)y(\tau(t))$. The main idea is to convert the semi-canonical operator into canonical form and then obtain some new sufficient conditions for the oscillation of all solutions. The obtained results essentially improve and complement to the known results. Examples are provided to illustrate the main results.

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1. Introduction

In this paper, we are concerned with the oscillation of solutions of the semi-canonical third-order neutral differential equation

$$\mathcal{L}z(t) + f(t)y^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L}z(t) = (p(t)(q(t)z'(t))')', \quad z(t) = y(t) + g(t)y(\tau(t)),$$

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and β is the ratio of odd positive integers. Throughout the paper, and without further mention, we will always assume that:

(H_1) $f, g \in C([t_0, \infty), \mathbb{R})$, $g(t) \geq 1$, $g(t) \not\equiv 1$ for large t, and $f(t) \geq 0$ is not identically zero for large t,

(H_2) $\tau, \sigma \in C^1([t_0, \infty), \mathbb{R})$, $\tau(t) \leq t$, τ is strictly increasing, σ is nondecreasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(H_3) the operator $\mathcal{L}$ is in semi-canonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{p(t)} dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{q(t)} dt = \infty,$$

where $p, q \in C([t_0, \infty), (0, \infty))$.

By a *solution* of (1.1), we mean a function $y \in C([t_y, \infty), \mathbb{R})$ for some $t_y \geq t_0$ such that $z \in C^1([t_y, \infty), \mathbb{R})$, $qz' \in C^1([t_y, \infty), \mathbb{R})$, $p(qz')' \in C^1([t_y, \infty), \mathbb{R})$ and y satisfies (1.1) on $[t_y, \infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_y, \infty)$ and satisfy the condition

$$\sup\{|y(t)| : T_1 \leq t < \infty\} > 0 \quad \text{for any} \quad T_1 \geq t_y;$$

we tacitly assume that (1.1) possesses such solutions. Such a solution $y(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_y, \infty)$, and it is called *nonoscillatory* otherwise. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

In the recent years many papers appeared in the literature dealing with the oscillatory and asymptotic behavior of solutions of various classes of third-order neutral type differential equations; see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, 19, 20] and the references cited therein. However, except for the papers [5, 6, 12, 19, 20], all the papers mentioned above were dealing with the case when $g(t)$ is bounded, that is, the cases when $0 \leq g(t) \leq g_0 < 1$, $-1 < g_0 \leq g(t) \leq 0$ and $0 < g(t) \leq g_0 < \infty$ were studied and so the criteria obtained in these papers cannot be applied to the case $g(t) \to \infty$ as $t \to \infty$.

Moreover, very recently in [5, 6, 20] the authors studied equation (1.1) and obtained oscillation criteria where $q(t) \equiv 1$ and $p(t) \equiv 1$ or $\int_{t_0}^{\infty} \frac{1}{p(t)} dt = \infty$. Based on these observations, the aim of this paper is to obtain some oscillation criteria that can be applied not only to the case where $g(t) \to \infty$ as $t \to \infty$ but also to the cases when $g(t)$ is bounded, $\int_{t_0}^{\infty} \frac{1}{p(t)} dt < \infty$ and $\int_{t_0}^{\infty} \frac{1}{q(t)} dt = \infty$. The main idea is to connect the semi-canonical equation (1.1) with that of canonical equations and then we obtain oscillation criteria for (1.1).

In the sequel, we deal only with positive solutions of (1.1), since if $y(t)$ is a solution of (1.1), then $-y(t)$ is also a solution.

2. Main results

Throughout the paper we employ the following notations:

$$A(t) := \int_t^{\infty} \frac{1}{p(s)} ds, \quad a(t) := p(t)A^2(t), \quad b(t) := \frac{q(t)}{A(t)},$$

$$F(t) := A(t)f(t), \quad \Pi(t) := \int_{t_0}^{t} \frac{1}{a(s)} ds, \quad B(t) := \int_{t_0}^{t} \frac{\Pi(s)}{b(s)} ds,$$

$$c(t) := \exp\left(\int_{t_1}^{t} \frac{\Pi(s)}{b(s)B(s)} ds\right) \quad \text{for } t \geq t_1 \quad \text{for some } t_1 \geq t_0,$$

$$h(t) := \tau^{-1}(\sigma(t)), \quad \lambda(t) := \tau^{-1}(\eta(t)), \quad \eta \in C^1([t_0, \infty), \mathbb{R}),$$

$$\psi_1(t) := \frac{1}{g(\tau^{-1}(t))}\left[1 - \frac{c(\tau^{-1}(\tau^{-1}(t)))}{g(\tau^{-1}(\tau^{-1}(t)))c(\tau^{-1}(t))}\right],$$

$$\psi_2(t) := \frac{1}{g(\tau^{-1}(t))}\left[1 - \frac{1}{g(\tau^{-1}(\tau^{-1}(t)))}\right],$$

and

$$R(t) := \int_{h(t)}^{\lambda(t)} \left(\frac{1}{b(u)} \int_{u}^{\lambda(t)} \frac{1}{a(v)} dv\right) du.$$

In order to ensure the nonnegativity of $\psi_1(t)$, we assume the following condition also holds:

(H_4) There exists a $t_1 \in [t_0, \infty)$ such that

$$\frac{c(\tau^{-1}(\tau^{-1}(t)))}{g(\tau^{-1}(\tau^{-1}(t)))c(\tau^{-1}(t))} \leq 1 \quad \text{for all } t \geq t_1. \tag{2.1}$$

Theorem 2.1. *Assume that*

$$\int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty. \tag{2.2}$$

Then the semi-canonical operator $\mathcal{L}$ has the following unique canonical representation

$$\mathcal{L}z(t) = \frac{1}{A(t)}\left(p(t)A^2(t)\left(\frac{q(t)}{A(t)}z'(t)\right)'\right)'. \tag{2.3}$$

Proof. Direct calculation shows that

$$\begin{aligned}\left(p(t)A^2(t)\left(\frac{q(t)}{A(t)}z'(t)\right)'\right)' &= (A(t)p(t)(q(t)z'(t))' + q(t)z'(t))' \\ &= A(t)(p(t)(q(t)z'(t))')'.\end{aligned}$$

Therefore

$$\frac{1}{A(t)}\left(p(t)A^2(t)\left(\frac{q(t)}{A(t)}z'(t)\right)'\right)' = (p(t)(q(t)z'(t))')'.$$

Taking (2.2) into account, we see that

$$\int_{t_0}^{\infty} \frac{A(t)}{q(t)} dt = \infty,$$

and since

$$\int_{t_0}^{\infty} \frac{1}{p(t)A^2(t)} dt = \lim_{t \to \infty}\left(\frac{1}{A(t)} - \frac{1}{A(t_0)}\right) = \infty,$$

we say that (2.3) is in the canonical form. However, Trench proved in [18] that there exists only one canonical representation of $\mathcal{L}$ (up to multiplicative constants with product 1) and so our canonical form is unique. This completes the proof. $\square$

From Theorem 2.1, it follows that (1.1) can be written in the canonical form as

$$(a(t)(b(t)z'(t))')' + F(t)y^{\beta}(\sigma(t)) = 0 \tag{2.4}$$

and the next result is immediate.

Theorem 2.2. *Assume that* (2.2) *holds. Then semi-canonical equation* (1.1) *possesses solution* $y(t)$ *if and only if canonical equation* (2.4) *has the solution* $y(t)$.

Corollary 2.3. *Assume that* (2.2) *holds. Then semi-canonical differential equation* (1.1) *has an eventually positive solution if and only if canonical equation* (2.4) *has an eventually positive solution.*

Corollary 2.3 clearly simplifies investigation of (1.1) since for (2.4) if $y(t)$ is an eventually positive solution, then the corresponding function $z(t)$ satisfies either

(I) $z(t) > 0, \quad b(t)z'(t) > 0, \quad a(t)(b(t)z'(t))' > 0, \quad (a(t)(b(t)z'(t))')' < 0$, or

(II) $z(t) > 0, \quad b(t)z'(t) < 0, \quad a(t)(b(t)z'(t))' > 0, \quad (a(t)(b(t)z'(t))')' < 0$

for sufficiently large t.

Lemma 2.4. *Assume that $z(t)$ satisfies case (I) for all $t \geq t_1$ for some $t_1 \geq t_0$. Then*

$$z'(t) \geq \frac{\Pi(t)}{b(t)} a(t)(b(t)z'(t))', \tag{2.5}$$

$$z(t) \geq B(t)a(t)(b(t)z'(t))', \tag{2.6}$$

$$z(t) \geq \frac{B(t)}{\Pi(t)} b(t)z'(t), \tag{2.7}$$

and

$$\frac{z(t)}{c(t)} \quad \text{is nonincreasing} \tag{2.8}$$

for all $t \geq t_1$.

Proof. Since $a(t)(b(t)z'(t))'$ is positive and decreasing, we see that

$$b(t)z'(t) = b(t_1)z'(t_1) + \int_{t_1}^{t} a(s)\frac{(b(s)z'(s))'}{a(s)} ds$$

or

$$z'(t) \geq \frac{a(t)}{b(t)}(b(t)z'(t))'\Pi(t),$$

i.e., (2.5) holds. Integrating the last inequality from t_1 to t yields

$$z(t) \geq a(t)(b(t)z'(t))' \int_{t_1}^{t} \frac{\Pi(s)}{b(s)} ds = B(t)a(t)(b(t)z'(t))',$$

i.e., (2.6) holds. From (2.5), we see that $b(t)z'(t)/\Pi(t)$ is decreasing for $t \geq t_2$ for some $t_2 \geq t_1$, and therefore

$$z(t) = z(t_2) + \int_{t_2}^{t} \frac{b(s)z'(s)}{\Pi(s)} \frac{\Pi(s)}{b(s)} ds \geq \frac{B(t)}{\Pi(t)} b(t)z'(t).$$

From the last inequality, we see that

$$\left(\frac{z(t)}{c(t)}\right)' = \frac{\left(z'(t) - \frac{\Pi(t)}{b(t)B(t)}z(t)\right)}{c(t)} \le 0$$

for $t \ge t_3$ for some $t_3 \ge t_2$. Hence, $z(t)/c(t)$ is non-increasing. This completes the proof. □

Theorem 2.5. *Let* (2.2) *holds. Assume that there exists a nondecreasing function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that* $\sigma(t) \le \eta(t) < \tau(t)$ *for all* $t \ge t_0$. *If both first-order delay differential equations*

$$X'(t) + F(t)\Psi_1^\beta(\sigma(t))B^\beta(h(t))X^\beta(h(t)) = 0 \tag{2.9}$$

and

$$W'(t) + F(t)\Psi_2^\beta(\sigma(t))R^\beta(t)W^\beta(\lambda(t)) = 0 \tag{2.10}$$

oscillate, then (1.1) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. From Corollary 2.3, $y(t)$ is also a positive solution of (2.4) for $t \ge t_1$. Then the corresponding function $z(t)$ satisfies either case (I) or case (II) for $t \ge t_2$ for some $t_2 \ge t_1$.

First, we consider case (I). From the definition of z, we get

$$\begin{aligned} y(t) &= \frac{1}{g(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - y(\tau^{-1}(t))\right] \\ &\ge \frac{z(\tau^{-1}(t))}{g(\tau^{-1}(t))} - \frac{z(\tau^{-1}(\tau^{-1}(t)))}{g(\tau^{-1}(t))g(\tau^{-1}(\tau^{-1}(t)))}. \end{aligned} \tag{2.11}$$

Now $\tau(t) \le t$ and τ is strictly increasing, so τ^{-1} is increasing and $t \le \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \le \tau^{-1}(\tau^{-1}(t)).$$

From this and the fact that $z(t)/c(t)$ is nonincreasing, we see that

$$z\left(\tau^{-1}(\tau^{-1}(t))\right) \le \frac{c(\tau^{-1}(\tau^{-1}(t)))z(\tau^{-1}(t))}{c(\tau^{-1}(t))}. \tag{2.12}$$

Using (2.12) in (2.11) yields

$$y(t) \ge \psi_1(t)z(\tau^{-1}(t)). \tag{2.13}$$

Since $\lim_{t\to\infty}\sigma(t) = \infty$, we can choose $t_3 \ge t_2$ such that $\sigma(t) \ge t_2$ for all $t \ge t_3$. Thus, it follows from (2.13) that

$$y(\sigma(t)) \ge \psi_1(\sigma(t))z(h(t)) \quad \text{for } t \ge t_3. \tag{2.14}$$

Combining (2.14) with (2.4) yields

$$(a(t)(b(t)z'(t))')' + F(t)\psi_1^\beta(\sigma(t))z^\beta(h(t)) \le 0 \quad \text{for } t \ge t_3. \tag{2.15}$$

From (2.6), we have

$$z(h(t)) \ge B(h(t))a(h(t))(b(h(t))z'(h(t)))'. \tag{2.16}$$

Using (2.16) in (2.15) and letting $X(t) = a(t)(b(t)z'(t))'$, we see that $X(t)$ is a positive solution of the first-order delay differential inequality

$$X'(t) + F(t)\psi_1^\beta(\sigma(t))B^\beta(h(t))X^\beta(h(t)) \le 0. \tag{2.17}$$

Therefore, by Corollary 1 of [14], we conclude that (2.9) also has a positive solution, which is a contradiction.

Next, we consider case (II). Since z is strictly decreasing and $\tau(t) \le t$, we have

$$z(\tau^{-1}(t)) \ge z(\tau^{-1}(\tau^{-1}(t)))$$

and using this in (2.11), we obtain

$$y(t) \ge \psi_2(t)z(\tau^{-1}(t)).$$

Hence,

$$y(\sigma(t)) \ge \psi_2(\sigma(t))z(h(t)) \tag{2.18}$$

for $t \ge t_3$ for some $t_3 \ge t_2$. Using (2.18) in (2.4) yields

$$(a(t)(b(t)z'(t))')' + F(t)\psi_2^\beta(\sigma(t))z^\beta(h(t)) \le 0 \quad \text{for } t \ge t_3. \tag{2.19}$$

For $t \ge s \ge t_3$, we have

$$b(t)z'(t) - b(s)z'(s) = \int_s^t \frac{a(u)(b(u)z'(u))'}{a(u)}du,$$

or

$$-z'(s) \ge \left(\frac{1}{b(s)}\int_s^t \frac{1}{a(u)}du\right) a(t)(b(t)z'(t))'.$$

Again integrating, we have

$$-z(t) + z(s) \ge \left(\int_s^t \frac{1}{b(u)}\left(\int_u^t \frac{1}{a(v)}dv\right)du\right) a(t)(b(t)z'(t))',$$

or

$$z(s) \ge \left[\int_s^t \frac{1}{b(u)}\left(\int_u^t \frac{1}{a(v)}dv\right)du\right] a(t)(b(t)z'(t))'. \tag{2.20}$$

Since $\sigma(t) \le \eta(t)$ and the fact that τ is strictly increasing, we have

$$\tau^{-1}(\sigma(t)) \le \tau^{-1}(\eta(t)).$$

Setting $s = \tau^{-1}(\sigma(t))$ and $t = \tau^{-1}(\eta(t))$ into (2.20), we obtain

$$z(h(t)) \ge \left(\int_{h(t)}^{\lambda(t)} \frac{1}{b(u)}\left(\int_u^{\lambda(t)} \frac{1}{a(v)}dv\right)du\right) a(\lambda(t))(b(\lambda(t))z'(\lambda(t)))'. \tag{2.21}$$

Using (2.21) in (2.19) and letting $W(t) = a(t)(b(t)z'(t))'$, we see that W is a positive solution of the first-order delay differential inequality

$$W'(t) + F(t)\psi_2^\beta(\sigma(t))R^\beta(t)W^\beta(\lambda(t)) \le 0. \tag{2.22}$$

The remaining part of the proof is similar to the case (I) and hence the details are not repeated. This completes the proof. □

Corollary 2.6. *Let* (2.2) *holds and* $\beta = 1$. *Assume that there exists a nondecreasing function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that* $\sigma(t) \leq \eta(t) < \tau(t)$ *for all* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} F(s)\psi_1(\sigma(s))B(h(s))ds > \frac{1}{e} \tag{2.23}$$

and

$$\liminf_{t\to\infty} \int_{\lambda(t)}^{t} F(s)\psi_2(\sigma(s))R(s)ds > \frac{1}{e}, \tag{2.24}$$

then (1.1) *is oscillatory.*

Proof. The proof follows from a well-known result in [11] and Theorem 2.5, and hence the details are omitted. □

Corollary 2.7. *Let* (2.2) *holds and* $0 < \beta < 1$. *Assume that there exists a nondecreasing function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that* $\sigma(t) \leq \eta(t) < \tau(t)$ *for all* $t \geq t_0$. *If*

$$\int_T^\infty F(t)\psi_1^\beta(\sigma(t))B^\beta(h(t))dt = \infty \tag{2.25}$$

and

$$\int_T^\infty F(t)\psi_2^\beta(\sigma(t))R^\beta(t)dt = \infty \tag{2.26}$$

for all sufficiently large $T \in [t_0, \infty)$ *with* $\sigma(t) \geq t_0$ *for all* $t \geq T$, *then* (1.1) *oscillates.*

Proof. Proceeding exactly as in the proof of Theorem 2.5, we again arrive at (2.17) and (2.22) for $t \geq t_3$. Since $h(t) < t$ and $X(t)$ is positive and decreasing, inequality (2.17) takes the form

$$X'(t) + F(t)\psi_1^\beta(\sigma(t))B^\beta(h(t))X^\beta(t) \leq 0,$$

or

$$\frac{X'(t)}{X^\beta(t)} + F(t)\psi_1^\beta(\sigma(t))B^\beta(h(t)) \leq 0. \tag{2.27}$$

Integrating (2.27) from t_3 to t yields

$$\int_{t_3}^{t} F(s)\psi_1^\beta(\sigma(s))B^\beta(h(s))ds \leq \frac{X^{1-\beta}(t_3)}{1-\beta} < \infty \quad \text{as } t \to \infty,$$

which contradicts (2.25). The remainder of the proof follows from $\lambda(t) < t$ and inequality (2.22). The proof is complete. □

In our final result, assume that $\sigma(t) = t - \delta_1$, $\tau(t) = t - \delta_3$ and $\eta(t) = t - \delta_2$, where δ_1, δ_2 and δ_3 are positive real numbers.

Corollary 2.8. *Let* (2.2) *holds and* $\beta > 1$. *If* $\delta_1 \geq \delta_2 > \delta_3$,

$$\liminf_{t\to\infty} \beta^{-t/(\delta_1-\delta_3)} \log\Big(F(t)\psi_1^\beta(t-\delta_1)B^\beta(t+\delta_3-\delta_1)\Big) > 0 \tag{2.28}$$

and

$$\liminf_{t\to\infty} \beta^{-t/(\delta_2-\delta_3)} \log\Big(F(t)\psi_2^\beta(t-\delta_1)R^\beta(t)\Big) > 0, \tag{2.29}$$

then (1.1) *oscillates.*

Proof. Application of (2.28) and (2.29) and Corollary 1.2 of [15] imply that (2.9) and (2.10) oscillate. Hence, by Theorem 2.5, equation (1.1) oscillates. □

3. Examples

In this section, we present some examples to show the importance of the main results.

Example 3.1. Consider the third-order linear neutral differential equation

$$\left(t^2\left(\frac{1}{t}\left(y(t)+16y\left(\frac{t}{2}\right)\right)'\right)'\right)'+\frac{f_0}{t^2}y\left(\frac{t}{4}\right)=0,\quad t\geq 1. \tag{3.1}$$

Here $p(t)=t^2$, $q(t)=1/t$, $g(t)=16$, $f(t)=f_0/t^2$ with $f_0>0$, $\tau(t)=t/2$, $\sigma(t)=t/4$ and $\beta=1$. Then $A(t)=1/t$, $a(t)=1$, $b(t)=1$, $F(t)=f_0/t^3$ and the transformed equation is

$$\left(y(t)+16y\left(\frac{t}{2}\right)\right)'''+\frac{f_0}{t^3}y\left(\frac{t}{4}\right)=0,\quad t\geq 1, \tag{3.2}$$

which is in canonical form. Simple calculation show that

$$\Pi(t)=t-1,\ B(t)=(t-1)^2/2,\ c(t)=(t-1)^2,\ \text{and}\ \psi_2(t)=15/256.$$

Since (2.1) holds, we have $\psi_1(t)\geq 0$ and

$$\psi_1(t)=\frac{1}{16}\left[1-\frac{(4t-1)^2}{16(2t-1)^2}\right]\geq\frac{7}{256}.$$

By choosing $\eta(t)=t/3$, we see that $h(t)=t/2$, $\lambda(t)=2t/3$ and $R(t)=t^2/72$. It is clear that condition (2.2) holds. Condition (2.23) becomes

$$\liminf_{t\to\infty}\int_{t/2}^{t}\frac{f_0}{2^9}\left(\frac{3}{s}-\frac{14}{s^2}+\frac{15}{s^3}\right)ds=\frac{3f_0\ln 2}{2^9},$$

and so condition (2.22) is satisfied if $f_0>\dfrac{2^9}{3e\ln 2}$.

Condition (2.24) becomes

$$\liminf_{t\to\infty}\int_{2t/3}^{t}\frac{5f_0}{3\times 2^{11}}\frac{1}{s}ds=\frac{5f_0\ln 3/2}{3\times 2^{11}},$$

that is, (2.24) is satisfied if $f_0>\dfrac{3\times 2^{11}}{5e\ln 3/2}$. Thus, by Corollary 2.6, equation (3.1) is oscillatory if $f_0>\dfrac{3\times 2^{11}}{5e\ln 3/2}$.

Note that canonical equation (3.2) is considered in [20] and proved that (3.2) is oscillatory if $f_0>\frac{3\times 2^{11}}{5\ln 3/2}$. Hence, Corollary 2.6 improves Theorem 2.7 of [20].

Example 3.2. Consider the third-order sublinear neutral differential equation

$$\left(t^2\left(\frac{1}{t}\left(y(t)+ty\left(\frac{t}{2}\right)\right)'\right)'\right)'+\frac{f_0}{t^\alpha}y^{3/5}\left(\frac{t}{10}\right)=0,\quad t\geq 16. \tag{3.3}$$

Here $p(t)=t^2$, $q(t)=1/t$, $g(t)=t$, $f(t)=f_0/t^\alpha$ with $f_0>0$ and $\alpha\leq 3/5$, $\tau(t)=t/2$, $\sigma(t)=t/10$ and $\beta=3/5$. Then $A(t)=1/t$, $a(t)=1$, $b(t)=1$, $F(t)=f_0/t^{\alpha+1}$ and the transformed equation is

$$\left(y(t)+ty\left(\frac{t}{2}\right)\right)'''+\frac{f_0}{t^{\alpha+1}}y^{3/5}\left(\frac{t}{10}\right)=0,\quad t\geq 16, \tag{3.4}$$

which is in canonical form. Simple calculation shows that

$$\Pi(t)=t-16,\ B(t)=(t-16)^2/2,\ c(t)=(t-16)^2,\ \text{and}\ \psi_2(t)=\frac{4t-1}{8t^2}>0.$$

Since (2.1) holds, we have $\psi_1(t)\geq 0$ and $\psi_1(t)\geq\dfrac{4t-9}{8t^2}$. By choosing $\eta(t)=t/8$, we see that $h(t)=t/5$, $\lambda(t)=t/4$ and $R(t)=t^2/800$. It is clear that condition (2.2) holds. For any $T\geq t_0$ with $\sigma(t)\geq t_0$, condition (2.25) becomes

$$\int_T^\infty\frac{f_0}{t^{\alpha+1}}\left(\frac{10t-225}{2t^2}\right)^{3/5}\left(\frac{t-80}{\sqrt{50}}\right)^{6/5}dt\geq d_1\int_{T_1}^\infty\frac{1}{t^{\alpha+2/5}}dt=\infty,$$

where $d_1>0$ is a constant and $T_1\geq T$.

Condition (2.26) becomes

$$\int_T^\infty\frac{f_0}{t^{\alpha+1}}\left(\frac{10t-25}{2t^2}\right)^{3/5}\left(\frac{t^2}{800}\right)^{3/5}dt\geq d_2\int_{T_1}^\infty\frac{1}{t^{\alpha+2/5}}=\infty,$$

where $d_2>0$ is a constant and $T_1\geq T$. Thus, by Corollary 2.7, equation (3.3) is oscillatory if $\alpha\leq 3/5$.

Note that canonical equation (3.4) is considered in [20] and proved that (3.4) is oscillatory if $\alpha=\frac{1}{5}$. Hence, Corollary 2.7 improves Theorem 2.8 of [20].

Example 3.3. Consider the third-order superlinear neutral differential equation

$$\left(t^2\left(\frac{1}{t}\left(y(t)+ty\left(t-2\right)\right)'\right)'\right)'+t\exp(4^t)y^3(t-4)=0,\quad t\geq 2. \tag{3.5}$$

Here $p(t)=t^2$, $q(t)=1/t$, $g(t)=t$, $f(t)=t\exp(4^t)$, $\tau(t)=t-2$, $\sigma(t)=t-4$ and $\beta=3$. Then $A(t)=1/t$, $a(t)=1$, $b(t)=1$, $F(t)=\exp(4^t)$ and the transformed equation is

$$(y(t)+ty(t-2))'''+\exp(4^t)y^3(t-4)=0, \tag{3.6}$$

which is in canonical form. A simple calculation show that

$$\Pi(t)=t-2,\ B(t)=(t-2)^2/2,\ c(t)=(t-2)^2,$$

$$\psi_1(t)=\frac{1}{t+2}\left[1-\frac{(t+2)^2}{(t+4)t^2}\right]\geq\frac{t}{(t+2)(t+4)}\geq 0\ \text{and}\ \psi_2(t)=\frac{t+3}{(t+2)(t+4)}\geq 0.$$

By choosing $\eta(t) = t - 3$, we see that $h(t) = t - 2$, $\lambda(t) = t - 1$, $R(t) = 1/2$, $\delta_1 = 4$, $\delta_2 = 3$, $\delta_3 = 2$. As in Examples 3.1 and 3.2, it is easy to see that conditions (2.2), (2.28) and (2.29) are satisfied. Thus, by Corollary 2.8, equation (3.5) is oscillatory.

4. Conclusion

In this paper, we have established some new oscillation criteria for (1.1). The results are obtained by converting (1.1) into canonical type equation. Hence, the results are new and complement to those in [5, 6, 12, 20]. Also we have shown that the results obtained here improve those in [20].

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Exponential dichotomy and invariant manifolds of semi-linear differential equations on the line

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Abstract. In this paper we investigate the homogeneous linear differential equation $v'(t) = A(t)v(t)$ and the semi-linear differential equation

$$v'(t) = A(t)v(t) + g(t, v(t))$$

in Banach space X, in which $A : \mathbb{R} \to \mathcal{L}(X)$ is a strongly continuous function, $g : \mathbb{R} \times X \to X$ is continuous and satisfies φ-Lipschitz condition. The first we characterize the exponential dichotomy of the associated evolution family with the homogeneous linear differential equation by space pair $(\mathcal{E}, \mathcal{E}_\infty)$, this is a Perron type result. Applying the achieved results, we establish the robustness of exponential dichotomy. The next we show the existence of stable and unstable manifolds for the semi-linear differential equation and prove that each a fiber of these manifolds is differentiable submanifold of class C^1.

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1. Introduction

The exponential dichotomy for the homogeneous linear differential equation $v'(t) = A(t)v(t)$ was extensively studied by mathematicians, for instance, Perron [15], Massera and Schäffer [13], Daleckii and Krein [5], Coppel [4], Chicone and Latushkin [3]. To characterize the exponential dichotomy for the homogeneous linear differential equation, Perron's method has played an underlying role up to now. Some

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efforts improved Perron's result by following two directions: one is to extend the notion of exponential dichotomy [7, 1], and the other is to change admissible space pair (input-output spaces) [16, 12, 19, 9, 17, 18].

Huy [9] had characterized the exponential dichotomy of evolution equations on a half-line by using the notion of admissible Banach function space. Through this notion, his group got some extended results for the existence of stable and unstable manifolds of evolution equations [10, 11].

In [1], the authors investigated the exponential dichotomy for the homogeneous linear differential equation $v'(t) = A(t)v(t)$, in which $A : \mathbb{R} \to \mathcal{L}(X)$ is a strongly continuous function. The notion of exponential dichotomy in [1] was with respect to the family of norms $\|\cdot\|_t$ on X for $t \in \mathbb{R}$. It was characterized by space pair (Y, Y), where $Y = C_b(\mathbb{R}, X)$ is equipped with the norm $\|v\|_\infty = \sup_{t\in\mathbb{R}} \|v(t)\|_t$, for $v \in Y$. So, the paper [1] has inspired us to investigate the exponential dichotomy for the homogeneous linear differential equation $v'(t) = A(t)v(t)$ in the present paper. Different from [1], in this paper we consider Banach space X with a fixed norm but our space pair is wider.

It is the aim of this paper to investigate the homogeneous linear differential equation $v'(t) = A(t)v(t)$ and the semi-linear differential equation $v'(t) = A(t)v(t) + g(t, v(t))$ in Banach space X, in which $A : \mathbb{R} \to \mathcal{L}(X)$ is a strongly continuous function, $g : \mathbb{R} \times X \to X$ is continuous and satisfies φ-Lipschitz condition. In Section 2 we use Perron's method to characterize the exponential dichotomy of the associated evolution family with the homogeneous linear differential equation by space pair $(\mathcal{E}, \mathcal{E}_\infty)$, the achieved result is a significant improvement compared to previous results for the homogeneous linear differential equation. As an application of this characterization, we get the robustness of exponential dichotomy.

The stable manifold theorem is one of the most important results in the local qualitative theory of autonomous nonlinear differential equations, see [2, 8, 14]. It was extended for the semi-linear differential equation $v'(t) = A(t)v(t) + g(t, v(t))$ in Banach space X, where g satisfies constant Lipschitz condition, i.e, there exists $q > 0$ such that $\|g(t, x) - g(t, y)\| \leq q\|x - y\|$ for all $t \in \mathbb{R}$ and $x, y \in X$, see [5]. In Section 3 we show the existence of stable and unstable manifolds for the semi-linear differential equation $v'(t) = A(t)v(t) + g(t, v(t))$, in which g satisfies φ-Lipschitz condition, i.e, $\|g(t, x) - g(t, y)\| \leq \varphi(t)\|x - y\|$ for all $t \in \mathbb{R}$ and $x, y \in X$. Different from the constant Lipschitz case, the semi-linear differential equation surely exists solution on positive semi-axis if initial value lies in a fiber of stable manifold and on negative semi-axis if initial value lies in a fiber of unstable manifold. The same as autonomous nonlinear differential equations, each a fiber of these manifolds is differentiable submanifold of class C^1 if the map $g(t, \cdot)$ is continuously differentiable (in the sense Fréchet derivative) on X for each fixed $t \in \mathbb{R}$.

The remainder in this section, we recall some notions on Banach function spaces on the line in the paper [6]. Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}$. The space $L_{1,loc}(\mathbb{R})$ of real-valued locally integrable functions on $\mathbb{R}$ becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{Z}$ (see [13, Chapt. 2, §20]).

Definition 1.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ is called *a Banach function space* (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if

1) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$, λ-a.e., then $\psi \in E$ and $\|\psi\|_E \leq \|\varphi\|_E$,
2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$,
3) $E \hookrightarrow L_{1,loc}(\mathbb{R})$, i.e., for each seminorm p_n of $L_{1,loc}(\mathbb{R})$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leq \beta_{p_n} \|f\|_E$ for all $f \in E$.

The following lemma is very useful in the later sections.

Lemma 1.2. *Let E be a Banach function space. Let φ and ψ be real-valued, measurable functions on $\mathbb{R}$ such that they coincide with each other outside a compact interval and they are essentially bounded on this compact interval. Then $\varphi \in E$ if only if $\psi \in E$.*

Definition 1.3. Let now E be a Banach function space and X a Banach space. The set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

is endowed the norm

$$\|f\|_{\mathcal{E}} := \|\|f(\cdot)\|\|_E.$$

Then, $\mathcal{E}$ is a Banach space and is called *Banach space corresponding to the Banach function space E.*

Definition 1.4. The Banach function space E is called *admissible* if

1. there is a constant $M \geq 1$ such that for every compact interval $[a, b] \subset \mathbb{R}$ we have
$$\int_a^b |\varphi(t)|dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \text{ for all } \varphi \in E, \tag{1.1}$$
2. for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1} \varphi(\tau)d\tau$ belongs to E.
3. E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined by
$$T_\tau^+\varphi(t) := \varphi(t-\tau) \text{ for } t \in \mathbb{R},$$
$$T_\tau^-\varphi(t) := \varphi(t+\tau) \text{ for } t \in \mathbb{R},$$
and there exists constants N_1, N_2 such that $\|T_\tau^+\| \leq N_1$, $\|T_\tau^-\| \leq N_2$ for all $\tau \in \mathbb{R}_+$.

Remark 1.5. It can be easily seen that if E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R})$, where

$$\mathbf{M}(\mathbb{R}) = \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(\tau)|d\tau < \infty \right\}.$$

We now collect some properties of admissible Banach function space in the following proposition, see [6, Proposition 2.3] for complete proof.

Proposition 1.6. *Let E be an admissible Banach function space. The following assertions hold.*

(a) Let $\varphi \in L_{1,\,loc}(\mathbb{R})$ such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$, where $\Lambda_1\varphi$ is defined as in Definition 1.4(ii). For $\sigma > 0$ we define functions $\Lambda_\sigma\varphi$ and $\bar{\Lambda}_\sigma\varphi$ by

$$\Lambda_\sigma\varphi(t) = \int_{-\infty}^{t} e^{-\sigma(t-s)}\varphi(s)ds,$$

$$\bar{\Lambda}_\sigma\varphi(t) = \int_{t}^{\infty} e^{-\sigma(s-t)}\varphi(s)ds.$$

Then, $\Lambda_\sigma\varphi$ and $\bar{\Lambda}_\sigma\varphi$ belong to E, and

$$\|\Lambda_\sigma\varphi\|_E \leq \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E, \quad \|\bar{\Lambda}_\sigma\varphi\|_E \leq \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E.$$

In particular, if $\sup_{t\in\mathbb{R}}\int_t^{t+1}|\varphi(\tau)|d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 1.5)) then $\Lambda_\sigma\varphi$ and $\bar{\Lambda}_\sigma\varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for sup-norm, we have

$$\|\Lambda_\sigma\varphi\|_\infty \leq \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \text{ and } \|\bar{\Lambda}_\sigma\varphi\|_\infty \leq \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty.$$

(b) E contains exponentially decaying functions $\psi(t) = e^{-\alpha|t|}$ for $t \in \mathbb{R}$ and $\alpha > 0$.

(c) E does not contain exponentially growing functions $f(t) = e^{bt}$ for $t \in \mathbb{R}$ and $b \neq 0$.

The associate space of Banach function space is defined as follows.

Definition 1.7. Let E be an admissible Banach function space and denote by $S(E)$ the unit sphere in E. Recall that $L_1(\mathbb{R}) = \{g : \mathbb{R} \to \mathbb{R} \,|\, g$ is Borel measurable and $\int_{-\infty}^{\infty}|g(t)|dt < \infty\}$. The set E' of all real-valued Borel-measureable functions ψ on $\mathbb{R}$ such that

$$\varphi\psi \in L_1(\mathbb{R}), \quad \int_{-\infty}^{\infty}|\varphi(t)\psi(t)|dt \leq k \quad \text{for all } \varphi \in S(E),$$

where k depends only on ψ. Then, E' is a normed space with the norm given by

$$\|\psi\|_{E'} := \sup\left\{\int_{-\infty}^{\infty}|\varphi(t)\psi(t)|dt : \varphi \in S(E)\right\} \qquad \text{for } \psi \in E'.$$

We call E' being *the associate space* of E.

Let E be an admissible Banach function space and E' be its associate space. Then, the following "Hölder-type inequality" holds:

$$\int_{-\infty}^{\infty}|\varphi(t)\psi(t)|dt \leq \|\varphi\|_E\|\psi\|_{E'} \quad \text{for all } \varphi \in E,\ \psi \in E'. \tag{1.2}$$

Definition 1.8. Let E be an admissible Banach function space and E' be its associate space. A positive function $\varphi \in E'$ is called *exponentially E-invariant* if for any fixed $\nu > 0$, the function h_ν defined by

$$h_\nu(t) := \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'} \quad \text{for } t \in \mathbb{R}$$

belongs to E.

2. Exponential dichotomy

Let $X = (X, \|\cdot\|)$ be a Banach space and $\mathcal{L}(X)$ be the set of all bounded linear operators on X. Assume that $A : \mathbb{R} \to \mathcal{L}(X)$ is strongly continuous function (that means the mapping $t \mapsto A(t)x$ is continuous on $\mathbb{R}$ for each $x \in X$). Then, the linear differential equation

$$v' = A(t)v, \quad t \in \mathbb{R} \tag{2.1}$$

generates an evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ on the Banach space X. This evolution family is strongly continuous, exponentially unbounded, differentiable and invertible (see [5] to more detailed informations), also called the associated evolution family with Eq. (2.1). In this section we characterize the exponential dichotomy of the associated evolution family with Eq. (2.1) and show that the exponential dichotomy is invariant under small perturbations. Firstly, we recall the concept of the exponential dichotomy of the evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ on the line.

Definition 2.1. The associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ is said to have an exponential dichotomy on the line if there exist bounded linear projections $P(t)$, $t \in \mathbb{R}$ on X and positive constants N, η, ν such that

(a)
$$T(t,\tau)P(\tau) = P(t)T(t,\tau), \quad t,\tau \in \mathbb{R}; \tag{2.2}$$

(b) for all $x \in X$ and $t \geq \tau$,
$$\begin{aligned} \|T(t,\tau)P(\tau)x\| &\leq Ne^{-\eta(t-\tau)}\|x\|, \\ \|T(\tau,t)Q(t)x\| &\leq Ne^{-\eta(t-\tau)}\|x\|; \end{aligned} \tag{2.3}$$

(c) for all $x \in X$ and $t \leq \tau$,
$$\begin{aligned} \|T(t,\tau)P(\tau)x\| &\leq Ne^{-\nu(t-\tau)}\|x\|, \\ \|T(\tau,t)Q(t)x\| &\leq Ne^{-\nu(t-\tau)}\|x\|; \end{aligned} \tag{2.4}$$

in which $Q(t) = I - P(t)$, $t \in \mathbb{R}$.

Note that Definition 2.1 is derived from the concept of the exponential dichotomy in [1] when the family of norms is a fixed norm for all $t \in \mathbb{R}$. This definition is also equivalent to the concept of the exponential dichotomy of a strongly continuous, exponentially bounded, and invertible evolution family.

To characterize the exponential dichotomy of the associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$, we define Banach space $\mathcal{E}_\infty$ as follows

$$\mathcal{E}_\infty = \mathcal{E} \cap C_b(\mathbb{R}, X) \ \text{ with the norm } \ \|f\|_{\mathcal{E}_\infty} = \max\{\|f\|_{\mathcal{E}}, \|f\|_\infty\}.$$

The next part we will characterize the exponential dichotomy of the associated evolution family with Eq. (2.1) by space pair $(\mathcal{E}, \mathcal{E}_\infty)$. From the properties of the admissible Banach function space, we see that the output solution has better information than the input function. So the output space is smaller than the input space in our results. We now give necessary condition for the exponential dichotomy in the following theorem.

Theorem 2.2. *Assume that the associated evolution family* $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ *has exponential dichotomy on the line. Then,*

1) *for each* $y \in \mathcal{E}$ *there exists a unique* $v \in \mathcal{E}_\infty$ *that is absolutely continuous on each* $[a,b] \subset \mathbb{R}$ *and satisfies*

$$v'(t) - A(t)v(t) = y(t) \quad \text{for a.e.} \quad t \in \mathbb{R}; \tag{2.5}$$

2) *there exist* $K, \alpha > 0$ *such that*

$$\|T(t,\tau)\| \leq K e^{\alpha|t-\tau|} \quad \text{for} \quad t,\tau \in \mathbb{R}. \tag{2.6}$$

Remark 2.3. The absolute continuity of v on each $[a,b] \subset \mathbb{R}$ guarantees that v is differentiable almost everywhere and furthermore the Newton-Leibniz formula for Bochner integral holds for v.

Proof. Take $y \in \mathcal{E}$, for $t \in \mathbb{R}$ we define

$$v(t) = \int_{-\infty}^{t} T(t,\tau)P(\tau)y(\tau)d\tau - \int_{t}^{\infty} T(t,\tau)Q(\tau)y(\tau)d\tau. \tag{2.7}$$

It follows from (2.3) and Proposition 1.6 that

$$\begin{aligned} \|v(t)\| &\leq \int_{-\infty}^{t} \|T(t,\tau)P(\tau)y(\tau)\|d\tau + \int_{t}^{\infty} \|T(t,\tau)Q(\tau)y(\tau)\|d\tau \\ &\leq N\int_{-\infty}^{t} e^{-\eta(t-\tau)}\|y(\tau)\|d\tau + N\int_{t}^{\infty} e^{-\eta(\tau-t)}\|y(\tau)\|d\tau \\ &= N\Lambda_\eta\varphi(t) + N\bar{\Lambda}_\eta\varphi(t), \end{aligned}$$

where $\varphi(t) = \|y(t)\|$. So $v(t)$ is well defined, continuous and bounded. On the other hand, by Banach lattice property of E we also obtain

$$\|v\|_{\mathcal{E}} \leq \frac{NN_1}{1-e^{-\eta}}\|\Lambda_1\varphi\|_E + \frac{NN_2}{1-e^{-\eta}}\|\Lambda_1\varphi\|_E.$$

Therefore, $v \in \mathcal{E}_\infty$ and $\|v\|_{\mathcal{E}_\infty} \leq N(N_1+N_2)(1-e^{-\eta})^{-1}\|\Lambda_1\varphi\|_{\mathcal{E}_\infty}$.
Moreover, given $t_0 \in \mathbb{R}$, by directly computing we have

$$\begin{aligned} v(t) &= T(t,t_0)v(t_0) + \int_{t_0}^{t} T(t,\tau)y(\tau)d\tau \\ &= T(t,t_0)\Big[v(t_0) + \int_{t_0}^{t} T(t_0,\tau)y(\tau)d\tau\Big], \end{aligned} \tag{2.8}$$

for $t \in \mathbb{R}$. Since $T(t,\tau)$ is the evolution family of Eq. (2.1) and property of Bochner integral, it follows from (2.8) that the function $v : \mathbb{R} \to X$ is differentiable almost everywhere and that identity (2.5) holds for a.e. $t \in \mathbb{R}$. Because $T(t_0,\tau)y(\tau)$ is locally Bochner-integrable function so

$$v(t_0) + \int_{t_0}^{t} T(t_0,\tau)y(\tau)d\tau, \quad t \in \mathbb{R}$$

is absolutely continuous function on each $[a,b] \subset \mathbb{R}$. On the other hand, $T(t,t_0)$ and $T(t_0,t)$ are continuously differentiable on $\mathbb{R}$ follow uniform topology in $\mathcal{L}(X)$. Therefore, $T(t,t_0)f(t)$ and $T(t_0,t)f(t)$ are absolutely continuous functions on each

$[a, b] \subset \mathbb{R}$ if so is f. This means that v is absolutely continuous on each $[a, b] \subset \mathbb{R}$. We now show that v is the unique function in $\mathcal{E}_\infty$ satisfying (2.5) for a.e. $t \in \mathbb{R}$.

Indeed, let $v_1 \in \mathcal{E}_\infty$ be absolutely continuous function on each $[a, b] \subset \mathbb{R}$ and satisfy (2.5) for a.e. $t \in \mathbb{R}$. So that

$$v_1'(t) - A(t)v_1(t) = y(t) \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Put $z(t) = T(t_0, t)v_1(t)$. Then, z is absolutely continuous on each $[a, b] \subset \mathbb{R}$, differentiable almost everywhere and

$$z'(t) = T(t_0, t)y(t) \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Thus,

$$z(t) - z(t_0) = \int_{t_0}^{t} z'(\tau)d\tau = \int_{t_0}^{t} T(t_0, \tau)y(\tau)d\tau.$$

This implies that

$$v_1(t) = T(t, t_0)z(t) = T(t, t_0)v_1(t_0) + \int_{t_0}^{t} T(t, \tau)y(\tau)d\tau.$$

Put $w(t) = v(t) - v_1(t)$, we have $w \in \mathcal{E}_\infty$ and $w(t) = T(t, t_0)w(t_0)$ for $t, t_0 \in \mathbb{R}$. For $\tau \geq 0$, using (2.2) and (2.3) we obtain

$$\begin{aligned} \|P(t)w(t)\| &= \|T(t, t - \tau)P(t - \tau)w(t - \tau)\| \leq Ne^{-\eta\tau}\|w\|_{\mathcal{E}_\infty}, \\ \|Q(t)w(t)\| &= \|T(t, t + \tau)Q(t + \tau)w(t + \tau)\| \leq Ne^{-\eta\tau}\|w\|_{\mathcal{E}_\infty}. \end{aligned}$$

Sending $\tau \to \infty$ yields that $P(t)w(t) = Q(t)w(t) = 0$ for $t \in \mathbb{R}$. Therefore, $w(t) = 0$ for $t \in \mathbb{R}$. So, v is unique.
In order to prove (2.6), we use (2.3) and (2.4). For $t \geq \tau$,

$$\begin{aligned} \|T(t, \tau)x\| &\leq \|T(t, \tau)P(\tau)x\| + \|T(t, \tau)Q(\tau)x\| \\ &\leq Ne^{-\eta(t-\tau)}\|x\| + Ne^{\nu(t-\tau)}\|x\| \leq 2Ne^{\nu(t-\tau)}\|x\|; \end{aligned}$$

and for $t \leq \tau$,

$$\begin{aligned} \|T(t, \tau)x\| &\leq \|T(t, \tau)P(\tau)x\| + \|T(t, \tau)Q(\tau)x\| \\ &\leq Ne^{\nu(\tau-t)}\|x\| + Ne^{-\eta(\tau-t)}\|x\| \leq 2Ne^{\nu(\tau-t)}\|x\|. \end{aligned}$$

Thus, (2.6) holds with $K = 2N$ and $\alpha = \nu$. □

The next we show that (2.5) and (2.6) are also sufficient condition for the exponential dichotomy of associated evolution family $(T(t, \tau))_{t,\tau\in\mathbb{R}}$.

Theorem 2.4. *Assume that the assertions 1) and 2) in Theorem 2.2 are true. Then, associated evolution family $(T(t, \tau))_{t,\tau\in\mathbb{R}}$ has exponential dichotomy on the line.*

Proof. The proof scheme is the same as [1, Theorem 2.3]. For the sake of completeness, we still present the complete proof.

Linear operator $H : \mathcal{D}(H) \subset \mathcal{E}_\infty \to \mathcal{E}$ is defined as follows:

$$\begin{aligned}(Hv)(t) &= v'(t) - A(t)v(t), \quad t \in \mathbb{R},\\ \mathcal{D}(H) &= \{v \in \mathcal{E}_\infty \text{ is absolutely continuous function}\\ &\qquad \text{on each } [a,b] \subset \mathbb{R} \text{ such that } Hv \in \mathcal{E}\}.\end{aligned} \tag{2.9}$$

Then, $(H, \mathcal{D}(H))$ is closed operator. Indeed, let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(H)$ such that $v_k \to v$ in $\mathcal{E}_\infty$ and $y_k := Hv_k \to y$ in $\mathcal{E}$. For each fixed $\tau \in \mathbb{R}$ and $t \geq \tau$, we have

$$\begin{aligned}v(t) - v(\tau) &= \lim_{k\to\infty}(v_k(t) - v_k(\tau)) = \lim_{k\to\infty}\int_\tau^t v_k'(s)ds\\ &= \lim_{k\to\infty}\int_\tau^t (y_k(s) + A(s)v_k(s))ds.\end{aligned}$$

On the other hand, by (1.1)

$$\left\|\int_\tau^t y_k(s)ds - \int_\tau^t y(s)ds\right\| \leq \int_\tau^t \|y_k(s) - y(s)\|ds \leq \frac{M(t-\tau)}{\|\chi_{[\tau,t]}\|_E}\|y_k - y\|_{\mathcal{E}}.$$

Therefore,

$$\lim_{k\to\infty}\int_\tau^t y_k(s)ds = \int_\tau^t y(s)ds.$$

Similarly,

$$\begin{aligned}\left\|\int_\tau^t A(s)v_k(s)ds - \int_\tau^t A(s)v(s)ds\right\| &\leq M_1\int_\tau^t \|v_k(s) - v(s)\|ds\\ &\leq M_1(t-\tau)\|v_k - v\|_{\mathcal{E}_\infty}\end{aligned}$$

with $M_1 = \sup\{\|A(s)\| : s \in [\tau, t]\}$. Thus,

$$\lim_{k\to\infty}\int_\tau^t A(s)v_k(s)ds = \int_\tau^t A(s)v(s)ds.$$

So,

$$v(t) - v(\tau) = \int_\tau^t (A(s)v(s) + y(s))ds.$$

This implies that $v(t)$ is absolutely continuous on each $[a,b] \subset \mathbb{R}$, differentiable almost everywhere and $v'(t) = A(t)v(t) + y(t)$ for a.e. $t \in \mathbb{R}$. So, $Hv = y$ and $v \in \mathcal{D}(H)$. Therefore, $(H, \mathcal{D}(H))$ is closed operator.

By the assumption, $H : \mathcal{D}(H) \to \mathcal{E}$ is bijective. So the operator H has an inverse operator $G : \mathcal{E} \to \mathcal{D}(H)$. Because G is closed operator and $\mathcal{D}(G) = \mathcal{E}$ is Banach space so G is bounded.

We now construct stable and unstable subspaces, for $\tau \in \mathbb{R}$

$$F_\tau^s = \{x \in X : \chi_{[\tau,\infty)}(\cdot)T(\cdot,\tau)x \in \mathcal{E} \quad \text{and} \quad \sup_{t\geq\tau}\|T(t,\tau)x\| < \infty\}, \tag{2.10}$$

$$F_\tau^u = \{x \in X : \chi_{(-\infty,\tau]}(\cdot)T(\cdot,\tau)x \in \mathcal{E} \quad \text{and} \quad \sup_{t\leq\tau}\|T(t,\tau)x\| < \infty\}. \tag{2.11}$$

Then, F^s_τ and F^u_τ are subspaces. The next we show that the associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ has exponential dichotomy corresponding to F^s_τ and F^u_τ subspaces. To track easily we will split the proof process into lemmas below.

Lemma 2.5. *$X = F^s_\tau \oplus F^u_\tau$ for each $\tau \in \mathbb{R}$.*

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[\tau, \infty)$ such that $0 \le \phi \le 1$, $\phi = 1$ on $[\tau+1, \infty)$ and $\sup_{t\in\mathbb{R}} |\phi'(t)| < \infty$. Given $x \in X$, put $g(t) = \phi'(t)T(t,\tau)x$. By (2.6) we get

$$\begin{aligned}\|g(t)\| &= \|\chi_{[\tau,\tau+1]}(t)\phi'(t)T(t,\tau)x\| \\ &\le \chi_{[\tau,\tau+1]}(t)\sup_{t\in\mathbb{R}}|\phi'(t)|Ke^{\alpha}\|x\| \qquad \text{for all } \ t\in\mathbb{R}.\end{aligned}$$

By Banach lattice property then $\|g(\cdot)\| \in E$. Thus, $g \in \mathcal{E}_\infty$. Because H is bijective so there exists unique $v \in \mathcal{D}(H) \subset \mathcal{E}_\infty$ such that $Hv = g$ for all $t \in \mathbb{R}$. Denoted $w(t) = (1-\phi(t))T(t,\tau)x + v(t)$ for $t \in \mathbb{R}$, we check easily $Hw = 0$. Therefore, w is a solution of Eq. (2.1). For $t \ge \tau$, we get

$$\|w(t)\| \le \chi_{[\tau,\tau+1]}(t)Ke^{\alpha}\|x\| + \|v(t)\|.$$

This implies $\chi_{[\tau,\infty)}(\cdot)\|w(\cdot)\| \in E$. Thus, $w(\tau) \in F^s_\tau$.
On the other hand, $w(t) - T(t,\tau)x$ is also a solution of Eq. (2.1). For $t \le \tau$, we have $w(t) - T(t,\tau)x = v(t)$ so thus $\chi_{(-\infty,\tau]}(\cdot)(w(\cdot) - T(\cdot,\tau)x) \in \mathcal{E}$ and

$$\sup_{t\le\tau}\|w(t) - T(t,\tau)x\| < \infty.$$

Therefore, $w(\tau) - x \in F^u_\tau$. Hence, $x \in F^s_\tau + F^u_\tau$ for all $x \in X$.

If $x \in F^s_\tau \cap F^u_\tau$ then $u(\cdot) := T(\cdot,\tau)x \in \mathcal{E}_\infty$. Furthermore, u is absolutely continuous function on each compact interval in $\mathbb{R}$. Therefore, $u \in \mathcal{D}(H)$. Since H is invertible and $Hu = 0$ so $u = 0$ for a.e. $t \in \mathbb{R}$. Because u is continuous function so $u = 0$ for all $t \in \mathbb{R}$. Thus, $x = 0$. So, $F^s_\tau \cap F^u_\tau = \{0\}$. ☐

The decomposition in Lemma 2.5 determines a complementary projection pair $P(\tau) : X \to F^s_\tau$ and $Q(\tau) : X \to F^u_\tau$ for each $\tau \in \mathbb{R}$. These projections are uniformly bounded.

Lemma 2.6. *There exists $M > 0$ such that*

$$\|P(\tau)x\| \le M\|x\| \tag{2.12}$$

for $x \in X$ and $\tau \in \mathbb{R}$.

Proof. Using the same notation as in the proof of Lemma 2.5, we get

$$\begin{aligned}\|P(\tau)x\| = \|w(\tau)\| &\le \|x\| + \|v(\tau)\| \le \|x\| + \|v\|_{\mathcal{E}_\infty} \\ &= \|x\| + \|Gg\|_{\mathcal{E}_\infty} \le \|x\| + \|G\|\|g\|_{\mathcal{E}}.\end{aligned}$$

Moreover, we have

$$\|g\|_{\mathcal{E}} \le \|\chi_{[\tau,\tau+1]}\|_E LKe^{\alpha}\|x\| \le \sup_{\tau\in\mathbb{R}}\|\chi_{[\tau,\tau+1]}\|_E LKe^{\alpha}\|x\|,$$

where $L = \sup_{t\in\mathbb{R}} |\phi'(t)| < \infty$. Therefore,

$$\|P(\tau)x\| \le (1 + \|G\|LKe^{\alpha} \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau,\tau+1]}\|_E)\|x\|. \qquad \square$$

We prove that property (2.2) holds in the following lemma.

Lemma 2.7.

$$T(t,\tau)P(\tau) = P(t)T(t,\tau) \quad \textit{for} \quad t,\, \tau \in \mathbb{R}.$$

Proof. Using the same notation as in the proof of Lemma 2.5. We prove this lemma in several steps.

Step 1. We show that $T(t,\tau)w(\tau) \in F_t^s$. Indeed,

$$\begin{aligned}\chi_{[t,\infty)}(\xi)T(\xi,t)T(t,\tau)w(\tau) &= \chi_{[t,\infty)}(\xi)T(\xi,\tau)w(\tau)\\ &= \begin{cases} 0 & \text{if} \quad \xi < t,\\ w(t) & \text{if} \quad \xi \ge t.\end{cases}\end{aligned}$$

Thus, $\chi_{[t,\infty)}(\cdot)T(\cdot,t)T(t,\tau)w(\tau) \in \mathcal{E}_\infty$. This implies $T(t,\tau)w(\tau) \in F_t^s$.

Step 2. We prove that $T(t,\tau)v(\tau) \in F_t^u$. Indeed, by $Hv = g$ we have

$$v(t) = T(t,\tau)v(\tau) + \int_\tau^t T(t,\xi)g(\xi)d\xi \quad \text{for} \quad t,\, \tau \in \mathbb{R}.$$

Therefore,

$$\begin{aligned}T(t,\tau)v(\tau) &= v(t) - \int_\tau^t T(t,\xi)g(\xi)d\xi\\ &= v(t) - \int_\tau^t T(t,\xi)\phi'(\xi)T(\xi,\tau)x d\xi\\ &= v(t) - \int_\tau^t \phi'(\xi)T(t,\tau)x d\xi\\ &= \begin{cases} v(t) & \text{if} \quad t \le \tau,\\ v(t) - \phi(t)T(t,\tau)x & \text{if} \quad t \ge \tau.\end{cases}\end{aligned}$$

Hence,

$$\begin{aligned}\chi_{(-\infty,t]}(\xi)T(\xi,t)T(t,\tau)v(\tau) &= \begin{cases} 0 & \text{if} \quad \xi > t,\\ T(\xi,\tau)v(\tau) & \text{if} \quad \xi \le t,\end{cases}\\ &= \begin{cases} 0 & \text{if} \quad \xi > t,\\ v(\xi) - \phi(\xi)T(\xi,\tau)x & \text{if} \quad \tau \le \xi \le t,\\ v(\xi) & \text{if} \quad \xi < \tau.\end{cases}\end{aligned}$$

Putting

$$f(\xi) = \begin{cases} 0 & \text{if} \quad \xi \ge \tau,\\ v(\xi) & \text{if} \quad \xi < \tau.\end{cases}$$

Then, $f \in \mathcal{E}_\infty$. By Lemma 1.2, we have $\chi_{(-\infty,t]}(\cdot)T(\cdot,t)T(t,\tau)v(\tau) \in \mathcal{E}_\infty$. Therefore, $T(t,\tau)v(\tau) \in F_t^u$.

Step 3. We also have

$$\begin{aligned}T(t,\tau)P(\tau)x &= T(t,\tau)w(\tau)\\ &= T(t,\tau)x + T(t,\tau)v(\tau).\end{aligned}$$

Let projection $P(t)$ act on both sides of the above equality, we obtain

$$T(t,\tau)P(\tau)x = P(t)T(t,\tau)x \quad \text{for all} \quad x \in X.$$ □

Lemma 2.8. *There exists constants $N, \eta > 0$ such that*

$$\|T(t,\tau)x\| \le Ne^{-\eta(t-\tau)}\|x\| \tag{2.13}$$

for $x \in P(\tau)X$ and $t \ge \tau$.

Proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function which has support on $[\tau, +\infty)$ such that $0 \le \psi \le 1$, $\psi = 1$ on $[\tau + 1, +\infty)$ and $\sup_{t\in\mathbb{R}} |\psi'(t)| \le 2$. Given $x \in F^s_\tau$, let u be a solution of Eq. (2.1) with $u(\tau) = x$, i.e, $u(t) = T(t,\tau)x$ for $t \in \mathbb{R}$. We have

$$\begin{aligned}\|\psi(t)u(t)\| = \|\psi(t)T(t,\tau)x\| &= \|\chi_{[\tau,+\infty)}(t)\psi(t)T(t,\tau)x\|\\ &\le \|\chi_{[\tau,+\infty)}(t)T(t,\tau)x\|.\end{aligned}$$

From $x \in F^s_\tau$ and using (2.10) we get $\chi_{[\tau,+\infty)}(\cdot)T(\cdot,\tau)x \in \mathcal{E}_\infty$. Therefore, by Banach lattice property then we have $\psi(\cdot)u(\cdot) \in \mathcal{E}_\infty$. Moreover, we have $H(\psi u) = \psi' u$ and

$$\|(\psi' u)(\xi)\| \le 2\chi_{[\tau,\tau+1]}(\xi)\|u(\xi)\| \le 2\chi_{[\tau,\tau+1]}(\xi)Ke^{\alpha}\|x\|, \quad \xi \in \mathbb{R}.$$

Thus,

$$\|\|\psi' u\|\|_E = \|\psi' u\|_{\mathcal{E}} \le 2Ke^{\alpha}\|\chi_{[\tau,\tau+1]}\|_E\|x\|.$$

- For $t \ge \tau + 1$,

$$\begin{aligned}\|u(t)\| = \|\psi(t)u(t)\| &= \|G(\psi' u)(t)\| \le \|G(\psi' u)\|_{\mathcal{E}_\infty}\\ &\le \|G\|\|\psi' u\|_{\mathcal{E}} \le 2\|G\|Ke^{\alpha}\|\chi_{[\tau,\tau+1]}\|_E\|x\|.\end{aligned}$$

- For $\tau \le t \le \tau + 1$,

$$\|u(t)\| = \|T(t,\tau)x\| \le Ke^{\alpha}\|x\|.$$

Therefore,

$$\|u(t)\| \le C\|x\| \quad \text{for} \quad t \ge \tau, \tag{2.14}$$

where $C = Ke^{\alpha}\max\{2\|G\|\sup_{\tau\in\mathbb{R}}\|\chi_{[\tau,\tau+1]}\|_E, 1\}$.

The next, we show that there exists $m \in \mathbb{N}$ such that

$$\|u(t)\| \le \frac{1}{2}\|x\| \quad \text{for} \quad t - \tau \ge m,\ \tau \in \mathbb{R}. \tag{2.15}$$

In order to prove (2.15), let

$$y(\xi) = \chi_{[\tau,t]}(\xi)u(\xi) \text{ and } v(\xi) = u(\xi)\int_{-\infty}^{\xi}\chi_{[\tau,t]}(s)ds.$$

It can be seen that $y \in \mathcal{E}$, $v \in \mathcal{D}(H) \subset \mathcal{E}_\infty$ and $Hv = y$. Therefore,

$$\|v\|_{\mathcal{E}_\infty} = \|Gy\|_{\mathcal{E}_\infty} \le \|G\|\|y\|_{\mathcal{E}}.$$

On the other hand,

$$\|y(\xi)\| \le \begin{cases} \chi_{[\tau,\tau+1]}(\xi) K e^{\alpha}\|x\|, \ \xi \in [\tau, \tau+1], \\ G(\psi' u)(\xi), \ \xi \in [\tau+1, +\infty). \end{cases}$$

Thus,

$$\|y\|_{\mathcal{E}} \le \max\{K e^{\alpha} \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau,\tau+1]}\|_E, 2Ke^{\alpha}\|G\| \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau,\tau+1]}\|_E\}\|x\| =: K_1\|x\|.$$

So,

$$\|v\|_{\mathcal{E}_\infty} \le \|G\| K_1 \|x\|.$$

We have

$$(t-\tau)\|u(t)\| = \|v(t)\| \le \|v\|_{\mathcal{E}_\infty} \le \|G\|K_1\|x\|.$$

Therefore,

$$\|u(t)\| \le \frac{\|G\|K_1}{t-\tau}\|x\|.$$

Hence, if $t-\tau \ge 2K_1\|G\|$ then $\|u(t)\| \le \frac{1}{2}\|x\|$. Taking $m > 2K_1\|G\|$, we obtain (2.15). Finally, take $t \ge \tau$ and write $t-\tau = km + r$ with $k \in \mathbb{N}$ and $0 \le r < m$. By (2.12), (2.14), (2.15), and Lemma 2.7 we get

$$\begin{aligned} \|T(t,\tau)P(\tau)x\| &= \|T(\tau+km+r,\tau)P(\tau)x\| \le C\|T(\tau+km,\tau)P(\tau)x\| \\ &\le \frac{C}{2^k}\|P(\tau)x\| \le 2CMe^{-(t-\tau)\frac{\ln 2}{m}}\|x\| \quad \text{for} \quad x \in X. \end{aligned}$$

□

Lemma 2.9. *There exists constants $N, \eta > 0$ such that*

$$\|T(t,\tau)x\| \le Ne^{-\eta(\tau-t)}\|x\| \tag{2.16}$$

for $x \in \operatorname{Ker}P(\tau) = Q(\tau)X$ and $t \le \tau$.

Proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $(-\infty, \tau]$ such that $0 \le \psi \le 1$, $\psi = 1$ on $(-\infty, \tau-1]$ and $\sup_{t\in\mathbb{R}} |\psi'(t)| \le 2$. Given $x \in F^u_\tau$, let u be a solution of Eq. (2.1) with $u(\tau) = x$. We have

$$\|\psi(t)u(t)\| = \|\chi_{(-\infty,\tau]}(t)\psi(t)T(t,\tau)x\| \le \|\chi_{(-\infty,\tau]}(t)T(t,\tau)x\|.$$

From $x \in F^u_\tau$ and using (2.11) we get $\chi_{(-\infty,\tau]}(\cdot)T(\cdot,\tau)x \in \mathcal{E}_\infty$. Therefore, $\psi(\cdot)u(\cdot) \in \mathcal{E}_\infty$. Furthermore, we can also easily verify that $H(\psi u) = \psi' u$. We have

$$|(\psi' u)(\xi)\| \le 2\chi_{[\tau-1,\tau]}(\xi)\|u(\xi)\| \le 2\chi_{[\tau-1,\tau]}(\xi)Ke^{\alpha}\|x\|, \quad \xi \in \mathbb{R}.$$

Thus,

$$\|\|\psi' u\|\|_E = \|\psi' u\|_{\mathcal{E}} \le 2Ke^{\alpha}\|\chi_{[\tau-1,\tau]}\|_E\|x\|.$$

- For $t \le \tau - 1$,

$$\begin{aligned} \|u(t)\| = \|\psi(t)u(t)\| &= \|G(\psi' u)(t)\| \le \|G(\psi' u)\|_{\mathcal{E}_\infty} \\ &\le \|G\|\|\psi' u\|_{\mathcal{E}} \le 2\|G\|Ke^{\alpha}\|\chi_{[\tau-1,\tau]}\|_E\|x\|. \end{aligned}$$

- For $\tau - 1 \le t \le \tau$,

$$\|u(t)\| = \|T(t,\tau)x\| \le Ke^{\alpha}\|x\|.$$

Therefore,

$$\|u(t)\| \leq C\|x\| \quad \text{for} \quad t \leq \tau, \tag{2.17}$$

where $C = Ke^{\alpha} \max\{2\|G\| \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau-1,\tau]}\|_E, 1\}$.
The next, we show that there exists $m \in \mathbb{N}$ such that

$$\|u(t)\| \leq \frac{1}{2}\|x\| \quad \text{for} \quad \tau - t \geq m,\, \tau \in \mathbb{R}. \tag{2.18}$$

In order to prove (2.18), let

$$y(\xi) = -\chi_{[t,\tau]}(\xi)u(\xi) \text{ and } v(\xi) = u(\xi)\int_{\xi}^{\infty} \chi_{[t,\tau]}(s)ds.$$

It can be seen that $y \in \mathcal{E}$, $v \in \mathcal{D}(H) \subset \mathcal{E}_\infty$ and $Hv = y$. Therefore,

$$\|v\|_{\mathcal{E}_\infty} = \|Gy\|_{\mathcal{E}_\infty} \leq \|G\|\|y\|_{\mathcal{E}}.$$

On the other hand,

$$\|y(\xi)\| \leq \begin{cases} \chi_{[\tau-1,\tau]}(\xi)Ke^{\alpha}\|x\|, \ \xi \in [\tau-1,\tau], \\ G(\psi' u)(\xi), \ \xi \in (-\infty, \tau-1]. \end{cases}$$

Thus,

$$\|y\|_{\mathcal{E}} \leq \max\{Ke^{\alpha} \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau-1,\tau]}\|_E, 2Ke^{\alpha}\|G\| \sup_{\tau\in\mathbb{R}} \|\chi_{[\tau-1,\tau]}\|_E\}\|x\| =: K_2\|x\|.$$

So,

$$\|v\|_{\mathcal{E}_\infty} \leq \|G\|K_2\|x\|.$$

We have

$$(\tau - t)\|u(t)\| = \|v(t)\| \leq \|v\|_{\mathcal{E}_\infty} \leq \|G\|K_2\|x\|.$$

Therefore,

$$\|u(t)\| \leq \frac{\|G\|K_2}{\tau - t}\|x\|.$$

Hence, if $\tau - t \geq 2K_2\|G\|$ then $\|u(t)\| \leq \frac{1}{2}\|x\|$. Taking $m > 2K_2\|G\|$, we obtain (2.18).
In order to complete the proof, take $t \leq \tau$ and write $\tau - t = km + r$ with $k \in \mathbb{N}$ and $0 \leq r < m$. By (2.12), (2.17), (2.18), and Lemma 2.7 we get

$$\begin{aligned} \|T(t,\tau)Q(\tau)x\| &= \|T(\tau - km - r, \tau)Q(\tau)x\| \leq C\|T(\tau - km, \tau)Q(\tau)x\| \\ &\leq \frac{C}{2^k}\|Q(\tau)x\| \leq 2C(1+M)e^{-(\tau-t)\frac{\ln 2}{m}}\|x\| \quad \text{for} \quad x \in X. \end{aligned}$$

□

So, we get (2.3) from (2.13) and (2.16). For $t \leq \tau$, using (2.6) and (2.12) we obtain (2.4) as follows.

$$\begin{aligned} \|T(t,\tau)P(\tau)x\| &\leq Ke^{\alpha|t-\tau|}\|P(\tau)x\| \leq Ke^{\alpha|t-\tau|}M\|x\| = KMe^{-\alpha(t-\tau)}\|x\|, \\ \|T(\tau,t)Q(t)x\| &\leq Ke^{\alpha|t-\tau|}\|Q(t)x\| \leq Ke^{\alpha|t-\tau|}(1+M)\|x\| \\ &= K(1+M)e^{-\alpha(t-\tau)}\|x\|. \end{aligned}$$

Thus, the associated evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ has exponential dichotomy on the line. □

In the remainder of this section we establish the robustness of the notion of exponential dichotomy. It is an application of Theorem 2.2 and Theorem 2.4.

Theorem 2.10. *Let $A, B : \mathbb{R} \to \mathcal{L}(X)$ be strongly continuous functions such that*

1. *the evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ of Eq.* (2.1) *has exponential dichotomy on the line;*
2. *there exists $\varphi \in E$ such that*

$$\|B(t) - A(t)\| \leq \varphi(t) \quad \text{for a.e.} \quad t \in \mathbb{R}. \tag{2.19}$$

Then, the evolution family $(U(t,\tau))_{t,\tau\in\mathbb{R}}$ of the equation $v' = B(t)v$ has exponential dichotomy on the line if $\|\varphi\|_E$ is sufficiently small.

Proof. Let H be the linear operator defined by (2.9) on the domain $\mathcal{D}(H)$. We define a linear operator $L : \mathcal{D}(L) \subset \mathcal{E}_\infty \to \mathcal{E}$ by

$$(Lv)(t) = v'(t) - B(t)v(t), \quad t \in \mathbb{R},$$

where $\mathcal{D}(L) = \{v \in \mathcal{E}_\infty$ is absolutely continuous function on each $[a,b] \subset \mathbb{R}$ such that $Lv \in \mathcal{E}\}$.

For $v \in \mathcal{E}_\infty$, denoted $(Pv)(t) := (B(t) - A(t))v(t)$. By (2.19) we get

$$\|(Pv)(t)\| \leq \varphi(t)\|v(t)\| \leq \varphi(t)\|v\|_{\mathcal{E}_\infty} \quad \text{for a.e} \quad t \in \mathbb{R}.$$

Therefore, $Pv \in \mathcal{E}$ and $\|Pv\|_{\mathcal{E}} \leq \|\varphi\|_E \|v\|_{\mathcal{E}_\infty}$. So, the mapping $P : \mathcal{E}_\infty \to \mathcal{E}$ is bounded linear operator and $\|P\| \leq \|\varphi\|_E$. Thus, $\mathcal{D}(H) = \mathcal{D}(L)$ and $L = H + P$. By Theorem 2.2 and Theorem 2.4, the operator H is invertible. Hence, if $\|\varphi\|_E$ is sufficiently small then L is also invertible.

Two evolution families $(U(t,\tau))_{t,\tau\in\mathbb{R}}$ and $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ have the relation as follows:

$$U(t,\tau)x = T(t,\tau)x + \int_\tau^t T(t,s)(B(s) - A(s))U(s,\tau)x\, ds$$

for $t, \tau \in \mathbb{R}$ and $x \in X$. Using Gronwall inequality and the relation above, we easily get

$$\|U(t,\tau)x\| \leq Ke^{\alpha|t-\tau| + K\left|\int_\tau^t \varphi(s)ds\right|}\|x\| \quad \text{for} \quad x \in X \ \text{ and } \ t, \tau \in \mathbb{R}.$$

On the other hand,

$$\left|\int_\tau^t \varphi(s)ds\right| \leq \|\Lambda_1\varphi\|_\infty(|t-\tau| + 1) \quad \text{for} \quad t, \tau \in \mathbb{R}.$$

Thus,

$$\|U(t,\tau)x\| \leq Ke^{K\|\Lambda_1\varphi\|_\infty} e^{(\alpha + K\|\Lambda_1\varphi\|_\infty)|t-\tau|}\|x\| \quad \text{for} \quad x \in X \ \text{ and } \ t, \tau \in \mathbb{R}.$$

By Theorem 2.4, we deduce that the evolution family $(U(t,\tau))_{t,\tau\in\mathbb{R}}$ of the equation $v' = B(t)v$ has exponential dichotomy on the line. $\square$

3. Stable and unstable manifolds

Let be semi-linear differential equation

$$v'(t) = A(t)v(t) + g(t, v(t)), \quad t \in \mathbb{R} \tag{3.1}$$

in Banach space X, in which A and g satisfy the following assumptions.
Assumption 1: $A : \mathbb{R} \to \mathcal{L}(X)$ is strongly continuous function and generates an evolution family $(T(t,\tau))_{t,\tau\in\mathbb{R}}$ having exponential dichotomy (that means the assertions 1) and 2) in Theorem 2.2 are satisfied).
Assumption 2: $g : \mathbb{R} \times X \to X$ is continuous and satisfies φ-Lipschitz condition, i.e,

(i) $\|g(t,0)\| \le \varphi(t)$ for $t \in \mathbb{R}$,
(ii) $\|g(t,x) - g(t,y)\| \le \varphi(t)\|x - y\|$ for $t \in \mathbb{R}$ and $x, y \in X$.

Assumption 3: E is admissible Banach function space such that its associate space E' is also admissible Banach function space and $\varphi \in E'$ is exponentially E-invariant (see Definition 1.8).

The these underlying assumptions, we show the existence of stable and unstable manifolds for the Eq. (3.1). Actually, these manifolds include trajectories of continuous solutions lying in the Banach space $\mathcal{E}$ (see Definition 1.3). We easily get the following result.

Lemma 3.1. *A function $v : \mathbb{R} \to X$ is solution of Eq.* (3.1) *if only if it is continuous on $\mathbb{R}$ and satisfies the integral equation*

$$v(t) = T(t,t_0)v(t_0) + \int_{t_0}^{t} T(t,\tau)g(\tau, v(\tau))d\tau, \quad t_0, t \in \mathbb{R}.$$

From now on we shall suppose that Assumption 1, Assumption 2 and Assumption 3 hold. For convenience, we define Green function as follows

$$\mathcal{G}(t,\tau) = \begin{cases} T(t,\tau)P(\tau) & \text{for } t > \tau, \\ -T(t,\tau)Q(\tau) & \text{for } t < \tau. \end{cases} \tag{3.2}$$

By (2.3), we have $\|\mathcal{G}(t,\tau)\| \le Ne^{-\eta|t-\tau|}$ for all $t, \tau \in \mathbb{R}$. Moreover, if a function v has the domain $D(v)$ then it can be extended on $\mathbb{R}$ by the characteristic function $\chi_{D(v)}$ as follows $(\chi_{D(v)}v)(t) = v(t)$ if $t \in D(v)$ and $(\chi_{D(v)}v)(t) = 0$ if otherwise. To construct stable and unstable manifolds we now give characteristic formula denoted solutions of Eq. (3.1) which belong to the Banach space $\mathcal{E}$.

Proposition 3.2. *The following assertions hold.*

i. The function $v \in \mathcal{E}$ is a solution of Eq. (3.1) *on $\mathbb{R}$ if only if it has the form*

$$v(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau, \quad t \in \mathbb{R}.$$

ii. For each fixed s, the function $\chi_{[s,\infty)}v \in \mathcal{E}$ is a solution of Eq. (3.1) *on $[s,\infty)$ if only if there is $\nu_0 \in \operatorname{Im}P(s)$ such that*

$$v(t) = T(t,s)\nu_0 + \int_{s}^{\infty} \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau, \quad t \ge s. \tag{3.3}$$

iii. For each fixed s, the function $\chi_{(-\infty,s]}v \in \mathcal{E}$ *is a solution of Eq.* (3.1) *on* $(-\infty, s]$ *if only if there is* $\mu_0 \in \mathrm{Im}Q(s)$ *such that*

$$v(t) = T(t,s)\mu_0 + \int_{-\infty}^{s} \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau, \quad t \le s. \tag{3.4}$$

Proof. The sufficient condition in the three assertions above is checked easily by simple computations. So we only prove the necessary condition in the these.

i. Put $y(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,\tau)g(\tau,v(\tau))d\tau,\ t \in \mathbb{R}$. Using Hölder-type inequality (1.2) and Assumption 3, we have

$$\begin{aligned}\|y(t)\| &\le N\int_{-\infty}^{\infty} e^{-\eta|t-\tau|}\varphi(\tau)(1+\|v(\tau)\|)d\tau \\ &\le Nh_{\frac{\eta}{2}}(t)\|e^{-\frac{\eta}{2}|t-\cdot|}\|_E + Nh_\eta(t)\|v\|_{\mathcal{E}}.\end{aligned}$$

By iii) in the Definition 1.4, we get $\|e^{-\frac{\eta}{2}|t-\cdot|}\|_E \le \max\{N_1, N_2\}\|e_{\frac{\eta}{2}}\|_E$, in which $e_{\frac{\eta}{2}}(\tau) = e^{-\frac{\eta}{2}|\tau|}$. Therefore,

$$\|y(t)\| \le N\max\{N_1,N_2\}\|e_{\frac{\eta}{2}}\|_E h_{\frac{\eta}{2}}(t) + N\|v\|_{\mathcal{E}}h_\eta(t).$$

Because E is the Banach lattice and $h_{\frac{\eta}{2}}, h_\eta \in E$ so $\|y(\cdot)\| \in E$. Thus, $y \in \mathcal{E}$. On the other hand, y also satisfies the integral equation

$$y(t) = T(t,t_0)y(t_0) + \int_{t_0}^{t} T(t,\tau)g(\tau,v(\tau))d\tau, \quad t_0, t \in \mathbb{R}.$$

Thus,

$$v(t) - y(t) = T(t,t_0)(v(t_0) - y(t_0)).$$

Because of $v - y \in \mathcal{E}$ so we obtain $v(t_0) = y(t_0)$. This deduces $v = y$ on $\mathbb{R}$.

ii. Put $y_2(t) = \int_{s}^{\infty} \mathcal{G}(t,\tau)g(\tau,v(\tau))d\tau,\ t \ge s$. The similar argumentation as above, we have

$$\begin{aligned}\|y_2(t)\| &\le N\int_{s}^{\infty} e^{-\eta|t-\tau|}\varphi(\tau)(1+\|v(\tau)\|)d\tau \\ &\le N\int_{-\infty}^{\infty} e^{-\eta|t-\tau|}\varphi(\tau)(1+\|(\chi_{[s,\infty)}v)(\tau)\|)d\tau \\ &\le N\max\{N_1,N_2\}\|e_{\frac{\eta}{2}}\|_E h_{\frac{\eta}{2}}(t) + Nh_\eta(t)\|\chi_{[s,\infty)}v\|_{\mathcal{E}}.\end{aligned}$$

Thus, $\chi_{[s,\infty)}y_2 \in \mathcal{E}$. On the other hand, y_2 also satisfies the integral equation

$$y_2(t) = T(t,s)y_2(s) + \int_{s}^{t} T(t,\tau)g(\tau,v(\tau))d\tau, \quad t \ge s.$$

Therefore, $v(t) - y_2(t) = T(t,s)(v(s) - y_2(s))$. Because of $\chi_{[s,\infty)}v - \chi_{[s,\infty)}y_2 \in \mathcal{E}$ so we obtain $v(s) - y_2(s) \in \mathrm{Im}P(s)$. So, there exists $\nu_0 \in \mathrm{Im}P(s)$ such that $v(t) = T(t,s)\nu_0 + y_2(t)$ with $t \ge s$. The last assertion is proved similarly. □

Using Proposition 3.2 and Banach fixed-point theorem we get the existence of solutions of Eq. (3.1) in the Banach space $\mathcal{E}$. The proof is basic, so we omit here.

Theorem 3.3. *Assume that* $N\|h_\eta\|_E < 1$. *Then:*

a) *The Eq.* (3.1) *has a unique solution in the Banach space* $\mathcal{E}$ *and this solution takes the form*

$$v^*(t) = \int_{-\infty}^{\infty} \mathcal{G}(t,\tau)g(\tau, v^*(\tau))d\tau, \quad t \in \mathbb{R}.$$

b) *For each fixed* s *and* $\nu_0 \in \mathrm{Im}P(s)$, *the Eq.* (3.1) *has a unique solution* v *on* $[s,\infty)$ *such that* $\chi_{[s,\infty)}v \in \mathcal{E}$ *and this solution is represented by the formula* (3.3).

c) *For each fixed* s *and* $\mu_0 \in \mathrm{Im}Q(s)$, *the Eq.* (3.1) *has a unique solution* v *on* $(-\infty, s]$ *such that* $\chi_{(-\infty,s]}v \in \mathcal{E}$ *and this solution is represented by the formula* (3.4).

The next, we show the existence of stable and unstable manifolds for the Eq. (3.1). These manifolds like bundles in $\mathbb{R} \times X$ space, each a fiber of these manifolds is a submanifold in X space. In precisely, it is graph of a Lipschitz map.

Theorem 3.4. *Assume that* $N^2 \max\{N_1, N_2\}\|e_\eta\|_E\|\varphi\|_{E'} + N\|h_\eta\|_E < 1$, *in which* $e_\eta(\tau) = e^{-\eta|\tau|}$. *Then, there exist an invariant stable manifold* $S = \bigsqcup_{s\in\mathbb{R}} S_s$ *and an invariant unstable manifold* $U = \bigsqcup_{s\in\mathbb{R}} U_s$ *of Eq.* (3.1). *Moreover, the stable manifold has the following properties*

(i) $S_s = \{\nu_0 + g_s^{st}(\nu_0) : \nu_0 \in \mathrm{Im}P(s)\}$, *where* $g_s^{st} : \mathrm{Im}P(s) \to \mathrm{Im}Q(s)$ *is a Lipschitz map having Lipschitz coefficient*

$$\mathrm{Lip}(g_s^{st}) \le \frac{N^2 \max\{N_1, N_2\}\|e_\eta\|_E\|\varphi\|_{E'}}{1 - N\|h_\eta\|_E} < 1$$

for all $s \in \mathbb{R}$;

(ii) S_s *is homeomorphic to* $\mathrm{Im}P(s)$ *for all* $s \in \mathbb{R}$;

(iii) *to each* $x_0 \in S_s$, *the Eq.* (3.1) *has a unique solution* v *on* $[s,\infty)$ *such that* $\chi_{[s,\infty)}v \in \mathcal{E}$, *and* $v(t) \in S_t$ *for all* $t \ge s$;

(iv) *if* $N(N_1 + N_2)\|\Lambda_1\varphi\|_\infty < 1$ *then the solution* v^* *attracts other solutions on* S *in the sense there exist* $\mu, C_\mu > 0$ *such that*

$$\|v(t) - v^*(t)\| \le C_\mu e^{-\mu(t-s)}\|P(s)v(s) - P(s)v^*(s)\| \quad \text{for all } t \ge s,\ v(s) \in S_s;$$

and the unstable manifold has the following properties

(i) $U_s = \{\mu_0 + g_s^{un}(\mu_0) : \mu_0 \in \mathrm{Im}Q(s)\}$, *where* $g_s^{un} : \mathrm{Im}Q(s) \to \mathrm{Im}P(s)$ *is a Lipschitz map having Lipschitz coefficient*

$$\mathrm{Lip}(g_s^{un}) \le \frac{N^2 \max\{N_1, N_2\}\|e_\eta\|_E\|\varphi\|_{E'}}{1 - N\|h_\eta\|_E} < 1$$

for all $s \in \mathbb{R}$;

(ii) U_s *is homeomorphic to* $\mathrm{Im}Q(s)$ *for all* $s \in \mathbb{R}$;

(iii) *to each* $x_0 \in U_s$, *the Eq.* (3.1) *has a unique solution* v *on* $(-\infty, s]$ *such that* $\chi_{(-\infty,s]}v \in \mathcal{E}$, *and* $v(t) \in U_t$ *for all* $t \le s$;

(iv) *if* $N(N_1 + N_2)\|\Lambda_1\varphi\|_\infty < 1$ *then the solution* v^* *attracts other solutions on* U *in the sense there exist* $\mu, C_\mu > 0$ *such that*

$$\|v(t) - v^*(t)\| \le C_\mu e^{\mu(t-s)}\|Q(s)v(s) - Q(s)v^*(s)\| \quad \text{for all } t \le s,\ v(s) \in U_s.$$

Proof. We shall prove the existence of stable manifold and its properties, the unstable manifold is done similarly.

By Theorem 3.3, for each $\nu_0 \in \mathrm{Im}P(s)$ then the Eq. (3.1) has a unique solution v on $[s,\infty)$ such that $\chi_{[s,\infty)}v \in \mathcal{E}$. So we define the map $g_s^{st} : \mathrm{Im}P(s) \to \mathrm{Im}Q(s)$ as follows

$$g_s^{st}(\nu_0) = \int_s^\infty \mathcal{G}(s,\tau)g(\tau, v(\tau))d\tau, \tag{3.5}$$

where $\mathcal{G}(s,\tau)$ is the Green function defined by (3.2). For $\nu_1, \nu_2 \in \mathrm{Im}P(s)$ we have

$$\begin{aligned}
\|g_s^{st}(\nu_1) - g_s^{st}(\nu_2)\| &\le N \int_s^\infty e^{-\eta|s-\tau|}\varphi(\tau)\|v_1(\tau) - v_2(\tau)\|d\tau \\
&\le N \int_{-\infty}^\infty e^{-\eta|s-\tau|}\varphi(\tau)\|(\chi_{[s,\infty)}v_1)(\tau) - (\chi_{[s,\infty)}v_2)(\tau)\|d\tau \\
&\le N \int_{-\infty}^\infty \varphi(\tau)\|(\chi_{[s,\infty)}v_1)(\tau) - (\chi_{[s,\infty)}v_2)(\tau)\|d\tau \\
&\le N\|\varphi\|_{E'}\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}} \quad (\text{ by } (1.2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|v_1(t) - v_2(t)\| &\le Ne^{-\eta(t-s)}\|\nu_1 - \nu_2\| \\
&\quad + N \int_s^\infty e^{-\eta|t-\tau|}\varphi(\tau)\|v_1(\tau) - v_2(\tau)\|d\tau \\
&\le Ne^{-\eta|t-s|}\|\nu_1 - \nu_2\| + Nh_\eta(t)\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}}
\end{aligned}$$

for $t \ge s$, and $\|e^{-\eta|\cdot - s|}\|_E \le \max\{N_1, N_2\}\|e_\eta\|_E$. Therefore, by the Banach lattice property of E we get

$$\begin{aligned}
\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}} &\le N\max\{N_1, N_2\}\|e_\eta\|_E\|\nu_1 - \nu_2\| \\
&\quad + N\|h_\eta\|_E\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}}.
\end{aligned}$$

This implies

$$\|\chi_{[s,\infty)}v_1 - \chi_{[s,\infty)}v_2\|_{\mathcal{E}} \le \frac{N\max\{N_1, N_2\}\|e_\eta\|_E}{1 - N\|h_\eta\|_E}\|\nu_1 - \nu_2\|.$$

So that

$$\|g_s^{st}(\nu_1) - g_s^{st}(\nu_2)\| \le \frac{N^2\max\{N_1, N_2\}\|e_\eta\|_E\|\varphi\|_{E'}}{1 - N\|h_\eta\|_E}\|\nu_1 - \nu_2\|.$$

Thus, g_s^{st} is a Lipschitz map with Lipschitz coefficient

$$\mathrm{Lip}(g_s^{st}) \le \frac{N^2\max\{N_1, N_2\}\|e_\eta\|_E\|\varphi\|_{E'}}{1 - N\|h_\eta\|_E} < 1$$

for all $s \in \mathbb{R}$. This also leads to that S_s is homeomorphic to $\mathrm{Im}P(s)$ for all $s \in \mathbb{R}$.

From the definition of S_s and Theorem 3.3, the solution v^* lies in the stable manifold S and the Eq. (3.1) has a unique solution v on $[s,\infty)$ such that $\chi_{[s,\infty)}v \in \mathcal{E}$ for each $x_0 \in S_s$. By the composition property of solution flows, we get $v(t) \in S_t$ for all $t \ge s$.

For $v(s) \in S_s$, the Eq. (3.1) has a unique solution v on $[s, \infty)$ such that $\chi_{[s,\infty)}v \in \mathcal{E}$ and this solution takes the form

$$v(t) = T(t,s)P(s)v(s) + \int_s^\infty \mathcal{G}(t,\tau)g(\tau, v(\tau))d\tau, \quad t \geq s.$$

Therefore,

$$\begin{aligned}\|v(t) - v^*(t)\| &\leq Ne^{-\eta(t-s)}\|P(s)v(s) - P(s)v^*(s)\| \\ &\quad + N\int_s^\infty e^{-\eta|t-\tau|}\varphi(\tau)\|v(\tau) - v^*(\tau)\|d\tau, \quad t \geq s.\end{aligned}$$

Put $w(t) = e^{\mu(t-s)}\|v(t) - v^*(t)\|$ for $t \geq s$ and $\mu \in (0, \eta)$. Then, w satisfies the integral equation

$$\begin{aligned}w(t) &\leq Ne^{-(\eta-\mu)(t-s)}\|P(s)v(s) - P(s)v^*(s)\| \\ &\quad + N\int_s^\infty e^{-\eta|t-\tau|+\mu(t-\tau)}\varphi(\tau)w(\tau)d\tau, \quad t \geq s.\end{aligned}$$

We shall find w in $C_b([s,\infty))$, consider the linear operator A on $C_b([s,\infty))$ as follows

$$(A\phi)(t) = N\int_s^\infty e^{-\eta|t-\tau|+\mu(t-\tau)}\varphi(\tau)\phi(\tau)d\tau, \quad t \geq s.$$

Then, $A\phi \in C_b([s,\infty))$ and $\|A\phi\|_\infty \leq N(N_1 + N_2)(1 - e^{-(\eta-\mu)})^{-1}\|\Lambda_1\varphi\|_\infty\|\phi\|_\infty$ by the property (a) in Proposition 1.6. So, we have

$$w(t) \leq z(t) + (Aw)(t),\ t \geq s \text{ and } z(t) = Ne^{-(\eta-\mu)(t-s)}\|P(s)v(s) - P(s)v^*(s)\|.$$

Take $\mu < \eta + \ln(1 - N(N_1 + N_2)\|\Lambda_1\varphi\|_\infty)$, we get

$$\|A\| \leq N(N_1 + N_2)(1 - e^{-(\eta-\mu)})^{-1}\|\Lambda_1\varphi\|_\infty < 1.$$

Therefore, by cone inequality theorem in Banach space (see [5, Chap. I, Theorem 9.3]) there exists $\phi \in C_b([s,\infty))$ such that $w(t) \leq \phi(t)$ for all $t \geq s$ and ϕ is a unique solution of the equation $\phi = z + A\phi$ in $C_b([s,\infty))$. Thus,

$$\begin{aligned}\|\phi\|_\infty = \|(I - A)^{-1}z\|_\infty &\leq \frac{1}{1 - \|A\|}\|z\|_\infty \\ &\leq \frac{N\|P(s)v(s) - P(s)v^*(s)\|}{1 - N(N_1 + N_2)(1 - e^{-(\eta-\mu)})^{-1}\|\Lambda_1\varphi\|_\infty}.\end{aligned}$$

So, there exist $\mu, C_\mu > 0$ such that

$$\|v(t) - v^*(t)\| \leq C_\mu e^{-\mu(t-s)}\|P(s)v(s) - P(s)v^*(s)\| \quad \text{for all } t \geq s,\ v(s) \in S_s. \quad \square$$

Remark 3.5. By the properties of stable and unstable manifolds, we get $S_s \cap U_s = \{v^*(s)\}$ for all $s \in \mathbb{R}$. Moreover, in Theorem 3.4 if we assume $g(t, 0) = 0$ for all $t \in \mathbb{R}$ then $v^* \equiv 0$. Therefore, $\lim_{t\to\infty} v(t) = 0$ for all $v(s) \in S_s$ and $\lim_{t\to-\infty} v(t) = 0$ for all $v(s) \in U_s$.

When the map $g(t,\cdot)$ is smooth on X for each fixed t then each a fiber of stable and unstable manifolds is also smooth in the sense the map determining this fiber is smooth.

Theorem 3.6. *Assume that*

$$\max\{N^2\max\{N_1,N_2\}\|e_\eta\|_E\|\varphi\|_{E'}+N\|h_\eta\|_E, N(N_1+N_2)\|\Lambda_1\varphi\|_\infty\}<1$$

and the map $g(t,\cdot)$ is continuously differentiable on X for each fixed $t\in\mathbb{R}$ such that $D_xg(t,v^*(t))=0$ for all $t\in\mathbb{R}$. Then, S_s and U_s are differentiable submanifolds of class C^1 and tangent to $v^*(s)+\mathrm{Im}P(s)$ and $v^*(s)+\mathrm{Im}Q(s)$ respectively at $v^*(s)$ for all $s\in\mathbb{R}$.

Proof. We need prove that the map g_s^{st} (see (3.5)) is continuously differentiable on closed subspace ImP(s). Because g satisfies φ-Lipschitz condition and $g(t,\cdot)$ is continuously differentiable on X so

$$\|D_xg(t,a)\|\le\varphi(t) \quad \text{for all } t\in\mathbb{R} \text{ and } a\in X. \tag{3.6}$$

For $\nu_0, h\in$ ImP(s), we have

$$\frac{g_s^{st}(\nu_0+h)-g_s^{st}(\nu_0)}{\|h\|}-\frac{1}{\|h\|}\int_s^\infty\mathcal{G}(s,\tau)D_xg(\tau,v(\tau))hd\tau$$
$$=\int_s^\infty\mathcal{G}(s,\tau)\Big(\frac{g(\tau,v_1(\tau))-g(\tau,v(\tau))-D_xg(\tau,v(\tau))h}{\|h\|}\Big)d\tau,$$

in which v_1 and v are solutions of Eq. (3.1) on $[s,\infty)$ corresponding to ν_0+h and ν_0, and by (3.6) then $\int_s^\infty\mathcal{G}(s,\tau)D_xg(\tau,v(\tau))d\tau$ is absolutely convergent in $\mathcal{L}(X)$. By the attractive property of stable manifold S, we have

$$\|v_1(\tau)-v(\tau)\|\le 2C_\mu\|h\|$$

for all $\tau\ge s$. Therefore,

$$\lim_{h\to0}\mathcal{G}(s,\tau)\Big(\frac{g(\tau,v_1(\tau))-g(\tau,v(\tau))-D_xg(\tau,v(\tau))h}{\|h\|}\Big)=0$$

for all $\tau\ge s$. On the other hand,

$$\Big\|\mathcal{G}(s,\tau)\Big(\frac{g(\tau,v_1(\tau))-g(\tau,v(\tau))-D_xg(\tau,v(\tau))h}{\|h\|}\Big)\Big\|$$
$$\le N(2C_\mu+1)e^{-\eta|s-\tau|}\varphi(\tau),\quad \tau\ge s.$$

According to Lebesgue's dominated convergence theorem, g_s^{st} is differentiable at ν_0 and

$$Dg_s^{st}(\nu_0)=\int_s^\infty\mathcal{G}(s,\tau)D_xg(\tau,v(\tau))d\tau.$$

From here deduces $Dg_s^{st}(P(s)v^*(s))=0$. By (3.6) and Lebesgue's dominated convergence theorem, Dg_s^{st} is continuous on $\mathrm{Im}P(s)$. So, S_s is differentiable submanifold of class C^1 and tangent to $v^*(s)+\mathrm{Im}P(s)$ at $v^*(s)$. Similarly, U_s is differentiable submanifold of class C^1 and tangent to $v^*(s)+\mathrm{Im}Q(s)$ at $v^*(s)$. □

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Dynamical behavior of q-deformed logistic map in superior orbit

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Abstract. In this paper, we study the q-deformed logistic map in Mann orbit (superior orbit) which is a two-step fixed point iterative algorithm. The main aim of this paper is to investigate the whole dynamical behavior of the proposed map through various techniques such as fixed point and stability approach, time-series analysis, bifurcation plot, Lyapunov exponent and cobweb diagram. We notice that the chaotic behavior of q-deformed logistic map can be controlled by choosing control parameters carefully. The convergence and stability range of the map can be increased substantially. Moreover, with the help of bifurcation diagrams, we prove that the stability performance of this map is larger than that of existing other one dimensional chaotic maps. This map may have better applications than that of classical logistic map in various situations as its stability performance is larger.

Mathematics Subject Classification (2010): 34H10, 37M10, 37B25, 37F45.

Keywords: Logistic map, q-deformation, Mann orbit, time series analysis, bifurcation plot, Lyapunov exponent (LE), cobweb plot.

1. Introduction

Dynamical systems, an interesting branch of mathematics is primarily devoted to the study of procedures in motion. Such procedures take place in various fields such as the motion of the stars and the galaxies in the heaven [11]. In general, the dynamical systems are expressed by differential or difference equations based on the time-varying parameters.

Starting from the work of Lorenz [22] and May [24], more or less, every scientific field has been filled by the concept of nonlinear differential and discrete difference

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equations. One of the popular discrete difference equation is the standard logistic map given by the relation

$$x_{n+1} = \mu x_n(1 - x_n), n = 0, 1, 2, .., \tag{1.1}$$

where $x_n \in [0, 1]$ denotes the population at time n and $\mu > 0$ represents the population growth rate.

This population growth model was originally given by P. F. Verhulst in 1845 and 1847 [15]. Nowadays, the logistic map has become a major breakthrough and has found wider applications in many fields such as image encryption in cryptography [9, 16], traffic control [2, 21] and secure communication system [29] etc. For more information about the behavior of dynamical systems one may refer Devaney [11, 10], Holmgren [15], Alligood et al. [1], Ausloos and Dirickx [3], Elagdi [13], Elhadj and Sprott [14], Chugh et al. [8], Diamond [12], Robinson [28], Wiggins [30], Kumari et al. [18, 19, 7, 17, 20] and various other references therein.

Thus the standard logistic map has become most popular nonlinear model which is used to describe various physical and natural systems. Banerjee and Parthasarathy [4] proposed a deformation of this standard logistic map. The resulting map is known as q-deformed logistic map which is given by the following discrete difference equation

$$[x_{n+1}] = \mu[x_n](1 - [x_n]), \tag{1.2}$$

where

$$[x] = \frac{1 - q^x}{1 - q}. \tag{1.3}$$

Here, q is real and $x_n \in [0, 1]$. This q- deformed logistic map is distinct from the standard logistic map.

In the recent past, the q-deformed physical systems have been the subject of enormous research [6]. Along with, the logistic map various other maps such as Henon map [25] and Gaussian Map [26] have also been analyzed using q-deformations. In 2011, Banerjee and Parthasarathy [4] propsed this q-deformation of logistic map, studied about its concavity, non-trivial fixed points and discussed its stability through Lyapunov exponent by changing the parameter q. The stability of this map was also studied in 2015 by Prasad and Katiyar [27]. In 2019, Canovas and Munoz-Guillermo [5] analyzed this map in which topological entropy was also computed to examine the chaos.

In q-deformation, there is some modification in the map in such a way that in the limiting case $q \to 1$, the modified map (q-deformed logistic map) changes to the original map (classical logistic map). The inspiration for this work comes from the recognition that the original logistic map considers only a saturation effect, that is, an interaction between the population as a whole and a global external constraint. The q-deformation introduces a real-valued parameter q, which models the interaction between individuals in the species - supraunitary q means interindividual competition, while subunitary q leads to cooperation.

Moreover, the Mann orbit models the “inertia” of the system, or the influence of the immediate past on the discrete dynamics. It introduces another parameter, $\alpha \in [0, 1]$, the smaller the value, the larger the inertia. Therefore, in the present paper we discuss various dynamical properties of the q-deformed logistic map using Mann orbit. The complete paper is divided into four sections. In Section 1, a brief introduction is given. Section 2 includes some basic definitions, results and notations which have been taken into consideration during our analysis. In Section 3, the whole dynamical behavior of the map is investigated. This section is further divided into six subsections which are mainly devoted to the study of this map through fixed point and stability analysis, time-series representation, bifurcation diagrams, Lyapunov exponent, combined bifurcation and Lyapunov exponent analysis and cobweb plots, respectively. In Section 4, we prove the superiority of q-deformed logistic map in superior orbit. At last, the conclusion of the paper is given in Section 5.

2. Preliminaries

In this section, we recollect some basic definitions, results and concepts which have been used in our study.

Definition 2.1. *(Mann iterative algorithm)*[23]: Let X be a non-empty set and $f\colon X \to X$ be an operator. Then for an arbitrary point $x_0 \in X$, the sequence $\{x_n\}$ of all iterates, defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(x_n), \tag{2.1}$$

where $\alpha_n \in [0,1], n \in N$, is known as Mann iterative algorithm. The sequence $\{x_n\}$ of iterates is also called Mann orbit. Further, for $\alpha_n = 1$, the Mann orbit reduces to the Picard orbit.

Definition 2.2. *(Fixed point)* [10] Let X be a non-empty set and $f\colon X \to X$ be an operator. Then, an arbitrary point $x_0 \in X$ is said to be a fixed point for the mapping f if it satisfies $f(x_0) = x_0$.

Definition 2.3. *(Periodic point)* [10] A point x_0 is said to be periodic for a mapping g if it satisfies $g^p(x_0) = x_0$, where p is the least positive integer and denotes the p^{th} iteration. The sequence of p^{th} iterates with initial choice x_0 is called periodic orbit of period-p.

Definition 2.4. *(Lyapunov exponent)*[1]: Let f be the mapping of reals $\mathbb{R}$. Then, the Lyapunov exponent (LE) of the mapping f for an orbit $\{x_n\}$ is given by

$$\sigma(x_1) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln(|f'(x_i)|), \tag{2.2}$$

provided that the limit on R.H.S. exists. Moreover, for $\sigma < 0$, the orbit of the map represents stable behavior and for $\sigma > 0$, the orbit represents unstable behavior.

3. Experimental analysis of q-deformed logistic map via Mann orbit

This entire section deals with an experimental study of the dynamical behavior of q-deformed logistic map using Mann orbit, which has nowadays become a significant

method for the study of various nonlinear maps.
Let us consider, the q-deformed logistic map given by

$$[x_{n+1}] = f(x_n) = \mu\left(\frac{1-q^{x_n}}{1-q}\right)\left(1-\left(\frac{1-q^{x_n}}{1-q}\right)\right), \tag{3.1}$$

where $x_n \in [0,1]$ and q is real.
By definition of Mann orbit (2.1), we have

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f(x_n)), \tag{3.2}$$

where $x_n \in [0,1]$ and $\alpha_n \in [0,1]$.
Using (3.1) in (3.2), we get

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left[\mu\left(\frac{1-q^{x_n}}{1-q}\right)\left(1-\left(\frac{1-q^{x_n}}{1-q}\right)\right)\right], \tag{3.3}$$

where $x_n \in [0,1]$, $\alpha_n \in [0,1]$ and q is real.
Further, it is noticed that in case of $\alpha_n = 1$, the system (3.3) reduces to (3.1) and for $\alpha_n = 0$, the system remains unchanged. For the sake of convenience, we take $\alpha_n = \alpha$ and $x_n = x$ throughout this paper. In this way, Eq. (3.3) takes the following form:

$$Q_{\mu,\alpha}(x) = f(x) = (1-\alpha)x + \alpha\left[\mu\left(\frac{1-q^{x}}{1-q}\right)\left(1-\left(\frac{1-q^{x}}{1-q}\right)\right)\right], \tag{3.4}$$

Here, α, μ and q are the control parameters. Now, we apply various experimental techniques one by one to describe the complete dynamical behavior of this map by using the matlab software.

3.1. Fixed point and stability analysis of q-deformed logistic map

The fixed points of this map (3.4) can be computed by using the definition (2.2). So, in order to get its fixed points, we have

$$Q_{\mu,\alpha}(x) = x,$$

$$i.e.,\ (1-\alpha)x + \alpha\left[\mu\left(\frac{1-q^{x}}{1-q}\right)\left(1-\left(\frac{1-q^{x}}{1-q}\right)\right)\right] = x, \tag{3.5}$$

Let $q^x = X$. Then $x\log q = logX$ and hence $x = \dfrac{\log X}{\log q}$. Using these in above Eq. (3.5) and after solving it, we obtain

$$(1-\alpha)\log X + \frac{\alpha\mu\log q(1-X)(X-q)}{(1-q)^2} = \log X, \tag{3.6}$$

Being a quadratic equation in X, the Eq. (3.6) has two roots. Out of which $X = 1$ is obvious or trivial root. This implies that one trivial fixed point of q-deformed logistic map in Mann orbit i.e., $Q_{\mu,\alpha}(x)$ is $x = 0$. But it is difficult to calculate the second fixed point because of the nonlinearity of this system. That fixed point depends on the parameters μ and q. To show this, a graphical representation is given in Fig. 1.

Here, the map $Q_{\mu,\alpha}(x)$ is iterated 100 times i.e., we observe 100 numbers of iterations of this map to compute its fixed points (see Table 1) for all $x \in [0,1]$. One parameter q is taken to be fixed as $q = 0.5$ (some other value can also be taken) throughout our study. In the table, along with fixed points, the maximum value of parameter μ is also given for which the system remains convergent and stable . Here, the fixed points are computed up to four decimal places by taking the values of μ up to two decimal places.

From Table 1, we observe that the complete dynmical behavior of this map depends on the parameter α. As we decrease the value of α, the system remains stable even for a larger value of μ. Thus, by decreasing the value of parameter α, the range of convergence and stability of $Q_{\mu,\alpha}(x)$ can be increased significantly up to $\mu = 20.81$.

α	Maximum value of μ	
	for convergence	for stability
0.9	3.05	3.92
0.8	3.40	4.39
0.7	3.85	4.96
0.6	4.45	5.48
0.5	5.35	6.18
0.4	6.68	6.78
0.3	8.91	8.92
0.2	12.36	12.36
0.1	20.81	20.81

TABLE 1. Range of convergence and stability of the map $Q_{\mu,\alpha}(x)$

Also, the fixed points exist when the diagonal line $y = x$ intersects the map $Q_{\mu,\alpha}(x)$ as shown in Fig. 1 at points a and b. Here, we have shown the fixed points a and b at $\mu = 3.05$.

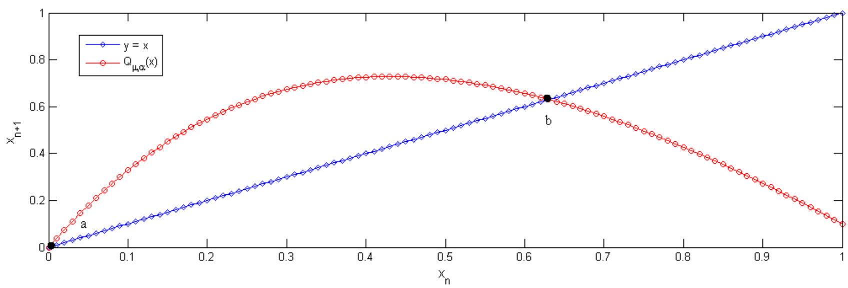

Fig. 1. Graphical representation of fixed points of $Q_{\mu,\alpha}(x)$.

3.2. Time series analysis of q-deformed logistic map for $\alpha = 0.9, 0.5$ and 0.1

In this section, using time series representation of q-deformed logistic map, we try to support the convergence and stability results given in Table 1 graphically. Here, for different values of α against some initial choices of $x \in [0, 1]$, the optimum value of μ is attained by using 100 numbers of iterations.

Example 3.1. Describe the complete dynamical behavior of q-deformed logistic map for $\alpha = 0.9$ and for all $x \in [0, 1]$ by using time series representation of dynamical systems.

Solution. We examine the complete dynamical behavior of q-deformed logistic map for $\alpha = 0.9$ by drawing Figs. 2, 3, 4, 5 and 6. We observe from Fig. 2 that the trajectory of $Q_{\mu,\alpha}(x)$ converges to a fixed point for $0 < \mu \leq 3.05$ for all values of x. This system oscillates between two fixed points for $3.21 < \mu \leq 3.74$ as shown in Fig. 3 at $\mu = 3.5$ for $x_0 = 0.5$. 4-stable oscillations exist for $3.80 < \mu \leq 3.88$ as shown at $\mu = 3.85$ in Fig. 4. The trajectory oscillates between 8-stable fixed points at $\mu = 3.92$ for all $x \in [0, 1]$ as depicted in Fig. 5 for $x_0 = 0.5$. Further, the system starts to show more and more irregular vibrations i.e. sensitive dependence on initials when parameter $\mu \geq 3.93$. This chaoticity of the system is shown at $\mu = 4$ for $x_0 = 0.5$ by Fig. 6.

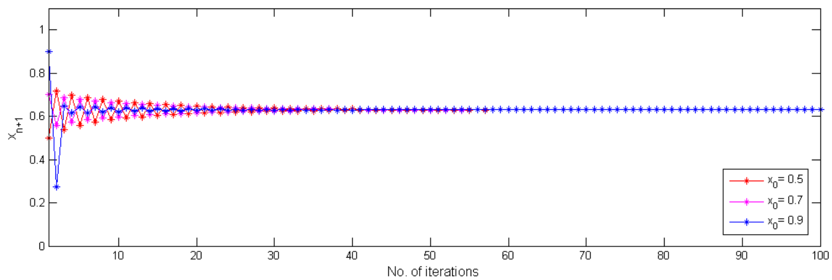

Fig. 2. Stable convergent solution of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9, \mu = 3.05$

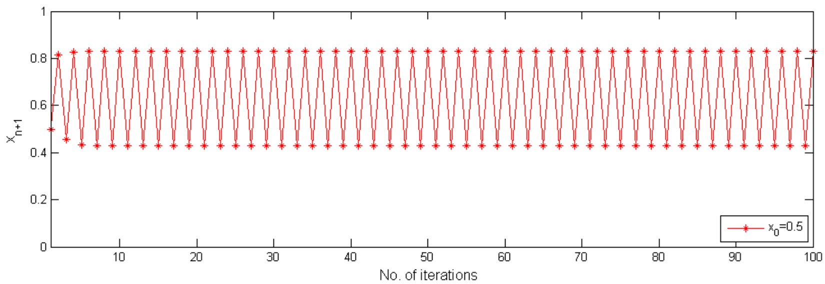

Fig. 3. 2-Stable fixed point oscillation of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9, \mu = 3.5$

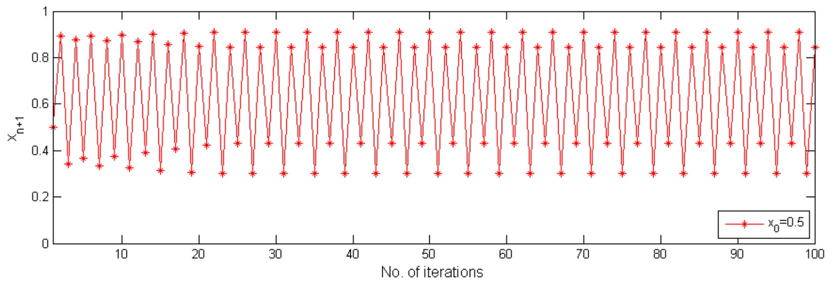

Fig. 4. 4-Stable fixed point oscillation of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9$, $\mu = 3.85$

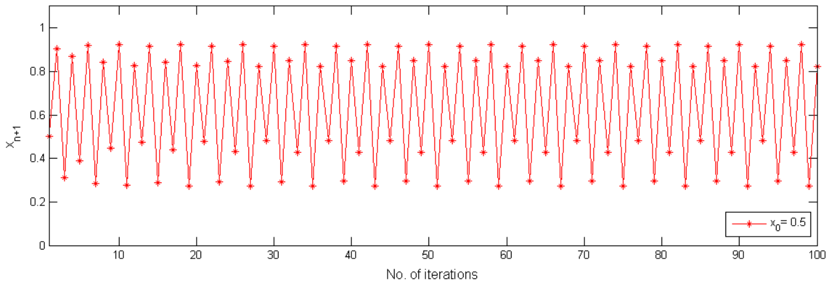

Fig. 5. 8-Stable fixed point oscillation of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9$, $\mu = 3.92$

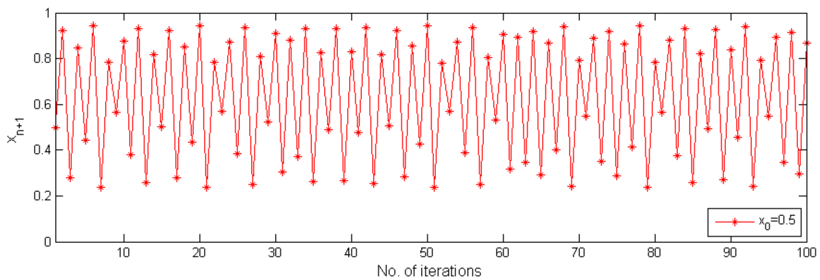

Fig. 6. Divergent behavior of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9$, $\mu = 4$

Example 3.2. By using time series analysis, describe the whole dynamical behavior of q-deformed logistic map $Q_{\mu,\alpha}(x)$ for $\alpha = 0.5$ and for all $x \in [0, 1]$ by taking 100 numbers of iterations.

Solution. For this particular value of parameter α, the system has stable fixed point for $0 < \mu \leq 5.35$ for all $x \in [0, 1]$, as shown in Fig. 7 at $x_0 = 0.5$. Also, the trajectory of the system oscillates between two fixed points for $5.63 < \mu \leq 6.18$ and for all $x \in [0, 1]$ as represented in Fig. 8 for $\mu = 6.12$ at $x_0 = 0.5$. Also, for $\mu \geq 6.19$, the system is undefined (see, Fig. 9 for $\mu = 6.19$).

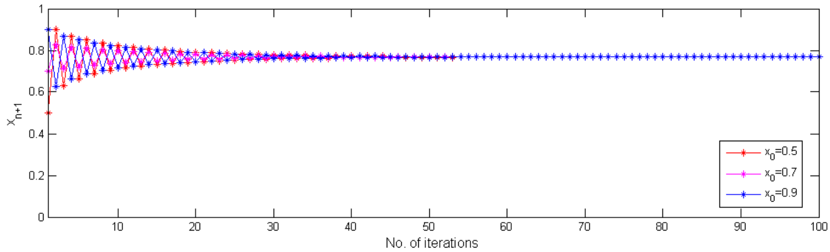

Fig. 7. Stable convergent solution of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.5$, $\mu = 5.35$

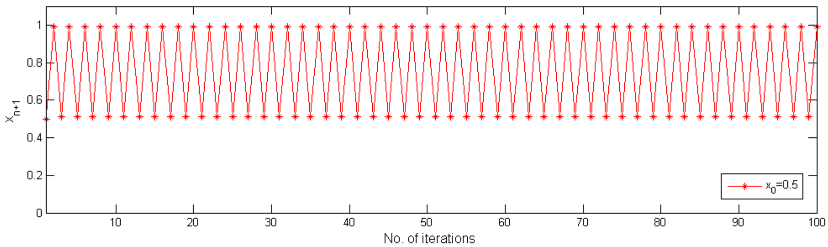

Fig. 8. 2-Stable fixed point oscillation of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.5$, $\mu = 6.12$

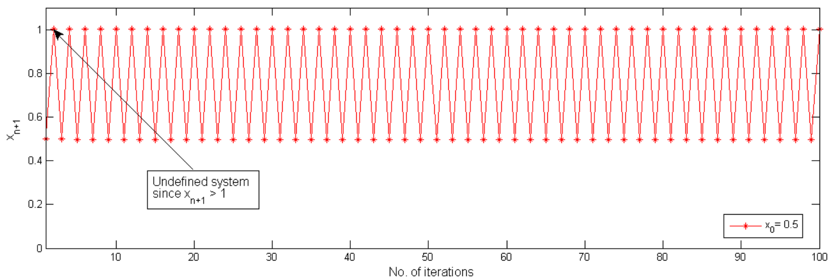

Fig. 9. Undefined $Q_{\mu,\alpha}(x)$ for $\alpha = 0.5$, $\mu = 6.19$

Example 3.3. Demonstrate that the stability of the map $Q_{\mu,\alpha}(x)$ can be extended by decreasing the value of parameter α. Explain this fact for all $x \in [0, 1]$ by taking $\alpha = 0.1$.

Solution. In this case, the q-deformed map $Q_{\mu,\alpha}(x)$ converges to a stable fixed point for $0 < \mu \leq 20.81$ and for all $x \in [0, 1]$. This convergent behavior is shown in Fig. 10 for $\mu = 20$. In addition, the map $Q_{\mu,\alpha}(x)$ cannot be defined for all $\mu > 20.81$, since in this range $x_{n+1} > 1$ as shown in Fig. 11 for $\mu = 21$ which represents the undefined behavior of the system.

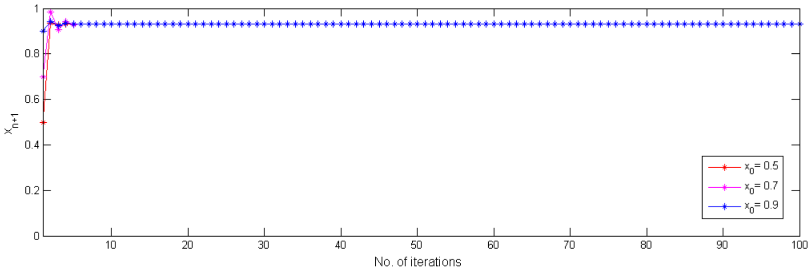

Fig. 10. Stable convergent solution of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.1$, $\mu = 20$

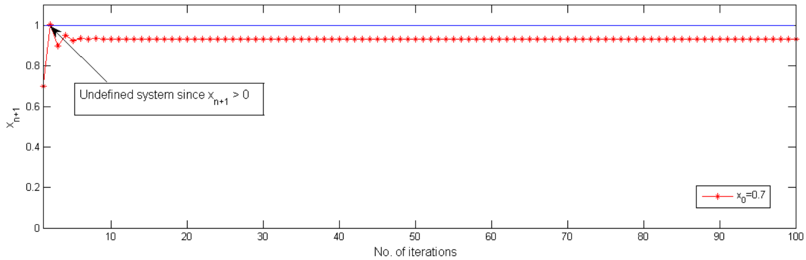

Fig. 11. Undefined $Q_{\mu,\alpha}(x)$ for $\alpha = 0.1$, $\mu = 21$

3.3. Bifurcation analysis of q-deformed logistic map for different choices of μ

In general, bifurcation diagrams are the tools mainly used to classify the dynamical systems in nonlinear regions. Bifurcation diagrams demonstrate an immediate change that occurs in the asymptotic solutions of a dynamical system.

Under this section, the complete dynamical behavior of $Q_{\mu,\alpha}(x)$ is presented by drawing bifurcation diagrams for $\alpha = 0.9$, 0.5 and 0.1. A route from periodic region to chaotic region has been shown in Figs. 12, 13, 14 by letting step size for parameter $\mu = 0.001$, initial choice $x_0 = 0.5$ and the number of iterations (N) = 800.

In Fig. 12, the entire dynamical system $Q_{\mu,\alpha}(x)$ has been divided into different regions which explain the complexity of the system. For $0 < \mu \leq 3.15$, the system $Q_{\mu,\alpha}(x)$ has a stable fixed point and period-doubling bifurcation occurs for $3.15 < \mu \leq 3.78$ as shown by regions of period-1 and period-2. Also, the system shows the route from 2-periods to more than 2-periods for $3.78 < \mu \leq 3.95$. The system becomes chaotic as parameter μ exceeds from 3.95, i.e., for $\mu > 3.95$, the system shows sensitive dependence on initials.

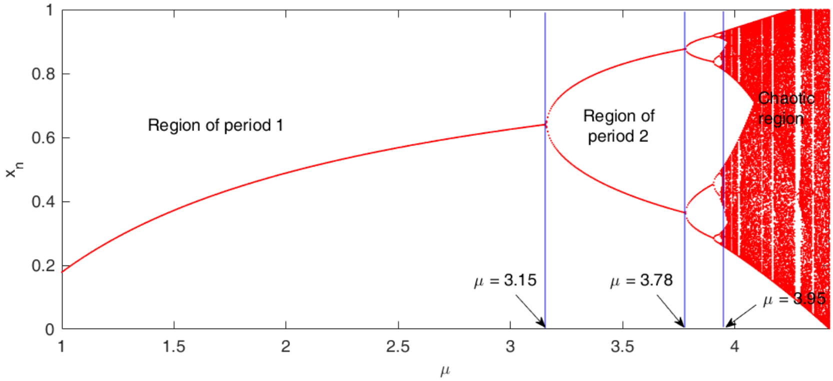

Fig. 12. Bifurcation plot of $Q_{\mu,\alpha}(x)$ for $3 \leq \mu \leq 4, \alpha = 0.9, x_0 = 0.5$

Moreover, period doubling bifurcation for the q-deformed logistic map is represented at $\alpha = 0.5$ in Fig. 13. For this, the system has stable solutions for $0 < \mu \leq 6.18$. Also, the system cannot be defined when the parameter μ exceeds from 6.18 as shown by undefined region.

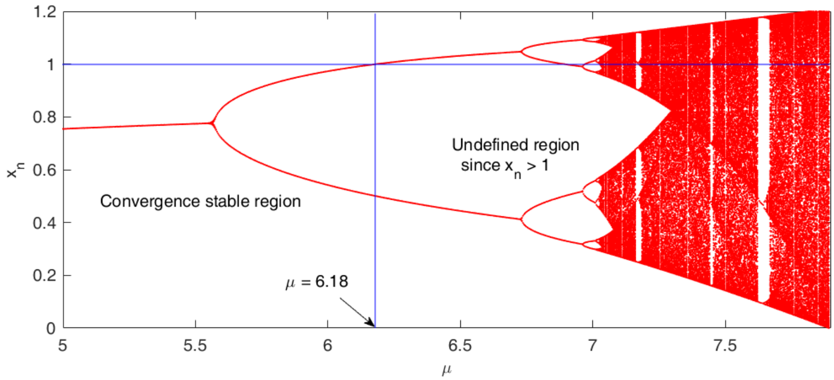

Fig. 13. Bifurcation plot of $Q_{\mu,\alpha}(x)$ for $5 \leq \mu \leq 7.5, \alpha = 0.5, x_0 = 0.5$

Further, from Fig. 14, we observe that the system $Q_{\mu,\alpha}(x)$ remains stable for an extended range of parameter μ, i.e., for $0 < \mu \leq 28.52$, the orbit is convergent to a fixed point. Also, this system cannot be defined for $\mu > 28.52$ as in this range $x_n > 1$. In other words, $x_n \notin [0, 1]$, which represents that the behavior of the system is undefined here.

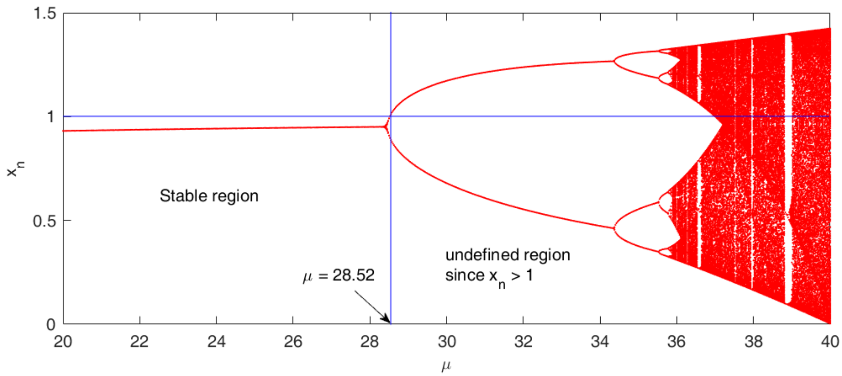

Fig. 14. Bifurcation plot of $Q_{\mu,\alpha}(x)$ for $20 \leq \mu \leq 38, \alpha = 0.1, x_0 = 0.5$

Remark 1. The system $Q_{\mu,\alpha}(x)$ gains more and more dynamical properties when the value of parameter $\alpha \in [0, 1]$ increases as shown by the bifurcation diagrams, i.e., for $\alpha = 0.1$, 0.5, the system demonstrates fixed point and periodic properties; for α= 0.9, system exhibits fixed points, periodicity and chaos.

3.4. Mathematical and experimental analysis of q-deformed logistic map by Lyapunov exponent

An another major characteristic of nonlinear dynamical systems is Lyapunov exponent, which determines the sensitive dependence of two distinct orbits beginning from very close initial positions. In case of stable periodic behavior, the rate of onvergence to stable point is determined by LE, whereas, in case of chaotic behavior, LE determines the rate of divergence between the orbits. For the q-deformed logistic map with Mann iteration ($Q_{\mu,\alpha}(x)$), Lyapunov exponent is defined as follows:

Let us begin the method by taking Mann orbits for two distinct initial choices x and $x + h$, where $0 < h < 1$. Here, Δ represents the divergence between these orbits, which is taken as the exponential growth $he^{n\sigma}$, where σ denotes the Lyapunov exponent of the map and n stands for the number of iterations. So, it can be written as

$$Q^n_{\mu,\alpha}(x + h) - Q^n_{\mu,\alpha}(x) = \Delta$$

$$Q^n_{\mu,\alpha}(x + h) - Q^n_{\mu,\alpha}(x) = he^{n\sigma}$$

$$\therefore \quad \frac{Q^n_{\mu,\alpha}(x + h) - Q^n_{\mu,\alpha}(x)}{h} = e^{n\sigma}. \tag{3.7}$$

Taking limit $h \to 0$, on both sides, we get

$$\lim_{h\to 0} \frac{Q^n_{\mu,\alpha}(x + h) - Q^n_{\mu,\alpha}(x)}{h} = e^{n\sigma}$$

$$i.e., \ (Q^n_{\mu,\alpha})'(x) = e^{n\sigma}. \tag{3.8}$$

Applying logarithm on both sides, we obtain

$$\sigma = \frac{1}{n} \log |(Q^n_{\mu,\alpha})'(x)|, \tag{3.9}$$

where $(Q^n_{\mu,\alpha})'(x)$ represents the first order derivative for the map $Q_{\mu,\alpha}(x)$. For nth degree polynomial, the derivative can be evaluated by applying the chain rule of differentiation.
So, for the succession $x_1, x_2 = Q_{\mu,\alpha}(x_1)$, $x_3 = Q_{\mu,\alpha}(x_2), \cdots$, $x_{n+1} = Q_{\mu,\alpha}(x_n), \cdots$, we have

$$(Q^n_{\mu,\alpha})'(x_1) = Q'_{\mu,\alpha}(x_n) \cdot Q'_{\mu,\alpha}(x_{n-1}) \cdots Q'_{\mu,\alpha}(x_2) \cdot Q'_{\mu,\alpha}(x_1). \tag{3.10}$$

Now, using (3.10) in (3.9), we get

$$\sigma = \frac{1}{n} \log |Q'_{\mu,\alpha}(x_n) \cdot Q'_{\mu,\alpha}(x_{n-1}) \cdots Q'_{\mu,\alpha}(x_2) \cdot Q'_{\mu,\alpha}(x_1)|,$$

$$= \frac{1}{n}[\log |Q'_{\mu,\alpha}(x_n)| + \log |Q'_{\mu,\alpha}(x_{n-1})| + \cdots + \log |Q'_{\mu,\alpha}(x_2)| + \log |Q'_{\mu,\alpha}(x_1)|],$$

$$\sigma = \frac{1}{n} \sum_{j=1}^{n} \log |Q'_{\mu,\alpha}(x_j)|, \tag{3.11}$$

which is the required Lyapunov exponent of $Q_{\mu,\alpha}(x)$.
In addition, if the map has fixed orbit, then (3.11) reduces to

$$\sigma = \ln(|Q'_{\mu,\alpha}(x_1)|). \tag{3.12}$$

Also, for perodic orbit of period- p, we get from (3.11)

$$\sigma = \frac{1}{p} \sum_{j=1}^{p} \ln(|Q'_{\mu,\alpha}(x_j)|). \tag{3.13}$$

Remark 2. In order to evaluate the Lyapunov exponent for aperiodic orbits, it is almost impossible to utilize the entire length of an orbit. So, only finite length of an orbit is used frequently to estimate the Lyapunov exponent.
Remark 3. Moreover, the fixed and periodic orbits of the map represent stable behavior for $\sigma < 0$ and unstable behavior for $\sigma > 0$. In this way, the Lyapunov exponent demonstrates the stable and unstable nature of various fixed and periodic orbits.

Example 3.4. Calculate the Lyapunov exponent of the map $Q_{\mu,\alpha}(x)$ for the following values of parameters α and μ :
(a) $\alpha = 0.9, \mu = 3$
(b) $\alpha = 0.9, \mu = 3.5$.
Also, examine the dynamical behavior of this map by plotting the Lyapunov exponent for $\alpha = 0.9, 1 \leq \mu \leq 4.4$.
Solution. (a) As discussed in Section 3.2, for $0 < \mu \leq 3.05$, the map $Q_{\mu,\alpha}(x)$ has a fixed orbit for all $x \in [0, 1]$. Also, the fixed point of the orbit for $\mu = 3$ is given as

0.6255. So, to compute the Lyapunov exponent of this orbit, it is enough to solve Eq. (3.12). For that, we have

$$Q_{\mu,\alpha}(x) = (1-\alpha)x + \alpha\left[\mu\left(\frac{1-q^x}{1-q}\right)\left(1-\left(\frac{1-q^x}{1-q}\right)\right)\right],$$

$$Q'_{\mu,\alpha}(x) = (1-\alpha) + \frac{\alpha\mu}{1-q}\cdot q^x \cdot \ln q\left[2\left(\frac{1-q^x}{1-q}\right)-1\right]. \tag{3.14}$$

Putting $\alpha = 0.9, \mu = 3, x = 0.6255$ in Eq. (3.14), we get

$$Q'_{3,0.9}(0.6255) = -0.3292. \tag{3.15}$$

Now, using (3.15) in (3.12), we obtain

$$\sigma = \ln(|-0.3292|) = -0.4825.$$

So, the Lyapunov exponent at $\mu = 3$ is -0.4825, which is a negative value and thus from the definition of Lyapunov exponent, this fixed point is a stable attractor.

(b) For $3.21 < \mu \leq 3.74$, the map $Q_{\mu,\alpha}(x)$ represents periodic orbit of period-2 for all $x \in [0,1]$. So, for $\mu = 3.5$, the periodic points are $x_1 = 0.4281$ nad $x_2 = 0.8297$. Thus, we get

$$Q'_{\mu,\alpha}(x_1) = 0.0617 \tag{3.16}$$

$$Q'_{\mu,\alpha}(x_2) = -0.6997. \tag{3.17}$$

Now, using Eqs. (3.16) and (3.17) in (3.13), we get

$$\begin{aligned}\sigma &= \frac{1}{2}\left[\ln|Q'_{\mu,\alpha}(x_1)| + \ln|Q'_{\mu,\alpha}(x_2)|\right]\\ &= \frac{1}{2}\left[\ln|0.0617| + \ln|-0.6997|\right]\\ &= \frac{1}{2}\left[(-1.2097) + (-0.1551)\right].\end{aligned}$$

This gives

$$\sigma = -0.6824$$

So, the Lyapunov exponent is less than zero in this case also. Thus, these periodic points are stable attractors.

In Fig. 15, we plot Lyapunov exponent (σ) to discover the behavior of dynamical system $Q_{\mu.\alpha}(x)$ for $1 \leq \mu \leq 4.4$ at $\alpha = 0.9$. To plot this, we consider 10,000 iterations, i.e., $N = 10,000$ and initiator $x_0 = 0.5$. It is clear from the figure that the system remains stable for $0 < \mu \leq 3.95$ since in this range σ <0, i.e., the system preserves stable orbits, Also, in the zoomed rectangular area, the chaotic behavior of the system is represented since here, $\sigma > 0$, i.e., the orbit shows sensitive dependence on initiators. Hence, chaos occurs in the system as we increase the parameter μ from $\mu = 3.95$.

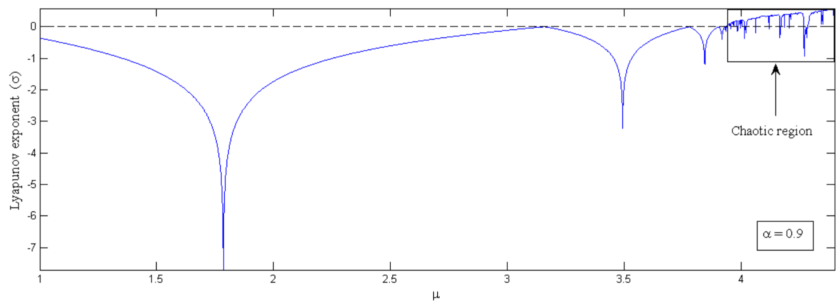

Fig. 15. Lyapunov exponent plot of $Q_{\mu,\alpha}(x)$ for $1 \leq \mu \leq 4.4, \alpha = 0.9, x_0 = 0.5$

Example 3.5. Explain the dynamical behavior of this q-deformed logistic map $Q_{\mu,\alpha}(x)$ by plotting the Lyapunov exponent for the following values of parameters μ and α:
(a) $1 \leq \mu \leq 7.9, \alpha = 0.5$,
(b) $1 \leq \mu \leq 28.52, \alpha = 0.1$.

Solution. (a) We investigate the dynamical behavior of $Q_{\mu,\alpha}(x)$ by drawing the Lyapunov exponent diagram as shown in Fig. 16, for the given values of parameters and initiator $x_0 = 0.5$. We observe that the Lyapunov exponent is negative, i.e. $\sigma < 0$ for $0 < \mu \leq 7.07$, which represents the stable behavior of the system. Also for $7.07 < \mu \leq 7.9$, the spectrum of Lyapunov exponent begins to approach to a positive value of σ, which indicates that there is chaos in the dynamical system.

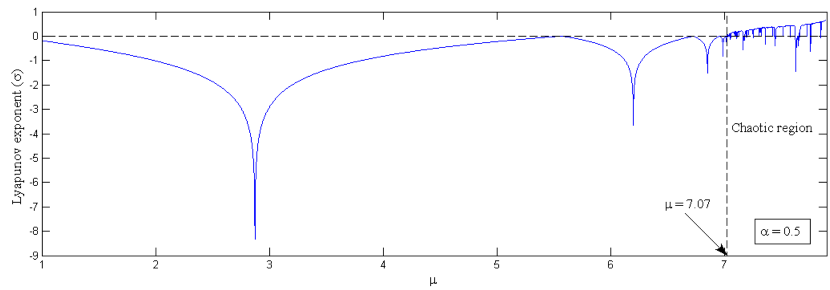

Fig. 16. Lyapunov exponent plot of $Q_{\mu,\alpha}(x)$ for $1 \leq \mu \leq 7.9, \alpha = 0.5, x_0 = 0.5$

(b) The stabilty of dynamical system can be increased by controlling the parameters. This fact is analyzed here by estimating the value of LE (σ) at a decreased value of parameter α, i.e., at $\alpha = 0.1$. For this particular value of α, the system shows stable behavior for an increased value of parameter μ, i.e., for $0 < \mu \leq 28.52$. We have explained this fact experimentally in Fig. 17. We observe that for $0 < \mu \leq 28.52$, the value of Lyapunov exponent (σ) is negative. Thus the system shows fixed stable behavior for this extended range of μ.

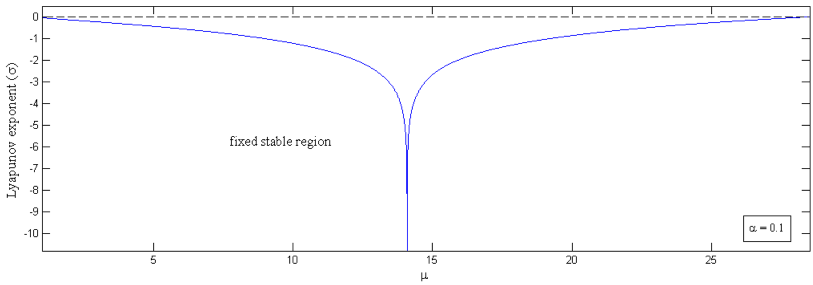

Fig. 17. Lyapunov exponent plot of $Q_{\mu,\alpha}(x)$ for $1 \leq \mu \leq 28.52$, $\alpha = 0.1$, $x_0 = 0.5$

3.5. A new experimental analysis of q-deformed logistic map via combined study of bifurcation and Lyapunov exponent

Under this section, we try to investigate the complex dynamical behavior of this system $Q_{\mu,\alpha}(x)$ with the help of combined bifurcation and Lyapunov exponent plots. This experimental technique enables us to investigate the exact value of parameter μ obtained in previous subsections at which the system changes its behavior. In these figures, the entire region of the dynamical system $Q_{\mu,\alpha}(x)$ is divided into distinct regions separated by a magenta dotted line.

Fig. 18 exhibits the combined representation of bifurcation and Lyapunov exponent for $1 \leq \mu \leq 4.4$ at $\alpha = 0.9$. Here, the system has two regions, stable periodic region and chaotic region, separated by a magenta dotted line at $\mu = 3.95$, which is the highest value of μ for which the system remains stable, afterwards chaos occurs.

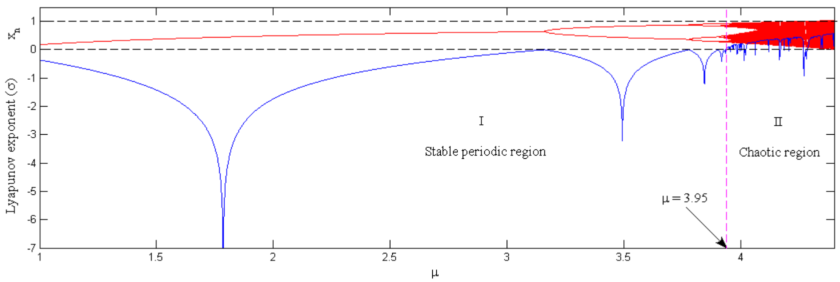

Fig. 18. Bifurcation plot v/s Lyapunov exponent plot of $Q_{\mu,\alpha}(x)$ for $1 \leq \mu \leq 4.4$ at $\alpha = 0.9$

The entire region of $Q_{\mu,\alpha}(x)$ is divided into three regions (stable, undefined and chaotic region) at particular values of parameter μ as shown in Figs. 19 and 20 for $\alpha = 0.5$ and $\alpha = 0.1$ respectively. Also, it can be noticed from the figures that the system preserves its stability for a larger value of parameter μ as we decrease the value of parameter α. Moreover, when $\sigma > 0$, the system represents chaotic behavior.

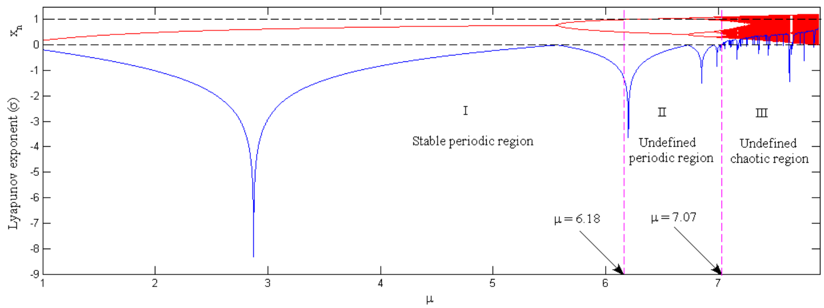

Fig. 19. Bifurcation plot v/s Lyapunov exponent plot of $Q_{\mu,\alpha}(x)$ for $1 \leq \mu \leq 7.9$ at $\alpha = 0.5$

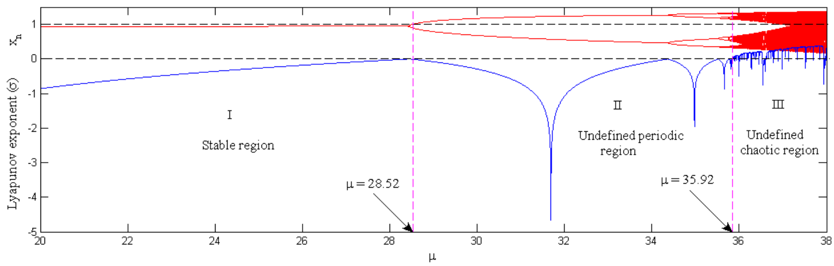

Fig. 20. Bifurcation plot v/s Lyapunov exponent plot of $Q_{\mu,\alpha}(x)$ for $20 \leq \mu \leq 38$ at $\alpha = 0.1$

3.6. Experimental analysis of q-deformed logistic map through cobweb plot

A cobweb diagram is generally a visual method which is used to examine the qualitative nature of the map in the field of dynamical systems. With the help of cobweb plot, we can predict the long term behavior of an initial condition under repeated application of a map.

Fig. 21 depicts the attracting behavior of the fixed point 0.6304 of the map $Q_{\mu,\alpha}(x)$ for the parameters $\alpha = 0.9$, $\mu = 3.05$ and for initiator $x_0 = 0.5$. Also, the periodic behavior of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9$, $\mu = 3.5$ and $x_0 = 0.5$ is shown in Fig. 22. In addition, Fig. 23 represents the unstable behavior of this map for $\alpha = 0.9, \mu = 4, x_0 = 0.5$.

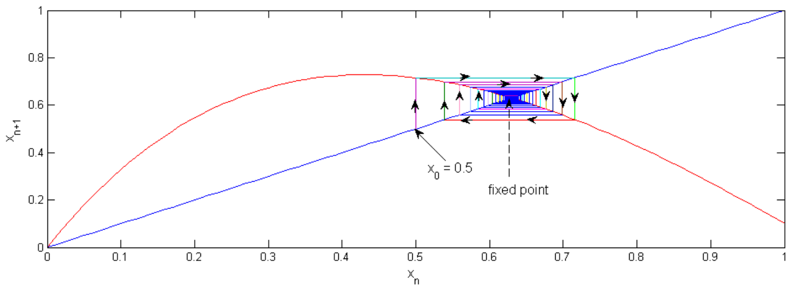

Fig. 21. Attracting behavior of fixed point of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9, \mu = 3.05$

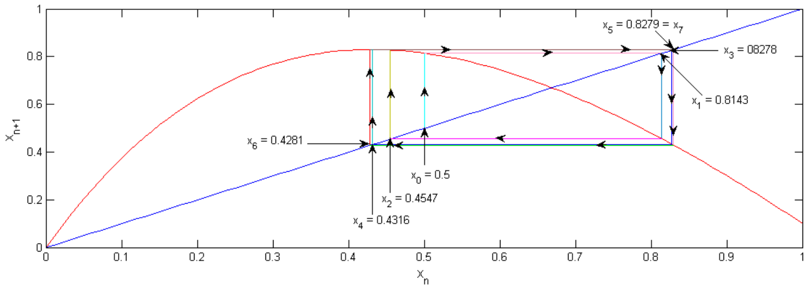

Fig. 22. Periodic behavior of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9, \mu = 3.5$

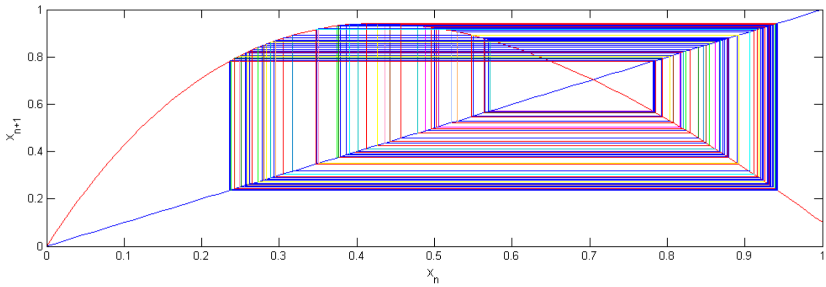

Fig. 23. Unstable behavior of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.9, \mu = 4$

Further, the attracting nature of fixed point 0.7686 for $\alpha = 0.5, \mu = 5.35$ and periodic nature for $\alpha = 0.5, \mu = 5.65$ of the q-deformed map $Q_{\mu,\alpha}(x)$ with initiator $x_0 = 0.5$ are represented in Figs. 24 and 25 respectively.

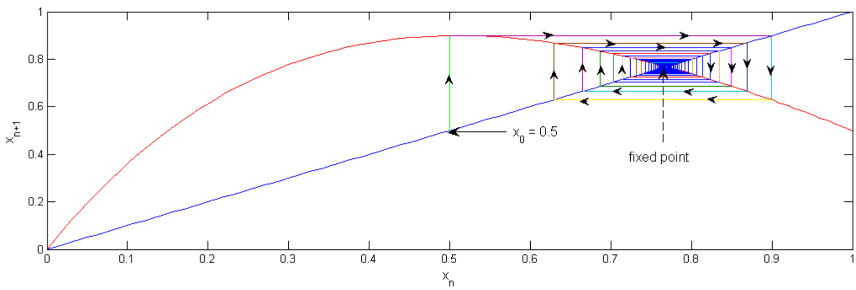

Fig. 24. Attracting behavior of fixed point of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.5, \mu = 5.35$

Moreover, Fig. 26 depicts the attracting behavior of the fixed point 0.9310 of this map $Q_{\mu,\alpha}(x)$ for the parameters $\alpha = 0.1, \mu = 20$ and $x_0 = 0.5$. Also, it is clear from the Fig. 27 that this q-deformed logistic map $Q_{\mu,\alpha}(x)$ is not defined for $\alpha = 0.1, \mu = 21$ and $x_0 = 0.5$, since $x_{n+1} > 1$ here.

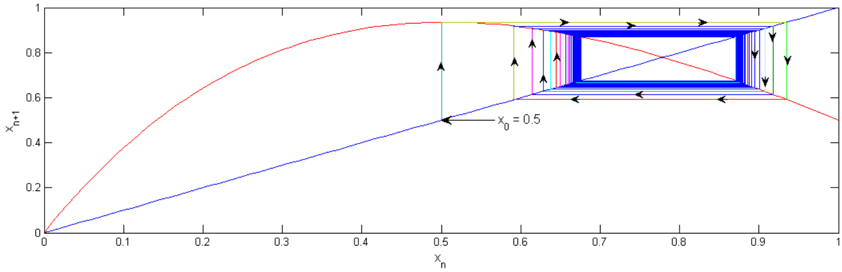

Fig. 25. Periodic behavior of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.5, \mu = 5.65$

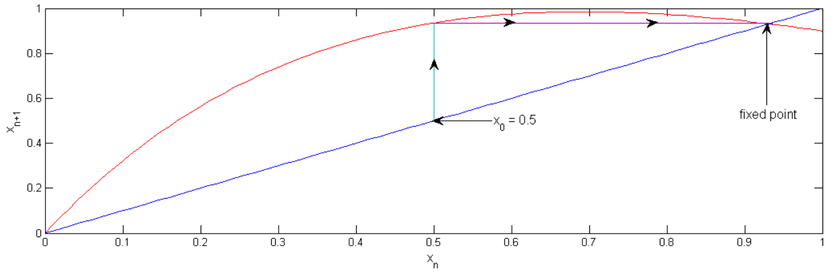

Fig. 26. Attracting behavior of fixed point of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.1, \mu = 20$

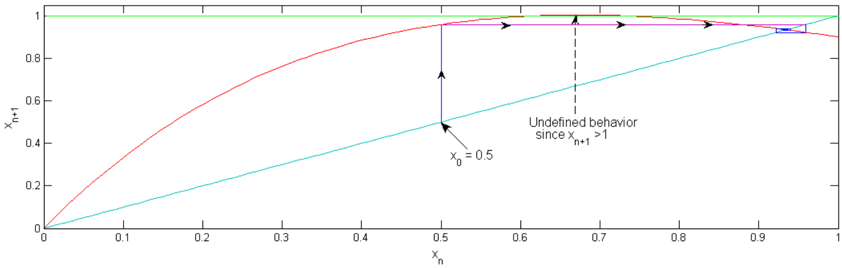

Fig. 27. Undefined behavior of $Q_{\mu,\alpha}(x)$ for $\alpha = 0.1, \mu = 21$

4. Superiority of q-deformed map in superior orbit

To prove the superiority of q-deformed map in superior orbit (3.3), we compare its stability performance with existing one dimensional maps using bifurcation plots.

4.1. Stability performance of q-deformed logistic map in superior orbit

In order to facilitate comparison, we compare the stability performance of the map (3.3) with existing one dimensional maps including classical logistic map, logistic map in superior orbit, sine map and q-deformed logistic map (3.1).

From Fig. 28, we observe that q-deformed logistic map considered in superior orbit (3.3) remains stable for $0 < \mu \leq 28.51$ which we have already shown in Subsection 3.3. In Subfigures 28a - 28d, we draw the bifurcation diagrams to study the stability performance of existing one dimensional chaotic maps. We notice that the classical logistic map is stable for $0 < \lambda \leq 3.57$ while logistic map in superior orbit remains stable for $0 < \mu \leq 21.2$. Also, the sine map shows its stable behavior for $0 < \mu \leq 0.86$ and the one dimensional q-deformed logistic map attains its stability performance for $0 < \mu \leq 3.58$. This proves that q-deformed logistic map in superior orbit has largest range of stability which is very higher than the existing other one dimensional chaotic maps.

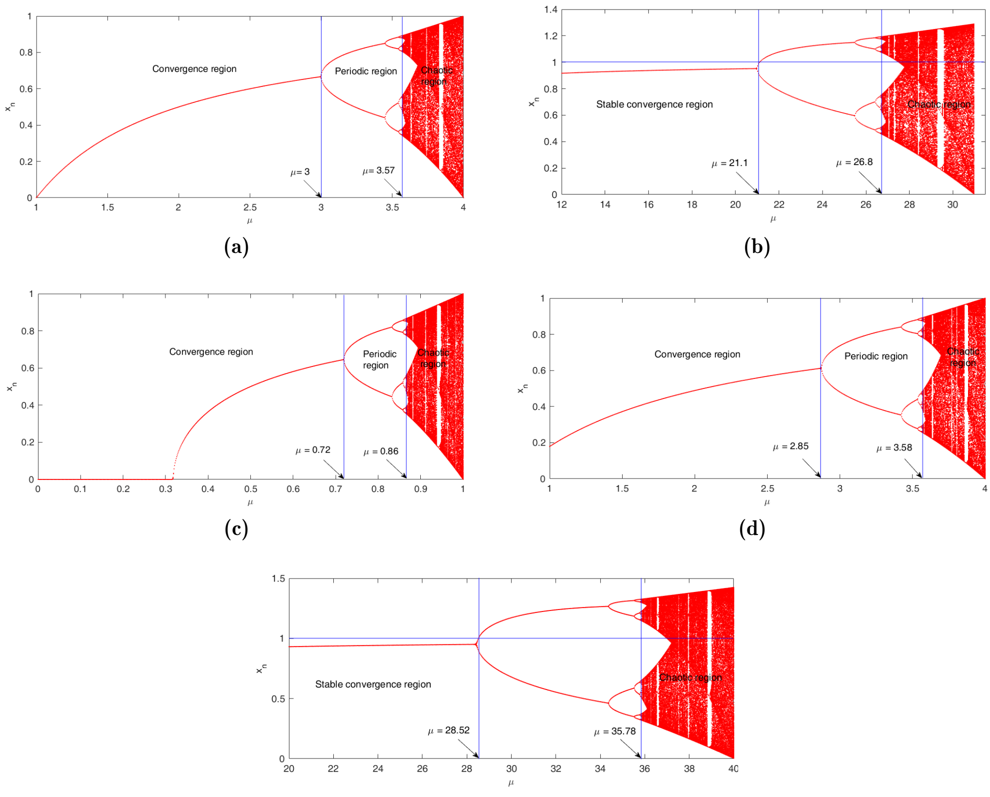

Fig. 28. Bifurcation plots (a) logistic map (b) logistic map in superior orbit (c) sine map (d) q-deformed logistic map and (e) q-deformed logistic map in superior orbit.

5. Conclusion

Here, a novel study of dynamical behavior of the q-deformed logistic map using Mann iterative algorithm is given. In this system, there are three control parameters denoted by α, μ and q. And it is quite interesting to notice that the entire dynamical behavior of this map depends on these three parameters. The q-deformed logistic map is studied via fixed point and stability analysis, time series representation, bifurcation analysis, Lyapunov exponent method, combined bifurcation and Lyapunov exponent analysis and cobweb plot. The following concluding remarks are drawn from our study:

1. The fixed point analysis approach has been used to compute the fixed points of the system (3.4). Also, the stability performance of the unrestricted system has been checked. The convergence and stability range of the q-deformed logistic map can be increased by choosing the parameters (μ, α) carefully (see, Table 1).
2. The complex dynamical behavior of this q-deformed logistic map has been further examined graphically by using time series representation for $\alpha = 0.9, 0.5$ and 0.1 to confirm the stability results obtained by fixed point analysis.
3. The bifurcation analysis is also used to investigate the various dynamical properties of the map such as fixed point, periodicity and chaos for different choices of μ.
4. The irregular behavior of dynamical system has also been analyzed numerically and experimentally by adopting Lyapunov exponent approach. Furthermore, combined bifurcation and Lyapunov exponent plots are shown to demonstrate various regions of this system. Also, cobweb plots have been used for further investigation.
5. It is strongly highlighted that the q-deformed logistic map has more stability performance than that of existing other one dimensional dynamical systems (see, Fig. 28).
6. For future research, an exhaustive search of the (μ, α) plane, followed by a graphical depiction of the $Q_{\mu,\alpha}(x)$, demarcating the areas of convergence, stability and sensitive dependence might be very interesting.

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On a coupled system of viscoelastic wave equation of infinite memory with acoustic boundary conditions

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Abstract. This work deals with a coupled system of viscoelastic wave equation of infinite memory with mixed Dirichlet-Neumann boundary conditions. The coupling is via by the acoustic boundary conditions on a portion of the boundary. The semigroup theory is used to show the well posedness and regularity of the initial and boundary value problem. Moreover, we investigate exponential stability of the system taking into account Gearhart-Prüss' theorem.

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1. Introduction

In this paper, we consider the following viscoelastic wave equation coupled with mixed boundary conditions

$$\begin{cases} u_{tt} - \operatorname{div}(\mathrm{A}\nabla u) + \int_0^{+\infty} g(s)\operatorname{div}(\mathrm{A}\nabla u(t-s))ds = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu_{\mathrm{A}}} - \int_0^{+\infty} g(t-s)\frac{\partial u}{\partial \nu_{\mathrm{A}}}(s)ds = z_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ hz_{tt} + fz_t + mz + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x) & \text{for} \quad x \in \Gamma_1, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, such that Γ_0 and Γ_1 are closed and disjoint and $\nu = (\nu_1, \cdots, \nu_N)$ represents the unit outward normal to Γ. The term $\int_0^{+\infty} g(s) \mathrm{div}(\mathrm{A}\nabla u(t-s)) \mathrm{d}s$ is the infinite memory (past history) responsible for the viscoelastic damping, where g is called the relaxation function. The functions $h, f, m : \Gamma_1 \to \mathbb{R}^+$ are essentially bounded. There exist three positive constants f_0, m_0, and h_0 such that $f(x) \geq f_0$, $m(x) \geq m_0$ and $h(x) \geq h_0$ for a.e. $x \in \Gamma_1$. The initial conditions $u_0, u_1 : \Omega \to \mathbb{C}$, $z_0, z_1 : \Gamma_1 \to \mathbb{C}$ are given functions. The operator $\mathrm{A} = (a_{ij}(x))_{i,j};\ i, j = 1, \ldots, N$; and $\frac{\partial u}{\partial \nu_{\mathrm{A}}} = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_j} \nu_i$.

The above model would be to describe the motion of fluid particles from rest in the domain Ω into part of the surface at a given point $x \in \Gamma_1$, which can be expressed by the pressure at that point. The relationship between the velocity potential $u_t = u_t(x,t)$ at a point on the surface and the normal displacement $z = z(x,t)$ is proportional to the pressure. It is called the acoustic impedance. This impedance may be complex in the case of the velocity potential was not in phase with the pressure. The coupling of our model (1.1) is via by the impenetrability boundary condition $(1.1)_3$ and the acoustic boundary condition $(1.1)_4$.

The partial differential equation (PDE) system of viscoelastic wave equation with acoustic boundary conditions was first introduced by Morse and Ingard [15] and developed by Beale [5]. In [5], the problem was formulated as an initial value problem in a Hilbert space and semigroup methods were used to solve it. The loss of decay has obtained by [5] provided that the term z_{tt} was included. Recently, the result concerning existence and asymptotic behavior of smooth, as well as weak solution of wave equation with acoustic boundary conditions have been established by many authors, see [10, 13]. Boukhatem and Benabderrahmane [8] studied the global existence and exponential decay of solution of finite memory of the system (1.1) in the absence of the second derivative z_{tt}. This absence brings us some difficulties in the study because of the abnormality of the system. It can not apply directly the semigroups or Faedo-Galerkin's theories. They added in the arguments the term εz_{tt} when $\varepsilon \to 0$ to overcome the difficulty. Mentionned the work of Peralta [16] who bringing an analysis of wave equation involving mixed Dirichlet-Neumann boundary conditions, delay and acoustic conditions where both are localized on a portion of the boundary

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu} - z_t = -a u_t(.,. - \tau) - k u_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ h z_{tt} + f z_t + m z + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ u(x,t) = \varphi(x,t) & \text{for} \quad (x,t) \in \Omega \times (-\tau, 0) \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x) & \text{for} \quad x \in \Gamma_1, \end{cases} \tag{1.2}$$

Here, $\tau > 0$ is a constant delay parameter and $a, k \geq 0$. He proved the existence and uniqueness of solutions of (1.2) using semigroup theory for bounded linear operators. Moreover, if the delay factor is less than the damping factor $(a < k)$, the exponential stability result is shown using the energy multiplier method. In the case of equality $(a = k)$, he showed that the energy decays to zero asymptotically using variational methods. In addition, the stability results have been considered in [16], where the term $-f_0 z_t$ was included in the right hand of side of the oscillator equation $(1.2)_4$. If $f > f_0$ then we show that the energy of the solution decays to zero exponentially. In the case $f = f_0$, the solutions have an asymptotically decaying energy. Moreover, Gao et al. [11] presented a new method to obtain uniform decay rates for (1.2) with nonlinear acoustic boundary conditions in the absene of delay. The system contains an internal localized damping term $w(x)u_t$ in $(1.2)_1$ and damping and potential in the boundary displacement equation are nonlinear, where the terms $f(z_t)$ and $m(z)$ are replaced by fz_t and mz in $(1.2)_4$, respectively.

The primary discussion touched upon by several authors is to use the integral term of relaxation function g instead the frictional damping term u_t. The question that have been focused their attention as an important works is the viscoelastic damping of memory effect should be strong enough to procreate the decay of the system.

One of important motivations to studying exponential stability of the associated semigroup comes from the spectral analysis. This purpose recalls the related results given by Gearhart-Prüss' theorem (see [14, 17]). It is shown all eigenvalues approach a line that parallel to the imaginary axis. Moreover, the resolvent operator is bounded for all eigenvalues of the generator associated. The proof is the combination of the contradiction argument with a PDE technique. Let us mention some papers on weakly dissipative coupled systems. In [12], the exponential decay is established for each of the wave equations that have been damped on the boundary. Prüss [18] gave the optimal results to characterize polynomial as well as exponential decay rates for viscoelastic materials. Apalara et al. [4] studied the exponential stability of laminated beams when the frictional damping acts on the effective rotation angle. For weak damping acting only one equation, the following coupled wave equation

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(s)\mathrm{d}s + \alpha v = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = v = 0 & \text{on} \quad \Gamma \times \mathbb{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) & \text{for} \quad x \in \Omega \end{cases} \tag{1.3}$$

has been considered by Almeida and Santos [3] (see also [9]). In [3, 9], they proved the lack of exponential decay to system (1.3). The authors obtained the optimal polynomial decay by using the recent results due to Borichev and Tomilov [7]. The method used in this contexts introduced by Alabau [1] and developed by Alabau-Cannarsa-Komornik [2]. For memory damping acting on the acoustic boundary, Benomar and Benaissa [6] established polynomial energy decay rates for system (1.2) without delay in one dimensional space.

Our main result is devoted to study the well posedness and exponential decay of the system (1.1), in which we analyze the spectral distribution in the complex plane. The semigroup theory is used to show, in Sect. 3, the global existence of energy-associated solution which its real part decreases with time. Motivated by the mentioned works above concerning Gearhart-Prüss' theorem, the exponential stability of the corresponding semigroup is concluded in Sect. 4.

2. Preliminary

In this section, we give some notations and we present some assumptions needed for our work. Let $\mathrm{H}(\mathrm{div},\Omega)=\{u\in \mathrm{H}^1(\Omega);\mathrm{div}(\mathrm{A}\nabla u)\in \mathrm{L}^2(\Omega)\}$ be the Hilbert space equipped with the norm

$$\|u\|_{\mathrm{H}(\mathrm{div},\Omega)}=\left(\|u\|^2_{\mathrm{H}^1(\Omega)}+\|\mathrm{div}(\mathrm{A}\nabla u)\|_2^2\right)^{1/2},$$

where $\mathrm{H}^1(\Omega)$ is the Sobolev space of first order, $\|.\|_2$ is an L^2-norm and $(.,.)$, $\langle .,.\rangle_{\Gamma_1}$ are the scalar product in $\mathrm{L}^2(\Omega)$, $\mathrm{L}^2(\Gamma_1)$, respectively.

Denoting $\gamma_0:\mathrm{H}^1(\Omega)\to \mathrm{L}^2(\Gamma)$ and $\gamma_1:\mathrm{H}(\mathrm{div},\Omega)\to \mathrm{L}^2(\Gamma)$ defined by $\gamma_0(u)=u_{|\Gamma}$ and $\gamma_1(u)=\left(\frac{\partial u}{\partial \nu_{\mathrm{A}}}\right)_\Gamma$ for all u in $\mathrm{H}(\mathrm{div},\Omega)$. Some times to simplify the notations we write u and $\frac{\partial u}{\partial \nu_{\mathrm{A}}}$ instead $\gamma_0(u)$ and $\gamma_1(u)$, respectively.

We denote by

$$\mathrm{H}^1_{\Gamma_0}(\Omega)=\{u\in \mathrm{H}^1(\Omega)\mid \ \gamma_0(u)=0 \text{ on } \Gamma_0\}$$

the closure subspace of $\mathrm{H}^1(\Omega)$ equipped with the norm equivalent to the usual norm in $\mathrm{H}^1_0(\Omega)$. The Poincaré inequality holds in $\mathrm{H}^1_{\Gamma_0}(\Omega)$.

In this study, we will need the following assumptions:

($\mathbf{A}_1$) The operator $\mathrm{A}=(a_{ij}(x))_{i,j},\ i,j=1,\dots,N$; where the coefficient a_{ij} in $\mathcal{C}^1(\overline{\Omega})$ is symmetric and there exists a constant $a_0>0$ such that

$$\sum_{i,j=1}^N a_{ij}(x)\zeta_i\zeta_j\geq a_0|\zeta|^2,\qquad \forall x\in\overline{\Omega},\ \ \forall\zeta\in\mathbb{C}^N. \tag{2.1}$$

($\mathbf{A}_2$) The kernel function $g:\mathbb{R}_+\to\mathbb{R}_+$ is a bounded $\mathcal{C}^1$ function satisfying

$$\lim_{t\to\infty} g(t)=0,\quad g(0)>0,\quad 1-\int_0^\infty g(s)\mathrm{d}s=\ell>0, \tag{2.2}$$

and there exists a constant $\alpha>0$ such that

$$g'(t)\leq -\alpha g(t),\quad \forall t\geq 0. \tag{2.3}$$

Furthermore, we define

$$a(u(t),v(t))=(Au(t),v(t))=\sum_{i,j=1}^N\int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_i}\overline{\frac{\partial v(t)}{\partial x_j}}\mathrm{d}x.$$

By using the hypothesis ($\mathbf{A}_1$), we verify that the sesquilinear form $a(.,.):\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{H}^1_{\Gamma_0}(\Omega)\to\mathbb{C}$ is continuous, and by (2.1), we deduce that a is coercive.

3. The well posedness

In this section, we will show the well posedness of the system (1.1).

Let us introduce a new variable η as follows

$$\eta(x,s,t)=u(x,t)-u(x,t-s),\quad x\in\Omega,\ t,s\in\mathbb{R}_+.$$

Then, the system (1.1) becomes

$$\begin{cases} u_{tt} - \ell \mathrm{div}(\mathrm{A}\nabla u) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta(s))\mathrm{d}s = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\ \eta_t + \eta_s - u_t = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ hz_{tt} + fz_t + mz + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\ \ell \frac{\partial u}{\partial \nu_{\mathrm{A}}} + \int_0^{+\infty} g(s) \frac{\partial \eta}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s = z_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\ u(0) = u_0, \ u_t(0) = u_1 & \text{in} \quad \Omega \\ \eta(s,0) = u_0 - u(-s) = \eta_0(s) & \text{for} \quad s \in \mathbb{R}_+ \\ z(0) = z_0, \ z_t(0) = z_1 & \text{in} \quad \Gamma_1 \end{cases} \tag{3.1}$$

In order to give a reformulation as first-order evolution system, we denote by

$$U = (u, v, \eta, z, \delta)^T \quad \text{with} \quad v = u_t \quad \text{and} \quad \delta = z_t.$$

We consider the product Hilbert spaces

$$\mathcal{H} = \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega)) \times \mathrm{L}^2(\Gamma_1) \times \mathrm{L}^2(\Gamma_1),$$

endowed with the following inner product

$$\begin{aligned} \left\langle U, \tilde{U} \right\rangle_{\mathcal{H}} &= \ell a(u(t), \tilde{u}(t)) + \int_\Omega v(t)\overline{\tilde{v}(t)}\mathrm{d}x + \langle \eta(t), \tilde{\eta}(t) \rangle_{\mathrm{L}^2_g} \\ &\quad + \langle mz(t), \tilde{z}(t) \rangle_{\Gamma_1} + \left\langle h\delta(t), \tilde{\delta}(t) \right\rangle_{\Gamma_1}, \end{aligned} \tag{3.2}$$

where $\mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))$ denotes the Hilbert space of $\mathrm{H}^1_{\Gamma_0}(\Omega)-$valued functions on $\mathbb{R}_+$, endowed with the inner product

$$\langle \eta(t), \tilde{\eta}(t) \rangle_{\mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))} = \int_0^{+\infty} g(s) a(\eta(s,t), \tilde{\eta}(s,t))\mathrm{d}s, \tag{3.3}$$

for every $U = (u, v, \eta, z, \delta)^T$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{z}, \tilde{\delta})^T$ in $\mathcal{H}$.
Thus, the system (3.1) can be rewritten in the following

$$\begin{cases} U_t(t) = \mathcal{A}U(t), \quad \forall t \geq 0 \\ U(0) = U_0 = (u_0, u_1, \eta_0, z_0, z_1)^T \end{cases} \tag{3.4}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U(t) = \begin{pmatrix} v(t) \\ \ell \mathrm{div}(\mathrm{A}\nabla u(t)) + \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta(t,s))\mathrm{d}s \\ v(t) - \eta_s(t,s) \\ \delta(t) \\ \frac{1}{h(x)}\left(-v(t) - m(x)z(t) - f(x)\delta(t)\right) \end{pmatrix} \tag{3.5}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ U \;\middle|\; \begin{array}{l} \ell u + \int_0^{+\infty} g(s)\eta(s)\mathrm{d}s \in \mathrm{H}(\mathrm{div}, \Omega); \ v \in \mathrm{H}^1_{\Gamma_0}(\Omega); \\ \eta \in \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega)); \ z, \delta \in \mathrm{L}^2(\Gamma_1); \\ \ell \frac{\partial u}{\partial \nu_{\mathrm{A}}} + \int_0^{+\infty} g(s)\frac{\partial \eta}{\partial \nu_{\mathrm{A}}}(s)\mathrm{d}s = \delta \ \text{ on } \Gamma_1 \end{array} \right\} \tag{3.6}$$

Set the energy functional E of the system (3.1)

$$E(t) = \frac{1}{2} \langle U, U \rangle_{\mathcal{H}} . \tag{3.7}$$

Lemma 3.1. *The energy functional (3.7), along the solution of (3.1), is a nonincreasing function satisfying, for all $t \geq 0$*

$$E'(t) = \frac{1}{2} \int_0^{+\infty} g'(s) a(\eta(s), \eta(s)) \mathrm{d}s - \| f^{1/2} \delta(t) \|_{2,\Gamma_1}^2 . \tag{3.8}$$

Proof. Taking the scalar product of $(3.1)_1$ with u_t and $(3.1)_3$ with z_t in $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^2(\Gamma_1)$, respectively, then adding it to the inner product (3.3) of $(3.1)_2$ with η. Using Green's formula and the properties of η. Taking its real part, we arrive at

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|u_t(t)\|_2^2 + \ell a(u(t), u(t)) + \|\eta(t)\|_{\mathrm{L}_g^2} + \|m^{1/2} z(t)\|_{2,\Gamma_1}^2 + \|h^{1/2} \delta(t)\|_{2,\Gamma_1}^2 \right) \\ = & \; - \int_0^{+\infty} g(s) a(\eta_s(t,s), \eta(t,s) ds - \|f^{1/2}\delta(t)\|_{2,\Gamma_1}^2 . \end{aligned} \tag{3.9}$$

Using (2.2) and the properties of η, we have

$$\int_0^{+\infty} g(s) a(\eta(s), \eta_s(s)) ds = -\frac{1}{2} \int_0^{+\infty} g'(s) a(\eta(s), \eta(s)) ds. \tag{3.10}$$

Combining (3.10) and (3.9), we get (3.8). □

Our aim is ensured by the following theorem:

Theorem 3.2. *The operator $\mathcal{A}$ is the infinitesimal generator of $\mathcal{C}_0-$semigroup of contractions over the Hilbert space $\mathcal{H}$. Thus, for any initial data $U_0 \in \mathcal{H}$, the problem (3.4) has a unique weak solution $U \in \mathcal{C}(\mathbb{R}_+; \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then the solution $U \in \mathcal{C}(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$.*

Proof. We will use the Hille-Yosida theorem. For this purpose, $\mathcal{A}$ is dissipative. Indeed, using (3.4) and (3.7), we have, for $U \in D(\mathcal{A})$

$$E'(t) = \Re \langle U_t(t), U(t) \rangle_{\mathcal{H}} = \Re \langle \mathcal{A} U(t), U(t) \rangle_{\mathcal{H}} .$$

Therefore, we deduce from Lemma 3.1 that

$$\Re \langle \mathcal{A} U, U \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^{+\infty} g'(s) a(\eta(t,s), \eta(t,s)) \mathrm{d}s - \|f^{1/2} \delta(t)\|_{\Gamma_1} \leq 0. \tag{3.11}$$

Next, $\mathrm{I} - \mathcal{A}$ is surjective. Indeed, for each $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, we show that there exists $U \in D(\mathcal{A})$ such that

$$(\mathrm{I} - \mathcal{A}) U = F.$$

Then, the previous equation reads

$$\begin{aligned} u - v &= f_1 && (3.12)\\ v - \ell \mathrm{div}(\mathrm{A}\nabla u) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta(s))\mathrm{d}s &= f_2 && (3.13)\\ \eta + \eta_s - v &= f_3 && (3.14)\\ z - \delta &= f_4 && (3.15)\\ h(x)\delta + v + f(x)\delta + m(x)z &= h(x)f_5. && (3.16)\end{aligned}$$

Suppose (u, z) are found in $\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)$. Thus, (3.12) and (3.15) yield

$$\begin{cases} v = u - f_1, \\ \delta = z - f_4, \end{cases} \tag{3.17}$$

Then,

$$v \in \mathrm{H}^1_{\Gamma_0}(\Omega), \quad \text{and} \quad \delta \in \mathrm{L}^2(\Gamma_1).$$

From (3.14), we can determine

$$\eta(s) = -ve^{-s} + v + \int_0^s f_3(\tau)e^{\tau - s}d\tau, \quad \forall s \in \mathbb{R}_+, \tag{3.18}$$

that is $\eta(0) = 0$. According (3.18) with $(3.17)_1$, we have

$$\eta(s) = -ue^{-s} + u + \eta_1(s), \quad \forall s \in \mathbb{R}_+, \tag{3.19}$$

with $\eta_1 \in \mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))$ defined by

$$\eta_1(s) = f_1 e^{-s} - f_1 + \int_0^s f_3(\tau)e^{\tau - s}d\tau.$$

Then, $(3.1)_5$ becomes

$$\ell_g \frac{\partial u}{\partial \nu_\mathrm{A}} + \int_0^{+\infty} g(s)\frac{\partial \eta_1}{\partial \nu_\mathrm{A}}(s)\mathrm{d}s = z - f_4 \ \text{ on } \Gamma_1 \times \mathbb{R}_+,$$

where

$$\ell_g = \left(\ell + \int_0^{+\infty} g(s)(1 - e^{-s})\mathrm{d}s\right) > 0.$$

Inserting (3.17) and (3.19) into (3.13)-(3.16) and adding the results, we get

$$u - \ell_g \mathrm{div}(\mathrm{A}\nabla u) = f_1 + f_2 + \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla \eta_1(s))\mathrm{d}s, \tag{3.20}$$

$$(h(x) + m(x) + f(x))z + u = h(x)f_5 + f_1 + (h(x) + f(x))f_4 \tag{3.21}$$

Taking the inner product of (3.20) with $\tilde{u}$ in $\mathrm{L}^2(\Omega)$, then adding it to the complex conjugate of the inner product of (3.21) with $\tilde{z}$ in $\mathrm{L}^2(\Gamma_1)$ and using Green's formula, we obtain the sesquilinear from $\mathfrak{B} : (\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)) \times (\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)) \to \mathbb{C}$ defined by

$$\begin{aligned} \mathfrak{B}((u, z)), (\tilde{u}, \tilde{z})) &= (u, \tilde{u}) + \ell_g a(u, \tilde{u}) - \langle z, \tilde{u}\rangle_{\Gamma_1} \\ &\quad + \langle u, \tilde{z}\rangle_{\Gamma_1} + \langle (h(x) + m(x) + f(x))z, \tilde{z}\rangle_{\Gamma_1} \end{aligned}$$

for every $(u,z),(\tilde{u},\tilde{z}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)$, and the antilinear from $\mathcal{G} : \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1) \to \mathbb{C}$ defined by

$$\begin{aligned}\mathcal{G}(\tilde{u},\tilde{z}) &= (f_1+f_2,\tilde{u}) - \int_0^{+\infty} g(s)a(\eta_1(s),\tilde{u})\mathrm{d}s \\ &\quad + \langle h(x)f_5 + f_1 + (h(x)+f(x))f_4, \tilde{z}\rangle_{\Gamma_1},\end{aligned}$$

for every $(\tilde{u},\tilde{z}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)$.

It's easy to see that $\mathfrak{B}$ is a continuous sesquilinear form and coercive on $(\mathrm{H}^1_{\Gamma_0}(\Omega)\times \mathrm{L}^2(\Gamma_1))\times(\mathrm{H}^1_{\Gamma_0}(\Omega)\times\mathrm{L}^2(\Gamma_1))$ and $\mathcal{G}$ is a continuous antilinear form on $\mathrm{H}^1_{\Gamma_0}(\Omega)\times\mathrm{L}^2(\Gamma_1)$. Using complex Lax-Milgram's theorem, then there exists a unique solution $(u,z) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)$, satisfying, for all $(\tilde{u},\tilde{z}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Gamma_1)$

$$\mathfrak{B}((u,z),(\tilde{u},\tilde{z})) = \mathcal{G}(\tilde{u},\tilde{z}). \tag{3.22}$$

Additionally, we proceed to get more regularity.
Taking $\tilde{z}=0$ in (3.22). Since $\mathcal{D}(\Omega)$ is dense in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, we deduce that

$$\ell_g\left(\mathrm{div}(\mathrm{A}\nabla u),\tilde{u}\right) = \left(\int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta_1(s))\mathrm{d}s,\tilde{u}\right) + \langle z,\tilde{u}\rangle_{\Gamma_1} - (u,\tilde{u}),$$

for every $\tilde{u} \in \mathrm{H}^1_{\Gamma_0}(\Omega)$. Hence, $u \in (\mathrm{H}(\mathrm{div},\Omega)\cap\mathrm{H}^1_{\Gamma_0}(\Omega))$.

Then $U \in D(\mathcal{A})$. Consequently, Lumper-Phillips' theorem guarantees the generator $\mathcal{A}$ of a $\mathcal{C}_0$−semigroup on $\mathcal{H}$. □

4. Exponential stability

Here we will show the exponential stability of (3.4). The method that we will use in the following theorem is based on Gearhart-Prüss' theorem [14, 17] to complex value dissipative systems.

Theorem 4.1. *Let $T(t) := e^{\mathcal{A}t}$ be a $\mathcal{C}_0$−semigroup of contractions on Hilbert space $\mathcal{H}$. Then $T(t)$ is exponentially stable if and only if*

(i) *The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains the imaginary axis ($i\mathbb{R} \subset \rho(\mathcal{A})$),*
(ii) $\limsup_{|\lambda|\to\infty} \|(i\lambda\mathrm{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Our starting point is to show that the semigroup associated to (3.4), generated by $\mathcal{A}$, is exponentially stable. The following Theorem gives our main result, that is to verify the conditions (i) and (ii) of Theorem 4.1.

Theorem 4.2. *Assume that $(\mathbf{A}_2)$ holds. Then, $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is exponentially stable, that is to say, there exist two constants $M \geq 1$ and $\epsilon > 0$ such that*

$$\|e^{\mathcal{A}t}\| \leq Me^{-\epsilon t}.$$

Proof. We first show that the resolvent of the system (3.4) is located on the imaginary axes. Note that the resolvent equation $(\mathrm{i}\lambda \mathrm{I}-\mathcal{A})U=F\in\mathcal{H}$ is given by

$$\begin{aligned}
\mathrm{i}\lambda u - v &= f_1 && (4.1)\\
\mathrm{i}\lambda v - \ell \mathrm{div}(\mathrm{A}\nabla u) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta(s))\mathrm{d}s &= f_2 && (4.2)\\
\mathrm{i}\lambda\eta + \eta_s - v &= f_3 && (4.3)\\
\mathrm{i}\lambda z - \delta &= f_4 && (4.4)\\
\mathrm{i}\lambda h(x)\delta + f(x)\delta + m(x)z + v &= h(x)f_5. && (4.5)
\end{aligned}$$

It's means to show that $\mathrm{i}\mathbb{R}\cap\sigma(\mathcal{A})=\emptyset$, where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$.
Using contradiction arguments. Let us suppose that $\mathcal{A}$ has an imaginary eigenvalue. Then, we have

$$\mathcal{A}U = i\lambda U, \qquad \lambda\in\mathbb{R}. \tag{4.6}$$

Thus, $F\equiv 0$ in (4.1)-(4.5). From (3.11) and (4.6), we can get

$$0=\Re\langle\mathcal{A}U,U\rangle_{\mathcal{H}} \le \frac{1}{2}\int_0^{+\infty} g'(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s - \|f^{1/2}\delta(t)\|_{\Gamma_1} \le 0.$$

It follows that $\delta=0$, and from the hypothesis of g that $\nabla\eta=0$. Using the fact $u=0$ in $\Gamma\times\mathbb{R}_+$ that $\eta=0$. This implies by (4.1) and (3.18) that $u=v=0$. From equation (4.5), we conclude that $z=0$. Hence, $U\equiv 0$. We obtain a contradiction.

We now prove (ii) by a contradiction argument again. Suppose that (ii) is not true. Then there exist a sequence λ_n with $|\lambda_n|\to+\infty$ and a sequence of functions

$$U_n=(u_n,v_n,\eta_n,z_n,\delta_n)^T\in D(\mathcal{A}) \qquad \text{with} \qquad \|U_n\|_{\mathcal{H}}=1, \tag{4.7}$$

such that, as $n\to+\infty$;

$$(\mathrm{i}\lambda_n\mathrm{I}-\mathcal{A})U_n\to 0 \quad \text{in} \quad \mathcal{H} \tag{4.8}$$

i.e,

$$\begin{aligned}
\mathrm{i}\lambda_n u_n - v_n &\to 0 \quad \text{in} \quad \mathrm{H}^1_{\Gamma_0}(\Omega) && (4.9)\\
\mathrm{i}\lambda_n v_n - \ell\mathrm{div}(\mathrm{A}\nabla u_n) - \int_0^{+\infty} g(s)\mathrm{div}(\mathrm{A}\nabla\eta_n(s))\mathrm{d}s &\to 0 \quad \text{in} \quad \mathrm{L}^2(\Omega) && (4.10)\\
\mathrm{i}\lambda_n\eta_n + \partial_s\eta_n - v_n &\to 0 \quad \text{in} \quad \mathrm{L}^2_g && (4.11)\\
\mathrm{i}\lambda_n z_n - \delta_n &\to 0 \quad \text{in} \quad \mathrm{L}^2(\Gamma_1) && (4.12)\\
\mathrm{i}\lambda_n h(x)\delta_n + f(x)\delta_n + m(x)z_n + v_n &\to 0 \quad \text{in} \quad \mathrm{L}^2(\Gamma_1) && (4.13)
\end{aligned}$$

Taking the inner product (3.2) of (4.8) with U_n and then taking its real part yields

$$-\Re\langle(\mathrm{i}\lambda_n\mathrm{I}-\mathcal{A})U_n,U_n\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^{+\infty} g'(s)a(\eta_n(s),\eta_n(s))\mathrm{d}s + \|f^{1/2}\delta_n\|^2_{2,\Gamma_1}\to 0. \tag{4.14}$$

Using (2.3), we find that

$$\begin{aligned}
\eta_n &\to 0 \qquad \text{in } \mathrm{L}^2_g(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega)), && (4.15)\\
\delta_n &\to 0 \qquad \text{in } \mathrm{L}^2(\Gamma_1). && (4.16)
\end{aligned}$$

On the other hand, taking the complex conjugate of the inner product of (4.9) with ℓu_n in $\mathrm{H}_0^1(\Omega)$, then adding it to the inner product of (4.10) with v_n in $\mathrm{L}^2(\Omega)$ and using Green's formula, we get

$$\mathrm{i}\left(-\ell a(u_n,u_n)+\|v_n\|_2^2\right)-\frac{1}{\lambda_n}\langle\delta_n,v_n\rangle_{\Gamma_1}+\frac{1}{\lambda_n}\int_0^{+\infty}g(s)a(\eta_n(s),v_n)\mathrm{d}s\to 0. \quad (4.17)$$

We can deduce from (4.9) that $\frac{1}{\lambda_n}\|\nabla v_n\|_2^2$ is uniformly bounded. By using (4.15) and (4.16), the last two terms in (4.17) converge to zero. Hence,

$$\ell a(u_n,u_n)-\|v_n\|_2^2\to 0. \quad (4.18)$$

Adding the complex conjugate of the inner product of (4.12) with $m(x)\delta_n$ to the inner product of (4.13) with z_n in $\mathrm{L}^2(\Gamma_1)$, we have

$$\mathrm{i}(\|m^{1/2}z_n\|_{2,\Gamma_1}^2+\|h^{1/2}\delta_n\|_{2,\Gamma_1}^2)+\frac{1}{\lambda_n}\|f^{1/2}\delta_n\|_{2,\Gamma_1}^2+\frac{1}{\lambda_n}\langle v_n,\delta_n\rangle_{\Gamma_1}\to 0.$$

By using (4.16) and the fact that $\frac{1}{\lambda_n}\|\nabla v_n\|_2^2$ and $\|f^{1/2}\delta_n\|_{2,\Gamma_1}^2$ are uniformly bounded, we obtain

$$\|m^{1/2}z_n\|_{2,\Gamma_1}^2\to 0. \quad (4.19)$$

Combining (4.7) with (4.15), (4.16) and (4.19). Then, using (4.18), we find that

$$a(u_n,u_n)\to\frac{1}{2}, \quad (4.20)$$

and

$$\|v_n\|_2^2\to\frac{1}{2}. \quad (4.21)$$

It's easy to see that $\frac{1}{\lambda_n}v_n\in\mathrm{L}_g^2(\mathbb{R}_+;\mathrm{H}_{\Gamma_0}^1(\Omega))$. Then, taking the inner product (3.3) of (4.11) with $\frac{1}{\lambda_n}v_n$, we have

$$\frac{1}{\lambda_n}\langle\eta_n(s),v_n\rangle_{\mathrm{L}_g^2}+\frac{1}{\lambda_n^2}\langle\partial_s\eta_n(s),v_n\rangle_{\mathrm{L}_g^2}-\frac{1}{\lambda_n^2}\langle v_n,v_n\rangle_{\mathrm{L}_g^2}\to 0. \quad (4.22)$$

Using again the fact that $\frac{v_n}{\lambda_n}$ is bounded in $\mathrm{H}_{\Gamma_0}^1(\Omega)$ and by using (4.15), we get that the first term of (4.22) converges to zero. This yields

$$\frac{(1-\ell)}{\lambda_n^2}a(v_n,v_n)-\underbrace{\frac{1}{\lambda_n^2}\int_0^{+\infty}g(s)a(\partial_s\eta_n(s),v_n)\mathrm{d}s}_{I_1}\to 0. \quad (4.23)$$

The second term (I_1) in (4.23) converges to zero. Indeed, from (2.3) and by using again that $\frac{v_n}{\lambda_n}$ is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, we have

$$\begin{aligned} |I_1| &= \frac{1}{|\lambda_n|}\left|\int_0^{+\infty} g'(s)a(\eta_n(s),\frac{v_n}{\lambda_n})\mathrm{d}s\right| \\ &\leq \frac{\alpha a_1}{|\lambda_n|}\left\|\frac{\nabla v_n}{\lambda_n}\right\|_2\left(\frac{(1-\ell)}{a_0}\int_0^{+\infty} g(s)a(\eta_n(s),\eta_n(s))\mathrm{d}s\right)^{1/2} \to 0, \end{aligned}$$

where $a_1 = \max\limits_{j=\overline{1,n}}\left(\sum\limits_{i=1}^{n}\|a_{ij}\|_\infty^2\right)$. This with (4.23), leads to

$$\frac{v_n}{\lambda_n} \to 0 \quad \text{in } \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{4.24}$$

Taking the inner product of (4.9) with ℓu_n in $\mathrm{H}^1_0(\Omega)$. Since u_n is bounded in $\mathrm{H}^1_{\Gamma_0}(\Omega)$. By using (4.24), we obtain

$$a(u_n,u_n) \to 0.$$

This contradicts (4.20). Therefore, the proof is completed. □

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A strong convergence algorithm for approximating a common solution of variational inequality and fixed point problems in real Hilbert space

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Abstract. In this paper, we propose an iterative algorithm for approximating a common solution of a variational inequality and fixed point problem. The algorithm combines the subgradient extragradient technique, inertial method and a modified viscosity approach. Using this algorithm, we state and prove a strong convergence algorithm for obtaining a common solution of a pseudomonotone variational inequality problem and fixed point of an η-demimetric mapping in a real Hilbert space. We give an application of this result to some theoretical optimization problems. Furthermore, we report some numerical examples to show the efficiency of our method by comparing with previous methods in the literature. Our result extend, improve and unify many other results in this direction in the literature.

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1. Introduction

In this paper, we consider the Variational Inequality Problem (VIP) which consists of finding a point $x^* \in K$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \ \forall \ x \in K, \tag{1.1}$$

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where K is a nonempty, closed and convex subset of a real Hilbert space H, $F : H \to H$ is a nonlinear single-valued mapping, $\langle \cdot, \cdot \rangle$ respectively $\| \cdot \|$ are inner products and norm defined on H. We denote by $VIP(K, F)$, the set of solutions of the VIP (1.1). A wide range of problems in science and engineering, optimization theory, equilibrium theory and differentiation equation leads to the study of the variational inequality problems. For this reason, there have been several researches into the study of iterative algorithms for approximating the solutions of VIP and related optimization problems, (see [1, 2, 4, 6, 30, 45, 48, 51, 50, 52]).

One of the simplest and earliest known method for solving VIP is the gradient projection method as a result of a fixed point formulation which involves the metric projection. The method is given as

$$x_{k+1} = P_K(x_k - \lambda F x_k),\ x_1 \in K,\ k \geq 1,$$

where P_K is the metric projection of H onto K and $\lambda \in (0, 1/L)$ with L the Lipschitz constant of the cost operator F. For the convergence of this method, it is required that the operator F is strongly monotone (see [22, 23, 21, 31]).

Another method for solving the VIP is the so-called Extragradient Method (EGM) initially proposed by Korpelevich for solving the saddle points problem (see also, Antipin [7]). For solving the VIP, the EGM is given as follows: $x_1 \in K$

$$\begin{cases} y_k = P_K(x_k - \lambda F x_k), \\ x_{k+1} = P_K(x_k - \lambda F y_k).\ k \geq 1 \end{cases} \tag{1.2}$$

The EGM (1.2) requires executing projection onto feasible set K twice per iteration. Considerable efforts have been made to modify and improve this method, one of which is to reduce the projection from two to one onto feasible sets. In particular, one of such modifications is the Subgradient Extragradient Method (SEGM) by Censor et. al (see [14, 15]). In this method, the second projection of the extragradient method was replaced by a projection onto a half-space whose formula can be easily executed. The SEGM is given as follows: $x_1 \in K$:

$$\begin{cases} y_k = P_K(x_k - \lambda F x_k), \\ T_k = \{x \in H : \langle x_k - \lambda F x_k - y_k, x - y_k \rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(x_k - \lambda F y_k),\ k \geq 1 \end{cases} \tag{1.3}$$

Another drawback of the EGM is the dependence of the constant λ on the Lipschitz constant of the associated cost operator. For this reason, many authors have proposed several methods which avoid the prior knowledge or use of the Lipschitz constant. One of such is the use of well defined linesearch rule (see [11]) and the references therein. One other popular method for avoiding the use of Lipschitz constant is to construct an adaptable step size (see, [51, 52]) for more.

On the other hand, the Fixed Point Problem (FPP) consists of finding a point $x^* \in K$ such that

$$x^* = Sx^*, \tag{1.4}$$

where K is a nonempty, closed and convex subset of a real Hilbert space H and $S : K \to K$ is a nonlinear mapping. We denote by $Fix(S)$, the fixed point set

of a mapping S. The FPP finds application in proving the existence of solution of many nonlinear problems arising in many real life problems. From the existence of solution of differential equation to integral equations and evolutionary equations. The approximation of fixed points of several nonlinear operators in Hilbert, Banach and Hadamard spaces have been considered in the literature (see [18, 20, 26, 36, 53]).
In this paper, we consider the problem of finding a common solution of the VIP (1.1) and FPP (1.4). That is, finding a point $x^* \in K$ such that

$$x^* \in VIP(K, F) \cap Fix(S). \tag{1.5}$$

The problem (1.5) has many real life applications which include signal recovery problems, beam-forming problems, power-control problems, bandwith allocation problems and optimal control problems (see [25, 43] and the references therein).
For obtaining a solution of (1.5) in the case where $F : H \to H$ is inverse strongly monotone and $S : K \to K$ is nonexpansive, Takahashi and Toyoda [49] introduced an algorithm whose sequence $\{x_k\}$ is generated by the following recursive formula:

$$\begin{cases} y_k = P_K(x_k - \lambda F x_k), \\ x_{k+1} = (1 - \alpha_k)x_k + \alpha_k S y_k, \end{cases} \tag{1.6}$$

where P_K is the metric projection of H onto K and $\{\alpha_k\}$ is a sequence in $(0, 1)$ satisfying some conditions.
Kraikaew and Saejung [34], for solving problem (1.5) combined the SEGM and Halpern method to propose an algorithm they called the Halpern Subgradient Extragradient Method (HSEGM). The HSEGM is given as

$$\begin{cases} x_1 \in H, \\ y_k = P_K(x_k - \lambda F x_k), \\ T_k = \{x \in H : \langle x_k - \lambda F x_k - y_k, x - y_k\rangle \le 0\}, \\ z_k = \alpha_k x_1 + (1 - \alpha_k)P_{T_k}(x_k - \lambda F y_k), \\ x_{k+1} = \beta_k x_k + (1 - \beta_k)S z_k, \end{cases} \tag{1.7}$$

where $\lambda \in (0, 1/L)$, $\alpha_k \subset (0, 1)$ satisfying $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\{\beta_k\} \subset [a, b] \subset (0, 1)$ and $S : H \to H$ is a quasi-nonexpansive mapping.
Recently, Thong and Hieu [50] introduced two viscosity-extragradient algorithms for approximating (1.5), where $S : H \to H$ is a η-demicontractive mapping and $F : H \to H$ is a L-Lipschitz monotone operator. The strong convergence of both algorithms were established under some mild conditions. One of these algorithms is presented as follows:

Algorithm 1.1. [50, Algorithm 3.1], *Viscosity-type Subgradient Extragradient Method (VSEM)*

Initialization: *Choose* $\lambda_0 > 0$, $\mu \in (0, 1)$, *and let* $x_0 \in K$ *be an arbitrary starting point.*

Iterative steps: *Calculate* x_{k+1} *as follows:*

Step 1: *Compute*

$$y_k = P_K(x_k - \lambda_k F x_k).$$

Step 2: *Compute*

$$z_k = P_{T_k}(x_k - \lambda_k F y_k),$$

where

$$T_k = \{w \in H : \langle x_k - \lambda_k F x_k - y_k, w - y_k \rangle \leq 0\}.$$

Step 3: *Compute*

$$\begin{cases} v_k = (1-\beta_k)z_k + \beta_k S z_k, \\ x_{n+1} = \alpha_k f(x_k) + (1-\alpha_k)v_k \end{cases}$$

and

$$\lambda_{k+1} = \begin{cases} \min\{\frac{\mu\|w_k - y_k\|}{\|Fw_k - Fy_k\|}, \ \lambda_k\} & \textit{if } Fw_k - Fy_k \neq 0, \\ \lambda_k, & \textit{otherwise.} \end{cases}$$

Stopping criterion *Set* $k := k+1$ *and return to* **Step 1**.

To speed up the convergence of iterative algorithm, the inertial technique has been widely employed (see [3, 8, 16, 38, 39, 47]). Inertial algorithms for variational inequality and other optimization problems have received due consideration by authors, see, e,g [51]. Very recently, Thong et al. [51] proposed the following inertial subgradient method:

Algorithm 1.2. *Inertial subgradient algorithm for VIP*

Initialization: *Choose* $\lambda_1 > 0$, $\mu \in (0,1)$, $\theta > 0$ *and let* $x_0, x_1 \in K$ *be an arbitrary starting point.*

Iterative steps: *Calculate* x_{k+1} *as follows:*

Step 1: *Given* x_k, x_{k-1}, $k \geq 1$. *Set*

$$w_k = x_k + \theta_k(x_k - x_{k-1}),$$

where

$$\theta_k = \begin{cases} \min\left\{\dfrac{1}{k^2\|x_k - x_{k-1}\|^2}, \theta\right\} & \textit{if } x_k \neq x_{k-1}, \\ \theta & \textit{otherwise.} \end{cases}$$

Step 2: *Calculate*

$$y_k = P_K(w_k - \lambda_k F w_k).$$

If $y_k = w_k$ *or* $Fy_k = 0$ *then stop (*y_k *is the solution of the VIP* (1.1) *). Otherwise go to* **Step 3**.

Step 3: *Compute*

$$z_k = P_{T_k}(w_k - \lambda_k F y_k),$$

where

$$T_k = \{w \in H : \langle w_k - \lambda_k F w_k - y_k, w - y_k \rangle \leq 0\}.$$

Step 4 *Compute*

$$x_{k+1} = \alpha_k f(z_k) + (1-\alpha_k) z_k.$$

Update

$$\lambda_{k+1} = \begin{cases} \min\left\{ \frac{\mu\|w_k - y_k\|}{\|Fw_k - Fy_k\|},\ \lambda_k \right\} & \text{if } Fw_k - Fy_k \neq 0, \\ \lambda_k, & \text{otherwise.} \end{cases}$$

Set $k := k+1$ and return to **Step1**.

In this paper, motivated by the works of Attouch and Alvarez [8], Censor et al. [14] and [51], we proposed an inertial self adaptive subgradient extragradient algorithm for approximating a solution of VIP and FPP in real Hilbert space. Combining this method with a modified viscosity approach, we proved a strong convergence theorem for approximating the solution of a pseudomonotone VIP and FPP for η-demimetric mapping. The following highlight some of the advantages of our method and work over previous ones in the literature.

(i) Unlike the work of Gang et al. [11] where the linesearch rule (a linesearch means that at each outer iteration, an inner loop is executed until some finite stopping criterion is reached which can be time consuming) was employed, we used a carefully chosen self adaptive step size.
(ii) Also, by using self adaptive step size, our work does not depend on the prior knowledge of the Lipschitz constant in practice which makes the execution of the algorithm easy for computation.
(iii) Our algorithm is used for approximating a common solution of a VIP for pseudomonotone operator and a fixed point of an η-demimetric mapping thus including the work of [51] as a special consideration.
(iv) We employed an inertial technique to speed up the convergence rate of the sequence generated by our method. Our numerical experiments confirm that our method perform better than some existing methods in literature.

The paper is organized as follows: In Section 2, we present some preliminary results and definitions that are useful in establishing our main result. We present the main result in Section 3, by first introducing our algorithm and then establishing the strong convergence of the sequence generated by this algorithm. In Section 4, we give two theoretical applications of our main result. We reported some numerical experiments in Section 5 to demonstrate the performance of our method as well as comparing it with some related methods in the literature. Finally, in Section 6, we gave a conclusion of the paper.

2. Preliminaries

Throughout this paper, we denote the set of positive integers and the set of real numbers by $\mathbb{N}$ and $\mathbb{R}$ respectively. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm given by $\|\cdot\|$ respectively. For a sequence $\{x_k\} \subset H$, we denote the weak and strong convergence of $\{x_k\}$ to a point $x \in H$ by $x_k \rightharpoonup x$ and $x_k \to x$ respectively.

Let K be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $S : K \to K$ is said to be:

(i) L-Lipschitz with a constant $L > 0$, if

$$\|Sx - Sy\| \leq L\|x - y\|, \ \forall \ x, y \in H;$$

(ii) a contraction respectively nonexpansive if $L \in (0, 1)$ respectively $L = 1$;

(iii) firmly nonexpansive, if

$$\langle Sx - Sy, x - y\rangle \geq \|Sx - Sy\|^2, \ \forall \ x, y \in H;$$

(iv) quasi-nonexpansive, if $Fix(S) \neq \emptyset$ and

$$\|Sx - Sx^*\| \leq \|x - x^*\|,$$

for any $x \in H$ and $x^* \in Fix(S)$;

(v) k-strictly pseudocontractive in the sense of Browder and Petryshyn [9], if there exists $k \in [0, 1)$, such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|x - y - (Sx - Sy)\|^2, \ \forall \ x, y \in H;$$

(vi) [41]. η-demimetric with $\eta \in (-\infty, 1)$, if $Fix(S) \neq \emptyset$ and

$$\langle x - x^*, x - Sx\rangle \geq \frac{1}{2}(1 - \eta)\|x - Sx\|^2, \text{for any } x \in K \text{ and } x^* \in Fix(S).$$

Equivalently, S is η-demimetric, if there exists $\eta \in (-\infty, 1)$ such that

$$||Sx - x^*||^2 \leq ||x - x^*||^2 + \eta||x - Tx||^2, \quad \forall x \in H \quad \text{and} \quad x^* \in Fix(S).$$

Remark 2.1. [41]. The class η-demimetric mappings covers the class of strictly pseudocontractive mappings with nonempty fixed points and many other important nonlinear mappings.

For each $x, y \in H$ and $t \in (0, 1)$, it is known that

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$$

and

$$||tx + (1 - t)y||^2 = t||x||^2 + (1 - t)||y||^2 - t(1 - t)||x - y||^2, \quad \text{(see, [28, 37])}.$$

Let K be a nonempty, closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point $P_K x \in K$, such that

$$||x - P_K x|| \leq ||x - y||, \quad \forall \ y \in K.$$

P_K is called the metric projection (also nearest point mapping) of H onto K, see [17, 29].

Lemma 2.2. [40]. *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Given $x \in H$ and $z \in K$. Then*

$$z = P_K x \iff \langle x - z, z - y\rangle \geq 0, \ \forall \ y \in K.$$

Lemma 2.3. [32, 40]. *Let K be be a nonempty, closed and convex subset of a real Hilbert space H. Given $x \in H$, then*

(a) $\|P_K x - P_K y\| \leq \langle P_K x - P_K y, x - y\rangle, \ \forall \ y \in K;$

(b) $\|x - y\| - \|x - P_K x\| \geq \|P_K x - y\|$;
(c) $\|(I - P_K)x - (I - P_K)y\|^2 \leq \langle (I - P_K)x - (I - P_K)y, x - y\rangle, \ \forall \ y \in K.$

Lemma 2.4. [32, Lemma 2.1]. *Consider* $VIP(K, F)$ (1.1) *with* K *being a nonempty, closed and convex subset of a real Hilbert space* H *and* $F : K \to H$ *being a pseudomonotone and continuous operator. Then* $x^* \in VIP(K, F)$ *if and only if*

$$\langle Fx, x - x^*\rangle \geq 0, \ \forall \ x \in K.$$

Lemma 2.5. [9]. *Let* H *be a real Hilbert space and* $S : H \to H$ *be a* η*-demimetric mapping with* $(-\infty, 1)$ *such that* $F(S) \neq \emptyset$. $S_\eta x := (1 - \eta)x + \eta S x$. *Then,* S_η *is a quasi-nonexpansive mapping and* $F(S_\eta) = F(S)$.

Lemma 2.6. [49]. *Let* $\{\alpha_k\}$ *be a sequence of nonnegative real numbers satisfying*

$$\alpha_{k+1} \leq (1 - \gamma_k)\alpha_k + \delta_k,$$

where $\{\gamma_k\}$ *is a sequence in (0,1) and* δ_k *is a sequence such that*

(i) $\sum\limits_{k=1}^{\infty} \gamma_k = \infty$ *and* $\lim\limits_{k\to\infty} \gamma_k = 0$;
(ii) $\sum\limits_{k=1}^{\infty} |\delta_k| < \infty$ *and* $\lim\limits_{k\to\infty} \dfrac{\delta_k}{\gamma_k} \leq 0$.

Then $\alpha_k \to 0$ *as* $k \to \infty$.

Lemma 2.7. [42, 46] *Let* $\{\Upsilon_k\}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Upsilon_{k_j}\}$ *of* $\{\Upsilon_k\}$ *such that* $\Upsilon_{k_j} < \Upsilon_{k_j+1}$ *for all* $j \geq 0$. *Also consider the sequence of integers* $\{\tau(k)\}_{k\geq k_0}$ *defined by*

$$\tau(k) = \max\{n \leq k : \Upsilon_k < \Upsilon_{k+1}\}.$$

Then, $\{\tau(k)\}_{k\geq k_0}$ *is a nondecreasing sequence verifying* $\lim\limits_{k\to\infty} \tau(k) = \infty$ *and, for all* $k \geq k_0$,

$$\max\{\Upsilon_{\tau(k)}, \Upsilon_k\} \leq \Upsilon_{\tau(k)+1}.$$

3. Main result

In this section, we present our main result of this paper.
For the convergence of our method, we assume the following conditions:

Assumption 3.1.

(C1) *The feasible set* K *is nonempty, closed and convex on* H.
(C2) *The mapping* $F : H \to H$ *is pseudomonotone, L-Lipschitz continuous on* H *and sequentially weakly continuous on* K.
(C3) *The solution set* $\Gamma = VIP(K, F) \cap Fix(S)$ *is nonempty, where* $S : H \to H$ *is an* η*-demimetric mapping.*

In addition to this, we assume that $\{\tau_k\}$ as used in Algorithm 3.2 is a positive sequence such that $\lim\limits_{k\to\infty} \dfrac{\tau_k}{\alpha_k} = 0$ (that is $\tau_k = \circ(\alpha_k)$), where $\{\alpha_k\} \subset (0, 1)$ such that

(C4) $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$,
(C5) $\alpha_k + \beta_k + \gamma_k = 1$.

Algorithm 3.2. *Iterative Algorithm*

Initialization: *Let $f : K \to K$ be a κ-contractive mapping. Choose $\lambda_1 > 0$, $\eta_k \subset (0,1)$, $\mu \in (0,1)$, $\theta > 0$ and let $x_0, x_1 \in K$ be an arbitrary starting point.*

Iterative steps: *Given x_k, x_{k-1}, choose θ_k such that $0 \leq \theta_k \leq \bar{\theta}_k$, where*

$$\bar{\theta}_k = \begin{cases} \min\left\{\theta, \dfrac{\tau_k}{\|x_k - x_{k-1}\|}\right\} & \text{if } x_k \neq x_{k-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Calculate x_{k+1} and λ_k for each $k \geq 1$ as follows:

Step 1: *Compute*

$$w_k = x_k + \theta_k(x_k - x_{k-1}). \tag{3.2}$$

Step : *Calculate*

$$y_k = P_K(w_k - \lambda_k F w_k). \tag{3.3}$$

Step 2: *Compute*

$$z_k = P_{T_k}(w_k - \lambda_k F y_k), \tag{3.4}$$

where

$$T_k = \{w \in H : \langle w_k - \lambda_k F w_k - y_k, w - y_k\rangle \leq 0\}.$$

Step 3: *We obtain x_{k+1} by*

$$x_{k+1} = \alpha_k f(x_k) + \beta_k x_k + \gamma_k S_{\eta_k} z_k \tag{3.5}$$

and

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\mu\|w_k - y_k\|}{\|Fw_k - Fy_k\|}, \ \lambda_k\right\} & \text{if } Fw_k - Fy_k \neq 0, \\ \lambda_k, & \text{otherwise.} \end{cases}$$

Stopping criterion *If $x_{k+1} = w_k = y_k = Sz_k$ for some $k \geq 1$ then stop. Otherwise set $k := k+1$ and return to* **Iterative step**.

The following result was stated and proved in [52]. It is easy to adapt for our situation. We state the lemma without proof.

Lemma 3.3. [52]. *The sequence $\{\lambda_k\}$ defined in Algorithm 3.2 is a nonincreasing sequence and*

$$\lim_{k\to\infty} \lambda_k = \lambda \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}.$$

The following is required for establishing the solution of the VIP (1.1).

Lemma 3.4. *Assume that Assumption 3.1 hold and $\{w_k\}$ is a sequence generated by Algorithm 3.2. If there exists a subsequence $\{w_{k_j}\}$ of $\{w_k\}$ convergent weakly to a point $\bar{x} \in H$ and $\lim_{j\to\infty} \|w_{k_j} - y_{k_j}\| = 0$, then $\bar{x} \in VIP(K, F)$.*

Proof. First we show that $\liminf_{j\to\infty}\langle Fy_{k_j}, z-y_{k_j}\rangle \geq 0$. Indeed, we have by the definition of $\{y_k\}$ and Lemma 2.2, that

$$\langle w_{k_j} - \lambda_{k_j} Fw_{k_j} - y_{k_j}, z - y_{k_j}\rangle \leq 0,\ \forall\ z \in K,$$

which implies

$$\frac{1}{\lambda_{k_j}}\langle w_{k_j} - y_{k_j}, z - y_{k_j}\rangle \leq \langle Fw_{k_j}, z - y_{k_j}\rangle\ \forall\ z \in K.$$

Consequence of this, we get that

$$\frac{1}{\lambda_{k_j}}\langle w_{k_j} - y_{k_j}\rangle + \langle Fw_{k_j}, y_{k_j} - w_{k_j}\rangle \leq \langle Fw_{k_j}, z - w_{k_j}\rangle,\ \forall\ z \in K. \tag{3.6}$$

Since $\{w_{k_j}\}$ is convergent, it is bounded. Then, since F is Lipschitz continuous, $\{Fw_{k_j}\}$ is bounded. We obtain also that $\{y_{k_j}\}$ is bounded since $\|w_{k_j} - y_{k_j}\| \to 0$ as $j \to \infty$ and $\lambda_{k_j} \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}$. Passing limit over (3.6) as $j \to \infty$, we obtain

$$\liminf_{j\to\infty}\langle Fw_{k_j}, z - w_{k_j}\rangle \geq 0.$$

Observe that

$$\langle Fw_{k_j}, z - y_{k_j}\rangle = \langle Fy_{k_j} - Fw_{k_j}, z - y_{k_j}\rangle + \langle Fy_{k_j}, z - w_{k_j}\rangle + \langle Fy_{k_j}, w_{k_j} - y_{k_j}\rangle. \tag{3.7}$$

We obtain from $\lim_{j\to\infty}\|w_{k_j} - y_{k_j}\| = 0$ and the Lipschitz continuity of F, that $\lim_{j\to\infty}\|Fw_{k_j} - Fy_{k_j}\| = 0$. Thus, we get from (3.7), that

$$\liminf_{j\to\infty}\langle Fy_{k_j}, z - y_{k_j}\rangle \geq 0.$$

Next we show that $\bar{x} \in VIP(K,F)$. We choose a subsequence $\{\epsilon_j\}$ of positive numbers decreasing such that $\epsilon_j \to 0$ as $j \to \infty$. For each j, let N_j be the smallest nonnegative integer such that

$$\langle Fy_{k_i}, z - y_{k_i}\rangle + \epsilon_j \geq 0,\ \forall\ i \geq N_j. \tag{3.8}$$

Since $\{\epsilon_j\}$ is decreasing, it is obvious that N_j is increasing. Further, for each $j \in \mathbb{N}$, $\{y_{N_j}\} \subset K$. Suppose $Fy_{N_j} \neq 0$ so that y_{N_j} is not a solution of the $VIP(K,F)$, set

$$\nu_{N_j} = \frac{Fy_{N_j}}{\|Fy_{N_j}\|^2},$$

so that $\langle Fy_{N_j}, \nu_{N_j}\rangle = 1$ for each j. We see from this and (3.8), that

$$\langle Fy_{N_j}, z + \epsilon_j\nu_{N_j} - y_{N_j}\rangle \geq 0.$$

Since F is pseudomonotone on H, we have

$$F(z + \epsilon_j\nu_{N_j}), z + \epsilon_j\nu_{N_j} - y_{N_j}\rangle \geq 0$$

and thus

$$\langle Fz, z - y_{N_j}\rangle \geq \langle Fz - F(z + \epsilon_j\nu_{N_j}), z + \epsilon_j\nu_{N_j} - y_{N_j}\rangle - \epsilon_j\langle Fz, \nu_{N_j}\rangle. \tag{3.9}$$

Now, we show that $\epsilon_j \nu_{N_j} \to 0$ as $j \to \infty$. To see this, from the hypothesis we get that $y_{N_j} \rightharpoonup \bar{x}$ as $j \to \infty$. By $\{y_k\} \subset K$, we have that $\bar{x} \in K$. Since F is sequentially weakly continuous on K, we have $Fy_{N_j} \rightharpoonup F\bar{x}$. Suppose that $F\bar{x} \neq 0$ so that $\bar{x} \in VIP(K, F)$. Since $\|\cdot\|$ is sequentially weakly continuous, we have

$$0 < \|F\bar{x}\| \leq \liminf_{j\to\infty} \|Fy_{N_j}\|.$$

From $\{y_{N_j}\} \subset \{y_{k_j}\}$ and $\epsilon_j \to 0$ as $j \to \infty$, we have

$$0 \leq \lim_{j\to\infty} \|\epsilon_j \nu_{N_j}\| = \lim_{j\to\infty} \left(\frac{\epsilon_j}{\|Fy_{k_j}\|} \right) \leq \frac{0}{\|F\bar{x}\|} = 0,$$

which shows that $\epsilon_j \nu_{N_j} \to 0$. Now letting $j \to \infty$, we obtain by the continuity of F that the right hand side of (3.9) tends to zero, $\{w_{N_j}\}$, $\{\nu_{N_j}\}$ are bounded and $\lim\limits_{j\to\infty} \epsilon_j \nu_{N_j} = 0$. Therefore,

$$\liminf_{j\to\infty} \langle Fz, z - y_{N_j} \rangle \geq 0.$$

Hence for all $z \in K$, we have

$$\langle Fz, z - \bar{x} \rangle = \lim_{j\to\infty} \langle Fz, z - y_{N_j} \rangle = \liminf_{j\to\infty} \langle Fz, z - y_{N_j} \rangle \geq 0.$$

By Lemma 2.4 we have $\bar{x} \in VIP(K, F)$. The proof is thus complete.

Lemma 3.5. *Let $\{z_k\}$ be given as in Algorithm 3.2 and $x^* \in \Gamma$, then there holds the inequality*

$$\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) [\|w_k - y_k\|^2 + \|y_k - z_k\|^2]. \tag{3.10}$$

Proof. Using Lemma 2.3 and (3.4), we have

$$\begin{aligned}
\|z_k - x^*\|^2 &= \|P_{T_k}(w_k - \lambda_k F y_k) - x^*\|^2 \\
&\leq \|w_k - \lambda_k F y_k - x^*\|^2 - \|w_k - \lambda_k F y_k - z_k\|^2 \\
&= \|w_k - x^*\|^2 - 2\lambda_k \langle w_k - x^*, F y_k \rangle - \|w_k - z_k\|^2 + 2\lambda_k \langle w_k - z_k, F y_k \rangle \\
&= \|w_k - x^*\|^2 - \|w_k - z_k\|^2 - 2\lambda_k \langle z_k - x^*, F y_k \rangle \\
&= \|w_k - x^*\|^2 - \|w_k - z_k\|^2 - 2\lambda_k \langle z_k - y_k, F y_k \rangle - 2\lambda_k \langle y_k - x^*, F y_k \rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k + y_k - z_k\|^2 \\
&- 2\lambda_k \langle z_k - y_k, F y_k \rangle - 2\lambda_k \langle y_k - x^*, F y_k \rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 + \langle z_k - y_k, w_k - y_k \rangle \\
&- 2\lambda_k \langle z_k - y_k, F y_k \rangle - 2\lambda_k \langle y_k - x^*, F y_k \rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\lambda_k \langle y_k - z_k, w_k - \lambda_k F y_k - y - k \rangle - 2\lambda_k \langle y_k - x^*, F y_k \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle + 2\lambda_k \langle z_k - y_k, F w_k - F y_k\rangle \\
&- 2\lambda_k \langle y_k - x^*, F y_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle + 2\lambda_k \|z_k - y_k\| \|F w_k - F y_k\| \\
&- 2\lambda_k \langle y_k - x^*, F y_k\rangle \\
&= \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle + 2\frac{\lambda_k}{\lambda_{k+1}} \|z_k - y_k\| \|F w_k - F y_k\| \\
&- 2\lambda_k \langle y_k - x^*, F y_k\rangle \\
&\leq \|w_k - x^*\|^2 - \|w_k - y_k\|^2 - \|y_k - z_k\|^2 \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle - 2\lambda_k \langle y_k - x^*, F y_k\rangle \\
&+ \frac{\lambda_k}{\lambda_{k+1}} (\|z_k - y_k\|^2 + \|F w_k - F y_k\|^2) \\
&= \|w_k - x^*\|^2 - \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) [\|w_k - y_k\|^2 + \|y_k - z_k\|^2] \\
&- 2\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle - 2\lambda_k \langle y_k - x^*, F y_k\rangle. \qquad (3.11)
\end{aligned}
$$

Since $x^* \in \Gamma$, $y_k \in K$ and the fact that F is pseudomonotone we have that

$$\langle y_k - x^*, F x^*\rangle \geq 0$$

which implies

$$\langle y_k - x^*, F y_k\rangle \geq 0.$$

Also from $z_k \in T_k$, we get that

$$\langle y_k - z_k, w_k - \lambda_k F w_k - y_k\rangle \geq 0.$$

Therefore, we obtain from (3.11) that

$$\|z_k - x^*\|^2 \leq \|w_k - x^*\|^2 - \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) [\|w_k - y_k\|^2 + \|y_k - z_k\|^2], \qquad (3.12)$$

as required.

Lemma 3.6. *The sequence $\{x_k\}$ generated by Algorithm 3.2 is bounded.*

Proof. From $x^* \in \Gamma$ and (3.2), we have

$$
\begin{aligned}
\|w_k - x^*\| &\| x_k + \theta_k (x_k - x_{k-1}) - x^*\| \\
&\leq \|x_k - x^*\| + \theta_k \|x_k - x_{k-1}\| \\
&= \|x_k - x^*\| + \alpha_k \cdot \frac{\theta_k}{\alpha_k} \|x_k - x_{k-1}\|.
\end{aligned}
$$

Since $\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \to 0$, there exists $M_1 > 0$ such that

$$\frac{\theta_k}{\alpha_k}\|x_k - x_{k-1}\| \leq M_1, \ \ k \geq 1,$$

hence

$$\|w_k - x^*\| \leq \|x_k - x^*\| + \alpha_k M_1.$$

It is easy to see from Lemma 3.5, that

$$\|z_k - x^*\| \leq \|w_k - x^*\| \leq \|x_k - x^*\| + \alpha_k M_1.$$

Furthermore, from (3.5), we have

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|\alpha_k f(x_k) + \beta_k x_k + \gamma_k S_{\eta_k} z_k - x^*\| \\
&\leq \alpha_k \|f(x_k) - x^*\| + \beta_k \|x_k - x^*\| + \gamma_k \|S_{\eta_k} z_k - x^*\| \\
&\leq \alpha_k \|f(x_k) - f(x^*)\| + \alpha_k \|f(x^*) - x^*\| + \beta_k \|x_k - x^*\| + \gamma_k \|z_k - x^*\| \\
&\leq \alpha_k \kappa \|x_k - x^*\| + \beta_k \|x_k - x^*\| \\
&\quad + \alpha_k \|f(x^*) - x^*\| + \gamma_k (\|x_k - x^*\| + \alpha_k M_1) \\
&= \alpha_k \kappa \|x_k - x^*\| + \beta_k \|x_k - x^*\| \\
&\quad + \alpha_k \|f(x^*) - x^*\| + \gamma_k \|x_k - x^*\| + \gamma_k \alpha_k M_1 \\
&= \alpha_k \kappa \|x_k - x^*\| + (1 - \alpha_k) \|x_k - x^*\| + \alpha_k \|f(x^*) - x^*\| + \gamma_k \alpha_k M_1 \\
&= [1 - \alpha_k (1 - \kappa)] \|x_k - x^*\| + \alpha_k \|f(x^*) - x^*\| + \gamma_k \alpha_k M_1 \\
&\leq \max \left\{ \|x_k - x^*\|, \frac{\|f(x^*) - x^*\| + \theta_k \alpha_k M_1}{1 - \kappa} \right\} \\
&\leq \vdots \\
&\leq \max \left\{ \|x_0 - x^*\|, \frac{\|f(x^*) - x^*\| + \theta_k \alpha_k M_1}{1 - \kappa} \right\}, \ \forall \ k \geq 1.
\end{aligned} \tag{3.13}$$

Therefore the sequence $\{x_k\}$ is bounded. Consequently, the sequences $\{z_k\}$, $\{y_k\}$ and $\{Sz_k\}$ are bounded.

Lemma 3.7. *Let $\{x_k\}$ be the sequence generated by Algorithm 3.2. Then, for $x^* \in \Gamma$, it holds that*

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &\leq \left(1 - \frac{2\alpha_k(1-\kappa)}{(1-\alpha_k\kappa)}\right) \|x_k - x^*\|^2 \\
&\quad + \frac{2\alpha_k(1-\kappa)}{(1-\alpha_k\kappa)} \left(\frac{\alpha_k}{1-\kappa} \|x_k - x^*\|^2 + \frac{1}{1-\kappa} \langle f(x^*) - x^*, x_{k+1} - x^* \rangle \right. \\
&\quad \left. + \frac{\theta_k \gamma_k}{\alpha_k(1-\kappa)} \|x_k - x^*\| \|x_k - x_{k-1}\| + \frac{\theta_k^2}{2\alpha_k(1-\kappa)} \|x_k - x_{k-1}\|^2 \right).
\end{aligned} \tag{3.14}$$

Proof. From (3.5) and $x^* \in \Gamma$, we have

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &\le \|\alpha_k f(x_k) + \beta_k x_k + \gamma_k S_{\eta_k} z_k - x^*\| \\
&\le \|\beta_k(x_k - x^*) + \gamma_k(S_{\eta_k} z_k - x^*)\|^2 + 2\alpha_k\langle f(x_k) - x^*, x_{k+1} - x^*\rangle \\
&\le \|\beta_k(x_k - x^*) + \gamma_k(z_k - x^*)\|^2 + 2\alpha_k\langle f(x_k) - x^*, x_{k+1} - x^*\rangle \\
&\le \|\beta_k(x_k - x^*) + \gamma_k(w_k - x^*)\|^2 + 2\alpha_k\langle f(x_k) - x^*, x_{k+1} - x^*\rangle \\
&\le [\beta_k\|x_k - x^*\| + \gamma_k(\|x_k - x^*\| + \theta_k\|x_k - x_{k-1}\|)]^2 \\
&+ 2\alpha_k\langle f(x_k) - f(x^*), x_{k+1} - x^*\rangle \\
&+ 2\alpha_k\langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\le [\beta_k\|x_k - x^*\| + \gamma_k\|x_k - x^*\| + \gamma_k\theta_k\|x_k - x_{k-1}\|]^2 \\
&+ 2\alpha_k\kappa\|x_k - x^*\| + \|x_{k+1} - x^*\| \\
&+ 2\alpha_k\langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\le [(1 - \alpha_k)\|x_k - x^*\| + \theta_k\gamma_k\|x_k - x_{k-1}\|]^2 \\
&+ \alpha_k\kappa(\|x_k - x^*\|^2 + \|x_{k+1} - x^*\|) \\
&+ 2\alpha_k\langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&\le (1 - \alpha_k)^2\|x_k - x^*\|^2 + 2\theta_k\gamma_k(1 - \alpha_k)\|x_k - x^*\|\|x_k - x_{k-1}\| \\
&+ \gamma_k\theta_k\|x_k - x_{k-1}\|^2 \\
&+ \alpha_k\kappa(\|x_k - x^*\|^2 + \|x_{k+1} - x^*\|) + 2\alpha_k\langle f(x^*) - x^*, x_{k+1} - x^*\rangle,
\end{aligned}$$

this implies that

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &\le \frac{(1 - \alpha_k)^2 + \alpha_k\kappa}{1 - \alpha_k\kappa}\|x_k - x^*\|^2 \\
&+ \frac{2\alpha_k}{1 - \alpha_k\kappa}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle + \frac{\theta_k}{1 - \alpha_k\kappa}\|x_k - x_{k-1}\|^2 \\
&+ \frac{2\gamma_k\theta_k}{1 - \alpha_k\kappa}\|x_k - x^*\|\|x_k - x_{k-1}\| \\
&= \frac{1 - 2\alpha_k + \alpha_k\kappa}{1 - \alpha_k\kappa}\|x_k - x^*\|^2 \\
&+ \frac{\alpha_k^2}{1 - \alpha_k\kappa}\|x_k - x^*\|^2 + \frac{2\alpha_k}{1 - \alpha_k\kappa}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle \\
&+ \frac{\theta_k}{1 - \alpha_k\kappa}\|x_k - x_{k-1}\|^2 + \frac{2\gamma_k\theta_k}{1 - \alpha_k\kappa}\|x_k - x^*\|\|x_k - x_{k-1}\| \\
&= \left(1 - \frac{2\alpha_k(1 - \kappa)}{1 - \alpha_k\kappa}\right)\|x_k - x^*\|^2 \\
&+ \frac{2\alpha_k^2(1 - \kappa)}{2(1 - \alpha_k\kappa)(1 - \kappa)}\|x_k - x^*\|^2 + \frac{2\theta_k^2(1 - \kappa)}{2(1 - \alpha_k\kappa)(1 - \kappa)}\|x_k - x_{k-1}\|^2 \\
&+ \frac{2\alpha_k(1 - \kappa)}{(1 - \alpha_k\kappa)(1 - \kappa)}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2\theta_k\gamma_k(1-\kappa)}{(1-\alpha_k)(1-\kappa)}\|x_k - x^*\|\|x_k - x_{k-1}\| \\
&= \left(1 - \frac{2\alpha_k(1-\kappa)}{1-\alpha_k\kappa}\right)\|x_k - x^*\|^2 \\
&+ \frac{2\alpha_k(1-\kappa)}{(1-\alpha_k\kappa)}\left(\frac{\alpha_k}{2(1-\kappa)}\|x_k - x^*\|^2 + \frac{\theta_k^2}{\alpha_k(1-\kappa)}\|x_k - x_{k-1}\|^2\right. \\
&\left. + \frac{\theta_k\gamma_k}{\alpha_k(1-\kappa)}\|x_k - x^*\|\|x_k - x_{k-1}\| + \frac{1}{(1-\kappa)}\langle f(x^*) - x^*, x_{k+1} - x^*\rangle\right). \qquad (3.15)
\end{aligned}
$$

Theorem 3.8. *Assume that condition $C1-C5$ hold. Then the sequence $\{x_k\}$ generated by Algorithm 3.2 converges to a common solution $x^* \in \Gamma$, which is also a unique solution of the variational inequality*

$$\langle f(x^*) - x^*, x^* - \bar{x}\rangle \geq 0, \ \forall \ \bar{x} \in \Gamma.$$

Proof. Let $x^* \in \Gamma$, the proof of this theorem is divided into two cases.

Case I: Suppose there exists $k_0 \in \mathbb{N}$ such that $\|\{x_k - x^*\|\}$ is monotonically non-increasing. Then, by Lemma 3.6, it follows that $\|\{x_k - x^*\|\}$ is a convergent sequence and thus

$$\|x_{k-1} - x^*\|^2 - \|x_k - x^*\|^2 \to 0 \text{ as } k \to \infty.$$

Consider

$$
\begin{aligned}
\|w_k - x^*\|^2 &= \|x_k - x^* + \theta_k(x_k - x_{k-1})\|^2 \\
&= \|x_k - x^*\|^2 + 2\theta_k\langle x_k - x^*, x_k - x_{k-1}\rangle + \theta_k^2\|x_k - x_{k-1}\|^2 \\
&\leq \|x_k - x^*\| + \theta_k\|x_k - x_{k-1}\|(2\|x_k - x^*\| + \theta_k\|x_k - x_{k-1}\|). \qquad (3.16)
\end{aligned}
$$

From (3.5) and Lemma 3.10, we have

$$
\begin{aligned}
\|x_{k+1} - x^*\|^2 &\leq \alpha_k\|f(x_k) - x^*\|^2 + \beta_k\|x_k - x^*\|^2 + \gamma_k\|S_{\eta_k}z_k - x^*\|^2 \\
&\leq \alpha_k\|f(x_k) - x^*\|^2 + \beta_k\|x_k - x^*\|^2 + \gamma_k\|z_k - x^*\|^2 \\
&\leq \alpha_k\|f(x_k) - x^*\|^2 + \beta_k\|x_k - x^*\|^2 \\
&+ \gamma_k(\|w_k - x^*\|^2 - \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k - y_k\|^2 + \|y_k - z_k\|^2]) \\
&\leq \alpha_k\|f(x_k) - x^*\|^2 + \beta_k\|x_k - x^*\|^2 + \gamma_k\|x_k - x^*\|^2 \\
&+ \frac{\gamma_k\theta_k\alpha_k}{\alpha_k}\|x_k - x_{k-1}\|(2\|x_k - x^*\| + \theta_k\|x_k - x_{k-1}\|) \\
&- \gamma_k\left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k - y_k\|^2 + \|y_k - z_k\|^2] \\
&= \alpha_k\|f(x_k) - x^*\|^2 + \|x_k - x^*\|^2 \\
&+ \frac{\gamma_k\theta_k\alpha_k}{\alpha_k}\|x_k - x_{k-1}\|(2\|x_k - x^*\| + \theta_k\|x_k - x_{k-1}\|) \\
&- \gamma_k\left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k - y_k\|^2 + \|y_k - z_k\|^2]), \qquad (3.17)
\end{aligned}
$$

which implies

$$\gamma_k\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)[\|w_k-y_k\|^2+\|y_k-z_k\|^2] \le \alpha_k\|f(x_k)-x^*\|^2+\|x_k-x^*\|^2 - \|x_{k+1}-x_k\|^2 + \frac{\gamma_k\theta_k\alpha_k}{\alpha_k}\|x_k-x_{k-1}\|(2\|x_k-x^*\| + \theta_k\|x_k-x_{k-1}\|) \to 0 \text{ as } k\to\infty. \quad (3.18)$$

Therefore, we obtain from the definition of λ_k, that

$$\lim_{k\to\infty}\|w_k-y_k\| = 0 = \lim_{k\to\infty}\|y_k-z_k\|. \quad (3.19)$$

Note also that

$$\|w_k-x_k\| = \theta_k\|x_k-x_{k-1}\| = \alpha_k\cdot\frac{\theta_k}{\alpha_k}\|x_k-x_{k-1}\| \to 0 \text{ as } k\to\infty. \quad (3.20)$$

It is easy to see from above that

$$\lim_{k\to\infty}\|y_k-x_k\| = 0 = \lim_{k\to\infty}\|z_k-x_k\|. \quad (3.21)$$

Next we show that $\|Sz_k-z_k\|\to 0$ as $k\to\infty$. From the definition of S_{η_k} and $x^*\in\Gamma$, we have

$$\begin{aligned}\|S_{\eta_k}z_k-x^*\|^2 & \|(1-\eta_k)(z_k-x^*)+\eta_k(Sz_k-x^*)\|^2\\ & (1-\eta_k)\|z_k-x^*\|^2+\eta_k\|Sz_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2\\ & \le (1-\eta_k)\|z_k-x^*\|^2+\eta_k\|z_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2\\ & = \|z_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2,\end{aligned}$$

which implies from Lemma 3.10, that

$$\|S_{\eta_k}z_k-x^*\|^2 \le \|w_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2.$$

Using this in (3.5), we get

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\le \alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2+\gamma_k\|S_{\eta_k}z_k-x^*\|^2\\ &\le \alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2\\ &+\gamma_k(\|w_k-x^*\|^2-\eta_k(1-\eta_k)\|Sz_k-z_k\|^2)\\ &\le \alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2\\ &+\gamma_k\|w_k-x^*\|^2-\eta_k(1-\eta_k)\gamma_k\|Sz_k-z_k\|^2\\ &\le \alpha_k\|f(x_k)-x^*\|^2+\beta_k\|x_k-x^*\|^2+\gamma_k\|x_k-x^*\|^2\\ &+\gamma_k\theta_k\|x_k-x_{k-1}\|(2\|x_k-x^*\|\|x_k-x_{k-1}\|)\\ &-\eta_k(1-\eta_k)\gamma_k\|Sz_k-z_k\|^2\\ &\le \alpha_k\|f(x_k)-x^*\|^2+\|x_k-x^*\|^2\\ &+\gamma_k\theta_k\|x_k-x_{k-1}\|(2\|x_k-x^*\|\|x_k-x_{k-1}\|)\\ &-\eta_k(1-\eta_k)\gamma_k\|Sz_k-z_k\|^2. \end{aligned} \quad (3.22)$$

We obtain from this that

$$\eta_k(1-\eta_k)\gamma_k\|Sz_k - z_k\|^2 \leq \alpha_k\|f(x_k) - x^*\|^2 + \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + \frac{\gamma_k\theta_k}{\alpha_k}\|x_k - x_{k-1}\|(2\|x_k - x^*\|\|x_k - x_{k-1}\|) \to 0 \text{ as } k \to \infty,$$

hence

$$\lim_{k\to\infty} \|Sz_k - z_k\| = 0. \tag{3.23}$$

It is not difficult to obtain from this, that

$$\lim_{k\to\infty} \|S_{\eta_k} z_k - z_k\| = 0. \tag{3.24}$$

Observe that

$$\|x_{k+1} - x^*\|^2 \leq \alpha_k\|f(x_k) - z_k\|^2 + \beta_k\|x_k - z_k\|^2 + \gamma_k\|S_{\eta_k} z_k - z_k\|^2, \tag{3.25}$$

thus, we have from (3.21), (3.24) and condition (i), that

$$\|x_{k+1} - z_k\| \to 0 \text{ as } k \to \infty.$$

Using this and (3.21), we obtain

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = \lim_{k\to\infty} (\|x_{k+1} - z_k\| + \|z_k - x_k\|) = 0. \tag{3.26}$$

By the conclusion of Lemma 3.6, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $\{x_{k_j}\}$ converge weakly to $\bar{x} \in H$ satisfying

$$\limsup_{k\to\infty} \langle f(x^*) - x^*, x_k - x^*\rangle = \lim_{j\to\infty} \langle f(x^*) - x^*, x_{k_j} - x^*\rangle. \tag{3.27}$$

By (3.19) and Lemma 3.5, we obtain $\bar{x} \in VIP(F, K)$. Also from (3.23), (3.24) and Lemma 2.5, we have $\bar{x} \in F(S_{\eta_k}) = F(S)$. Hence $x^* \in \Gamma$. It is clear that $P_\Gamma f$ is a contraction. Using Banach's principle of contraction, $P_\Gamma f$ has a unique fixed point, say $x^* \in H$. That is $x^* = P_\Gamma f(x^*)$. It follows from Lemma 2.2, that

$$\langle f(x^*) - x^*, \bar{x} - x^*\rangle \leq 0. \tag{3.28}$$

Thus, we have that

$$\limsup_{k\to\infty} \langle f(x^*) - x^*, x_k - x^*\rangle = \lim_{j\to\infty} \langle f(x^*) - x^*, x_{k_j} - x^*\rangle \langle f(x^*) - x^*, \bar{x} - x^*\rangle \leq 0. \tag{3.29}$$

Hence by (3.26) and (3.29), we have

$$\limsup_{k\to\infty} \langle f(x^*) - x^*, x_{k+1} - x^*\rangle \leq \limsup_{k\to\infty} \langle f(x^*) - x^*, x_{k+1} - x_k\rangle + \limsup_{k\to\infty} \langle f(x^*) - x^*, x_k - x^*\rangle \leq 0. \tag{3.30}$$

By applying Lemma 2.6, Lemma 3.7, and (3.30), we have $x_k \to 0$ as $k \to \infty$.
Case II: There exists a subsequence $\{\|x_{k_j} - x^*\|\}$ of $\{\|x_k - x^*\|\}$ such that

$$\|x_{k_j} - x^*\|^2 \leq \|x_{k_j+1} - x^*\|^2$$

for all $j \in \mathbb{N}$. By Lemma 2.7, there exists a nondecreasing sequence $\{m_n\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} m_n = \infty$ and there hold

$$\|x_{m_n} - x^*\|^2 \leq \|x_{m_n+1} - x^*\|^2 \quad \text{and} \quad \|x_k - x^*\|^2 \leq \|x_{m_n+1} - x^*\|^2,\ \forall\, n \in \mathbb{N}. \tag{3.31}$$

By (3.17) and Lemma 3.7, we have

$$\begin{aligned}
\|x_{m_n} - x^*\|^2 \leq \|x_{m_n+1} - x^*\|^2 &\leq \alpha_{m_n}\|f(x_{m_n}) - x^*\|^2 \\
&+ \beta_{m_n}\|x_{m_n} - x^*\|^2 + \gamma_{m_n}\Bigg(\|w_{m_n} - x^*\|^2 \\
&- \left(1 - \frac{\lambda_{m_n}}{\lambda_{m_n}+1}\right)[\|w_{m_n} - y_{m_n}\|^2 + \|y_{m_n} - z_{m_n}\|^2]\Bigg) \\
&\leq \alpha_{m_n}\|f(x_{m_n}) - x^*\|^2 + \beta_{m_n}\|x_{m_n} - x^*\|^2 + \gamma_{m_n}\|x_{m_n} - x^*\|^2 \\
&+ \gamma_{m_n}\theta_{m_n}\|x_{m_n} - x_{m_n-1}\|(2\|x_{m_n} - x^*\| + \theta_{m_n}\|x_{m_n} - x_{m_n-1}\|) \\
&- \gamma_{m_n}\left(1 - \frac{\lambda_{m_n}}{\lambda_{m_n}+1}\right)[\|w_{m_n} - y_{m_n}\|^2 + \|y_{m_n} - z_{m_n}\|^2] \\
&= \alpha_{m_n}\|f(x_{m_n}) - x^*\|^2 + (1-\alpha_{m_n})\|x_{m_n} - x^*\| \\
&- \gamma_{m_n}\left(1 - \frac{\lambda_{m_n}}{\lambda_{m_n}+1}\right)[\|w_{m_n} - y_{m_n}\|^2 + \|y_{m_n} - z_{m_n}\|^2] \\
&+ \gamma_{m_n}\theta_{m_n}\|x_{m_n} - x_{m_n-1}\|(2\|x_{m_n} - x^*\| + \theta_{m_n}\|x_{m_n} - x_{m_n-1}\|).
\end{aligned}$$

Since $\alpha_{m_n} \to 0$ as $n \to \infty$, it follows from above that

$$\lim_{n\to\infty} \gamma_{m_n}\left(1 - \frac{\lambda_{m_n}}{\lambda_{m_n}+1}\right)[\|w_{m_n} - y_{m_n}\|^2 + \|y_{m_n} - z_{m_n}\|^2] = 0,$$

hence

$$\lim_{n\to\infty} \|w_{m_n} - y_{m_n}\| = \|y_{m_n} - z_{m_n}\| = 0. \tag{3.32}$$

By using similar arguments as in **Case I**, the following are easy to establish:

$$\lim_{n\to\infty} \|S_{\eta_{m_n}} z_{m_n} - z_{m_n}\| = \|S z_{m_n} - z_{m_n}\| = 0, \tag{3.33}$$

$$\lim_{n\to\infty} \|w_{m_n} - x_{m_n}\| = \|x_{m_n+1} - x_{m_n}\| = 0. \tag{3.34}$$

and

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_{m_n+1} - x^*\rangle \leq 0.$$

It follows from (3.14), that

$$\begin{aligned}\|x_{m_n+1}-x^*\|^2 \leq& \left(1-\frac{2\alpha_{m_n}(1-\kappa)}{1-\alpha_{m_n}\kappa}\right)\|x_{m_n}-x^*\|^2\\&+\frac{2\alpha_{m_n}(1-\kappa)}{1-\alpha_{m_n}\kappa}\left(\frac{\alpha_{m_n}}{1-\kappa}\|x_{m_n}-x^*\|^2\right.\\&+\frac{1}{1-\kappa}\langle f(x^*)-x^*,x_{m_n+1}-x^*\rangle+\frac{\theta_{m_n}^2}{2\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x_{m_n-1}\|^2\\&\left.+\frac{\theta_{m_n}\gamma_{m_n}}{\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x^*\|\|x_{m_n}-x_{m_n-1}\|\right),\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{m_n+1}-x^*\|^2 \leq& \frac{\alpha_{m_n}}{1-\kappa}\|x_{m_n}-x^*\|^2+\frac{1}{1-\kappa}\langle f(x^*)-x^*,x_{m_n+1}-x^*\rangle\\&+\frac{\theta_{m_n}^2}{2\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x_{m_n-1}\|^2\\&+\frac{\theta_{m_n}\gamma_{m_n}}{\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x^*\|\|x_{m_n}-x_{m_n-1}\|.\end{aligned}$$

By (3.31), we obtain

$$\begin{aligned}\|x_k-x^*\|^2 &\leq \|x_{m_n+1}-x^*\|^2\\&\leq \frac{\alpha_{m_n}}{1-\kappa}\|x_{m_n}-x^*\|^2+\frac{1}{1-\kappa}\langle f(x^*)-x^*,x_{m_n+1}-x^*\rangle\\&+\frac{\theta_{m_n}^2}{2\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x_{m_n-1}\|^2\\&+\frac{\theta_{m_n}\gamma_{m_n}}{\alpha_{m_n}(1-\kappa)}\|x_{m_n}-x^*\|\|x_{m_n}-x_{m_n-1}\|.\end{aligned}$$

Thus, we get that $\limsup_{k\to\infty}\|x_n-x^*\|^2=0$, which means that $\lim_{n\to\infty}\|x_n\|=x^*$. The proof is therefore complete.

4. Application

In this section, we give some applications of our main result.

4.1. Constrained optimization problem

Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $h: H\to\mathbb{R}$ be a differentiable function on K with its gradient ∇h. The Constrained Optimization Problem (COP) is given as: Find $x^*\in K$ such that

$$h(x^*)\leq h(x),\ \forall\, x\in K. \tag{4.1}$$

We denote by $Sol(h)$ the solution set of (4.1). It is well known (see [44]), that a point x^* is a minimizer of (4.1) if and only if x^* is a solution of the VIP (1.1) with $F=\nabla h$.

Thus by applying this formulations and substituting $F = \nabla h$ in Algorithm 3.2, we have the following result for finding a common solution of a COP and a FPP.

Theorem 4.1. *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $h : H \to \mathbb{R}$ be a differentiable function on K with its gradient ∇h. Let $S : H \to H$ be an η-demimetric mapping. Assume $Sol(h) \cap Fix(S) \neq \emptyset$. Then, the sequence $\{x_k\}$ generated by Algorithm 3.2 with F replaced by ∇h converges strongly to a point $x^* = P_{Sol(h)\cap Fix(S)} f(x^*)$.*

4.2. Split feasibility problem

Let K and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) in the sense of Censor and Elfving [13] is to find

$$x \in K \text{ such that } Ax \in Q. \tag{4.2}$$

We denote by Ω the solution set of (4.2). Many authors have considered the solution of the SFP (4.2). We note that whenever the SFP (4.2) is consistent (i.e, has a solution), then $x^* \in \Omega$ solves the fixed point equation

$$x^* = P_K(x - \lambda A^*(I - P_Q)Ax), \ \forall \ x \in K,$$

where P_K and P_Q are orthogonal projection of H_1 and H_2 onto K and Q respectively $\lambda > 0$ and A^* is the adjoint of A. One of the most popular method for solving the SFP was the algorithm proposed by Bryne [10]. He gave a recursive formula $\{x_k\}$ generated by x_1 and

$$x_{k+1} = P_K(x_k - \lambda A^*(I - P_Q)Ax_k), \ k \in \mathbb{N}, \tag{4.3}$$

where $\lambda \in [0, 2/\gamma]$ with γ the spectral radius of the operator A^*A.
For the adaptation of our main result to the solution of the SFP, we need the following proposition (see Ceng et al. [12]).

Proposition 4.2. [12] *Given $x^* \in H_1$, the following are equivalent*

(i) $x^* \in \Omega$;
(ii) x^* *solves* (4.3);
(iii) x^* *solves the system of variational inequality problem: find $x^* \in K$ such that*

$$\langle A^*(I - P_Q)Ax^*, x - x^* \rangle \geq 0, \ \forall \ x \in K,$$

where A^* is the adjoint of A.

By these adaptations, we have the following theorem for approximating a solution of an SFP and a FPP.

Theorem 4.3. *Let K and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H \to H$ be an η-demimetric mapping. Assume $\Omega \cap Fix(S) \neq \emptyset$. Then, the sequence $\{x_k\}$ generated by Algorithm 3.2 with $F := A^*(I - P_Q)A$ converges strongly to*

$$x^* = P_{\Omega\cap Fix(S)} f(x^*).$$

5. Numerical examples

We next provide some numerical experiments to illustrate the performance of our method as well as comparing it with some related methods in the literature.

Example 5.1. Let $H = \mathbb{R}^m$ with the standard topology. Consider a mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ given in the form $F(x) = Mx + q$ (see [19], also [35]) where

$$M = BB^T + P + Q$$

q is a vector in $\mathbb{R}^m$, B is an $m \times m$ matrix, P is an $m \times m$ skew-symmetric matrix, Q is a positive definite diagonal matrix, hence the variational inequality is consistent with a unique solution. We define the feasible set K by $K := \{x \in H : \|x\| \leq 1\}$. Let $S : H \to H$ be defined by $S(x) = \frac{-3x}{2}$ for all $x \in H$ and $f(x) = x$. In this example, we choose $\alpha_k = \frac{1}{k+3}$, $\beta_k = \gamma_k = 0.5(1 - \alpha_k)$, $\eta_k = 0.8 - \alpha_k$, $\theta = \frac{1}{3}$, $\lambda_0 = \mu = 0.95$ and $\tau_k = \frac{1}{k^{1.9}}$. For VSEGM and HSEGM, we choose $\beta_k = 0.8 - \alpha_k$ and $\lambda_k = 0.75/L$ where $L = ||F||$. We terminate the iterations at $Tol = ||x_k - P_C(x_k - Fx_k)||_2 \leq \epsilon$ with $\epsilon = 10^{-4}$.
We compare Algorithm 3.2, VSEGM [50] and HSEGM [34] for different values of m. The results are presented in Figure 1.

Example 5.2. The following example was taken from [24],

$$\min g(x) = \frac{x^T Px + a^T x + a_0}{b^T x + b_0}$$
$$\text{subject to } x \in X = \{x \in \mathbb{R}^5 : b^T x + b_0 > 0\},$$

where

$$P = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \ a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} \ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2, \ b_0 = 4.$$

Since P is symmetric and positive definite, g is pseudoconvex on X. We minimize g on $K = \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 10\} \subset X$.
Following our consideration in Theorem 4.1, we have

$$F(x) = \nabla g(x) = \frac{(b^T x + b_0)(2Px + a) - b(x^T Px + a^T x + a_0)}{(b^T x + b_0)^2}. \tag{5.1}$$

We define the mapping $S : H \to H$ by $S(x) = \frac{-3x}{2}$ and the function f by $f(x) = \frac{x}{2}$. Since the Lipschitz constant of F given by (5.1) is unknown, we compare Algorithm 3.2 with the VSEGM [50]. The following choices of parameters are made: $\alpha_k = \frac{1}{k+3}$, $\beta_k = \gamma_k = 0.5(1 - \alpha_k)$, $\eta_k = 0.5$, $\theta = \frac{1}{3}$, $\lambda_0 = \mu = 0.5$ and $\tau_k = \frac{1}{k^{1.5}}$.
We terminate the iterations at $Tol = ||x_k - P_C(x_k - Fx_k)||_2 \leq \epsilon$ with $\epsilon = 10^{-4}$. The results are presented in Figure 2 for varying initial values x_0 and x_1.

Case1: $x_0 = (10, 10, 10, 10)'$ and $x_1 = (5, 5, 5, 5)'$;
Case2: $x_0 = (5, 5, 5, 5)'$ and $x_1 = (20, 20, 20, 20)'$;
Case3: $x_0 = (1, 1, 1, 1)'$ and $x_1 = (-4, -4, -4, -4)'$.

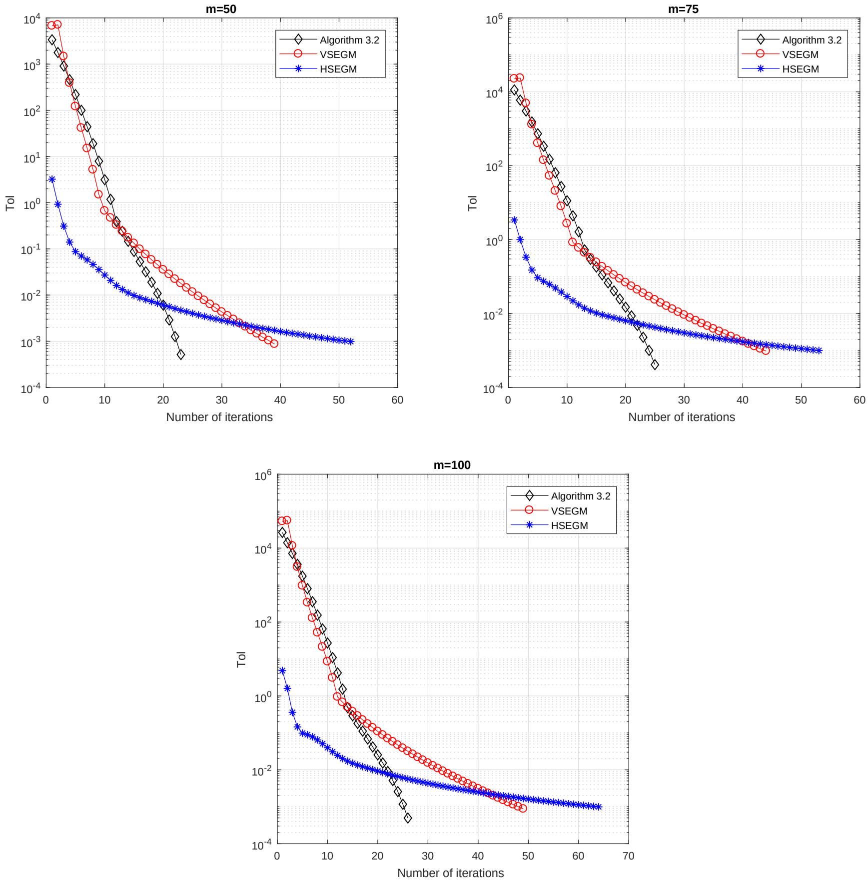

FIGURE 1. Performance of Algorithm 3.2 compared with VSEGM [50] and HSEGM [34].

Example 5.3. Let $H = L^2([0,1])$ with the inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \ \forall\ x, y \in H$$

and the induced norm

$$\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}.$$

Let the mapping $F : H \to H$ be defined by $F(x) = \max\{0, x(t)\}$, $\forall x \in L^2([0,1])$, $t \in [0,1]$ for all $x \in H$ and the feasible $K := \{x \in H : \|x\| \leq 1\}$. Define the mapping T

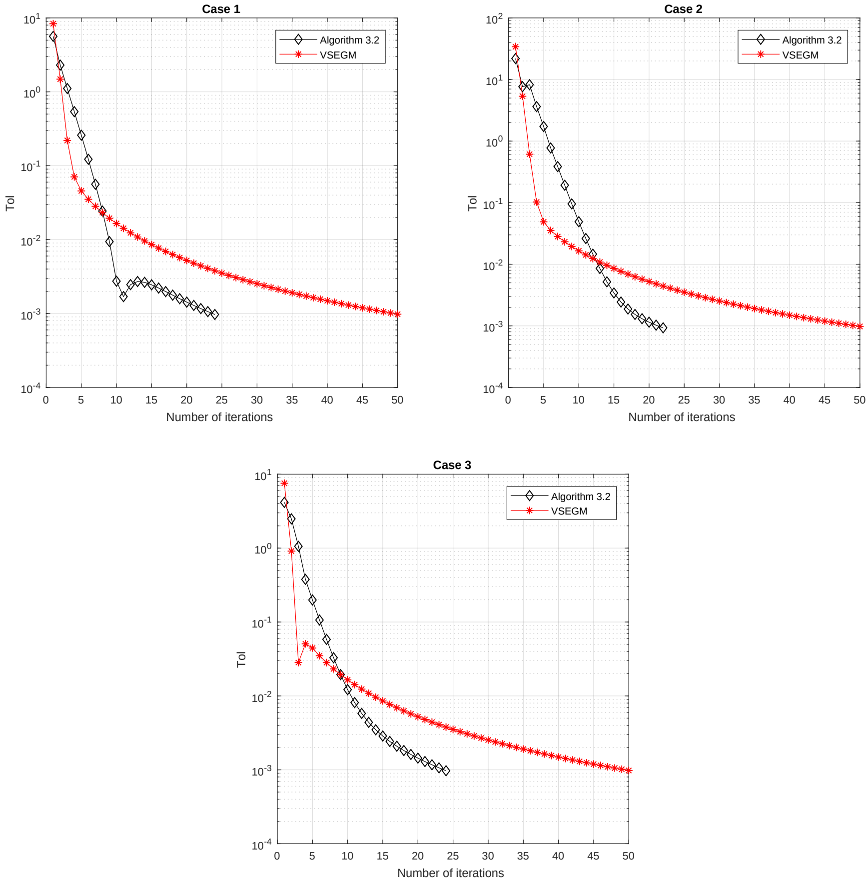

FIGURE 2. Performance of Algorithm 3.2 compared with VSEGM [50].

by

$$Tx(t) = \int_0^1 tx(t)dt, \ \forall x \in L^2([0,1]), \ t \in [0,1],$$

then T is 0-demimetric. Also, let $f : H \to H$ be given by $f(x) = \frac{x}{2}$. For this example, we choose parameters for Algorithm 3.2, HSEGM [34] and VSEGM [50] as follows: $\alpha_k = \frac{1}{k+3}$, $\beta_k = \gamma_k = 0.5(1-\alpha_k)$, $\eta_k = \frac{1}{2k+1}$, $\theta = \frac{1}{3}$, $\lambda_0 = 0.25$, $\mu = 0.5$ and $\tau_k = \frac{1}{k^{1.9}}$. For the VSEGM and HSEGM, we choose $\beta_k = \frac{1}{2k+1}$. We make our comparisons with different initial values and present the result in Figure 3.

Case i: $x_0 = -5t$ and $x_1 = 2t$;
Case ii: $x_0 = 9t^3 + 11t$ and $x_1 = t^2$;
Case iii: $x_0 = \cos(2t) + 5$ and $x_1 = e^{-3t}$.

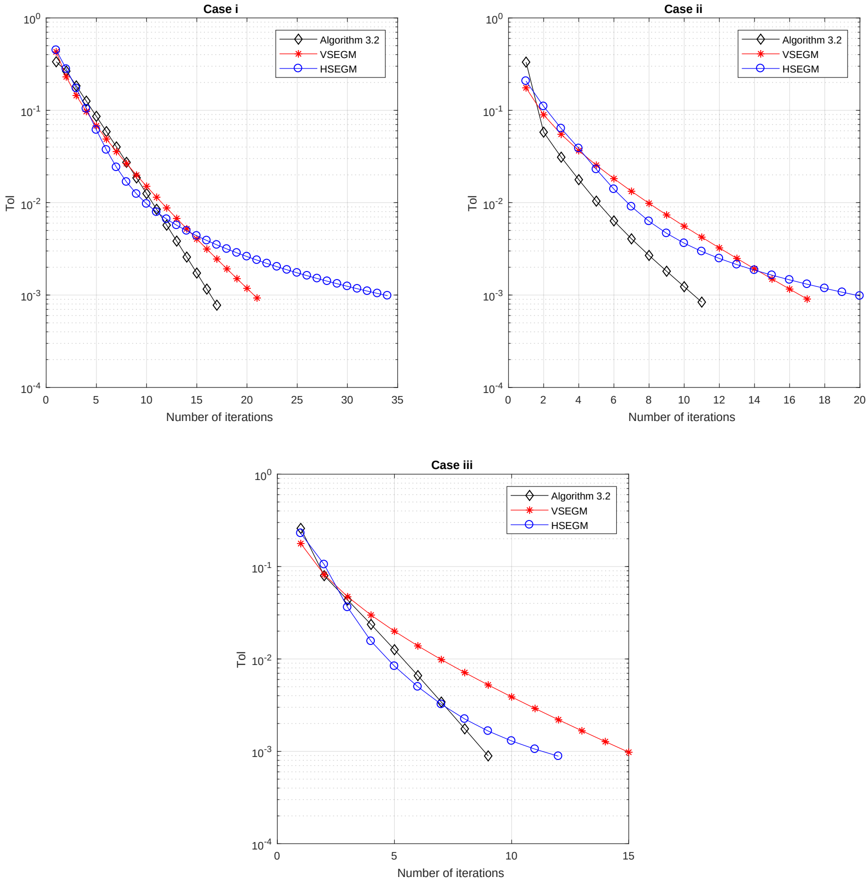

FIGURE 3. Performance of Algorithm 3.2 compared with VSEGM [50] and HSEGM [34].

6. Conclusion

In this paper, we considered the problem of finding a common element of the set of solution of VIP and FPP for η-demimetric mapping in real Hilbert space. We proposed a new iterative algorithm of inertial form and proved a strong convergence theorem under some mild conditions. Our proposed method uses a combination of subgradient extragradient method and a modified viscosity approach with self adaptable step size which avoids the knowledge of the Lipschitz constant of the cost operator in practice. Some applications to constrained optimization and split feasibility problems were considered. We finally gave some numerical experiments to illustrate the behaviour of our method and compare it with some related methods in the literature.

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Weakly Picard mappings: Retraction-displacement condition, quasicontraction notion and weakly Picard admissible perturbation

Ioan A. Rus

Abstract. Let (X, d) be a metric space, $f : X \to X$ be a mapping and $G(\cdot, f(\cdot))$ be an admissible perturbation of f. In this paper we study the following problems: In which conditions imposed on f and G we have the following:

(DDE) data dependence estimate for the mapping f perturbation;
(UH) Ulam-Hyers stability for the equation, $x = f(x)$;
(WP) well-posedness of the fixed point problem for f;
(OP) Ostrowski property of the mapping f.
Some research directions are suggested.

Mathematics Subject Classification (2010): 47H25, 54H25, 47H09, 65J15, 37N30, 39A30.

Keywords: Metric space, fixed point equation, Picard mapping, weakly Picard mapping, admissible perturbation, retraction-displacement condition, data dependence estimate, Ulam-Hyers stability, well-posedness, Ostrowski property, quasicontraction.

1. Introduction

Let X be a nonempty set and $f : X \to X$ be a mapping. To define a perturbation of f we consider a mapping $G : X \times X \to X$ with the following properties:

(A_1) $G(x, x) = x$, $\forall\ x \in X$;
(A_2) $x, y \in X$, $G(x, y) = x$ implies $y = x$.

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Now, we consider the operator, $f_G : X \to X$ defined by,

$$f_G(x) := G(x, f(x)).$$

It is clear that, $F_f = F_{f_G}$, i.e., the fixed point equations,

$$x = f(x) \text{ and } x = f_G(x)$$

are equivalent.

By definition, the mapping f_G is an admissible perturbation of the mapping f corresponding to the mapping G.

Let us consider an example. For other examples see [53].

Example 1.1. Let $\mathbb{B}$ be a Banach space, $f : \mathbb{B} \to \mathbb{B}$ be a mapping and $G : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ be defined by,

$$G(x, y) := (1 - \lambda)x + \lambda y$$

for some $\lambda \in \mathbb{R}^*$. Then f_G is an admissible perturbation of f. We denote it by, f_λ.

Remark 1.2. If $X \subset \mathbb{B}$ is a nonempty convex subset of $\mathbb{B}$, $f : X \to X$ is a mapping and $G(x, y) := (1 - \lambda)x + \lambda y$ for some $\lambda \in]0, 1[$, then f_λ is an admissible perturbation of f, i.e., Krasnoselskii perturbation of f. For more considerations of this perturbation see [52], [3], [12], [20], [21].

Let (X, d) be a metric space, $f : X \to X$ be a mapping and $G(\cdot, f(\cdot))$ be an admissible perturbation of f. In this paper we shall study the following problems:

In which conditions imposed on f and G we have the following (all or one!) :

(DDE) data dependence estimate for the general perturbation of f;

(UH) Ulam-Hyers stability for the equation, $x = f(x)$;

(WP) well-posedness of the fixed point problem for f;

(OP) Ostrowski property of the mapping f.

Some research direction are suggested.

Throughout this paper the notations and terminology given in [8], [38], [56] and [57] are used.

Instead of long preliminaries we give the following references:

- Picard and weakly Picard mappings: [48], [56], [57], [61], [64];
- Ulam-Hyers stability: [55], [56], [57], [64];
- Well-posedness of fixed point problem: [56], [57], [9], [10], [35], [50], [33];
- Ostrowski property of a mapping (limit shadowing property): [35], [17], [22], [46], [56], [57], [61], [64], [13], [34], [32].

2. Retractions on the fixed point set and retraction-displacement conditions

Let (X, d) be a metric space and $f : X \to X$ be a mapping with $F_f \neq \varnothing$. Let $r : X \to F_f$ be a set retraction, i.e., $r|_{F_f} = 1_{F_f}$. Then,

$$X = \bigcup_{x \in F_f} r^{-1}(x)$$

is a partition of X. If $x^* \in F_f$ then we denote, $X_{x^*} := r^{-1}(x^*)$. By definition, the partition $X = \bigcup_{x^* \in F_f} X_{x^*}$ is a fixed point partition of X corresponding to the retraction r (see [59]).

Remark 2.1. In general, X_{x^*} is not an invariant subset for f.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function with $\psi(0) = 0$ and continuous at 0. By definition, the condition,

$$d(x, r(x)) \leq \psi(d(x, f(x))),\ \forall\ x \in X,$$

is a retraction-displacement condition on f corresponding to the retraction r.

Example 2.2. (see [57]; see also [42], [37], [36]). Let (X, d) be a complete metric space and $f : X \to X$ be a graphic l-contraction. In addition we suppose that,

$$d(f(f^n(x)), f(f^\infty(x))) \to 0 \text{ as } n \to \infty,$$

for all $x \in X$. Then f is weakly Picard mapping.

The mapping $f^\infty : X \to F_f$ is a set-retraction and

$$d(x, f^\infty(x)) \leq \frac{1}{1-l} d(x, f(x)),\ \forall\ x \in X.$$

In this case, $f(X_{x^*}) \subset X_{x^*}$, $\forall\ x^* \in F_f$, i.e., $X = \bigcup_{x^* \in F_f} X_{x^*}$ is an invariant fixed point partition of X corresponding to the retraction f^∞.

Example 2.3. (Browder [11] and Bruck [14], pp. 6, 33). Let H be a Hilbert space, $X \subset H$ be a convex, closed and bounded subset of H and $f : X \to X$ be a nonexpansive mapping. Let $r_1(x) = \lim_{n\to\infty} x_n(x)$, where x_n is the unique solution of,

$$x_n(x) = \frac{1}{n}x + \big(1 - \frac{1}{n}\big) f(x_n(x)),\ n \in \mathbb{N}^*,\ x \in X,$$

and

$$r_2(x) = w - lim \frac{1}{n}(1_X + f + \ldots + f^{n-1})(x),\ n \in \mathbb{N}^*,\ x \in X.$$

Then the mappings, $r_1, r_2 : X \to F_f$ are nonexpansive retractions. In general, $r_1 \neq r_2$.

In this case we have two distinct fixed point partitions of X corresponding to r_1 and to r_2.

Remark 2.4. The notion *fixed point partition of the space with respect to a retraction* is a relevant one. For example, in terms of this notion we can give the following definitions.

Let (X, d) be a metric space, $f : X \to X$ be a mapping with $F_f \neq \varnothing$, $r : X \to F_f$ be a set retraction and $X = \bigcup_{x^* \in F_f} X_{x^*}$ be the fixed point partition of X, corresponding to the retraction r.

Definition 2.5. The fixed point problem for the mapping f is well-posed with respect to the partition $X = \bigcup_{x^* \in F_f} X_{x^*}$ if the following implication holds:

$$x^* \in F_f,\ x_n \in X_{x^*},\ n \in \mathbb{N},\ d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty$$
$$\Rightarrow\ x_n \to x^* \text{ as } n \to \infty.$$

Definition 2.6. The mapping f has the Ostrowski property with respect to the partition, $X = \bigcup_{x^* \in F_f} X_{x^*}$, if the following implication holds:

$$x^* \in F_f,\ x_n \in X_{x^*},\ n \in \mathbb{N},\ d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty$$
$$\Rightarrow\ x_n \to x^* \text{ as } n \to \infty.$$

3. Results for (DDE), (UH) and (WP) problems

3.1. (DDE) problem

Let (X, d) be a metric space, $f : X \to X$ be a mapping and f_G be an admissible perturbation. Let $g : X \to X$ be a mapping such that,

$$d(f(x), g(x)) \leq \eta,\ \forall\ x \in X,\ \text{for some } \eta \in \mathbb{R}_+^*.$$

We suppose that, $F_f = \{x^*\}$ and $F_g \neq \varnothing$.

The problem is to find in which conditions imposed on f and G, there exists an increasing, $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, with $\theta(0) = 0$ and continuous in 0 such that,

$$d(y^*, x^*) \leq \theta(\eta),\ \forall\ y^* \in F_g.$$

We have the following result.

Theorem 3.1. *We suppose that:*

(1) *f_G is a ψ-Picard mapping ($F_{f_G} = \{x^*\}$);*
(2) *$d(x, f_G(x)) \leq cd(x, f(x))$, $\forall\ x \in X$ with some $c \in \mathbb{R}_+^*$;*
(3) *$d(g(x), f(x)) \leq \eta$, $\forall\ x \in X$ with some $\eta \in \mathbb{R}_+^*$.*

Then we have that:

(*i*) *$d(x, x^*) \leq \psi(cd(x, f(x)))$, $\forall\ x \in X$;*
(*ii*) *$d(y^*, x^*) \leq \psi(c\eta)$, $\forall\ y^* \in F_g$.*

Proof. Since f_G is a Picard mapping and an admissible perturbation of f we have that, $F_f = \{x^*\}$ and from (1),

$$d(x, x^*) \leq \psi(d(x, f_G(x))),\ \forall\ x \in X.$$

From (2) we have (i).

If we take $x = y^* \in F_g$, then from (i) and (3),

$$d(y^*, x^*) \leq \psi(cd(y^*, f(y^*))) = \psi(cd(g(y^*), f(y^*))) \leq \psi(c\eta). \qquad \square$$

Example 3.2. Let $X := \mathbb{B}$ be a Banach space and $G(x, y) := (1 - \lambda)x + \lambda y$, with $\lambda \in \mathbb{R}_+^*$. We suppose that f_λ is an l-contraction for some $\lambda \in \mathbb{R}_+^*$. Then f_λ is $\frac{1}{1-l}$ -Picard mapping and $d(x, f_\lambda(x)) = \|x - f_\lambda(x)\| \leq |\lambda| \|x - f(x)\|$.

Let, $\|f(x) - g(x)\| \leq \eta$, $\forall\ x \in \mathbb{B}$. Then by Theorem 3.1 we have that:

$$\|y^* - x^*\| \leq \frac{|\lambda|}{1-l}\eta,\ \forall\ y^* \in F_g.$$

Remark 3.3. For the mappings f_λ which are contractions or which satisfy other metric conditions, see Berinde [4] and Berinde-Păcurar [7].

Remark 3.4. With similar proof as the one given for Theorem 3.1, we have the following result.

Theorem 3.5. *We suppose that:*

(1) f_G *is a* ψ*-weakly Picard mapping;*
(2) $d(x, f_G(x)) \leq cd(x, f(x))$, $\forall\ x \in X$ *with some* $c \in \mathbb{R}_+^*$*;*
(3) $d(g(x), f(x)) \leq \eta$, $\forall\ x \in X$ *with some* $\eta \in \mathbb{R}_+^*$.

Then we have that:

(*i*) $d(x, f_G^\infty(x)) \leq \psi(cd(x, f(x)))$, $\forall\ x \in X$*;*
(*ii*) *if* $x^* \in F_f$*, then* $d(y^*, x^*) \leq \psi(c\eta)$, $\forall\ y^* \in F_g \cap X_{x^*}$*, where* $X = \bigcup_{x^* \in F_f} X_{x^*}$ *is a fixed point partition of* X *corresponding to the retraction* f_G^∞.

3.2. (UH) problem

Let (X, d) be a metric space, $f : X \to X$ be a mapping and $f_G : X \to X$ be an admissible perturbation of f. For $\varepsilon \in \mathbb{R}_+^*$ we consider the inequation

$$d(y, f(y)) \leq \varepsilon.$$

Let y^* be a solution of this inequation. We suppose that f_G is a ψ-weakly Picard mapping and

$$d(x, f_G(x)) \leq cd(x, f(x)),\ \forall\ x \in X,\ \text{with some } c \in \mathbb{R}_+^*.$$

There exists $x^* \in F_f$ such that $y^* \in X_{x^*}$. For a such x^* we have that

$$d(y^*, x^*) \leq \psi(c\varepsilon).$$

So, we have the following result.

Theorem 3.6. *In the above conditions the fixed point equation,* $x = f(x)$ *is Ulam-Hyers stable.*

3.3. (WP) problem

By standard proof (see [56], [57]) and the above considerations, we have the following result for this problem.

Theorem 3.7. *Let* (X, d) *be a metric space,* $f : X \to X$ *be a mapping and* f_G *be an admissible perturbation. We suppose that:*

(1) f_G *is* ψ*-weakly Picard mapping;*
(2) $d(x, f_G(x)) \leq cd(x, f(x))$, $\forall\ x \in X$*, for some* $c \in \mathbb{R}_+^*$.

Then the fixed point problem for f *is well-posed.*

4. Notion of quasicontraction and (OP) problem

4.1. Quasicontractions

In [8] the following definition is given:

Let (X,d) be a metric space and $f : X \to X$ be a mapping with $F_f \neq \varnothing$. By definition f is a quasicontraction if there exists $l \in]0,1[$ such that

$$d(f(x),x^*) \leq ld(x,x^*),\ \forall\, x \in X,\ \forall\, x^* \in F_f.$$

It is clear that if f is a quasicontraction then $F_f = \{x^*\}$.

If $F_f \neq \varnothing$ and $r : X \to F_f$ is a set-retraction then we have the following definition.

Definition 4.1. Let (X,d) be a metric space, $f : X \to X$ be a mapping with $F_f \neq \varnothing$ and $r : X \to F_f$ be a set retraction. Then f is a quasicontraction with respect to the retraction r if there exists $l \in]0,1[$ such that,

$$d(f(x),r(x)) \leq ld(x,r(x)),\ \forall\, x \in X.$$

For example, if f is a weakly Picard mapping then f is a quasicontraction if,

$$d(f(x),f^{\infty}(x)) \leq ld(x,f^{\infty}(x)),\ \forall\, x \in X.$$

For more considerations on quasicontractions, see: [3], [17], [46], [56], [57], [67], [14], [13].

4.2. (OP) problem

Let (X,d) be a metric space, $f : X \to X$ be a mapping with $F_f \neq \varnothing$ and $r : X \to F_f$ be a set retraction. Let $X = \bigcup_{x^* \in F_f} X_{x^*}$ be the partition of X corresponding to the retraction r. Let $x^* \in F_f$ and $x_n \in X_{x^*}$, $n \in \mathbb{N}$ such that,

$$d(x_{n+1},f(x_n)) \to 0 \text{ as } n \to \infty.$$

Let us suppose that the mapping f is a quasi l-contraction with respect to the retraction r, i.e.,

$$d(f(x),x^*) \leq ld(x,x^*),\ \forall\, x \in X_{x^*},\ \forall\, x^* \in F_f.$$

From this condition we have that,

$$\begin{aligned}
d(x_{n+1},x^*) &\leq d(x_{n+1},f(x_n)) + d(f(x_n),x^*)\\
&\leq d(x_{n+1},f(x_n)) + ld(x_n,x^*)\\
&\leq d(x_{n+1},f(x_n)) + ld(x_n,f(x_{n-1})) + l^2 d(x_{n-1},x^*)\\
&\vdots\\
&\leq d(x_{n+1},f(x_n)) + ld(x_n,f(x_{n-1})) + \ldots + l^n d(x_1,f(x_0)) \to 0,
\end{aligned}$$

as $n \to \infty$, from a Cauchy-Toeplitz lemma [63].

So we have,

Theorem 4.2. *Let (X,d) be a metric space, $f : X \to X$ be a mapping with $F_f \neq \varnothing$ and $r : X \to F_f$ be a set retraction. We suppose that f is a quasicontraction with respect to the retraction r. Then the mapping f has the Ostrowski property.*

For example let f_G be an admissible perturbation of f. If f_G is a weakly Picard mapping and the mapping f is a quasicontraction with respect to f_G^∞, then the mapping f has the Ostrowski property with respect to f_G^∞.

5. Research directions

5.1. To give relevant examples of iterative fixed point algorithms which generate retractions on a fixed point set.

References: [3], [10], [12], [17], [28], [31], [35], [45], [53], [58], [66], [65], [11].

5.2. To give relevant examples of quasicontractions with respect to retractions defined by iterative algorithms.

For theoretical and applicative point of view, from the considerations of this article, the following problems arise:

To give similar results for:

5.3. Zero point equations
References: [16], [43], [19], [3], [35], [55].

5.4. Coincidence point equations
References: [15], [55], [60].

5.5. Equations with nonself mappings
References: [6], [9], [18], [35], [54], [55], [61].

5.6. Equations in $\mathbb{R}_+^m$-metric spaces
References: [35], [47], [61], [48], [56], [63], [27], [34].

5.7. Equations in $s(\mathbb{R}_+)$-metric spaces
References: [68], [56], [57], [61], [63], [27].

5.8. Equations in dislocated metric spaces
References: [31], [51], [24], [25], [29], [2], [1], [5].

5.9. Equations in a set with two metrics
References: [48], [61], [49], [24], [47].

5.10. Equations in a set with an order relation and a metric
References: [41] and the references therein.

5.11. Equations with multivalued mappings
References: [40], [44], [61], [55], [62], [14], [30].

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The Pólya f-curvature of plane curves

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Abstract. We introduce and study a new curvature function for plane curves inspired by the weighted mean curvature of M. Gromov. We call it *Pólya*, being the difference between the usual curvature and the inner product of the normal vector field with the Pólya vector field of a given planar function f. We computed it for several examples, since the general problem of vanishing or constant values of this new curvature involves the general expression of f.

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Keywords: Plane curve, Pólya vector field, f-curvature, reverse potential.

1. Introduction

The last forty years known an intensive research in the area of geometric flows. The most simple of them is the *curve shortening flow* and already the excellent survey [4] is almost twenty years old. Recall that the main geometric tool in this last flow is the well-known curvature of plane curves. Hence, to give a re-start to this problem seams to search for variants of the curvature or in terms of [11], deformations of the usual curvature. The goal of this short note is to propose such a deformation using a type of planar vector fields introduced by George Pólya (1887-1985). The life and research of this brilliant mathematician is exposed in the book [1].

The contents of this paper is as follows. In the following section we introduce our new curvature, using an idea of Mikhael Gromov. This curvature function, denoted k_f, is defined with respect to a given planar function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ through its associated Pólya vector field. Starting from the given curve C we compute k_f in some examples in order to determine the complexity of computation. At this level, due to the generality of function f, it is impossible to determine cases when k_f is zero or another real constant. For the examples of this section we choose in particular a

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holomorphic function, namely the square function $f(z) = z^2$ and hence we denote the corresponding curvature as k_{square}. At the end of the section we use the Fermi-Walker derivative to express k_f.

In the third section we start from the given f and define a notion of *reverse potential* F which involves the paracomplex structure of $\mathbb{R}^2$; hence we change the notation of our introduced curvature in k_F. Now, we can point out cases when k_F is zero or another constant and an interesting example is provided by the harmonic radial function $F(x,y) = \frac{1}{2}\ln(x^2+y^2)$.

2. The Pólya f-curvature for a plane curve

Fix $I \subseteq \mathbb{R}$ an open interval and $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)), \quad \|r'(t)\| > 0, \quad t \in I. \tag{2.1}$$

The ambient setting, namely $\mathbb{R}^2$, is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = u^1v^1 + u^2v^2, \quad u = (u^1, u^2), \quad v = (v^1, v^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2.2}$$

The infinitesimal generator of the rotations in $\mathbb{R}^2$ is the linear vector field, called *angular*:

$$\xi(u) := -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi(u) = i \cdot u = i \cdot (u^1 + iu^2). \tag{2.3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O, R)$:

$$\begin{cases} \gamma^{\xi}_{u_0}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix} = SO(2) \cdot u_0, \\ R = \|u_0\| = \|(u^1_0, u^2_0)\|, t \in \mathbb{R}, R(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2) = S^1 \end{cases} \tag{2.4}$$

and since the rotations are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2$ it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2, g_{can})$. The first integrals of ξ are the Gaussian functions i.e. multiples of the square norm:

$$f_C(x, y) = C(x^2 + y^2), \ C \in \mathbb{R}.$$

For an arbitrary vector field $X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X, \xi] = (yA_x - xA_y - B)\frac{\partial}{\partial x} + (A + yB_x - xB_y)\frac{\partial}{\partial y}$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ commutes with *the radial* (or Euler) vector field:

$$E(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

which is also a complete vector field having as integral curves the homotheties $\gamma^E_{u_0}(t) = e^t u_0$ for all $t \in \mathbb{R}$. The vector field E is the basis of the 1-dimensional annihilator of the Liouville (or tautological) 1-form $\lambda = \frac{1}{2}(-ydx + xdy)$ whose exterior derivative is the area 2-form $dx \wedge dy$. We point out also that the opposite vector field $W = -E$ is exactly the wind in the Zermelo navigation problem corresponding to the Funk metric in the unit disk of $\mathbb{R}^2$, [5]. For an arbitrary Euclidean space $\mathbb{R}^n$ with $n \geq 2$ the radial vector field $E = x^i \frac{\partial}{\partial x^i}$ defines the notion of *horizontal* 1-*form* ρ as satisfying $i_E \rho = 0$ with i_E the interior product.

The Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \dfrac{r'(t)}{\|r'(t)\|} \in S^1, \\ N(t) = i \cdot T(t) = \dfrac{1}{\|r'(t)\|}(-y'(t), x'(t)) \in S^1 \\ k(t) = \dfrac{1}{\|r'(t)\|}\langle T'(t), N(t)\rangle = \dfrac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{2.5}$$

Hence, if C is naturally parametrized (or parametrized by arc-length) i.e. $\|r'(s)\| = 1$ for all $s \in I$ then $r''(s) = k(s)ir'(s)$. In a complex approach based on

$$z(t) = x(t) + iy(t) \in \mathbb{C} = \mathbb{R}^2$$

we have $2\lambda = Im(\bar{z}dz)$ and

$$\begin{cases} k(t) = \dfrac{1}{|z'(t)|^3} Im(\bar{z}'(t) \cdot z''(t)) = \dfrac{1}{|z'(t)|} Im\left(\frac{z''(t)}{z'(t)}\right), \\ Re(\bar{z}'(t) \cdot z''(t)) = \dfrac{1}{2}\dfrac{d}{dt}\|r'(t)\|^2, \quad f_C(z) = C|z|^2. \end{cases} \tag{2.6}$$

This note defines a new curvature function for C inspired by a notion introduced by M. Gromov in [8, p. 213] and concerning with hypersurfaces M^n in a weighted Riemannian manifold $(\tilde{M}, g, f \in C^\infty_+(\tilde{M}))$. More precisely, the *weighted mean curvature* of M is the difference:

$$H^f := H - \langle \tilde{N}, \tilde{\nabla} f\rangle_g \tag{2.7}$$

where H is the usual mean curvature of M and $\tilde{N}$ is the unit normal to M. This curvature was studied in several papers; for example if H^f is the constant $\lambda \in \mathbb{R}$ then M is called λ-*hypersurface* and the influence of a shrinking Ricci soliton on the geometry of such a hypersurface is studied in [2].

Suppose that the geometric image of the given curve is contained in a domain $\Omega \subseteq \mathbb{R}^2$ and we have also a given function $f : \Omega \to \mathbb{R}^2 = \mathbb{C}$, $f = (u, v) = u + iv$ for $u, v \in C^\infty(\Omega)$. This function has an associated vector field, called *Pólya*:

$$V_f := u\frac{\partial}{\partial x} - v\frac{\partial}{\partial y} \tag{2.8}$$

whose Lie bracket with ξ and E is:

$$\begin{cases} [V_f, \xi] = (yu_x - xu_y + v)\dfrac{\partial}{\partial x} + (u - yv_x + xv_y)\dfrac{\partial}{\partial y}, \\ [V_f, E] = (u - xu_x - yu_y)\dfrac{\partial}{\partial x} + (xv_x + yv_y - v)\dfrac{\partial}{\partial y}. \end{cases} \tag{2.9}$$

For details concerning this type of vector fields see [3] and [9]. Hence we follow this path and we consider:

Definition 2.1. The *Pólya f-curvature* of C is the smooth function $k_f : I \to \mathbb{R}$ given by:

$$k_f(t) := k(t) - \langle N(t), V_f(r(t)) \rangle. \tag{2.10}$$

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and, besides its theory of *Myller configuration* ([13]), he generalizes also a type of curvature for space curves in [12]. It is worth to remark that for its meaningfully contribution to the geometry, the Romanian edition (1966) of the book ([13]) has received the "Gheorghe Ţiţeica" Prize of the Romanian Academy in 1968. Obviously, we can present on several pages the enormous contributions of Academician Radu Miron to the theory of space curves (e.g. by extensions of the celebrated Gauss-Bonnet theorem) but due to the planar character of our study we stop here our commemorative discourse.

Returning to our subject we note:

Theorem 2.2. *(i) The expression of the Pólya f-curvature is:*

$$k_f(t) = k(t) + \frac{x'(t)v(x(t), y(t)) + y'(t)u(x(t), y(t))}{\|r'(t)\|}. \tag{2.11}$$

(ii) Moreover:

$$k_f(t) \leq k(t) + \sqrt{[u(x(t), y(t))]^2 + [v(x(t), y(t))]^2} = k(t) + \|V_f(r(t))\| \tag{2.12}$$

with equality if and only if the vector field $V_f \circ r$ is parallel to N but in the opposite direction.
(iii) In particular, if C is an integral curve of V_f then k_f is exactly k.
(iv) If the normal projection of $V_f \circ r$ is invariant with respect to the orientation preserving parameter changes on C then k_f is invariant too, and conversely.
(v) If the angle made by $V_f \circ r$ with the normal is invariant w.r.t. positively oriented isometries then k_f is invariant too, and conversely.

Proof. We have directly:

$$\langle N(t), V_f(r(t)) \rangle = \langle iT(t), V_f(r(t)) \rangle \tag{2.13}$$

and the conclusion (2.11) follows. The inequality (2.12) is the direct application of the CBS inequality. The claimed consequence follows from the ODE system:

$$x' = u,$$
$$y' = -v.$$

□

Theorem 2.3. *With the previous notations, let $I \subseteq \mathbb{R}$ be an open subset and let $h : I \to \mathbb{R}$ be a smooth function. Fix $t_0 \in I$, $(x_0, y_0) \in \mathbb{R}^2$ and an orthonormal pair $\{T_0 \in S^1, N_0 \in S^1\}$ of $\mathbb{R}^2$. Then there exists a maximal open interval $J \subseteq I$ around t_0 and a unique parameterized curve $C : J \to \mathbb{R}^2$, such that $k_f = h$, $C(t_0) = (x_0, y_0)$ and $T(t_0) = T_0$, $N(t_0) = N_0$.*

Proof. This result is an analogue of the fundamental theorem of plane curves ([10], 1.3.6) and the proof is similar. Consider the ODEs system:

$$X'(t) = (h(t) + \langle Y(t), V_f(x(t), y(t))\rangle) \cdot Y(t)$$
$$Y'(t) = -\,(h(t) + \langle Y(t), V_f(x(t), y(t))\rangle) \cdot X(t)$$
$$X(t) = \frac{1}{(x'(t))^2 + (y'(t))^2} \cdot (x'(t), y'(t)),$$
$$Y(t) = \frac{1}{(x'(t))^2 + (y'(t))^2} \cdot (-y'(t), x'(t))$$

with the initial conditions $(x(t_0), y(t_0)) = (x_0, y_0)$ and $(x'(t_0), y'(t_0)) = (T_0, N_0)$. The existence and uniqueness theorem for ODEs ensures there exists a solution $C(t) = (x(t), y(t))$ on a maximal open interval $J \subseteq I$ around t_0. A short computation proves that $\left\{X, Y\right\}$ is the Frenet frame along C and the that the first two formulas of the previous system are the Frenet equations. As the function $(h(t) + \langle Y(t), V_f(x(t), y(t))\rangle)$ must be the curvature function $k = k(t)$ of C, we obtain the relation (2.10), hence the equality $k_f = h$. □

Example 2.4. i) If C is the line $r_0 + tU, t \in \mathbb{R}$ with the vector $U = (U^1, U^2) \neq \bar{0} = (0, 0)$ then k_f is the constant:

$$k_f(t) = \frac{U^1 v(x_0 + tU^1, y_0 + tU^2) + U^2 u(x_0 + tU^1, y_0 + tU^2)}{\|U\|}. \tag{2.14}$$

In particular, if $O \in C$ then

$$k_f(t) = \frac{U^1 v(tU^1, tU^2) + U^2 u(tU^1, tU^2)}{\sqrt{(U^1)^2 + (U^2)^2}}$$

and for $f(z) = z^2$ we have:

$$k_{square}(t) = \frac{U^2[3(U^1)^2 - (U^2)^2]}{\sqrt{(U^1)^2 + (U^2)^2}} t^2. \tag{2.15}$$

ii) If C is the circle $\mathcal{C}(O, R) : r(t) = Re^{it}$ then:

$$k_f(t) = \frac{1}{R} - v(R\cos t, R\sin t)\sin t + u(R\cos t, R\sin t)\cos t. \tag{2.16}$$

For $f(z) = z^2$ we have:

$$k_{square}(t) = \frac{1}{R} + R^2\cos 3t \in \left[\frac{1}{R} - R^2, \frac{1}{R} + R^2\right]. \tag{2.17}$$

iii) For the case of logarithmic spiral expressed in polar coordinates as $\rho_{R,\alpha}(t) = Re^{\alpha t}$, $R, \alpha > 0$ and $t \in \mathbb{R}$ we have the f-curvature:

$$\sqrt{\alpha^2 + 1}k_f(t) = R^{-1}e^{-\alpha t} + (\alpha\cos t - \sin t)v(Re^{\alpha t}\cos t, Re^{\alpha t}\sin t)$$
$$+(\alpha\sin t + \cos t)u(Re^{\alpha t}\cos t, Re^{\alpha t}\sin t) \tag{2.18}$$

and for $\alpha \to 0$ we re-obtain the f-curvature of the circle $\mathcal{C}(O, R)$. Again for $f(z) = z^2$ we have:

$$\sqrt{\alpha^2 + 1}k_{square}(t) = R^{-1}e^{-\alpha t} + R^2 e^{2\alpha t}[\cos 3t + \alpha\sin 3t]. \tag{2.19}$$

In the following since the problem of vanishing or of constant values for k_f can not be treated due to the generality of f we continue to present some concrete examples in order to remark the computational aspects of our approach.

Example 2.5. We study completely a curve with non-constant rotational curve. Namely, the involute of the unit circle S^1 is:

$$C : r(t) = (\cos t + t \sin t, \sin t - t \cos t) = (1 - it)e^{it}, \quad t \in (0, +\infty). \tag{2.20}$$

A direct computation gives:

$$r'(t) = (t \cos t, t \sin t) = te^{it}, \quad k(t) = \frac{1}{t} > 0, \quad \|r'(t)\| = t \tag{2.21}$$

and then the f-curvature is:

$$k_f(t) = \frac{1}{t} + v(\cos t + t \sin t, \sin t - t \cos t) \cos t + u(\cos t + t \sin t, \sin t - t \cos t) \sin t \tag{2.22}$$

which for $f(z) = z^2$ becomes:

$$k_{square}(t) = \frac{1}{t} + 3(1 - t^2) \sin t \cos^2 t - 2t \cos^3 t - \sin^3 t + 6t \sin^2 t \cos t. \tag{2.23}$$

Example 2.6. For the square function $f(z) = z^2$ the integral curves of its Pólya vector field are the solutions of the ODE system:

$$\dot{x} = x^2 - y^2, \quad \dot{y} = -2xy \tag{2.24}$$

having the first integral:

$$F_{square}(z = x + iy) = 3x^2 y - y^3 = Im(z^3). \tag{2.25}$$

Fix then a arbitrary real number $a \neq 0$; the implicit plane curve:

$$C(a) : F(z) = a \tag{2.26}$$

has the usual curvature:

$$k(C(a)) = -\frac{a}{27(x^2 + y^2)^2}. \tag{2.27}$$

We end this section with an approach in terms of Fermi-Walker derivative. Let $\mathcal{X}_\gamma$ be the set of vector fields along the curve γ. Then the Fermi-Walker derivative is the map ([7]) $\nabla^{FW}_\gamma : \mathcal{X}_\gamma \to \mathcal{X}_\gamma$:

$$\nabla^{FW}_\gamma(X) := \frac{d}{dt} X + k\|r'(\cdot)\|[\langle X, N\rangle T - \langle X, T\rangle N] = \frac{d}{dt} X + k[X^\flat(N)T - X^\flat(T)N] \tag{2.28}$$

with $X^\flat$ the differential 1-form dual to X with respect to the Euclidean metric. For $X = V_f \circ r$ we have:

$$\nabla^{FW}_\gamma(V_f \circ r)(t) = \frac{d}{dt} V_f(r(t)) + \|r'(t)\|k(t)[\langle V_f \circ r(t), N(t)\rangle T(t) - \langle V_f \circ r(t), T(t)\rangle N(t)] \tag{2.29}$$

and then we restrict to the tangential component of this equation:

$$\langle(\nabla^{FW}_\gamma V_f \circ r)(t) - \frac{d}{dt} V_f(r(t)), T(t)\rangle = \|r'(t)\|k(t)\langle V_f \circ r(t), N(t)\rangle. \tag{2.30}$$

Hence, if C is not a line we have:

$$k_f(t) = k(t) - \frac{\langle (\nabla_\gamma^{FW} V_f \circ r)(t) - \frac{d}{dt} V_f(r(t)), T(t) \rangle}{\|r'(t)\| k(t)}. \tag{2.31}$$

3. A reverse potential for f and the corresponding Pólya curvature

Usually, the smooth function $F \in C^\infty(\Omega)$ is called *a potential of* f if the gradient relation holds $f = \nabla F$ which means $u = F_x$ and $v = F_y$. But for our formulae (2.11) another object seems more naturally:

Definition 3.1. F is a *reverse-potential* of f if $u = F_y$ and $v = F_x$.

In a matrix form we express this condition as:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Gamma \cdot \nabla F, \quad \Gamma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Sym(2). \tag{3.1}$$

We point out that since $\Gamma^2 = I_2$ and $\dim Ker(I_2 + \Gamma) = \dim Ker(I_2 - \Gamma) = 1$ the endomorphism Γ is exactly the paracomplex structure of the plane $\mathbb{R}^2$, [6]. The kernel of $I_2 + \Gamma$ is the second bisectrix $B_2 : x + y = 0$ while the kernel of $I_2 - \Gamma$ is the first bisectrix $B_1 : x - y = 0$. The paracomplex structure Γ and the complex structure $J := R\left(\frac{\pi}{2}\right)$ of the plane commute:

$$\Gamma \cdot J = J \cdot \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = diag(1, -1). \tag{3.2}$$

In fact, in [9, p. 5] there is another vector field associated to f, namely

$$V_f^\perp := -V_{if} = v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}$$

and hence if F is a reverse potential of f then its gradient is exactly $V_f^\perp$.

It results immediately that our considered curvature, denoted now k_F, is:

$$k_F(t) = k(t) + \frac{1}{\|r'(t)\|}\frac{d}{dt}F(r(t)), \quad k_F(t) \leq k(t) + \|\nabla F(r(t))\| \tag{3.3}$$

since $\|V_f\| = \|\nabla F\|$.

Remark 3.2. An useful formalism is that of [14, p. 2]; if $r : S^1 \simeq [0, 2\pi) \to \mathbb{R}^2$ is naturally parametrized then there exists the smooth function $\theta : S^1 \to \mathbb{R}$, called *normal angle*, such that:

$$N(s) = e^{i\theta(s)} = (\cos\theta(s), \sin\theta(s)), \quad T(s) = -iN(s) = -ie^{i\theta(s)} = e^{i(\theta(s) - \frac{\pi}{2})} \tag{3.4}$$

and then the Frenet equations yield:

$$\frac{d\theta}{ds}(s) = k(s). \tag{3.5}$$

Then k_F is a derivative:

$$k_F(s) = \frac{d}{ds}\left(\theta(s) + F(r(s))\right) \tag{3.6}$$

and hence k_F is vanishing if and only if the function $\theta + F \circ r$ is a constant.

Example 3.3. Suppose that f is a holomorphic function i.e. its real and imaginary components satisfy the Cauchy-Riemann equations: $u_x = v_y$, $u_y = -v_x$. If f is provided by the reverse potential F then the first equation holds directly while the second equation implies the harmonicity of F i.e. the vanishing of the Euclidean Laplacian: $\Delta F = 0$. If we restrict the class of F to radial (i.e. S^1-invariant) ones $F = \tilde{F}(x^2+y^2)$ we have the solution $F(x,y) = \frac{1}{2}\ln(x^2+y^2) = \frac{1}{2}\ln f_1(x,y)$ for $0 \notin \Omega$ and then:

$$\begin{cases} f(z) = \dfrac{i}{z} = \frac{y}{x^2+y^2} + i\dfrac{x}{x^2+y^2} = \left(\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right), \\ V_f = \dfrac{y}{x^2+y^2}\dfrac{\partial}{\partial x} - \dfrac{x}{x^2+y^2}\dfrac{\partial}{\partial y} \\ k_F(t) = k(t) + \dfrac{\langle r(t), r'(t)\rangle}{\|r(t)\|^2\|r'(t)\|} \leq k(t) + \dfrac{1}{\|r(t)\|}. \end{cases} \tag{3.7}$$

The circles $\mathcal{C}(O,R) : r(t) = Re^{it}$ are exactly the integral curves of V_f and applying the last part of proposition 2.2 we get: $k_F(t) = k(\mathcal{C}(O,R)) = \frac{1}{R}$=constant. For the more general example of logarithmic spiral $r(t) = Re^{i\alpha t}$, $\alpha > 0$ we obtain:

$$k_F(t) = \frac{\alpha+1}{Re^{\alpha t}\sqrt{\alpha^2+1}}, \quad \lim_{\alpha\to 0} k_F = k(\mathcal{C}(O,R)). \tag{3.8}$$

We have

$$V_f^{\perp}(x,y) = \frac{1}{\|(x,y)\|^2}E(x,y)$$

and then

$$\|V_f\| = \|V_f^{\perp}\| = \frac{1}{\sqrt{x^2+y^2}}.$$

For a harmonic function f the Lie brackets (2.9) can be expressed only with the partial derivatives of u:

$$\begin{cases} [V_f, \xi] = (yu_x - xu_y + v)\dfrac{\partial}{\partial x} + (u + xu_x + yu_y)\dfrac{\partial}{\partial y}, \\ [V_f, E] = (u - xu_x - yu_y)\dfrac{\partial}{\partial x} + (uu_x - xu_y - v)\dfrac{\partial}{\partial y} \end{cases} \tag{3.9}$$

and then V_f commutes with ξ while $[V_f, E] = 2V_F$, equality which follows also from the (-1)-homogeneity of coefficients of f.

4. Pólya related curves

Let $f(x,y) = u(x,y) + iv(x,y)$ be an arbitrary function on the complex plane and $C : I \to \mathbb{R}^2$ be a regular parameterized curve, as in Section 2. Denote by k and k_f the curvature function and the Pólya curvature function of C, respectively. From the fundamental theorem of the theory of plane curves, we know there exists a regular parameterized curve $\tilde{C} : I \to \mathbb{R}^2$, whose curvature $\tilde{k}$ is exactly k_f; moreover, this curve is unique, up to a positively oriented isometry and an orientation preserving parameter change.

Definition 4.1. We say $\tilde{C}$ is *the Pólya mate* of C w.r.t. the function f.

Example 4.2. Let again $C = \mathcal{C}(O,R) : r(t) = Re^{it}$ and consider $f(z) = \bar{z}$. Then, from the formula (2.16) it results $k_f = \frac{1}{R} + R$ and then $\tilde{C} = \mathcal{C}(O,\tilde{R})$ is the Pólya mate of C for:

$$\tilde{R} = \frac{R}{R^2+1} \leq \min\{\frac{1}{2}, R\}. \tag{4.1}$$

Continuing this process with the fixed f we obtain the Pólya mate of $\tilde{C}$ as being the circle $\hat{C} = \mathcal{C}(O,\hat{R})$ with:

$$\hat{R} = \frac{R(R^2+1)}{R^2+(R^2+1)^2} \tag{4.2}$$

which proves that the "Pólya mate" relation for a fixed f is not a symmetric one in general.

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A polynomial algorithm for some instances of NP-complete problems

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Abstract. In this paper, given a fixed reference point and a fixed intersection of finitely many equal radii balls, we consider the problem of finding a point in the said set which is the most distant, under Euclidean distance, to the said reference point. This proble is NP-complete in the general setting. We give sufficient conditions for the existence of an algorithm of polynomial complexity which can solve the problem, in a particular setting. Our algorithm requires that any point in the said intersection to be no closer to the given reference point than the radius of the intersecting balls. Checking this requirement is a convex optimization problem hence one can decide if running the proposed algorithm enjoys the presented theoretical guarantees. We also consider the problem where a fixed initial reference point and a fixed polytope are given and we want to find the farthest point in the polytope to the given reference point. For this problem we give sufficient conditions in which the solution can be found by solving a linear program. Both these problems are known to be NP-complete in the general setup, i.e the existence of an algorithm which solves any of the above problem without restrictions on the given reference point and search set is undecided so far.

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1. Introduction

In this paper we begin by presenting a novel framework for asserting the feasibility of the intersection of convex sets. Our approach is to synthesize the information

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in the given convex sets in a non-smooth convex function whose unconstrained minimizer can be used to assert the feasibility of the intersection. This problem is known in the literature as the so called "convex feasibility problem". Classic algorithms for this problem exist and can be found in [16], [18], [17], [19], while more novel approaches are found here [5], [13]. Our approach to this problem is the presentation of a simple and elegant criterion for asserting the feasibility of the intersection of two convex sets. Unlike (some of) the references above, we do not focus on the convex minimization problem itself, but on the formation of the convex function to be minimized and on the interpretation of the resulting minimizer.

Next we extend the presented method to a particular case of mathematical programming: the assertion of the inclusion of an intersection of equal radii balls in another, bigger, ball. We are able to give meaningful results under some requirements regarding the distance between the center of the bigger ball and the the intersection of the balls.

We will use throughout the paper the symbol $d(\cdot, \times)$ where $\cdot$ can be a point and $\times$ can be a point or a convex set of points, to designate the Euclidean distance between $\cdot$ and $\times$. For a vector $u \in \mathbb{R}^n$, $u = (u_1, ..., u_n)^T$ and $r > 0$, we denote by $\mathcal{B}(u, r)$ the open ball centered at u and of radius r and we denote by

$$\bar{\mathcal{B}}(u, r) = \{x \in \mathbb{R}^n | \|x - u\| \leq r\}$$

the closed ball centered in u and of radius r. We also denote by $\|u\|$, $\|u\|^2 = u^T u$, the Euclidean norm of the vector u.

Finally, for a function $f : \mathbb{R}^n \to \mathbb{R}$ we denote by

$$f^+(x) = \max\{f(x), 0\} \qquad f^-(x) = \min\{f(x), 0\} \tag{1.1}$$

Note that $f(x) = f^+(x) + f^-(x)$.

1.1. Convex domains of interest

Let $x \in \mathbb{R}^n$, $n, m \in \mathbb{N}_+$ and let $g_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions for $k \in \{1, \ldots, m\}$. We define the convex sets:

$$S_k = \left\{ x \in \mathbb{R}^n \middle| g_k(x) \leq 0 \right\}$$

and we are interested if the set

$$\mathcal{S} = \bigcap_{k=1}^{m} S_k \tag{1.2}$$

is empty or not.

For this we define the following function $\tilde{G}(x) : \mathbb{R}^n \to \mathbb{R}$:

$$\widetilde{G}(x) = \sum_{k=1}^{m} g_k^+(x)$$

2. Main results

In this section we present a novel feasibility criteria for the finite intersection of certain convex sets. One classic method from the literature for solving this problem is the method of alternating projections, [4], [13], for finding a feasible solution in the intersection of convex sets. Below, we give a projection-free method for solving set intersection problems. Our approach reformulates the feasibility problem as a non-smooth convex minimization problem.

2.1. Convex feasibility

The following result is a characterization of the set $\mathcal{S}$ in terms of a global minimum of $\widetilde{G}(x)$.

Lemma 2.1. *Let*

$$x^\star \in \underset{x\in\mathbb{R}^n}{\operatorname{argmin}}\, \widetilde{G}(x). \tag{2.1}$$

Then the following are equivalent:

1. *The set $\mathcal{S}$ is not empty, i.e $\exists\, x_0 \in \mathbb{R}^n$ such that*

$$g_k(x_0) \leq 0 \qquad \forall k \in \{1,\ldots,m\}$$

2. *The point $x^\star$ defined by (2.1) satisfies*

$$g_k(x^\star) \leq 0 \qquad \forall k \in \{1,\ldots,m\}$$

Proof. The part $2 \Rightarrow 1$ follows immediately from $g_k(x^\star) \leq 0$ for all $k \in \{1,\ldots,m\}$ which implies $x^\star \in \mathcal{S}$ and therefore $\mathcal{S} \neq \emptyset$. To prove $1 \Rightarrow 2$, let x_0 such that $g_k(x_0) \leq 0$ for all $k \in \{1,\ldots,m\}$ and assume that $\exists k$ such that $g_k(x^\star) > 0$. This implies

$$0 = \widetilde{G}(x_0) < \widetilde{G}(x^\star)$$

which contradicts the fact that $x^\star$ is a global minimum of $\widetilde{G}$. □

Remark 2.2. The simple result above shows that the feasibility of the intersection of m convex sets (sub-level sets of convex functions) can be asserted by examining the global minimum of a non-smooth convex function.

Encouraged by the simplicity of the above result we propose a somewhat similar approach to study the following problem: assert if a fixed intersection of finitely many equal radii balls is included in another given ball.

2.2. Test for the inclusion of an intersection of balls into another ball

We want to solve the following non-convex optimization problem:

$$\begin{aligned} \max \quad & \|x-c\|^2 \\ \text{s.t} \quad & \|x-c_k\|^2 \leq R^2, \quad \forall k \in \{1,\ldots,m\}, \end{aligned} \tag{2.2}$$

where $c_k, c \in \mathbb{R}^n$ and $R \in \mathbb{R}$, $R > 0$. Problem (2.2) is equivalent to finding a point in the intersection of the balls centered at c_k and of radius R which is the farthest away from the point c. Please note that for any polytope one can choose c_k and R in such a way that the intersection of the balls provide an approximation of the polytope.

Although we will not expand this approximation here, this is the main reason for considering problem (2.2).

It is obviously a quadratically constrained quadratic maximization problem. Algorithms for such, or similar problem, have been proposed in the literature, see [7], [12], [15], [1]. These treat a similar problem, i.e optimizing a quadratic function with box constraints. The *S-procedure*, [10], is a well known algorithm for solving programs with quadratic objective and quadratic constraints. However, the presented problem is fundamentally different to the problems which the S-procedure can solve in polynomial time. That is, we are interested if an intersection of more balls is included in another ball, whereas the S-procedure can be used for testing ellipsoid containment, i.e to assert if an ellipsoid is included in another. The S-procedure cannot be used to assert if an intersection of ellipsoids is included in another ellipsoid. Also, the presented problem is fundamentally different to the sphere/ellipsoid packing problem, as we are not interested in finding the maximum number of non-overlapping spheres/ellipsoids which can be included in a given sphere/ellipsoid. In our case all the geometrical objects (the balls) are fixed and given. We are just supposed to answer with YES or NO to the question: "is the intersection of the these given balls included in this other ball?". Here is is worth noting the work done in [6] which finds the smallest ball enclosing an intersection of balls. This problem is somewhat similar to ours as one would, in absence of other choices, propose an "approximate" solution to our problem by simply computing the smallest ball enclosing the intersection of balls, then asserting if that is or not included in the bigger ball. Unfortunately, in [6] the number of intersecting balls is required to be strictly smaller than the dimension of the search space. Finally the work presented here [3] treats a slightly more general problem to what we will be discussing in the next section, i.e maximizing a quadratic function over an intersection of half spaces. However, we limit ourselves to analyzing the simpler to understand problem of maximizing the distance to an external point over an intersection of half spaces. The authors of [3] approach is to cover the search space with ellipsoids then to maximize over each to finally obtain an approximation to the initial problem. Unfortunately, covering the search space (or at least its frontier) with small enough ellipsoids (as required by the precision requirements) requires an exponential number of ellipsoids [2], so this approach does not seem to be able to provide a polynomial complexity algorithm for arbitrary small tolerances.

Our approach is different to those presented above and focuses on solving a non-smooth minimization problem.

Given $R, r > 0$, we consider the following sets:

$$\begin{aligned}
\mathcal{B}_0 &= \overline{B}(c, r) = \left\{ x \in \mathbb{R}^n \middle| \|x - c\|^2 \leq r^2 \right\}, \\
\mathcal{B}_k &= \overline{B}(c^k, R) = \left\{ x \in \mathbb{R}^n \middle| \|x - c_k\| \leq R^2 \right\}, \\
\mathcal{C}_1 &= \bigcap_{k=1}^{m} \mathcal{B}_k, \qquad \mathcal{C}_0 = \mathcal{B}_0 \qquad (2.3)
\end{aligned}$$

. In order to solve the problem (2.2), we keep R fixed and design a test which can assert if $\mathcal{C}_1 \subseteq \mathcal{C}_0$ for various values of r.

We start by defining the functions $f, f_k : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} f_k(x) &= \|x - c_k\|^2 - R^2 \\ f(x) &= \|x - c\|^2 - r^2 \end{aligned} \tag{2.4}$$

and the function $G_k : \mathbb{R}^n \to \mathbb{R}$, given by

$$G_k(x) = f_k(x) - f^-(x) + \sum_{i=1, i\neq k}^{m} f_i^+(x)$$

for $k \in \{1, ..., m\}$.

Remark 2.3. It can be seen that G_k is a convex function. First the *"sum"*–term $\sum_{i=1,i\neq k}^{m} f_i^+(x)$ is convex, since each term in the sum is convex. On the other hand, the remaining term of $G_k(x)$, namely $f_k(x) - f^-(x)$, can be written as

$$f_k(x) - f^-(x) = f_k(x) - f(x) + f(x) - f^-(x) = f_k(x) - f(x) + f^+(x)$$

which is convex since it is the sum of the convex function $f^+(x)$ and the affine function $f_k(x) - f(x) = \|x - c_k\|^2 - \|x - c\|^2 - R^2 + r^2 = (c - c_k)^T \cdot (2 \cdot x - c - c_k) - R^2 + r^2$.

We take $G(x)$ to be the maximum of $G_k(x)$, when k ranges from 1 to m. That is,

$$G(x) = \max \left\{ G_k(x) \middle| k \in \{1, \dots, m\} \right\} = \max_{k=\overline{1,m}} G_k(x)$$

Remark 2.4. We note that, since $G : \mathbb{R}^n \to \mathbb{R}$ is defined as the pointwise maximum of the convex functions $G_k : \mathbb{R}^n \to \mathbb{R}$, it follows that G is convex.

Finally we use $x^\star$, a global minimizer of $G(x)$, i.e.,

$$x^\star \in \underset{x\in\mathbb{R}^n}{\operatorname{argmin}} \;\; G(x) \tag{2.5}$$

Before giving our main result, we present a few simple, but usefull lemmas.

Lemma 2.5. *Let $a, b \in \mathbb{R}^n$ and $r > 0$ such that $b \notin B(a, r)$. Then $\forall x \in B(a, r)$ the inequality*

$$(x - b)^T (a - b) > 0$$

holds.

Proof. Using the Euclidean norm properties over $\mathbb{R}^n$, we write

$$\begin{aligned} \|x - a\|^2 &= \|(x - b) + (b - a)\|^2 \\ &= \|x - b\|^2 + \|b - a\|^2 - 2(x - b)^T (a - b). \end{aligned} \tag{2.6}$$

For $x \in B(a, r)$, $b \notin B(a, r)$, we have $\|x - a\|^2 < r^2$ and $\|b - a\|^2 \geq r^2$. Combining these together with $\|x - b\|^2 \geq 0$ in (2.6), leads to $(x - b)^T (a - b) > 0$ and concludes the proof. □

Lemma 2.6. *Let $x \in \mathcal{C}_1$, with $\mathcal{C}_1$ defined by (2.3). Then for $y \in \mathbb{R}^n$ such that $d(y, \mathcal{C}_1) > R$ one has*

$$(x - y)^T (c_k - y) > 0, \;\; \forall k \in \{1, \dots, m\}$$

Proof. For $x \in \mathcal{C}_1$, one has $d(x, c_k) \leq R$ and therefore $c_k \in B(x, R)$. From $d(y, \mathcal{C}_1) > R$, it follows that $d(x, y) > R$, hence $y \notin B(x, R)$. Applying 2.5, with $a := x$, $b := y$, $r := R$, and $x := c_k$, one obtains the desired conclusion. □

Lemma 2.7. *Let $z, y, c^1, c^2 \in \mathbb{R}^n$ with $\|y - c_1\| = \|y - c_2\|$. Assume, without loss of generality, that $\|z - c_1\|^2 \geq \|z - c_2\|^2$ then*

$$\|y + t(z - y) - c_1\|^2 \geq \|y + t(z - y) - c_2\|^2, \qquad \forall t \geq 0.$$

Proof. Let

$$h(t) = \|y + t(z - y) - c_1\|^2 - \|y + t(z - y) - c_2\|^2.$$

From the identity above, it can be seen that $h(t)$ is a polynomial of degree at most 1 in t. Since $\|y - c_1\| = \|y - c_2\|$ gives $h(0) = 0$ and $\|z - c_1\| \geq \|z - c_2\|$ gives $h(1) \geq h(0) = 0$, it follows that $h(t)$ is a non-decreasing first order polynomial in t and therefore

$$h(t) \geq 0 = h(0), \ \ \forall t \geq 0,$$

which completes the proof. □

Lemma 2.8. *Let $y, c^1, \ldots, c^m \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ such that $\|v\| = 1$. Let $p \in \{1, ..., m-1\}$ be such that*

$$\|y - c_i\| = \|y - c_j\| > \|y - c_l\| \tag{2.7}$$

for all $i, j \in \{1, \ldots, p\}$ and $l \in \{p + 1, \ldots, m\}$. Then $\exists k_v \in \{1, ..., p\}$ and $\delta_v > 0$ such that for all $i \in \{1, \ldots, m\}$ one has

$$\|y + tv - c_{k_v}\| \geq \|y + tv - c^i\| \qquad \forall t \in (0, \delta_v), \tag{2.8}$$

which is stating that there is a small segment starting at y in the direction of v, such that for all the points on this segment, c^{k_v} remains the furthest away. For the case $p = m$, (2.8) holds without any additional requirements.

Proof. First, we consider the case $p \in \{1, ..., m - 1\}$. We define

$$\rho := \|y - c_1\| = \ldots = \|y - c_p\|.$$

Let $\delta > 0$ and $z \in B(y, \delta)$. The triangle inequality gives

$$\begin{aligned} \|z - c_k\| &\geq \|c_k - y\| - \|z - y\|, \\ \|z - c_i\| &\leq \|c_i - y\| + \|z - y\|. \end{aligned}$$

Using the above inequalities with arbitrary $k \in \{1, ..., p\}$ and $i \in \{p + 1, ..., m\}$, gives

$$\begin{cases} d(z, c_k) \geq \rho - \delta, \\ d(z, c_i) \leq \eta + \delta. \end{cases} \tag{2.9}$$

where $\eta = \|y - c^i\| < \|y - c^k\| = \rho$ Following (2.9), we will pick $\delta > 0$ such that $\rho - \delta > \eta + \delta$. Since (2.7) implies $\rho - \eta > 0$, it follows that any $\delta \in \left(0, \frac{\rho - \eta}{2}\right)$ will satisfy this requirement. Thus, for any $\delta \in \left(0, \frac{\rho - \eta}{2}\right)$ and any $z \in B(y, \delta)$, we have

$$d(z, c_k) > d(z, c_i) \quad \ \forall k \in \{1, \ldots, p\}, \ \forall i \in \{p + 1, \ldots, m\}. \tag{2.10}$$

Let $\delta_v = \frac{\delta}{2}$, $z = y + \delta_v v$ and $k_v \in \underset{k\in\{1,...,p\}}{\operatorname{argmax}} \|z - c_k\|$. For the points c_k, $k \in \{1, ..., p\}$, we apply Lemma 2.7 to obtain

$$\|y + (t\delta_v)v - c_{k_v}\|^2 \geq \|y + (t\delta_v)v - c_k\|^2, \ \forall t \geq 0, \ \forall k \in \{1, ..., p\}. \tag{2.11}$$

On the other hand, for the points c_i, $i \in \{p+1, ..., m\}$ we let $z := y + tv$ in (2.10) which gives

$$\|y + tv - c_{k_v}\|^2 > \|y + tv - c_i\|^2 \ \ \forall i \in \{p+1, ..., m\}, \ \forall t \in (0, \delta_v). \tag{2.12}$$

Combining (2.11) and (2.12) leads to the desired conclusion (2.8). For the case $p = m$, (2.8) follows immediately. □

The following theorem represents our **main result**. This is a localization result for $x^\star$ using the balls intersection denoted by $\mathcal{C}_1$ and the *"outside"* ball denoted $\mathcal{C}_0$.

Theorem 2.9. *For $R, \mathcal{C}, \mathcal{C}_1$ defined by (2.3), if $d(C, \mathcal{C}_1) > R$ then*

$$\mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \neq \emptyset \iff x^\star \in \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \tag{2.13}$$

where $x^\star$ is defined by (2.5)

Proof. Clearly the implication $x^\star \in \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \Rightarrow \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \neq \emptyset$ is trivial. We now assume that $\mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0) \neq \emptyset$ and first show that in such a case $x^\star \in \mathcal{C}_1$.

Indeed, for $x \notin \mathcal{C}_1$ $(= \bigcap_{k=1}^m \mathcal{B}_k)$ i.e. it is not in the intersection of congruent balls, follows that $\|x - c_k\| > R^2$ for some $k \in \{1, ..., m\}$ or equivalently $f_k(x) > 0$ for some $k \in \{1, ..., m\}$. From the definitions of f^- and f_i^+, we have $-f^-(x) \geq 0$ and $f_i^+(x) \geq 0$. Combining this with $f_k(x) > 0$, leads to the fact that for $x \notin \mathcal{C}_1$ we have $G_k(x) > 0$, hence $G(x) = \max_{k\in\{1,...,m\}} G_k(x) > 0$ as well. On the other hand if $x \in \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0)$, we have $-f^-(x) = 0$, $f_k(x) \leq 0$, $\forall k \in \{1, ..., m\}$, implying $G(x) \leq 0$ therefore $x^\star$, a minimizer of G, is not outside of $\mathcal{C}_1$ since there are "better" points in $\mathcal{C}_1$.

From the observations above, it follows that $x^\star \in \mathcal{C}_1$. Next, we will show that $x^\star \notin \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$, leading to the desired conclusion. Let $y \in \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$. It follows that there exists $\delta_y > 0$ such that $B(y, \delta_y) \subseteq \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$. We can assume without loss of generality that $\exists p \in \{1, ..., m-1\}$ such that

$$\|y - c_1\| = ... = \|y - c_p\| > \|y - c_l\|, \ \forall l \in \{p+1, ..., m\}.$$

This implies

$$G(y) = G_1(y) = ... = G_p(y).$$

From Lemma 2.8 follows that $\forall\, v \in \mathbb{R}^n$ with $\|v\| = 1$, $\exists k_v \in \{1, ..., p\}$ and $\delta_v > 0$ such that

$$G(y + tv) = G_{k_v}(y + tv) \qquad \forall t \in [0, \delta_v). \tag{2.14}$$

Let $\delta := \min\{\delta_y, \delta_v\}$, $v = \frac{y-c}{\|y-c\|}$ and $z = y + \frac{\delta}{2}$. Clearly $z \in \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$. Let $h(t) := G(y + tv)$, $\forall t \in [0, \delta_v)$. From (2.14), it follows that $h(t) = G_{k_v}(y + tv)$, or equivalently

$$h(t) = r^2 - \|y - c + tv\|^2 + \|y - c_{k_v} + tv\|^2 - R^2, \ \forall t \in [0, \delta_v). \tag{2.15}$$

Differentiating (2.15) with respect to t gives

$$\begin{aligned} h'(t) &= -(y-c+tv)^T\, v + (y - c_{k_v} + tv)^T v \\ &= -(c_{k_v} - c)^T \frac{y-c}{\|y-c\|}. \end{aligned} \tag{2.16}$$

Since $d(c, \mathcal{C}_1) > R$, it follows from Lemma 2.6 and (2.16) that $h'(t) < 0$, $\forall t \in [0, \delta_v)$ implying that $h(t)$ is strictly decreasing. Therefore $h(0) > h(\frac{\delta}{2})$, which is equivalent to $G(z) < G(y)$, for $z = y + \frac{\delta}{2}v \in \operatorname{int}(\mathcal{C}_0 \cap \mathcal{C}_1)$. It follows that $x^\star \in \underset{x\in\mathbb{R}^n}{\operatorname{argmin}}\, G(x) \notin \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$. Since $\mathcal{C}_1$ can be partitioned as

$$\mathcal{C}_1 = \mathcal{C}_1 \setminus \mathcal{C}_0 \cup \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0) \cup \partial(\mathcal{C}_1 \cap \mathcal{C}_0)$$

and we showed that $x^\star \in \mathcal{C}_1$, $x^\star \notin \operatorname{int}(\mathcal{C}_1 \cap \mathcal{C}_0)$, we have

$$\begin{aligned} x^\star &\in \mathcal{C}_1 \setminus \mathcal{C}_0 \cup \partial(\mathcal{C}_1 \cap \mathcal{C}_0) \\ &\subseteq \mathcal{C}_1 \setminus \mathcal{C}_0 \cup \partial\mathcal{C}_0, \end{aligned}$$

implying that $x^\star \in \mathcal{C}_1 \setminus \operatorname{int}(\mathcal{C}_0)$. This concludes our proof. □

2.3. Complexity Analysis

Theorem 2.9 allows one to solve (2.2) if $d(c, \mathcal{C}_1) > R$. Indeed, let $x_0 \in \mathcal{C}_1$ (this can be found initially by the use of Section 2.1 assuming that $\mathcal{C}_1 \neq \emptyset$). Then one can show that $\mathcal{C}_1 \subseteq B(x_0, 2R)$. Let $\underline{r} = R$ and $\bar{r} = 2R + \|x_0 - c\|$. It is obvious that $\mathcal{C}_1 \setminus B(c, \underline{r}) \neq \emptyset$ and $\mathcal{C}_1 \setminus B(c, \bar{r}) = \emptyset$.

We can now search for $r^\star \in [\underline{r}, \bar{r}]$ such that $\mathcal{C}_1 \setminus B(c, r^\star - \epsilon) \neq \emptyset$ and $\mathcal{C}_1 \setminus B(c, r^\star + \epsilon) = \emptyset$ for some arbitrarily fixed precision $\epsilon > 0$, using Theorem 2.9 and the bisection algorithm.

From the computational complexity point of view, each bisection step involves the application of Theorem 2.9 for some $r \in [\underline{r}, \bar{r}]$. For this, one has to solve (2.5) to find $x^\star$. Once $x^\star$ is found, asserting its membership to $\mathcal{C}_1 \setminus B(c, r)$ involves computing $m+1$ distances in $\mathbb{R}^n$, that is $(m+1)n$ flops (for the square of the distances) and comparing them to some real numbers, hence another $m+1$ flops. Finally the computational complexity analysis for each step is completed by analyzing the cost of finding $x^\star$. This basically involves an unconstrained minimization of a continuous, non-differentiable convex function. The starting point can be considered x_0 and the search radius can be taken $2R$. There are various algorithms (of sub-gradient, [9] or ellipsoid type, [18]) which are known to have polynomial deterministic worst case complexity for such a problem. Let Λ (a polynomial in $n, m, \log(R), -\log(\epsilon)$) denote the number of floating point operations required to solve (2.5). Then solving (2.2) requires

$$\mathcal{O}\left((\Lambda + (m+1)\cdot n) \cdot \log_2\left(\frac{R + \|X_0 - C\|}{\epsilon} \right) \right),$$

where $\epsilon > 0$ is the precision used to find $r^\star$.

3. Results regarding polytopes

In this section we tackle a similar problem as in the previous section but instead of considering a finite intersection of balls, we will consider a polytope $\mathcal{P}$ (i.e a finite intersection of half-spaces) and find a vertex that is the farthest away from a point of the form $c+\alpha d$ with $c, d \in \mathbb{R}^n$, for all sufficiently large values of the scalar α. Without any restrictions on α this is also known to be an NP-hard problem, i.e maximizing the distance to a point over a polytope, but under certain restrictions, we prove that this problem can be reduced to a linear program over the polytope $\mathcal{P}$.

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax + b \leq 0\}$ a given polytope (closed, bounded polyhedral set). Let $b = [b_1, ..., b_m]^T$ and $A_{(i,:)}$, $i = \overline{1, m}$ denote the rows of the matrix A, viewed as column vectors, i.e., $A^T = \left[A_{(1,:)}, ..., A_{(m,:)}\right]$. In what follows, we give several results related to polytopes.

Theorem 3.1. *Let $c, d \in \mathbb{R}^n$. Then there exists $\alpha_0 \in \mathbb{R}_+^*$ such that if v^* is a vertex of $\mathcal{P}$, with $v^* \in argmax_{x\in\mathcal{P}}\ \|c + \alpha_0 d - x\|^2$, then*

$$v^* \in \underset{x\in\mathcal{P}}{argmax}\ \|c + \alpha d - x\|^2 \tag{3.1}$$

for all $\alpha \geq \alpha_0$.

Proof. Since we are maximizing a continuous function over the compact subset $\mathcal{P}$ of $\mathbb{R}^n$, the maximum is attained for any value of α. For an arbitrarly selected α, writing (3.1) as a minimization problem, leads to a concave quadratic program (QP), which is known to attain its minimum in a vertex of the polytope $\mathcal{P}$ (see for example [14]). If $v_1, ..., v_p$ are the vertices of the polytope $\mathcal{P}$, it then follows that $\forall\alpha$, $\exists i_\alpha \in \{1, ..., p\}$ such that

$$v_{i_\alpha} \in \underset{x\in\mathcal{P}}{\operatorname{argmax}}\ \|c + \alpha d - x\|^2.$$

Let $\underline{\alpha} > 0$ and $\overline{\alpha} > \underline{\alpha}$ be such that

$$v_{i_{\underline{\alpha}}} \in \underset{x\in\mathcal{P}}{\operatorname{argmax}}\ \|c + \underline{\alpha} d - x\|^2 \text{ and } v_{i_{\underline{\alpha}}} \notin \underset{x\in\mathcal{P}}{\operatorname{argmax}}\ \|c + \overline{\alpha} d - x\|^2. \tag{3.2}$$

If (3.2) does not hold, then the conclusion automatically follows, i.e $\nexists\ \overline{\alpha} > \underline{\alpha}$ such that $v_{i_{\underline{\alpha}}} \notin \operatorname{argmax}_{x\in\mathcal{P}}\ \|c + \overline{\alpha} d - x\|^2$, hence simply take $\alpha_0 = \underline{\alpha}$ and $v^\star = v_{i_{\underline{\alpha}}}$. Otherwise, if (3.2) holds, then we show that

$$v_{i_{\underline{\alpha}}} \notin \underset{x\in\mathcal{P}}{\operatorname{argmax}}\ \|c + \alpha d - x\|^2,\ \forall\alpha,\ \alpha \geq \overline{\alpha}, \tag{3.3}$$

i.e., $\forall\alpha,\ \alpha \geq \overline{\alpha}$, $v_{i_{\underline{\alpha}}}$ is not the vertex furthest away from $c + \alpha d$. To see this, let $i_{\overline{\alpha}} \in \{1, ..., p\} \setminus i_{\underline{\alpha}}$ be such that

$$v_{i_{\overline{\alpha}}} \in \underset{x\in\mathcal{P}}{\operatorname{argmax}}\ \|c + \overline{\alpha} d - x\|^2.$$

Clearly, we have

$$\|c + \overline{\alpha} d - v_{i_{\underline{\alpha}}}\| < \|c + \overline{\alpha} d - v_{i_{\overline{\alpha}}}\|. \tag{3.4}$$

We define

$$f(\alpha) = \|c + \alpha d - v_{i_{\underline{\alpha}}}\|^2 - \|c + \alpha d - v_{i_{\overline{\alpha}}}\|^2.$$

From (3.2) and (3.4), it follows that

$$f(\underline{\alpha}) \geq 0 \text{ and } f(\overline{\alpha}) < 0,$$

which together with $\underline{\alpha} < \overline{\alpha}$ and the fact that f is affine, implies that f is a strictly decreasing function of α. This leads to $f(\alpha) < f(\overline{\alpha})$, $\forall \alpha > \overline{\alpha}$, which implies (3.3).

To finish the proof, assume that the conclusion of the theorem does not hold. This is to say that for any $\alpha_0 > 0$, there exists $\alpha_1 > \alpha_0$ such that

$$v_{i_{\alpha_0}} \in \underset{x \in \mathcal{P}}{\text{argmax}} \;\; \|c + \alpha_0 d - x\|^2 \text{ and } v_{i_{\alpha_0}} \notin \underset{x \in \mathcal{P}}{\text{argmax}} \;\; \|c + \alpha_1 d - x\|^2.$$

According to what we have shown above, $v_{i_{\alpha_0}}$ will never be the furthest point away from $c + \alpha d$ for any $\alpha \geq \alpha_1$. We can repeat this reasoning now with α_0 replaced by α_1 and i_{α_0} replaced by $i_{\alpha_1} \in \{1, .., p\} \setminus i_{\alpha_0}$. After $p - 1$ such repetitions, we are exhausting all the vertices from the solution set, which is a contradiction to the fact that the problem attains its maximum in a vertex for any value of α. □

The next result shows that the point $v^\star$ of Theorem 3.1 can be found as the solution of a linear program (LP), whenever the solution set of this LP is a singleton.

Theorem 3.2. *Let $\mathcal{P}$ be a polytope, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$ with $\|d\| = 1$ such that*

$$x^\star = \underset{x \in \mathcal{P}}{\textit{argmin}} \;\; d^T x$$

is unique. Let α_0 and $v^\star$ be given by Theorem 3.1, i.e.,

$$v^\star = \underset{x \in \mathcal{P}}{\textit{argmax}} \;\; \|c + \alpha d - x\|^2 = \underset{x \in \mathcal{P}}{\textit{argmax}} \;\; \|c + \alpha_0 d - x\|^2 \qquad \forall \alpha \geq \alpha_0$$

Then $v^\star = x^\star$.

Proof. To show that $v^\star = x^\star$, it is enough to prove that

$$(v^\star)^T d \leq x^T d, \forall x \in \mathcal{P}. \tag{3.5}$$

Now assume, that (3.5) does not hold. It follows that there exists $\widetilde{x} \in \mathcal{P}$, such that $\widetilde{x}^T d < (v^\star)^T d$. Define $f(\alpha) = \|c + \alpha d - v^\star\|^2 - \|c + \alpha d - \widetilde{x}\|^2$. A simple calculation leads to

$$f'(\alpha) = (\widetilde{x} - v^\star)^T d < 0,$$

implying that the linear function $f(\alpha)$ is decreasing and therefore $\lim\limits_{\alpha \to \infty} f(\alpha) = -\infty$. The latter implies that there exists $\alpha_1 > 0$, such that $f(\alpha) < 0$, $\forall \alpha \geq \alpha_1$ or equivalently $\|c + \alpha d - v^\star\|^2 < \|c + \alpha d - \widetilde{x}\|$, $\forall \alpha > \alpha_1$, which is a contradiction to the way $v^\star$ is defined. Therefore $v^\star$ must satisfy (3.5) or equivalently $v^\star \in \text{argmax}_{x \in \mathcal{P}}$. Since by assumption, the argmin–set is a singleton, we are led to $v^\star = x^\star$, which concludes our proof. □

4. Conclusion and future work

In this paper we have considered two known NP hard problems namely maximizing the distance to a reference point over (i) an intersection of balls and (ii) an intersection of half-spaces. We have provided some particular cases of the above mentioned problems where algorithms of polynomial complexity exist. In both cases, our restrictions are in the form of some relation between the given fixed reference point and the set over which the maximum is searched for.

Consider the first problem (i): for a given finite intersection of equal radii balls, one can choose the reference point anywhere in the $\mathbb{R}^{n\times 1}$ to form a problem. Our algorithm provides a P time solution to all these choices except for a finite measure set "near" the search space, that is, this paper does not offer guarantees for the reference points whom distance to the search space is less than the radius of the intersecting balls. It is not known if "conquering" this last region is even possible, but obviously reducing it might be the subject of future work. As a first improvement one can try to provide an P time algorithm which allows the given reference point to be anywhere outside of the convex hull of the centers of the intersecting balls.

The approach to the first problem is based on a novel feasibility criteria for the intersection of convex sets which we apply to a non-convex optimization problem. The restrictions we obtain, are sufficient to actually transform the non-convex problem in a convex one.

The approach to the second problem, maximizing the distance to a point over a polytope, is somehow inspired from the first problem, by observing that if the exterior point is far enough, then in some situations the optimal point is actually obtained by solving a linear program.

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