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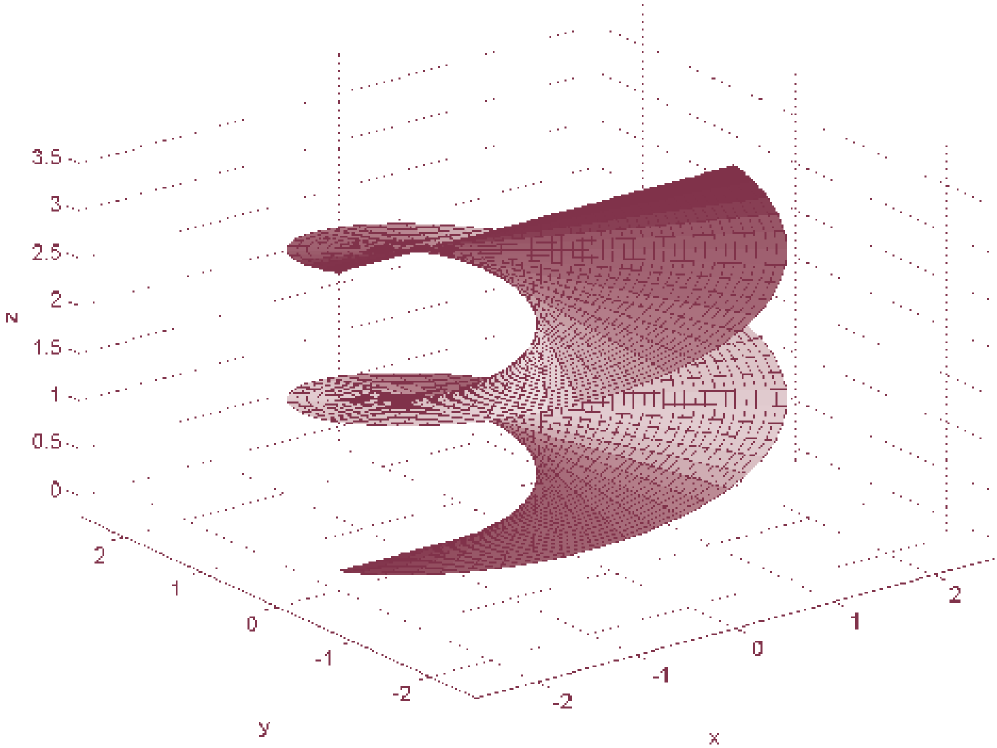

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Identification of induction curves

Levente Lócsi

Abstract. Induction curves (induction surfaces, induction sets in general) were recently introduced to provide a visual aid to examine the fractions defining the norm of a matrix, along with the discovery and description of p-eigenvectors. In our current investigation we delve into an inverse problem, the identification of induction curves. Namely: could the elements of the matrix and the used power parameter p be reconstructed given the induction curve, i.e. the case of 2×2 matrices is examined. The analytic solution is not possible in most cases already in this planar setting, therefore numerical approximation methods shall be applied.

Mathematics Subject Classification (2010): 15A83, 47A30, 65F20, 65F35.

Keywords: Matrix norms, power norm, p-norm, induction curves, identification, optimization, p-eigenvectors, Nelder–Mead method.

1. Introduction

A common way to define a norm of a matrix is to take the supremum of the fraction of the vector norms of the matrix-vector product and the non-zero vector, with respect to a given vector norm, i.e. the least upper bound for the norm of the vectors of the transformed unit sphere. Recently induction curves (induction surfaces, induction sets in general) were introduced to provide a visual aid to examine these fractions defining the norm of a matrix, along with the discovery and description of p-eigenvectors [12, 13]. The study of different phenomena in relation to various norms (most importantly with $p = 1$, 2 and ∞) is a traditional and still active topic [2, 3, 6, 8, 16].

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In our current investigation we delve into the inverse problem, the identification of induction curves, posed in [12]. Namely: could the elements of the matrix and the used parameter p be reconstructed given the induction curve/surface/manifold, or a sampled subset of such an object. For now we restrict ourselves to induction curves, i.e. the case of 2×2 matrices.

The analytic solution is not possible in most cases already in this planar setting, therefore numerical approximation methods shall be applied. In this work our experiences using the well-known Nelder–Mead algorithm [14] are summarized. We have already successfully applied this method earlier to identification problems related to ECG curves and also examined a hyperbolic variant of it [7, 9, 11]. Of course several further optimization methods exist the application of which shall be also investigated for our problem at hand in the future. We have recently seen advances in related topics concerning e.g. Newton-type solvers, conjugate gradient (BFGS) and gradient projection methods [1, 5, 15, 17].

The software package of Matlab/Octave programs available at

`http://locsi.web.elte.hu/indsets/`

will be extended with new components for the task of identification.

2. Formulating the problem

Let us now consider a matrix $A \in \mathbb{R}^{n \times n}$. The p-norm of A is defined as

$$\|.\|_p : \mathbb{R}^{n\times n} \to \mathbb{R}, \qquad \|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \qquad (p \in [1,\infty]),$$

with the usual power norms for vectors $x \in \mathbb{R}^n$

$$\|.\|_p : \mathbb{R}^{n} \to \mathbb{R}, \qquad \|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \qquad (p \in [1,\infty))$$

and

$$\|x\|_\infty = \max_{k=1}^{n} |x_k| .$$

It is well known that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ $(x \in \mathbb{R}^n)$. Notable examples for the above matrix norms include the column norm for $p = 1$, the spectral norm for $p = 2$ and the row norm for $p = \infty$.

Definition 2.1. (c.f. [12], Def. 1.) Given a matrix $A \in \mathbb{R}^{n\times n}$ with $2 \leq n \in \mathbb{N}$ and $p \in [1, \infty]$, the set of points

$$\mathcal{I}_p(A) := \left\{ \frac{\|Ax\|_p}{\|x\|_p} \cdot \frac{x}{\|x\|_2} \in \mathbb{R}^n \,:\, 0 \neq x \in \mathbb{R}^n \right\} \subset \mathbb{R}^n$$

is called the *induction set* of A with parameter p. The induction set may be called induction curve for $n = 2$, induction surface for $n = 3$, induction manifold in general.

Remark 2.2. An induction set basically describes the effect of the multiplication with the matrix on the norm of the vectors in each direction (independent of the length of the vector). The properties of induction sets are discussed in detail in [12]. Here we recall the following. For each direction the set contains exactly one point and its distance from the origin depends continuously on the direction, so in case of 2×2 matrices the set is a closed curve around the origin. These sets are always symmetric with respect to the origin. The values $p \in (0, 1)$ may be also allowed. These sets are not to be confused with the transformed unit sphere by multiplication with the matrix.

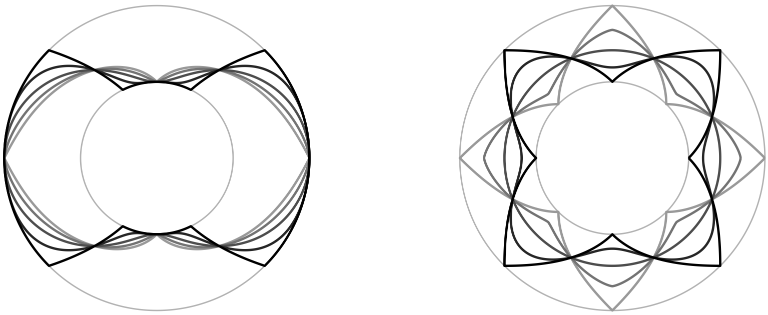

FIGURE 1. Some examples of induction curves.

Example 2.3. Fig. 1 shows examples of induction curves. On the left-hand side the diagonal matrix $\operatorname{diag}(2, 1)$ is used, on the right-hand side the rotation matrix (with scaling) $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Shades of gray represent different p values, namely 1, 4/3, 2, 4 and ∞. Circles denote radial units 1 and 2.

Remark 2.4. Note the intersection points of the induction curves for different p values in case of a fixed matrix. These are common intersection points for all values $p \in [1, \infty]$. Since eigenvectors provide such directions, these are called p-eigenvectors. In some cases these can be expressed explicitly with the matrix elements, and in general they can be found by computing eigenvectors of the matrix with permuted rows as detailed in [12] and [13]. The case of p-eigenvectors should be considered also in our current task of identification, see Section 4.3.

2.1. The identification problem

Now our task is to identify an induction curve, i.e. given some points on the curve, can we find the elements of the matrix and the used parameter value p? As a motivation we provide one more example plot on Fig. 2 and will aim to identify it during this research.[1]

[1] This problem was posed by the Author on the presentation about [12] on the conference *Harmonic Analysis and Related Fields* in Visegrd, Hungary, June 11–13, 2019.

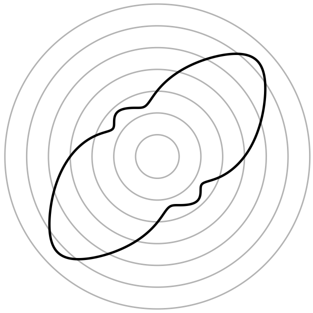

FIGURE 2. The originally posed problem to identify an induction curve. What could be the used matrix and p value?

Consider input data given in polar form, i.e. we are handed pairs (φ_i, r_i) $(i = 1, \dots, k)$ for a fixed value $k \in \mathbb{N}$. Let us first formalize the problem in the simple case of diagonal 2×2 matrices with positive diagonal elements. Denote by

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{thus} \quad Ax = \begin{pmatrix} ax_1 \\ bx_2 \end{pmatrix},$$

with $0 < a, b \in \mathbb{R}$. Introduce $f\colon \mathbb{R} \to \mathbb{R}$ as

$$f_{a,b,p}(\varphi) := n(v(\varphi)) := \frac{\|A \cdot v(\varphi)\|_p}{\|v(\varphi)\|_p} \quad \text{where} \quad v(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}. \tag{2.1}$$

Thus

$$f(\varphi) = f_{a,b,p}(\varphi) = \frac{\big(\,|a\,\cos\varphi|^p + |b\,\sin\varphi|^p\,\big)^{1/p}}{\big(\,|\cos\varphi|^p + |\sin\varphi|^p\,\big)^{1/p}} \tag{2.2}$$

and we are to find parameters a, b and p such that $f(\varphi_i) = r_i$ $(i = 1, \dots, k)$ holds with respect to the input data. Contemplating the formula for f we conclude that the problem is strongly non-linear (mostly with respect to p), but to find 3 parameters $k = 3$ should be minimally prescribed for a unique solution. Solving the problem analytically does not seem to be a promising path, therefore numerical methods shall be applied.

For numerical optimization consider the least squares problem

$$F(a, b, p) := \sum_{i=1}^{k} \big(f_{a,b,p}(\varphi_i) - r_i\big)^2 \quad \longrightarrow \quad \min_{a,b,p}.$$

Without noise at the exact solution the minimum $F(a, b, p) = 0$ could be achieved and is desirable. Add penalty terms to ensure non-negativity of parameters a and b, and tame also $p \in [1, +\infty]$ using $p = w(q)$ and $\mathrm{in}(q)$:

$$\Phi(a, b, q) := F(a, b, w(q)) + \mathrm{nn}(a) + \mathrm{nn}(b) + \mathrm{in}(q) \tag{2.3}$$

with

$$w(q) = \begin{cases} 1, & q < 1 \\ (q-1)^2 + 1, & 1 \le q \le 2 \\ 2/(3-q), & 2 < q < 3 \\ +\infty & q \ge 3 \end{cases}, \quad \mathrm{in}(q) = \begin{cases} (q-1)^2, & q < 1 \\ 0, & 1 \le q \le 3 \\ (q-3)^2, & q > 3 \end{cases}$$

and

$$\mathrm{nn}(x) = \begin{cases} x^2, & x < 0 \\ 0, & x \ge 0 \end{cases}.$$

The functions w and in used for p serve the purpose to transform the optimization of this variable from the domain $[1, \infty]$ to $[1, 3]$ which would be easier to handle for any numerical method considering constraints. Note that this way the extreme parameter values $p = 1$ (corresponding to $q = 1$) and $p = \infty$ ($q = 3$) can both be reached and will not be exceeded, furthermore $w(2) = 2$. The choice of $[1, 3]$ may be modified to a different compact interval. Observe that with this choice of w we don't have to put a constraint on q, and that the function w provides a spline-like smooth (continuously differentiable) map $q \mapsto p$ at least on $q \in (-\infty, 3)$. Fig. 3 illustrates functions w and in, the latter being basically a square penalty function, similarly to nn.

Therefore the task of identification is reduced to an unconstrained optimization problem with the objective function Φ of (2.3):

$$\Phi(a, b, q) = F(a, b, w(q)) + \mathrm{nn}(a) + \mathrm{nn}(b) + \mathrm{in}(q) \quad \longrightarrow \quad \min_{a,b,q}. \tag{2.4}$$

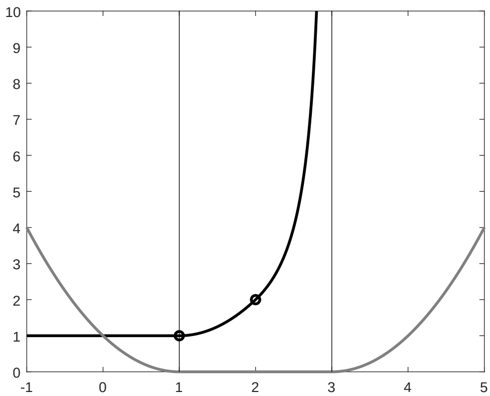

FIGURE 3. The functions w and in plotted on the interval $[-1, 5]$. These maps are used to reduce the optimization of parameter $p \in [1, +\infty]$ to $q \in [1, 3]$ in an unconstrained manner.

This formalization of the problem treats the number of input data points $k \ge 3$ generally. Furthermore the method also generalizes straightforward to arbitrary 2×2

matrices with the main difference in the notation of the matrix A and the function f, namely (c.f. (2.1))

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad f_{A,p}(\varphi) = f_{a,b,c,d,p} := n(v(\varphi)) = \frac{\|A \cdot v(\varphi)\|_p}{\|v(\varphi)\|_p}.$$

In an actual implementation using a high-level programming language we don't need to expand the form of f such as in (2.2). This is left as an exercise to the Reader. Several further terms arise, the formula is much more complicated, but the non-linear nature of the problem still persists. However in case of non-diagonal matrices, the signs of the elements results in different induction curves (unlike for diagonal matrices), therefore the constraints to keep the parameters positive should be dropped.

2.2. Conditions for non-uniqueness

In the case of diagonal matrices it is trivial to observe that varying the signs of the diagonal elements would result in the same induction curve, i.e. with the notation of Def. 2.1. (parentheses simplified)

$$\mathcal{I}_p \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} -a & 0 \\ 0 & b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix}.$$

Therefore we only consider diagonal matrices with positive diagonal elements in the identification task.

But in the case of arbitrary 2×2 matrices we can not neglect the variations with signs since different induction curves arise which need to be identified. However we still experienced that a seemingly perfectly fitting approximation arises from a "completely" different matrix, which led to the following observation about the possible ill-posedness of the problem.

Proposition 2.5. *Let $A \in \mathbb{R}^{2\times 2}$. The matrix A and the new matrix that we get by switching the two rows of A or multiplying a row (or both rows) of A by -1 (or performing both operations) have the same induction curve.*

Proof. Consider the matrices with switched rows

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Then following the definition of the induction sets, for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ and $p \in [1, \infty]$:

$$Ax = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}, \qquad Bx = \begin{pmatrix} cx_1 + dx_2 \\ ax_1 + bx_2 \end{pmatrix}$$

therefore

$$\|Ax\|_p = \left(|ax_1 + bx_2|^p + |cx_1 + dx_2|^p \right)^{1/p} = \|Bx\|_p$$

and hence indeed $\mathcal{I}_p(A) = \mathcal{I}_p(B)$. Clearly multiplying a row with -1 does not effect the p-norm values either. $\square$

The above proposition can be generalized to arbitrary dimensions. Following the notation in [13] let $P \in S_n$ be an element of the symmetric group over n elements represented by a permutation matrix of $\mathbb{R}^{n\times n}$ and $I^{\pm} = \operatorname{diag}(\pm 1, \pm 1, \dots, \pm 1) \in \mathbb{R}^{n\times n}$.

Theorem 2.6. *Let $A \in \mathbb{R}^{n\times n}$, then with the above notations*

$$\mathcal{I}_p(A) = \mathcal{I}_p(I^{\pm}PA) \qquad (p \in [1,\infty]).$$

Proof. The vital observation for this proof is the same as for Proposition 2.5, that the p-norm of the matrix-vector product present in the definition of induction sets is unaffected by the operations of permuting the rows, or multiplying them with -1, i.e.

$$\|Ax\|_p = \|PAx\|_p = \left\|I^{\pm}Ax\right\|_p = \left\|I^{\pm}PAx\right\|_p \qquad (x \in \mathbb{R}^n)$$

Therefore the statement of the theorem holds. $\square$

Example 2.7. The below matrices all have the same induction curve.

$$\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \begin{pmatrix} -3 & 4 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}.$$

There would be 8 possibilities in this case.

Remark 2.8. The transformations of the matrix of the type $I^{\pm}PA$ were used in [13] to deduce the relation of p-eigenvectors the regular eigenvectors of the transformed matrices. It is left to examine the reason behind having the same transform behind two seemingly different but clearly closely related phenomena.

Remark 2.9. Obviously matrices with the same induction curve cannot be told apart using any identification technique.

Remark 2.10. A related fact in linear algebra is that the only matrices that preserve the p-norm of a real vector (for any p) are also the *signed permutation matrices* $I^{\pm}P$ as in the above theorem [4, 10].

3. Optimization method

For the numerical optimization now we have used the *Nelder–Mead simplex method* [14]. This is a general unconstrained, derivative-free method for the optimization of an arbitrary objective function. Unfortunately it has very few proven convergence properties, but is widely used in practice which is highlighted by the fact that it is method behind the `fminsearch` command of the Matlab software package for mathematical modeling, programming and numerical computation.

We already have significant experience using this algorithm [7, 9, 11] and we have our own implementation which allows us e.g. to create animations to examine the progress of the optimization. Fig. 4. illustrates how this method works in two dimensions, basically relying on the function values at the vertices of a simplex and applying the steps of reflection, expansion, (inner and outer) shrink and contraction.

In the problem of induction curve identification we have used the starting parameters $(a, b, q) = (1, 1, 2)$ in case of diagonal matrices and $(a, b, c, d, q) = (1, 1, 1, 1, 2)$

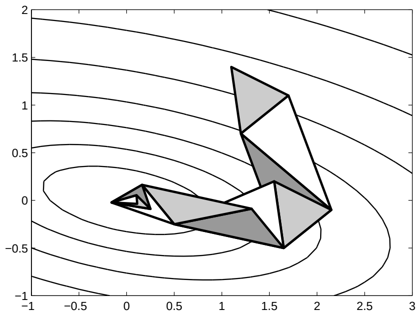

Figure 4. The progress of the Nelder–Mead method in case of the optimization of a quadratic function of two variables.

with slight variation in the parameters for further vertices of the simplex. The optimization process was terminated if the mean objective function value at the vertices comes below a prescribed $\varepsilon = 10^{-6}$ or 10^{-8} threshold (or a step count limit has been reached).

A direction of future research can be the investigation of further optimization methods applied to our problem at hand.

4. Results and experiences

In this section we will summarize the results of the identification process carried out according to the formalization and optimization method discussed in Sections 2 and 3. Furthermore we discuss some findings with respect to the case of p-eigenvectors.

4.1. Diagonal matrices

We have many options to analyze the efficiency of the identification already in the case of diagonal matrices. We have the parameters a and b as positive numbers, the parameters q (or equivalently p) and also the effect of the number of input points can be considered. Furthermore the angles of the points can be chosen randomly or uniformly distributed. Many experiments were carried out for various parameter settings.

To present an overall impression of the identification results we have chosen the following method. First we have selected a number of random points from induction curves generated with parameters a and b randomly chosen from a uniform distribution on $[1, 8]$ (these already provide a wide range of possible induction curves) and q also similarly chosen from $[1, 3]$. Since $a = b$ would result in a multiple of the identity matrix generating a circle as induction curve independent of q we required that $|a - b| > 0.1$. We used the values $k = 3, 5, 10, 20$ for the number of points sampled from the induction curves. Furthermore we ensured that the angle difference of the samples are at least $0.2\pi, 0.2\pi, 0.1\pi, 0.05\pi$ respectively to avoid points too close to each other.

For each value of k we performed $N = 1000$ tests with random a, b and q values as detailed above. Denote by a', b' and q' the approximated values given by the optimization method, we collected the values $\|(a, b) - (a', b')\|_2$ and $|q - q'|$. Finally we plotted the sorted approximation errors on a logarithmic scale as seen on Fig. 5. On the left-hand side one can observe the errors of (a, b), on the right-hand side the errors of q. (The results are very similar.) Shades of gray correspond to the values of k, the number of input points, the lightest for $k = 3$, the darkest for $k = 20$.

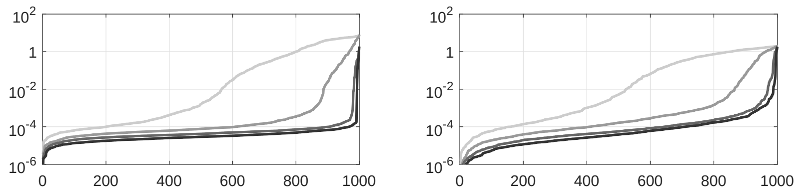

FIGURE 5. Measurement results about the identification of induction curves of diagonal matrices. The error is plotted on a logarithmic scale versus the measurement number (results are sorted). Darker lines correspond to higher number of input points.

On one hand we can conclude that in case of only $k = 3$ input points, in about half of the test cases the identification errors are below 10^{-2}. Such few points may not prove sufficient to identify the matrix and the parameter for this algorithm, although theoretically the solution is unique. It is known that the Nelder–Mead algorithm may also get stuck in local minima, here this phenomena would correspond to very similar induction curves considering only 3 given points.

On the other hand if at least $k = 10$ points are given, the approximation error rises above 10^{-4} only in very few cases. So in practice (when we could observe many points of the curve) already our current method performs very well.

4.2. General matrices

Since in case of arbitrary matrices the generating matrix may be significantly different then the matrix resulting from the optimization process, a representative of their equivalence classes (based on induction curves) must be chosen to measure the approximation results. A representative is chosen based on the signs and ordering of matrix elements.

With the above in sight we have carried out very similar measurements to those in case of diagonal matrices described in Section 4.1, and the presentation of the results is also analogous as seen on Fig. 6.

In this case the matrix elements were all randomly chosen from a uniform distribution on the interval $[-10, 10]$, q again from $[1, 3]$. Now we did not rule out any special matrices (such as possible multiples of the identity matrix). The values for the number of sample points k were $5, 10, 20, 30$ with the minimal angle differences

$0.2\pi, 0.1\pi, 0.05\pi, 0.01\pi$ respectively. We measured $\|A - A'\|_2$ (seen on the left-hand side of the figure) and again $|q - q'|$ (right-hand side).

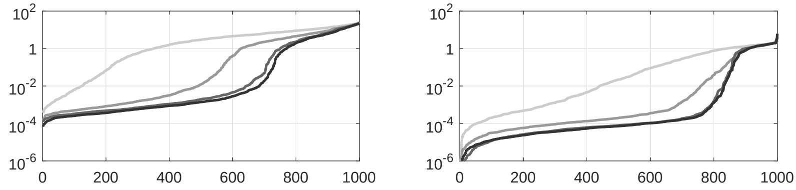

FIGURE 6. Measurement results about the identification of induction curves of general 2×2 matrices.

In this case the approximation results are not as good as in case of diagonal matrices, but still acceptable. Again the results in case of $k = 5$ are not good, in many cases the difference is considerable. But in case of at least $k = 20$ points about 70% of the tests the difference in the matrix is less than 10^{-2}, and the error in identifying q is less than 10^{-3} in about 80% of the tests.

Possible directions of improvement include using different optimization methods, maybe even creating a dictionary for better starting points based on some similarity measure on induction curves. Also it would be interesting to examine the problematic cases (matrices and parameters) in more detail.

Finally in this section on Fig. 7 we present some steps of the optimization progress in case of the original example for the identification problem as on Fig. 2. The sample points in the amount of $k = 20$ were selected randomly with a minimum angle difference of 0.05π. The images also show the induction curves corresponding to the vertices of the simplex, and the matrix and parameter by the centroid of the simplex is written on the lower right parts rounded to 2 decimal digits. Steps $1, 20, 50, 100, 200$ and 270 are shown. We have arrived at the result

$$A' = \begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad p' = 7,$$

which is correct in light of Proposition 2.5 and Theorem 2.6. The original parameters for generating Fig. 2 and the sample points for the optimization were

$$A = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad p = 7.$$

4.3. On the case of p-eigenvectors

A corner case of induction curve identification is when we are given exactly the common intersection points for the p values. In this case any value for p is good and will fit. E.g. in case of the diagonal matrix $\mathrm{diag}(2, 1)$ of Fig. 1 we are handed polar values $(0, 2)$, $(\pi/2, 1)$ and $(\mathrm{atan}\sqrt{2}, \sqrt{2})$ (c.f. [12], Ex. 2.).

Plotting the objective function values $\Phi(a, b, q)$ as in (2.3) for fixed $(a, b) = (2, 1)$, $q \in [1, 3]$, two input points fixed at $(0, 2)$, $(\pi/2, 1)$ but the third input point moving

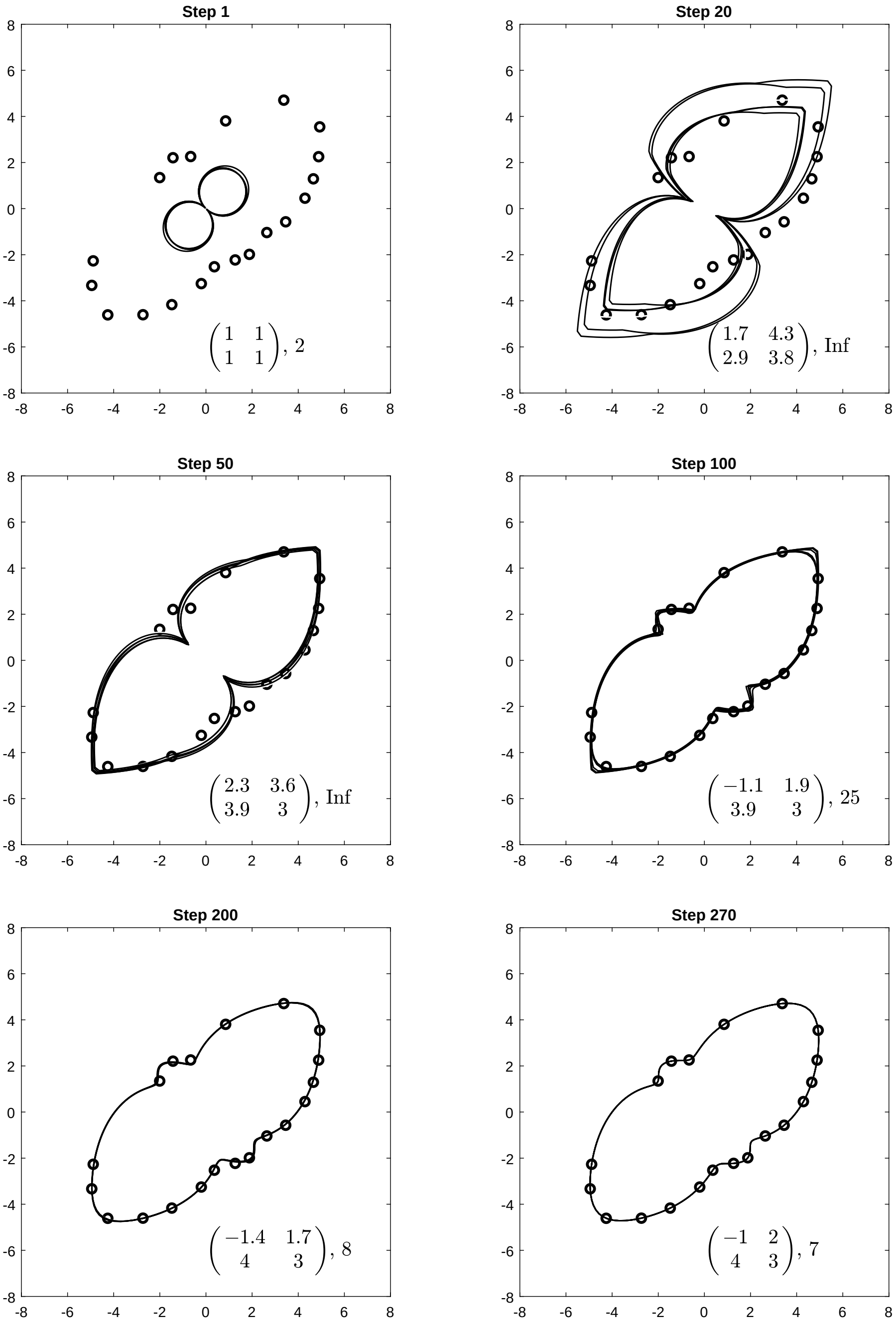

FIGURE 7. Some steps of the optimization progress in case of a sampled version of the originally posed identification problem as seen on Fig. 2.

along a short curve segment passing through $(\mathrm{atan}\sqrt{2}, \sqrt{2})$ with varying φ and r we get a result as depicted on Fig. 8.

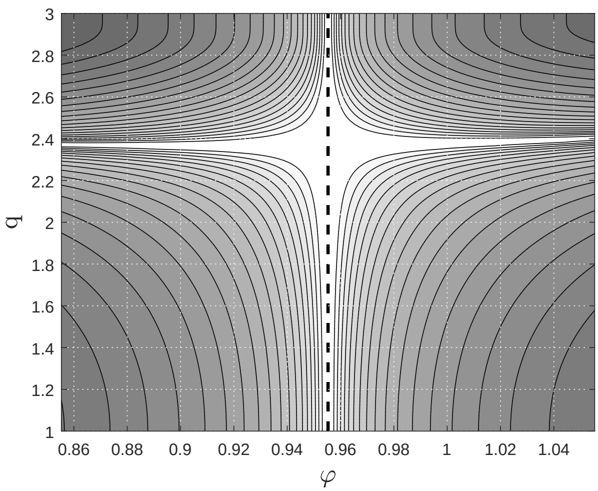

Figure 8. An illustration of the singularity near p-eigenvectors. If only the common intersection points are given as input, then the objective function does not have a unique minimizer, the parameter value p cannot be decided.

This contour plot confirms our expectations: At the critical value of φ corresponding to the p-eigenvector the q (and p) parameter values all give the same minimal objective function value 0 (as shown along the black dashed line). But already slightly away from the critical point with φ — where the induction curves for different p values start to spread — the optimal q parameter is unique, the farther from the critical angle the easier to identify.

In practice we would usually have far more points to start the identification process with. So this phenomena would not cause problems when considering an actual induction curve plot. It is just of theoretical importance.

5. Conclusions and further research

In this paper we have shown that the automatic identification of induction curves based on a sampled subset of them is possible using the Nelder–Mead simplex method in an appropriate setting. The accuracy in case of diagonal matrices is very high, a bit lower in case of general matrices. Some mathematical reasons behind non-unique identification were uncovered in general: the signed permutations of the rows of a matrix results in the same induction curve. Experiments were also made near and in the extreme case of providing input values only in p-eigendirections.

As possible directions of further research we list:

- Experiment with further optimization methods in hope to improve identification accuracy and speed.
- Detailed analysis of the cases when this method does not give a proper approximation.
- Identification of induction surfaces (or their 2D projections). Even higher dimensional identification problems.
- The effect of noise in the input data on the precision of identification.
- Describing induction curves with exactly 2 common points.
- A possible topic may be the development of an *induction curve identification application* for mobile devices, such that a user can easily identify a curve of this family using the camera of the device.

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New integral inequalities involving generalized Riemann-Liouville fractional operators

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Abstract. In this paper, using a generalized operator of the Riemann-Liouville type, defined and studied in a previous work, several integral inequalities for synchronous functions are established.

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Keywords: Generalized fractional Riemann-Liouville integral, fractional integral inequality, synchronous functions.

1. Introduction

One of the most developed mathematical areas in the last 20 years is that of Integral Inequalities, associated with different functional notions: convex, synchronous functions among other, within the framework of Riemann, fractional and generalized integral operators. A detail that we want to point out is the fact of the appearance in recent years of various integral operators, natural extensions of the fractional integral of Riemann-Liouville, this together with the attention received by Integral Inequalities, make more and more researchers and research is devoted to this topic. To get a more complete idea in this regard, we recommend consulting the works [1, 2, 6, 8, 10, 11, 12, 13, 16] and the references cited therein.

In this direction, one of the most fruitful notions is the following (see [4]).

Definition 1.1. If χ and ψ are two integrable functions on $[a, b]$, they are synchronous on $[a_1, a_2]$ if $(\chi(x) - \chi(y))(\psi(x) - \psi(y)) \geq 0$, for any $x, y \in [a_1, a_2]$.

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Various integral inequalities have been obtained using this notion, within the framework of different integral operators (see [4], [7], [9], [15], [18], [20], and the references cited there).

Throughout the work we use the functions Γ (see [17, 19, 22, 23]) and Γ_k (cf. defined by [5]):

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau} \, d\tau, \quad \Re(z) > 0, \tag{1.1}$$

$$\Gamma_k(z) = \int_0^\infty \tau^{z-1} e^{-\tau^k/k} \, d\tau, k > 0. \tag{1.2}$$

It is clear that if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$. As well, we define the k-beta function as follows

$$B_k(u,v) = \frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1} d\tau,$$

notice that $B_k(u,v) = \frac{1}{k}B(\frac{u}{k}, \frac{v}{k})$ and $B_k(u,v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

In [7] the following fractional integral operator of the Riemann-Liouville type is defined.

Definition 1.2. The k-generalized fractional Riemann-Liouville integral of order α with $\alpha \in \mathbb{R}$, and $s \neq -1$ of an integrable function $\chi(u)$ on $[0,\infty)$, are given as follows:

$$^{s}J_{F,a_1}^{\frac{\alpha}{k}}\chi(u) = \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u} \frac{F(\tau,s)\chi(\tau)d\tau}{[\mathbb{F}(u,\tau)]^{1-\frac{\alpha}{k}}}, \tag{1.3}$$

with $F(\tau,0) = 1$ and $\mathbb{F}(u,\tau) = \int_\tau^u F(\theta,s)d\theta$.

Remark 1.3. In the aforementioned paper, the main properties of this operator (boundedness, conmutatity, etc.) and various inequalities associated with it were studied.

Remark 1.4. If in Definition 1.2 we consider the kernel $F(t,s) = 1$ and $k = 1$, we obtain the classic fractional Integral Riemann-Liouville, used in the work [4]; in the case of the same kernel but $k \neq 1$ then the k-fractional integral of the Riemann-Liouville type of [14] is obtained (see also [15, 18, 20]). If, on the contrary, we consider the kernel $F(t,s) = t^s$, we obtain the (k;s)-Riemann-Liouville fractional integral of [21]. In the case of taking the kernel as $F(t,s) = h'(t)$, we obtain the (k;h)-Riemann-Liouville integral fractional used in [9]. It is clear then, that the results obtained in our work, generalize those of the works mentioned before.

The main purpose of this paper, using the generalized fractional integral operator of the Riemann-Liouville type of Definition 1.2, is to establish several integral inequalities, which contain as particular cases, several of those reported in the literature.

2. Main results

Our first fundamental result is the following.

Theorem 2.1. *Let φ, ψ be two synchronous functions on $[0,\infty)$ and let $f, \phi, \omega \geq 0$, then for all $\tau > a_1 \geq 0$, $\alpha > 0$, and $s \neq -1$, we have*

$$\begin{aligned}
&2\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau)\Big] \\
&\qquad +2\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau) \\
&\geq {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)\Big] \\
&+ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\Big] \\
&+ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau) + {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\phi)(\tau)\Big]. \quad (2.1)
\end{aligned}$$

To prove the previous Theorem, we need the following lemma.

Lemma 2.2. *Let φ, ψ be two synchronous functions on $[0,\infty)$ and let $h, g \geq 0$, then for all $\tau > a_1 \geq 0$, $\alpha > 0$, and $s \neq -1$, we have the following inequality*

$$\begin{aligned}
{}^sJ^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\varphi\psi)(\tau) + {}^sJ^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau) &\geq {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\psi)(\tau) \\
&+ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau). \quad (2.2)
\end{aligned}$$

Proof. Since φ, ψ are synchronous functions on $[0,\infty)$, then for all $u, v \geq 0$, we have

$$\varphi(u)\psi(u) + \varphi(v)\psi(v) \geq \varphi(u)\psi(v) + \varphi(v)\psi(u). \quad (2.3)$$

Thus, if we multiple both sides of (2.3) by $\frac{F(u,s)h(u)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, and then we integrate the resulting inequality with respect to u over (a_1, τ), it holds that

$${}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau) + \varphi(v)\psi(v)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}h(\tau) \geq \psi(v)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau) + \varphi(v)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau). \quad (2.4)$$

Now, multiplying both sides of (2.4) by $\frac{F(v,s)g(v)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,v)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to v over (a_1, τ), we get

$$\begin{aligned}
{}^sJ^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\varphi\psi)(\tau) + {}^sJ^{\frac{\alpha}{k}}_{F,a_1}g(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau) &\geq {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\psi)(\tau) \\
&+ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(g\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau).
\end{aligned}$$

Thus, we conclude the result. □

Remark 2.3. If we consider the kernel $F(t,s) = 1$ and $k = 1$ of this result, we obtain the Lemma 3 of [4], in the case that $F(t,s) = h'(t)$, this result covers Lemma 1 of [9].

Let us now prove the Theorem 2.1.

Proof. By setting $h = f$ and $g = \phi$ in (2.2), then we multiple the resulting inequality by ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)$, we get

$$\begin{aligned} &{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau)\right] \\ &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)\right]. \quad (2.5) \end{aligned}$$

By setting $h = \omega$ and $g = \phi$ in (2.2), and multiplying the result by ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)$, we get

$$\begin{aligned} &{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\right] \\ &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\right]. \quad (2.6) \end{aligned}$$

Now, using the same idea, we put $h = \omega$ and $g = f$, and then we multiple the result by ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)$, we find

$$\begin{aligned} &{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\right] \\ &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\left[{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\right]. \quad (2.7) \end{aligned}$$

Finally, by adding the inequalities (2.5), (2.6) and (2.7) we obtain the result. □

Remark 2.4. If we take the kernel $F(t,s) = 1$ and $k = 1$ we obtain Theorem 2 of [4], and if $F(t,s) = h'(t)$, this result reduces to Theorem 3 of [9].

The next Lemma will be from very useful for proving the last theorem.

Lemma 2.5. *Let φ, ψ be two synchronous functions on $[0,\infty)$ and let $h, g \geq 0$, then for all $\tau > a_1 \geq 0$, $\alpha, \beta > 0$, and $s \neq -1$, we have the following inequality*

$$\begin{aligned} {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\varphi\psi)(\tau) + {}^{s}J^{\frac{\beta}{k}}_{F,a_1}g(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau) &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)\ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\psi)(\tau) \\ &+ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau). \quad (2.8) \end{aligned}$$

Proof. Multiplying both sides of (2.4) by $\frac{F(v,s)g(v)}{k\Gamma_k(\beta)[\mathbb{F}(\tau,v)]^{1-\frac{\beta}{k}}}$, then we integrate the resulting inequality with respect to v over (a_1, τ), we conclude that

$$\begin{aligned} {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}h(\tau)\ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\varphi\psi)(\tau) + {}^{s}J^{\frac{\beta}{k}}_{F,a_1}g(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\varphi\psi)(\tau) &\geq {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\varphi)(\tau)\ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\psi)(\tau) \\ &+ {}^{s}J^{\frac{\beta}{k}}_{F,a_1}(g\varphi)(\tau)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(h\psi)(\tau). \end{aligned}$$

Thus, the Lemma is proved. □

Remark 2.6. If we take the kernel $F(t,s) = 1$ and $k = 1$ we obtain Lemma 6 of [4], and if $F(t,s) = h'(t)$, this result reduces to Lemma 2 of [9].

Theorem 2.7. *Let φ, ψ be two synchronous functions on $[0, \infty)$ and let $f, \phi, \omega \geq 0$, then for all $\tau > a_1 \geq 0$, $\alpha, \beta > 0$, and $s \neq -1$, we get*

$$\begin{aligned}
&{}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\varphi\psi)(\tau) + 2\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) \\
&\qquad + {}^sJ^{\frac{\beta}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau)\Big] \\
&+\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}\phi(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big]\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau) \\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)\Big] \\
&+ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\Big] \\
&+ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\phi)(\tau)\Big]. \qquad (2.9)
\end{aligned}$$

Proof. Substituting in (2.8): $h = f$ and $g = \phi$,then we multiple the resulting inequality by ${}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)$, we have

$$\begin{aligned}
&{}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi\psi)(\tau)\Big] \\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(f\psi)(\tau)\Big]. \qquad (2.10)
\end{aligned}$$

Replacing again in (2.8): $h = \omega$ and $g = \phi$, and multiplying the result by ${}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)$, we get

$$\begin{aligned}
&{}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}\phi(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\Big] \\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}f(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(\phi\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\Big]. \qquad (2.11)
\end{aligned}$$

Now, by using the same idea, we put $h = \omega$ and $g = f$, and then we multiply the result by ${}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)$, we conclude that

$$\begin{aligned}
&{}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\omega(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\varphi\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}f(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi\psi)(\tau)\Big] \\
&\geq\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}\phi(\tau)\Big[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\varphi)(\tau)\ {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\psi)(\tau) + {}^sJ^{\frac{\beta}{k}}_{F,a_1}(f\varphi)(\tau)\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1}(\omega\psi)(\tau)\Big]. \qquad (2.12)
\end{aligned}$$

Finally, by adding the inequalities (2.10), (2.11) and (2.12) we obtain (2.9). □

Remark 2.8. In the case of the classical Riemann-Liouville Integral, this result extends the Theorem 4 of [4], and if we use the (k; h) -Riemann-Liouville integral fractional, we obtain the Theorem 4 of [9].

Remark 2.9. Remark 2 and Remark 3 of [9], and Remark 7 of [4] are true for the general kernel used in our work.

3. Conclusions

In our work we obtained several generalized integral inequalities, which contain, as a particular case, some of those known in the literature, for example, if in Theorem 2.7 we consider $\alpha = \beta = 1$, $k = 1$ and $F(t, s) = 1$, we obtain the well-known Chebishev Inequality (see [3]). In the same direction, we can add that one of the strengths of our results lies in the fact that by suitably choosing of F, i.e., if we consider kernels different than those indicated in the various remarks, one can further easily obtain additional integral inequalities involving the various types of fractional integral operators from our main results.

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A nonlocal Cauchy problem for nonlinear generalized fractional integro-differential equations

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Abstract. In this paper, we study the existence of solutions of a nonlocal Cauchy problem for nonlinear fractional integro-differential equations involving generalized Katugampola fractional derivative. By using fixed point theorems, the results are obtained in weighted space of continuous functions. In the last, results are illustrated with suitable examples.

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Keywords: Fractional derivative, fractional integration, fractional integro-differential equation, existence of solution, fixed point theorem.

1. Introduction

The idea of fractional differentiation was introduced by Riemann and Liouville in the nineteenth century. It is the generalization of ordinary differentiation and integration to arbitrary non-integer order, for details, see [1, 2, 4, 5, 6, 15, 16] and the references therein.

The area of fractional differential equations is now considered to be very important due to its various applications in different fields of science and technology such as control theory, rheology, signal processing, modelling, fractals, chaotic dynamics, bioengineering and biomedical and so on, for example see [6, 13, 17] and the references therein. Recently, many researchers studied the fractional differential and integro-differential equations and obtained many interesting existence and uniqueness results, see [3, 7, 12, 18, 20, 19, 21, 22, 23] and the references therein.

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Recently, the authors in [8] discussed the existence and stability of solution of the initial value problem (IVP):

$$({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f(t, x(t)),\ t \in J := (a, T], \tag{1.1}$$

$$({}^{\varrho}I^{1-\gamma}_{a+}x)(a) = c_2, \gamma = \alpha + \beta(1-\alpha), c_2 \in \mathbb{R}, \tag{1.2}$$

for generalized Katugampola fractional differential equation by using Schauder fixed point theorem and the equivalence between IVP (1.1)-(1.2) and the integral equation

$$x(t) = \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^{\varrho} - a^{\varrho}}{\varrho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s, x(s))\mathrm{d}s. \tag{1.3}$$

In [9], using Krasnoselskii's fixed point theorem, Schauder fixed point theorem and Schaefer fixed point theorem, authors discussed the existence of solution of IVP with nonlocal initial condition:

$$({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f(t, x(t)),\ t \in J := (a, T], \tag{1.4}$$

$$({}^{\varrho}I^{1-\gamma}_{a+}x)(a+) = \sum_{j=1}^{m} \eta_j x(\xi_j), \alpha \le \gamma = \alpha + \beta(1-\alpha),\ \xi_j \in (a, T], \tag{1.5}$$

where ${}^{\varrho}D^{\alpha,\beta}_{a+}$ is the generalized Katugampola fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and ${}^{\varrho}I^{1-\gamma}_{a+}$ is the generalized Katugampola fractional integral with $\varrho > 0$. Authors also proved the equivalence between (1.4)-(1.5) and the integral equation

$$\begin{aligned} x(t) =& \frac{K}{\Gamma(\alpha)}\left(\frac{t^{\varrho} - a^{\varrho}}{\varrho}\right)^{\gamma-1} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s, x(s))\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s, x(s))\mathrm{d}s, \end{aligned} \tag{1.6}$$

where

$$K = \left(\Gamma(\gamma) - \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho}\right)^{\gamma-1}\right)^{-1}. \tag{1.7}$$

The above results motivate us and therefore, in this paper, we obtain the existence of solution of the following Nonlinear Generalized Fractional Integro–Differential Equation (NGFIDE) of order α $(0 < \alpha < 1)$ and type $\beta \in [0,1]$:

$$({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f\left(t, x(t), \int_a^t h(t,s)x(s)\mathrm{d}s\right),\ t \in J := (a, T], \tag{1.8}$$

$$({}^{\varrho}I^{1-\gamma}_{a+}x)(a+) = \sum_{j=1}^{m} \eta_j x(\xi_j), \alpha \le \gamma = \alpha + \beta(1-\alpha),\ \xi_j \in (a, T], \tag{1.9}$$

where ${}^{\varrho}D^{\alpha,\beta}_{a+}$ is the generalized Katugampola fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and ${}^{\varrho}I^{1-\gamma}_{a+}$ is the generalized Katugampola fractional integral with $\varrho > 0$. Function $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a given function, ξ_j are pre–fixed points satisfy $0 < a < \xi_1 \le \ldots \le \xi_m < T$ and η_j, $j = 1, 2, \ldots, m$ are real numbers.

First, we establish an equivalent mixed-type nonlinear Volterra integral equation

$$
\begin{aligned}
x(t) =&\frac{K}{\Gamma(\alpha)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\
&\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s\\
&+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f\left(s,x(s),\int_a^t h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \qquad (1.10)
\end{aligned}
$$

where

$$
K=\left(\Gamma(\gamma)-\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\right)^{-1}, \qquad (1.11)
$$

for NGFIDE (1.8)-(1.9) in the weighted space of continuous functions $C_{1-\gamma,\varrho}[a,T]$ presented in the next section. We use the Krasnoselskii's fixed point theorem and Schauder fixed point theorem to prove the existence results for NGFIDE (1.8)-(1.9).

The rest of the paper is organized as follows. In Section 2, some definitions, notations and basic results are given. We prove the equivalent integral equation in Section 2 and the existence results are proved in Section 3. Illustrative examples are given in the last section.

2. Preliminaries

Here we introduce some definitions and present preliminary results needed in our proofs later.
Let the Euler gamma and beta functions be defined, respectively, by

$$
\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}\mathrm{d}x,\quad \mathbf{B}(\alpha,\beta)=\int_0^1(1-x)^{\alpha-1}x^{\beta-1}\mathrm{d}x,\ \alpha>0,\ \beta>0.
$$

It is well known that $\mathbf{B}(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ for $\alpha>0,\ \beta>0$, see [13]. Throughout the paper, we consider $[a,T],\ 0<a<T<\infty$ being a finite interval on $\mathbb{R}^+$ and $\varrho>0$.

Definition 2.1 ([13])**.** The space $X_c^p(a,T),\ c\in\mathbb{R},\ p\geq 1$ consists of those real valued Lebesgue measurable functions g on (a,T) for which $\|g\|_{X_c^p}<\infty$, where

$$
\|g\|_{X_c^p}=\left(\int_a^b|t^c g(t)|^p\frac{\mathrm{d}t}{t}\right)^{1/p},\quad p\geq 1\quad\text{and}\quad \|g\|_{X_c^{\infty}}=\operatorname*{ess\,sup}_{a\leq t\leq T}|t^c g(t)|
$$

In particular, when $c=1/p$, we see that $X_{1/p}^c(a,T)=L_p(a,T)$.

Definition 2.2 ([14])**.** We denote by $C[a,T]$ a space of continuous functions g on $(a,T]$ with the norm

$$
\|g\|_C=\max_{t\in[a,T]}|g(t)|
$$

The weighted space $C_{\gamma,\varrho}[a,T],\ 0\leq\gamma<1$ of functions g on $(a,T]$ is defined as

$$C_{\gamma,\varrho}[a,T]=\left\{g:(a,T]\to\mathbb{R}:\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma}g(t)\in C[a,T]\right\} \tag{2.1}$$

with the norm

$$\|g\|_{C_{\gamma,\varrho}}=\left\|\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma}g(t)\right\|_{C}=\max_{t\in[a,t]}\left|\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma}g(t)\right|,$$

and $C_{0,\varrho}[a,T]=C[a,T]$

Definition 2.3 ([14]). Let $\delta_{\varrho}=\left(t^{\varrho-1}\mathrm{d}/\mathrm{d}t\right),\ 0\leq\gamma<1$. Denote $C^{n}_{\delta_{\varrho}\gamma}[a,T]$ the Banach space of functions g which are continuously differentinble, with δ_{ϱ}, on $[a,T]$ upto order $(n-1)$ and have the derivative $\delta^{n}_{\varrho}g$ on $(a,T]$ such that $\delta^{n}_{\varrho}g\in C_{\gamma,\varrho}[a,T]$:

$$C^{n}_{\delta_{\varrho,\gamma}}[a,T]=\left\{\delta^{k}_{\varrho}g\in C[a,T],k=0,1,\ldots,n-1,\delta^{n}_{\varrho}g\in C_{\gamma,\varrho}[a,T]\right\},\quad n\in\mathrm{N}$$

with the norm

$$\|g\|_{C^{n}_{\delta_{\varrho,\gamma}}}=\sum_{k=0}^{n-1}\left\|\delta^{k}_{\varrho}g\right\|_{C}+\left\|\delta^{n}_{\varrho}g\right\|_{C_{\gamma,\varrho}},\quad \|g\|_{C^{n}_{\delta_{\varrho}}}=\sum_{k=0}^{n}\max_{t\in\Omega}\left|\delta^{k}_{\varrho}g(t)\right|.$$

In particular, for $n=0$ we have $C^{0}_{\delta_{\varrho}\gamma}[a,T]=C_{\gamma,\varrho}[a,T]$.

Definition 2.4 ([10]). Let $\alpha>0$ and $f\in X^{p}_{c}(a,T)$, where X^{p}_{c} is as in Definition 2.1. The left-sided Katugampola fractional integral ${}^{\varrho}I^{\alpha}_{a+}$ of order α is defined as

$${}^{\varrho}I^{\alpha}_{a+}f(t)=\int_{a}^{t}s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{f(s)}{\Gamma(\alpha)}\mathrm{d}s,\ t>a. \tag{2.2}$$

Definition 2.5 ([11]). Let $\alpha\in\mathbb{R}^{+}\backslash N$ and $n=[\alpha]+1$, where $[\alpha]$ is the integer part of α. The left-sided Katugampola fractional derivative ${}^{\varrho}D^{\alpha}_{a+}$ is defined as

$$\begin{aligned}{}^{\varrho}D^{\alpha}_{a+}f(t)&=\delta^{n}_{\varrho}\left({}^{\varrho}I^{n-\alpha}_{a+}f(s)\right)(t)\\&=\left(t^{\varrho-1}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_{a}^{t}s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{n-\alpha-1}\frac{f(s)}{\Gamma(n-\alpha)}\mathrm{d}s.\end{aligned} \tag{2.3}$$

Definition 2.6 ([14]). The left-sided generalized Katugampola fractional derivative ${}^{\varrho}D^{\alpha,\beta}_{a+}$ of order $0<\alpha<1$ and type $0\leq\beta\leq1$ is defined as

$$\left({}^{\varrho}D^{\alpha,\beta}_{a+}f\right)(t)=\left({}^{\varrho}I^{\beta(1-\alpha)}_{a+}\delta_{\varrho}\ {}^{\varrho}I^{(1-\beta)(1-\alpha)}_{a+}f\right)(t), \tag{2.4}$$

for the functions for which the right-hand side expression exists.

Lemma 2.7 ([9]). *Suppose that $\alpha>0,\ \beta>0,\ p\geq1$ and $\varrho,c\in\mathbb{R}$ such that $\varrho\geq c$. Then for $f\in X^{p}_{c}(a,T)$, the semigroup property of Katugampola integral is valid. This is*

$${}^{\varrho}I^{\alpha}_{a+}\,{}^{\varrho}I^{\beta}_{a+}f(t)={}^{\varrho}I^{\alpha+\beta}_{a+}f(t). \tag{2.5}$$

Lemma 2.8 ([11]). *Suppose that $\alpha>0$, $0\leq\gamma<1$ and $f\in C_{\gamma,\varrho}[a,T]$. Then for all $t\in(a,T]$,*

$${}^{\varrho}D^{\alpha}_{a+}\,{}^{\varrho}I^{\alpha}_{a+}f(t)=f(t). \tag{2.6}$$

Lemma 2.9 ([11]). *Suppose that $\alpha > 0$, $0 \leq \gamma < 1$, $f \in C_{\gamma,\varrho}[a,T]$ and ${}^{\varrho}I_{a+}^{1-\alpha} f \in C_{\gamma,\varrho}^{1}[a,T]$. Then*

$$ {}^{\varrho}I_{a+}^{\alpha}\, {}^{\varrho}D_{a+}^{\alpha} f(t) = f(t) - \frac{{}^{\varrho}I_{a+}^{1-\alpha} f(a)}{\Gamma(\alpha)} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\alpha-1}. \tag{2.7} $$

Lemma 2.10 ([9]). *Suppose ${}^{\varrho}I_{a+}^{\alpha}$ and ${}^{\varrho}D_{a+}^{\alpha}$ are defined as in Definitions 2.4 and 2.5, respectively. Then*

$$ {}^{\varrho}I_{a+}^{\alpha} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma+1)} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\alpha+\sigma-1}, \ \alpha \leq 0,\ \sigma > 0,\ t > a, \tag{2.8} $$

$$ {}^{\varrho}D_{a+}^{\alpha} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\alpha-1} = 0,\ 0 < \alpha < 1. \tag{2.9} $$

Remark 2.11. For $0 < \alpha < 1$, $0 \leq \beta \leq 1$, the generalized Katugampola fractional derivative ${}^{\varrho}D_{a+}^{\alpha,\beta}$ can be written in terms of Katugampola fractional derivative as

$$ {}^{\varrho}D_{a+}^{\alpha,\beta} = {}^{\varrho}I_{a+}^{\beta(1-\alpha)} \delta_{\varrho} {}^{\varrho}I_{a+}^{1-\gamma} = {}^{\varrho}I_{a+}^{\beta(1-\alpha)}\, {}^{\varrho}D_{a+}^{\gamma}, \quad \gamma = \alpha + \beta(1-\alpha). $$

Lemma 2.12 ([14]). *Let $\alpha > 0$, $0 < \gamma \leq 1$ and $f \in C_{1-\gamma,\varrho}[a,b]$. If $\alpha > \gamma$, then*

$$ \left({}^{\varrho}I_{a+}^{\alpha} f\right)(a) = \lim_{x \to a+} \left({}^{\varrho}I_{a+}^{\alpha} f\right)(t) = 0. $$

To discuss the existence of a solution of NGFIDE (1.8)-(1.9), we need the following spaces:

$$ C_{1-\gamma,\varrho}^{\alpha,\beta}[a,T] = \left\{ g \in C_{1-\gamma,\varrho}[a,T] : {}^{\varrho}D_{a+}^{\alpha,\beta} g \in C_{1-\gamma,\varrho}[a,T] \right\},\ 0 < \gamma \leq 1 \tag{2.10} $$

and

$$ C_{1-\gamma,\varrho}^{\gamma}[a,T] = \left\{ g \in C_{1-\gamma,\varrho}[a,T] : {}^{\varrho}D_{a+}^{\gamma} g \in C_{1-\gamma,\varrho}[a,T] \right\},\ 0 < \gamma \leq 1. \tag{2.11} $$

Since ${}^{\varrho}D_{a+}^{\alpha,\beta} g = {}^{\varrho}I_{a+}^{\beta(1-\alpha)}\, {}^{\varrho}D_{a+}^{\gamma} g$, it is obvious that $C_{1-\gamma,\varrho}^{\gamma}[a,T] \subset C_{1-\gamma,\varrho}^{\alpha,\beta}[a,T]$.

Lemma 2.13 ([9]). *Let $\alpha > 0$, $\beta > 0$ and $\gamma = \alpha + \beta - \alpha\beta$. If $g \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$, then*

$$ {}^{\varrho}I_{a+}^{\gamma}\, {}^{\varrho}D_{a+}^{\gamma} g(t) = {}^{\varrho}I_{a+}^{\alpha}\, {}^{\varrho}D_{a+}^{\alpha,\beta} g(t) = {}^{\varrho}D_{a+}^{\beta(1-\alpha)} g(t). $$

To prove the equivalence between NGFIDE (1.8)-(1.9) with Volterra integral equation (1.10), we note the following lemmas.

Lemma 2.14 ([14]). *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$. If $f : (a,T] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ for any $x(\cdot) \in C_{1-\gamma,\varrho}[a,T]$, then $x(\cdot) \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$ satisfies IVP* (1.1)-(1.2) *if and only if $x(\cdot)$ satisfies the nonlinear Volterra integral equation.* (1.3)

Lemma 2.15 ([9]). *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$. If $f : (a,T] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ for any $x(\cdot) \in C_{1-\gamma,\varrho}[a,T]$, then $x \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$ satisfies IVP* (1.4)-(1.5) *if and only if x satisfies the nonlinear Volterra integral equation* (1.6).

Using the aforementioned equivalence, we prove a new equivalent mixed-type integral equation for NGFIDE (1.8)-(1.9).

Lemma 2.16. *Let* $0 < \alpha < 1$, $0 \leq \beta \leq 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *Suppose that* $f : (a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a function such that* $f(\cdot, x(\cdot), y(\cdot)) \in C_{1-\gamma,\varrho}[a, T]$ *for any* $x(\cdot) \in C_{1-\gamma,\varrho}[a, T]$. *Function* $x(\cdot) \in C^{\gamma}_{1-\gamma,\varrho}[a, T]$ *is a solution of NGFIDE* (1.8)–(1.9) *if and only if* $x(\cdot)$ *is a solution of the mixed-type nonlinear Volterra integral equation.* (1.10)

Proof. First, we start with necesssary part. By appling Lemma 2.14 and Lemma 2.15, a solution of NGFIDE (1.8)-(1.9) can be expressed as

$$\begin{aligned} x(t) = {}& \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho} \right)^{\gamma-1} \\ & + \int_a^t s^{\varrho-1} \left(\frac{t^{\varrho} - s^{\varrho}}{\varrho} \right)^{\alpha-1} \frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)} \mathrm{d}s. \end{aligned} \tag{2.12}$$

By putting $t = \xi_j$ in (2.12), we obtain

$$\begin{aligned} x(\xi_j) = {}& \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)} \left(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho} \right)^{\gamma-1} \\ & + \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho} \right)^{\alpha-1} \frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)} \mathrm{d}s, \end{aligned} \tag{2.13}$$

and by multiplying both sides of (2.13) by η_j, we get

$$\begin{aligned} \eta_j x(\xi_j) = {}& \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)} \eta_j \left(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho} \right)^{\gamma-1} \\ & + \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho} \right)^{\alpha-1} \frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)} \mathrm{d}s. \end{aligned} \tag{2.14}$$

Using the initial condition of NGFIDE (1.8)-(1.9), we have

$$\begin{aligned} ({}^{\varrho}I^{1-\gamma}_{a+}x)(a+) = \sum_{j=1}^{m} \eta_j x(\xi_j) = {}& \frac{{}^{\varrho}I^{1-\gamma}_{a+}x(a+)}{\Gamma(\gamma)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^{\rho} - a^{\ell}}{\varrho} \right)^{\gamma-1} \\ & + \sum_{j-1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho} \right)^{\alpha-1} \frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)} \mathrm{d}s, \end{aligned}$$

which gives

$$\begin{aligned} & ({}^{\varrho}I^{1-\gamma}_{a+}x)(a+) \left(\Gamma(\gamma) - \sum_{j-1}^{m} \eta_j \left(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho} \right)^{\gamma-1} \right) \\ & = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{j-1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho^{-1}} \left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho} \right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right) \mathrm{d}s, \end{aligned} \tag{2.15}$$

i.e.

$$\left({}^{\varrho}I_{a+}^{1-\gamma}x\right)(a+) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j-1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho^{-1}}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \tag{2.16}$$

where K is as in (1.11). Substituting (2.16) into (2.12), we obtain the integral equation (1.10).

Secondly, we prove the sufficient part.

Applying ${}^{\varrho}I_{a+}^{1-\gamma}$ on both sides of the integral equation (1.10), we get

$$\begin{aligned}{}^{\varrho}I_{a+}^{1-\gamma}x(t) =& \frac{K}{\Gamma(\alpha)}{}^{\varrho}I_{a+}^{1-\gamma}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s\\ &+{}^{\varrho}I_{a+}^{1-\gamma}{}^{\varrho}I_{a+}^{\alpha}f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s,\end{aligned}$$

using Lemmas 2.7 and 2.10 , we have

$$\begin{aligned}{}^{\varrho}I_{a+}^{1-\gamma}x(t) =& \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s\\ &+{}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}f\left(t,x(t),\int_a^t h(t,s)x(s)\mathrm{d}s\right).\end{aligned} \tag{2.17}$$

Since $1-\gamma<1-\beta(1-\alpha)$, Lemma 2.12 can be utilized and limit $t\to a+$ gives

$$\begin{aligned}{}^{\varrho}I_{a+}^{1-\gamma}x(a) =& \frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s.\end{aligned} \tag{2.18}$$

By putting $t=\xi_j$ in (1.10), we have

$$\begin{aligned}x(\xi_j) =& \frac{K}{\Gamma(\alpha)}\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s.$$

Further,

$$\begin{aligned}\sum_{j=1}^m \eta_j x\left(\xi_j\right) &= \frac{K}{\Gamma(\alpha)}\sum_{j-1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\ &\quad \times f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s \sum_{j=1}^m \eta_j\left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \\ &+\sum_{j=1}^m \eta_j \frac{1}{\Gamma(\alpha)}\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s \\ &=\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \frac{f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s \\ &\quad \times\left(1 + K\sum_{j=1}^m \eta_j\left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}\right) \\ &=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}K\sum_{j=1}^m \eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s. \end{aligned}\tag{2.19}$$

Equations (2.18) and (2.19), implies that

$${}^\varrho I_{a+}^{1-\gamma} x(a+) = \sum_{j=1}^m \eta_j x\left(\xi_j\right).$$

Applying ${}^\varrho D_{a+}^{\gamma}$ to both sides of (1.10), from Lemmas 2.10 and 2.14 if follows that

$${}^\varrho D_{a+}^{\gamma} x(t) = {}^\varrho D_{a+}^{\beta(1-\alpha)} f\left(t, x(t), \int_a^t h(t,s)x(s)\mathrm{d}s\right), \tag{2.20}$$

since $x \in C_{1-\gamma,\varrho}^{\gamma}[a,T]$, from the definition of $C_{1-\gamma,\varrho}^{\gamma}[a,T]$ we have ${}^\varrho D_{a+}^{\gamma} x \in C_{1-\gamma,\varrho}[a,T]$ then ${}^\varrho D_{a+}^{\beta(1-\alpha)} f = \delta_\varrho {}^\varrho I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}[a,T]$.

For $f \in C_{1-\gamma,\varrho}[a,T]$, obviously ${}^\varrho I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}[a,T]$, then ${}^\varrho I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\varrho}^{\delta_\varrho}[a,T]$. This means f and ${}^\varrho I_{a+}^{1-\beta(1-\alpha)} f$ satisfy the conditions of Lemma 2.9. Lastly, applying ${}^\varrho I_{a+}^{1-\beta(1-\alpha)}$ to both sides of (2.20), Lemma 2.9 helps us to obtain

$${}^\varrho D_{a+}^{a,\beta} x(t) = f\left(t, x(t), \int_a^t h(t,s)x(s)\mathrm{d}s\right) - \frac{{}^\varrho I_{a+}^{1-\beta(1-\alpha)} f(a)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{\beta(1-\alpha)-1}.$$

By Lemma 2.12 it is easy to see that ${}^\varrho I_{a+}^{1-\beta(1-\alpha)} f(a) = 0$. Hence, it reduces to

$${}^\varrho D_{a+}^{\alpha,\beta} x(t) = f\left(t, x(t), \int_a^t h(t,s)x(s)\mathrm{d}s\right).$$

Hence, the sufficiency is proved. This completes the proof of the lemma. □

3. Existence of solutions

In this section, we state and prove the main results concerning the existence of a solution of NGFIDE (1.8)-(1.9).

By using Krasnoselskii's fixed point theorem we prove the first existence result for NGFIDE (1.8)-(1.9).

Theorem 3.1. *Suppose that:*

(H_{01}) *$f:(a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function such that $f(\cdot,x(\cdot),y(\cdot))\in C^{\beta(1-\alpha)}_{1-\gamma,\varrho}[a,T]$ for any $x\in C_{1-\gamma,\varrho}[a,T]$ and there exists a positive constant $L>0$ such that for all $x,y,\bar{x},\bar{y}\in\mathbb{R}$,*
$|f(t,x,y)-f(t,\bar{x},\bar{y})|\leq L(|x-\bar{x}|+|y-\bar{y}|)$.

(H_{02}) *The constant*

$$\theta=\frac{\Gamma(\gamma)L(1+h_T(T-a))}{\Gamma(\gamma+\alpha)}\left(|K|\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^\varrho-a^\varrho}{\varrho}\right)^{\alpha+\gamma-1}+\left(\frac{T^\varrho-a^\varrho}{\varrho}\right)^{\alpha}\right)$$
<1,

where K is as in (1.11) *and $h_T=Sup\{|h(t,s)|\,|a\leq s\leq t\leq T\}$.*

Then NGFIDE (1.8)-(1.9) *has at least one solution in $C^{\gamma}_{1-\gamma,\varrho}[a,T]\subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T]$.*

Proof. From Lemma 2.16 it is sufficient to prove the existence of a solution for mixed-type integral equation (1.10). Define $N:C_{1-\gamma,\varrho}[a,T]\to C_{1-\gamma,\varrho}[a,T]$ by

$$\begin{aligned}(Nx)(t)=&\frac{K}{\Gamma(\alpha)}\left(\frac{t^\varrho-a^\varrho}{\varrho}\right)^{\gamma-1}\sum_{j-1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}\\&\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s\\&+\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^\varrho-s^\varrho}{\varrho}\right)^{\alpha-1}f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s.\end{aligned}\tag{3.1}$$

Obviously, the operntor N is well defined. Set $\bar{f}(s)=f(s,0,0)$ and

$$\varpi=\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(|K|\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^\varrho-a^\varrho}{\varrho}\right)^{a+\gamma-1}+\left(\frac{T^\varrho-a^\varrho}{\varrho}\right)^{\alpha}\right)\|\bar{f}\|_{C_{1-\gamma,\varrho}}.$$

Consider

$$B_r=\left\{x\in C_{1-\gamma,\varrho}[a,T]:\|x\|_{C_{1-\gamma,\varrho}}\leq r\right\},\quad\text{where } r\geq\frac{\varpi}{1-\theta},\ \theta<1.$$

Now, we subdivide the operator N into two operators P and Q on B_r as follows:

$$(Px)(t) = \frac{K}{\Gamma(\alpha)}\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \times f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \tag{3.2}$$

and

$$(Qx)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s. \tag{3.3}$$

The proof is divided into several steps:

Step 1. For any $x, \bar{x} \in B_r$ we prove $Px + Q\bar{x} \in B_r$. For operator P, multiplying both sides of (3.2) by $\left((t^\varrho - a^\varrho)/\varrho\right)^{1-\gamma}$, we have

$$(Px)(t)\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} = \frac{K}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \times f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s,$$

then

$$\begin{aligned}
&\left|(Px)(t)\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}\right| \\
&\leq \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\left|f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\right|\mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\
&\quad\times\left(\left|f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right) - f(s,0,0)\right| + \left|f(s,0,0)\right|\right)\mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\
&\quad\times\left(L\left(|x(s)| + h_T\int_a^s |x(\tau)|d\tau\right) + |\bar{f}(s)|\right)\mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \\
&\quad\times\left(\left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} L(1 + h_T(T-a))|x(s)| + \left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}|\bar{f}(s)|\right)\mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)}\sum_{j-1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}
\end{aligned}$$

$$\times\left(L(1+h_T(T-a))\|x\|_{C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{\gamma,\varrho}}\right)\mathrm{d}s$$

$$\leq \frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}\mathbf{B}(\alpha,\gamma)\times\left(L(1+h_T(T-a))\|x\|_{C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{1-\gamma,\varrho}}\right),$$

which implies

$$\|Px\|_{C_{1-\gamma,\varrho}}\leq \frac{\Gamma(\gamma)|K|}{\Gamma(\alpha+\gamma)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}$$
$$\times\left(L(1+h_T(T-a))\|x\|_{[C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{1-\gamma,\varrho}}\right). \tag{3.4}$$

For operator Q,

$$\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}(Qx)(t)-\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}$$
$$\times f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\mathrm{d}s, \tag{3.5}$$

using the same fact that we used in the case of operator P again, we obtain

$$\left|(Qx)(t)\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}$$
$$\times\left|f\left(s,x(s),\int_a^s h(s,\tau)x(\tau)d\tau\right)\right|\mathrm{d}s$$
$$\leq\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}$$
$$\times\left(L(1+h_T(T-a))|x(s)|+|\bar{f}(s)|\right)\mathrm{d}s$$
$$\leq\frac{\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)}\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}\left(L(1+h_T(T-a))\|x\|_{C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{1-\gamma,\varrho}}\right).$$

This gives

$$\|Qx\|_{C_{1-\gamma,\varrho}}\leq\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{T^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}$$
$$\times\left(L(1+h_T(T-a))\|x\|_{C_{1-\gamma,\varrho}}+\|\bar{f}\|_{C_{1-\gamma,\varrho}}\right). \tag{3.6}$$

From equations (3.4) and (3.6) for every $x,\bar{x}\in B_r$ we obtain

$$\|Px+Q\bar{x}\|_{C_{1-\gamma,\varrho}}\leq\|Px\|_{C_{1-\gamma,\varrho}}+\|Q\bar{x}\|_{C_{1-\gamma,\varrho}}\leq\theta r+\varpi\leq r$$

which implies that $Px+Q\bar{x}\in B_r$.

Step 2. Now we prove that operator P is a contraction mapping.
Let $x, \bar{x} \in B_r$, for operator P we have,

$$\begin{aligned}
&\Big((Px)(t) - (P\bar{x})(t)\Big)\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} \\
&= \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left(f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\right. \\
&\left. - f\left(s, \bar{x}(s), \int_a^s h(s,\tau)\bar{x}(\tau)d\tau\right)\right)\mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left(\left|f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\right.\right. \\
&\left.\left. - f\left(s, \bar{x}(s), \int_a^s h(s,\tau)\bar{x}(\tau)d\tau\right)\right|\right)\mathrm{d}s \\
&\leq \frac{|K|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_s} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} L(1 + h_T(T-a))|x(s) - \bar{x}(s)|\mathrm{d}s \\
&\leq \frac{L(1 + h_T(T-a))|K|\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \|x - \bar{x}\|_{C_{1-\gamma,\varrho}},
\end{aligned}$$

which is

$$\begin{aligned}
\|Px - P\bar{x}\|_{C_{1-\gamma,\varrho}} \leq{}& \frac{L(1 + h_T(T-a))|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \\
& \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \|x - \bar{x}\|_{C_{1-\gamma,\varrho}} \leq \theta\|x - \bar{x}\|_{C_{1-\gamma,\varrho}}.
\end{aligned}$$

Thus, by assumption (H_{02}), operator P is a contraction mapping.

Step 3. Operator Q is compact and continuous.
Since $f \in C_{1-\gamma,\varrho}[a,T]$, by the definition of $C_{1-\gamma,\varrho}[a,T]$, it is obvious that Q is continuous. By Step 1, we have

$$\begin{aligned}
\|Qx\|_{C_{1-\gamma,\varrho}} \leq{}& \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \left(\frac{T^\varrho - a^\varrho}{\varrho}\right)^{\alpha} \\
&\times \left(L(1 + h_T(T-a))\|x\|_{[C_{1-\gamma,\varrho}} + \|\bar{f}\|_{C_{1-\gamma,\varrho}}\right),
\end{aligned}$$

this means Q is uniformly bounded on B_r.

To prove the compactness of Q, for any $0 < a < t_1 < t_2 \leq T$ we have

$$\begin{aligned}
&\Big|(Qx)(t_1) - (Qx)(t_2)\Big| \\
&\quad = \Bigg| \int_a^{t_1} s^{\varrho-1} \left(\frac{t_1^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \frac{f(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau)}{\Gamma(\alpha)} \mathrm{d}s \\
&\quad - \int_a^{t_2} s^{\varrho-1} \left(\frac{t_2^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \frac{f(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau)}{\Gamma(\alpha)} \mathrm{d}s \Bigg| \\
&\quad \leq \frac{\|f\|_{C_{1-\gamma,\varrho}}}{\Gamma(\alpha)} \Bigg| \int_a^{t_1} s^{\varrho-1} \left(\frac{t_1^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \mathrm{d}s \\
&\quad - \int_a^{t_2} s^{\varrho-1} \left(\frac{t_2^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \mathrm{d}s \Bigg| \\
&\quad \leq \frac{\|f\|_{C_{1-\gamma,\varrho}}\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left| \left(\frac{t_1^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} - \left(\frac{t_2^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \right|. \qquad (3.7)
\end{aligned}$$

The right-hand side of inequality (3.7) tends to zero as $t_2 \to t_1$ either $\alpha + \gamma < 1$ or $\alpha + \gamma \geq 1$. Therefore, Q is equicontinuous. Hence, by Arzelà-Ascoli theorem, Q is compact on B_r.

By applying Krasnoselskii's fixed point theorem, NGFIDE (1.8)-(1.9) has at least one solution $x \in C_{1-\gamma,\varrho}[a,T]$. One can easily show that this solution is actually in $C^\gamma_{1-\gamma,\varrho}[a,T]$ by repeating the process from the proof of Lemma 2.16. Thus, we complete the proof. □

Now, we will discuss the next existence result by using Schauder fixed point theorem. For this, we consider the following hypothesis:

(H_{11}) $f : (a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot), y(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\varrho}[a,T]$ for any $x, y \in C_{1-\gamma,\varrho}[a,T]$, and for all $x, y \in \mathbb{R}$ there exist $L > 0$ and $M \geq 0$ such that

$$|f(t,x,y)| \leq L(|x| + |y|) + M.$$

Theorem 3.2. *Suppose that (H_{11}) and (H_{02}) hold. Then NGFIDE* (1.8)-(1.9) *has at least one solution in* $C^\gamma_{1-\gamma,\varrho}[a,T] \subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T]$.

Proof. Let $B_\varepsilon = \left\{x \in C_{1-\gamma,\varrho}[a,T] : \|x\|_{C_{1-\gamma,\varrho}} \leq \varepsilon\right\}$ with $\varepsilon \geq \Omega/(1-\theta)$ for $\theta < 1$, where

$$\Omega = \frac{M|K|}{\Gamma(\alpha+1)} \sum_{j=1}^m \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^\alpha + \frac{M}{\Gamma(\alpha+1)} \left(\frac{T^\varrho - a^\varrho}{\varrho}\right)^{\alpha-\gamma+1}.$$

Consider the operator N on B_ε defined in (3.1). We prove the theorem in the following three steps:

Step 1. First we prove that $N(B_\varepsilon) \subset B_\varepsilon$. By hypotheses (H_{11}) and (H_{02}), for any $x \in C_{1-\gamma,\varrho}[a,T]$ and $\|x\|_{C_{1-\gamma,\varrho}}$ we have

$$\begin{aligned}
&\left|(Nx)(t)\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}\right| \\
&\leq \Bigg(\frac{L(1+h_T(T-a))\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\sum_{j=1}^m \eta_j\left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \\
&+ \frac{L(1+h_T(T-a))\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(\frac{T^\varrho - a^\varrho}{\varrho}\right)^{\alpha}\Bigg)\|x\|_{C_{1-\gamma,\varrho}} \\
&+ \frac{M}{\Gamma(\alpha+1)}\sum_{j=1}^m \eta_j\left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha} + \frac{M}{\Gamma(\alpha+1)}\left(\frac{T^\varrho - a^\varrho}{\varrho}\right)^{\alpha-\gamma+1}.
\end{aligned}$$

This is $\|Nx\|_{C_{1-\gamma,\varrho}} \leq \theta\varepsilon + \Omega \leq \varepsilon$, which gives $N(B_\varepsilon) \subset B_\varepsilon$.

Next we shall prove that N is completely continuous.

Step 2. N is continuous. Let x_n be a sequence such that $x_n \to x$ in B_ε. Then for each $t \in (a,T]$, we have

$$\begin{aligned}
&\left|((Nx)(x_n) - (Nx)(t))\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}\right| \\
\leq &\frac{|K|}{\Gamma(\alpha)}\sum_{j=1}^m \eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\Bigg|f\left(s, x_n(s), \int_a^s h(s,\tau)x_n(\tau)d\tau\right) \\
&- f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\Bigg|\mathrm{d}s \\
&+ \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\
&\times\left|f\left(s, x_n(s), \int_a^s h(s,\tau)x_n(\tau)d\tau\right) - f\left(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau\right)\right|\mathrm{d}s \\
\leq &\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(|K|\sum_{j=1}^m \eta_j\left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} + \left(\frac{T^\varrho - a^\varrho}{\varrho}\right)^{\alpha}\right) \\
&\times\left\|f\left(\cdot, x_n(\cdot), \int_a^s h(s,\tau)x_n(\cdot)d\tau\right) - f\left(\cdot, x(\cdot), \int_a^s h(s,\tau)x(\cdot)d\tau\right)\right\|_{C_{1-\gamma,\varrho}},
\end{aligned}$$

this implies

$$\begin{aligned}
\left\|Nx_n - Nx\right\|_{C_{1-\gamma,\varrho}} \leq &\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(|K|\sum_{j=1}^m \eta_j\left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} + \left(\frac{T^\varrho - a^\varrho}{\varrho}\right)^{\alpha}\right) \\
&\times\left\|f\left(\cdot, x_n(\cdot), \int_a^s h(s,\tau)x_n(\tau)d\tau\right) - f\left(\cdot, x(\cdot), \int_a^s h(s,\tau)x(\cdot)d\tau\right)\right\|_{C_{1-\gamma,\alpha}}.
\end{aligned}$$

Thus, N is a continuous operator.

Step 3. Finally, we prove that $N(B_\varepsilon)$ is relatively compact.
Since $N(B_\varepsilon) \subset B_\varepsilon$, it follows that $N(B_\varepsilon)$ is uniformly bounded.

By repeating the same process as in Step 3 in Theorem 3.1 , one can easily prove that N is equicontinuous on B_ε.

As $\alpha \leq \gamma < 1$ and noting (3.7), for any $0 < a < t_1 < t_2 \leq T$ one has

$$\begin{aligned}
&|(Nx)(t_1) - (Nx)(t_2)| \\
&\leq \frac{\|f\|_{C_{1-\gamma,\varrho}}|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \\
&\times \left(\left(\frac{t_1^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} - \left(\frac{t_2^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}\right) + |(Qx)(t_1) - (Qx)(t_2)| \\
&\leq \frac{\|f\|_{C_{1-\gamma,\varrho}}|K|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \left|\frac{t_2^\varrho - t_1^\varrho}{(t_1^\varrho - a^\varrho)(t_2^\varrho - a^\varrho)}\right|^{1-\gamma} \\
&+ \frac{\|f\|_{C_{1-\gamma,\varrho}}\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \left|\left(\frac{t_1^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} - \left(\frac{t_2^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1}\right| \to 0,
\end{aligned}$$

as $t_2 \to t_1$. Thus, Q is equicontinuous.

Hence, $N(B_\varepsilon)$ is an equicontinuous set and therefore $N(B_\varepsilon)$ is relatively compact. As a consequence of Steps 1 to 3 together with Arzelà-Ascoli theorem, we can conclude that $N : B_\varepsilon \to B_\varepsilon$ is completely continuous. By applying Schauder fixed point theorem, we complete the proof. □

4. Example

In this section, we will show the applications of our main results with two examples.

Example 4.1. Consider the nonlocal problem

$$\left({}^\varrho D_{a+}^{\alpha,\beta}\right)x(t) = f(t, x(t), Hx(t)),\ t \in (1, 2], \tag{4.1}$$

$$\left({}^\varrho I_{a+}^{1-\gamma}x\right)(1+) = 2x\left(\frac{5}{3}\right), \quad \gamma = \alpha + \beta(1-\alpha). \tag{4.2}$$

Denoting $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$ gives $\gamma = \frac{7}{8}$. Let $\varrho = \frac{1}{2} > 0$ and set

$$f(t, x(t), Hx(t)) = \left(\frac{t^\varrho - 1}{\varrho}\right)^{-1/16} + \frac{1}{4}\left(\frac{t^\varrho - 1}{\varrho}\right)^{15/16} \sin x(t) + \frac{1}{4}Hx(t),$$

where

$$Hx(t) = \int_1^t \frac{1}{(3+t)^2} x(s)\mathrm{d}s.$$

We can see that

$$\left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{1/8} f(t,x(t),Hx(t)) = \left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{1/16} + \frac{1}{4}\left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{17/16} \sin x(t) + \frac{1}{4}\left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{1/8} Hx(t) \in C[1,2] \tag{4.3}$$

i.e. $f(t,x,Hx(t)) \in C_{1/8,1/2}[1,2]$.
Moreover,

$$|f(t,x,Hx(t)) - f(t,\bar{x},H\bar{x}(t))| \leq \frac{1}{4}\Big(|x-\bar{x}| + |Hx(t) - H\bar{x}(t)|\Big).$$

So, we have $L = \frac{1}{4}$, $h_T = \frac{1}{16}$.
Some elementary computations gives us

$$|K| = \left|\left(\Gamma(0.875) - 2\left(\frac{\left(\frac{5}{3}\right)^{1/2}-1}{\frac{1}{2}}\right)^{-1/8}\right)^{-1}\right| \approx 0.9521 < 1$$

and

$$\begin{aligned}\theta &= \frac{\Gamma(0.875)\frac{1}{4}(1+\frac{1}{16}(2-1))}{4\Gamma(1.625)} \\ &\times \left(|K| \times 2\left(\frac{\left(\frac{5}{3}\right)^{1/2}-1}{\frac{1}{2}}\right)^{5/8} + \left(\frac{2^{1/2}-1}{\frac{1}{2}}\right)^{3/4}\right) \\ &\approx 0.17964219 < 1.\end{aligned}$$

All the assumptions of Theorem 3.1 are satisfied with

$$|K| \approx 0.9521 \text{ and } \theta \approx 0.17964219.$$

Therefore, problem (4.1)-(4.2) has at least one solution in $C_{1/8,1/2}[1,2]$.

Example 4.2. Consider the nonlocal problem

$$\left({}^{\varrho}D^{\alpha,\beta}_{a+}x\right)(t) = f(t,x(t),Hx(t)),\ t \in (1,2], \tag{4.4}$$

$$\left({}^{\varrho}I^{1-\gamma}_{a+}x\right)(1+) = 3x\left(\frac{8}{7}\right) + 2x\left(\frac{4}{3}\right). \tag{4.5}$$

Denote $\alpha = \frac{1}{2}$, $\beta = \frac{3}{4}$ and $\varrho = \frac{1}{2} > 0$. So $\gamma = \frac{7}{6}$ and $\left(\xi_1 = \frac{8}{7}\right) \leq \left(\xi_2 = \frac{4}{3}\right)$. Set

$$f(t,x(t),Hx(t)) = \sin\left(\frac{1}{3}|x(t)|\right) + \frac{1}{3}Hx(t),\ t \in (1,2],$$

where

$$Hx(t) = \int_1^t \frac{1}{(3+t)^2}x(s)\mathrm{d}s.$$

It is easy to see that $f(t,x(t),Hx(t)) \in C_{1/8,1/2}[1,2]$ and

$$|f(t,x,Hx(t))| \le \frac{1}{3}\Big(|x| + |Hx(t)|\Big).$$

So, we have $L = \frac{1}{3}$, $M = 0$, $h_T = \frac{1}{16}$. Moreover,

$$|K| = \left|\left(\Gamma(0.875) - \left(3\left(\frac{\left(\frac{8}{7}\right)^{1/2} - 1}{\frac{1}{2}}\right)^{-1/8} + 2\left(\frac{\left(\frac{4}{3}\right)^{1/2} - 1}{\frac{1}{2}}\right)^{-1/8}\right)\right)^{-1}\right| \approx 0.1973 < 1$$

and

$$\begin{aligned}\theta = {} & \frac{\Gamma(0.875)\frac{1}{3}(1 + \frac{1}{16}(2-1))}{3\Gamma(1.375)} \\ & \times \left(|K| \times 3\left(\frac{\left(\frac{8}{7}\right)^{1/2} - 1}{\frac{1}{2}}\right)^{3/8} + 2\left(\frac{\left(\frac{4}{3}\right)^{1/2} - 1}{\frac{1}{2}}\right)^{3/8}\right) \approx 0.2515 < 1\end{aligned}$$

With the values of $|K|$ and θ, problem (4.4)-(4.5) satisfies all the conditions of Theorem 3.2. Thus, problem (4.4)-(4.5) has at least one solution in $C_{1/8,1/2}[1,2]$.

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An extension of Wirtinger's inequality to the complex integral

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Abstract. In this paper we establish a natural extension of the Wirtinger inequality to the case of complex integral of analytic functions. Applications related to the trapezoid inequalities are also provided. Examples for logarithmic and exponential complex functions are given as well.

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1. Introduction

It is well known that, see for instance [4], or [7], if $u \in C^1([a,b],\mathbb{R})$ satisfies $u(a) = u(b) = 0$, then we have the *Wirtinger inequality*

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K \in \mathbb{R}$.

If $u \in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a) = 0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t) = L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L \in \mathbb{R}$.

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If $h \in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a) = h(b) = 0$, then $\mathrm{Re}h(a) = \mathrm{Re}h(b) = 0$ and $\mathrm{Im}h(a) = \mathrm{Im}h(b) = 0$ and by writing (1.1) for $\mathrm{Re}h$ and $\mathrm{Im}h$ and adding the obtained inequalities, we get

$$\int_a^b |h(t)|^2 \, dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |h'(t)|^2 \, dt \tag{1.3}$$

with the equality holding if and only if

$$h(t) = K \sin\left[\frac{\pi (t-a)}{b-a}\right]$$

for some complex constant $K \in \mathbb{C}$.

Similarly, if $h \in C^1([a,b],\mathbb{C})$ with $h(a) = 0$, then by (1.2) we have

$$\int_a^b |h(t)|^2 \, dt \leq \frac{4(b-a)^2}{\pi^2} \int_a^b |h'(t)|^2 \, dt \tag{1.4}$$

and the equality holds if and only if

$$h(t) = L \sin\left[\frac{\pi (t-a)}{2(b-a)}\right]$$

for some complex constant $L \in \mathbb{R}$.

For some related Wirtinger type integral inequalities see [1], [2], [4] and [6]-[9].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\, |dz| := \int_a^b f(z(t))\, |z'(t)|\, dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\, dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z) g'(z)\, dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z)\, dz. \tag{1.5}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_{\gamma} f(z)\, dz \right| \le \int_{\gamma} |f(z)|\, |dz| \le \|f\|_{\gamma,\infty} \ell(\gamma) \tag{1.6}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^p\, |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_{\gamma} |f(z)|\, |dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$

In this paper we establish a natural extension of the Wirtinger inequality to the case of complex integral of analytic functions. Applications related to the trapezoid inequalities are also provided. Examples for logarithmic and exponential complex functions are given as well.

2. Wirtinger type inequalities

We have the following weighted version of Wirtinger inequality:

Lemma 2.1. *Let $g : [a,b] \to [g(a), g(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b).*

(i) *If $h \in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a) = h(b) = 0$, then*

$$\int_a^b |h(t)|^2 g'(t)\, dt \le \frac{[g(b) - g(a)]^2}{\pi^2} \int_a^b \frac{|h'(t)|^2}{g'(t)} dt. \tag{2.1}$$

The equality holds in (2.1) iff

$$h(t) = K \sin\left[\frac{\pi (g(t) - g(a))}{g(b) - g(a)} \right],\ K \in \mathbb{C}.$$

(ii) *If $h \in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a) = 0$, then*

$$\int_a^b |h(t)|^2 g'(t)\, dt \le \frac{4[g(b) - g(a)]^2}{\pi^2} \int_a^b \frac{|h'(t)|^2}{g'(t)} dt. \tag{2.2}$$

The equality holds in (2.2) iff

$$h(t)=K\sin\left[\frac{\pi(g(t)-g(a))}{2(g(b)-g(a))}\right],\ K\in\mathbb{C}.$$

Proof. (i) We write the inequality (1.3) for the function $h=h\circ g^{-1}$ on the interval $[g(a),g(b)]$ to get

$$\int_{g(a)}^{g(b)}\left|\left(h\circ g^{-1}\right)(z)\right|^2dz\le\frac{(g(b)-g(a))^2}{\pi^2}\int_{g(a)}^{g(b)}\left|\left(h\circ g^{-1}\right)'(z)\right|^2dz. \tag{2.3}$$

If $h:[c,d]\to\mathbb{C}$ is absolutely continuous on $[c,d]$, then $h\circ g^{-1}:[g(c),g(d)]\to\mathbb{C}$ is absolutely continuous on $[g(c),g(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(h\circ g^{-1}\right)'(z)=\left(h'\circ g^{-1}\right)(z)\left(g^{-1}\right)'(z)=\frac{\left(h'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)} \tag{2.4}$$

for almost every (a.e.) $z\in[g(c),g(d)]$.

Using the inequality (2.3) we then get

$$\int_{g(a)}^{g(b)}\left|\left(h\circ g^{-1}\right)(z)\right|^2dz\le\frac{(g(b)-g(a))^2}{\pi^2}\int_{g(a)}^{g(b)}\left|\frac{\left(h'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)}\right|^2dz, \tag{2.5}$$

provided $\left(h\circ g^{-1}\right)(g(a))=h(a)=0$ and $\left(h\circ g^{-1}\right)(g(b))=h(b)=0.$

Observe also that, by the change of variable $t=g^{-1}(z)$, $z\in[g(a),g(b)]$, we have $z=g(t)$ that gives $dz=g'(t)\,dt$ and

$$\int_{g(a)}^{g(b)}\left|\left(h\circ g^{-1}\right)(z)\right|^2dz=\int_a^b|h(t)|^2g'(t)\,dt. \tag{2.6}$$

We also have

$$\int_{g(a)}^{g(b)}\left|\frac{\left(h'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)}\right|^2dz=\int_a^b\left|\frac{h'(t)}{g'(t)}\right|^2g'(t)\,dt=\int_a^b\frac{|h'(t)|^2}{g'(t)}dt.$$

By making use of (2.5) we get (2.1).

The equality holds in (2.5) provided

$$\left(h\circ g^{-1}\right)(z)=K\sin\left[\frac{\pi(z-g(a))}{g(b)-g(a)}\right],\ K\in\mathbb{C}$$

for $z\in[g(a),g(b)]$. If we take $t\in[a,b]$ and $z=g(t)$, we then get

$$h(t)=K\sin\left[\frac{\pi(g(t)-g(a))}{g(b)-g(a)}\right],\ K\in\mathbb{C}.$$

(ii) Follows in a similar way by (1.4). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 2.2. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $h \in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a) = h(b) = 0$, then*

$$\int_a^b |h(t)|^2 w(t)\,dt \le \frac{1}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|h'(t)|^2}{w(t)}dt. \tag{2.7}$$

The equality holds in (2.7) iff

$$h(t) = K\sin\left[\frac{\pi\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\right],\ K \in \mathbb{C}.$$

If $h(a) = 0$, then

$$\int_a^b |h(t)|^2 w(t)\,dt \le \frac{4}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|h'(t)|^2}{w(t)}dt \tag{2.8}$$

with equality iff

$$h(t) = K\sin\left[\frac{\pi\int_a^t w(s)\,ds}{2\int_a^b w(s)\,ds}\right],\ K \in \mathbb{C}.$$

We have the following Wirtinger type inequality for complex functions:

Theorem 2.3. *Let f be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ and $z'(t) \neq 0$ for $t \in (a,b)$.*

(i) *If $f(u) = f(w) = 0$, then*

$$\int_\gamma |f(z)|^2 |dz| \le \frac{1}{\pi^2}\ell^2(\gamma)\int_\gamma |f'(z)|^2 |dz|. \tag{2.9}$$

The equality holds in (2.9) iff

$$f(v) = K\sin\left[\frac{\pi\ell(\gamma_{u,v})}{\ell(\gamma)}\right],\ K \in \mathbb{C} \tag{2.10}$$

where $v = z(t)$, $t \in [a,b]$ and $\ell(\gamma_{u,v}) = \int_a^t |z'(s)|\,ds$.

(ii) *If $f(u) = 0$, then*

$$\int_\gamma |f(z)|^2 |dz| \le \frac{4}{\pi^2}\ell^2(\gamma)\int_\gamma |f'(z)|^2 |dz|. \tag{2.11}$$

The equality holds in (2.11) iff

$$f(v) = K\sin\left[\frac{\pi\ell(\gamma_{u,v})}{2\ell(\gamma)}\right],\ K \in \mathbb{C} \tag{2.12}$$

where $v = z(t)$, $t \in [a,b]$.

Proof. (i) Consider the function $h(t) = f(z(t))$ and $w(t) = |z'(t)|$, $t \in [a,b]$. Then $h'(t) = (f(z(t)))' = f'(z(t))z'(t)$ for $t \in (a,b)$. Also $h(a) = f(z(a)) = f(u) = 0$

and $h(b) = f(z(b)) = f(w) = 0$. By utilising the inequality (2.7) for these choices, we get

$$\begin{aligned}\int_a^b |f(z(t))|^2 |z'(t)| \, dt &\le \frac{1}{\pi^2} \left(\int_a^b |z'(s)| \, ds \right)^2 \int_a^b \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt \\ &= \frac{1}{\pi^2} \left(\int_a^b |z'(s)| \, ds \right)^2 \int_a^b \frac{|f'(z(t))|^2 |z'(t)|^2}{|z'(t)|} dt \\ &= \frac{1}{\pi^2} \left(\int_a^b |z'(s)| \, ds \right)^2 \int_a^b |f'(z(t))|^2 |z'(t)| \, dt,\end{aligned}$$

which is equivalent to (2.9).

The equality (2.10) follows by the corresponding equality in Corollary 2.2.

(ii) Follows by the corresponding result from Corollary 2.2. □

3. Some trapezoid type inequalities

We have:

Proposition 3.1. *Let g be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$, $w \neq u$ and $z'(t) \neq 0$ for $t \in (a,b)$. Then*

$$\begin{aligned}&\left| \frac{1}{w-u} \int_\gamma g(z) \, dz - \frac{g(u) + g(w)}{2} \right| \\ &\qquad \le \frac{1}{\pi} \frac{\ell(\gamma)}{|w-u|} \left(\frac{1}{\ell(\gamma)} \int_\gamma \left| g'(z) - \frac{g(w) - g(u)}{w-u} \right|^2 |dz| \right)^{1/2}. \quad (3.1)\end{aligned}$$

Proof. Consider the function $f : G \to \mathbb{C}$ defined by

$$f(z) := g(z) - \frac{g(u)(w-z) + g(w)(z-u)}{w-u}, \ z \in \gamma.$$

Observe that $f(u) = f(w) = 0$ and by (2.9) we get

$$\begin{aligned}&\int_\gamma \left| g(z) - \frac{g(u)(w-z) + g(w)(z-u)}{w-u} \right|^2 |dz| \\ &\qquad \le \frac{1}{\pi^2} \ell^2(\gamma) \int_\gamma \left| g'(z) - \frac{g(w) - g(u)}{w-u} \right|^2 |dz|. \quad (3.2)\end{aligned}$$

Using Cauchy-Bunyakovsky-Schwarz integral inequality we also have

$$\left|\int_\gamma \left[g(z) - \frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right] dz\right|^2$$
$$\leq \int_\gamma |dz| \int_\gamma \left|g(z) - \frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2 |dz|$$
$$= \ell(\gamma) \int_\gamma \left|g(z) - \frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2 |dz|$$

and since

$$\int_\gamma \left[g(z) - \frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right] dz$$
$$= \int_\gamma g(z)\,dz - \frac{g(u)\int_\gamma (w-z)\,dz + g(w)\int_\gamma (z-u)\,dz}{w-u}$$
$$= \int_\gamma g(z)\,dz - \frac{g(u)+g(w)}{2}(w-u),$$

hence

$$\left|\int_\gamma g(z)\,dz - \frac{g(u)+g(w)}{2}(w-u)\right|^2$$
$$\leq \ell(\gamma) \int_\gamma \left|g(z) - \frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2 |dz|$$

and by (3.2) we get

$$\left|\int_\gamma g(z)\,dz - \frac{g(u)+g(w)}{2}(w-u)\right|^2$$
$$\leq \ell(\gamma) \int_\gamma \left|g(z) - \frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2 |dz|$$
$$\leq \frac{1}{\pi^2}\ell^3(\gamma) \int_\gamma \left|g'(z) - \frac{g(w)-g(u)}{w-u}\right|^2 |dz|,$$

which implies the desired result (3.1). □

We also have:

Proposition 3.2. *Let g be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$,*

$w \neq u$ and $z'(t) \neq 0$ for $t \in (a,b)$. If $u+w-z \in G$ for $z \in \gamma$, then

$$\left|\frac{1}{w-u}\int_{\gamma}\widehat{g(z)}dz - \frac{g(u)+g(w)}{2}\right| \leq \frac{1}{2\pi}\frac{\ell(\gamma)}{|w-u|}\left(\frac{1}{\ell(\gamma)}\int_{\gamma}|g'(z)-g'(u+w-z)|^{2}|dz|\right)^{1/2}, \quad (3.3)$$

where

$$\widehat{g(z)} := \frac{g(z)+g(u+w-z)}{2}, \ z \in \gamma.$$

Proof. Consider the function $f : G \to \mathbb{C}$ defined by

$$f(z) := \frac{g(z)+g(u+w-z)}{2} - \frac{g(u)+g(w)}{2}, \ z \in \gamma.$$

Observe that $f(u) = f(w) = 0$ and by (2.9) we get

$$\int_{\gamma}\left|\frac{g(z)+g(u+w-z)}{2} - \frac{g(u)+g(w)}{2}\right|^{2}|dz| \leq \frac{1}{4\pi^{2}}\ell^{2}(\gamma)\int_{\gamma}|g'(z)-g'(u+w-z)|^{2}|dz|. \quad (3.4)$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\left|\int_{\gamma}\left[\frac{g(z)+g(u+w-z)}{2} - \frac{g(u)+g(w)}{2}\right]dz\right|^{2} \leq \int_{\gamma}|dz|\int_{\gamma}\left|\frac{g(z)+g(u+w-z)}{2} - \frac{g(u)+g(w)}{2}\right|^{2}|dz|,$$

which gives

$$\left|\int_{\gamma}\frac{g(z)+g(u+w-z)}{2}dz - \frac{g(u)+g(w)}{2}(w-u)\right|^{2} \leq \ell(\gamma)\int_{\gamma}\left|\frac{g(z)+g(u+w-z)}{2} - \frac{g(u)+g(w)}{2}\right|^{2}|dz|.$$

By (3.4) we then get

$$\left|\int_{\gamma}\frac{g(z)+g(u+w-z)}{2}dz - \frac{g(u)+g(w)}{2}(w-u)\right|^{2} \leq \frac{1}{4\pi^{2}}\ell^{3}(\gamma)\int_{\gamma}|g'(z)-g'(u+w-z)|^{2}|dz|,$$

which is equivalent to (3.3). □

Corollary 3.3. *With the assumption of Proposition 3.2 and if*

$$|g'(z)-g'(u+w-z)| \leq |2z-u-w|\,L$$

for some $L>0$ *and for any* $z\in\gamma$, *then*

$$\left|\frac{1}{w-u}\int_\gamma \widehat{g(z)}dz-\frac{g(u)+g(w)}{2}\right|$$
$$\leq \frac{1}{\pi}\frac{\ell(\gamma)L}{|w-u|}\left(\frac{1}{\ell(\gamma)}\int_\gamma\left|z-\frac{u+w}{2}\right|^2|dz|\right)^{1/2}. \quad (3.5)$$

Remark 3.4. If $\|g''\|_{\infty,G}:=\sup_{z\in G}|g''(z)|<\infty$, then we can take above
$$L=\|g''\|_{\infty,G}\,.$$

4. Some examples for logarithmic and exponential functions

Consider the function $g(z)=\frac{1}{z}$, $z\in\mathbb{C}\backslash\{0\}$. Then
$$g^{(k)}(z)=\frac{(-1)^k k!}{z^{k+1}}\text{ for }k\geq 0,\ z\in\mathbb{C}\backslash\{0\}$$
and suppose $\gamma\subset\mathbb{C}_\ell:=\mathbb{C}\backslash\{x+iy:x\leq 0,\ y=0\}$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where $u,\ w\in\mathbb{C}_\ell$ and $z'(t)\neq 0$ for $t\in(a,b)$. Then
$$\int_\gamma g(z)\,dz=\int_{\gamma_{u,w}}g(z)\,dz=\int_{\gamma_{u,w}}\frac{dz}{z}=\operatorname{Log}(w)-\operatorname{Log}(u)$$
for $u,\ w\in\mathbb{C}_\ell$.

By making use of the inequality (3.1) we get
$$\left|\frac{\operatorname{Log}(w)-\operatorname{Log}(u)}{w-u}-\frac{u+w}{2uw}\right|$$
$$\leq\frac{1}{\pi}\frac{\ell(\gamma)}{|w-u||wu|}\left(\frac{1}{\ell(\gamma)}\int_\gamma\left|\frac{z^2-uw}{z^2}\right|^2|dz|\right)^{1/2}. \quad (4.1)$$

Observe also that
$$\int_{\gamma_{u,w}}\frac{dz}{u+w-z}=-\operatorname{Log}(u+w-z)\big|_u^w=-\operatorname{Log}(u)+\operatorname{Log}(w),$$
therefore, by the inequality (3.3) we get
$$\left|\frac{\operatorname{Log}(w)-\operatorname{Log}(u)}{w-u}-\frac{u+w}{2uw}\right|$$
$$\leq\frac{1}{\pi}\frac{|w+u|\,\ell(\gamma)}{|w-u|}\left(\frac{1}{\ell(\gamma)}\int_\gamma\left|\frac{z-\frac{u+w}{2}}{z^2(u+w-z)^2}\right|^2|dz|\right)^{1/2}. \quad (4.2)$$

Consider the function $g(z)=\exp z$, $z\in\mathbb{C}$. Suppose γ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where $u,\ w\in\mathbb{C}$ and $z'(t)\neq 0$ for $t\in(a,b)$.

By making use of the inequality (3.1) we get

$$\left|\frac{\exp w-\exp u}{w-u}-\frac{\exp u+\exp w}{2}\right| \\ \leq \frac{1}{\pi}\frac{\ell(\gamma)}{|w-u|}\left(\frac{1}{\ell(\gamma)}\int_{\gamma}\left|\exp z-\frac{\exp w-\exp u}{w-u}\right|^{2}|dz|\right)^{1/2}, \quad (4.3)$$

while from (3.3) we get

$$\left|\frac{\exp w-\exp u}{w-u}-\frac{\exp u+\exp w}{2}\right| \\ \leq \frac{1}{2\pi}\frac{\ell(\gamma)}{|w-u|}\left(\frac{1}{\ell(\gamma)}\int_{\gamma}|\exp z-\exp(u+w-z)|^{2}|dz|\right)^{1/2}. \quad (4.4)$$

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Theorems regarding starlikeness and convexity

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Abstract. Giving the sharp version of an univalence condition means to give the final response to an open question. We prove in this paper the sharp version of a starlikeness condition. The basic tool in our study is the convolution theory. We present a short history of the problem before the proof of the main result.

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1. Introduction

The class $\mathcal{A}$ is the set of analytic functions f defined in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

The subset of $\mathcal{A}$ which contains univalent functions is denoted by S.
The class K consists of all functions $f \in S$ for which the domain $f(\mathbb{D})$ is convex in $\mathbb{C}$. We have

$$K = \{f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D}\}.$$

The class of starlike functions is defined as follows

$$S^* = \{f \in S : f(\mathbb{D})\text{is starlike with respect to the origin}\}.$$

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We have

$$S^* = \{f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D}\}.$$

The class of strongly starlike functions of order α, $\alpha \in (0,1]$ is defined by

$$SS^*(\alpha) = \{f \in \mathcal{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{D}\}.\text{ We have } SS^*(\alpha) \subset S^*.$$

In [8] it is proved that

$$f \in \mathcal{A}, \quad \mathrm{Re}[f'(z) + zf''(z)] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^* \tag{1.1}$$

and in [6] this result is improved as follows

$$f \in \mathcal{A}, \quad \mathrm{Re}[f'(z) + zf''(z)] > -\frac{\pi^2 - 6}{24 - \pi^2} \approx -0.27,\ z \in \mathbb{D} \Rightarrow f \in S^*.$$

These two results lead to a big number of papers regarding these questions and analogous ones. We present a few improvements in the followings.

In [6] the conjecture is stated that the biggest c for which

$$f \in \mathcal{A}, \quad \mathrm{Re}\left[f'(z) + zf''(z)\right] > -c, \forall z \in \mathbb{D} \Rightarrow f \in S^*$$

is $c = \frac{\ln 4 - 1}{2 - \ln 4} \approx 0.629$.
This conjecture was proved in [10].
Professor P. Mocanu improved the result (1.1) in an other direction. He proved in [4] the implication

$$f \in \mathcal{A}, \quad \mathrm{Re}\left[f'(z) + \frac{z}{2}f''(z)\right] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^*. \tag{1.2}$$

In [3] Corollary 5.5j.1 it is given the following improvement of (1.2).

Theorem 1.1. *If $f \in \mathcal{A}$ and $-1 < \gamma < \gamma_0 = 1.869\ldots$, then*

$$\mathrm{Re}\left[f'(z) + \frac{1}{1+\gamma}zf''(z)\right] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^*, \tag{1.3}$$

$(\frac{1}{1+\gamma_0} \approx 0.348)$.

Two improvements of (1.2) and (1.3) are given in [9] and [12].

Theorem 1.2. [9] *The biggest value of c for which the implication holds*

$$f \in \mathcal{A}, \quad \mathrm{Re}\left[f'(z) + \frac{z}{2}f''(z)\right] > -c,\ z \in \mathbb{D} \Rightarrow f \in S^*, \tag{1.4}$$

it is $c = \frac{3 - \ln 16}{\ln 16 - 2} \approx 0.294$.

Theorem 1.3. [12] *The implication holds*

$$f \in \mathcal{A}, \quad \mathrm{Re}\left[f'(z) + \frac{z}{7}f''(z)\right] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^*. \tag{1.5}$$

In [12] it is also proved that if λ_0 is the smallest positive value for which the implication holds

$$f \in \mathcal{A}, \quad \Big[f'(z) + \lambda_0 z f''(z)\Big] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^*,$$

then we have $\lambda_0 \in (\frac{1}{8}, \frac{1}{7})$.

The following improvements of (1.1) are proved in [5].

Theorem 1.4. *If $f \in \mathcal{A}$ then the implications hold:*

$$\operatorname{Re}\Big[f'(z) + zf''(z)\Big] > 0,\ z \in \mathbb{D} \Rightarrow \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{3}; \tag{1.6}$$

$$\Big|\arg[f'(z) + zf''(z)]\Big| < \frac{2\pi}{3},\ z \in \mathbb{D} \Rightarrow \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D}; \tag{1.7}$$

$$\operatorname{Re}\Big[f'(z) + \frac{z}{2} f''(z)\Big] > 0,\ z \in \mathbb{D} \Rightarrow \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{4\pi}{9},\ z \in \mathbb{D}; \tag{1.8}$$

$$\Big|\arg[f'(z) + zf''(z)]\Big| < \frac{5\pi}{9},\ z \in \mathbb{D} \Rightarrow \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D}. \tag{1.9}$$

In [3] it is proved the following result. (Corollary5.2d.1)

Theorem 1.5. [3] *If $f \in \mathcal{A}$ and $\operatorname{Re}[zf''(z)] > -\frac{3}{7}$, $z \in \mathbb{D}$, then the function F defined by $F(z) = \frac{2}{z}\int_0^z f(t)dt$ belongs to the class K.*

The sharp version of this theorem is the next result and it is given in [11].

Theorem 1.6. [11] *If $f \in \mathcal{A}$ and*

$$\operatorname{Re}[zf''(z)] > -1,\ z \in \mathbb{D},$$

then the function F defined by $F(z) = \frac{2}{z}\int_0^z f(t)dt$ belongs to the class S^*. If $\operatorname{Re}[zf''(z)] > -c$, $z \in \mathbb{D}$ and $c > 1$, then there are functions $f \in \mathcal{A}$ which verify this inequality and F does not belong to S.

This theorem can be given in the following equivalent form.

Theorem 1.7. *If $F \in \mathcal{A}$ is a function with the property*

$$\operatorname{Re}\Big(zF''(z) + \frac{z^2}{3} F'''(z)\Big) > -\frac{2}{3},\ z \in \mathbb{D}, \tag{1.10}$$

then F belongs to the class K.

2. Preliminaries

Our main result it is analogous to Theorem 1.7 and Theorem 1.6. In order to prove it, we need the following lemmas.

Lemma 2.1. [7] *If $f \in \mathcal{A}$, then f is starlike if and only if*

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0 \text{ for every } z \in \mathbb{D} \text{ and } T \in \mathbb{R},$$

where

$$h_T(z) = \frac{iT\frac{z}{1-z} + \frac{z}{(1-z)^2}}{1+iT} = 1 + \sum_{n=1}^{\infty} \frac{n+1+iT}{1+iT} z^n, \ T \in \mathbb{R}, z \in \mathbb{D}.$$

Lemma 2.2. [7] *Let $\mathcal{H}(\mathbb{D})$ be the set of analytic functions in $\mathbb{D}$. The class of functions with positive real part is denoted by $\mathcal{P}$ and it is defined by the equality*

$$\mathcal{P} = \{f \in \mathcal{H}(\mathbb{D}) : f(0) = 1, \ \operatorname{Re} f(z) > 0, \ z \in \mathbb{D}\}.$$

Provided that $g(0) = 1$ we have $f(z) * g(z) \neq 0$, $(\forall) z \in \mathbb{D}$, $(\forall)$ $f \in \mathcal{P}$ if

$$\operatorname{Re} g(z) > \frac{1}{2}, \ z \in \mathbb{D}.$$

3. Main result

Theorem 3.1. *Let $f \in \mathcal{A}$. If*

$$\operatorname{Re}[zf''(z) + \frac{z^2}{2} f'''(z)] > -c, \ z \in \mathbb{D}, \tag{3.1}$$

where $c = \frac{1}{4(2\ln 2 - 1)}$, then we have $f \in S^*$. The result is sharp.

Proof. The condition (3.1) is equivalent to

$$\frac{c + zf''(z) + \frac{z^2}{2} f'''(z)}{c} \in \mathcal{P},$$

and the Herglotz representation formula gives

$$\frac{c + zf''(z) + \frac{z^2}{2} f'''(z)}{c} = \int_0^{2\pi} \frac{1 + e^{-it}z}{1 - e^{-it}z} d\mu(t), z \in \mathbb{D}. \tag{3.2}$$

If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

then (3.2) leads to

$$1 + \frac{1}{c} \sum_{n=2}^{\infty} a_n \frac{n^2(n-1)}{2} z^{n-1} = 1 + 2 \sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} d\mu(t), z \in \mathbb{D}.$$

Thus we get

$$a_n = \frac{4c}{n^2(n-1)} \int_0^{2\pi} e^{-i(n-1)t} d\mu(t), \ n \in N, \ n \geq 2$$

and

$$f(z) = z + 4c \sum_{n=2}^{\infty} \frac{z^n}{n^2(n-1)} \int_0^{2\pi} e^{-i(n-1)t} d\mu(t). \tag{3.3}$$

According to Lemma 2.1 the condition of starlikeness can be rewritten as follows

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} = \left(1 + 4c \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)^2} \int_0^{2\pi} e^{-int} d\mu(t)\right) * \left(1 + \sum_{n=1}^{\infty} \frac{n+1+iT}{1+iT} z^n\right)$$

$$= \left(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} d\mu(t)\right) * \left(1 + 2c\sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)^2 n} z^n\right) \neq 0. \tag{3.4}$$

Lemma 2.2 implies that the condition of starlikeness (3.4) holds if

$$\operatorname{Re}\left(1 + 2c\sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)^2 n} z^n\right) > \frac{1}{2}, \quad (\forall)\ z \in \mathbb{D}, T \in \mathbb{R} \tag{3.5}$$

or equivalently

$$\operatorname{Re}\left(\frac{1}{4c} + \sum_{n=1}^{\infty} \frac{n(1-iT)+1+T^2}{(1+T^2)(n+1)^2 n} e^{in\theta}\right) \geq 0, \qquad (\forall)\ z \in \mathbb{D}, T \in \mathbb{R}. \tag{3.6}$$

Our aim is to prove the equality:

$$\min_{\substack{\theta\in[0,2\pi]\\ T\in\mathbb{R}}} M(\theta, T) = \frac{1}{4c} - (2\ln 2 - 1), \tag{3.7}$$

where

$$M(\theta, T) = \operatorname{Re}\left(\frac{1}{4c} + \sum_{n=1}^{\infty} \frac{n(1-iT)+1+T^2}{(1+T^2)(n+1)^2 n} e^{in\theta}\right)$$

$$= \frac{1}{4c} + \frac{1}{1+T^2} \operatorname{Re}\left[\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)} - iT\sum_{n=1}^{\infty} \frac{e^{in\theta}}{(n+1)^2} + T^2 \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)^2}\right].$$

We use the integral representations

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{(n+1)^2} = \int_0^1\int_0^1 tu \frac{e^{i\theta} - tu}{1 + t^2u^2 - 2tu\cos\theta} dtdu, \theta \in (0, 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)} = \int_0^1\int_0^1 u \frac{e^{i\theta} - tu}{1 + t^2u^2 - 2tu\cos\theta} dtdu,\ \ \theta \in (0, 2\pi)$$

and we obtain

$$\begin{aligned} M(\theta,T) \quad &= \quad \frac{1}{4c} + \frac{1}{1+T^2}\left(\int_0^1\int_0^1 u\frac{\cos\theta - tu}{1+t^2u^2-2tu\cos\theta}dtdu + \right. \\ &\quad +T\int_0^1\int_0^1 \frac{tu\sin\theta}{1+t^2u^2-2tu\cos\theta}dtdu + \\ &\quad \left. +T^2\int_0^1\int_0^1 u(1-t)\frac{\cos\theta - tu}{1+t^2u^2-2tu\cos\theta}dtdu\right). \end{aligned}$$

$M(\theta,T)$ can be written in the following equivalent form

$$\begin{aligned}
M(\theta,T) &= \frac{1}{4c}-\frac{1}{1+T^2}\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu-\frac{T^2}{1+T^2}\int_0^1\!\!\int_0^1\frac{u(1-t)}{1+tu}dtdu\\
&+\frac{1}{1+T^2}\Bigg[\int_0^1\!\!\int_0^1 u\frac{\cos\theta-tu}{1+t^2u^2-2tu\cos\theta}dtdu+\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu\\
&+T\int_0^1\!\!\int_0^1\frac{tu\sin\theta}{1+t^2u^2-2tu\cos\theta}dtdu\\
&+T^2\Bigg(\int_0^1\!\!\int_0^1 u(1-t)\frac{\cos\theta-tu}{1+t^2u^2-2tu\cos\theta}dtdu+\int_0^1\!\!\int_0^1\frac{u(1-t)}{1+tu}dtdu\Bigg)\Bigg].
\end{aligned}$$

$$\begin{aligned}
M(\theta,T) &= \frac{1}{4c}-\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu+\frac{T^2}{1+T^2}\int_0^1\!\!\int_0^1\frac{tu}{1+tu}dtdu\\
&+\frac{1}{1+T^2}\Bigg((1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dt\\
&+T\sin\theta\int_0^1\!\!\int_0^1\frac{tu}{1+t^2u^2-2tu\cos\theta}dtdu\\
&+T^2(1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-t)(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\Bigg).
\end{aligned}$$

$$\begin{aligned}
M(\theta,T) &= \frac{1}{4c}-\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu\\
&+\frac{T^2}{1+T^2}(1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-t)(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\\
&+\frac{1}{1+T^2}\Bigg((1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\\
&+T\sin\theta\int_0^1\!\!\int_0^1\frac{tu}{1+t^2U^2-2tu\cos\theta}dtdu+T^2\int_0^1\!\!\int_0^1\frac{tu}{1+tu}dtdu\Bigg).
\end{aligned}$$

A simple calculation leads to

$$M(\theta,T)=\frac{1}{4c}-\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu+L_1(\theta,T)+\frac{1}{1+T^2}L_2(\theta,T). \tag{3.8}$$

where

$$L_1(\theta,T)=\frac{T^2}{1+T^2}(1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-t)(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu$$

and

$$\begin{aligned}
L_2(\theta,T) &= (1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu+\\
&+T\sin\theta\int_0^1\!\!\int_0^1\frac{tu}{1+t^2u^2-2tu\cos\theta}dtdu+T^2\int_0^1\!\!\int_0^1\frac{tu}{1+tu}dtdu.
\end{aligned}$$

It is easily seen that $L_1(\theta,T)\geq 0,\ (\forall)\ \theta\in(0,2\pi)$ and $T\in\mathbb{R}$. We will prove that $L_2(\theta,T)\geq 0,\ (\forall)\ \ \theta\in(0,2\pi)$ and $T\in\mathbb{R}$.
$L_2(\theta,T)$ is a polynomial of degree two with respect to the variable T.
We have

$$\begin{aligned}\Delta_2(\theta)=\sin^2\theta&\left(\int_0^1\!\!\int_0^1\frac{tu}{1+t^2u^2-2tu\cos\theta}dtdu\right)^2\\ &-4(1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\int_0^1\!\!\int_0^1\frac{tu}{1+tu}dtdu.\end{aligned}$$

We calculate the integrals and we get

$$\int_0^1\!\!\int_0^1\frac{tu}{1+tu}dtdu=1-\frac{\pi^2}{12}>\frac{1-\log 2}{2}=\frac{1}{2}\int_0^1\!\!\int_0^1\frac{t^2u}{1-t^2u^2}dtdu.$$

This inequality implies

$$\begin{aligned}\Delta_2(\theta)\leq\sin^2\theta&\left(\int_0^1\int_0^1\frac{tu}{1+t^2u^2-2tu\cos\theta}dtdu\right)^2\\ &-4(1+\cos\theta)\int_0^1\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\int_0^1\int_0^1\frac{t^2u}{1-t^2u^2}dtdu\\ &=4L_3(\theta)\cos^2\frac{\theta}{2}.\end{aligned}$$

We have

$$L_3(\theta)=\left(\int_0^1\!\!\int_0^1\frac{tu\sin\frac{\theta}{2}}{1+t^2u^2-2tu\cos\theta}dtdu\right)^2$$

$$-2\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\int_0^1\!\!\int_0^1\frac{t^2u}{1-t^2u^2}dtdu. \tag{3.9}$$

The inequality Cauchy - Schwarz implies

$$\begin{aligned}&\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\int_0^1\!\!\int_0^1\frac{t^2u}{1-t^2u^2}dtdu\\ &\geq\left(\int_0^1\!\!\int_0^1\frac{tu}{(1+tu)\sqrt{1+t^2u^2-2tu\cos\theta}}dtdu\right)^2.\end{aligned}\tag{3.10}$$

Thus in order to prove $\Delta_2(\theta)\leq 0,\ \theta\in(0,2\pi)$ it is enough to show that

$$\int_0^1\!\!\int_0^1\frac{tu\sin\frac{\theta}{2}}{1+t^2u^2-2tu\cos\theta}dtdu\leq\int_0^1\!\!\int_0^1\frac{tu}{(1+tu)\sqrt{1+t^2u^2-2tu\cos\theta}}dtdu. \tag{3.11}$$

The inequality (3.11) holds if we show that

$$\frac{tu}{(1+tu)\sqrt{1+t^2u^2-2tu\cos\theta}}\geq\frac{tu\sin\frac{\theta}{2}}{1+t^2u^2-2tu\cos\theta} \tag{3.12}$$

in case of $t,u\in[0,1]$ and $\theta\in(0,2\pi)$. A short calculation shows that the inequality (3.12) is equivalent to

$$(1+\cos\theta)(1-t)^2\geq 0,t,u\in[0,1],\theta\in(0,2\pi)$$

and we have proved $L_3(\theta,T)\leq 0$ for $\theta\in(0,2\pi)$, $T\in\mathbb{R}$ and this is equivalent to $\Delta_2(\theta)\leq 0$, $\theta\in(0,2\pi)$.
Thus the inequalities hold $L_1(\theta,T)\geq 0$ $L_2(\theta,T)\geq 0$ for $\theta\in(0,2\pi)$, $T\in\mathbb{R}$ and $L_1(\pi,0)=L_2(\pi,0)=0$. This implies that

$$\inf_{\substack{\theta\in(0,2\pi)\\ T\in\mathbb{R}}} M(\theta,T)=\frac{1}{4c}-\int_0^1\int_0^1\frac{u}{1+tu}dtdu=\frac{1}{4c}-(2\ln 2-1). \tag{3.13}$$

Finally if $\frac{1}{4c}-(2\ln 2-1)=0\Leftrightarrow c=\frac{1}{4(2\ln 2-1)}$, then

$$M(\theta,T)\geq 0 \text{ for every } \theta\in(0,2\pi),\ T\in\mathbb{R}$$

and this implies the starlikeness of the function f. □

The integral version of the proved theorem is given in the next corollary.

Corollary 3.2. *If $f\in\mathcal{A}$ and $\operatorname{Re}(zf''(z))>-c, z\in\mathbb{D}$, where $c=\frac{1}{2(2\ln 2-1)}$, then the image function F defined by the Alexander operator*

$$F(z)=A(f)(z)=\int_0^z\frac{f(t)}{t}dt,\ z\in\mathbb{D}$$

belongs to the class S^*.

Proof. Differentiating the equality $F(z)=\int_0^z\frac{f(t)}{t}dt$ three times we get

$$zF''(z)+\frac{z^2}{2}F'''(z)=\frac{1}{2}zf''(z), z\in\mathbb{D}. \tag{3.14}$$

The conditions of the corollary and the equality (3.14) imply

$$zF''(z)+\frac{z^2}{2}F'''(z)>\frac{-1}{4(2\ln 2-1)}, z\in\mathbb{D}. \tag{3.15}$$

The inequality (3.15) and Theorem 3.1 show that $F\in S^*$. □

The next theorem shows that $c=\frac{1}{4(2\ln 2-1)}$ can not be replaced by a bigger number in case of Theorem 3.1.

Theorem 3.3. *If $c_1>\frac{1}{4(2\ln 2-1)}$, then there are functions $f\in\mathcal{A}$ which verify the condition*

$$zf''(z)+\frac{z^2}{2}f'''(z)>-c_1,\ z\in\mathbb{D} \tag{3.16}$$

and are not univalent.

Proof. The condition (3.16) implies

$$f(z)=z+4c_1\sum_{n=2}^{\infty}\frac{z^n}{n^2(n-1)}d\mu(t),\ z\in\mathbb{D}$$

and it follows that

$$f'(z)=1+4c_1\sum_{n=2}^{\infty}\frac{z^n}{n(n-1)}d\mu(t),\ z\in\mathbb{D}.$$

The condition $f'(z) \neq 0,\ z \in \mathbb{D}$ is a necessary condition of the univalence. On the other hand we have

$$f'(z) = 1 + 4c_1 \sum_{n=1}^{\infty} \frac{z^n}{n(n-1)} \int_0^{2\pi} e^{-int} d\mu(t) =$$

$$\left(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} d\mu(t)\right) * \left(1 + 2c_1 \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}\right).$$

Lemma 2.2 implies that $f'(z) \neq 0, z \in \mathbb{D}$ if and only if

$$\mathrm{Re}\left[1 + 2c_1 \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}\right] > \frac{1}{2}, z \in \mathbb{D},$$

and this is equivalent to

$$\mathrm{Re}\left[1 + 4c_1 \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}\right] > 0, z \in \mathbb{D}.$$

In particular from radial continuity we get

$$1 + 4c_1 \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} \geq 0,$$

or equivalently

$$\frac{1}{4(2\ln 2 - 1)} = \frac{1}{4\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}\right)} \geq c_1$$

and this contradicts the condition of the theorem $c_1 > \frac{1}{4(2\ln 2 - 1)}$. This contradiction shows that there are points $z^* \in \mathbb{D}$ such that $f'(z^*) = 0$ and consequently the function f is not univalent. $\square$

4. Concluding remarks

The method of convolution leads to sharp results in case of linear starlikeness and convexity conditions. In the presented theorems all the sharp ones are proved by convolution. Theorem 1.4 is an example, which was proved by differential subordination and we think that this method it is much useful in this case. It would be interesting to give the sharp version of an implication from Theorem 1.4.

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New subclasses of bi-univalent functions connected with a q-analogue of convolution based upon the Legendre polynomials

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Abstract. In this paper, we introduce new subclasses of analytic and bi-univalent functions connected with a q-analogue of convolution by using the Legendre polynomials. Furthermore, we find estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses and obtain Fekete-Szegő problem for these subclasses.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; z \in \mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}, \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent functions in $\mathbb{E}$.

If $h \in \mathcal{A}$ is given by

$$h(z) = z + \sum_{k=2}^{\infty} b_k z^k, \; z \in \mathbb{E}, \tag{1.2}$$

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then, the *Hadamard (or convolution) product* of f and h is defined by

$$(f * h)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \ z \in \mathbb{E}. \tag{1.3}$$

If f and F are analytic functions in $\mathbb{E}$, we say that f *is subordinate to* F, written $f \prec F$, if there exists a *Schwarz function* w, which is analytic in $\mathbb{E}$, with $w(0) = 0$, and, $|w(z)| < 1$ for all $z \in \mathbb{E}$, such that $f(z) = F(w(z))$, $z \in \mathbb{E}$. Furthermore, if the function F is univalent in $\mathbb{E}$, then we have the following equivalence (see [5] and [17]):

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathbb{E}) \subset F(\mathbb{E}).$$

In [23] Srivastava presented and motivated about brief expository overview of the classical q-analysis versus the so-called (p, q)-analysis with an obviously redundant additional parameter p. We also briefly consider several other families of such extensivelyand widely-investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Attiya linear convolution operators, together with their extended and generalized versions. The theory of (p, q)-analysis has important role in many areas of mathematics and physics. Our usages here of the q-calculus and the fractional qcalculus in geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see [1, 14, 15, 21, 22, 26]).

Srivastava [23] made use of various operators of q-calculus and fractional q-calculus and recalling the definition and notations. The q-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\lambda; q)_k = \begin{cases} 1 & k = 0, \\ (1-\lambda)(1-\lambda q)\ldots\left(1-\lambda q^{k-1}\right) & k \in \mathbb{N}. \end{cases}$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$\left(q^{\lambda}; q\right)_k = \frac{(1-q)^k \ \Gamma_q(\lambda + k)}{\Gamma_q(\lambda)}, \qquad (k \in \mathbb{N}_0),$$

where (see [13])

$$\Gamma_q(z) = (1-q)^{1-z} \, \frac{(q; q)_\infty}{(q^z; q)_\infty}, \qquad (|q| < 1).$$

Also, we note that

$$(\lambda; q)_\infty = \prod_{k=0}^{\infty} \left(1 - \lambda q^k\right), \qquad (|q| < 1),$$

and, the q-gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1) = [z]_q \ \Gamma_q(z),$$

where $[k]_q$ denotes the basic q-number defined as follows

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ 1 + \sum\limits_{j=1}^{k-1} q^j, & k \in \mathbb{N}. \end{cases} \tag{1.4}$$

Using the definition formula (1.4), we have the next two products:

(i) For any non negative integer k, the *q-shifted factorial* is given by

$$[k]_q! := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{n=1}^{k} [n]_q, & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{n=r}^{r+k-1} [n]_q, & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \qquad \text{as } q \to 1^-.$$

Also, we observe that

$$\lim_{q \to 1^-} \left\{ \frac{(q^\lambda; q)_k}{(1-q)^k} \right\} = (\lambda)_k .$$

For $0 < q < 1$, the *q-derivative operator* for $f * h$ is defined by

$$\begin{aligned} D_q (f * h)(z) &= D_q \left[z + \sum_{k=2}^{\infty} a_k b_k z^k \right] \\ &= \frac{(f * h)(z) - (f * h)(qz)}{z(1-q)} \\ &= 1 + \sum_{k=2}^{\infty} [k, q] a_k b_k z^{k-1}, \ z \in \mathbb{E}, \end{aligned}$$

where

$$[k, q] := \frac{1 - q^k}{1 - q} = 1 + \sum_{j=1}^{k-1} q^j, \qquad [0, q] := 0. \tag{1.5}$$

Using the definition formula (1.5), we will define the next two products:
(i) For any non negative integer k, the *q-shifted factorial* is given by

$$[k, q]! := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{i=1}^{k} [i, q], & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r, q]_k := \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod\limits_{i=1}^{k} [r + i - 1, q], & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

For $\lambda > -1$ and $0 < q < 1$, El-Deeb et al. [12] defined the linear operator $\mathcal{H}_h^{\lambda,q} : \mathcal{A} \to \mathcal{A}$ as follows

$$\mathcal{H}_h^{\lambda,q} f(z) * \mathcal{M}_{q,\lambda+1}(z) = z\, D_q (f * h)(z), \ z \in \mathbb{E},$$

where the function $\mathcal{M}_{q,\lambda+1}$ is given by

$$\mathcal{M}_{q,\lambda+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda+1,q]_{k-1}}{[k-1,q]!} z^k,\ z \in \mathbb{E}.$$

A simple computation shows that

$$\mathcal{H}_h^{\lambda,q} f(z) := z + \sum_{k=2}^{\infty} \phi_k a_k z^k,\ (\lambda > -1,\ 0 < q < 1,\ z \in \mathbb{E}), \tag{1.6}$$

where

$$\phi_k = \frac{[k,q]!}{[\lambda+1,q]_{k-1}} b_k. \tag{1.7}$$

Remark 1.1. [12] From the definition relation (1.6), we can easily verify that the next relations hold for all $f \in \mathcal{A}$:

(i) $[\lambda+1,q]\mathcal{H}_h^{\lambda,q} f(z) = [\lambda,q]\mathcal{H}_h^{\lambda+1,q} f(z) + q^{\lambda}\ zD_q\left(\mathcal{H}_h^{\lambda+1,q} f(z)\right),\ z \in \mathbb{E};$

(ii)
$$\lim_{q\to 1^-} \mathcal{H}_h^{\lambda,q} f(z) = \mathcal{H}_h^{\lambda,1} f(z) := \mathcal{I}_h^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\lambda+1)_{k-1}}\, a_k b_k z^k, z \in \mathbb{E}. \tag{1.8}$$

Remark 1.2. By taking special cases b_k in the operator $\mathcal{H}_h^{\lambda,q}$, we obtain

(i) Taking $b_k = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}$ $(\upsilon > 0)$, we get the operator $\mathcal{N}_{\upsilon,q}^{\lambda}$ studied by El-Deeb and Bulboaca [8] and El-Deeb [7], as follows:

$$\begin{aligned} \mathcal{N}_{\upsilon,q}^{\lambda} f(z) &= z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k,\ z \in \mathbb{E}, \\ &= z + \sum_{k=2}^{\infty} \psi_k\ a_k z^k,\quad (\upsilon > 0,\ \lambda > -1,\ 0 < q < 1), \end{aligned} \tag{1.9}$$

where

$$\psi_k = \frac{[k,q]!}{[\lambda+1,q]_{k-1}} \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}; \tag{1.10}$$

(ii) Taking $b_k = \left(\frac{n+1}{n+k}\right)^{\delta}$ $(\delta > 0,\ n \geq 0)$,we find the operator $\mathcal{N}_{n,1,q}^{\lambda,\delta} = \mathcal{M}_{n,q}^{\lambda,\delta}$ studied by El-Deeb and Bulboaca [9] and Srivastava and El-Deeb [24] as follows:

$$\mathcal{M}_{n,q}^{\lambda,\delta} f(z) := z + \sum_{k=2}^{\infty} \left(\frac{n+1}{n+k}\right)^{\delta} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k,\ z \in \mathbb{E}; \tag{1.11}$$

(iii) Taking $b_k = 1$, we have the operator $\mathfrak{J}_q^{\lambda}$ studied by Arif et al. [2] and Srivastava et al. [27] as follows:

$$\mathfrak{J}_q^{\lambda} f(z) := z + \sum_{k=2}^{\infty} \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k,\ z \in \mathbb{E}; \tag{1.12}$$

(iv) Taking $b_k = \frac{m^{k-1}}{(k-1)!}e^{-m}$ $(m > 0)$ (see [19]), we get a q-analogue of poission operator $\mathcal{I}_q^{\lambda,m}$ studied by El-Deeb et al. [12] as follows:

$$\mathcal{I}_q^{\lambda,m} f(z) := z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}; \tag{1.13}$$

(v) Taking $b_k = \left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m$ $(m \in \mathbb{Z},\ \ell \geq 0,\ \delta \geq 0)$ (see [20]), we get a q-analogue of Prajapat operator $\mathcal{J}_{q,\ell,\delta}^{\lambda,m}$ as follows:

$$\mathcal{J}_{q,\ell,\delta}^{\lambda,m} f(z) := z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}; \tag{1.14}$$

(vi) Taking $b_k = \binom{k+m-2}{m-1} \theta^{k-1} (1-\theta)^m$ $(m \geq 1,\ 0 \leq \theta \leq 1)$ (see [10, 11]), we get a q-analogue of Pascal distribution series $\Psi_{q,\theta}^{\lambda,m}$ defined by Srivastava and El-deeb [25] as follows:

$$\Psi_{q,\theta}^{\lambda,m} f(z) := z + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} \theta^{k-1} (1-\theta)^m \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k, \ z \in \mathbb{E}. \tag{1.15}$$

Definition 1.3. Let $P_k(x)$ be the Legendre polynomials of the first kind of order $k = 0, 1, 2, ...$ for which, the recurrence formula is

$$P_{k+1}(x) = \frac{2k+1}{k+1} x P_k(x) - \frac{k}{k+1} P_{k-1}(x), \tag{1.16}$$

with

$$P_0(x) = 1 \quad \text{and } P_1(x) = x$$

For $|x| < 1$. The generating function for Legendre Polynomials is given by (see [16])

$$G(x,z) = \frac{1}{\sqrt{1-2xz+z^2}} = \sum_{k=0}^{\infty} P_k(x) z^k.$$

The Koebe one quarter theorem (see [6]) proves that the image of $\mathbb{E}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfied

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{E})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \dots . \tag{1.17}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{E}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{E}$. Let Σ denote the class of bi-univalent functions in $\mathbb{E}$ given by (1.1). For a brief history and interesting examples in the class Σ (see [3]). Brannan and Taha [4] (see also [28]) introduced certain subclasses of the bi-univalent functions class Σ similar to the familiar subclasses $S^*(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex functions of order β $(0 \leq \beta < 1)$, respectively (see [3]). Thus, following Brannan and Taha [4] a function $f \in \mathcal{A}$ is said to be in the class $S_{\Sigma}^{*}(\beta)$ of strongly bi-starlike functions of order β $(0 < \beta \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \quad \text{and} \quad \left| \arg\left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\beta\pi}{2} \ \ (0 < \beta \leq 1;\ z \in \mathbb{E}) \tag{1.18}$$

and

$$\left| \arg\left(\frac{zg'(w)}{g(w)} \right) \right| < \frac{\beta\pi}{2} \ \ (0 < \beta \leq 1;\ w \in \mathbb{E}), \tag{1.19}$$

where h is the extension of f^{-1} to $\mathbb{E}$ is given by (1.17). The classes $S_{\Sigma}^{*}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$ of bi-starlike functions of order β and bi-convex functions of order β $(0 < \beta \leq 1)$, corresponding to the function classes $S^{*}(\beta)$ and $\mathcal{K}(\beta)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^{*}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [4] and [28]).

The object of the present paper is to introduce new classes of the function class Σ involving the $q-$analogue of convolution based upon the Legendre polynomials previous defined classes, and find estimates on the coefficients $|a_2|$, and $|a_3|$ for functions in these new subclasses of the function class Σ.

Definition 1.4. Let $\eta \neq 0$ be a complex number and $f(z)$ given by (1.1) and $h(z)$ given by (1.2), then $f(z)$ is said to be in the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x)$ if the following conditions are satisfied:

$$f \in \Sigma,$$

$$1+\frac{1}{\eta}\left(\frac{\alpha z\ D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q} f(z)\right)\right) + \alpha D_q\left(\mathcal{H}_h^{\lambda,q} f(z)\right) + 1 - \alpha}{D_q\left(\mathcal{H}_h^{\lambda,q} f(z)\right)} - 1 \right) \prec G(x,z), \tag{1.20}$$

and

$$1+\frac{1}{\eta}\left(\frac{\alpha w D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q} g(w)\right)\right) + \alpha D_q\left(\mathcal{H}_h^{\lambda,q} g(w)\right) + 1 - \alpha}{D_q\left(\mathcal{H}_h^{\lambda,q} g(w)\right)} - 1 \right) \prec G(x,w), \tag{1.21}$$

with $\alpha > 0$, $\lambda > -1;\ 0 < q < 1;\ \eta \in \mathbb{C}^{*} = \mathbb{C}\backslash\{0\}$, where the function $g = f^{-1}$ is given by (1.17).

Remark 1.5. (i) For $q \to 1^{-}$ we obtain that $\lim_{q \to 1^{-}} \mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x) =: \mathcal{N}_{\Sigma}^{\lambda}(\eta, \alpha, h, x)$, where $\mathcal{N}_{\Sigma}^{\lambda}(\eta, \alpha, h, x)$ represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{I}_h^{\lambda}$ (see (1.8)).
(ii) For $b_k = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}$ $(\upsilon > 0)$, we obtain the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta, \alpha, \upsilon, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{N}_{\upsilon,q}^{\lambda}$ (see (1.9)).

(iii) For $b_k = \left(\frac{n+1}{n+k}\right)^{\delta}$ $(\delta > 0,\ n \geq 0)$, we obtain the class $\mathcal{I}_{\Sigma}^{q,\lambda}(\eta, \alpha, \delta, n, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{M}_{n,q}^{\lambda,\delta}$ (see (1.11)).

(iv) For $b_k = \frac{m^{k-1}}{(k-1)!}e^{-m}$ $(m > 0)$ we obtain the class $\mathcal{P}_{\Sigma}^{q,\lambda}(\eta, \alpha, m, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{I}_{\lambda,m}^{q}$ (see (1.13)).

(v) For $b_k = \left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m$ $(m \in \mathbb{Z},\ \ell \geq 0,\ \delta \geq 0)$, we obtain the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta, \alpha, m, \ell, \delta, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\mathcal{J}_{q,\ell,\delta}^{\lambda,m}$ (see (1.14)).

(vi) For $b_k = \binom{k+m-2}{m-1}\theta^{k-1}(1-\theta)^m$ $(m \geq 1,\ 0 < \theta < 1)$, we obtain the class $\Psi_{\Sigma}^{q,\lambda}(\eta, \alpha, m, \theta, x)$, that represents the functions $f \in \Sigma$ that satisfies (1.20) and (1.21) for $\mathcal{H}_h^{\lambda,q}$ replaced with $\Psi_{q,\theta}^{\lambda,m}$ (see (1.15)).

The following Lemma will be needed later.

Lemma 1.6. [18, p. 172] *If* $w(z) = \sum_{k=1}^{\infty} p_k z^k$ *is a Schwarz function for* $z \in E$*, then*

$$|p_1| \leq 1, \quad |p_k| \leq 1 - |p_1|^2,\ k \geq 1.$$

2. Coefficient bounds for the function class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x)$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\alpha \geq 0,\ \lambda > -1,\ 0 < q < 1,\ \eta \in \mathbb{C}^*,\ x \in \mathbb{R}$ and h is given by (1.2), the powers are understood as principle values.

Theorem 2.1. *Let the function* f *given by* (1.1) *belongs to the class* $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x)$*, then*

$$|a_2| \leq \frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\phi_3 - [\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2\phi_2^2\right|}},$$

and

$$|a_3| \leq \frac{|\eta|\,|x|}{(\alpha(2+q)-1)\,(1+q+q^2)\,\phi_3} + \frac{|\eta|^2 x^2}{(1+q)^2\,(2\alpha-1)^2\,\phi_2^2},$$

where ϕ_k*,* $k \in \{2,3\}$*, are given by* (1.7)*.*

Proof. Since $f \in \mathcal{F}_{\Sigma}^{q,\lambda}(\eta, \alpha, h, x)$. Then there exist two analytic functions R and S in $\mathbb{E}$ with $R(0) = S(0) = 0$, and $|R(z)| < 1$, $|S(w)| < 1$ for all $z, w \in \mathbb{E}$ given by

$$R(z) = \sum_{k=1}^{\infty} r_k z^k \qquad \text{and} \qquad S(w) = \sum_{k=1}^{\infty} s_k w^k, \qquad z, w \in \Delta,$$

from Lemma 1.6 we have

$$|r_k| \leq 1 \quad \text{and} \quad |s_k| \leq 1,\ k \in \mathbb{N}. \tag{2.1}$$

In view of (1.20) and (1.21), we get

$$\frac{\alpha z\ D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)\right)+\alpha D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)}-1=\eta\left(G(x,R(z))-1\right), \tag{2.2}$$

and

$$\frac{\alpha w D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)\right)+\alpha D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)}-1=\eta\left(G(x,S(w))-1\right). \tag{2.3}$$

Since

$$\begin{aligned}&\frac{\alpha z\ D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)\right)+\alpha D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}f(z)\right)}-1\\ =&\ (1+q)(2\alpha-1)\phi_2a_2z\\ +&\ \left[(\alpha(2+q)-1)\left(1+q+q^2\right)\phi_3a_3-(2\alpha-1)(1+q)^2\phi_2^2a_2^2\right]z^2+\ldots,\end{aligned}$$

$$\begin{aligned}&\frac{\alpha w D_q\left(D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)\right)+wD_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)+1-\alpha}{D_q\left(\mathcal{H}_h^{\lambda,q}g(w)\right)}-1\\ =&\ -(1+q)(2\alpha-1)\phi_2a_2w\\ &+\left[(\alpha(2+q)-1)\left(1+q+q^2\right)\phi_3\left(2a_2^2-a_3\right)-(2\alpha-1)(1+q)^2\phi_2^2a_2^2\right]w^2+\ldots,\end{aligned}$$

and

$$\eta\left(G(x,R(z))-1\right)=\eta P_1(x)r_1z+\left(P_1(x)r_2+P_2(x)r_1^2\right)\eta z^2+\ldots,$$

$$\eta\left(G(x,S(w))-1\right)=\eta P_1(x)s_1w+\left(P_1(x)s_2+P_2(x)s_1^2\right)\eta w^2+\ldots.$$

Next, equating the corresponding coefficients of z and w in (2.2) and (2.3), we get

$$(1+q)(2\alpha-1)\phi_2a_2=\eta P_1(x)r_1, \tag{2.4}$$

$$(\alpha(2+q)-1)\left(1+q+q^2\right)\phi_3a_3-(2\alpha-1)(1+q)^2\phi_2^2a_2^2=\eta P_1(x)r_2+\eta P_2(x)r_1^2 \tag{2.5}$$

$$-(1+q)(2\alpha-1)\phi_2a_2=\eta P_1(x)s_1, \tag{2.6}$$

$$(\alpha(2+q)-1)(1+q+q^2)\phi_3(2a_2^2-a_3)-(2\alpha-1)(1+q)^2\phi_2^2a_2^2=\eta P_1(x)s_2+\eta P_2(x)s_1^2. \tag{2.7}$$

From (2.4) and (2.6), we have

$$r_1=-s_1 \tag{2.8}$$

By squaring (2.4) and (2.6), then adding the new relations we get

$$2(1+q)^2(2\alpha-1)^2a_2^2\phi_2^2=\eta^2P_1^2(x)\left(r_1^2+s_1^2\right). \tag{2.9}$$

If we add (2.5) and (2.7) we obtain

$$2[(\alpha(2+q)-1)(1+q+q^2)\phi_3-(2\alpha-1)(1+q)^2\phi_2^2]a_2^2=\eta P_1(x)(r_2+s_2)+\eta P_2(x)(r_1^2+s_1^2).$$

We can rewrite (2.9) as

$$r_1^2 + s_1^2 = \frac{2(1+q)^2\,(2\alpha-1)^2}{\eta^2 P_1^2(x)} a_2^2\phi_2^2.$$

From above equation, we get

$$2[(\alpha(2+q)-1)(1+q+q^2)\eta P_1^2(x)\phi_3 - [\eta P_1^2(x) + (2\alpha-1)P_2(x)](2\alpha-1)(1+q)^2\phi_2^2]a_2^2$$
$$= \eta^2 P_1^3(x)(r_2+s_2),$$

it follows that

$$a_2^2 = \frac{\eta^2 P_1^3(x)(r_2+s_2)}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta P_1^2(x)\phi_3 - \left(\eta P_1^2(x)+(2\alpha-1)P_2(x)\right)(2\alpha-1)(1+q)^2\phi_2^2\right]}. \tag{2.10}$$

Then taking the absolute value to the above equation and from (1.16) and (2.1), we obtain

$$|a_2| \le \frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\phi_3 - [\eta P_1^2(x) + \frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2\phi_2^2\right|}},$$

which gives the bound for $|a_2|$ as we asserted in our theorem. To find the bound for $|a_3|$. Using (2.5) from (2.7), we have

$$2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3\left(a_3-a_2^2\right) = \eta\left[P_1(x)\left(r_2-s_2\right) + P_2(x)\left(r_1^2-s_1^2\right)\right]. \tag{2.11}$$

Form (2.8), (2.9) and (2.11), we obtain

$$a_3 = \frac{\eta P_1(x)\left(r_2-s_2\right)}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3} + \frac{\eta^2 P_1^2(x)\left(r_1^2+s_1^2\right)}{2(1+q)^2\left(2\alpha-1\right)^2\phi_2^2}. \tag{2.12}$$

Using (1.16) and (2.1), we get

$$|a_3| \le \frac{|\eta|\,|x|}{\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3} + \frac{|\eta|^2 x^2}{(1+q)^2\left(2\alpha-1\right)^2\phi_2^2}. \qquad \square$$

In view of Theorem 2.1 we obtain the following results.
Putting $q \to 1^-$ we get the following corollary:

Corollary 2.2. *Let the function f given by* (1.1) *belongs to the class $f \in \mathcal{N}_\Sigma^\lambda(\eta,\alpha,h,x)$, then*

$$|a_2| \le \frac{|\eta|\,|x|\sqrt{x}}{\sqrt{\left|\frac{18(3\alpha-1)\eta x^2 b_3}{(\lambda+1)_2} - 16\left[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)\right]\frac{(2\alpha-1)b_2^2}{(\lambda+1)^2}\right|}},$$

and

$$|a_3| \le \frac{|\eta|\,|x|\left(\lambda+1\right)_2}{18\left(3\alpha-1\right)b_3} + \frac{|\eta|^2\left(x\left(\lambda+1\right)\right)^2}{16\left(2\alpha-1\right)^2 b_2^2}.$$

Considering $b_k = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}$ $(\upsilon > 0)$, we obtain the following result.

Corollary 2.3. *Let the function f given by (1.1) belongs to the class $f \in \mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\alpha,\upsilon,x)$, then*

$$|a_2| \leq \frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\psi_3-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\psi_2^2\right|}},$$

and

$$|a_3| \leq \frac{|\eta|\,|x|}{(\alpha(2+q)-1)\,(1+q+q^2)\,\psi_3}+\frac{|\eta|^2\,x^2}{(1+q)^2\,(2\alpha-1)^2\,\psi_2^2}.$$

where ψ_k, $k\in\{2,3\}$, are given by (1.10).

For $b_k=\left(\frac{n+1}{n+k}\right)^{\delta}$ $(\delta>0,\ n\geq 0)$, we obtain the following corollary.

Corollary 2.4. *Let the function f given by (1.1) belongs to the class $f \in \mathcal{I}_{\Sigma}^{q,\lambda}(\eta,\alpha,\delta,n,x)$, then*
$|a_2| \leq$

$$\frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{[\lambda+1,q]_2}\left(\frac{n+1}{n+3}\right)^{\delta}-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}\left(\frac{n+1}{n+2}\right)^{2\delta}\right|}},$$

and

$$|a_3| \leq \frac{|\eta||x|[\lambda+1,q]_2(n+3)^{\delta}}{(\alpha(2+q)-1)(1+q+q^2)[3,q]!(n+1)^{\delta}}+\frac{|\eta|^2(x[\lambda+1,q])^2(n+2)^{2\delta}}{(1+q)^2(2\alpha-1)^2([2,q]!)^2(n+1)^{2\delta}}.$$

If we take $b_k=\frac{m^{k-1}}{(k-1)!}e^{-m}$ $(m>0)$ we get the following special case.

Corollary 2.5. *Let the function f given by (1.1) belongs to the class $f \in \mathcal{P}_{\Sigma}^{q,\lambda}(\eta,\alpha,m,x)$, then*
$|a_2| \leq$

$$\frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{2[\lambda+1,q]_2}m^2e^{-m}-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}m^2e^{-2m}\right|}},$$

and

$$|a_3| \leq \frac{2|\eta||x|[\lambda+1,q]_2}{(\alpha(2+q)-1)(1+q+q^2)[3,q]!m^2e^{-m}}+\frac{|\eta|^2x^2([\lambda+1,q])^2}{(1+q)^2(2\alpha-1)^2([2,q]!)^2m^2e^{-2m}}.$$

Putting $b_k=\left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m$ $(m\in\mathbb{Z},\ \ell\geq 0,\ \delta\geq 0)$ we get the following result.

Corollary 2.6. *Let the function f given by (1.1) belongs to the class $f \in \mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\alpha,m,\ell,\delta,x)$, then*
$|a_2| \leq$

$$\frac{|\eta||x|\sqrt{x}}{\sqrt{\left|(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{[\lambda+1,q]_2}[\frac{1+\ell+2\delta}{1+\ell}]^m-[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}[\frac{1+\ell+\delta}{1+\ell}]^{2m}\right|}},$$

and

$$|a_3| \leq \frac{|\eta||x|[\lambda+1,q]_2[1+\ell]^m}{(\alpha(2+q)-1)(1+q+q^2)[3,q]![1+\ell+2\delta]^m}+\frac{|\eta|^2x^2([\lambda+1,q])^2[1+\ell]^{2m}}{(1+q)^2(2\alpha-1)^2([2,q]!)^2[1+\ell+\delta]^{2m}}.$$

For $b_k=\binom{k+m-2}{m-1}\theta^{k-1}(1-\theta)^m$ $(m\geq 1,\ 0<\theta<1)$, we obtain the following corollary.

Corollary 2.7. *Let the function f given by (1.1) belongs to the class $f \in \Psi_{\Sigma}^{q,\lambda}(\eta,\alpha,m,\theta,x)$, then*

$$|a_2| \leq \frac{|\eta||x|\sqrt{x}}{\sqrt{A}},$$

where

$$A = \Big|(\alpha(2+q)-1)(1+q+q^2)\eta x^2 \frac{[3,q]!}{2[\lambda+1,q]_2} m(m+1)\theta^2(1-\theta)^m$$
$$-[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2 \frac{([2,q]!)^2}{([\lambda+1,q])^2} m^2\theta^2(1-\theta)^{2m}\Big|$$

and

$$|a_3| \leq \frac{2|\eta||x|[\lambda+1,q]_2}{(\alpha(2+q)-1)(1+q+q^2)[3,q]!m(m+1)\theta^2(1-\theta)^m} + \frac{|\eta|^2x^2([\lambda+1,q])^2}{(1+q)^2(2\alpha-1)^2m^2\theta^2(1-\theta)^{2m}([2,q]!)^2}.$$

3. Fekete-Szegő problem for the function class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta;\alpha,h;x)$

Theorem 3.1. *If the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta,\alpha,h,x)$, and $\eta \in \mathbb{C}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \leq |\eta||x|\left(|K+L| + |K-L|\right), \tag{3.1}$$

where

$$K = \frac{(1-\mu)\eta x^2}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta x^2\phi_3 - \left[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\phi_2^2\right]}, \tag{3.2}$$

and

$$L = \frac{1}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3},$$

where $\mu \in \mathbb{C}$, and ϕ_k, $k \in \{2,3\}$, are given by (1.7).

Proof. If $f \in \mathcal{F}_{\Sigma}^{q,\lambda}(\eta,\alpha,h,x)$. As in the proof of Theorem 2.1, from (2.8) and (2.11), we have

$$a_3 - a_2^2 = \frac{\eta P_1(x)\left(r_2 - s_2\right)}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3}, \tag{3.3}$$

and multiplying (2.10) by $(1-\mu)$ we get

$$(1-\mu)\,a_2^2 = \frac{(1-\mu)\eta^2 P_1^3(x)(r_2+s_2)}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta P_1^2(x)\phi_3 - \left[\eta P_1^2(x) + (2\alpha-1)P_2(x)\right](2\alpha-1)(1+q)^2\phi_2^2\right]}. \tag{3.4}$$

Adding (3.3) and (3.4) leads to

$$a_3 - \mu a_2^2 = \eta h_2\left[\left(K+L\right)r_2 + \left(K-L\right)s_2\right], \tag{3.5}$$

where K and L are given by (3.2), and taking the absolute value of (3.5), from (2.1) we obtain the inequality (3.1). The proof is complete. □

Remark 3.2. A simple computation shows that the inequality $|K| \leq L$ is equivalent to

$$|\mu - 1| \leq \left|1 - \frac{\left[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)\right]\left(2\alpha-1\right)(1+q)^2\phi_2^2}{\eta x^2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3}\right|,$$

therefore, from Theorem 3.1 we get the next result. If the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta;\alpha,h;x)$, and $\eta\in\mathbb{C}^*$, then

$$\left|a_3-\mu a_2^2\right|\leq\frac{\eta x}{\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3},$$

where $\mu\in\mathbb{C}$, with

$$|\mu-1|\leq\left|1-\frac{\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right]\left(2\alpha-1\right)\left(1+q\right)^2\phi_2^2}{\eta x^2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\phi_3}\right|,$$

and ϕ_k, $k\in\{2,3\}$, are given by (1.7).

We conclude our result with the following consequence of Theorem 3.1. Putting $q\to1^-$, we obtain the following corollary.

Corollary 3.3. *If the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta;\alpha,h;x)$, and $\mu\in\mathbb{C},\ \eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\leq|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{\left(1-\mu\right)\eta x^2}{\frac{36(3\alpha-1)\eta x^2 b_3}{(\lambda+1)_2}-32\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right]\frac{(2\alpha-1)b_2^2}{(\lambda+1)^2}},$$

and

$$L=\frac{\eta x\left(\lambda+1\right)_2}{36\left(3\alpha-1\right)b_3}.$$

If we put $b_k=\frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)}$ $(\upsilon>0)$, we obtain the following result.

Corollary 3.4. *If the function f given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\alpha,\upsilon,x)$, and $\eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\leq|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{(1-\mu)\eta x^2}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta x^2\psi_3-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\psi_2^2\right]},$$

and

$$L=\frac{1}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\psi_3},$$

where $\mu\in\mathbb{C}$, and ψ_k, $k\in\{2,3\}$, are given by (1.10).

Considering $b_k=\left(\frac{n+1}{n+k}\right)^{\delta}$ $(\delta>0,\ n\geq0)$, we get the following corollary.

Corollary 3.5. *If the function f given by (1.1) belongs to the class $\mathcal{I}_{\Sigma}^{q,\lambda}(\eta,\alpha,\delta,n,x)$, and $\mu\in\mathbb{C},\ \eta\in\mathbb{C}^*$, then*

$$\left|a_3-\mu a_2^2\right|\leq|\eta||x|\left(|K+L|+|K-L|\right),$$

where

$$K=\frac{(1-\mu)\eta x^2}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta x^2\frac{[3,q]!}{[\lambda+1,q]_2}\left(\frac{n+1}{n+3}\right)^{\delta}-\left[\eta x^2+\frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2\frac{([2,q]!)^2}{([\lambda+1,q])^2}\left(\frac{n+1}{n+2}\right)^{2\delta}\right]},$$

and

$$L = \frac{[\lambda+1,q]_2 \, (n+3)^{\delta}}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)[3,q]! \, (n+1)^{\delta}}.$$

If we take $b_k = \frac{m^{k-1}}{(k-1)!} e^{-m}$ $(m > 0)$, we get the following case.

Corollary 3.6. *If the function f given by (1.1) belongs to the class $\mathcal{P}_{\Sigma}^{q,\lambda}(\eta, \alpha, m, x)$, and $\mu \in \mathbb{C}$, $\eta \in \mathbb{C}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \le |\eta||x| \left(|K+L| + |K-L|\right),$$

where

$$K = \frac{(1-\mu)\eta x^2}{2\left[(\alpha(2+q)-1)(1+q+q^2)\eta x^2 \frac{[3,q]!}{2[\lambda+1,q]_2} m^2 e^{-m} - \left[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2 \frac{([2,q]!)^2}{([\lambda+1,q])^2} m^2 e^{-2m}\right]},$$

and

$$L = \frac{[\lambda+1,q]_2}{\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)[3,q]! m^2 e^{-m}}.$$

Putting $b_k = \left[\frac{1+\ell+\delta(k-1)}{1+\ell}\right]^m$ $(m \in \mathbb{Z},\ \ell \ge 0,\ \delta \ge 0)$, we obtain the following result.

Corollary 3.7. *If the function f given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta, \alpha, m, \ell, \delta, x)$, and $\mu \in \mathbb{C}$, $\eta \in \mathbb{C}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \le |\eta||x| \left(|K+L| + |K-L|\right),$$

where

$$K = \frac{(1-\mu)\,\eta x^2}{B},$$

where

$$B = 2\Bigg[\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)\eta x^2 \frac{[3,q]!}{[\lambda+1,q]_2}\left[\tfrac{1+\ell+2\delta}{1+\ell}\right]^m - \left[\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)\right](2\alpha-1)(1+q)^2 \frac{([2,q]!)^2}{([\lambda+1,q])^2}\left[\tfrac{1+\ell+\delta}{1+\ell}\right]^{2m}\Bigg]$$

and

$$L = \frac{[\lambda+1,q]_2 \, [1+\ell]^m}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)[3,q]! \, [1+\ell+2\delta]^m}.$$

For $b_k = \binom{k+m-2}{m-1}\theta^{k-1}(1-\theta)^m$ $(m \ge 1,\ 0 < \theta < 1)$, we get the following special case.

Corollary 3.8. *If the function f given by (1.1) belongs to the class $\Psi_{\Sigma}^{q,\lambda}(\eta, \alpha, m, \theta, x)$, and $\mu \in \mathbb{C}$, $\eta \in \mathbb{C}^*$, then*

$$\left|a_3 - \mu a_2^2\right| \le |\eta||x| \left(|K+L| + |K-L|\right),$$

where

$$K = \frac{(1-\mu)\eta x^2}{C},$$

where

$$C = 2[(\alpha(2+q)-1)(1+q+q^2)\eta x^2 \frac{[3,q]!}{2[\lambda+1,q]_2} m(m+1)\theta^2(1-\theta)^m - [\eta x^2 + \frac{(2\alpha-1)}{2}(3x^2-1)](2\alpha-1)(1+q)^2 \frac{([2,q]!)^2}{([\lambda+1,q])^2} m^2\theta^2(1-\theta)^{2m}]$$

and

$$L = \frac{[\lambda+1,q]_2}{2\left(\alpha(2+q)-1\right)\left(1+q+q^2\right)[3,q]! m(m+1)\theta^2\left(1-\theta\right)^m},$$

Now, the following examples are presented here to illustrate our results. For $\eta = 1$ and $\alpha = 1$. Therefore, from Theorem 2.1 and Theorem 3.1.

Example 3.9. Let the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(1;1,h;x)$, then

$$|a_2| \leq \frac{|x|\sqrt{x}}{\sqrt{\left|(1+q)\left(1+q+q^2\right)x^2\phi_3 - \frac{1}{2}(5x^2-1)(1+q)^2\phi_2^2\right|}},$$

$$|a_3| \leq \frac{|x|}{(1+q)\left(1+q+q^2\right)\phi_3} + \frac{x^2}{(1+q)^2\phi_2^2},$$

and

$$\left|a_3 - \mu a_2^2\right| \leq |x|\left(|K+L| + |K-L|\right),$$

with

$$K = \frac{(1-\mu)x^3}{2\left[(1+q)\left(1+q+q^2\right)x^2\phi_3 - \frac{1}{2}(5x^2-1)(1+q)^2\phi_2^2\right]},$$

and

$$L = \frac{x}{2(1+q)\left(1+q+q^2\right)\phi_3},$$

where $\mu \in \mathbb{C}$ and ϕ_k, $k \in \{2,3\}$, are given by (1.7).

For $\eta = 1$ and $\alpha = 0$. Therefore, from Theorem 2.1 and Theorem 3.1.

Example 3.10. Let the function f given by (1.1) belongs to the class $\mathcal{F}_{\Sigma}^{q,\lambda}(1;0,h;x)$, then

$$|a_2| \leq \frac{|x|\sqrt{x}}{\sqrt{\left|\left[-\left(1+q+q^2\right)x^2\phi_3 + \frac{1}{2}(1-x^2)(1+q)^2\phi_2^2\right]\right|}},$$

$$|a_3| \leq -\frac{|x|}{\left(1+q+q^2\right)\phi_3} + \frac{x^2}{(1+q)^2\phi_2^2},$$

and

$$\left|a_3 - \mu a_2^2\right| \leq |x|\left(|K+L| + |K-L|\right),$$

with

$$K = \frac{(1-\mu)x^3}{2\left[-\left(1+q+q^2\right)x^2\phi_3 + \frac{1}{2}(1-x^2)(1+q)^2\phi_2^2\right]},$$

and

$$L = -\frac{x}{2\left(1+q+q^2\right)\phi_3},$$

where $\mu \in \mathbb{C}$ and ϕ_k, $k \in \{2,3\}$, are given by (1.7).

Remark 3.11. We mention that all the above estimations for the coefficients $|a_2|$, $|a_3|$, and Fekete-Szegő problem for the function class $\mathcal{F}_{\Sigma}^{q,\lambda}(\eta;\alpha,h;x)$ are not sharp. To find the sharp upper bounds for the above functionals remains an interesting open problem, as well as those for $|a_n|$, $n \geq 4$.

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Non-instantaneous impulsive fractional integro-differential equations with proportional fractional derivatives with respect to another function

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Abstract. This paper concerns the existence and uniqueness of solutions of non-instantaneous impulsive fractional integro-differential equations with proportional fractional derivatives with respect to another function. By the aid of the nonlinear alternative of Leray-Schauder type and the Banach contraction mapping principle, the main results are demonstrated. Two examples are inserted to illustrate the applicability of the theoretical results.

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Keywords: Non-instantaneous impulses, proportional fractional derivatives, Leray-Schauder alternative.

1. Introduction

The theory of fractional differential equations has recently acquired plentiful circulation and great significance because of its rife applications in fields of science and engineering, see, for example [10, 17, 18, 19] and references cited therein. The field of fractional differential equations with instantaneous impulses has become a valuable tool for the description of sudden changes or discontinuous jumps in the evolution progress of dynamical systems such as the shocks, disturbance and natural disasters, see [1, 2] and references cited therein. In the instantaneous impulses the duration of impulsive effect is relatively short as compared to the overall duration of the whole process, see [15]. But many times it has been observed that some certain dynamics of

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evolution processes cannot be described by instantaneous impulsive dynamic systems. For example, the injecting drugs in the bloodstream, and the consequent absorption for the body are gradual and continuous process. In this case the impulsive action begins at any arbitrary fixed point and continues with a finite time interval. Such types of systems are known as non-instantaneous impulsive systems which are more suitable to study the dynamics of evolution processes. Hernándaz et al. [6] introduced a new class of evolution equations with non-instantaneous impulses of the form

$$\begin{cases} x'(t) = Ax(t) + f\left(t, x(t)\right), \ t \in (s_k, t_{k+1}], \ k = 0, 1, \cdots, m, \\ \\ y(t) = g_k\left(t, x(t)\right), \ t \in (t_k, s_k], \ k = 1, \cdots, m, \\ \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq E \to E$, is the generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ on a Banach space E.

Recently, Agarwal et al. in [3] constructed monotone successive approximations for solutions to initial value problems for a scalar nonlinear Caputo fractional differential equation with non-instantaneous impulses of the form

$$\begin{cases} {}_0^C D^q x(t) = f\left(t, x(t)\right), \ t \in (t_k, s_k], \ k = 0, 1, \cdots, p, p+1 \\ \\ x(t) = \phi_k\left(t, x(t), x(s_k - 0)\right), \ t \in (s_k, t_{k+1}], \ k = 1, \cdots, p, \\ \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where ${}_0^C D^q$ is the Caputo fractional derivative of order $0 < q < 1$.

In [12], Kumar et al. studied the sufficient conditions for the existence of mild solution of Atangana-Baleanu fractional differential system with non-instantaneous impulses of the form

$$\begin{cases} {}^{ABC}D^\rho x(t) = Ax(t) + f\left(t, x(t)\right), \ t \in \bigcup_{k=0}^m (s_k, t_{k+1}], \\ \\ x(t) = \gamma_k\left(t, x(t)\right), \ t \in \bigcup_{k=1}^m (t_k, s_k], \\ \\ x(0) = x_0 - g(x), \end{cases} \tag{1.3}$$

where ${}^{ABC}D^\rho$ is the Atangana-Baleanu-Caputo fractional derivative of order $0 < \rho < 1$ and $A : D(A) \subseteq X \to X$, is the generator of ρ-resolvent operator $\{S_\rho(t) : t \geq 0\}$.

In [14], Luo et al. considered the existence of solutions for a kind of ψ-Hilfer fractional differential inclusions involving non-instantaneous impulses of the form

$$\begin{cases} {}^H D_{t_0^+}^{\alpha,\beta;\psi} x(t) \in A(t)x(t) + G\left(t, x(t)\right), \ t \in (s_k, t_{k+1}] \cap [t_0, T], k = 0, 1, \cdots, p, \\ x(t) = \dfrac{\phi_k\left(t, x(t), x(t_k - 0)\right)}{\Gamma(\gamma)\Gamma(2-\gamma)}, \ t \in \bigcup_{k=1}^m (t_k, s_k] \cap [t_0, T], k = 1, \cdots, p, \\ x(t_0) = x_0, \end{cases} \tag{1.4}$$

where ${}^{H}D^{\alpha,\beta;\psi}_{t_0^+}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $0 < \beta \leq 1$, with respect to function ψ, $A(t) : D \subseteq X \to X$ is a bounded operator and $G : (s_k, t_{k+1}] \cap [t_0, T] \times X \to P(X)$ is a multi-valued mapping, $P(X)$ is the family of all nonempty subsets of a real separable Banach space X.
For more recent contributions relevant to non-instantaneous impulsive fractional differential equations, we refer the reader to the papers [11, 13, 16, 20, 21] and references cited therein.

Motivated by the above papers, we investigate the following non-instantaneous impulsive fractional integro-differential equation:

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t) = f\left(t, y(t), {}_aI^{\beta,\rho,g}y(t)\right), \ t \in (s_k, t_{k+1}] \subset J, \ k = 0, 1, \cdots, m, \\ y(t) = \psi_k\left(t, y\left(t_k^+\right)\right), \ t \in (t_k, s_k], \ k = 1, \cdots, m, \\ {}_aI^{1-\alpha,\rho,g}y(a) = y_0 \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $J = [a, T], T > a$, $0 < \alpha \leq 1$, $\beta, \rho > 0$, ${}_aD^{\alpha,\rho,g}$ is the proportional fractional derivative with respect to another function g, ${}_aI^{\beta,\rho,g}$ is the proportional fractional integral with respect to another function g, and $f \in C(J \times \mathbb{R}^2, \mathbb{R})$.
Here, $a = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_{m-1} \leq s_m \leq t_m \leq t_{m+1} = T$ are fixed numbers, $y\left(t_k^+\right) = \lim_{\epsilon \to 0^+} y\left(t_k + \epsilon\right)$, and $\psi_k \in C([t_k, s_k], \mathbb{R})$, $k = 1, \cdots, m$.

Remark 1.1.

- For the non-instantaneous impulsive fractional integro-differential equation (1.5), the $(t_k, s_k]$, $k = 1, \cdots, m$ are called intervals of non-instantaneous impulses and $\psi_k\left(t, y\right)$, $k = 1, \cdots, m$ are called non-instantaneous impulsive functions.
- If $t_k = s_{k-1}$, $k = 1, \cdots, m$, then the non-instantaneous impulsive fractional integro-differential equation (1.5) reduces to an impulsive fractional integro-differential equation.

In recent years, there are various new definitions of fractional derivatives, among these new definitions the so-called fractional conformable derivative, which is introduced by Khalil et al. [9]. Unfortunately, this new definition has an obstacle that it does not tend to the original function as the order ρ tends to zero. Anderson et al. [4] were able to define the proportional (conformable) derivative of order ρ by

$$^{P}D_t^{\rho} f(t) = \kappa_1(\rho, t) f(t) + \kappa_0(\rho, t) f'(t),$$

where f is differentiable function and $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ are continuous functions of the variable t and the parameter $\rho \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$\lim_{\rho \to 0^+} \kappa_0(\rho, t) = 0, \ \lim_{\rho \to 1^-} \kappa_0(\rho, t) = 1, \ \kappa_0(\rho, t) \neq 0, \ \rho \in (0, 1], \tag{1.6}$$

$$\lim_{\rho \to 0^+} \kappa_1(\rho, t) = 1, \ \lim_{\rho \to 1^-} \kappa_1(\rho, t) = 0, \ \kappa_1(\kappa, t) \neq 0, \ \rho \in [0, 1). \tag{1.7}$$

This newly defined local derivative tends to the original function as the order ρ tends to zero and hence improved the conformable derivatives. In [7, 8], Jarad et al. proposed more general forms and properties of proportional derivative of a function f with respect to another function g. The kernel obtained in their investigation contains an exponential function and is function dependent (more details can be seen in Section 2).

The novelty of the current work is that, to the best knowledge of the author, no one has yet been treated with non-instantaneous impulsive fractional differential equations involving the proportional fractional derivative with respect to another function.

2. Preliminaries

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_C = \sup_{t\in J} |y(t)|.$$

We consider the Banach space

$$PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} : \ y \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, \cdots, m \text{ and there exist } y\left(t_k^-\right)$$

$$\text{and } y\left(t_k^+\right), k = 1, \cdots, m \text{ with } y\left(t_k^-\right) = y\left(t_k\right)\},$$

with the norm

$$\|y\|_{PC} = \sup_{t\in J} |y(t)|.$$

Now, we recall some basic definitions and properties of fractional proportional derivative and integral of a function with respect to another function. The terms and notations are adopted from [7, 8].

Definition 2.1. (The proportional derivative of a function with respect to another function) For $\rho \in [0,1]$, let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have

$$\lim_{\rho\to 0^+} \kappa_1(\rho,t) = 1, \ \lim_{\rho\to 0^+} \kappa_0(\rho,t) = 0, \ \lim_{\rho\to 1^-} \kappa_1(\rho,t) = 0, \ \lim_{\rho\to 1^-} \kappa_0(\rho,t) = 1,$$

and $\kappa_1(\rho,t) \neq 0, \rho \in [0,1]$, $\kappa_0(\rho,t) \neq 0, \rho \in [0,1]$. Let $g(t)$ be a strictly increasing continuous function. Then the proportional differential operator of order ρ of f with respect to g is defined by

$$D^{\rho,g} f(t) = \kappa_1(\rho,t) f(t) + \kappa_0(\rho,t) \frac{f'(t)}{g'(t)}. \tag{2.1}$$

For the restricted case when $\kappa_1(\rho,t) = 1-\rho$ and $\kappa_0(\rho,t) = \rho$, (2.1) becomes

$$D^{\rho,g} f(t) = (1-\rho) f(t) + \rho \frac{f'(t)}{g'(t)}. \tag{2.2}$$

Definition 2.2. (The proportional integral of a function with respect to another function) For $\rho \in (0,1]$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $g \in C[a,b]$, $g'(t) > 0$, we define the left and right fractional integrals of f with respect to g by

$$\begin{aligned} {}_aI^{\alpha,\rho,g} f(t) &= \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{\alpha-1} f(s) g'(s) ds, \quad (2.3) \\ I_b^{\alpha,\rho,g} f(t) &= \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_t^b e^{\frac{\rho-1}{\rho}(g(s)-g(t))} (g(s)-g(t))^{\alpha-1} f(s) g'(s) ds, \quad (2.4) \end{aligned}$$

respectively.

Definition 2.3. For $\rho \in (0,1]$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, we define the left fractional derivative of f with respect to g as

$$\begin{aligned} {}_aD^{\alpha,\rho,g} f(t) &= D^{n,\rho,g}\, {}_aI^{n-\alpha,\rho,g} f(t) \\ &= \frac{D_t^{n,\rho,g}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{n-\alpha-1} f(s) g'(s) ds, \quad (2.5) \end{aligned}$$

and the right fractional derivative of f with respect to g as

$$\begin{aligned} D_b^{\alpha,\rho,g} f(t) &= {}_\ominus D^{n,\rho,g} I_b^{n-\alpha,\rho,g} f(t) \\ &= \frac{{}_\ominus D^{n,\rho,g}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_t^b e^{\frac{\rho-1}{\rho}(g(s)-g(t))} (g(s)-g(t))^{n-\alpha-1} f(s) g'(s) ds, \quad (2.6) \end{aligned}$$

where $n = [\Re(\alpha)] + 1$, $D^{n,\rho,g} = \underbrace{D^{\rho,g} D^{\rho,g} \cdots D^{\rho,g}}_{n \text{ times}}$ and

$$ {}_\ominus D^{\rho,g} := (1-\rho) f(t) - \rho \frac{f'(t)}{g'(t)}, \quad {}_\ominus D^{n,\rho,g} = \underbrace{{}_\ominus D^{\rho,g}\, {}_\ominus D^{\rho,g} \cdots {}_\ominus D^{\rho,g}}_{n \text{ times}}. $$

Lemma 2.4. ([8]) *If $\rho \in (0,1]$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. Then, for f is continuous and defined for $t \geq a$, we have*

$$\begin{aligned} {}_aI^{\alpha,\rho,g} \left({}_aI^{\beta,\rho,g} f\right)(t) &= {}_aI^{\beta,\rho,g} \left({}_aI^{\alpha,\rho,g} f\right)(t) = \left({}_aI^{\alpha+\beta,\rho,g} f\right)(t), \quad (2.7) \\ I_b^{\alpha,\rho,g} \left(I_b^{\beta,\rho,g} f\right)(t) &= I_b^{\beta,\rho,g} \left(I_b^{\alpha,\rho,g} f\right)(t) = \left(I_b^{\alpha+\beta,\rho,g} f\right)(t). \quad (2.8) \end{aligned}$$

Lemma 2.5. ([7]) *Let $\Re[\alpha] > 0$, $n = -[-\Re(\alpha)]$, $f \in L_1(a,b)$ and $({}_aI^{\alpha,\rho,g} f)(t) \in AC^n[a,b]$. Then*

$$ {}_aI^{\alpha,\rho,g}\, {}_aD^{\alpha,\rho,g} f(t) = f(t) - e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \sum_{j=1}^{n} ({}_aI^{j-\alpha,\rho,g} f)(a^+) \frac{(g(t)-g(a))^{\alpha-j}}{\rho^{\alpha-j}\Gamma(\alpha+1-j)}. \quad (2.9) $$

For $0 < \alpha \leq 1$, we have

$$ {}_aI^{\alpha,\rho,g}\, {}_aD^{\alpha,\rho,g} f(t) = f(t) - e^{\frac{\rho-1}{\rho}(g(t)-g(a))} ({}_aI^{1-\alpha,\rho,g} f)(a^+) \frac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}. \quad (2.10) $$

Lemma 2.6. *Let $\alpha, \beta > 0$. Then, for any $a, b \in \mathbb{R}$, we get*

$$ I_g := \int_a^b (g(b)-g(s))^{\beta-1} (g(s)-g(a))^{\alpha-1} g'(s) ds = (g(b)-g(a))^{\alpha+\beta-1} \mathbf{B}(\alpha,\beta), $$

where $\mathbf{B}(\cdot,\cdot)$ *is the well-known beta function defined as*

$$\mathbf{B}(m,n)=\int_0^1(1-s)^{m-1}s^{n-1}ds,\ m>0,n>0.$$

Proof. By the substitution $g(s)=g(b)z$, the integral I_g becomes

$$I_g=(g(b))^{\alpha+\beta-1}\int_{\frac{g(a)}{g(b)}}^1(1-z)^{\beta-1}\left(z-\frac{g(a)}{g(b)}\right)^{\alpha-1}dz.$$

Using the following well-known integral

$$\int_a^b(s-a)^{m-1}(b-s)^{n-1}ds=(b-a)^{m+n-1}\mathbf{B}(m,n)=(b-a)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

$m>0$, $n>0$, we get

$$I_g=(g(b))^{\alpha+\beta-1}\left(1-\frac{g(a)}{g(b)}\right)^{\alpha+\beta-1}\mathbf{B}(\alpha,\beta).$$

The proof is finished. □

3. Existence and uniqueness results

In order to investigate the existence of solution for (1.5), we consider the following auxiliary lemma

Lemma 3.1. *Let* $0<\alpha\le 1$ *and let* $h:J\to\mathbb{R}$ *be an integrable function. Then the linear problem*

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t)=h(t),\ t\in(s_k,t_{k+1}]\subset J,\ k=0,1,\cdots,m,\\ y(t)=\psi_k\left(t,y\left(t_k^+\right)\right),\ t\in(t_k,s_k],\ k=1,\cdots,m,\\ {}_aI^{1-\alpha,\rho,g}y(a)=y_0\in\mathbb{R}, \end{cases}\tag{3.1}$$

has a solution given by

$$y(t)=\begin{cases} e^{\frac{\rho-1}{\rho}(g(t)-g(a))}\dfrac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}y_0\\ +\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\left(g(t)-g(s)\right)^{\alpha-1}h(s)g'(s)ds\ ,t\in[a,t_1],\\ \psi_k\left(t,y\left(t_k^+\right)\right),\ t\in(t_k,s_k],\ k=1,\cdots,m,\\ e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))}\left(\dfrac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\\ \times\Big[\psi_k\left(s_k,y\left(t_k^+\right)\right)-\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^{s_k}e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))}\left(g(s_k)-g(s)\right)^{\alpha-1}h(s)g'(s)ds\Big]\\ +\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\left(g(t)-g(s)\right)^{\alpha-1}h(s)g'(s)ds\ ,t\in(s_k,t_{k+1}],\\ k=1,\cdots,m. \end{cases}\tag{3.2}$$

Proof. Let $t \in (0, t_1]$. Then, using Lemma 2.5, the problem

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t) = h(t), \ t \in (a, t_1], \\ {}_aI^{1-\alpha,\rho,g}y(a) = y_0 \in \mathbb{R}, \end{cases} \tag{3.3}$$

has a solution given by

$$\begin{aligned} y(t) &= e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \frac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} y_0 \\ &+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{\alpha-1} h(s)g'(s)ds \ , t \in [0, t_1]. \end{aligned}$$

For $t \in (t_1, s_1]$, $y(t) = \psi_1\left(t, y\left(t_1^+\right)\right)$. Again, using Lemma 2.5 and applying the proportional fractional integral ${}_aI^{\alpha,\rho,g}$ over $(a, t_2]$ to both sides of the problem

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t) = h(t), \ t \in (s_1, t_2], \\ y(s_1) = \psi_1\left(s_1, y\left(t_1^+\right)\right), \end{cases} \tag{3.4}$$

we get

$$\begin{aligned} y(t) &= e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \frac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} {}_aI^{1-\alpha,\rho,g}y(a) \\ &+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{\alpha-1} h(s)g'(s)ds. \end{aligned} \tag{3.5}$$

Substituting $t = s_1$ in (3.5), we get

$$\begin{aligned} y(s_1) &= e^{\frac{\rho-1}{\rho}(g(s_1)-g(a))} \frac{(g(s_1)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} {}_aI^{1-\alpha,\rho,g}y(a) \\ &+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_1} e^{\frac{\rho-1}{\rho}(g(s_1)-g(s))} (g(s_1)-g(s))^{\alpha-1} h(s)g'(s)ds. \end{aligned} \tag{3.6}$$

From the second equation of (3.3), we get

$$\begin{aligned} {}_aI^{1-\alpha,\rho,g}y(a) &= e^{-\frac{\rho-1}{\rho}(g(s_1)-g(a))} \frac{\Gamma(\alpha)\left(g(s_1)-g(a)\right)^{1-\alpha}}{\rho^{1-\alpha}} \\ &\times \left[\psi_1\left(s_1, y\left(t_k^+\right)\right) - \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_1} e^{\frac{\rho-1}{\rho}(g(s_1)-g(s))} (g(s_1)-g(s))^{\alpha-1} h(s)g'(s)ds\right]. \end{aligned} \tag{3.7}$$

Therefore, by substituting (3.7) in (3.5), we get

$$\begin{aligned} y(t) &= e^{\frac{\rho-1}{\rho}(g(t)-g(s_1))} \left(\frac{(g(t)-g(a))}{(g(s_1)-g(a))}\right)^{\alpha-1} \\ &\times \left[\psi_1\left(s_1, y\left(t_1^+\right)\right) - \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_1} e^{\frac{\rho-1}{\rho}(g(s_1)-g(s))} (g(s_1)-g(s))^{\alpha-1} h(s)g'(s)ds\right] \\ &+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{\alpha-1} h(s)g'(s)ds. \end{aligned}$$

For $t \in (t_2, s_2]$,

$$y(t) = \psi_2\left(t, y\left(t_2^+\right)\right).$$

Performing the same process, we deduce when $t \in (s_2, t_3]$ that the solution of the problem

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t) = h(t), \ t \in (s_2, t_3], \\ y(s_2) = \psi_2\left(s_2, y\left(t_2^+\right)\right), \end{cases} \tag{3.8}$$

is given by

$$\begin{aligned} y(t) &= e^{\frac{\rho-1}{\rho}(g(t)-g(s_2))}\left(\frac{(g(t)-g(a))}{(g(s_2)-g(a))}\right)^{\alpha-1} \\ &\times\left[\psi_2\left(s_2, y\left(t_2^+\right)\right) - \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^{s_2} e^{\frac{\rho-1}{\rho}(g(s_2)-g(s))}\left(g(s_2)-g(s)\right)^{\alpha-1}h(s)g'(s)ds\right] \\ &+ \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\left(g(t)-g(s)\right)^{\alpha-1}h(s)g'(s)ds. \end{aligned}$$

In general, when $t \in (s_k, t_{k+1}]$, the solution of the problem

$$\begin{cases} {}_aD^{\alpha,\rho,g}y(t) = h(t), \ t \in (s_k, t_{k+1}], \\ y(s_k) = \psi_k\left(s_k, y\left(t_k^+\right)\right), \end{cases} \tag{3.9}$$

is given by

$$\begin{aligned} y(t) &= e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))}\left(\frac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \\ &\times\left[\psi_k\left(s_k, y\left(t_k^+\right)\right) - \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^{s_k} e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))}\left(g(s_k)-g(s)\right)^{\alpha-1}h(s)g'(s)ds\right] \\ &+ \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\left(g(t)-g(s)\right)^{\alpha-1}h(s)g'(s)ds. \end{aligned}$$

This shows that $y(t)$ satisfies (3.2). This completes the proof. $\square$

By virtue of Lemma 3.1, we deduce that the solution of the non-instantaneous impulsive fractional integro-differential equation (1.5) is given by

$$y(t)=\begin{cases} e^{\frac{\rho-1}{\rho}(g(t)-g(a))}\dfrac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}y_0 \\ +\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}(g(t)-g(s))^{\alpha-1}f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)g'(s)ds, \\ t\in[a,t_1], \\ \\ \psi_k\left(t,y\left(t_k^+\right)\right),\ t\in(t_k,s_k],\ k=1,\cdots,m, \\ \\ e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))}\left(\dfrac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\Big[\psi_k\left(s_k,y\left(t_k^+\right)\right) \\ -\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^{s_k}e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))}(g(s_k)-g(s))^{\alpha-1}f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)g'(s)ds\Big] \\ +\dfrac{1}{\rho^\alpha\Gamma(\alpha)}\displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))}(g(t)-g(s))^{\alpha-1}f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)g'(s)ds, \\ t\in(s_k,t_{k+1}],\ k=1,\cdots,m. \end{cases} \tag{3.10}$$

For ease of handling later, we will use the following brief constants:

$$\begin{cases} \Theta_1:=(g(t_1)-g(a))^\alpha,\ \Theta_2:=\max\left\{(g(t_{k+1})-g(a))^\alpha,\ k=1,\cdots,m\right\}, \\ \Theta_3:=\dfrac{(g(t_1)-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)},\ \Theta_4:=\dfrac{(g(t_1)-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}, \\ \Xi_1:=\dfrac{(g(t_1)-g(a))^{\alpha+\beta}}{\rho^\beta\Gamma(\beta+1)}, \\ \Xi_2:=\max\left\{\left(\dfrac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1},\ k=1,\cdots,m\right\}, \\ \Xi_3:=\max\left\{\dfrac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^\beta\Gamma(\beta+1)},\ k=1,\cdots,m\right\}, \\ \Xi_4:=\max\left\{\dfrac{(g(t_{k+1})-g(a))^{\alpha}}{\rho^\alpha\Gamma(\alpha+1)},\ k=1,\cdots,m\right\}, \\ \Xi_5:=\max\left\{\dfrac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)},\ k=1,\cdots,m\right\}. \end{cases} \tag{3.11}$$

In order to investigate the main results, the following hypotheses will be imposed.

(H1). The function $f:J\times\mathbb{R}^2\to\mathbb{R}$ is continuous and $\psi_k\in C([t_k,s_k],\mathbb{R})$, $k=1,\cdots,m$.

(H2). There exists a constant $L_f>0$ such that

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\le L_f\left(|u_1-u_2|+|v_1-v_2|\right),$$

for each $t\in[s_k,t_{k+1}],k=0,1,\cdots,m$, for all $u_i,v_i\in\mathbb{R},\ i=1,2$.

(H3). There exist constants $L_k > 0, k = 1, \cdots, m$ such that

$$|\psi_k(t, u_1) - \psi_k(t, u_2)| \leq L_k |u_1 - u_2|,$$

for each $t \in [t_k, s_k], k = 1, \cdots, m$, for all $u_1, u_2 \in \mathbb{R}$.

(H4). There exist positive constants ℓ_0, ℓ_1 and ℓ_2 such that

$$|f(t, u, v)| \leq \ell_0 + \ell_1 |u| + \ell_2 |v|,$$

for each $t \in [s_k, t_{k+1}], k = 0, 1, \cdots, m$, for all $u, v \in \mathbb{R}$.

(H5). There exist positive constants $\aleph_0$ and $\aleph_1 0$ such that

$$|\psi_k(t, u)| \leq \aleph_0 + \aleph_1 |u|,$$

for each $t \in [t_k, s_k], k = 1, \cdots, m$, for all u.

(H6). There exists a constant $M > 0$ such that

$$\max\left\{ \frac{M}{\Theta_3(\ell_0 + \ell_1 M) + \Theta_4 \ell_2 M}, \frac{M}{\aleph_0 + \aleph_1 M}, \right.$$

$$\left. \frac{M}{\Xi_2 \left[\aleph_0 + \aleph_1 M + \Xi_4(\ell_0 + \ell_1 M) + \Xi_5 \ell_2 M \right] + \Xi_4(\ell_0 + \ell_1 M) + \Xi_5 \ell_2 M} \right\} > 1.$$

For the purpose of convenience, for each $t \in [a, T]$ and each $y_1, y_2 \in PC(J, \mathbb{R})$, we have

$$\begin{aligned}
&|{}_a I^{\beta,\rho,g} y_1(t) - {}_a I^{\beta,\rho,g} y_2(t)| \\
&\leq \frac{1}{\rho^\beta \Gamma(\beta)} \int_a^t \left| e^{\frac{\rho-1}{\rho}(g(t)-g(s))} \right| (g(t) - g(s))^{\beta-1} |y_1(s) - y_2(s)| g'(s) ds \\
&\leq \frac{(g(T) - g(a))^\beta}{\rho^\beta \Gamma(\beta + 1)} \|y_1 - y_2\|_{PC}.
\end{aligned} \tag{3.12}$$

Also, since g is monotonic increasing, then $\forall t > a,\ \rho \in (0, 1)$, we have

$$\left| e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \right| < 1.$$

The following result is based on the Banach contraction mapping principle.

Theorem 3.2. *Assume that the hypotheses* **(H1)**-**(H3)** *are satisfied. If*

$$\Omega := \max\left\{ \frac{L_f(\Theta_1 + \Xi_1)}{\rho^\alpha \Gamma(\alpha + 1)}, \Xi_2 \left(L_k + \frac{L_f(\Theta_2 + \Xi_3)}{\rho^\alpha \Gamma(\alpha + 1)} \right) + \frac{L_f(\Theta_2 + \Xi_3)}{\rho^\alpha \Gamma(\alpha + 1)} \right\} < 1, \tag{3.13}$$

then the non-instantaneous impulsive fractional integro-differential equation (1.5) has a unique solution on J.

Proof. We transform the problem of non-instantaneous impulsive fractional integro-differential equation (1.5) into a fixed point problem.

Define an operator $\mathcal{N} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$(\mathcal{N}y)(t) = \begin{cases} e^{\frac{\rho-1}{\rho}(g(t)-g(a))} \dfrac{(g(t)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} y_0 \\ + \dfrac{1}{\rho^\alpha\Gamma(\alpha)} \displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{\alpha-1} f(s, y(s), {}_aI^{\beta,\rho,g}y(s)) g'(s)ds, \\ t \in [a, t_1]; \\ \psi_k\left(t, y\left(t_k^+\right)\right),\ t \in (t_k, s_k],\ k = 1, \cdots, m; \\ e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))} \left(\dfrac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \Big[\psi_k\left(s_k, y\left(t_k^+\right)\right) \\ - \dfrac{1}{\rho^\alpha\Gamma(\alpha)} \displaystyle\int_a^{s_k} e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))} (g(s_k)-g(s))^{\alpha-1} f(s, y(s), {}_aI^{\beta,\rho,g}y(s)) g'(s)ds\Big] \\ + \dfrac{1}{\rho^\alpha\Gamma(\alpha)} \displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(g(t)-g(s))} (g(t)-g(s))^{\alpha-1} f\left(s, y(s), {}_aI^{\beta,\rho,g}y(s)\right) g'(s)ds, \\ t \in (s_k, t_{k+1}], k = 1, \cdots, m. \end{cases} \tag{3.14}$$

Obviously, it is easy to see that the operator $\mathcal{N}$ is well defined according to the continuity hypotheses of f and ψ_k. Next, we shall show that $\mathcal{N}$ is a contraction.

Case I. For each $t \in [a, t_1]$ and each $y_1, y_2 \in PC(J,\mathbb{R})$, using (3.11) and (3.12), we have

$$\begin{aligned} &|(\mathcal{N}y_1)(t) - (\mathcal{N}y_2)(t)| \\ \le& \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t \left|e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\right| (g(t)-g(s))^{\alpha-1} \left|f\left(s, y_1(s), {}_aI^{\beta,\rho,g}y_1(s)\right)\right. \\ &\left.-f\left(s, y_2(s), {}_aI^{\beta,\rho,g}y_2(s)\right)\right| g'(s)ds \\ \le& \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} L_f\left(|y_1(s)-y_2(s)|\right. \\ &\left.+|{}_aI^{\beta,\rho,g}y_1(s) - {}_aI^{\beta,\rho,g}y_2(s)|\right) g'(s)ds \\ \le& \frac{L_f}{\rho^\alpha\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\|y_1-y_2\|_{PC} + \frac{(g(t_1)-g(a))^\beta}{\rho^\beta\Gamma(\beta+1)}\|y_1-y_2\|_{PC}\right) g'(s)ds \\ \le& \frac{L_f}{\rho^\alpha\Gamma(\alpha+1)} (\Theta_1 + \Xi_1) \|y_1-y_2\|_{PC}. \end{aligned}$$

Case II. For each $t \in (t_k, s_k], k = 1, \cdots, m$ and each $y_1, y_2 \in PC(J,\mathbb{R})$, we obtain

$$|(\mathcal{N}y_1)(t) - (\mathcal{N}y_2)(t)| \le L_k\|y_1-y_2\|_{PC}.$$

Case III. For each $t \in (s_k, t_{k+1}], k = 1, \cdots, m$ and each $y_1, y_2 \in PC(J, \mathbb{R})$, using (3.12), we get

$$
\begin{aligned}
&\left|(\mathcal{N}y_1)(t) - (\mathcal{N}y_2)(t)\right| \\
&\leq \left| e^{\frac{\rho-1}{\rho}(g(t)-g(s_k))} \left(\frac{(g(t)-g(a))}{(g(s_k)-g(a))} \right)^{\alpha-1} \right| \Bigg[\left|\psi_k\left(s_k, y_1\left(t_k^+\right)\right) - \psi_k\left(s_k, y_2\left(t_k^+\right)\right)\right| \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_k} \left| e^{\frac{\rho-1}{\rho}(g(s_k)-g(s))} \right| (g(s_k)-g(s))^{\alpha-1} \left| f\left(s, y_1(s), {}_aI^{\beta,\rho,g} y_1(s)\right) \right. \\
&\left. - f\left(s, y_2(s), {}_aI^{\beta,\rho,g} y_2(s)\right) \right| g'(s) ds \Bigg] \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{t} \left| e^{\frac{\rho-1}{\rho}(g(t)-g(s))} \right| (g(t)-g(s))^{\alpha-1} \left| f\left(s, y_1(s), {}_aI^{\beta,\rho,g} y_1(s)\right) \right. \\
&\left. - f\left(s, y_2(s), {}_aI^{\beta,\rho,g} y_2(s)\right) \right| g'(s) ds \\
&\leq \Bigg[\left(\frac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))} \right)^{\alpha-1} \\
&\times \left(L_k + \frac{L_f}{\rho^\alpha \Gamma(\alpha+1)} \left[(g(t_{k+1})-g(a))^\alpha + \frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^\beta \Gamma(\beta+1)} \right] \right) \\
&+ \frac{L_f}{\rho^\alpha \Gamma(\alpha+1)} \left[(g(t_{k+1})-g(a))^\alpha + \frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^\beta \Gamma(\beta+1)} \right] \Bigg] \|y_1 - y_2\|_{PC} \\
&\leq \left[\Xi_2 \left(L_k + \frac{L_f(\Theta_2 + \Xi_3)}{\rho^\alpha \Gamma(\alpha+1)} \right) + \frac{L_f(\Theta_2 + \Xi_3)}{\rho^\alpha \Gamma(\alpha+1)} \right] \|y_1 - y_2\|_{PC}
\end{aligned}
$$

Therefore, one has

$$\|\mathcal{N}y_1 - \mathcal{N}y_2\|_{PC} \leq \Omega \|y_1 - y_2\|_{PC}.$$

Since, by (3.13), $\Omega < 1$. Then, the operator $\mathcal{N}$ is a contraction and there exists a unique solution $y \in PC(J, \mathbb{R})$ of the non-instantaneous impulsive fractional integro-differential equation (1.5). This completes the proof. □

Now, we prove the existence of solutions of the non-instantaneous impulsive fractional integro-differential equation (1.5) by applying the following Leray-Schauder nonlinear alternative.

Theorem 3.3. [5] **(Leray-Schauder nonlinear alternative)** *Let $\mathbb{E}$ be a Banach space, D a closed convex subset of $\mathbb{E}$ and $\mathcal{S} \subset D$ an open subset with $0 \in \mathcal{S}$. Then each continuous compact mapping $\mathcal{N} : \overline{\mathcal{S}} \to D$ has at least one of the following properties:*

i. *$\mathcal{N}$ has a fixed point in $\overline{\mathcal{S}}$, or*

ii. *there exists $w \in \partial\mathcal{S}$ (the boundary of $\mathcal{S}$ in D) and $\xi \in (0,1)$ with $w = \xi\mathcal{N}(w)$.*

Theorem 3.4. *Assume that the hypotheses* **(H4)**-**(H6)** *are satisfied.If*

$$\max\left\{\Theta_3\ell_1 + \Theta_4\ell_2,\ \aleph_1,\ \Xi_2\left[\aleph_1 + \Xi_4\ell_1 + \Xi_5\ell_2\right] + \Xi_4\ell_1 + \Xi_5\ell_2\right\} < 1. \tag{3.15}$$

Then the non-instantaneous impulsive fractional integro-differential equation (1.5) has at least one solution on J.

Proof. Let $\mathcal{N}$ be defined by (3.14) and $\mathcal{B}_r = \{y \in PC(J,\mathbb{R}) : \|y\|_{PC} \leq r\}$ be a closed convex subset of $PC(J,\mathbb{R})$, where

$$r \geq \max\left\{\frac{\Theta_3\ell_0}{1-(\Theta_3\ell_1+\Theta_4\ell_2)},\ \frac{\aleph_0}{1-\aleph_1},\ \frac{\Xi_2\left[\aleph_0+\Xi_4\ell_0\right]+\Xi_4\ell_0}{1-\left(\Xi_2\left[\aleph_1+\Xi_4\ell_1+\Xi_5\ell_2\right]+\Xi_4\ell_1+\Xi_5\ell_2\right)}\right\}. \tag{3.16}$$

The proof will be given in several steps.

Step 1. $\mathcal{N}$ is continuous.

Let y_n be a sequence such that $y_n \to y$ in $PC(J,\mathbb{R})$.

Case I. For each $t \in [a,t_1]$, we have

$$\begin{aligned}
&|(\mathcal{N}y_n)(t)-(\mathcal{N}y)(t)| \\
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t \left|e^{\frac{\rho-1}{\rho}(g(t)-g(s))}\right|(g(t)-g(s))^{\alpha-1}\left|f\left(s,y_n(s),{}_aI^{\beta,\rho,g}y_n(s)\right)\right. \\
&\left.-f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)\right|g'(s)ds \\
&\leq \frac{(g(t_1)-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\left\|f\left(\cdot,y_n(\cdot),{}_aI^{\beta,\rho,g}y_n(\cdot)\right)-f\left(\cdot,y(\cdot),{}_aI^{\beta,\rho,g}y(\cdot)\right)\right\|_{PC}.
\end{aligned}$$

Case II. For each $t \in (t_k,s_k], k=1,\cdots,m$, we get

$$|(\mathcal{N}y_n)(t)-(\mathcal{N}y)(t)| \leq \|\psi_k\left(\cdot,y_n\left(\cdot\right)\right)-\psi_k\left(\cdot,y\left(\cdot\right)\right)\|_{PC}.$$

Case III. For each $t \in (s_k,t_{k+1}], k=1,\cdots,m$, we obtain that

$$\begin{aligned}
&|(\mathcal{N}y_n)(t)-(\mathcal{N}y)(t)| \\
&\leq \left(\frac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\left[\left|\psi_k\left(s_k,y_n\left(t_k^+\right)\right)-\psi_k\left(s_k,y\left(t_k^+\right)\right)\right|\right. \\
&+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^{s_k}(g(s_k)-g(s))^{\alpha-1}\left|f\left(s,y_n(s),{}_aI^{\beta,\rho,g}y_n(s)\right)\right. \\
&\left.\left.-f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)\right|g'(s)ds\right] \\
&+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t(g(t)-g(s))^{\alpha-1}\left|f\left(s,y_n(s),{}_aI^{\beta,\rho,g}y_n(s)\right)\right. \\
&\left.-f\left(s,y(s),{}_aI^{\beta,\rho,g}y(s)\right)\right|g'(s)ds \\
&\leq \left(\frac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\left[\|\psi_k\left(\cdot,y_n\left(\cdot\right)\right)-\psi_k\left(\cdot,y\left(\cdot\right)\right)\|_{PC}\right. \\
&\left.+\frac{(g(s_k)-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\left\|f\left(\cdot,y_n(\cdot),{}_aI^{\beta,\rho,g}y_n(\cdot)\right)-f\left(\cdot,y(\cdot),{}_aI^{\beta,\rho,g}y(\cdot)\right)\right\|_{PC}\right] \\
&+\frac{(g(t_{k+1})-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\left\|f\left(\cdot,y_n(\cdot),{}_aI^{\beta,\rho,g}y_n(\cdot)\right)-f\left(\cdot,y(\cdot),{}_aI^{\beta,\rho,g}y(\cdot)\right)\right\|_{PC}.
\end{aligned}$$

Since the functions f and ψ_k are continuous. Then, from above inequalities, we deduce that $\|\mathcal{N}y_n - \mathcal{N}y\|_{PC} \to 0$ as $n \to \infty$.

Step 2. $\mathcal{N}$ is uniformly bounded.

Case I. For each $t \in [a, t_1]$ and for any $y \in \mathcal{B}_r$, using (3.11) and Lemma 2.6, we have

$$\begin{aligned}
|(\mathcal{N}y)(t)| &\leq \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left|f\left(s, y(s), {}_aI^{\beta,\rho,g}y(s)\right)\right| g'(s)ds \\
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1|y(s)| + \ell_2|{}_aI^{\beta,\rho,g}y(s)|\right) g'(s)ds \\
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1|y(s)| + \ell_2\, {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds \\
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \Bigg(\ell_0 + \ell_1\|y\|_{PC} \\
&+ \frac{\ell_2}{\rho^\beta\Gamma(\beta)} \int_a^s (g(s)-g(\tau))^{\beta-1} \|y\|_{PC}\; g'(\tau)d\tau\Bigg) g'(s)ds \\
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\ell_0+\ell_1 r + \frac{\ell_2 r}{\rho^\beta\Gamma(\beta+1)}(g(s)-g(a))^\beta\right) g'(s)ds \\
&\leq \Theta_3(\ell_0+\ell_1 r) + \Theta_4\ell_2 r \leq r.
\end{aligned}$$

Case II. For each $t \in (t_k, s_k], k = 1, \cdots, m$, and for any $y \in \mathcal{B}_r$, we get

$$\begin{aligned}
|(\mathcal{N}y)(t)| &\leq \left|\psi_k\left(t, y\left(t_k^+\right)\right)\right| \\
&\leq \aleph_0 + \aleph_1|y(t_k^+)| \\
&\leq \aleph_0 + \aleph_1\|y\|_{PC} \\
&\leq \aleph_0 + \aleph_1 r \leq r.
\end{aligned}$$

Case III. For each $t \in (s_k, t_{k+1}], k = 1, \cdots, m$, and for any $y \in \mathcal{B}_r$, using (3.11) and Lemma 2.6, we obtain

$$\begin{aligned}
|(\mathcal{N}y)(t)| &\leq \left(\frac{(g(t)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \Bigg[\aleph_0 + \aleph_1|y(t_k^+)| \\
&+ \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^{s_k} (g(s_k)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1|y(s)| + \ell_2|{}_aI^{\beta,\rho,g}y(s)|\right) g'(s)ds\Bigg] \\
&+ \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1|y(s)| + \ell_2|{}_aI^{\beta,\rho,g}y(s)|\right) g'(s)ds \\
&\leq \left(\frac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1} \left[\aleph_0+\aleph_1 r+\frac{(g(t_{k+1})-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}(\ell_0+\ell_1 r)+\frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2 r\right] \\
&+ \frac{(g(t_{k+1})-g(a))^\alpha}{\rho^\alpha\Gamma(\alpha+1)}(\ell_0+\ell_1 r) + \frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2 r \\
&\leq \Xi_2\left[\aleph_0 + \aleph_1 r + \Xi_4(\ell_0+\ell_1 r) + \Xi_5\ell_2 r\right] + \Xi_4(\ell_0+\ell_1 r) + \Xi_5\ell_2 r \leq r.
\end{aligned}$$

From the above three inequalities, using (3.16), we infer that $\|\mathcal{N}y\|_{PC} \leq r$. Hence, the operator $\mathcal{N}$ maps bounded sets into bounded sets of $PC(J,\mathbb{R})$.

Step 3. $\mathcal{N}$ maps bounded sets into equicontinuous sets.

Case I. For the interval $t \in [a,t_1]$, $a \leq \vartheta_1 < \vartheta_2 \leq t_1$ and for any $y \in \mathcal{B}_r$, we have

$$\begin{aligned}
&|(\mathcal{N}y)(\vartheta_2) - (\mathcal{N}y)(\vartheta_1)| \\
&\leq \left| e^{\frac{\rho-1}{\rho}(g(\vartheta_2)-g(a))} \frac{(g(\vartheta_2)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} - e^{\frac{\rho-1}{\rho}(g(\vartheta_1)-g(a))} \frac{(g(\vartheta_1)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} \right| |y_0| \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{\vartheta_1} \left| (g(\vartheta_2)-g(s))^{\alpha-1} - (g(\vartheta_1)-g(s))^{\alpha-1} \right| \\
&\times \left(\ell_0 + \ell_1 |y(s)| + \ell_2 \ {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{\vartheta_1}^{\vartheta_2} (g(\vartheta_2)-g(s))^{\alpha-1} \left(\ell_0 + \ell_1 |y(s)| + \ell_2 \ {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds \\
&\leq \left| e^{\frac{\rho-1}{\rho}(g(\vartheta_2)-g(a))} \frac{(g(\vartheta_2)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} - e^{\frac{\rho-1}{\rho}(g(\vartheta_1)-g(a))} \frac{(g(\vartheta_1)-g(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)} \right| |y_0| \\
&+ \frac{\ell_0 + \ell_1 r}{\rho^\alpha \Gamma(\alpha+1)} \left(2\,(g(\vartheta_2)-g(\vartheta_1))^\alpha + |(g(\vartheta_2)-g(a))^\alpha - (g(\vartheta_1)-g(a))^\alpha|\right) \\
&+ \frac{\ell_2 r}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+1)} \Bigg(\int_a^{\vartheta_1} \left| (g(\vartheta_2)-g(s))^{\alpha-1} - (g(\vartheta_1)-g(s))^{\alpha-1} \right| \\
&\times (g(s)-g(a))^\beta g'(s)ds \\
&+ \int_{\vartheta_1}^{\vartheta_2} (g(\vartheta_2)-g(s))^{\alpha-1} (g(s)-g(a))^\beta g'(s)ds \Bigg) \to 0, \ as\ \vartheta_2 \to \vartheta_1.
\end{aligned}$$

Case II. For each $t \in (t_k, s_k], k = 1, \cdots, m$, $a \leq \vartheta_1 < \vartheta_2 \leq t_1$ and for any $y \in \mathcal{B}_r$, one has

$$|(\mathcal{N}y)(\vartheta_2) - (\mathcal{N}y)(\vartheta_1)| \quad \leq \quad \left| \psi_k\left(\vartheta_2, y\left(t_k^+\right)\right) - \psi_k\left(\vartheta_1, y\left(t_k^+\right)\right) \right| \to 0, \ as\ \vartheta_2 \to \vartheta_1.$$

Case III. For each $t \in (s_k, t_{k+1}], k = 1, \cdots, m$, $a \leq \vartheta_1 < \vartheta_2 \leq t_1$ and for any $y \in \mathcal{B}_r$, using Lemma 2.6, one has

$$\begin{aligned}
&|(\mathcal{N}y)(\vartheta_2) - (\mathcal{N}y)(\vartheta_1)| \\
&\leq \left| e^{\frac{\rho-1}{\rho}(g(\vartheta_2)-g(s_k))} \left(\frac{(g(\vartheta_2)-g(a))}{(g(s_k)-g(a))} \right)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(g(\vartheta_1)-g(s_k))} \left(\frac{(g(\vartheta_1)-g(a))}{(g(s_k)-g(a))} \right)^{\alpha-1} \right| \\
&\times \Big[\left|\psi_k\left(s_k, y\left(t_k^+\right)\right)\right| + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{s_k} (g(s_k)-g(s))^{\alpha-1} \\
&\times \left(\ell_0 + \ell_1 |y(s)| + \ell_2 \ {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds \Big] \\
&+ \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{\vartheta_1} \left| (g(\vartheta_2)-g(s))^{\alpha-1} - (g(\vartheta_1)-g(s))^{\alpha-1} \right| \\
&\times \left(\ell_0 + \ell_1 |y(s)| + \ell_2 \ {}_aI^{\beta,\rho,g}|y(s)|\right) g'(s)ds
\end{aligned}$$

$$+\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{\vartheta_1}^{\vartheta_2}(g(\vartheta_2)-g(s))^{\alpha-1}\left(\ell_0+\ell_1|y(s)|+\ell_2\ {}_aI^{\beta,\rho,g}|y(s)|\right)g'(s)ds$$

$$\leq\left|e^{\frac{\rho-1}{\rho}(g(\vartheta_2)-g(s_k))}\left(\frac{(g(\vartheta_2)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}-e^{\frac{\rho-1}{\rho}(g(\vartheta_1)-g(s_k))}\left(\frac{(g(\vartheta_1)-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\right|$$

$$\times\left[\left|\psi_k\left(s_k,y\left(t_k^+\right)\right)\right|+\frac{(g(s_k)-g(a))^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}(\ell_0+\ell_1 r)+\frac{(g(s_k)-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2 r\right]$$

$$+\frac{\ell_0+\ell_1 r}{\rho^{\alpha}\Gamma(\alpha+1)}\left(2\left(g(\vartheta_2)-g(\vartheta_1)\right)^{\alpha}+|(g(\vartheta_2)-g(a))^{\alpha}-(g(\vartheta_1)-g(a))^{\alpha}|\right)$$

$$+\frac{\ell_2 r}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+1)}\left(\int_a^{\vartheta_1}|(g(\vartheta_2)-g(s))^{\alpha-1}-(g(\vartheta_1)-g(s))^{\alpha-1}|(g(s)-g(a))^{\beta}g'(s)ds\right.$$

$$\left.+\int_{\vartheta_1}^{\vartheta_2}(g(\vartheta_2)-g(s))^{\alpha-1}(g(s)-g(a))^{\beta}g'(s)ds\right)\to 0,\ as\ \vartheta_2\to\vartheta_1.$$

In view of the above three inequalities, we infer that $\|(\mathcal{N}y)(\vartheta_2)-(\mathcal{N}y)(\vartheta_1)\|_{PC}\to 0$ independently of $y\in\mathcal{B}_r$, as $\vartheta_2\to\vartheta_1$. Consequently, the operator $\mathcal{N}$ is equicontinuous and uniformly bounded. Hence, by Arzelà-Ascoli Theorem, the operator $\mathcal{N}: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is is completely continuous.

Step 4. We show that there exists an open set $\mathcal{S}\subset PC(J,\mathbb{R})$ with $y\neq\xi\mathcal{N}y$ for $\xi\in(0,1)$ and $y\in\partial\mathcal{S}$.

In other words, we shall show that the part **(i)** in Theorem 3.3 is verified.

Consider the equation $y=\xi\ \mathcal{N}y$, for $\xi\in(0,1)$. Then, in view of **Step 2**, we have the following cases:

Case I. For the interval $t\in[a,t_1]$, we have

$$\begin{aligned}|y(t)| &= |\xi\ \mathcal{N}y(t)|\\ &\leq \frac{(g(t_1)-g(a))^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}(\ell_0+\ell_1\|y\|_{PC})+\frac{(g(t_1)-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2\|y\|_{PC},\end{aligned}$$

which implies that:

$$\frac{\|y\|_{PC}}{\Theta_3(\ell_0+\ell_1\|y\|_{PC})+\Theta_4\ell_2\|y\|_{PC}}\leq 1. \tag{3.17}$$

Case II. For each $t\in(t_k,s_k], k=1,\cdots,m$, one has

$$\begin{aligned}|y(t)| &= |\xi\ \mathcal{N}y(t)|\\ &\leq \aleph_0+\aleph_1\|y\|_{PC},\end{aligned}$$

which implies that:

$$\frac{\|y\|_{PC}}{\aleph_0+\aleph_1\|y\|_{PC}}\leq 1. \tag{3.18}$$

Case III. For each $t \in (s_k, t_{k+1}], k = 1, \cdots, m$, one has

$$\begin{aligned}|y(t)| &= |\xi\ \mathcal{N}y(t)| \\ &\le \left(\frac{(g(t_{k+1})-g(a))}{(g(s_k)-g(a))}\right)^{\alpha-1}\left[\aleph_0+\aleph_1\|y\|_{PC}+\frac{(g(t_{k+1})-g(a))^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}(\ell_0+\ell_1\|y\|_{PC})\right. \\ &\left.+\frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2\|y\|_{PC}\right] \\ &+\frac{(g(t_{k+1})-g(a))^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}(\ell_0+\ell_1\|y\|_{PC})+\frac{(g(t_{k+1})-g(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\ell_2\|y\|_{PC},\end{aligned}$$

which implies, by (3.11), that:

$$\frac{\|y\|_{PC}}{\Xi_2[\aleph_0+\aleph_1\|y\|_{PC}+\Xi_4(\ell_0+\ell_1\|y\|_{PC})+\Xi_5\ell_2\|y\|_{PC}]+\Xi_4(\ell_0+\ell_1\|y\|_{PC})+\Xi_5\ell_2\|y\|_{PC}}\le 1. \tag{3.19}$$

By combining (3.17),(3.18) and (3.19) together with **(H6)**, there exists M such that:

$$M \neq \|y\|_{PC}.$$

Let us set

$$\mathcal{S} = \{y \in PC(J, \mathbb{R} : \|y\|_{PC} < M)\}.$$

Note that the operator $\mathcal{N} : \overline{\mathcal{S}} \to PC(J, \mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{S}$, there is no $y \in \partial\mathcal{S}$ such that $y = \xi\mathcal{N}y$ for $\xi \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 3.3), we deduce that $\mathcal{N}$ has a fixed point $y \in \overline{\mathcal{S}}$ which is a solution of (1.5). This completes the proof. $\square$

4. Illustrative examples

Example 4.1. Consider the following non-instantaneous impulsive fractional problem:

$$\begin{cases} {}_aD^{\frac{1}{2},2,t^2}y(t) = \dfrac{e^{-2t}\left(|y(t)|+\left|{}_{0^+}I^{\frac{3}{4},2,t^2}y(t)\right|\right)}{(1+7e^t)\left(1+|y(t)|+\left|{}_{0^+}I^{\frac{3}{4},2,t^2}y(t)\right|\right)}, \ t \in (0,\frac{1}{3}] \cup (\frac{2}{3},1], \\ y(t) = \dfrac{|y(\frac{1}{3}^+)|}{(3+7e^{2t})(1+|y(\frac{1}{3}^+)|)}, \ t \in (\frac{1}{3},\frac{2}{3}], \\ {}_{0^+}I^{\frac{1}{2},2,t^2}y(0) = 0, \end{cases} \tag{4.1}$$

Here, $J = [0,1],\ 0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$, and $\alpha = \frac{1}{2}, \beta = \frac{3}{4}, \rho = 1, m = 1$. Set

$$g(t) = t^2, \quad f(t,u,v) = \frac{e^{-2t}\left(|u|+|v|\right)}{\left(1+7e^t\right)\left(1+|u|+|v|\right)}$$

and

$$\psi_1(t,u) = \frac{|u|}{(3+7e^{2t})(1+|u|)}.$$

Let $u_i, v_i \in \mathbb{R}, i = 1,2$ and $t \in [0, \frac{1}{3}] \cup (\frac{2}{3}, 1]$. Then, we get

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq \frac{1}{8}\left(|u_1 - u_2| + |v_1 - v_2|\right).$$

Let $u_1, u_2 \in \mathbb{R}$ and $t \in (\frac{1}{3}, \frac{2}{3}]$. Then, we obtain

$$|\psi_1(t,u_1) - \psi_1(t,u_2)| \leq \frac{1}{10}|u_1 - u_2|.$$

Thus, the hypotheses **(H1)**, **(H2)** and **(H3)** in Theorem 3.2 are satisfied with $L_f = \frac{1}{8}$ and $L_k = L_1 = \frac{1}{10}$. Therefore, by (3.13), one can deduce that:

$$\Omega = \max\{0.0568660825, 0.55752695\} = 0.55752695 < 1.$$

Hence, the non-instantaneous impulsive fractional problem (4.1) has a unique solution on $[0,1]$.

Example 4.2. Consider

$$\begin{cases} {}_aD^{\frac{1}{2},2,t^2}y(t) = \frac{\sin t}{5\sqrt{9+t^2}} + \frac{|y(t)|}{10e^t(1+|y(t)|)} + \frac{\left|{}_{0^+}I^{\frac{3}{4},2,t^2}y(t)\right|}{25+t^2}, \ t \in (0, \frac{1}{3}] \cup (\frac{2}{3}, 1], \\ y(t) = \frac{e^{-t}}{16+t^4} + \frac{\cos y\left(\frac{1}{3}^+\right)}{4\sqrt{49+t^2}}, \ t \in (\frac{1}{3}, \frac{2}{3}], \\ {}_{0^+}I^{\frac{1}{2},2,t^2}y(0) = 0, \end{cases} \tag{4.2}$$

Here, $J = [0,1]$, $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$, and $\alpha = \frac{1}{2}, \beta = \frac{3}{4}, \rho = 1, m = 1$. Set

$$g(t) = t^2, \quad f(t,u,v) = \frac{\sin t}{5\sqrt{9+t^2}} + \frac{|u|}{10e^t(1+|u|)} + \frac{|v|}{25+t^2}$$

and

$$\psi_1(t,u) = \frac{e^{-t}}{16+t^4} + \frac{\cos u}{4\sqrt{49+t^2}}.$$

For all $u, v \in \mathbb{R}$ and each $t \in [0, \frac{1}{3}] \cup (\frac{2}{3}, 1]$, we get

$$|f(t,u,v)| \leq \frac{1}{15} + \frac{1}{10}|u| + \frac{1}{25}|v|.$$

For all $u \in \mathbb{R}$ and each $t \in (\frac{1}{3}, \frac{2}{3}]$, we get

$$|\psi_1(t,u)| \leq \frac{1}{16} + \frac{1}{28}|u|.$$

Thus, the hypotheses **(H4)** and **(H5)** hold with $\ell_0 = \frac{1}{15}, \ell_1 = \frac{1}{10}, \ell_2 = \frac{1}{25}, \aleph_0 = \frac{1}{16}$ and $\aleph_1 = \frac{1}{28}$. Moreover, from (3.15), we get

$$\begin{aligned} &\max\left\{\Theta_3\ell_1 + \Theta_4\ell_2, \ \aleph_1, \ \Xi_2\left[\aleph_1 + \Xi_4\ell_1 + \Xi_5\ell_2\right] + \Xi_4\ell_1 + \Xi_5\ell_2\right\} \\ &= \{0.039877417, 0.03571428571, 0.500165849\} \\ &= 0.0.500165849 < 1. \end{aligned}$$

By Theorem 3.4, we conclude that our theoretical results are applicable to the problem (4.2).

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A fixed point approach to the semi-linear Stokes problem

David Brumar

Abstract. The aim of this paper is to study the Dirichlet problem for semi-linear Stokes equations. The approach of this study is based on the operator method, using abstract results of nonlinear functional analysis. We first study the problem using Schauder's fixed point theorem and we prove the existence of a solution in case that the nonlinear term has a linear growth. Next we establish whether the existence of solutions can still be obtained without this linear growth restriction. Such a result is obtained by applying the Leray-Schauder fixed point theorem.

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Keywords: Stokes system, semi-linear problem, operator method, fixed point theorem, Sobolev space.

1. Introduction

The field of fluid dynamics does not only engage the attention of mathematicians and physicists but also of astrophysicists, oceanographers and many others and this is due to the fact that it addresses real-world natural phenomena and tries to come up with mathematical models that help us to understand them.

An inertial fluid flow that is Newtonian, incompressible and homogeneous follows the Navier-Stokes equations, which are essentially derived from Newton's second law of motion applied to the fluid and the law of mass conservation in the context of constant density flow. If the velocity field is not time-dependent, then the flow is called steady, and it means that the fluid particles follow the streamlines, which do not change in time. Neglecting the nonlinear term in the Navier-Stokes system we get the Stokes system, which is in fact, the one that we are here interested in.

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The aim of this paper is to study the existence of solutions of the semi-linear Dirichlet problem for the steady Stokes system

$$\begin{cases} -\mu\Delta u + \nabla p = f(x, u(x)) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

2. Preliminaries

In this section we briefly recall without proof, some important results from functional analysis and some basic results regarding the Stokes system that are used in the forthcoming material. For additional details, we refer the reader to the following works [1, 2, 3, 4, 7, 8, 9, 10].

2.1. The Nemytskii operator

First we recall some properties of the Nemytskii superposition operator (see, e.g., [6]).

Definition 2.1 (Nemytskii operator). Let $\Omega \subset \mathbb{R}^N, N \geq 1$, be an open set and let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m, n, m \geq 1$. By the Nemystkii operator associated to f we understand the operator N_f which, to each function $u : \Omega \to \mathbb{R}^n$, assigns $f \circ u$, that is

$$N_f u(x) = (f \circ u)(x) = f(x, u(x)), \text{ for } x \in \Omega.$$

Definition 2.2 (Carathéodory function). Let $\Omega \subset \mathbb{R}^N, N \geq 1$, be an open set. We say that $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m, n, m \geq 1$ is a Carathéodory function if it satisfies the following conditions:

(i) $x \mapsto f(x, y)$ is measurable in Ω for every $y \in \mathbb{R}^n$;
(ii) $y \mapsto f(x, y)$ is continuous on $\mathbb{R}^n$ for a.e. $x \in \Omega$.

Proposition 2.3 (see [7, Proposition 9.1]). *If f is a Carathéodory function, then the Nemystkii operator associated to the function f maps measurable functions into measurable functions.*

Theorem 2.4 (see [7, Theorem 9.1]). *Let $\Omega \subset \mathbb{R}^N$ be an open set, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ and $1 \leq p, q < +\infty$. If f satisfies the Carathéodory conditions and there exists $a \in \mathbb{R}_+$ and $h \in L^q(\Omega; \mathbb{R}_+)$ such that*

$$||f(x, y)|| \leq a||y||^{\frac{p}{q}} + h(x)$$

for every $y \in \mathbb{R}^n$ and a.e. $x \in \Omega$, then the operator

$$N_f : L^p(\Omega; \mathbb{R}^n) \to L^q(\Omega; \mathbb{R}^m) \quad \text{given by } N_f(u) = f(\cdot, u)$$

is well defined, continuous and bounded. Moreover, the following inequality holds:

$$||N_f(u)||_{L^q} \leq a||u||_{L^p}^{\frac{p}{q}} + ||h||_{L^q} \quad \text{for all } u \in L^p(\Omega; \mathbb{R}^n).$$

2.2. Embedding results

The purpose of our work, namely the study of the existence of solutions for a semi-linear boundary valued problem, is achieved by looking for a weak solution which lead us to use continuous or compact embeddings of function spaces. In particular, we use the following embedding results due to Sobolev and Rellich-Kondrachov regarding the continuous and compact embeddings of Sobolev spaces into Lebesgues spaces.

Let $1 \leq q \leq +\infty$. Then the critical exponent associated to q is denoted by q^* and is defined by

$$\begin{cases} \dfrac{1}{q^*} = \dfrac{1}{q} - \dfrac{1}{n}, & q < n \\ q^* = +\infty, & q \geq n, \end{cases}$$

where by n is denoted the dimension of the space.

Theorem 2.5 (Sobolev). *Let $\Omega \subset \mathbb{R}^n$ be an open set of class C^1 (or $\Omega = \mathbb{R}^n$). Then the following continuous embeddings hold:*

a) $H^1(\Omega) \subset L^q(\Omega)$ *for every $q \in [2, 2^*]$, where $n \geq 3$.*
b) $H^1(\Omega) \subset L^q(\Omega)$ *for every $q \in [2, +\infty)$, if $n = 2$.*

Theorem 2.6 (Rellich-Kondrachov). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1.*

a) *If $n \geq 3$, then the embedding $H^1(\Omega) \subset L^q(\Omega)$ is compact for $q \in [1, 2^*)$, where $2^* := 2n/(n-2)$.*
b) *If $n = 2$, then the embedding $H^1(\Omega) \subset L^q(\Omega)$ is compact for every $q \in [1, +\infty)$.*

We recall that a real number λ is said to be an eigenvalue of the Dirichlet problem for $-\Delta$ if the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has nonzero weak solutions.

Theorem 2.7 (Poincaré's inequality). *Let Ω be a bounded open set of $\mathbb{R}^n$. Then there exists a constant C that depends on Ω such that*

$$||u||_{L^2(\Omega)} \leq C||\nabla u||_{L^2}, \quad \textit{for every } u \in H_0^1(\Omega).$$

Due to this result, the Sobolev space $H_0^1(\Omega)$ can be endowed with an equivalent norm

$$||u||_{H_0^1} := ||\nabla u||_{L^2} = \left(\int_\Omega ||\nabla u||^2 \right)^{1/2}$$

that comes from the scalar product in $H_0^1(\Omega)$

$$(u, v)_{H_0^1} = (\nabla u, \nabla v)_{L^2} = \int_\Omega \nabla u \cdot \nabla v.$$

Hence in terms of the new norm, Poincaré's inequality can be written as

$$||u||_{L^2} \leq C||u||_{H_0^1}, \quad u \in H_0^1(\Omega).$$

Since the first eigenvalue of the Dirichlet problem for $-\Delta$ is

$$\lambda_1 = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{||u||^2_{H_0^1}}{||u||^2_{L^2}},$$

it follows that the smallest constant C for which the Poincaré's inequality holds, is in fact $\dfrac{1}{\sqrt{\lambda_1}}$. Therefore,

$$||u||_{L^2} \leq \frac{1}{\sqrt{\lambda_1}} ||u||_{H_0^1}, \quad \text{for all } u \in H_0^1(\Omega).$$

Moreover, the Poincaré's inequality also holds for the embedding $L^2(\Omega) \subset H^{-1}(\Omega)$ with the same constant, namely

$$||u||_{H^{-1}} \leq \frac{1}{\sqrt{\lambda_1}} ||u||_{L^2}, \quad \text{for all } u \in L^2(\Omega).$$

For a more detailed exposition of these results we refer the reader to [7, Chapter 3].

Remark 2.8. Since we are concerned with n-dimensional vector-valued functions, we shall use the notations

$$\mathbf{L}^p(\Omega) := (L^p(\Omega))^n, \qquad \mathbf{H}^m(\Omega) := (H^m(\Omega))^n, \qquad \mathbf{H}_0^m(\Omega) := (H_0^m(\Omega))^n.$$

2.3. The variational form of the Stokes system

Let us consider the Dirichlet problem for the steady non-homogeneous Stokes system

$$\begin{cases} -\mu\Delta u + \nabla p = f & \text{in } \Omega \\ \text{div } u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\mu > 0$ is a constant representing the kinematic viscosity, $u : \Omega \to \mathbb{R}^n$ is the velocity field, p is the pressure and $f \in \mathbf{L}^2(\Omega)$ is the external force. In this subsection we give the variational formulation of problem (2.1). For a very detailed way of getting to the variational form of the Stokes equation we refer the reader to [10].

We define the Hilbert space

$$V := \{v \in \mathbf{H}_0^1(\Omega) : \text{ div } v = 0\},$$

endowed with the scalar product

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v, \quad \text{for } u, v \in V$$

and the corresponding norm

$$\|u\|_V = \left(\int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}.$$

We can now state the variational formulation of problem (2.1):
Given $f \in \mathbf{L}^2(\Omega)$ find $u \in V$ such that

$$\mu(u, v)_V = (f, v)_{L^2}, \quad \text{for all } v \in V. \tag{2.2}$$

Definition 2.9 (Weak solution). Let $f \in L^2(\Omega)$. By the *weak solution* of the Stokes problem (2.1) we mean a function $u_f \in V$ that satisfies (2.2).

One has the following embeddings:

$$V \subset H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) \subset V'.$$

Then, by the Riesz's representation theorem, we can extend (2.2) so that for any $f \in V'$ there exist a unique $u_f \in V$ such that

$$\mu(u_f, v)_V = (f, v), \quad \text{for all } v \in V. \tag{2.3}$$

Notice that the notation (f, v), for $f \in V'$ and $v \in V$, stands for the value at v of the linear functional f.

Definition 2.10 (Solution operator). The operator $S : V' \to V$ defined by $Sf := u_f$ for any $f \in V'$ is called the *solution operator*.

If in (2.3) we take in particular $v := u_f$ we obtain

$$\|u_f\|_V^2 = \frac{1}{\mu}(f, u_f) \leq \frac{1}{\mu}\|f\|_{V'}\|u_f\|_V.$$

Hence we have $\|u_f\|_V \leq \mu^{-1}\|f\|_{V'}$, that is

$$\|Sf\|_V \leq \frac{1}{\mu}\|f\|_{V'}.$$

Thus the linear operator S is continuous from V' to V.

Remark 2.11. Note that the existence of the pressure p is guaranteed as a consequence of De Rham's Lemma.

3. Main results

Let us now turn back to the semi-linear problem (1.1), where $\Omega \subset \mathbb{R}^n, n \geq 2$ is a bounded open set, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $p : \Omega \to \mathbb{R}$.
We seek weak solutions, i.e., functions $u \in V$ such that

$$f(\cdot, u(\cdot)) \in H^{-1}(\Omega)$$

and

$$\mu(u, v)_V = (f(\cdot, u), v) \ \text{ for all } v \in V.$$

The system (1.1) can be written as an equivalent fixed point equation

$$u = T(u), \quad u \in V,$$

where

$$T := S \circ F,$$

where $F : V \to V'$, $F(u) = f(\cdot, u(\cdot))$.

3.1. Application of Schauder's fixed point theorem

In this section we find sufficient conditions that assure the existence of a solution of problem (1.1), having in mind Schauder's fixed point theorem on the space V.

First we show that T *is a completely continuous operator*. In order to do so we would like to have the representation of $F = I \circ N_f \circ P$, where

- $P : V \to L^2(\Omega),\ Pu = u;$
- $N_f : L^2(\Omega) \to L^2(\Omega),\ N_f(w) = f(\cdot, w(\cdot));$
- $I : L^2(\Omega) \to V',\ I(v) = (v, \cdot)_{L^2}.$

Let us observe that by the Theorem 2.6, the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is continuous. Then it follows that P is a continuous linear operator, hence bounded. Also, since the embedding $L^2(\Omega) \subset H^{-1}(\Omega)$ is compact it follows that operator I is a completely continuous linear operator. It remains to see whether the operator $N_f : L^2(\Omega) \to L^2(\Omega)$ is well-defined. For this purpose let us assume that f is a Carathéodory function. Hence, for any $w \in L^2(\Omega),\ N_f(w)$ is also measurable. We impose a linear growth condition on f, that is

$$||f(x,u)|| \leq a||u|| + k(x), \quad \text{for all } u \in \mathbb{R}^n \text{ and a.e. } x \in \Omega, \tag{3.1}$$

for some $k \in L^2(\Omega, \mathbb{R}_+)$ and $a \in \mathbb{R}_+$. Then, we have

$$||N_f(w)(x)|| \leq a||w(x)|| + k(x), \quad \text{for a.e. } x \in \Omega.$$

Hence, by these assumptions over f, it follows that N_f is well-defined, continuous and bounded.

Due to the boundedness of the operators P and N_f, it follows that $N_f \circ P$ is bounded too. Therefore, since I is completely continuous, it follows that the operator F is completely continuous from V to V'. Next, by the linearity and continuity of the solution operator S we have that $T = S \circ F$ is completely continuous from V to itself.

Secondly, we show that T *is a self-map of a closed ball of* V. To this purpose, let $u \in V$. Notice that for every $h \in H^{-1}(\Omega)$ one has

$$||h||_{V'} \leq ||h||_{H^{-1}}.$$

Indeed, since $V \subset H_0^1(\Omega)$ we have that

$$||h||_{V'} = \sup_{v \in V} \frac{|(h,v)|}{||v||_V} \leq \sup_{v \in H_0^1(\Omega)} \frac{|(h,v)|}{||v||_{H_0^1}} = ||h||_{H^{-1}}.$$

Then, since the operator S is linear and continuous and also by the Poincaré´s inequality we have

$$\begin{aligned} \|T(u)\|_V = \|S \circ F(u)\|_V &\leq \frac{1}{\mu}\|F(u)\|_{V'} \leq \frac{1}{\mu}||F(u)||_{H^{-1}} \\ &\leq \frac{1}{\mu\sqrt{\lambda_1}}\|F(u)\|_{L^2} = \frac{1}{\sqrt{\mu\lambda_1}}\|f(\cdot, u(\cdot))\|_{L^2}. \end{aligned}$$

By the growth condition (3.1), we deduce that

$$||T(u)||_V \leq \frac{a}{\mu\sqrt{\lambda_1}}||u||_{L^2} + \frac{1}{\mu\sqrt{\lambda_1}} \cdot ||k||_{L^2}.$$

Since $u \in V$, we can apply again the Poincaré inequality and we obtain that

$$||T(u)||_V \leq \frac{a}{\mu\lambda_1}||u||_V + \frac{1}{\mu\sqrt{\lambda_1}} \cdot ||k||_{L^2}.$$

In the end, we assume that $\frac{a}{\mu\lambda_1} < 1$ so that there exists a radius $r > 0$ such that if $||u||_V \leq r$ then $||T(u)||_V \leq r$. Indeed, from $\frac{a}{\mu\lambda_1} < 1$ and $||u||_V \leq r$ we have that

$$||T(u)||_V \leq \frac{a}{\mu\lambda_1} r + \frac{1}{\mu\lambda_1}||k||_{L^2} \leq r \quad \text{for } r > 0 \text{ large enough.}$$

Hence, $||T(u)||_V \leq r$.
Therefore, based on Schauder's fixed point theorem we can state the following result:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping such that:*

(a) *f is a Carathéodory function;*
(b) *there is a positive constant a and $k \in L^2(\Omega, \mathbb{R}_+)$ such that*

$$||f(x, u)|| \leq a||u|| + k(x) \quad \textit{for all } u \in \mathbb{R}^n \textit{ and a.e. } x \in \Omega.$$

Also, assume that $\frac{a}{\mu\lambda_1} < 1$. Then the semi-linear Stokes problem (1.1) has at least one solution (u, p) with $u \in V$.

3.2. Application of Lerray-Schauder's fixed point theorem

In this section we consider more generally that the right hand side of the problem (1.1) is of the form $f_0 + f_1(\cdot, u(\cdot))$, where $f_0 \in H^{-1}(\Omega)$ and $f_1 : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$. As before, the problem can be written as an equivalent fixed point equation

$$u = T(u),$$

where this time

$$T = S \circ (F + f_0),$$

with

$$F : V \to V', \quad F(u) = f_1(\cdot, u(\cdot)).$$

We are now interested if one can still obtain the existence of the solution of the new problem without a linear growth restriction. We shall see this is possible due to the Lerray-Schauder's fixed point theorem (see [5]).

We first guarantee the *complete continuity* of the operator T. The idea we follow here is similar to the one in the previous section: since the operator S is linear and bounded, in order for $T = S \circ F$ to be completely continuous, we need that the operator F is completely continuous. To this end, we write the operator F as $F = I \circ N_{f_1} \circ P$, where

- $P : V \to L^{2^*}(\Omega), \quad Pu = u;$
- $N_{f_1} : L^{2^*}(\Omega) \to L^q(\Omega), \quad N_{f_1}(\omega) = f_1(\cdot, \omega);$
- $I : L^q(\Omega) \to V', \quad I(v) = (v, \cdot)_{L^2}, \quad \text{for some } q \in ((2^*)', +\infty).$

Due to Theorem 2.6 it follows that the embedding $(V \subset) H_0^1(\Omega) \subset L^{2^*}(\Omega)$ is continuous. Therefore the operator P is a continuous linear operator, hence bounded. Since $H_0^1(\Omega) \subset L^p(\Omega)$ is compact for $p \in [1, 2^*)$, passing to duals, we get that $L^q(\Omega) \subset H^{-1}(\Omega)$ is also compact for $q \in ((2^*)', +\infty)$, where $(2^*)' = 2n/(n+2)$ is the conjugate of 2^*. Therefore, if $q > (2^*)'$, the inclusion operator I is completely continuous. We now show the operator N_{f_1} is well-defined. For this purpose we will make use of Theorem 2.4. In view of this result, we need to impose a growth condition on the function f_1, namely, for some $a \in \mathbb{R}_+$ and $\overline{h} \in L^q(\Omega)$ to have

$$||f_1(x,\omega)|| \le a||\omega||^{\frac{2^*}{q}} + \overline{h}(x),$$

for every $\omega \in \mathbb{R}^n$ and a.e. $x \in \Omega$. To this aim, it sufficies to have

$$||f_1(x,\omega)|| \le a||\omega||^{\alpha} + h(x). \tag{3.2}$$

for some $\alpha \in [1, 2^*/q]$ and $h \in L^q(\Omega)$. Note that from $\alpha \le 2^*/q$ it follows that $q \le 2^*/\alpha$. Together with the condition $q > (2^*)'$, this shows that

$$(2^*)' < q \le \frac{2^*}{\alpha},$$

and so, such a q exists if

$$\alpha < \frac{2^*}{(2^*)'} = \frac{n+2}{n-2}.$$

Thus, the condition (3.2) holds for $\alpha \in [1, (n+2)/(n-2))$; hence, we can let $h \in L^{2^*/\alpha}(\Omega)$.

Then from Theorem 2.4 it follows that the Nemytskii operator N_{f_1} is well defined, continuous and bounded. Therefore, the operator F is well-defined and completely continuous. Hence the operator T is completely continuous.

Finally, we carry on with the *a priori bounds* of solutions, that is to show there is a positive constant $R > 0$ such that $||u||_V < R$ for any solution $u \in V$ of the equation $\lambda T(u) = u$ and any $\lambda \in (0,1)$. Let $u \in V$ be any solution of the equation $\lambda T(u) = u$ for some $\lambda \in (0,1)$. Thus, u is a weak solution of the problem

$$\begin{cases} -\mu\Delta u = -\nabla p + \lambda f_0(x) + \lambda f_1(x,u) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Therefore

$$(u,v)_V = \frac{\lambda}{\mu}(f_0(\cdot) + f_1(\cdot, u(\cdot)), v), \quad \text{for any } v \in V.$$

If we take in particular $v = u$ we obtain

$$||u||_V^2 = \frac{\lambda}{\mu}(f_0 + f_1(\cdot,u), u) = \frac{\lambda}{\mu}(f_0, u) + \frac{\lambda}{\mu}(f_1(\cdot,u), u).$$

Note that since $f_1(\cdot, u(\cdot)) \in L^{(2^*)'}(\Omega)$, one has

$$(f_1(\cdot,u),u) = \int_\Omega u(x) \cdot f_1(x, u(x)).$$

Let us now assume that there exists a positive constant k such that

$$y \cdot f_1(x,y) \le k||y||^2, \quad \text{for all } y \in \mathbb{R}^n \text{ and a.e. } x \in \Omega.$$

Then $(f_1(\cdot,u),u) \le k||u||^2_{L^2}$ and using Poincaré's inequality we obtain

$$\begin{aligned}
||u||_V^2 &= \frac{\lambda}{\mu}(f_0,u) + \frac{\lambda}{\mu}\int_\Omega u(x)\cdot f_1(x,u(x)) \\
&\le \frac{\lambda}{\mu}\big(||f_0||_{H^{-1}}||u||_V + k||u||^2_{L^2}\big) \\
&< \frac{1}{\mu}||f_0||_{H^{-1}}||u||_V + \frac{k}{\mu}||u||^2_{L^2} \\
&\le \frac{1}{\mu}||f_0||_{H^{-1}}||u||_V + \frac{k}{\mu\lambda_1}||u||^2_V.
\end{aligned}$$

Hence, we have

$$||u||_V\left(1 - \frac{k}{\mu\lambda_1}\right) \le \frac{1}{\mu}||f_0||_{H^{-1}}.$$

Assuming that $k < \mu\lambda_1$ it follows that

$$||u||_V < \frac{\lambda_1}{\mu\lambda_1 - k}||f_0||_{H^{-1}} := R.$$

Therefore, we can state the following result:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f = f_0 + f_1$ with $f_0 \in H^{-1}(\Omega)$ and f_1 a function such that*

(a) *f_1 is a Carathéodory function;*

(b) *there is a positive constant a and $\alpha \in [1,(n+2)/(n-2))$ and a function $h \in L^{2^*/\alpha}(\Omega)$ such that*

$$||f_1(x,u)|| \le a||u||^\alpha + h(x),$$

for any $u \in \mathbb{R}^n$ and a.e. $x \in \Omega$;

(c) *there is a positive constant $k < \mu\lambda_1$ such that the condition*

$$y \cdot f_1(x,y) \le k||y||^2$$

holds for any $y \in \mathbb{R}^n$ and a.e. $x \in \Omega$.

Then the problem (3.3) has at least one solution (u,p) with $u \in V$ and

$$||u||_V \le \frac{\lambda_1}{\mu\lambda_1 - k}||f_0||_{H^{-1}}.$$

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Existence of solutions for fractional q-difference equations

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Abstract. In this paper, we obtain some existence results for the integral boundary value problems of nonlinear fractional q-difference equations. The differential operator is taken in the Riemann-Liouville sense.

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Keywords: Riemann-Liouville fractional derivative, fractional q-difference equations, integral boundary value problems, the fixed point theorem, positive solution, upper and lower solutions.

1. Introduction

In this paper we will study the existence and uniqueness of solutions for the following singular boundary value problem of fractional q-difference equations

$$\left(D_q^{\alpha}u\right)(t)+\varphi(t)\,f(t,u(t))=0\ ,\qquad 0<t<1, \tag{1.1}$$

$$u(0)=0,\quad u(1)=a\int_0^1 h(s)\,u(s)\,d_qA(s)+b, \tag{1.2}$$

where D_q^{α} is a fractional q-derivative of Riemann-Liouville type with $1<\alpha\leq 2$, $\int_0^1 x(t)\,d_qA(t)$, is the Riemann-Stieltjes q-integral of x with respect to $A(t)$ such that $d_qA(t)=D_qA(t)d_qt$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $h:[0,1]\to\mathbb{R}$ is a continuous function, φ is defined on the interval $(0,1)$ and φ may be singular at 0 or 1.

In the last few years, fractional differential equations have been studied extensively, because of their demonstrated applications in various fields of science and

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engineering; see [5, 16, 19, 27, 36, 39]. Recently, many researchers study the existence of solutions of fractional differential equations such as the Riemann-Liouville fractional derivative problem [3, 12, 17, 31, 32, 34, 35, 37, 38, 40, 41, 42] the Caputo fractional boundary value problem [3, 33], the Hadamard fractional boundary value problem [28, 30], conformable fractional boundary value problem [20, 24, 25] etc.

Quantum calculus is ordinary calculus without limits. There are several types of quantum calculus: h-calculus, q-calculus and Hahn's calculus. In this paper we are concerned with the q-calculus. The q-derivative and the q-integral were first defined by Jackson [14, 15]. For some recent existence results on q-difference equations see [2, 6, 10, 13, 22, 26] and the references there in.

There has also been a growing interest on the subject of discrete fractional equations. Fractional q-difference equations have recently attracted the attention of several researchers for the applications of fields such as physics, chemistry, biology, economics, control theory, signal and image processing, electricity etc. Some recent work on the existence theory of fractional q-difference equations can be found in [4, 7, 8, 9, 23]. Motivated by all the works above, in this paper we discuss the problem (1.1)-(1.2) and we will give the existence results for this problem.

The paper is organized as follows. In Section 2, we give some preliminary results that will be used in the proof of our main results. In Section 3, we establish the existence of a solution for the nonlinear fractional q-difference boundary value problems (1.1)-(1.2).

2. Preliminaries

In this section, we list some useful definitions and preliminaries, which will be used in the proofs of the main results.
Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^k$, $k \in N_0 = \{0,1,2,\dots\}$ is

$$(a-b)^0 = 1 \ , \qquad (a-b)^{(k)} = \prod_{i=0}^{k-1} \left(a - bq^i\right), \quad k \in N, \ a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$.
The q-gama function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \qquad x \in \mathbb{R} \backslash \{0,-1,-2,\dots\}$$

then

$$\Gamma_q(x+1) = [x] \, \Gamma_q(x).$$

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)^x}, (D_q f)(0) = \lim_{x \to 0} (D_q f)(x) \quad \text{for} \quad x \neq 0$$

and q-derivatives of higher order by

$$\left(D_q^0 f\right)(x) = f(x) \qquad \text{and} \qquad \left(D_q^n f\right)(x) = D_q\left(D_q^{n-1} f\right)(x), \quad n \in N.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \qquad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, i.e.,

$$\left(I_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x)\ , \ n \in N.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x)$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

We now point out two formulas that will be used later (${}_tD_q$ denotes the derivative with respect to variable t)

$$\begin{aligned} {}_tD_q(t-s)^{(\alpha)} &= [\alpha]_q (t-s)^{(\alpha-1)}, \\ \left({}_xD_q \int_0^x f(x,t)\, d_q t\right)(x) &= \int_0^x {}_xD_q f(x,t) d_q t + f(qx, x). \end{aligned}$$

Remark 2.1. If $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$.

Definition 2.2. [1] *Let $\alpha \ge 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type is*

$$(I_q^0 f)(x) = f(x)$$

and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t\ , \qquad x \in [0, 1].$$

Definition 2.3. [29] *The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by*

$$(D_q^\alpha f)(x) = f(x)$$

and

$$(D_q^\alpha f)(x) = \left(D_q^p I_q^{p-\alpha} f\right)(x), \qquad \alpha > 0,$$

where p is the smallest integer greater than or equal to α.

Next, we list some properties about q-derivative and q-integral that are already known in the literature, which are helpful in proofs of our main results.

Lemma 2.4. [21]
(1) *If f and g are q-integral on the interval $[a,b]$, $\alpha \in \mathbb{R}$, $c \in [a,b]$, then*

1. $\int_a^b (f(t) + g(t))\, d_q t = \int_a^b f(t)\, d_q t + \int_a^b g(t) d_q t$
2. $\int_a^b \alpha f(t)\, d_q t = \alpha \int_a^b f(t) d_q t$
3. $\int_a^b f(t) d_q t = \int_a^c f(t) d_q t + \int_c^b f(t) d_q t$
4. $\int x^\alpha d_q s = \frac{x^{\alpha+1}}{[\alpha+1]}$, $(\alpha \neq -1)$;

(2) *If $|f|$ is q-integral on the interval $[0,x]$, then*

$$\left| \int_0^x f(t) d_q t \right| \leq \int_0^x |f(t)|\, d_q t;$$

(3) *If f and g are q-integral on the interval $[0,x]$, $f(t) \leq g(t)$, $\forall t \in [0,x]$, then*

$$\int_0^x f(t) d_q t \leq \int_0^x g(t) d_q t.$$

Lemma 2.5. [9] *Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:*

$$\left(I_q^\alpha D_q^p f\right)(x) = \left(D_q^p I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Now, we will give the existence theorems used in our main results.

Theorem 2.6. [11] *Let $T : X \to X$ be a map on a complete non-empty metric space. If some iterate T^n of T is a contraction, then T has a unique fixed point.*

Theorem 2.7. [18] *Let X be a Banach space and $P \subseteq X$ be a cone. Suppose that Ω_1 and Ω_2 are bounded open sets contained in X such that $0 \subseteq \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ is a completely continuous operator. If either*

1. $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*
2. $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *then*

T has at least one fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Theorem 2.8. [4] *(Nonlinear alternative for single valued maps) Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

1. *F has a fixed point in $\overline{U}$, or*
2. *There is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda\, F(u)$.*

The next result is important in the sequel.

Lemma 2.9. *Let $g(t) : [0,1] \to [0,\infty)$ be a given continuous function. Then the boundary value problem*

$$\left(D_q^\alpha u\right)(t) + g(t) = 0,\ 0 < t < 1, \tag{2.1}$$

$$u\left(0\right)=0,\quad u\left(1\right)=a\int_{0}^{1}h\left(s\right)u\left(s\right)d_{q}A\left(s\right)+b \tag{2.2}$$

has a unique solution

$$u\left(t\right)=\int_{0}^{1}H\left(t,qs\right)g\left(s\right)d_{q}s+\frac{b}{k}\ t^{\alpha-1}$$

where

$$H\left(t,s\right)=G\left(t,s\right)+\frac{a\ t^{\alpha-1}}{k}G_{A}(s)$$

such that

$$G\left(t,s\right)=\frac{1}{\Gamma_{q}(\alpha)}\begin{cases}\left(t\left(1-s\right)\right)^{\left(\alpha-1\right)}-\left(t-s\right)^{\left(\alpha-1\right)}, & s\leq t\\ \left(t\left(1-s\right)\right)^{\left(\alpha-1\right)}, & s\geq t,\end{cases}$$

$$G_{A}\left(s\right)=\int_{t=0}^{1}h\left(t\right)G\left(t,s\right)d_{q}A(t)$$

and

$$k=1-a\int_{0}^{1}h\left(s\right)s^{\alpha-1}\ d_{q}A(s)\neq 0.$$

Proof. From *Lemma 2.5* and *Definition 2.2*, we have

$$u(t)=-\frac{1}{\Gamma_{q}\left(\alpha\right)}\int_{0}^{t}\left(t-qs\right)^{\left(\alpha-1\right)}g\left(s\right)d_{q}s+c_{1}t^{\alpha-1}+c_{2}t^{\alpha-2}.$$

Since $u\left(0\right)=0$ we get $c_{2}=0$. Thus, we have

$$u(t)=-\frac{1}{\Gamma_{q}\left(\alpha\right)}\int_{0}^{t}\left(t-qs\right)^{\left(\alpha-1\right)}g\left(s\right)d_{q}s+c_{1}t^{\alpha-1}.$$

Using the second boundary condition we get

$$\begin{aligned}&-\frac{1}{\Gamma_{q}\left(\alpha\right)}\int_{0}^{1}\left(1-qs\right)^{\left(\alpha-1\right)}g\left(s\right)d_{q}s+c_{1}\\&\qquad=a\int_{0}^{1}h\left(s\right)\left[-\frac{1}{\Gamma_{q}\left(\alpha\right)}\int_{0}^{s}\left(s-qw\right)^{\left(\alpha-1\right)}g\left(w\right)d_{q}w+c_{1}s^{\alpha-1}\right]d_{q}A\left(s\right)+b\end{aligned}$$

Thus, we have

$$\begin{aligned}c_{1}\left[1-a\int_{0}^{1}h\left(s\right)s^{\alpha-1}d_{q}A\left(s\right)\right]&=\frac{1}{\Gamma_{q}\left(\alpha\right)}\int_{0}^{1}\left(1-qs\right)^{\left(\alpha-1\right)}g\left(s\right)d_{q}s\\&-\frac{a}{\Gamma_{q}\left(\alpha\right)}\int_{0}^{1}\left[\int_{wq}^{1}h\left(s\right)\left(s-qw\right)^{\left(\alpha-1\right)}d_{q}A\left(s\right)\right]g\left(w\right)d_{q}w+b\end{aligned}$$

and

$$\begin{aligned}c_{1}=\frac{1}{k}\Bigg\{&\frac{1}{\Gamma_{q}\left(\alpha\right)}\int_{0}^{1}\left(1-qs\right)^{\left(\alpha-1\right)}g\left(s\right)d_{q}s\\&-\frac{a}{\Gamma_{q}\left(\alpha\right)}\int_{0}^{1}\left[\int_{wq}^{1}h\left(s\right)\left(s-qw\right)^{\left(\alpha-1\right)}d_{q}A\left(s\right)\right]g\left(w\right)d_{q}w\Bigg\}+\frac{b}{k}\end{aligned}$$

so

$$c_1 = \frac{1}{k\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s$$
$$-\frac{a}{k\Gamma_q(\alpha)}\int_0^1 \left[\int_{sq}^1 h(t)(t-qs)^{(\alpha-1)} d_q A(t)\right] g(s)\, d_q s \; + \frac{b}{k}.$$

Therefore, we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} g(s)\, d_q s + \frac{t^{\alpha-1}}{k}\left\{\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s\right. \\ &\left. -\frac{a}{\Gamma_q(\alpha)}\int_0^1 \left[\int_{sq}^1 h(t)(t-qs)^{(\alpha-1)} d_q A(t)\right] g(s)\, d_q s\right\} + \frac{b}{k}t^{\alpha-1} \\ &= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} g(s)\, d_q s + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s \\ &- \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s + \frac{t^{\alpha-1}}{k}\left\{\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)} g(s)\, d_q s\right. \\ &\left. -\frac{a}{\Gamma_q(\alpha)}\int_0^1 \left[\int_{sq}^1 h(t)(t-qs)^{(\alpha-1)} d_q A(t)\right] g(s)\, d_q s \right\} + \frac{b}{k}t^{\alpha-1} \end{aligned}$$

$$\begin{aligned} u(t) &= \int_0^1 G(t,qs)\, g(s)\, d_q s \\ &+ \frac{a t^{\alpha-1}}{k\Gamma_q(\alpha)}\left\{\int_{s=0}^1 \left[\int_{t=0}^1 h(t)\, t^{\alpha-1} d_q A(t)\right](1-qs)^{(\alpha-1)} g(s)\, d_q s\right. \\ &\qquad \left. -\int_{s=0}^1 \left[\int_{t=sq}^1 h(t)(t-qs)^{(\alpha-1)} d_q A(t)\right] g(s)\, d_q s\right\} + \frac{b}{k}t^{\alpha-1}. \end{aligned}$$

Thus

$$u(t) = \int_0^1 G(t,qs)\, g(s)\, d_q s + \frac{a t^{\alpha-1}}{k}\int_0^1 G_A(s)\, g(s)\, d_q s + \frac{b}{k}t^{\alpha-1},$$

where

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} (t(1-s))^{(\alpha-1)} - (t-s)^{(\alpha-1)}, & s \le t \\ (t(1-s))^{(\alpha-1)}, & s \ge t \end{cases}$$

$$G_A(s) = \int_{t=0}^1 h(t)\, G(t,s)\, d_q A(t),$$

and

$$k = 1 - a\int_0^1 h(s)\, s^{\alpha-1}\, d_q A(s) \neq 0.$$

Consequently, we can write

$$u(t) = \int_0^1 H(t,qs)\, g(s)\, d_q s + \frac{b}{k}t^{\alpha-1}$$

where

$$H(t,s) = G(t,s) + \frac{a\ t^{\alpha-1}}{k} G_A(s). \qquad \square$$

Lemma 2.10. *Assume that* $0 < k < 1$ *and* $G_A(s) \geq 0$ *for* $s \in [0,1]$*, then* $H(t,s)$ *satisfies followings:*

1. $H(t,s) \geq 0, \quad \forall t, s \in [0,1]$
2. *There exist a constant*

$$L = \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{a.H}{k}\right)$$

such that

$$\frac{at^{\alpha-1}}{k} G_A(s) \leq H(t,s) \leq L\ (1-s)^{(\alpha-1)} t^{\alpha-1},$$

where

$$H = \int_0^1 h(t)\, t^{\alpha-1}\ d_qA(t).$$

Proof. 1. (i) For $s \leq t$, we know that

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \left[(t(1-s))^{(\alpha-1)} - (t-s)^{(\alpha-1)}\right].$$

Since

$$t < 1 \Rightarrow \frac{1}{t} > 1 \Rightarrow -\frac{s}{t} < -s \Rightarrow \left(1 - \frac{s}{t}\right)^{(\alpha-1)} < (1-s)^{(\alpha-1)},$$

we get

$$\begin{aligned} &\frac{1}{\Gamma_q(\alpha)} \left[t^{\alpha-1}(1-s)^{(\alpha-1)} - t^{\alpha-1}\left(1 - \frac{s}{t}\right)^{(\alpha-1)}\right] \\ &> \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1} \left[(1-s)^{(\alpha-1)} - \left(1 - \frac{s}{t}\right)^{(\alpha-1)}\right] > 0, \end{aligned}$$

so $G(t,s) \geq 0$.
(ii) For $s \geq t$, it is clear that $G(t,s) > 0$.

Thus we get $G(t,s) \geq 0, \quad \forall t, s \in [0,1]$.
Since $G_A(s) = \int_{t=0}^1 h(t)\, G(t,s)\, d_qA(t) > 0$, then $H(t,s) \geq 0$, for $t, s \in [0,1]$.

2. Since

$$G(t,s) \leq \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}(1-s)^{(\alpha-1)} < \frac{1}{\Gamma_q(\alpha)} (1-s)^{(\alpha-1)},$$

we have

$$\begin{aligned} G_A(s) &= \int_{t=0}^1 h(t)\, G(t,s)\, d_qA(t) < \int_{t=0}^1 h(t) \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}(1-s)^{(\alpha-1)} d_qA(t) \\ &= \frac{(1-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} \int_{t=0}^1 h(t)\ t^{\alpha-1} d_qA(t) = \frac{(1-s)^{(\alpha-1)}}{\Gamma_q(\alpha)}\ H. \end{aligned}$$

Also, we know

$$H(t,s) = G(t,s) + \frac{a\ t^{\alpha-1}}{k} G_A(s)$$

that

$$\begin{aligned}H(t,s) &\leq \frac{1}{\Gamma_q(\alpha)}t^{\alpha-1}(1-s)^{(\alpha-1)} + \frac{a}{k\Gamma_q(\alpha)}t^{\alpha-1}(1-s)^{(\alpha-1)}H \\ &= \frac{1}{\Gamma_q(\alpha)}\left(1+\frac{aH}{k}\right)t^{\alpha-1}(1-s)^{(\alpha-1)} \\ &\leq L\,(1-s)^{(\alpha-1)}t^{\alpha-1}.\end{aligned}$$

In conclusion, we have

$$\frac{at^{\alpha-1}}{k}G_A(s) \leq H(t,s) \leq \frac{1}{\Gamma_q(\alpha)}\left(1+\frac{aH}{k}\right)(1-s)^{(\alpha-1)}t^{\alpha-1}. \qquad \square$$

3. Main results

We are now in a position to state and prove our main results in this paper. Transform the problem $(1.1)-(1.2)$ into a fixed point problem. We define the operator $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$(Tu)(t) = \int_0^1 H(t,qs)\varphi(s) f(s,u(s))\, d_qs + \frac{b}{k}t^{\alpha-1}. \tag{3.1}$$

It's easy to show that, from *Lemma 2.9*, the fixed points of operator T coincide with the solutions of boundary value problems $(1.1)-(1.2)$.

Suppose that the following conditions are satisfied.

(H_1) $\varphi(t)$ is nonnegative on $(0,1)$ and

$$\int_0^1 (1-qs)^{(\alpha-1)}\varphi(s)\, d_qs < \infty$$

(H_2) $|f(t,u) - f(t,v)| \leq K.\,|u-v|$, for all $t \in [0,1]$, $u,v \in \mathbb{R}$

(H_3) $f \in C([0,1]\times\mathbb{R},\ [0,\infty))$, $C \subset B$, $C = \{u \in C[0,1] : u(t) \geq 0\}$

(H_4) $f \in C([0,1]\times[0,\infty),[0,\infty))$, $f(t,u_1) \leq f(t,u_2)$ for $0 \leq u_1 < u_2$ and any $t \in [0,1]$.

Let $B = C([0,1],R)$ is the Banach space with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$ and $C = \{u \in B : u(t) \geq 0\}$. Then C is a normal cone on B. Also we denote $u_1 \preccurlyeq u_2$ if and only if $u_{2-}u_1 \in C$ for $u_1,u_2 \in B$.

Lemma 3.1 *If there holds (H_1) and f meets (H_3). Then the operator $T : C \to B$*

$$(Tu)(t) = \int_0^1 H(t,qs)\varphi(s) f(s,u(s))\, d_qs + \frac{b}{k}t^{\alpha-1}$$

satisfies $T(C) \subset C$ and T is completely continuous.

Proof. It follows from (H_1) and the nonnegativeness and continuity of $H(t,qs)$ and $f(t,u(t))$ that T has definition and satisfies $T(C) \subset C$. The next proof will be given in several steps.

Step 1. T is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$. Then for each $t \in [0,1]$, according to Lebesgue control convergence theorem and *Lemma 2.10*, we have

$$\|Tu_n - Tu\| = \sup_{t \in [0,1]} |(Tu_n)(t) - (Tu)(t)|$$

$$\begin{aligned}
&= \sup_{t \in [0,1]} \left| \int_0^1 H(t,qs)\varphi(s) f(s,u_n(s))\, d_q s - \int_0^1 H(t,qs)\varphi(s) f(s,u(s))\, d_q s \right| \\
&\le \sup_{t \in [0,1]} \int_0^1 H(t,qs)\ \varphi(s)\ |f(s,u_n(s)) - f(s,u(s))|\ d_q s \\
&\le \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{aH}{k}\right) t^{\alpha-1} \int_0^1\ (1-qs)^{(\alpha-1)} \varphi(s)\ |f(s,u_n(s)) - f(s,u(s))|\ d_q s \\
&\to\ 0\ , \quad n \to \infty
\end{aligned}$$

Therefore, T is continuous.
Step 2. T maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.
Indeed, it is enough to show that for any $\mu > 0$, there exists a positive constant

$$r = M \int_0^1 \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{aH}{k}\right) (1-qs)^{(\alpha-1)} \varphi(s)\ d_q s + \frac{b}{k}.$$

Such that for each $u \in B_\mu = \{u \in C([0,1],\mathbb{R}) : \|u\| \le \mu\}$, we have $\|T_u\| \le r$.
Denote $M = max_{t\in[0,1],\|u\|\le\mu}\ \{f(t,u(t)) + 1\}$. We have for each $t \in [0,1]$,

$$\begin{aligned}
|Tu(t)| &= \int_0^1 H(t,qs)\varphi(s) f(s,u(s))\, d_q s + \frac{b}{k} t^{\alpha-1} \\
&\le \int_0^1 H(t,qs)\varphi(s) f(s,u(s))\, d_q s + \frac{b}{k} \\
&\le M \int_0^1 H(t,qs)\varphi(s)\, d_q s + \frac{b}{k} \\
&\le M \int_0^1 \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{aH}{k}\right) (1-qs)^{(\alpha-1)} \varphi(s)\ d_q s + \frac{b}{k} = r.
\end{aligned}$$

Thus we get $\|Tu\| \le r$.
Step 3. T maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.
Let $t_1, t_2 \in [0,1],\ t_1 < t_2,\ B_\mu$ be bounded set of $C([0,1],\mathbb{R})$ as in Step 2 and let $u \in B_\mu$. Then

$$\begin{aligned}
|(Tu)(t_2) - (Tu)(t_1)| = \Bigg| &\int_0^1 (H(t_2,qs) - H(t_1,qs))\varphi(s) f(s,u(s))\, d_q s \\
&+ \frac{b}{k}\left({t_2}^{\alpha-1} - {t_1}^{\alpha-1}\right) \Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \Bigg| -\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s)\, f(s,u(s))\, d_q s \\
&\quad + \frac{t_2{}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \varphi(s)\ \ f(s,u(s))\, d_q s \\
&\quad - \frac{a t_2{}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 (t_2-qs)^{(\alpha-1)} h(t_2)\, d_q A(t_2) \right] \varphi(s)\ \ f(s,u(s))\, d_q s + \frac{b}{k} t_2{}^{\alpha-1} \\
&\quad + \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s)\, f(s,u(s))\, d_q s \\
&\quad - \frac{t_1{}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \varphi(s)\ \ f(s,u(s))\, d_q s \\
&\quad + \frac{a t_1{}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 (t_1-qs)^{(\alpha-1)} h(t_1)\, d_q A(t_1) \right] \varphi(s)\ \ f(s,u(s))\, d_q s - \frac{b}{k} t_1{}^{\alpha-1} \Bigg|
\end{aligned}
$$

Furthermore, we deduce that

$$
\begin{aligned}
\left|(Tu)(t_2)-(Tu)(t_1)\right| \leq &\Bigg| \int_0^{t_1} \frac{(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s)\, f(s,u(s))\, d_q s \\
&+ \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s)\, f(s,u(s))\, d_q s \Bigg| \\
&+ \Bigg| \frac{t_2{}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \varphi(s)\ \ f(s,u(s))\, d_q s \\
&- \frac{t_1{}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} \varphi(s)\ \ f(s,u(s))\, d_q s \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+ \Bigg| \frac{a t_2{}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 (t_2-qs)^{(\alpha-1)} h(t_2)\, d_q A(t_2) \right] \varphi(s)\ \ f(s,u(s))\, d_q s \\
&- \frac{a t_1{}^{\alpha-1}}{k\Gamma_q(\alpha)} \int_0^1 \left[\int_{sq}^1 (t_1-qs)^{(\alpha-1)} h(t_1)\, d_q A(t_1) \right] \varphi(s)\ \ f(s,u(s))\, d_q s \Bigg| \\
&+ \frac{b}{k} \left| t_2{}^{\alpha-1} - t_1{}^{\alpha-1} \right| \\
&\leq M \int_0^{t_1} \frac{(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi(s)\, d_q s
\end{aligned}
$$

$$+\frac{M}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}(1-qs)^{(\alpha-1)}\varphi(s)\,d_qs+\frac{M\left|t_2{}^{\alpha-1}-t_1{}^{\alpha-1}\right|}{k}\int_0^1\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varphi(s)\ d_qs$$

$$+\frac{aM}{k\Gamma_q(\alpha)}\int_0^1\left\{t_2{}^{\alpha-1}\int_{sq}^1(t_2-qs)^{(\alpha-1)}h(t_2)\,d_qA(t_2)\right.$$

$$\left.-t_1{}^{\alpha-1}\int_{sq}^1(t_1-qs)^{(\alpha-1)}h(t_1)\,d_qA(t_1)\right\}$$

$$+\frac{b}{k}\left|t_2{}^{\alpha-1}-t_1{}^{\alpha-1}\right|. \tag{3.2}$$

Obviously,

$$\int_0^{t_1}\left((t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right)\ \varphi(s)\,d_qs$$

$$\leq\int_0^1\left(\frac{(t_2-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}-\frac{(t_1-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}\right)(1-qs)^{(\alpha-1)}\ \varphi(s)\,d_qs.$$

The function $\frac{(t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}$ is continuous with respect to t and s on $[0,1]\times[0,1]$ and so it is uniformly continuous on $[0,1]\times[0,1]$.

Therefore, for any $t_1,t_2\in[0,1]\,,\ t_1<t_2,\ s\in[0,1]\,,$ as $t_1\to t_2$, we can conclude that

$$\frac{(t_2-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}-\frac{(t_1-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}\to 0.$$

So we can see

$$\int_0^{t_1}\left((t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right)\ \varphi(s)\,d_qs$$

$$\leq\int_0^1\left(\frac{(t_2-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}-\frac{(t_1-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-1)}}\right)(1-qs)^{(\alpha-1)}\ \varphi(s)\,d_qs$$

$$\to 0\ ,\qquad t_1\to t_2.$$

For

$$\int_{t_1}^{t_2}(1-qs)^{(\alpha-1)}\varphi(s)\,d_qs,$$

according to Cauchy criterion for convergence of an improper integral, as $t_2\to t_1$,

$$\int_{t_1}^{t_2}(1-qs)^{(\alpha-1)}\varphi(s)\,d_qs\to 0.$$

In conclusion, as $t_2\to t_1$, the right-hand side of the above inequality (3.2) tends to zero. As a consequence of Step 1 to 3 together with the Arzela-Ascoli theorem. Hence T is completely continuous. The proof is complete. □

Our first result is based on the generalization of Banach contraction principle.

Theorem 3.2. *Assume* (H_1) *and* (H_2) *hold. Let*

$$M = \int_0^1 s^{\alpha-1} (1-qs)^{(\alpha-1)} \varphi(s)\, d_q s$$

and

$$MK \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{aH}{k}\right) < 1$$

Then the boundary value problems $(1.1) - (1.2)$ *have a unique solution.*

Proof. We shall prove that under the conditions (H_1) and (H_2), the operator T^n is a contraction map in the space $C[0,1]$ for sufficiently large n.

$$T : C[0,1] \to C[0,1]$$

By *Lemma 2.10* we have

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| = \Bigg| \int_0^1 & H(t,qs)\, \varphi(s)\, f(s,u(s))\, d_q s + \frac{b}{k} t^{\alpha-1} \\ & - \int_0^1 H(t,qs)\, \varphi(s)\, f(s,v(s))\, d_q s - \frac{b}{k} t^{\alpha-1} \Bigg| \end{aligned}$$

$$\begin{aligned} &\le \int_0^1 |H(t,qs)|\, |\varphi(s)|\ |f(s,u(s)) - f(s,v(s))|\, d_q s \\ &\le \int_0^1 \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{aH}{k}\right) (1-qs)^{(\alpha-1)} t^{\alpha-1} \varphi(s)\ K\, |u(s) - v(s)|\, d_q s \\ &\le \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{aH}{k}\right)\ K\, \|u-v\|\ t^{\alpha-1} \underbrace{\int_0^1 (1-qs)^{(\alpha-1)} \varphi(s)\, d_q s}_{l} \\ &\le \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{aH}{k}\right)\ K\, \|u-v\|\ t^{\alpha-1}\ l \end{aligned}$$

$$\left|\left(T^2 u\right)(t) - \left(T^2 v\right)(t)\right| \le \int_0^1 |H(t,qs)|\, |\varphi(s)|\ |f(s,(Tu)(s)) - f(s,(Tv)(s))|\, d_q s$$

$$\begin{aligned} &\le \int_0^1 \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{aH}{k}\right) (1-qs)^{(\alpha-1)} t^{\alpha-1} \varphi(s)\ K\, |Tu - Tv|\, d_q s \\ &\le \int_0^1 \frac{1}{\Gamma_q{}^2(\alpha)} \left(1 + \frac{aH}{k}\right)^2 (1-qs)^{(\alpha-1)} t^{\alpha-1} \varphi(s)\ K^2\, \|u-v\|\ s^{\alpha-1}\ l\ d_q s \\ &< \frac{1}{\Gamma_q{}^2(\alpha)} \left(1 + \frac{aH}{k}\right)^2 K^2\ \|u-v\|\ t^{\alpha-1}\ l\ \underbrace{\int_0^1 s^{\alpha-1} (1-qs)^{(\alpha-1)} \varphi(s)\, d_q s}_{M} \end{aligned}$$

By the induction method, we have

$$\begin{aligned} \left|\left(T^n u\right)(t)-\left(T^n v\right)(t)\right| & \leq \frac{\|u-v\|}{{\Gamma_q}^n(\alpha)}\left(1+\frac{aH}{k}\right)^n K^n\ l\ M^{n-1}\ t^{\alpha-1} \\ & \leq \frac{K^n\ l\ M^{n-1}}{{\Gamma_q}^n(\alpha)}\left(1+\frac{aH}{k}\right)^n\|u-v\|, \end{aligned}$$

we can choose enough large n, such that

$$\frac{K^n\ l\ M^{n-1}}{{\Gamma_q}^n(\alpha)}\left(1+\frac{aH}{k}\right)^n<\frac{1}{2},$$

then it follows that

$$\left|\left(T^n u\right)(t)-\left(T^n v\right)(t)\right| \leq \frac{1}{2}\|u-v\|.$$

By means of *Theorem 2.6*, we claim that the operator T has a unique fixed point. $\square$

Theorem 3.3. *If there holds* (H_1)*, define two constants*

$$W=\max_{(t,s)\in[0,1]\times[0,1]} H(t,qs) \quad \text{and} \quad Q=\int_0^1 W\ \varphi(s)d_qs.$$

If there exist two positive constants $r_2>r_1$ *such that*

$$\frac{b}{k}+Q\max_{(t,u)\in[0,1]\times[0,r_2]} f(t,u)\leq r_2$$

and

$$\frac{b}{k}+Q\min_{(t,u)\in[0,1]\times[0,r_1]} f(t,u)\geq r_1$$

then the boundary value problems $(1.1)-(1.2)$ *have at least one solution satisfying* $r_1\leq\|u\|\leq r_2$.

Proof. It follows from continuity of $H(t,qs)$ and $f(t,u)$ that $H(t,qs)$, $f(t,u)$ has a maximum on any closed field.

Let $\Omega_1=\{u\in C:\ \|u\|<r_1\}$. For $u\in C\cap\partial\Omega_1$, we have $0\leq u(t)\leq r_1$ on $[0,1]$,

$$\begin{aligned} \|Tu\| &= \sup_{t\in[0,1]}\left(\int_0^1 H(t,qs)\,\varphi(s)\,f(s,u(s))\,d_qs+\frac{b}{k}t^{\alpha-1}\right) \\ &= \int_0^1 \max_{t\in[0,1]} H(t,qs)\,\varphi(s)\,f(s,u(s))d_qs+\frac{b}{k} \\ &= \int_0^1 W\ \varphi(s)\,f(s,u(s))\,d_qs+\frac{b}{k} \\ &\geq \min_{(t,u)\in[0,1]\times[0,r_1]} f(t,u)\int_0^1 W\ \varphi(s)\,d_qs+\frac{b}{k} \\ &\geq r_1=\|u\|. \end{aligned}$$

Let $\Omega_2=\{u\in C:\ \|u\|<r_2\}$. For $u\in C\cap\partial\Omega_2$, we have $0\leq u(t)\leq r_2$ on $[0,1]$,

$$\begin{aligned}
\|Tu\| &= \underset{t\in[0,1]}{sup}\left(\int_0^1 H(t,qs)\,\varphi(s)\,f(s,u(s))\,d_qs+\frac{b}{k}t^{\alpha-1}\right)\\
&\le \max_{(t,u)\in[0,1]\times[0,r_2]} f(t,u)\int_0^1 W\ \varphi(s)\,d_qs+\frac{b}{k}\\
&\le r_2=\|u\|\,.
\end{aligned}$$

By *Theorem 2.7* and *Lemma 3.1*, we can conclude that the operator equation $Tu = u$ has a solution satisfying $r_1 \le \|u\| \le r_2$. The proof is complete. □

Theorem 3.4. *Assume that* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function. Let* (H_1) *and* (H_3) *be satisfied. If there exists a constant* R *such that*

$$\frac{R}{r} > 1. \tag{3.3}$$

Then the boundary value problems $(1.1) - (1.2)$ *have at least one solution, where* r *is given in Lemma 3.1.*

Proof. Let u be a solution. Then for $t \in [0,1]$, using the computations in proving that T is bounded, we have $|u(t)| = |\lambda Tu(t)| \le r$ and thus we have

$$\frac{\|u\|}{r} \le 1.$$

In view of (3.3) there exists R such that $\|u\| \ne R$. Let us set

$$U = \{u \in C([0,1],\mathbb{R}) :\ \|u\| < R+1\}$$

Note that the operator $T : \overline{U} \to C([0,1],\mathbb{R})$ is completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda Tu(t)$ for some $\lambda \in (0,1)$.

Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that T has a fixed point $u \in \overline{U}$ which is a solution of $(1.1) - (1.2)$. □

Now we will give the upper and lower solutions result.

Definition 3.5. *Let* $x \in C^2[0,1]$, *we say that* x *is a lower solution of the boundary value problems* $(1.1) - (1.2)$, *if*

$$\left(D_q^\alpha x\right)(t) + \varphi(t) f(t,x(t)) \ge 0, \qquad t \in (0,1)$$

$$x(0) = 0, \quad x(1) \le a\int_0^1 h(s)\,x(s)\,d_qA(s) + b$$

Let $y \in C^2[0,1]$, *we say that* y *is a upper solution of the boundary value problems* $(1.1) - (1.2)$, *if*

$$\left(D_q^\alpha y\right)(t) + \varphi(t) f(t,y(t)) \le 0, \qquad t \in (0,1)$$

$$y(0) = 0, \quad y(1) \ge a\int_0^1 h(s)\,y(s)\,d_qA(s) + b$$

Theorem 3.6. *Assume that* (H_4) *holds, boundary value problems* $(1.1) - (1.2)$ *has a lower solution* $u_0 \in C$ *and an upper solution* $v_0 \in C$ *such that* $u_0 \preccurlyeq v_0$. *The boundary*

value problems $(1.1)-(1.2)$ *has the maximal lower solution* u^* *and the minimal upper solution* v^* *on* $[u_0,\ v_0] \subset C$, *both* u^* *and* v^* *are positive solutions of boundary value problems* $(1.1)-(1.2)$.
Furthermore,

$$0 \leq u_0 \leq u^* \leq v^* \leq v_0.$$

Proof. The proof will be given with three steps.
Step 1. We will obtain the lower solution sequence $\{u_k\}$ and the upper solution sequence $\{v_k\}$. According to *Lemma 2.9* for given $u_0 \in C$,

$$D_q^\alpha u_1(t) + \varphi(t) f(t, u_0(t)) = 0, \qquad t \in (0,1)$$

$$u_1(0) = 0, \quad u_1(1) = a \int_0^1 h(s) u_0(s) d_q A(s) + b$$

has a unique solution u_1.
Since u_0 is a lower solution of boundary value problems $(1.1)-(1.2)$ then

$$D_q^\alpha u_0(t) + \varphi(t) f(t, u_0(t)) \geq 0, \qquad t \in (0,1)$$

$$u_0(0) = 0, \quad u_0(1) = a \int_0^1 h(s) u_0(s) d_q A(s) + b.$$

Thus we can get that

$$D_q^\alpha (u_1(t) - u_0(t)) \leq 0$$

and

$$(u_1 - u_0)(0) = 0, \quad (u_1 - u_0)(1) \geq a \int_0^1 h(s)(u_0 - u_0)(s) d_q A(s) \geq 0.$$

If we define $u_1(t) - u_0(t) = k(t)$, we get

$$D_q^\alpha k(t) = g(t)$$

$$k(0) = 0, \quad k(1) = \gamma$$

so we know that

$$k(t) = -\int_0^1 G(t, qs) g(s) d_q s + \gamma t^{\alpha-1},$$

since $g(t) \leq 0$ and $\gamma \geq 0$ we say that $k(t) \geq 0$ and so $u_1(t) \geq u_0(t)$.
So we can get that if $u_0 \preccurlyeq u_1$ than $f(t, u_1) \geq f(t, u_0)$, from the condition (H_4).
Using this, we get

$$D_q^\alpha u_1(t) = -\varphi(t) f(t, u_0(t)) \geq -\varphi(t) f(t, u_1(t))$$

$$u_1(0) = 0, u_1(1) = a \int_0^1 h(s) u_0(s) d_q A(s) + b \leq a \int_0^1 h(s) u_1(s) d_q A(s) + b.$$

Since

$$D_q^\alpha u_1(t) + \varphi(t) f(t, u_1(t)) \geq 0, \qquad t \in (0,1)$$

$$u_1(0) = 0, \quad u_1(1) \leq a \int_0^1 h(s)\ u_1(s) d_q A(s) + b,$$

then $u = u_1(t)$ is a lower solution of boundary value problems $(1.1)-(1.2)$.

Starting from the initial function u_0 by the following iterative scheme

$$D_q^{\alpha} u_k(t) + \varphi(t) f(t, u_{k-1}(t)) = 0, \qquad t \in (0,1), \quad k = 1, 2, \ldots$$

$$u_k(0) = 0, \quad u_k(1) = a \int_0^1 h(s) u_{k-1}(s) d_q A(s) + b, \tag{3.4}$$

we can obtain the sequence $\{u_k\}$, where $u = u_k(t)$ are lower solutions of boundary value problems (1.1) – (1.2) and $u_{k-1} \preccurlyeq u_k$, so that $\{u_k\}$ is monotonically increasing.

Starting from the initial function v_0 by the following iterative scheme

$$D_q^{\alpha} v_k(t) + \varphi(t) f(t, v_{k-1}(t)) = 0, \qquad t \in (0,1), \quad k = 1, 2, \ldots$$

$$v_k(0) = 0, \quad v_k(1) = a \int_0^1 h(s) v_{k-1}(s) d_q A(s) + b, \tag{3.5}$$

we can get the sequence $\{v_k\}$, where $v = v_k(t)$ are upper solutions of boundary value problems (1.1) – (1.2) and $\{v_k\}$ is monotonically decreasing.

Step 2. We prove that $u_k \preccurlyeq v_k$ if $u_{k-1} \preccurlyeq v_{k-1}$, $k = 1, 2, \ldots$
Since $u_{k-1} \preccurlyeq v_{k-1}$, then $u_{k-1}(t) \leq v_{k-1}(t)$ and $D_q^{\alpha} u_{k-1}(t) \geq D_q^{\alpha} v_{k-1}(t)$ and from (H_4), we have

$$f(t, u_{k-1}(t)) \leq f(t, v_{k-1}(t)).$$

Thus, by (3.4) and (3.5), we get

$$D_q^{\alpha}(v_k(t) - u_k(t)) = -\varphi(t)(f(t, v_{k-1}(t)) - f(t, u_{k-1}(t))) \leq 0,$$

$$v_k(0) - u_k(0) = 0,$$

$$v_k(1) - u_k(1) = a \int_0^1 h(s) v_{k-1}(s) d_q A(s) - a \int_0^1 h(s) u_{k-1}(s) d_q A(s) \geq 0.$$

Similarly we can show that $u_k \preccurlyeq v_k$ in the same way as the above.
Therefore,

$$u_0 \preccurlyeq u_1 \preccurlyeq \cdots \preccurlyeq u_k \preccurlyeq \cdots \preccurlyeq \cdots \preccurlyeq v_k \preccurlyeq \cdots \preccurlyeq v_1 \preccurlyeq v_0.$$

Since C is a normal cone on B, the $\{u_k\}$ is uniformly bounded. Because H, G, φ and f are continuous, we can easily get that $\{u_k\}$ is equicontinuous. Hence the $\{u_k\}$ is relatively compact. Then there exist u^* and v^* such that

$$\lim_{k\to\infty} u_k = u^*, \qquad \lim_{k\to\infty} D_q^{\alpha} u_k = D_q^{\alpha} u^* \tag{3.6}$$

$$\lim_{k\to\infty} v_k = v^*, \qquad \lim_{k\to\infty} D_q^{\alpha} v_k = D_q^{\alpha} v^* \tag{3.7}$$

which imply that u^* is the maximal lower solution, v^* is the minimal upper solution of boundary value problems (1.1) – (1.2) in $[u_0, v_0] \subset C$ and $u^* \preccurlyeq v^*$.

Step 3. We prove that u^* and v^* are the solution of boundary value problems (1.1) – (1.2).
According to *Lemma 2.9* and (3.4), we can get that

$$u_k(t) = \int_0^1 H(t, qs) \varphi(s) f(s, u_{k-1}(s)) d_q s + \frac{b}{k} t^{\alpha-1}.$$

From (3.6) and by the continuity of H, f and Lebesgue dominated convergence theorem, we have

$$u^*(t) = \int_0^1 H(t, qs)\varphi(s) f(s, u^*(s)) d_q s + \frac{b}{k} t^{\alpha-1},$$

which implies that u^* is a solution of boundary value problems $(1.1)-(1.2)$. In the same way, we can show that v^* is a solution of boundary value problems $(1.1)-(1.2)$, too.
Furthermore,

$$0 \le u_0(t) \le u^*(t) \le v^*(t) \le v_0(t). \qquad \square$$

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Linear delay-differential operator of a meromorphic function sharing two sets or small function together with values with its c-shift or q-shift

Arpita Roy and Abhijit Banerjee

Abstract. The paper is devoted to study the uniqueness problem of linear delay-differential operator of a meromorphic function sharing two sets or small function together with values with its c-shift and q-shift operator. Results of this paper drastically improve two recent results of Meng-Liu [J. Appl. Math. Inform. 37(1-2)(2019), 133-148] and Qi-Li-Yang [Comput. Methods Funct. Theory, 18(2018), 567-582]. In addition to this, one of our results improves and extends that of Qi-Yang [Comput. Methods Funct. Theory, 20(2020), 159-178].

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1. Introduction, Definitions and Results

Throughout the paper we use standard notations of Nevanlinna theory as stated in [7] and by any meromorphic function f we always mean that it is defined on $\mathbb{C}$. Let f and g be such two non-constant meromorphic functions. For $a \in \mathbb{C} \cup \{\infty\}$, the following two quantities

$$\delta(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \longrightarrow \infty} \frac{m(r, a; f)}{T(r, f)}$$

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and

$$\Theta(a; f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r, a; f)}{T(r, f)}$$

are respectively known as Nevanlinna deficiency of the value a and ramification index.

In the beginning of the nineteenth century R. Nevanlinna inaugurated the value distribution theory with his famous Five value and Four value theorems which can be considered as the backbone of the modern uniqueness theory. Illuminated by these two basic results initially the research were performed on the value sharing of meromorphic functions. After five decades, uniqueness theory moved to a new direction led by F. Gross [4], who transformed the traditional value sharing problem to a more general set up namely shared set problems. Now we recall the definition of set sharing.

Definition 1.1. For some $a \in \mathbb{C}$, we denote by $E_f(a)$, the collection of the zeros of $f - a$, where a zero is counted according to its multiplicity. In addition to this, when $a = \infty$, the above notation implies that we are considering the poles. In the same manner, by $\overline{E}_f(a)$, we denote the collection of the distinct zeros or poles of $f - a$ according as $a \in \mathbb{C}$ or $a = \infty$ respectively.

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$. For a non-constant meromorphic function f, let $E_f(S) = \bigcup_{a\in S} E_f(a)$ $\big(\overline{E}_f(S) = \bigcup_{a\in S} \overline{E}_f(a)\big)$. Then we say f, g share the set S CM(IM) if $E_f(S) = E_g(S)$ $\big(\overline{E}_f(S) = \overline{E}_g(S)\big)$.

Evidently, if S is a singleton, then it coincides with the traditional definition of CM(IM) sharing of values, which are known to the readers.

In 2001, due to a revolutionary approach by Lahiri [8, 9], the notion of weighted sharing of values or sets appeared in the literature and expedite the research work there in. Though now-a-days the definition is widely circulated, we invoke the definition.

Definition 1.2. [8, 9] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k and denote it by (a, k). The IM and CM sharing corresponds to $(a, 0)$ and (a, ∞) respectively.

Definition 1.3. [8] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S, k)$ the set $\cup_{a\in S} E_k(a; f)$. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

If $E_f(S, k) = E_g(S, k)$, then we say that f, g share the set S with weight k and write it as f, g share (S, k).

By $N(r, a; f \mid< m)$ we mean the counting function of those a-points of f whose multiplicities are less than m where each a-point is counted according to its multiplicity and by $\overline{N}(r, a; f \mid\geq m)$ we mean the counting function of those a-points of f whose multiplicities are not less than m where each a-point is counted ignoring multiplicity. We also denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.

Usually, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set of finite linear measure. Also $S_1(r, f)$ denotes any quantity satisfying $S_1(r,f) = o(T(r,f))$ for all r on a set of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap F} \frac{dt}{t}.$$

Throughout the paper for a positive integer n, S_1, S_1^* and S_2 represents respectively the sets $\{1, \omega, \dots, \omega^{n-1}\}$, $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\{\infty\}$, where $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and α_i, $i = 1, 2, \dots, n$ are non-zero constants.

Let $a_{t-1}(\neq 0)$, a_{t-2}, $\dots$, a_0 and $C(\neq 0)$ be complex numbers. We define

$$P(z) = CzQ(z) = Cz(a_{t-1}z^{t-1} + a_{t-2}z^{t-2} + \dots + a_1 z + a_0). \tag{1.1}$$

For the polynomial $P(z)$ as given in (1.1) let us define the following two functions:

$$\chi_0^{t-1} = \begin{cases} 1, & if\ a_0 \neq 0 \\ 0, & if\ a_0 = 0 \end{cases}$$

and

$$\mu_0^{t-1} = \begin{cases} 1, & if\ a_0 = 0, a_1 \neq 0 \\ 0, & otherwise. \end{cases}$$

In view of (1.1), corresponding to the set S_1^*, let us consider the polynomial $P_*(z)$ as follows:

$$P_*(z) = CzQ_*(z), \text{ where } C = \frac{1}{(-1)^{n+1}\alpha_1\alpha_2\dots\alpha_n} \text{ and} \tag{1.2}$$

$$Q_*(z) = \sum_{r=0}^{n-1}(-1)^r \sum \alpha_1\alpha_2\dots\alpha_r\ z^{n-r-1},$$

$\sum \alpha_1\alpha_2\dots\alpha_r =$ sum of the products of the values $\alpha_1, \alpha_2, \dots, \alpha_n$ taken r into account. We also denote by m_1 and m_2 as the number of simple and multiple zeros of $Q_*(z)$ respectively.

Next we define linear shift operator, delay operator and differential operator respectively as follows:

$$L_1(f(z)) = a_k f(z+c_k) + a_{k-1}f(z+c_{k-1}) + \dots + a_1 f(z+c_1) + a_0 f(z),$$
$$L_2(f(z)) = b_s f^{(s)}(z+c_s) + b_{s-1}f^{(s-1)}(z+c_{s-1}) + \dots + b_1 f'(z+c_1),$$
$$L_3(f(z)) = d_t f^{(t)}(z) + d_{t-1}f^{(t-1)}(z) + \dots + d_1 f'(z),$$

where a_k, b_s and d_t are non-zero and k, s, t are natural numbers and all $c_i's$ are non-zero. For the sake of convenience we shall call $L_2(f(z)) + L_3(f(z))$ as delay-differential operator which is denoted by $\tilde{L}(f(z))$.

As far as the knowledge of the authors are concerned, Qi-Li-Yang [13] were the first authors who initiated two shared set problems for the derivative of a meromorphic function $f(z)$ with its shift $f(z+c)$ as follows:

Theorem A. [13] *Let $f(z)$ be a non-constant meromorphic function of finite order, $n \geq 9$ be an integer and a be a non-zero complex constant. If $[f'(z)]^n$ and $f^n(z+c)$ share (a,∞) and (∞,∞), then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

Recently employing the notion of weighted sharing, Meng-Liu [12] further investigated *Theorem A* to obtain the following result.

Theorem B. [12] *Let $f(z)$ be a non-constant meromorphic function of finite order, $n \geq 10$ be an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $(1,2)$ and $(\infty,0)$, then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

Considering $f(z) = e^z$ and $\omega = e^{-c}$ satisfying $\omega^n = 1$, it is easy to see that f' and $f(z+c)$ share the sets (S_1, ∞), (∞, ∞) and $f'(z) = \omega f(z+c)$ for each n. So it is natural to conjecture that in *Theorem A* and *Theorem B* the cardinality of n could further be reduced. To this end, we have performed our investigations and have been able to reduce the cardinality of n in *Theorem B* up to 6. In fact, we have proved our theorem for a more general setting S_1^* rather than to consider only the set S_1.

Theorem 1.1. *Let $f(z)$ be a non-constant meromorphic function of finite order such that $\tilde{L}(f(z))$ and $f(z+c)$ share $(S_1^*, 2)$ and $(S_2, 0)$. If*

$$n > 2(\chi_0^{n-1} + \mu_0^{n-1} + m_1 + 2m_2) + \frac{15}{(2n-3)}(\chi_0^{n-1} + m_1 + m_2), \quad \textit{then}$$

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(z+c) - \alpha_i).$$

Remark 1.1. From the definitions, we easily can calculate the value of χ_0^{n-1}, μ_0^{n-1}, m_1 and m_2 for a particular set S_1^*. Clearly for the set S_1, $\chi_0^{n-1} = 0$; $\mu_0^{n-1} = 0$; $m_1 = 0$ and $m_2 = 1$. Therefore in above theorem for the set S_1 if $n > 4 + \frac{15}{(2n-3)}$ i.e., if $n \geq 6$ then $\tilde{L}(f(z)) = tf(z+c)$, for a constant t that satisfies $t^n = 1$. For a particular choices of coefficients of $\tilde{L}(f(z))$ we can easily make $\tilde{L}(f(z)) = f'$.

Corresponding to q-shift Meng-Liu [12] also investigated the same result like *Theorem B* as follows :

Theorem C. [12] *Let $f(z)$ be a non-constant meromorphic function of zero order, $n \geq 10$ be an integer. If $[f'(z)]^n$ and $f^n(qz)$ share $(1,2)$ and $(\infty,0)$, then $f'(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.*

In connection to *Theorem C* below we present our result which improves the same.

Theorem 1.2. *Let $f(z)$ be a non-constant meromorphic function of zero order such that $\tilde{L}(f(z))$ and $f(qz)$ share $(S_1^*, 2)$ and $(S_2, 0)$. If*

$$n > 2(\chi_0^{n-1} + \mu_0^{n-1} + m_1 + 2m_2) + \frac{15}{(2n-3)}(\chi_0^{n-1} + m_1 + m_2) \quad \textit{then}$$

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(qz) - \alpha_i).$$

In the next theorem we shall show that the lower bound of n can further be reduced at the expense of allowing both the range sets S_1^*, S_2 to be shared CM.

Theorem 1.3. *Let $f(z)$ be a non-constant meromorphic function of finite order such that $\tilde{L}(f(z))$ and $f(z+c)$ share (S_1^*,∞) and (S_2,∞) with*

$$T(r,f)=N\left(r,\frac{1}{\tilde{L}(f(z))}\right)+S(r,f)$$

then for $n>2(\chi_0^{n-1}+m_1+m_2)+1$,

$$\prod_{i=1}^{n}(\tilde{L}(f(z))-\alpha_i)\equiv\prod_{i=1}^{n}(f(z+c)-\alpha_i).$$

Remark 1.2. In connection of *Remark 1.1*, for the set S_1 in *Theorem 1.3* the result holds for $n\geq 4$.

Our next theorem is analogous theorem of *Theorem 1.3* corresponding to q-shift.

Theorem 1.4. *Let $f(z)$ be a non-constant meromorphic function of zero order such that $\tilde{L}(f(z))$ and $f(qz)$ share (S_1^*,∞) and (S_2,∞). If $n>2(\chi_0^{n-1}+m_1+m_2)+1$ then*

$$\prod_{i=1}^{n}(\tilde{L}(f(z))-\alpha_i)\equiv\prod_{i=1}^{n}(f(qz)-\alpha_i).$$

Recently, corresponding to *Theorem A*, Qi-Yang [14] obtained the value sharing problem for entire function as follows:

Theorem D. [14] *Let $f(z)$ be a transcendental entire function of finite order and let $(a\neq 0)\in\mathbb{C}$. If $f'(z)$ and $f(z+c)$ share $(0,\infty)$ and $(a,0)$, then $f'(z)\equiv f(z+c)$.*

In view of *Theorem 1.1*, [14] we know that $f(z)$ actually becomes a transcendental entire function. Since we are dealing with $\tilde{L}(f(z))$ instead of f', it will be reasonable to consider the above theorem for meromorphic function under small function sharing category. In this respect we prove the following theorem.

Theorem 1.5. *Let $f(z)$ be a transcendental meromorphic function of finite order and let $a(z)(\not\equiv 0)\in S(f)$ be an entire function. If $\tilde{L}(f(z))$ and $f(z+c)$ share $(0,\infty)$, (∞,∞) and $(a(z),0)$ with $\Theta(0;f)+\Theta(\infty;f)>0$, then $\tilde{L}(f(z))\equiv f(z+c)$.*

From *Theorem 1.5* we can immediately deduce the following corollary.

Corollary 1.1. *Let $f(z)$ be a transcendental entire function of finite order and let $a(z)(\not\equiv 0)\in S(f)$. If $\tilde{L}(f(z))$ and $f(z+c)$ share $(0,\infty)$ and $(a(z),0)$, then $\tilde{L}(f(z))\equiv f(z+c)$.*

Following example shows that in *Theorem 1.5* the CM pole sharing can not be replaced by IM.

Example 1.1. Let $f(z)=\frac{2e^{2\sqrt{2}iz}-8e^{\sqrt{2}iz}+2}{(e^{\sqrt{2}iz}+1)^2}$ and $c=\sqrt{2}\pi$. Choose the coefficients of $\tilde{L}(f(z))$ in such a way that $\tilde{L}(f(z))=f''$. Then

$$\tilde{L}(f(z))\left(=\frac{24e^{\sqrt{2}iz}[e^{2\sqrt{2}iz}-4e^{\sqrt{2}iz}+1]}{(e^{\sqrt{2}iz}+1)^4}\right)$$

and $f(z+c)$ share $(0,\infty)$, $(1,0)$ and $(\infty,0)$ and $\Theta(0;f)+\Theta(\infty;f)=\frac{1}{2}>0$ but $\tilde{L}(f(z))\not\equiv f(z+c)$.

From the next example we can show that in Theorem 1.5 sharing of 0 can not be replaced by sharing of a non-zero value.

Example 1.2. Let $f(z)=(e^{\lambda z}-1)^2+1$. Choose $e^{\lambda c}=1$,

$$\sum_{i=1}^{s} b_i(2\lambda)^i e^{2\lambda c_i}+\sum_{i=1}^{t} d_i(2\lambda)^i=0$$

and

$$\sum_{i=1}^{s} b_i(\lambda)^i e^{\lambda c_i}+\sum_{i=1}^{t} d_i(\lambda)^i=-\frac{1}{2}.$$

Then $f(z+c)=(e^{\lambda z}-1)^2+1$ and $\tilde{L}(f(z))=e^{\lambda z}$. Clearly $f(z+c)$ and $\tilde{L}(f(z))$ share $(2,\infty)$, (∞,∞) and $(1,0)$ with $\Theta(0;f)+\Theta(\infty;f)>0$. But $\tilde{L}(f(z))\neq f(z+c)$.

In *Theorem 1.5*, sharing of the value 0 can be removed at the cost of slightly manipulating the deficiency condition. In this respect, we state the following theorem for transcendental meromorphic function.

Theorem 1.6. *Let $f(z)$ be a transcendental meromorphic function of finite order and let $a(z)(\not\equiv 0)\in S(f)$ be an entire function. If $\tilde{L}(f(z))$ and $f(z+c)$ share $(a(z),\infty)$ and (∞,∞) with $\delta(0;f)>0$, then $\tilde{L}(f(z))\equiv f(z+c)$.*

By an example we now show that $a(z)$ CM sharing can not be replaced by IM in *Theorem 1.6*.

Example 1.3. Let $f(z)=\frac{-2e^z-1}{e^{2z}}$ and $c=\pi i$. Choose $\tilde{L}(f(z))=L_3(f(z))$ with

$$2\sum_{i=1}^{t}(-1)^{i+1}d_i=1 \text{ and } \sum_{i=1}^{t}(-2)^i d_i=0.$$

Then $\tilde{L}(f(z))=\frac{1}{e^z}$ and $f(z+c)=\frac{2e^z-1}{e^{2z}}$ share $(1,0)$, (∞,∞) and $\delta(0;f)=\frac{1}{2}>0$. Clearly $\tilde{L}(f(z))\neq f(z+c)$.

Our next example shows that $a(z)\not\equiv 0$ in *Theorem 1.6* can not be dropped as well as $(a(z),0)$ sharing in *Theorem 1.5* can not be removed.

Example 1.4. Let $f(z)=e^{\frac{\pi i z}{c}}$. Choose $\tilde{L}(f(z))=f'$. Then clearly $f(z+c)$ and $\tilde{L}(f(z))$ share $(0,\infty)$, (∞,∞) and $\delta(0;f)>0$. But $\tilde{L}(f(z))\neq f(z+c)$.

Following two examples show that $\delta(0;f)>0$ in *Theorem 1.6* can not be removed.

Example 1.5. In *Example 1.2* though $f(z+c)$ and $\tilde{L}(f(z))$ share $(2,\infty)$, (∞,∞) but $\delta(0;f)=0$. Here $\tilde{L}(f(z))\neq f(z+c)$.

Example 1.6. Let $f(z) = \frac{e^z+z}{2}$ and $a(z) = z$. Choose $\tilde{L}(f(z)) = L_3(f(z))$ with $d_1 = 2c$ and

$$\sum_{j=2}^{t} d_j = 2(e^c - c).$$

Then

$$f(z+c)\left(= \frac{e^c e^z + z + c}{2}\right) \text{ and } \tilde{L}(f(z)) \ (= e^c e^z + c)$$

share $(a(z), \infty)$ and (∞, ∞) but $\delta(0; f) = 0$. Clearly $\tilde{L}(f(z)) \neq f(z+c)$.

2. Lemmas

In this section some lemmas will be presented which will be needed in the sequel.

Lemma 2.1. [3] *Let $f(z)$ be a meromorphic function of finite order ρ and let $c \in \mathbb{C}\setminus\{0\}$ be fixed. Then, for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

Lemma 2.2. [5] *Let $f(z)$ be a meromorphic function of finite order and $c \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.3. [6] *Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N\left(r, \frac{1}{f(z+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f),$$

$$N\left(r, f(z+c)\right) \leq N\left(r, f(z)\right) + S(r, f),$$

$$\overline{N}\left(r, \frac{1}{f(z+c)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f)$$

and

$$\overline{N}\left(r, f(z+c)\right) \leq \overline{N}\left(r, f(z)\right) + S(r, f).$$

Lemma 2.4. [2] *Let $f(z)$ be a meromorphic function of zero order and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.5. [16] *Let $f(z)$ be a non-constant zero order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$, then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z))$$

and

$$N(r, f(qz)) = (1 + o(1))N(r, f(z))$$

on a set of lower logarithmic measure 1.

Using *Lemma 2.4* and *Lemma 2.5* and by the help of simple transformation one can easily prove the next lemma.

Lemma 2.6. *Let $f(z)$ be a meromorphic function of zero order and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(z)}{f(qz)}\right) = S_1(r, f).$$

Lemma 2.7. [15] *Let $f(z)$ be a non-constant meromorphic function in the complex plane, and let $R(f) = \frac{P(f)}{Q(f)}$, where*

$$P(f) = \sum_{k=0}^{p} a_k(z) f^k \quad \text{and} \quad Q(f) = \sum_{j=0}^{q} b_j(z) f^j$$

are two mutually prime polynomials in f. If the coefficients $a_k(z)$ for $k = 0, 1, \ldots, p$ and $b_j(z)$ for $j = 0, 1, \ldots, q$ are small functions of f with $a_p(z) \not\equiv 0$ and $b_q(z) \not\equiv 0$, then

$$T(r, P(f)) = \max\{p, q\} T(r, f) + S(r, f).$$

Lemma 2.8. [11] *Suppose that h is a non-constant meromorphic function satisfying*

$$N(r, h) + N\left(r, \frac{1}{h}\right) = S(r, h).$$

Let $f = a_0 h^p + a_1 h^{p-1} + \ldots + a_p$, and $g = b_0 h^q + b_1 h^{q-1} + \ldots + b_q$ be polynomials in h with coefficients $a_0, a_1, \ldots, a_p$; $b_0, b_1, \ldots, b_q$ being small functions of h and $a_0 b_0 a_p \not\equiv 0$. If $q \le p$, then $m\left(r, \frac{g}{f}\right) = S(r, h)$.

Lemma 2.9. [10] *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \le k\overline{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\overline{N}(r, 0; f \mid \ge k) + S(r, f).$$

Lemma 2.10. *Let F be a meromorphic function. Then*

$$\overline{N}(r, 1; F \mid \ge k+1) \le \frac{1}{k}\{\overline{N}(r, 0; F) + \overline{N}(r, \infty; F)\} + S(r, F).$$

Since the proof is straight forward, it is omitted.

Lemma 2.11. [1] *Let F, G be two meromorphic functions sharing $(1, 2)$ and (∞, k), where $0 \le k \le \infty$. Then one of the following cases holds*

$$\begin{aligned} (i)\ T(r, F) + T(r, G) \le\ & 2\{N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) \\ & + \overline{N}_*(r, \infty; F, G)\} + S(r, F) + S(r, G), \end{aligned}$$

where $\overline{N}_*(r, \infty; f, g)$ is the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G,
(ii) $F \equiv G$,
(iii) $FG \equiv 1$.

Lemma 2.12. *Let $P_*(f)$ and $P_*(g)$ be defined in (1.2), for two non-constant meromorphic functions f and g. Then*

$$\overline{N}(r,0;P_*(f)) \leq (\chi_0^{n-1}+m_1+m_2)T(r,f);$$
$$N_2(r,0;P_*(f)) \leq (\chi_0^{n-1}+\mu_0^{n-1}+m_1+2m_2)T(r,f).$$

Similar results occur for $P_*(g)$.

Proof. Rewrite $P_*(f)$ and $P_*(g)$ as

$$P_*(f) = Cf(f-\beta_1)\ldots(f-\beta_{m_1})(f-\beta_{m_1+1})^{n_{m_1+1}}\ldots(f-\beta_{m_1+m_2})^{n_{m_1+m_2}} \quad (2.1)$$

and

$$P_*(g) \;=\; Cg(g-\beta_1)\ldots(g-\beta_{m_1})(g-\beta_{m_1+1})^{n_{m_1+1}}\ldots(g-\beta_{m_1+m_2})^{n_{m_1+m_2}},$$

where $\beta_i' s$ $(i=1,2,\ldots,m_1+m_2)$ are distinct complex constants and n_i is the multiplicity of the factor $(z-\beta_i)$ in $P_*(z)$ for $i=1,2,\ldots,m_1+m_2$ with $n_1=n_2=\ldots=n_{m_1}=1$ and $n_{m_1+1},\ldots,n_{m_1+m_2}\geq 2$.

Here we have to consider two cases:

Case 1. Suppose none of $\beta_i' s$ $(i=1,2,\ldots,m_1+m_2)$ be zero. Then

$$\overline{N}(r,0;P_*(f)) \;\leq\; \overline{N}(r,0;f)+\sum_{i=1}^{m_1+m_2}\overline{N}(r,\beta_i;f) \;\leq\; (1+m_1+m_2)T(r,f);$$

$$N_2(r,0;P_*(f)) \leq N(r,0;f)+\sum_{i=1}^{m_1}N(r,\beta_i;f)+2\sum_{i=m_1+1}^{m_1+m_2}\overline{N}(r,\beta_i;f) \leq (1+m_1+2m_2)T(r,f).$$

Case 2. Next let one of $\beta_i' s$ $(i=1,2,\ldots,m_1+m_2)$ be zero.

Subcase 1: Suppose one among $\beta_i' s$ $(i=1,2,\ldots,m_1)$ be zero. Without loss of generality let us assume that $\beta_1=0$. Then

$$\overline{N}(r,0;P_*(f)) \;\leq\; \overline{N}(r,0;f)+\sum_{i=2}^{m_1+m_2}\overline{N}(r,\beta_i;f) \;\leq\; (m_1+m_2)T(r,f)\;;$$

$$\begin{aligned}N_2(r,0;P_*(f)) &\leq 2\overline{N}(r,0;f)+\sum_{i=2}^{m_1}N(r,\beta_i;f)+2\sum_{i=m_1+1}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq (1+m_1+2m_2)T(r,f).\end{aligned}$$

Subcase 2: Next suppose one among $\beta_i' s$ $(i=m_1+1,m_1+2,\ldots,m_1+m_2)$ be zero. Without loss of generality let us assume that $\beta_{m_1+1}=0$. Then

$$\begin{aligned}\overline{N}(r,0;P_*(f)) &\leq \overline{N}(r,0;f)+\sum_{i=1}^{m_1}\overline{N}(r,\beta_i;f)+\sum_{i=m_1+2}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq (m_1+m_2)T(r,f);\end{aligned}$$

$$\begin{aligned}N_2(r,0;P_*(f)) &\leq 2\overline{N}(r,0;f)+\sum_{i=1}^{m_1}N(r,\beta_i;f)+2\sum_{i=m_1+2}^{m_1+m_2}\overline{N}(r,\beta_i;f)\\ &\leq (m_1+2m_2)T(r,f).\end{aligned}$$

Combining all cases we can write

$$\begin{aligned}&\overline{N}(r,0;P_*(f)) \leq (\chi_0^{n-1}+m_1+m_2)T(r,f);\\&N_2(r,0;P_*(f)) \leq (\chi_0^{n-1}+\mu_0^{n-1}+m_1+2m_2)T(r,f).\end{aligned}$$

Similarly we can obtain the same conclusions for the function g. □

Lemma 2.13. *Let $P_*(f)$ and $P_*(g)$ for two non-constant meromorphic functions f and g (as defined in (1.2)) share $(1,2)$ and $(\infty,0)$. If*

$$n > 2(\chi_0^{n-1}+\mu_0^{n-1}+m_1+2m_2)+\frac{15}{(2n-3)}(\chi_0^{n-1}+m_1+m_2),$$

then either $P_*(f)(z) \equiv P_*(g)(z)$ or $P_*(f)(z).P_*(g)(z) \equiv 1$.

Proof. Set

$$\Phi = \frac{P_*(f)(P_*(g)-1)}{P_*(g)(P_*(f)-1)}.$$

Clearly $S(r,\Phi)$ can be replaced by $S(r,f)+S(r,g)$. It is obvious that $\Phi \not\equiv 0$. If $\Phi \equiv 0$ then either $P_*(f)=0$ or $P_*(g)=1$, which gives f and g are constants, a contradiction. First suppose that $\Phi \not\equiv 1$. So $P_*(f) \not\equiv P_*(g)$.
Therefore, using *Lemma 2.10* we get

$$\begin{aligned}&\overline{N}(r,0;\Phi)+\overline{N}(r,\infty;\Phi)\\ \leq\;& \overline{N}(r,1;P_*(f)\mid\geq 3)+\overline{N}(r,0;P_*(f))+\overline{N}(r,0;P_*(g))\\ \leq\;& \frac{1}{2}\left(\overline{N}(r,0;P_*(f))+\overline{N}(r,\infty;P_*(f))\right)+\overline{N}(r,0;P_*(f))\\ &+\overline{N}(r,0;P_*(g))+S(r,P_*(f))\\ \leq\;& \frac{3}{2}\overline{N}(r,0;P_*(f))+\frac{1}{2}\overline{N}(r,\infty;f)+\overline{N}(r,0;P_*(g))+S(r,f).\end{aligned}$$

Now,

$$\Phi-1=\frac{P_*(g)-P_*(f)}{P_*(g)(P_*(f)-1)} \text{ and } \Phi' = \left[\frac{P_*(g)'}{P_*(g)(P_*(g)-1)}-\frac{P_*(f)'}{P_*(f)(P_*(f)-1)}\right]\Phi.$$

If $\Phi' \equiv 0$ then

$$\left[\frac{P_*(g)'}{P_*(g)(P_*(g)-1)}-\frac{P_*(f)'}{P_*(f)(P_*(f)-1)}\right] \equiv 0.$$

Integrating we have

$$\frac{P_*(f)-1}{P_*(f)} \equiv A\,\frac{P_*(g)-1}{P_*(g)},$$

where A is non-zero constant. i.e.,

$$1-\frac{1}{P_*(f)} \equiv A-\frac{A}{P_*(g)}.$$

Since $P_*(f)$ and $P_*(g)$ share $(\infty,0)$ so $A=1$. Then $P_*(f) \equiv P_*(g)$ which gives $\Phi \equiv 1$, a contradiction. Therefore $\Phi' \not\equiv 0$. Clearly all poles of $P_*(f)$ and $P_*(g)$ are multiple poles which are multiple zeros of $\Phi - 1$ and so zeros of Φ' with multiplicity at least $(n-1)$ but not zeros of Φ. Therefore by *Lemma 2.9*,

$$\begin{aligned}(n-1)\overline{N}(r,\infty;f) &= (n-1)\overline{N}(r,\infty;P_*(f)) = (n-1)\overline{N}(r,\infty;P_*(f) \mid\geq n)\\ &\leq N(r,0;\Phi^{'} \mid \Phi \neq 0) \leq \overline{N}(r,0;\Phi) + \overline{N}(r,\infty;\Phi) + S(r,\Phi).\end{aligned}$$

So,

$$(2n-3)\overline{N}(r,\infty;f) \leq 3\overline{N}(r,0;P_*(f)) + 2\overline{N}(r,0;P_*(g)) + S(r,f).$$

Applying *Lemma 2.12* we obtain

$$\begin{aligned}\overline{N}(r,\infty;f) \leq & \frac{3(\chi_0^{n-1}+m_1+m_2)}{2n-3}T(r,f)\\ &+ \frac{2(\chi_0^{n-1}+m_1+m_2)}{2n-3}T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

Similarly

$$\begin{aligned}\overline{N}(r,\infty;g) \leq & \frac{3(\chi_0^{n-1}+m_1+m_2)}{2n-3}T(r,g)\\ &+ \frac{2(\chi_0^{n-1}+m_1+m_2)}{2n-3}T(r,f) + S(r,f) + S(r,g).\end{aligned}$$

That is

$$\begin{aligned}\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) \quad \leq \quad & \frac{5(\chi_0^{n-1}+m_1+m_2)}{2n-3}\left(T(r,f)+T(r,g)\right)\\ &+S(r,f)+S(r,g).\end{aligned} \tag{2.2}$$

If possible, we suppose that (i) of *Lemma 2.11* holds. Therefore

$$\begin{aligned}& T(r,P_*(f)) + T(r,P_*(g))\\ \leq\; & 2\{N_2(r,0;P_*(f)) + N_2(r,0;P_*(g)) + \overline{N}(r,\infty;P_*(f)) + \overline{N}(r,\infty;P_*(g))\\ & +\overline{N}_*(r,\infty;P_*(f),P_*(g))\} + S(r,P_*(f)) + S(r,P_*(g)).\end{aligned}$$

Then using *Lemma 2.7*, *Lemma 2.12* and (2.2) we have

$$\begin{aligned}& n\left(T(r,f)+T(r,g)\right)\\ \leq\; & \left(2(\chi_0^{n-1}+\mu_0^{n-1}+m_1+2m_2) + \frac{15(\chi_0^{n-1}+m_1+m_2)}{2n-3}\right)\left(T(r,f)+T(r,g)\right)\\ & +S(r,f)+S(r,g),\end{aligned}$$

which contradicts our assumption. So by *Lemma 2.11* we have

$$P_*(f)(z).P_*(g)(z) \equiv 1.$$

If $\Phi \equiv 1$, then $P_*(f)(z) \equiv P_*(g)(z)$.
Hence the lemma is proved. $\square$

Lemma 2.14. *Let f and g be two non-constant meromorphic functions of finite order. Let $n \geq 2$, and let $\{a_1(z), a_2(z), \dots, a_n(z)\} \in S(f)$ be distinct meromorphic periodic functions with period c. If $m\left(r, \frac{g}{f-a_k}\right) = S(r,f)$, for $k = 1, 2, \dots, n$, then*

$$\sum_{k=1}^{n} m\left(r, \frac{1}{f-a_k}\right) \leq m\left(r, \frac{1}{g}\right) + S(r,f),$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Proof. Set

$$P(f) = \prod_{k=1}^{n}(f - a_k).$$

Rewriting we have

$$\frac{1}{P(f)} = \sum_{k=1}^{n} \frac{\alpha_k}{f - a_k},$$

where $\alpha_k \in S(f)$ are certain periodic function with period c. Now,

$$m\left(r, \frac{g}{P(f)}\right) \leq \sum_{k=1}^{n} m\left(r, \frac{g}{f-a_k}\right) + S(r,f) = S(r,f),$$

and so

$$m\left(r, \frac{1}{P(f)}\right) = m\left(r, \frac{g}{P(f)}\right) + m\left(r, \frac{1}{g}\right) \leq m\left(r, \frac{1}{g}\right) + S(r,f).$$

By the first fundamental theorem and using the above inequation we get

$$m\left(r, \frac{1}{g}\right) \geq m\left(r, \frac{1}{P(f)}\right) + S(r,f) = T(r, P(f)) - N\left(r, \frac{1}{P(f)}\right) + S(r,f)$$

$$\geq nT(r,f) - \sum_{k=1}^{n} N\left(r, \frac{1}{f-a_k}\right) + S(r,f) = \sum_{k=1}^{n} m\left(r, \frac{1}{f-a_k}\right) + S(r,f). \qquad \square$$

Lemma 2.15. *If f be a meromorphic function of finite order then $\tilde{L}(f(z))$ is of finite order and*

$$m\left(r, \frac{\tilde{L}(f(z))}{f(z+c)}\right) = S(r,f), \; m\left(r, \frac{\tilde{L}(f(z))}{f(z) - \beta_i}\right) = S(r,f)$$

and

$$m\left(r, \frac{\tilde{L}(f(z))}{f(qz)}\right) = S_1(r,f).$$

Proof. Using logarithmic derivative lemma and *Lemma 2.2* we have

$$\begin{aligned} m\left(r, \frac{\tilde{L}(f(z))}{f(z+c)}\right) &= m\left(r, \frac{\sum_{j=1}^{s} b_j f^{(j)}(z+c_j) + \sum_{j=1}^{t} d_j f^{(j)}(z)}{f(z+c)}\right) \\ &\leq \sum_{j=1}^{s} m\left(r, \frac{f^{(j)}(z+c_j)}{f^{(j)}(z)}\right) + \sum_{j=1}^{s} m\left(r, \frac{f^{(j)}(z)}{f(z)}\right) \\ &\quad + \sum_{j=1}^{t} m\left(r, \frac{f^{(j)}(z)}{f(z)}\right) + (s+t)\, m\left(r, \frac{f(z)}{f(z+c)}\right) + O(1) \\ &= S(r,f). \end{aligned} \tag{2.3}$$

Also,

$$\begin{aligned} m\left(r, \frac{\tilde{L}(f(z))}{f(z)-\beta_i}\right) &= m\left(r, \frac{\sum_{j=1}^{s} b_j f^{(j)}(z+c_j) + \sum_{j=1}^{t} d_j f^{(j)}(z)}{f(z)-\beta_i}\right) \\ &\leq \sum_{j=1}^{s} m\left(r, \frac{f^{(j)}(z+c_j)}{f^{(j)}(z)}\right) + \sum_{j=1}^{t} m\left(r, \frac{f^{(j)}(z)}{f(z)-\beta_i}\right) \\ &\quad + \sum_{j=1}^{s} m\left(r, \frac{f^{(j)}(z)}{f(z)-\beta_i}\right) + O(1) = S(r,f). \end{aligned}$$

Using (2.3) and *Lemma 2.1* we have

$$T(r, \tilde{L}(f(z))) \leq \frac{s^2+t^2+3(s+t)+2}{2}\, T(r,f) + S(r,f).$$

As f is of finite order so $\tilde{L}(f(z))$ and $f(z+c)$ is of finite order and $S(r, \tilde{L}(f(z)))$ can be replaced by $S(r,f)$.

Similarly by using *Lemma 2.4*, *Lemma 2.5* and *Lemma 2.6* as and when required we can prove $f(qz)$ and $\tilde{L}(f(z))$ are zero order when f is of zero order and

$$m\left(r, \frac{\tilde{L}(f(z))}{f(qz)}\right) = S_1(r,f). \qquad \square$$

3. Proofs of the theorems

Proof of Theorem 1.1. Since

$$E_{f(z+c)}(S_1^*, 2) = E_{\tilde{L}(f(z))}(S_1^*, 2) \text{ and } E_{f(z+c)}(S_2, 0) = E_{\tilde{L}(f(z))}(S_2, 0),$$

it follows that $P_*(f(z+c))$, $P_*(\tilde{L}(f(z)))$ share $(1,2)$ and $(\infty,0)$. So by *Lemma 2.13* we have either $P_*(f(z+c)) \equiv P_*(\tilde{L}(f(z)))$ or $P_*(f(z+c)).P_*(\tilde{L}(f(z))) \equiv 1$. Suppose that

$$P_*(f(z+c)).P_*(\tilde{L}(f(z))) \equiv 1. \tag{3.1}$$

Noting that $P_*(f(z+c))$, $P_*(\tilde{L}(f(z)))$ share $(\infty,0)$, so we can conclude that $P_*(f(z+c))$, $P_*(\tilde{L}(f(z)))$ both are entire functions.
So

$$N\left(r,\infty;\frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right) = N(r,0;P_*(f(z+c))).$$

Therefore using *Lemma 2.12* and *Lemma 2.1*, we get

$$N\left(r,\infty;\frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right) \leq (\chi_0^{n-1}+m_1+m_2)T(r,f(z+c)) \leq nT(r,f)+S(r,f).$$

Using *Lemma 2.2* and *Lemma 2.15* we have

$$\begin{aligned}
m\left(r,\ \frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right) &= m\left(r,\ \frac{\tilde{L}(f(z))}{f(z+c)}\prod_{i=1}^{m_1+m_2}\left(\frac{\tilde{L}(f(z))-\beta_i}{f(z+c)-\beta_i}\right)^{n_i}\right)\\
&\leq m\left(r,\frac{\tilde{L}(f(z))}{f(z+c)}\right) + m\left(r,\ \prod_{i=1}^{m_1+m_2}\left(\frac{\tilde{L}(f(z))-\beta_i}{f(z+c)-\beta_i}\right)^{n_i}\right)+O(1)\\
&\leq \sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{\tilde{L}(f(z))-\beta_i}{f(z+c)-\beta_i}\right)+S(r,f)\\
&\leq \sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{\tilde{L}(f(z))}{f(z)-\beta_i}\right)+\sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{1}{f(z)-\beta_i}\right)\\
&+\sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{f(z)-\beta_i}{f(z+c)-\beta_i}\right)+S(r,f)\\
&\leq \sum_{i=1}^{m_1+m_2} n_i\, m\left(r,\frac{1}{f(z)-\beta_i}\right)+S(r,f)\\
&\leq (n_1+n_2+\ldots+n_{m_1+m_2})T(r,f)+S(r,f)\\
&\leq (n-1)T(r,f)+S(r,f).
\end{aligned}$$

By *Lemma 2.1*, *Lemma 2.7* and (3.1),

$$\begin{aligned}
2nT(r,f) &= 2nT(r,f(z+c))+S(r,f) = 2T(r,P_*(f(z+c)))+S(r,f)\\
&\leq T\left(r,\ \frac{1}{P_*(f(z+c))^2}\right)+S(r,f) \leq T\left(r,\ \frac{P_*(\tilde{L}(f(z)))}{P_*(f(z+c))}\right)+S(r,f)\\
&\leq (2n-1)T(r,f)+S(r,f),
\end{aligned}$$

which is a contradiction.

Therefore $P_*(\tilde{L}(f(z))) \equiv P_*(f(z+c))$, which yields

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(z+c) - \alpha_i). \qquad \square$$

Proof of Theorem 1.2. By proceeding in a similar way of the proof of *Theorem 1.1* we can prove this theorem using *Lemma 2.4*, *Lemma 2.5* and *Lemma 2.6* as and when required instead of *Lemma 2.1 and Lemma 2.2.* $\square$

Proof of Theorem 1.3. Since the finite order meromorphic functions $f(z+c)$ and $\tilde{L}(f(z))$ share (S_1^*, ∞), (S_2, ∞), it follows that $P_*(f(z+c))$, $P_*(\tilde{L}(f(z)))$ share $(1, \infty)$ and (∞, ∞) which yields

$$N(r, \tilde{L}(f(z))) = N(r, f(z+c)) \tag{3.2}$$

and

$$\frac{P_*(\tilde{L}(f(z))) - 1}{P_*(f(z+c)) - 1} = e^{\gamma(z)}, \tag{3.3}$$

where $\gamma(z)$ is a polynomial.
Now,

$$T(r, e^{\gamma(z)}) = m(r, e^{\gamma(z)}) = m\left(r, \frac{P_*(\tilde{L}(f(z))) - 1}{P_*(f(z+c)) - 1}\right).$$

Using the definition of $P_*(z)$ we have

$$\begin{aligned}
T(r, e^{\gamma(z)}) &= m\left(r, \frac{(\tilde{L}(f(z))) - \alpha_1)(\tilde{L}(f(z))) - \alpha_2) \dots (\tilde{L}(f(z))) - \alpha_n)}{(f(z+c) - \alpha_1)(f(z+c) - \alpha_2) \dots (f(z+c) - \alpha_n)}\right) \\
&\leq \sum_{j=1}^{n} m\left(r, \frac{\tilde{L}(f(z))) - \alpha_j}{f(z+c) - \alpha_j}\right) + O(1) \\
&\leq \sum_{j=1}^{n} m\left(r, \frac{\tilde{L}(f(z))}{f(z) - \alpha_j}\right) + \sum_{j=1}^{n} m\left(r, \frac{1}{f(z) - \alpha_j}\right) + \sum_{j=1}^{n} m\left(r, \frac{f(z) - \alpha_j}{f(z+c) - \alpha_j}\right) \\
&\quad + O(1).
\end{aligned}$$

In view of *Lemma 2.2*, *Lemma 2.14*, *Lemma 2.15* and then by the first fundamental theorem and (3.2) we have

$$\begin{aligned}
T(r, e^{\gamma(z)}) &= \sum_{j=1}^{n} m\left(r, \frac{1}{f(z) - \alpha_j}\right) + S(r,f) \leq m\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f) \\
&\leq T(r, \tilde{L}(f(z))) - N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f) \\
&\leq m\left(r, \frac{\tilde{L}(f(z))}{f(z+c)}\right) + m(r, f(z+c)) + N(r, \tilde{L}(f(z))) - N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f) \\
&\leq T(r, f(z+c)) - N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f)
\end{aligned}$$

$$\leq T(r,f) - N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f).$$

According to the given condition

$$T(r,f) = N\left(r, \frac{1}{\tilde{L}(f(z))}\right) + S(r,f),$$

so $T(r, e^{\gamma(z)}) = S(r,f)$.
Now from (3.3) we have

$$P_*(\tilde{L}(f(z))) = e^{\gamma(z)}\left(P_*(f(z+c)) - 1 + e^{-\gamma(z)}\right).$$

Set

$$W(z) = \frac{P_*(f(z+c))}{1 - e^{-\gamma(z)}}.$$

If $e^{\gamma(z)} \not\equiv 1$, then by applying Nevanlinna's second fundamental theorem to $W(z)$ and using (3.2) and *Lemma 2.12* we obtain

$$\begin{aligned}
&T(r, P_*(f(z+c))) \leq T(r,W) + S(r,f)\\
&\leq \overline{N}(r,0;W) + \overline{N}(r,\infty;W) + \overline{N}(r,0;W-1) + S(r,f)\\
&\leq \overline{N}(r,0;P_*(f(z+c))) + \overline{N}(r,\infty;P_*(f(z+c))) + \overline{N}(r,0;P_*(\tilde{L}(f(z)))) + S(r,f)\\
&\leq (\chi_0^{n-1} + m_1 + m_2)\left(T(r,f(z+c)) + T(r,\tilde{L}(f(z)))\right) + N(r,\infty;f(z+c)) + S(r,f)\\
&\leq (\chi_0^{n-1} + m_1 + m_2)\left(T(r,f(z+c)) + m(r,f(z+c)) + m\left(r, \frac{\tilde{L}(f(z))}{f(z+c)}\right)\right.\\
&\left. + N(r,\infty;f(z+c))\right) + N(r,\infty;f) + S(r,f).
\end{aligned}$$

Using *Lemma 2.1* and *Lemma 2.15* we get

$$nT(r,f) \leq (2\chi_0^{n-1} + 2m_1 + 2m_2 + 1)T(r,f) + S(r,f),$$

which contradicts $n > 2(\chi_0^{n-1} + m_1 + m_2) + 1$. This gives $e^{\gamma(z)} \equiv 1$, that yields

$$\prod_{i=1}^{n}(\tilde{L}(f(z)) - \alpha_i) \equiv \prod_{i=1}^{n}(f(z+c) - \alpha_i). \qquad \square$$

Proof of Theorem 1.4. Here $\tilde{L}(f(z))$ and $f(qz)$ are of zero order. Since $f(qz)$ and $\tilde{L}(f(z))$ share (S_1^*, ∞) and (S_2, ∞), it follows that $P_*(f(qz))$ and $P_*(\tilde{L}(f(z)))$ share $(1,\infty)$ and (∞,∞). Therefore

$$\frac{P_*(\tilde{L}(f(z))) - 1}{P_*(f(qz)) - 1} = A,$$

where A is a non-zero constant.
This gives

$$P_*(\tilde{L}(f(z))) = A\left(P_*(f(qz)) - 1 + \frac{1}{A}\right).$$

Set $W_1(z) = \frac{P_*(f(qz))}{1-\frac{1}{A}}$. If $A \not\equiv 1$, then applying Nevanlinna's second fundamental theorem to $W_1(z)$ and using *Lemmas 2.4* and *2.5* and *2.15* as and when required we can calculate the rest of the proof similar to *Theorem 1.3*. □

Proof of Theorem 1.5. Here $f(z+c)$ and $\tilde{L}(f(z))$ are of finite order. Since $f(z+c)$ and $\tilde{L}(f(z))$ share $(0,\infty)$ and (∞,∞), so

$$\frac{\tilde{L}(f(z))}{f(z+c)} = e^{\delta(z)}, \tag{3.4}$$

where $\delta(z)$ is a polynomial.
Clearly by *Lemma 2.15* we get

$$T(r, e^{\delta(z)}) = S(r, f).$$

When $e^{\delta(z)} \equiv 1$ then $\tilde{L}(f(z)) \equiv f(z+c)$.
When $e^{\delta(z)} \not\equiv 1$, using the fact that $f(z+c)$ and $\tilde{L}(f(z))$ share $(a(z), 0)$ we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{\tilde{L}(f(z)) - a(z)}\right) = \overline{N}\left(r, \frac{1}{f(z+c) - a(z)}\right) &\le \overline{N}\left(r, \frac{1}{e^{\delta(z)} - 1}\right) + \overline{N}\left(r, \frac{1}{a(z)}\right)\\ &\le T(r, e^{\delta(z)}) + S(r, f) = S(r, f).\end{aligned}$$

Rewriting (3.4) we get

$$\tilde{L}(f(z)) - a(z) = e^{\delta(z)}(f(z+c) - a(z)e^{-\delta(z)}).$$

Clearly $a(z)e^{-\delta(z)} \not\equiv a(z)$. So,

$$\overline{N}\left(r, \frac{1}{f(z+c) - a(z)e^{-\delta(z)}}\right) = \overline{N}\left(r, \frac{1}{\tilde{L}(f(z)) - a(z)}\right) = S(r, f).$$

Using *Lemma 2.1*, *2.3* and the second fundamental theorem we obtain

$$\begin{aligned} & 2T(r, f) = 2T(r, f(z+c)) + S(r, f)\\ \le\ & \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}\left(r, \frac{1}{f(z+c) - a(z)}\right)\\ & + \overline{N}\left(r, \frac{1}{f(z+c) - a(z)e^{-\delta(z)}}\right) + S(r, f) \le \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f),\end{aligned}$$

which is a contradiction to $\Theta(0; f) + \Theta(\infty; f) > 0$. Hence $\tilde{L}(f(z)) \equiv f(z+c)$. □

Proof of Theorem 1.6. Here $f(z+c)$ and $\tilde{L}(f(z))$ are of finite order. Since $f(z+c)$ and $\tilde{L}(f(z))$ share $(a(z),\infty)$ and (∞,∞), so

$$\frac{\tilde{L}(f(z)) - a(z)}{f(z+c) - a(z)} = e^{\zeta(z)}, \tag{3.5}$$

where $\zeta(z)$ is a polynomial. Using logarithmic derivative lemma, *Lemma 2.1* and *Lemma 2.2* we get

$$\begin{aligned}
&T(r,e^{\zeta(z)}) = m(r,e^{\zeta(z)}) = m\left(r,\ \frac{\tilde{L}(f(z))-a(z)}{f(z+c)-a(z)}\right)\\
\leq\ & m\left(r,\ \frac{\tilde{L}(f(z))-\tilde{L}(a(z-c))}{f(z+c)-a(z)}\right)+m\left(r,\ \frac{\tilde{L}(a(z-c))-a(z)}{f(z+c)-a(z)}\right)+O(1)\\
\leq\ & m\left(r,\ \frac{\tilde{L}(f(z))-\tilde{L}(a(z-c))}{f(z)-a(z-c)}\right)+m\left(r,\ \frac{f(z)-a(z-c)}{f(z+c)-a(z)}\right)\\
&+m\left(r,\ \frac{1}{f(z+c)-a(z)}\right)+S(r,f)\\
\leq\ & m\left(r,\ \frac{\sum_{j=1}^{s} b_j(f^{(j)}(z+c_j)-a^{(j)}(z-c+c_j))+\sum_{j=1}^{t} d_j(f^{(j)}(z)-a^{(j)}(z-c))}{f(z)-a(z-c)}\right)\\
&+T(r,f(z+c))+S(r,f)\\
\leq\ & \sum_{j=1}^{s} m\left(r,\ \frac{f^{(j)}(z+c_j)-a^{(j)}(z-c+c_j)}{f^{(j)}(z)-a^{(j)}(z-c)}\right)+\sum_{j=1}^{t} m\left(r,\ \frac{f^{(j)}(z)-a^{(j)}(z-c)}{f(z)-a(z-c)}\right)\\
&+\sum_{j=1}^{s} m\left(r,\ \frac{f^{(j)}(z)-a^{(j)}(z-c)}{f(z)-a(z-c)}\right)+T(r,f)+S(r,f)\\
\leq\ & T(r,f)+S(r,f).
\end{aligned}$$

So $S(r,e^{\zeta(z)})$ can be replaced by $S(r,f)$. When $e^{\zeta(z)}\equiv 1$ then $\tilde{L}(f(z))\equiv f(z+c)$. Suppose $e^{\delta(z)}\not\equiv 1$. Now rewriting (3.5) we can obtain

$$\frac{1}{f(z+c)} = -\frac{\tilde{L}(f(z))}{a(z)f(z+c)(e^{\zeta(z)}-1)}+\frac{e^{\zeta(z)}}{a(z)(e^{\zeta(z)}-1)}.$$

Therefore in view of *Lemma 2.15* we have

$$m\left(r,\frac{1}{f(z+c)}\right)\leq 2\, m\left(r,\frac{1}{e^{\zeta(z)}-1}\right)+S(r,f).$$

If $\zeta(z)$ is constant then automatically $m\left(r,\frac{1}{f(z+c)}\right)=S(r,f)$. If $\zeta(z)$ is non-constant then by *Lemma 2.8* we get

$$m\left(r,\frac{1}{f(z+c)}\right)=S(r,e^{\zeta(z)})=S(r,f).$$

By *Lemma 2.1* and *Lemma 2.3* we have

$$\begin{aligned} T(r,f) &= T(r,f(z+c)) + S(r,f) = T\left(r, \frac{1}{f(z+c)}\right) + S(r,f) \\ &\leq N\left(r, \frac{1}{f(z+c)}\right) + S(r,f) \leq N\left(r, \frac{1}{f}\right) + S(r,f) \leq T(r,f) + S(r,f). \end{aligned}$$

Therefore,

$$N\left(r, \frac{1}{f}\right) = T(r,f) + S(r,f),$$

which contradicts the fact that $\delta(0,f) > 0$. Hence $\tilde{L}(f(z)) \equiv f(z+c)$. □

4. Observation

Take $\tilde{L}(f(z)) = L_3$ with all coefficients are 1. Then we see that choosing

$$c = \frac{\log(\alpha + \alpha^2 + \ldots + \alpha^t)}{\alpha},$$

where $1 + \alpha + \ldots + \alpha^{t-1} \neq 0$, we somehow get a solution $f(z) = e^{\alpha z}$ $(\alpha \neq 0)$ of

$$\tilde{L}(f(z)) = f(z+c). \tag{4.1}$$

However choosing $c = \frac{\pi}{2}$, we can present the solution of $f' = f(z+c)$ as the linear combination of two independent solutions. e.g., $f(z) = d_1 e^{iz} + d_2 e^{-iz}$. So it is a matter of concern that how the solutions of (4.1) looks like. Unfortunately we can not elucidated in this matter.

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Hardy-Littlewood-Stein-Weiss type theorems for Riesz potentials and their commutators in Morrey spaces

Canay Aykol and Javanshir J. Hasanov

Abstract. In this paper we consider weighted Morrey spaces $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$. We prove the Hardy-Littlewood-Stein-Weiss type $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ theorems for Riesz potential I^α and its commutators $[b, I^\alpha]$ and $|b, I^\alpha|$, where $0 < \alpha < n$, $0 \leq \lambda < n - \alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n + \lambda \leq \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$, $b \in BMO(\mathbb{R}^n)$. As a result of these we obtain the conditions for the boundedness of the commutator $|b, I^\alpha|$ from Besov-Morrey spaces $B^s_{p,\theta,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $B^s_{q,\theta,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$. Furthermore, we consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ and obtain weighted Morrey $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ estimates for the operators $V^s(-\Delta + V)^{-\beta}$ and $V^s\nabla(-\Delta + V)^{-\beta}$. Finally we apply our results to various operators which are estimated from above by Riesz potentials.

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1. Introduction

The well known Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$ introduced by Charles Morrey (see [24]) in 1938 in relation to the study of partial differential equations, and presented in various books, see e.g. [11, 16, 39]. They were widely investigated during the last decades, including the study of classical operators of harmonic analysis maximal, singular and potential operators on Morrey spaces and their various generalizations

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have found wide applications in many problems of real analysis and partial differential equations. Morrey spaces are defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} = \sup_{x,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$ and $B(x,t)$ is the open ball in $\mathbb{R}^n$ of radius t centered at x. In the theory of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces play an important role. Later, Morrey spaces found important applications to Navier-Stokes ([22], [39]) and Schrödinger ([28], [29], [30], [33], [34]) equations, elliptic problems with discontinuous coefficients ([5], [8]), and potential theory ([1], [2]).

The results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [27], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [6].

Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces was proved by H.G. Hardy and J.E. Littlewood [12] in the one-dimensional case and by E.M. Stein and G. Weiss [37] in the case $n > 1$. In the Lebesgue and Morrey spaces with variable exponent the Hardy-Littlewood-Stein-Weiss inequality was proved by S.G. Samko [31] and J.J. Hasanov [13], respectively.

Let f be a locally integrable function on $\mathbb{R}^n$. The so-called fractional maximal function is defined by the formula

$$M^\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)|dy, \; 0 \le \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$ such that $|B(x,t)| = \omega_n t^n$ in which ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$. Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods.

Fractional maximal operator is intimately related to the Riesz potential

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n,$$

such that

$$M^\alpha f(x) \le \omega_n^{\frac{\alpha}{n}-1} (I^\alpha |f|(x)).$$

The aim of this paper is to give the necessary and sufficient conditions for the boundedness of Riesz potential I^α and its commutators from weighted Morrey spaces $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{p,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$. We also obtain the necessary conditions for the boundedness of the commutator $|b, I^\alpha|$ from Besov-Morrey spaces $B^s_{p,\theta,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $B^s_{q,\theta,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$. Furthermore, we consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ and obtain weighted Morrey $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ estimates for the operators $V^s(-\Delta + V)^{-\beta}$ and $V^s \nabla (-\Delta + V)^{-\beta}$. Finally we apply our results to various operators which are estimated from above by Riesz potentials.

Throughout the paper we use the letters c, C for positive constants, independent of appropriate parameters and not necessarily the same at each occurrence. If $A \le CB$ and $B \le CA$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries

We use the following notation. For $1 \le p < \infty$, $L_p(\mathbb{R}^n)$ is the space of all classes of measurable functions on $\mathbb{R}^n$ for which

$$\|f\|_{L_p} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

up to the equivalence of the norms

$$\|f\|_{L_p} \sim \sup_{\|g\|_{L^{p'}} \le 1} \left| \int_{\mathbb{R}^n} f(y)g(y)dy \right| \tag{2.1}$$

and also $WL_p(\mathbb{R}^n)$, the weak L_p space defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{WL_p} = \sup_{r>0} r \left|\{x \in \mathbb{R}^n \ : \ |f(x)| > r\}\right|^{1/p} < \infty.$$

For $p = \infty$ the space $L_\infty(\mathbb{R}^n)$ is defined by means of the usual modification

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

For $1 \le p < \infty$ let $L_{p,\omega}(\mathbb{R}^n)$ be the space of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L_{p,\omega}} = \|f\omega^{1/p}\|_{L_p(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty,$$

and for $p = \infty$ the space $L_{\infty,\omega}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

Definition 2.1. The weight function ω belongs to the class $A_p(\mathbb{R}^n)$ for $1 \le p < \infty$, if the following statement

$$\sup_{x \in \mathbb{R}^n, t>0} \frac{1}{|B(x,t)|} \int\limits_{B(x,t)} \omega(y)dy \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \omega^{-\frac{1}{p-1}}(y)dy \right)^{p-1}$$

is finite and ω belongs to $A_1(\mathbb{R}^n)$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n$ and $t > 0$

$$|B(x,t)|^{-1} \int\limits_{B(x,t)} \omega(y)dy \le C \operatorname*{ess\,sup}_{y \in B(x,t)} \frac{1}{\omega(y)}.$$

The following theorem was proved in [37].

Theorem 2.2. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $\alpha p - n < \gamma < n(p-1)$, $\mu = \frac{q\gamma}{p}$. *Then the operators* M^α *and* I^α *are bounded from* $L_{p,|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $L_{q,|\cdot|^\mu}(\mathbb{R}^n)$.

Theorem 2.3. [36] *Let $1 < p < \infty$ and $-n < \gamma < n(p-1)$. Then the operator M is bounded on $L_{p,|\cdot|^\gamma}(\mathbb{R}^n)$.*

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| dy,$$

where $f_{B(x,t)}(x) = |B(x,t)|^{-1} \int_{B(x,t)} f(y)dy$.

Definition 2.4. We define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n,\, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| dy$$

or

$$\|f\|_{BMO} = \inf_C \sup_{x\in\mathbb{R}^n,\, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - C| dy.$$

Definition 2.5. We define the $BMO_{p,\omega}(\mathbb{R}^n)$ $(1 \le p < \infty)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in\mathbb{R}^n,\, t>0} \frac{\|(f(\cdot) - f_{B(x,t)})\chi_{B(x,t)}\|_{L_{p,\omega}(\mathbb{R}^n)}}{\|\chi_{B(x,t)}\|_{L_{p,\omega}(\mathbb{R}^n)}}.$$

Theorem 2.6. [14, Theorem 4.4] *Let $1 \le p < \infty$ and ω be a Lebesgue measurable function. If $\omega \in A_p(\mathbb{R}^n)$, then the norms $\|\cdot\|_{BMO_{p,\omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

We find it convenient to define the Morrey and weighted Morrey spaces in the form as follows.

Definition 2.7. Let $1 \le p < \infty$. Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ and weighted Morrey spaces $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ are defined by the norms

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))}$$

and

$$\|f\|_{L_{p,\lambda,|\cdot|^\gamma}} = \sup_{x\in\mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_{p,|\cdot|^\gamma}(B(x,t))},$$

respectively.

For $1 \le p, \theta \le \infty$ and $0 < s < 1$, Besov-Morrey space $B^s_{p,\theta,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ consists of all functions $f \in L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,\theta,\lambda,|\cdot|^\gamma}} = \|f\|_{L_{p,\lambda,|\cdot|^\gamma}} + \left(\int_{\mathbb{R}^n} \frac{\|f(x-\cdot) - f(\cdot)\|^\theta_{L_{p,\lambda,|\cdot|^\gamma}}}{|x|^{n+s\theta}} dx \right)^{1/\theta} < \infty.$$

3. Riesz potential operator in the spaces $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$

In this section we prove the Hardy-Littlewood-Stein-Weiss type $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ -theorem for Riesz potential I^α, where $-n+\lambda \le \gamma < n(p-1)+\lambda$, $1 < p < \frac{n-\lambda}{\alpha}$, $\mu = \frac{q\gamma}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.
First we give following theorems which we use while proving our main results.

Theorem 3.1. [25] *Let $1 < p < \infty$, then $M : L_{p,\varphi}(\mathbb{R}^n) \to L_{p,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_p(\mathbb{R}^n)$.*

Theorem 3.2. [15] *Let $1 < p < \infty$, $0 \le \lambda < n$, $\varphi \in A_p(\mathbb{R}^n)$, then $M : L_{p,\lambda,\varphi}(\mathbb{R}^n) \to L_{p,\lambda,\varphi}(\mathbb{R}^n)$.*

Theorem 3.3. *Let $0 < \alpha < n$, $0 \le \lambda < n-\alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n+\lambda \le \gamma < n(p-1)+\lambda$ and $\mu = \frac{q\gamma}{p}$. Then the operator I^α is bounded from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ if and only if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.*

Proof. Sufficiency: Let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ and $f \in L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$. Then

$$\begin{aligned} |I^\alpha f(x)| &= \left(\int\limits_{B(x,t)} + \int\limits_{\mathbb{R}^n \setminus B(x,t)} \right) |f(y)||x-y|^{\alpha-n} dy \\ &\equiv F_1(x,t) + F_2(x,t). \end{aligned}$$

First we estimate $F_1(x,t)$. By using Hölder's inequality we have

$$\begin{aligned} F_1(x,t) &= \int\limits_{B(x,t)} |f(y)||x-y|^{\alpha-n} dy \\ &\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-n} \int\limits_{B(x,2^{j+1}t) \setminus B(x,2^j t)} |f(y)| dy \\ &\le C t^\alpha M f(x). \end{aligned} \tag{3.1}$$

Now we estimate $F_2(x,t)$. By using Hölder's inequality we get

$$\begin{aligned} F_2(x,t) &\le \int\limits_{\mathbb{R}^n \setminus B(x,t)} |f(y)||x-y|^{\alpha-n} dy \\ &\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-n} \int\limits_{B(x,2^{j+1}t) \setminus B(x,2^j t)} |f(y)| dy \\ &\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-n} \left\| \chi_{B(x,2^{j+1}t)} \right\|_{L_{p'(\cdot),|\cdot|^{\gamma/(1-p)}}} \left\| f \chi_{B(x,2^{j+1}t)} \right\|_{L_{p,|\cdot|^\gamma}} \\ &\le C t^{\alpha - \frac{n-\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}} \sum_{j=0}^{\infty} 2^{j\left(\alpha - \frac{n-\lambda}{p}\right)} \\ &\le C t^{\alpha - \frac{n-\lambda}{p}} |x|^{-\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,|\cdot|^\gamma}} \end{aligned}$$

Thus

$$F_2(x,t) \le Ct^{\alpha-\frac{n-\lambda}{p}}|x|^{-\frac{\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}. \tag{3.2}$$

Therefore from (3.1) and (3.2) we get

$$|I^\alpha f(x)| \le Ct^\alpha Mf(x) + Ct^{\alpha-\frac{n-\lambda}{p}}|x|^{-\frac{\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Minimizing with respect to $t = \left[(Mf(x))^{-1}\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}\right]^{\frac{p}{n-\lambda}}|x|^{-\frac{\gamma}{n-\lambda}}$ we arrive at

$$|I^\alpha f(x)| \le C\left(\frac{Mf(x)}{\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}}\right)^{1-\frac{p\alpha}{n-\lambda}}|x|^{-\frac{\gamma\alpha}{n-\lambda}}.$$

It is obvious that

$$|x|^\gamma = |x|^{\mu-\frac{\gamma\alpha q}{n-\lambda}}.$$

From Theorem 3.2, taking $\varphi(x) = |x|^\gamma$ we get

$$\begin{aligned}\int\limits_{B(x,t)} |I^\alpha f(y)|^q|y|^\mu dy &\le C\|f\|^{q-p}_{L_{p,\lambda,|\cdot|^\gamma}} \int\limits_{B(x,t)} (Mf(y))^p|y|^\gamma dy \\ &\le Ct^\lambda\|f\|^{q-p}_{L_{p,\lambda,|\cdot|^\gamma}}\|f\|^p_{L_{p,\lambda,|\cdot|^\gamma}} \\ &= Ct^\lambda\|f\|^q_{L_{p,\lambda,|\cdot|^\gamma}}.\end{aligned}$$

Therefore $I^\alpha f \in L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ and we obtain

$$\|I^\alpha f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Necessity: Let I^α be bounded from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$, $1 < p < \frac{n-\lambda}{\alpha}$. Define $f_t(x) =: f(tx)$, $t > 0$. Then

$$\begin{aligned}\left(r^{-\lambda}\int_{B(x,r)}|f_t(y)|^p|y|^\gamma dy\right)^{1/p} &= t^{-\frac{n+\gamma}{p}}\left(r^{-\lambda}\int_{B(x,tr)}|f(y)|^p|y|^\gamma dy\right)^{1/p} \\ &= t^{-\frac{n-\lambda+\gamma}{p}}\left((tr)^{-\lambda}\int_{B(x,tr)}|f(y)|^p|y|^\gamma dy\right)^{1/p} \\ &\le t^{-\frac{n-\lambda+\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.\end{aligned}$$

Therefore we get

$$\|f_t\|_{L_{p,\lambda,|\cdot|^\gamma}} \le t^{-\frac{n-\lambda+\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Since

$$I^\alpha f_t(x) = t^{-\alpha}I^\alpha f(tx),$$

we obtain

$$\left(r^{-\lambda}\int_{B(x,r)}|I^{\alpha}f_t(y)|^q\,|y|^{\mu}dy\right)^{1/q}=t^{-\alpha}\left(r^{-\lambda}\int_{B(x,r)}|I^{\alpha}f(ty)|^q\,|y|^{\mu}dy\right)^{1/q}$$
$$=t^{-\alpha-\frac{n-\lambda+\mu}{q}}\left((tr)^{-\lambda}\int_{B(x,tr)}|I^{\alpha}f(y)|^q\,|y|^{\mu}dy\right)^{1/q}$$
$$\le t^{-\alpha-\frac{n-\lambda+\mu}{q}}\,\|I^{\alpha}f\|_{L_{q,\lambda,|\cdot|^{\mu}}}.$$

Therefore we get

$$\|I^{\alpha}f_t\|_{L_{q,\lambda,|\cdot|^{\mu}}}\le t^{-\alpha-\frac{n-\lambda+\mu}{q}}\,\|I^{\alpha}f\|_{L_{q,\lambda,|\cdot|^{\mu}}}.$$

Since the operator I^{α} is bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$, we have

$$\|I^{\alpha}f_t\|_{L_{q,\lambda,|\cdot|^{\mu}}}\le Ct^{-\alpha-\frac{n-\lambda+\mu}{q}+\frac{n-\lambda+\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^{\gamma}}},\tag{3.3}$$

where C depends on p,q,λ,γ,μ and n.
If $\frac{1}{p}>\frac{1}{q}+\frac{\alpha}{n-\lambda}$, from the inequality (3.3), $\|I^{\alpha}f_t\|_{L_{q,\lambda,|\cdot|^{\mu}}}=0$ for all $f\in L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ as $t\to 0$.
If $\frac{1}{p}<\frac{1}{q}+\frac{\alpha}{n-\lambda}$, from the inequality (3.3), $\|I^{\alpha}f_t\|_{L_{q,\lambda,|\cdot|^{\mu}}}=0$ for all $f\in L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ as $t\to\infty$. Therefore $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. □

Remark 3.4. The proof of the sufficiency part of Theorem 3.3 is also given with different methods in [26].

Corollary 3.5. [26] *Let $0<\alpha<n$, $0\le\lambda<n-\alpha$, $1<p<\frac{n-\lambda}{\alpha}$, $-n+\lambda\le\gamma<n(p-1)+\lambda$, $\mu=\frac{q\gamma}{p}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. Then the operator M^{α} is bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$.*

4. Commutators of the Riesz potential operator in the spaces $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In this section we consider commutators of the Riesz potential defined by the following equality

$$[b,I^{\alpha}]f(x)=\int_{\mathbb{R}^n}(b(x)-b(y))|x-y|^{\alpha-n}f(y)dy,\quad 0<\alpha<n.$$

Given a measurable function b the operator $|b,I^{\alpha}|$ is defined by

$$|b,I^{\alpha}|f(x)=\int_{\mathbb{R}^n}|b(x)-b(y)|\,|x-y|^{\alpha-n}\,|f(y)|dy,\quad 0<\alpha<n.$$

The following statement holds:

Lemma 4.1. [9] *Let $1 < s < \infty$ and $b \in BMO(\mathbb{R}^n)$. Then there exists a positive constant C, independent of f and x, such that*

$$M^\sharp([b, I^\alpha]f(x)) \le C\|b\|_{BMO}\left[(M|I^\alpha f(x)|^s)^{\frac{1}{s}} + (M^{s\alpha}|f(x)|^s)^{\frac{1}{s}}\right].$$

Proposition 4.2. *([36], Lemma 3.5) Let $1 < p < \infty$. Then for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$ there exists a positive constant C such that*

$$\left|\int_{\mathbb{R}^n} f(y)g(y)dy\right| \le C\left|\int_{\mathbb{R}^n} M^\sharp f(y)Mg(y)dy\right|.$$

The following lemma is valid.

Lemma 4.3. *Let $1 < p < \infty$, $\varphi \in A_p(\mathbb{R}^n)$. Then there exists a positive constant C, independent of f, such that*

$$\|f\varphi^{\frac{1}{p}}\|_{L_p(\mathbb{R}^n)} \le C\|\varphi^{\frac{1}{p}} M^\sharp f\|_{L_p(\mathbb{R}^n)}.$$

Proof. By (2.1) we have

$$\|f\varphi^{\frac{1}{p}}\|_{L_p(\mathbb{R}^n)} \le C \sup_{\|g\|_{L_{p'}(\mathbb{R}^n)}\le 1}\left|\int_{\mathbb{R}^n} f(y)g(y)\varphi^{\frac{1}{p}}(y)dy\right|.$$

According to Proposition 4.2,

$$\|f\varphi^{\frac{1}{p}}\|_{L_p(\mathbb{R}^n)} \le C \sup_{\|g\|_{L_{p'}(\mathbb{R}^n)}\le 1}\left|\int_{\mathbb{R}^n} M^\sharp f(y)M(g\varphi^{\frac{1}{p}})(y)dy\right|.$$

From Hölder inequality and Theorem 3.1, we obtain

$$\|f\varphi^{\frac{1}{p}}\|_{L_p(\mathbb{R}^n)} \le C \sup_{\|g\|_{L_{p'}(\mathbb{R}^n)}\le 1} \|\varphi^{\frac{1}{p}} M^\sharp f\|_{L_p(\mathbb{R}^n)}\|\varphi^{-\frac{1}{p}} M(g\varphi^{\frac{1}{p}})\|_{L_{p'}(\mathbb{R}^n)}$$

$$\le C \sup_{\|g\|_{Lp'(\mathbb{R}^n)}\le 1} \|\varphi^{\frac{1}{p}} M^\sharp f\|_{L_p(\mathbb{R}^n)}\|g\|_{L_{p'}(\mathbb{R}^n)} \le C\|\varphi^{\frac{1}{p}} M^\sharp f\|_{L_p(\mathbb{R}^n)}. \qquad \square$$

Corollary 4.4. *Let $1 < p < \infty$, $\varphi = \psi|\cdot|^\gamma \in A_p(\mathbb{R}^n)$. Then there exists a positive constant C, independent of f, such that*

$$\|f\psi^{\frac{1}{p}}\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)} \le C\|\psi^{\frac{1}{p}} M^\sharp f\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)}.$$

Lemma 4.5. *Let $1 < p < \infty$, $0 \le \lambda < n$. Then the following inequality holds*

$$\|f\|_{L_{p,\lambda,|\cdot|^\gamma}} \le C\left\|M^\sharp f\right\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Proof. If $0 < \theta < 1$, $\psi(x) = (M\chi_{B(x,r)})^\theta \in A_p(\mathbb{R}^n)$, from Lemma 4.3 we have

$$\|f\|_{L_{p,|\cdot|^\gamma}(B(x,r))} \le \|f\psi^{\frac{1}{p}}\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)} \le C\|\psi^{\frac{1}{p}} M^\sharp f\|_{L_{p,|\cdot|^\gamma}(\mathbb{R}^n)} \le C\|M^\sharp f\|_{L_{p,|\cdot|^\gamma}(B(x,r))}.$$

Therefore we get

$$\|f\|_{L_{p,\lambda,|\cdot|^\gamma}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}}\|f\|_{L_{p,|\cdot|^\gamma}(B(x,t))}$$

$$\le C \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}}\|M^\sharp f\|_{L_{p,|\cdot|^\gamma}(B(x,r))} = C\|M^\sharp f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Thus the lemma has been proved. $\square$

In the following theorem we give the necessary and sufficient conditions for the boundedness of the commutator $[b, I^\alpha]$ from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$.

Theorem 4.6. *Let* $0<\alpha<n$, $0\le\lambda<n-\alpha$, $1<p<\frac{n-\lambda}{\alpha}$, $-n+\lambda\le\gamma<n(p-1)+\lambda$, $\mu=\frac{q\gamma}{p}$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. *Then the commutator* $[b, I^\alpha]$ *is bounded from* $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ *to* $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$ *if and only if* $b\in BMO$.

Proof. Let $f\in L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ and $b\in BMO(\mathbb{R}^n)$. From Lemma 4.5, we have

$$\|[b,I^\alpha]f\|_{L_{q,\lambda,|\cdot|^\mu}}\le C_1\|M^\sharp([b,I^\alpha]f)\|_{L_{q,\lambda,|\cdot|^\mu}}.$$

From Lemma 4.1, we get

$$\begin{aligned}\|M^\sharp([b,I^\alpha]f)\|_{L_{q,\lambda,|\cdot|^\mu}}&\le C_2\|b\|_{BMO}\left\|(M|I^\alpha f|^s)^{\frac{1}{s}}+(M^{\alpha s}|f|^s)^{\frac{1}{s}}\right\|_{L_{q,\lambda,|\cdot|^\mu}}\\ &\le C_3\|b\|_{BMO}\left[\left\|(M|I^\alpha f|^s)^{\frac{1}{s}}\right\|_{L_{q,\lambda,|\cdot|^\mu}}+\left\|(M^{\alpha s}|f|^s)^{\frac{1}{s}}\right\|_{L_{q,\lambda,|\cdot|^\mu}}\right].\end{aligned}$$

From Theorem 3.2 and Theorem 3.3, we have

$$\begin{aligned}\left\|(M|I^\alpha f|^s)^{\frac{1}{s}}\right\|_{L_{q,\lambda,|\cdot|^\mu}}&=\|M|I^\alpha f|^s\|^{\frac{1}{s}}_{L_{\frac{q}{s},\lambda,|\cdot|^\mu}}\\ \le C\,\||I^\alpha f|^s\|^{\frac{1}{s}}_{L_{\frac{q}{s},\lambda,|\cdot|^\mu}}&=C\,\|I^\alpha f\|_{L_{q,\lambda,|\cdot|^\mu}}\le C\,\|f\|_{L_{p,\lambda,|\cdot|^\mu}}.\end{aligned}$$

Similarly it can be shown that

$$\left\|(M^{\alpha s}|f|^s)^{\frac{1}{s}}\right\|_{L_{q,\lambda,|\cdot|^\mu}}\le C\,\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Therefore we obtain

$$\|[b,I^\alpha]f\|_{L_{q,\lambda,|\cdot|^\mu}}\le C_2\|b\|_{BMO}\,\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

$(i)\Rightarrow(ii)$ Now, let us prove the "only if" part. Let $[b, I^\alpha]$ be bounded from $L_{p,\lambda,|\cdot|^\gamma}$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$, $1<p<\frac{n-\lambda}{\alpha}$. Now we consider $f=\chi_{B(x,r)}$. It is easy to compute that

$$\begin{aligned}\left\|\chi_{B(x,r)}\right\|_{L_{p,\lambda,|\cdot|^\gamma}}&\approx\sup_{t>0,\,x\in\mathbb{R}^n}\left(t^{-\lambda}\int\limits_{B(y,t)}\chi_{B(x,r)}(y)|y|^\gamma dy\right)^{1/p}\\ &\approx\sup_{B(y,t)\subset B(x,r)}\left(t^{-\lambda}\int\limits_{B(y,t)}|y|^\gamma dy\right)^{1/p}\approx r^{\frac{n-\lambda+\gamma}{p}}.\end{aligned}$$

Then

$$\frac{1}{|B(x,t)|}\int_{B(x,t)}|b(z)-b_{B(x,t)}|dz$$

$$=\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\left|b(z)-\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}b(y)dy\right|dz$$

$$\leq\frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}}\int\limits_{B(x,t)}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\left|\int\limits_{B(x,t)}(b(z)-b(y))\,dy\right|dz$$

$$\leq\frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}}\int\limits_{B(x,t)}\left|\int\limits_{B(x,t)}(b(z)-b(y))\,|x-y|^{\alpha-n}dy\right|dz$$

$$\leq\frac{1}{|B(x,t)|^{1+\frac{\alpha}{n}}}\int\limits_{B(x,t)}\left|[b,I^{\alpha}]\chi_{B(x,t)}(z)\right|dz$$

$$\leq Ct^{-n-\alpha+\lambda}\|[b,I^{\alpha}]\chi_{B(x,t)}\|_{L_{q,\lambda,|\cdot|^{\mu}}}\|\chi_{B(x,t)}\|_{L_{q',\lambda,|\cdot|^{\frac{\mu}{1-q}}}}$$

$$\leq Ct^{-n-\alpha+\frac{n-\lambda+\gamma}{p}+n-\frac{n-\lambda+\mu}{q}}\leq C.$$

Hence we get

$$|B(x,t)|^{-1}\int_{B(x,t)}|b(y)-b_{B(x,t)}|dy\leq C,$$

which shows that $b\in BMO(\mathbb{R}^n)$.
Thus the theorem has been proved. □

Theorem 4.7. *Let $0<\alpha<n$, $0\leq\lambda<n-\alpha$, $1<p<\frac{n-\lambda}{\alpha}$, $-n+\lambda\leq\gamma<n(p-1)+\lambda$, $\mu=\frac{q\gamma}{p}$ and $b\in BMO$. Then the commutator $|b,I^{\alpha}|$ is bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$ if and only if $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$.*

Proof. 1) The sufficiency follows from Theorem 4.6.
Necessity: Let $1<p<\frac{n-\lambda}{\alpha}$ and $|b,I^{\alpha}|$ be bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$. Define $f_t(x)=:f(tx)$, $t>0$. Then

$$\left(r^{-\lambda}\int_{B(x,r)}|f_t(y)|^p|y|^{\gamma}dy\right)^{1/p}=t^{-\frac{n+\gamma}{p}}\left(r^{-\lambda}\int_{B(x,tr)}|f(y)|^p|y|^{\gamma}dy\right)^{1/p}$$

$$=t^{-\frac{n-\lambda+\gamma}{p}}\left((tr)^{-\lambda}\int_{B(x,tr)}|f(y)|^p|y|^{\gamma}dy\right)^{1/p}$$

$$\leq t^{-\frac{n-\lambda+\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^{\gamma}}}.$$

Therefore we get

$$\|f_t\|_{L_{p,\lambda,|\cdot|^{\gamma}}}\leq t^{-\frac{n-\lambda+\gamma}{p}}\|f\|_{L_{p,\lambda,|\cdot|^{\gamma}}}.$$

Since

$$|b, I^{\alpha}| f_t(x) = t^{-\alpha} |b, I^{\alpha}| f(tx),$$

we obtain

$$\begin{aligned}
&\left(r^{-\lambda} \int_{B(x,r)} \left[||b, I^{\alpha}| f_t| \right]^q (y) |y|^{\mu} dy \right)^{1/q} \\
&= t^{-\alpha} \left(r^{-\lambda} \int_{B(x,r)} \left[||b, I^{\alpha}| f| \right]^q (ty) |y|^{\mu} dy \right)^{1/q} \\
&= t^{-\alpha - \frac{n-\lambda+\mu}{q}} \left((tr)^{-\lambda} \int_{B(x,tr)} \left[||b, I^{\alpha}| f| \right]^q (y) |y|^{\mu} dy \right)^{1/q} \\
&\le t^{-\alpha - \frac{n-\lambda+\mu}{q}} \left\| |b, I^{\alpha}| f \right\|_{L_{q,\lambda,|\cdot|^{\mu}}} .
\end{aligned}$$

Therefore we get

$$\left\| |b, I^{\alpha}| f_t \right\|_{L_{q,\lambda,|\cdot|^{\mu}}} \le t^{-\alpha - \frac{n-\lambda+\mu}{q}} \left\| |b, I^{\alpha}| f \right\|_{L_{q,\lambda,|\cdot|^{\mu}}} .$$

Since the operator $|b, I^{\alpha}|$ is bounded from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$, we have

$$\left\| |b, I^{\alpha}| f_t \right\|_{L_{q,\lambda,|\cdot|^{\mu}}} \le C t^{-\alpha - \frac{n-\lambda+\mu}{q} + \frac{n-\lambda+\gamma}{p}} \|b\|_{BMO} \|f\|_{L_{p,\lambda,|\cdot|^{\gamma}}}, \tag{4.1}$$

where C depends on p,q,λ,γ,μ and n.
If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, from the inequality (4.1), $\left\| |b, I^{\alpha}| f_t \right\|_{L_{q,\lambda,|\cdot|^{\mu}}} = 0$ for all $f \in L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ as $t \to 0$.
If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n-\lambda}$, from the inequality (4.1), $\left\| |b, I^{\alpha}| f_t \right\|_{L_{q,\lambda,|\cdot|^{\mu}}} = 0$ for all $f \in L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ as $t \to \infty$. Therefore $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. □

The following theorem gives the conditions for the boundedness of the commutator $|b, I^{\alpha}|$ from $B^s_{p,\theta,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $B^s_{q,\theta,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$.

Theorem 4.8. *Let $0 < \alpha < n$, $0 \le \lambda < n-\alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n+\lambda \le \gamma < n(p-1)+\lambda$, $\mu = \frac{q\gamma}{p}$, $0 < s < 1$, $1 \le \theta \le \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ and $b \in BMO(\mathbb{R}^n)$. Then the commutator $|b, I^{\alpha}|$ is bounded from $B^s_{p,\theta,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $B^s_{q,\theta,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$.*

Proof. From the definition of the Besov-Morrey type spaces it suffices to show that

$$\left\| |b, I^{\alpha}| f(x - \cdot) - |b, I^{\alpha}| f(\cdot) \right\|_{L_{p,\lambda,|\cdot|^{\gamma}}} \le C \|b\|_{BMO} \|f(x - \cdot) - f(\cdot)\|_{L_{p,\lambda,|\cdot|^{\gamma}}} .$$

Hence we have

$$|[b, I^{\alpha}] f(x - \cdot) - |b, I^{\alpha}| f| \le |b, I^{\alpha}|(|f(x - \cdot) - f|).$$

Taking $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ norm of both sides of the above inequality, from the boundedness of $|b, I^{\alpha}|$ from $L_{p,\lambda,|\cdot|^{\gamma}}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^{\mu}}(\mathbb{R}^n)$, we obtain the desired result. Thus Theorem 4.8 has been proved. □

5. The weighted Morrey estimates for the operators $V^s(-\Delta+V)^{-\beta}$ and $V^s\nabla(-\Delta+V)^{-\beta}$

In this section we consider the Schrödinger operator $-\Delta+V$ on $\mathbb{R}^n$, where the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for some $q_1 \geq n$. We obtain weighted Morrey $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ estimates for the operators $V^s(-\Delta+V)^{-\beta}$ and $V^s\nabla(-\Delta+V)^{-\beta}$.

Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class have been studied by many authors (see [10, 32, 40]). Shen [32] studied the Schrödinger operator $-\Delta+V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \geq n/2$ and he proved the L_p boundedness of the operators $(-\Delta+V)^{is}$, $\nabla^2(-\Delta+V)^{-1}$, $\nabla(-\Delta+V)^{-\frac{1}{2}}$ and $\nabla(-\Delta+V)^{-1}$. Kurata and Sugano generalized Shens' results to uniformly elliptic operators in [18]. Sugano [38] also extended some results of Shen to the operator $V^s(-\Delta+V)^{-\beta}$, $0 \leq s \leq \beta \leq 1$ and $V^s\nabla(-\Delta+V)^{-\beta}$, $0 \leq s \leq \frac{1}{2} \leq \beta \leq 1$ and $\beta - s \geq \frac{1}{2}$. Later, Lu [21] and Li [19] investigated the Schrödinger operators in a more general setting.
We investigate the weighted Morrey $L_{p,\lambda,|\cdot|^\gamma} - L_{q,\lambda,|\cdot|^\mu}$ boundedness of the operators

$$T_1 = V^s(-\Delta+V)^{-\beta},\ 0 \leq s \leq \beta \leq 1,$$

$$T_2 = V^s\nabla(-\Delta+V)^{-\beta},\ 0 \leq s \leq \frac{1}{2} \leq \beta \leq 1,\ \beta - s \geq \frac{1}{2}.$$

Note that the operators $V(-\Delta+V)^{-1}$ and $V^{\frac{1}{2}}\nabla(-\Delta+V)^{-1}$ in [19] are the special case of T_1 and T_2, respectively.

It is worth pointing out that we need to establish pointwise estimates for T_1, T_2 and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^n$ in [19]. And we give the Morrey estimates by using $L_{p,\lambda,|\cdot|^\gamma} - L_{q,\lambda,|\cdot|^\mu}$ boundedness of the fractional maximal operators.

Definition 5.1. 1) A nonnegative locally L_p integrable function V on $\mathbb{R}^n$ is said to belong to the reverse Hölder class B_p $(1 < p < \infty)$ if there exists a positive constant C such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^p dx\right)^{\frac{1}{p}} \leq \frac{C}{|B|}\int_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

2) Let $V \geq 0$. We say $V \in B_\infty$, if there exists a positive constant C such that the inequality

$$\|V\|_{L_\infty(B)} \leq \frac{C}{|B|}\int_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

Clearly, $B_\infty \subset B_p$ for $1 < p < \infty$. But it is important that the B_p class has a property of "self-improvement"; that is, if $V \in B_p$, then $V \in B_{p+\varepsilon}$ for some $\varepsilon > 0$ (see [19]).

The following two pointwise estimates for T_1 and T_2 were proved in [40] with the potential $V \in B_\infty$.

Theorem A. *Suppose $V \in B_\infty$ and $0 \le s \le \beta \le 1$. Then there exists a positive constant C such that*

$$|T_1 f(x)| \le C M^\alpha f(x),\ \ f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - s)$.

Theorem B. *Suppose $V \in B_\infty$, $0 \le s \le \frac{1}{2} \le \beta \le 1$ and $\beta - s \ge \frac{1}{2}$. Then there exists a positive constant C such that*

$$|T_2 f(x)| \le C M^\alpha f(x),\ \ f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - s) - 1$.

Note that the similar estimates for the adjoint operators T_1^* and T_2^* with the potential $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$ are also valid (see [20]).

Theorem C. *Suppose $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$, $0 \le s \le \beta \le 1$ and let $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$. Then there exists a positive constant C such that*

$$|T_1^* f(x)| \le C \left(M_{\alpha q_2}\left(|f|^{q_2}\right)(x)\right)^{\frac{1}{q_2}},\ \ f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - s)$.

Theorem D. *Suppose $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$, $0 \le s \le \frac{1}{2} \le \beta \le 1$ and $\beta - s \ge \frac{1}{2}$. And let*

$$\frac{1}{q_1} = \begin{cases} 1 - \frac{s}{q_1}, & \text{if } q_1 > n, \\ 1 - \frac{\alpha+1}{q_1} + \frac{1}{n}, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

Then there exists a positive constant C such that

$$|T_2^* f(x)| \le C \left(M_{\alpha q_2}\left(|f|^{q_2}\right)(x)\right)^{\frac{1}{q_2}},\ \ f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - s) - 1$.

The above theorems will yield the weighted Morrey estimates for T_1 and T_2.

Corollary 5.2. *Assume that $V \in B_\infty$, and $0 \le s \le \beta \le 1$. Let $1 < p < \frac{n}{s}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ and $0 \le \lambda < n$, where $\alpha = 2(\beta - s) < n$.*

Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a positive constant C such that

$$\|T_1 f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Corollary 5.3. *Let $V \in B_\infty$, $0 \le s \le \frac{1}{2} \le \beta \le 1$, $\beta - s \ge \frac{1}{2}$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$ and $0 \le \lambda < n$, where $\alpha = 2(\beta - s) - 1 < n$.*

Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a positive constant C such that

$$\|T_2 f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Corollary 5.4. *Assume that $V \in B_{q_1}$ for $q_1 > \frac{n}{2}$, and $0 \le s \le \beta \le 1$.*

Let $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$, $1 < p < \frac{1}{\frac{\alpha}{q_1} + \frac{\alpha}{n}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\frac{n}{q_2} - \lambda}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$ and $0 \le \lambda < nq_2$, where $\alpha = 2(\beta - s) < n$.

Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a positive constant C such that

$$\|T_1 f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

Corollary 5.5. *Assume that $V \in B_{q_1}$ for $q_1 > \frac{n}{2}$, and*

$$\begin{cases} 0 \le s \le \frac{1}{2} \le \beta \le 1, & \text{if } q_1 > n, \\ 0 \le s \le \frac{1}{2} < \beta \le 1, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

Let $\alpha = 2(\beta - s) - 1 < n$ and $\beta - s \ge \frac{1}{2}$, and let $1 < p < \frac{1}{\frac{\alpha}{q_1} + \frac{\alpha}{n}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\frac{n}{q_2} - \lambda}$, $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$, $-n + \lambda \le \gamma < n(p-1) + \lambda$, $\mu = \frac{q\gamma}{p}$ and $0 \le \lambda < nq_2$, where

$$\frac{1}{p_1} = \begin{cases} \frac{\alpha}{q_1}, & \text{if } q_1 > n, \\ \frac{\alpha+1}{q_1} + \frac{1}{n}, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

Then for any $f \in C_0^\infty(\mathbb{R}^n)$ there exists a positive constant C such that

$$\|T_2 f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}.$$

6. Some applications

The theorems of the Section 3 can be applied to various operators which are estimated from above by Riesz potentials. Now we give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \le \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{6.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I^α. (See, for example, Chapter 5 in [36].)

Theorem 6.1. *Let $0 < \alpha < n$, $0 \le \lambda < n - \alpha$, $1 < p < \frac{n-\lambda}{\alpha}$, $-n+\lambda \le \gamma < n(p-1)+\lambda$, $\mu = \frac{q\gamma}{p}$ and condition (6.1) be satisfied. Then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is sufficient for the boundedness of $L^{-\alpha/2}$ from $L_{p,\lambda,|\cdot|^\gamma}(\mathbb{R}^n)$ to $L_{q,\lambda,|\cdot|^\mu}(\mathbb{R}^n)$.*

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (6.1), it follows that

$$|L^{-\alpha/2} f(x)| \le C I^\alpha |f|(x)$$

for all $x \in \mathbb{R}^n$ (see [7]). Therefore from the aforementioned theorems we have

$$\|L^{-\alpha/2} f\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|I^\alpha |f|\|_{L_{q,\lambda,|\cdot|^\mu}} \le C\|f\|_{L_{p,\lambda,|\cdot|^\gamma}}. \qquad \square$$

Large classes of differential operators satisfies condition (6.1). Now we investigate two of them:

(i) Let us consider a magnetic potential $\vec{a}$, i. e., a real-valued vector potential $\vec{a} = (a_1, a_2, \dots, a_n)$, and an electric potential V. Assume that for any $k = 1, 2, \dots, n$, $a_k \in L_2^{loc}$ and $0 \le V \in L_1^{loc}$. The magnetic Schrödinger operator, L, is defined by

$$L = -(\nabla - i\vec{a})^2 + V(x).$$

From the well-known diamagnetic inequality (see [35], Theorem 2.3) we have the following pointwise estimate. For any $t > 0$ and $f \in L_2$,

$$|e^{-tL}f| \leq e^{-t\triangle}|f|,$$

which implies that the semigroup e^{-tL} has the kernel $p_t(x,y)$ that satisfies upper bound (6.1).

(ii) Let $A = (a_{ij}(x))_{1\leq i,j\leq n}$ be an $n \times n$ matrix with complex-valued entries $a_{ij} \in L_\infty$ satisfying

$$\mathrm{Re} \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \geq \lambda|\zeta|^2$$

for all $x \in \mathbb{R}^n, \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n$ and some $\lambda > 0$. Consider the divergence form operator

$$Lf \equiv -\mathrm{div}(A\nabla f),$$

which is interpreted in the usual weak sense via the appropriate sesquilinear form.

It is known that the Gaussian bound (6.1) for the kernel of e^{-tL} holds when A has real-valued entries (see, for example, [3]), or when $n = 1$, 2 in the case of complex-valued entries (see [4, Chapter 1]).

Finally we note that under the appropriate assumptions (see [23]; [36], Chapter 5; [4], pp. 58-59) one can obtain results similar to Theorem 6.1 for a homogeneous elliptic operator L in L_2 of order $2m$ in the divergence form

$$Lf = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \left(a_{\alpha\beta} D^\beta f\right).$$

In this case estimate (6.1) should be replaced by

$$|p_t(x,y)| \leq \frac{c_3}{t^{n/2m}} e^{-c_4\left(\frac{|x-y|}{t^{1/(2m)}}\right)^{2m/(2m-1)}}$$

for all $t > 0$ and all x, $y \in \mathbb{R}^n$.

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Expansion-compression fixed point theorem of Leggett-Williams type for the sum of two operators and applications for some classes of BVPs

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Abstract. The purpose of this work is to establish an extension of a Leggett-Williams type expansion-compression fixed point theorem for a sum of two operators. As illustration, our approach is applied to prove the existence of non trivial nonnegative solutions for two-point BVPs and three-point BVPs.

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Keywords: Fixed point index, cone, sum of operators, expansion, compression, nonnegative solution.

1. Introduction

For applicability reasons, we often search for existence and localization of positive fixed points which may represent positive solutions for various nonlinear problems posed in a Banach space.

One of the main results in fixed point theory is the cone expansion and compression theorem proved by Krasnosel'skii in 1964 (see, e.g., [8, 14, 15]). It represents a powerful existence tool in studying operator equations and showing existence of nonnegative solutions to various boundary value problems. Then, many researchers have been intersted in the extension of the above theorem in various directions (see, e.g., [1, 6, 7, 9, 16, 18, 19]).

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Throughout this paper, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$. Let Ψ and δ be nonnegative continuous functionals on $\mathcal{P}$; then, for positive real numbers a and b, we define the sets:

$$\mathcal{P}(\Psi, b) = \{x \in \mathcal{P} : \Psi(x) \leq b\},$$

and

$$\mathcal{P}(\Psi, \delta, a, b) = \{x \in \mathcal{P} : a \leq \Psi(x) \text{ and } \delta(x) \leq b\}.$$

Krasnosel'skii type compression-expansion fixed point theorems gives us fixed points localized in a conical shell of the form $\{x \in \mathcal{P} : a \leq \|x\| \leq b\}$, where $a, b \in (0, \infty)$, while with the Leggett-Williams type they are localized in a conical shell of the form $\mathcal{P}(\alpha, \beta, a, b)$, where α is a concave nonnegative functional and β a convex nonnegative functional.

The original Leggett-Williams fixed point theorem (see [17, Theorem 3.2]) discuss the existence of at least one fixed point in a conical shell of the form $\{x \in \mathcal{P} : a \leq \alpha(x) \text{ and } \|x\| \leq b\}$, where $a, b \in (0, +\infty)$. Noting that this result has been widely extended in many directions, (see, e.g., [2, 3, 10, 11, 17]). In [2, Theorem 4.1], Anderson et al. have discussed the existence of at least one solution in $\mathcal{P}(\beta, \alpha, r, R)$ or in $\mathcal{P}(\alpha, \beta, r, R)$ for the nonlinear operational equation

$$Ax = x \tag{1.1}$$

where A is a completely continuous nonlinear map acting in $\mathcal{P}$, α is a nonnegative continuous concave functional on $\mathcal{P}$ and β is a nonnegative continuous convex functional on $\mathcal{P}$. In this result, the authors have used techniques similar to those of Leggett-Williams that require only subsets of both boundaries to be mapped inward and outward, respectively. They thus provide more general results than those obtained by using the Krasnosel'skii's cone compression and expansion one. Noting that, in [2], the authors provided more general results than those obtained in [1, 4, 11, 12, 17, 19] for completely continuous mappings.

In this paper, we use the fixed point index theory developed in [6] to generalize the main result of [2] for the sum $T+F$ where T is an expansive mapping with constant $h > 1$ and $I - F$ is a k-set contraction with $k < h$. The concept of set contraction is related to that of the Kuratowski measure of noncompactness (see [5, 13]).

The paper is organized as follows. In Section 2 we give some auxiliary results used for the proof of the main result. In Section 3, we present our main result. As application, the existence of non trivial nonnegative solution for two-point BVPs and three-point BVPs are considered in Section 4.

2. Auxiliary results

Let Ω be any subset of $\mathcal{P}$, and U be a bounded open subset of $\mathcal{P}$.
Consider $T : \Omega \to E$ an expansive mapping with constant $h > 1$, and $I - F : \overline{U} \to E$ a k-set contraction with $0 \leq k < h$. So, the operator T^{-1} is $\frac{1}{h}$-Lipschtzian on $T(\Omega)$. Assume that

$$(I - F)\left(\overline{U}\right) \subset T(\Omega).$$

Then the mapping $T^{-1}(I-F):\overline{U}\to\mathcal{P}$ is a strict $\frac{k}{h}$-set contraction.
Hence, by Djebali et al. in [6], the fixed point index of the sum $T+F$ on $U\cap\Omega$ with respect to the cone $\mathcal{P}$, noted $i_*\left(T+F,U\cap\Omega,\mathcal{P}\right)$, is well defined.

The proof of our theorical result invokes the following main properties of the fixed point index i_*.

(i). *(Normalization)* If $U=\mathcal{P}(\Psi,R),\ 0\in\Omega$, and $(I-F)x=z_0$ for all $x\in\overline{U}$, where $z_0\in\mathcal{P},\Psi$ is a nonnegative continuous functionals on $\mathcal{P}$ satisfying $\Psi(x)\le\|x\|$ for all $x\in\mathcal{P}$ and $\|z_0-T0\|<hR,$ then

$$i_*\left(T+F,U\cap\Omega,\mathcal{P}\right)=1.$$

(ii). *(Additivity)* For any pair of disjoint open subsets $U_1,U_2\subset U$ such that $T+F$ has no fixed point on $(\overline{U}\setminus(U_1\cup U_2))\cap\Omega$, we have

$$i_*(T+F,U\cap\Omega,\mathcal{P})=i_*(T+F,U_1\cap\Omega,\mathcal{P})+i_*(T+F,U_2\cap\Omega,\mathcal{P}).$$

(iii). *(Homotopy invariance)* The fixed point index $i_*(T+H(.,t),U\cap\Omega,\mathcal{P})$ does not depend on the parameter $t\in[0,1]$, where

- **(a).** $(I-H):[0,1]\times\overline{U}\to E$ is continuous and $H(t,x)$ is uniformly continuous in t with respect to $x\in\overline{U}$,
- **(b).** $(I-H)([0,1]\times\overline{U})\subset T(\Omega)$,
- **(c).** $(I-H(t,.)):\overline{U}\to E$ is a ℓ-set contraction with $0\le\ell<h$ for all $t\in[0,1]$,
- **(d).** $Tx+H(t,x)\ne x$ for all $t\in[0,1]$ and $x\in\partial U\cap\Omega$.

(iv). *(Solvability)* If $i_*(T+F,U\cap\Omega,\mathcal{P})\ne 0,$ then $T+F$ has a fixed point in $U\cap\Omega$.

For proof and more details see [6, Theorem 3.1].

3. Main result

Let Ω be a subset of $\mathcal{P}$ such that $0\in\Omega$. We consider the nonlinear equation

$$Tx+Fx=x, \tag{3.1}$$

where $T:\Omega\to E$ an expansive mapping with constant $h>1$, and $I-F:\mathcal{P}\to E$ a k-set contraction with $0\le k<h$.

In what follows, we will establish an extension of [2, Theorem 4.1], which guarantees the existence of at least one non trivial nonnegative solution of equation (3.1).

Theorem 3.1. *Let α be a nonnegative continuous concave functional on $\mathcal{P}$ and β be a nonnegative continuous convex functional on $\mathcal{P}$ with $\beta(x)\le\|x\|$ for all $x\in\mathcal{P}$. Assume that there exists nonnegative numbers a, b, c, d and $z_0\in\mathcal{P}$ such that $\|T0\|<h\min(b,d)$ and $\alpha(T^{-1}z_0)>\max(a,c)$.*
Suppose that:

($A1$). *if $x\in\mathcal{P}$ with $\beta(x)=b$, then $\alpha(Tx+x)\ge a$;*
($A2$). *if $x\in\mathcal{P}$ with $\beta(x)=b$ and $\alpha(x)\ge a$, then $\beta(Tx+Fx)<b$ and $\beta(Tx+x)\le b$;*
($A3$). *if $x\in\mathcal{P}$ with $\beta(x)=b$ and $\alpha(Tx+Fx)<a$, then $\beta(Tx+Fx)<b$ and $\beta(Tx+x)\le b$;*
($A4$). *if $x\in\mathcal{P}$ with $\alpha(x)=c$, then $\beta(Tx+x-z_0)\le d$;*

($A5$). *if* $x \in \mathcal{P}$ *with* $\alpha(x) = c$ *and* $\beta(x) \le d$, *then* $\alpha(Tx + Fx) > c$ *and* $\alpha(Tx + x - z_0) \ge c$;

($A6$). *if* $x \in \mathcal{P}$ *with* $\alpha(x) = c$ *and* $\beta(Tx + Fx) > d$, *then* $\alpha(Tx + Fx) > c$ *and* $\alpha(Tx + x - z_0) \ge c$.

Then,

1. *(Expansive form)* $T + F$ *has a fixed point* x^* *in* $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$ *if*

   ($H1$). $a < c$, $b < d$, $\{x \in \mathcal{P} : b < \beta(x)$ *and* $\alpha(x) < c\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$, $\mathcal{P}(\beta, b) \cap \Omega \neq \emptyset$ *and* $\mathcal{P}(\alpha, c)$ *is bounded and*

$$t(I - F)(\mathcal{P}(\beta, b)) \subset T(\Omega), \text{ for all } t \in [0, 1], \tag{3.2}$$

$$t(I - F)(\mathcal{P}(\alpha, c)) + (1 - t)z_0 \subset T(\Omega), \text{ for all } t \in [0, 1]. \tag{3.3}$$

2. *(Compressive form)* $T + F$ *has a fixed point* x^* *in* $\mathcal{P}(\alpha, \beta, a, d) \cap \Omega$ *if*

   ($H2$). $c < a$, $d < b$, $\{x \in \mathcal{P} : a < \alpha(x)$ *and* $\beta(x) < d\} \cap \Omega \neq \emptyset$, $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$, $\mathcal{P}(\alpha, a) \cap \Omega \neq \emptyset$, *and* $\mathcal{P}(\beta, d)$ *is bounded and*

$$t(I - F)(\mathcal{P}(\beta, d)) \subset T(\Omega), \text{ for all } t \in [0, 1], \tag{3.4}$$

$$t(I - F)(\mathcal{P}(\alpha, a)) + (1 - t)z_0 \subset T(\Omega), \text{ for all } t \in [0, 1]. \tag{3.5}$$

Proof. We will prove the expansion form. The proof of the compression form is nearly identical.
If we list

$$U = \{x \in \mathcal{P} : \beta(x) < b\}, \tag{3.6}$$

$$V = \{x \in \mathcal{P} : \alpha(x) < c\}, \tag{3.7}$$

then, the interior of $V - U$ is given by

$$W = (V - U)^o = \{x \in \mathcal{P} : \ b < \beta(x) \text{ and } \alpha(x) < c\}.$$

Thus U, V and W are bounded (they are subsets of V which is bounded by condition ($H1$)), not empty (by condition ($H1$)) and open subsets of $\mathcal{P}$. To prove the existence of a fixed point for the sum $T + F$ in $\mathcal{P}(\beta, \alpha, b, c) \cap \Omega$, it is enough for us to show that $i_*(T + F, W \cap \Omega, \mathcal{P}) \neq 0$ since W is the interior of $\mathcal{P}(\beta, \alpha, b, c)$.

Claim 1. $Tx + Fx \neq x$ for all $x \in \partial U \cap \Omega$.
Let $x_0 \in \partial U \cap \Omega$, then $\beta(x_0) = b$. Suppose that $x_0 = Tx_0 + Fx_0$, then $\beta(Tx_0 + Fx_0) = b$. If $\alpha(x_0) \ge a$, then $\beta(Tx_0 + Fx_0) < b$ by condition ($A2$), and if $\alpha(x_0) < a$, then $\alpha(Tx_0 + Fx_0) < a$, then $\beta(Tx_0 + Fx_0) < b$ by condition ($A3$). This is a contradiction. Thus we have $Tx + Fx \neq x$ for all $x \in \partial U \cap \Omega$.

Claim 2. $Tx + Fx \neq x$ for all $x \in \partial V \cap \Omega$.
Let $x_1 \in \partial V \cap \Omega$, then $\alpha(x_1) = c$. Suppose that $x_1 = Tx_1 + Fx_1$, then $\alpha(Tx_1 + Fx_1) = c$. If $\beta(x_1) \le d$, then $\alpha(Tx_1 + Fx_1) > c$ by condition ($A5$), and if $\beta(x_1) > d$, then $\beta(Tx_1 + Fx_1) > d$, then $\alpha(Tx_1 + Fx_1) > c$ by condition ($A6$).
This is a contradiction. Thus we have $Tx + Fx \neq x$ for all $x \in \partial V \cap \Omega$.

Claim 3. Let $H_1 : [0, 1] \times \overline{U} \to E$ be defined by

$$H_1(t, x) = tFx + (1 - t)\, x.$$

Clearly H_1 is uniformly continuous in t with respect to $x \in \overline{U}$ and $(I - H_1)$ is continuous, and from (3.2) we easily see that $(I - H_1([0,1] \times \overline{U})) \subset T(\Omega)$. Moreover $(I - H_1(t,.)) : \overline{U} \to E$ is a k-set contraction for all $t \in [0,1]$ and $Tx + H_1(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U \cap \Omega$. Otherwise, there would exists $(t_2, x_2) \in [0,1] \times \partial U \cap \Omega$ such that $Tx_2 + H_1(t_2, x_2) = x_2$. Since $x_2 \in \partial U$, $\beta(x_2) = b$. Either $\alpha(Tx_2 + Fx_2) < a$ or $\alpha(Tx_2 + Fx_2) \geq a$.

Case (1): If $\alpha(Tx_2 + Fx_2) < a$, the convexity of β and the condition $(A3)$ lead

$$\begin{aligned} b = \beta(x_2) &= \beta\left(Tx_2 + H_1(t_2, x_2)\right) \\ &= \beta\left(Tx_2 + t_2 Fx_2 + (1 - t_2)x_2\right) \\ &\leq t_2 \beta\left(Tx_2 + Fx_2\right) + (1 - t_2)\, \beta(Tx_2 + x_2) \\ &< b, \end{aligned}$$

which is a contradiction.

Case (2): If $\alpha(Tx_2 + Fx_2) \geq a$, from the concavity of α and the condition $(A1)$, we obtain $\alpha(x_2) \geq a$. Indeed,

$$\begin{aligned} \alpha(x_2) &= \alpha\left(Tx_2 + H_1(t_2, x_2)\right) \\ &\geq t_2 \alpha\left(Tx_2 + Fx_2\right) + (1 - t_2)\, \alpha(Tx_2 + x_2) \\ &\geq a, \end{aligned}$$

and thus by condition $(A2)$, we have $\beta(Tx_2 + Fx_2) < b$ and $\beta(Tx_2 + x_2) < b$, which is the same contradiction we arrived at in the previous case.

Being $T^{-1}0 \in U$ (we have $h\beta(T^{-1}0) \leq h\|T^{-1}0\| \leq \|T0\| < hb$), the homotopy invariance property (iii) and the normality property (i) of the fixed point index i_* lead

$$i_*(T + F, U \cap \Omega, \mathcal{P}) = i_*(T + I, U \cap \Omega, \mathcal{P}) = 1.$$

Claim 4. Let $H_2 : [0,1] \times \overline{V} \to E$ be defined by

$$H_2(t,x) = tFx + (1 - t)(x - z_0).$$

Clearly H_2 is uniformly continuous in t with respect to $x \in \overline{V}$ and $(I - H_2)$ is continuous, and from (3.3) we easily see that $(I - H_2([0,1] \times \overline{V})) \subset T(\Omega)$. Moreover $I - H_2(t,.) : \overline{V} \to E$ is a k-set contraction for all $t \in [0,1]$ and $Tx + H_2(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial V \cap \Omega$. Otherwise, there would exists $(t_3, x_3) \in [0,1] \times \partial V \cap \Omega$ such that $H_2(t_3, x_3) = x_3$. Since $x_3 \in \partial V$ we have that $\alpha(x_3) = c$. Either $\beta(Tx_3 + Fx_3) \leq d$ or $\beta(Tx_3 + Fx_3) > d$.

Case (1): If $\beta(Tx_3 + Fx_3) > d$. the concavity of α and the condition $(A6)$ lead

$$\begin{aligned} c = \alpha(x_3) &= \alpha(Tx_3 + H_2(t_3, x_3)) \\ &= \alpha(Tx_3 + t_3 Fx_3 + (1 - t_3)(x_3 - z_0)) \\ &\geq t_3 \alpha(Tx_3 + Fx_3) + t_3 \alpha(Tx_3 + x_3 - z_0) \\ &> c. \end{aligned}$$

This is a contradiction.

Case (2): If $\beta(Tx_3 + Fx_3) \le d$, from the convexity of β and the condition $(A4)$, we obtain $\beta(x_3) \le d$. Indeed,

$$\begin{aligned}\beta(x_3) &= \beta(Tx_3 + H_2(t_3, x_3)) \\ &\le t_3\beta(Tx_3 + Fx_3) + (1-t_3)\beta(Tx_3 + x_3 - z_0) \\ &\le d,\end{aligned}$$

and thus by condition $(A5)$, we have $\alpha(Tx_3+Fx_3) > c$, which is the same contradiction we arrived at in the previous case.

The homotopy invariance property (iii) of the fixed index i_* yields

$$i_*(T+F, V\cap\Omega, \mathcal{P}) = i_*(T+I-z_0, V\cap\Omega, \mathcal{P}),$$

and by the solvability property (iv) of the index i_* (since $T^{-1}z_0 \notin V$ the index cannot be nonzero) we have

$$i_*(T+F, V\cap\Omega, \mathcal{P}) = i_*(T+I-z_0, V\cap\Omega, \mathcal{P}) = 0.$$

Since U and W are disjoint open subsets of V and $T+F$ has no fixed points in $\overline{V} - (U\cup W)$ (by claims 1 and 2), from the additivity property (ii) of the index i_*, we deduce

$$i_*(T+F, V\cap\Omega, \mathcal{P}) = i_*(T+F, U\cap\Omega, \mathcal{P}) + i_*(T+F, W\cap\Omega, \mathcal{P}).$$

Consequently, we have

$$i(T+F, W\cap\Omega, \mathcal{P}) = -1,$$

and thus by the solvability property (iv) of the fixed point index i_*, the sum $T+F$ has a fixed point $x^* \in W \subset \mathcal{P}(\beta,\alpha,b,c)\cap\Omega$. □

4. Applications

In this section we will apply our main result Theorem 3.1 for two-point BVPs and for three-point BVPs and will show that, using Theorem 3.1, some well-known results can be enriched.

4.1. A Three-Point BVP

In this subsection, we will investigate the three-point BVP

$$\begin{aligned} &y'' + f(t,y) = 0, \quad t\in(0,1), \\ &y(0) = ky(\eta), \quad y(1) = 0, \end{aligned} \tag{4.1}$$

where

(B1). $f \in \mathcal{C}([0,1]\times\mathbb{R}^+)$, $0 < \widetilde{A} \le f(t,u) \le A$, $t\in[0,1]$, $u\in[0,\infty)$, for some positive constants $A \ge \widetilde{A}$.

(B2). $\eta\in(0,1)$, $k>0$, $k(1-\eta)<1$, $B = \frac{1+k\eta}{1-k(1-\eta)}$, $\epsilon\in(1,2)$, $c=0$ and there exist $a,b,d,z_0>0$ so that $z_0 = a$ and

$$a<d<b, \quad 2z_0 < \epsilon d, \quad (\epsilon-1)b + 2z_0 < \tfrac{d}{2},$$

$$(\epsilon-1)b + \epsilon AB < d, \quad a < \tfrac{\epsilon AB + 2z_0}{\epsilon} \le d.$$

After the proof of the main result in this subsection, we will give an example for a function f and constants A, $\widetilde{A}$, B, η, k, a, b, d, ϵ, z_0 which satisfy $(B1)$ and $(B2)$. We will investigate the BVP (4.1) for existence of at least one non trivial nonnegative solution. Our main result is as follows.

Theorem 4.1. *Suppose* $(B1)$ *and* $(B2)$*. Then the BVP* (4.1) *has at least one non trivial nonnegative solution* $y \in \mathcal{C}^2([0,1])$.

To prove our main result, we will use Theorem 3.1.

In [20] the BVP (4.1) is investigated when the function f satisfies the following conditions

(B3). $f(t,u)$ is nonnegative and continuous on $(0,1)\times[0,\infty)$, $f(t,u)$ is monotone increasing on u for fixed $t \in (0,1)$, there exists $q \in (0,1)$ such that

$$f(t,ru) > r^q f(t,u), \quad 0 < r < 1, \quad (t,u) \in (0,1)\times[0,\infty),$$

and in [20] it is proved that the BVP (4.1) has a unique solution $u \in \mathcal{C}([0,1])\bigcap\mathcal{C}^2((0,1))$. We will note that there are cases for the function f for which we can apply Theorem 4.1 and we can not apply Theorem 4.1 in [20] and conversely. For example, if $f(t,u) = 1 + \frac{1}{1+u}$, $t,u \in [0,\infty)$, then it is bounded below and above and we can apply Theorem 4.1. At the same time, it is decreasing with respect to u for $t,u \in [0,\infty)$ and we can not apply Theorem 4.1 in [20]. If $f(t,u) = \sum\limits_{j=1}^{m} a_j(t)u^{\alpha_j}$, where $a_j \in \mathcal{C}([0,\infty))$ are nonnegative functions and $\alpha_j \in (0,1)$, $j \in \{1,\ldots,m\}$, as it is shown in [20], it satisfies $(B3)$. On the other hand, it is unbounded above and we can not apply Theorem 4.1. Thus, our result Theorem 4.1 and Theorem 4.1 in [20] are complementary.

4.1.1. Proof of Theorem 4.1. Set

$$H(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

and

$$G(t,s) = H(t,s) + \frac{k(1-t)}{1-k(1-\eta)}H(\eta,s), \quad t,s \in [0,1].$$

Note that $0 \le H(t,s) \le 1$, $t,s \in [0,1]$. Hence,

$$0 \le G(t,s) \le 1 + \frac{k}{1-k(1-\eta)} = \frac{1-k+k\eta+k}{1-k(1-\eta)} = \frac{1+k\eta}{1-k(1-\eta)} = B,$$

$t,s \in [0,1]$. Moreover, for $t,s \in \left[\frac{\eta}{3},\frac{\eta}{2}\right]$, we have

$$H(t,s) \ge \frac{\eta}{3}\left(1-\frac{\eta}{2}\right)$$

and

$$G(t,s) \ge H(t,s) \ge \frac{\eta}{3}\left(1-\frac{\eta}{2}\right).$$

Next,

$$H_t(t,s) = \begin{cases} -s, & 0 \le s \le t \le 1, \\ \\ 1-s, & 0 \le t \le s \le 1. \end{cases}$$

Hence, $|H_t(t,s)| \le 1$, $t, s \in [0,1]$, and

$$\begin{aligned} |G_t(t,s)| &= \left| H_t(t,s) - \frac{k}{1-k(1-\eta)} H(\eta, s) \right| \\ &\le |H_t(t,s)| + \frac{k}{1-k(1-\eta)} H(\eta, s) \\ &\le 1 + \frac{k}{1-k(1-\eta)} = \frac{1+k\eta}{1-k(1-\eta)} = B, \quad t, s \in [0,1]. \end{aligned}$$

Let $E = \mathcal{C}([0,1])$ be endowed with the maximum norm

$$\|y\| = \max_{t\in[0,1]} |y(t)|.$$

On E, define

$$\alpha(y) = \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} |y(t)| + z_0, \quad \beta(y) = \max_{t\in[0,1]} |y(t)|.$$

In [20] it is proved that the solution of the BVP (4.1) can be expressed in the following form

$$y(t) = \int_0^1 G(t,s) f(s, y(s)) ds, \quad t \in [0,1].$$

Set

$$k_1 = \frac{\min\left\{\epsilon \frac{\eta^2}{18}\left(1 - \frac{\eta}{2}\right) \widetilde{A}, z_0\right\}}{d\epsilon}.$$

Define

$$\begin{aligned} \mathcal{P} &= \{y \in E : y(t) \ge 0, \quad t \in [0,1], \quad \min_{t \in \left[\frac{\eta}{3}, \frac{\eta}{2}\right]} y(t) \ge k_1 \max_{t\in[0,1]} y(t)\}, \\ \Omega &= \{y \in \mathcal{P} : \|y\| \le \frac{2z_0 + \epsilon AB}{\epsilon}\}. \end{aligned}$$

Note that $0 \in \Omega$ and $\Omega \subset \mathcal{P}$. For $y \in \mathcal{P}$, define the operators

$$\begin{aligned} Ty(t) &= -\epsilon y(t) + 2z_0, \\ Fy(t) &= y(t) - 2z_0 + \epsilon \int_0^1 G(t,s) f(s, y(s)) ds, \quad t \in [0,1]. \end{aligned}$$

Note that if $y \in \mathcal{P}$ is a fixed point of the operator $T + F$, then it is a solution to the BVP (4.1). Next, if $y \in \mathcal{P}$ and $\beta(y) \leq b$, we have

$$\begin{aligned} |Ty(t) + y(t)| &\leq (\epsilon - 1)y(t) + 2z_0 \\ &\leq (\epsilon - 1)b + 2z_0 \\ &< \frac{d}{2}, \quad t \in [0,1], \end{aligned}$$

and

$$\begin{aligned} |Ty(t) + Fy(t)| &= \left| -(\epsilon - 1)y(t) + \epsilon \int_0^1 G(t,s) f(s, y(s)) ds \right| \\ &\leq (\epsilon - 1)y(t) + \epsilon \int_0^1 G(t,s) f(s, y(s)) ds \\ &\leq (\epsilon - 1)b + \epsilon A \int_0^1 G(t,s) ds \\ &\leq (\epsilon - 1)b + \epsilon AB \\ &< d. \end{aligned}$$

Therefore, if $y \in \mathcal{P}$ and $\beta(y) \leq b$, we have

$$\beta(Ty + y) < d \tag{4.2}$$

and

$$\beta(Ty + Fy) < d. \tag{4.3}$$

For $y, z \in \mathcal{P}$, we have

$$|Ty(t) - Tz(t)| = \epsilon |y(t) - z(t)|, \quad t \in [0,1].$$

Hence,

$$\|Ty - Tz\| = \epsilon \|y - z\|.$$

Thus, $T : \mathcal{P} \to E$ is an expansive operator with constant $h = \epsilon$.
Let now, $y \in \mathcal{P}$. Then

$$\begin{aligned} |(I - F)y(t))| &= \epsilon \left| \int_0^1 G(t,s) f(s, y(s)) ds \right| \\ &\leq \epsilon A \int_0^1 G(t,s) ds \\ &\leq \epsilon AB, \quad t \in [0,1], \end{aligned}$$

whereupon

$$\|(I - F)y\| \leq \epsilon AB$$

and $I-F:\mathcal{P}\to E$ is uniformly bounded. Moreover,

$$\begin{aligned}\left|\frac{d}{dt}(I-F)y(t)\right| &= \left|\int_0^1 G_t(t,s)f(s,y(s))ds\right|\\ &\leq \int_0^1 |G_t(t,s)|f(s,y(s))ds\\ &\leq AB, \quad t\in[0,1].\end{aligned}$$

Consequently, $I-F:\mathcal{P}\to E$ is completely continuous. Then $I-F:\mathcal{P}\to E$ is a 0-set contraction.
Note that

$$\|T0\| = 2z_0 < \epsilon\min\{b,d\}.$$

For $y\in E$, we have

$$T^{-1}y = -\frac{y-2z_0}{\epsilon}.$$

Hence,

$$\alpha\left(T^{-1}z_0\right) = \alpha\left(\frac{z_0}{\epsilon}\right) = \frac{z_0}{\epsilon}+z_0 > \max\{a,c\}.$$

Suppose that $y\in\mathcal{P}$ with $\beta(y)=b$. Then

$$\alpha(Ty+y) = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]}|Ty(t)+y(t)|+z_0 \geq z_0 = a.$$

Consequently $(A1)$ holds.
Now, we take $y\in\mathcal{P}$ with $\beta(y)=b$, $\alpha(y)\geq a$. Then, using $d<b$, (4.2) and (4.3), we obtain

$$\beta(Ty+y)<b \quad \text{and} \quad \beta(Ty+Fy)<b.$$

Consequently $(A2)$ holds.
Observe that, if $y\in\mathcal{P}$, $\beta(y)=b$ and $\alpha(Ty+Fy)<a$, using $d<b$ and (4.2), (4.3), we find

$$\beta(Ty+Fy)<b \quad \text{and} \quad \beta(Ty+y)<b.$$

Thus, $(A3)$ holds.
Since $c=0$ and $\alpha(y)>0$ for any $y\in\mathcal{P}$, the case $\alpha(y)=c$ is impossible.
Let now, $a_1\in\left(a,\frac{\epsilon AB+z_0}{\epsilon}\right)$ be arbitrarily chosen. Then

$$\alpha(a_1) = a_1+z_0 > a$$

and

$$\beta(a_1) = a_1 < \frac{\epsilon AB+2z_0}{\epsilon} \leq d.$$

Therefore

$$\{y\in\mathcal{P} : a<\alpha(y) \quad \text{and} \quad \beta(y)<d\}\cap\Omega\neq\emptyset.$$

Let $y\in\mathcal{P}(\alpha,a)$. Then $y\in\mathcal{P}$ and $\alpha(y)\leq a$. Hence,

$$a \geq \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t)+z_0 = \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t)+a.$$

Therefore $\min\limits_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) = 0$ and using the definition of the cone $\mathcal{P}$, we find

$$\beta(y) = \max_{t\in[0,1]} y(t) \leq \frac{1}{k_1} \min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} y(t) = 0 \leq d.$$

Thus, $y \in \mathcal{P}(\beta, d)$ and $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$.
Since $0 \in \mathcal{P}(\alpha, a)$, we have $\mathcal{P}(\alpha, a) \cap \Omega \neq \emptyset$.
Note that $\mathcal{P}(\beta, d)$ is bounded.
Let $\lambda \in [0, 1]$ is fixed and $u \in \mathcal{P}(\alpha, a)$ is arbitrarily chosen. Then $\beta(u) \leq d < b$. Set

$$v(t) = \frac{\lambda\epsilon \int_0^1 G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon}, \quad t \in [0,1].$$

We have that $v(t) \geq 0$, $t \in [0, 1]$, and

$$v(t) \leq \frac{\epsilon AB + z_0}{\epsilon} \leq d, \quad t \in [0,1],$$

and

$$\|v\| \leq \frac{\epsilon AB + z_0}{\epsilon} \leq d.$$

$$\begin{aligned}
\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} v(t) &\geq \frac{\lambda\epsilon \int_{\frac{\eta}{3}}^{\frac{\eta}{2}} \min\limits_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\lambda\epsilon\left(\frac{\eta}{2} - \frac{\eta}{3}\right)\frac{\eta}{3}\left(1 - \frac{\eta}{2}\right)\widetilde{A} + (1-\lambda)z_0}{\epsilon} \\
&\geq \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right\}}{\epsilon} \\
&= \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A}, z_0\right\}}{d\epsilon} d \\
&\geq k_1 \max_{t\in[0,1]} v(t).
\end{aligned}$$

Thus, $v \in \Omega$. Next,

$$\begin{aligned}
\lambda(I-F)u(t) + (1-\lambda)z_0 &= 2\lambda z_0 - \lambda\epsilon\int_0^1 G(t,s)f(s,u(s))ds + z_0 - \lambda z_0 \\
&= -\lambda\epsilon\int_0^1 G(t,s)f(s,u(s))ds + (1+\lambda)z_0 \\
&= -\epsilon\frac{\lambda\epsilon\int_0^1 G(t,s)f(s,u(s))ds + (1-\lambda)z_0}{\epsilon} + 2z_0 \\
&= Tv(t), \quad t \in [0,1].
\end{aligned}$$

Therefore

$$\lambda(I-F)(\mathcal{P}(\alpha,a))+(1-\lambda)z_0\subset T(\Omega).$$

Let $\lambda\in[0,1]$ be fixed and $u\in\mathcal{P}(\beta,d)$ be arbitrarily chosen. Take

$$w(t)=\frac{2(1-\lambda)z_0+\lambda\epsilon\int_0^1 G(t,s)f(s,u(s))ds}{\epsilon},\quad t\in[0,1].$$

We have $v(t)\geq 0$, $t\in[0,1]$, and

$$w(t)\leq\frac{\epsilon AB+2z_0}{\epsilon}\leq d,\quad t\in[0,1].$$

Moreover,

$$\begin{aligned}
\min_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} w(t) &\geq \frac{\lambda\epsilon\int_{\frac{\eta}{3}}^{\frac{\eta}{2}}\min\limits_{t\in\left[\frac{\eta}{3},\frac{\eta}{2}\right]} G(t,s)f(s,u(s))ds+2(1-\lambda)z_0}{\epsilon}\\
&\geq \frac{\lambda\epsilon\left(\frac{\eta}{2}-\frac{\eta}{3}\right)\frac{\eta}{3}\left(1-\frac{\eta}{2}\right)\widetilde{A}+(1-\lambda)z_0}{\epsilon}\\
&\geq \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A},z_0\right\}}{\epsilon}\\
&= \frac{\min\left\{\epsilon\frac{\eta^2}{18}\left(1-\frac{\eta}{2}\right)\widetilde{A},z_0\right\}}{d\epsilon}d\\
&\geq k_1\max_{t\in[0,1]} w(t).
\end{aligned}$$

Therefore $w\in\Omega$. Also,

$$\begin{aligned}
\lambda(I-F)u(t) &= \lambda\left(2z_0-\epsilon\int_0^1 G(t,s)f(s,u(s))ds\right)\\
&= -\epsilon\frac{\epsilon\int_0^1 G(t,s)f(s,u(s))ds+2(1-\lambda)z_0}{\epsilon}+2z_0\\
&= -\epsilon w(t)+2z_0\\
&= Tw(t),\quad t\in[0,1].
\end{aligned}$$

Therefore

$$\lambda(I-F)(\mathcal{P}(\beta,d))\subset T(\Omega).$$

By Theorem 3.1, it follows that the BVP (4.1) has at least one solution in $\{y\in\mathcal{P}: a<\alpha(y) \text{ and } \beta(y)<d\}\cap\Omega\subset P(\alpha,\beta,a,d)\cap\Omega$.

4.1.2. An Example. Consider the BVP

$$\begin{aligned} y'' + \tfrac{1}{300(1+t^2)(1+y)} + \tfrac{1}{300} &= 0, \quad t \in (0,1), \\ y(0) = y\left(\tfrac{1}{2}\right), \quad y(1) &= 0. \end{aligned} \tag{4.4}$$

Here

$$f(t,y) = \frac{1}{300(1+t^2)(1+y)} + \frac{1}{300}, \quad t \in (0,1), \quad y \in [0,\infty), \quad k=1, \quad \eta = \frac{1}{2}.$$

Note that for the function f we can not apply Theorem 4.1 in [20] because it is a decreasing function with respect to y for $t, y \in [0,\infty)$. Take the constants

$$\epsilon = \tfrac{41}{40}, \quad B = 3, \quad A = \tfrac{1}{123}, \quad \widetilde{A} = \tfrac{1}{300}, \quad b = 1, \quad d = \tfrac{1}{2},$$

$$z_0 = \tfrac{1}{400}, \quad a = \tfrac{1}{400}.$$

We have

$$a < d < b, \quad 2z_0 = 2a = \frac{1}{200} < \frac{41}{80} = \epsilon d,$$

$$(\epsilon - 1)b + 2z_0 = \frac{1}{40} + \frac{1}{200} = \frac{3}{100} < \frac{1}{4} = \frac{d}{2},$$

$$(\epsilon - 1)b + \epsilon AB = \frac{1}{40} + \frac{41}{40} \cdot \frac{3}{123} = \frac{1}{40} + \frac{1}{40} = \frac{1}{20} < \frac{1}{2} = d,$$

$$\frac{1}{400} = a < \frac{\epsilon AB + 2z_0}{\epsilon} = \frac{40}{41} \cdot \left(\frac{41}{40} \cdot \frac{3}{123} + \frac{1}{200}\right) < \frac{1}{2} = d.$$

Thus, $(B2)$ holds. Next, $f \in \mathcal{C}([0,1] \times \mathbb{R}^+)$ and

$$\frac{1}{300} \le f(t,y) = \frac{1}{300(1+t^2)(1+y)} + \frac{1}{300} \le \frac{1}{150} \le \frac{1}{123} = A,$$

i.e., $(B1)$ holds. By Theorem 3.1, it follows that the BVP (4.4) has at least one nonnegative solution.

4.2. A Two-Point BVP

In this subsection, we will investigate the following BVP

$$\begin{aligned} x''(t) + g(x(t)) &= 0, \quad t \in (0,1), \\ x(0) = 0 &= x'(1), \end{aligned} \tag{4.5}$$

where

(C1). $g \in \mathcal{C}(\mathbb{R}^+)$, $0 < \widetilde{A}_1 \le g(x) \le A_1$, $x \in [0,\infty)$, for some positive constants $A_1 \ge \widetilde{A}_1$.

(C2). The nonnegative constants z_1, a_1, b_1, c_1, d_1, ϵ_1 satisfy

$$\epsilon_1 \in (1,2), \quad (\epsilon_1 - 1)b_1 + 2z_1 < \frac{d_1}{2}, \quad (\epsilon_1 - 1)b_1 + \epsilon_1 A_1 < d_1,$$

$$c_1 = 0, \quad 2z_1 < \epsilon_1 \min\{b_1, d_1\}, \quad \frac{z_1}{\epsilon_1} + z_1 > \max\{a_1, c_1\}, \quad z_1 = a_1,$$

$$a_1 < d_1 < b_1, \quad a_1 < \frac{\epsilon_1 A_1 + 2z_1}{\epsilon_1} \le d_1.$$

Our main result in this subsection is as follows.

Theorem 4.2. *Suppose* $(C1)$ *and* $(C2)$. *Then the BVP* (4.5) *has at least one non trivial nonnegative solution.*

The BVP (4.5) is investigated in [2] under the following conditions

(C1.1). $\tau \in (0,1)$ is fixed, b and c are positive constants with $3b \le c$, $g : [0,\infty) \to [0,\infty)$ is a continuous function such that

1. $g(w) > \frac{c}{\tau(1-\tau)}, \quad w \in \left[c, \frac{c}{\tau}\right]$,
2. g is decreasing on $[a, b\tau]$ with $g(b\tau) \ge g(w)$ for $w \in [b\tau, b]$.
3. $\int_0^\tau sg(s)ds \le \frac{2b-g(b\tau)(1-\tau^2)}{2}$,

and it is proved that the BVP (4.5) has at least one nonnegative solution. Note that there are cases for the function g for which we can apply Theorem 4.2 and we can not apply Theorem 5.1 in [2] and conversely. For instance, if $g(x) = \frac{x}{1+x} + 1$, $x \in [0,\infty)$, then it is bounded above and below and we can apply Theorem 4.2. On the other hand, g is an increasing function on $[0,\infty)$ and we can not apply Theorem 5.1 in [2]. If $g(x) = \frac{1}{\sqrt{x}} + e^{x-2}$, $x \in (0,\infty)$, as it is shown in [2], we can apply for it Theorem 5.1 in [2]. Since it is unbounded above, we can not apply Theorem 4.2. Therefore our main result Theorem 3.1 and the main result Theorem 4.1 in [2] are complementary.

After the proof of Theorem 4.2, we will give an example for a function g and constants A_1, $\widetilde{A}_1$, z_1, a_1, b_1, c_1, d_1, ϵ_1 that satisfy $(C1)$ and $(C2)$.

4.2.1. Proof of Theorem 4.2. Let $E = \mathcal{C}([0,1])$ be endowed with the maximum norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

Define

$$G_1(t,s) = \min\{t,s\}, \quad (t,s) \in [0,1]\times[0,1].$$

Note that

$$0 \le G_1(t,s) \le 1, \quad (t,s) \in [0,1]\times[0,1],$$

and

$$G_1(t,s) \ge \frac{1}{3}, \quad t,s \in \left[\frac{1}{3}, \frac{1}{2}\right].$$

On E, define the following functionals

$$\alpha_1(x) = \min_{t\in[0,1]} |x(t)| + z_1, \quad \beta_1(x) = \max_{t\in[0,1]} |x(t)|.$$

In [2] it is proved that the solution of the BVP (4.5) can be represented in the form

$$x(t) = \int_0^1 G_1(t,s)g(x(s))ds, \quad t \in [0,1].$$

Set

$$k_2 = \frac{\min\left\{\frac{\epsilon_1 \widetilde{A}_1}{3}, z_1\right\}}{d_1\epsilon_1}.$$

Define

$$\begin{aligned}
\mathcal{P}_1 &= \{x \in E : x(t) \geq 0, \quad t \in [0,1], \quad \min_{t \in \left[\frac{1}{3}, \frac{1}{2}\right]} x(t) \geq k_2 \max_{t \in [0,1]} x(t)\},\\
\Omega_1 &= \left\{x \in \mathcal{P}_1 : \|x\| \leq \frac{2z_1 + \epsilon_1 A_1}{\epsilon_1}\right\}.
\end{aligned}$$

Note that $0 \in \Omega_1$ and $\Omega_1 \subset \mathcal{P}_1$. For $x \in \mathcal{P}_1$, define the following operators.

$$\begin{aligned}
T_1 x(t) &= -\epsilon_1 x(t) + 2z_1,\\
F_1 x(t) &= x(t) - 2z_0 + \epsilon_1 \int_0^1 G_1(t,s) g(x(s)) ds, \quad t \in [0,1].
\end{aligned}$$

Now, the proof of Theorem 4.2 follows similar arguments to those in the proof of Theorem 4.1.

4.2.2. An Example. Consider the BVP

$$\begin{aligned}
x''(t) + \tfrac{x(t)}{400(1+x(t))} + \tfrac{1}{400} &= 0, \quad t \in (0,1),\\
x(0) = 0 &= x'(1).
\end{aligned} \tag{4.6}$$

Here

$$g(x) = \frac{x}{400(1+x)} + \frac{1}{400}, \quad x \in [0, \infty).$$

Note that the function g is an increasing function on $[0,\infty)$ and then we can not apply Theorem 5.1 in [2]. Take

$$\begin{aligned}
\epsilon_1 &= \frac{41}{40}, \quad A_1 = \frac{1}{123}, \quad \widetilde{A}_1 = \frac{1}{400}, \quad b_1 = 1, \quad d_1 = \frac{1}{2},\\
z_1 &= \frac{1}{400}, \quad a_1 = \frac{1}{400}, \quad c_1 = 0.
\end{aligned}$$

Then, $\epsilon_1 > 1$ and

$$(\epsilon_1 - 1) b_1 + 2z_1 = \frac{1}{40} + \frac{1}{200} < \frac{1}{4} = \frac{d_1}{2},$$

$$(\epsilon_1 - 1) b_1 + \epsilon_1 A_1 = \frac{1}{40} + \frac{41}{40} \cdot \frac{1}{123} = \frac{1}{40} + \frac{1}{120} < \frac{1}{2} = d_1,$$

$$\epsilon_1 \min\{b_1, d_1\} = \frac{41}{40} \cdot \frac{1}{2} = \frac{41}{80} > \frac{1}{200} = 2z_1,$$

$$\frac{z_1}{\epsilon_1} + z_1 = \frac{\frac{1}{400}}{\frac{41}{40}} = \frac{1}{410} + \frac{1}{400} > \frac{1}{400} = \max\{a_1, c_1\},$$

$$a_1 < d_1 < b_1,$$

$$\begin{aligned}
a_1 = \frac{1}{400} < \frac{\epsilon_1 A_1 + 2z_1}{\epsilon_1} &= \frac{\frac{41}{40} \cdot \frac{1}{123} + \frac{1}{200}}{\frac{41}{40}} = \frac{\frac{1}{120} + \frac{1}{200}}{\frac{41}{40}}\\
&= \frac{\frac{1}{3} + \frac{1}{5}}{41} = \frac{8}{615} < \frac{1}{2} = d_1.
\end{aligned}$$

Thus, $(C2)$ holds. Next,

$$\frac{1}{400} \leq g(x) \leq \frac{1}{200}, \quad x \in [0, \infty).$$

So, $(C1)$ holds. Hence, applying Theorem 4.2, we conclude that the BVP (4.6) has at least one nonnegative solution.

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Reducing the complexity of equilibrium problems and applications to best approximation problems

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Abstract. We consider the scalar equilibrium problems governed by a bifunction in a finite-dimensional framework and we characterize the solutions by means of extreme or exposed points.

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Keywords: Extreme points, exposed points, equilibrium points.

1. Introduction

In this article, we focus on scalar equilibrium problems governed by a bifunction within a finite-dimensional framework. Through the use of classical arguments and techniques from Convex Analysis, we show that under suitable generalized convexity assumptions imposed on the bifunction, the solutions of the equilibrium problem can be characterized by means of extreme points (Corollary 4.13) or exposed points (Corollary 4.16) of the feasible domain. Our findings have significant implications for various particular instances, including variational inequalities and optimization problems, and are particularly relevant to best approximation problems, as seen in the examples of Section 4.

This paper is organized as follows. In Section 2, we introduce our general notations and we recall some useful facts from Convex Analysis, primarily focused on the best approximation problem. In Section 3, following up on the same problem, from a geometric point of view, we proved that if S is a nonempty convex subset of $\mathbb{R}^n$,

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†Meanwhile, professor Nicolae Popovici passed away unexpectedly and prematurely.

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then the elements of best approximation to an arbitrary element in $\mathbb{R}^n$ from S, can be characterized by means of the Gauss map (Remark 3.2). In fact, if S is also closed, then it is known that x^0 is the element of best approximation to an arbitrary element x^* from S, if and only if x^* is an element of the translated normal cone to S at x^0 by the vector x^0 (Proposition 3.3). This led us to Proposition 3.4, where we have proved that the set $\{x + N_S(x) \setminus \{0\} \mid x \in \operatorname{bd} S\}$ is a partition of $\mathbb{R}^n \setminus S$, where $N_S(x)$ is the normal cone to S at x.

In Section 4, we move onto equilibrium problems and reducing their complexity (Theorem 4.7), as follows. For an arbitrary nonempty set A and an arbitrary nonempty subset M of $\mathbb{R}^n$, we consider the bifunction $g : A \times \operatorname{conv}M \to \mathbb{R}$, which is assumed to be quasiconvex in the second argument. For such a bifunction, we show that the equilibrium points of g are precisely the equilibrium points of the restriction $g|_{A\times M}$. A consequence of this is Corollary 4.13, which for a nonempty convex and compact subset S of $\mathbb{R}^n$, and a bifunction $g : A \times S \to \mathbb{R}$, also assumed to be quasiconvex in the second argument, shows that the equilibrium points of g are precisely the equilibrium points of $g|_{A\times \operatorname{ext} S}$. We also point out the particular case, when $M = \operatorname{ext}S$ for some Minkowski set S, which shows that the previous hypothesis of boundedness of S is not crucial. Finally, Theorem 4.7 led us to our main result, Corollary 4.16, where we have obtained that for a nonempty convex and compact subset S of $\mathbb{R}^n$, if $g : A \times S \to \mathbb{R}$ is quasiconvex and lower semicontinous in the second argument, then the equilibrium points of g are precisely the equilibrium points of $g|_{A\times \exp S}$.

2. Notations and preliminaries

Throughout this paper $\mathbb{R}^n$ stands for the n-dimensional real Euclidean space, whose norm $\|\cdot\|$ is induced by the usual inner product $\langle\cdot,\cdot\rangle$. For any points $x, y \in \mathbb{R}^n$, we use the notations

$$\begin{aligned}
[x, y] &:= \{(1-t)x + ty \mid t \in [0, 1]\,\},\\
]x, y[&:= \{(1-t)x + ty \mid t \in]0, 1[\,\}.
\end{aligned}$$

Recall that a set $S \subseteq \mathbb{R}^n$ is called convex if $[x, y] \subseteq S$, for all $x, y \in S$. Of course, this is equivalent to say that $]x, y[\subseteq S$, for all $x, y \in S$.

Given a convex set $S \subseteq \mathbb{R}^n$ we denote the set of extreme points of S by

$$\operatorname{ext} S = \{x^0 \in S \mid \forall x, y \in S \,:\, x^0 = \tfrac{1}{2}(x + y) \;\Rightarrow x = y = x^0\}.$$

A point x^0 is said to be an exposed point of S if there is a supporting hyperplane H which supports S at x^0 such that $\{x^0\} = H \cap S$. We denote the set of exposed points of S by

$$\exp S = \{x^0 \in S \mid \exists c \in \mathbb{R}^n \setminus \{0\} \text{ such that } \operatorname*{argmin}_{x\in S}\langle c, x\rangle = \{x^0\}\}.$$

It is well-known that $\exp S \subseteq \operatorname{ext} S$.

The convex hull of a set $M \subseteq \mathbb{R}^n$, i.e., the smallest convex set in $\mathbb{R}^n$ containing M is denoted by $\operatorname{conv}M$.

Next, we recall the following well-known theorems (see for example [5] and [7]):

Theorem 2.1 (Minkowski (Krein-Milman)). *Every compact convex set in* $\mathbb{R}^n$ *is the convex hull of its extreme points.*

Theorem 2.2 (Straszewicz). *Every compact convex subset* M *of* $\mathbb{R}^n$ *admits the representation*:

$$M \quad = \quad \mathrm{cl}(\mathrm{conv}(\exp M)).$$

In the book by Breckner and Popovici [1, C 5.2.7, p. 82] we have the following remark:

Remark 2.3 (Minkowski). Let $S \subseteq \mathbb{R}^n$ be a compact convex set. Then, for each subset M of S, the following equivalence holds:

$$S = \mathrm{conv}\, M \quad \Longleftrightarrow \quad \mathrm{ext}\, S \subseteq M.$$

Definition 2.4. Let S be a nonempty subset of $\mathbb{R}^n$ and let $x^* \in \mathbb{R}^n$. A point $x^0 \in S$ is said to be *an element of best approximation to* x^* *from* S (or *a nearest point to* x^* *from* S) if

$$\|x^0 - x^*\| \leq \|x - x^*\|, \text{ for all } x \in S.$$

The problem of best approximation of x^* by elements of S consists in finding all elements of best approximation to x^* from S. The solution set

$$P_S(x^*) := \{x^0 \in S \mid \ \|x^0 - x^*\| \leq \|x - x^*\|, \text{ for all } x \in S\}$$

is called *the metric projection of* x^* *on* S.

Remark 2.5. The problem of best approximation is an optimization problem,

$$\begin{cases} f(x) \longrightarrow \min \\ x \in S, \end{cases}$$

whose objective function $f : \mathbb{R}^n \to \mathbb{R}$ is defined for all $x \in \mathbb{R}^n$ by

$$f(x) := \|x - x^*\|.$$

Actually, we have

$$P_S(x^*) = \operatorname*{argmin}_{x \in S} f(x).$$

Definition 2.6. Let S be a nonempty subset of $\mathbb{R}^n$ and let $x^* \in \mathbb{R}^n$, we say that $x^0 \in S$ is a farthest point from S to x^* if

$$\|x^0 - x^*\| \geq \|x - x^*\|, \text{ for all } x \in S,$$

i.e.,

$$x^0 \in \operatorname*{argmax}_{x \in S} \|x - x^*\|.$$

In this paper we will use the following well known results from Convex Analysis (see for example [1]).

Proposition 2.7. *Any farthest point from a nonempty set* $S \subseteq \mathbb{R}^n$ *to a point* $x^* \in \mathbb{R}^n$ *is an exposed point of* S, *i.e.,*

$$\operatorname*{argmax}_{x \in S} \|x - x^*\| \subseteq \exp S.$$

Theorem 2.8 (existence of elements of best approximation). *If S is a nonempty closed subset of $\mathbb{R}^n$, then for every $x^* \in \mathbb{R}^n$ there is an element of best approximation to x^* from S. In other words, we have*

$$P_S(x^*) \neq \emptyset, \text{ i.e., } \operatorname{card}(P_S(x^*)) \geq 1.$$

Theorem 2.9 (unicity of the element of best approximation). *If $S \subseteq \mathbb{R}^n$ is a nonempty convex set and $x^* \in \mathbb{R}^n$, then there exists at most one element of best approximation to x^* from S. In other words, we have*

$$\operatorname{card}(P_S(x^*)) \leq 1.$$

Theorem 2.10 (characterization of elements of best approximation). *Let $S \subseteq \mathbb{R}^n$, let $x^0 \in S$, and let $x^* \in \mathbb{R}^n$. Then the following hold:*

(a) *If $\langle x - x^0, x^* - x^0 \rangle \leq 0$ for all $x \in S$, then x^0 is an element of best approximation to x^* from S.*
(b) *If S is convex and x^0 is an element of best approximation to x^* from S, then we have that $\langle x - x^0, x^* - x^0 \rangle \leq 0$ for all $x \in S$.*

Corollary 2.11. *Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and let $x^* \in \mathbb{R}^n$. Then*

$$P_S(x^*) = \{x^0 \in S \mid \langle x - x^0, x^* - x^0 \rangle \leq 0, \text{ for all } x \in S\}.$$

3. The inverse images of the metric projection

Our main results from Section 4 generate some interesting examples with regard to the best approximation problem. A further analysis of the characterization of the elements of best approximation (Theorem 2.10) led us to a geometric approach involving the inverse images of the metric projection, which will be presented in this section.

Remark 3.1. From a geometric point of view, the property $\langle x - x^0, x^* - x^0 \rangle \leq 0$ for all $x \in S$ in assertion (b) of Theorem 2.10 shows that $x^* - x^0$ belongs to the so-called normal cone to S at x^0, i.e.,

$$N_S(x^0) = \{d \in \mathbb{R}^n \mid \langle x - x^0, d \rangle \leq 0, \text{ for all } x \in S\}.$$

For $S \subseteq \mathbb{R}^n$ a nonempty convex set, we recall the *Gauss map* of S, introduced in [4], which is a set-valued map, defined as follows:

$$G_S : \mathbb{R}^n \rightrightarrows S^{n-1}, \qquad G_S(x) := N_S(x) \cap S^{n-1},$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

Remark 3.2. Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and let $x^* \in \mathbb{R}^n \setminus S$. Then x^0 is an element of best approximation to x^* from S if and only if

$$\frac{x^* - x^0}{\|x^* - x^0\|} \in G_S(x^0).$$

Indeed, x^0 is an element of best approximation to x^* from S if and only if for all $x \in S$

$$\begin{aligned}\langle x - x^0, x^* - x^0\rangle \leq 0 \quad &\Leftrightarrow \quad \left\langle x - x^0, \frac{x^* - x^0}{\|x^* - x^0\|}\right\rangle \leq 0 \\ &\Leftrightarrow \quad \frac{x^* - x^0}{\|x^* - x^0\|} \in N_S(x^0) \\ &\Leftrightarrow \quad \frac{x^* - x^0}{\|x^* - x^0\|} \in G_S(x^0).\end{aligned}$$

Let $S \subseteq \mathbb{R}^n$ be a nonempty closed convex set. By Theorems 2.8 and 2.9 it follows that, for all $x^* \in \mathbb{R}^n$, $P_S(x^*)$ is a singleton. So, in this case, P_S can be considered as a single valued mapping.

The following result is well-known, yet we include it because it is one of the main ingredient of our main result of this section, Proposition 3.4.

Proposition 3.3. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then, for all $x^* \in \mathbb{R}^n$, we have that $P_S(x^*) = x^0$ if and only if $x^* \in x^0 + N_S(x^0)$.*

Proof. Let $x^* \in \mathbb{R}^n$. Since S is closed and convex, there exists $x^0 \in S$ such that $P_S(x^*) = x^0$. It follows by Corollary 2.11 that

$$\langle x - x^0, x^* - x^0\rangle \leq 0, \text{ for all } x \in S$$

which, by Remark 3.1, is equivalent to

$$x^* - x^0 \in N_S(x^0) \iff x^* \in x^0 + N_S(x^0),$$

and the statement is completely proved. □

Proposition 3.4. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then, the family*

$$\{x + N_S(x) \setminus \{0\} \mid x \in \operatorname{bd} S\}$$

is a partition of $\mathbb{R}^n \setminus S$.

Proof. We need to show that,

$$\mathbb{R}^n \setminus S = \bigcup_{x \in \operatorname{bd} S} (x + N_S(x) \setminus \{0\})$$

and $(x + N_S(x) \setminus \{0\}) \cap (y + N_S(y) \setminus \{0\}) \neq \emptyset$ implies $x = y$.

Let $x^* \in \mathbb{R}^n \setminus S$ and $x^0 \in S$ such that $P_S(x^*) = x^0$. By Proposition 3.3, we obtain that $x^* \in x^0 + N_S(x^0) \setminus \{0\}$, yet

$$x^0 + N_S(x^0) \setminus \{0\} \subseteq \bigcup_{x \in S}(x + N_S(x) \setminus \{0\}).$$

Subsequently, we get that

$$\mathbb{R}^n \setminus S \subseteq \bigcup_{x \in S}(x + N_S(x) \setminus \{0\}).$$

In order to prove the opposite inclusion, let us consider $x \in S$ and $u \in N_S(x) \setminus \{0\}$. If we assume that $x + u \in S$ then, by the definition of $N_S(x) \setminus \{0\}$,

$$\langle x + u - x, u\rangle = \langle u, u\rangle \leq 0 \implies u = 0$$

which contradicts the fact that $u \in N_S(x) \setminus \{0\}$. Thus $x + N_S(x) \setminus \{0\} \subseteq \mathbb{R}^n \setminus S$, for all $x \in S$, i.e.,

$$\bigcup_{x \in S} (x + N_S(x) \setminus \{0\}) \subseteq \mathbb{R}^n \setminus S.$$

Therefore, we have proved that

$$\mathbb{R}^n \setminus S = \bigcup_{x \in S} (x + N_S(x) \setminus \{0\}).$$

However, since $x + N_S(x) \setminus \{0\}$ is nonempty if and only if x is an boundary point of S, we obtain that

$$\mathbb{R}^n \setminus S = \bigcup_{x \in \operatorname{bd} S} (x + N_S(x) \setminus \{0\}).$$

If $(x + N_S(x) \setminus \{0\}) \cap (y + N_S(y) \setminus \{0\}) \neq \emptyset$, then there is an $u \in N_S(x) \setminus \{0\}$ and $v \in N_S(y) \setminus \{0\})$ such that $x + u = y + v$. Furthermore, since $u \in N_S(x) \setminus \{0\}$, we obtain $\langle y - x, u \rangle = \langle u - v, u \rangle \leq 0$, therefore $\|u\|^2 \leq \langle u, v \rangle$. By similar reasoning, since $v \in N_S(y) \setminus \{0\}$, we obtain $\|v\|^2 \leq \langle u, v \rangle$. Thus,

$$0 \leq \|x - y\|^2 = \|u - v\|^2 = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2 \leq 0$$

Henceforth, $\|x - y\| = 0$, which implies $x = y$.

□

Remark 3.5. Alternatively, one may argue as follows. By Proposition 3.3, we have that for all $x \in S$, the set $x + N_S(x)$ is the inverse image $P_S^{-1}(x)$, of x through P_S. If we consider the restriction $P_S|_{\mathbb{R}^n \setminus S}$ of the mapping P_S, then for all $x \in S$, we have that the set $x + N_S(x) \setminus \{0\}$ is the inverse image of x through the restriction $P_S|_{\mathbb{R}^n \setminus S}$. By the equivalence relation induced by $\ker P_S|_{\mathbb{R}^n \setminus S}$, we obtain that the family

$$\{x + N_S(x) \setminus \{0\} \mid x \in \operatorname{bd} S\}$$

is a partition of $\mathbb{R}^n \setminus S$.

Example 3.6. For $n = 2$, consider the set

$$M = \{x_1 = (1, 1), x_2 = (-1, 1), x_3 = (-1, -1), x_4 = (1, -1)\} \subseteq \mathbb{R}^2$$

and

$$S = \operatorname{conv} M = [-1, 1] \times [-1, 1],$$

as in Figure 1. Obviously, S is a nonempty closed convex subset of $\mathbb{R}^2$ and

$$\operatorname{bd} S =]x_1, x_2[\cup]x_2, x_3[\cup]x_3, x_4[\cup]x_1, x_4[\cup \{x_1, x_2, x_3, x_4\}.$$

On the four vertices of the square, we have

$$\begin{array}{lcl} N_S(x_1) \setminus \{0\} & = & \{(v_1, v_2) \mid v_1 \geq 0, v_2 \geq 0\} \setminus \{(0, 0)\} \\ N_S(x_2) \setminus \{0\} & = & \{(v_1, v_2) \mid v_1 \leq 0, v_2 \geq 0\} \setminus \{(0, 0)\} \\ N_S(x_3) \setminus \{0\} & = & \{(v_1, v_2) \mid v_1 \leq 0, v_2 \leq 0\} \setminus \{(0, 0)\} \\ N_S(x_4) \setminus \{0\} & = & \{(v_1, v_2) \mid v_1 \geq 0, v_2 \leq 0\} \setminus \{(0, 0)\}. \end{array}$$

On the other points of the boundary, we have

$$N_S(x) \setminus \{0\} = \begin{cases} \{(0,v) \mid v > 0\}, \text{ for all } x \in]x_1, x_2[\\ \{(v,0) \mid v < 0\}, \text{ for all } x \in]x_2, x_3[\\ \{(0,v) \mid v < 0\}, \text{ for all } x \in]x_3, x_4[\\ \{(v,0) \mid v > 0\}, \text{ for all } x \in]x_1, x_4[. \end{cases}$$

It is easy to see that $\mathbb{R}^n \setminus S = \bigcup_{x \in \operatorname{bd} S} (x + N_S(x) \setminus \{0\})$.

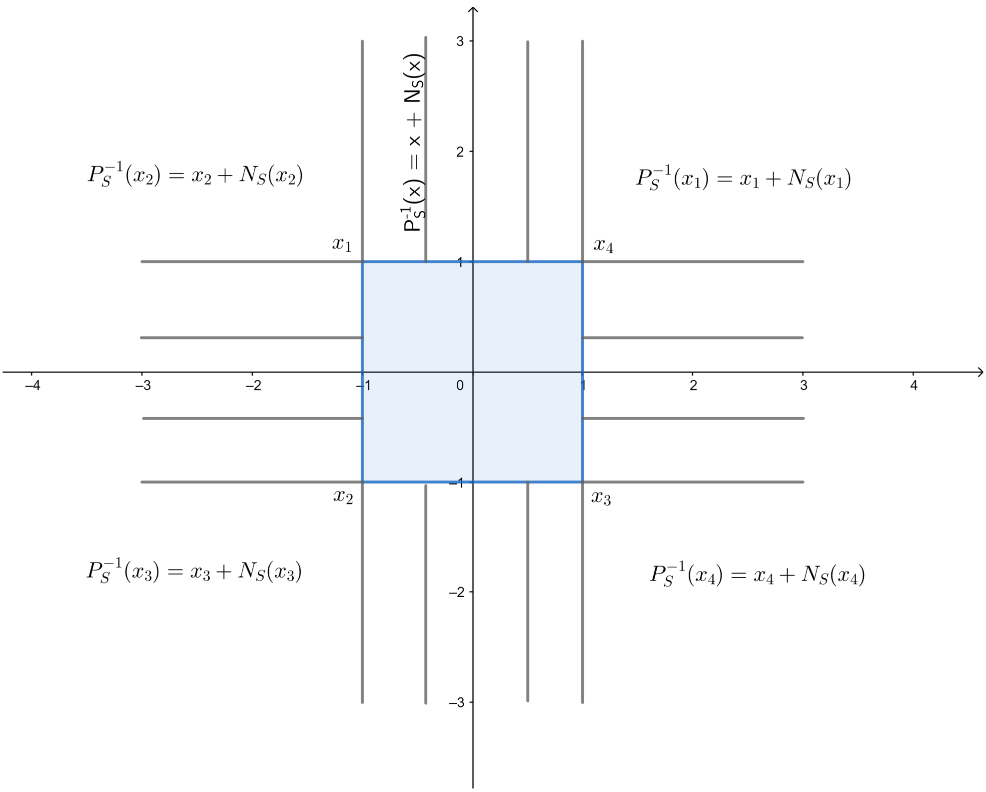

FIGURE 1. Proposition 3.4 applied for the particular case of a square in $\mathbb{R}^2$

4. Equilibrium problems

The equilibrium problem, introduced in [6], has been formulated in a more general way in [2, p. 18]. We propose a slightly modified definition. Let $g : A \times B \to \mathbb{R}$ be a "bifunction", where A and B are nonempty sets.

Definition 4.1. The *equilibrium problem* with respect to $g : A \times B \to \mathbb{R}$ and a couple of subsets $A' \subseteq A$ and $B' \subseteq B$, consists in finding the elements $x^0 \in A'$ satisfying

$$g(x^0, x) \leq 0 \text{ for all } x \in B'.$$

The set of all solutions of the equilibrium problem will be denoted by

$$\mathrm{eq}(g \mid A', B') := \{x^0 \in A' \mid g(x^0, x) \leq 0, \ \forall\, x \in B'\}.$$

Remark 4.2. It is easy to see that

$$\mathrm{eq}(g \mid A', \emptyset) = A'$$

and that

$$\mathrm{eq}(g \mid A', B') \subseteq \mathrm{eq}(g \mid A', B''), \, \forall B'' \subseteq B'$$

Example 4.3 (optimization problems). Consider a minimization problem

$$\begin{cases} f(x) \longrightarrow \min \\ x \in S, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a function and $S \subseteq \mathbb{R}^n$ is a nonempty set. By defining the bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$g(u, v) := f(u) - f(v), \ \forall\, (u, v) \in \mathbb{R}^n \times \mathbb{R}^n,$$

we obtain

$$\mathrm{eq}(g \mid S, S) = \operatorname*{argmin}_{x \in S} f(x).$$

Example 4.4 (variational inequalities). Let $T : S \to \mathbb{R}^n$ be a function defined on a nonempty set $S \subseteq \mathbb{R}^n$. The problem of finding $x^0 \in S$ such that

$$\langle T(x^0), x - x^0 \rangle \geq 0, \forall\, x \in S$$

is called a variational inequality. Denote by sol(VI) the set of its solutions. By defining the bifunction $g : S \times S \to \mathbb{R}$ as

$$g(u, v) := \langle T(u), u - v \rangle, \ \forall\, (u, v) \in S \times S,$$

we obtain

$$\mathrm{eq}(g \mid S, S) = \mathrm{sol(VI)}$$

Example 4.5 (the best approximation problem).

1. The problem of best approximation of x^* by elements of S fits the model described in Example 4.3, where

   $$f(x) = \|x - x^*\| \, \text{and} \, g(u, v) := f(u) - f(v)$$

   hence

   $$\mathrm{eq}(g \mid S, S) = \operatorname*{argmin}_{x \in S} \|x - x^*\|.$$

2. Another way of seeing the best approximation problem as an equilibrium problem is to consider the bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$g(u, v) := \langle v - u, x^* - u\rangle, \ \forall\, (u, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

According to Theorem 2.10,

$$\mathrm{eq}(g \mid S, S) = \{x^0 \in S \mid \langle x - x^0, x^* - x^0\rangle \le 0, \ \forall\, x \in S\} \subseteq P_S(x^*),$$

the equality being true whenever S is convex, i.e.,

$$\mathrm{eq}(g \mid S, S) = P_S(x^*).$$

Actually, by considering the function $T : S \to \mathbb{R}^n$ defined by $T(x) = x - x^*$ for all $x \in S$, we recover

$$g(u, v) = \langle T(u), u - v\rangle, \forall\, (u, v) \in S \times S,$$

hence, under the convexity assumption on S we can reduce the best approximation problem to a variational inequality:

$$P_S(x^*) = \mathrm{sol(VI)}.$$

Example 4.6 (the farthest point problem). Let $S \subseteq \mathbb{R}^n$ be a nonempty set and let $x^* \in \mathbb{R}^n$. The problem of finding the farthest points from S to x^* fits the model described in Example 4.3, where

$$f(x) = -\|x - x^*\| \,\text{and}\, g(u, v) := f(u) - f(v),$$

hence

$$\mathrm{eq}(g \mid S, S) = \operatorname*{argmax}_{x\in S} \|x - x^*\|$$

Theorem 4.7. *Let A be a nonempty set, let $S = \mathrm{conv}M$ for some nonempty set $M \subseteq \mathbb{R}^n$ and let $g : A \times S \to \mathbb{R}$ be a bifunction. If for every $u \in A$, the function $h = g(u, \cdot) : S \to \mathbb{R}$ is quasiconvex, i.e.,*

$$h((1 - t)v' + tv'') \le \max\{h(v'), h(v'')\}$$

for all $v, v' \in S$ and $t \in [0, 1]$, then

$$\mathrm{eq}(g \mid A, S) = \{x^0 \in A \mid g(x^0, x) \le 0, \ \forall\, x \in M\},$$

i.e.,

$$\mathrm{eq}(g \mid A, S) = \mathrm{eq}(g \mid A, M).$$

Proof. We denote by

$$E := \{x^0 \in A \mid g(x^0, x) \le 0, \ \forall\, x \in M\}.$$

It is obvious that $\mathrm{eq}(g \mid A, S) \subseteq E$. In order to prove the converse, let $x^0 \in E$ and $x \in S$, arbitrary chosen. Since $x \in S = \mathrm{conv}\, M$, this implies that there exists $k \in \mathbb{N}$, $x^1, x^2, ..., x^k \in M$ and $t_1, t_2, ..., t_k \ge 0$ such that $\sum_{i=1}^k t_i = 1$ and that $x = \sum_{i=1}^k t_i x^i$. Therefore, given that $g(x^0, \cdot)$ is quasiconvex, we have

$$g(x^0, x) = g(x^0, \sum_{i=1}^{k} t_i x^i) \le \max\{g(x^0, x^i) \mid i = 1, \dots, k\} \le 0,$$

since $x^1, x^2, ..., x^k \in M$. □

Remark 4.8. Consider the minimization problem described in Example 4.3, where $g(u,v) = f(u) - f(v)$ for all $u, v \in S$. Since for every $u \in S$ we have $g(u,\cdot) = f(u) - f$, the quasiconvexity of $g(u,\cdot)$ for some $u \in S$ reduces to the the quasiconcavity of f. Thus we deduce from the Theorem 4.7 the following result.

Corollary 4.9. *Assume that* $S = \operatorname{conv} M$ *for some nonempty set* $M \subseteq \mathbb{R}^n$. *If* $f : S \to \mathbb{R}$ *is a quasiconcave function, then*

$$\operatorname*{argmin}_{x\in S} f(x) = \{x^0 \in S \mid f(x^0) \leq f(x), \forall\, x \in M\} \supseteq \operatorname*{argmin}_{x\in M} f(x).$$

Moreover, $\operatorname*{argmin}_{x\in S} f(x)$ *is nonempty if and only if so is* $\operatorname*{argmin}_{x\in M} f(x)$, *hence*

$$\min\, f(S) = \min\, f(M).$$

The assumptions on quasiconvexity of $g(u,\cdot)$ in Theorem 4.7 and quasiconcavity of f in Corollary 4.9 are essential, as shown by the next example (Example 4.10). Moreover, under the hypothesis of Corollary 4.9, the inclusion

$$\operatorname*{argmin}_{x\in S} f(x) \subseteq \operatorname*{argmin}_{x\in M} f(x)$$

does not hold in general, as shown by Example 4.11.

Example 4.10. Let $n = 1$, $M = \{-1, 1\}$, $S = \operatorname{conv} M = [-1, 1]$. Consider the function

$$\begin{cases} f : S \to \mathbb{R} \\ f(x) = x^2 \end{cases}$$

and the bifunction

$$\begin{cases} g : \mathbb{R}^2 \to \mathbb{R} \\ g(u,v) = f(u) - f(v),\ \forall\, (u,v) \in \mathbb{R}^2. \end{cases}$$

Clearly, the f is not quasiconcave, hence function $g(u,\cdot) : S \to \mathbb{R}$ is not quasiconvex for any $u \in S$, in view of Remark 4.8. It is easy to see that

$$\begin{array}{rcl} \operatorname{eq}(g \mid S, S) & = & \operatorname*{argmin}_{x\in S} f(x) \\ & = & \{0\} \\ & \not\supseteq & \operatorname{eq}(g \mid S, M) \\ & = & \{x^0 \in S \mid g(x^0, x) \leq 0, \forall\, x \in M\} \\ & = & S. \end{array}$$

Of course, this example also shows that the quasiconcavity assumption imposed on f in Corollary 4.8 is essential, because

$$\operatorname*{argmin}_{x\in S} f(x) = \{0\} \not\supseteq \operatorname*{argmin}_{x\in M} f(x) = M.$$

Example 4.11. Let $n = 1$, $M = \{-1, 1\}$, $S = \operatorname{conv} M = [-1, 1]$. Consider the function $f : S \to \mathbb{R}$ defined as $f(x) = \max\{0, x\}$, for all $x \in S$. Obviously, f is nondecreasing, hence quasiconcave. However,

$$\operatorname*{argmin}_{x\in S} f(x) = [-1, 0] \not\subseteq \operatorname*{argmin}_{x\in M} f(x) = \{-1\}.$$

Corollary 4.12. *Assume that* $S = \operatorname{conv} M$ *for some nonempty set* $M \subseteq \mathbb{R}^n$*. and let* $T : S \to \mathbb{R}^n$ *be an arbitrary function. Then the set of solutions*

$$\operatorname{sol(VI)} := \{x^0 \in S \mid \langle T(x^0), x - x^0 \rangle \geq 0, \forall\, x \in S\}$$

to the variational inequality introduced in Example 4.4, admits the following representation

$$\operatorname{sol(VI)} = \{x^0 \in S \mid \langle T(x^0), x - x^0 \rangle \geq 0, \forall\, x \in M\}.$$

Corollary 4.13. *If* $S \subseteq \mathbb{R}^n$ *is a nonempty convex compact set and function* $g(u, \cdot)$ *is quasiconvex on* S *for every* $u \in A$*, then*

$$\operatorname{eq}(g \mid A, S) = \{x^0 \in A \mid g(x^0, x) \leq 0, \ \forall\, x \in \operatorname{ext} S\}$$

i.e.,

$$\operatorname{eq}(g \mid A, S) = \operatorname{eq}(g \mid A, \operatorname{ext} S).$$

Proof. Follows by Theorem 4.7 and Minkowski's theorem (Theorem 2.1). □

Note that the conclusion of Corollary 4.13 still holds if S is a so-called "Minkowski set" (sets which were introduced in [3], i.e., closed, possibly unbounded sets which can be represented as the convex hull of their extreme points), yet the hypothesis of closeness is crucial as it shown in the next example.

Example 4.14. Let $n = 1, S =]-1, 1[$. Consider the function $f : S \to \mathbb{R}$ defined by $f(x) = -x^2$ and the bifunction $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(u, v) := f(u) - f(v), \ \forall\, (u, v) \in \mathbb{R}^2.$$

Clearly, $\forall\, u \in S$ the function $g(u, \cdot) : S \to \mathbb{R}$ is quasiconvex (even convex). It is easy to see that

$$\operatorname{eq}(g \mid S, S) = \operatorname*{argmin}_{x \in S} f(x) = \emptyset \neq \operatorname{eq}(g \mid S, \operatorname{ext} S) = \operatorname{eq}(g \mid S, \emptyset) = S.$$

Theorem 4.15. *Let* A *be a nonempty set, let* $S = \operatorname{cl}(\operatorname{conv}(M))$ *for some nonempty set* $M \subseteq \mathbb{R}^n$ *and let* $g : A \times S \to \mathbb{R}$ *be a bifunction. If function* $g(u, \cdot)$ *is quasiconvex and lower semicontinous on* S *for every* $u \in A$*, then*

$$\operatorname{eq}(g \mid A, S) = \{x^0 \in A \mid g(x^0, x) \leq 0, \ \forall\, x \in M\},$$

i.e.,

$$\operatorname{eq}(g \mid A, S) = \operatorname{eq}(g \mid A, M).$$

Proof. Let $x^0 \in A$ such that $g(x^0, x) \leq 0, \forall\, x \in M$. We prove that

$$g(x^0, y) \leq 0, \forall\, y \in S.$$

Let $y \in S$. By Theorem 4.7, it follows that

$$\begin{array}{rl} & \{x^0 \in A \mid g(x^0, x) \leq 0, \ \forall\, x \in M\} \\ = & \{x^0 \in A \mid g(x^0, x) \leq 0, \ \forall\, x \in \operatorname{conv} M\}. \end{array}$$

Hence

$$g(x^0, x) \leq 0, \ \forall\, x \in \operatorname{conv} M. \tag{$*$}$$

Since $S = \mathrm{cl}(\mathrm{conv}\, M)$ and $y \in S$, it follows that there exists a sequence $(y^k)_{k\in\mathbb{N}}$ in $\mathrm{conv}\, M$ which converges to y.
According to $(*)$, we have $g(x^0, y^k) \le 0, \forall\, k \in \mathbb{N}$, i.e.,

$$y^k \in L := \{z \in S \mid g(x^0, z) \le 0\},\ \forall\, k \in \mathbb{N}.$$

On the other hand, the function $g(x^0, \cdot) : S \to \mathbb{R}$ is lower semicontinuous so, the level set L is closed with respect to the induced topology in S from $\mathbb{R}^n$ and, since S is closed, we deduce that L is a closed subset of $\mathbb{R}^n$, hence

$$y = \lim_{k\to\infty} y^k \in \mathrm{cl}\, L = L.$$

Thereby, $g(x^0, y) \le 0$ and, since y was arbitrary chosen from S, we obtain that

$$\mathrm{eq}(g \mid A, S) \supseteq \{x^0 \in S \mid g(x^0, x) \le 0,\ \forall\, x \in M\}.$$

The reverse inclusion is obvious. □

An immediate consequence of Theorem 4.15 and Straszewicz's theorem (Theorem 2.2) is the following corollary (Corollary 4.16), where we characterize solutions of an equilibrium problem by means of exposed points. Finally, another consequence of Theorem 4.15, by also using Remark 4.8 is given in Corollary 4.17.

Corollary 4.16. *Let A be a nonempty set, let $S \subseteq \mathbb{R}^n$ be a nonempty convex compact set and let $g : A \times S \to \mathbb{R}$ be a bifunction. If function $g(u, \cdot)$ is quasiconvex and lower semicontinous on S for every $u \in A$, then*

$$\mathrm{eq}(g \mid A, S) = \{x^0 \in A \mid g(x^0, x) \le 0,\ \forall\, x \in \exp S\},$$

i.e.,

$$\mathrm{eq}(g \mid A, S) = \mathrm{eq}(g \mid A, \exp S).$$

Corollary 4.17. *Assume that $S = \mathrm{cl}(\mathrm{conv}\, M)$ for some nonempty set $M \subseteq \mathbb{R}^n$. If $f : S \to \mathbb{R}$ is a quasiconcave upper semicontinous function, then*

$$\operatorname*{argmin}_{x\in S} f(x) = \{x^0 \in S \mid f(x^0) \le f(x), \forall\, x \in M\} \supseteq \operatorname*{argmin}_{x\in M} f(x).$$

Moreover, $\operatorname*{argmin}_{x\in S} f(x)$ *is nonempty if and only if so is* $\operatorname*{argmin}_{x\in M} f(x)$, *hence*

$$\min\, f(S) = \min\, f(M).$$

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Transmission problem between two Herschel-Bulkley fluids in thin layer

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Abstract. The paper is devoted to the study of steady-state transmission problem between two Herschel-Bulkley fluids in thin layer.

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1. Introduction

The rigid viscoplastic and incompressible fluid of Herschel-Bulkley has been studied and used by many mathematicians, physicists and engineers, to model the flow of metals, plastic solids and a variety of polymers. Due to the existence of the yield limit, the model can capture phenomena connected with the development of discontinuous stresses. A particularity of Herschel-Bulkley fluid lies in the presence of rigid zones located in the interior of the flow and as yield limit increases, the rigid zones become larger and may completely block the flow, this phenomenon is known as the blockage property. The literature concerning this topic is extensive; see e.g. [7, 8, 9, 11]. The purpose of this paper is to study the asymptotic behavior of the steady flow of Herschel-Bulkley fluid in a two-dimensional thin layer. The paper is organized as follows. In section 2 we present the mechanical problem of the steady flow of Herschel-Bulkley fluid in a two-dimensional thin layer. We introduce some notations and preliminaries. Moreover, we define some function spaces and we recall the variational formulation. In Section 3, we are interested in the asymptotic behavior, to this aim we prove some convergence results concerning the velocity and pressure when the thickness tends to zero. Besides, the uniqueness of a limit solution has been also established.

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2. Problem statement

Denoting by I the open interval $I =]0,1[$. Introducing the functions $h_i : I \to \mathbb{R}_+^*$ such that $h_i \in C^1(I)$, $i = 1,2$.

Considering the following domains

$$\begin{aligned}
\Omega_1 &= \left\{(x,y) \in \mathbb{R}^2 / \ x \in I \text{ and } 0 < y < h_1(x)\right\},\\
\Omega_1^\varepsilon &= \left\{(x_1,x_2) \in \mathbb{R}^2 / \ x_1 \in I \text{ and } 0 < x_2 < \varepsilon h_1(x_1)\right\},\\
\Omega_2 &= \left\{(x,y) \in \mathbb{R}^2 / \ x \in I \text{ and } h_1(x) < y < h_2(x)\right\},\\
\Omega_2^\varepsilon &= \left\{(x_1,x_2) \in \mathbb{R}^2 / \ x_1 \in I \text{ and } \varepsilon h_1(x_1) < x_2 < \varepsilon h_2(x_1)\right\},
\end{aligned}$$

where $\varepsilon > 0$. Remark that if $(x_1,x_2) \in \Omega_i^\varepsilon$ then $(x,y) = (x_1, \frac{x_2}{\varepsilon}) \in \Omega_i$. This permits us to define, for every function $\varphi_i^\varepsilon : \Omega_i^\varepsilon \to \mathbb{R}$, the function $\widehat{\varphi_i^\varepsilon} : \Omega_i \to \mathbb{R}$ given by $\widehat{\varphi_i^\varepsilon}(x,y) = \varphi_i^\varepsilon(x_1,x_2)$, $i = 1,2$. Let $1 < p \leq 2$, p' the conjugate p, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ and $f_i = (f_{i1}, f_{i2}) \in L^{p'}(\Omega_i)^2$ a given functions. We define the functions $f_i^\varepsilon \in L^{p'}(\Omega_i^\varepsilon)^2$ such that $\widehat{f_i^\varepsilon} = f_i$, $i = 1,2$. We consider a mathematical problem modeling the steady flow of a rigid viscoplastic and incompressible Herschel-Bulkley fluid. We suppose that the consistency and yield limit of the fluid are respectively $\mu_i \varepsilon^p$, $g_i \varepsilon$ where μ_i, $g_i > 0$, $i = 1,2$ and p represents the power-law index. The first fluid occupies a bounded domain $\Omega_1^\varepsilon \subset R^2$ with the boundary $\partial\Omega_1^\varepsilon$ of class C^1. The second one occupies a bounded domain $\Omega_2^\varepsilon \subset \mathbb{R}^2$ with the boundary $\partial\Omega_2^\varepsilon$ of class C^1. We denote by Ω^ε the domain $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$ and we suppose that $\partial\Omega_1^\varepsilon = \Gamma_0 \cup \Gamma_1$ and $\partial\Omega_2^\varepsilon = \Gamma_0 \cup \Gamma_2$ the velocity is known and equal to zero, where $\Gamma_0, \Gamma_1, \Gamma_2$ are measurable domains and $meas(\Gamma_1)$, $meas(\Gamma_2) > 0$. The fluids are acted upon by given volume forces of densities f_1, f_2 respectively. We denote by S_2 the space of symmetric tensors on $\mathbb{R}^2$. We define the inner product and the Euclidean norm on $\mathbb{R}^2$ and S_2, respectively, by

$$\begin{aligned}
u.v &= u_l v_l \quad \forall u,v \in \mathbb{R}^2 \quad \text{and} \quad \sigma.\tau = \sigma_{lm}\tau_{lm} \ \forall \sigma,\tau \in S_2.\\
|u| &= (u.u)^{\frac{1}{2}} \quad \forall u \in \mathbb{R}^2 \quad \text{and} \quad |\sigma| = (\sigma.\sigma)^{\frac{1}{2}} \ \forall \sigma \in S_2.
\end{aligned}$$

Here and below, the indices l and m run from 1 to 2 and the summation convention over repeated indices is used. We denote by $\widetilde{\sigma_i^\varepsilon}$ the deviator of σ_i^ε given by

$$\sigma_i^\varepsilon = -p_i^\varepsilon I_2 + \widetilde{\sigma_i^\varepsilon},$$

where p_i^ε, $i = 1,2$ represents the hydrostatic pressure and I_2 denotes the identity matrix of size 2. We consider the rate of deformation operator defined for every $v_i^\varepsilon \in W^{1,p}(\Omega_i^\varepsilon)^2$ by

$$D(v_i^\varepsilon) = (D_{lm}(v_i^\varepsilon)), \ \ D_{lm}(v_i^\varepsilon) = \frac{1}{2}((v_i^\varepsilon)_{l,m} + (v_i^\varepsilon)_{m,l}), \ i = 1,2.$$

We denote by n the unit outward normal vector on the boundary Γ_0 oriented to the exterior of Ω_1^ε and to the interior of Ω_2^ε, see the figure below. For every vector field $v_i^\varepsilon \in W^{1,p}(\Omega_i^\varepsilon)^2$ we also write v_i^ε for its trace on $\partial\Omega_i^\varepsilon$, $i = 1,2$.

The steady-state transmission problem for the Herschel-Bulkley fluids in thin layer is given by the following mechanical problem.

Problem P_ε. Find the velocity field $u_i^\varepsilon = (u_{i1}^\varepsilon, u_{i2}^\varepsilon) : \Omega_i^\varepsilon \to \mathbb{R}^2$, the stress field $\sigma_i^\varepsilon = (\sigma_{i1}^\varepsilon, \sigma_{i2}^\varepsilon) : \Omega_i^\varepsilon \to S_2$ and the pressure $p_i^\varepsilon : \Omega_i^\varepsilon \to \mathbb{R}$, $i = 1, 2$ such that

$$\operatorname{div} \sigma_1^\varepsilon + f_1^\varepsilon = 0 \text{ in } \Omega_1^\varepsilon. \tag{2.1}$$

$$\operatorname{div} \sigma_2^\varepsilon + f_2^\varepsilon = 0 \text{ in } \Omega_2^\varepsilon. \tag{2.2}$$

$$\left.\begin{array}{ll} \widetilde{\sigma_1^\varepsilon} = \mu_1 \varepsilon^p \ |D(u_1^\varepsilon)|^{p-2} D(u_1^\varepsilon) + g_1 \varepsilon \frac{D(u_1^\varepsilon)}{|D(u_1^\varepsilon)|} & \text{if } |D(u_1^\varepsilon)| \neq 0 \\ \left|\widetilde{\sigma_1^\varepsilon}\right| \leq g_1 \varepsilon & \text{if } |D(u_1^\varepsilon)| = 0 \end{array}\right\} \text{ in } \Omega_1^\varepsilon, \tag{2.3}$$

$$\left.\begin{array}{ll} \widetilde{\sigma_2^\varepsilon} = \mu_2 \varepsilon^p \ |D(u_2^\varepsilon)|^{p-2} D(u_2^\varepsilon) + g_2 \varepsilon \frac{D(u_2^\varepsilon)}{|D(u_2^\varepsilon)|} & \text{if } |D(u_2^\varepsilon)| \neq 0 \\ \left|\widetilde{\sigma_2^\varepsilon}\right| \leq g_2 \varepsilon & \text{if } |D(u_2^\varepsilon)| = 0 \end{array}\right\} \text{ in } \Omega_2^\varepsilon, \tag{2.4}$$

$$\operatorname{div} u_1^\varepsilon = 0 \text{ in } \Omega_1^\varepsilon, \tag{2.5}$$

$$\operatorname{div} u_2^\varepsilon = 0 \text{ in } \Omega_2^\varepsilon \ , \tag{2.6}$$

$$u_1^\varepsilon = 0 \text{ on } \Gamma_1 \ , \tag{2.7}$$

$$u_2^\varepsilon = 0 \text{ on } \Gamma_2, \tag{2.8}$$

$$u_1^\varepsilon - u_2^\varepsilon = 0 \text{ on } \Gamma_0, \tag{2.9}$$

$$\sigma_1^\varepsilon.\mathbf{n} - \sigma_2^\varepsilon.\mathbf{n} = 0 \text{ on } \Gamma_0. \tag{2.10}$$

Here, the flow is given by the equations (2.1) and (2.2). Equations (2.3) and (2.4) represent the constitutive law of Herschel-Bulkley fluid. Equations (2.5) and (2.6) represents the incompressibility condition. Equality (2.7), (2.8) give the velocities on the boundaries Γ_1 and Γ_2, respectively. Finally, on the boundary part Γ_0, equations (2.9) and (2.10) represent the transmission condition for liquid-liquid interface. Let us define now the following Banach spaces

$$W_{\Gamma i}^{1,p}(\Omega_i^\varepsilon) = \left\{ v_i \in W^{1,p}(\Omega_i^\varepsilon)^2 : v_i = 0 \text{ on } \Gamma_i,\, i = 1, 2 \right\}, \tag{2.11}$$

$$W_{\operatorname{div}}^{p,\varepsilon}(\Omega_i^\varepsilon) = \left\{ v_i \in W^{1,p}(\Omega_i^\varepsilon)^2 : \operatorname{div}(v_i) = 0 \text{ in } \Omega_i^\varepsilon,\, i = 1, 2 \right\}, \tag{2.12}$$

$$W_{\operatorname{div}}^{p}(\Omega_i) = \left\{ v_i \in W^{1,p}(\Omega_i)^2 : \operatorname{div}(v_i) = 0 \text{ in } \Omega_i,\, i = 1, 2 \right\}, \tag{2.13}$$

$$W_p = \left\{ \begin{array}{c} (\varphi_1, \varphi_2) \in L^p(\Omega_1) \times L^p(\Omega_2) : \\ \frac{\partial \varphi_1}{\partial y} \in L^p(\Omega_1),\ \frac{\partial \varphi_2}{\partial y} \in L^p(\Omega_2) \end{array} \right\}, \tag{2.14}$$

$$L_0^p(\Omega_i^\varepsilon) = \left\{ \begin{array}{c} \varphi_i^\varepsilon \in L^p(\Omega_i^\varepsilon) : \\ \int_{\Omega_i^\varepsilon} \varphi_i^\varepsilon(x_1, x_2) dx_1 dx_2 = 0,\, i = 1, 2 \end{array} \right\}, \tag{2.15}$$

$$L_0^p(\Omega_i) = \left\{ \varphi_i \in L^p(\Omega_i) \ : \int_{\Omega_i} \varphi_i(x, y) dx dy = 0,\, i = 1, 2 \right\}, \tag{2.16}$$

$$W^\varepsilon = \left\{ \begin{array}{c} (v_1, v_2) \in W_{\operatorname{div}}^{p,\varepsilon}(\Omega_1^\varepsilon) \times W_{\operatorname{div}}^{p,\varepsilon}(\Omega_2^\varepsilon) : v_1 = v_2 \\ \text{on } \Gamma_0,\, v_1 = 0 \text{ on } \Gamma_1,\, v_2 = 0 \text{ on } \Gamma_2 \end{array} \right\}. \tag{2.17}$$

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem.

The use of Green's formula permits us to derive the following variational formulation of the mechanical problem (P_ε), see [10, 11].

Problem PV$_\varepsilon$. For prescribed data $(f_1^\varepsilon, f_2^\varepsilon) \in L^{p'}(\Omega_1^\varepsilon)^2 \times L^{p'}(\Omega_2^\varepsilon)^2$. Find $(u_1^\varepsilon, u_2^\varepsilon) \in W^\varepsilon$ and $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$ satisfying the variational inequality

$$\begin{aligned}
&\mu_1\varepsilon^p \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|^{p-2} D(u_1^\varepsilon) D(v_1 - u_1^\varepsilon) dx_1 dx_2 + g_1\varepsilon \int_{\Omega_1^\varepsilon} |D(v_1)| \, dx_1 dx_2 \\
&-g_1\varepsilon \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)| \, dx_1 dx_2 + \mu_2\varepsilon^p \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|^{p-2} D(u_2^\varepsilon) D(v_2 - u_2^\varepsilon) dx_1 dx_2 \\
&+g_2\varepsilon \int_{\Omega_2^\varepsilon} |D(v_2)| \, dx_1 dx_2 - g_2\varepsilon \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)| \, dx_1 dx_2 \\
&\geq \int_{\Omega_1^\varepsilon} f_1^\varepsilon.(v_1 - u_1^\varepsilon) dx_1 dx_2 + \int_{\Omega_1^\varepsilon} p_1^\varepsilon \operatorname{div}(v_1 - u_1^\varepsilon) dx_1 dx_2 \\
&+ \int_{\Omega_2^\varepsilon} f_2^\varepsilon.(v_2 - u_2^\varepsilon) dx_1 dx_2 + \int_{\Omega_2^\varepsilon} p_2^\varepsilon \operatorname{div}(v_2 - u_2^\varepsilon) dx_1 dx_2, \; \forall (v_1, v_2) \in W^\varepsilon. \qquad (2.18)
\end{aligned}$$

It is known that this variational problem has a unique solution $(u_1^\varepsilon, u_2^\varepsilon) \in W^\varepsilon$ and $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$, see for more details [7, 10, 11].

3. Asymptotic behavior

In this section, we establish some results concerning the asymptotic behavior of the solution when ε tends to zero. We begin by recalling the following lemmas (see [12, 1, 3, 6])

Lemma 3.1. 1. *Poincaré's inequality. For every $v_i \in W_{\Gamma i}^{1,p}(\Omega_i^\varepsilon)$ we have*

$$\|v_i^\varepsilon\|_{L^p(\Omega_i^\varepsilon)^2} \leq \varepsilon \left\| \frac{\partial v_i^\varepsilon}{\partial x_2} \right\|_{L^p(\Omega_i^\varepsilon)^2}, \; i = 1, 2. \qquad (3.1)$$

2. *Korn's inequality. For every $v_i \in W_{\Gamma i}^{1,p}(\Omega_i^\varepsilon)$ there exists a positive constant C_0 independent on ε, such that*

$$\|\nabla v_i^\varepsilon\|_{L^p(\Omega_i^\varepsilon)^4} \leq C_0 \|D(v_i^\varepsilon)\|_{L^p(\Omega_i^\varepsilon)^4}, \; i = 1, 2. \qquad (3.2)$$

Lemma 3.2. *Let E be a Banach space, $A : E \to E'$ a monotone and hemi-continuous operator, $J : E \to]-\infty, +\infty]$ a proper and convex functional. Let $u \in E$ and $f \in E'$. The following assertions are equivalent:*

1. $\langle Au; v - u\rangle_{E' \times E} + J(v) - J(u) \geq \langle f; v - u\rangle_{E' \times E} \quad \forall v \in E.$
2. $\langle Av; v - u\rangle_{E' \times E} + J(v) - J(u) \geq \langle f; v - u\rangle_{E' \times E} \quad \forall v \in E.$

The main results of this section are stated by the following proposition.

Proposition 3.3. *Let $(u_1^\varepsilon, u_2^\varepsilon) \in W^\varepsilon$ and $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$ be the solution of variational problem (PV_ε). Then, there exists $(\widehat{u_1}, \widehat{u_2}) \in W_p^2$ and $(\widehat{p_1}, \widehat{p_2}) \in L_0^{p'}(\Omega_1) \times$*

$L_0^{p'}(\Omega_2)$ *such that*

$$(\widehat{u_1^\varepsilon}, \widehat{u_2^\varepsilon}) \to (\widehat{u_1}\ , \widehat{u_2}) \ \textit{in } W_p^2 \ \textit{weakly,} \tag{3.3}$$

$$\left(\frac{\partial \widehat{u_{12}^\varepsilon}}{\partial y}, \frac{\partial \widehat{u_{22}^\varepsilon}}{\partial y}\right) \to (0,0) \ \textit{in } L^p(\Omega_1) \times L^p(\Omega_2) \ \textit{weakly,} \tag{3.4}$$

$$(\widehat{p_1^\varepsilon}, \widehat{p_2^\varepsilon}) \to (\widehat{p_1}\ , \widehat{p_2}) \ \textit{in } L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2) \ \textit{weakly.} \tag{3.5}$$

Proof. Choosing $(v_1, v_2) = (0,0)$ as test function in inequality (2.18), we deduce that

$$\begin{aligned} &\mu_1 \varepsilon^p \left\| D(u_1^\varepsilon) \right\|_{L^p(\Omega_1^\varepsilon)^4}^p \ + \mu_2 \varepsilon^p \left\| D(u_2^\varepsilon) \right\|_{L^p(\Omega_2^\varepsilon)^4}^p \\ \leq \quad & \int_{\Omega_1^\varepsilon} f_1^\varepsilon . u_1^\varepsilon dx_1 dx_2 + \int_{\Omega_2^\varepsilon} f_2^\varepsilon . u_2^\varepsilon dx_1 dx_2 . \end{aligned}$$

this permits us to obtain, making use of Poincaré's and Korn's inequalities and by passage to variables x and y

$$\begin{aligned} \left\| \widehat{u_1^\varepsilon} \right\|_{L^p(\Omega_1)^2} + \left\| \widehat{u_2^\varepsilon} \right\|_{L^p(\Omega_2)^2} &\leq c, & (3.6) \\ \left\| \frac{\partial \widehat{u_1^\varepsilon}}{\partial y} \right\|_{L^p(\Omega_1)^2} + \left\| \frac{\partial \widehat{u_2^\varepsilon}}{\partial y} \right\|_{L^p(\Omega_2)^2} &\leq c, & (3.7) \\ \left\| \frac{\partial \widehat{u_1^\varepsilon}}{\partial x} \right\|_{L^p(\Omega_1)^2} + \left\| \frac{\partial \widehat{u_2^\varepsilon}}{\partial x} \right\|_{L^p(\Omega_2)^2} &\leq \frac{c}{\varepsilon}. & (3.8) \end{aligned}$$

Moreover, we get using the incompressibility condition (2.5), (2.6) and Green's formula, for any function $(\varphi_1^\varepsilon, \varphi_2^\varepsilon) \in W_{\Gamma_1}^{1,p}(\Omega_1^\varepsilon) \times W_{\Gamma_2}^{1,p}(\Omega_2^\varepsilon)$

$$\begin{aligned} &\int_{\Omega_1} \frac{\partial \widehat{u_{12}^\varepsilon}}{\partial y} \widehat{\varphi_1^\varepsilon} dx dy + \int_{\Omega_2} \frac{\partial \widehat{u_{22}^\varepsilon}}{\partial y} \widehat{\varphi_2^\varepsilon} dx dy \\ = \varepsilon & \int_{\Omega_1} \widehat{u_{11}^\varepsilon} \frac{\partial \widehat{\varphi_1^\varepsilon}}{\partial x} dx dy + \varepsilon \int_{\Omega_2} \widehat{u_{21}^\varepsilon} \frac{\partial \widehat{\varphi_2^\varepsilon}}{\partial x} dx dy . \end{aligned}$$

Which gives, making use of (2.14)

$$\left\| \frac{\partial \widehat{u_{12}^\varepsilon}}{\partial y} \right\|_{W^{-1,p'}(\Omega_1)} + \left\| \frac{\partial \widehat{u_{22}^\varepsilon}}{\partial y} \right\|_{W^{-1,p'}(\Omega_2)} \leq c\varepsilon. \tag{3.9}$$

We can then extract a subsequences still denoted by $(\widehat{u_1^\varepsilon}, \widehat{u_2^\varepsilon})$ such that

$$(\widehat{u_1^\varepsilon}, \widehat{u_2^\varepsilon}) \to (\widehat{u_1}, \widehat{u_2}) \text{ in } L^p(\Omega_1)^2 \times L^p(\Omega_2)^2 \text{ weakly,} \tag{3.10}$$

$$\left(\frac{\partial \widehat{u_1^\varepsilon}}{\partial y}, \frac{\partial \widehat{u_2^\varepsilon}}{\partial y}\right) \to \left(\frac{\partial \widehat{u_1}}{\partial y}, \frac{\partial \widehat{u_2}}{\partial y}\right) \text{ in } L^p(\Omega_1)^2 \times L^p(\Omega_2)^2 \text{ weakly,} \tag{3.11}$$

$$\left(\frac{\partial \widehat{u_{12}^\varepsilon}}{\partial y}, \frac{\partial \widehat{u_{22}^\varepsilon}}{\partial y}\right) \to (0,0) \text{ in } L^p(\Omega_1) \times L^p(\Omega_2) \text{ weakly.} \tag{3.12}$$

Let now $(v_1^\varepsilon, v_2^\varepsilon) \in W^{1,p}_{\Gamma_1}(\Omega_1^\varepsilon) \times W^{1,p}_{\Gamma_2}(\Omega_2^\varepsilon)$, we obtain by setting $(u_1^\varepsilon - v_1^\varepsilon, u_2^\varepsilon - v_2^\varepsilon)$ as test function in inequality (2.18), using the incompressibility conditions (2.5) and (2.6) as well as the Green formula and Hölder's inequality

$$\begin{aligned}
&\int_{\Omega_1^\varepsilon} \nabla p_1^\varepsilon v_1^\varepsilon dx_1 dx_2 + \int_{\Omega_2^\varepsilon} \nabla p_2^\varepsilon v_2^\varepsilon dx_1 dx_2 \\
&\le \mu_1 \varepsilon^p \left(\int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|^p \, dx_1 dx_2 \right)^{\frac{1}{p'}} \left(\int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}} \\
&\quad + g_1 \varepsilon^{\frac{1}{p'}+1} meas(\Omega_1^\varepsilon)^{\frac{1}{p'}} \left(\int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}} \\
&\quad + \varepsilon \left\| \widehat{f_1^\varepsilon} \right\|_{L^{p'}(\Omega_1^\varepsilon)^2} \left\| \widehat{v_1^\varepsilon} \right\|_{W^{1,p}_{\Gamma_1}(\Omega_1)} + \varepsilon \left\| \widehat{f_2^\varepsilon} \right\|_{L^{p'}(\Omega_2^\varepsilon)^2} \left\| \widehat{v_2^\varepsilon} \right\|_{W^{1,p}_{\Gamma_2}(\Omega_2)} \\
&\quad + \mu_2 \varepsilon^p \left(\int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|^p \, dx_1 dx_2 \right)^{\frac{1}{p'}} \left(\int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}} \\
&\quad + g_2 \varepsilon^{\frac{1}{p'}+1} meas(\Omega_2^\varepsilon)^{\frac{1}{p'}} \left(\int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}} . \qquad (3.13)
\end{aligned}$$

On the other hand, it is easy to check that, after some algebraic manipulations, we find

$$\left(\int_{\Omega_i^\varepsilon} |D(v_i^\varepsilon)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}} \le \varepsilon^{\frac{1}{p}-1} \left\| \widehat{v_i^\varepsilon} \right\|_{W^{1,p}_{\Gamma_i}(\Omega_i)}, \; i = 1, 2. \qquad (3.14)$$

Hence, from (3.7), (3.8), (3.13) and (3.14) it follows that

$$\begin{aligned}
&\int_{\Omega_1^\varepsilon} \nabla p_1^\varepsilon v_1^\varepsilon dx_1 dx_2 + \int_{\Omega_2^\varepsilon} \nabla p_2^\varepsilon v_2^\varepsilon dx_1 dx_2 \\
&\le \; c\varepsilon \left(\left\| \widehat{v_1^\varepsilon} \right\|_{W^{1,p}_{\Gamma_1}(\Omega_1)} + \left\| \widehat{v_2^\varepsilon} \right\|_{W^{1,p}_{\Gamma_2}(\Omega_2)} \right). \qquad (3.15)
\end{aligned}$$

Passing to the variables x and y in the left hand side of (3.15) we find the following estimates

$$\left\| \widehat{p_1^\varepsilon} \right\|_{L_0^{p'}(\Omega_1)} + \left\| \widehat{p_2^\varepsilon} \right\|_{L_0^{p'}(\Omega_2)} \le c, \qquad (3.16)$$

$$\left\| \frac{\partial \widehat{p_1^\varepsilon}}{\partial x} \right\|_{W^{-1,p'}(\Omega_1)} + \left\| \frac{\partial \widehat{p_2^\varepsilon}}{\partial x} \right\|_{W^{-1,p'}(\Omega_2)} \le c, \qquad (3.17)$$

$$\left\| \frac{\partial \widehat{p_1^\varepsilon}}{\partial y} \right\|_{W^{-1,p'}(\Omega_1)} + \left\| \frac{\partial \widehat{p_2^\varepsilon}}{\partial y} \right\|_{W^{-1,p'}(\Omega_2)} \le \varepsilon c. \qquad (3.18)$$

Consequently, we can extract a subsequence still denoted by $\left(\widehat{p_1^\varepsilon}, \widehat{p_2^\varepsilon}\right)$ such that

$$\left(\widehat{p_1^\varepsilon}, \widehat{p_2^\varepsilon}\right) \to (\widehat{p_1}, \widehat{p_2}) \text{ in } L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2) \text{ weakly,} \tag{3.19}$$

which achieves the proof. This proof permits also to deduce that the limit pressure verify $(\widehat{p_1}(x,y), \widehat{p_2}(x,y)) = (\widehat{p_1}(x), \widehat{p_2}(x))$. □

Proposition 3.4. *The velocity limit given by (3.3) verifies*

$$\int_0^{h_1(x)} \widehat{u_{11}}(x,y)dy + \int_{h_1(x)}^{h_2(x)} \widehat{u_{21}}(x,y)dy = 0 \quad \forall x \in I. \tag{3.20}$$

Proof. We know from incompressibility conditions (2.5) and (2.6) that

$$\int_{\Omega_1^\varepsilon} \operatorname{div} u_1^\varepsilon(x_1,x_2)\varphi_1(x_1)dx_1dx_2 + \int_{\Omega_2^\varepsilon} \operatorname{div} u_2^\varepsilon(x_1,x_2)\varphi_2(x_1)dx_1dx_2$$
$$= 0 \quad \text{for all} \quad (\varphi_1, \varphi_2) \in D(I)^2.$$

This implies, using Green's formula

$$\int_{\Omega_1^\varepsilon} u_{11}^\varepsilon(x_1,x_2)\frac{d\varphi_1}{dx_1}(x_1)dx_1dx_2 + \int_{\Omega_2^\varepsilon} u_{21}^\varepsilon(x_1,x_2)\frac{d\varphi_2}{dx_1}(x_1)dx_1dx_2$$
$$= \int_{\Omega_1^\varepsilon} \frac{\partial u_{12}^\varepsilon}{\partial x_2}(x_1,x_2)\varphi_1(x_1)dx_1dx_2 + \int_{\Omega_2^\varepsilon} \frac{\partial u_{22}^\varepsilon}{\partial x_2}(x_1,x_2)\varphi_2(x_1)dx_1dx_2.$$

Hence, by passage to the variables x and y using Fubini's theorem and Green's formula, we can infer

$$\int_0^1 \varphi_1(x)\left(\frac{d}{dx}\int_0^{h_1(x)} \widehat{u_{11}^\varepsilon}(x,y)dy\right)dx + \int_0^1 \varphi_2(x)\left(\frac{d}{dx}\int_{h_1(x)}^{h_2(x)} \widehat{u_{21}^\varepsilon}(x,y)dy\right)dx$$
$$= 0 \quad \forall(\varphi_1,\varphi_2) \in D(I)^2.$$

Then,

$$\int_0^1 \varphi(x)\left(\frac{d}{dx}\left(\int_0^{h_1(x)} \widehat{u_{11}^\varepsilon}(x,y)dy + \int_{h_1(x)}^{h_2(x)} \widehat{u_{21}^\varepsilon}(x,y)dy\right)\right)dx = 0, \ \forall\varphi \in D(I).$$

Then,

$$\frac{d}{dx}\left(\int_0^{h_1(x)} \widehat{u_{11}^\varepsilon}(x,y)dy + \int_{h_1(x)}^{h_2(x)} \widehat{u_{21}^\varepsilon}(x,y)dy\right) = 0.$$

Moreover, the fact that $(\widehat{u_{11}^\varepsilon}, \widehat{u_{21}^\varepsilon}) \in L^p(\Omega_1) \times L^p(\Omega_2)$ and $(h_1,h_2) \in C^1(I)^2$ gives, using the Sobolev embedding $W^{1,p}(I) \subset C^0(\overline{I})$

$$\int_0^{h_1(x)} \widehat{u_{11}^\varepsilon}(x,y)dy + \int_{h_1(x)}^{h_2(x)} \widehat{u_{21}^\varepsilon}(x,y)dy \in C^0(\overline{I}).$$

Thus, by passage to the limit when ε tends to zero, taking into account the boundaries conditions (2.7), (2.8) and (2.9), the assertion (3.20) can be deduced. □

We derive in the proposition below the strong equation verified by the limit solution $(\widehat{u_1}, \widehat{u_2}) \in W_p^2$ and $(\widehat{p_1}, \widehat{p_2}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$.

Proposition 3.5. *If* $\left(\frac{\partial \widehat{u_{11}}}{\partial y}, \frac{\partial \widehat{u_{21}}}{\partial y}\right) \neq (0,0)$ *then the limit point* $(\widehat{u_{11}}, \widehat{u_{21}})$ *and* $(\widehat{p_1}, \widehat{p_2})$ *given by (3.3) and (3.5) verify the limit problem*

$$-\frac{\partial}{\partial y}\left(\begin{array}{c} \frac{\mu_1}{2^{\frac{p}{2}}}\left|\frac{\partial \widehat{u_{11}}}{\partial y}\right|^{p-2}\frac{\partial \widehat{u_{11}}}{\partial y} + \frac{\sqrt{2}}{2} g_1 sign\left(\frac{\partial \widehat{u_{11}}}{\partial y}\right) + \frac{\mu_2}{2^{\frac{p}{2}}}\left|\frac{\partial \widehat{u_{21}}}{\partial y}\right|^{p-2}\frac{\partial \widehat{u_{21}}}{\partial y} \\ + \frac{\sqrt{2}}{2} g_2 sign\left(\frac{\partial \widehat{u_{21}}}{\partial y}\right) \end{array}\right)$$
$$= \widehat{f_{11}} - \frac{d\widehat{p_1}}{dx} + \widehat{f_{21}} - \frac{d\widehat{p_2}}{dx} \ \text{in } W^{-1,p'}(\Omega). \tag{3.21}$$

Proof. Introducing the operator Φ defined as follows

$$\Phi : W^{\varepsilon} \to W^{\varepsilon'},$$

$$\langle \Phi(u_1^\varepsilon, u_2^\varepsilon), (v_1^\varepsilon, v_2^\varepsilon)\rangle_{W^{\varepsilon'} \times W^{\varepsilon}} = \mu_1 \varepsilon^p \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|^{p-2} D(u_1^\varepsilon) D(v_1^\varepsilon) dx_1 dx_2$$
$$+ \mu_2 \varepsilon^p \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|^{p-2} D(u_2^\varepsilon) D(v_2^\varepsilon) dx_1 dx_2.$$

It is easy to verify that Φ is monotone and hemi-continuous (see for more details the reference [11, 4, 2]). Moreover, we know that the functional

$$(v_1^\varepsilon, v_2^\varepsilon) \in W^\varepsilon \to g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|\, dx_1 dx_2 + g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|\, dx_1 dx_2$$

is proper and convex. Then, the use of Minty's lemma permits us to affirm that (2.18) is equivalent to the following inequality

$$\mu_1 \varepsilon^p \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|^{p-2} D(v_1^\varepsilon) D(v_1^\varepsilon - u_1^\varepsilon) dx_1 dx_2 + g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|\, dx_1 dx_2$$
$$- g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|\, dx_1 dx_2 + \mu_2 \varepsilon^p \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|^{p-2} D(v_2^\varepsilon) D(v_2^\varepsilon - u_2^\varepsilon) dx_1 dx_2$$
$$+ g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|\, dx_1 dx_2 - g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|\, dx_1 dx_2$$
$$\geq \int_{\Omega_1^\varepsilon} f_1^\varepsilon . (v_1^\varepsilon - u_1^\varepsilon) dx_1 dx_2 + \int_{\Omega_1^\varepsilon} p_1^\varepsilon \operatorname{div}(v_1^\varepsilon - u_1^\varepsilon) dx_1 dx_2$$
$$+ \int_{\Omega_2^\varepsilon} f_2^\varepsilon . (v_2^\varepsilon - u_2^\varepsilon) dx_1 dx_2 + \int_{\Omega_2^\varepsilon} p_2^\varepsilon \operatorname{div}(v_2^\varepsilon - u_2^\varepsilon) dx_1 dx_2 \ \forall (v_1^\varepsilon, v_2^\varepsilon) \in W^\varepsilon.$$

Our object now is to pass to the limit when ε tends to zero**.** To this aim, we use Proposition (3.3) and the weak lower semi-continuity of the convex and continuous functional

$$(v_1^\varepsilon, v_2^\varepsilon) \in W^\varepsilon \to g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|\, dx_1 dx_2 + g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|\, dx_1 dx_2.$$

We find the following limit inequality

$$
\begin{aligned}
&\mu_1 \int_{\Omega_1} \frac{1}{2^{\frac{p-2}{2}}} \left[\left| \frac{\partial \widehat{v_{11}}}{\partial y} \right|^2 + \left| \frac{\partial \widehat{v_{12}}}{\partial y} \right|^2 \right]^{\frac{p-2}{2}} \times \frac{1}{2} \left[\begin{array}{c} \frac{\partial \widehat{v_{11}}}{\partial y} \frac{\partial (\widehat{v_{11}} - \widehat{u_{11}})}{\partial y} \\ + \frac{\partial \widehat{v_{12}}}{\partial y} \frac{\partial (\widehat{v_{12}} - \widehat{u_{12}})}{\partial y} \end{array} \right] dxdy \\
&+ g_1 \int_{\Omega_1} \frac{1}{\sqrt{2}} \left[\left| \frac{\partial \widehat{v_{11}}}{\partial y} \right|^2 + \left| \frac{\partial \widehat{v_{12}}}{\partial y} \right|^2 \right]^{\frac{1}{2}} dxdy - g_1 \int_{\Omega_1} \frac{1}{\sqrt{2}} \left[\begin{array}{c} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^2 \\ + \left| \frac{\partial \widehat{u_{12}}}{\partial y} \right|^2 \end{array} \right]^{\frac{1}{2}} dxdy \\
&+ \mu_2 \int_{\Omega_2} \frac{1}{2^{\frac{p-2}{2}}} \left[\left| \frac{\partial \widehat{v_{21}}}{\partial y} \right|^2 + \left| \frac{\partial \widehat{v_{22}}}{\partial y} \right|^2 \right]^{\frac{p-2}{2}} \times \frac{1}{2} \left[\begin{array}{c} \frac{\partial \widehat{v_{21}}}{\partial y} \frac{\partial (\widehat{v_{21}} - \widehat{u_{21}})}{\partial y} \\ + \frac{\partial \widehat{v_{22}}}{\partial y} \frac{\partial (\widehat{v_{22}} - \widehat{u_{22}})}{\partial y} \end{array} \right] dxdy \\
&+ g_2 \int_{\Omega_2} \frac{1}{\sqrt{2}} \left[\left| \frac{\partial \widehat{v_{21}}}{\partial y} \right|^2 + \left| \frac{\partial \widehat{v_{22}}}{\partial y} \right|^2 \right]^{\frac{1}{2}} dxdy - g_2 \int_{\Omega_2} \frac{1}{\sqrt{2}} \left[\begin{array}{c} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^2 \\ + \left| \frac{\partial \widehat{u_{22}}}{\partial y} \right|^2 \end{array} \right]^{\frac{1}{2}} dxdy \\
&\geq \int_{\Omega_1} \widehat{f_1}.(\widehat{v_1} - \widehat{u_1}) dxdy + \int_{\Omega_1} \widehat{p_1} \operatorname{div}(\widehat{v_1} - \widehat{u_1}) dxdy + \int_{\Omega_2} \widehat{f_2}.(\widehat{v_2} - \widehat{u_2}) dxdy \\
&\qquad + \int_{\Omega_2} \widehat{p_2} \operatorname{div}(\widehat{v_2} - \widehat{u_2}) dxdy \quad \forall (v_1^\varepsilon, v_2^\varepsilon) \in W^\varepsilon. \qquad (3.22)
\end{aligned}
$$

Furthermore, from (3.3) and (3.4) we find

$$
\left(\frac{\partial \widehat{u_{12}}}{\partial y}, \frac{\partial \widehat{u_{22}}}{\partial y} \right) = (0,0) \text{ in } \Omega_1 \times \Omega_2.
$$

It follows, keeping in mind (3.20), that $\widehat{u_1}(x,y) = (\widehat{u_{11}}(x,y), 0)$ and, $\widehat{u_2}(x,y) = (\widehat{u_{21}}(x,y), 0)$, this permits also to choose $(\widehat{v_{12}}, \widehat{v_{22}}) = (0,0)$ in (3.22). Considering now the operator Φ such that

$$
\begin{gathered}
\Phi : W_p \to W_p', \\
\langle \Phi(\widehat{u_{11}}, \widehat{u_{21}}), (\widehat{v_{11}}, \widehat{v_{21}}) \rangle_{W_p' \times W_p} \\
= \frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \frac{\partial \widehat{v_{11}}}{\partial y} dxdy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \frac{\partial \widehat{v_{21}}}{\partial y} dxdy.
\end{gathered}
$$

It is clear that the operator Φ is monotone and hemi-continuous and the functional

$$
(\widehat{v_{11}}, \widehat{v_{21}}) \in W_p \to \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{v_{11}}}{\partial y} \right| dxdy + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{v_{21}}}{\partial y} \right| dxdy
$$

is proper and convex. Hence, we deduce using again Minty's lemma

$$
\begin{aligned}
&\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \frac{\partial (\widehat{v_{11}} - \partial \widehat{u_{11}})}{\partial y} dxdy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{v_{11}}}{\partial y} \right| dxdy \\
&- \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right| dxdy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \frac{\partial (\widehat{v_{21}} - \partial \widehat{u_{21}})}{\partial y} dxdy \\
&\qquad + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_1} \left| \frac{\partial \widehat{v_{21}}}{\partial y} \right| dxdy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right| dxdy
\end{aligned}
$$

$$\geq \int_{\Omega_1} \widehat{f_{11}}.(\widehat{v_{11}} - \widehat{u_{11}})dxdy - \int_{\Omega_1} \frac{d\widehat{p_1}}{dx}(\widehat{v_{11}} - \widehat{u_{11}})dxdy$$

$$+ \int_{\Omega_2} \widehat{f_{21}}.(\widehat{v_{21}} - \widehat{u_{21}})dxdy - \int_{\Omega_2} \frac{d\widehat{p_2}}{dx}(\widehat{v_{21}} - \widehat{u_{21}})dxdy \quad \forall(\widehat{v_{11}}, \widehat{v_{21}}) \in W_p \ . \tag{3.23}$$

This yields, via Green's formula

$$-\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \right) (\widehat{v_{11}} - \widehat{u_{11}})dxdy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{v_{11}}}{\partial y} \right| dxdy$$

$$-\frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right| dxdy - \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \right) (\widehat{v_{21}} - \widehat{u_{21}})dxdy$$

$$+\frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{v_{21}}}{\partial y} \right| dxdy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right| dxdy$$

$$\geq \int_{\Omega_1} \widehat{f_{11}}(\widehat{v_{11}} - \widehat{u_{11}})dxdy - \int_{\Omega_1} \frac{d\widehat{p_1}}{dx}(\widehat{v_{11}} - \widehat{u_{11}})dxdy$$

$$+ \int_{\Omega_2} \widehat{f_{21}}(\widehat{v_{21}} - \widehat{u_{21}})dxdy - \int_{\Omega_2} \frac{d\widehat{p_2}}{dx}(\widehat{v_{21}} - \widehat{u_{21}})dxdy \quad \forall(\widehat{v_{11}}, \widehat{v_{21}}) \in W_p. \tag{3.24}$$

Due to the fact that $W^{1,p}_{\Gamma_i}(\Omega_i)$ is dense in $W_p(\Omega_i)$, see [1, 5] , we can take $\widehat{v_{11}} = \widehat{u_{11}} \pm \varphi_1$ and $\widehat{v_{21}} = \widehat{u_{21}} \pm \varphi_2$ in (3.24), where $(\varphi_1, \varphi_2) \in W^{1,p}_{\Gamma_1}(\Omega_1) \times W^{1,p}_{\Gamma_2}(\Omega_2)$ to obtain the following inequalities

$$-\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \right) \varphi_1 dxdy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial (\widehat{u_{11}} + \varphi_1)}{\partial y} \right| dxdy$$

$$-\frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right| dxdy - \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \right) \varphi_2 dxdy$$

$$+\frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial (\widehat{u_{21}} + \varphi_2)}{\partial y} \right| dxdy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right| dxdy$$

$$\geq \int_{\Omega_1} \widehat{f_{11}} \varphi_1 dxdy - \int_{\Omega_1} \frac{d\widehat{p_1}}{dx} \varphi_1 dxdy + \int_{\Omega_2} \widehat{f_{21}} \varphi_2 dxdy$$

$$- \int_{\Omega_2} \frac{d\widehat{p_2}}{dx} \varphi_2 dxdy \quad \forall(\varphi_1, \varphi_2) \in W^{1,p}_{\Gamma_1}(\Omega_1) \times W^{1,p}_{\Gamma_2}(\Omega_2).$$

and

$$\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \right) \varphi_1 dxdy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial (\widehat{u_{11}} - \varphi_1)}{\partial y} \right| dxdy$$

$$-\frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right| dxdy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \right) \varphi_2 dxdy$$

$$+\frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial (\widehat{u_{21}} - \varphi_2)}{\partial y} \right| dxdy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right| dxdy$$

$$\geq -\int_{\Omega_1} \widehat{f_{11}}\varphi_1 dxdy + \int_{\Omega_1} \frac{d\widehat{p_1}}{dx}\varphi_1 dxdy - \int_{\Omega_2} \widehat{f_{21}}\varphi_2 dxdy$$
$$+\int_{\Omega_2} \frac{d\widehat{p_2}}{dx}\varphi_2 dxdy \quad \forall(\varphi_1,\varphi_2) \in W^{1,p}_{\Gamma_1}(\Omega_1) \times W^{1,p}_{\Gamma_2}(\Omega_2).$$

Replacing in these two inequalities the test function (φ_1, φ_2) by $(\lambda\varphi_1, \lambda\varphi_2)$, $\lambda > 0$, dividing the obtained inequalities by λ. The passage to the limit when λ tends to 0 implies, under the hypothesis $\left(\frac{\partial \widehat{u_{11}}}{\partial y}, \frac{\partial \widehat{u_{21}}}{\partial y}\right) \neq (0,0)$, that

$$-\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \right) \varphi_1 dxdy$$
$$+\frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} sign\left(\frac{\partial \widehat{u_{11}}}{\partial y}\right) \left(\frac{\partial \varphi_1}{\partial y}\right) dxdy$$
$$-\frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \right) \varphi_2 dxdy$$
$$+\frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} sign\left(\frac{\partial \widehat{u_{21}}}{\partial y}\right) \left(\frac{\partial \varphi_2}{\partial y}\right) dxdy$$
$$\geq \int_{\Omega_1} \widehat{f_{11}}\varphi_1 dxdy - \int_{\Omega_1} \frac{d\widehat{p_1}}{dx}\varphi_1 dxdy + \int_{\Omega_2} \widehat{f_{21}}\varphi_2 dxdy$$
$$-\int_{\Omega_2} \frac{d\widehat{p_2}}{dx}\varphi_2 dxdy \quad \forall(\varphi_1,\varphi_2) \in W^{1,p}_{\Gamma_1}(\Omega_1) \times W^{1,p}_{\Gamma_2}(\Omega_2),$$

and

$$\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \right) \varphi_1 dxdy$$
$$-\frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} sign\left(\frac{\partial \widehat{u_{11}}}{\partial y}\right) \left(\frac{\partial \varphi_1}{\partial y}\right) dxdy$$
$$+\frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left(\left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \right) \varphi_2 dxdy$$
$$-\frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} sign\left(\frac{\partial \widehat{u_{21}}}{\partial y}\right) \left(\frac{\partial \varphi_2}{\partial y}\right) dxdy$$
$$\geq -\int_{\Omega_1} \widehat{f_{11}}\varphi_1 dxdy + \int_{\Omega_1} \frac{d\widehat{p_1}}{dx}\varphi_1 dxdy - \int_{\Omega_2} \widehat{f_{21}}\varphi_2 dxdy$$
$$+\int_{\Omega_2} \frac{d\widehat{p_2}}{dx}\varphi_2 dxdy \quad \forall(\varphi_1,\varphi_2) \in W^{1,p}_{\Gamma_1}(\Omega_1) \times W^{1,p}_{\Gamma_2}(\Omega_2).$$

Consequently, we get by combining these two inequalities and using a simple integration by parts

$$-\int_{\Omega_1} \frac{\partial}{\partial y} \left[\left(\frac{\mu_1}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_1 sign \left(\frac{\partial \widehat{u_{11}}}{\partial y} \right) \right] \varphi_1 dxdy$$

$$- \int_{\Omega_2} \frac{\partial}{\partial y} \left[\left(\frac{\mu_2}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_2 sign \, (\frac{\partial \widehat{u_{21}}}{\partial y}) \right] \varphi_2 dxdy$$
$$= \int_{\Omega_1} \left(\widehat{f_{11}} - \frac{d\widehat{p_1}}{dx} \right) \varphi_1 dxdy + \int_{\Omega_2} \left(\widehat{f_{21}} - \frac{d\widehat{p_2}}{dx} \right) \varphi_2 dxdy$$
$$\forall (\varphi_1, \varphi_2) \in W^{1,p}_{\Gamma_1}(\Omega_1) \times W^{1,p}_{\Gamma_2}(\Omega_2).$$

Let us consider

$$\varphi \in W^{1,p}_0(\Omega) : \varphi = \begin{cases} \varphi_1 \text{ in } \Omega_1 \\ \varphi_2 \text{ in } \Omega_2 \end{cases},$$

and

$$\widetilde{a_1} = \begin{cases} -\frac{\partial}{\partial y} \left[\left(\frac{\mu_1}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_1 sign \left(\frac{\partial \widehat{u_{11}}}{\partial y} \right) \right] & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2, \end{cases}$$
$$\widetilde{a_2} = \begin{cases} 0 & \text{in } \Omega_1, \\ -\frac{\partial}{\partial y} \left[\left(\frac{\mu_2}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_2 sign \left(\frac{\partial \widehat{u_{21}}}{\partial y} \right) \right] & \text{in } \Omega_2, \end{cases}$$
$$\widetilde{b_1} = \begin{cases} \widehat{f_{11}} - \frac{d\widehat{p_1}}{dx} & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2, \end{cases}$$
$$\widetilde{b_2} = \begin{cases} 0 & \text{in } \Omega_1, \\ \widehat{f_{21}} - \frac{d\widehat{p_2}}{dx} & \text{in } \Omega_2. \end{cases}$$

Then,

$$\int_{\Omega} (\widetilde{a_1} + \widetilde{a_2}) \varphi dxdy = \int_{\Omega_1} (\widetilde{a_1} + \widetilde{a_2}) \varphi_1 dxdy + \int_{\Omega_2} (\widetilde{a_1} + \widetilde{a_2}) \varphi_2 dxdy$$
$$= \int_{\Omega_1} \widetilde{a_1} \varphi_1 dxdy + \int_{\Omega_2} \widetilde{a_2} \varphi_2 dxdy$$
$$= \int_{\Omega_1} -\frac{\partial}{\partial y} \left[(\frac{\mu_1}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u_{11}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{11}}}{\partial y}) + \frac{\sqrt{2}}{2} g_1 sign \, (\frac{\partial \widehat{u_{11}}}{\partial y}) \right] \varphi_1 dxdy$$
$$+ \int_{\Omega_2} -\frac{\partial}{\partial y} \left[(\frac{\mu_2}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u_{21}}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_{21}}}{\partial y}) + \frac{\sqrt{2}}{2} g_2 sign \, (\frac{\partial \widehat{u_{21}}}{\partial y}) \right] \varphi_2 dxdy$$
$$= \int_{\Omega_1} \left(\widehat{f_{11}} - \frac{d\widehat{p_1}}{dx} \right) \varphi_1 dxdy + \int_{\Omega_2} \left(\widehat{f_{21}} - \frac{d\widehat{p_2}}{dx} \right) \varphi_2 dxdy$$
$$= \int_{\Omega_1} \widetilde{b_1} \varphi_1 dxdy + \int_{\Omega_2} \widetilde{b_2} \varphi_2 dxdy$$
$$= \int_{\Omega} (\widetilde{b_1} + \widetilde{b_2}) \varphi dxdy \quad \forall \varphi \in W^{1,p}_0(\Omega).$$

Which eventually gives (3.21).

From now on we will denote by $(\widehat{u_{11}}, \widehat{u_{21}}) \in W_p$ and $(\widehat{p_1}, \widehat{p_2}) \in L^{p'}_0(\Omega_1) \times L^{p'}_0(\Omega_2)$ the solution of the limit problem (3.21). □

The following proposition shows the uniqueness of the limit solution $(\widehat{u_{11}}, \widehat{p_1})$ and $(\widehat{u_{21}}, \widehat{p_2})$.

Proposition 3.6. *The limit strong problem (3.21) has a unique, solution* $(\widehat{u_{11}}, \widehat{u_{21}}) \in W_p$ *and* $(\widehat{p_1}, \widehat{p_2}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$ *with the condition (3.20) .*

Proof. Suppose that the limit problem (3.21) has at least two solutions $(\widehat{u_{11}}, \widehat{u_{21}}) \in W_p$, $(\widehat{p_1}, \widehat{p_2}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$ and $(\overline{\widehat{u_{11}}}, \overline{\widehat{u_{21}}}) \in W_p$, $(\overline{\widehat{p_1}}, \overline{\widehat{p_2}}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$. In particular, $(\widehat{u_{11}}, \widehat{u_{21}})$, $(\widehat{p_1}, \widehat{p_2})$ and $(\overline{\widehat{u_{11}}}, \overline{\widehat{u_{21}}})$, $(\overline{\widehat{p_1}}, \overline{\widehat{p_2}})$ are solutions of the weak formulation (3.23). Then

$$\begin{aligned}
&\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial\widehat{u_{11}}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_{11}}}{\partial y}\frac{\partial(\widehat{v_{11}}-\partial\widehat{u_{11}})}{\partial y}dxdy+\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial\widehat{v_{11}}}{\partial y}\right|dxdy\\
&-\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial\widehat{u_{11}}}{\partial y}\right|dxdy+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial\widehat{u_{21}}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_{21}}}{\partial y}\frac{\partial(\widehat{v_{21}}-\partial\widehat{u_{21}})}{\partial y}dxdy\\
&\qquad+\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial\widehat{v_{21}}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial\widehat{u_{21}}}{\partial y}\right|dxdy\\
&\qquad\geq\int_{\Omega_1}\widehat{f_{11}}(\widehat{v_{11}}-\widehat{u_{11}})dxdy-\int_{\Omega_1}\frac{d\widehat{p_1}}{dx}(\widehat{v_{11}}-\widehat{u_{11}})dxdy\\
&+\int_{\Omega_2}\widehat{f_{21}}(\widehat{v_{21}}-\widehat{u_{21}})dxdy-\int_{\Omega_2}\frac{d\widehat{p_2}}{dx}(\widehat{v_{21}}-\widehat{u_{21}})dxdy\quad\forall(\widehat{v_{11}},\widehat{v_{21}})\in W_p,
\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}
&\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial\overline{\widehat{u_{11}}}}{\partial y}\right|^{p-2}\frac{\partial\overline{\widehat{u_{11}}}}{\partial y}\frac{\partial(\widehat{v_{11}}-\partial\overline{\widehat{u_{11}}})}{\partial y}dxdy+\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial\widehat{v_{11}}}{\partial y}\right|dxdy\\
&-\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial\overline{\widehat{u_{11}}}}{\partial y}\right|dxdy+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial\overline{\widehat{u_{21}}}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_{21}}}{\partial y}\frac{\partial(\widehat{v_{21}}-\partial\overline{\widehat{u_{21}}})}{\partial y}dxdy\\
&\qquad+\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial\widehat{v_{21}}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial\overline{\widehat{u_{21}}}}{\partial y}\right|dxdy\\
&\qquad\geq\int_{\Omega_1}\widehat{f_{11}}(\widehat{v_{11}}-\overline{\widehat{u_{11}}})dxdy-\int_{\Omega_1}\frac{d\overline{\widehat{p_1}}}{dx}(\widehat{v_{11}}-\overline{\widehat{u_{11}}})dxdy\\
&+\int_{\Omega_2}\widehat{f_{21}}(\widehat{v_{21}}-\overline{\widehat{u_{21}}})dxdy-\int_{\Omega_2}\frac{d\overline{\widehat{p_2}}}{dx}(\widehat{v_{21}}-\overline{\widehat{u_{21}}})dxdy\quad\forall(\widehat{v_{11}},\widehat{v_{21}})\in W_p.
\end{aligned}\tag{3.26}$$

Setting $\widehat{v_{11}} = \overline{\widehat{u_{11}}}$, $\widehat{v_{21}} = \overline{\widehat{u_{21}}}$ and $\widehat{v_{11}} = \widehat{u_{11}}$, $\widehat{v_{21}} = \widehat{u_{21}}$ as test functions in (3.25) and (3.26), respectively. Subtracting the two obtained inequalities, we can infer

$$\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left(\left|\frac{\partial\overline{\widehat{u_{11}}}}{\partial y}\right|^{p-2}\frac{\partial\overline{\widehat{u_{11}}}}{\partial y}-\left|\frac{\partial\widehat{u_{11}}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_{11}}}{\partial y}\right)\frac{\partial(\overline{\widehat{u_{11}}}-\widehat{u_{11}})}{\partial y}dxdy$$
$$+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left(\left|\frac{\partial\overline{\widehat{u_{21}}}}{\partial y}\right|^{p-2}\frac{\partial\overline{\widehat{u_{21}}}}{\partial y}-\left|\frac{\partial\widehat{u_{21}}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_{21}}}{\partial y}\right)\frac{\partial(\overline{\widehat{u_{21}}}-\widehat{u_{21}})}{\partial y}dxdy$$
$$\leq\int_{\Omega_1}\frac{d(\widehat{p_1}-\overline{\widehat{p_1}})}{dx}(\overline{\widehat{u_{11}}}-\widehat{u_{11}})dxdy+\int_{\Omega_2}\frac{d(\widehat{p_2}-\overline{\widehat{p_2}})}{dx}(\overline{\widehat{u_{21}}}-\widehat{u_{21}})dxdy. \tag{3.27}$$

Observe that for every $x,y\in\mathbb{R}^n$,

$$\left(|x|^{p-2}x-|y|^{p-2}y\right)(x-y)\geq c\frac{|x-y|^2}{(|x|+|y|)^{2-p}},\ 1<p\leq 2.$$

This leads, making use (3.27), to

$$\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\frac{\left|\frac{\partial(\overline{\widehat{u_{11}}}-\widehat{u_{11}})}{\partial y}\right|^2}{\left(\left|\frac{\partial\overline{\widehat{u_{11}}}}{\partial y}\right|+\left|\frac{\partial\widehat{u_{11}}}{\partial y}\right|\right)^{2-p}}dxdy+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\frac{\left|\frac{\partial(\overline{\widehat{u_{21}}}-\widehat{u_{21}})}{\partial y}\right|^2}{\left(\left|\frac{\partial\overline{\widehat{u_{21}}}}{\partial y}\right|+\left|\frac{\partial\widehat{u_{21}}}{\partial y}\right|\right)^{2-p}}dxdy$$
$$\leq\int_{\Omega_1}\frac{d(\widehat{p_1}-\overline{\widehat{p_1}})}{dx}(\overline{\widehat{u_{11}}}-\widehat{u_{11}})dxdy+\int_{\Omega_2}\frac{d(\widehat{p_2}-\overline{\widehat{p_2}})}{dx}(\overline{\widehat{u_{21}}}-\widehat{u_{21}})dxdy$$
$$=\int_0^1\left[\frac{d(\widehat{p_1}-\overline{\widehat{p_1}})}{dx}\int_0^{h_1(x)}(\overline{\widehat{u_{11}}}-\widehat{u_{11}})dy\right]dx$$
$$+\int_0^1\left[\frac{d(\widehat{p_2}-\overline{\widehat{p_2}})}{dx}\int_{h_1(x)}^{h_2(x)}(\overline{\widehat{u_{21}}}-\widehat{u_{21}})dy\right]dx.$$

The use of (3.20) gives

$$\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\frac{\left|\frac{\partial(\overline{\widehat{u_{11}}}-\widehat{u_{11}})}{\partial y}\right|^2}{\left(\left|\frac{\partial\overline{\widehat{u_{11}}}}{\partial y}\right|+\left|\frac{\partial\widehat{u_{11}}}{\partial y}\right|\right)^{2-p}}dxdy+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\frac{\left|\frac{\partial(\overline{\widehat{u_{21}}}-\widehat{u_{21}})}{\partial y}\right|^2}{\left(\left|\frac{\partial\overline{\widehat{u_{21}}}}{\partial y}\right|+\left|\frac{\partial\widehat{u_{21}}}{\partial y}\right|\right)^{2-p}}dxdy=0. \tag{3.28}$$

Which gives, keeping in mind (3.28)

$$\left(\frac{\partial(\overline{\widehat{u_{11}}}-\widehat{u_{11}})}{\partial y},\frac{\partial(\overline{\widehat{u_{21}}}-\widehat{u_{21}})}{\partial y}\right)=(0,0).$$

Since $\left(\overline{\widehat{u_{11}}}(x,h_1(x)),\overline{\widehat{u_{21}}}(x,h_2(x))\right)=(\widehat{u_{11}}(x,h_1(x)),\widehat{u_{21}}(x,h_2(x)))=(0,0)$, we deduce that $(\overline{\widehat{u_{11}}},\overline{\widehat{u_{21}}})=(\widehat{u_{11}},\widehat{u_{21}})$ a.e. in $\Omega_1\times\ \Omega_2$. Finally, to prove the uniqueness of the pressure, we use equation (3.21), with the two pressures $(\widehat{p_1}\ ,\overline{\widehat{p_1}}\ )$ and $(\widehat{p_2}\ ,\overline{\widehat{p_2}}\ )$. We find

$$\frac{d(\widehat{p_1}-\overline{\widehat{p_1}})}{dx}=0\ \text{ and }\ \frac{d(\widehat{p_2}-\overline{\widehat{p_2}})}{dx}=0.$$

Then, due to fact that $(\widehat{p_1}\ , \overline{\widehat{p_1}}\ ) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_1)$, $(\widehat{p_2}\ , \overline{\widehat{p_2}}\ ) \in L_0^{p'}(\Omega_2) \times L_0^{p'}(\Omega_2)$ the result can be easily deduced. □

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A compartmental model for COVID-19 to assess effects of non-pharmaceutical interventions with emphasis on contact-based quarantine

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Abstract. Relative to the number of casualties, COVID-19 ranks among the ten most devastating plagues in history. The pandemic hit the South Asian nation of Bangladesh in early March 2020 and has greatly impacted the socio-economic status of the country. In this article, we propose a compartmental model for COVID-19 dynamics, introducing a separate class for quarantined susceptibles, synonymous to isolation of individuals who have been exposed and are suspected of being infected. The current model assumes a perfect quarantine based on contact with infectious individuals. Numerical simulation is conducted to investigate the efficiency of disease control by segregating suspected individuals and other non-pharmaceutical interventions. In addition, we assort quantitatively the importance of parameters that influence the dynamics of the system. Fitting the system to the early phase of COVID-19 outbreaks in Bangladesh, by taking into account the cumulative number of cases with the data of the first 17-week period, the basic reproduction number is estimated as 1.69.

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Keywords: COVID-19, compartmental model, quarantine, data fitting.

1. Background

In December 2019, a novel coronavirus was first detected in Wuhan, Hubei Province, China, following reports of highly infectious pneumonia cases of unidentified origin [38]. The pathogen was later officially named SARS-CoV-2 (severe acute

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respiratory syndrome coronavirus) by the WHO and the disease caused by this virus is referred to as COVID-19 (coronavirus disease). Since its emergence, SARS-CoV-2 infection rapidly spread to many other countries [14]. It has been detected in 190 countries, is accountable for over 5 million deaths, and has clinically affected over 270 million people globally as of 19 December 2021 [31, 35]. SARS-CoV-2 is also extremely pervasive and currently poses a major public health concern. COVID-19 outbreak was declared a pandemic on 11 March 2020 and is of the most devastating nature since the 1918 H1N1 influenza pandemic [13].

The primary mode of transmission of SARS-CoV-2 is respiratory droplets, but it can also be transmitted via human contact or aerial droplets [19]. Epidemiological inspection states the latency period to range from 3 to 7 days with a maximum of 14 days, during which it remains contagious unlike the SARS-CoV (SARS-related coronaviruses). also displays a wide spectrum of clinical manifestations in infected patients, which may be mild, moderate, or severe. Fever, fatigue, dry cough, shortness of breath, etc are some of its most typical symptoms [15]. The complications of SARS-CoV-2 involve acute respiratory distress syndrome (ARDS) distinguishable by the hyper-inflammatory response, often leading to extensive lung damage, or probable death [30]. Additionally, it has also impacted the social and economic condition of people on a huge scale. There are currently no certified antiviral drugs for SARS-CoV-2 infection. The safety and efficiency of most drugs indicated or suggested for the treatment of COVID-19 are controversial, or under the experimental phase [4]. At present, vaccines are seen as the most competent in controlling disease transmission. The first mass vaccination program started in early December 2020 and at least 13 different vaccines (across 4 platforms) have been administered by WHO so far [36].

The majority of the countries adopted stringent containment measures as early as March 2020 to lessen the transmission of SARS-CoV-2, which included the introduction of non-pharmaceutical interventions such as physical distancing measures, prohibition of social gatherings, proposing work-from-home schemes, etc. The concept of two special control strategies, namely mitigation and suppression has also been proposed and adopted. The UK was following mitigation in the early phase of the pandemic prior to the publication of this report but later started following suppression. It has been suggested that combining multiple intervention strategies is most efficient and optimal in curtailing disease transmission compared to when a single strategy is in force [12].

The first confirmed case of COVID-19 in Bangladesh was recorded on March 8, 2020. Following reports of one or two average cases in the subsequent days, COVID-19 cases have rapidly escalated since the initial outbreak. Several measures have been adopted by the government of Bangladesh to control the spread of disease. The very first step to this intent was declaration of a general holiday from 26 March to 4 April 2020, which was later extended to 30 May, with some relaxation due to financial challenges [37]. Later came the second phase of control strategy, typically known as lockdown, with strict policies such as mandatory use of masks, home quarantine, social distancing, and banning or restrictions on national or international flights which were imposed from 12 April 2021 [34]. Despite these plans of action, the spread of disease could not be curtailed, possibly as a result of low-income people moving actively for their livelihood, religious festivals, social programs, and overall, lack of public awareness, etc. As of 19 December 2021, there are more than 1,580,000 confirmed COVID-19 cases in Bangladesh and more than 28,000 related deaths [33].

One of the most essential features of COVID-19 is its ability to remain asymptomatic and hence remain undetected in infected individuals. However, the severity of the outbreak lies in the fact that the infectious individuals are capable of transmitting the disease with a positive probability during the course of infection while showing no symptoms [23, 26]. In order to best capture this feature of asymptomatic disease transmission in compartmental disease modeling, a class of unidentified infected and infectious individuals is introduced. Numerous mathematical models for the spread of COVID-19 have been formulated to study the transmission dynamics and control since its outbreak, which emphasize this additional feature [5, 32, 18, 25] also focusing particularly on disease outbreak in specific regions or countries [2, 22, 17]. Various papers have appeared that study the effects of non-pharmaceutical interventions on COVID-19 transmission, see e.g. [8, 27]. The situation in Bangladesh has also been described and analyzed by various modeling works. Masud et al. [22] introduced a compartmental model, where they found that the reproduction number is strongly associated with the time and pick of the epidemic. A modified SIR model was theoretically analyzed and validated the result using fourth-order polynomial regression by Shahrear et al. [29].

As quarantine was one of the main tools of non-pharmaceutical intervention at the beginning of the epidemic, it is important to consider this phenomenon in mathematical models describing the spread of COVID-19. A significant number of research articles have documented well the impact of quarantine at a population level that has been investigated using mathematical models, typically involving deterministic systems of nonlinear differential equations. It is imperative to mention that, the term quarantine here refers to the temporary isolation of susceptible individuals who are suspected to have been exposed to an infectious disease, rather than to the removal of individuals who have already been confirmed to being infected with the disease. These individuals are removed in the interim from actively associating with the rest of the population, until after the incubation period of the disease, at the very least. Following this, they are tested to determine if they have contracted the disease, in which case they are further isolated. Alternatively, they return to the class of susceptibles in case they do not exhibit any clinical symptoms.

Quarantine is described in most models in a way that is rather suitable to model isolation, meaning the removal of individuals who are known to be infected. We express quarantine as a temporary separation of susceptibles who are feared to have contracted the disease due to exposure to the disease via contact with an infectious individual. Few articles that have correctly included quarantine in their models are noteworthy here. Lipsitch et al. [21] introduced a model where susceptible individuals are moved to quarantine based on their contact with infected individuals. Mubayi et al. [24], Safi and Gumel [28], Dénes and Gumel [9] followed a similar way to include quarantine in their models. Furthermore, the quarantine models in Lipsitch et al. [21]; Mubayi et al. [24] do not allow for breakthrough infection to occur during quarantine. To be specific, they assume a state of perfect quarantine, which we incorporate into the current model.

The objective of the current work is to develop a dynamic model for the spread of coronavirus (COVID-19) with the inclusion of a new class of quarantined susceptible individuals, that aims for a more realistic capture of the spread of the disease. The rest of the paper is structured as follows: in Section 2, we introduce a compartmental model including quarantine, calculate the basic reproduction number and determine some basic properties of the model. In Section 3, we fit the model to data from the first period of the epidemic in Bangladesh identifying the most probable values for the model parameters and perform sensitivity analysis. Section 4 is devoted to numerical simulations concerning the effect of possible intervention measures. The paper is closed by a short discussion of the results.

2. Formulation of the compartmental model

In order to develop a compartmental model for COVID-19 transmission, including the above-described confinement of those feared exposed, we introduce quarantine for susceptible individuals, with the assumption that quarantine is perfect. In the current model, $N(t)$, which denotes the total human population at time t, is partitioned into the population of those in quarantine (denoted by $N_q(t)$) and those not in quarantine (denoted by $N_u(t)$), so that

$$N(t) = N_u(t) + N_q(t).$$

Additionally, the total population in quarantine at time t is divided into the following compartments: susceptibles in quarantine ($S_q(t)$), exposed ($E_q(t)$, that is, infected but not yet infectious), infected ($I_q(t)$) and recovered ($R_q(t)$). Hence,

$$N_q(t) = S_q(t) + E_q(t) + I_q(t) + R_q(t).$$

Similarly, the total population of individuals not in quarantine at time t is subdivided into the following subpopulations of susceptibles ($S_u(t)$), exposed ($E_u(t)$), infected who do not show any symptoms or have only mild symptoms ($I_u^m(t)$), symptomatically infected ($I_u^s(t)$), treated ($I_t(t)$), recovered ($R_u(t)$) so that,

$$N_u(t) = S_u(t) + E_u(t) + I_u^m(t) + I_u^s(t) + I_t(t) + R_u(t).$$

We introduce the auxiliary compartment $D(t)$ to take care of the number of individuals at time t who have passed away due to COVID-19. The force of infection (denoted by $\Lambda(t)$) associated with this model to be developed is given by

$$\Lambda(t) = \frac{I_u^s(t) + \beta_m I_u^m(t) + \beta_q I_q(t) + \beta_t I_t(t)}{N(t)},$$

where β_m, β_q and β_t are modification parameters accounting for the variability of the infectiousness of infected individuals in the $I_u^m(t)$, $I_q(t)$ and $I_t(t)$ classes, compared to those in the $I_u^s(t)$, respectively.

In this model, we do not consider demography, however, there is a disease-induced death rate, denoted by d_s, d_q, and d_t for those in the compartments I_u^s, I_q and I_t, respectively. As noted above, we follow Lipsitch et al. [21] to introduce quarantine.

In our model, human-to-human transmission rate is split into the product of the average number of contacts (κ) and the probability of transmission per contact (b), while q stands for the fraction of those susceptible individuals who are feared exposed and hence moved to quarantine.

TABLE 1. Description of parameters of model (2.1).

Parameters	**Description**
b	Probability of disease transmission
κ	Average number of contacts
q	Fraction of those moved to quarantine
$1/\nu$	Average length of incubation period
γ	Recovery rate for mildly infected
$1/\epsilon$	Average time from symptoms onset until treatment
ξ	Recovery rate in quarantine
θ	Fraction of asymptomatic cases among non-quarantined
$\beta_m, \beta_q, \beta_t$	Relative transmissibilities for I_u^m, I_q, I_t compartments
$1/\sigma$	Average length of quarantine
ζ	Recovery rate for treated individuals
d_s, d_q, d_t	Disease-induced death rate for I_u^s, I_q and I_t respectively

Upon contacting an infectious individual (i.e. an individual from the compartments I_u^m, I_u^s, I_q or I_t) a susceptible (quarantined or non-quarantined) human may contract the disease (with probability b) and hence move to one of the two exposed classes, depending on whether this person is moved to quarantine: a fraction q arrive in the exposed quarantined compartment E_q, while the remaining fraction $1-q$ of those who have contracted the disease arrive in the non-quarantined exposed compartment E_u. A fraction $1-b$ of those in contact with infectious individuals will not be infected. Though, based on their contacts, a fraction q will be quarantined and hence arrive in compartment S_q. The remaining fraction $1-q$ will stay in the S_u compartment. Individuals in quarantine who turn out to be healthy will move back to the S_u class at a rate σ at the end of their quarantine period. Hence, transition rate from S_u to E_u takes the form $(1-q)\kappa b$, from S_u to S_q takes the form $(1-b)\kappa q$, while transition rate from S_u to E_q is given by $q\kappa b$. The parameter θ is a fraction of non-quarantined exposed people who have shown mild symptoms and move to compartment I_u^m, while the rest have shown all symptoms of the disease. $1/\nu$ denotes the length of the incubation period. The duration of the infectious period for mildly symptomatic individuals and people under treatment is represented by $1/\gamma$ and $1/\zeta$ respectively. The progression tenure of the patients from symptomatically infected is denoted by $1/\epsilon$.

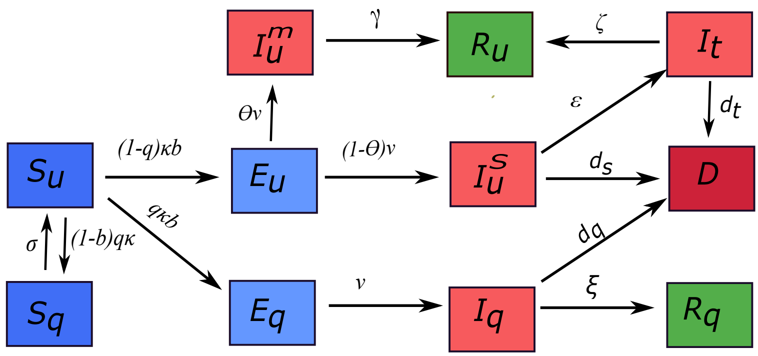

FIGURE 1. Transmission diagram. Arrows indicate transition from one compartment to another

Using the notations for compartments and parameters as described in the methods section (see Figure 1, Table 1), our model takes the form:

$$
\begin{aligned}
\frac{\mathrm{d}S_u(t)}{\mathrm{d}t} &= -\big((1-b)\kappa q + q\kappa b + (1-q)\kappa b\big)\Lambda(t)S_u(t) + \sigma S_q(t),\\
\frac{\mathrm{d}S_q(t)}{\mathrm{d}t} &= (1-b)\kappa q\Lambda(t)S_u(t) - \sigma S_q(t),\\
\frac{\mathrm{d}E_u(t)}{\mathrm{d}t} &= (1-q)\kappa b\Lambda(t)S_u(t) - \nu E_u(t),\\
\frac{\mathrm{d}E_q(t)}{\mathrm{d}t} &= q\kappa b\Lambda(t)S_u(t) - \nu E_q(t),\\
\frac{\mathrm{d}I_u^m(t)}{\mathrm{d}t} &= \theta\nu E_u(t) - \gamma I_u^m(t),\\
\frac{\mathrm{d}I_u^s(t)}{\mathrm{d}t} &= (1-\theta)\nu E_u(t) - \epsilon I_u^s(t) - d_s I_u^s(t),\\
\frac{\mathrm{d}I_q(t)}{\mathrm{d}t} &= \nu E_q(t) - \xi I_q(t) - d_q I_q(t),\\
\frac{\mathrm{d}I_t(t)}{\mathrm{d}t} &= \epsilon I_u^s(t) - \zeta I_t(t) - d_t I_t(t),\\
\frac{\mathrm{d}R_u(t)}{\mathrm{d}t} &= \gamma I_u^m(t) + \zeta I_t(t),\\
\frac{\mathrm{d}R_q(t)}{\mathrm{d}t} &= \xi I_q(t),\\
\frac{\mathrm{d}D(t)}{\mathrm{d}t} &= d_s I_u^s(t) + d_q I_q(t) + d_t I_t(t),
\end{aligned}
\tag{2.1}
$$

where the force of infection $\Lambda(t)$ is given as above.

2.1. Basic reproduction number

For the analytical computation of the basic reproduction number $\mathscr{R}_0$ of (2.1), we follow the general approach established in [10, 11]. Splitting the system into vectors $x = \big(E_u, E_q, I_u^m, I_u^s, I_q, I_t\big)$, composed of infectious compartments, and $y =$

$\big(S_u, S_q, R_u, R_q, D\big)$, composed of non-infectious compartments, the system can be expressed as

$$\begin{aligned} x_i' &= \mathcal{F}_i(x,y) - \mathcal{V}_i(x,y), \qquad & i &= 1,\dots,6,\\ y_j' &= g_j(x,y), & j &= 1,\dots,5, \end{aligned}$$

where $\mathcal{F}_i$ represents the new infections due to COVID-19 and $\mathcal{V}_i$ contains the transitions between infected compartments, and are given as follows:

$$\mathcal{F}_i = \begin{bmatrix} (1-q)\kappa b \frac{(I_u^s(t)+\beta_m I_u^m(t)+\beta_q I_q(t)+\beta_t I_t(t))}{N(t)} S_u(t) \\ q\kappa b \frac{(I_u^s(t)+\beta_m I_u^m(t)+\beta_q I_q(t)+\beta_t I_t(t))}{N(t)} S_u(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathcal{V}_i = \begin{bmatrix} \nu E_u(t) \\ \nu E_q(t) \\ -\theta\nu E_u(t) + \gamma I_u^m(t) \\ -(1-\theta)\nu E_u + \epsilon I_u^s(t) + d_s I_u^s(t) \\ -\nu E_q(t) + \xi I_q(t) + d_q I_q(t) \\ -\epsilon I_u^s(t) + \zeta I_t(t) + d_t I_t(t) \end{bmatrix}.$$

In the (x,y)-notation, the disease-free equilibrium (DFE) of the system is $\mathcal{E}_0 = (0, y_*)$ where $y_* = (N,0,0,0,0)$. By means of linearization at the DFE $\mathcal{E}_0$, we obtain the equation

$$x' = Ax$$

where A is the Jacobian matrix. Next, we take the decomposition $A = F - V$, where

$$F_{i,j} = \left[\frac{\partial \mathcal{F}_i}{\partial x_j}(DFE)\right], \qquad V_{i,j} = \left[\frac{\partial \mathcal{V}_i}{\partial x_j}(DFE)\right].$$

In the case of our model, the transmission and transition matrices take the form

$$F = \begin{bmatrix} 0 & 0 & (1-q)\kappa b\beta_m & (1-q)\kappa b & (1-q)\kappa b\beta_q & (1-q)\kappa b\beta_t \\ 0 & 0 & q\kappa b\beta_m & q\kappa b & q\kappa b\beta_q & q\kappa b\beta_t \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \nu & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 & 0 \\ -\theta\nu & 0 & \gamma & 0 & 0 & 0 \\ -(1-\theta)\nu & 0 & 0 & \epsilon + d_s & 0 & 0 \\ 0 & -\nu & 0 & 0 & \xi + d_q & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & \zeta + d_t \end{bmatrix}.$$

Thus, we have the reproduction number as of the co-infection model $\rho(FV^{-1})$, given by the formula

$$\mathscr{R}_0 = \frac{(1-q)\kappa b[\gamma(1-\theta)(\beta_t\epsilon+\zeta+d_t)+\theta\beta_m(d_s+\epsilon)(d_t+\zeta)]}{\gamma(d_s+\epsilon)(d_t+\zeta)} + \frac{q\kappa b\beta_q}{d_q+\xi}, \tag{2.2}$$

where ρ represents the spectral radius.

2.2. Basic qualitative properties

We close this section by showing some basic qualitative properties of the model.

Proposition 2.1. *Any solution of* (2.1) *starting from nonnegative initial values will remain nonnegative for all forward time.*

Proof. Suppose the assertion is false, then there exists a minimal time T when (at least) one of the compartments reaches zero. Let X be a compartment ($X \in \{S_u, S_q, E_u, E_q, I_u^m, I_u^s, I_q, I_t, R_u, R_q\}$) for which $X(T) = 0$. One can see that at this time point, we have $X'(t) \geq 0$, hence, $X(t)$ cannot drop below zero. This shows the nonnegativity of all solutions. □

Proposition 2.2. *All solutions of* (2.1) *are bounded.*

Proof. By adding all equations in (2.1), we obtain that the sum of the total population and the number of deceased is constant. Hence, the total population – which is decreasing – is bounded. □

Lemma 2.3. *The infected compartments $E_u(t), E_q(t), I_u^m(t), I_u^s(t), I_q(t), I_t(t)$ as well as the compartment for quarantined susceptibles $S_q(t)$ will eventually go extinct as $t \to \infty$.*

Proof. To see that $E_u(t) \to 0$ and $E_q(t) \to 0$ as $t \to \infty$, we consider

$$(S_u(t) + S_q(t) + E_u(t) + E_q(t))' = -\nu(E_u(t) + E_q(t)).$$

If either $E_u(t)$ or $E_q(t)$ does not tend to zero, then $S_u(t) + S_q(t) + E_u(t) + E_q(t)$ drops below 0, which contradicts the nonnegativity of the compartments. To see that the compartments $I_u^m(t), I_u^s(t), I_q(t)$ extinct, we consider

$$(I_u^m(t) + I_u^s(t) + I_q(t))' = \nu(E_u(t) + E_q(t)) - \gamma I_u^m(t) - (\epsilon + d_s)I_u^s(t) - (\xi + d_q)I_q(t).$$

If any of $I_u^m(t), I_u^s(t)$ or $I_q(t)$ remain positive then, considering that the two exposed classes have been shown to tend to 0, $I_u^m(t) + I_u^s(t) + I_q(t)$ drops below 0, which is not possible. The statement for compartment $I_t(t)$ follows from the previous assertions. In a similar way, the compartment S_q of quarantined susceptibles will also tend to zero. □

3. Data fitting and sensitivity analysis

3.1. Data fitting

First and foremost, it should be noted that, as enumerated in Table 2, good approximations are available in the literature for a set of parameter values. Thus, our task is to find good estimates for the remaining parameters. For that purpose, and to consequently validate the model, our model has been fitted using the available data for the COVID-19 outbreaks in Bangladesh [33].

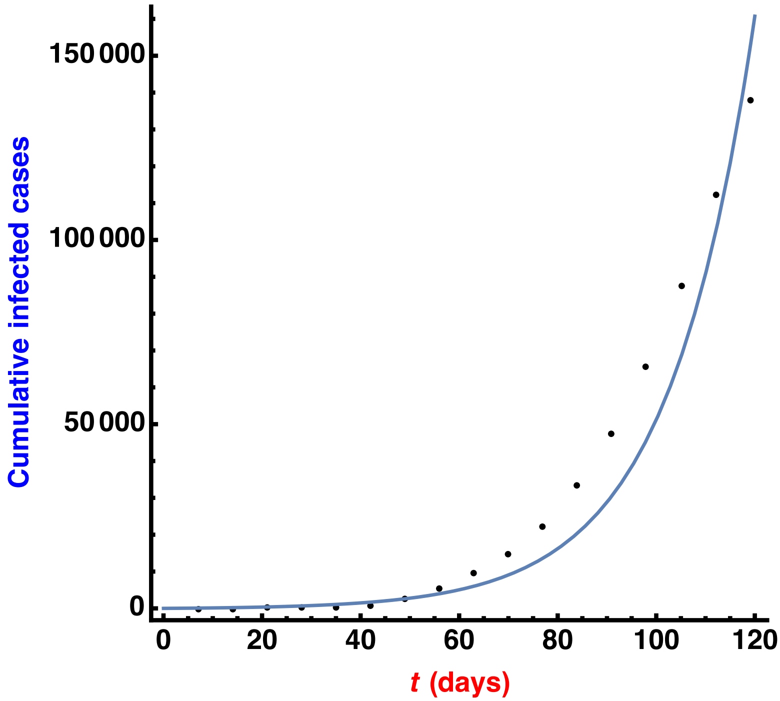

FIGURE 2. The best fitting solution plotted with 17-week WHO data from Bangladesh started on March 2, 2020. Parameter values obtained in the fitting are given in Table 2.

Using the baseline values for the available parameter values as listed in Table 2, we utilize the Latin Hypercube Sampling method to find the parameter values which provide the best fit to the data. This is a computational technique used in statistics to estimate the simultaneous variation of various model parameters to construct a representative sample set of n-tuples of parameters (n is the number of parameters fitted) taking values from given ranges. The estimated values of the fitted parameters of the model are given in Table 2. Figure 2 illustrates the simulation results obtained by fitting the model for the cumulative number of cases with the data of the first 17-week period.

Table 2. Parameters for model (2.1) providing the best fit.

Parameters	**Baseline (Range)**	**Units**	**Sources**
ν	$1/5.2$	Days^{-1}	[7]
γ	$1/7$	Days^{-1}	[3, 2, 1]
ϵ	$1/3$	Days^{-1}	[1]
ξ	$1/8$	Days^{-1}	Assumed
θ	0.4	–	[20]
κ	$8.317\ (5, 16)$	–	Fitted
b	$0.064\ (0.01, 0.08)$	Days^{-1}	[6]
q	$0.195\ (0.007, 0.2)$	–	[6]
β_m	$0.603\ (0.1, 0.7)$	–	[16]
β_q	$0.168\ (0.1, 0.4)$	–	Fitted
β_t	$0.127\ (0.1, 0.7)$	–	Fitted
σ	$0.163\ (1/21, 1/3)$	Days^{-1}	Fitted
ζ	$0.064\ (1/21, 1/7)$	Days^{-1}	[1]
d_s	$0.047\ (0.01, 0.08)$	Days^{-1}	Fitted
d_q	$0.047\ (0.01, 0.07)$	Days^{-1}	Fitted
d_t	$0.085\ (0.05, 0.1)$	Days^{-1}	Fitted

3.2. Sensitivity analysis

In order to assess the population-level impact of possible intervention parameters of the model (2.1), and to evaluate their significance, a sensitivity analysis is conducted. The current analysis is conducted using the Latin Hypercube Sampling (LHS) and Partial Rank Correlation Coefficient (PRCC). with $15,000$ Monte Carlo simulations per run.

With simultaneously varying parameter values, the PRCC method facilitates us to quantify the effect of the various parameter values on the model's feedback, hence, establishing the statistical relationships between the input parameters and the outcome value. The sign (positive or negative) of the parameter's PRCC characterizes the qualitative association with the model response(s). While increasing parameters with positive PRCC values results in the growth of the number of cumulative cases, increasing parameters with negative PRCC will result in a smaller number of cumulative cases. It is to be noted that parameters with larger PRCC values are regarded to be most critical for the model.

The input parameters for which the PRCC analysis was performed are: average number of contacts (κ), transmission probability per contact (b), the fraction of people moved to quarantine among those with contacts with infected individuals (q), incubation time ($1/\nu$), recovery rate for asymptomatically infected (γ), average time until hospitalization ($1/\epsilon$), recovery rate for quarantined (ξ), fraction of asymptomatic cases (θ), length of quarantine ($1/\sigma$), recovery rate among treated individuals (ζ), disease-induced death rate for symptomatically infected (d_s), disease-induced death rate for quarantined individuals (d_q) and disease-induced death rate for treated individuals (d_t), while the output parameter in our work is the cumulative number of infected

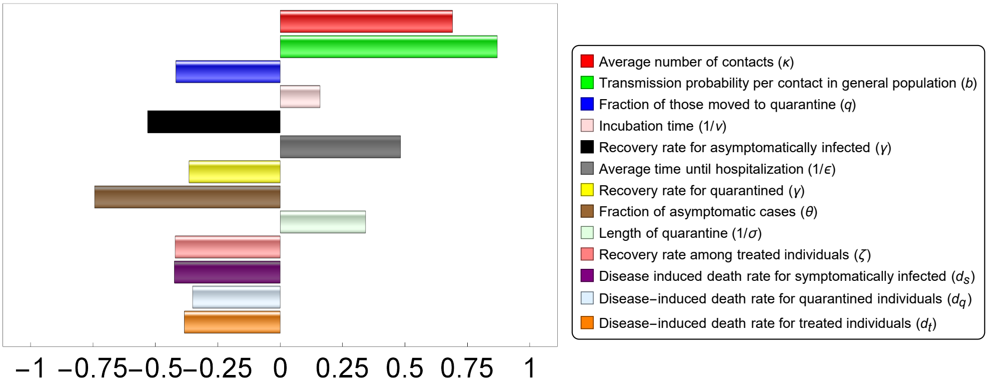

FIGURE 3. Partial rank correlation coefficients (PRCCs) of model parameters. Increasing parameters with positive PRCC value will increase the number of cases, and increasing ones with negative PRCC will decrease the number of cases.

until the time period under consideration in the fitting. The results obtained in Figure 3 demonstrate that the parameters with the largest effect are the average number of contacts κ, transmission probability per contact b, and the fraction of asymptomatic cases θ, hence, the first two are shown to be the most important parameters that might be subject to control measures. The parameter corresponding to the third important intervention measure, namely the efficiency of quarantining individuals who are feared to have contracted the disease (q) is suggested to have a lower effect than the other two parameters related to non-pharmaceutical intervention methods, however, the effect of changing this parameter is still remarkable and can be compared to that of parameters such as recovery rates γ, ζ, ξ and death rates d_s, d_q, d_t. The average time until hospitalization of severe cases is also shown to have a significant impact, though, in comparison with the above-mentioned intervention parameters, decreasing this time period is more difficult than the implementation of the other control measures.

4. Effect of possible control measures

Numerical simulations were performed to observe the probable effects of the most straightforward non-pharmaceutical interventions. As long as no vaccines against a given disease are available, the most easily applicable interventions are the reduction of contacts (e.g. by introducing partial or complete curfew or closing schools), decreasing transmission probability (e.g. by wearing masks and increasing hygiene), as well as quarantining those who are feared to have contracted the disease. Moreover, PRCC analysis in the previous section showed that these are efficient tools to reduce the number of infected.

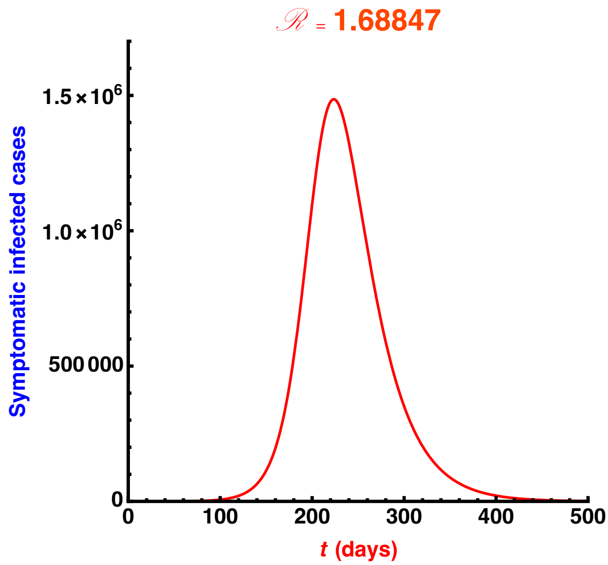

FIGURE 4. Number of symptomatic cases with the fitted parameters.

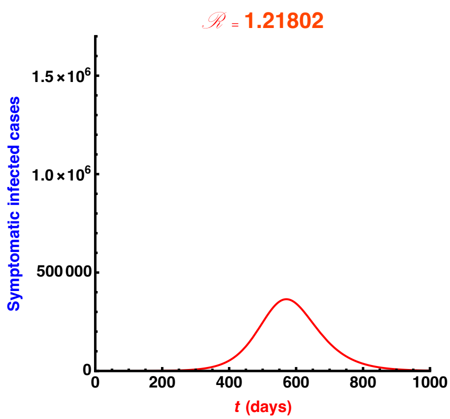

FIGURE 5. Number of symptomatic cases with $\kappa = 6$.

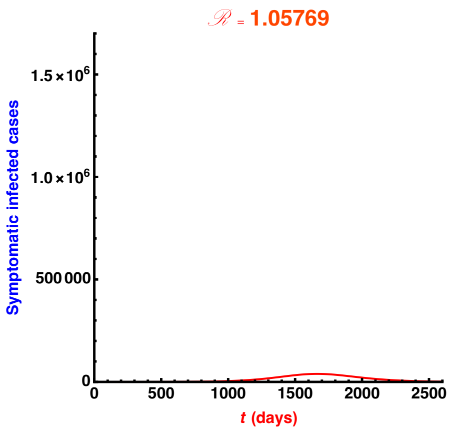

FIGURE 6. Number of symptomatic cases with $b = 0.04$.

Hence, we selected the three parameters of our model which correspond to these three types of intervention measures, namely the average number of contacts, transmission probability per contact, and the fraction of suspected individuals being quarantined, since these are most likely to be altered due to some control measures. Our goal was to see what degree of change in these parameters might turn out to be

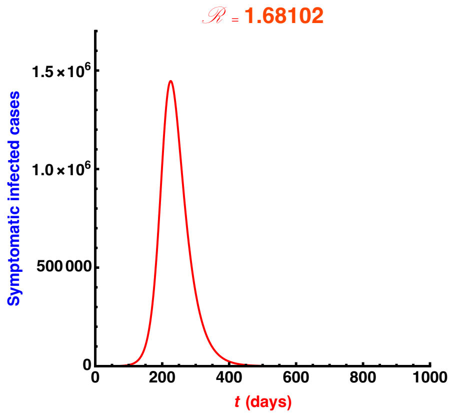

FIGURE 7. Number of symptomatic cases with $q = 0.2$.

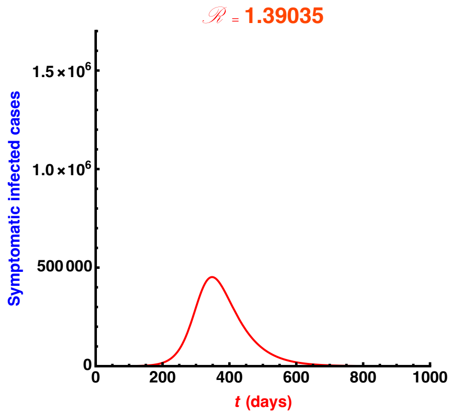

FIGURE 8. Number of symptomatic cases with $q = 0.4$.

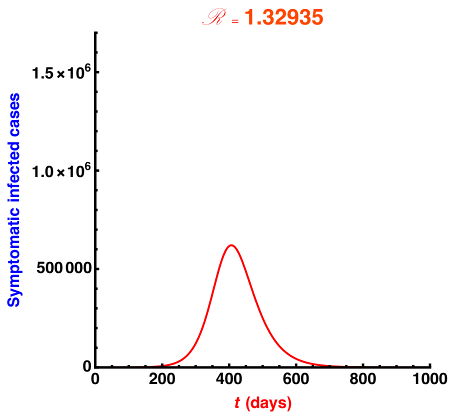

FIGURE 9. Number of symptomatic cases with $\kappa = 7$, $b = 0.06$, and $q = 0.2$.

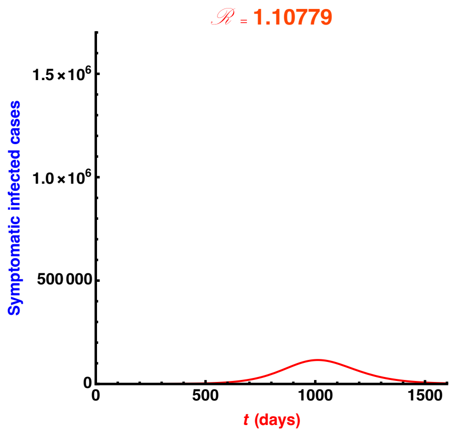

FIGURE 10. Number of symptomatic cases with $\kappa = 7$, $b = 0.05$, and $q = 0.2$.

sufficient to reduce the peak of the epidemic. Initially starting the simulations with the fitted parameters up to week 17, we applied the modification of one or more parameters to observe the changes.

To gain further insight into how intervention measures may reduce disease burden, in each subfigure of Figure 11 we show the contour plot of the reproduction number as a function of two of the three parameters subject to intervention measures, namely the average number of contacts (κ), transmission probability per contact (b)

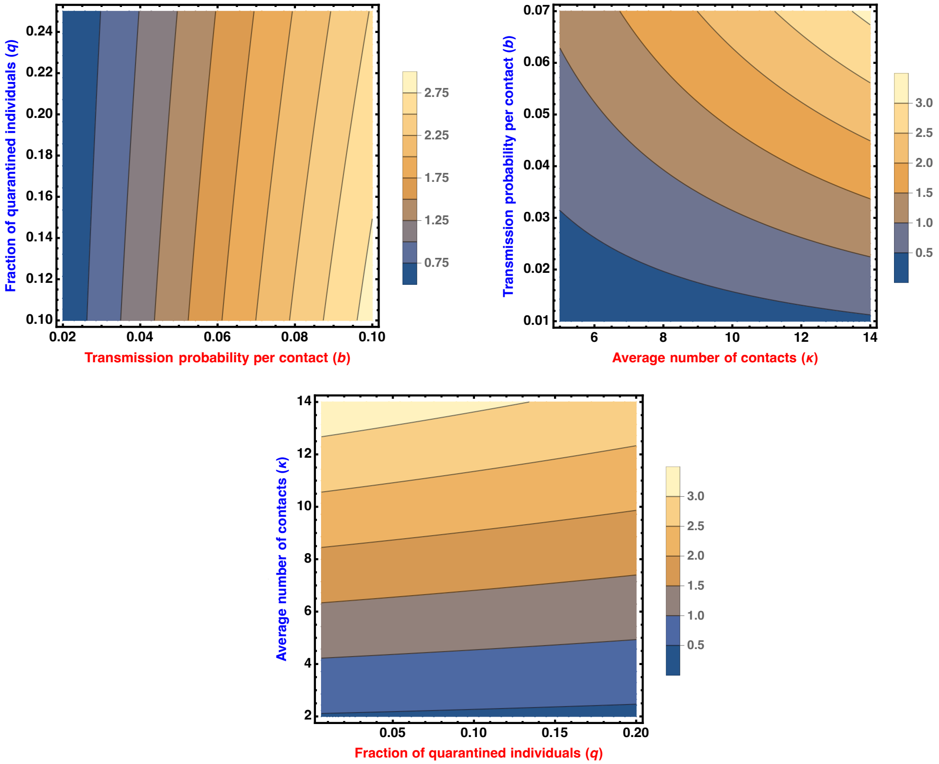

Figure 11. Contour plot of the basic reproduction number as a function of the three parameters subject to intervention measures.

and the fraction of those moved to quarantine (q). The rest of the parameters are set to the values used in the fitting as shown in Table 2. These plots support earlier results shown in the sensitivity analysis: reducing transmission probability and the number of contacts are the most powerful tools to decrease the reproduction number, while increasing the fraction of those moved to quarantine has a milder effect, though adjusting this parameter will strengthen the mitigating effect of decreasing the other two parameters or enables us to perform a smaller change on these parameters to obtain the same result.

The simulations suggest that all three interventions might significantly reduce the number of infected. In accordance with the results of the PRCC analysis, decreasing the number of contacts and reducing transmission probability are shown to be the most effective ways to reduce disease burden. As shown in Figures 5 and 6, even a moderate reduction of these two parameters can significantly contribute to a decrease in cases. At the same time, increasing the fraction of quarantine for those with contact with infectious individuals seems to be less efficient (see Figure 7). Changing this parameter can also turn out to have a positive effect, however, to achieve really significant changes, one would need to increase this parameter to ranges that are rather improbable in a real-life situation as putting into quarantine a high number of healthy people has a negative effect on the economy (see Figure 8). The most efficient way to reduce the number of infected is of course the parallel application of the three measures which is shown to be rather effective in Figures 9 and 10.

5. Conclusions

The current study proposes a new deterministic model for evaluating the population-level impact of implementing quarantine on the control of the COVID-19 outbreaks in Bangladesh. We attempted to include the features most substantial in reflecting the disease transmission, paying attention to the special characteristics of COVID-19. The effect of quarantine – one of the most important tools to hinder disease spread at the beginning of a new epidemic – implemented in this model which we described following [21], reflects the isolation of suspected individuals that is based on contacts with infected individuals, unlike in many other models including quarantine. We have also included separate compartments for the mildly symptomatic and severely symptomatic cases considering the large percentage of those infected who are asymptomatic or exhibit only mild symptoms.

The main novelty of the model introduced in this work is the incorporation of quarantine of individuals suspected of being exposed to COVID-19, which is one of the earliest public health policies for combating the spread of such infectious diseases in populations. The inclusion of quarantine has been modeled in a way that, up to our knowledge, had not been previously incorporated in models for COVID-19 transmission. This approach of defining quarantine facilitates us to keep track of those individuals who are moved into quarantine on the account of being feared to have contracted the disease, but ultimately turn out to be healthy, and those who are actually infected.

To validate our model as well as to obtain a starting point for our numerical assessment of the effect of various intervention parameters, we fitted the model to data from the early stage of the epidemic in Bangladesh. Using Latin Hypercube Sampling and the least squares method, we obtained a fairly good fit. Using the baseline parameter values obtained this way, we performed sensitivity analysis and numerical experiments to determine what kind of results may be achieved by applying intervention measures that are the most natural at the beginning of a novel epidemic, namely reducing contact rates and transmission probability as well as introducing quarantine. The results of the sensitivity analysis and the numerical experiments equally suggest that decreasing contact numbers and transmission probability are both efficient ways to reduce the number of infections, while quarantine alone – though an effective method – is a less powerful tool to reduce the number of infections.

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