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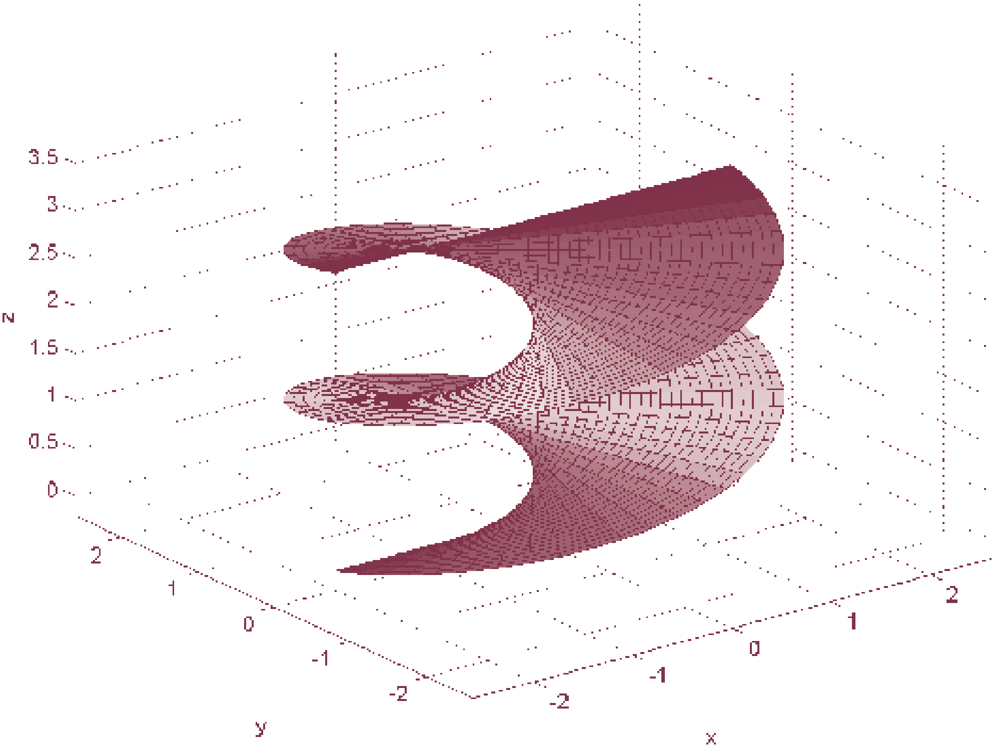

MATHEMATICA

4/2015

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA

4/2015

YEAR **(LX) 2015**
MONTH **DECEMBER**
ISSUE **4**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
4

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 493–507

$\aleph_1$-A-coseperable groups

Ulrich Albrecht

Abstract. Let A be a countable self-small Abelian group with a right Noetherian right hereditary endomorphism ring. We show that the question whether strongly-$\aleph_1$-A-generated groups are $\aleph_1$-A-coseparable is undecidable in ZFC. Our main focus is on the algebraic aspect of the proof, not on the underlying set-theory.

Mathematics Subject Classification (2010): 20K20, 20K40.

Keywords: $\aleph_1$-a-coseparable groups, Martin's axiom, endomorphism rings.

1. Introduction

Let A be an Abelian group with endomorphism ring $E = E(A)$. Associated with A are the functors $H_A(.) = \mathrm{Hom}(A, .)$ and $T_A(.) = . \otimes_E A$ which induce natural maps $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(x)](a) = x \otimes a$ for all $\alpha \in H_A(G)$, $x \in M$ and $a \in A$. The *A-solvable groups* are the Abelian groups G such that θ_G is an isomorphism. Finally, a sequence $0 \to G \to H \to L \to 0$ of Abelian group is *A-balanced* if the induced sequence $0 \to H_A(G) \to H_A(H) \to H_A(L) \to 0$ of right E-modules is exact

An important class of A-solvable groups are the *(finitely) A-projective groups*, i.e. groups which are isomorphic to a direct summand of $\oplus_I A$ for some (finite) index-set I. Finitely A-projective groups are always A-solvable [8], and the same holds for arbitrary A-projective groups [9] if A is *self-small*, i.e. if H_A preserves direct sums of copies of A. Arnold and Murley showed in [9, Corollary 2.3] that a countable Abelian group is self-small if and only if E is countable.

Epimorphic images of A-projective groups are called *A-generated*, but need not be A-solvable. It is easy to see that a group G is A-generated if and only if θ_G is onto. Moreover, if A is self-small, then a group G is A-solvable if and only if there is an A-balanced exact sequence $0 \to U \to F \to G \to 0$ in which F is A-projective and U is A-generated [3]. Finally, G is *A-torsion-free* if every finitely A-generated subgroup of G is isomorphic to a subgroup of a finitely A-projective group, and an A-generated subgroup U of an A-torsion-free group G is *A-pure* if $(U + P)/U$ is A-torsion-free for all finitely A-generated subgroups P of G. If A is flat as an E-module, then A-torsion-free groups are A-solvable [4]. We want to remind the reader that a right E-module M

is *non-singular* if $xI \neq 0$ for all non-zero x in M and all essential right ideals I of E. The ring R is *right non-singular* if R_R is a non-singular module. If U is a submodule of a non-singular right E-module M, then the S-closure of U in M consists of all $x \in M$ such that $xI \subseteq M$ for some essential right ideal I of E [14]. Non-singularity is closely related to A-torsion-freeness whenever A is a self-small Abelian group whose endomorphism ring is right non-singular [5]:

a) If an A-generated group G is A-torsion-free, then $H_A(G)$ is non-singular.
b) An A-generated subgroup U of an A-torsion-free group G is contained in a smallest A-pure subgroup V of G which is obtained as $\theta_G(T_A(W))$ where W is the S-closure of $H_A(U)$ in $H_A(G)$.

The focus of this paper are A-torsion-free groups G such that all A-generated subgroups U of G with $|U| < |G|$ are A-projective. Since A-generated subgroups of A-projective groups need not be A-projective in general ([4] and [8]), some immediate restrictions on A are needed to guarantee the existence of non-trivial groups with the above property.

2. Hereditary Endomorphism Rings and κ-A-projective groups

An Abelian group is *κ-A-generated*, where κ is an infinite cardinal, if it is an epimorphic image of $\oplus_I A$ for some index-set I with $|I| < \kappa$. The $\aleph_0$-A-generated groups are referred to as *finitely A-generated groups*. An A-generated group G is *κ-A-projective* if every κ-A-generated subgroup U of G is A-projective. If $|A| < \kappa$, then this is equivalent to the condition that all A-generated subgroups U with $|U| < \kappa$ are A-projective. Since every finitely A-generated subgroup of a κ-A-projective group G is A-projective, G is A-solvable. In particular, an A-generated group G is $\aleph_0$-A-projective if every finitely A-generated subgroup is A-projective. If A is faithfully flat as a left E-module, then finitely A-generated A-projective groups are finitely A-projective [4].

Theorem 2.1. *The following conditions are equivalent for a self-small torsion-free Abelian group A:*

a) i) *A-projective groups are κ-A-projective for all infinite cardinals κ.*
   ii) *Every exact sequence $0 \to U \to G \to H \to 0$, in which G and H is κ-A-projective for some infinite cardinal κ, is A-balanced.*

b) *E is a right hereditary ring.*

In this case, A is faithfully flat as an E-module.

Proof. *a)* $\Rightarrow$ *b)*: To see that A is flat as an E-module, observe that A^n is $\aleph_0$-A-projective for all $n < \omega$, from which we obtain that $G = \alpha(A^n)$ is A-projective for all $\alpha : A^n \to A$. By a.ii), the exact sequence $0 \to U \to A^n \to G \to 0$ with $U = \ker\ \alpha$ is A-balanced which yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(U) & \longrightarrow & T_AH_A(A^n) & \longrightarrow & T_AH_A(G) & \longrightarrow & 0 \\ & & \downarrow \theta_U & & \wr\downarrow \theta_{A^n} & & \wr\downarrow \theta_G & & \\ 0 & \longrightarrow & U & \longrightarrow & A^n & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Thus, θ_U is an isomorphism. By Ulmer's Theorem [17], A is E-flat.

Consider a right ideal I of E. Because A is E-flat, $T_A(I) \cong IA \subseteq A$. Since IA is an A-generated subgroup of A, and A is $|IA|^+$-A-projective by a.i), IA is A-projective. Thus, $H_AT_A(I)$ is a projective module fitting into the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H_AT_A(I) & \longrightarrow & H_AT_A(E) \\ & & \uparrow \phi_I & & \wr\uparrow \phi_E \\ 0 & \longrightarrow & I & \longrightarrow & E \end{array}$$

from which we obtain that ϕ_I is one-to-one.

On the other hand, consider an exact sequence $0 \to V \to F \to I \to 0$ where F is a free right E-module. It induces the exact sequence

$$0 \to T_A(V) \to T_A(F) \to T_A(I) \to 0.$$

The latter sequence is A-balanced by a.ii). Hence, the top-row in the commutative diagram

$$\begin{array}{ccccc} H_AT_A(F) & \longrightarrow & H_AT_A(I) & \longrightarrow & 0 \\ \wr\uparrow \phi_F & & \uparrow \phi_I & & \\ F & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

is exact, which yields that ϕ_I is onto. Consequently, I is projective, and E is right hereditary.

b) $\Rightarrow$ *a)*: Let M be a right E-module. Since E is right hereditary, we can find an exact sequence $0 \to P \to F \to M \to 0$ in which P and F are projective. It induces exact sequence

$$0 = \mathrm{Tor}_1^R(F, A) \to \mathrm{Tor}_1^R(M, A) \to T_A(P) \to T_A(F) \to T_A(M) \to 0.$$

We obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_A(\mathrm{Tor}_1^R(M, A)) & \longrightarrow & H_AT_A(P) & \longrightarrow & H_AT_A(F) \\ & & & & \wr\uparrow \phi_P & & \wr\uparrow \phi_F \\ & & 0 & \longrightarrow & P & \longrightarrow & F. \end{array}$$

Therefore, $H_A(\mathrm{Tor}_1^R(M, A)) = 0$ for all right R-modules M.

If M^+ is torsion-free, then it is isomorphic to a submodule of $\mathbb{Q}M = \mathbb{Q} \otimes_{\mathbb{Z}} M$. Since $\mathrm{Tor}_1^R(\mathbb{Q}M, A)$ is torsion-free and divisible, $H_A(\mathrm{Tor}_1^R(\mathbb{Q}M, A)) = 0$ is only possible if $\mathrm{Tor}_1^R(\mathbb{Q}M, A) = 0$. However, because E is right hereditary, we have the exact sequence $0 \to \mathrm{Tor}_1^R(M, A) \to \mathrm{Tor}_1^R(\mathbb{Q}M, A) = 0$, and $\mathrm{Tor}_1^R(M, A) = 0$.

If M^+ is torsion, then we select an exact sequence $0 \to U \to F_1 \to A \to 0$ in which F_1 is a free left E-module. It induces

$$0 = \mathrm{Tor}_1^R(M, F_1) \to \mathrm{Tor}_1^R(M, A) \to M \otimes_E V.$$

Since $M \otimes_E V$ is torsion, the same holds for $\mathrm{Tor}_1^R(M, A)$. But, the latter also is isomorphic to a subgroup of the torsion-free group $T_A(P)$. Thus, $\mathrm{Tor}_1^R(M, A) = 0$.

For an arbitrary M, we consider the exact sequence

$$0 = \operatorname{Tor}_1^R(tM, A) \to \operatorname{Tor}_1^R(M, A) \to \operatorname{Tor}_1^R(M/tM, A) = 0$$

where the first and the last term vanish but what has already been shown. Thus, A is E-flat.

To show that A is faithful as a left E-module, suppose that $T_A(M) = 0$. The sequence $0 \to P \to F \to M \to 0$ yields the exact sequence

$$0 \to T_A(P) \to T_A(F) \to T_A(M) = 0$$

since A is flat as an E-module. Hence, the top-row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_AT_A(P) & \longrightarrow & H_AT_A(F) & \longrightarrow & 0 & & \\ & & \wr\uparrow \phi_P & & \wr\uparrow \phi_F & & & & \\ 0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

is exact. A simple diagram chase shows $M = 0$.

Finally, A-generated subgroups of A-projective groups are A-projective if A is faithfully flat and E is right hereditary [4], and a.i) holds. Finally, a.ii) is a direct consequence of [6] since κ-A-projective groups are A-solvable. □

In particular, the last result shows that A-generated subgroups of self-small groups with right hereditary endomorphism ring are A-projective. Our next results summarizes other properties of such groups which we use frequently in this paper:

Corollary 2.2. *Let A be a self-small torsion-free Abelian group whose endomorphism ring is right hereditary:*

a) *Every exact sequence $G \to P \to 0$ such that G is A-generated and P is A-projective splits.*
b) *An A-generated group is A-torsion-free if and only if it is $\aleph_0$-A-projective.*
c) *An A-generated subgroup of an A-torsion-free group is A-pure if and only if U is a direct summand of $U + V$ for all finitely A-generated subgroups V of G.*

Proof. a) follows directly from the fact that A is faithfully flat which was established in Theorem 2.1.

b) It remains to show that A-torsion-free groups are $\aleph_0$-A-projective. Suppose that G is A-torsion-free, and let U be a finitely A-generated subgroup of G. Then U can be embedded into an A-projective group, and thus is A-projective by Theorem 2.1.

c) Let U be an A-pure subgroup of an A-torsion-free group G. If V is a finitely A-generated subgroup of G, then $(U + V)/U$ can be embedded into an A-projective group by Theorem 2.1. Thus, $(U+V)/U$ is A-projective. By a), U is a direct summand of $U + V$. □

However, the S-closure of a countable submodule of a non-singular module does not need to be countable even if R is countable. For instance, if $Q = \mathbb{Q}^\omega$ and $R = \mathbb{Z}1_S + \mathbb{Z}^{(\omega)}$, then Q is the maximal ring of quotients of R and $|Q| > |R|$ although Q is an essential extension of R. We want to remind the reader that a right E-module M has *Goldie-dimension* $m < \infty$ if it contains an essential submodule which is the direct sum of m non-zero uniform submodules where a module $X \neq 0$ is uniform if all its non-zero submodules are essential.

Proposition 2.3. *Let R be a countable right non-singular ring which has finite right Goldie-dimension. The S-closure of a countable submodule U of a non-singular right R-module M is countable.*

Proof. Let V be S-closure of U, and assume that V is uncountable. Let

$$\mathcal{F} = \{X \subseteq R \mid |X| < \infty \text{ and } \sum_{x \in X} xR \text{ is an essential right ideal}\}.$$

Then, $V = \{y \in M \mid yX \subseteq U \text{ for some } x \in \mathcal{F}\}$ since R has finite right Goldie-dimension. Since V is uncountable and $\mathcal{F}$ is countable, we can find $X_0 \in \mathcal{F}$ such that $yX_0 \subseteq U$ for uncountably many $y \in V$. Let $Y_0 = \{y \in V \mid yX_0 \subseteq U\}$. Write $X_0 = \{x_1, \ldots, x_n\}$, and consider $Y_0 x_1 \subseteq U$. There is an uncountable subset Y_1 of Y_0 such that $yx_1 = y'x_1$ for all $y, y' \in Y_1$ since U is countable. Repeating this argument with x_2 and Y_1 yields an uncountable subset Y_2 of Y_1 such that $yx_2 = y'x_2$ for all $y, y' \in Y_2$. By induction, we can find an uncountable subset Y_n of Y_0 such that $yx_i = y'x_i$ for all $i = 1, \ldots, n$ and all $y, y' \in Y_n$. Thus, $(y-y')(x_1R+\ldots+x_nR) = 0$ for all $y, y' \in Y_n$ which contradicts the fact that M is non-singular because $x_1R+\ldots+x_nR$ is essential. Thus V has to be countable. □

By Sandomierski's Theorem [11], a right finite dimensional, right hereditary ring is right Noetherian.

Corollary 2.4. *The following conditions are equivalent for a self-small torsion-free Abelian group A whose endomorphism ring is right hereditary:*

a) *E is right Noetherian.*
b) *An A-generated subgroup U of a finitely A-projective group G is finitely A-projective.*

Proof. *a) ⇒ b)*: Suppose that U is an A-generated subgroup of a finitely A-projective group P. Then $H_A(U)$ is a submodule of $H_A(P)$, and hence a finitely generated projective module. By Theorem 2.1, U is A-solvable, and $U \cong T_A H_A(U)$ is finitely A-projective.

b) ⇒ a): Let I be a right ideal of E. Arguing as in the proof of Theorem 2.1, ϕ_I is an isomorphism. Moreover $T_A(I) \cong IA$ since A is flat as an E-module. By b), IA is finitely A-projective, from which we obtain that I is finitely generated. □

In view of the results of this section, we assume from this point on that A is a self-small torsion-free group with a right Noetherian right hereditary endomorphism ring. Huber and Warfield showed in [16] that E is a right and left Noetherian ring whenever A is a torsion-free reduced group of finite rank with a right hereditary endomorphism ring. Moreover, no generality is lost if we restrict our discussion to the case that κ is a regular cardinal because Shelah's singular compactness theorem applies to A-projective groups [2].

3. $\aleph_1$-A-Coseparable Groups

Let $\kappa > \aleph_0$ be a regular cardinal, and A be a torsion-free Abelian group with $|A| < \kappa$. An A-projective subgroup U of an $\aleph_0$-A-projective group G is *κ-A-closed* provided that $(U+V)/U$ is A-projective for all κ-A-generated subgroups V of G. If $|U| < \kappa$, then this is equivalent to saying that G/U is κ-A-projective. The group G is *strongly κ-A-projective* if it is κ-A-projective and every κ-A-generated subgroup U of G is contained in an κ-A-generated, κ-A-closed subgroup V of G. By [1], $S_A(A^I)$ is $\aleph_1$-A-projective, but not strongly $\aleph_1$-A-projective since $\oplus_I A$ is not an $\aleph_1$-A-closed subgroup.

In the following we focus on the case $\kappa = \aleph_1$ since we are mainly interested in the algebraic aspects instead of the underlying set-theory. However, most results of this section carry over to the general case. In order to avoid immediate difficulties, we restrict our discussion to the case that A is countable.

Lemma 3.1. *Let A be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring.*

a) *If G is $\aleph_1$-A-projective, then G/U is $\aleph_1$-A-projective for all $\aleph_1$-A-closed subgroups U of G.*
b) *If G is strongly $\aleph_1$-A-projective, then G/U is strongly $\aleph_1$-A-projective for all countable $\aleph_1$-A-closed subgroups U of G.*

Proof. a) Let $\{\phi_n | n < \omega\} \subseteq H_A(G/U)$. Since $\Sigma_{n<\omega}\phi_n(A)$ is countable, there is a countable subgroup K of G such that $\Sigma_{n<\omega}\phi_n(A) \subseteq (K+U)/U$. Because A is countable, we can choose K to be A-generated. Since U is $\aleph_1$-A-closed in G, the group $(K+U)/U$ is U-projective, and the same holds $\Sigma_{n<\omega}\phi_n(A)$. Therefore, G/U is $\aleph_1$-A-projective.

b) Let V/U be a countable A-generated subgroup of G/U. Without loss of generality, we may assume that V is A-generated. Then, V is contained in an $\aleph_1$-A-closed subgroup W is a $\aleph_1$-A-closed subgroup of G. Since U is countable this means that G/W is $\aleph_1$-A-projective. Since $G/W \cong (G/U)/(W/U)$ and G/U is $\aleph_1$-A-projective, we obtain that G/U is strongly $\aleph_1$-A-projective. □

An A-generated group G $\aleph_1$-*A-coseparable* if it is $\aleph_1$-A-projective and every A-generated subgroup U of G such that G/U is countable contains a direct summand V of G such that G/V is countable. Our next results describes $\aleph_1$-A-coseparable group. Although our arguments follow the general outline of [13], significant modifications are necessary in our setting.

Theorem 3.2. *Let A be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring. A group G is $\aleph_1$-A-coseparable if and only if G is A-solvable and every exact sequence*

$$0 \to P \to X \to G \to 0$$

with P a direct summand of $\oplus_\omega A$ and X A-generated splits.

Proof. Suppose that G is $\aleph_1$-A-coseparable, and consider an exact sequence

$$0 \to P \stackrel{\alpha}{\to} X \stackrel{\beta}{\to} G \to 0$$

with P a direct summand of $\oplus_\omega A$ and X A-generated. Since A is faithfully flat, X is A-generated and G is A-solvable, the induced sequence

$$0 \to H_A(P) \xrightarrow{\alpha} H_A(X) \overset{H_A(\beta)}{\longrightarrow} H_A(G) \to 0$$

of right E-modules is exact by Theorem 2.1. Since $H_A(G)$ is non-singular by the remarks in the introduction, the same holds for $H_A(X)$. Observe that $H_A(P)$ is countable since it is a direct summand of $H_A(\oplus_\omega A)$, and the latter is countable because A is self-small. We choose a complement W of $im(H_A(\alpha))$ in $H_A(X)$, and observe that $H_A(X)/W$ is nonsingular. Since

$$M = H_A(X)/(im(H_A(\alpha) \oplus W) \cong [H_A(X)/W][(im(H_A(\alpha) \oplus W)/W]$$

is singular and $(im(H_A(\alpha) \oplus W)/W$ is countable, $H_A(X)/W$ is countable as the S-closure of a countable submodule by Proposition 2.3 because E is right Noetherian and countable. Applying T_A yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(P) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(X) & \xrightarrow{T_AH_A(\beta)} & T_AH_A(G) & \longrightarrow & 0 \\ & & \wr\big\downarrow \theta_P & & \big\downarrow \theta_X & & \wr\big\downarrow \theta_G & & \\ 0 & \longrightarrow & P & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & G & \longrightarrow & 0. \end{array}$$

Therefore, X is A-solvable, and $U = \theta_X(T_A(W))$ is an A-generated subgroup of X such that $\alpha(P) \cap X = 0$ and

$$X/[\alpha(P) \oplus U] \cong T_A(M)$$

is countable. If $H = \beta(U)$, then $\beta|U$ is one-to-one. Since $\beta(U) \cong U \cong T_A(W)$ is A-generated and $G/\beta(U)$ is countable, there is a subgroup K of U such that $G = \beta(K) \oplus B$ for some countable subgroup B of G using the fact that G is $\aleph_1$-A-coseparable. Select a subgroup V of X containing $\alpha(P)$ such that $\beta(V) = B$. Clearly, V is countable.

To show $X = K \oplus V$, take $x \in X$ and write $\beta(x) = \beta(k) + \beta(v)$ for some $k \in K$ and $v \in V$. Then $x - k - b \in \alpha(P) \subseteq V$. On the other hand, suppose that $y \in K \cap V$. Then $\beta(y) \in \beta(K) \cap B = 0$, from which we obtain

$$y \in \alpha(P) \cap K \subseteq \alpha(P) \cap U = 0.$$

Moreover, V is A-generated since it is a direct summand of X, while $\beta(V) \cong V/\alpha(P)$ is A-projective as a countable subgroup of G. Therefore, $\alpha(P)$ is a direct summand of V.

Conversely, assume that G is an A-solvable group such that every exact sequence $0 \to P \to X \to G \to 0$ with P a direct summand of $\oplus_\omega A$ and X A-generated splits. Suppose that G contains a countable A-generated subgroup U which is not A-projective. Since U is A-solvable because A is E-flat by Theorem 2.1, $H_A(U)$ is not projective. Looking at projective resolutions of $H_A(U)$, we can find a countable projective module P with $\mathrm{Ext}^1_E(H_A(U), P) \neq 0$. Since E is right hereditary, we have an exact sequence

$$\mathrm{Ext}^1_E(H_A(G), P) \to \mathrm{Ext}^1_E(H_A(U), P) \to 0.$$

Thus, there is a non-splitting sequence $0 \to P \to M \to H_A(G) \to 0$ which induces $0 \to T_A(P) \to T_A(M) \to T_AH_A(G) \to 0$ which splits since $G \cong T_AH_A(G)$. We therefore obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_AT_A(P) & \longrightarrow & H_AT_A(M) & \longrightarrow & H_AT_AH_A(G) & \longrightarrow & 0 \\ & & \wr\Big\uparrow \phi_P & & \Big\uparrow \phi_M & & \wr\Big\uparrow \phi_{H_A(G)} & & \\ 0 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & H_A(G) & \longrightarrow & 0 \end{array}$$

in which ϕ_M is an isomorphism by the 3-Lemma. Since the top-row splits, the same has to hold for the bottom, which contradicts its choice. Therefore, G is $\aleph_1$-A-projective.

Consider an A-generated subgroup C of G such that G/C is countable. We can find a countable subgroup B such that $G = C + B$, and no generality is lost if we assume in addition that B is A-generated. By what was shown in the last paragraph, B is A-projective. We consider the exact sequence $0 \to K \to B \oplus C \xrightarrow{\pi} G \to 0$ with $\pi(b, c) = b + c$. Since G is A-solvable, and C is an A-generated subgroup of G, the group $B \oplus C$ is A-solvable. By Theorem 2.1, $K = \{(b, -b) \mid b \in B \cap C\}$ is A-generated, and hence A-solvable since A is E-flat. Since K is isomorphic to a subgroup of the countable A-projective group B, another application of Theorem 2.1 yields that K is a countable A-projective group. By our hypotheses, the map π splits, say $\pi\delta = 1_G$ for some homomorphism $\delta : G \to B \oplus C$. Let $\rho : B \oplus C \to B$ be the projection onto B with kernel C, and consider $D = \ker(\rho\delta)$. Since G/D is A-generated and isomorphic to a subgroup of the countable A-projective group B, it is A-projective itself. By Theorem 2.1, D is a direct summand of G. Moreover, every $d \in D$ satisfies $\delta(d) = (0, c)$ for some $c \in C$ since $\rho\delta(d) = 0$ yields $\delta(d) \in \ker\ \rho = C$. Then $d = \pi\delta(d) = \pi(0, c) = c$, and $D \subseteq C$. □

A group W is an *A-Whitehead group* if it admits an A-balanced exact sequence $0 \to U \to F \to W \to 0$ in which F is A-projective and U is A-generated with the property that

$$0 \to \mathrm{Hom}(W, A) \to \mathrm{Hom}(F, A) \to Hom(U, A) \to 0$$

is exact.

Corollary 3.3. *Let A be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring.*

a) *Every $\aleph_1$-A-coseparable group W is an A-Whitehead group.*
b) *It is consistent with ZFC that there exists a strongly $\aleph_1$-A-projective group G which is not $\aleph_1$-A-coseparable.*

Proof. a) By [7], an A-solvable group W is an A-Whitehead group if every exact sequence $0 \to A \to X \to W \to 0$ with $S_A(X) = X$ splits which is satisfied by W because of Theorem 3.2.

b) If we assume $V = L$, then all A-Whitehead groups are A-projective. However, there exist strongly $\aleph_1$-A-projective group G with $\mathrm{Hom}(G, A) = 0$ [7]. □

4. Strongly $\aleph_1$-A-Projective Groups and Martin's Axiom

We use the formulation of Martin's Axiom given in [13, Definition VI.4.2]. A partially ordered set $(P, \leq)$ satisfies the *countable chain condition (ccc)* if any antichain in $(P, \leq)$ is countable. An *antichain* is a subset A of P such that any two distinct members of A are *incompatible*, i.e., whenever $p, q \in A$, then there does not exist $r \in P$ such that $r \geq p$ and $r \geq q$. A subset D of P is dense if, for every $p \in P$ there is $q \in D$ such that $p \leq q$. Finally, a subset $\mathcal{F}$ of P is directed, if, for all $p, q \in \mathcal{F}$, there is $r \in \mathcal{F}$ such that $r \geq p$ and $r \geq q$.

For a cardinal κ, let MA(κ) denote the statement:

Let $(P, \leq)$ be a partially ordered set satisfying the *countable chain condition (ccc)*. For every family $\mathcal{D} = \{D_\alpha \mid \alpha < \kappa\}$ of dense subsets of P, there is a directed subset $\mathcal{F}$ of P such that $\mathcal{F} \cap D_\alpha \neq \emptyset$ for all α, i.e. $\mathcal{F}$ is *$\mathcal{D}$-generic.*

Martin's axiom (MA) stipulates that MA(κ) holds for every $\kappa < 2^{\aleph_0}$ [13].

Theorem 4.1. *$(MA + \aleph_1 < 2^{\aleph_0})$ Let A be a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring. If G is a strongly $\aleph_1$-A-projective group and $0 \to U \to \oplus_I A \to G \to 0$ is an A-balanced exact sequence such that $S_A(U) = U$ and $|I| < 2^{\aleph_0}$, then the induced sequence*

$$0 \to \mathrm{Hom}(G, B) \to Hom(\oplus_I A, B) \to \mathrm{Hom}(G, B) \to 0$$

is exact for all countable A-solvable group B.

Proof. We consider an A-balanced exact sequence $0 \to U \to \oplus_I A \to G \to 0$ where $U \to \oplus_I A$ is the inclusion map. Let $\mathcal{P}(U)$ be the collection of A-generated A-pure subgroups V of $F = \oplus_I A$ containing U such that V/U is finitely A-projective. Since V is A-generated and A is faithfully flat, U is a direct summand of V by Corollary 2.2, say $V = U \oplus R_V$ for some finitely A-projective group R_V.

To show that the sequence $Hom(\oplus_I A, B) \to \mathrm{Hom}(G, B) \to 0$ is exact whenever B is a countable A-solvable group, let $\phi \in \mathrm{Hom}(U, B)$, and consider

$$P = \{(V, \psi) \mid V \in \mathcal{P}(U), \psi \in \mathrm{Hom}(V, B), \text{ and } \psi|U = \phi\}.$$

We partially order P by $(V_1, \psi_1) \geq (V_2, \psi_2)$ if and only if $V_2 \subseteq V_1$ and $\psi_1|V_2 = \psi_2$. Once we have shown that P and $\mathcal{D} = \{D(J)|J \subseteq I \;\text{ finite}\}$ satisfy the hypotheses of Martin's Axiom, then we can find a $\mathcal{D}$-generic directed directed subset $\mathcal{F}$ of P. Define a map $\psi : \oplus_I A \to B$ as follows. For $x \in \oplus_I A$, choose a finite subset J of I such that $x \in \oplus_J A$. Since $\mathcal{F}$ is $\mathcal{D}$-generic, there is $(V, \delta) \in D(J) \cap F$ with $x \in V$. Define $\psi(x) = \delta(x)$. Moreover, if (V_1, δ_1) and (V_2, δ_2) are two choices, then there is $(V_3, \delta_3) \in \mathcal{F}$ such that $(V_i, \delta_i) \leq (V_3, \delta_3)$ for $i = 1, 2$ since $\mathcal{F}$ is directed. Thus, $\delta_1(x) = \delta_3(x) = \delta_2(x)$. □

The key towards showing that $(P, \leq)$ satisfies the countable chain condition is

Theorem 4.2. *Every uncountable subset P' of P contains an uncountable subset P'' for which we can find an A-pure A-projective subgroup X of F containing U as a direct summand such that $V \subseteq X$ whenever $(V, \psi) \in P''$.*

Proof. We may assume that $P' = \{(V_\nu, \psi_\nu) | \nu < \omega_1\}$. Since U is a direct summand of V_ν, we obtain that $H_A(V_\nu/U) \cong H_A(V_\nu)/H_A(U)$ is a finitely generated right E-module. In particular, it has finite right Goldie dimension since E is right Noetherian. Therefore, no generality is lost if we assume that there is $m < \omega$ such that $H_A(V_\nu/U)$ has Goldie dimension m for all $\nu < \omega_1$.

Let $0 \leq k \leq m$ be maximal with respect to the property that there exists an A-pure A-projective subgroup T of F containing U such that $H_A(T/U)$ has Goldie dimension k and T is contained in uncountable many V_ν. This k exists since U is the choice for T in the case $k = 0$. Observe that T/U is A-solvable as an A-generated subgroup of the A-solvable group $G = F/U$. Thus, $0 \to U \to T \to T/U \to 0$ is A-balanced, and $H_A(T/U) \cong H_A(T)/H_A(U)$ has finite Goldie-dimension and is non-singular. Thus, it contains a finitely generated essential submodule. Since E is right Noetherian and countable, any essential extension of a non-singular finite dimensional right E-module is countable by Proposition 2.3. In particular, $H_A(T/U)$ is countable, and hence $T/U \cong T_A H_A(T/U)$ is countable. Since G is $\aleph_1$-A-projective, T/U is A-projective, and $T = U \oplus W$ because A is faithfully flat by Corollary 2.2.

Suppose that T' is an A-generated subgroup of F containing T such that $T \neq T'$. There exists $\alpha \in H_A(T')$ with $\alpha(A) \not\subseteq T$. Since T is A-pure in F, we obtain $T + \alpha(A) = T \oplus C$ with $C \neq 0$. Thus, the Goldie-dimension of $H_A(T')$ is at least $k+1$, and T' is contained in only countably many of the V_ν. No generality is lost if we assume that T is contained in V_ν for all ν. Since T is A-pure in F and $V_\nu = U \oplus R_{V_\nu} = T + R_{V_\nu}$ for some finitely A-projective subgroup R_{V_ν} of F, we obtain decompositions $V_\nu = T \oplus W_\nu$. Observe that W_ν is finitely A-projective.

We construct X as the union of a smooth ascending chain $\{X_\nu | \nu < \omega_1\}$ of A-pure A-projective subgroups of F containing T and an ascending chain of ordinals $\{\sigma_\nu | \nu < \omega_1\}$ such that $X_{\nu+1}/X_\nu$ is A-projective, $W_{\sigma_{\nu+1}} \subseteq X_{\nu+1}$, and X_ν/U is an image of $\oplus_\omega A$ for all $\nu < \omega_1$. We set $X_0 = T$, and $X_\alpha = \cup_{\nu<\alpha} X_\nu$ if α is a limit ordinal. Then, X_α/U is a countable subgroup of F/U, and hence A-projective. Set $\sigma_\alpha = sup(\sigma_\nu | \nu < \alpha)$.

If $\alpha = \nu + 1$, then there exists a subgroup C_ν of F containing X_ν such that the group C_ν/U is an A-projective countable $\aleph_1$-A-closed subgroup of F/U since F/U is strongly $\aleph_1$-A-projective. In particular, $F/C_\nu \cong (F/U)/(C_\nu/U)$ is A-solvable. Since A is flat, C_ν is A-generated by Theorem 2.1. If K is a countable A-generated subgroup of F, then $(K+U)/U$ is a countable subgroup of F/U. Hence,

$$(K + C_\nu)/C_\nu \cong [(K + C_\nu)/U]/[C_\nu/U]$$

is A-projective.

To construct σ_α, assume $W_\mu \cap C_\nu \neq 0$ for all $\mu > \sigma_\nu$. Then,

$$W_\mu/(W_\mu \cap C_\nu) \cong (W_\mu + C_\nu)/C_\nu$$

is A-projective by the last paragraph. Since A is faithfully flat, $W_\mu \cap C_\nu$ is A-generated, and there is a map $0 \neq \alpha_\mu \in H_A(W_\mu \cap C_\nu) \subseteq H_A(C_\nu)$. Since C_ν/U is a countable subgroup of F/U, it is A-projective, and $C_\nu = U \oplus P_\nu$ since A is faithfully flat. Observe that P_ν is countable and A-projective. Write $\alpha_\mu = \beta_\mu + \epsilon_\mu$ with $\beta_\mu \in H_A(U)$ and $\epsilon_\mu \in H_A(P_\nu)$. Since E is countable, the same holds for $H_A(P_\nu)$, and there is

$\epsilon \in H_A(P_\nu)$ such that $\epsilon_\mu = \epsilon$ for uncountably many μ. For all these μ, we have $\epsilon(A) \subseteq W_\mu + U \subseteq V_\mu$. Hence, $T + \epsilon(A) \subseteq V_\mu$ for uncountably many μ. However, this is only possible if $\epsilon(A) \subseteq T$. But then, $\alpha_\mu(A) \subseteq T \cap W_\mu = 0$, a contradiction.

Therefore, we can find an ordinal $\sigma_\alpha > \sigma_\nu$ with $C_\nu \cap W_{\sigma_\alpha} = 0$. In particular, $X_\nu \subseteq C_\nu$ yields $X_\nu \cap W_{\sigma_\alpha} = 0$. Let Y be the S-closure of

$$H_A(X_\nu \oplus W_{\sigma_\alpha}) = H_A(X_\nu \oplus H_A(W_{\sigma_\alpha}) \supseteq H_A(U)$$

in $H_A(F)$ and let $X_\alpha = \theta_F(Y \otimes A) = YA$. As an A-generated subgroup of F, X_α is A-solvable. Then, the inclusion $Y \subseteq H_A(X_\alpha)$ induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_A(Y) & \longrightarrow & T_AH_A(X_\alpha) & \longrightarrow & T_A(H_A(X_\alpha)/Y) & \longrightarrow & 0 \\ & & \wr\big\downarrow \theta_F|T_A(Y) & & \wr\big\downarrow \theta_{X_\alpha} & & & & \\ 0 & \longrightarrow & YA & \xrightarrow{1_{YA}} & YA & \longrightarrow & 0 & & \end{array}$$

from which we get $T_A(H_A(X_\alpha)/Y) = 0$. Since A is faithfully flat, $Y = H_A(X_\alpha)$, and $H_A(X_\alpha)/[H_A(X_\nu) \oplus H_A(W_{\sigma_\alpha})]$ is singular.

Observe that $Y/H_A(U)$ is the S-closure of $[H_A(X_\nu) + H_A(W_{\sigma_\alpha})]/H_A(U)$ in $H_A(F)/H_A(U)$ because

$$H_A(F)/Y \cong [H_A(F)/H_A(U)]/[Y/H_A(U)]$$

is non-singular and

$$Y/H_A(X_\nu \oplus W_{\sigma_\alpha}) \cong [Y/H_A(U)]/[H_A((X_\nu \oplus W_{\sigma_\alpha})/H_A(U)]$$

is singular. Since F/U is A-solvable, and X_ν/U is countable, $H_A(X_\nu)/H_A(U)$ is countable. Moreover, W_μ is finitely A-projective. Hence, the E-module $H_A(W_{\sigma_\alpha})$ is countable too, and

$$[H_A(X_\nu) + H_A(W_{\sigma_\alpha})]/H_A(U)$$

is countable. Thus, $Y/H_A(U)$ is an essential extension of a countable non-singular right E-module. By Proposition 2.3, we obtain that $Y/H_A(U)$ is countable. Thus, there is a countable submodule Y' of Y with $Y = Y' + H_A(U)$. Then X_α/U is countable and $X_\alpha = Y'A + X_\nu$, and. Consequently, X_α/U has to be A-projective, and the same holds for $X_\alpha \cong X_\alpha/U \oplus U$.

It remains to show that X_α/X_ν is A-projective. For this, observe that the group

$$X_\alpha/(X_\alpha \cap C_\nu) \cong (X_\alpha + C_\nu)/C_\nu$$

is countable since it is an epimorphic image of $(X_\alpha + C_\nu)/U$ which is countable because X_α and C_ν/U are countable. Since C_ν/U is $\aleph_1$-A-closed in F/U, we have that

$$X_\alpha/(X_\alpha \cap C_\nu) \cong [(X_\alpha + C_\nu)/U]/[C_\nu/U]$$

is A-projective. Since A is flat, $X_\alpha \cap C_\nu$ is A-generated as in Theorem 2.1. For τ in $H_A(X_\alpha \cap C_\nu)$, choose a regular element $c \in E$ such that $\tau c \in H_A(X_\nu) \oplus H_A(W_{\sigma_\alpha})$, say $\tau c = \beta + \gamma$ for some $\beta \in H_A(X_\nu)$ and $\gamma \in H_A(W_{\sigma_\alpha})$. Then

$$\gamma = \tau c - \beta \in H_A(W_{\sigma_\alpha}) \cap H_A(C_\nu) = 0.$$

Hence, $\tau c \in H_A(X_\nu)$. Since X_ν is A-pure in F, we obtain $\tau \in H_A(X_\nu)$. Therefore, $H_A(X_\alpha \cap C_\nu) \subseteq H_A(X_\nu)$, and $X_\alpha \cap C_\nu \subseteq X_\nu$. Since X_ν is contained in X_α and in

C_ν, we obtain $X_\alpha \cap C_\nu = X_\nu$. Then $X_\alpha/X_\nu \cong (X_\alpha + C_\nu)/C_\nu$ is A-projective by what we have already shown. In particular, X_ν is a direct summand of X_α.

Consequently, $X = \cup_{\nu<\omega_1} X_\nu$ is A-pure and A-projective. Because

$$X_{\nu+1}/X_\nu \cong [X_{\nu+1}/T]/[X_\nu/T]$$

is A-projective for all ν, the group X/T is A-projective. This yields $X = T \oplus S$. However, $T = U \oplus W$, so that $X = U \oplus W \oplus S$. Finally,

$$V_{\sigma_{\nu+1}} = T \oplus W_{\sigma_{\nu+}} \subseteq X_{\nu+1} \subseteq X$$

for all $\nu < \omega_1$. Let $P'' = \{(V_{\sigma_{\nu+1}}, \psi_{\sigma_{\nu+1}})|\nu < \omega_1\}$. □

Corollary 4.3. *P satisfies the countable chain condition.*

Proof. Since B is a countable A-solvable group, there is an exact sequence

$$0 \to V \to \oplus_\omega A \to B \to 0$$

which is A-balanced by Theorem 2.1. Thus, $H_A(B)$ is countable as an epimorphic image of $H_A(\oplus_\omega A) \cong \oplus_\omega E$ using the self-smallness of A.

Let P' be an uncountable subset of P. By the previous Lemma, we may assume $P' = \{(V_\nu, \psi_\nu)|\nu < \omega_1\}$ such that there is an A-pure A-projective subgroup X containing U as a direct summand satisfying $V_\nu \subseteq X$ for all $\nu < \omega_1$. We can write $X = U \oplus Y$ and $Y = \oplus_J Y_j$ where each Y_j is isomorphic to a subgroup of A. This is possible since E is hereditary.

For $\nu < \omega_1$, we have $V_\nu = U \oplus (Y \cap V_\nu)$. Since $Y \cap V_\nu$ is finitely A-projective, there is a finite subset J_ν of J such that $H_A(Y \cap V_\nu) \subseteq H_A(\oplus_{J_\nu} Y_j)$, and $Y \cap V_\nu \subseteq \oplus_{J_\nu} Y_j$. Therefore, V_ν is an A-pure subgroup of

$$V_\nu + (\oplus_{J_\nu} Y_j) = U \oplus (\oplus_{J_\nu} Y_j).$$

Because $\oplus_{J_\nu} Y_j$ is finitely A-generated, V_ν is a direct summand of $U \oplus (\oplus_{J_\nu} Y_j)$, say $V_\nu + (\oplus_{J_\nu} Y_j) = V_\nu \oplus X_\nu$. Since $V_\nu + (\oplus_{J_\nu} Y_j)$ is A-projective, the same holds for X_ν. Thus, X_ν is isomorphic to a direct summand of $\oplus_{J_\nu} Y_j$. Moreover, $\psi_\nu : V_\nu \to B$ extends to a map $\lambda_\nu : U \oplus (\oplus_{J_\nu} Y_j) \to B$. By the Adjoint-Functor-Theorem,

$$\mathrm{Hom}(\oplus_{J_\nu} Y_j, B) \cong \mathrm{Hom}_E(H_A(\oplus_{J_\nu} Y_j), H_A(B))$$

is countable since $H_A(B)$ is countable as was shown in the first paragraph of the proof and J_ν is finite. Consequently, there are at most countably many different extensions of ϕ to $U \oplus (\oplus_{J_\nu})$.

If there are only countably many different J_ν's, then there is ν_0 such that $J_{\nu_0} = J_\mu$ for uncountable μ. Thus, there are μ_1 and μ_2 with $J_{\nu_0} = J_{\mu_1} = J_{\mu_2}$ and $\lambda_{\mu_1} = \lambda_{\mu_2}$. Thus, ψ_{μ_1} and ψ_{μ_2} have a common extension. Therefore, $P' = \{(V_\nu, \psi_\nu)|\nu < \omega_1\}$ cannot be an antichain. On the other hand, if there are uncountably many J_ν's, then we may assume without loss of generality that $J_\nu \neq J_\mu$ for $\mu \neq \nu$. Finally, we can impose the requirement that all the J_ν have the same order. Thus, J_ν cannot be contained in J_μ for $\mu \neq \nu$. Since $(V_\nu, \psi_\nu) \leq (V_\nu \oplus X_\nu, \lambda_\nu)$, we may assume that $V_\nu = U \oplus (\oplus_{J_\nu} Y_j)$ and $\lambda_\nu = \psi_\nu$.

There is a subset T of J which is maximal with respect to the property that it is contained in uncountably many of the J_ν. We may assume that T is actually contained in all of the J_ν. Observe that T is finite and a proper subset of all the J_ν. Otherwise, all the J_ν would have to coincide with T since they have the same finite order. Since $\mathrm{Hom}(\oplus_T Y_j, B) \cong \mathrm{Hom}_E(H_A(\oplus_T Y_j), H_A(B))$ is countable by the Adjoint-Functor-Theorem, there are uncountably many ψ_ν which have the same restriction to $W = U \oplus (\oplus_T Y_j)$. Without loss of generality, we may assume that this happens for all ν.

Let $j \in J_0 \setminus T$. The maximality of T guarantees that j is contained in only countably many of the J_ν. Since $J_0 \setminus T$ is finite, there is $\mu < \omega_1$ with $J_\mu \cap J_0 = T$. The maps ψ_μ and ψ_0 have a common extension $\sigma : U \oplus (\oplus_{J_0 \cup J_\nu} Y_j) \to B$ since they coincide on W. Since $U \oplus (\oplus_{J_0 \cup J_\nu} Y_j)$ is a direct summand of X, and X is A-pure in F, we have that $U \oplus (\oplus_{J_0 \cup J_\nu} Y_j)$ is A-pure in F. Because $J_0 \cup J_\nu$ is finite,

$$(U \oplus (\oplus_{J_0 \cup J_\nu} Y_j), \sigma) \in P.$$

Thus,

$$(U \oplus (\oplus_{J_0 \cup J_\nu} Y_j), \sigma) \geq (V_\mu, \psi_\mu), (V_0, \psi_0).$$

Consequently, P' cannot be an anti-chain. □

For every finite subset J of I, let $D(J) = \{(V, \psi) \in P | \oplus_J A \subseteq V\}$.

Proposition 4.4. *P and $\mathcal{D} = \{D(J) | J \subseteq I$ finite$\}$ satisfy the hypotheses of Martin's Axiom.*

Proof. By Corollary 4.3, it remains to show that $D(J)$ is dense in P. For this, let $(V, \psi) \in P$. We have to find $(W, \alpha) \in P$ such that $\oplus_J A$ and V are contained in W and $\alpha | V = \psi$. Since V/U is finitely A-projective and $G \cong F/U$ is strongly $\aleph_1$-A-projective, there is a subgroup X of F containing V and $\oplus_J A$ such that X/U is a $\aleph_1$-A-closed, A-projective countable subgroup of F/U. Since

$$[F/U]/[X/U] \cong F/X$$

is $\aleph_1$-A-projective, it is A-solvable by Theorem 2.1. Using the same result once more, we obtain that the sequence $0 \to X \to F \to F/X \to 0$ is A-balanced. In particular, $S_A(X) = X$ and X is A-projective. Moreover,

$$H_A(F)/H_A(X) \cong H_A(F/X) \cong H_A([F/U]/[X/U]) \cong H_A(F/U)/H_A(X/U)$$

since X in F and X/U in F/U are A-balanced by the faithful flatness of A. But the latter is non-singular, since $[F/U]/[X/U]$ is $\aleph_1$-Aprojective. Therefore, X is A-pure in F.

Since the group X/U is A-projective, we have a decomposition $X = U \oplus P$. Hence, $V = U \oplus (V \cap P)$ and $V \cap P$ is finitely A-projective. In the same way,

$$(\oplus_J A) + U = U \oplus [((\oplus_J A) + U) \cap P]$$

yields that $((\oplus_J A) + U) \cap P$ is A-generated and an image of $\oplus_J A$.
Therefore, $((\oplus_J A) + U) \cap P$ and $V \cap P$ are finitely A-projective subgroups of P. Thus, $H_A(((\oplus_J A) + U) \cap P)$ and $H_A(V \cap P)$ are finitely generated submodule of $H_A(P)$. Since E is right hereditary, $H_A(P)$ is a direct sum of right ideals of E, which yields that $H_A(((\oplus_J A) + U) \cap P)$ and $H_A(V \cap P)$ are contained in a finitely generated direct summand of $H_A(P)$. Hence, there is a finitely A-projective summand D of P

which contains $V \cap P$ and $((\oplus_J A) + U) \cap P$. Since $U \oplus D = V + D$ and V is A-pure in F, we obtain that V is a direct summand of $U \oplus D$. Thus, ψ extends to a map $\alpha : U \oplus D \to B$. Clearly, $(U \oplus D, \alpha) \in P$ and $(U \oplus D, \alpha) \geq (V, \psi)$. □

An A-generated group G $\aleph_1$-*A-separable* if every countable subset of G is contained in an A-projective direct summand of G.

Corollary 4.5. $(MA + \aleph_1 < 2^{\aleph_0})$ *If A is a self-small countable torsion-free group with a right Noetherian right hereditary endomorphism ring, then every strongly $\aleph_1$-A-projective group is $\aleph_1$-A-separable and $\aleph_1$-A-coseparable.*

Proof. By Theorem 3.2 and Theorem 4.1, a strongly $\aleph_1$-A-projective group G is $\aleph_1$-A-coseparable. It remains to show that is $\aleph_1$-A-separable too. For a countable subset X of G select a countable $\aleph_1$-A-closed subgroup U of G containing X. Since G/U is strongly $\aleph_1$-A-projective, the sequence $0 \to U \to G \to G/U \to 0$ splits by Theorem 4.1. □

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 509–525

Some new integral inequalities of Hermite-Hadamard type for $(\log, (\alpha, m))$-convex functions on co-ordinates

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Abstract. In the paper, the authors introduce a new concept "$(\log, (\alpha, m))$-convex functions on the co-ordinates on the rectangle of the plane" and establish some new integral inequalities of Hermite-Hadamard type for $(\log, (\alpha, m))$-convex functions on the co-ordinates on the rectangle from the plane.

Mathematics Subject Classification (2010): 26A51, 26D15, 26D20, 26E60, 41A55.

Keywords: Co-ordinates, $(\log, (\alpha, m))$-convex functions on co-ordinates, Hermite-Hadamard's inequality.

1. Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2. If a positive function $f : I \subseteq \mathbb{R} \to (0, \infty)$ satisfies

$$f(\lambda x + (1-\lambda)y) \le f^{\lambda}(x) f^{1-\lambda}(y),$$

for all $x, y \in I$ and $\lambda \in [0, 1]$, then we call f a logarithmically convex function on I.

This work was partially supported by the National Natural Science Foundation of China under Grant No. 11361038 and by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZY14192 and by the Inner Mongolia Autonomous Region Natural Science Foundation Project under Grant No. 2015MS0123, China.

Definition 1.3 ([8]). For $f : [0,b] \to \mathbb{R}$ and $m \in (0,1]$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that f is an m-convex function on $[0,b]$.

Definition 1.4. [(9)] For $f : [0,b] \to \mathbb{R}$ and $(\alpha, m) \in (0,1] \times (0,1]$, if

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that f is an (α, m)-convex function on $[0,b]$.

Definition 1.5 ([4, 5]). A function $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$, with $a < b$ and $c < d$, is said to be convex on the co-ordinates on Δ if the partial functions

$$f_y : [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y) \quad \text{and} \quad f_x : [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v)$$

are convex for all $x \in (a,b)$ and $y \in (c,d)$.

Definition 1.6 ([4, 5]). A function $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$, with $a < b$ and $c < d$, is said to be convex on the co-ordinates on Δ if the partial functions

$$\begin{aligned} f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ \le t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda)f(z,w) \end{aligned}$$

holds for all $t, \lambda \in [0,1], (x,y), (z,w) \in \Delta$.

Definition 1.7 ([3]). For some $(\alpha_1, m_1), (\alpha_2, m_2) \in (0,1]^2$, a function $f : [0,b] \times [0,d] \to \mathbb{R}$ is said to be (α_1, m_1)-(α_2, m_2)-convex on the co-ordinates on $[0,b] \times [0,d]$, if

$$\begin{aligned} f(ta + m_1(1-t)b, \lambda c + m_2(1-\lambda)d) &\le t^{\alpha_1}\lambda^{\alpha_2} f(a,c) + m_2 t^{\alpha_1}(1-\lambda^{\alpha_2})f(a,d) \\ &+ m_1(1-t^{\alpha_1})\lambda^{\alpha_2} f(b,c) + m_1 m_2(1-t^{\alpha_1})(1-\lambda^{\alpha_2})f(b,d) \quad (1.1) \end{aligned}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in [0,b] \times [0,d]$.

Now we recite some integral inequalities of Hermite-Hadamard type for the above-mentioned convex functions.

Theorem 1.1 ([6]). *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be m-convex and $m \in (0,1]$. If $f \in L([a,b])$ for $0 \le a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \min\left\{\frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2}\right\}.$$

Theorem 1.2 ([7, Theorem 3.1]). *Let $I \supseteq \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $[f'(x)]^q$ is (α, m)-convex on $[a,b]$ for some given numbers $\alpha, m \in (0,1]$ and $q \ge 1$, then*

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q} \min\Biggl\{\biggl[v_1[f'(a)]^q \\ + v_2 m\left[f'\left(\frac{b}{m}\right)\right]^q\biggr]^{1/q}, \left[v_2 m\left[f'\left(\frac{a}{m}\right)\right]^q + v_1[f'(b)]^q\right]^{1/q}\Biggr\}, \end{aligned}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right)$$

and

$$v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right).$$

Theorem 1.3 **(**[4, 5, Theorem 2.2]**).** *Let $f : \Delta = [a,b] \times [c,d]$ be convex on the co-ordinates on Δ with $a < b$ and $c < d$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x \right.\\ &\quad \left. + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y) \,\mathrm{d}\, y \,\mathrm{d}\, x \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c) \,\mathrm{d}\, x + \int_a^b f(x,d) \,\mathrm{d}\, x\right)\right. \\ &\quad \left. + \frac{1}{d-c}\left(\int_c^d f(a,y) \,\mathrm{d}\, y + \int_c^d f(b,y) \,\mathrm{d}\, y\right)\right] \\ &\le \frac{1}{4}\big[f(a,c) + f(b,c) + f(a,d) + f(b,d)\big]. \end{aligned}$$

For more information on this topic, please refer to [1, 2, 10, 11, 12, 13, 14, 15] and closely related references therein.

In this paper, we will introduce a new concept "$(\log, (\alpha, m))$-convex function on the co-ordinates" and establish some integral inequalities of Hermite-Hadamard type for functions whose derivatives are of "co-ordinated $(\log, (\alpha, m))$-convexity".

2. A definition and a lemma

Motivated by Definitions 1.2 to 1.4, we introduce the notion "co-ordinated $(\log, (\alpha, m))$-convex function".

Definition 2.1. A mapping $f : [0,b] \times [c,d] \to \mathbb{R}_+ = (0, +\infty)$ is called co-ordinated $(\log, (\alpha, m))$-convex on $[0,b] \times [c,d]$ for $b > 0$ and $c, d \in \mathbb{R}$ with $c < d$, if

$$\begin{aligned} f(tx + (1-t)z, \lambda y + m(1-\lambda)w) &\le [\lambda^\alpha f(x,y) \\ &\quad + m(1-\lambda^\alpha) f(x,w)]^t [\lambda^\alpha f(z,y) + m(1-\lambda^\alpha) f(z,w)]^{1-t} \end{aligned} \tag{2.1}$$

holds for all $t, \lambda \in [0,1]$, for all $(x,y), (z,w) \in [0,b] \times [c,d]$, and for all $m, \alpha \in (0,1]$.

Remark 2.1. It is clear that, for all $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in [0,b] \times [c,d]$ and for some $m, \alpha \in (0,1]$,

$$\begin{aligned} &[\lambda^\alpha f(x,y) + m(1-\lambda^\alpha) f(x,w)]^t [\lambda^\alpha f(z,y) + m(1-\lambda^\alpha) f(z,w)]^{1-t} \\ &\le t\lambda^\alpha f(x,y) + mt(1-\lambda^\alpha) f(x,w) + (1-t)\lambda^\alpha f(z,y) + m(1-t)(1-\lambda^\alpha) f(z,w). \end{aligned}$$

If the function f is co-ordinated $(\log,(\alpha,m))$-convex on $[0,b]\times[c,d]$, then, by taking $(\alpha_1,m_1)=(1,1)$ and $(\alpha_2,m_2)=(\alpha,m)$ in Definition 1.7, we easily see that it is also co-ordinated $(1,1)$-(α,m)-convex on $[0,b]\times[c,d]$.

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ have partial derivatives of the second order. If $\frac{\partial^2 f}{\partial x\partial y}\in L(\Delta)$, then*

$$\begin{aligned}S(f)\triangleq{}&\frac{4}{(b-a)(d-c)}\biggl\{\frac{9f(a,c)-3f(a,d)-3f(b,c)+f(b,d)}{4}\\&-\frac{1}{2(b-a)}\int_a^b[3f(x,c)-f(x,d)]\,\mathrm{d}\,x-\frac{1}{2(d-c)}\int_c^d[3f(a,y)-f(b,y)]\,\mathrm{d}\,y\\&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}\,y\,\mathrm{d}\,x\biggr\}\\&=\int_0^1\int_0^1(1+2t)(1+2\lambda)f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda.\end{aligned}\tag{2.2}$$

Proof. By integration by parts, we have

$$\begin{aligned}&\int_0^1\int_0^1(1+2t)(1+2\lambda)f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&=-\frac{1}{b-a}\int_0^1(1+2\lambda)\biggl[(1+2t)f'_y(ta+(1-t)b,\lambda c+(1-\lambda)d)\Big|_{t=0}^{t=1}\\&\qquad-2\int_0^1 f'_y(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\biggr]\,\mathrm{d}\,\lambda\\&=-\frac{1}{b-a}\biggl\{\int_0^1\Bigl[3(1+2\lambda)f'_y(a,\lambda c+(1-\lambda)d)\\&\qquad-(1+2\lambda)f'_y(b,\lambda c+(1-\lambda)d)\Bigr]\,\mathrm{d}\,\lambda\\&\quad-2\int_0^1\int_0^1(1+2\lambda)f'_y(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\biggr\}\\&=\frac{1}{(b-a)(d-c)}\biggl\{3(1+2\lambda)f(a,\lambda c+(1-\lambda)d)\\&\qquad-(1+2\lambda)f(b,\lambda c+(1-\lambda)d)\Big|_{\lambda=0}^{\lambda=1}\\&\quad-6\int_0^1 f(a,\lambda c+1-\lambda)d)\,\mathrm{d}\,\lambda+2\int_0^1 f(b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,\lambda\\&\quad-2\int_0^1(1+2\lambda)f(ta+(1-t)b,\lambda c+(1-\lambda)d)\Big|_{\lambda=0}^{\lambda=1}\mathrm{d}\,t\\&\quad+4\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\biggr\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(b-a)(d-c)}\Big[9f(a,c)-3f(b,c)-3f(a,d)+f(b,d)\\
&-6\int_0^1 f(a,\lambda c+(1-\lambda)d)\,\mathrm{d}\,\lambda+2\int_0^1 f(b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,\lambda\\
&-6\int_0^1 f(ta+(1-t)b,c)\,\mathrm{d}\,t+2\int_0^1 f(ta+(1-t)b,d)\,\mathrm{d}\,t\\
&+4\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\Big].
\end{aligned}
$$

After further making use of the substitutions $x = ta+(1-t)b$ and $y = \lambda c+(1-\lambda)d$ for $t,\lambda \in [0,1]$, we obtain (2.2). Lemma 2.1 is thus proved. □

3. Some integral inequalities of Hermite-Hadamard type

Now we turn our attention to establish inequalities of Hermite-Hadamard type for $(\log,(\alpha,m))$-convex functions on the co-ordinates.

Theorem 3.1. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}$ be a partial differentiable mapping on $\mathbb{R}_0\times\mathbb{R}$ and $f''_{xy}\in L_1([a,\frac{b}{m}]\times[c,d])$ with $0\le a<b$ and $c<d$ for some fixed $m\in(0,1]$. If $\left|f''_{xy}\right|^q$ is co-ordinated $(\log,(\alpha,m))$-convex on $[0,\frac{b}{m}]\times[c,d]$ for $q\ge 1$ and $\alpha\in(0,1]$, then*

$$
\begin{aligned}
|S(f)| \le{}& \frac{2^{2(1-1/q)}}{[6(\alpha+1)(\alpha+2)]^{1/q}}\Big[7(3\alpha+4)\left|f''_{xy}(a,c)\right|^q+7m\alpha(2\alpha\\
&+3)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q+5(3\alpha+4)\left|f''_{xy}(b,c)\right|^q+5m(2\alpha+3)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Big]^{1/q}.
\end{aligned}
$$

Proof. By Lemma 2.1, Hölder's integral inequality, the $(\log,(\alpha,m))$-convexity of $\left|f''_{xy}\right|^q$, and the GA-inequality, we obtain

$$
\begin{aligned}
|S(f)| &\le \int_0^1\int_0^1 (1+2t)(1+2\lambda)\left|f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&\le \left(\int_0^1\int_0^1 (1+2t)(1+2\lambda)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\Big[\int_0^1\int_0^1 (1+2t)(1+2\lambda)\\
&\qquad\times\left|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q\mathrm{d}\,t\,\mathrm{d}\,\lambda\Big]^{1/q}\\
&\le 2^{2(1-1/q)}\Big\{\int_0^1\int_0^1 (1+2t)(1+2\lambda)\Big[\lambda^\alpha\left|f''_{xy}(a,c)\right|^q\\
&\qquad+m(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\Big]^t\Big[\lambda^\alpha\left|f''_{xy}(b,c)\right|^q\\
&\qquad+m(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Big]^{1-t}\mathrm{d}\,t\,\mathrm{d}\,\lambda\Big\}^{1/q}\\
&\le 2^{2(1-1/q)}\Big\{\int_0^1\int_0^1 (1+2t)(1+2\lambda)\Big[t\lambda^\alpha\left|f''_{xy}(a,c)\right|^q
\end{aligned}
$$

$$
+mt(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q+(1-t)\lambda^\alpha\left|f''_{xy}(b,c)\right|^q
$$

$$
+m(1-t)(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Bigg]\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\Bigg\}^{1/q}
$$

$$
=\frac{2^{2(1-1/q)}}{[6(\alpha+1)(\alpha+2)]^{1/q}}\Bigg[7(3\alpha+4)\left|f''_{xy}(a,c)\right|^q
$$

$$
+7m\alpha(2\alpha+3)\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q
$$

$$
+5(3\alpha+4)\left|f''_{xy}(a,d)\right|^q+5m(2\alpha+3)\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\Bigg]^{1/q}.
$$

This completes the proof of Theorem 3.1. □

Corollary 3.1.1. *Under the assumptions of Theorem 3.1, if $q=1$, we have*

$$
|S(f)|\le\frac{1}{6(\alpha+1)(\alpha+2)}\Bigg[7(3\alpha+4)\left|f''_{xy}(a,c)\right|+7m\alpha(2\alpha+3)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|
$$

$$
+5(3\alpha+4)\left|f''_{xy}(b,c)\right|+5m(2\alpha+3)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|\Bigg].
$$

Corollary 3.1.2. *Under the assumptions of Corollary 3.1.1,*

1. *if $m=1$, then*

$$
|S(f)|\le\frac{1}{6(\alpha+1)(\alpha+2)}\Big[7(3\alpha+4)\left|f''_{xy}(a,c)\right|+7\alpha(2\alpha+3)\left|f''_{xy}(a,d)\right|
$$

$$
+5(3\alpha+4)\left|f''_{xy}(b,c)\right|+5(2\alpha+3)\left|f''_{xy}(b,d)\right|\Big];
$$

2. *if $\alpha=1$, then*

$$
|S(f)|\le\frac{1}{36}\Bigg[49\left|f''_{xy}(a,c)\right|+35m\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|
$$

$$
+35\left|f''_{xy}(b,c)\right|+25m\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|\Bigg];
$$

3. *if $m=\alpha=1$, then*

$$
|S(f)|\le\frac{1}{36}\Big[49\left|f''_{xy}(a,c)\right|+35\left|f''_{xy}(a,d)\right|+35\left|f''_{xy}(b,c)\right|+25\left|f''_{xy}(b,d)\right|\Big].
$$

Corollary 3.1.3. *Under the assumptions of Theorem 3.1,*

1. *if $m=1$, then*

$$
|S(f)|\le\frac{2^{2(1-1/q)}}{[6(\alpha+1)(\alpha+2)]^{1/q}}\Big[7(3\alpha+4)\left|f''_{xy}(a,c)\right|^q
$$

$$
+7\alpha(2\alpha+3)\left|f''_{xy}(b,c)\right|^q+5(3\alpha+4)\left|f''_{xy}(a,d)\right|^q+5(2\alpha+3)\left|f''_{xy}(b,d)\right|^q\Big]^{1/q};
$$

2. *if* $\alpha = 1$, *then*

$$|S(f)| \le \frac{4}{12^{2/q}} \left[49 \left| f''_{xy}(a,c) \right|^q + 35m \left| f''_{xy}\left(\frac{b}{m}, c\right) \right|^q + 35 \left| f''_{xy}(a,d) \right|^q + 25m \left| f''_{xy}\left(\frac{b}{m}, d\right) \right|^q \right]^{1/q}.$$

Theorem 3.2. *Let* $f : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ *be a partial differentiable mapping on* $\mathbb{R}_0 \times \mathbb{R}$ *and* $f''_{xy} \in L_1([a, \frac{b}{m}] \times [c,d])$ *for* $0 \le a < b$, $c < d$ *and some fixed* $m \in (0,1]$. *If* $\left| f''_{xy} \right|^q$ *is co-ordinated* $(\log, (\alpha, m))$*-convex on* $[0, \frac{b}{m}] \times [c,d]$ *for* $q > 1$ *and some* $\alpha \in (0,1]$ *with* $q \ge r > -1$, *then*

$$\begin{aligned} |S(f)| \le & \left[\frac{(3^{(2q-r-1)/(q-1)} - 1)(q-1)}{2q-r-1} \right]^{1-1/q} \\ & \times \left[\frac{1}{4(\alpha+1)(\alpha+2)(r+1)(r+2)} \right]^{1/q} \Big\{ (2r3^{r+1} + 3^{r+1} + 1) \\ & \times \left[(3\alpha+4) \left| f''_{xy}(a,c) \right|^q + m\alpha(2\alpha+3) \left| f''_{xy}\left(a, \frac{d}{m}\right) \right|^q \right] + (3^{r+2} - 5 \\ & - 2r) \left[(3\alpha+4) \left| f''_{xy}(b,c) \right|^q + m(2\alpha+3) \left| f''_{xy}\left(b, \frac{d}{m}\right) \right|^q \right] \Big\}^{1/q}. \end{aligned}$$

Proof. By Lemma 2.1, Hölder's integral inequality, the $(\log, (\alpha, m))$-convexity of $\left| f''_{xy} \right|^q$, and the well known GA-inequality, we obtain

$$\begin{aligned} |S(f)| \le & \int_0^1 \int_0^1 (1+2t)(1+2\lambda) \left| f''_{xy}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right| \mathrm{d}\,t \, \mathrm{d}\,\lambda \\ \le & \left(\int_0^1 \int_0^1 (1+2t)^{(q-r)/(q-1)} (1+2\lambda) \, \mathrm{d}\,t \, \mathrm{d}\,\lambda \right)^{1-1/q} \left[\int_0^1 \int_0^1 (1+2t)^r \right. \\ & \left. \times (1+2\lambda) \left| f''_{xy} f(ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^q \mathrm{d}\,t \, \mathrm{d}\,\lambda \right]^{1/q} \\ \le & \left(\frac{(3^{(2q-r-1)/(q-1)} - 1)(q-1)}{2q-r-1} \right)^{1-1/q} \left\{ \int_0^1 \int_0^1 (1+2t)^r (1+2\lambda) \right. \\ & \times \left[\lambda^\alpha \left| f''_{xy}(a,c) \right|^q + m(1-\lambda^\alpha) \left| f''_{xy}\left(a, \frac{d}{m}\right) \right|^q \right]^t \\ & \left. \times \left[\lambda^\alpha \left| f''_{xy}(b,c) \right|^q + m(1-\lambda^\alpha) \left| f''_{xy}\left(b, \frac{d}{m}\right) \right|^q \right]^{1-t} \mathrm{d}\,t \, \mathrm{d}\,\lambda \right\}^{1/q} \\ \le & \left(\frac{(3^{(2q-r-1)/(q-1)} - 1)(q-1)}{2q-r-1} \right)^{1-1/q} \left\{ \int_0^1 \int_0^1 (1+2t)^r \right. \\ & \times (1+2\lambda) \left[t\lambda^\alpha \left| f''_{xy}(a,c) \right|^q + mt(1-\lambda^\alpha) \left| f''_{xy}\left(a, \frac{d}{m}\right) \right|^q \right. \end{aligned}$$

$$+(1-t)\lambda^\alpha\big|f''_{xy}(b,c)\big|^q+m(1-t)(1-\lambda^\alpha)\bigg|f''_{xy}\bigg(b,\frac{d}{m}\bigg)\bigg|^q\bigg]\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg\}^{1/q}$$

$$=\bigg(\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\bigg)^{1-1/q}$$

$$\times\bigg(\frac{1}{4(\alpha+1)(\alpha+2)(r+1)(r+2)}\bigg)^{1/q}\bigg[(2r3^{r+1}+3^{r+1}+1)$$

$$\times\bigg((3\alpha+4)\big|f''_{xy}(a,c)\big|^q+m\alpha(2\alpha+3)\bigg|f''_{xy}\bigg(a,\frac{d}{m}\bigg)\bigg|^q\bigg)+(3^{r+2}-5$$

$$-2r)\bigg((3\alpha+4)\big|f''_{xy}(b,c)\big|^q+m(2\alpha+3)\bigg|f''_{xy}\bigg(b,\frac{d}{m}\bigg)\bigg|^q\bigg)\bigg]^{1/q}.$$

The proof of Theorem 3.2 is complete. □

Corollary 3.2.1. *Under the conditions of Theorem 3.2, if $r=0$, we have*

$$|S(f)|\le\bigg(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\bigg)^{1-1/q}\bigg(\frac{1}{2(\alpha+1)(\alpha+2)}\bigg)^{1/q}$$

$$\times\bigg[(3\alpha+4)\big|f''_{xy}(a,c)\big|^q+m\alpha(2\alpha+3)\bigg|f''_{xy}\bigg(a,\frac{d}{m}\bigg)\bigg|^q$$

$$+(3\alpha+4)\big|f''_{xy}(b,c)\big|^q+m(2\alpha+3)\bigg|f''_{xy}\bigg(b,\frac{d}{m}\bigg)\bigg|^q\bigg]^{1/q}.$$

Corollary 3.2.2. *Under the conditions of Theorem 3.2,*

1. *if $m=1$, then*

$$|S(f)|\le\bigg[\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\bigg]^{1-1/q}$$

$$\times\bigg[\frac{1}{4(\alpha+1)(\alpha+2)(r+1)(r+2)}\bigg]^{1/q}\bigg\{(2r3^{r+1}+3^{r+1}+1)\Big[(3\alpha$$

$$+4)\big|f''_{xy}(a,c)\big|^q+\alpha(2\alpha+3)\big|f''_{xy}(a,d)\big|^q\Big]$$

$$+(3^{r+2}-5-2r)\Big[(3\alpha+4)\big|f''_{xy}(b,c)\big|^q+(2\alpha+3)\big|f''_{xy}(b,d)\big|^q\Big]\bigg\}^{1/q};$$

2. *if $\alpha=1$, then*

$$|S(f)|\le\bigg(\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\bigg)^{1-1/q}\bigg(\frac{1}{24(r+1)(r+2)}\bigg)^{1/q}$$

$$\times\bigg\{(2r3^{r+1}+3^{r+1}+1)\bigg[7\big|f''_{xy}(a,c)\big|^q+5m\bigg|f''_{xy}\bigg(\frac{b}{m},c\bigg)\bigg|^q\bigg]$$

$$+(3^{r+2}-5-2r)\bigg[7\big|f''_{xy}(a,d)\big|^q+5m\bigg|f''_{xy}\bigg(\frac{b}{m},d\bigg)\bigg|^q\bigg]\bigg\}^{1/q};$$

3. *if* $m = \alpha = 1$, *then*

$$|S(f)| \le \left(\frac{(3^{(2q-r-1)/(q-1)}-1)(q-1)}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{24(r+1)(r+2)}\right)^{1/q}$$

$$\times\left\{(2r3^{r+1}+3^{r+1}+1)\Big[7\big|f''_{xy}(a,c)\big|^q+5\Big|f''_{xy}(b,c)\Big|^q\Big]\right.$$

$$\left.+(3^{r+2}-5-2r)\Big[7\big|f''_{xy}(a,d)\big|^q+5\big|f''_{xy}(b,d)\big|^q\Big]\right\}^{1/q}.$$

Theorem 3.3. *Let* $f : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ *be a partial differentiable mapping on* $\mathbb{R}_0 \times \mathbb{R}$ *and* $f''_{xy} \in L_1([a, \frac{b}{m}] \times [c,d])$ *for* $0 \le a < b$, $c < d$ *and some fixed* $m \in (0,1]$. *If* $\big|f''_{xy}\big|^q$ *is co-ordinated* $(\log,(\alpha,m))$*-convex on* $[0,\frac{b}{m}]\times[c,d]$ *for* $q > 1$ *and some* $\alpha \in (0,1]$, *then*

$$|S(f)| \le \left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}\left(\frac{1}{2(\alpha+1)}\right)^{1/q}\Bigg[\big|f''_{xy}(a,c)\big|^q$$

$$+m\alpha\left|f''_{xy}\left(a,\frac{b}{m}\right)\right|^q+\big|f''_{xy}(b,c)\big|^q+m\alpha\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Bigg]^{1/q}.$$

Proof. By Lemma 2.1, Hölder's integral inequality, the $(\log,(\alpha,m))$-convexity of $\big|f''_{xy}\big|^q$, and the GA-inequality, we obtain

$$|S(f)| \le \int_0^1\int_0^1 (1+2t)(1+2\lambda)\big|f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\big|\,\mathrm{d}\,t\,\mathrm{d}\,\lambda$$

$$\le \left(\int_0^1\int_0^1 (1+2t)^{q/(q-1)}(1+2\lambda)^{q/(q-1)}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}$$

$$\times\left[\int_0^1\int_0^1 \big|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\big|^q\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}$$

$$\le \left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}$$

$$\times\left\{\int_0^1\int_0^1\left[\lambda^\alpha\big|f''_{xy}(a,c)\big|^q+m(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\right]^t\right.$$

$$\left.\times\left[\lambda^\alpha\big|f''_{xy}(b,c)\big|^q+m(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\right]^{1-t}\mathrm{d}\,t\,\mathrm{d}\,\lambda\right\}^{1/q}$$

$$\le \left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}$$

$$\times\left\{\int_0^1\int_0^1\left[t\lambda^\alpha\big|f''_{xy}(a,c)\big|^q+mt(1-\lambda^\alpha)\left|f''_{xy}\left(a,\frac{d}{m}\right)\right|^q\right.\right.$$

$$\left.\left.+(1-t)\lambda^\alpha\big|f''_{xy}(b,c)\big|^q+m(1-t)(1-\lambda^\alpha)\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\right]\mathrm{d}\,t\,\mathrm{d}\,\lambda\right\}^{1/q}$$

$$= \left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}\left(\frac{1}{2(\alpha+1)}\right)^{1/q}\Bigg[\big|f''_{xy}(a,c)\big|^q$$

$$+m\alpha\left|f''_{xy}\left(a,\frac{b}{m}\right)\right|^q+\left|f''_{xy}(b,c)\right|^q+m\alpha\left|f''_{xy}\left(b,\frac{d}{m}\right)\right|^q\Bigg]^{1/q}.$$

The proof of Theorem 3.3 is complete. □

Corollary 3.3.1. *Under the conditions of Theorem 3.3, if $m=\alpha=1$, then*

$$|S(f)|\le\left(\frac{(3^{(2q-1)/(q-1)}-1)(q-1)}{2q-1}\right)^{2(1-1/q)}\left(\frac{1}{4}\right)^{1/q}\Big[\left|f''_{xy}(a,c)\right|^q+\left|f''_{xy}(a,d)\right|^q+\left|f''_{xy}(b,c)\right|^q+\left|f''_{xy}(b,d)\right|^q\Big]^{1/q}.$$

Theorem 3.4. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}_+$ be integrable on $[0,\frac{b}{m^2}]\times[c,d]$ for $0\le a<b$, $c<d$, and some $m\in(0,1]$. If f is co-ordinated $(\log,(\alpha,m))$-convex on $[0,\frac{b}{m^2}]\times[c,d]$ for $\alpha\in(0,1]$, then*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le\frac{1}{2(\alpha+1)(b-a)}\int_a^b\left[f(x,c)+m(2^\alpha-1)f\left(x,\frac{d}{m}\right)\right]\mathrm{d}\,x+\frac{1}{2^{\alpha+1}(d-c)}\\
&\quad\times\int_c^d L\left(f(a,y)+m(2^\alpha-1)f\left(a,\frac{y}{m}\right),f(b,y)+m(2^\alpha-1)f\left(b,\frac{y}{m}\right)\right)\mathrm{d}\,y\\
&\le\frac{1}{2(\alpha+1)(b-a)}\int_a^b\left[f(x,c)+m(2^\alpha-1)f\left(x,\frac{d}{m}\right)\right]\mathrm{d}\,x+\frac{1}{2^{\alpha+2}(d-c)}\\
&\quad\times\int_c^d\left\{f(a,y)+f(b,y)+m(2^\alpha-1)\left[f\left(a,\frac{y}{m}\right)+f\left(b,\frac{y}{m}\right)\right]\right\}\mathrm{d}\,y\\
&\le\frac{1}{2^\alpha(\alpha+1)}\left\{L\left(f(a,c)+m(2^\alpha-1)f\left(a,\frac{c}{m}\right),f(b,c)\right.\right.\\
&\quad\left.+m(2^\alpha-1)f\left(b,\frac{c}{m}\right)\right)+m(2^\alpha-1)L\left(f\left(a,\frac{d}{m}\right)+m(2^\alpha-1)f\left(a,\frac{d}{m^2}\right),\right.\\
&\quad\left.\left.f\left(b,\frac{d}{m}\right)+m(2^\alpha-1)f\left(b,\frac{d}{m^2}\right)\right)\right\}\le\frac{1}{2^{\alpha+1}(\alpha+1)}\\
&\quad\times\left\{f(a,c)+f(b,c)+m(2^\alpha-1)\left[f\left(a,\frac{c}{m}\right)+f\left(b,\frac{c}{m}\right)\right]+m(2^\alpha-1)\right.\\
&\quad\left.\times\left[f\left(a,\frac{d}{m}\right)+f\left(b,\frac{d}{m}\right)+m(2^\alpha-1)\left(f\left(a,\frac{d}{m^2}\right)+f\left(b,\frac{d}{m^2}\right)\right)\right]\right\},
\end{aligned}$$

where $L(u,v)$ is the logarithmic mean defined by

$$L(u,v)=\begin{cases}\dfrac{v-u}{\ln v-\ln u}, & u\ne v,\\ u, & u=v.\end{cases}$$

Proof. Putting $y = \lambda c + (1-\lambda)d$ for $0 \le \lambda \le 1$ and using the $(\log, (\alpha, m))$-convexity of f, we have

$$f(x,y) = f(x, \lambda c + (1-\lambda)d) \le \lambda^\alpha f(x,c) + m(1-\lambda^\alpha) f\left(x, \frac{d}{m}\right) \tag{3.1}$$

for all $(x,y) \in [a,b] \times [c,d]$, $t = \frac{1}{2}$, and $0 \le \lambda \le 1$.

Similarly, setting $x = ta + (1-t)b$ for $0 \le t \le 1$ and using the $(\log, (\alpha, m))$-convexity of f with $0 \le t \le 1$ and $\lambda = \frac{1}{2}$ in (2.1), we obtain

$$\begin{aligned} f(x,c) &= f(ta + (1-t)b, c) \\ &\le \frac{1}{2^\alpha}\left[f(a,c) + m(2^\alpha - 1) f\left(a, \frac{c}{m}\right)\right]^t \left[f(b,c) + m(2^\alpha - 1) f\left(b, \frac{c}{m}\right)\right]^{1-t} \end{aligned}$$

and

$$\begin{aligned} f\left(x, \frac{d}{m}\right) &= f\left(ta + (1-t)b, \frac{d}{m}\right) \le \frac{1}{2^\alpha}\left[f\left(a, \frac{d}{m}\right)\right. \\ &\quad \left. + m(2^\alpha - 1) f\left(a, \frac{d}{m^2}\right)\right]^t \left[f\left(b, \frac{d}{m}\right) + m(2^\alpha - 1) f\left(b, \frac{d}{m^2}\right)\right]^{1-t}. \end{aligned} \tag{3.2}$$

From inequalities (3.1) to (3.2), we have

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y) \,\mathrm{d}\,x \,\mathrm{d}\,y \le \frac{1}{b-a} \int_0^1 \int_a^b \left[\lambda^\alpha f(x,c) + m(1\right. \\ &\left. - \lambda^\alpha) f\left(x, \frac{d}{m}\right)\right] \mathrm{d}\,x \,\mathrm{d}\,\lambda = \frac{1}{(\alpha+1)(b-a)} \int_a^b \left[f(x,c) + m(2^\alpha - 1)\right. \\ &\left. \times f\left(x, \frac{d}{m}\right)\right] \mathrm{d}\,x \le \frac{1}{2^\alpha(\alpha+1)} \int_0^1 \left\{\left[f(a,c) + m(2^\alpha - 1) f\left(a, \frac{c}{m}\right)\right]^t\right. \\ &\times \left[f(b,c) + m(2^\alpha - 1) f\left(b, \frac{c}{m}\right)\right]^{1-t} + m(2^\alpha - 1)\left[f\left(a, \frac{d}{m}\right) + m(2^\alpha\right. \\ &\left.\left. - 1) f\left(a, \frac{d}{m^2}\right)\right]^t \times \left[f\left(b, \frac{d}{m}\right) + m(2^\alpha - 1) f\left(b, \frac{d}{m^2}\right)\right]^{1-t}\right\} \mathrm{d}\,t. \end{aligned} \tag{3.3}$$

It is obvious that

$$\int_0^1 u^t v^{1-t} \,\mathrm{d}\,t = L(u,v) \quad \text{and} \quad L(u,v) \le \frac{u+v}{2}. \tag{3.4}$$

By (3.3) and (3.4), we acquire

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y) \,\mathrm{d}\,x \,\mathrm{d}\,y \\ &\le \frac{1}{2^\alpha(\alpha+1)}\left\{L\left(f(a,c) + m(2^\alpha - 1) f\left(a, \frac{c}{m}\right), f(b,c) + m(2^\alpha - 1) f\left(b, \frac{c}{m}\right)\right)\right. \\ &\left. + m(2^\alpha - 1) L\left(f\left(a, \frac{d}{m}\right) + m(2^\alpha - 1) f\left(a, \frac{d}{m^2}\right), f\left(b, \frac{d}{m}\right) + m(2^\alpha - 1) f\left(b, \frac{d}{m^2}\right)\right)\right\} \end{aligned}$$

$$\le \frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f(a,c)+f(b,c)+m(2^\alpha-1)\bigg[f\bigg(a,\frac{c}{m}\bigg)+f\bigg(b,\frac{c}{m}\bigg)\bigg]$$

$$+m(2^\alpha-1)\bigg[f\bigg(a,\frac{d}{m}\bigg)+f\bigg(b,\frac{d}{m}\bigg)+m(2^\alpha-1)\bigg(f\bigg(a,\frac{d}{m^2}\bigg)+f\bigg(b,\frac{d}{m^2}\bigg)\bigg)\bigg]\bigg\}.$$

Similarly, one has

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y=\frac{1}{d-c}\int_c^d\int_0^1 f(ta+(1-t)b,y)\,\mathrm{d}\,t\,\mathrm{d}\,y$$

$$\le \frac{1}{2^\alpha(d-c)}\int_c^d\int_0^1\bigg[f(a,y)+m(2^\alpha-1)f\bigg(a,\frac{y}{m}\bigg)\bigg]^t\bigg[f(b,y)+m(2^\alpha-1)$$

$$\times f\bigg(b,\frac{y}{m}\bigg)\bigg]^{1-t}\,\mathrm{d}\,t\,\mathrm{d}\,y=\frac{1}{2^\alpha(d-c)}\int_c^d L\bigg(f(a,y)+m(2^\alpha-1)f\bigg(a,\frac{y}{m}\bigg),$$

$$f(b,y)+m(2^\alpha-1)f\bigg(b,\frac{y}{m}\bigg)\bigg)\,\mathrm{d}\,y\le\frac{1}{2^{\alpha+1}(d-c)}\int_c^d\bigg\{f(a,y)+f(b,y)$$

$$+m(2^\alpha-1)\bigg[f\bigg(a,\frac{y}{m}\bigg)+f\bigg(b,\frac{y}{m}\bigg)\bigg]\bigg\}\mathrm{d}\,y\le\frac{1}{2^{\alpha+1}}\int_0^1\bigg\{\bigg[\lambda^\alpha f(a,c)$$

$$+m(2^\alpha-1)f\bigg(a,\frac{d}{m}\bigg)+\lambda^\alpha f(b,c)+m(1-\lambda^\alpha)f\bigg(b,\frac{d}{m}\bigg)+m(2^\alpha-1)$$

$$\times\bigg[\lambda^\alpha f\bigg(a,\frac{c}{m}\bigg)+m(1-\lambda^\alpha)f\bigg(a,\frac{d}{m^2}\bigg)+\lambda^\alpha f\bigg(b,\frac{c}{m}\bigg)+m(1-\lambda^\alpha)$$

$$\times f\bigg(b,\frac{d}{m^2}\bigg)\bigg]\bigg\}\,\mathrm{d}\,\lambda=\frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f(a,c)+f(b,c)+m(2^\alpha$$

$$-1)\bigg[f\bigg(a,\frac{c}{m}\bigg)+f\bigg(b,\frac{c}{m}\bigg)\bigg]+m(2^\alpha-1)\bigg[f\bigg(a,\frac{d}{m}\bigg)+f\bigg(b,\frac{d}{m}\bigg)$$

$$+m(2^\alpha-1)\bigg(f\bigg(a,\frac{d}{m^2}\bigg)+f\bigg(b,\frac{d}{m^2}\bigg)\bigg)\bigg]\bigg\}.$$

Theorem 3.4 is thus proved. □

Theorem 3.5. *Let* $f:[0,\frac{b}{m}]\times[c,d]\subseteq\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}_+$ *be integrable on* $[0,\frac{b}{m^2}]\times[c,d]$ *for* $0\le a<b$, $c<d$, *and some fixed* $m\in(0,1]$. *If* f *is co-ordinated* $(\log,(\alpha,m))$*-convex on* $[0,\frac{b}{m^2}]\times[c,d]$ *for* $\alpha\in(0,1]$, *then*

$$f\bigg(\frac{a+b}{2},\frac{c+d}{2}\bigg)\le\frac{1}{2^{\alpha+1}(b-a)}\int_a^b\bigg[f\bigg(x,\frac{c+d}{2}\bigg)+m(2^\alpha-1)f\bigg(x,\frac{c+d}{2m}\bigg)\bigg]^{1/2}$$

$$\times\bigg[f\bigg(a+b-x,\frac{c+d}{2}\bigg)+m(2^\alpha-1)f\bigg(a+b-x,\frac{c+d}{2m}\bigg)\bigg]^{1/2}\,\mathrm{d}\,x$$

$$+\frac{1}{2^{\alpha+1}(d-c)}\int_c^d\bigg[f\bigg(\frac{a+b}{2},y\bigg)+m(2^\alpha-1)f\bigg(\frac{a+b}{2},\frac{y}{m}\bigg)\bigg]\,\mathrm{d}\,y$$

$$\le\frac{1}{2^{\alpha+1}(b-a)}\int_a^b\bigg[f\bigg(x,\frac{c+d}{2}\bigg)+m(2^\alpha-1)f\bigg(x,\frac{c+d}{2m}\bigg)\bigg]\,\mathrm{d}\,x$$

$$+\frac{1}{2^{\alpha+1}(d-c)}\int_c^d\left[f\left(\frac{a+b}{2},y\right)+m(2^\alpha-1)f\left(\frac{a+b}{2},\frac{y}{m}\right)\right]\mathrm{d}\,y$$
$$\le\frac{1}{2^{2\alpha+1}(b-a)(d-c)}\int_c^d\int_a^b\left\{f(x,y)+2m(2^\alpha-1)f\left(x,\frac{y}{m}\right)\right.$$
$$+m^2(2^\alpha-1)^2f\left(x,\frac{y}{m^2}\right)+\left[f(x,y)+m(2^\alpha-1)f\left(x,\frac{y}{m}\right)\right]^{1/2}$$
$$\times\left[f(a+b-x,y)+m(2^\alpha-1)f\left(a+b-x,\frac{y}{m}\right)\right]^{1/2}$$
$$+m(2^\alpha-1)\left[f\left(x,\frac{y}{m}\right)+m(2^\alpha-1)f\left(x,\frac{y}{m^2}\right)\right]^{1/2}$$
$$\left.\times\left[f\left(a+b-x,\frac{y}{m}\right)+m(2^\alpha-1)f\left(a+b-x,\frac{y}{m^2}\right)\right]^{1/2}\right\}\mathrm{d}\,x\,\mathrm{d}\,y$$
$$\le\frac{1}{2^{2\alpha}(b-a)(d-c)}\int_c^d\int_a^b\left[f(x,y)+2m(2^\alpha-1)f\left(x,\frac{y}{m}\right)\right.$$
$$\left.+m^2(2^\alpha-1)^2f\left(x,\frac{y}{m^2}\right)\right]\mathrm{d}\,x\,\mathrm{d}\,y.$$

Proof. Using the $(\log,(\alpha,m))$-convexity of f, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=f\left(\frac{1}{2}(ta+(1-t)b+(1-t)a+tb),\frac{c+d}{2}\right)$$
$$\le\frac{1}{2^\alpha}\left[f\left(ta+(1-t)b,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(ta+(1-t)b,\frac{c+d}{2m}\right)\right]^{1/2}$$
$$\times\left[f\left((1-t)a+tb,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left((1-t)a+tb,\frac{c+d}{2m}\right)\right]^{1/2}$$

for all $t\in[0,1]$.

Integrating on both sides of the above inequality on $[0,1]$, from the GA-inequality, and by the $(\log,(\alpha,m))$-convexity of f, we reveals

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\int_0^1f\left(\frac{1}{2}(ta+(1-t)b+(1-t)a+tb),\frac{c+d}{2}\right)\mathrm{d}\,t$$
$$\le\frac{1}{2^\alpha}\int_0^1\left[f\left(ta+(1-t)b,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(ta+(1-t)b,\frac{c+d}{2m}\right)\right]^{1/2}$$
$$\times\left[f\left((1-t)a+tb,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left((1-t)a+tb,\frac{c+d}{2m}\right)\right]^{1/2}\mathrm{d}\,t$$
$$=\frac{1}{2^\alpha(b-a)}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(x,\frac{c+d}{2m}\right)\right]^{1/2}$$
$$\times\left[f\left(a+b-x,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(a+b-x,\frac{c+d}{2m}\right)\right]^{1/2}\mathrm{d}\,x$$

$$
\begin{aligned}
&\le \frac{1}{2^\alpha(b-a)}\int_a^b\biggl[f\biggl(x,\frac{c+d}{2}\biggr)+m(2^\alpha-1)f\biggl(x,\frac{c+d}{2m}\biggr)\biggr]\operatorname{d}x\\
&=\frac{1}{2^\alpha(b-a)}\int_0^1\int_a^b\biggl[f\biggl(x,\frac{1}{2}[\lambda c+(1-\lambda)d+(1-\lambda)c+\lambda d]\biggr)\\
&\quad+m(2^\alpha-1)f\biggl(x,\frac{1}{2m}[\lambda c+(1-\lambda)d+(1-\lambda)c+\lambda d]\biggr)\biggr]\operatorname{d}x\operatorname{d}\lambda\\
&\le \frac{1}{2^{2\alpha}(b-a)}\int_0^1\int_a^b\biggl\{f(x,\lambda c+(1-\lambda)d)+m(2^\alpha-1)f\biggl(x,\frac{(1-\lambda)c+\lambda d}{m}\biggr)\\
&\quad+m(2^\alpha-1)\biggl[f\biggl(x,\frac{\lambda c+(1-\lambda)d}{m}\biggr)+m(2^\alpha-1)f\biggl(x,\frac{(1-\lambda)c+\lambda d}{m^2}\biggr)\biggr]\biggr\}\operatorname{d}x\operatorname{d}\lambda\\
&=\frac{1}{2^{2\alpha}(b-a)(d-c)}\int_c^d\int_a^b\biggl[f(x,y)\\
&\quad+2m(2^\alpha-1)f\biggl(x,\frac{y}{m}\biggr)+m^2(2^\alpha-1)^2f\biggl(x,\frac{y}{m^2}\biggr)\biggr]\operatorname{d}x\operatorname{d}y.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
f\biggl(\frac{a+b}{2},\frac{c+d}{2}\biggr)&\le\frac{1}{2^\alpha}\int_0^1\biggl[f\biggl(\frac{a+b}{2},\lambda c+(1-\lambda)d\biggr)\\
&\quad+m(2^\alpha-1)f\biggl(\frac{a+b}{2},\frac{\lambda c+(1-\lambda)d}{m}\biggr)\biggr]\operatorname{d}\lambda\\
&=\frac{1}{2^\alpha(d-c)}\int_c^d\biggl[f\biggl(\frac{a+b}{2},y\biggr)+m(2^\alpha-1)f\biggl(\frac{a+b}{2},\frac{y}{m}\biggr)\biggr]\operatorname{d}y\\
&\le\frac{1}{2^{2\alpha}(d-c)}\int_c^d\int_0^1\biggl\{\biggl[f(ta+(1-t)b,y)+m(2^\alpha-1)f\biggl(ta+(1-t)b,\frac{y}{m}\biggr)\biggr]^{1/2}\\
&\quad\times\biggl[f((1-t)a+tb,y)+m(2^\alpha-1)f\biggl((1-t)a+tb,\frac{y}{m}\biggr)\biggr]^{1/2}\\
&\quad+m(2^\alpha-1)\biggl[f\biggl(ta+(1-t)b,\frac{y}{m}\biggr)+m(2^\alpha-1)f\biggl(ta+(1-t)b,\frac{y}{m^2}\biggr)\biggr]^{1/2}\\
&\quad\times\biggl[f\biggl((1-t)a+tb,\frac{y}{m}\biggr)+m(2^\alpha-1)f\biggl((1-t)a+tb,\frac{y}{m^2}\biggr)\biggr]^{1/2}\biggr\}\operatorname{d}t\operatorname{d}y\\
&=\frac{1}{2^{2\alpha}(b-a)(d-c)}\int_c^d\int_a^b\biggl\{\biggl[f(x,y)+m(2^\alpha-1)f\biggl(x,\frac{y}{m}\biggr)\biggr]^{1/2}\\
&\quad\times\biggl[f(a+b-x,y)+m(2^\alpha-1)f\biggl(a+b-x,\frac{y}{m}\biggr)\biggr]^{1/2}\\
&\quad+m(2^\alpha-1)\biggl[f\biggl(x,\frac{y}{m}\biggr)+m(2^\alpha-1)f\biggl(x,\frac{y}{m^2}\biggr)\biggr]^{1/2}\\
&\quad\times\biggl[f\biggl(a+b-x,\frac{y}{m}\biggr)+m(2^\alpha-1)f\biggl(a+b-x,\frac{y}{m^2}\biggr)\biggr]^{1/2}\biggr\}\operatorname{d}x\operatorname{d}y
\end{aligned}
$$

$$\leq \frac{1}{2^{2\alpha}(b-a)(d-c)} \int_c^d \int_a^b \left[f(x,y) + 2m(2^\alpha - 1) f\left(x, \frac{y}{m}\right) \right.$$
$$\left. + m^2(2^\alpha - 1)^2 f\left(x, \frac{y}{m^2}\right) \right] \mathrm{d}\, x \, \mathrm{d}\, y.$$

The proof of Theorem 3.5 is complete. □

Corollary 3.5.1. *Under the conditions of Theorems 3.4 and 3.5, if $m = 1$, then*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2}\left\{ \frac{1}{b-a} \int_a^b \left[f\left(x, \frac{c+d}{2}\right) f\left(a+b-x, \frac{c+d}{2}\right) \right]^{1/2} \mathrm{d}\, x \right.$$
$$\left. + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y \right\}$$
$$\leq \frac{1}{2}\left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y \right]$$
$$\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y) \, \mathrm{d}\, x \, \mathrm{d}\, y$$
$$\leq \frac{1}{2(\alpha+1)(b-a)} \int_a^b \left[f(x,c) + (2^\alpha - 1) f(x,d) \right] \mathrm{d}\, x$$
$$+ \frac{1}{2(d-c)} \int_c^d L\big(f(a,y), f(b,y)\big) \, \mathrm{d}\, y$$
$$\leq \frac{1}{2(\alpha+1)(b-a)} \int_a^b \left[f(x,c) + (2^\alpha - 1) f(x,d) \right] \mathrm{d}\, x$$
$$+ \frac{1}{4(d-c)} \int_c^d \left[f(a,y) + f(b,y) \right] \mathrm{d}\, y$$
$$\leq \frac{1}{2(\alpha+1)} \Big\{ L\big(f(a,c), f(b,c)\big) + (2^\alpha - 1) L\big(f(a,d), f(b,d)\big)$$
$$+ f(a,c) + f(b,c) + (2^\alpha - 1)\big[f(a,d) + f(b,d)\big] \Big\}$$
$$\leq \frac{1}{2(\alpha+1)} \Big\{ f(a,c) + f(b,c) + (2^\alpha - 1)\big[f(a,d) + f(b,d)\big] \Big\}.$$

If $m = \alpha = 1$, then

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2}\left\{ \frac{1}{b-a} \int_a^b \left[f\left(x, \frac{c+d}{2}\right) f\left(a+b-x, \frac{c+d}{2}\right) \right]^{1/2} \mathrm{d}\, x \right.$$
$$\left. + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y \right\}$$
$$\leq \frac{1}{2}\left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y \right]$$
$$\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y) \, \mathrm{d}\, x \, \mathrm{d}\, y$$

$$\le \frac{1}{4(b-a)}\int_a^b \big[f(x,c)+f(x,d)\big]\,\mathrm{d}\,x+\frac{1}{2(d-c)}\int_c^d L\big(f(a,y),f(b,y)\big)\,\mathrm{d}\,y$$

$$\le \frac{1}{4}\bigg\{\frac{1}{b-a}\int_a^b \big[f(x,c)+f(x,d)\big]\,\mathrm{d}\,x+\frac{1}{d-c}\int_c^d \big[f(a,y)+f(b,y)\big]\,\mathrm{d}\,y\bigg\}$$

$$\le \frac{1}{4}\Big\{f(a,c)+f(a,d)+f(b,c)+f(b,d)$$

$$+L\big(f(a,c),f(b,c)\big)+L\big(f(a,d),f(b,d)\big)\Big\}$$

$$\le \frac{1}{4}\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big].$$

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 527–534

Some Hermite-Hadamard type inequalities for functions whose exponentials are convex

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Abstract. Some inequalities of Hermite-Hadamard type for functions whose exponentials are convex are obtained.

Mathematics Subject Classification (2010): 26D15, 25D10.

Keywords: Convex functions, Hermite-Hadamard inequality, special means.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f : [a,b] \to \mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [1] and the references therein.

We denote by $\mathfrak{Expconv}(I)$ the class of all functions defined on the interval I of real numbers such that $\exp(f)$ is convex on I. If $\mathfrak{Conv}(I)$ is the class of convex functions defined on I then we have the following fact:

Proposition 1.1. *We have the strict inclusion*

$$\mathfrak{Conv}(I) \subsetneq \mathfrak{Expconv}(I).$$

Proof. If f is convex , then $\exp(f)$ is log-convex and therefore convex on I and the inclusion is proved.

For $r \geq 1$ the function $f_r(x) = r\ln x$, $x > 0$ is concave on $(0,\infty)$. We have $\exp(f_r(x)) = x^r$ is a convex function, therefore $f_r \in \mathfrak{Expconv}(I) \setminus \mathfrak{Conv}(I)$. □

We observe that for twice differentiable functions g on $\mathring{I}$, the interior of I we have that

$$(\exp(g(x)))'' = \left([g'(x)]^2 + g''(x)\right)\exp g(x), \ x \in \mathring{I},$$

therefore $g \in \mathfrak{Expconv}(I)$ if and only if

$$[g'(x)]^2 + g''(x) \geq 0 \text{ for any } x \in \mathring{I}.$$

2. Some Hermite-Hadamard type inequalities

Now, if $g \in \mathfrak{Expconv}(I)$, then by the Hermite-Hadamard inequality for $\exp(g)$ we have for $a, b \in I$ with $a < b$ that

$$\exp g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \exp g(t)\,dt \leq \frac{1}{2}[\exp g(a) + \exp g(b)]. \tag{2.1}$$

By Jensen's integral inequality for the exp function we also have for any integrable function $h : [a, b] \to \mathbb{R}$ that

$$\exp\left(\frac{1}{b-a}\int_a^b h(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b \exp h(t)\,dt. \tag{2.2}$$

We define the logarithmic mean as

$$L = L(a, b) := \begin{cases} a & \text{if} \quad a = b, \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b, \end{cases} \quad a, b > 0.$$

We can improve the inequality (2.1) for convex functions as follows:

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then we have for $f(b) \neq f(a)$ the inequalities*

$$\begin{aligned}\exp f\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b \exp f(t)\,dt \\ &\leq \frac{\exp f(b) - \exp f(a)}{f(b) - f(a)}\left(\leq \frac{1}{2}[\exp f(a) + \exp f(b)]\right).\end{aligned} \tag{2.3}$$

Proof. The first inequality follows by Hermite-Hadamard inequality for the convex function f. The second inequality follows by (2.2).

It is know that if g is log convex, then by [2]

$$\frac{1}{b-a}\int_a^b g(t)\,dt \leq L(g(a), g(b)). \tag{2.4}$$

Since f is convex, then $g = \exp(f)$ is log-convex and by (2.4) we get the third inequality in (2.3). □

A recent paper connected with such results is [4].

Consider the *identric mean* of two positive numbers

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, \end{cases} \qquad a, b > 0.$$

We observe that

$$\ln I(a,b) = \frac{1}{b-a}\int_a^b \ln u du$$

for $a, b > 0$, $a \neq b$.

The following result holds:

Theorem 2.2. *Assume that $f \in \mathfrak{Expconv}(I)$ and $a, b \in I$ with $a < b$. Then we have*

$$\exp\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq I(\exp f(a), \exp f(b)) \tag{2.5}$$

and

$$\begin{aligned} &\exp f\left(\frac{a+b}{2}\right) \\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \ln\left[\frac{\exp f(x) + \exp f(a+b-x)}{2}\right] dx\right) \\ &\leq \frac{1}{b-a}\int_a^b \exp f(x)\,dx. \end{aligned} \tag{2.6}$$

Proof. Since $f \in \mathfrak{Expconv}(I)$, then

$$\exp f((1-\lambda)a + \lambda b) \leq (1-\lambda)\exp f(a) + \lambda \exp f(b)$$

for any $\lambda \in [0,1]$, which is equivalent to

$$f((1-\lambda)a + \lambda b) \leq \ln[(1-\lambda)\exp f(a) + \lambda \exp f(b)] \tag{2.7}$$

for any $\lambda \in [0,1]$.

Integrating (2.7) on $[0,1]$ we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &= \int_0^1 f((1-\lambda)a + \lambda b)\,d\lambda \\ &\leq \int_0^1 \ln[(1-\lambda)\exp f(a) + \lambda \exp f(b)]\,d\lambda \\ &= \frac{1}{\exp f(b) - \exp f(a)}\int_{\exp f(a)}^{\exp f(b)} \ln u du \\ &= \ln I(\exp f(a), \exp f(b)) \end{aligned} \tag{2.8}$$

and the inequality in (2.5) is proved.

From (2.7) we have

$$f\left(\frac{x+y}{2}\right) \le \ln\left[\frac{\exp f(x) + \exp f(y)}{2}\right] \tag{2.9}$$

for any $x, y \in I$.

From (2.9) we have

$$f\left(\frac{a+b}{2}\right) \le \ln\left[\frac{\exp f(x) + \exp f(a+b-x)}{2}\right] \tag{2.10}$$

for any $x \in [a, b]$.

Integrating the inequality (2.10) over x on $[a, b]$ we get the first inequality in (2.6).

By the Jensen's inequality for the concave function ln we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \ln\left[\frac{\exp f(x) + \exp f(a+b-x)}{2}\right] dx \\ &\le \ln\left(\frac{1}{b-a}\int_a^b \left[\frac{\exp f(x) + \exp f(a+b-x)}{2}\right] dx\right) \\ &= \ln\left(\frac{1}{2(b-a)}\int_a^b [\exp f(x) + \exp f(a+b-x)]\, dx\right) \\ &= \ln\left(\frac{1}{b-a}\int_a^b \exp f(x)\, dx\right) \end{aligned} \tag{2.11}$$

and the second inequality in (2.6) is proved. □

If we consider *Toader's mean* defined as (see for instance [5] and [7] for many relations of this mean with other means)

$$E = E(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \log I(\exp a, \exp b) & \text{if} \quad a \ne b, \end{cases} \qquad a, b \in \mathbb{R}.$$

we can write (2.5) in an equivalent form as

$$\frac{1}{b-a}\int_a^b f(t)\, dt \le E(\exp f(a), \exp f(b)). \tag{2.12}$$

Remark 2.3. If the function $g : I \to (0, \infty)$ is convex on I, then $f = \ln g \in \mathfrak{Expconv}(I)$ and for $a, b \in I$ with $a < b$ we have, by (2.5) and (2.6), the following inequalities

$$\exp\left(\frac{1}{b-a}\int_a^b \ln g(t)\, dt\right) \le I(g(a), g(b)) \tag{2.13}$$

and

$$g\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln\left[\frac{g(x)+g(a+b-x)}{2}\right]dx\right) \tag{2.14}$$
$$\leq \frac{1}{b-a}\int_a^b g(x)\,dx.$$

3. Related results

The following related result also holds:

Theorem 3.1. *Assume that* $f \in \mathfrak{Expconv}(I)$ *and* $a, b \in I$ *with* $a < b$. *Then we have*

$$\frac{f(a)(x-a)+f(b)(b-x)}{b-a} - \frac{1}{b-a}\int_a^b f(y)\,dy \tag{3.1}$$
$$\geq \exp f(x)\left[\exp(-f(x)) - \frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right]$$

for any $x \in [a, b]$.

In particular, we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(y)\,dy \tag{3.2}$$
$$\geq \exp f\left(\frac{a+b}{2}\right)\left[\exp\left(-f\left(\frac{a+b}{2}\right)\right) - \frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right].$$

Proof. Since the function $\exp(f)$ is convex, it has lateral derivatives in each point of (a, b) and $f = \ln(\exp f)$ does the same. Then for any $x, y \in (a, b)$ we have

$$\exp f(x) - \exp f(y) \geq f'_-(y)(x-y)\exp f(y)$$

and dividing by $\exp f(y) > 0$ we get

$$\exp f(x)\exp[-f(y)] - 1 \geq f'_-(y)(x-y) \tag{3.3}$$

for any $x, y \in (a, b)$.

Integrating (3.3) over y on $[a, b]$ and dividing by $b - a$ we get

$$\exp f(x)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy - 1 \tag{3.4}$$
$$\geq \frac{1}{b-a}\int_a^b f'_-(y)(x-y)\,dy$$
$$= \frac{1}{b-a}\left[f(y)(x-y)\big|_a^b + \int_a^b f(y)\,dy\right]$$
$$= \frac{1}{b-a}\left[\int_a^b f(y)\,dy - f(a)(x-a) - f(b)(b-x)\right]$$

for any $x \in [a, b]$, which is equivalent to the desired inequality (3.1). □

Corollary 3.2. *With the assumptions of Theorem* 3.1 *we have*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{\exp f(a)+\exp f(b)}{2}\left[1-\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right]. \tag{3.5}$$

Proof. If we take $x=a$ and $x=b$ in (3.4) we get

$$\exp f(a)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\geq \frac{1}{b-a}\int_a^b f(y)\,dy-f(b)$$

and

$$\exp f(b)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\geq \frac{1}{b-a}\int_a^b f(y)\,dy-f(a).$$

Adding these inequalities and dividing by two we get

$$\frac{\exp f(a)+\exp f(b)}{2}\left[\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\right] \geq \frac{1}{b-a}\int_a^b f(y)\,dy-\frac{f(a)+f(b)}{2},$$

which is equivalent to the desired inequality (3.5). □

Corollary 3.3. *With the assumptions of Theorem* 3.1 *and if*

$$x_0:=\frac{f(b)\,b-f(a)\,a-\int_a^b f(y)\,dy}{f(b)-f(a)}\in[a,b], \tag{3.6}$$

where $f(b)\neq f(a)$, *then we have*

$$\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\geq \exp\left(-f\left(\frac{f(b)\,b-f(a)\,a-\int_a^b f(y)\,dy}{f(b)-f(a)}\right)\right). \tag{3.7}$$

Proof. Follows by (3.1) by taking $x=x_0$ defined in (3.6). □

The inequality (3.7) can be found in Sándor's paper [3] where x_0 considered in (3.6) is in fact a mean called by him as "*generated by derivatives of functions*". This mean is extended in [9] (see also [6]), and generalized many results related to integral inequalities. See also [8] for more results.

Remark 3.4. Since

$$x_0=\frac{\int_a^b f'(y)\,y\,dy}{\int_a^b f'(y)\,dy},$$

then a sufficient condition for (3.6) to hold is that f is monotonic nondecreasing or nonincreasing on the whole interval $[a,b]$.

Remark 3.5. If the function $g : I \to (0,\infty)$ is convex on I, then $f = \ln g \in \mathfrak{Expconv}\,(I)$ and for $a, b \in I$ with $a < b$ we have, by (3.1), (3.2) and (3.5), the following inequalities

$$\ln\left([g(a)]^{\frac{x-a}{b-a}}\,[g(b)]^{\frac{b-x}{b-a}}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \tag{3.8}$$
$$\geq g(x)\left[\frac{1}{g(x)} - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right],$$

$$\ln\left(\sqrt{g(a)\,g(b)}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \tag{3.9}$$
$$\geq g\left(\frac{a+b}{2}\right)\left[\frac{1}{g\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right],$$

and

$$\ln\left(\sqrt{g(a)\,g(b)}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \tag{3.10}$$
$$\geq \frac{g(a)+g(b)}{2}\left[1 - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right].$$

If

$$x_0 := \frac{\ln\left(\frac{[g(b)]^b}{[g(a)]^a}\right) - \int_a^b \ln g(y)\,dy}{\ln\left(\frac{g(b)}{g(a)}\right)} \in [a,b], \tag{3.11}$$

where $g(b) \neq g(a)$, then we have

$$\frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy \geq \frac{1}{g\left(\frac{\ln\left(\frac{[g(b)]^b}{[g(a)]^a}\right) - \int_a^b \ln g(y)dy}{\ln\left(\frac{g(b)}{g(a)}\right)}\right)}. \tag{3.12}$$

Acknowledgement. The authors would like to warmly thank the anonymous referee for pointing out some essential references and making valuable comments that have been implemented in the final version of the paper.

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 535–542

On the univalence of an integral operator

Virgil Pescar

Abstract. In this paper we introduce a new general integral operator for analytic functions in the open unit disk and we derive some criteria for univalence of this integral operator.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic, Schwarz lemma, integral operator, univalence.

1. Introduction

Let $\mathcal{P}$ be the class of functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$, with positive real part in $\mathcal{U}$. We denote by $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U}$ and we consider $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$.

In this work we introduce a new integral operator, which is defined by

$$K_{\gamma_1,\ldots,\gamma_n}(z) = \int_0^z \prod_{j=1}^{n} (p_j(u))^{\gamma_j}\, du, \tag{1.1}$$

for functions $p_j \in \mathcal{P}$ and γ_j be complex numbers, $j = \overline{1,n}$.

2. Preliminary results

In order to prove our main results we will use the lemmas.

Lemma 2.1. [1]. *If the function f is analytic in $\mathcal{U}$ and*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 2.2. [4]. *Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{2.2}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\beta \ge Re\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{2.3}$$

is regular and univalent in $\mathcal{U}$.

Lemma 2.3. (Schwarz [2]). *Let f be the function regular in the disk*

$$\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$$

with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\ge m$, then

$$|f(z)| \le \frac{M}{R^m}|z|^m,\ (z \in \mathcal{U}_R), \tag{2.4}$$

the equality (in the inequality (2.4) for $z \neq 0$) can hold if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

Lemma 2.4. [3]. *If the function f is regular in $\mathcal{U}$ and $|f(z)| < 1$ in $\mathcal{U}$, then for all $\xi \in \mathcal{U}$ and $z \in \mathcal{U}$ the following inequalities hold*

$$\left|\frac{f(\xi)-f(z)}{1-\overline{f(z)}f(\xi)}\right| \le \frac{|\xi - z|}{|1-\overline{z}\xi|}, \tag{2.5}$$

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}, \tag{2.6}$$

the equalities hold only in the case $f(z) = \frac{\epsilon(z+u)}{1+\overline{u}z}$, where $|\epsilon| = 1$ and $|u| < 1$.

Remark 2.5. [3]. For $z = 0$, from inequality (2.5)

$$\left|\frac{f(\xi)-f(0)}{1-\overline{f(0)}f(\xi)}\right| \le |\xi| \tag{2.7}$$

and, hence

$$|f(\xi)| \le \frac{|\xi|+|f(0)|}{1+|f(0)||\xi|}. \tag{2.8}$$

Considering $f(0) = a$ and $\xi = z$, we have

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|}, \tag{2.9}$$

for all $z \in \mathcal{U}$.

3. Main results

Theorem 3.1. *Let γ_j be complex numbers, M_j positive real numbers, $p_j \in \mathcal{P}$,*

$$p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots, \ j = \overline{1, n}.$$

If

$$\left| \frac{zp_j'(z)}{p_j(z)} \right| \leq M_j, \quad (j = \overline{1, n}; z \in \mathcal{U}) \tag{3.1}$$

and

$$|\gamma_1|M_1 + |\gamma_2|M_2 + \ldots + |\gamma_n|M_n \leq \frac{3\sqrt{3}}{2}, \tag{3.2}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n}$ defined by (1.1), is in the class $\mathcal{S}$.

Proof. The function $K_{\gamma_1,\ldots,\gamma_n}$ is regular in $\mathcal{U}$ and

$$K_{\gamma_1,\ldots,\gamma_n}(0) = K'_{\gamma_1,\ldots,\gamma_n}(0) - 1 = 0.$$

We have

$$\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} = \sum_{j=1}^{n} \gamma_j \frac{zp_j'(z)}{p_j(z)}, \quad (z \in \mathcal{U}), \tag{3.3}$$

and hence, we obtain

$$(1 - |z|^2) \left| \frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} \right| \leq (1 - |z|^2) \sum_{j=1}^{n} |\gamma_j| \left| \frac{zp_j'(z)}{p_j(z)} \right|, \tag{3.4}$$

for all $z \in \mathcal{U}$.

From (3.1) and Lemma 2.3 we get

$$\left| \frac{zp_j'(z)}{p_j(z)} \right| \leq M_j|z|, \quad (j = \overline{1, n}; z \in \mathcal{U}) \tag{3.5}$$

and by (3.4) we have

$$(1 - |z|^2) \left| \frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} \right| \leq (1 - |z|^2)|z| \sum_{j=1}^{n} |\gamma_j|M_j, \tag{3.6}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z| \leq 1}(1 - |z|^2)|z| = \frac{2}{3\sqrt{3}},$$

from (3.2) and (3.6) we obtain that

$$(1-|z|^2)\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \le 1, \tag{3.7}$$

for all $z \in \mathcal{U}$ and by Lemma 2.1, it results that the integral operator $K_{\gamma_1,\ldots,\gamma_n}$ belongs to the class $\mathcal{S}$. □

Theorem 3.2. *Let α, γ_j be complex numbers, $j=\overline{1,n}$, $0 < Re\ \alpha \le 1$ and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j=\overline{1,n}$.*

If

$$\left|\frac{zp'_j(z)}{p_j(z)}\right| \le \frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}, \quad (j=\overline{1,n}; z \in \mathcal{U}) \tag{3.8}$$

and

$$|\gamma_1| + |\gamma_2| + \ldots | + \gamma_n| \le 1, \tag{3.9}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n}$, defined by (1.1), is in the class $\mathcal{S}$.

Proof. From (3.3) we obtain

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \le \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\sum_{j=1}^{n}|\gamma_j|\left|\frac{zp'_j(z)}{p_j(z)}\right|, \tag{3.10}$$

for all $z \in \mathcal{U}$.

By (3.8) and Lemma 2.3, we get

$$\left|\frac{zp'_j(z)}{p_j(z)}\right| \le \frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}|z|, \quad (j=\overline{1,n}; z \in \mathcal{U}) \tag{3.11}$$

and hence, by (3.10) we have

$$\begin{aligned}\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} \quad &\cdot \quad \left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \le \\ &\le \quad \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}\sum_{j=1}^{n}|\gamma_j|,\end{aligned} \tag{3.12}$$

for all $z \in \mathcal{U}$.

We have

$$\max_{|z|\le 1}\left[\frac{(1-|z|)^{2Re\ \alpha}|z|}{Re\ \alpha}\right] = \frac{2}{(Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}$$

and from (3.9) and (3.12) we get

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \le 1, \tag{3.13}$$

for all $z \in \mathcal{U}$. By (3.13) and Lemma 2.2, for $\beta = 1$, $f = K_{\gamma_1,\ldots,\gamma_n}$, it results that the integral operator $K_{\gamma_1,\ldots,\gamma_n}$ is in the class $\mathcal{S}$. □

Theorem 3.3. *Let γ_j be complex numbers,*

$$p_j \in \mathcal{P}, p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots, j = \overline{1,n}.$$

If

$$|\gamma_1| + |\gamma_2| + \ldots | + \gamma_n| \leq \frac{1}{2}, \tag{3.14}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n}$ defined by (1.1) belongs to the class $\mathcal{S}$.

Proof. Since $p_j \in \mathcal{P}$, $j = \overline{1,n}$ we have

$$\left|\frac{zp_j'(z)}{p_j(z)}\right| \leq \frac{2|z|}{1-|z|^2}, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{3.15}$$

by (3.3) we obtain

$$(1-|z|^2)\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \leq 2\sum_{j=1}^{n}|\gamma_j|, \quad (z \in \mathcal{U}). \tag{3.16}$$

From (3.14) and (3.16) we get

$$(1-|z|^2)\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \leq 1, \tag{3.17}$$

for all $z \in \mathcal{U}$.

By (3.17) and Lemma 2.1 we obtain that the integral operator $K_{\gamma_1,\ldots,\gamma_n}$ belongs to the class $\mathcal{S}$. □

Theorem 3.4. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < Re\ \alpha \leq 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| < M_j, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{3.18}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \leq \frac{1}{max_{|z|\leq 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\frac{|z|+|c|}{1+|c||z|}\right]}, \tag{3.19}$$

where

$$c = \frac{b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}, \tag{3.20}$$

then the integral operator $K_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ defined by (1.1) is in the class $\mathcal{S}$.

Proof. We have

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}\right| \leq \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\sum_{j=1}^{n}|\gamma_j|\left|\frac{p_j'(z)}{p_j(z)}\right|, \tag{3.21}$$

for all $z \in \mathcal{U}$. We consider the function

$$f_n(z) = \frac{1}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}\frac{K''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)}, \quad (z \in \mathcal{U}) \tag{3.22}$$

and from (1.1) we obtain

$$f_n(z) = \frac{\gamma_1}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|} \cdot \frac{p_1'(z)}{p_1(z)} + \ldots +$$

$$+ \frac{\gamma_n}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|} \cdot \frac{p_n'(z)}{p_n(z)}, \tag{3.23}$$

for all $z \in \mathcal{U}$.

From (3.18) and (3.23) we obtain $|f_n(z)| < 1$, $z \in \mathcal{U}$.

We have

$$f_n(0) = \frac{b_{11}\gamma_1 + \ldots + b_{1n}\gamma_n}{M_1|\gamma_1| + \ldots + M_n|\gamma_n|} = c$$

and by Remark 2.5 we get

$$|f_n(z)| \leq \frac{|z| + |c|}{1 + |c||z|}, \quad (z \in \mathcal{U}), \tag{3.24}$$

where

$$|c| = \frac{|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n|}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}.$$

From (3.22) and (3.24) we obtain

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} \right| \leq$$

$$\leq (M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|) \max_{|z|\leq 1} \left[\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} |z| \frac{|z| + |c|}{1 + |c||z|} \right], \tag{3.25}$$

for all $z \in \mathcal{U}$.

By (3.19) and (3.25) we have

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{zK''_{\gamma_1,\ldots,\gamma_n}(z)}{K'_{\gamma_1,\ldots,\gamma_n}(z)} \right| \leq 1, \quad (z \in \mathcal{U}). \tag{3.26}$$

From (3.26) and Lemma 2.2 for $\beta = 1$, it results that the integral operator $K_{\gamma_1,\ldots,\gamma_n} \in \mathcal{S}$. □

Corollary 3.5. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < Re\ \alpha \leq 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left| \frac{p_j'(z)}{p_j(z)} \right| < M_j, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{3.27}$$

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}, \tag{3.28}$$

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| = M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|, \tag{3.29}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n} \in \mathcal{S}$.

Proof. From (3.29) and (3.20) we obtain $|c| = 1$. Using the inequality (3.19) we have

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le \frac{1}{max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\right]}, \tag{3.30}$$

Since

$$max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\right] = \frac{2}{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}, \tag{3.31}$$

from (3.30) and (3.29) we obtain (3.28).

The conditions of Theorem 3.4 are satisfied. □

Corollary 3.6. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < Re\ \alpha \le 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$, $b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n = 0$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| < M_j, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{3.32}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le (Re\ \alpha+1)^{\frac{Re\ \alpha+1}{Re\ \alpha}}, \tag{3.33}$$

then the integral operator $K_{\gamma_1,\ldots,\gamma_n} \in \mathcal{S}$.

Proof. From Theorem 3.4, by (3.20), we obtain $c = 0$ and using the inequality (3.19) we get

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le \frac{1}{max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|^2\right]}. \tag{3.34}$$

We have

$$max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|^2\right] = \frac{1}{(Re\ \alpha+1)^{\frac{Re\ \alpha+1}{Re\ \alpha}}}$$

and from (3.34) we obtain the inequality (3.33). Since the conditions of Theorem 3.4 are verified it results that $K_{\gamma_1,\ldots,\gamma_n}$ belongs to $\mathcal{S}$. □

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 543–552

Some new subclasses of bi-univalent functions

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Abstract. The purpose of the present paper is to obtain the initial coefficients for normalized analytic functions f in the open unit disk U with its inverse $g = f^{-1}$ belonging to the classes $H^n_\sigma(\phi)$, $ST^n_\sigma(\alpha, \phi)$, $M^n_\sigma(\alpha, \phi)$ and $L^n_\sigma(\alpha, \phi)$. Relevant connections of the results presented here with various known results are briefly indicated. Finally, we give an open problem for the readers.

Mathematics Subject Classification (2010): 30C45.

Keywords: Univalent functions, bi-univalent functions, subordination, Salagean derivative.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of functions of the form (1.1) which are also univalent in U. The Koebe one-quarter theorem [4] ensures that the image of U under every univalent function $f \in A$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq 1/4)$. A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let σ denote the class of bi-univalent functions defined in the unit disk U.

A domain $U \subset C$ is convex if the line segment joining any two points in U lies entirely in U, while a domain is starlike with respect to a point $w_0 \in U$ if the line segment joining any point of U to w_0 lies inside U. A function $f \in A$ is starlike if $f(U)$ is a starlike domain with respect to origin, and convex if $f(U)$ is convex. Analytically $f \in A$ is starlike if and only if $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$, whereas $f \in A$ is convex if and only if $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$. The classes consisting of starlike and convex functions are denoted by ST and CV respectively. The classes $ST(\alpha)$ and $CV(\alpha)$ of starlike and convex functions of order α , $0 \leq \alpha < 1$, are respectively characterized by $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$ and $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$. Ma and Minda [8] unified various subclasses of starlike and convex functions by using subordination. Now we recall the definition of subordination

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on U with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$.

Lewin [7] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Several researchers have subsequently studied similar problems in this direction (see [2], [5], [6], [10], [12], [13]). Brannan and Taha [2] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Recently, Srivastava *et al.* [12] introduced and investigated subclasses of the bi-univalent functions and obtained bounds for the initial coefficients. The results of [12] were generalized in [5], [6], [10] and [13].

Very recently Ali *et al.* [1] estimates on the initial coefficients for bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type functions are obtained. In this paper, we generalized these results by using Salagean operator and obtain sharp estimates on coefficient for function classes $H_\sigma^n(\phi)$, $ST_\sigma^n(\alpha,\phi)$, $M_\sigma^n(\alpha,\phi)$ and $L_\sigma^n(\alpha,\phi)$.

2. Coefficient estimates

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the disk U, satisfying $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots,\ (B_1 > 0)\,. \tag{2.1}$$

A function $f \in \sigma$ is said to be in the class $H_\sigma^n(\phi)$ if the following subordination hold:

$$\frac{D^n f(z)}{z} \prec \phi(z)$$

and

$$\frac{D^n g(w)}{w} \prec \phi(w)\,,\ g(w) = f^{-1}(w)\,,$$

where D^n stands for the Salagean operator introduced by Salagean [11] for function f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

analytic in the open unit disk U as follows

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = z f'(z)$$
$$\ldots\ldots$$
$$D^n f(z) = D(D^{n-1} f(z))$$

Thus

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

For functions in the class $H_\sigma^n(\phi)$, we obtain the following result.

Theorem 2.1. *If $f \in H_\sigma^n(\phi)$ is given by*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{2.2}$$

then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|3^n B_1^2 - 2^{2n} B_2 + 2^{2n} B_1|}} \tag{2.3}$$

and

$$|a_3| \leq \left(\frac{1}{3^n} + \frac{B_1}{2^{2n}} \right) B_1. \tag{2.4}$$

Proof. Let $f \in H_\sigma^n(\phi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : U \to U$, with $u(0) = v(0) = 0$, satisfying

$$\frac{D^n f(z)}{z} = \phi(u(z)) \text{ and } \frac{D^n g(w)}{w} = \phi(v(w)). \tag{2.5}$$

Define the functions $p_1(z)$ and $p_2(z)$ by

$$p_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \ldots$$

and

$$p_2(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + b_1 z + b_2 z^2 + \ldots$$

or, equivalently,

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \ldots \right) \tag{2.6}$$

and

$$v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left(b_1 z + \left(b_2 - \frac{b_1^2}{2} \right) z^2 + \ldots \right). \tag{2.7}$$

Then $p_1(z)$ and $p_2(z)$ are analytic in U with $p_1(0) = 1 = p_2(0)$. Since $u, v : U \to U$, the functions $p_1(z)$ and $p_2(z)$ have a positive real part in U, and $|b_i| \leq 2$ and $|c_i| \leq 2$. In view of (2.5)-(2.7), clearly

$$\frac{D^n f(z)}{z} = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) \tag{2.8}$$

and

$$\frac{D^n g(w)}{w} = \phi\left(\frac{p_2(w)-1}{p_2(w)+1}\right). \tag{2.9}$$

Using (2.5) and (2.7) together with (2.1), it is evident that

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \dots \tag{2.10}$$

and

$$\phi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) = 1 + \frac{1}{2}B_1b_1w + \left(\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2\right)w^2 + \dots \tag{2.11}$$

Since $f \in \sigma$ has the Maclaurin series given by (2.2), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 + \dots$$

Since

$$\frac{D^n f(z)}{z} = 1 + 2^n a_2 z + 3^n a_3 z^2 + \dots$$

and

$$\frac{D^n g(w)}{w} = 1 - 2^n a_2 w + \left(2a_2^2 - a_3\right)3^n w^2 + \dots,$$

it follows from (2.8)-(2.11) that

$$2^n a_2 = \frac{1}{2}B_1c_1, \tag{2.12}$$

$$3^n a_3 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2 \tag{2.13}$$

$$-2^n a_2 = \frac{1}{2}B_1b_1 \tag{2.14}$$

and

$$3^n\left(2a_2^2 - a_3\right) = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2. \tag{2.15}$$

From (2.12) and (2.14), it follows that

$$c_1 = -b_1. \tag{2.16}$$

Now (2.13)-(2.16) yield

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{4\left(3^n B_1^2 - 2^{2n}B_2 + 2^{2n}B_1\right)}$$

which, in view of the well-known inequalities $|b_2| \le 2$ and $|c_2| \le 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.3). By subtracting (2.15) from (2.13), further computations using (2.12) and (2.16) lead to

$$a_3 = \frac{B_1(c_2 - b_2)}{4.3^n} + \frac{B_1^2c_1^2}{4.2^{2n}},$$

and this yields the estimates given in (2.4). □

Remark 2.2. If we put $n = 1$ in Theorem 2.1, then we obtain the corresponding result of Ali *et al.* [1].

Remark 2.3. If we put $n = 1$ with $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$ in Theorem 2.1, then we obtain the corresponding result of Srivastava *et al.* [12].

Remark 2.4. If we put $n = 1$ with $\phi(z) = \frac{1+(1-2\gamma)z}{1-z}$ in Theorem 2.1, then we obtain the corresponding result of Srivastava *et al.* [12].

A function $f \in \sigma$ is said to be in the class $ST_{\sigma}^{n}(\alpha, \phi)$, $n \in N_0, \alpha \geq 0$, if the following subordinations hold:

$$\frac{(1-\alpha) D^{n+1} f(z) + \alpha D^{n+2} f(z)}{D^n f(z)} \prec \phi(z)$$

and

$$\frac{(1-\alpha) D^{n+1} g(w) + \alpha D^{n+2} g(w)}{D^n g(w)} \prec \phi(w); \; g(w) = f^{-1}(w).$$

Note that $ST_{\sigma}^{n}(\phi) \equiv ST_{\sigma}^{n}(0, \phi)$. For the functions in the class $ST_{\sigma}^{n}(\alpha, \phi)$, the following coefficient estimates are obtained.

Theorem 2.5. *Let f given by (2.2) be in the class $ST_{\sigma}^{n}(\alpha, \phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2\left(3^n 2(1+3\alpha) - 2^{2n}(1+2\alpha)\right) + (B_1 - B_2)\, 2^{2n}(1+2\alpha)^2\right|}}$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{3^n \cdot 2(1+3\alpha) - 2^{2n}(1+2\alpha)}.$$

Proof. Let $f \in ST_{\sigma}^{n}(\alpha, \phi)$. Then there are analytic functions $u, v : U \to U$, with $u(0) = v(0) = 0$, satisfying

$$\frac{(1-\alpha) D^{n+1} f(z) + \alpha D^{n+2} f(z)}{D^n f(z)} = \phi(u(z)) \tag{2.17}$$

and

$$\frac{(1-\alpha) D^{n+1} g(w) + \alpha D^{n+2} g(w)}{D^n g(w)} = \phi(v(w)), \;\; \left(g = f^{-1}\right). \tag{2.18}$$

Since

$$\begin{aligned}\frac{(1-\alpha) D^{n+1} f(z) + \alpha D^{n+2} f(z)}{D^n f(z)} &= 1 + (1+2\alpha)\, 2^n a_2 z \\ &+ \left(3^n \cdot 2(1+3\alpha) a_3 - 2^{2n}(1+2\alpha) a_2^2\right) z^2 + \ldots\end{aligned}$$

and

$$\frac{(1-\alpha) D^{n+1} g(w) + \alpha D^{n+2} f(w)}{D^n g(w)} = 1 - (1+2\alpha)\, 2^n a_2 w$$
$$+ \left(\left(3^n \cdot 4(1+3\alpha) - 2^{2n}(1+2\alpha)\right) a_2^2 - 3^n \cdot 2(1+3\alpha) a_3\right) w^2 + \ldots,$$

then (2.10),(2.11), (2.17) and (2.18) yield

$$2^n(1+2\alpha)a_2 = \frac{1}{2}B_1c_1, \tag{2.19}$$

$$3^n2(1+3\alpha)a_3 - 2^{2n}(1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.20}$$

$$-2^n(1+2\alpha)a_2 = \frac{1}{2}B_1b_1, \tag{2.21}$$

and

$$\left(3^n \cdot 4\left(1+3\alpha\right) - 2^{2n}\left(1+2\alpha\right)\right)a_2^2 - 3^n \cdot 2\left(1+3\alpha\right)a_3 = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2. \tag{2.22}$$

Now, the required result follows by using the techniques as used in Theorem 2.1. □

Remark 2.6. If we put $n = 0$ in Theorem 2.5, then we obtain the corresponding result of Ali *et al.* [1].
Next, if we put $n = 0$, $\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma$ with $\alpha = 0$, then we obtain corresponding result of Brannan and Taha [2].

Next, a function $f \in \sigma$ belongs to the class $M_\sigma^n(\alpha,\phi)$, $n \in N_0, \alpha \geq 0$, if the following subordinations hold:

$$(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec \phi(z)$$

and

$$(1-\alpha)\frac{D^{n+1}g(w)}{D^ng(w)} + \alpha\frac{D^{n+2}g(w)}{D^{n+1}g(w)} \prec \phi(w),\ g(w) = f^{-1}(w).$$

For function in the class $M_\sigma^n(\alpha,\phi)$, the following coefficient estimates hold.

Theorem 2.7. *Let f given by (2.2) be in the class $M_\sigma^n(\alpha,\phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{B_1^2\left(2\cdot 3^n\left(1+2\alpha\right) - 2^{2n}\left(1+3\alpha\right) + 2^{2n}(1+\alpha)^2\left(B_1 - B_2\right)\right)}} \tag{2.23}$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{2\left(1+2\alpha\right)3^n - \left(1+3\alpha\right)2^{2n}}. \tag{2.24}$$

Proof. If $f \in M_\sigma^n(\alpha,\phi)$, then there are analytic functions $u, v : U \to U$, with $u(0) = v(0) = 0$, such that

$$(1-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = \phi(u(z)) \tag{2.25}$$

and

$$(1-\alpha)\frac{D^{n+1}g(w)}{D^ng(w)} + \alpha\frac{D^{n+2}g(w)}{D^{n+1}g(w)} = \phi(v(w)). \tag{2.26}$$

Since

$$(1-\alpha)\frac{D^{n+1}f(z)}{D^{n}f(z)}+\alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}=1+(1+\alpha)\,2^{n}a_{2}z$$
$$+\left(2(1+2\alpha)\,3^{n}a_{3}-(1+3\alpha)\,2^{2n}a_{2}^{2}\right)z^{2}+\ldots$$

and

$$(1-\alpha)\frac{D^{n+1}g(w)}{D^{n}g(w)}+\alpha\frac{D^{n+2}g(w)}{D^{n+1}g(w)}=1-(1+\alpha)\,2^{n}a_{2}w$$
$$+\left(\left(4(1+2\alpha)\,3^{n}-(1+3\alpha)\,2^{2n}\right)a_{2}^{2}-2(1+2\alpha)\,3^{n}a_{3}\right)w^{2}+\ldots.$$

From (2.10), (2.11), (2.25) and (2.26) it follows that

$$(1+\alpha)\,2^{n}a_{2}=\frac{1}{2}B_{1}c_{1} \tag{2.27}$$

$$2(1+2\alpha)\,3^{n}a_{3}-(1+3\alpha)\,2^{2n}a_{2}^{2}=\frac{1}{2}B_{1}\left(c_{2}-\frac{c_{1}^{2}}{2}\right)+\frac{1}{4}B_{2}c_{1}^{2} \tag{2.28}$$

$$-(1+\alpha)\,2^{n}a_{2}=\frac{1}{2}B_{1}b_{1} \tag{2.29}$$

and

$$\left(4(1+2\alpha)\,3^{n}-(1+3\alpha)\,2^{2n}\right)a_{2}^{2}-2(1+2\alpha)\,3^{n}a_{3}=\frac{1}{2}B_{1}\left(b_{2}-\frac{b_{1}^{2}}{2}\right)+\frac{1}{4}B_{2}b_{1}^{2}. \tag{2.30}$$

Equation (2.27) and (2.29) yield

$$c_{1}=-b_{1}. \tag{2.31}$$

From (2.28), (2.30) and (2.31), it follows that

$$a_{2}^{2}=\frac{B_{1}^{3}(b_{2}+c_{2})}{4\left(B_{1}^{2}\left(2\cdot 3^{n}(1+2\alpha)-2^{2n}(1+3\alpha)\right)+2^{2n}(1+\alpha)^{2}(B_{1}-B_{2})\right)}$$

which yields the describe estimate on as describe in (2.23). As in the earlier proofs, use of (2.28)-(2.31) shows that

$$a_{3}=\frac{(B_{1}/2)\left(\left(4(1+2\alpha)\,3^{n}-(1+3\alpha)\,2^{n}\right)c_{2}+(1+3\alpha)\,2^{n}b_{2}\right)+b_{1}^{2}(1+2\alpha)(B_{2}-B_{1})}{4\cdot 3^{n}(1+2\alpha)\left(2(1+2\alpha)\,3^{n}-(1+3\alpha)\,2^{2n}\right)}.$$

Thus the proof of Theorem 2.7 is complete. □

Next, a function $f\in\sigma$ is said to be in the class $L_{\sigma}^{n}(\alpha,\phi)$ $n\in N_{0},\ \alpha\geq 0$, if the following subordinations hold:

$$\left(\frac{D^{n+1}f(z)}{D^{n}f(z)}\right)^{\alpha}\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right)^{1-\alpha}\prec\phi(z)$$

and

$$\left(\frac{D^{n+1}g(w)}{D^{n}g(w)}\right)^{\alpha}\left(\frac{D^{n+2}g(w)}{D^{n+1}g(w)}\right)^{1-\alpha}\prec\phi(w)$$

$$g(w)=f^{-1}(w).$$

For function in this class, the following coefficient estimates are obtained

Theorem 2.8. *Let f given by (2.2) be in the class $L^n_\sigma(\alpha,\phi)$. Then*

$$|a_2| \le \frac{2B_1\sqrt{B_1}}{\sqrt{\left|2\left(4\left(3-\alpha\right)3^n+\left(\alpha^2+5\alpha-8\right)2^{2n}B_1^2\right)+4\cdot 2^{2n}(\alpha-2)^2\left(B_1-B_2\right)\right|}}, \tag{2.32}$$

and

$$|a_3| \le \frac{2\left(3-2\alpha\right)3^n\left(B_1+|B_1-B_2|\right)}{|3^n\left(3-2\alpha\right)\left(4\left(3-2\alpha\right)3^n+\left(\alpha^2+5\alpha-8\right)2^{2n}\right)|}. \tag{2.33}$$

Proof. Let $f \in L^n_\sigma(\alpha,\phi)$. Then there are analytic functions $u, v : U \to U$, with $u(0) = v(0) = 0$, such that

$$\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^\alpha \left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right)^{(1-\alpha)} = \phi\left(u(z)\right) \tag{2.34}$$

and

$$\left(\frac{D^{n+1}g(w)}{D^n g(w)}\right)^\alpha \left(\frac{D^{n+2}g(w)}{D^{n+1}g(w)}\right)^{(1-\alpha)} = \phi\left(v(w)\right). \tag{2.35}$$

Since

$$\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^\alpha \left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right)^{(1-\alpha)} = 1+2^n\left(2-\alpha\right)a_2 z$$
$$+\left(3^n\cdot 2\left(3-2\alpha\right)a_3+\left(\frac{\alpha^2-5\alpha+8}{2}\right)2^{2n}a_2^2\right)z^2+\dots$$

and

$$\left(\frac{D^{n+1}g(w)}{D^n g(w)}\right)^\alpha \left(\frac{D^{n+2}g(w)}{D^{n+1}g(w)}\right)^{(1-\alpha)} = 1-2^n\left(2-\alpha\right)a_2 w$$
$$+\left(\left(4\cdot\left(3-2\alpha\right)3^n+\frac{\alpha^2+5\alpha-8}{2}\right)a_2^2-3^n\cdot 2\left(3-2\alpha\right)a_3\right)w^2+\dots$$

from (2.10), (2.11), (2.34) and (2.35) it follows that

$$2^n\cdot\left(2-\alpha\right)a_2 = \frac{1}{2}B_1c_1 \tag{2.36}$$

$$3^n\ 2\left(3-2\alpha\right)a_3+\left(\alpha^2+5\alpha-8\right)2^{2n}\cdot\frac{a_2^2}{2} = \frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2 \tag{2.37}$$

$$-2^n\left(2-\alpha\right)a_2 = \frac{1}{2}B_1b_1 \tag{2.38}$$

and

$$\left(4\left(3-2\alpha\right)3^n+2^n\frac{\left(\alpha^2+5\alpha-8\right)}{2}\right)a_2^2-3^n\cdot 2\left(3-2\alpha\right)a_3 = \frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2. \tag{2.39}$$

Now (2.36) and (2.38) clearly yield

$$c_1 = -b_1. \tag{2.40}$$

Equation (2.37), (2.39) and (2.40) lead to

$$a_2^2 = \frac{B_1^3\,(b_2+c_2)}{2\left(4\,(3-2\alpha)\,3^n + (\alpha^2+5\alpha-8)\,2^{2n}\right)B_1^2 + 4\cdot 2^{2n}(\alpha-2)^2\,(B_1-B_2)}$$

which yields the desired estimate on $|a_2|$ as asserted in (2.32). Proceeding similarly as in the earlier proof, using (2.37)-(2.40), it following that

$$a_3 = \frac{(B_1/2)((8(3-2\alpha)3^n + 2^{2n}(\alpha^2+5\alpha-8))c_2 - 2^{2n}(\alpha^2+5\alpha-8)b_2) + 3^n 2b_1^2(3-2\alpha)(B_1-B_2)}{4\cdot 3^n(3-2\alpha)(4(3-2\alpha)3^n + (\alpha^2+5\alpha-8)2^{2n})}$$

which yields the estimate (2.33). □

Remark 2.9. If we put $n = 0$ in Theorem 2.7-2.8 then we obtain the corresponding result of Ali *et al.* [1].

Remark 2.10. Sharp estimates for the coefficients $|a_2|$, $|a_3|$ and other coefficients of functions belonging to the classes investigated in this paper are yet open problems. Indeed it would be of interest even to find estimates (not necessarily sharp) for $|a_n|$, $n \geq 4$.

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 553–559

A new class of (j, k)-symmetric harmonic starlike functions

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Abstract. Using the concepts of (j, k)-symmetrical functions we define the class of sense-preserving harmonic univalent functions $\mathcal{SH}_s^{j,k}(\beta)$, and prove certain interesting results.

Mathematics Subject Classification (2010): 30C45.

Keywords: (j, k)-Symmetric functions, harmonic functions.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \in \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$, [see 3].

Denote by $\mathcal{SH}$ the class of functions $f(z) = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$, for which $f(0) = f_z(0) - 1 = 0$. Then for $f(z) = h + \overline{g} \in \mathcal{SH}$, we may express the analytic functions f and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

Note that $\mathcal{SH}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the coanalytic part of its members is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.2}$$

A function $f(z) = h + \overline{g}$ with h and g given by (1.1) is said to be harmonic starlike of order β for $0 \leq \beta < 1$, for $|z| = r < 1$ if

$$\frac{\partial}{\partial \theta}(\arg f(re^{i\theta})) = \Im \left\{ \frac{\frac{\partial}{\partial \theta} f(re^{i\theta})}{f(re^{i\theta})} \right\} = \Re \left\{ \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} \geq \beta.$$

The class of all harmonic starlike functions of order β is denoted by $\mathcal{S}^*_{\mathcal{H}}(\beta)$ and extensively studied by Jahangiri [4]. The cases $\beta = 0$ and $b_1 = 1$ were studied by Silverman and Silvia [8] and Silverman [7].

Definition 1.1. *Let k be a positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$*

$$f(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} f(z).$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$ we get class of odd univalent functions.

The notion of (j,k)-symmetrical functions ($k = 2, 3, ...\ ; j = 0, 1, 2, ..., k-1$) is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.
The theory of (j,k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings [5].

Definition 1.2. *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0, 1, 2, .., k-1$ where $k \geq 2$ is a natural number. A function $f : \mathcal{D} \mapsto \mathbb{C}$ and $\mathcal{D}$ is a k-fold symmetric set, f is called (j,k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z),\ \ z \in \mathcal{U}.$$

We note that the family of all (j,k)-symmetric functions is denoted be $\mathcal{S}^{(j,k)}$. Also, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are called even, odd and k-symmetric functions respectively. We have the following decomposition theorem.

Theorem 1.3. *[5] For every mapping $f : \mathcal{D} \mapsto \mathbb{C}$, and $\mathcal{D}$ is a k-fold symmetric set, there exists exactly the sequence of (j,k)- symmetrical functions $f_{j,k}$,*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z), \tag{1.3}$$

$$(f \in \mathcal{A};\ \ k = 1, 2, ...; j = 0, 1, 2, ..., k-1)$$

From (1.3) we can get

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} \left(\sum_{n=1}^{\infty} a_n(\varepsilon^v z)^n\right),$$

then

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \delta_{n,j} a_n z^n,\ a_1 = 1, \quad \delta_{n,j} = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{(n-j)v} = \begin{cases} 1, & n = lk + j; \\ 0, & n \neq lk + j; \end{cases}, \tag{1.4}$$

Ahuja and Jahangiri [2] discussed the class $\mathcal{SH}(\beta)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1.1) and satisfying

$$\Im\left\{\frac{2\frac{\partial}{\partial\theta} f(re^{i\theta})}{f(re^{i\theta}) - f(-re^{i\theta})}\right\} \geq \beta, \quad 0 \leq \beta < 1.$$

Al-Shaqsi and and Maslina Darus [1] discussed the class $\mathcal{SH}^k_s(\beta)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1.1) and satisfying

$$\Im\left\{\frac{\frac{\partial}{\partial\theta} f(re^{i\theta})}{f_k(re^{i\theta})}\right\} \geq \beta, \quad 0 \leq \beta < 1.$$

Now using the concepts of $(j,k)-$symmteric points we define the following

Definition 1.4. *For $0 \leq \beta < 1$ and $k = 1, 2, 3, ...,\ j = 0, 1, ..., k-1$, let $\mathcal{SH}^{j,k}_s(\beta)$ denote the class of sense-preserving, harmonic univalent functions f of the form* (1.1) *which satisfy the condition*

$$\Im\left\{\frac{\frac{\partial}{\partial\theta} f(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} \geq \beta. \tag{1.5}$$

Where $z = re^{i\theta},\ 0 \leq r < 1,\ 0 \leq \theta < 2\pi$ and $f_{j,k} = h_{j,k} + \overline{g_{j,k}}$, where $h_{j,k},\ g_{j,k}$ given by

$$h_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} h(\varepsilon^v z), \quad g_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} g(\varepsilon^v z). \tag{1.6}$$

The following special cases are of interest
(i) $\mathcal{SH}^{1,k}_s(\beta) = \mathcal{SH}^k_s(\beta)$ the class introduced by Al-Shaqsi and Darus [1].
(ii) $\mathcal{SH}^{1,2}_s(\beta) = \mathcal{SH}(\beta)$ the class introduced by Ahuja and Jahangiri [2].
(iii) $\mathcal{SH}^{1,1}_s(\beta) = \mathcal{S_H}^*(\beta)$ the class introduced by Jahangiri [4].
(iv) $\mathcal{SH}^{1,1}_s(0) = \mathcal{S_H}^*$ the class introduced by Silverman and Silvia [8].
We need the following lemma to prove our main results.

Lemma 1.5. [4] *Let $f = f + \overline{g}$ with h and g are given by* (1.1). *If*

$$\sum_{n=1}^{\infty}\left\{\frac{n-\beta}{1-\beta}|a_n| + \frac{n+\beta}{1-\beta}|b_n|\right\} \leq 2,\ a_1 = 1,\ 0 \leq \beta < 1.$$

Then f is sense-preserving, harmonic univalent and starlike of order β in $\mathcal{U}$.

2. Main result

Theorem 2.1. *Let* $f \in \mathcal{SH}_s^{(j,k)}(\beta)$ *where* f *given by* (1.1), *then* $f_{j,k}(z)$ *is in* $\mathcal{S_H}^*(\beta)$, *where* $f_{j,k}$ *given by* (1.6).

Proof. Suppose that $f \in \mathcal{SH}_s^{(j,k)}(\beta)$. Then we get

$$\Im\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} \geq \beta. \tag{2.1}$$

replacing $re^{i\theta}$ by $\varepsilon^v re^{i\theta}$ in (2.1), we get

$$\Im\left\{\frac{\frac{\partial}{\partial\theta}f(\varepsilon^v re^{i\theta})}{f_{j,k}(\varepsilon^v re^{i\theta})}\right\} \geq \beta.$$

According to the definition of $f_{j,k}$ and $\varepsilon^k = 1$, we know $f_{j,k}(\varepsilon^v re^{i\theta}) = \varepsilon^{vj} f_{j,k}(re^{i\theta})$, we get

$$\Im\left\{\frac{\varepsilon^{-vj}\frac{\partial}{\partial\theta}f(\varepsilon^v re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} \geq \beta, \tag{2.2}$$

letting $(v = 0, 1, 2, ...k-1)$ in (2.2) and summing them we can get

$$\Im\left\{\frac{\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}\frac{\partial}{\partial\theta}f(\varepsilon^v re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} = \Im\left\{\frac{\frac{\partial}{\partial\theta}f_{j,k}(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\} > \beta, \tag{2.3}$$

that is $f_{j,k}(z) \in \mathcal{S_H}^*(\beta)$. □

Theorem 2.2. *If* $f = h + \overline{g}$ *with* h *and* g *given by* (1.1) *and* $f_{j,k} = h_{j,k} + \overline{g_{j,k}}$ *with* $h_{j,k}$ *and* $g_{j,k}$ *given by* (1.6). *Let*

$$\sum_{n=1}^{\infty}\left\{\frac{(n-1)k+j-\beta}{1-\delta_{1,j}\beta}|a_{(n-1)k+j}| + \frac{(n-1)k+j+\beta}{1-\delta_{1,j}\beta}|b_{(n-1)k+j}|\right\} \tag{2.4}$$

$$+\sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}\{|a_n|+|b_n|\} \leq 2,$$

where $a_1 = 1,\ 0 \leq \beta < 1,\ k = 1, 2, 3, ...,\ j = 0, 1, ..., k-1,\ l \in \mathbb{N}$ *and* $\delta_{1,j}$ *is defined by* (1.4). *Then* f *is sense-preserving harmonic univalent in* $\mathcal{U}$ *and* $f \in \mathcal{SH}_s^{(j,k)}(\beta)$.

Proof. Since

$$\sum_{n=1}^{\infty}\left[\frac{n-\beta}{1-\beta}|a_n| + \frac{n+\beta}{1-\beta}|b_n|\right] \leq \sum_{n=1}^{\infty}\left\{\frac{n-\delta_{n,j}\beta}{1-\delta_{1,j}\beta}|a_n| + \frac{n+\delta_{n,j}\beta}{1-\delta_{1,j}\beta}|b_n|\right\},$$

where $\delta_{n,j}$ is given by (1.4),

$$= \sum_{n=1}^{\infty}\left\{\frac{(n-1)k+j-\beta}{1-\delta_{1,j}\beta}|a_{(n-1)k+j}| + \frac{(n-1)k+j+\beta}{1-\delta_{1,j}\beta}|b_{(n-1)k+j}|\right\}$$

$$+\sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}\{|a_n|+|b_n|\} \leq 2.$$

By Lemma 1.5, we conclude that f is sense-preserving, harmonic univalent and starlike in $\mathcal{U}$. To prove $f \in \mathcal{SH}_s^{(j,k)}(\beta)$, according to condition (1.5), we need to show that

$$\Im\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_k(re^{i\theta})}\right\} = \Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h_{j,k}(z) + g_{j,k}(z)}\right\} = \Re\left\{\frac{A(z)}{B(z)}\right\} \geq \beta.$$

Where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$, $0 \leq \beta < 1$ and $k = 1, 2, 3, ...,$ $j = 0, 1, ..., k-1$.

$$A(z) = zh'(z) - \overline{zg'(z)} = z + \sum_{n=2}^{\infty} na_n z^n - \overline{\sum_{n=1}^{\infty} nb_n z^n} \tag{2.5}$$

and

$$B(z) = f_{j,k}(z) = \sum_{n=1}^{\infty} a_n\delta_{n,j} z^n + \overline{\sum_{n=1}^{\infty} \delta_{n,j} b_n z^n}, \tag{2.6}$$

where $\delta_{n,j}$ is defined by (1.4), and $\varepsilon^k = 1$.
Using the fact that $\Re\{w\} \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffices to show that

$$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \geq 0.$$

For $A(z)$ and $B(z)$ as given in (2.5) and (2.6) respectively, we get

$$\begin{aligned}
&|A(z) - (1-\beta)B(z)| - |A(z) - (1-\beta)B(z)| \\
&= |(1-\beta)h_{j,k} + zh'(z) + \overline{(1-\beta)g_{j,k} - zg'(z)}| - |(1+\beta)h_{j,k} - zh'(z) + \overline{(1+\beta)g_{j,k} + zg'(z)}| \\
&= \left|[1 + (1-\beta)\delta_{1,j}]z + \sum_{n=2}^{\infty}[n + (1-\beta)\delta_{n,j}]a_n z^n - \overline{\sum_{n=1}^{\infty}[n - (1-\beta)\delta_{n,j}]b_n z^n}\right| \\
&\quad - \left|[1 - (1+\beta)\delta_{1,j}]z + \sum_{n=2}^{\infty}[n - (1+\beta)\delta_{n,j}]a_n z^n - \overline{\sum_{n=1}^{\infty}[n + (1+\beta)\delta_{n,j}]b_n z^n}\right| \\
&\geq [1 + (1-\beta)\delta_{1,j}]|z| - \sum_{n=2}^{\infty}[n + (1-\beta)\delta_{n,j}]|a_n||z|^n - \sum_{n=1}^{\infty}[n - (1-\beta)\delta_{n,j}]|b_n||z|^n \\
&\quad -[1 - (1+\beta)\delta_{1,j}]|z| - \sum_{n=2}^{\infty}[n - (1+\beta)\delta_{n,j}]|a_n||z|^n - \sum_{n=1}^{\infty}[n + (1+\beta)\delta_{n,j}]|b_n||z|^n \\
&= 2(1-\beta\delta_{1,j})|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\beta\delta_{n,j}}{1-\beta\delta_{1,j}}|a_n||z|^{n-1} - \sum_{n=1}^{\infty}\frac{n+\beta\delta_{n,j}}{1-\beta\delta_{1,j}}|b_n||z|^{n-1}\right\} \\
&\geq 2(1-\delta_{1,j})|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\beta\delta_{n,j}}{1-\beta\delta_{1,j}}|a_n| - \sum_{n=1}^{\infty}\frac{n+\beta\delta_{n,j}}{1-\beta\delta_{1,j}}|b_n|\right\}.
\end{aligned}$$

From the definition of $\delta_{n,j}$ in (1.4), we have

$$\begin{aligned}
&|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \\
&\geq 2(1-\beta\delta_{1,j})|z|\left\{1 - \sum_{n=1}^{\infty}\frac{nk+j-\beta}{1-\beta\delta_{1,j}}|a_{nk+j}| - \sum_{n=1}^{\infty}\frac{nk+j+\beta}{1-\delta_{1,j}\beta}|b_{nk+j}|\right\}
\end{aligned}$$

$$-(1-\beta\delta_{1,j})|z|\left\{\sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}|a_n|+\sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}|b_n|+\frac{1+\beta}{1-\delta_{1,j}\beta}|b_1|\right\}$$

$$\geq 2(1-\beta\delta_{1,j})|z|\left\{1-\sum_{n=1}^{\infty}\left[\frac{(n-1)k+j-\beta}{1-\delta_{1,j}\beta}|a_{(n-1)k+j}|-\frac{(n-1)k+j+\beta}{1-\delta_{1,j}\beta}|b_{(n-1)k+j}|\right]\right\}$$

$$-(1-\beta\delta_{1,j})|z|\left\{\sum_{n\neq lk+j}^{\infty}\frac{n}{1-\delta_{1,j}\beta}\left[|a_n|+|b_n|\right]\right\}\geq 0,$$

we note that in (2.4). This concludes the proof of the theorem. □

For $j = 1$ we get the result introduced by Al-Shaqsi and Darus in [1].

Corollary 2.3. *If* $f = h + \overline{g}$ *with* h *and* g *given by* (1.1) *and* $f_k = h_k + \overline{g_k}$ *. Let*

$$\sum_{n=1}^{\infty}\left\{\frac{(n-1)k+1-\beta}{1-\beta}|a_{(n-1)k+1}|+\frac{(n-1)k+1+\beta}{1-\beta}|b_{(n-1)k+1}|\right\} \tag{2.7}$$

$$+\sum_{n\neq lk+1}^{\infty}\frac{n}{1-\beta}\left\{|a_n|+|b_n|\right\}\leq 2,$$

where $a_1 = 1,\ 0 \leq \beta < 1,\ k = 1, 2, 3, \ldots,\ l \in \mathbb{N}$. *Then* f *is sense-preserving harmonic univalent in* $\mathcal{U}$ *and* $f \in \mathcal{SH}_s^{(k)}(\beta)$.

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On close-to-convex functions satisfying a differential inequality

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Abstract. Let $\mathcal{C}_\alpha(\beta)$ denote the class of normalized functions f, analytic in the open unit disk $\mathbb{E}$ which satisfy the condition

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\frac{(zf'(z))'}{\phi'(z)}\right]>\beta,\ z\in\mathbb{E},$$

where $\dfrac{f(z)f'(z)}{z}\neq 0,\ z\in\mathbb{E}$, ϕ is starlike and $\alpha,\ \beta$ are pre-assigned real numbers. In 1977, Chichra, P. N. [1] introduced and studied the class $\mathcal{C}_\alpha=\mathcal{C}_\alpha(0)$. He proved the members of class $\mathcal{C}_\alpha$ are close-to-convex for $\alpha\geq 0$. We here prove that functions in class $\mathcal{C}_\alpha(\beta)$ are close-to-convex for $-\dfrac{\alpha}{2}\Re\left(\dfrac{z\phi'(z)}{\phi(z)}\right)\leq\beta<1,\ \alpha\geq 0$ and the result is sharp in the sense that the constant β cannot be replaced by a real number smaller than $-\dfrac{\alpha}{2}\Re\left(\dfrac{\phi(z)}{z\phi'(z)}\right)$. We claim that our result improves the result of Chichra, P. N. [1].

Mathematics Subject Classification (2010): 30C80, 30C45.

Keywords: Analytic function, convex function, starlike function, close-to-convex.

1. Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E}=\{z:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the classes of starlike and convex functions respectively analytically defined as follows:

$$\mathcal{S}^*=\left\{f\in\mathcal{A}:\Re\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E}\right\},$$

and

$$\mathcal{K}=\left\{f\in\mathcal{A}:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ z\in\mathbb{E}\right\}.$$

This is well-known that

$$f(z)\in\mathcal{K}\Leftrightarrow zf'(z)\in\mathcal{S}^*. \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be close to convex if there is a real number $\alpha, -\pi/2 < \alpha < \pi/2$ and a convex function g (not necessarily normalized) such that

$$\Re\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0,\ z \in \mathbb{E}.$$

In view of the relation (1.1), the above definition takes the following form in case g is normalized. A function $f \in \mathcal{A}$ is said to be close to convex if there is a real number $\alpha, -\pi/2 < \alpha < \pi/2$, and a starlike function ϕ such that

$$\Re\left(e^{i\alpha}\frac{zf'(z)}{\phi(z)}\right) > 0,\ z \in \mathbb{E}.$$

It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [3] and Warchawski [4] obtained a simple but elegant criterion for univalence of analytic functions. They proved that if an analytic function f satisfies $\Re f'(z) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.
Let $\mathcal{C}_\alpha(\beta)$ denote the class of normalized analytic functions f which satisfy the condition

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right] > \beta,\ z \in \mathbb{E},$$

where $\dfrac{f(z)f'(z)}{z} \neq 0,\ z \in \mathbb{E}$, ϕ is starlike and $\alpha,\ \beta$ are pre-assigned real numbers. The class $\mathcal{C}_\alpha = \mathcal{C}_\alpha(0)$ was introduced and studied by Chichra, P. N. [1] in 1977. He called the members of class $\mathcal{C}_\alpha$ as $\alpha-$ close-to-convex functions. Infact, he proved the following result.

Theorem 1.1. *Let $f(z) \in \mathcal{C}_\alpha$ and $\alpha \geq 0$. Then $f(z)$ is close-to-convex in $\mathbb{E}$.*

In the present paper, we establish the result that functions in $\mathcal{C}_\alpha(\beta)$ are close-to-convex for $-\dfrac{\alpha}{2}\Re\left(\dfrac{z\phi'(z)}{\phi(z)}\right) \leq \beta < 1,\ \alpha \geq 0$. Our result is the best possible in the sense that the constant β cannot be replaced by a real number smaller than $-\dfrac{\alpha}{2}\Re\left(\dfrac{\phi(z)}{z\phi'(z)}\right)$. We also claim that our result improves the result of Chichra, P. N. [1]. To prove our main result, we shall use the following lemma of Miller [2].

Lemma 1.2. *Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let ϕ satisfy the following conditions:*
(i) $\phi(u, v)$ is continuous in $\mathbb{D}$
(ii) $(1, 0) \in \mathbb{D}$ and $\Re[\phi(1, 0)] > 0$ and
(iii) $\Re[\phi(iu_2, v_1)] \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -(1 + u_2^2)/2$.
Let $p(z) = 1 + p_1z + p_2z^2 + \cdots$ be regular in the open unit disk $\mathbb{E}$, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If

$$\Re[\phi(p(z), zp'(z))] > 0,\ z \in \mathbb{E},$$

then $\Re p(z) > 0,\ z \in \mathbb{E}$.

2. Main result

Theorem 2.1. *Let α and β be real numbers such that $\alpha \geq 0$ and*

$$-\frac{\alpha}{2}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right) \leq \beta < 1$$

for a starlike function ϕ. Assume that $f \in \mathcal{A}$ satisfies

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right] > \beta, \ z \in \mathbb{E}, \tag{2.1}$$

then $\Re\left(\frac{zf'(z)}{\phi(z)}\right) > 0$ in $\mathbb{E}$ and hence f is close-to-convex and hence univalent in $\mathbb{E}$. The result is sharp in the sense that the constant β on the right hand side of (2.1) *cannot be replaced by a real number smaller than $-\frac{\alpha}{2}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right)$.*

Proof. Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in $\mathbb{E}$ such that for all $z \in \mathbb{E}$, we write

$$\frac{zf'(z)}{\phi(z)} = p(z).$$

Then,

$$(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)} = p(z) + \alpha z p'(z)\frac{\phi(z)}{z\phi(z)}.$$

Therefore, condition (2.1) is equivalent to

$$\Re\left(\frac{1}{1-\beta}p(z) + \frac{\alpha}{1-\beta}zp'(z)\frac{\phi(z)}{z\phi'(z)} - \frac{\beta}{1-\beta}\right) > 0, \ z \in \mathbb{E}. \tag{2.2}$$

For $\mathbb{D} = \mathbb{C} \times \mathbb{C}$, define $\Phi(u,v) : \mathbb{D} \to \mathbb{C}$ as under:

$$\Phi(u,v) = \frac{1}{1-\beta}u + \frac{\alpha}{1-\beta}v\frac{\phi(z)}{z\phi'(z)} - \frac{\beta}{1-\beta}, \ z \in \mathbb{E}.$$

Then $\Phi(u,v)$ is continuous in $\mathbb{D}$, $(1,0) \in \mathbb{D}$ and $\Re\left(\Phi(1,0)\right) = 1 > 0$. Further, in view of (2.2), we get, $\Re[\Phi(p(z), zp'(z))] > 0, \ z \in \mathbb{E}$. Let $u = u_1 + iu_2, v = v_1 + iv_2$ where u_1, u_2, v_1 and v_2 are all real numbers. Then, for $(iu_2, v_1) \in \mathbb{D}$, with $v_1 \leq -\frac{1+u_2^2}{2}$, we have

$$\begin{aligned}
\Re\Phi(iu_2, v_1) &= \Re\left(\frac{1}{1-\beta}u_2 i + \frac{\alpha}{1-\beta}v_1\frac{\phi(z)}{z\phi'(z)} - \frac{\beta}{1-\beta}\right) \\
&\leq -\left[\frac{\alpha}{1-\beta}\frac{1+u_2^2}{2}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right) + \frac{\beta}{1-\beta}\right] \\
&\leq -\left[\frac{\alpha}{2(1-\beta)}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right) + \frac{\beta}{1-\beta}\right] \\
&\leq 0.
\end{aligned}$$

In view of (2.2) and Lemma 1.2, proof now follows.

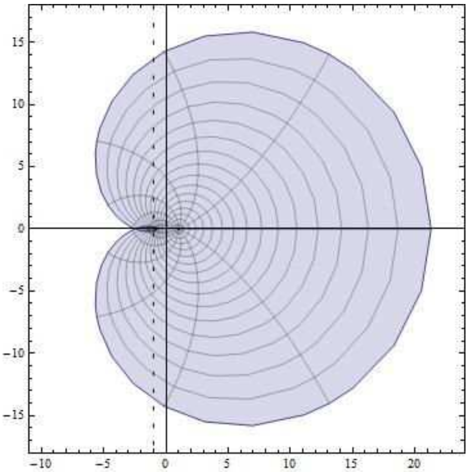

Figure 2.1

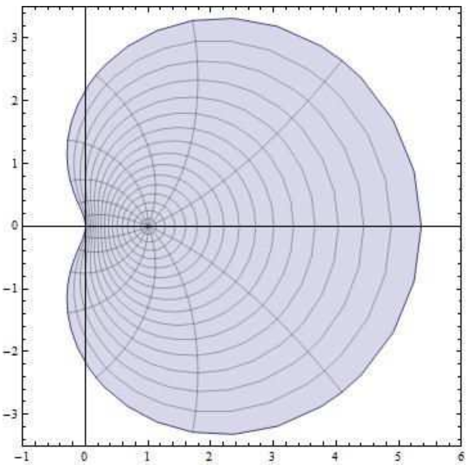

Figure 2.2

To show that the constant β on the right hand side of (2.1) cannot be replaced by a real number smaller than $-\dfrac{\alpha}{2}\Re\left(\dfrac{\phi(z)}{z\phi'(z)}\right)$, we consider the function $f(z) = z\,e^z \in \mathcal{A}$ and $\phi(z) = z \in \mathcal{S}^*$. Using Mathematica 9.0, we plot, in Figure 2.1, the image of the unit disk under the operator $(1-\alpha)\dfrac{zf'(z)}{\phi(z)} + \alpha\dfrac{(zf'(z))'}{\phi'(z)}$ taking $\alpha = 2$. From this figure, we notice that minimum real part of $(1-\alpha)\dfrac{zf'(z)}{\phi(z)} + \alpha\dfrac{(zf'(z))'}{\phi'(z)}$ is smaller than -1 (the calculated value of $-\frac{\alpha}{2}\Re\left(\frac{\phi(z)}{z\phi'(z)}\right)$ for $\alpha = 2$ and $\phi(z) = z$). In Figure 2.2, we plot the image of unit disk under the operator $\frac{zf'(z)}{\phi(z)}$. It is obvious that $\Re\left(\frac{zf'(z)}{\phi(z)}\right) \not> 0$ for all z in $\mathbb{E}$. For example, the point $z = -\frac{1}{2} + i\frac{\pi}{4}$ is an interior point of $\mathbb{E}$, but at this point $\Re\left(\frac{zf'(z)}{\phi(z)}\right) = -\frac{\pi - 2}{4\sqrt{2e}} = -0.1224\cdots < 0$. This justifies our claim. □

Remark 2.2. We claim that our result improves the result of Chichra, P. N. [1]. In fact, when we take $f(z) = -z - 2\log(1 - z) \in \mathcal{A}$, $\phi(z) = z$ and $\alpha = 2$ in Theorem 2.1, we notice that at $z = -1$,

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right] = -1.$$

Thus the function f does not satisfy the hypothesis of Theorem 1.1 due to Chichra, P. N. [1] i.e. $f \notin \mathcal{C}_\alpha$ although $\Re\left(\frac{zf'(z)}{\phi(z)}\right) = \Re\left(\frac{1+z}{1-z}\right) > 0$ in $\mathbb{E}$. Hence the result of Chichra, P. N. [1] fails to conclude the close-to-convexity in this case whereas Theorem 2.1 concludes the same.

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Generalized and numerical solution for a quasilinear parabolic equation with nonlocal conditions

Fatma Kanca and Irem Baglan

Abstract. In this paper we study the one dimensional mixed problem with nonlocal boundary conditions, for the quasilinear parabolic equation. We prove an existence, uniqueness of the weak generalized solution and also continuous dependence upon the data of the solution are shown by using the generalized Fourier method. We construct an iteration algorithm for the numerical solution of this problem. We analyze computationally convergence of the iteration algorithm, as well as on test example.

Mathematics Subject Classification (2010): 35K55.

Keywords: Quasilinear parabolic equation, nonlocal boundary condition, finite difference method.

1. Introduction

In this study, we consider the following mixed problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t,u), \; D := \{0 < x < 1, 0 < t < T\} \tag{1.1}$$

$$u(0,t) = u(1,t), \qquad t \in [0,T] \tag{1.2}$$

$$u_x(1,t) = 0, \qquad t \in [0,T] \tag{1.3}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,1] \tag{1.4}$$

for a quasilinear parabolic equation with the nonlinear source term $f = f(x,t,u)$.

The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,1]$ and $\bar{D} \times (-\infty, \infty)$, respectively.

Denote the solution of the problem (1.1)-(1.4) by $u = u(x,t)$.

This problem was investigated with different boundary conditions by various researchers by using Fourier or different methods [2, 4].

In this study, we consider the initial-boundary value problem (1.1)-(1.4) with nonlocal boundary conditions (1.2)-(1.3). The periodic nature of (1.2)-(1.3) type boundary conditions is demonstrated in [10]. In this study, we prove the existence, uniqueness, convergence of the weak generalized solution continuous dependence upon the data of the solution and we construct an iteration algorithm for the numerical solution of this problem. We analyze computationally convergence of the iteration algorithm, as well as on test example. We demonstrate a numerical procedure for this problem on concrete examples, and also we obtain numerical solution by using the implicit finite difference algorithm [11].

We will use the weak solution approach from [3] for the considered problem (1.1)-(1.4).

According to [1, 5] assume the following definitions.

Definition 1.1. *The function $v(x,t) \in C^2(\bar{D})$ is called test function if it satisfies the following conditions:*

$$v(x,T) = 0,\ v(0,t) = v(1,t),\ v_x(1,t) = 0, \forall t \in [0,T] \text{ and } \forall x \in [0,1].$$

Definition 1.2. *The function $u(x,t) \in C(\bar{D})$ satisfying the integral identity*

$$\int_0^T \int_0^1 \left[\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} \right) u - f(x,t,u)v \right] dxdt$$

$$- \int_0^T \left[u(0,t)v_x(0,t) - v(0,t)u_x(0,t) \right] dt - \int_0^1 \varphi(x)v(x,0)dx = 0, \tag{1.5}$$

for arbitrary test function $v = v(x,t)$, is called a generalized (weak) solution of the problem (1)-(4).

2. Reducing the problem to countable system of integral equations

Consider the following system of functions on the interval $[0,1]$:

$$X_0(x) = 2,\ X_{2k-1}(x) = 4\cos(2\pi kx),\ X_{2k}(x) = 4(1-x)\sin(2\pi kx),\ k = 1,2,\ldots, \tag{2.1}$$

$$Y_0(x) = x,\ Y_{2k-1}(x) = x\cos(2\pi kx),\ Y_{2k}(x) = \sin(2\pi kx),\ k = 1,2,\ldots \tag{2.2}$$

The system of functions (2.1) and (2.2) arise in [6] for the solution of a nonlocal boundary value problem in heat conduction.

It is easy to verify that the system of function (2.1) and (2.2) are biorthonormal on $[0,1]$. They are also Riesz bases in $L_2[0,1]$ (see [7, 8]).

We will use the Fourier series representation of the weak solution to transform the initial-boundary value problem to the infinite set of nonlinear integral equations.

Any solution of the equation (1.1)-(1.4) can be represented as

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x), \tag{2.3}$$

where the functions $u_k(t)$, $k = 0, 1, 2, \ldots$ satisfy the following system of equations:

$$\begin{aligned} u_0(t) &= \varphi_0 + \int_0^t f_0(\tau) d\tau, \\ u_{2k}(t) &= \varphi_{2k}\, e^{-(2\pi k)^2 t} + \int_0^t f_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau)} d\tau, \\ u_{2k-1}(t) &= (\varphi_{2k-1} - 4\pi k \varphi_{2k}) e^{-(2\pi k)^2 t} \\ &\quad + \int_0^t e^{-(2\pi k)^2 (t-\tau)} [f_{2k-1}(\tau) - 4\pi k (t-\tau) f_{2k}(\tau)] d\tau, \end{aligned} \tag{2.4}$$

where

$$\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx,$$

$$f_k(x) = \int_0^1 f(x,t,u) Y_k(x) dx.$$

Definition 2.1. *Denote the set*

$$\{u(t)\} = \{u_0(t), u_{2k}(t), u_{2k-1}(t),\ k = 1, 2, \ldots, \},$$

of continuous on $[0, T]$ *satisfying the following condition*

$$underset 0 \leq t \leq T \max 2\,|u_0(t)| + 4 \sum_{k=1}^{\infty} \left(\max_{0 \leq t \leq T} |u_{2k}(t)| + \max_{0 \leq t \leq T} |u_{2k-1}(t)| \right) < \infty,$$

by B*. Let*

$$\|u(t)\| = \max_{0 \leq t \leq T} 2\,|u_0(t)| + 4 \sum_{k=1}^{\infty} \left(\max_{0 \leq t \leq T} |u_{2k}(t)| + \max_{0 \leq t \leq T} |u_{2k-1}(t)| \right),$$

be the norm in B*. It can be shown that* B *is the Banach space* [9].

We denote the solution of the nonlinear system (2.4) by $\{u(t)\}$.

Theorem 2.2. *a) Let the function* $f(x,t,u)$ *is continuous with respect to all arguments in* $\bar{D} \times (-\infty, \infty)$ *and satisfies the following condition*

$$|f(x,t,u) - f(x,t,\tilde{u})| \leq b(x,t)\,|u - \tilde{u}|,$$

where $b(x,t) \in L_2(D)$, $b(x,t) \geq 0$,

b) $f(x,t,0) \in C^2[0,1]$, $t \in [0,1]$,

c) $\varphi(x) \in C^2[0,1]$.

Then the system (2.4) has a unique solution in D*.*

Proof. For $N = 0, 1, \ldots$ let's define an iteration for the system (2.4) as follows:

$$\begin{aligned} u_0^{(N+1)}(t) &= u_0^{(0)}(t) + \int_0^t \int_0^1 f(\xi, \tau, Au^{(N)}(\xi, \tau))\xi d\xi d\tau, \\ u_{2k}^{(N+1)}(t) &= u_{2k}^{(0)}(t) + \int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi, \tau, Au^{(N)}(\xi, \tau)) \sin 2\pi k\xi d\xi d\tau, \\ u_{2k-1}^{(N+1)}(t) &= u_{2k-1}^{(0)}(t) + \int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi, \tau, Au^{(N)}(\xi, \tau))\xi \cos 2\pi k\xi d\xi d\tau \\ &\quad -4\pi k \int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi, \tau, Au^{(N)}(\xi, \tau))(t-\tau) \sin 2\pi k\xi d\xi d\tau, \end{aligned} \tag{2.5}$$

where, for simplicity, let

$$Au^{(N)}(\xi, \tau) = 2u_0^{(N)}(\tau) + 4\sum_{k=1}^{\infty} \left(u_{2k}^{(N)}(\tau)(1-\xi) \sin 2\pi k\xi + u_{2k-1}^{(N)}(\tau) \cos 2\pi k\xi \right).$$

where,

$$u_0^{(0)}(t) = \varphi_0 \;, u_{2k}^{(0)}(t) = \varphi_{2k}\; e^{-(2\pi k)^2 t}, u_{2k-1}^{(0)}(t) = (\varphi_{2k-1} \; - 4\pi k\; \varphi_{2k}\;)\; e^{-(2\pi k)^2 t}.$$

From the condition of the theorem we have $u^{(0)}(t) \in B$. We will prove that the other sequentially approximations satisfy this condition.

Let us write $N = 0$ in (2.5).

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \int_0^t \int_0^1 f(\xi, \tau, Au^{(0)}(\xi, \tau)) d\xi d\tau.$$

Adding and subtracting $\int_0^t \int_0^1 f(\xi, \tau, 0) d\xi d\tau$, applying Cauchy inequality, Lipschitz condition, taking the maximum of both sides of the last inequality yields the following:

$$\max_{0 \le t \le T} \left| u_0^{(1)}(t) \right| \le |\varphi_0| + \sqrt{T}\, \|b(x,t)\|_{L_2(D)} \left\| u^{(0)}(t) \right\| + \sqrt{T}\, \|f(x,t,0)\|_{L_2(D)}.$$

$$u_{2k}^{(1)}(t) = \varphi_{2k} e^{-(2\pi k)^2 t} + \int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi, \tau, Au^{(N)}(\xi, \tau)) \sin 2\pi k\xi d\xi d\tau.$$

Adding and subtracting $\int_0^t \int_0^1 e^{-(2\pi k)^2(t-\tau)} f(\xi, \tau, 0) \sin 2\pi k\xi d\xi d\tau$, applying Cauchy inequality, taking the summation of both sides respect to k and using Hölder inequality, Bessel inequality, Lipschitz condition and taking maximum of both sides of the last inequality yields the following:

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}|\varphi_{2k}| + \frac{1}{4\sqrt{3}}\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\| + \frac{1}{4\sqrt{3}}\|f(x,t,0)\|_{L_2(D)}.$$

In the same way, we obtain:

$$\begin{aligned}\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right| \le{}& \sum_{k=1}^{\infty}|\varphi_{2k-1}| + \frac{1}{\sqrt{6}}\sum_{k=1}^{\infty}\left|\varphi_{2k}^{''}\right| \\ &+ \frac{1}{4\sqrt{3}}\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\| + \frac{1}{4\sqrt{3}}\|f(x,t,0)\|_{L_2(D)} \\ &+ \sqrt{2}\,|T|\,\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\| + \sqrt{2}\,|T|\,\|f(x,t,0)\|_{L_2(D)}.\end{aligned}$$

Finally we have the following inequality:

$$\begin{aligned}\left\|u^{(1)}(t)\right\|_B ={}& 2\max_{0\le t\le T}\left|u_0^{(1)}(t)\right| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right| + \max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right|\right) \\ \le{}& 2|\varphi_0| + 4\sum_{k=1}^{\infty}(|\varphi_{2k}| + |\varphi_{2k-1}|) + \frac{2\sqrt{6}\,|T|}{3}\sum_{k=1}^{\infty}\left|\varphi_{2k}^{''}\right| \\ &+ \left(2\sqrt{T} + \frac{2\sqrt{3}}{3} + 4\sqrt{2}\,|T|\right)\left(\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_B\right) \\ &+ \left(2\sqrt{T} + \frac{2\sqrt{3}}{3} + 4\sqrt{2}\,|T|\right)\|f(x,t,0)\|_{L_2(D)}.\end{aligned}$$

Hence $u^{(1)}(t) \in B$. In the same way, for a general value of N we have

$$\begin{aligned}\left\|u^{(N)}(t)\right\|_B ={}& 2\max_{0\le t\le T}\left|u_0^{(N)}(t)\right| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(N)}(t)\right| + \max_{0\le t\le T}\left|u_{2k-1}^{(N)}(t)\right|\right) \\ \le{}& 2|\varphi_0| + 4\sum_{k=1}^{\infty}(|\varphi_{2k}| + |\varphi_{2k-1}|) + \frac{2\sqrt{6}\,|T|}{3}\sum_{k=1}^{\infty}\left|\varphi_{2k}^{''}\right| \\ &+ \left(2\sqrt{T} + \frac{2\sqrt{3}}{3} + 4\sqrt{2}\,|T|\right)\left(\|b(x,t)\|_{L_2(D)}\left\|u^{(N-1)}(t)\right\|_B\right) \\ &+ \left(2\sqrt{T} + \frac{2\sqrt{3}}{3} + 4\sqrt{2}\,|T|\right)\|f(x,t,0)\|_{L_2(D)},\end{aligned}$$

$u^{(N-1)}(t) \in B$, we deduce that $u^{(N)}(t) \in B$, we obtain

$$\{u(t)\} = \{u_0(t), u_{2k}(t), u_{2k-1}(t),\ k = 1,2,\ldots\} \in B.$$

Now we prove that the iterations $u^{(N+1)}(t)$ converge in B, as $N \to \infty$.

$$u^{(1)}(t) - u^{(0)}(t) = 2(u_0^{(1)}(t) - u_0^{(0)}(t)) + 4\sum_{k=1}^{\infty}[(u_{2k}^{(1)}(t) - u_{2k}^{(0)}(t)) + (u_{2k-1}^{(1)}(t) - u_{2k-1}^{(0)}(t))]$$

$$= 2\int_0^t\int_0^1 \left[f(\xi,\tau,Au^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]\xi d\xi d\tau$$

$$+4\int_0^t\int_0^1 \left[f(\xi,\tau,Au^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right] e^{-(2\pi k)^2(t-\tau)}\sin 2\pi k\xi d\xi d\tau$$

$$+4\int_0^t\int_0^1 \left[f(\xi,\tau,Au^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right] e^{-(2\pi k)^2(t-\tau)}\xi\cos 2\pi k\xi d\xi d\tau$$

$$-16\pi k\int_0^t\int_0^1 (t-\tau)\left[f(\xi,\tau,Au^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right] e^{-(2\pi k)^2(t-\tau)}\sin 2\pi k\xi d\xi d\tau.$$

Applying Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right side of $u^{(1)}(t) - u^{(0)}(t)$ respectively, we obtain:

$$\left|u^{(1)}(t) - u^{(0)}(t)\right| \le 2\left|u_0^{(1)}(t) - u_0^{(0)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(1)}(t) - u_{2k}^{(0)}(t)\right| + \left|u_{2k-1}^{(1)}(t) - u_{2k-1}^{(0)}(t)\right|\right)$$

$$\le \left(2\sqrt{T} + 4\sqrt{2}\left|T\right| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\left|u^{(0)}(t)\right|$$

$$+\left(2\sqrt{T} + 4\sqrt{2}\left|T\right| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}},$$

$$A_T = \left[\left(2\sqrt{T} + 4\sqrt{2}\left|T\right| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\left|u^{(0)}(t)\right|\right.$$

$$\left.+\left(2\sqrt{T} + 4\sqrt{2}\left|T\right| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}\right].$$

Applying Cauchy unequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right hand side of $u^{(2)}(t) - u^{(1)}(t)$ respectively, we obtain:

$$\left|u^{(2)}(t) - u^{(1)}(t)\right| \le 2\left|u_0^{(2)}(t) - u_0^{(1)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(2)}(t) - u_{2k}^{(1)}(t)\right| + \left|u_{2k-1}^{(2)}(t) - u_{2k-1}^{(1)}(t)\right|\right)$$

$$\le \left(2\sqrt{T} + 4\sqrt{2}\left|T\right| + \frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}} A_T.$$

In the same way, for a general value of N we have

$$\left|u^{(N+1)}(t)-u^{(N)}(t)\right| \leq 2\left|u_0^{(N+1)}(t)-u_0^{(N)}(t)\right|$$
$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(N+1)}(t)-u_{2k}^{(N)}(t)\right|+\left|u_{2k-1}^{(N+1)}(t)-u_{2k-1}^{(N)}(t)\right|\right)$$
$$\leq (2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3})^N \frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}$$
$$\leq (2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3})^N A_T\frac{1}{\sqrt{N!}}\left\|b(x,t)\right\|_{L_2(D)}^N . \tag{2.6}$$

Then the last inequality shows that the $u^{(N+1)}(t)$ convergence in B.

Now let us show $\lim\limits_{N\to\infty} u^{(N+1)}(t)=u(t)$. It follows that if we prove

$$\lim_{N\to\infty}\left\|u(\tau)-u^{(N)}(\tau)\right\|_B=0,$$

then we may deduce that $u(t)$ satisfies (2.4). For this aim we estimate the difference $\left\|u(t)-u^{(N+1)}(t)\right\|_B$, after some transformation we obtain:

$$\left|u(t)-u^{(N+1)}(t)\right| = 2\left|u_0(t)-u_0^{(N+1)}(t)\right|$$
$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}(t)-u_{2k}^{(N+1)}(t)\right|+\left|u_{2k-1}(t)-u_{2k-1}^{(N+1)}(t)\right|\right)$$
$$\leq 2\left|\int_0^t\int_0^1\left\{f[\xi,\tau,Au(\xi,\tau)]-f[\xi,\tau,Au^{(N)}(\xi,\tau)]\right\}\xi d\xi d\tau\right|$$
$$+4\left|\sum_{k=1}^{\infty}\int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\left\{f[\xi,\tau,Au(\xi,\tau)]-f[\xi,\tau,Au^{(N)}(\xi,\tau)]\right\}\sin 2\pi k\xi d\xi d\tau\right|$$
$$+4\left|\sum_{k=1}^{\infty}\int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\left\{f[\xi,\tau,Au(\xi,\tau)]-f[\xi,\tau,Au^{(N)}(\xi,\tau)]\right\}\xi\cos 2\pi k\xi d\xi d\tau\right|$$
$$+16\pi k\left|\int_0^t\int_0^1(t-\tau)\left[f(\xi,\tau,Au(\xi,\tau))-f(\xi,\tau,Au^{(N)}(\xi,\tau))\right]\right.$$
$$\left.\cdot e^{-(2\pi k)^2(t-\tau)}\sin 2\pi k\xi d\xi d\tau\right|.$$

Adding and subtracting $f(\xi,\tau,Au^{(N+1)}(\xi,\tau))$ under appropriate integrals to the right hand side of the inequality we obtain

$$\left|u(t)-u^{(N+1)}(t)\right|\leq\left(2\sqrt{T}+4\sqrt{2}|T|+\frac{2\sqrt{3}}{3}\right)\left\{\int_0^t\int_0^1 b^2(\xi,\tau)\left|u(\tau)-u^{(N+1)}(\tau)\right|^2 d\xi d\tau\right\}^{\frac{1}{2}}$$

$$+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\left\{\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right\}^{\frac{1}{2}}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B.$$

Applying Gronwall's inequality to the last inequality and using inequality (2.6), we have

$$\left\|u(t)-u^{(N+1)}(t)\right\|_B\leq\sqrt{\frac{2}{N!}}A_T\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^{(N+1)}\|b(x,t)\|_{L_2(D)}^{(N+1)}$$

$$\times\exp\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^2\|b(x,t))\|_{L_2(D)}^2. \tag{2.7}$$

For the uniqueness, we assume that the problem (1.1)-(1.4) has two solutions u,v. Applying Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right hand side of $|u(t)-v(t)|$ respectively, we obtain:

$$|u(t)-v(t)|^2\leq\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^2\int_0^t\int_0^1 b^2(\xi,\tau)\,|u(\tau)-v(\tau)|^2\,d\xi d\tau,$$

applying Gronwall's inequality to the last inequality we have $u(t)=v(t)$.
The theorem is proved. □

3. Solution of the problem (1.1)-(1.4)

Using the solution of the system (2.4) we compose the series

$$2u_0(t)+4\sum_{k=1}^{\infty}[u_{2k}(t)(1-x)sin2\pi kx+u_{2k-1}(t)cos2\pi kx].$$

It is evident that these series convergence uniformly on D. Therefore the sum

$$u(\xi,\tau)=2u_0(\tau)+4\sum_{k=1}^{\infty}[u_{2k}(\tau)(1-\xi)sin2\pi k\xi+u_{2k-1}(\tau)cos2\pi k\xi],$$

continuous on D.

$$u_l(\xi,\tau)=2u_0(\tau)+4\sum_{k=1}^{l}[u_{2k}(\tau)(1-\xi)sin2\pi k\xi+u_{2k-1}(\tau)cos2\pi k\xi]. \tag{3.1}$$

From the conditions of Theorem 2.2 and from

$$\lim_{l\to\infty}u_l(\xi,\tau)=u(\xi,\tau),$$

it follows

$$\lim_{l\to\infty} f(\xi,\tau,u_l(\tau,\xi)) = f(\xi,\tau,u(\xi,\tau)).$$

Using $u_l(\xi,\tau)$ and

$$\varphi_l(x) = 2\varphi_0 + 4\sum_{k=1}^{l}[\varphi_{2k}(1-x)sin2\pi kx + \varphi_{2k-1}cos2\pi kx],\ x\in[0,1]$$

on the left hand side of (1.5) we denote the obtained expression by J$_l$:

$$J_l = \int_0^T\int_0^1 \left[\left(\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2}\right) u_{(l)}(x,t) + f(x,t,u_{(l)}(x,t))v(x,t)\right] dxdt$$
$$+ \int_0^1 \varphi_{(l)}(x)v(x,0)dx. \qquad (3.2)$$

Applying the integration by part formula to the right hand side the last equation and using the conditions of Theorem 2.2 , we can show that

$$\lim_{l\to\infty} J_l = 0.$$

This shows that the function $u(x,t)$ is a generalized(weak) solution of the problem (1.1)-(1.4).
The following theorem shows the existence and uniqueness results for the generalized solutions of problem (1.1)-(1.4).

Theorem 3.1. *Under the assumptions of Theorem 2.2, Problem (1.1)-(1.4) possesses a unique generalized solution* $u = u(x,t) \in C(\overline{D})$ *in the following form*

$$u(x,t) = 2u_0(t) + 4\sum_{k=1}^{\infty}[u_{2k}(t)(1-x)sin2\pi kx + u_{2k-1}(t)cos2\pi kx].$$

4. Continuous dependence upon the data

In this section, we shall prove the continuous dependence of the solution

$$u = u(x,t)$$

using an iteration method.

Theorem 4.1. *Under the conditions of Theorem 2.2, the solution* $u = u(x,t)$ *depends contiunously upon the data.*

Proof. Let $\phi = \{\varphi, f\}$ and $\overline{\phi} = \left\{\overline{\varphi}, \overline{f}\right\}$ be two sets of data which satisfy the conditions of Theorem 1.Let $u = u(x,t)$ and $v = v(x,t)$ be the solutions of the problem (1.1)-(1.4) corresponding to the data ϕ and $\overline{\phi}$ respectively and

$$\left|f(t,x,0) - \overline{f}(t,x,0)\right| \le \varepsilon,\quad for\ \varepsilon \ge 0.$$

The solution $v=v(x,t)$ is in the following form

$$v_0(t)=\overline{\varphi}_0+\int_0^t \overline{f}_0(\tau)d\tau,$$

$$v_{2k}(t)=\overline{\varphi}_{2k}\, e^{-(2\pi k)^2 t}+\int_0^t \overline{f}_{2k}(\tau)e^{-(2\pi k)^2(t-\tau)}d\tau,$$

$$v_{2k-1}(t)=(\overline{\varphi}_{2k-1}-4\pi k t\overline{\varphi}_{2k})e^{-(2\pi k)^2 t}$$

$$+\int_0^t e^{-(2\pi k)^2(t-\tau)}[\overline{f}_{2k-1}(\tau)-4\pi k(t-\tau)\overline{f}_{2k}(\tau)]d\tau,$$

where, for simplicity, let

$$Av^{(N)}(\xi,\tau)=2v_0^{(N)}(\tau)+4\sum_{k=1}^{\infty}\left(v_{2k}^{(N)}(\tau)(1-\xi)\sin 2\pi k\xi+v_{2k-1}^{(N)}(\tau)\cos 2\pi k\xi\right)$$

$$v_0^{(N+1)}(t)=v_0^{(0)}(t)+\int_0^t\int_0^1 \overline{f}(\xi,\tau,Av^{(N)}(\xi,\tau))\xi d\xi d\tau,$$

$$v_{2k}^{(N+1)}(t)=v_{2k}^{(0)}(t)+\int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\overline{f}(\xi,\tau,Av^{(N)}(\xi,\tau))\sin 2\pi k\xi d\xi d\tau,$$

$$v_{2k-1}^{(N+1)}(t)=v_{2k-1}^{(0)}(t)+\int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\overline{f}(\xi,\tau,Av^{(N)}(\xi,\tau))\xi\cos 2\pi k\xi d\xi d\tau$$

$$-4\pi k\int_0^t\int_0^1 e^{-(2\pi k)^2(t-\tau)}\overline{f}(\xi,\tau,Av^{(N)}(\xi,\tau))(t-\tau)\sin 2\pi k\xi d\xi d\tau,$$

where

$$v_0^{(0)}(t)=\overline{\varphi}_0\ , v_{2k}^{(0)}(t)=\overline{\varphi}_{2k}\, e^{-(2\pi k)^2 t}, u_{2k-1}^{(0)}(t)=(\overline{\varphi}_{2k-1}\ -4\pi k\overline{\varphi}_{2k}\ )\ e^{-(2\pi k)^2 t}.$$

From the condition of the theorem we have $v^{(0)}(t)\in B$. We will prove that the other sequentially approximations satisfy this condition.

First of all , we consider $u^{(1)}(t)-v^{(1)}(t)$, applying Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the $\left|u^{(1)}(t)-v^{(1)}(t)\right|$ respectively, we obtain:

$$\left|u^{(1)}(t)-v^{(1)}(t)\right|\le 2\left|u_0^{(1)}(t)-v_0^{(1)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(1)}(t)-v_{2k}^{(1)}(t)\right|+\left|u_{2k-1}^{(1)}(t)-v_{2k-1}^{(1)}(t)\right|\right)\le 2\max|\varphi_0-\overline{\varphi_0}|$$

$$+4\sum_{k=1}^{\infty}\max|\varphi_{2k}-\overline{\varphi_{2k}}|+\max|\varphi_{2k-1}-\overline{\varphi_{2k-1}}|+\frac{2\sqrt{6}\,|T|}{3}\sum_{k=1}^{\infty}\max\left|\varphi_{2k}^{''}-\overline{\varphi_{2k}^{''}}\right|$$

$$\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\left|\overline{u}^{(0)}(t)\right|$$

$$+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 \overline{b}^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\left|\overline{v}^{(0)}(t)\right|$$

$$+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)-\overline{f}^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}},$$

$$A_T=\|\varphi-\overline{\varphi}\|+\left[\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\|b(x,t)\|\left|u^{(0)}(t)\right|\right.$$

$$\left.+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\|b(x,t)\|\left|\overline{v}^{(0)}(t)\right|\right]+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\left\|f-\overline{f}\right\|.$$

$$\|\varphi-\overline{\varphi}\|=2\max|\varphi_0-\overline{\varphi_0}|$$

$$+4\sum_{k=1}^{\infty}\max|\varphi_{2k}-\overline{\varphi_{2k}}|+\max|\varphi_{2k-1}-\overline{\varphi_{2k-1}}|+\frac{2\sqrt{6}\,|T|}{3}\sum_{k=1}^{\infty}\max\left|\varphi_{2k}''-\overline{\varphi_{2k}''}\right|.$$

Applying Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right hand side of $u^{(2)}(t)-v^{(2)}(t)$ respectively, we obtain:

$$\left|u^{(2)}(t)-v^{(2)}(t)\right|\le 2\left|u_0^{(2)}(t)-v_0^{(2)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(2)}(t)-v_{2k}^{(2)}(t)\right|+\left|u_{2k-1}^{(2)}(t)-v_{2k-1}^{(2)}(t)\right|\right)$$

$$\le\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}A_T$$

$$+\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)\left(\int_0^t\int_0^1 \overline{b}^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}A_T.$$

In the same way, for a general value of N we have

$$\left|u^{(N+1)}(t)-v^{(N+1)}(t)\right|\le 2\left|u_0^{(N+1)}(t)-v_0^{(N+1)}(t)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(N+1)}(t)-v_{2k}^{(N+1)}(t)\right|+\left|u_{2k-1}^{(N+1)}(t)-v_{2k-1}^{(N+1)}(t)\right|\right)$$

$$\le\left(2\sqrt{T}+4\sqrt{2}\,|T|+\frac{2\sqrt{3}}{3}\right)^N\frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}$$

$$+\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)^{N}\frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1\overline{b}^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}$$

$$\leq\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)^{N}A_T\frac{1}{\sqrt{N!}}\left\|b(x,t)\right\|^N_{L_2(D)}$$

$$+\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)^{N}A_T\frac{1}{\sqrt{N!}}\left\|\overline{b}(x,t)\right\|^N_{L_2(D)}$$

$$\leq A_T\cdot a_N=a_N\ \left(\left\|\varphi-\overline{\varphi}\right\|+C(t)+M_1\left\|f-\overline{f}\right\|\right)$$

where

$$a_N=\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)^{N}\frac{1}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}$$

$$+\left(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3}\right)^{N}\frac{1}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1\overline{b}^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}}.$$

and

$$M_1=(2\sqrt{T}+4\sqrt{2}\left|T\right|+\frac{2\sqrt{3}}{3})^N.$$

(The sequence a_N is convergent then we can write $a_N\leq M$, $\forall N$). It follows from the estimation ([2], page 76-77) that $\lim\limits_{N\to\infty}u^{(N+1)}(t)=u(t)$, then let $N\to\infty$ for last equation

$$\left|u(t)-v(t)\right|\leq M\left\|\varphi-\overline{\varphi}\right\|+M_2\left\|f-\overline{f}\right\|)$$

where $M_2=M.M_1$. If $\left\|f-\overline{f}\right\|\leq\varepsilon$ and $\left\|\varphi-\overline{\varphi}\right\|\leq\varepsilon$ then $\left|u(t)-v(t)\right|\leq\varepsilon$. □

5. Numerical procedure for the nonlinear problem (1.1)-(1.4)

We construct an iteration algorithm for the linearization of the problem (1.1)-(1.4):

$$\frac{\partial u^{(n)}}{\partial t}-\frac{\partial^2 u^{(n)}}{\partial x^2}=f(x,t,u^{(n-1)}),\ (x,t)\in D \tag{5.1}$$

$$u^{(n)}(0,t)=u^{(n)}(1,t),\qquad t\in[0,T] \tag{5.2}$$

$$u_x^{(n)}(1,t)=0,\qquad t\in[0,T] \tag{5.3}$$

$$u^{(n)}(x,0)=\varphi(x),\qquad x\in[0,1]. \tag{5.4}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $f(x,t,u^{(n-1)}) = \widetilde{f}(x,t)$. Then the problem (5.1)-(5.4) can be written as a linear problem:

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \widetilde{f}(x,t) \quad (x,t) \in D \tag{5.5}$$

$$v(0,t) = v(1,t), \quad t \in [0,T] \tag{5.6}$$

$$v_x(1,t) = 0, \quad t \in [0,T] \tag{5.7}$$

$$v(x,0) = \varphi(x), \quad x \in [0,1]\ . \tag{5.8}$$

We use the finite difference method to solve (5.5)-(5.8).
We subdivide the intervals $[0,1]$ and $[0,T]$ into M and N subintervals of equal lengths $h = \frac{1}{M}$ and $\tau = \frac{T}{N}$, respectively. Then, we add a line $x = (M+1)\,h$ to generate the fictitious point needed for the second boundary condition.
We choose the implicit scheme, which is absolutely stable and has a second order accuracy in h and a first order accuracy in τ.
The implicit monotone difference scheme for (5.5)-(5.8) is as follows:

$$\frac{v_{i,j+1} - v_{i,j}}{\tau} = \frac{a^2}{h^2}(v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}) + \widetilde{f}_{i,j+1}$$

$$v_{i,0} = \varphi_i,\ v_{0,j} = v_{M,j},\ v_{x,M_j} = 0$$

where $0 \le i \le M$ and $1 \le j \le N$ are the indices for the spatial and time steps, respectively, $v_{i,j}$ is the approximation to $v(x_i,t_j)$, $f_{i,j} = f(x_i,t_j)$, $v_i = v(x_i)$, $x_i = ih$, $t_j = j\tau$. [12]

At the $t = 0$ level, adjustment should be made according to the initial condition and the compatibility requirements.

6. Numerical example

In this section, we will consider an example of numerical solution of the problem (1.1)-(1.4).

These problems were solved by applying the iteration scheme and the finite difference scheme which were explained in the Section 5. The condition

$$error(i,j) := \left\| u_{i,j}^{(n+1)} - u_{i,j}^{(n)} \right\|_\infty$$

with $error(i,j) := 10^{-3}$ was used as a stopping criteria for the iteration process.

Example 6.1. Consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \left[1 - (2\pi)^2(\cos 2\pi x + (\sin 2\pi x)^2)\right] u$$

$$u(x,0) = \exp(-\cos 2\pi x),\ x \in [0,1]$$

$$u(0,t) = u(1,t), \quad t \in [0,T]\,, u_x(1,t) = 0, \quad t \in [0,T]\,.$$

It is easy to see that the analytical solution of this problem is

$$u(x,t) = \exp(t - \cos 2\pi x).$$

The comparisons between the analytical solution and the numerical finite difference solution f when $T = 1$ are shown in Figure 1 for the step sizes $h = 0.0025$, $\tau = 0.0025$.

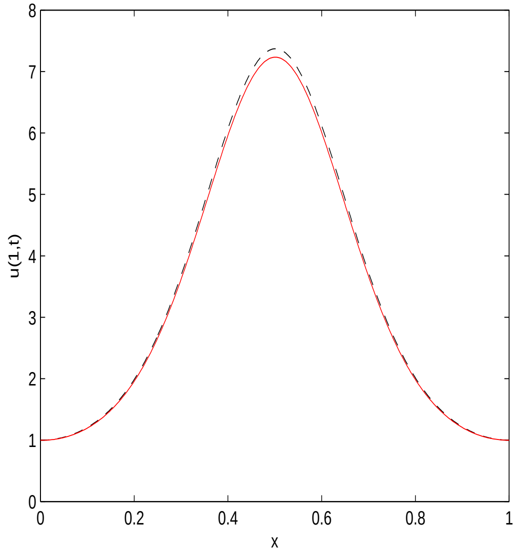

FIGURE 1. The exact and numerical solutions of $u(x,1)$, the exact solution is shown with dashes line.

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 583–595

Lines in the three-dimensional Bolyai-Lobachevskian hyperbolic geometry

Zoltán Gábos and Ágnes Mester

Abstract. The purpose of this paper is to describe the geodesics of the three-dimensional Bolyai-Lobachevskian hyperbolic space. We also determine the equation of the orthogonal surfaces and the scalar curvature of the surfaces of revolution. The metric applied is the Lobachevskian metric extended into three dimensions. During the analysis we use Cartesian and cylindrical coordinates. This article is a continuation of the paper [4].

Mathematics Subject Classification (2010): 53A35.

Keywords: Hyperbolic geometry, geodesics, orthogonal surfaces, cylindrical coordinates.

1. General context

In the literature exists several models for hyperbolic geometry, see [1]-[10]. The aim of this paper is to present a three dimensional model using [8] to describe some classical and new properties.

We consider the following metric

$$ds^2 = \cosh^2 \frac{z}{k} \left(\cosh^2 \frac{y}{k} dx^2 + dy^2 \right) + dz^2, \tag{1.1}$$

where k is the parameter of the three-dimensional hyperbolic space, and x, y, z are the Cartesian coordinates of any $P(x, y, z)$ point. The usage of Cartesian coordinates is justified by the existence of such hyperbolic lines which can also be considered Euclidean lines. These lines include the coordinate axes illustrated in figure 1. Note that the x-value can only be determined by axis Ox. Figure 1 also represents how the coordinates of any $P(x, y, z)$ point are determined: $x = \overline{OP_2},\ y = \overline{P_1P_2},\ z = \overline{PP_1}$.

From metric (1.1) we can obtain two possible symmetry operations. These consist of the reflections across the coordinate planes and the translation of the origin along the direction of the x-axis (the values y and z are not modified).

Based on metric (1.1), the geodesic lines verify

$$\cosh^2 \frac{z}{k} \cosh^2 \frac{y}{k} \frac{dx}{ds} = C_1, \tag{1.2}$$

where C_1 is constant. From this we obtain

$$\frac{d}{ds}\left(\cosh^2 \frac{z}{k} \frac{dy}{ds}\right) - \frac{1}{k} \cosh^2 \frac{z}{k} \sinh \frac{y}{k} \cosh \frac{y}{k} \left(\frac{dx}{ds}\right)^2 = 0, \tag{1.3}$$

$$\frac{d^2 z}{ds^2} - \frac{1}{k} \sinh \frac{z}{k} \cosh \frac{z}{k} \left[\cosh^2 \frac{y}{k} \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right] = 0. \tag{1.4}$$

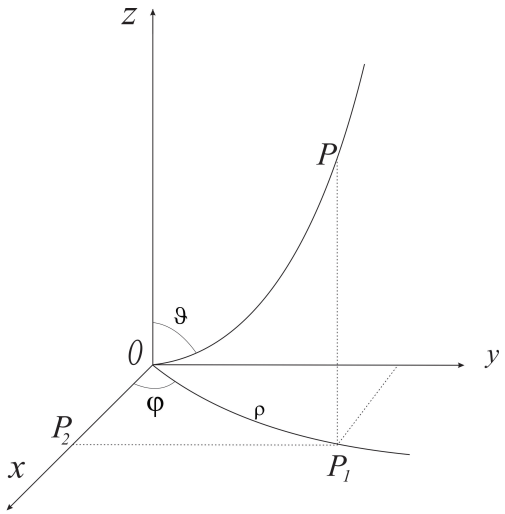

FIGURE 1

If we use x instead of variable s in (1.2), we can write equations (1.3) and (1.4) in the following form:

$$\frac{d^2 \tanh \frac{y}{k}}{dx^2} - \frac{1}{k^2} \tanh \frac{y}{k} = 0, \tag{1.5}$$

$$\frac{d}{dx}\left(\frac{1}{\cosh^2 \frac{y}{k}} \frac{d \tanh \frac{z}{k}}{dx}\right) - \frac{1}{k^2}\left[1 + k^2 \cosh^2 \frac{y}{k} \left(\frac{d \tanh \frac{y}{k}}{dx}\right)^2\right] \tanh \frac{z}{k} = 0. \tag{1.6}$$

If we use variable x, we can apply the results obtained in the hyperbolic plane by determining the function $y = y(x)$. Moreover, we claim that the projections of the geodesics in the three-dimensional space to the xOy plane are geodesics of the two-dimensional plane.

Using (1.1) and (1.2), we get

$$\frac{1}{C_1^2} = \frac{1}{\cosh^2 \frac{z}{k}} \left[\frac{1}{\cosh^2 \frac{y}{k}} + k^2 \left(\frac{d \tanh \frac{y}{k}}{dx}\right)^2\right] + k^2 \frac{1}{\cosh^4 \frac{y}{k}} \left(\frac{d \tanh \frac{z}{k}}{dx}\right)^2. \tag{1.7}$$

The curvature of the geodesics equals zero,

$$\frac{1}{r_g} = 0. \tag{1.8}$$

Metric (1.1) can also be obtained by using the metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_0^2 \tag{1.9}$$

defined in the four-dimensional pseudo-Euclidean space, with the help of the following equations:

$$x_1^2 + x_2^2 + x_3^2 - x_0^2 = -k^2, \tag{1.10}$$

$$x_1 = k \sinh\frac{x}{k}\cosh\frac{y}{k}\cosh\frac{z}{k}, \quad x_2 = k\sinh\frac{y}{k}\cosh\frac{z}{k}, \tag{1.11}$$
$$x_3 = k\sinh\frac{z}{k}, \quad x_0 = k\cosh\frac{x}{k}\cosh\frac{y}{k}\cosh\frac{z}{k}.$$

If we use equations

$$x_1 = k\cos\varphi\sinh\frac{\rho}{k}\cosh\frac{z}{k}, \quad x_2 = k\sin\varphi\sinh\frac{\rho}{k}\cosh\frac{z}{k}, \tag{1.12}$$
$$x_3 = k\sinh\frac{z}{k}, \quad x_0 = k\cosh\frac{\rho}{k}\cosh\frac{z}{k},$$

we obtain metric

$$ds^2 = \cosh^2\frac{z}{k}\left(d\rho^2 + k^2\sinh^2\frac{\rho}{k}d\varphi^2\right) + dz^2, \tag{1.13}$$

where ρ, φ and z represent cylindrical coordinates (figure 1).

Metric (1.13) justifies that the rotation around axis Oz (the constant choices for ρ and z) is a symmetry operation.

By choosing s as variable, the geodesic lines verify

$$\sinh^2\frac{\rho}{k}\cosh^2\frac{z}{k}\frac{d\rho}{ds} = C_2, \tag{1.14}$$

where C_2 is constant. We can also write

$$\frac{d^2\rho}{ds^2} + \frac{2}{k}\tanh\frac{z}{k}\frac{d\rho}{ds}\frac{dz}{ds} - k\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}\left(\frac{d\varphi}{ds}\right)^2 = 0, \tag{1.15}$$

$$\frac{d^2z}{ds^2} - \frac{1}{k}\sinh\frac{z}{k}\cosh\frac{z}{k}\left[\left(\frac{d\rho}{ds}\right)^2 + k^2\sinh^2\frac{\rho}{k}\left(\frac{d\varphi}{ds}\right)^2\right] = 0. \tag{1.16}$$

If we consider φ as variable, we will use the following differential equations:

$$\frac{d^2\coth\frac{\rho}{k}}{d\varphi^2} + \coth\frac{\rho}{k} = 0, \tag{1.17}$$

$$\frac{d}{d\varphi}\left(\frac{1}{\sinh^2\frac{\rho}{k}}\frac{d\tanh\frac{z}{k}}{d\varphi}\right) - \tanh\frac{z}{k}\left[1 + k^2\left(\frac{d\coth\frac{\rho}{k}}{d\varphi}\right)^2\right] = 0. \tag{1.18}$$

Using (1.13) and (1.14), we obtain

$$\frac{1}{C_2^2} = k^2\left\{\frac{1}{\cosh^2\frac{z}{k}}\left[\frac{1}{\sinh^2\frac{\rho}{k}} + k^2\left(\frac{d\coth\frac{\rho}{k}}{d\varphi}\right)^2\right] + \frac{1}{\sinh^4\frac{\rho}{k}}\left(\frac{d\tanh\frac{z}{k}}{d\varphi}\right)^2\right\}. \tag{1.19}$$

In the following sections we describe the different types of lines. Note that each line verifies the differential equations which characterize the geodesics. Also, C_1 and C_2 are constant values. During the analysis our choice of coordinates may vary depending on the form of calculations.

2. Lines crossing the origin

Let us consider the line passing through points O and P represented in figure 1, where ϑ is the angle of intersection with axis Oz. In the OP_1P right triangle we can write

$$\tanh \frac{z}{k} = \cot \vartheta \sinh \frac{\rho}{k}. \tag{2.1}$$

The projection of line OP onto the xOy plane satisfies the following equation:

$$\tanh \frac{y}{k} = \tan \varphi \sinh \frac{x}{k}. \tag{2.2}$$

Using (1.11) and (1.12), we obtain

$$\cosh \frac{x}{k} \cosh \frac{y}{k} = \cosh \frac{\rho}{k}.$$

These formulas imply

$$\tanh \frac{z}{k} = \frac{\cot \vartheta \sinh \frac{x}{k}}{\cos \varphi \sqrt{1 - \tan^2 \varphi \sinh^2 \frac{x}{k}}}. \tag{2.3}$$

The lines verifying equations (2.2) and (2.3) also satisfy the (1.5) and (1.6) differential equations. Using (1.7), we get

$$C_1 = \cos \varphi \sin \vartheta$$

constant. Therefore, the lines crossing the origin satisfy the conditions mentioned in the previous section.

In the two-dimensional hyperbolic plane the orthogonal curves of lines crossing the origin are circles. Based on the rotational symmetry operation, we claim that in the three-dimensional case the orthogonal surfaces are spheres. By the use of cylindrical coordinates we can write

$$\cosh \frac{\rho}{k} \cosh \frac{z}{k} = \cosh \frac{R}{k}. \tag{2.4}$$

In order to determine the curvature of the sphere surface, we use the metric

$$ds^2 = E(\rho) d\rho^2 + G(\rho) d\varphi^2$$

obtained from equations (1.13) and (2.4), where

$$E(\rho) = \frac{\sinh^2 \frac{R}{k} \cosh^2 \frac{R}{k}}{\cosh^2 \frac{\rho}{k} \left(\cosh^2 \frac{R}{k} - \cosh^2 \frac{\rho}{k}\right)}, \quad G(\rho) = k^2 \cosh^2 \frac{R}{k} \tanh^2 \frac{\rho}{k}.$$

The Christoffel symbols of the second kind are as follows:

$$\Gamma^1_{11} = \frac{2\sinh\frac{\rho}{k}\cosh\frac{\rho}{k} - \cosh^2\frac{R}{k}\tanh\frac{\rho}{k}}{k\left(\cosh^2\frac{R}{k} - \cosh^2\frac{\rho}{k}\right)}, \quad \Gamma^2_{12} = \frac{1}{k\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}},$$

$$\Gamma^1_{22} = -\frac{k}{\sinh^2\frac{R}{k}\cosh^2\frac{R}{k}}\tanh\frac{\rho}{k}\left(\cosh^2\frac{R}{k} - \cosh^2\frac{\rho}{k}\right),$$

where we used index 1 for ρ and index 2 for φ.

The components of the Ricci curvature tensor are

$$R_{11} = \frac{d\Gamma^2_{12}}{d\rho} + \Gamma^2_{12}\left(\Gamma^2_{12} - \Gamma^1_{11}\right), \quad R_{22} = -\frac{d\Gamma^1_{22}}{d\rho} + \Gamma^1_{22}\left(\Gamma^2_{12} - \Gamma^1_{11}\right).$$

Using the expressions above, we obtain for the scalar curvature

$$R = \frac{1}{E}R_{11} + \frac{1}{G}R_{22} = -\frac{2}{k^2\sinh^2\frac{R}{k}}. \tag{2.5}$$

3. Lines crossing the x-axis

Let us consider the line passing through points $P_0(a,0,0)$ and $P_1(0,b,c)$ illustrated in figure 2. If we project this line onto the xOy plane, we get the line passing through points $P_0(a,0,0)$ and $P_2(0,b,0)$, which verifies

$$\tanh\frac{y}{k} = \tanh\frac{b}{k}\frac{\sinh\frac{a-x}{k}}{\sinh\frac{a}{k}}. \tag{3.1}$$

The angle of intersection between the lines P_0P_1 and P_0P_2 is denoted by δ.

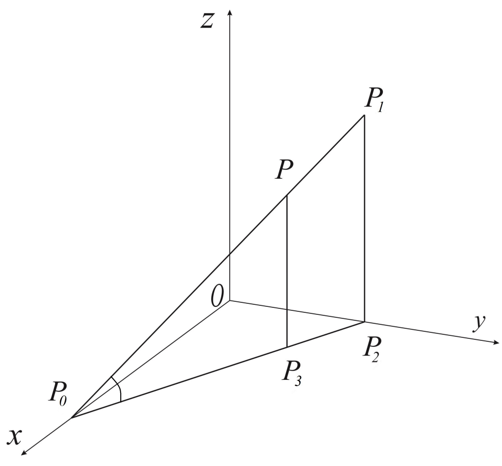

FIGURE 2

We can obtain the distance d between the points $P_0(a,0,0)$ and $P_2(0,b,0)$ from

$$\cosh\frac{d}{k} = \cosh\frac{a}{k}\cosh\frac{b}{k}.$$

Furthermore, distance d_1 between $P_0(a,0,0)$ and $P_3(x,y,0)$ verifies

$$\cosh\frac{d_1}{k}=\cosh\frac{a-x}{k}\cosh\frac{y}{k}.$$

If we consider the P_0PP_3 right triangle, we can write

$$\tanh\frac{z}{k}=\tan\delta\sinh\frac{d_1}{k},$$

while in the right triangle $P_0P_1P_2$

$$\tanh\frac{c}{k}=\tan\delta\sinh\frac{d}{k}.$$

These formulas imply

$$\tanh\frac{z}{k}=\tanh\frac{c}{k}\frac{B\sinh\frac{a-x}{k}}{A\sqrt{1-\tanh^2\frac{b}{k}\frac{\sinh^2\frac{a-x}{k}}{\sinh^2\frac{a}{k}}}},\tag{3.2}$$

where

$$B=\sqrt{1+\frac{\tanh^2\frac{b}{k}}{\sinh^2\frac{a}{k}}},\quad A=\sqrt{\cosh^2\frac{a}{k}\cosh^2\frac{b}{k}-1}.$$

Using (1.5) and (1.6) one can easily prove that equations (3.1) and (3.2) determine geodesic lines. Also, formula (1.13) implies

$$\frac{1}{C_1^2}=\left(1+\frac{\tanh^2\frac{b}{k}}{\sinh^2\frac{a}{k}}\right)\left(1+\frac{\tanh^2\frac{c}{k}}{\cosh^2\frac{a}{k}\cosh^2\frac{b}{k}-1}\right),\tag{3.3}$$

thus C_1 is constant.

Now we determine the orthogonal surface of the family of lines crossing point $P_0\in Ox$. As the translation of the origin along the direction of the x-axis into point P_0 is a symmetry operation, we obtain spheres with center P_0. If we use Cartesian coordinates, these spheres verify

$$\cosh\frac{x-a}{k}\cosh\frac{y}{k}\cosh\frac{z}{k}=\cosh\frac{R}{k}.\tag{3.4}$$

The curvature of the orthogonal surface is determined by formula (2.5).

As $a\longrightarrow\infty$, we obtain lines being parallel to the x-axis:

$$\tanh\frac{y}{k}=\tanh\frac{b}{k}e^{-\frac{x}{k}},\quad \tanh\frac{z}{k}=\frac{\tanh\frac{c}{k}}{\sqrt{\cosh^2\frac{b}{k}e^{2x}-\sinh^2\frac{b}{k}}}.$$

Thus we get

$$C_1=1.$$

If

$$R=a,\tag{3.5}$$

by applying equation (3.4), we obtain the equation of a parasphere containing the origin:

$$\cosh\frac{y}{k}\cosh\frac{z}{k}=e^{\frac{x}{k}}.$$

Therefore the parasphere which contains point $P(x_0, 0, 0)$ verifies

$$\cosh \frac{y}{k} \cosh \frac{z}{k} = e^{\frac{x-x_0}{k}}. \tag{3.6}$$

By using condition (3.5) and equation (2.5), the curvature of the parasphere becomes

$$R = 0.$$

This implies that we can use Euclidean geometry in order to study the surface of the parasphere, fact which was also mentioned by Bolyai in his main work [1].

4. Lines crossing the z-axis

If we consider the set of lines crossing the z-axis, we can differentiate three types of lines. The first set contains lines crossing the xOy plane, the second set consists of lines which do not cross the xOy plane, finally, the lines of the third family are parallel to the xOy plane.

In each case the projections of the lines contain the origin. Note that the rotation around the z-axis is a symmetry operation. Therefore, we can determine the relevant lines by using surfaces of revolution which are created by rotating the curves around the z-axis in the xOy plane (the role of x is taken by ρ). On the other hand, the lines which cross a projected line onto the xOy plane while being parallel to the z-axis determine an orthogonal surface perpendicular to the xOy plane. The intersection of this orthogonal surface and the surface of revolution determines the lines in question.

For fixed φ we obtain from metric (1.13)

$$ds^2 = \cosh^2 \frac{z}{k} d\rho^2 + dz^2, \tag{4.1}$$

which describes the orthogonal surfaces.

If we use s as variable, we can write

$$\cosh^2 \frac{z}{k} \frac{d\rho}{ds} = C, \quad \frac{d^2 z}{ds^2} - \frac{1}{k} \sinh \frac{z}{k} \cosh \frac{z}{k} \left(\frac{d\rho}{ds} \right)^2 = 0. \tag{4.2}$$

Then by substituting s with ρ, we obtain the following differential equation:

$$\frac{d^2 \tanh \frac{z}{k}}{d\rho^2} - \frac{1}{k^2} \tanh \frac{z}{k} = 0. \tag{4.3}$$

From (4.1) and (4.2) we get

$$\frac{1}{C^2} = \frac{1}{\cosh^2 \frac{z}{k}} + k^2 \left(\frac{d \tanh \frac{z}{k}}{d\rho} \right)^2 .$$

Applying (2.1), we obtain the condition

$$C = \sin \vartheta$$

for the lines passing through the origin.

a. Lines parallel to the xOy plane and lines crossing the xOy plane

Using the rotational symmetry operation, we obtain for the line passing through points $P_1(0,0,z_0)$ and $P_2(a,b,0)$

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}\frac{\sinh\frac{\rho_0-\rho}{k}}{\sinh\frac{\rho_0}{k}}. \tag{4.4}$$

This formula satisfies equation (4.3), where

$$\cosh\frac{\rho_0}{k}=\cosh\frac{a}{k}\cosh\frac{b}{k}.$$

Also

$$\frac{1}{C^2}=1+\frac{\tanh^2\frac{z_0}{k}}{\sinh^2\frac{\rho_0}{k}}. \tag{4.5}$$

If $\rho_0\longrightarrow\infty$, it follows that

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}e^{-\frac{\rho}{k}} \tag{4.6}$$

and

$$C=1.$$

b. Lines not crossing the xOy plane

In this case the lines have a minimum point. If we apply the rotational symmetry, we obtain

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}\frac{\cosh\frac{\rho_m-\rho}{k}}{\cosh\frac{\rho_m}{k}}, \tag{4.7}$$

where ρ_m denotes the value of ρ determined by the minimum point. For the value of C we have

$$\frac{1}{C^2}=1-\frac{\tanh^2\frac{z_0}{k}}{\cosh^2\frac{\rho_m}{k}}. \tag{4.8}$$

If $\rho_m=0$, which means that the intersection coincides with the minimum point, we can write

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}\cosh\frac{\rho}{k} \tag{4.9}$$

and

$$C=\cosh\frac{z_0}{k}.$$

If we consider the lines passing through $P(0,0,z_0)$, we get for the orthogonal curves circles with center P in the xOz plane. Therefore, because of the rotational symmetry, the orthogonal surfaces of the lines containing $P(0,0,z_0)$ are spheres with center P, which verify

$$\cosh\frac{z_0}{k}\cosh\frac{z}{k}\cosh\frac{\rho}{k}-\sinh\frac{z_0}{k}\sinh\frac{z}{k}=\cosh\frac{R}{k}. \tag{4.10}$$

Now let us consider the orthogonal surface which is perpendicular to the xOy plane and contains the projected line. Here we use variables ρ and z. Furthermore, we will use indexes 1 and 2 for two arbitrary lines which intersect in point $P(\rho,z)$ on this surface. Thus we obtain

$$\left(\frac{dz_1}{d\rho_1}\right)_P\left(\frac{dz_2}{d\rho_2}\right)_P+\cosh^2\frac{z}{k}=0,$$

where the lower index P means that we need to substitute the coordinates of the intersection.

Using equations (4.4), (4.6) and (4.7), we get for the lines crossing the z-axis

$$\tanh\frac{z_1}{k} = \tanh\frac{z_0}{k}\left(\cosh\frac{\rho_1}{k} - p\sinh\frac{\rho_1}{k}\right). \tag{4.11}$$

In the three different cases (lines crossing the xOy plane, lines not crossing the xOy plane, lines parallel to the xOy plane) the required values are as follows:

$$\coth\frac{\rho_0}{k}, \quad \tanh\frac{\rho_m}{k} \text{ and } \quad 1.$$

By deriving equation (4.11) we obtain

$$\frac{dz_1}{d\rho_1} = \tanh\frac{z_0}{k}\cosh^2\frac{z_1}{k}\left(\sinh\frac{\rho_1}{k} - p\cosh\frac{\rho_1}{k}\right). \tag{4.12}$$

Then, using (4.11) and (4.12), we eliminate variable p. Thus we get

$$\frac{dz_1}{d\rho_1} = \frac{\tanh\frac{z_0}{k}\cosh^2\frac{z_1}{k}}{\sinh\frac{\rho_1}{k}}\left(\coth\frac{z_0}{k}\tanh\frac{z_1}{k}\cosh\frac{\rho_1}{k} - 1\right).$$

After differentiating equation (4.10) we obtain

$$\frac{dz_2}{d\rho_2} = -\frac{\coth\frac{z_0}{k}\sinh\frac{\rho_2}{k}}{\coth\frac{z_0}{k}\tanh\frac{z_2}{k}\coth\frac{\rho_2}{k} - 1}.$$

In the point of intersection we have $\rho_1 = \rho_2 = \rho$ and $z_1 = z_2 = z$. Thus the orthogonality condition holds, which proves the validity of equation (4.10).

Equation (4.10) can be written in the following form:

$$\cos\frac{\rho}{k} = \frac{\cosh\frac{R}{k} + \sinh\frac{z_0}{k}\sinh\frac{z}{k}}{\cosh\frac{z_0}{k}\cosh\frac{z}{k}} = F(\rho). \tag{4.13}$$

Furthermore, equation (1.13) yields metric

$$ds^2 = \frac{1 - F^2 + \cosh^2\frac{z}{k}\left(\frac{dF}{d\rho}\right)^2}{F^2 - 1}d\rho^2 + k^2\cosh^2\frac{z}{k}\left(F^2 - 1\right)d\varphi^2. \tag{4.14}$$

Using metric (4.14) and formula (4.13), we can obtain the curvature of the orthogonal surface. The Ricci scalar is determined by formula (2.5).

5. Family of lines not having common point

In this section we consider two sets of lines.

a. Lines parallel to the z-axis

In this case, on the orthogonal surfaces the value of z is constant, $z = z_0$. Indeed, the lines verify $d\rho_1 = 0$, while on the orthogonal surface $dz_2 = 0$. Thus we obtain the following orthogonality condition:

$$\cosh\frac{z}{k}d\rho_1 d\rho_2 + dz_1 dz_2 = 0.$$

The orthogonal surface called hypersphere verifies

$$ds^2 = \cosh^2 \frac{z_0}{k} \left(d\rho^2 + k^2 \sinh^2 \frac{\rho}{k} d\varphi^2 \right), \tag{5.1}$$

the scalar curvature is

$$R = \frac{2}{k^2}.$$

b. Lines having minimum point on the z-axis

Here we use formula (4.9), where the parameter is $\tanh \frac{z_0}{k}$.

By deriving (4.9) and eliminating the parameter, we obtain

$$\frac{dz_1}{d\rho_1} = \sinh \frac{z_1}{k} \cosh \frac{z_1}{k} \tanh \frac{\rho_1}{k}.$$

The rotational symmetry operation induces for the orthogonal surface equation

$$\sinh \frac{\rho}{k} \cosh \frac{z}{k} = \sinh \frac{\rho_0}{k}, \tag{5.2}$$

where ρ_0 is constant. Hence we get

$$\frac{dz_2}{d\rho_2} = -\coth \frac{z_2}{k} \coth \frac{\rho_2}{k}.$$

This and the orthogonality condition proves formula (5.2).
From equations (1.13) and (5.2) we obtain metric

$$ds^2 = \sinh^2 \frac{\rho_0}{k} \frac{1 + \coth^2 \frac{\rho}{k}}{\sinh^2 \frac{\rho}{k}} d\rho_0 + k^2 \sinh^2 \frac{\rho_0}{k} d\varphi^2.$$

Hence the scalar curvature of the orthogonal surface is

$$R = 0.$$

This means that this orthogonal surface is the dual of the parasphere.

6. Surfaces with constant curvature

For lines crossing axis Oz we applied equations of type

$$\tanh \frac{z}{k} = \Phi(\rho), \tag{6.1}$$

which were as follows: equation (2.2), (4.4), (4.7) and (4.6).

Using formulas (5.1) and (6.1), we obtain

$$ds^2 = E(\rho) d\rho^2 + G(\rho) d\varphi^2$$

for the metric, where

$$E(\rho) = \cosh^2 z(\rho) + \left(\frac{dz}{d\rho} \right)^2 = \frac{A}{(1 - \Phi^2)^2},$$

$$G(\rho) = k^2 \cosh^2 z(\rho) \sinh^2 \frac{\rho}{k} = k^2 \frac{\sinh^2 \frac{\rho}{k}}{1 - \Phi^2}.$$

In the four different cases (lines crossing the origin, lines crossing the xOy plane, lines not crossing the xOy plane, lines parallel to the xOy plane) the values of the constant A are as follows:

$$\frac{1}{\sin^2\vartheta},\quad 1+\frac{\tanh^2\frac{z_0}{k}}{\sinh^2\frac{z_0}{k}},\quad 1-\frac{\tanh^2\frac{z_0}{k}}{\cosh^2\frac{\rho_m}{k}}\ \text{and}\quad 1.$$

The Christoffel symbols of the second kind are as follows:

$$\Gamma^1_{11}=\frac{1}{2E}\frac{dE}{d\rho}=2\frac{\Phi\frac{d\Phi}{d\rho}}{1-\Phi^2},$$

$$\Gamma^1_{22}=-\frac{1}{2E}\frac{dG}{d\rho}=-\frac{k}{A}\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}\left(1-\Phi^2\right)-\frac{k^2}{A}\sinh^2\frac{\rho}{k}\Phi\frac{d\Phi}{d\rho},$$

$$\Gamma^2_{12}=\frac{1}{2G}\frac{dG}{d\rho}=\frac{1}{1-\Phi^2}\left[\frac{1}{k}\left(1-\Phi^2\right)\coth\frac{\rho}{k}+\Phi\frac{d\Phi}{d\rho}\right],$$

while the components of the Ricci curvature tensor are

$$R_{11}=\frac{d\Gamma^2_{12}}{d\rho}+\Gamma^2_{12}\left(\Gamma^2_{12}-\Gamma^1_{11}\right)=\frac{1}{k^2\left(1-\Phi^2\right)^2}\left[1-\Phi^2+k^2\left(\frac{d\Phi}{d\rho}\right)^2\right],$$

$$R_{22}=-\frac{d\Gamma^1_{22}}{d\rho}+\Gamma^1_{22}\left(\Gamma^2_{12}-\Gamma^1_{11}\right)=\frac{\sinh^2\frac{\rho}{k}}{A\left(1-\Phi^2\right)}\left[1-\Phi^2+k^2\left(\frac{d\Phi}{d\rho}\right)^2\right].$$

Therefore the scalar curvature is

$$R=\frac{1}{E}R_{11}+\frac{1}{G}R_{22}=\frac{2}{k^2}$$

constant for all surfaces.

7. Lines not crossing the z-axis and the xOy plane

As the projection of these lines to the xOy plane verifies

$$\tanh\frac{y}{k}=\tanh\frac{b}{k}\cosh\frac{x}{k},\tag{7.1}$$

from equations (7.1) and (4.9) it follows that

$$\tanh\frac{z}{k}=\tanh\frac{z_0}{k}\frac{\cosh\frac{x}{k}}{\sqrt{1-\tanh^2\frac{b}{k}\cosh^2\frac{x}{k}}}.\tag{7.2}$$

By deriving (7.2) we obtain

$$\frac{d\tanh\frac{z}{k}}{dx}=\frac{\tanh\frac{z_0}{k}}{k}\frac{\sinh\frac{x}{k}}{\left(1-\tanh^2\frac{b}{k}\cosh^2\frac{x}{k}\right)^{\frac{3}{2}}}.\tag{7.3}$$

Using (1.7) and (7.3), we get for the value of C_1

$$\frac{1}{C_1^2}=\frac{1}{\cosh^2\frac{b}{k}}-\tanh^2\frac{z_0}{k}.$$

C_1 is real if and only if $\cosh\frac{b}{k} \le \coth\frac{z_0}{k}$. Indeed, using formula (4.9) as $z_0 \longrightarrow \infty$, we get for the maximal value of ρ

$$\cosh\frac{\rho_m}{k} = \coth\frac{z_0}{k}.$$

From

$$\frac{d}{dx}\left(\frac{1}{\cosh^2\frac{y}{k}}\frac{d\tanh\frac{z}{k}}{dx}\right) = \frac{\cosh\frac{x}{k}}{k^2\left(1-\tanh\frac{y_0}{k}\cosh^2\frac{x}{k}\right)^{\frac{3}{2}}\cosh^2\frac{y_0}{k}}$$

and

$$1 + k^2\cosh^2\frac{y}{k}\left(\frac{d\tanh\frac{z}{k}}{dx}\right)^2 = \frac{1}{\cosh^2\frac{y_0}{k}\left(1-\tanh\frac{y_0}{k}\cosh^2\frac{x}{k}\right)}$$

it follows that the lines verifying (7.1) and (7.2) satisfy differential equations (1.5) and (1.6).

If we use cylindrical coordinates, from

$$\coth\frac{\rho}{k} = \coth\frac{b}{k}\sin\varphi \tag{7.4}$$

and equation (4.9) we get

$$\tanh\frac{z}{k} = \tanh\frac{z_0}{k}\coth\frac{b}{k}\frac{\sin\varphi}{\sqrt{\coth^2\frac{b}{k}\sin^2\varphi - 1}}. \tag{7.5}$$

These lines verify differential equations (1.17) and (1.18). Applying (1.19), we get for the value of C_2

$$\frac{1}{C_2^2} = \frac{k^2}{\sinh^2\frac{b}{k}}\left(1-\tanh^2\frac{z_0}{k}\cosh^2\frac{b}{k}\right).$$

Thus C_2 is real if and only if $\cosh\frac{b}{k} \le \coth\frac{z_0}{k}$.

If we use equation (4.9) and formula

$$\coth\frac{\rho}{k} = \coth\frac{a}{k}\left(\sin\varphi + \cos\varphi\right), \tag{7.6}$$

we obtain a different line which satisfies

$$\tanh\frac{z}{k} = \tanh\frac{z_0}{k}\coth\frac{a}{k}\frac{\sin\varphi+\cos\varphi}{\sqrt{\coth^2\frac{a}{k}\left(\sin\varphi+\cos\varphi\right)^2 - 1}}. \tag{7.7}$$

The curves verifying (7.6) and (7.7) also satisfy the differential equations (1.17) and (1.18). Also, from

$$\frac{1}{C_2^2} = k^2\frac{\cosh^2\frac{a}{k} + \cosh^2\frac{z_0}{k}}{\sinh^2\frac{a}{k}\cosh^2\frac{z_0}{k}}$$

we obtain a constant value for C_2. Thus these lines are lines of the hyperbolic space.

If we use Cartesian coordinates, instead of (7.6) and (7.7) we may write

$$\tanh\frac{y}{k} = \tanh\frac{a}{k}\cosh\frac{x}{k} - \sinh\frac{x}{k}$$

and

$$\tanh\frac{z}{k} = \tanh\frac{z_0}{k}\frac{\cosh\frac{x}{k}}{\sqrt{1-\left(\tanh\frac{a}{k}\cosh\frac{x}{k}-\sinh\frac{x}{k}\right)^2}}.$$

For each surface of revolution the scalar curvature equals the curvature of the xOy plane and xOz is a plane of symmetry. Hence we obtain surfaces on the left and the right side of the xOz plane. However, only equation (4.9) provides a necessary condition. Let us consider a line crossing axis Oz, which connects two distinct surfaces. The transitions between the line and the surfaces are smooth (the tangent vector field is continuous) only in the case of (4.9). Therefore, new lines can only be derived by the surface of revolution (4.9).

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 597–601

Multisymplectic connections on supermanifolds

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Abstract. In this paper we show that on any multisymplectic supermanifold there exist a connection compatible to the multisymplectic form.

Mathematics Subject Classification (2010): 58A50, 53D05.

Keywords: Multisymplectic supermanifolds, multisymplectic connections.

1. Introduction

Multisymplectic structures in field theory play a role similar to that of symplectic structures in classical mechanics. In the other hand supergeometry plays an important role in physics. In [2] and [3], the authors studied geometry of symplectic connections and in [1], the author studied symplectic connections on supermanifold. In this paper we study multisymplectic connections on supermanifolds.
A supermanifold $\mathcal{M}$ of dimension $n|m$ is a pair $(M, \mathcal{O}_{\mathcal{M}})$, where M is a Hausdorff topological space and $\mathcal{O}_{\mathcal{M}}$ is a sheaf of commutative superalgebras with unity over $\mathbb{R}$ locally isomorphic to $\mathbb{R}^{m|n} = (\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n} \otimes \Lambda_{\eta^1,\ldots,\eta^m})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions on $\mathbb{R}^n$ and $\Lambda_{\eta^1,\ldots,\eta^m}$ is the grassmann superalgebra of m generators (for more details see [5]).

If $\mathcal{M}$ is a supermanifold of dimension $n|m$, we define the tangent sheaf as follows,

$$\mathcal{T}_{\mathcal{M}}(U) = Der(\mathcal{O}_{\mathcal{M}}(U)),$$

the $\mathcal{O}_{\mathcal{M}}(U)$-supermodule of derivations of $\mathcal{O}_{\mathcal{M}}(U)$. $\mathcal{T}_{\mathcal{M}}$ is locally free of dimension $n|m$. The sections of $\mathcal{T}_{\mathcal{M}}$ are called vector fields.

Definition 1.1. *If ξ be a locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-supermodules on $\mathcal{M}$, a connection on ξ is a morphism $\nabla : \mathcal{T}_{\mathcal{M}} \otimes_{\mathbb{R}} \xi \to \xi$ of sheaves of supermodules over $\mathbb{R}$ such that*

$$\nabla_{fX} v = f\nabla_X v, \ \nabla_X fv = (Xf) + (-1)^{\widetilde{X}\widetilde{f}} f\nabla_X v \text{ and } \widetilde{\nabla_X v} = \widetilde{v} + \widetilde{X},$$

for all homogeneous function f, vector fields X and section v of ξ. (In the case $\xi = \mathcal{T}_{\mathcal{M}}$ we speak of a connection on $\mathcal{M}$).

We define the torsion of a connection ∇ on $\mathcal{T}_{\mathcal{M}}$ by

$$T(X,Y) = \nabla_X Y - (-1)^{\widetilde{X}\widetilde{Y}}\nabla_Y X - [X,Y].$$

Definition 1.2. *A graded Riemannian metric on supermanifold $\mathcal{M}$ is a graded-symmetric non-degenerate $\mathcal{O}_{\mathcal{M}}$-linear morphism of sheaves*

$$g : \mathcal{T}_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}.$$

A supermanifold equipped with graded Riemannian metric is called a Riemannian supermanifold. If $\mathcal{M}$ is a Riemannian supermanifold with Riemannian metric g, we call a connection ∇ metric if $\nabla g = 0$.

On a suppermanifold M with a Riemannian metric g, there exist a unique torsion free and metric connection ∇^0, which will be called the Levi-Civita connection of the metric(see [4]).

2. Multisymplectic connections on supermanifolds

Let us consider a multisymplectic supermanifold of degree k $(\mathcal{M},\omega)$, i.e. a supermanifold $\mathcal{M}$ with a closed non-degenerate graded differential k-form ω.

Definition 2.1. *A multisymplectic connection on $\mathcal{M}$ is a connection for which:*
i) The torsion tensor vanishes, i.e.

$$\nabla_X Y - (-1)^{\widetilde{X}\widetilde{Y}}\nabla_Y X = [X,Y].$$

ii) It is compatible to the multisymplectic form, i.e. $\nabla\omega = 0$.

To prove the existence of such a connection, take ∇^0 to be the Levi-Civita connection associated to a metric g on $\mathcal{M}$. Consider tensor N on $\mathcal{M}$ defined by

$$\nabla^0_{Y_0}\omega(Y_1,Y_2,...,Y_k) = (-1)^{\widetilde{\omega}\widetilde{Y_0}}\omega(N(Y_0,Y_1),Y_2,...,Y_k).$$

We shall proof some properties of N.

Lemma 2.2. *We have*
i) $\omega(N(Y_0,Y_1),Y_2,...,Y_k) = -(-1)^{\widetilde{Y_1}\widetilde{Y_2}}\omega(N(Y_0,Y_2),Y_1,...,Y_k)$;
ii) $\omega(N(Y_0,Y_1),Y_2,...,Y_k) + \Sigma_{i=1}^k(-1)^{i+\sum_{p<i}\widetilde{Y_p}\widetilde{Y_i}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y_i},...,Y_k) = 0$,
where the hats indicate omitted arguments.

Proof. We first prove (i)

$$\begin{aligned}\omega(N(Y_0,Y_1),Y_2,...,Y_k) &= (-1)^{\widetilde{Y_0}\widetilde{\omega}}\nabla^0_{Y_0}\omega(Y_1,Y_2,...,Y_k)\\ &= -(-1)^{\widetilde{Y_0}\widetilde{\omega}+\widetilde{Y_1}\widetilde{Y_2}}\nabla^0_{Y_0}\omega(Y_2,Y_1,...,Y_k)\\ &= -(-1)^{\widetilde{Y_1}\widetilde{Y_2}}\omega(N(Y_0,Y_2),Y_1,...,Y_k).\end{aligned}$$

For proof (ii) we know $d\omega = 0$ so

$$0 = d\omega(Y_0,Y_1,...,Y_k) = \sum_{i=0}^{k}(-1)^{i+\widetilde{Y_i}(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y_i},...,Y_k))$$

$$+\sum_{i<j}(-1)^{j+\sum_{i<p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},[Y_i,Y_j],Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y}_i,...,Y_k))$$

$$+\sum_{i<j}(-1)^{j+\sum_{i<p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\nabla^0_{Y_i}Y_j-(-1)^{\widetilde{Y}_i\widetilde{Y_j}}\nabla^0_{Y_j}Y_i,Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y}_i,...,Y_k))$$

$$+\sum_{i<j}(-1)^{j+\sum_{i<p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\nabla^0_{Y_i}Y_j,Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$-\sum_{i<j}(-1)^{j+\sum_{i\le p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\nabla^0_{Y_j}Y_i,Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y}_i,...,Y_k))$$

$$+\sum_{i<j}(-1)^{j+\sum_{i<p<j}\widetilde{Y_j}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\nabla^0_{Y_i}Y_j,Y_{i+1},...,\hat{Y}_j,...,Y_k)$$

$$-\sum_{j<i}(-1)^{i+\sum_{j\le p<i}\widetilde{Y_i}\widetilde{Y_p}}\omega(Y_0,...,Y_{j-1},\nabla^0_{Y_i}Y_j,Y_{j+1},...,\hat{Y}_i,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}Y_i(\omega(Y_0,...,\hat{Y}_i,...,Y_k))$$

$$-\sum_{i<j}(-1)^{i+\sum_{i<p<j}\widetilde{Y_i}\widetilde{Y_p}}\omega(Y_0,...,Y_{i-1},\hat{Y}_i,...,Y_{j-1},\nabla^0_{Y_i}Y_j,Y_{j+1},...,Y_k)$$

$$-\sum_{j<i}(-1)^{i+\sum_{j\le p<i}\widetilde{Y_i}\widetilde{Y_p}}\omega(Y_0,...,Y_{j-1},\nabla^0_{Y_i}Y_j,Y_{j+1},...,\hat{Y}_i,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}(Y_i(\omega(Y_0,....\hat{Y}_i,...,Y_k))$$

$$-\sum_{j}(-1)^{\widetilde{Y}_i(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\omega(Y_0,...,Y_{j-1},\nabla^0_{Y_i}Y_j,...,\hat{Y}_i,...,Y_k)$$

$$=\sum_{i=0}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{w}+\sum_{p<i}\widetilde{Y_p})}\nabla^0_{Y_i}\omega(Y_0,...,\hat{Y}_i,...,Y_k)$$

$$=(-1)^{\widetilde{Y_0}\widetilde{\omega}}\nabla^0_{Y_0}\omega(Y_1,...,Y_k)$$

$$+\sum_{i=1}^{k}(-1)^{i+\widetilde{Y}_i(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\nabla^0_{Y_i}\omega(Y_0,...,\hat{Y}_i,...,Y_k)$$

$$=\omega(N(Y_0,Y_1),Y_2,...,Y_k)+\Sigma_{i=1}^{k}(-1)^{i+\sum_{p<i}\widetilde{Y_p}\widetilde{Y_i}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y}_i,...,Y_k). \qquad \square$$

Now we show that on any multisymplectic supermanifold there exist a connection compatible to the multisymplectic form.

Theorem 2.3. *Let $(\mathcal{M},\omega)$ be a multisymplectic supermanifold. Then on $\mathcal{M}$ there is at least a multisymplectic connection.*

Proof. We define now a new connection ∇ as follows

$$\nabla_X Y = \nabla^0_X Y + \frac{1}{k+1}N(X,Y) + \frac{(-1)^{\widetilde{X}\widetilde{Y}}}{k+1}N(Y,X).$$

It is easy to show that ∇ is a torsion free connection. We show that the connection is compatible with the multisymplectic form ω, i.e. $\nabla\omega = 0$. We have

$$\nabla_{Y_0}\omega(Y_1,...,Y_k) = Y_0(\omega(Y_1,...,Y_k))$$

$$-\sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},\nabla_{Y_0}Y_i,Y_{i+1},...,Y_k)$$

$$= Y_0(\omega(Y_1,...,Y_k)) - \sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},\nabla^0_{Y_0}Y_i$$

$$+\frac{1}{k+1}N(Y_0,Y_i) + \frac{(-1)^{\widetilde{Y_0}\widetilde{Y_i}}}{k+1}N(Y_i,Y_0),Y_{i+1},...,Y_k)$$

$$= Y_0(\omega(Y_1,...,Y_k)) - \sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{1\le p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},\nabla^0_{Y_0}Y_i,Y_{i+1},...,Y_k)$$

$$-\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{1\le p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},N(Y_0,Y_i),Y_{i+1},...,Y_k)$$

$$-\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{1\le p\le i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},N(Y_i,Y_0),Y_{i+1},...,Y_k)$$

$$= \nabla^0_{Y_0}\omega(Y_1,...,Y_k)$$

$$-\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{i-1}(-1)^{\widetilde{Y_0}\widetilde{\omega}+\widetilde{Y_i}\sum_{1\le p<i}\widetilde{Y_p}}\omega(N(Y_0,Y_i),Y_1,...,\hat{Y_i},..,Y_k)$$

$$-\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{i-1}(-1)^{\widetilde{Y_0}\widetilde{\omega}+\widetilde{Y_i}\sum_{0\le p<i}\widetilde{Y_p}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y_i},...,Y_k)$$

$$= (-1)^{\widetilde{Y_0}\widetilde{\omega}}\omega(N(Y_0,Y_1),Y_2,...,Y_k) - \frac{k}{k+1}(-1)^{\widetilde{Y_0}\widetilde{\omega}}\omega(N(Y_0,Y_1),Y_2,...,Y_k)$$

$$+\frac{1}{k+1}\sum_{i=1}^{k}(-1)^{i+\widetilde{Y_0}\widetilde{\omega}+\widetilde{Y_i}\sum_{0\le p<i}\widetilde{Y_p}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y_i},...,Y_k)$$

$$= \frac{1}{k+1}(-1)^{\widetilde{Y_0}\widetilde{\omega}}(\omega(N(Y_0,Y_1),Y_2,...,Y_k)$$

$$+\sum_{i=1}^{k}(-1)^{i+\widetilde{Y_i}\sum_{p<i}\widetilde{Y_p}}\omega(N(Y_i,Y_0),Y_1,...,\hat{Y_i},...,Y_k))=0.\qquad\square$$

Let now ∇ be a multisymplectic connection and $\nabla'_X Y=\nabla_X Y+S(X,Y)$, where S is a tensor field on $\mathcal{M}$. We have

Theorem 2.4. *∇' is a multisymplelectic connection if and only if S is supersymmetric and*

$$\sum_i(-1)^{\sum_{p<i}\widetilde{Y_0}\widetilde{Y_p}}\omega(Y_1,...,Y_{i-1},S(Y_0,Y_i),Y_{i+1},...,Y_k)=0.$$

Proof. If we want ∇' to be torsion free then

$$\nabla_Y X+S(X,Y)-(-1)^{\widetilde{X}\widetilde{Y}}\nabla_Y X-(-1)^{\widetilde{X}\widetilde{Y}}S(Y,X)=[X,Y].$$

So $S(X,Y)=-(-1)^{\widetilde{X}\widetilde{Y}}S(Y,X)$. If ∇' be compatible to the multisymplectic form ω. We have

$$0=\nabla'_{Y_0}\omega(Y_1,...,Y_k)=Y_0(\omega(Y_1,...,Y_k))$$
$$-\sum_i(-1)^{\widetilde{Y_0}(\widetilde{\omega}+\sum_{p<i}\widetilde{Y_p})}\omega(Y_1,...,Y_{i-1},\nabla'_{Y_0}Y_i,Y_{i+1},..,Y_k)$$
$$=\nabla_{Y_0}\omega(Y_1,...,Y_k)-(-1)^{\widetilde{Y_0}\widetilde{\omega}}(\Sigma_i(-1)^{\sum_{p<i}\widetilde{Y_0}\widetilde{Y_p}}\omega(Y_1,...,Y_{i-1},S(Y_0,Y_i),Y_{i+1},...,Y_k)).$$

So

$$\sum_i(-1)^{\sum_{p<i}\widetilde{Y_0}\widetilde{Y_p}}\omega(Y_1,...,Y_{i-1},S(Y_0,Y_i),Y_{i+1},...,Y_k)=0.\qquad\square$$

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On applications of Andrica-Badea and Nagy inequalities in spectral graph theory

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Abstract. Applications of Andrica-Badea and Nagy inequalities for determining bounds of graph invariants of undirected, connected graphs are investigated. We consider bounds of the following invariants: the first Zagreb index, general Randić index, Laplacian linear spread and normalized Laplacian spread of graphs.

Mathematics Subject Classification (2010): 60E15, 05C50.

Keywords: Inequalities, first Zagreb index, spread of graph.

1. Introduction

Andrica and Badea (see [1]) have proved the following result.
Let $p_1, p_2, \dots, p_n$ be non-negative real numbers and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ real numbers with the properties

$$0 < r_1 \le a_i \le R_1 < +\infty \qquad \text{and} \qquad 0 < r_2 \le b_i \le R_2 < +\infty$$

for each $i = 1, 2, \dots, n$. Further, let S be a subset of $I_n = \{1, 2, \dots, n\}$ which minimizes the expression

$$\left| \sum_{i \in S} p_i - \frac{1}{2} \sum_{i=1}^{n} p_i \right| . \tag{1.1}$$

Then

$$\left| \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right|$$
$$\le (R_1 - r_1)(R_2 - r_2) \sum_{i \in S} p_i \left(\sum_{i=1}^{n} p_i - \sum_{i \in S} p_i \right) \tag{1.2}$$

In [17] Nagy has proved the following result:

Let $a_1, a_2, \ldots, a_n$ are real numbers with the property $r \le a_i \le R$, for each $i = 1, 2, \ldots, n$. Then

$$n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2 \ge \frac{n}{2}(R-r)^2. \tag{1.3}$$

In this paper we consider bounds of some graph invariants and prove that they are direct corollaries of inequalities (1.2) and (1.3). Some of the obtained bounds are better than those obtained in the literature so far.

In the next section we recall some results from spectral graph theory needed for our work.

2. Laplacian and normalized laplacian spectrum of graph

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, be undirected connected graph with n vertices and m edges, with sequence of vertex degrees $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, $d_i = d(i)$, $i = 1, 2, \ldots, n$. Denote with $\mathbf{A}$ adjacency matrix of G. Its eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ represent ordinary eigenvalues of graph G. If $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is diagonal matrix of vertex degrees, then $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of the G. Eigenvalues of $\mathbf{L}$, $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1} > \mu_n = 0$ represent Laplacian eigenvalues of graph G. The main properties of these eigenvalues are (see [3, 7, 8, 15])

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i = 2m \quad \text{and} \quad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m, \tag{2.1}$$

where $M_1 = \sum_{i=1}^{n} d_i^2$ is the first Zagreb index (see [13]).

Because the graph G is assumed to be connected, it has no isolated vertices and therefore the matrix $\mathbf{D}^{-1/2}$ is well–defined. Then $\mathbf{L}^* = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$ is the normalized Laplacian matrix of the graph G. Its eigenvalues are $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_{n-1} > \rho_n = 0$. Main properties of these eigenvalues are given by (see [19])

$$\sum_{i=1}^{n-1} \rho_i = n \quad \text{and} \quad \sum_{i=1}^{n-1} \rho_i^2 = n + 2R_{-1}, \tag{2.2}$$

where $R_{-1} = \sum_{\{i,j\} \in E} (d_i d_j)^{-1}$ is the general Randić index (see [6, 18]).

3. Main result

In the following theorem we prove the inequality that establishes lower and upper bounds for invariant M_1 in terms of parameters n, m, d_1 and d_n.

Theorem 3.1. *Let $G = (V, E)$ be undirected connected graph with $n, n \ge 2$, vertices and m edges. Then*

$$\frac{4m^2}{n} + \frac{1}{2}(d_1 - d_n)^2 \le M_1 \le \frac{4m^2}{n} + n\alpha(n)(d_1 - d_n)^2, \tag{3.1}$$

where

$$\alpha(n) = \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{(-1)^{n+1}+1}{2n^2}\right) = \begin{cases}\frac{1}{4}, & \textit{if } n \textit{ is even}\\ \frac{n^2-1}{4n^2}, & \textit{if } n \textit{ is odd}\end{cases}$$

Equality holds if and only if G *is regular graph.*

Proof. For $a_i = d_i$, $i = 1, 2, \ldots, n$, $R = d_1$ and $r = d_n$, the inequality (1.3) transforms into

$$n\sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} d_i\right)^2 \geq \frac{n}{2}(d_1 - d_n)^2,$$

i.e. according to (2.1), into

$$nM_1 - 4m^2 \geq \frac{n}{2}(d_1 - d_n)^2, \tag{3.2}$$

wherefrom the left part of inequality (3.1) is obtained.

For $p_i = 1$, $i = 1, 2, \ldots, n$ and $S = \{1, 2, \ldots, k\} \subset I_n$, the expression (1.1) reaches the minimum for $k = \left\lfloor\frac{n}{2}\right\rfloor$. Now for $S = \{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor\}$, $p_i = 1$, $a_i = b_i = d_i$, $i = 1, 2, \ldots, n$, $r_1 = r_2 = d_n$ and $R_1 = R_2 = d_1$, the inequality (1.2) becomes

$$n\sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} d_i\right)^2 \leq (d_1 - d_n)^2\left\lfloor\frac{n}{2}\right\rfloor\left(n - \left\lfloor\frac{n}{2}\right\rfloor\right)$$

i.e.

$$nM_1 - 4m^2 \leq n^2(d_1 - d_n)^2\alpha(n) \tag{3.3}$$

where

$$\alpha(n) = \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{(-1)^{n+1}+1}{2n^2}\right) = \begin{cases}\frac{1}{4}, & \text{if } n \text{ is even}\\ \frac{n^2-1}{4n^2}, & \text{if } n \text{ is odd}\end{cases}.$$

From (3.3) right side of inequality (3.1) immediately follows.

Equalities in (3.2) and (3.3) hold if and only if $d_1 = d_2 = \cdots = d_n$, so the equalities in (3.1) hold if and only if G is a regular graph. □

Remark 3.2. Since $(d_1 - d_n)^2 \geq 0$, left inequality in (3.1) is stronger than

$$M_1 \geq \frac{4m^2}{n} \tag{3.4}$$

which was proved in [9].

Remark 3.3. In [2] the invariant $irr_{EB}(G) = \sqrt{\frac{nM_1}{4m^2} - 1}$ as the irregularity measure of graph was introduced. In [11] another irregularity measure $irr_g(G) = \frac{d_1}{d_n} - 1$ was defined. According to (3.1) we can establish the following relationship between these two measures

$$\sqrt{\frac{nd_n}{8m^2}}irr_g(G) \leq irr_{EB}(G) \leq \sqrt{\frac{n^2 d_n \alpha(n)}{4m^2}}irr_g(G).$$

The linear Laplacian spread of graph G is defined as $\mu_1 - \mu_{n-1}$. The following theorem establishes lower and upper bounds for this invariant in terms of parameters n, m and M_1.

Theorem 3.4. *Let $G = (V, E)$ be undirected connected graph with $n, n \geq 2$, vertices and m edges. Then*

$$\sqrt{\frac{(n-1)(M_1+2m)-4m^2}{(n-1)^2\alpha(n-1)}} \leq \mu_1 - \mu_{n-1} \leq \sqrt{\frac{2((n-1)(M_1+2m)-4m^2)}{n-1}}. \quad (3.5)$$

where

$$\alpha(n-1) = \frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\left(1-\frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\right) = \frac{1}{4}\left(1-\frac{(-1)^n+1}{2(n-1)^2}\right).$$

Equalities hold if and only if G is a complete graph, $G \cong K_n$.

Proof. For $n := n-1$, $a_i = \mu_i$, $i = 1, 2, \ldots, n-1$, $R = \mu_1$ and $r = \mu_{n-1}$, the inequality (1.3) becomes

$$(n-1)\sum_{i=1}^{n-1}\mu_i^2 - \left(\sum_{i=1}^{n-1}\mu_i\right)^2 \geq \frac{(n-1)}{2}(\mu_1-\mu_{n-1})^2$$

i.e. according to (2.1) we have

$$(n-1)(M_1+2m)-4m^2 \geq \frac{(n-1)}{2}(\mu_1-\mu_{n-1})^2 \quad (3.6)$$

wherefrom right side of (3.5) is obtained.

For $n := n-1$, $p_i = 1$, $a_i = b_i = \mu_i$, $i = 1, 2, \ldots, n-1$, $r_1 = r_2 = \mu_{n-1}$ and $R_1 = R_2 = \mu_1$, the inequality (1.2) transforms into

$$(n-1)\sum_{i=1}^{n-1}\mu_i^2 - \left(\sum_{i=1}^{n-1}\mu_i\right)^2 \leq (\mu_1-\mu_{n-1})^2\left\lfloor\frac{n-1}{2}\right\rfloor\left(n-1-\left\lfloor\frac{n-1}{2}\right\rfloor\right)$$

i.e.

$$(n-1)(M_1+2m)-4m^2 \leq (\mu_1-\mu_{n-1})^2(n-1)^2\alpha(n), \quad (3.7)$$

where

$$\alpha(n-1) = \frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\left(1-\frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\right) = \begin{cases}\frac{1}{4} & \text{if } n \text{ is even}\\ \frac{n(n-2)}{4(n-1)^2}, & \text{if } n \text{ is odd}\end{cases}.$$

From (3.7) left part of inequality (3.5) is directly obtained.

Equalities (3.6) and (3.7) hold if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$, hence equalities in (3.5) hold if and only if G is a complete graph, $G \cong K_n$. □

Remark 3.5. Right side of inequality (3.5) was proved in [16]. Since $\alpha(n-1) \leq \frac{1}{4}$, for each n, left side of inequality (3.5) is stronger than

$$\mu_1 - \mu_{n-1} \geq \frac{2}{n-1}\sqrt{(n-1)(M_1+2m)-4m^2},$$

for even n. The above inequality was proved in [10] and [20].

From inequality (3.4) and $M_1 \leq m\left(\frac{2m}{n-1} + (n-2)\right)$, proved in [5], and inequality $M_1 \geq \frac{4m^2}{n}$, proved in [9], the following corollary of Theorem 3.4 holds.

Corollary 3.6. *Let $G = (V, E)$ be undirected connected graph with $n, n \geq 2$, vertices and m edges. Then*

$$\frac{1}{n-1}\sqrt{\frac{2m(n(n-1)-2m)}{n\alpha(n-1)}} \leq \mu_1 - \mu_{n-1} \leq \sqrt{\frac{2m(n(n-1)-2m)}{n-1}}.$$

Equalities hold if and only if G is a complete graph, $G \cong K_n$.

In the following theorem we determine lower and upper bounds for graph invariant R_{-1} in terms of parameters n, ρ_1 and ρ_{n-1}.

Theorem 3.7. *Let $G = (V, E)$ be undirected connected graph with $n, n \geq 2$, vertices and m edges. Then*

$$\frac{n}{2(n-1)} + \frac{1}{4}(\rho_1 - \rho_{n-1})^2 \leq R_{-1} \leq \frac{n}{2(n-1)} + \frac{(n-1)\alpha(n-1)}{2}(\rho_1 - \rho_{n-1})^2. \quad (3.8)$$

Equalities hold if and only if G is a complete graph, $G \cong K_n$.

Proof. for $n := n-1$, $a_i = \rho_i$, $i = 1, 2, \ldots, n-1$, $r = \rho_{n-1}$ and $R = \rho_1$ the inequality (1.3) becomes

$$(n-1)\sum_{i=1}^{n-1}\rho_i^2 - \left(\sum_{i=1}^{n-1}\rho_i\right)^2 \geq \frac{n-1}{2}(\rho_1 - \rho_{n-1})^2$$

i.e. according to (2.2)

$$(n-1)(n + 2R_{-1}) - n^2 \geq \frac{n-1}{2}(\rho_1 - \rho_{n-1})^2, \quad (3.9)$$

wherefrom left side of inequality (3.8) is obtained.

For $n := n-1$, $p_i = 1$, $a_i = b_i = \rho_i$, $i = 1, 2, \ldots, n-1$, $r_1 = r_2 = \rho_{n-1}$ and $R_1 = R_2 = \rho_1$, inequality (1.3) transforms into

$$(n-1)\sum_{i=1}^{n-1}\rho_i^2 - \left(\sum_{i=1}^{n-1}\rho_i\right)^2 \leq (\rho_1 - \rho_{n-1})^2 \left\lfloor\frac{n-1}{2}\right\rfloor \left(n - 1 - \left\lfloor\frac{n-1}{2}\right\rfloor\right),$$

i.e.

$$(n-1)(n + 2R_{-1}) - n^2 \leq (n-1)^2\alpha(n-1)(\rho_1 - \rho_{n-1})^2, \quad (3.10)$$

wherefrom right part of inequality (3.8) is obtained.

Equalities in (3.9) and (3.10) hold if and only if $\rho_1 = \rho_2 = \cdots = \rho_{n-1}$, therefore equalities in (3.8) hold if and only if G is complete graph, $G \cong K_n$. □

Remark 3.8. Since $(\rho_1 - \rho_{n-1})^2 \geq 0$, it follows that left side of inequality (3.8) is stronger than inequality

$$R_{-1} \geq \frac{n}{2(n-1)}$$

which was proved in [14].

Remark 3.9. Inequalities in (3.8) can be presented in an equivalent form as

$$\sqrt{\frac{2(n-1)R_{-1}-n}{(n-1)^2\alpha(n-1)}} \leq \rho_1 - \rho_{n-1} \leq \sqrt{\frac{2(2(n-1)R_{-1}-n)}{n-1}}. \tag{3.11}$$

For even n, left side of inequality (3.11) is stronger than inequality

$$\rho_1 - \rho_{n-1} \geq \frac{2}{n-1}\sqrt{2(n-1)R_{-1}-n},$$

which was proved in [4].

Acknowledgement. This work was supported by the Serbian Ministry of Education and Science, Project No TR32012 and TR32009.

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Conjugate free convection in a vertical channel filled with nanofluid

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Abstract. The steady natural conjugate convection in a long vertical channel filled with a nanofluid and including internal heat generation is presented in this paper. A new mathematical model is proposed for the momentum, energy and nanoparticles' concentration equations. The system of partial differential equations is written in terms of dimensionless velocity, temperature and concentration of the nanoparticles and is solved analytically. The effects of the governing parameters, such as the ratio between the thermophoresis parameter and the Brownian motion parameter, R, and the buoyancy ratio parameter, Nr, on the velocity, temperature and nanoparticles' concentration are studied. It is found that the addition of the nanoparticles into the fluid reduces the temperature and enhances the heat transfer. A limit case when the thermal conductivity of the nanoparticles is much larger than the thermal conductivity of the base fluid has been also studied.

Mathematics Subject Classification (2010): 76R10, 82D15, 82D80.

Keywords: Free convection, heat transfer, nanofluid, Brownian motion, thermophoresis, heat generation, thermal energy.

1. Introduction

General fluids used in industrial processes involving heat transfer (energy generation, insulation, cooling of microelectronic components) are water, mineral oil, ethylene glycol, etc. (see [3], [11]). Low values of the physical properties of these fluids (thermal conductivity, density, expansion coefficient, etc.) limit the efficiency of heat transport and it is necessary to obtain new type of fluids, having improved heat transfer characteristics (see [8]). In order to enhance the thermal characteristics of the fluids, one can form mixtures by adding ultra-fine solid particles (metallic, nonmetallic or polymeric) to the fluid. Choi [7] was probably, the first one who called the fluids with particles of nanometer dimensions nanofluids.

Over the last twenty years, many industrial processes, biology, medicine, catalytic chemistry and environmental applications started to use nanotechnologies (see

[13] and [17]). Different mathematical models were used by several authors to describe heat transfer in nanofluids. Among these models the most used are those where the concentration of nanoparticles is constant and the addition of nanoparticles into the base fluid improved their physical properties (see [18]). Moreover, other models based on physical properties variation include thermal dispersion (see [12]) or Brownian motion (see [14]). A more complex mathematical model (see [4]) considers that nanoparticles' concentration is variable and incorporates the effects of Brownian motion and thermophoresis. Recently, Celli [6] had the idea to combine the model proposed by Buongiorno [4] and the model based on improved physical properties considering for the last one an average concentration of nanoparticles.

Many times theoretical problems as well as industrial processes and natural phenomena are modelled using simple geometries such as infinite channels (see [1], [2]). However, in real simulation it is necessary to take into account the interaction between the convective heat transfer in nanofluid and conductive heat transfer in the thick solid walls. Such situations (i.e. conjugate heat transfer) appears in cellular structures, cavities or channels with solid walls, etc.

Several authors such as Pătrulescu and Groşan [16], Groşan [9], Groşan and Pop [10] and Li [15] have studied the fully developed flow in a vertical channel filled by a nanofluid using different mathematical models for nanofluid and different boundary conditions. In the present paper, the fully developed conjugate heat transfer in a vertical channel filled with a nanofluid when heat generation in the solid wall is considered has been studied analytically.

2. Notations and preliminaries

Consider the fully developed steady conjugate free convection flow of an incompressible nanofluid in vertical channel differentially heated. The left wall of the channel is kept at a constant temperature T_H, while the right wall has a constant temperature T_C. We consider a two-dimensional coordinate frame in which x-axis is aligned vertically upwards, see Figure 1. The left wall is at $y = 0$ and has thickness b, $b > 0$. The right wall is at $y = L$, $L > b$.

The field variables are the temperature in solid domain T_s, the temperature in nanofluid domain T_f, the velocity $\boldsymbol{v} = (u, v)$ and the nanoparticle volume fraction C.

As in [15] it is assumed that the nanoparticles' flux

$$q_c = D_B \nabla C + (D_T/T_f)\, \nabla T_f \tag{2.1}$$

is zero on the solid vertical walls. In (2.1) D_B and D_T are the Brownian and thermophoretic diffusion coefficients.

To define the effective viscosity we use a model proposed in [5], namely

$$\mu_{nf} = \frac{\mu_f}{(1 - C_0)^{2.5}}, \tag{2.2}$$

where μ_f represents the dynamic viscosity of base fluid and C_0 is the reference nanoparticles volume fraction concentration. Moreover, the effective thermal conductivity is approximated by a model introduced in [19], namely

$$k_{nf} = k_f \frac{k_p + 2k_f - 2C_0(k_f - k_p)}{k_p + 2k_f + C_0(k_f - k_p)}, \tag{2.3}$$

where k_f and k_p are the thermal conductivity of the base fluid and thermal conductivity of the nanoparticles, respectively. The behavior of μ_{nf} and k_{nf} as functions of C_0 was studied in [6] for a side heated square cavity. There, the nanofluid is composed of water as base fluid and Alumina as nanoparticles dispersed inside the base fluid.

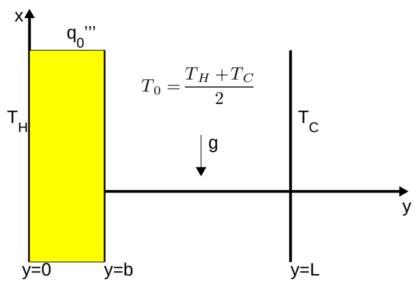

FIGURE 1. Geometry of the problem and the co-ordinate system

The limit case of k_{nf} is

$$k_{nf}^{lim} = k_f \frac{1 + 2C_0}{1 - C_0} \tag{2.4}$$

and it is obtained when the ratio k_f/k_p is very small. Physically this is possible when high thermal conductivity nanopraticles such as cooper, gold, carbon nanotubes are used. Thus, it is possible to predict the maximum achievable temperature in this mathematical model for different kind of fluids and concentrations of nanoparticles. Some examples concerning nanofluids' thermo-physical properties are shown in the following table.

Physical properties	Fluid phase (water)	Cu	Al_2O_3	TiO_3
C_p (J/kgK)	4179	385	765	686.2
ρ (kg/m^3)	997.1	8933	3970	4250
k (W/mK)	0.613	400	40	8.9538
$\beta \times 10^{-5}$ (1/K)	21	1.67	0.85	0.9

3. Basic equations

In this section we provide the governing equations for the flow and heat transfer. Thus, as in [16], we consider the following equation for temperature in solid domain

$$\alpha_s \nabla^2 T_s + \frac{q_0'''}{(\rho c)_s} = 0, \tag{3.1}$$

where q_0''' represents the heat generation, α_s is the thermal diffusivity coefficient and $(\rho c)_s$ represents the heat capacity. As in [10], in the fluid domain we consider the following four field equations in the vectorial form. More exactly, the equations embody the conservation of total mass, momentum, thermal energy and nanoparticles' concentration. Thus, we have

$$\nabla \cdot \boldsymbol{v} = 0, \tag{3.2}$$

$$\rho_f(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \mu_{nf} \nabla^2 \boldsymbol{v} + \{\rho_p C + (1 - C)[\rho_f(1 - \beta(T_f - T_0))]\}\boldsymbol{g}, \tag{3.3}$$

$$(\rho c)_f(\boldsymbol{v} \cdot \nabla T_f) = k_{nf} \nabla^2 T_f + (\rho c)_p[D_B \nabla T_f \cdot \nabla C + (D_T/T_f)\nabla T_f \cdot \nabla T_f], \tag{3.4}$$

$$\boldsymbol{v} \cdot \nabla C = \nabla(D_B \nabla C + (D_T/T_f)\nabla T_f). \tag{3.5}$$

Here ρ_f is the fluid density, ρ_p is the nanoparticle mass density, β represents the thermal expansion coefficient, $\boldsymbol{g}$ is the gravitational acceleration. Finally, $(\rho c)_f$ and $(\rho c)_p$ are the heat capacity of the base fluid and of the nanoparticle material, respectively.

In the rest of the paper we use the following linearized version of the momentum equation (see [10])

$$\rho_f(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \mu_{nf} \nabla^2 \boldsymbol{v} + [(\rho_p - \rho_{f_0})(C - C_0) - (1 - C_0)\rho_{f_0}\beta(T_f - T_0)]\boldsymbol{g}, \tag{3.6}$$

where ρ_{f_0} represents the reference fluid density.

Based on the fact that the flow is fully developed we introduce the following assumptions

$$v = 0, \ \frac{\partial T_f}{\partial x} = 0, \ \frac{\partial T_s}{\partial x} = 0, \ \frac{\partial C}{\partial x} = 0. \tag{3.7}$$

Taking into account (2.1) and (3.7) the governing equations for the flow and heat transfer (3.1)–(3.6) become

$$\alpha_s \frac{d^2 T_s}{dy^2} + \frac{q_0'''}{(\rho c)_s} = 0, \tag{3.8}$$

$$\frac{d^2 T_f}{dy^2} = 0, \tag{3.9}$$

$$D_B \frac{dC}{dy} + \frac{D_T}{T_f}\frac{dT_f}{dy} = 0, \tag{3.10}$$

$$\mu_{nf} \frac{d^2 u}{dy^2} + (1 - C_0)\rho_{f_0}\beta(T_f - T_0)g - (\rho_p - \rho_{f_0})(C - C_0)g = 0, \tag{3.11}$$

subject to the boundary conditions

$$T_s|_{y=0} = T_H, \quad T_f|_{y=L} = T_C, \tag{3.12}$$

$$T_f|_{y=b} = T_s|_{y=b}, \tag{3.13}$$

$$k_s \frac{dT_s}{dy}|_{y=b} = k_{nf}\frac{dT_f}{dy}|_{y=b}, \tag{3.14}$$

$$u(b) = u(L) = 0. \tag{3.15}$$

In (3.14) k_s denotes the thermal conductivity of solid domain. To complete the set of equations and boundary conditions we add the following nanoparticles' conservation relation

$$\int_b^L C(y)\,dy = Q_0, \tag{3.16}$$

where Q_0 is defined in the next section.

4. Dimensionless equations

In this section we solve equations (3.8)–(3.11) subject to (3.12)– (3.16). To this end, we consider the following dimensionless variables used in [16].

$$Y = \frac{y}{L},\ \Theta_s = \frac{k_s(T_s - T_0)}{q_0''' L^2},\ \Theta_f = \frac{k_s(T_f - T_0)}{q_0''' L^2},\ \phi = \frac{C - C_0}{C_0},\ U = \frac{u}{U_c}, \tag{4.1}$$

where U_c is the characteristic velocity given by

$$U_c = \frac{g\beta(\frac{q_0''' L^2}{k_s})L^2}{\nu_f}. \tag{4.2}$$

In (4.2) ν_f represents the kinematic viscosity of base fluid.
We substitute dimensionless variables (4.1) into equations (3.8)–(3.11) and we obtain the following ordinary differential equations

$$\frac{d^2\Theta_s}{dY^2} + 1 = 0, \tag{4.3}$$

$$\frac{d^2\Theta_f}{dY^2} = 0, \tag{4.4}$$

$$\frac{d\phi}{dY} + \frac{R}{w\Theta_f + 1}\frac{d\Theta_f}{dY} = 0, \tag{4.5}$$

$$\frac{d^2U}{dY^2} = N_r(1 - C_0)^{2.5}\phi - (1 - C_0)^{3.5}\Theta_f, \tag{4.6}$$

where w is a dimensionless constant given by

$$w = \frac{q_0''' L^2}{k_s T_0} = \frac{2q_0''' L^2}{k_s(T_H + T_C)} \tag{4.7}$$

and, as in [10], Nr is the buoyancy ratio parameter defined by

$$Nr = \frac{g(\rho_p - \rho_{f_0})C_0 L^2}{\mu_f U_c}. \tag{4.8}$$

Moreover, R is given by

$$R = \frac{N_t}{N_b}, \tag{4.9}$$

and it represents the ration between the thermophoresis parameter and the Brownian motion parameter (see [10]).

The boundary conditions (3.12)–(3.15) become

$$\Theta_s|_{Y=0} = q, \tag{4.10}$$

$$\Theta_s|_{Y=r} = \Theta_f|_{Y=r}, \tag{4.11}$$

$$\frac{d\Theta_s}{dY}|_{Y=r} = K\frac{d\Theta_f}{dY}|_{Y=r}, \tag{4.12}$$

$$\Theta_f|_{Y=1} = -q, \tag{4.13}$$

$$U(r) = U(1) = 0, \tag{4.14}$$

where r, q, K are given by

$$r = \frac{b}{L}, \quad q = \frac{k_s(T_H - T_C)}{2q_0'''L^2}, \quad K = \frac{k_{nf}}{k_s}. \tag{4.15}$$

Finally, we choose $Q_0 = C_0L(1-r)$ such that (3.16) takes the form

$$\int_r^1 \phi(Y)dY = 0. \tag{4.16}$$

5. Results and discussions

In this section we determine the exact solutions of equations (4.3)–(4.6) and discuss the results. Integrating equations (4.3), (4.4) and taking into account boundary conditions (4.10)–(4.13) we obtain

$$\Theta_s(Y) = -\frac{Y^2}{2} + A_1Y + A_2, \tag{5.1}$$

and

$$\Theta_f(Y) = A_3Y + A_4. \tag{5.2}$$

The constants A_1, A_2, A_3, A_4 are given by

$$A_1 = r + K\frac{\frac{1}{2}r^2 + 2q}{r(1-K)-1}, \quad A_2 = q, \tag{5.3}$$

and

$$A_3 = \frac{\frac{1}{2}r^2 + 2q}{r(1-K)-1}, \quad A_4 = -q - A_3. \tag{5.4}$$

Moreover, integrating (4.5) and taking into account (5.2), (5.4) and (4.16) we obtain

$$\phi(Y) = -\frac{R}{w}\Big(\ln(wA_3Y + wA_4 + 1) - \frac{\mathcal{C}_a}{1-r}\Big). \tag{5.5}$$

The constant $\mathcal{C}_a$ is given by

$$\mathcal{C}_a = (1+z)\ln(wA_3 + wA_4 + 1) - (r+z)\ln(wA_3r + wA_4 + 1) - (1-r),$$

where

$$z = \frac{1 + wA_4}{wA_3}.$$

Finally, integrating (4.6) and taking into account (5.2), (5.4), (5.5) and boundary conditions (4.14) we obtain

$$U(Y) = G(Y) + \mathcal{C}_1Y + \mathcal{C}_2. \tag{5.6}$$

The constants $\mathcal{C}_1$, $\mathcal{C}_2$ are given by

$$\mathcal{C}_1 = \frac{G(r) - G(1)}{1 - r}, \quad \mathcal{C}_2 = \frac{rG(1) - G(r)}{1 - r}, \tag{5.7}$$

and function G is defined in the following way

$$G(Y) = (1 - C_0)^{2.5}\Big[- \mathcal{C}_b\Big(\frac{Y^2}{2} + zY + \frac{z^2}{2}\Big)\ln(wA_3Y + wA_4 + 1)$$
$$-(1 - C_0)A_3\frac{Y^3}{6} + \Big(\frac{3}{2}\mathcal{C}_b + \frac{\mathcal{C}_a\mathcal{C}_b}{1 - r} - (1 - C_0)A_4\Big)\frac{Y^2}{2} + z\mathcal{C}_b\frac{Y}{2}\Big],$$

where

$$\mathcal{C}_b = \frac{RN_r}{w}.$$

Next we determine the Nusselt number. For the conjugate wall is defined as:

$$Nu = \frac{hL}{k_f}|_{y=b}, \tag{5.8}$$

where the convective heat transfer coefficient, h, is obtained from relation (see [16]):

$$-k_{nf}\frac{dT_f}{dy}|_{y=b} = h(T_f|_{y=b} - T_0). \tag{5.9}$$

Substituting (5.9) in (5.8) the dimensionless form of Nusselt number becomes:

$$Nu = -\frac{k_{nf}}{k_f}\frac{1}{\Theta_f|_{Y=r}}\frac{d\Theta_f}{dY}|_{Y=r}. \tag{5.10}$$

Tacking into account (5.2) we deduce that

$$Nu = -K\frac{A_3}{A_3r + A_4}. \tag{5.11}$$

Finally, we present the effects of the governing parameters on the velocity, temperature and nanoparticles' concentration. In this study we consider the following fixed values $r = 0.3$, $T_H = 300$, $T_C = 15$, $k_s = 1.2$, $k_f = 0.613$, $C_0 = 0.08$, $q = 1$. Figure 2 presents the variation of the solid temperature profiles for different kind of nanoparticles. It is obvious that the addition of the nanoparticles leads to a decrease of the temperature in solid, specially on the solid-fluid interface and the results for copper nanoparticles are the most close to the limit case. The influence of the parameter R on the nanoparticles' concentration profiles in the limit case is depicted in Figure 3. The concentration profile is almost flat for small values of the parameter R, while for large values large differences between the concentration values on the left and on the right walls appear. The variation of the velocity profiles with R and Nr is given in Figures 4 and 5, respectively. In both cases there is a reversed flow near the right wall. The maximum velocity increases with the increase of R and Nr, respectively.

Acknowledgement. The work of the first author has been partially supported by project POSDRU/159/1.5/S/132400: *Young successful researchers-professional development in an international and interdisciplinary environment* at Babeş-Bolyai University.

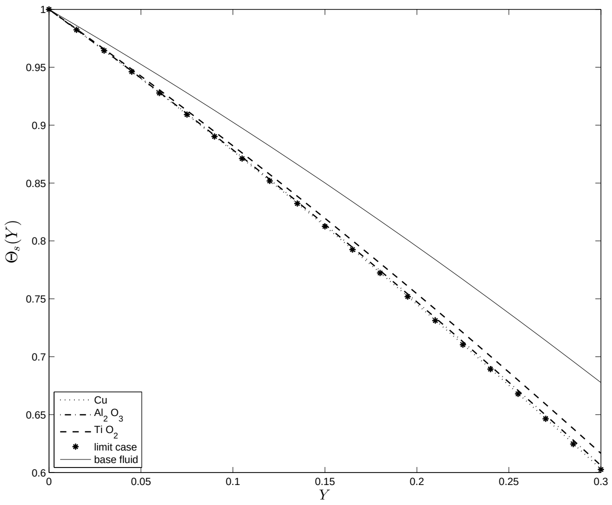

FIGURE 2. Solid temperature

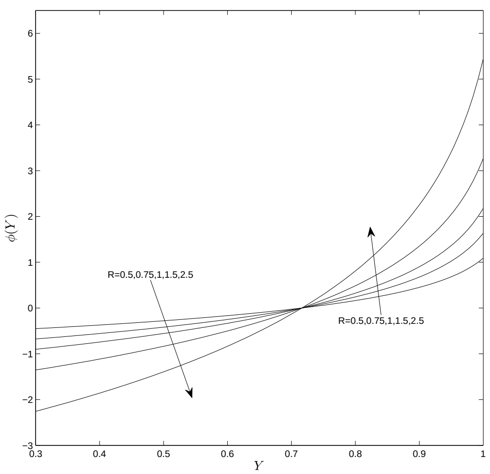

FIGURE 3. Nanoparticles' concentration

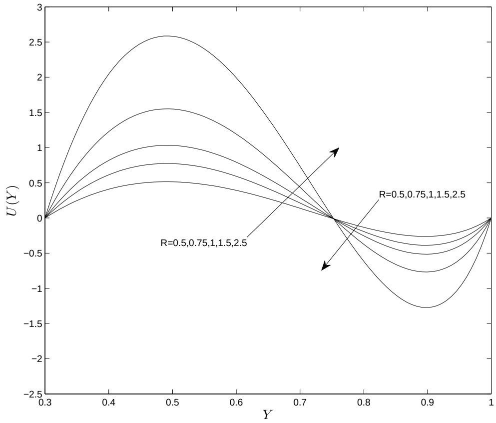

FIGURE 4. Velocity profile, $N_r = 100$

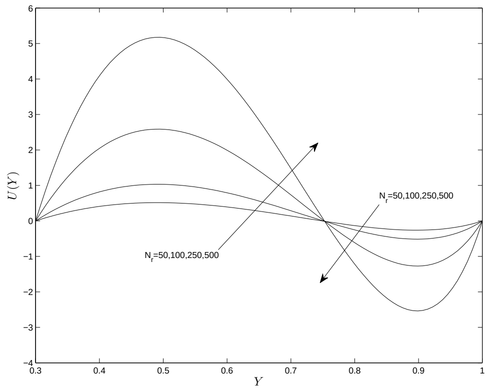

FIGURE 5. Velocity profile, $R = 1$

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 4, 623–628

Book reviews

Thomas Bedürftig und Roman Murawski, Philosophie der Mathematik, 3rd expanded and revised edition, xvii+465 pp, Walter de Gruyter, Berlin, 2015, ISBN: 978-3-11-033117-2/pbk; 978-3-11-033118-9/ebook.

Any person, well acquainted with the history of mathematics, could grasp the fact that the development of mathematics is not a linear one. Indisputably, the most notable conceptual changing happened in the 19 century, a moment in which the analysis of the real numbers, the set theory and the axiomatic approach impose a new paradigm, in which much of the old problems get a mathematical solution and in which new topics do generate new problems. This is the conceptual ground on which, in a very inspired and extremely flexible mode, the mathematics and the philosophy encounter each other in an excellent book on the *philosophy of mathematics*.

A "philosophy of mathematics" in a time in which the mathematics is separated from philosophy? What is the good of it? The answer to this question is also the leading idea of the whole analysis carried out in the chapters of the book and that can be outlined in the following terms: the mathematical solutions to some problems do not necessarily dissolve the philosophical solutions; "we come to decisions but the philosophical questions are not answered. They remained and will remain" (372). This is the key subject organizing in a perfectly coherent totality the six chapters of the book.

Although the starting point, Chapter 1, *On the road to the real numbers*, is a usual one in the present mathematical practice, that of the real number, the analysis reveals a whole range of fundamental philosophical problems connected to this notion: irrationality of a number, incommensurability of some quantities, the geometrical continuum and the real line, the infinite. Surely, since in the new paradigm the set R of the real numbers gives the solutions, these are the problems mathematically decided, but philosophically some of them are still important problems. According to Cantor and Dedekind, for example, the linear continuum is R, but can the set R be the linear continuum? Mathematically, the small infinity isn't a problem any more, but if R doesn't express the intuitive, geometrical continuum, then does not the intuitive meaning be captured, at least partially, by resorting to the classical notion of infinitesimals? The actual infinity, stipulated by the axiom of infinity in the set theory, claims to work with infinite sets as given mathematical objects, but can be understood an infinite series like an usual given object? Moreover, why such abstract, theoretical constructions, as actual infinity and *eo ipso* real numbers, find their applications in the solving of a number of *concrete* problems?

All these matters are the subject of a historical analysis in Chapter 2, *From the history of philosophy and mathematics*, from the antiquity to the recent realism-antirealism controversy in the philosophy of mathematics. This analysis paves the way to some possible answers to the above questions.

The effectiveness of the concept "philosophy of mathematics", proposed by the authors, is masterly illustrated in the next chapters. No doubt, the standard is given by the analysis of the real number, based on the idea of continuum and on the related notion of infinity, the topic of Chapter 3, *About fundamental questions of the philosophy of mathematics*. Sketched, the idea is this. The actual understanding of the continuum (Cantor/Dedekind) is a form of the atomism, the lines and their parts being uncountable sets of points, different from the infinitesimal understanding (Leibniz), in which the lines are made up of an infinite number of infinitely small segments. Is this identification *Continuum* = R the answer to the question about the very nature of the continuum? Mathematically, yes! Philosophically, no, if by "philosophical" we understand the capture of the intuitive meaning of this notion, in which the continuum (e.g. the line) has parts and the parts obtained by division are again the continua. Since the points have no parts, they represent the idea of discontinuity and can form only something discontinuous. Therefore, R is not the intuitive continuum, a notion that, given the above identification, did not survive mathematically, but that is still living philosophically. Can this intuitive meaning be recovered also in a formal mathematical way? On the authors view, the solution is given by the vindication of the infinitesimals, in a non-standard analysis (D.Laugwitz, C.Schmieden, A.Robinson), in which the notion of real number is extended to a non-Archimedean field of the "hyperreal numbers" with infinitely large and infinitely small numbers. In this field besides the real "standard" numbers, at an infinite distance there are "nonstandard" numbers; and this new understanding occurs by reintroducing Leibniz's infinitesimal quantities.

Undoubtedly, one key problem of the philosophy of mathematics is that concerning the *nature* of mathematical objects. Since the mathematical concepts can be defined in set-theoretic terms, they are reducible to the notion of set. And, in this way, the question referring to the nature of mathematical objects (e.g. real number) reduces to the question of what kind of entities the sets are. Chapter 4, *The sets and the theories of sets*, integrates therefore in the structure of the book, in a perfectly coherent way, the *ontological* problems of what actually the sets are (hard to say!) in the terms of the way they are defined, i.e. in the terms of the axioms describing the relations between them. Two axiomatic theories of sets (Zermelo-Frenkel and Neuman-Bernays-Gödel), in an elegant comparative analysis, characterize ontological this central notion of mathematical thinking that, by integration of the notion of set, becomes set-theoretic thinking.

Certainly, the new methodological paradigm of the present mathematics is represented by the set theory and logic, *axiomatically organized.* Then, in a natural way, the analysis continues with a distinct Chapter 5, *The axiomatic and the logic*, focused on the new mathematical and philosophical problems, as the problem of the consistency of mathematics, that of completeness, the problem of the nature of mathematical truth and the re-thinking of the continuum. Directly or indirectly, all these problems are originated in the axiomatic setting of the present mathematical theories. They are those which reveals the “distance” between the provability and the truth and, at last, the limits of this kind of mathematical thinking.

A *Retrospection*, in Chapter 6, about the topics developed in the book and some technical considerations on *To think and to calculate infinitesimally*, concludes this excellent book on the philosophy of mathematics.

The historical, mathematical and philosophical perspectives from which the key notions are investigated are very different. But the authors avoid to give “definitive” solutions to these problems. Often we find the expressions of the following form: ”We don’t know what is the point” (203), ”R is not the continuum, R is a model” (264), “What exactly are the sets, nobody knows” (310), “We do not find an effective answer to the question about numbers” (377), “For the set theory, for the theory of infinite, the infinite remains enigmatical” (314), “Even mathematically, the continuum remains transcendent” (227). Therefore, the fundamental pieces of the philosophical and mathematical investigation “remain transcendent” (386). In a perfect consonance with the project of the construction of the book, the source of the transcendence is to be viewed in the essential difference between a philosophical notion and its mathematical description, and from here the necessity of the *philosophical analysis* of the fundamental mathematical meanings.

The text of the book has an exemplary clearness, all the key notions are indicated in the distinct paragraphs and the bibliographical list of the papers to which the authors are referred is impressively vast and relevant.

The conceptual strictness, the quality of argumentation and the originality of the analysis make of this book one of the profoundest and more elegant conceptual construction in the philosophy of mathematics.

Virgil Drăghici

Gerard Walschap, Multivariable Calculs and Differential Geometry, Walter de Gruyter (De Gruyter Graduate), 2015, ix+355 pages, Paperback, ISBN 978–3-11-036949-6.

The book under review is a rigorous introduction to differential geometry (mainly of hypersurfaces in a Euclidean space) based on a solid foundation of calculus and linear algebra.

I give below a short description of the contents. The first chapter is a short introduction to the topology of a Euclidean space. The second chapter (called, generically, *Differentiation*) is concerned with a study of the mappings between two Euclidean spaces, including, among the classical stuff (Taylor series, implicit functions), a thorough introduction to vector field and Lie brackets, as well as the partition of unity on an open set of a Euclidean space. What these two chapter have in common is the consistent use of the tools of linear algebra (for instance, à la manière de *Calculus on Manifolds*, Addison Wesley, 1965, by Michael Spivak). This language is not, usually, the favorite of the analysts, although we can hardly call it "modern" (the book of Spivak has been published fifty years ago). The third chapter is the first one with a geometric flavor. Although the name of the chapter is, simply, *Manifolds*, in reality, it is a short introduction to the geometry of the manifolds (well, actually submanifolds of a Euclidean space). It includes, beside the standard notions of differential topology (manifolds, Lie groups, maps, vector fields, the tangent bundle), also differential geometric concepts as: covariant derivative, geodesics, the second fundamental form, curvatures, isometries (with respect to the Euclidean scalar product on the ambient Euclidean spaces). The following chapter is devoted to the theory of integration on a Euclidean space. It includes the classical subjects (the definition of the integral and of the integrability, the Fubini theorem, the classical integral theorems), but also some applications to physics and the Sard theorem for Euclidean spaces, a quite unexpected (but not inappropriate) presence here. The chapter five, *Differential forms*, has, actually, as subject the theory of integration on manifolds (tensors and forms, differential forms on manifolds, integration on manifolds and manifolds with boundary, Stokes' theorem). The author takes time to discuss the connection between the Stokes' theorem and the classical theorems of integral calculus. The chapter ends with a short look at the de Rham cohomology. For the last two chapters, the author returns to geometry. Thus, chapter six treats standard subjects from Riemannian geometry (extremal properties of geodesics, Jacobi fields, the variation of the length functional, Hopf-Rinow theorem, comparison theorems a.o.), while the final chapter focuses on the geometry of hypersurfaces in a Euclidean space, with the induced Riemannian metric (orientation, Gauss map, curvature a.o.). In this context, the general concepts from manifold theory become more intuitive. In particular, there are also discussed some classical examples of surfaces (ruled surface, surfaces of revolution).

Multivariable calculus and geometry always meet (well, at least in textbooks). The question is *where* do they meet. I could mention several such intersection points:

- analytical geometry that goes together with calculus (this is common, especially, in the American universities);
- the calculus as a prerequisite to a textbook of manifolds;
- the other way around: (sub)manifolds used as a foundation for multiple integration (for instance).

What I haven't seen elsewhere (or, if I have, I don't remember) is a marriage between calculus and differential geometry, where the two subjects to be, more or less, on equal footing. From this point of view, the author did an excellent job. The book can be used as a textbook for a course in differential geometry for advanced undergraduate or beginning graduate student in mathematics or physics, or for self-study.

The book includes a large number of worked examples and exercises, as well as a number of excellent drawings that improve the presentation.

Paul A. Blaga

The Princeton Companion to Applied Mathematics, Edited by Nicholas J. Higham, Associate Editors: Mark R. Dennis, Paul Glendinning, Paul A. Martin, Fadil Santosa and Jared Tanner, Princeton University Press, Princeton, NJ, 2015, xvii + 994 pp, ISBN: 978-0-691-15039-0/hbk; 978-1-4008-7447-7/ebook.

As the editor writes in the Preface "*The Princeton Companion to Applied Mathematics* describes what applied mathematics is about, why it is important, its connections with other disciplines, and some of the main areas of current research." We also reproduce here the nice words of Paul Halmos quoted in Editor's (NJH) article *What is applied mathematics*?

> Pure mathematics can be practically useful and applied mathematics can be artistically elegant... Just as pure mathematics can be useful, applied mathematics can be more beautifully useless than is sometimes recognized...

On the other side, it is not easy to give a precise definition of applied mathematics and, in some cases, it is difficult to say whether a specific domain (or topic) belongs to pure or to applied mathematics. Also, over time, a domain of pure mathematics finds its applications, as, e.g., the applications of number theory to cryptography. Pure mathematics was presented in another Princeton volume *The Princeton companion to mathematics* (Editors: Timothy Gowers, June Barrow-Green, Imre Leader; Princeton University Press, 2008), which contains some topics (Mathematics and Chemistry, Mathematical Biology, Mathematical Statistics, Optimization and Lagrange Multipliers) which can be considered to belong to applied mathematics and could be included in the present volume as well. In order to avoid overlapping the topics presented in the previous Companion are excluded from the present one, and in the case of some crucial concepts (e.g. algebraic geometry, fast Fourier transform), the approach here is different, with emphasis on applications and computational aspects. In some cases, particular aspects of topics treated in the previous companion are included here.

The book is divided into eight parts.

Part I, *Introduction to Applied Mathematics*, contains some general results about applied mathematics as: what is it, the language, algorithms, goals and history.

Part II, *Concepts*, contains short articles explaining specific concepts as convexity, chaos, floating-point arithmetic, Markov chains, etc. This part is not a comprehensive, other concepts being presented in other articles.

Part III, *Equations, Laws, and Functions of Applied Mathematics*, contains short presentations of some functions and equations encountered in applied mathematics, as, e.g., Bessel functions, Mathieu functions, Euler functions, Black-Scholes law, Hooke's law, the equations of Cauchy-Riemann, Laplace, Korteweg-de Vries, Dirac, etc.

Part IV, *Areas of Applied Mathematics*, contains longer articles giving an overview of some domains of applied mathematics as complex analysis, ordinary and partial differential equations, data mining, random matrices, control theory, information theory, etc.

Part V, *Modeling*, presents some mathematical models in chemistry, biology, financial mathematics, wheather prediction, etc.

Part VI, *Example Problems*, contains short articles on various interesting problems as bubbles, foams, the inverted pendulum, robotics, random number generation, etc.

Part VII, *Application Areas* contains articles on connections of mathematics with other domains as aircraft noise, social networks, chip design, digital imaging, medical imaging, radar imaging, etc.

The final part, VIII, *Final Perspectives*, contains some longer articles of general interest on mathematical writing, the reading of mathematical papers, teaching applied mathematics, mathematics in the media, mathematics and policy (how mathematicians can influence political decisions).

In the last years, partly due to high performance computers, the area of applied mathematics enlarged considerably, so that the present volume is a welcome addition to the existing publications and a guide for the researchers interested to apply mathematics in their domains (an unavoidable option, as it can be seen).

Together with its elder brother, *The Princeton companion to mathematics*, which turned to be a very successful enterprise, the present volume covers a lot of topics in applied mathematics, and surely it will also have great success and impact. It is dedicated to students (starting with the undergraduate level), teachers, researchers in various areas and specialists in various domains desiring to know what mathematics can offer and, as it is proved in this volume, it has a lot to offer.

S. Cobzaş