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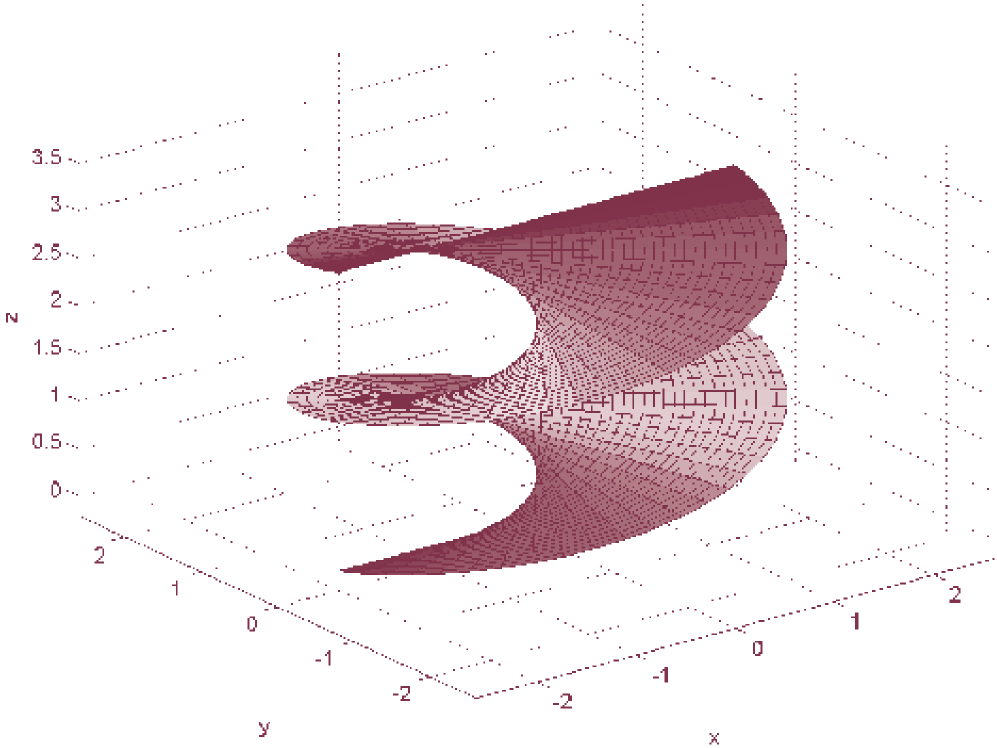

MATHEMATICA

3/2015

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA

3/2015

YEAR **(LX) 2015**
MONTH **SEPTEMBER**
ISSUE **3**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
3

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CONTENTS

Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 355–366

Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals

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Abstract. In this paper, firstly we have established Hermite–Hadamard-Fejér inequality for fractional integrals. Secondly, an integral identity and some Hermite-Hadamard-Fejér type integral inequalities for the fractional integrals have been obtained. The some results presented here would provide extensions of those given in earlier works.

Mathematics Subject Classification (2010): 26A51, 26A33, 26D10.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejér inequality, Riemann-Liouville fractional integral, convex function.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [4].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [3], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a + b)/2$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [1, 5, 6, 7, 12, 16].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.2. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt,\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt \text{ and } J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x).$$

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [2, 8, 9, 10, 14, 15, 17, 18].

In [14], Sarıkaya et. al. represented Hermite–Hadamard's inequalities in fractional integral forms as follows.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha > 0$.

In [14] some Hermite-Hadamard type integral inequalities for fractional integral were proved using the following lemma.

Lemma 1.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$ then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \tag{1.4}$$

$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)\,dt.$$

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$ then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \\ \leq\; & \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}} \right) \left[|f'(a)| + |f'(b)| \right]. \end{aligned} \tag{1.5}$$

Lemma 1.6 ([11, 18])**.** *For $0 < \alpha \leq 1$ and $0 \leq a < b$, we have*

$$|a^{\alpha} - b^{\alpha}| \leq (b-a)^{\alpha}.$$

In this paper, we firstly represented Hermite-Hadamard-Fejér inequality in fractional integral forms which is the weighted generalization of Hermite-Hadamard inequality (1.3). Secondly, we obtained some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for the fractional integrals.

2. Main results

Throughout this section, let $\|g\|_{\infty} = \sup_{t \in [a,b]} |g(x)|$, for the continuous function $g : [a,b] \to \mathbb{R}$.

Lemma 2.1. *If $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $(a+b)/2$ with $a < b$, then*

$$J_{a+}^{\alpha} g(b) = J_{b-}^{\alpha} g(a) = \frac{1}{2} \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a) \right]$$

with $\alpha > 0$.

Proof. Since g is symmetric to $(a+b)/2$, we have $g(a+b-x) = g(x)$, for all $x \in [a,b]$. Hence, in the following integral setting $x = a+b-t$ and $dx = -dt$ gives

$$\begin{aligned} J_{a+}^{\alpha} g(b) &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} g(x) dx \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} g(a+b-t) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} g(t) dt = J_{b-}^{\alpha} g(a). \end{aligned}$$

This completes the proof. □

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a+b)/2$, then the following* *inequalities for fractional integrals hold*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] &\leq \left[J_{a+}^{\alpha}\left(fg\right)\left(b\right)+J_{b-}^{\alpha}\left(fg\right)\left(a\right)\right] \\
&\leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]
\end{aligned} \tag{2.1}
$$

with $\alpha>0$.

Proof. Since f is a convex function on $[a,b]$, we have for all $t\in[0,1]$

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) \\
&\leq \frac{f\left(ta+(1-t)b\right)+f\left(tb+(1-t)a\right)}{2}.
\end{aligned} \tag{2.2}
$$

Multiplying both sides of (2.2) by $2t^{\alpha-1}g\left(tb+(1-t)a\right)$ then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$
\begin{aligned}
& 2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}g\left(tb+(1-t)a\right)dt \\
\leq\ & \int_0^1 t^{\alpha-1}\left[f\left(ta+(1-t)b\right)+f\left(tb+(1-t)a\right)\right]g\left(tb+(1-t)a\right)dt \\
=\ & \int_0^1 t^{\alpha-1}f\left(ta+(1-t)b\right)g\left(tb+(1-t)a\right)dt \\
& +\int_0^1 t^{\alpha-1}f\left(tb+(1-t)a\right)g\left(tb+(1-t)a\right)dt.
\end{aligned}
$$

Setting $x=tb+(1-t)a$, and $dx=(b-a)\,dt$ gives

$$
\begin{aligned}
& \frac{2}{(b-a)^{\alpha}}f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\alpha-1}g\left(x\right)dx \\
\leq\ & \frac{1}{(b-a)^{\alpha}}\left\{\int_a^b (x-a)^{\alpha-1}f\left(a+b-x\right)g\left(x\right)dx+\int_a^b (x-a)^{\alpha-1}f\left(x\right)g\left(x\right)dx\right\} \\
=\ & \frac{1}{(b-a)^{\alpha}}\left\{\int_a^b (b-x)^{\alpha-1}f\left(x\right)g\left(a+b-x\right)dx+\int_a^b (x-a)^{\alpha-1}f\left(x\right)g\left(x\right)dx\right\} \\
=\ & \frac{1}{(b-a)^{\alpha}}\left\{\int_a^b (b-x)^{\alpha-1}f\left(x\right)g\left(x\right)dx+\int_a^b (x-a)^{\alpha-1}f\left(x\right)g\left(x\right)dx\right\}.
\end{aligned}
$$

Therefore, by Lemma 2.1 we have

$$
\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\leq\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\left(fg\right)\left(b\right)+J_{b-}^{\alpha}\left(fg\right)\left(a\right)\right]
$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if f is a convex function, then, for all $t\in[0,1]$, it yields

$$
f\left(ta+(1-t)b\right)+f\left(tb+(1-t)a\right)\leq f(a)+f(b). \tag{2.3}
$$

Then multiplying both sides of (2.3) by $2t^{\alpha-1}g(tb+(1-t)a)$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} f(ta+(1-t)b)\, g(tb+(1-t)a)\, dt \\ & + \int_0^1 t^{\alpha-1} f(tb+(1-t)a)\, g(tb+(1-t)a)\, dt \\ \leq\ & [f(a)+f(b)] \int_0^1 t^{\alpha-1} g(tb+(1-t)a)\, dt \end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right] \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right]$$

The proof is completed. □

Remark 2.3. In Theorem 2.2,
(i) if we take $\alpha = 1$, then inequality (2.1) becomes inequality (1.2) of Theorem 1.1.
(ii) if we take $g(x) = 1$, then inequality (2.1) becomes inequality (1.3) of Theorem 1.3.

Lemma 2.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $(a+b)/2$ then the following equality for fractional integrals holds*

$$\begin{aligned} & \left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right] \\ =\ & \frac{1}{\Gamma(\alpha)} \int_a^b \left[\int_a^t (b-s)^{\alpha-1} g(s)ds - \int_t^b (s-a)^{\alpha-1} g(s)ds\right] f'(t)dt \qquad (2.4) \end{aligned}$$

with $\alpha > 0$.

Proof. It suffices to note that

$$\begin{aligned} I &= \int_a^b \left[\int_a^t (b-s)^{\alpha-1} g(s)ds - \int_t^b (s-a)^{\alpha-1} g(s)ds\right] f'(t)dt \\ &= \int_a^b \left(\int_a^t (b-s)^{\alpha-1} g(s)ds\right) f'(t)dt + \int_a^b \left(-\int_t^b (s-a)^{\alpha-1} g(s)ds\right) f'(t)dt \\ &= I_1 + I_2. \end{aligned}$$

By integration by parts and Lemma 2.1 we get

$$\begin{aligned}
I_1 &= \left(\int_a^t (b-s)^{\alpha-1} g(s)ds\right) f(t)\Big|_a^b - \int_a^b (b-t)^{\alpha-1} g(t)f(t)dt \\
&= \left(\int_a^b (b-s)^{\alpha-1} g(s)ds\right) f(b) - \int_a^b (b-t)^{\alpha-1} (fg)(t)dt \\
&= \Gamma(\alpha)\left[f(b)J_{a+}^{\alpha}g(b) - J_{a+}^{\alpha}(fg)(b)\right] \\
&= \Gamma(\alpha)\left[\frac{f(b)}{2}\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - J_{a+}^{\alpha}(fg)(b)\right]
\end{aligned}$$

and similarly

$$\begin{aligned}
I_2 &= \left(-\int_t^b (s-a)^{\alpha-1} g(s)ds\right) f(t)\Big|_a^b - \int_a^b (t-a)^{\alpha-1} g(t)f(t)dt \\
&= \left(\int_a^b (s-a)^{\alpha-1} g(s)ds\right) f(a) - \int_a^b (t-a)^{\alpha-1} (fg)(t)dt \\
&= \Gamma(\alpha)\left[\frac{f(a)}{2}\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - J_{b-}^{\alpha}\left(fg\right)(a)\right].
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
I &= I_1 + I_2 \\
&= \Gamma(\alpha)\left\{\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right\}.
\end{aligned}$$

Multiplying the both sides by $(\Gamma(\alpha))^{-1}$ we obtain (2.4) which completes the proof. □

Remark 2.5. In Lemma 2.4, if we take $g(x) = 1$, then equality (2.4) becomes equality (1.4) of Lemma 1.4.

Theorem 2.6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds*

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\
\leq\ & \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[|f'(a)| + |f'(b)|\right] \qquad (2.5)
\end{aligned}$$

with $\alpha > 0$.

Proof. From Lemma 2.4 we have

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_a^b \left|\int_a^t (b-s)^{\alpha-1} g(s)ds - \int_t^b (s-a)^{\alpha-1} g(s)ds\right| |f'(t)|\, dt. \qquad (2.6)
\end{aligned}$$

Since $|f'|$ is convex on $[a,b]$, we know that for $t \in [a,b]$

$$|f'(t)| = \left| f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right| \leq \frac{b-t}{b-a}\left|f'(a)\right| + \frac{t-a}{b-a}\left|f'(b)\right|, \tag{2.7}$$

and since $g : [a,b] \to \mathbb{R}$ is symmetric to $(a+b)/2$ we write

$$\int_t^b (s-a)^{\alpha-1} g(s)ds = \int_a^{a+b-t} (b-s)^{\alpha-1} g(a+b-s)ds = \int_a^{a+b-t} (b-s)^{\alpha-1} g(s)ds,$$

then we have

$$\begin{aligned} & \left| \int_a^t (b-s)^{\alpha-1} g(s)ds - \int_t^b (s-a)^{\alpha-1} g(s)ds \right| \\ = & \left| \int_t^{a+b-t} (b-s)^{\alpha-1} g(s)ds \right| \\ \leq & \begin{cases} \int_t^{a+b-t} \left|(b-s)^{\alpha-1} g(s)\right| ds, & t \in \left[a, \frac{a+b}{2}\right] \\ \int_{a+b-t}^t \left|(b-s)^{\alpha-1} g(s)\right| ds, & t \in \left[\frac{a+b}{2}, b\right] \end{cases}. \end{aligned} \tag{2.8}$$

A combination of (2.6), (2.7) and (2.8), we get

$$\begin{aligned} & \left| \left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] - \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right] \right| \\ \leq & \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \left|(b-s)^{\alpha-1} g(s)\right| ds \right) \left(\frac{b-t}{b-a}\left|f'(a)\right| + \frac{t-a}{b-a}\left|f'(b)\right| \right) dt \\ & + \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \left|(b-s)^{\alpha-1} g(s)\right| ds \right) \left(\frac{b-t}{b-a}\left|f'(a)\right| + \frac{t-a}{b-a}\left|f'(b)\right| \right) dt \\ \leq & \frac{\|g\|_\infty}{(b-a)\Gamma(\alpha+1)} \left\{ \int_a^{\frac{a+b}{2}} \left[(b-t)^\alpha - (t-a)^\alpha\right]\left((b-t)\left|f'(a)\right| + (t-a)\left|f'(b)\right|\right) dt \right. \\ & \left. + \int_{\frac{a+b}{2}}^b \left[(t-a)^\alpha - (b-t)^\alpha\right]\left((b-t)\left|f'(a)\right| + (t-a)\left|f'(b)\right|\right) dt \right\} \end{aligned} \tag{2.9}$$

Since

$$\begin{aligned} & \int_a^{\frac{a+b}{2}} \left[(b-t)^\alpha - (t-a)^\alpha\right](b-t)\, dt \\ = & \int_{\frac{a+b}{2}}^b \left[(t-a)^\alpha - (b-t)^\alpha\right](t-a)\, dt \\ = & \frac{(b-a)^{\alpha+2}}{(\alpha+1)} \left(\frac{\alpha+1}{\alpha+2} - \frac{1}{2^{\alpha+1}} \right) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left[(b-t)^{\alpha}-(t-a)^{\alpha}\right](t-a)\,dt \\
&= \int_{\frac{a+b}{2}}^{b} \left[(t-a)^{\alpha}-(b-t)^{\alpha}\right](b-t)\,dt \\
&= \frac{(b-a)^{\alpha+2}}{(\alpha+1)}\left(\frac{1}{\alpha+2}-\frac{1}{2^{\alpha+1}}\right) \qquad (2.11)
\end{aligned}$$

Hence, if we use (2.10) and (2.11) in (2.9), we obtain the desired result. This completes the proof. □

Remark 2.7. In Theorem 2.6, if we take $g(x)=1$, then equality (2.5) becomes equality (1.5) of Theorem 1.5.

Theorem 2.8. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^{q}, q>1$, is convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \qquad (2.12)$$

$$\leq \frac{2(b-a)^{\alpha+1}\|g\|_{\infty}}{(b-a)^{1/q}(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{1/q}$$

where $\alpha>0$ and $1/p+1/q=1$.

Proof. Using Lemma 2.4, Hölder's inequality, (2.8) and the convexity of $|f'|^{q}$, it follows that

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)ds\right|dt\right)^{1-1/q} \\
&\times\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)ds\right|\left|f'(t)\right|^{q}dt\right)^{1/q} \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\right]^{1-1/q} \\
&\times\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left|f'(t)\right|^{q}dt\right.
\end{aligned}$$

$$+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left|f'(t)\right|^{q}dt\Bigg]^{1/q}$$

$$\leq\frac{2^{1-1/q}\left\|g\right\|_{\infty}}{(b-a)^{1/q}\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{1-1/q}$$

$$\times\left\{\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]\left((b-t)\left|f'(a)\right|^{q}+(t-a)\left|f'(b)\right|^{q}\right)dt\right.$$

$$\left.+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]\left((b-t)\left|f'(a)\right|^{q}+(t-a)\left|f'(b)\right|^{q}\right)dt\right\}^{1/q} \tag{2.13}$$

where it is easily seen that

$$\int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{a+b-t}(b-s)^{\alpha-1}ds\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}(b-s)^{\alpha-1}ds\right)dt$$

$$=\frac{2\,(b-a)^{\alpha+1}}{\alpha\,(\alpha+1)}\left[1-\frac{1}{2^{\alpha}}\right].$$

Hence, if we use (2.10) and (2.11) in (2.13), we obtain the desired result. This completes the proof. □

We can state another inequality for $q>1$ as follows:

Theorem 2.9. *Let $f:I\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^{q}, q>1$, is convex on $[a,b]$ and $g:[a,b]\rightarrow\mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals hold:*

(i) $$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq\frac{2^{1/p}\left\|g\right\|_{\infty}(b-a)^{\alpha+1}}{(\alpha p+1)^{1/p}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)^{1/p}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{1/q} \tag{2.14}$$

with $\alpha>0$.

(ii) $$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq\frac{\left\|g\right\|_{\infty}(b-a)^{\alpha+1}}{(\alpha p+1)^{1/p}\Gamma(\alpha+1)}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{1/q} \tag{2.15}$$

for $0<\alpha\leq1$, where $1/p+1/q=1$.

Proof. (i) Using Lemma 2.4, Hölder's inequality, (2.8) and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)ds\right|^p dt\right)^{1/p}\left(\int_a^b|f'(t)|^q dt\right)^{1/q}. \\
\leq\ & \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^\alpha-(t-a)^\alpha\right]^p dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^\alpha-(b-t)^\alpha\right]^p dt\right)^{1/p} \\
& \times\left(\int_a^b\left(\frac{b-t}{b-a}\left|f'(a)\right|^q+\frac{t-a}{b-a}\left|f'(b)\right|^q\right)dt\right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
=\ & \frac{\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^\alpha-t^\alpha\right]^p dt+\int_{\frac{1}{2}}^1\left[t^\alpha-(1-t)^\alpha\right]^p dt\right)^{1/p} \\
& \times\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{1/q} \qquad (2.16)
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right]dt\right)^{1/p} \\
& \times\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{1/q}
\end{aligned}$$

$$\leq \frac{\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\frac{2}{\alpha p+1}\left[1-\frac{1}{2^{\alpha p}}\right]\right)^{1/p}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{1/q}.$$

Here we use

$$\left[(1-t)^\alpha-t^\alpha\right]^p\leq(1-t)^{\alpha p}-t^{\alpha p}$$

for $t\in[0,1/2]$ and

$$\left[t^\alpha-(1-t)^\alpha\right]^p\leq t^{\alpha p}-(1-t)^{\alpha p}$$

for $t\in[1/2,1]$, which follows from

$$(A-B)^q\leq A^q-B^q,$$

for any $A\geq B\geq 0$ and $q\geq 1$. Hence the inequality (2.14) is proved.

(ii) The inequality (2.15) is easily proved using (2.16) and Lemma 1.6. □

Remark 2.10. In Theorem 2.9, if we take $\alpha=1$, then equality (2.15) becomes equality in [18, Corollary 13].

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 367–377

On some generalized integral inequalities for φ-convex functions

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Abstract. The main goal of the paper is to state and prove some new general inequalities for φ-convex function.

Mathematics Subject Classification (2010): 26D15, 41A55, 26D10.

Keywords: Hermite-Hadamard's inequality, convex function, φ-convex function, Hölder's inequality.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[4], [8, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([3]-[15]) and the references cited therein.

Let us consider a function $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the φ-convex functions in [16], but we work here with the improved definition, according to [1]:

Definition 1.1. *A function $f : [a, b] \to \mathbb{R}$ is said to be φ- convex on $[a, b]$ if for every two points $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)).$$

Obviously, if function φ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the φ-convex functions can be found, for instance, in [1], [2], [16], [17], [18].

Moreover in [2], Cristescu have presented a version Hermite-Hadamard type inequality for the φ-convex functions as follows:

Theorem 1.2. *If a function $f:[a,b]\to\mathbb{R}$ is φ- convex for the continuous function $\varphi:[a,b]\to[a,b]$, then*

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)}f(x)dx\leq\frac{f(\varphi(a))+f(\varphi(b))}{2}. \tag{1.2}$$

In this article, we will consider two parts which within the first section we give some new general inequalities for φ-convex function. In the second part, using functions whose derivatives absolute values are φ-convex function, we obtained new inequalities related to the left and the right sides of Hermite-Hadamard inequality are given with (2.1).

2. Hermite-Hadamard type inequality for φ-convex function

Theorem 2.1. *Let J be an interval $a,b\in J$ with $a<b$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a φ-convex function on $I=[a,b]$, then we have*

$$\begin{aligned}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)&\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(\varphi(x))\,d\varphi(x)\\&\leq\frac{f(\varphi(a))+f(\varphi(b))}{2}.\end{aligned}\tag{2.1}$$

Proof. By definition of the φ-convex function

$$\begin{aligned}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)&=\int_0^1 f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)dt\\&=\int_0^1 f\left(\frac{(1-t)\varphi(a)+t\varphi(b)+t\varphi(a)+(1-t)\varphi(b)}{2}\right)dt\\&\leq\frac{1}{2}\int_0^1\left[f((1-t)\varphi(a)+t\varphi(b))+f(t\varphi(a)+(1-t)\varphi(b))\right]dt.\end{aligned}$$

Using the change of the variable in last integrals, we get

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(\varphi(x))\,d\varphi(x). \tag{2.2}$$

By similar way, we have

$$\begin{aligned}\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x) &= \int_0^1 f((1-t)\varphi(a)+t\varphi(b))\,dt \\ &\le \int_0^1 [(1-t)f(\varphi(a))+tf(\varphi(b))]\,dt \\ &= \frac{f(\varphi(a))+f(\varphi(b))}{2}. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), it is obtained desired result. □

Remark 2.2. If we choose $\varphi(x) = x$ for all $x \in [a,b]$ in Theorem 2.1, the (2.1) inequality reduce to the inequality (1.1).

Theorem 2.3. *Let J be an interval $a, b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let f be a φ-convex function on $I = [a,b]$ and let $w : [\varphi(a), \varphi(b)] \to \mathbb{R}$ be nonnegative, integrable and symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$. Then*

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\int_{\varphi(a)}^{\varphi(b)} w(\varphi(x))\,d\varphi(x) \le \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)$$

$$\le \frac{f(\varphi(a))+f(\varphi(b))}{2}\int_{\varphi(a)}^{\varphi(b)} w(\varphi(x))\,d\varphi(x). \tag{2.4}$$

Proof. Since f be a φ-convex function and $w : [\varphi(a), \varphi(b)] \to \mathbb{R}$ be nonnegative, integrable and symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$, then we obtain

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\int_{\varphi(a)}^{\varphi(b)} w(\varphi(x))\,d\varphi(x) = \int_{\varphi(a)}^{\varphi(b)} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) w(\varphi(x))\,d\varphi(x)$$

$$\le \frac{1}{2}\int_{\varphi(a)}^{\varphi(b)} [f(\varphi(a)+\varphi(b)-\varphi(x))+f(\varphi(x))]\,w(\varphi(x))\,d\varphi(x)$$

$$= \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)$$

$$= \frac{1}{2}\int_{\varphi(a)}^{\varphi(b)} [f(\varphi(a)+\varphi(b)-\varphi(x))]\,w(\varphi(x))\,d\varphi(x) + \frac{1}{2}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)$$

$$\le \frac{1}{2}\int_{\varphi(a)}^{\varphi(b)} [f(\varphi(a))+f(\varphi(b))]\,w(\varphi(x))\,d\varphi(x)$$

$$= \frac{f(\varphi(a))+f(\varphi(b))}{2}\int_{\varphi(a)}^{\varphi(b)} w(\varphi(x))\,d\varphi(x)$$

which completes the proof of Theorem 2.3. □

Corollary 2.4. *Under the same assumptions of Theorem 2.3 with $\varphi(x) = x$ for all $x \in [a,b]$, we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)\,w(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx.$$

Theorem 2.5. *Let J be an interval $a,b \in J$ with $a<b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f,w : I \subseteq \mathbb{R} \to \mathbb{R}$ be a φ-convex and nonnegative function on $I=[a,b]$. Then, for all $t \in [0,1]$, we have*

$$\begin{aligned}2f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)w\left(\frac{\varphi(a)+\varphi(b)}{2}\right) &\leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)\\ &\leq \frac{1}{6}M(\varphi(a),\varphi(b))+\frac{1}{3}N(\varphi(a),\varphi(b)) \qquad (2.5)\end{aligned}$$

where

$$\begin{aligned}M(\varphi(a),\varphi(b)) &= f(\varphi(a))\,w(\varphi(a))+f(\varphi(b))\,w(\varphi(b)),\\ N(\varphi(a),\varphi(b)) &= f(\varphi(a))\,w(\varphi(b))+f(\varphi(b))\,w(\varphi(a)).\end{aligned} \qquad (2.6)$$

Proof. Since f and w be φ-convex functions, then we have

$$\begin{aligned}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)w\left(\frac{\varphi(a)+\varphi(b)}{2}\right) &= f\left(\frac{t\varphi(a)+(1-t)\varphi(b)+(1-t)\varphi(a)+t\varphi(b)}{2}\right)\\ &\quad\times w\left(\frac{t\varphi(a)+(1-t)\varphi(b)+(1-t)\varphi(a)+t\varphi(b)}{2}\right)\\ &\leq \frac{1}{2}\left[f(t\varphi(a)+(1-t)\varphi(b))+f((1-t)\varphi(a)+t\varphi(b))\right]\\ &\quad\times\frac{1}{2}\left[w(t\varphi(a)+(1-t)v(b))+w((1-t)\varphi(a)+t\varphi(b))\right]\\ &\leq \frac{1}{4}\{2t(1-t)f(\varphi(a))w(\varphi(a))+2t(1-t)f(\varphi(b))w(\varphi(b))\\ &\quad+\left(t^2+(1-t)^2\right)[f(\varphi(a))w(\varphi(b))+f(\varphi(b))w(\varphi(a)]\Big\}.\end{aligned}$$

Integrating with respect to on $[0,1]$, we get

$$\begin{aligned}&f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)w\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\\ &\leq \frac{1}{4}\left[\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,w(\varphi(x))\,d\varphi(x)\right]\\ &\quad+\frac{1}{2}\left[\frac{1}{6}M(\varphi(a),\varphi(b))+\frac{1}{3}N(\varphi(a),\varphi(b))\right]\end{aligned}$$

which completes the proof of Theorem 2.5. ∎

Remark 2.6. If we choose $\varphi(x) = x$ for all $x \in [a, b]$ in Theorem 2.5, the inequality (2.5) reduce to the inequality

$$2f\left(\frac{a+b}{2}\right) w\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\, w(x)\, dx \leq \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

which is proved by Cristescu in [2].

Theorem 2.7. *Let J be an interval $a, b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ be a φ-convex on and nonnegative function on $I = [a, b]$. If w is symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$, then, for all $t \in [0, 1]$, we have*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, w(\varphi(x))\, d\varphi(x) \leq \frac{1}{6}M(\varphi(a), \varphi(b)) + \frac{1}{3}N(\varphi(a), \varphi(b))$$

where $M(\varphi(a), \varphi(b))$ and $N(\varphi(a), \varphi(b))$ are given by (2.6).

Proof. Since w is symmetric about $\frac{\varphi(a)+\varphi(b)}{2}$, and f, w be φ-convex functions, then we have

$$\begin{aligned}
&\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, w(\varphi(x))\, d\varphi(x)\\
=\;& \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, w(\varphi(a)+\varphi(b)-\varphi(x))\, d\varphi(x)\\
=\;& \int_0^1 f(t\varphi(a)+(1-t)\varphi(b)) w((1-t)\varphi(a)+t\varphi(b)) dt\\
\leq\;& \int_0^1 \left[tf(\varphi(a))+(1-t)f(\varphi(b))\right]\left[(1-t)w(\varphi(a))+tw(\varphi(b))\right] dt\\
=\;& \int_0^1 \{t(1-t)\left[f(\varphi(a))w(\varphi(a))+f(\varphi(b))w(\varphi(b))\right]\\
& + t^2 f(\varphi(a))w(\varphi(b)) + (1-t)^2 f(\varphi(b))w(\varphi(a)\}\, dt\\
=\;& \frac{1}{6}M(\varphi(a), \varphi(b)) + \frac{1}{3}N(\varphi(a), \varphi(b)).
\end{aligned}$$

This completes the proof. □

Remark 2.8. If we choose $\varphi(x) = x$ for all $x \in [a, b]$ in Theorem 2.7, the inequality (2.5) reduce to the inequality

$$\frac{1}{b-a}\int_a^b f(x)\, w(x)\, dx \leq \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

which is proved by Cristescu in [2].

3. The left and right sides of Hermite-Hadamard type inequality

In order to prove our results, we need the following lemma (see, [11]):

Lemma 3.1. *Let J be an interval $a,b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differantiable function on I° (the interior I). If $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$, then the following equality holds:*

$$\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \tag{3.1}$$

$$= \frac{\varphi(b)-\varphi(a)}{2} \int_0^1 p(t) f'(t\varphi(a) + (1-t)\varphi(b))\, dt$$

where

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right) \\ t-1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Proof. A simple proof of the equality can be done by performing integration by parts. □

Let us begin with the following Theorem.

Theorem 3.2. *Let J be an interval $a,b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior I) and $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$. If $|f'|$ is the φ- convex on $[a,b]$, then the following inequality holds:*

$$\left| \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \right| \tag{3.2}$$

$$\leq \frac{(\varphi(b)-\varphi(a))}{8} \left[|f'(\varphi(a))| + |f'(\varphi(b))| \right].$$

Proof. The proof of this Theorem is done with a similar method of proof Noor et al. in [11]. □

Remark 3.3. If we take $\varphi(x) = x$ for all $x \in [a,b]$, then inequality (3.2) coincide with the left sides of Hermite-Hadamard inequality proved by Kirmanci in [10].

Theorem 3.4. *Let J be an interval $a,b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior*

I) and $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$. If $|f'|^q$ is the φ- convex on $[a,b]$, $q > 1$, then the following inequalities hold:

$$\begin{aligned}
&\left|\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right| \\
\leq\ & \frac{(g(b)-g(a))}{4(p+1)^{\frac{1}{p}}} \left[\left(\frac{|f'(\varphi(a))|^q + 3|f'(\varphi(b))|^q}{8}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{3|f'(\varphi(a))|^q + |f'(\varphi(b))|^q}{8}\right)^{\frac{1}{q}}\right] \qquad (3.3) \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{(p+1)^{\frac{1}{p}}} \left(\frac{1}{8}\right)^{\frac{1}{q}} \left(|f'(\varphi(a))| + |f'(\varphi(b))|\right),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$

Proof. From Lemma 3.1 , using Hölder's inequality and the φ-convexity of $|f'|^q$, we find

$$\begin{aligned}
&\left|\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right| \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{2} \left\{\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |f'(t\varphi(a) + (1-t)\varphi(b))|^q\, dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_{\frac{1}{2}}^{1} (1-t)^p\, dt\right) \left(\int_{\frac{1}{2}}^{1} |f'(t\varphi(a) + (1-t)\varphi(b))|^q\, dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{(\varphi(b)-\varphi(a))}{4(p+1)^{\frac{1}{p}}} \left\{\left(\int_0^{\frac{1}{2}} \left[t|f'(\varphi(a))|^q + (1-t)|f'(\varphi(b))|^q\right] dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_{\frac{1}{2}}^{1} \left[t|f'(\varphi(a))|^q + (1-t)|f'(b)|^q\right] dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{4(p+1)^{\frac{1}{p}}} \\
&\times \left\{\left(\frac{|f'(\varphi(a))|^q + 3|f'(\varphi(b))|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|f'(\varphi(a))|^q + |f'(\varphi(b))|^q}{8}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Let $a_1 = |f'(a)|^q$, $b_1 = 3|f'(b)|^q$, $a_2 = 3|f'(a)|^q$, $b_2 = |f'(b)|^q$. Here , $0 < \frac{1}{q} < 1$ for $q > 1$. Using the fact that,

$$\sum_{k=1}^{n}(a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s.$$

For $(0 \leq s < 1)$, $a_1, a_2, ..., a_n \geq 0$, $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\left|\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{8}\right)^{\frac{1}{q}}\left[\left(|f'(\varphi(a))| + 3^{\frac{1}{q}}|f'(\varphi(b))|\right) + \left(3^{\frac{1}{q}}|f'(\varphi(a))| + |f'(\varphi(b))|\right)\right]$$
$$= \frac{\varphi(b)-\varphi(a)}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{8}\right)^{\frac{1}{q}}\left[\left(1+3^{\frac{1}{q}}\right)(|f'(\varphi(a))| + |f'(\varphi(b))|)\right]$$
$$\leq \frac{\varphi(b)-\varphi(a)}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{8}\right)^{\frac{1}{q}}(|f'(\varphi(a))| + |f'(\varphi(b))|).$$

This completes the proof. □

Remark 3.5. If we thake $\varphi(x) = x$ for all $x \in [a,b]$, then inequality (3.3) coincide with the left sides of Hermite-Hadamard inequality proved by Kirmanci in [10].

Lemma 3.6. *Let J be an interval $a, b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differantiable function on I° (the interior I). If $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$, then the following equality holds:*

$$\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) \tag{3.4}$$
$$= \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 (2t-1)\left[f'(t\varphi(b) + (1-t)\varphi(a))\right] dt.$$

Proof. A simple proof of the equality can be done by performing integration by parts. □

Let us begin with the following Theorem.

Theorem 3.7. *Let J be an interval $a, b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differantiable function on I° (the interior I) and $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$. If $|f'|$ is the φ- convex on $[a,b]$, then* *the following inequaliy holds:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right| \tag{3.5}$$
$$\leq \frac{\varphi(b)-\varphi(a)}{4}\left(\frac{|f'(\varphi(b))|+|f'(\varphi(a))|}{2}\right).$$

Proof. From Lemma 3.6 and by using $\varphi-$convexity function of $|f'|$, we have

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right|$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int\limits_{0}^{1}|2t-1|\,|f'(t\varphi(b)+(1-t)\varphi(a))|\,dt$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int\limits_{0}^{1}|2t-1|\,[t\,|f'(\varphi(b))|+(1-t)\,|f'(\varphi(a))|]\,dt$$
$$= \frac{\varphi(b)-\varphi(a)}{2}\left[\frac{|f'(\varphi(b))|+|f'(\varphi(a))|}{4}\right]$$

which completes the proof. □

Remark 3.8. If we thake $\varphi(x)=x$ for all $x\in[a,b]$, then inequality (3.5) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in [5].

Theorem 3.9. *Let J be an interval $a,b\in J$ with $a<b$ and $\varphi: J\to\mathbb{R}$ a continuous increasing function. Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differantiable function on I° (the interior I) and $f'\in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b)\in I$. If $|f'|^q$ is the φ- convex on $[a,b]$, $q>1$, then the following inequaliy holds:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int\limits_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right| \tag{3.6}$$
$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(\varphi(b))|^q+|f'(\varphi(a))|^q}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.6 and by using Hölder's integral inequality, we have

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^1 |2t-1|^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left|f'\left(t\varphi(b)+(1-t)\varphi(a)\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is $\varphi-$convex on $[a,b]$, we get

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\,d\varphi(x)\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left[t\left|f'(\varphi(b))\right|^q+(1-t)\left|f'(\varphi(a))\right|^q\right]dt\right)^{\frac{1}{q}}$$

which completes the proof. □

Remark 3.10. If we thake $\varphi(x)=x$ for all $x\in[a,b]$, then inequality (3.6) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in [5].

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 379–384

Extension of Karamata inequality for generalized inverse trigonometric functions

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Abstract. Discussing Ramanujan's Question 294, Karamata established the inequality

$$\frac{\log x}{x-1} \le \frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}, \qquad (x>0,\, x\neq 1)\,, \tag{*}$$

which is the cornerstone of this paper. We generalize the above inequality transforming into terms of arctan and artanh. Moreover, we expand the established result to the class of generalized inverse p–trigonometric arctan_p and to hyperbolic artanh_p functions.

Mathematics Subject Classification (2010): 26D99, 39B62, 39B72.

Keywords: Karamata's inequality, Ramanujan's question 294, zero-balanced hypergeometric functions, generalized inverse trigonometric functions, rational upper bounds.

1. Introduction

The monumental *Analytical Inequalities* monograph by Mitrinović [6] contains several results by the famous Serbian mathematician Jovan Karamata. The first (Serbo–Croatian) edition's page 267 presents two Karamata's inequalities [6, **3.6.15.**, **3.6.16.**]

$$\frac{\log x}{x-1} \le \begin{cases} \dfrac{1}{\sqrt{x}} \\ \dfrac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}} \end{cases}, \tag{1.1}$$

which hold for all $x\in\mathbb{R}_+\setminus\{1\}$. Both estimates Karamata [4] applied in showing the monotone decreasing behavior of a sequence occurring in the famous Ramanujan's

Question 294 [7, p. 128] *Show that* [*if x is a positive integer*]

$$\frac{1}{2}\,\mathrm{e}^x = \sum_{k=0}^{x-1}\frac{x^k}{k!} + \frac{x^x}{x!}\,\theta\,,$$

where θ *lies between* $\frac{1}{3}$ *and* $\frac{1}{2}$.

For further information about Question 294 consult [2, p. 46 *et seq.*], while subsequent results concerning (1.1) belong also to Simić [8], see also the related references therein.

Being $\sqrt{x} \le (x + \sqrt[3]{x})(1 + \sqrt[3]{x})^{-1}$, the second Karamata's upper bound is more accurate on the whole range of their validity, therefore we concentrate to (*). In Mitrinović's monograph the proofs of inequalities (1.1) belong to B. Mesihović; we present the sketch of the proof's idea for the cubic–root–bound. By putting

$$(1+x)^3(1-x)^{-3} \mapsto x,$$

the radicals disappear in (*), and it transforms into

$$\frac{3}{2x}\log\frac{1+x}{1-x} - \frac{x^2+3}{1-x^4} < 0, \qquad (0 < |x| < 1)\,. \tag{1.2}$$

Expanding this expression into a power series, we get

$$K_{3,1}^{(2)}(4;x) := 3\sum_{k\ge 0}\left(1-\frac{1}{4k+1}\right)x^{4k} + \sum_{k\ge 0}\left(1-\frac{3}{4k+3}\right)x^{4k+2} > 0\,,$$

which finishes in an elegant way the proof.

However, summing up $K_{3,1}^{(2)}(4;x)$, we can write

$$K_{3,1}^{(2)}(4;x) = \frac{x^2+3}{1-x^4} - 3\cdot {}_2F_1\left[\begin{matrix}1,\frac14\\ \frac54\end{matrix};x^4\right] - x^2\,{}_2F_1\left[\begin{matrix}1,\frac34\\ \frac74\end{matrix};x^4\right],$$

such that gives the new form of (1.2):

$$3\cdot {}_2F_1\left[\begin{matrix}1,\frac14\\ \frac54\end{matrix};x^4\right] + x^2\,{}_2F_1\left[\begin{matrix}1,\frac34\\ \frac74\end{matrix};x^4\right] < \frac{3+x^2}{1-x^4},$$

which simplifies into

$$\frac{3}{x}\operatorname{arctanh} x < \frac{3+x^2}{1-x^4}, \qquad (0 < |x| < 1)\,, \tag{1.3}$$

since

$$\begin{aligned} {}_2F_1\left[\begin{matrix}1,\frac14\\ \frac54\end{matrix};z\right] &= \frac{1}{\sqrt[4]{z}}\left(\operatorname{arctanh}\sqrt[4]{z} + \arctan\sqrt[4]{z}\right)\\ {}_2F_1\left[\begin{matrix}1,\frac34\\ \frac74\end{matrix};z\right] &= \frac{3}{2\sqrt[4]{z^3}}\left(\operatorname{arctanh}\sqrt[4]{z} - \arctan\sqrt[4]{z}\right)\,. \end{aligned}$$

Here by using the shifted factorial

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

for $a > 0$, the power series

$${}_2F_1\left[\begin{matrix}a,\,b\\ a+b\end{matrix};x\right] = \sum_{n\ge 0}\frac{(a)_n(b)_n}{(a+b)_n}\frac{x^n}{n!},$$

stands for the zero-balanced Gaussian hypergeometric series, which converges for $|x| < 1$.

It is worth to mention that as $x \to 0$, we have the strong asymptotic relation

$$K_{3,1}^{(2)}(4;x) = \frac{12}{5}\, x^4 + \mathcal{O}(x^6)\,, \tag{1.4}$$

compare [6, p. 267].

In the sequel our aim is to extend Mesihović's method to general weighted sum of zero–balanced Gaussian hypergeometric functions getting appropriate extensions of Karamata's inequality (*).

2. Extending $K_{3,1}^{(2)}(4;x)$

In this section we are going to investigate the sum

$$K_{p,q}^{(\gamma)}(\mu;x) := p\sum_{k\geq 0}\left(1-\frac{q}{\mu k+q}\right) x^{\mu k} + q\sum_{k\geq 0}\left(1-\frac{p}{\mu k+p}\right) x^{\mu k+\gamma}\,,$$

for the widest possible range of the variable x and its representation in a form of a weighted sum of two zero-balanced hypergeometric terms.

Theorem 2.1. *For all $p,q,\mu>0, \gamma\in\mathbb{R}$ and $0<x<1$ we have*

$$K_{p,q}^{(\gamma)}(\mu;x) = \frac{p+qx^\gamma}{1-x^\mu} - p\,{}_2F_1\left[\begin{matrix} 1,\ \frac{q}{\mu} \\ \frac{q}{\mu}+1\end{matrix}\,;x^\mu\right] - q\,x^\gamma\,{}_2F_1\left[\begin{matrix} 1,\ \frac{p}{\mu} \\ \frac{p}{\mu}+1\end{matrix}\,;x^\mu\right]. \tag{2.1}$$

Also, there holds

$$p\,{}_2F_1\left[\begin{matrix} 1,\ \frac{q}{\mu} \\ \frac{q}{\mu}+1\end{matrix}\,;x^\mu\right] + q\,x^\gamma\,{}_2F_1\left[\begin{matrix} 1,\ \frac{p}{\mu} \\ \frac{p}{\mu}+1\end{matrix}\,;x^\mu\right] < \frac{p+q\,x^\gamma}{1-x^\mu}\,. \tag{2.2}$$

Proof. The following conclusion–chain lead us to the asserted expression (2.1) for $K_{p,q}^{(\gamma)}(\mu;x)$, assuming that $a,b>0$ and $0<x<1$ (which enables the convergence of the following power series):

$$\begin{aligned} L_b(\mu;x) &:= \sum_{k\geq 0}\left(1-\frac{b}{\mu\,k+b}\right) x^{\mu\,k} = \frac{1}{1-x^\mu} - A\sum_{k\geq 0}\frac{x^{\mu\,k}}{k+A} \\ &= \frac{1}{1-x^\mu} - A\sum_{k\geq 0}\frac{(1)_k\,\Gamma(k+A)}{\Gamma(k+A+1)}\,\frac{x^{\mu\,k}}{k!} = \frac{1}{1-x^\mu} - \sum_{k\geq 0}\frac{(1)_k\,(A)_k}{(A+1)_k}\,\frac{x^{\mu\,k}}{k!} \\ &= \frac{1}{1-x^\mu} - {}_2F_1\left[\begin{matrix} 1,\ A \\ A+1\end{matrix}\,;x^\mu\right] = \frac{1}{1-x^\mu} - {}_2F_1\left[\begin{matrix} 1,\ \frac{b}{\mu} \\ \frac{b}{\mu}+1\end{matrix}\,;x^\mu\right], \end{aligned}$$

where $A := b\,\mu^{-1}$. Thus, for $p,q>0$, because

$$K_{p,q}^{(\gamma)}(\mu;x) = p\,L_q(\mu;x) + q\,x^\gamma\,L_p(\mu;x),$$

relation (2.1) is proved. Finally, since we have $K_{p,q}^{(\gamma)}(\mu;x)>0$, we deduce the inequality (2.2) and this completes the proof. □

Remark 2.2. For even positive integer values of μ and γ, the results achieved in Theorem 2.1 one extends to all $x \in (-1,1)$. Moreover, it is worth to mention that if $p, q, \mu < 0$, $x \in (0,1)$ and $\gamma \in \mathbb{R}$, then we get that

$$K_{p,q}^{(\gamma)}(\mu; x) = p \sum_{k \geq 0} \frac{k}{k + q/\mu} x^{\mu k} + q \sum_{k \geq 0} \frac{k}{k + p/\mu} x^{\mu k + \gamma} < 0,$$

that is, the inequality (2.2) is reversed.

The generalized trigonometric and generalized inverse trigonometric functions were introduced by Lindqvist [5]. For $p > 0$ the inverse p–trigonometric and p–hyperbolic functions are defined as special zero-balanced hypergeometric series, that is,

$$\operatorname{arctan}_p(x) = \int_0^x (1 + t^p)^{-1}\, \mathrm{d}t = x\, {}_2F_1 \left[\begin{matrix} 1, \frac{1}{p} \\ \frac{1}{p} + 1 \end{matrix} ; -x^p \right],$$

$$\operatorname{artanh}_p(x) = \int_0^x (1 - t^p)^{-1}\, \mathrm{d}t = x\, {}_2F_1 \left[\begin{matrix} 1, \frac{1}{p} \\ \frac{1}{p} + 1 \end{matrix} ; x^p \right].$$

Note that these functions were investigated by many authors in the recent years, see for example [1, 3] and the references therein. The following result is a variant of Theorem 2.1 in terms of generalized inverse trigonometric functions.

Theorem 2.3. *For all $p, q, \mu > 0$, $\gamma \in \mathbb{R}$ and $x \in (0,1)$ we have*

$$p x^{-q} \operatorname{artanh}_{\frac{\mu}{q}}(x^q) + q x^{\gamma - p} \operatorname{artanh}_{\frac{\mu}{p}}(x^p) < \frac{p + q x^\gamma}{1 - x^\mu}. \tag{2.3}$$

Also for all $p > 0$ and $x \in (0,1)$ it follows

$$\operatorname{artanh}_p(x) < \frac{x}{1 - x^p}. \tag{2.4}$$

Moreover, we have the asymptotic relation as $x \to 0$

$$\frac{p + q x^\gamma}{1 - x^\mu} - \frac{p}{x^q} \operatorname{artanh}_{\frac{\mu}{q}}(x^q) - \frac{q}{x^{p-\gamma}} \operatorname{artanh}_{\frac{\mu}{p}}(x^p) = \frac{p\,\mu}{q + \mu}\, x^\mu + \mathcal{O}\left(x^{\mu + \min(\gamma, \mu)} \right). \tag{2.5}$$

Proof. Transforming

$${}_2F_1 \left[\begin{matrix} 1, \frac{p}{\mu} \\ \frac{p}{\mu} + 1 \end{matrix} ; x^p \right] = {}_2F_1 \left[\begin{matrix} 1, \frac{1}{\mu/p} \\ \frac{1}{\mu/p} + 1 \end{matrix} ; (x^p)^{\frac{\mu}{p}} \right],$$

by means of (2.2) we deduce (2.3). Now, taking $p = q$ in (2.3) and then substituting $x \mapsto x^{1/p}$, $\mu = p^2$, we get (2.4). Finally, expanding (2.1), we have for $x \to 0$:

$$K_{p,q}^{(\gamma)}(\mu; x) = \frac{p\,\mu}{q + \mu}\, x^\mu + \mathcal{O}\left(x^{\mu + \min(\gamma, \mu)} \right).$$

Since $K_{p,q}^{(\gamma)}(\mu; x)$ coincides with the left hand side expression in (2.5), the assertion is proved. □

Now, in establishing the companion inequality associated with (1.3), we study the expression

$$\overline{K}^{(2)}_{3,1}(4;x) := 3\sum_{k\geq 0}\left(1-\frac{1}{4k+1}\right)x^{4k} - \sum_{k\geq 0}\left(1-\frac{3}{4k+3}\right)x^{4k+2} > 0\,.$$

To establish the positivity of $\overline{K}^{(2)}_{3,1}(4;x)$ for all $0<|x|<1$, it is enough to observe that

$$\begin{aligned}\overline{K}^{(2)}_{3,1}(4;x) &= 12\sum_{k\geq 0}\frac{k}{4k+1}\,x^{4k} - 4x^2\sum_{k\geq 0}\frac{k}{4k+3}\,x^{4k}\\ &> 4\sum_{k\geq 0}\left(\frac{3k}{4k+1}-\frac{k}{4k+3}\right)x^{4k}\,.\end{aligned}$$

Thus, rewriting $\overline{K}^{(2)}_{3,1}(4;x)$ in terms of hypergeometric series, and then in inverse trigonometric and hyperbolic terms, we conclude that

$$\overline{K}^{(2)}_{3,1}(4;x) = \frac{3-x^2}{1-x^4} - \frac{3}{x}\arctan x\,.$$

Having in mind that $\overline{K}^{(2)}_{3,1}(4;x)>0$, we get

$$\frac{3}{x}\arctan x < \frac{3-x^2}{1-x^4}\,, \qquad \big(0<|x|<1\big).$$

Also, the following asymptotic behavior holds true

$$\overline{K}^{(2)}_{3,1}(4;x) = \frac{12}{5}\,x^4 + \mathcal{O}(x^6), \qquad (x\to 0)$$

which coincides with the one in (1.4).

Now, the counterpart result of Theorem 2.1 reads as follows.

Theorem 2.4. *For all $p,q,\mu,\gamma>0$ such that $p\geq q$ and for all $0<x<1$ we have*

$$px^{-q}\,\mathrm{artanh}_{\frac{\mu}{q}}\,(x^q) - qx^{\gamma-p}\,\mathrm{artanh}_{\frac{\mu}{p}}\,(x^p) < \frac{p-qx^\gamma}{1-x^\mu}\,. \tag{2.6}$$

Proof. Consider the linear combination of power series

$$\overline{K}^{(\gamma)}_{p,q}(\mu;x) := p\sum_{k\geq 0}\left(1-\frac{q}{\mu k+q}\right)x^{\mu k} - q\sum_{k\geq 0}\left(1-\frac{p}{\mu k+p}\right)x^{\mu k+\gamma}\,.$$

For all $x\in(0,1)$ and $\gamma>0$ it follows

$$\begin{aligned}\overline{K}^{(\gamma)}_{p,q}(\mu;x) &> \mu\sum_{k\geq 0}\left(\frac{pk}{\mu k+q}-\frac{qk}{\mu k+p}\right)x^{\mu k}\\ &= \mu(p-q)\sum_{k\geq 0}\frac{k(\mu k+p+q)}{(\mu k+q)(\mu k+p)}\,x^{\mu k}\,;\end{aligned}$$

the last estimate is non–negative for $p\geq q$. Transforming the constituting sums of $\overline{K}^{(\gamma)}_{p,q}(\mu;x)$ into hypergeometric expressions, and following the lines of the proof of Theorem 2.3, we arrive at the desired inequality (2.6). □

We mention that the expression $L_b(\mu;x)$ can be expressed also in another way as

$$\begin{aligned}L_b(\mu;x) &= \sum_{k\geq 0}\frac{\mu k}{\mu k+b}\,x^{\mu k} = x\sum_{k\geq 0}\frac{\mu k}{\mu k+b}\,x^{\mu k-1} = \frac{x}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\geq 0}\frac{x^{\mu k}}{k+\frac{b}{\mu}}\\ &= \frac{x}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\geq 0}\frac{\Gamma(k+\frac{b}{\mu})\,\Gamma(k+1)}{(k+\frac{b}{\mu})\Gamma(k+\frac{b}{\mu})}\,\frac{x^{\mu k}}{k!}\\ &= \frac{x\Gamma(\frac{b}{\mu})}{\mu\,\Gamma(1+\frac{b}{\mu})}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k\geq 0}\frac{(\frac{b}{\mu})_k\,(1)_k}{(1+\frac{b}{\mu})_k}\,\frac{x^{\mu k}}{k!}\\ &= \frac{x}{b}\,\frac{\mathrm{d}}{\mathrm{d}x}\,{}_2F_1\left[\begin{matrix}\frac{b}{\mu},\,1\\ \frac{b}{\mu}+1\end{matrix}\,;x^{\mu}\right] = \frac{\mu}{b+\mu}\,x^{\mu}\,{}_2F_1\left[\begin{matrix}\frac{b}{\mu}+1,\,2\\ \frac{b}{\mu}+2\end{matrix}\,;x^{\mu}\right].\end{aligned}$$

However, by this expression we cannot reach any rational upper bound for $K_{p,q}^{(\gamma)}(\mu;x)$.

Acknowledgment. The work of Á. Baricz was supported by a research grant of the Babeş-Bolyai University for young researchers, project number GTC_34021. The research of T. K. Pogány has been supported in part by Croatian Science Foundation under the project No. 5435 (2014).

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 385–394

On certain subclasses of meromorphic functions defined by convolution with positive and fixed second coefficients

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Abstract. In this paper we consider the class $M(f, g; \alpha, \beta, \lambda, c)$ of meromorphic univalent functions defined by convolution with positive coefficients and fixed second coefficients. We obtained coefficient inequalities, distortion theorems, closure theorems, the radii of meromorphic starlikeness, and convexity for functions of this class.

Mathematics Subject Classification (2010): 30C45.

Keywords: Meromorphic, coefficient inequality, fixed second coefficient, distortion theorem, radii of starlikeness and convexity.

1. Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \backslash \{0\}$. Let $g \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

A function $f \in \Sigma$ is meromorphically starlike of order β $(0 \leq \beta < 1)$ if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \ (z \in U), \tag{1.4}$$

the class of all such functions is denoted by $\Sigma^*(\beta)$. A function $f \in \Sigma$ is meromorphically convex of order β $(0 \leq \beta < 1)$ if

$$-\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta \ (z \in U), \tag{1.5}$$

the class of such functions is denoted by $\Sigma_k(\beta)$. The classes $\Sigma^*(\beta)$ and $\Sigma_k(\beta)$ were introduced and studied by Pommerenke [18], Miller [15], Mogra et al. [16], Cho [9], Cho et al. [10] and Aouf ([1] and [2]).
It is easy to observe from (1.4) and (1.5) that

$$f \in \Sigma_k(\beta) \Longleftrightarrow -zf^{'} \in \Sigma^*(\beta).$$

For $\alpha \geq 0,\ 0 \leq \beta < 1,\ 0 \leq \lambda < \frac{1}{2}$ and g given by (1.2) with $b_k > 0$ $(k \geq 1)$, Aouf et al. [3] defined the class $M(f,g;\alpha,\beta,\lambda)$ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\Re\left\{\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+\beta\right\}$$
$$\geq \alpha\left|\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+1\right| (z \in U). \tag{1.6}$$

We note that for suitable choices of $g,\ \alpha$ and λ, we obtain the following subclasses of the class $M(f,g;\alpha,\beta,\lambda)$:

(1) $M\left(f,\dfrac{1}{z(1-z)};0,\beta;0\right) = \Sigma^*(\beta)\ (0 \leq \beta < 1)$ (see Pommerenke [18]);

(2) $M\left(f,\dfrac{1}{z}+\sum\limits_{k=1}^{\infty} D_k(\gamma)z^k;\alpha,\beta,\lambda\right) = \Sigma_\gamma(\alpha,\beta,\lambda)$ (see Atshan and Kulkarni [7] and Atshan [6]) $(\alpha \geq 0,\ 0 \leq \beta < 1,\ \gamma > -1,\ 0 \leq \lambda < \dfrac{1}{2})$, where

$$D_k(\gamma) = \frac{(\gamma+1)(\gamma+2)...(\gamma+k+1)}{(k+1)!}; \tag{1.7}$$

(3) $M\left(f,\dfrac{1}{z}+\sum\limits_{k=1}^{\infty} \Gamma_k(\alpha_1)z^k;\alpha,\beta,\lambda\right) = \Sigma(\beta,\alpha,\lambda)$ (see Magesh et al. [14]) $(\alpha \geq 0,$ $0 \leq \beta < 1, 0 \leq \lambda < \dfrac{1}{2})$, where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k+1}...(\alpha_q)_{k+1}}{(\beta_1)_{k+1}...(\beta_s)_{k+1}}\frac{1}{(k+1)!}; \tag{1.8}$$

(4) $M\left(f,\dfrac{1}{z}+\sum\limits_{k=1}^{\infty} \Gamma_k(\alpha_1)z^k;0,\beta,\lambda\right) = M_s^q(\lambda,\beta)$ (see Murugusundaramoorthy et al. [17]) $(0 \leq \beta < 1,\ 0 \leq \lambda < \dfrac{1}{2},\ q \leq s+1,\ q,s \in \mathbb{N}_0 = \mathbb{N}\cup\{0\},\ \mathbb{N} = \{1,2,...\})$, where $\Gamma_k(\alpha_1)$ is defined by (1.8).

Also, we note that

(1) $M(f,g;\alpha,\beta,0) = N(f,g;\alpha,\beta)$

$$= \left\{f \in \Sigma : -\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}+\beta\right) \geq \alpha\left|\frac{z(f*g)'(z)}{(f*g)(z)}+1\right|\right\} (z \in U); \tag{1.9}$$

(2) Putting $g(z)=\frac{1}{z}+\sum\limits_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k$ in (1.6), then the class

$$M\left(f,\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k;\alpha,\beta,\lambda\right)$$

reduces to the class

$$M_{\delta,\ell}(m;\alpha,\beta,\lambda)=\left\{f\in\Sigma:-\Re\left\{\frac{z(I^m(\delta,\ell)f(z))'+\lambda z^2(I^m(\delta,\ell)f(z))''}{(1-\lambda)(I^m(\delta,\ell)f(z))+\lambda z(I^m(\delta,\ell)f(z))'}+\beta\right\}\geq\alpha\right.$$

$$\left.\left|\frac{z(I^m(\delta,\ell)f(z))'+\lambda z^2(I^m(\delta,\ell)f(z))''}{(1-\lambda)(I^m(\delta,\ell)f(z))+\lambda z(I^m(\delta,\ell)f(z))'}+1\right|\ (\delta\geq 0,\ \ell>0,\ m\in\mathbb{N}_0,\ z\in U)\right\},$$

where the operator

$$I^m(\delta,\ell)(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k, \tag{1.10}$$

was introduced and studied by Bulboacă et al. [8], El-Ashwah [11 with $p=1$] and El-Ashwah et al. [12 with $p=1$].

Unless otherwise mentioned, we shall assume in the reminder of this paper that $0\leq\beta<1,\ 0\leq\lambda<\frac{1}{2},\ \alpha\geq 0$, g is given by (1.2) with $b_k>0$ and $b_k\geq b_1\,(k\geq 1)$.

We begin by recalling the following lemma due to Aouf et al. [4].

Lemma 1.1. *Let the function f be defined by (1.1). Then f is in the class $M(f,g;\alpha,\beta,\lambda)$ if and only if*

$$\sum_{k=1}^{\infty}[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k a_k\leq(1-\beta)(1-2\lambda). \tag{1.11}$$

Proof. In view of (1.11), we can see that the functions f defined by (1.1) in the class $M(f,g;\alpha,\beta,\lambda)$ and satisfy the coefficient inequality

$$a_1\leq\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\,b_1}. \tag{1.12}$$

Hence we may take

$$a_1=\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1},\ 0<c<1. \tag{1.13}$$

Making use of (1.13), we now introduce the following class of functions:

Let $M(f,g;\alpha,\beta,\lambda,c)$ denote the subclass of $M(f,g;\alpha,\beta,\lambda)$ consisting of functions of the form:

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}z+\sum_{k=2}^{\infty}a_k z^k\ \ (a_k\geq 0;0<c<1). \tag{1.14}$$

Motivated by the works of Aouf and Darwish [3], Aouf and Joshi [5], Ghanim and Darus [13] and Uralegaddi [19], we now introduce the following class of meromorphic functions with fixed second coefficients.

2. Coefficient estimates

Theorem 2.1. *Let the function f be defined by* (1.14)*. Then f is in the class $M(f,g;\alpha,\beta,\lambda,c)$, if and only if,*

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k a_k \leq (1-\beta)(1-2\lambda)(1-c). \tag{2.1}$$

Proof. Putting

$$a_1=\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}, \qquad 0<c<1, \tag{2.2}$$

in (1.11) and simplifying we get the required result. The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}z+\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k}z^k, k\geq 2. \tag{2.3}$$

Corollary 2.1. *Let the function f defined by* (1.13) *be in the class $M(f,g;\alpha,\beta,\lambda,c)$, then*

$$a_k \leq \frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k}, \qquad k\geq 2. \tag{2.4}$$

The result is sharp for the function f given by (2.3)*.*

3. Growth and Distortion theorems

Theorem 3.1. *If the function f defined by* (1.14) *is in the class $M(f,g;\alpha,\beta,\lambda,c)$ for $0<|z|=r<1$, then we have*

$$\frac{1}{r}-\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}r-\frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)\,b_2}r^2 \leq |f(z)|$$

$$\leq \frac{1}{r}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}r+\frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)\,b_2}r^2. \tag{3.1}$$

The result is sharp for the function f given by

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}z+\frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)\,b_2}z^2. \tag{3.2}$$

Proof. Since $f\in M(f,g;\alpha,\beta,\lambda,c)$, then Theorem 2.1 yields

$$a_k \leq \frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\,b_k}, \qquad k\geq 2. \tag{3.3}$$

Thus, for $0<|z|=r<1$,

$$|f(z)| \leq \frac{1}{|z|}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}|z|+\sum_{k=2}^{\infty}a_k|z|^k$$

$$\leq \frac{1}{r}+\frac{(1-\beta)(1-2\lambda)\,c}{(2\alpha+\beta+1)\,b_1}r+r^2\sum_{k=2}^{\infty}a_k$$

$$\leq \frac{1}{r}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r+\frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2}r^2, \text{ by } (3.3).$$

Also we have

$$|f(z)| \geq \frac{1}{|z|}-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}|z|-\sum_{k=2}^{\infty}a_k|z|^k$$
$$\geq \frac{1}{r}-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r-r^2\sum_{k=2}^{\infty}a_k$$
$$\geq \frac{1}{r}-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r-\frac{(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2}r^2.$$

Thus the proof of Theorem 3.1 is completed.

Theorem 3.2. *If the function f defined by* (1.14) *is in the class* $M(f,g;\alpha,\beta,\lambda,c)$ *for* $0<|z|=r<1$, *then we have*

$$\frac{1}{r^2}-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}-\frac{2(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2}r$$
$$\leq \left|f'(z)\right| \leq \frac{1}{r^2}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}+\frac{2(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2}r. \tag{3.4}$$

The result is sharp for the function f given by (3.2).

Proof. In view of Theorem 2.1, it follows that

$$ka_k \leq \frac{k(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}, \quad k\geq 2. \tag{3.5}$$

Thus, for $0<|z|=r<1$, and making use of (3.5), we obtain

$$\left|f'(z)\right| \leq \frac{1}{|z^2|}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}+\sum_{k=2}^{\infty}ka_k|z|^{k-1}$$
$$\leq \frac{1}{r^2}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}+r\sum_{k=2}^{\infty}ka_k$$
$$\leq \frac{1}{r^2}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}+\frac{2(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2}r, \text{ by } (3.5).$$

Also we have

$$\left|f'(z)\right| \geq \frac{1}{|z^2|}-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}-\sum_{k=2}^{\infty}ka_k|z|^{k-1}$$
$$\geq \frac{1}{r^2}-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}-r\sum_{k=2}^{\infty}ka_k$$
$$\geq \frac{1}{r^2}-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}-\frac{2(1-\beta)(1-2\lambda)(1-c)}{(1+\lambda)(3\alpha+\beta+2)b_2}r.$$

Hence the result follows.

4. Closure theorems

In this section we shall show that the class $M(f,g;\alpha,\beta,\lambda,c)$ is closed under convex linear combination.

Theorem 4.1. *Let*

$$f_1(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z, \tag{4.1}$$

and

$$f_k(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+ \tag{4.2}$$

$$\sum_{k=2}^{\infty}\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}z^k \qquad (k\geq 2).$$

Then $f\in M(f,g;\alpha,\beta,\lambda,c)$, if and only if it can expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z), \tag{4.3}$$

where $\mu_k\geq 0$ and $\sum\limits_{k=1}^{\infty}\mu_k\leq 1$.

Proof. Let

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z),$$

then from (4.2) and (4.3), we have

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+\sum_{k=2}^{\infty}\frac{(1-\beta)(1-2\lambda)(1-c)\mu_k}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}z^k. \tag{4.4}$$

Since

$$\sum_{k=2}^{\infty}\frac{(1-\beta)(1-2\lambda)(1-c)\mu_k}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}{(1-\beta)(1-2\lambda)(1-c)}$$

$$=\sum_{k=2}^{\infty}\mu_k=1-\mu_1\leq 1,$$

hence by using Lemma 1.1, we have $f\in M(f,g;\alpha,\beta,\lambda,c)$.
Conversely, suppose that f defined by (1.14) is in the class $M(f,g;\alpha,\beta,\lambda,c)$. Then by using (2.4) ,we get

$$a_k\leq\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}, \qquad k\geq 2. \tag{4.5}$$

Setting

$$\mu_k=\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}{(1-\beta)(1-2\lambda)(1-c)}, \quad k\geq 2 \tag{4.6}$$

and

$$\mu_1=1-\sum_{k=2}^{\infty}\mu_k, \tag{4.7}$$

we can see that f can be expressed in the form (4.3) .This completes the proof of Theorem 4.1.

Theorem 4.2. *The class $M(f,g;\alpha,\beta,\lambda,c)$ is closed under linear combination.*

Proof. Suppose that the function f given by (1.14) , and the function g given by

$$g(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+\sum_{k=2}^{\infty}d_kz^k, \qquad d_k\geq 0. \tag{4.8}$$

Assuming that f and g are in the class $M(f,g;\alpha,\beta,\lambda,c)$, it is enough to prove that the function h defined by

$$h(z)=\mu f(z)+(1-\mu)g(z), \qquad 0\leq\mu\leq 1, \tag{4.9}$$

is also in the class $M(f,g;\alpha,\beta,\lambda,c)$. Since

$$h(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+\sum_{k=2}^{\infty}[a_k\mu+(1-\mu)d_k]z^k, \tag{4.10}$$

we observe that

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k[a_k\mu+(1-\mu)d_k] \leq (1-\beta)(1-2\lambda)(1-c), \tag{4.11}$$

with the aid of Theorem 2.1. Thus, $h\in M(f,g;\alpha,\beta,\lambda,c)$.

5. Radii of Meromorphically Starlikeness and Convexity

Theorem 5.1. *Let the function f defined by (1.14) be in the class $M(f,g;\alpha,\beta,\lambda,c)$. Then f is meromorphically starlike of order $\delta\,(0\leq\delta<1)$ in $0<|z|<r_1(\alpha,\beta,\lambda,c,\delta)$, where $r_1(\alpha,\beta,\lambda,c,\delta)$ is the largest value for which*

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\frac{(k+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}r^{k+1}\leq(1-\delta), \tag{5.1}$$

for $k\geq 2$. The result is sharp for the function

$$f_k(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}z+\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}z^k \tag{5.2}$$

for some k.

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\leq 1-\delta\,(0\leq\delta<1) \text{ for } 0<|z|<r_1. \tag{5.3}$$

Note that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\leq\frac{\frac{2(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\sum\limits_{k=2}^{\infty}(k+1)a_kr^{k+1}}{1-\frac{(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2-\sum\limits_{k=2}^{\infty}a_kr^{k+1}}\leq 1-\delta \tag{5.4}$$

for $(0 \leq \delta < 1)$ if and only if

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\sum_{k=2}^{\infty}(k+2-\delta)a_k r^{k+1}\leq(1-\delta). \tag{5.5}$$

Since f is in the class $M(f,g;\alpha,\beta,\lambda,c)$, from (2.4), we may take

$$a_k=\frac{(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}\mu_k \qquad (k\geq 2), \tag{5.6}$$

where $\mu_k \geq 0\,(k \geq 2)$ and $\sum\limits_{k=2}^{\infty}\mu_k \leq 1$.

For each fixed r, we choose the positive integer $k_0 = k_0(r)$ for which

$$\frac{(k+2-\delta)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]}r^{k+1}$$

is maximal. Then it follows that

$$\sum_{k=2}^{\infty}(k+2-\delta)a_k r^{k+1}\leq\frac{(k_0+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k_0-1)][k_0(1+\alpha)+(\alpha+\beta)]b_{k_0}}r^{k_0+1}. \tag{5.7}$$

Then f is starlike of order δ in $0<|z|<r_1(\alpha,\beta,\lambda,c,\delta)$ provided that

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\frac{(k_0+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k_0-1)][k_0(1+\alpha)+(\alpha+\beta)]b_{k_0}}r^{k_0+1}\leq(1-\delta). \tag{5.8}$$

We find the value $r_0 = r_0(\alpha,\beta,\lambda,c,\delta)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r_0^2+\frac{(k_0+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k_0-1)][k_0(1+\alpha)+(\alpha+\beta)]b_{k_0}}r_0^{k_0+1}=(1-\delta). \tag{5.9}$$

Then this value r_0 is the radius of meromorphically starlike of order δ for functions belonging to the class $M(f,g;\alpha,\beta,\lambda,c)$.

Corollary 5.1. *Let the function f defined by* (1.14) *be in the class $M(f,g;\alpha,\beta,\lambda,c)$. Then f is meromorphically convex of order $\delta\,(0\leq\delta<1)$ in $0<|z|<r_2(\alpha,\beta,\lambda,c,\delta)$, where $r_2(\alpha,\beta,\lambda,c,\delta)$ is the largest value for which*

$$\frac{(3-\delta)(1-\beta)(1-2\lambda)c}{(2\alpha+\beta+1)b_1}r^2+\frac{k(k+2-\delta)(1-\beta)(1-2\lambda)(1-c)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]b_k}r^{k+1}\leq(1-\delta), \tag{5.10}$$

$(k\geq 2)$. *The result is sharp for function f given by* (5.2) *for some k.*

Remark. Specializing the function g, in(1.6) , we have results for the subclasses maintain in the introduction in the case of fixed second coefficients.

Acknowledgment. The authors thank the referees for their valuable suggestions which led to the improvement of this paper.

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 395–402

On sandwich theorems for p-valent functions involving a new generalized differential operator

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Abstract. A new differential operator $F^m_{\alpha,\beta,\lambda}f(z)$ is introduced for functions of the form $f(z) = z^p + \sum_{n=2}^{\infty} a_n z^n$ which are p-valent in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. The main object of this paper is to derive some subordination and superordination results involving differential operator $F^m_{\alpha,\beta,\lambda}f(z)$.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic functions, differential subordination, differential superordination, sandwich theorems.

1. Introduction

Let $H(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and let $H[a, b]$ denote the subclass of the functions $f \in H(\mathbb{U})$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + ... \quad (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, ...\}). \tag{1.1}$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}(p)$ be the subclass of $H(\mathbb{U})$ consisting of functions of the form:

$$f(z) = z^p + \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0; p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{1.2}$$

which are p-valent in $\mathbb{U}$. If $f,\ g \in H(\mathbb{U})$, we say that f is subordinate to g or g is subordinate to f, written $f(z) \prec g(z)$, if there exists an analytic function w on $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$, such that $g(z) = h(w(z))$ for $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [5] and [13]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination:

$$\phi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{1.3}$$

then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.3) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in $\mathbb{U}$ and if $p(z)$ satisfies second-order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z), z^2p''(z); z\right), \tag{1.4}$$

then $p(z)$ is a solution of the differential supordination (1.4). An univalent function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.4) is called the best subordinant. Using the results of Miller and Mocanu [14], Bulboaca [4] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [5]. Ali et al. [1], have used the results of Bulboaca [4] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}(1)$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$.

Also, Tuneski [23] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}(1)$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$.

Recently, Shanmugam et al. [18], [19] and [21] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}(1)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{f^2(z)} \prec q_2(z).$$

Recently, Shanmugam et al. [19] obtained the such called sandwich results for certain classes of analytic functions.

For the function $f \in \mathcal{A}(p)$, we define the following new differential operator:

$$\begin{aligned} F^0 f(z) &= f(z); \\ F^1_{\alpha,\beta,\lambda} f(z) &= (1 - p\beta(\lambda-\alpha))f(z) + \beta(\lambda-\alpha)zf'(z); \\ F^2_{\alpha,\beta,\lambda} f(z) &= (1 - p\beta(\lambda-\alpha))(F^1_{\alpha,\beta,\lambda} f(z)) + \beta(\lambda-\alpha)z(F^1_{\alpha,\beta,\lambda} f(z))' \end{aligned}$$

and for $m = 1, 2, 3, ...$

$$\begin{aligned} F^m_{\alpha,\beta,\lambda} f(z) &= (1 - p\beta(\lambda - \alpha))(F^{m-1}_{\alpha,\beta,\lambda} f(z)) + \beta(\lambda - \alpha)z(F^{m-1}_{\alpha,\beta,\lambda} f(z))' \\ &= F^1_{\alpha,\beta,\lambda}(F^{m-1}_{\alpha,\beta,\lambda} f(z)) \\ &= z^p + \sum_{n=2}^{\infty} [1 + \beta(\lambda - \alpha)(n - p)]^m a_n z^n, \end{aligned} \tag{1.5}$$

for $\alpha \geq 0, \beta \geq 0, \lambda \geq 0$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It easily verified from (1.5) that

$$\beta(\lambda - \alpha)z(F^m_{\alpha,\beta,\lambda} f(z))' = F^{m+1}_{\alpha,\beta,\lambda} f(z) - (1 - p\beta(\lambda - \alpha))F^m_{\alpha,\beta,\lambda} f(z). \tag{1.6}$$

Remark 1.1. (i) When $\delta = 0$ and $p = 1$, we have the operator introduced and studied by Rabha (see [7]).

(ii) When $\alpha = 0$ and $\beta = p = 1$, we have the operator introduced and studied by Al-Oboudi (see [3]).

(iii) And when $\alpha = 0$ and $\lambda = \beta = p = 1$, we have the operator introduced and studied by Sălăgean (see [17]).

In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator $F^m_{\lambda,p} f(z)$.

2. Definitions and preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 2.1. [14] *Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial\mathbb{U}\colon \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(f)$.

Lemma 2.2. [14] *Let $q(z)$ be univalent in $\mathbb{U}$ and let θ and φ be analytic in a domain D containing $q(\mathbb{U})$, with $\varphi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $\psi(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + \psi(z)$. Suppose that*

(i) ψ is a starlike function in $\mathbb{U}$,

(ii) $\text{Re}\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$, $z \in \mathbb{U}$.

If $p(z)$ is a analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant of (2.1).

Lemma 2.3. [4] *Let $q(z)$ be convex univalent in $\mathbb{U}$ and let ϑ and ϕ be analytic in a domain D containing $q(\mathbb{U})$. Suppose that*

(i) $Re\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} > 0$, $z \in \mathbb{U}$,

(ii) $\Psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in $\mathbb{U}$.

If $p(z) \in H[q(0),1] \cap Q$, with $p(\mathbb{U}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\mathbb{U}$ and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant of (2.2).

3. Subordination and superordination for p-valent functions

We begin with the following result involving differential subordination between analytic functions.

Theorem 3.1. *Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0) = 1$, Further, assume that*

$$\mathrm{Re}\left\{\frac{2(\delta+\alpha)q(z)}{\delta} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{3.1}$$

If $f \in \mathcal{A}(p)$ satisfy the following subordination condition:

$$\Upsilon(m,\lambda,p,\delta;z) \prec \delta zq'(z) + (\delta+\alpha)\left(q(z)\right)^2, \tag{3.2}$$

where

$$\Upsilon(m,\lambda,p,\delta;z) = \frac{\delta F_{\lambda,p}^{m+2}f(z)}{\beta(\lambda-\alpha)F_{\lambda,p}^{m}f(z)} + \left(\delta+\alpha-\frac{\delta}{\beta(\lambda-\alpha)}\right)\frac{\left(F_{\lambda,p}^{m+1}f(z)\right)^2}{\left(F_{\lambda,p}^{m}f(z)\right)^2}, \tag{3.3}$$

then

$$\frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z) = \frac{F_{\lambda,p}^{m+1}f(z)}{F_{\lambda,p}^{m}f(z)} \quad (z \in \mathbb{U}). \tag{3.4}$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Therefore, differentiating (3.4) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

$$\frac{\delta F_{\lambda,p}^{m+2}f(z)}{\beta(\lambda-\alpha)F_{\lambda,p}^{m}f(z)} + \left(\delta+\alpha-\frac{\delta}{\beta(\lambda-\alpha)}\right)\frac{\left(F_{\lambda,p}^{m+1}f(z)\right)^2}{\left(F_{\lambda,p}^{m}f(z)\right)^2} = (\delta+\alpha)\left(p(z)\right)^2 + \delta zp'(z), \tag{3.5}$$

that is,

$$(\delta+\alpha)\left(p(z)\right)^2 + \delta zp'(z) \prec (\delta+\alpha)\left(q(z)\right)^2 + \delta zq'(z).$$

Therefore, Theorem 3.1 now follows by applying Lemma 2.2 by setting

$$\theta(w) = (\delta+\alpha)w^2 \text{ and } \varphi(w) = \delta. \qquad \square$$

Corollary 3.2. *Let* $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ *in Theorem* 3.1, *further assuming that (*3.1*) holds.*

If $f \in \mathcal{A}(p)$ *satisfy the following subordination condition:*

$$\Upsilon(m, \lambda, p, \delta; z) \prec \frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2,$$

then

$$\frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ *is the best dominant.*

In particular, if $q(z) = \frac{1+z}{1-z}$, *then for* $f \in \mathcal{A}(p)$ *we have,*

$$\Upsilon(m, \lambda, p, \delta; z) \prec \frac{2\delta z}{(1-z)^2} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^2,$$

then

$$\frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \prec \frac{1+z}{1-z}$$

and the function $\frac{1+z}{1-z}$ *is the best dominant.*

Furthermore, if we take $q(z) = \left(\frac{1+z}{1-z}\right)^\mu$, $(0 < \mu \le 1)$, *then for* $f \in \mathcal{A}(p)$ *we have,*

$$\Upsilon(m, \lambda, p, \delta; z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^{2\mu},$$

then

$$\frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \prec \left(\frac{1+z}{1-z}\right)^\mu$$

and the function $\left(\frac{1+z}{1-z}\right)^\mu$ *is the best dominant.*

Next, by applying Lemma 2.3 we prove the following.

Theorem 3.3. *Let* $q(z)$ *be convex univalent in* $\mathbb{U}$ *with* $q(0) = 1$. *Assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\alpha)q(z)q'(z)}{\delta}\right\} > 0. \tag{3.6}$$

Let $f \in \mathcal{A}(p)$ *such that* $\frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \in H[q(0), 1] \cap Q$, $\Upsilon(m, \lambda, p, \delta; z)$ *is univalent in* $\mathbb{U}$ *and the following superordination condition*

$$(\delta+\alpha)(q(z))^2 + \delta z q'(z) \prec \Upsilon(m, \lambda, p, \delta; z) \tag{3.7}$$

holds, then

$$q(z) \prec \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \tag{3.8}$$

and $q(z)$ *is the best subordinant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)}.$$

Then from the assumption of Theorem 3.3, the function $p(z)$ is analytic in $\mathbb{U}$ and (3.5) holds. Hence, the subordination (3.7) is equivalent to

$$(\delta+\alpha)\left(q(z)\right)^2 + \delta z q'(z) \prec (\delta+\alpha)\left(p(z)\right)^2 + \delta z p'(z)$$

The assertion (3.8) of Theorem 3.3 now follows by an application of Lemma 2.3. □

Corollary 3.4. *Let* $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ *in Theorem* 3.3*, further assuming that (*3.6*) holds.*

If $f \in \mathcal{A}(p)$ such that $\frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)} \in H[q(0),1] \cap Q$, $\Upsilon(m,\lambda,p,\delta;z)$ is univalent in $\mathbb{U}$ and the following superordination condition

$$\frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2 \prec \Upsilon(m,\lambda,p,\delta;z)$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)}$$

and $q(z)$ is the best subordinant.

Also, let $q(z) = \frac{1+z}{1-z}$, then for $f \in \mathcal{A}(p)$ we have,

$$\frac{2\delta z}{(1-z)^2} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^2 \prec \Upsilon(m,\lambda,p,\delta;z),$$

then

$$\frac{1+z}{1-z} \prec \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)}$$

and the function $\frac{1+z}{1-z}$ is the best subordinant.

Finally, by taking $q(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$, $(0 < \mu \leq 1)$, then for $f \in \mathcal{A}(p)$ we have,

$$\frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^{2\mu} \prec \Upsilon(m,\lambda,p,\delta;z),$$

then

$$\left(\frac{1+z}{1-z}\right)^{\mu} \prec \frac{F_{\lambda,p}^{m+1} f(z)}{F_{\lambda,p}^{m} f(z)}$$

and the function $\left(\frac{1+z}{1-z}\right)^{\mu}$ is the best subordinant.

Combining Theorem 3.1 and Theorem 3.3, we get the following sandwich theorem.

Theorem 3.5. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$ and satisfies (3.1) and (3.6) respectively. If $f \in \mathcal{A}(p)$ such that $\frac{F^{m+1}_{\lambda,p} f(z)}{F^{m}_{\lambda,p} f(z)} \in H[q(0), 1] \cap Q$, $\Upsilon(m, \lambda, p, \delta; z)$ is univalent in $\mathbb{U}$ and*

$$(\delta + \alpha) (q_1(z))^2 + \delta z q_1'(z) \prec \Upsilon(m, \lambda, p, \delta; z) \prec (\delta + \alpha) (q_2(z))^2 + \delta z q_2'(z),$$

holds, then $q_1(z) \prec \frac{F^{m+1}_{\lambda,p} f(z)}{F^{m}_{\lambda,p} f(z)} \prec q_2(z)$ and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Corollary 3.6. *Let $q_i(z) = \frac{1+A_i z}{1+B_i z}$ $(i = 1, 2; -1 \leq B_2 < B_1 < A_1 \leq A_2 \leq 1)$ in Theorem 3.5. If $f \in \mathcal{A}(p)$ such that $\frac{F^{m+1}_{\lambda,p} f(z)}{F^{m}_{\lambda,p} f(z)} \in H[q(0), 1] \cap Q$, $\Upsilon(m, \lambda, p, \delta; z)$ is univalent in $\mathbb{U}$ and*

$$\frac{\delta(A_1 - B_1)z}{(1 + B_1 z)^2} + (\delta+\alpha)\left(\frac{1 + A_1 z}{1 + B_1 z}\right)^2 \prec \Upsilon(m, \lambda, p, \delta; z) \prec \frac{\delta(A_2 - B_2)z}{(1 + B_2 z)^2} + (\delta+\alpha)\left(\frac{1 + A_2 z}{1 + B_2 z}\right)^2$$

holds, then $\frac{1+A_1 z}{1+B_1 z} \prec \frac{F^{m+1}_{\lambda,p} f(z)}{F^{m}_{\lambda,p} f(z)} \prec \frac{1+A_2 z}{1+B_2 z}$ and $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are, respectively, the best subordinant and the best dominant.

Remarks. Other works related to differential subordination or superordination can be found in [2], [6], [8]-[12], [15], [16], [20], [22].

Acknowledgement. The work here was fully supported by AP-2013-009.

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 403–411

Pascu-type p-valent functions associated with the convolution structure

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Abstract. Making use of convolution structure, we introduce a new class of p-valent functions. Among the results presented in this paper include the coefficient bounds, distortion inequalities, extreme points and integral means inequalities for this generalized class of functions are discussed.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: p-valent functions, coefficient bounds, Hadamard product (or convolution), extreme points, distortion bounds, integral means, Sălăgean operator.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=2p+1}^{\infty} a_k z^k. \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.1}$$

which are *analytic* and *p-valent* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \,\text{and}\, |z| < 1\}$.

A function $f \in \mathcal{A}_p$ is β-Pascu convex of order α if

$$\frac{1}{p} Re \left\{ \frac{(1-\beta) z f'(z) + \frac{\beta}{p} z \left(z f'(z) \right)'}{(1-\beta) f(z) + \frac{\beta}{p} z f'(z)} \right\} > \alpha \qquad (0 \leq \beta \leq 1,\ 0 \leq \alpha < 1)\,.$$

In the other words $(1-\beta) f(z) + \frac{\beta}{p} z f'(z)$ is in $f \in \mathcal{S}_p^*$ the class of p-valent starlike functions (for details [5], see also [1], [3]).

Given two functions $f, g \in \mathcal{A}_p$, where f is given by (1.1) and g is given by

$$g(z) = z^p + \sum_{k=2p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N})\,,$$

the *Hadamard product* (or *convolution*) $f * g$ is defined (as usual) by

$$(f * g)(z) = z^p + \sum_{k=2p+1}^{\infty} a_k b_k z^k = (g * f)(z)\ ,\ z \in \mathbb{U}. \tag{1.2}$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

See also Duren [2].

On the other hand, Sălăgean [6] introduced the following operator which is popularly known as the *Sălăgean derivative operator*:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

and, in general,

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

We easily find from (1.1) that

$$D^n f(z) = p^n z^p + \sum_{k=2p+1}^{\infty} k^n a_k z^k \qquad (f \in \mathcal{A}_p\ ;\ n \in \mathbb{N}_0).$$

We denote by $\mathcal{T}_p$ the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=2p+1}^{\infty} a_k z^k, \quad (a_k \geq 0,\ p \in \mathbb{N}) \tag{1.3}$$

which are p-valent in $\mathbb{U}$.

For a given function $g \in \mathcal{A}_p$ defined by

$$g(z) = z^p + \sum_{k=2p+1}^{\infty} b_k z^k \quad (b_k > 0,\ p \in \mathbb{N}), \tag{1.4}$$

we introduce here a new class $\mathcal{AS}_g^*(n, p, \alpha, \beta)$ of functions belonging to the subclass of $\mathcal{T}_p$ which consists of functions $f(z)$ of the form (1.3) satisfying the following inequality:

$$\frac{1}{p} Re \left\{ \frac{(1-\beta) D^{n+1}(f * g)(z) + \frac{\beta}{p} D^{n+2}(f * g)(z)}{(1-\beta) D^n (f * g)(z) + \frac{\beta}{p} D^{n+1}(f * g)(z)} \right\} > \alpha \tag{1.5}$$

$$(0 \leq \alpha < 1,\ 0 \leq \beta \leq 1,\ n, p \in \mathbb{N})$$

In this paper, we obtain the coefficient inequalities, distortion theorems as well as integral means inequalities for functions in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$.

We first prove a necessary and sufficient condition for functions to be in $\mathcal{AS}_g^*(n,p,\alpha,\beta)$ in the following:

2. Coefficient inequalities

Theorem 2.1. *A function $f(z)$ given by (1.3) is in $\mathcal{AS}_g^*(n,p,\alpha,\beta)$ if and only if for $0 \le \alpha < 1$, $0 \le \beta \le 1$, $n, p \in \mathbb{N}$,*

$$\sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\, k^n a_k b_k \le p^{n+2}(1-\alpha). \tag{2.1}$$

Proof. Assume that $f \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$. Then, in view of (1.3) to (1.5), we have

$$\frac{1}{p} Re\left\{\frac{(1-\beta)D^{n+1}(f*g)(z)+\frac{\beta}{p}D^{n+2}(f*g)(z)}{(1-\beta)D^{n}(f*g)(z)+\frac{\beta}{p}D^{n+1}(f*g)(z)}\right\}$$

$$=\frac{1}{p} Re\left\{\frac{p^{n+1}-\sum\limits_{k=2p+1}^{\infty}\left[(1-\beta+\frac{\beta}{p}k)\right]k^{n+1}a_k b_k z^{k-p}}{p^{n}-\sum\limits_{k=2p+1}^{\infty}\left[(1-\beta+\frac{\beta}{p}k)\right]k^{n}a_k b_k z^{k-p}}\right\} > \alpha \qquad (z\in\mathbb{U}).$$

If we choose z to be real and let $r \to 1^-$, the last inequality leads us to desired assertion (2.1) of Theorem 2.1.

Conversely, assume that (2.1) holds. For $f(z) \in \mathcal{A}_p$, let us define the function $F(z)$ by

$$F(z)=\frac{1}{p}\frac{(1-\beta)D^{n+1}(f*g)(z)+\frac{\beta}{p}D^{n+2}(f*g)(z)}{(1-\beta)D^{n}(f*g)(z)+\frac{\beta}{p}D^{n+1}(f*g)(z)}-\alpha$$

It suffices to show that

$$\left|\frac{F(z)-1}{F(z)+1}\right| < 1 \qquad (z\in\mathbb{U}).$$

We note that

$$\left|\frac{F(z)-1}{F(z)+1}\right|$$

$$=\left|\frac{(1-\beta)D^{n+1}(f*g)(z)+\frac{\beta}{p}D^{n+2}(f*g)(z)-p(\alpha+1)\left[(1-\beta)D^{n}(f*g)(z)+\frac{\beta}{p}D^{n+1}(f*g)(z)\right]}{(1-\beta)D^{n+1}(f*g)(z)+\frac{\beta}{p}D^{n+2}(f*g)(z)-p(\alpha-1)\left[(1-\beta)D^{n}(f*g)(z)+\frac{\beta}{p}D^{n+1}(f*g)(z)\right]}\right|$$

$$=\left|\frac{-\alpha p^{n+1}-\sum\limits_{k=2p+1}^{\infty}\left[(k-\alpha p-p)(1-\beta+\frac{\beta}{p}k)\right]k^{n}a_k b_k z^{k-p}}{(2-\alpha)p^{n+1}-\sum\limits_{k=2p+1}^{\infty}\left[(k-\alpha p+p)(1-\beta+\frac{\beta}{p}k)\right]k^{n}a_k b_k z^{k-p}}\right|$$

$$\leq \frac{\alpha p^{n+2} + \sum_{k=2p+1}^{\infty} \left[(k-\alpha p - p)(p-\beta p+\beta k)\right] k^n a_k b_k}{(2-\alpha)p^{n+2} - \sum_{k=2p+1}^{\infty} \left[(k-\alpha p + p)(p-\beta p+\beta k)\right] k^n a_k b_k}$$

The last expression is bounded above by 1, if

$$\alpha p^{n+2} + \sum_{k=2p+1}^{\infty} \left[(k-\alpha p - p)(p-\beta p+\beta k)\right] k^n a_k b_k$$

$$\leq (2-\alpha)p^{n+2} - \sum_{k=2p+1}^{\infty} \left[(k-\alpha p + p)(p-\beta p+\beta k)\right] k^n a_k b_k$$

which is equivalent to our condition (2.1). This completes the proof of our theorem. □

Corollary 2.2. *Let $f(z)$ given by (1.3). If $f \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$, then*

$$a_k \leq \frac{p^{n+2}(1-\alpha)}{\left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n b_k} \tag{2.2}$$

with equality for functions of the form

$$f_k(z) = z^p - \frac{p^{n+2}(1-\alpha)}{\left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n b_k} z^k$$

Proof. If $f \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$, then by making use of (2.1), we obtain

$$\left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n a_k b_k \leq \sum_{k=2p+1}^{\infty} \left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n a_k b_k$$
$$\leq p^{n+2}(1-\alpha)$$

or

$$a_k \leq \frac{p^{n+2}(1-\alpha)}{\left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n b_k}.$$

Clearly for

$$f_k(z) = z^p - \frac{p^{n+2}(1-\alpha)}{\left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n b_k} z^k,$$

we have

$$a_k = \frac{p^{n+2}(1-\alpha)}{\left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n b_k}.$$

□

3. Distortion inequalities

In this section, we shall prove distortion theorems for functions belonging to the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$.

Theorem 3.1. *Let the function $f(z)$ of the form (1.3) be in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$|f(z)| \geq r^p - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} r^{2p+1} \tag{3.1}$$

and

$$|f(z)| \leq r^p + \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} r^{2p+1}, \tag{3.2}$$

provided $b_k \geq b_{2p+1}$ $(k \geq 2p+1)$. The result is sharp with equality for

$$f(z) = z^p - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} z^{2p+1}.$$

at $z = r$ and $z = re^{\frac{i(2m+1)\pi}{p+1}}$, $m \in \mathbb{Z}$.

Proof. Since $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$, we apply Theorem 2.1, we obtain

$$(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1} \sum_{k=2p+1}^{\infty} a_k$$

$$\leq \sum_{k=2p+1}^{\infty} \left[(k-\alpha p)(p-\beta p+\beta k)\right] k^n a_k b_k \leq p^{n+2}(1-\alpha).$$

Thus, we obtain

$$\sum_{k=2p+1}^{\infty} a_k \leq \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}}. \tag{3.3}$$

From (1.3) and (3.3), we have

$$|f(z)| \leq |z|^p + |z|^{2p+1} \sum_{k=2p+1}^{\infty} a_k \leq r^p + \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} r^{2p+1}$$

and

$$|f(z)| \geq |z|^p - |z|^{2p+1} \sum_{k=2p+1}^{\infty} a_k \geq r^p - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^n b_{2p+1}} r^{2p+1}.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let the function $f(z)$ of the form (1.3) be in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$|f'(z)| \geq pr^{p-1} - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n-1} b_{2p+1}} r^{2p} \tag{3.4}$$

and

$$|f'(z)| \leq pr^p + \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n-1}b_{2p+1}}r^{2p}, \tag{3.5}$$

provided $b_k \geq b_{2p+1}$ $(k \geq 2p+1)$. *The result is sharp with equality for*

$$f(z) = z^p - \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n-1}b_{2p+1}}z^{2p}$$

at $z = r$ *and* $z = re^{\frac{i(2m+1)\pi}{p}}$, $m \in \mathbb{Z}$.

Proof. From Theorem 2.1 and (3.3), we have

$$\sum_{k=2p+1}^{\infty} ka_k \leq \frac{p^{n+2}(1-\alpha)}{(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n-1}b_{2p+1}}.$$

and the remaining part of the proof is similar to the proof of Theorem 3.1. □

4. Extreme points

Theorem 4.1. *Let* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k}z^k$$

$$(b_k > 0, 0 \leq \alpha < 1, 0 \leq \beta \leq 1,\ n, p \in \mathbb{N},\ k = 2p+1, 2p+2, \ldots).$$

Then $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$ *if and only if it can be expressed in the following form:*

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z),$$

where $\lambda_p \geq 0$, $\lambda_k \geq 0$ *and* $\lambda_p + \sum_{k=2p+1}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z) = z^p - \sum_{k=2p+1}^{\infty} \lambda_k \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k}z^k.$$

Then from Theorem 2.1, we have

$$\sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\,k^n \lambda_k \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k} b_k$$

$$= \sum_{k=2p+1}^{\infty} \lambda_k p^{n+2}(1-\alpha) = p^{n+2}(1-\alpha)(1-\lambda_p) \leq p^{n+2}(1-\alpha)$$

Thus, in view of Theorem 2.1, we find that $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$.
Conversely, suppose that $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$. Then, since

$$a_k \leq \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k} \qquad (p \in \mathbb{N}),$$

we may set

$$\lambda_k = \frac{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k}{p^{n+2}(1-\alpha)} a_k \quad (p \in \mathbb{N})$$

and

$$\lambda_p = 1 - \sum_{k=2p+1}^{\infty} \lambda_k.$$

Thus, clearly, we have

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z).$$

This completes the proof of theorem. □

Corollary 4.2. *The extreme points of the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$ are given by*

$$f_p(z) = z^p$$

and

$$f_k(z) = z^p - \frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^n b_k} z^k, \qquad (k \geq 2p+1,\ p \in \mathbb{N})\,. \tag{4.1}$$

Theorem 4.3. *The class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$ is a convex set.*

Proof. Suppose that each of the functions $f_i(z)$, $(i=1,2)$ given by

$$f_i(z) = z^p - \sum_{k=2p+1}^{\infty} a_{k,i} z^k, \qquad (a_{k,i} \geq 0)$$

is in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \eta f_1(z) + (1-\eta) f_2(z), \qquad (0 \leq \eta < 1)$$

is also in the class $\mathcal{AS}_g^*(n,p,\alpha,\beta)$. Since

$$g(z) = \eta \left(z^p - \sum_{k=2p+1}^{\infty} a_{k,1} z^k \right) + (1-\eta) \left(z^p - \sum_{k=2p+1}^{\infty} a_{k,2} z^k \right)$$

$$= z^p - \sum_{k=2p+1}^{\infty} [\eta a_{k,1} + (1-\eta) a_{k,2}]\, z^k$$

with the aid of Theorem 2.1, we have

$$\sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\,k^n\,[\eta a_{k,1} + (1-\eta)a_{k,2}]\,b_k$$

$$= \eta \sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\,k^n a_{k,1} b_k$$

$$+(1-\eta) \sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\,k^n a_{k,2} b_k$$

$$\leq \eta p^{n+2}(1-\alpha)+(1-\eta)p^{n+2}(1-\alpha)=p^{n+2}(1-\alpha). \qquad \square$$

5. Integral means inequalities

In 1925, Littlewood proved the following subordination theorem.

Theorem 5.1. (Littlewood [4]) *If f and g are analytic in $\mathbb{U}$ with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leqq \int_0^{2\pi} |g\,(z)|^{\mu}\, d\theta.$$

We will make use of Theorem 5.1 to prove

Theorem 5.2. *Let $f(z) \in \mathcal{AS}_g^*(n,p,\alpha,\beta)$ and $f_k(z)$ is defined by (4.1). If there exists an analytic function $w(z)$ given by*

$$w(z)^{k-p} = \frac{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k}{p^{n+2}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p},$$

then for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|f_k(re^{i\theta})\right|^{\mu} d\theta \qquad (\mu > 0).$$

Proof. We must show that

$$\int_0^{2\pi} \left|1-\sum_{k=2p+1}^{\infty} a_k z^{k-p}\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|1-\frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k} z^{k-p}\right|^{\mu} d\theta.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1-\sum_{k=2p+1}^{\infty} a_k z^{k-p} \prec 1-\frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k} z^{k-p}.$$

By setting

$$1-\sum_{k=2p+1}^{\infty} a_k z^{k-p} = 1-\frac{p^{n+2}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k} \left[w(z)\right]^{k-p},$$

we find that

$$\left[w(z)\right]^{k-p} = \frac{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k}{p^{n+2}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p}$$

which readily yields $w(0) = 0$.

Furthermore, using (2.1), we obtain

$$|w(z)|^{k-p} \leq \left|\frac{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k}{p^{n+2}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p}\right|$$

$$\leq \frac{[(k-\alpha p)(p-\beta p+\beta k)]\, k^n b_k}{p^{n+2}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k\, |z|^{k-p} \leq |z|^{k-p} < 1.$$

This completes the proof of the theorem. □

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Second Hankel determinant for the class of Bazilevic functions

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Abstract. The objective of this paper is to obtain a sharp upper bound to the second Hankel determinant $H_2(2)$ for the function f when it belongs to the class of Bazilevic functions, using Toeplitz determinants. The result presented here include two known results as their special cases.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic function, Bazilevic function, upper bound, second Hankel functional, positive real function, Toeplitz determinants.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke ([15]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, (a_1 = 1). \tag{1.2}$$

This determinant has been considered by many authors in the literature. Noonan and Thomas ([13]) studied about the second Hankel determinant of areally mean p-valent functions. Ehrenborg ([5]) studied the Hankel determinant of exponential polynomials. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali ([2]) found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$$

when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. In this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2. \tag{1.3}$$

Janteng, Halim and Darus ([8]) have considered the functional $|a_2a_4 - a_3^2|$ and found sharp upper bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor ([11]). In their work, they have shown that if $f \in RT$ then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. Janteng, Halim and Darus ([7]) also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and have shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors ([1], [3], [9], [12], [18]).

Motivated by the results obtained by different authors in this direction mentioned above, in this paper, we seek an upper bound to the functional $|a_2a_4 - a_3^2|$ for the function f when it belongs to the class of Bazilevic functions denoted by B_γ $(0 \leq \gamma \leq 1)$, defined as follows.

Definition 1.1. *A function $f(z) \in A$ is said to be Bazilevic function, if it satisfies the condition*

$$Re\left\{z^{1-\gamma}\frac{f'(z)}{f^{1-\gamma}(z)}\right\} > 0,\ \forall z \in E \tag{1.4}$$

where the powers are meant for principal values. This class of functions was denoted by B_γ, studied by Ram Singh ([16]). It is observed that for $\gamma = 0$ and $\gamma = 1$ respectively, we get $B_0 = ST$ and $B_1 = RT$.

Some preliminary Lemmas required for proving our result are as follows:

2. Preliminary results

Let $\mathcal{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ... = 1 + \sum_{n=1}^{\infty} c_nz^n, \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\text{Re}p(z) > 0$ for any $z \in E$. Here $p(z)$ is called Carathéodory function [4].

Lemma 2.1. ([14], [17]) *If $p \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. ([6]) *The power series for p given in (2.1) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. These are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z),$$

$\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in ([6]) is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we obtain

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2] \geq 0,$$

it is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \text{ for some } x, |x| \leq 1. \tag{2.2}$$

$$\text{and } D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

Simplifying the relations (2.2) and (2.3), we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}, \text{ with } |z| \leq 1. \tag{2.4}$$

To obtain our result, we refer to the classical method devised by Libera and Zlotkiewicz ([10]).

3. Main result

Theorem 3.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\gamma$ $(0 \leq \gamma \leq 1)$ then*

$$|a_2 a_4 - a_3^2| \leq \left[\frac{2}{2+\gamma}\right]^2$$

and the inequality is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\gamma$, by virtue of Definition 1.1, there exists an analytic function $p \in \mathcal{P}$ in the open unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$z^{1-\gamma} \frac{f'(z)}{f^{1-\gamma}(z)} = p(z) \Leftrightarrow z^{1-\gamma} f'(z) = f^{1-\gamma}(z) p(z). \tag{3.1}$$

Replacing the values of $f(z)$, $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$z^{1-\gamma} \left\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right\} = \left\{z + \sum_{n=2}^{\infty} a_n z^n\right\}^{1-\gamma} \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}. \tag{3.2}$$

Using the binomial expansion on the right-hand side of (3.2) subject to the condition

$$\left|\sum_{n=2}^{\infty} a_n z^n\right| < 1 - \gamma,$$

upon simplification, we obtain

$$\begin{aligned} 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + ... &= 1 + \{c_1 + (1-\gamma)a_2\} z \\ &+ \left[c_2 + (1-\gamma)\left\{c_1 a_2 + a_3 + \frac{(-\gamma)}{2} a_2^2\right\}\right] z^2 \\ &+ \left[c_3 + (1-\gamma)\left\{c_2 a_2 + c_1 a_3 + a_4 + (-\gamma)\left\{\frac{1}{2} c_1 a_2^2 + a_2 a_3 + \frac{(-1-\gamma)}{6} a_2^3\right\}\right\}\right] z^3 + \dots \end{aligned} \tag{3.3}$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively on both sides of (3.3), after simplifying, we get

$$\begin{aligned} a_2 &= \frac{c_1}{(1+\gamma)}; \; a_3 = \frac{1}{2(1+\gamma)^2(2+\gamma)} \left\{2(1+\gamma)^2 c_2 + (1-\gamma)(2+\gamma) c_1^2\right\}; \\ a_4 &= \frac{1}{6(1+\gamma)^3(2+\gamma)(3+\gamma)} \times \{6(1+\gamma)^2(2+\gamma) c_3 \\ &\quad + 6(1-\gamma)(1+\gamma)^2(3+\gamma) c_1 c_2 + (\gamma-1)(2+\gamma)(2\gamma^2 + 5\gamma - 3) c_1^3\}. \end{aligned} \tag{3.4}$$

Substituting the values of a_2, a_3 and a_4 from (3.4) in the second Hankel functional $|a_2 a_4 - a_3^2|$ for the function $f \in B_\gamma$, which simplifies to

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{1}{12(1+\gamma)^3(2+\gamma)^2(3+\gamma)} |12(1+\gamma)^2(2+\gamma)^2 c_1 c_3 \\ &\quad - 12(1+\gamma)^3(3+\gamma) c_2^2 + (2+\gamma)^2(3+\gamma)(\gamma-1) c_1^4|. \end{aligned}$$

The above expression is equivalent to

$$|a_2 a_4 - a_3^2| = \frac{1}{12(1+\gamma)^3(2+\gamma)^2(3+\gamma)} \left|d_1 c_1 c_3 + d_2 c_2^2 + d_3 c_1^4\right|, \tag{3.5}$$

where

$$\begin{aligned} d_1 &= 12(1+\gamma)^2(2+\gamma)^2; \quad d_2 = -12(1+\gamma)^3(3+\gamma); \\ d_3 &= (2+\gamma)^2(3+\gamma)(\gamma-1). \end{aligned} \tag{3.6}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 on the right-hand side of (3.5), we have

$$\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| = |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} + d_2 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 + d_3c_1^4|.$$

Using the facts that $|z| < 1$ and $|xa + yb| \leq |x||a| + |y||b|$, where x, y, a and b are real numbers, after simplifying, we get

$$4\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \leq |(d_1 + d_2 + 4d_3)c_1^4 + 2d_1c_1(4-c_1^2) +2(d_1+d_2)c_1^2(4-c_1^2)|x| - \left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\}(4-c_1^2)|x|^2|. \tag{3.7}$$

With the values of $d_1,\ d_2$ and d_3 from (3.6), we can write

$$\begin{aligned} d_1 + d_2 + 4d_3 &= 4(\gamma^4 + 6\gamma^3 + 12\gamma^2 + 2\gamma - 9); \\ d_1 &= 12(1+\gamma)^2(2+\gamma)^2;\ d_1 + d_2 = 12(1+\gamma)^2 \end{aligned} \tag{3.8}$$

and

$$(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2 = 12(1+\gamma)^2\left\{c_1^2 + 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\}. \tag{3.9}$$

Consider

$$\begin{aligned} &\left\{c_1^2 + 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\} \\ =&\left[\left\{c_1 + (2+\gamma)^2\right\}^2 - (2+\gamma)^4 + 4(1+\gamma)(3+\gamma)\right] \\ =&\left[\left\{c_1 + (2+\gamma)^2\right\}^2 - \left\{\sqrt{\gamma^4 + 8\gamma^3 + 20\gamma^2 + 16\gamma + 4}\right\}^2\right] \\ =&\left[c_1 + \left\{(2+\gamma)^2 + \sqrt{\gamma^4 + 8\gamma^3 + 20\gamma^2 + 16\gamma + 4}\right\}\right] \\ \times&\left[c_1 + \left\{(2+\gamma)^2 - \sqrt{\gamma^4 + 8\gamma^3 + 20\gamma^2 + 16\gamma + 4}\right\}\right] \end{aligned} \tag{3.10}$$

Since $c_1 \in [0,2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ on the right-hand side of (3.10), after simplifying, we get

$$\begin{aligned} &\left\{c_1^2 + 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\} \\ \geq&\left\{c_1^2 - 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\}. \end{aligned} \tag{3.11}$$

From the relations (3.9) and (3.11), we can write

$$\begin{aligned} &-\left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\} \\ \leq&-12(1+\gamma)^2\left\{c_1^2 - 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\}. \end{aligned} \tag{3.12}$$

Substituting the calculated values from (3.8) and (3.12) on the right-hand side of (3.7), we have

$$\begin{aligned} \left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \leq& |(\gamma^4 + 6\gamma^3 + 12\gamma^2 + 2\gamma - 9)c_1^4 \\ &+ 6(1+\gamma)^2(2+\gamma)^2c_1(4-c_1^2) + 6(1+\gamma)^2c_1^2(4-c_1^2)|x| \\ &- 3(1+\gamma)^2\left\{c_1^2 - 2(2+\gamma)^2c_1 + 4(1+\gamma)(3+\gamma)\right\}(4-c_1^2)|x|^2|. \end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we obtain

$$\begin{aligned}\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| &\le [(-\gamma^4 - 6\gamma^3 + 12\gamma^2 - 2\gamma + 9)c^4 \\ &+ 6(1+\gamma)^2(2+\gamma)^2c(4-c^2) + 6(1+\gamma)^2c^2(4-c^2)\mu \\ &+ 3(1+\gamma)^2\left\{c^2 - 2(2+\gamma)^2c + 4(1+\gamma)(3+\gamma)\right\}(4-c^2)\mu^2] \\ &= F(c,\mu), \text{ for } 0 \le \mu = |x| \le 1, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned}F(c,\mu) = [(&-\gamma^4 - 6\gamma^3 + 12\gamma^2 - 2\gamma + 9)c^4 \\ &+ 6(1+\gamma)^2(2+\gamma)^2c(4-c^2) + 6(1+\gamma)^2c^2(4-c^2)\mu \\ &+ 3(1+\gamma)^2\left\{c^2 - 2(2+\gamma)^2c + 4(1+\gamma)(3+\gamma)\right\}(4-c^2)\mu^2]. \end{aligned} \tag{3.14}$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ in (3.14) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 6(1+\gamma)^2[c^2 + \left\{c^2 - 2(2+\gamma)^2c + 4(1+\gamma)(3+\gamma)\right\}\mu] \times (4-c^2). \tag{3.15}$$

For $0 < \mu < 1$, for any fixed c with $0 < c < 2$ and $o \le \gamma \le 1$, from (3.15), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ and hence it cannot have maximum value any point in the interior of the closed region $[0,2] \times [0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0 \le \mu \le 1} F(c,\mu) = F(c,1) = G(c). \tag{3.16}$$

In view of (3.16), replacing μ by 1 in (3.14), upon simplification, we obtain

$$\begin{aligned}G(c) = F(c,1) = -\gamma(\gamma^3 + 6\gamma^2 - 3\gamma + 20)c^4 &- 12\gamma(1+\gamma)^2(4+\gamma)c^2 \\ &+ 48(1+\gamma)^3(3+\gamma), \end{aligned} \tag{3.17}$$

$$G'(c) = -4\gamma c\left\{(\gamma^3 + 6\gamma^2 - 3\gamma + 20)c^2 + 6(1+\gamma)^2(4+\gamma)\right\}. \tag{3.18}$$

From the expression (3.18), we observe that $G'(c) \le 0$, for every $c \in [0,2]$ and for fixed γ with $0 \le \gamma \le 1$. Therefore, $G(c)$ is a decreasing function of c in the interval [0,2], whose maximum value occurs at $c = 0$ only. For $c = 0$ in (3.17), the maximum value of $G(c)$ is given by

$$G_{max} = G(0) = 48(1+\gamma)^3(3+\gamma). \tag{3.19}$$

From the expressions (3.13) and (3.19), we have

$$\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \le 48(1+\gamma)^3(3+\gamma). \tag{3.20}$$

Simplifying the relations (3.5) and (3.20), we obtain

$$|a_2a_4 - a_3^2| \le \left[\frac{2}{2+\gamma}\right]^2. \tag{3.21}$$

Choosing $c_1 = c = 0$ and selecting $x = 1$ in (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$. Substituting these values in (3.20), we observe that equality is attained which shows that our result is sharp. For these values, we derive that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + 2z^2 + 2z^4 + ... = \frac{1+z^2}{1-z^2}. \tag{3.22}$$

Therefore, the extremal function in this case is

$$z^{1-\gamma}\frac{f'(z)}{f^{1-\gamma}(z)} = 1 + 2z^2 + 2z^4 + ... = \frac{1+z^2}{1-z^2}. \tag{3.23}$$

This completes the proof of our Theorem. □

Remark 3.2. Choosing $\gamma = 0$, from (3.21), we get $|a_2 a_4 - a_3^2| \leq 1$, this inequality is sharp and coincides with that of Janteng, Halim, Darus ([7]).

Remark 3.3. For the choice of $\gamma = 1$ in (3.21), we obtain $|a_2 a_4 - a_3^2| \leq \frac{4}{9}$ and is sharp, coincides with the result of Janteng, Halim, Darus ([8]) .

Acknowledgement. The authors express their sincere thanks to the esteemed Referee(s) for their careful readings, valuable suggestions and comments, which helped to improve the presentation of the paper.

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Some extensions of the Open Door Lemma

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Abstract. Miller and Mocanu proved in their 1997 paper a greatly useful result which is now known as the Open Door Lemma. It provides a sufficient condition for an analytic function on the unit disk to have positive real part. Kuroki and Owa modified the lemma when the initial point is non-real. In the present note, by extending their methods, we give a sufficient condition for an analytic function on the unit disk to take its values in a given sector.

Mathematics Subject Classification (2010): 30C45, 30C80.

Keywords: Open door function, subordination.

1. Introduction

We denote by $\mathcal{H}$ the class of holomorphic functions on the unit disk

$$\mathbb{D} = \{z : |z| < 1\}$$

of the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ denote the subclass of $\mathcal{H}$ consisting of functions h of the form $h(z) = a + c_n z^n + c_{n+1} z^{n+1} + \cdots$. Here, $\mathbb{N} = \{1, 2, 3, \dots\}$. Let also $\mathcal{A}_n$ be the set of functions f of the form $f(z) = zh(z)$ for $h \in \mathcal{H}[1, n]$.

A function $f \in \mathcal{A}_1$ is called *starlike* (resp. *convex*) if f is univalent on $\mathbb{D}$ and if the image $f(\mathbb{D})$ is starlike with respect to the origin (resp. convex). It is well known (cf. [1]) that $f \in \mathcal{A}_1$ is starlike precisely if $q_f(z) = zf'(z)/f(z)$ has positive real part on $|z| < 1$, and that $f \in \mathcal{A}_1$ is convex precisely if $\varphi_f(z) = 1+zf''(z)/f'(z)$ has positive real part on $|z| < 1$. Note that the following relation holds for those quantities:

$$\varphi_f(z) = q_f(z) + \frac{zq_f'(z)}{q_f(z)}.$$

It is geometrically obvious that a convex function is starlike. This, in turn, means the implication

$$\mathrm{Re}\left[q(z) + \frac{zq'(z)}{q(z)}\right] > 0 \text{ on } |z| < 1 \quad \Rightarrow \quad \mathrm{Re}\, q(z) > 0 \text{ on } |z| < 1$$

The authors were supported in part by JSPS Grant-in-Aid for Scientific Research (B) 22340025.

for a function $q \in \mathcal{H}[1,1]$. Interestingly, it looks highly nontrivial. Miller and Mocanu developed a theory (now called *differential subordination*) which enables us to deduce such a result systematically. See a monograph [4] written by them for details.

The set of functions $q \in \mathcal{H}[1,1]$ with $\mathrm{Re}\, q > 0$ is called the Carathéodory class and will be denoted by $\mathcal{P}$. It is well recognized that the function

$$q_0(z) = (1+z)/(1-z)$$

(or its rotation) maps the unit disk univalently onto the right half-plane and is extremal in many problems. One can observe that the function

$$\varphi_0(z) = q_0(z) + \frac{zq_0'(z)}{q_0(z)} = \frac{1+z}{1-z} + \frac{2z}{1-z^2} = \frac{1+4z+z^2}{1-z^2}$$

maps the unit disk onto the slit domain $V(-\sqrt{3}, \sqrt{3})$, where

$$V(A,B) = \mathbb{C} \setminus \{iy : y \le A \text{ or } y \ge B\}$$

for $A, B \in \mathbb{R}$ with $A < B$. Note that $V(A,B)$ contains the right half-plane and has the "window" (Ai, Bi) in the imaginary axis to the left half-plane. The Open Door Lemma of Miller and Mocanu asserts for a function $q \in \mathcal{H}[1,1]$ that, if $q(z) + zq'(z)/q(z) \in V(-\sqrt{3}, \sqrt{3})$ for $z \in \mathbb{D}$, then $q \in \mathcal{P}$. Indeed, Miller and Mocanu [3] (see also [4]) proved it in a more general form. For a complex number c with $\mathrm{Re}\, c > 0$ and $n \in \mathbb{N}$, we consider the positive number

$$C_n(c) = \frac{n}{\mathrm{Re}\, c}\left[|c|\sqrt{\frac{2\mathrm{Re}\, c}{n} + 1} + \mathrm{Im}\, c \right].$$

In particular, $C_n(c) = \sqrt{n(n+2c)}$ when c is real. The following is a version of the Open Door Lemma modified by Kuroki and Owa [2].

Theorem A (Open Door Lemma). *Let c be a complex number with positive real part and n be an integer with $n \ge 1$. Suppose that a function $q \in \mathcal{H}[c,n]$ satisfies the condition*

$$q(z) + \frac{zq'(z)}{q(z)} \in V(-C_n(c), C_n(\bar{c})), \quad z \in \mathbb{D}.$$

Then $\mathrm{Re}\, q > 0$ on $\mathbb{D}$.

Remark 1.1. In the original statement of the Open Door Lemma in [3], the slit domain was erroneously described as $V(-C_n(c), C_n(c))$. Since $C_n(\bar{c}) < C_n(c)$ when $\mathrm{Im}\, c > 0$, we see that $V(-C_n(\bar{c}), C_n(\bar{c})) \subset V(-C_n(c), C_n(\bar{c})) \subset V(-C_n(c), C_n(c))$ for $\mathrm{Im}\, c \ge 0$ and the inclusions are strict if $\mathrm{Im}\, c > 0$. As the proof will suggest us, seemingly the domain $V(-C_n(c), C_n(\bar{c}))$ is maximal for the assertion, which means that the original statement in [3] and the form of the associated open door function are incorrect for a non-real c. This, however, does not decrease so much the value of the original article [3] by Miller and Mocanu because the Open Door Lemma is mostly applied when c is real. We also note that the Open Door Lemma deals with the function $p = 1/q \in \mathcal{H}[1/c, n]$ instead of q. The present form is adopted for convenience of our aim.

The Open Door Lemma gives a sufficient condition for $q \in \mathcal{H}[c,n]$ to have positive real part. We extend it so that $|\arg q| < \pi\alpha/2$ for a given $0 < \alpha \le 1$. First we note that the Möbius transformation

$$g_c(z) = \frac{c + \bar{c}z}{1-z}$$

maps $\mathbb{D}$ onto the right half-plane in such a way that $g_c(0) = c$, where c is a complex number with $\operatorname{Re} c > 0$. In particular, one can take an analytic branch of $\log g_c$ so that $|\operatorname{Im} \log g_c| < \pi/2$. Therefore, the function $q_0 = g_c^\alpha = \exp(\alpha \log g_c)$ maps $\mathbb{D}$ univalently onto the sector $|\arg w| < \pi\alpha/2$ in such a way that $q_0(0) = c^\alpha$. The present note is based mainly on the following result, which will be deduced from a more general result of Miller and Mocanu (see Section 2).

Theorem 1.2. *Let c be a complex number with $\operatorname{Re} c > 0$ and α be a real number with $0 < \alpha \le 1$. Then the function*

$$R_{\alpha,c,n}(z) = g_c(z)^\alpha + \frac{n\alpha z g_c'(z)}{g_c(z)} = \left(\frac{c+\bar{c}z}{1-z}\right)^\alpha + \frac{2n\alpha(\operatorname{Re} c)z}{(1-z)(c+\bar{c}z)}$$

is univalent on $|z| < 1$. If a function $q \in \mathcal{H}[c^\alpha, n]$ satisfies the condition

$$q(z) + \frac{zq'(z)}{q(z)} \in R_{\alpha,c,n}(\mathbb{D}), \quad z \in \mathbb{D},$$

then $|\arg q| < \pi\alpha/2$ on $\mathbb{D}$.

We remark that the special case when $\alpha = 1$ reduces to Theorem A (see the paragraph right after Lemma 3.3 below. Also, the case when $c = 1$ is already proved by Mocanu [5] even under the weaker assumption that $0 < \alpha \le 2$ (see Remark 3.6). Since the shape of $R_{\alpha,c,n}(\mathbb{D})$ is not very clear, we will deduce more concrete results as corollaries of Theorem 1.2 in Section 3. This is our principal aim in the present note.

2. Preliminaries

We first recall the notion of subordination. A function $f \in \mathcal{H}$ is said to be *subordinate* to $F \in \mathcal{H}$ if there exists a function $\omega \in \mathcal{H}[0,1]$ such that $|\omega| < 1$ on $\mathbb{D}$ and that $f = F \circ \omega$. We write $f \prec F$ or $f(z) \prec F(z)$ for subordination. When F is univalent, $f \prec F$ precisely when $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

Miller and Mocanu [3, Theorem 5] (see also [4, Theorem 3.2h]) proved the following general result, from which we will deduce Theorem 1.2 in the next section.

Lemma 2.1 (Miller and Mocanu). *Let $\mu, \nu \in \mathbb{C}$ with $\mu \ne 0$ and n be a positive integer. Let $q_0 \in \mathcal{H}[c,1]$ be univalent and assume that $\mu q_0(z) + \nu \ne 0$ for $z \in \mathbb{D}$ and $\operatorname{Re}(\mu c + \nu) > 0$. Set $Q(z) = zq_0'(z)/(\mu q_0(z) + \nu)$, and*

$$h(z) = q_0(z) + nQ(z) = q_0(z) + \frac{nzq_0'(z)}{\mu q_0(z) + \nu}. \tag{2.1}$$

Suppose further that

(a) $\operatorname{Re}[zh'(z)/Q(z)] = \operatorname{Re}[h'(z)(\mu q_0(z) + \nu)/q_0'(z)] > 0$, *and*
(b) *either h is convex or Q is starlike.*

If $q \in \mathcal{H}[c,n]$ *satisfies the subordination relation*

$$q(z) + \frac{zq'(z)}{\mu q(z) + \nu} \prec h(z), \tag{2.2}$$

then $q \prec q_0$, *and* q_0 *is the best dominant. An extremal function is given by*

$$q(z) = q_0(z^n).$$

In the investigation of the generalized open door function $R_{\alpha,c,n}$, we will need to study the positive solution to the equation

$$x^2 + Ax^{1+\alpha} - 1 = 0, \tag{2.3}$$

where $A > 0$ and $0 < \alpha \le 1$ are constants. Let $F(x) = x^2 + Ax^{1+\alpha} - 1$. Then $F(x)$ is increasing in $x > 0$ and $F(0) = -1 < 0$, $F(+\infty) = +\infty$. Therefore, there is a unique positive solution $x = \xi(A,\alpha)$ to the equation. We have the following estimates for the solution.

Lemma 2.2. *Let* $0 < \alpha \le 1$ *and* $A > 0$. *The positive solution* $x = \xi(A,\alpha)$ *to equation* (2.3) *satisfies the inequalities*

$$(1+A)^{-1/(1+\alpha)} \le \xi(A,\alpha) \le (1+A)^{-1/2} \ (<1).$$

Here, both inequalities are strict when $0 < \alpha < 1$.

Proof. Set $\xi = \xi(A,\alpha)$. Since the above $F(x)$ is increasing in $x > 0$, the inequalities $F(x_1) \le 0 = F(\xi) \le F(x_2)$ imply $x_1 \le \xi \le x_2$ for positive numbers x_1, x_2 and the inequalities are strict when $x_1 < \xi < x_2$. Keeping this in mind, we now show the assertion. First we put $x_2 = (1+A)^{-1/2}$ and observe

$$F(x_2) = \frac{1}{1+A} + \frac{A}{(1+A)^{(1+\alpha)/2}} - 1 \ge \frac{1}{1+A} + \frac{A}{1+A} - 1 = 0,$$

which implies the right-hand inequality in the assertion.

Next put $x_1 = (1+A)^{-1/(1+\alpha)}$. Then

$$F(x_1) = \frac{1}{(1+A)^{2/(1+\alpha)}} + \frac{A}{1+A} - 1 \le \frac{1}{1+A} + \frac{A}{1+A} - 1 = 0,$$

which implies the left-hand inequality. We note also that $F(x_1) < 0 < F(x_2)$ when $\alpha < 1$. The proof is now complete. □

3. Proof and corollaries

Theorem 1.2 can be rephrased in the following.

Theorem 3.1. *Let* c *be a complex number with* $\operatorname{Re} c > 0$ *and* α *be a real number with* $0 < \alpha \le 1$. *Then the function*

$$R_{\alpha,c,n}(z) = g_c(z)^\alpha + \frac{n\alpha z g_c'(z)}{g_c(z)}$$

is univalent on $|z| < 1$. *If a function* $q \in \mathcal{H}[c^\alpha, n]$ *satisfies the subordination condition*

$$q(z) + \frac{zq'(z)}{q(z)} \prec R_{\alpha,c,n}(z)$$

on $\mathbb{D}$, *then* $q(z) \prec g_c(z)^\alpha$ *on* $\mathbb{D}$. *The function* g_c^α *is the best dominant.*

Proof. We first show that the function $Q(z) = \alpha z g_c'(z)/g_c(z)$ is starlike. Indeed, we compute

$$\frac{zQ'(z)}{Q(z)} = 1 - \frac{\bar{c}z}{c+\bar{c}z} + \frac{z}{1-z} = \frac{1}{2}\left[\frac{c-\bar{c}z}{c+\bar{c}z} + \frac{1+z}{1-z}\right].$$

Thus we can see that $\mathrm{Re}\,[zQ'(z)/Q(z)] > 0$ on $|z| < 1$. Next we check condition (a) in Lemma 2.1 for the functions $q_0 = g_c^\alpha, h = R_{\alpha,c,n}$ with the choice $\mu = 1, \nu = 0$. We have the expression

$$\frac{zh'(z)}{Q(z)} = q_c(z)^\alpha + n\frac{zQ'(z)}{Q(z)}.$$

Since both terms in the right-hand side have positive real part, we obtain (a). We now apply Lemma 2.1 to obtain the required assertion up to univalence of $h = R_{\alpha,c,n}$. In order to show the univalence, we have only to note that the condition (a) implies that h is close-to-convex, since Q is starlike. As is well known, a close-to-convex function is univalent (see [1]), the proof has been finished. □

We now investigate the shape of the image domain $R_{\alpha,c,n}(\mathbb{D})$ of the generalized open door function $R_{\alpha,c,n}$ given in Theorem 1.2. Let $z = e^{i\theta}$ and $c = re^{it}$ for $\theta \in \mathbb{R}, r > 0$ and $-\pi/2 < t < \pi/2$. Then we have

$$\begin{aligned} R_{\alpha,c,n}(e^{i\theta}) &= \left(\frac{re^{it} + re^{-it}e^{i\theta}}{1-e^{i\theta}}\right)^\alpha + \frac{2n\alpha e^{i\theta}\cos t}{(1-e^{i\theta})(e^{it}+e^{-it}e^{i\theta})} \\ &= \left(\frac{r\cos(t-\theta/2)}{\sin(\theta/2)}i\right)^\alpha + \frac{i}{2}\cdot\frac{n\alpha\cos t}{\sin(\theta/2)\cos(t-\theta/2)} \\ &= r^\alpha e^{\pi\alpha i/2}\left(\cos t\cot(\theta/2) + \sin t\right)^\alpha + \frac{i}{2}\cdot\frac{n\alpha(1+\cot^2(\theta/2))\cos t}{\cos t\cot(\theta/2)+\sin t}. \end{aligned}$$

Let $x = \cot(\theta/2)\cos t + \sin t$. When $x > 0$, we write $R_{\alpha,c,n}(e^{i\theta}) = u_+(x) + iv_+(x)$ and get the expressions

$$\begin{cases} u_+(x) = a(rx)^\alpha, \\ v_+(x) = b(rx)^\alpha + \dfrac{n\alpha}{2\cos t}\left(x - 2\sin t + \dfrac{1}{x}\right), \end{cases}$$

where

$$a = \cos\frac{\alpha\pi}{2} \quad \text{and} \quad b = \sin\frac{\alpha\pi}{2}.$$

Taking the derivative, we get

$$v_+'(x) = \frac{n\alpha}{2x^2\cos t}\left[x^2 + \frac{2br^\alpha\cos t}{n}x^{\alpha+1} - 1\right].$$

Hence, the minimum of $v_+(x)$ is attained at $x = \xi(A, \alpha)$, where $A = 2br^\alpha n^{-1}\cos t$. By using the relation (2.3), we obtain

$$\min_{0<x} v_+(x) = v_+(\xi) = \frac{n}{2\cos t}\left(A\xi^\alpha + \alpha\xi + \frac{\alpha}{\xi}\right) - n\alpha\tan t$$
$$= \frac{n}{2\cos t}\left((\alpha-1)\xi + \frac{\alpha+1}{\xi}\right) - n\alpha\tan t = U(\xi),$$

where

$$U(x) = \frac{n}{2\cos t}\left((\alpha-1)x + \frac{\alpha+1}{x}\right) - n\alpha\tan t.$$

Since the function $U(x)$ is decreasing in $0 < x < 1$, Lemma 2.2 yields the inequality

$$v_+(\xi) = U(\xi) \geq U((1+A)^{-1/2})$$
$$= \frac{n}{2\cos t}\left(\frac{\alpha-1}{\sqrt{1+A}} + (\alpha+1)\sqrt{1+A}\right) - n\alpha\tan t.$$

We remark here that

$$U((1+A)^{-1/2}) > U(1) = \frac{n\alpha(1-\sin t)}{\cos t} > 0;$$

namely, $v_+(x) > 0$ for $x > 0$. When $x < 0$, letting $y = -x = -\cot(\theta/2)\cos t - \sin t$, we write $R_{\alpha,c,n}(e^{i\theta}) = u_-(y) + iv_-(y)$. Then, with the same a and b as above,

$$\begin{cases} u_-(y) = a(ry)^\alpha, \\ v_-(y) = -b(ry)^\alpha - \dfrac{n\alpha}{2\cos t}\left(y + 2\sin t + \dfrac{1}{y}\right), \end{cases}$$

We observe here that $u_+ = u_- > 0$ and, in particular, we obtain the following.

Lemma 3.2. *The left half-plane $\Omega_1 = \{w : \operatorname{Re} w < 0\}$ is contained in $R_{\alpha,c,n}(\mathbb{D})$.*

We now look at $v_-(y)$. Since

$$v_-'(y) = -\frac{n\alpha}{2y^2\cos t}\left[y^2 + \frac{2br^\alpha\cos t}{n}y^{\alpha+1} - 1\right],$$

in the same way as above, we obtain

$$\max_{0<y} v_-(y) = v_-(\xi) = -\frac{n}{2\cos t}\left((\alpha-1)\xi + \frac{\alpha+1}{\xi}\right) - n\alpha\tan t$$
$$\leq -\frac{n}{2\cos t}\left(\frac{\alpha-1}{\sqrt{1+A}} + (\alpha+1)\sqrt{1+A}\right) - n\alpha\tan t,$$

where $\xi = \xi(A, \alpha)$ and $A = 2br^\alpha n^{-1}\cos t$. Note also that $v_-(y) < 0$ for $y > 0$.

Since the horizontal parallel strip $v_-(\xi) < \operatorname{Im} w < v_+(\xi)$ is contained in the image domain $R_{\alpha,c,n}(\mathbb{D})$ of the generalized open door function, we obtain the following.

Lemma 3.3. *The parallel strip Ω_2 described by*

$$|\operatorname{Im} w + n\alpha\tan t| < \frac{n}{2\cos t}\left(\frac{\alpha-1}{\sqrt{1+A}} + (\alpha+1)\sqrt{1+A}\right)$$

is contained in $R_{\alpha,c,n}(\mathbb{D})$. Here, $t = \arg c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $A = \frac{2}{n}|c|^\alpha \sin\frac{\pi\alpha}{2}\cos t$.

When $\alpha = 1$, we have $u_\pm = 0$, that is, the boundary is contained in the imaginary axis. Since $\xi(A,1) = (1+A)^{-1/2}$ by Lemma 2.2, the above computations tell us

$$\min v_+ = (n/\cos t)(\sqrt{1+A} - \sin t) = C_n(\bar{c}).$$

Similarly, we have

$$\max v_- = -(n/\cos t)(\sqrt{1+A} + \sin t) = -C_n(c).$$

Therefore, we have

$$R_{1,c,n}(\mathbb{D}) = V(-C_n(c), C_n(\bar{c})).$$

Note that the open door function then takes the following form

$$\begin{aligned} R_{1,c,n}(z) &= \frac{c+\bar{c}z}{1-z} + \frac{2n(\operatorname{Re} c)z}{(1-z)(c+\bar{c}z)} \\ &= \frac{2\operatorname{Re} c + n}{1 + cz/\bar{c}} - \frac{n}{1-z} - \bar{c}, \end{aligned}$$

which is the same as given by Kuroki and Owa [2, (2.2)]. In this way, we see that Theorem 1.2 contains Theorem A as a special case.

Remark 3.4. In [2], they proposed another open door function of the form

$$R(z) = \frac{2n|c|}{\operatorname{Re} c}\sqrt{\frac{2\operatorname{Re} c}{n} + 1}\,\frac{(\zeta - z)(1-\bar{\zeta}z)}{(1-\bar{\zeta}z)^2 - (\zeta - z)^2} - \frac{\operatorname{Im} c}{\operatorname{Re} c} i,$$

where

$$\zeta = 1 - \frac{2}{\omega}, \quad \omega = \frac{c}{|c|}\sqrt{\frac{2\operatorname{Re} c}{n} + 1} + 1.$$

It can be checked that $R(z) = R_{1,c,n}(-\omega z/\bar{\omega})$. Hence, R is just a rotation of $R_{1,c,n}$.

We next study the argument of the boundary curve of $R_{\alpha,c,n}(\mathbb{D})$. We will assume that $0 < \alpha < 1$ since we have nothing to do when $\alpha = 1$.

As we noted above, the boundary is contained in the right half-plane $\operatorname{Re} w > 0$. When $x > 0$, we have

$$\frac{v_+(x)}{u_+(x)} = \frac{b}{a} + \frac{n\alpha}{2ar^\alpha x^\alpha \cos t}\left[x + \frac{1}{x} - 2\sin t\right].$$

We observe now that $v_+(x)/u_+(x) \to +\infty$ as $x \to 0+$ or $x \to +\infty$. We also have

$$\left(\frac{v_+}{u_+}\right)'(x) = \frac{n\alpha}{2ar^\alpha x^{\alpha+2}\cos t}\left[(1-\alpha)x^2 + 2\alpha x \sin t - (1+\alpha)\right].$$

Therefore, $v_+(x)/u_+(x)$ takes its minimum at $x = \xi$, where

$$\xi = \frac{-\alpha \sin t + \sqrt{1 - \alpha^2 \cos^2 t}}{1-\alpha}$$

is the positive root of the equation $(1-\alpha)x^2 + 2\alpha x \sin t - (1+\alpha) = 0$. It is easy to see that $1 < \xi$ and that

$$T_+ := \min_{0<x} \frac{v_+(x)}{u_+(x)} = \frac{v_+(\xi)}{u_+(\xi)} = \frac{b}{a} + \frac{n\alpha}{2ar^\alpha \xi^\alpha \cos t}\left[\xi + \frac{1}{\xi} - 2\sin t\right]$$
$$= \tan\frac{\pi\alpha}{2} + \frac{n(\xi - \xi^{-1})}{2ar^\alpha \xi^\alpha \cos t}.$$

When $x = -y < 0$, we have

$$\frac{v_-(y)}{u_-(y)} = -\frac{b}{a} - \frac{n\alpha}{2ar^\alpha y^\alpha \cos t}\left[y + \frac{1}{y} + 2\sin t\right]$$

and

$$\left(\frac{v_-}{u_-}\right)'(y) = \frac{-n\alpha}{2ar^\alpha y^{\alpha+2}\cos t}\left[(1-\alpha)y^2 - 2\alpha y \sin t - (1+\alpha)\right].$$

Hence, $v_-(y)/u_-(y)$ takes its maximum at $y = \eta$, where

$$\eta = \frac{\alpha \sin t + \sqrt{1 - \alpha^2 \cos^2 t}}{1-\alpha}.$$

Note that

$$T_- := \max_{0<y} \frac{v_-(y)}{u_-(y)} = \frac{v_-(\eta)}{u_-(\eta)} = -\tan\frac{\pi\alpha}{2} - \frac{n(\eta - \eta^{-1})}{2ar^\alpha \eta^\alpha \cos t}.$$

Therefore, the sector $\{w : T_- < \arg w < T_+\}$ is contained in the image $R_{\alpha,c,n}(\mathbb{D})$. It is easy to check that $T_- < -\tan(\pi\alpha/2) < \tan(\pi\alpha/2) < T_+$. In particular $T_- < \arg c^\alpha = \alpha t < T_+$. We summarize the above observations, together with Theorem 1.2, in the following form.

Corollary 3.5. *Let $0 < \alpha < 1$ and $c = re^{it}$ with $r > 0, -\pi/2 < t < \pi/2$, and n be a positive integer. If a function $q \in \mathcal{H}[c^\alpha, n]$ satisfies the condition*

$$-\Theta_- < \arg\left(q(z) + \frac{zq'(z)}{q(z)}\right) < \Theta_+$$

on $|z| < 1$, then $|\arg q| < \pi\alpha/2$ on $\mathbb{D}$. Here,

$$\Theta_\pm = \arctan\left[\tan\frac{\pi\alpha}{2} + \frac{n(\xi_\pm - \xi_\pm^{-1})}{2r^\alpha \xi_\pm^\alpha \cos(\pi\alpha/2)\cos t}\right],$$

and

$$\xi_\pm = \frac{\mp\alpha \sin t + \sqrt{1 - \alpha^2\cos^2 t}}{1-\alpha}.$$

It is a simple task to check that $x^{1-\alpha} - x^{-1-\alpha}$ is increasing in $0 < x$. When $\operatorname{Im} c > 0$, we see that $\xi_- > \xi_+$ and thus $\Theta_- > \Theta_+$. It might be useful to note the estimates $\xi_- < \sqrt{(1+\alpha)/(1-\alpha)} < \xi_+$ and $\xi_- < 1/\sin t$ for $\operatorname{Im} c > 0$.

Remark 3.6. When $c = 1$ and $n = 1$, we have

$$\xi := \xi_\pm = \sqrt{(1+\alpha)/(1-\alpha)},\ \xi - \xi^{-1} = 2\alpha/\sqrt{1-\alpha^2},$$

and thus

$$\begin{aligned}\Theta_{\pm} &= \arctan\left[\tan\frac{\pi\alpha}{2} + \frac{\xi-\xi^{-1}}{2\xi^{\alpha}\cos\frac{\pi\alpha}{2}}\right] \\ &= \arctan\left[\tan\frac{\pi\alpha}{2} + \frac{\alpha}{\cos\frac{\pi\alpha}{2}(1-\alpha)^{\frac{1-\alpha}{2}}(1+\alpha)^{\frac{1+\alpha}{2}}}\right] \\ &= \frac{\pi\alpha}{2} + \arctan\left[\frac{\alpha\cos\frac{\pi\alpha}{2}}{(1-\alpha)^{\frac{1-\alpha}{2}}(1+\alpha)^{\frac{1+\alpha}{2}} + \alpha\sin\frac{\pi\alpha}{2}}\right].\end{aligned}$$

Therefore, the corollary gives a theorem proved by Mocanu [6].

Since the values Θ_+ and Θ_- are not given in an explicitly way, it might be convenient to have a simpler sufficient condition for $|\arg q| < \pi\alpha/2$.

Corollary 3.7. *Let* $0 < \alpha \le 1$ *and* c *with* $\operatorname{Re} c > 0$ *and* n *be a positive integer. If a function* $q \in \mathcal{H}[c^{\alpha}, n]$ *satisfies the condition*

$$q(z) + \frac{zq'(z)}{q(z)} \in \Omega,$$

then $|\arg q| < \pi\alpha/2$ *on* $\mathbb{D}$. *Here,* $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, *and* Ω_1 *and* Ω_2 *are given in Lemmas* 3.2 *and* 3.3, *respectively, and* $\Omega_3 = \{w \in \mathbb{C} : |\arg w| < \pi\alpha/2\}$.

Proof. Lemmas 3.2 and 3.3 yield that $\Omega_1 \cup \Omega_2 \subset R_{\alpha,c,n}(\mathbb{D})$. Since $\Theta_{\pm} > \pi\alpha/2$, we also have $\Omega_3 \subset R_{\alpha,c,n}(\mathbb{D})$. Thus $\Omega \subset R_{\alpha,c,n}(\mathbb{D})$. Now the result follows from Theorem 1.2. □

See Figure 1 for the shape of the domain Ω together with $R_{\alpha,c,n}(\mathbb{D})$. We remark that $\Omega = R_{\alpha,c,n}(\mathbb{D})$ when $\alpha = 1$.

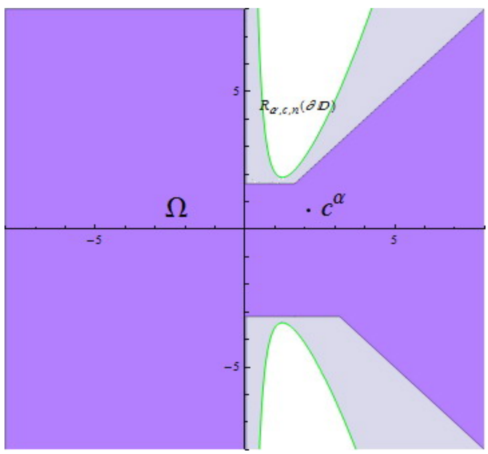

FIGURE 1. The image $R_{\alpha,c,n}(\mathbb{D})$ and Ω for $\alpha = 1/2, c = 4+3i, n = 2$.

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 431–436

On a functional differential inclusion

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Abstract. We consider a Cauchy problem associated to a nonconvex functional differential inclusion and we prove a Filippov type existence result. This result allows to obtain a relaxation theorem for the problem considered.

Mathematics Subject Classification (2010): 34A60, 34K05, 34K15, 47H10.

Keywords: set-valued map, functional differential inclusion, relaxation.

1. Introduction

In this note we study functional differential inclusions of the form

$$x'(t) \in F(t, x(t), x(\lambda t)), \quad x(0) = x_0, \tag{1.1}$$

where $F(.,.,.) : [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map with non-empty values, $\lambda \in (0, 1)$ and $x_0 \in \mathbf{R}$. The present note is motivated by a recent paper [5], where it was studied problem (1.1) with F single valued and several results were obtained using fixed point techniques: existence, uniqueness and differentiability with respect with the delay of the solutions. The study in [5] contains, as a particular case, the problem

$$x'(t) = -ax(t) + a\lambda x(\lambda t), \quad x(0) = x_0,$$

which appears in the radioactive propagation theory ([2]).

The aim of this note is to consider the multivalued framework and to show that Filippov's ideas ([3]) can be suitably adapted in order to obtain the existence of solutions of problem (1.1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([3]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion.

As an application of our main result we obtain a relaxation theorem for the problem considered. Namely, we prove that the solution set of the problem (1.1) is dense in the set of the relaxed solutions; i.e. the set of solutions of the differential inclusion whose right hand side is the convex hull of the original set-valued map.

The paper is organized as follows: in Section 2 we briefly recall some preliminary results that we will use in the sequel and in Section 3 we prove the main results of the paper.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},\ \ d^*(A,B)=\sup\{d(a,B);\, a\in A\},$$

where $d(x,B)=\inf\{d(x,y);y\in B\}$. Let $T>0,\ I:=[0,T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Denote by $\mathcal{P}(\mathbf{R})$ the family of all nonempty subsets of $\mathbf{R}$ and by $\mathcal{B}(\mathbf{R})$ the family of all Borel subsets of $\mathbf{R}$. For any subset $A\subset\mathbf{R}$ we denote by cl$A$ the closure of A and by $\overline{\mathrm{co}}(A)$ the closed convex hull of A.

As usual, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions $x(.):I\to\mathbf{R}$ endowed with the norm

$$|x|_C=\sup_{t\in I}|x(t)|$$

and by $L^1(I,\mathbf{R})$ the Banach space of all integrable functions $x(.):I\to\mathbf{R}$ endowed with the norm

$$|x|_1=\int_0^T|x(t)|\mathrm{d}t.$$

The Banach space of all absolutely continuous functions $x(.):I\to\mathbf{R}$ will be denoted by $AC(I,\mathbf{R})$. We recall that for a set-valued map $U:I\to\mathcal{P}(\mathbf{R})$ the Aumann integral of U, denoted by $\int_I U(t)\mathrm{d}t$, is the set

$$\int_I U(t)\mathrm{d}t=\left\{\int_I u(t)\mathrm{d}t;\ u(.)\in L^1(I,\mathbf{R}),\ u(t)\in U(t)\, a.e.\,(I)\right\}.$$

We recall two results that we are going to use in the next section. The first one is a selection result (e.g., [1]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem. The proof of the second one may be found in [4].

Lemma 2.1. *Consider X a separable Banach space, B is the closed unit ball in X, $H:I\to\mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g:I\to X, L:I\to\mathbf{R}_+$ are measurable functions. If*

$$H(t)\cap(g(t)+L(t)B)\neq\emptyset\quad a.e.(I),$$

then the set-valued map $t\to H(t)\cap(g(t)+L(t)B)$ has a measurable selection.

Lemma 2.2. *Let $U:I\to\mathcal{P}(\mathbf{R})$ be a measurable set-valued map with closed nonempty images and having at least one integrable selection. Then*

$$cl\left(\int_0^T\overline{co}U(t)\,dt\right)=cl\left(\int_0^T U(t)\,dt\right).$$

3. The main results

In what follows we assume the following hypotheses.

Hypothesis. i) $F(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.

ii) There exist $l_1(.), l_2(.) \in L^1(I, \mathbf{R}_+)$ such that, for almost all $t \in I$,

$$\mathrm{d}_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2| \;\forall\, x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

Theorem 3.1. *Assume that Hypothesis is satisfied and* $|l_1|_1 + |l_2|_1 < 1$. *Let* $y(.) \in AC(I, \mathbf{R})$ *be such that there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *verifying*

$$d(y(t), F(t, y(t), y(\lambda t))) \leq p(t) \text{ a.e. } (I).$$

Then there exists $x(.)$ *a solution of problem (1.1) satisfying for all* $t \in I$

$$|x - y|_C \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)}(|x_0 - y(0)| + |p|_1). \tag{3.1}$$

Proof. We set $x_0(.) = y(.)$, $f_0(.) = y'(.)$. It follows from Lemma 2.1 and Hypothesis that there exists a measurable function $f_1(.)$ such that $f_1(t) \in F(t, x_0(t), x_0(\lambda t))$ a.e. (I) and, for almost all $t \in I$, $|f_1(t) - y'(t)| \leq p(t)$. Define

$$x_1(t) = x_0 + \int_0^t f_1(s)ds$$

and one has

$$|x_1(t) - y(t)| \leq |x_0 - y(0)| + \int_0^t p(s)ds \leq |x_0 - y(0)| + |p|_1.$$

Thus $|x_1 - y|_C \leq |x_0 - y(0)| + |p|_1$.

From Lemma 2.1 and Hypothesis we deduce the existence of a measurable function $f_2(.)$ such that $f_2(t) \in F(t, x_1(t), x_1(\lambda t))$ a.e. (I) and for almost all $t \in I$

$$|f_1(t) - f_2(t)| \leq d(f_1(t), F(t, x_1(t), x_1(\lambda t))) \leq d_H(F(t, x_0(t), x_0(\lambda t)),$$
$$F(t, x_1(t), x_1(\lambda t))) \leq l_1(t)|x_1(t) - x_2(t)| + l_2(t)|x_1(\lambda t) - x_2(\lambda t)|.$$

Define

$$x_2(t) = x_0 + \int_0^t f_2(s)ds$$

and one has

$$|x_1(t) - x_2(t)| \leq \int_0^t |f_1(s) - f_2(s)|ds$$
$$\leq \int_0^t [l_1(s)|x_1(s) - x_2(s)| + l_2(s)|x_1(\lambda s) - x_2(\lambda s)|]ds$$
$$\leq (|l_1|_1 + |l_2|_1)|x_1 - x_2|_C \leq (|l_1|_1 + |l_2|_1)(|x_0 - y(0)| + |p|_1).$$

Assume that for some $p \geq 1$ we have constructed $(x_i)_{i=1}^p$ with x_p satisfying

$$|x_p - x_{p-1}|_C \leq (|l_1|_1 + |l_2|_1)^p(|x_0 - y(0)| + |p|_1).$$

Using Lemma 2.1 and Hypothesis we deduce the existence of a measurable function $f_{p+1}(.)$ such that $f_{p+1}(t) \in F(t, x_p(t), x_p(\lambda t))$ a.e. (I) and for almost all $t \in I$

$$\begin{aligned} |f_{p+1}(t) - f_p(t)| &\leq d(f_{p+1}(t), F(t, x_{p-1}(t), x_{p-1}(\lambda t))) \\ &\leq d_H(F(t, x_p(t), x_p(\lambda t)), F(t, x_{p-1}(t), x_{p-1}(\lambda t))) \\ &\leq l_1(t)|x_p(t) - x_{p-1}(t)| + l_2(t)|x_p(\lambda t) - x_{p-1}(\lambda t)|. \end{aligned}$$

Define

$$x_{p+1}(t) = x_0 + \int_0^t f_{p+1}(s)ds. \tag{3.2}$$

We have

$$|x_{p+1}(t) - x_p(t)| \leq \int_0^t |f_{p+1}(s) - f_p(s)|ds$$

$$\leq \int_0^t [l_1(s)|x_p(s) - x_{p-1}(s)| + l_2(s)|x_p(\lambda s) - x_{p-1}(\lambda s)|]ds$$

$$\leq (|l_1|_1 + |l_2|_1)|x_p - x_{p-1}|_C \leq (|l_1|_1 + |l_2|_1)^p(|x_0 - y(0)| + |p|_1).$$

Therefore $(x_p(.))_{p\geq 0}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, so it converges to $x(.) \in C(I, \mathbf{R})$. Since, for almost all $t \in I$, we have

$$\begin{aligned} |f_{p+1}(t) - f_p(t)| &\leq l_1(t)|x_p(t) - x_{p-1}(t)| + l_2(t)|x_p(\lambda t) - x_{p-1}(\lambda t)| \\ &\leq [l_1(t) + l_2(t)]|x_p - x_{p-1}|_C, \end{aligned}$$

$\{f_p(.)\}$ is a Cauchy sequence in the Banach space $L^1(I, \mathbf{R})$ and thus it converges to $f(.) \in L^1(I, \mathbf{R})$. Passing to the limit in (3.2) and using Lebesgue's dominated convergence theorem we get $x(t) = x_0 + \int_0^t f(s)ds$, which shows, in particular, that $x(.)$ is absolutely continuous.

Moreover, since the values of $F(., ., .)$ are closed and $f_{p+1}(t) \in F(t, x_p(t), x_p(\lambda t))$ passing to the limit we obtain $f(t) \in F(t, x(t), x(\lambda t))$ a.e. (I).

It remains to prove the estimate (3.2). One has

$$|x_p - x_0|_C \leq |x_p - x_{p-1}|_C + ... + |x_2 - x_1|_C + |x_1 - x_0|_C$$

$$\leq (|l_1|_1+|l_2|_1)^p(|x_0-y(0)|+|p|_1)+...+(|l_1|_1+|l_2|_1)(|x_0-y(0)|+|p|_1)+(|x_0-y(0)|+|p|_1)$$

$$\leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)}(|x_0 - y(0)| + |p|_1).$$

Passage to the limit in the last inequality completes the proof. □

Remark 3.2. a) If we consider the space $C(I, \mathbf{R})$ endowed with a Bielecki type norm of the form $|x|_B = \sup_{t\in I} e^{-at}|x(t)|$ with an appropriate choice of $a \in \mathbf{R}$, the condition $|l_1|_1 + |l_2|_1 < 1$ can be removed from the assumptions of Theorem 3.1.

b) The statement in Theorem 3.1 remains valid for the more general problem

$$x'(t) \in F(t, x(t), x(g(t))), \quad x(0) = x_0,$$

with $g(.) : I \to I$ a continuous function.

As we already pointed out, Theorem 3.1 allows to obtain a relaxation theorem for problem (1.1). In what follows, we are concerned also with the convexified (relaxed) problem

$$x'(t) \in \overline{\text{co}}F(t, x(t), x(\lambda t)), \quad x(0) = x_0. \tag{3.3}$$

Note that if $F(.,.,.)$ satisfies Hypothesis, then so does the set-valued map

$$(t, x, y) \to \overline{\text{co}}F(t, x, y).$$

Theorem 3.3. *We assume that Hypothesis is satisfied and* $|l_1|_1 + |l_2|_1 < 1$. *Let* $\overline{x}(.) : I \to \mathbf{R}$ *be a solution to the relaxed inclusion (3.3) such that the set-valued map* $t \to F(t, \overline{x}(t), \overline{x}(\lambda t))$ *has at least one integrable selection.*

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of problem (1.1) such that*

$$|x - \overline{x}|_C < \varepsilon.$$

Proof. Since $\overline{x}(.)$ is a solution of the relaxed inclusion (3.3), there exists $\overline{f}(.) \in L^1(I, \mathbf{R})$, $\overline{f}(t) \in \overline{\text{co}}F(t, \overline{x}(t), \overline{x}(\lambda t))$ a.e. (I) such that

$$\overline{x}(t) = x_0 + \int_0^t \overline{f}(s)ds.$$

From Lemma 2.2, for $\delta > 0$, there exists $\tilde{f}(t) \in F(t, \overline{x}(t), \overline{x}(\lambda t))$ a.e. (I) such that

$$\sup_{t \in I} \left| \int_0^t (\tilde{f}(s) - \overline{f}(s))ds \right| \le \delta.$$

Define

$$\tilde{x}(t) = x_0 + \int_0^t \tilde{f}(s)ds.$$

Therefore, $|\tilde{x} - \overline{x}|_C \le \delta$.

We apply, now, Theorem 3.1 for the "quasi" solution $\tilde{x}(.)$ of (1.1). One has

$$p(t) = d(\tilde{f}(t), F(t, \tilde{x}(t), \tilde{x}(\lambda t))) \le d_H(F(t, \overline{x}(t), \overline{x}(\lambda t)),$$

$$F(t, \tilde{x}(t), \tilde{x}(\lambda t))) \le l_1(t)|\overline{x}(t) - \tilde{x}(t)| + l_2(t)|\overline{x}(\lambda t) - \tilde{x}(\lambda t)|$$

$$\le l_1(t)|\tilde{x} - \overline{x}|_C + l_2(t)|\tilde{x} - \overline{x}|_C \le (l_1(t) + l_2(t))\delta,$$

which shows that $p(.) \in L^1(I, \mathbf{R})$.

From Theorem 3.1 there exists $x(.)$ a solution of (1.1) such that

$$|x - \tilde{x}|_C \le \frac{1}{1 - (|l_1|_1 + |l_2|_1)}|p|_1 \le \frac{|l_1|_1 + |l_2|_1}{1 - (|l_1|_1 + |l_2|_1)}\delta.$$

It remains to take $\delta = [1 - (|l_1|_1 + |l_2|_1)]\varepsilon$ and to deduce that

$$|x - \overline{x}|_C \le |x - \tilde{x}|_C + |\tilde{x} - \overline{x}|_C \le \varepsilon.$$

□

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 437–448

Helicoidal surfaces with $\Delta^J r = Ar$ in 3-dimensional Euclidean space

Bendehiba Senoussi and Mohammed Bekkar

Abstract. In this paper we study the helicoidal surfaces in the 3-dimensional Euclidean space under the condition $\Delta^J r = Ar$; $J = I, II, III$, where $A = (a_{ij})$ is a constant 3×3 matrix and Δ^J denotes the Laplace operator with respect to the fundamental forms I, II and III.

Mathematics Subject Classification (2010): 53A05, 53A07, 53C40.

Keywords: Surfaces of coordinate finite type, helicoidal surfaces, Laplace operator.

1. Introduction

Let $r = r(u, v)$ be an isometric immersion of a surface M^2 in the Euclidean space $\mathbb{E}^3$.

The inner product on $\mathbb{E}^3$ is

$$g(X, Y) = x_1y_1 + x_2y_2 + x_3y_3,$$

where $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The Euclidean vector product $X \wedge Y$ of X and Y is defined as follows:

$$X \wedge Y = \big(x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1\big).$$

The notion of finite type immersion of submanifolds of a Euclidean space has been widely used in classifying and characterizing well known Riemannian submanifolds [6]. B.-Y. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. An Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ [6]. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space. Since then the theory of submanifolds of finite type has been studied by many geometers.

A well known result due to Takahashi [18] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition

$$\Delta r = \lambda r, \;\; \lambda \in \mathbb{R}.$$

In [10] Ferrandez, Garay and Lucas proved that the surfaces of $\mathbb{E}^3$ satisfying

$$\Delta H = AH,\ A \in Mat(3,3)$$

are either minimal, or an open piece of sphere or of a right circulaire cylindre.

In [7] M. Choi and Y. H. Kim characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. In [2] M. Bekkar and H. Zoubir classified the surfaces of revolution with non zero Gaussian curvature K_G in the 3-dimensional Lorentz-Minkowski space $\mathbb{E}^3_1$, whose component functions are eigenfunctions of their Laplace operator, i.e.

$$\Delta^{II} r_i = \lambda_i r_i,\ \lambda_i \in \mathbb{R}.$$

In [9] F. Dillen, J. Pas and L. Verstraelen proved that the only surfaces in $\mathbb{E}^3$ satisfying

$$\Delta r = Ar + B,\ A \in Mat(3,3),\ B \in Mat(3,1),$$

are the minimal surfaces, the spheres and the circular cylinders.

In [1] Ch. Baba-Hamed and M. Bekkar studied the helicoidal surfaces without parabolic points in $\mathbb{E}^3_1$, which satisfy the condition

$$\Delta^{II} r_i = \lambda_i r_i,$$

where Δ^{II} is the Laplace operator with respect to the second fundamental form.

In [13] G. Kaimakamis and B.J. Papantoniou classified the first three types of surfaces of revolution without parabolic points in the 3-dimensional Lorentz–Minkowski space, which satisfy the condition

$$\Delta^{II} r = Ar,\ A \in Mat(3,3).$$

We study helicoidal surfaces M^2 in $\mathbb{E}^3$ which are of finite type in the sense of B.-Y. Chen with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u,v)$ satisfies the condition

$$\Delta^J r = Ar;\ J = I, II, III, \tag{1.1}$$

where $A = (a_{ij})$ is a constant 3×3 matrix and Δ^J denotes the Laplace operator with respect to the fundamental forms I, II and III. Then we shall reduce the geometric problem to a simpler ordinary differential equation system.

In [14] G. Kaimakamis, B.J. Papantoniou and K. Petoumenos classified and proved that such surfaces of revolution in the 3-dimensional Lorentz-Minkowski space $\mathbb{E}^3_1$ satisfying

$$\Delta^{III} \vec{r} = A\vec{r}$$

are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius, where Δ^{III} is the Laplace operator with respect to the third fundamental form. S. Stamatakis and H. Al-Zoubi in [17] classified the surfaces of revolution with non zero Gaussian curvature in $\mathbb{E}^3$ under the condition

$$\Delta^{III} r = Ar,\ A \in Mat(3, \mathbb{R}).$$

On the other hand, a helicoidal surface is well known as a kind of generalization of some ruled surfaces and surfaces of revolution in a Euclidean space $\mathbb{E}^3$ or a Minkowski space $\mathbb{E}^3_1$ ([5], [8], [12]).

2. Preliminaries

Let $\gamma : I \subset \mathbb{R} \to P$ be a plane curve in $\mathbb{E}^3$ and let l be a straight line in P which does not intersect the curve γ (axis). A helicoidal surface in $\mathbb{E}^3$ is a surface invariant by a uniparametric group $G_{L,c} = \{g_v /\ g_v : \mathbb{E}^3 \to \mathbb{E}^3;\ v \in \mathbb{R}\}$ of helicoidal motions. The motion g_v is called a helicoidal motion with axis l and pitch c. If we take $c = 0$, then we obtain a rotations group about the axis l.

A helicoidal surface in $\mathbb{E}^3$ which is spanned by the vector $(0, 0, 1)$ and with pitch $c \in \mathbb{R}^*$ as follows:

$$r(u, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \\ \varphi(u) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ cv \end{pmatrix}, c \in \mathbb{R}^*.$$

Next, we will use the parametrization of the profile curve γ as follows:

$$\gamma(u) = (u, 0, \varphi(u)).$$

Therefore, the surface M^2 may be parameterized by

$$r(u, v) = \big(u \cos v, u \sin v, \varphi(u) + cv\big) \tag{2.1}$$

in $\mathbb{E}^3$, where $(u, v) \in I \times [0, 2\pi]$, $c \in \mathbb{R}^*$.

A surface M^2 is said to be of finite type if each component of its position vector field r can be written as a finite sum of eigenfunctions of the Laplacian Δ of M^2, that is, if

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

where r_i are $\mathbb{E}^3$ -valued eigenfunctions of the Laplacian of (M^2, r): $\Delta r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, .., k$ [6]. If λ_i are different, then M^2 is said to be of k-type.
The coefficients of the first fundamental form and the second fundamental form are

$$\begin{aligned} E &= g_{11} = g(r_u, r_u),\ F = g_{12} = g(r_u, r_v),\ G = g_{22} = g(r_v, r_v); \\ L &= h_{11} = g(r_{uu}, \mathbf{N}),\ M = h_{12} = g(r_{uv}, \mathbf{N}),\ N = h_{22} = g(r_{vv}, \mathbf{N}), \end{aligned}$$

where $\mathbf{N}$ is the unit normal vector to M^2.

The Laplace-Beltrami operator of a smooth function

$$\varphi : M^2 \to \mathbb{R}, (u, v) \mapsto \varphi(u, v)$$

with respect to the first fundamental form of the surface M^2 is the operator Δ^I, defined in [15] as follows:

$$\Delta^I \varphi = \frac{-1}{\sqrt{|EG - F^2|}} \left[\frac{\partial}{\partial u} \left(\frac{G\varphi_u - F\varphi_v}{\sqrt{|EG - F^2|}} \right) - \frac{\partial}{\partial v} \left(\frac{F\varphi_u - E\varphi_v}{\sqrt{|EG - F^2|}} \right) \right]. \tag{2.2}$$

The second differential parameter of Beltrami of a function

$$\varphi : M^2 \to \mathbb{R}, (u, v) \longmapsto \varphi(u, v)$$

with respect to the second fundamental form of M^2 is the operator Δ^{II} which is defined by [15]

$$\Delta^{II}\varphi = \frac{-1}{\sqrt{|LN-M^2|}}\left[\frac{\partial}{\partial u}\left(\frac{N\varphi_u - M\varphi_v}{\sqrt{|LN-M^2|}}\right) + \frac{\partial}{\partial v}\left(\frac{L\varphi_v - M\varphi_u}{\sqrt{|LN-M^2|}}\right)\right], \tag{2.3}$$

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

In the classical literature, one write the third fundamental form as

$$III = e_{11}du^2 + 2e_{12}dudv + e_{22}dv^2.$$

The second Beltrami differential operator with respect to the third fundamental form III is defined by

$$\Delta^{III} = \frac{-1}{\sqrt{|e|}}\left(\frac{\partial}{\partial x^i}\left(\sqrt{|e|}e^{ij}\frac{\partial}{\partial x^j}\right)\right), \tag{2.4}$$

where $e = \det(e_{ij})$ and e^{ij} denote the components of the inverse tensor of e_{ij}.

If $r = r(u,v) = (r_1 = r_1(u,v), r_2 = r_2(u,v), r_3 = r_3(u,v))$ is a function of class C^2 then we set

$$\Delta^J r = (\Delta^J r_1, \Delta^J r_2, \Delta^J r_3);\ J = I, II, III.$$

The mean curvature H and the Gauss curvature K_G are, respectively, defined by

$$H = \frac{1}{2(EG-F^2)}\left(EN + GL - 2FM\right)$$

and

$$K_G = \frac{LN - M^2}{EG - F^2}.$$

Suppose that M^2 is given by (2.1).

3. Helicoidal surfaces with $\Delta^I r = Ar$ in $\mathbb{E}^3$

The main result of this section states that the only helicoidal surfaces M^2 of $\mathbb{E}^3$ satisfying the condition

$$\Delta^I r = Ar \tag{3.1}$$

on the Laplacian are open pieces of helicoidal minimal surfaces.

The coefficients of the first and the second fundamental forms are:

$$E = 1 + \varphi'^2,\ F = c\varphi',\ G = c^2 + u^2; \tag{3.2}$$

$$L = \frac{u\varphi''}{W},\ M = -\frac{c}{W},\ N = \frac{u^2\varphi'}{W}, \tag{3.3}$$

where $W = \sqrt{EG - F^2} = \sqrt{u^2(1+\varphi'^2) + c^2}$ and the prime denotes derivative with respect to u.

The unit normal vector of M^2 is given by

$$\mathbf{N} = \frac{1}{W}(u\varphi'\cos v - c\sin v, c\cos v + u\varphi'\sin v, -u).$$

From these we find that the mean curvature H and the curvature K_G of (3.2) are given by

$$\begin{aligned} H &= \frac{1}{2W^3}\big(u^2\varphi'(1+\varphi'^2) + 2c^2\varphi' + u\varphi''(c^2+u^2)\big) \\ &= \frac{1}{2u}\left(\frac{u^2\varphi'}{W}\right)' \end{aligned}$$

and

$$K_G = \frac{1}{W^4}\big(u^3\varphi'\varphi'' - c^2\big). \tag{3.4}$$

If a surface M^2 in $\mathbb{E}^3$ has no parabolic points, then we have

$$u^3\varphi'\varphi'' - c^2 \neq 0.$$

The Laplacian Δ^I of M^2 can be expressed as follows:

$$\begin{aligned} \Delta^I &= -\frac{1}{W^2}((c^2+u^2)\frac{\partial^2}{\partial u^2} - 2c\varphi'\frac{\partial^2}{\partial u \partial v} + (1+\varphi'^2)\frac{\partial^2}{\partial v^2}) \\ &\quad -\frac{1}{W^4}(u^3(1+\varphi'^2) + c^2u(1-\varphi'^2) - u^2\varphi'\varphi''(c^2+u^2))\frac{\partial}{\partial u} \\ &\quad -\frac{1}{W^4}(-c\varphi''(c^2+u^2) + cu\varphi'(1+\varphi'^2))\frac{\partial}{\partial v}. \end{aligned}$$

Accordingly, we get

$$\Delta^I r = -2H\mathbf{N}. \tag{3.5}$$

The equation (3.1) by means of (3.2) and (3.5) gives rise to the following system of ordinary differential equations

$$(u\varphi' A(u) - a_{11}u)\cos v - (cA(u) + a_{12}u)\sin v = a_{13}(\varphi + cv) \tag{3.6}$$

$$(u\varphi' A(u) - a_{22}u)\sin v + (cA(u) - a_{21}u)\cos v = a_{23}(\varphi + cv) \tag{3.7}$$

$$-uA(u) = a_{31}u\cos v + a_{32}u\sin v + a_{33}(\varphi + cv), \tag{3.8}$$

where

$$A(u) = \frac{2H}{W}. \tag{3.9}$$

On differentiating (3.6), (3.7) and (3.8) twice with respect to v we have

$$a_{13} = a_{23} = a_{33} = 0, \; A(u) = 0. \tag{3.10}$$

From (3.10) we obtain

$$\begin{aligned} -a_{11}u\cos v - a_{12}u\sin v &= 0 \\ -a_{22}u\sin v - a_{21}u\cos v &= 0 \\ a_{31}u\cos v + a_{32}u\sin v &= 0. \end{aligned} \tag{3.11}$$

But cos and sin are linearly independent functions of v, so we finally obtain $a_{ij} = 0$. From (3.9) we obtain $H = 0$. Consequently M^2, being a minimal surface.

Theorem 3.1. *Let $r : M^2 \to \mathbb{E}^3$ be an isometric immersion given by (2.1). Then $\Delta^I r = Ar$ if and only if M^2 has zero mean curvature.*

4. Helicoidal surfaces with $\Delta^{II} r = Ar$ in $\mathbb{E}^3$

In this section we are concerned with non-degenerate helicoidal surfaces M^2 without parabolic points satisfying the condition

$$\Delta^{II} r = Ar. \tag{4.1}$$

By a straightforward computation, the Laplacian Δ^{II} of the second fundamental form II on M^2 with the help of (3.3) and (2.3) turns out to be

$$\begin{aligned}\Delta^{II} &= -\frac{W}{R}\left(u^2\varphi'\frac{\partial^2}{\partial u^2} + u\varphi''\frac{\partial^2}{\partial v^2} + 2c\frac{\partial^2}{\partial u\partial v}\right)\\ &\quad -\frac{W}{2R^2}u\big(-\varphi'(\varphi'\varphi''' - \varphi''^2)u^4 + \varphi'^2\varphi''u^3 - 2c^2\varphi''u - 4c^2\varphi'\big)\frac{\partial}{\partial u}\\ &\quad +\frac{W}{2R^2}cu^2\big((\varphi'\varphi''' + \varphi''^2)u + 3\varphi'\varphi''\big)\frac{\partial}{\partial v},\end{aligned}$$

where $R = u^3\varphi'\varphi'' - c^2$.

Accordingly, we get

$$\Delta^{II} r(u,v) = \begin{pmatrix} (u\varphi'\cos v - c\sin v)P(u) \\ (u\varphi'\sin v + c\cos v)P(u) \\ u\varphi'^2 P(u) - u^2 Q(u)\end{pmatrix}, \tag{4.2}$$

where

$$P(u) = \frac{W}{2R^2}\big((\varphi''^2 + \varphi'\varphi''')u^4 - \varphi'\varphi''u^3 + 4c^2\big) \tag{4.3}$$

$$Q(u) = \frac{W}{2R^2}\big(4\varphi'^2\varphi''^2u^3 - c^2(\varphi''^2 + \varphi'\varphi''')u - 7c^2\varphi'\varphi''\big). \tag{4.4}$$

Therefore, the problem of classifying the helicoidal surfaces M^2 given by (2.1) and satisfying (4.1) is reduced to the integration of this system of ordinary differential equations

$$\begin{aligned}(u\varphi' P(u) - a_{11}u)\cos v - (cP(u) + a_{12}u)\sin v &= a_{13}(\varphi + cv)\\ (u\varphi' P(u) - a_{22}u)\sin v + (cP(u) - a_{21}u)\cos v &= a_{23}(\varphi + cv)\\ u\varphi'^2 P(u) - u^2Q(u) &= a_{31}u\cos v + a_{32}u\sin v + a_{33}(\varphi + cv).\end{aligned}$$

Remark 4.1. We observe that

$$c^2P(u) + u^3Q(u) = 2W. \tag{4.5}$$

But $\cos v$ and $\sin v$ are linearly independent functions of v, so we finally obtain

$$a_{32} = a_{31} = a_{33} = a_{13} = a_{23} = 0.$$

We put $a_{11} = a_{22} = \alpha$ and $a_{21} = -a_{12} = \beta$, $\alpha, \beta \in \mathbb{R}$. Therefore, this system of equations is equivalently reduced to

$$\begin{cases} \varphi' P(u) = \alpha \\ cP(u) = \beta u \\ \varphi'^2 P(u) - uQ(u) = 0. \end{cases} \tag{4.6}$$

Now, let us examine the system of equations (4.6) according to the values of the constants α and β.

Case 1. Let $\alpha = 0$ and $\beta \neq 0$.

In this case the system (4.6) is reduced equivalently to

$$\begin{cases} \varphi' = 0 \\ cP(u) = -\beta u \\ Q(u) = 0. \end{cases} \tag{4.7}$$

From (4.7) we have $P''(u) = 0$. From (4.5) and the fact that $c \neq 0$ we have a contradiction. Hence there are no helicoidal surfaces of $\mathbb{E}^3$ in this case which satisfy (4.1).

Case 2. Let $\alpha \neq 0$ and $\beta = 0$.

In this case the system (4.6) is reduced equivalently to

$$\begin{cases} \varphi' P(u) = \alpha \\ P(u) = 0. \end{cases}$$

But this is not possible. So, in this case there are no helicoidal surfaces of $\mathbb{E}^3$.

Case 3. Let $\alpha = \beta = 0$.

In this case the system (4.6) is reduced equivalently to

$$\begin{cases} P(u) = 0 \\ Q(u) = 0. \end{cases}$$

From (4.5) we have $W = 0$, which is a contradiction. Consequently, there are no helicoidal surfaces of $\mathbb{E}^3$ in this case.

Case 4. Let $\alpha \neq 0$ and $\beta \neq 0$.

In this case the system (4.6) is reduced equivalently to

$$\varphi(u) = \frac{\alpha c}{\beta} \ln(u) + k, \quad k \in \mathbb{R}. \tag{4.8}$$

By using (4.6) and (4.8), we obtain

$$\begin{cases} P(u) = \frac{\beta}{c} u \\ Q(u) = \frac{\alpha^2 c}{\beta u^2}. \end{cases} \tag{4.9}$$

Substituting (4.9) into (4.5), we get

$$\frac{c^2(\alpha^2 + \beta^2)^2}{\beta^2} u^2 = 4(u^2 + \frac{c^2\alpha^2}{\beta^2} + c^2).$$

Then

$$\begin{cases} c^2(\alpha^2 + \beta^2) = 0 \\ c^2(\alpha^2 + \beta^2)^2 = 4\beta^2. \end{cases}$$

From the first equation we have $\alpha = \beta = 0$, which is a contradiction. Hence, there are no helicoidal surfaces of $\mathbb{E}^3$ in this case.

Consequently, we have:

Theorem 4.2. *Let $r : M^2 \to \mathbb{E}^3$ be an isometric immersion given by (2.1). There are no helicoidal surfaces in $\mathbb{E}^3$ without parabolic points, satisfying the condition $\Delta^{II} r = Ar$.*

Theorem 4.3. *If* $K_G = a \in \mathbb{R}\backslash\{0\}$, *then*

$$\Delta^{II} r(u,v) = -2N. \tag{4.10}$$

Proof. If $K_G = a \in \mathbb{R}\backslash\{0\}$, then $\frac{\partial K_G}{\partial u} = 0$.

From (3.4) we obtain

$$-\varphi'\varphi''u^4 + \varphi''^2u^5 + 7c^2\varphi'\varphi''u^2 + c^2\varphi''^2u^3 - 3\varphi'^2\varphi''^2u^5 + 4c^2u - \varphi'^3\varphi''u^4 + 4c^2\varphi'^2u$$
$$= -(\varphi'\varphi''' + \varphi'^3\varphi''')u^5 - c^2\varphi'\varphi'''u^3 \tag{4.11}$$

By using (4.3), (4.4) and (4.11) we get

$$\begin{aligned} uP(u) - u^2Q(u) &= \frac{W}{2R^2}\left(\varphi'^3\varphi''u^4 - 4c^2\varphi'^2u - \varphi'^3\varphi'''u^5 - \varphi'^2\varphi''^2u^5\right) \\ &= -\varphi'^2uP(u). \end{aligned} \tag{4.12}$$

From (4.2) and (4.12) we deduce that

$$\Delta^{II} r(u,v) = WP(u)N. \tag{4.13}$$

From (4.5) and (4.12) we have that

$$P(u) = \frac{2}{W}. \tag{4.14}$$

By using (4.13) and (4.14) we get (4.10). □

5. Helicoidal surfaces with $\Delta^{III} r = Ar$ in $\mathbb{E}^3$

In this section we are concerned with non-degenerate helicoidal surfaces M^2 without parabolic points satisfying the condition

$$\Delta^{III} r = Ar. \tag{5.1}$$

The components of the third fundamental form of the surface M^2 is given by

$$\begin{aligned} e_{11} &= \frac{1}{W^4}(c^2(\varphi' + u\varphi'')^2 + c^2 + u^4\varphi''^2), \\ e_{12} &= -\frac{c}{W^2}(\varphi' + u\varphi''), \\ e_{22} &= \frac{1}{W^2}(c^2 + u^2\varphi'^2), \end{aligned} \tag{5.2}$$

hence

$$e = \frac{1}{W^6}(u^3\varphi'\varphi'' - c^2)^2.$$

The Laplacian of M^2 can be expressed as follows:

$$\begin{aligned} \Delta^{III} = &-\frac{1}{\sqrt{|e|}}\Big(W\Big(\frac{c^2 + u^2\varphi'^2}{c^2 - u^3\varphi'\varphi''}\Big)\frac{\partial^2}{\partial u^2} + 2cW\Big(\frac{\varphi' + u\varphi''}{c^2 - u^3\varphi'\varphi''}\Big)\frac{\partial^2}{\partial u\partial v} + \\ &\frac{1}{W}\Big(\frac{c^2(\varphi' + u\varphi'')^2 + c^2 + u^4\varphi''}{c^2 - u^3\varphi'\varphi''}\Big)\frac{\partial^2}{\partial v^2} + \frac{d}{du}W\Big(\frac{c^2 + u^2\varphi'^2}{c^2 - u^3\varphi'\varphi''}\Big)\frac{\partial}{\partial u} \\ &+ c\frac{d}{du}W\Big(\frac{\varphi' + u\varphi''}{c^2 - u^3\varphi'\varphi''}\Big)\frac{\partial}{\partial v}\Big). \end{aligned} \tag{5.3}$$

By using (5.1) and (5.3) we get

$$\begin{cases} \Delta^{III}(u\cos v) = -u\varphi' Q(u)\cos v - cQ(u)\sin v \\ \Delta^{III}(u\sin v) = cQ(u)\cos v - u\varphi' Q(u)\sin v \\ \Delta^{III}(\varphi(u)+cv) = P(u). \end{cases}$$

Hence

$$\Delta^{III} r(u,v) = \begin{pmatrix} -u\varphi' Q(u)\cos v - cQ(u)\sin v \\ cQ(u)\cos v - u\varphi' Q(u)\sin v \\ P(u) \end{pmatrix}, \tag{5.4}$$

where

$$\begin{aligned} Q(u) \;&=\; \frac{W^2}{(c^2-u^3\varphi'\varphi'')^3}\big(W^2u^2(c^2+u^2\varphi'^2)\varphi''' + 3c^2u^2\varphi' + 3c^2u^2\varphi'^3 \\ &\quad +7c^2u^3\varphi'^2\varphi'' + 5c^2u^3\varphi'' + c^2u^4\varphi'\varphi''^2 + 4c^4u\varphi'' - u^6\varphi'\varphi''^2 \\ &\quad +u^7\varphi''^3 + c^2u^5\varphi''^3 + 2c^4\varphi' - 2u^6\varphi'^3\varphi''^2\big), \end{aligned} \tag{5.5}$$

$$\begin{aligned} P(u) \;&=\; \frac{-W^2}{(c^2-u^3\varphi'\varphi'')^3}\big(W^2u(c^2+u^2\varphi'^2)^2\varphi''' + 4c^4u^2\varphi'' + \\ &\quad 7c^2u^4\varphi'^2\varphi'' - 2u^7\varphi'^3\varphi''^2 + 3c^6\varphi'' + 15c^4u^2\varphi'^2\varphi'' \\ &\quad -3c^2u^5\varphi'^3\varphi''^2 + 9c^2u^4\varphi'^4\varphi'' - 3u^7\varphi'^5\varphi''^2 + 2c^4u\varphi' \\ &\quad +4c^4u\varphi'^3 + 3c^2u^3\varphi'^3 + 3c^2u^3\varphi'^5 \\ &\quad +3c^4u^3\varphi'\varphi''^2 + c^2u^5\varphi'\varphi''^2 + c^2u^6\varphi''^3 + c^4u^4\varphi''^3\big). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we have

$$\begin{aligned} P(u) \;&=\; \frac{-u}{W}L(u) - \left(\frac{c^2+u^2\varphi'^2}{u^3\varphi'\varphi''-c^2}\right)WL'(u) \\ Q(u) \;&=\; \frac{-1}{W}L(u) + \left(\frac{u}{u^3\varphi'\varphi''-c^2}\right)WL'(u), \end{aligned} \tag{5.7}$$

where $L(u) = h_{11}e^{11} + 2h_{12}e^{12} + h_{22}e^{22} = \frac{2H}{K_G}$.

Remark 5.1. We observe that

$$uP(u) + (c^2+u^2\varphi'^2)Q(u) = -W\left(\frac{2H}{K_G}\right). \tag{5.8}$$

The equation (5.1) by means of (2.1) and (5.4) gives rise to the following system of ordinary differential equations

$$\begin{cases} -u\varphi' Q(u)\cos v - cQ(u)\sin v = a_{11}u\cos v + a_{12}u\sin v + a_{13}(\varphi+cv) \\ cQ(u)\cos v - u\varphi' Q(u)\sin v = a_{21}u\cos v + a_{22}u\sin v + a_{23}(\varphi+cv) \\ P(u) = a_{31}u\cos v + a_{32}u\sin v + a_{33}(\varphi+cv). \end{cases}$$

But $\cos v$ and $\sin v$ are linearly independent functions of v, so we finally obtain

$$a_{32} = a_{31} = a_{33} = a_{13} = a_{23} = 0.$$

We put $-a_{11} = -a_{22} = \lambda_1$ and $a_{21} = -a_{12} = \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$. Therefore, this system of equations is equivalently reduced to

$$\begin{cases} \varphi' Q(u) = \lambda_1 \\ cQ(u) = \lambda_2 u \\ P(u) = 0. \end{cases} \tag{5.9}$$

Therefore, the problem of classifying the surfaces M^2 given by (2.1) and satisfying (5.1) is reduced to the integration of this system of ordinary differential equations.

Case 1. Let $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

In this case the system (5.9) is reduced equivalently to

$$\begin{cases} \varphi' Q(u) = 0 \\ cQ(u) = \lambda_2 u \\ P(u) = 0. \end{cases} \tag{5.10}$$

Differentiating (5.10), we obtain $P''(u) = 0$, which is a contradiction. Hence there are no helicoidal surfaces of $\mathbb{E}^3$ in this case which satisfy (5.1).

Case 2. Let $\lambda_1 \neq 0$ and $\lambda_2 = 0$.

In this case the system (5.9) is reduced equivalently to

$$\begin{cases} \varphi' Q(u) = \lambda_1 \\ cQ(u) = 0 \\ P(u) = 0. \end{cases}$$

But this is not possible. So, in this case there are no helicoidal surfaces of $\mathbb{E}^3$.

Case 3. Let $\lambda_1 = \lambda_2 = 0$.

In this case the system (5.9) is reduced equivalently to

$$\begin{cases} \varphi' Q(u) = 0 \\ Q(u) = 0. \end{cases}$$

From (5.8) we have $H = 0$. Consequently M^2, being a minimal surface.

Case 4. Let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

In this case the system (5.9) is reduced equivalently to

$$\varphi(u) = \frac{\lambda_1 c}{\lambda_2} \ln(u) + a, \quad a \in \mathbb{R}. \tag{5.11}$$

If we substitute (5.11) in (5.5) we get $Q(u) = 0$. So we have a contradiction and therefore, in this case there are no helicoidal surfaces of $\mathbb{E}^3$.

Consequently, we have:

Theorem 5.2. *Let $r : M^2 \to \mathbb{E}^3$ be an isometric immersion given by (2.1). Then $\Delta^{III} r = Ar$ if and only if M^2 has zero mean curvature.*

Theorem 5.3. *If $\frac{2H}{K_G} = \alpha \in \mathbb{R}\backslash\{0\}$, then*

$$\Delta^{III} r(x, y) = -\frac{2H}{K_G} N.$$

Proof. From (5.7) we have

$$P(u) = uQ(u). \tag{5.12}$$

Finally, (5.12) and (5.4) give

$$\begin{aligned} \Delta^{III} r(u,v) &= Q(u)(-u\varphi' \cos v + c\sin v, -c\cos v - u\varphi' \sin v, u) \\ &= \frac{-1}{W}(\frac{2H}{K_G})(-u\varphi' \cos v + c\sin v, -c\cos v - u\varphi' \sin v, u) \\ &= -\frac{2H}{K_G}N. \end{aligned}$$

□

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 449–462

Curves with constant geodesic curvature in the Bolyai-Lobachevskian plane

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Abstract. The aim of this note is to present the curves with constant geodesic curvature of the Bolyai-Lobachevskian hyperbolic plane. By using the Lobachevskian metric the equations of the circle, paracycloid and hipercycloid are given. Furthermore, we determine a new family of curves with constant curvature which was not emphasized before. During the analysis we use Cartesian and polar coordinates.

Mathematics Subject Classification (2010): 53A35.

Keywords: hyperbolic geometry, geodesics, orthogonal trajectory, polar coordinates.

1. General formulas in Cartesian coordinates

We consider the Lobachevskian metric

$$ds^2 = \cosh^2 \frac{y}{k} dx^2 + dy^2 \ , \tag{1.1}$$

where k is the parameter of the two-dimensional hyperbolic plane.

Among the geodesics we can find so-called Euclidean lines too, which can be used as coordinate axes. Therefore we can define a Cartesian coordinate system in the hyperbolic plane. If $dx = 0$, then $ds^2 = dy^2$, thus we can use the euclidean method when determining the value of y. The x-value of a point can only be determined by the x-axis, because when $dy = 0$, then the formula $ds^2 = dx^2$ can only be used in the case of $y = 0$. Now let us consider a point $P(x, y)$ in the hyperbolic plane. The foot of the perpendicular from P to the x-axis is denoted by $P_1(x, 0)$. Then distance $\overline{OP_1}$ corresponds with the x-coordinate of P, while the length of $\overline{PP_1}$ equals the y-coordinate of P.

As the reflection over the coordinate axes is a symmetry operation, during the analysis we will consider only the first quadrant of the plane. Note that the lines we obtain in the first quadrant have segments in the other quadrants, too. From metric (1.1) it follows that the translation of the origin along the direction of the x-axis is also a symmetry operation.

If we use s as variable, the functions characterizing the geodesics are of form $x = x(s), y = y(s)$. The geodesic lines are determined by the following differential equations:

$$\frac{d^2x}{ds^2} + \frac{2}{k}\tanh\frac{y}{k}\frac{dx}{ds}\frac{dy}{ds} = 0, \tag{1.2}$$

$$\frac{d^2y}{ds^2} - \frac{1}{k}\sinh\frac{y}{k}\cosh\frac{y}{k}\left(\frac{dx}{ds}\right)^2 = 0. \tag{1.3}$$

If we replace variable s with x, equation (1.2) can be written in the following equivalent form:

$$\cosh^2\frac{y}{k}\frac{dx}{ds} = C_1, \tag{1.4}$$

where C_1 is constant. Using (1.3) and (1.4) we obtain the

$$\frac{d^2\tanh\frac{y}{k}}{dx^2} - \frac{1}{k^2}\tanh\frac{y}{k} = 0 \tag{1.5}$$

differential equation. If we use variable x, we only need to determine the constants a_1 and a_2 which appear in the equation

$$\tanh\frac{y}{k} = a_1\cosh\frac{x}{k} + a_2\sinh\frac{x}{k}. \tag{1.6}$$

By using (1.1), (1.4) and (1.6), for the value of C_1 we get

$$\frac{1}{C_1^2} = 1 - \tanh^2\frac{y}{k} + k^2\left(\frac{d\tanh\frac{y}{k}}{dx}\right)^2 = 1 - a_1^2 + a_2^2. \tag{1.7}$$

In order to determine the geodesic curvature, we use the formula given by Schlesinger:

$$\frac{1}{r_g} =$$

$$= \cosh\frac{y}{k}\left\{\left(\frac{d^2x}{ds^2} + \frac{2}{k}\tanh\frac{y}{k}\frac{dx}{ds}\frac{dy}{ds}\right)\frac{dy}{ds} - \left[\frac{d^2y}{ds^2} - \frac{1}{k}\sinh\frac{y}{k}\cosh\frac{y}{k}\left(\frac{dx}{ds}\right)^2\right]\frac{dx}{ds}\right\}. \tag{1.8}$$

From (1.2), (1.3) and (1.8) it follows that

$$\frac{1}{r_g} = 0. \tag{1.9}$$

Metric (1.1) can also be obtained by using the metric

$$ds^2 = dx_1^2 + dx_2^2 - dx_0^2 \tag{1.10}$$

defined in the three-dimensional pseudo-Euclidean space, with the help of the following formulas:

$$x_1 = k\sinh\frac{x}{k}\cosh\frac{y}{k}, \quad x_2 = k\sinh\frac{y}{k}, \quad x_0 = k\cosh\frac{x}{k}\cosh\frac{y}{k}. \tag{1.11}$$

2. General formulas in polar coordinates

Using (1.10) and equalities

$$x_1 = k\cos\varphi\sinh\frac{\rho}{k}, \quad x_2 = k\sin\varphi\sinh\frac{\rho}{k}, \quad x_0 = k\cosh\frac{\rho}{k}, \tag{2.1}$$

we obtain the metric

$$ds^2 = d\rho^2 + k^2\sinh^2\frac{\rho}{k}d\varphi^2, \tag{2.2}$$

where ρ and φ represent polar coordinates.

If we use s as variable, we obtain the following differential equations which determine the lines of the hyperbolic plane.

$$\frac{d^2\varphi}{ds^2} + \frac{2}{k}\coth\frac{\rho}{k}\frac{d\varphi}{ds}\frac{d\rho}{ds} = 0, \tag{2.3}$$

$$\frac{d^2\rho}{ds^2} - k\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}\left(\frac{d\rho}{ds}\right)^2 = 0. \tag{2.4}$$

If we replace variable s with φ, equation (2.3) can be written in the following equivalent form:

$$\sinh^2\frac{\rho}{k}\frac{d\varphi}{ds} = C_2. \tag{2.5}$$

Also, from (2.4) we get

$$\frac{d^2\coth\frac{\rho}{k}}{d\varphi^2} + \coth\frac{\rho}{k} = 0. \tag{2.6}$$

The geodesics satisfy

$$\coth\frac{\rho}{k} = b_1\sin\varphi + b_2\cos\varphi, \tag{2.7}$$

where b_1 and b_2 are constant values. From (1.6) and (2.7) it follows that the values x_1, x_2 and x_0 admit a linear connection.

Using (2.2), (2.5) and (2.7), we get for the value of C_2

$$\frac{1}{C_2^2} = k^2\left[\coth^2\frac{\rho}{k} - 1 + \left(\frac{d\coth\frac{\rho}{k}}{d\varphi}\right)^2\right] = k^2\left(b_1^2 + b_2^2 - 1\right). \tag{2.8}$$

The geodesic curvature verifies the formula given by Schlesinger:

$$\frac{1}{r_g} = k\sinh\frac{\rho}{k}\left\{\left(\frac{d^2\varphi}{ds^2} + \frac{2}{k}\coth\frac{\rho}{k}\frac{d\varphi}{ds}\frac{d\rho}{ds}\right)\frac{d\rho}{ds} - \left[\frac{d^2\rho}{ds^2} - k\sinh\frac{\rho}{k}\cosh\frac{\rho}{k}\left(\frac{d\varphi}{ds}\right)^2\right]\frac{d\varphi}{ds}\right\}. \tag{2.9}$$

Using (1.11) and (2.1) it follows that the connection between the Cartesian and polar coordinates is determined by the following equations:

$$\sinh\frac{x}{k}\cosh\frac{y}{k} = \cos\varphi\sinh\frac{\rho}{k}, \tag{2.10}$$

$$\sinh\frac{y}{k} = \sin\varphi\sinh\frac{\rho}{k}, \tag{2.11}$$

$$\cosh\frac{x}{k}\cosh\frac{y}{k} = \cosh\frac{\rho}{k}. \tag{2.12}$$

Furthermore, from (2.10) and (2.12) we obtain

$$\tanh\frac{x}{k} = \cos\varphi \tanh\frac{\rho}{k}. \tag{2.13}$$

3. The Bolyai-Lobachevskian lines

As the geometry in discussion is based on metric (1.1), we differentiate four types of lines. The first family contains lines crossing the origin. The second set consists of lines which cross the x-axis, while the lines of the third family do not cross the x-axis. Then there are the lines which are parallel to the x-axis.

a) Based on (1.6), the lines crossing the origin satisfy

$$\tanh\frac{y}{k} = a_2 \sinh\frac{x}{k}.$$

Let us consider a point $P(x, y)$ on a line in question, then the tangent vector to the line in P admits

$$\tan\alpha = \frac{1}{\cosh\frac{y}{k}}\frac{dy}{dx}.$$

In this case

$$\tan\alpha = a_2 \cosh\frac{x}{k}\cosh\frac{y}{k},$$

thus the value of a_2 determines the tangent vector in the origin. The angle of intersection between the line and the x-axis is denoted by φ, which verifies

$$\tanh\frac{y}{k} = \tan\varphi \sinh\frac{x}{k}. \tag{3.1}$$

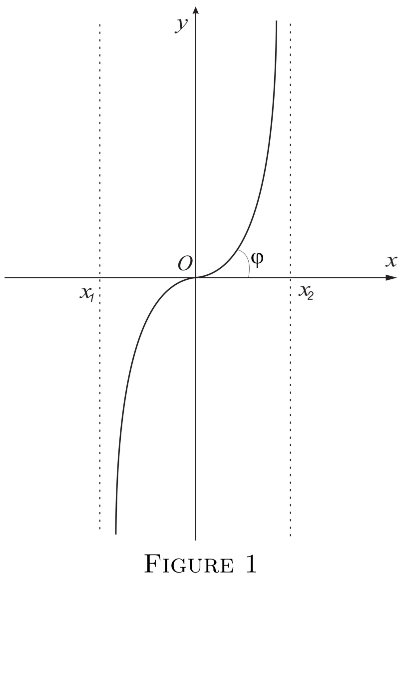

FIGURE 1

Obviously, φ is the polar angle of point P. If φ is set as constant, we get $ds^2 = d\rho^2$, thus the value of ρ determines the distance between the origin and point P measured along the geodesic.

The values of x_1 and x_2 represented in figure 1 are determined by equation (3.1), when $y \longrightarrow -\infty$ and $y \longrightarrow \infty$.

Using equation (1.7), we obtain

$$C_1 = \cos\varphi.$$

b) From (1.6) it follows that the lines crossing the x-axis and passing through points $P_0(a,0)$ and $P_1(0,b)$ verify

$$\tanh\frac{y}{k} = \tanh\frac{b}{k}\left(\cosh\frac{x}{k} - \coth\frac{a}{k}\sinh\frac{x}{k}\right). \tag{3.2}$$

The values x_1 and x_2 are determined by equation (3.1) as $y \longrightarrow -\infty$ and $y \longrightarrow \infty$ (figure 2).

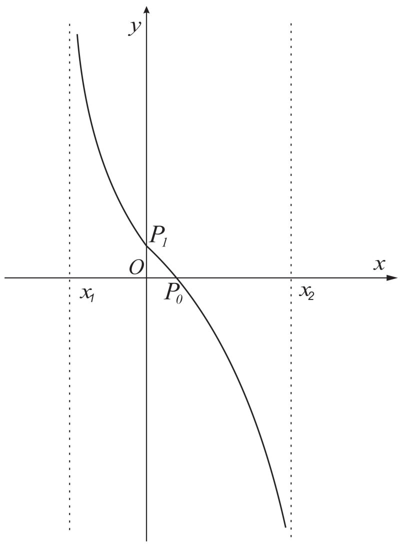

FIGURE 2

In the case of polar coordinates we use

$$\coth\frac{\rho}{k} = \coth\frac{b}{k}\sin\varphi + \coth\frac{a}{k}\cos\varphi \tag{3.3}$$

obtained from formula (2.7). The value of φ is given by equation (3.3) as $\rho \longrightarrow \infty$.

Using (2.6) and (2.8), we get for the constants C_1 and C_2 the following formulas:

$$\frac{1}{C_1} = \sqrt{1 + \frac{\tanh^2\frac{b}{k}}{\sinh^2\frac{a}{k}}}, \quad \frac{1}{C_2} = k\sqrt{\coth^2\frac{b}{k} + \coth^2\frac{a}{k} - 1}. \tag{3.4}$$

c) Figure 3 illustrates that the line passing through the y-axis in point $P_1(0,b)$ while not crossing the x-axis has a minimum point.

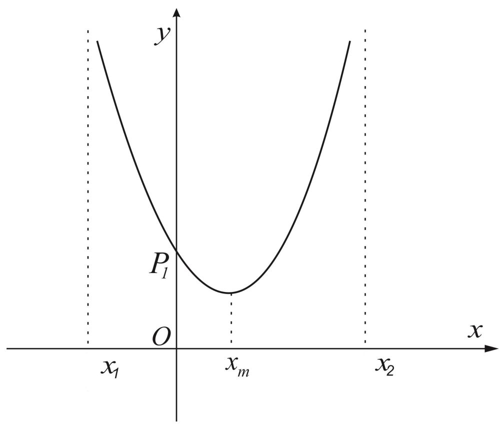

FIGURE 3

The line admits

$$\tanh\frac{y}{k} = \tanh\frac{b}{k}\left(\cosh\frac{x}{k} - \tanh\frac{x_m}{k}\sinh\frac{x}{k}\right), \tag{3.5}$$

where x_m denotes the value of x determined by the minimum point.

The domain of the line is determined by equation (3.5) as $y \longrightarrow \infty$.

In the case of polar coordinates, by using equations (2.9), (2.10), (2.11) and (2.12), we get

$$\coth\frac{\rho}{k} = \coth\frac{b}{k}\sin\varphi + \tanh\frac{x_m}{k}\cos\varphi. \tag{3.6}$$

For the values of C_1 and C_2, we obtain formulas

$$\frac{1}{C_1} = \sqrt{1 - \frac{\tanh^2\frac{b}{k}}{\cosh^2\frac{x_m}{k}}} \quad\text{and}\quad \frac{1}{C_2} = k\sqrt{\coth^2\frac{b}{k} + \tanh^2\frac{x_m}{k} - 1}. \tag{3.7}$$

If $x_m = 0$, the line admits

$$\tanh\frac{y}{k} = \tanh\frac{b}{k}\cosh\frac{x}{k}, \quad \coth\frac{\rho}{k} = \coth\frac{b}{k}\sin\varphi, \tag{3.8}$$

while the values C_1 and C_2 verify

$$C_1 = \cosh\frac{b}{k}, \quad C_2 = \frac{1}{k}\sinh\frac{b}{k}. \tag{3.9}$$

d) If $a \longrightarrow \infty$ and $x_m \longrightarrow \infty$, we obtain the line parallel to the x-axis, passing through $P_1(0,b)$. The lines of this family satisfy

$$\tanh\frac{y}{k} = \tanh\frac{b}{k}e^{-\frac{x}{k}}, \quad \coth\frac{\rho}{k} = \coth\frac{b}{k}\sin\varphi + cos\varphi, \tag{3.10}$$

while $C_1 = 1, \quad C_2 = \frac{1}{k}\tanh\frac{b}{k}$.

4. The orthogonal curves

If in the formulas obtained in the previous section we set a variable as a varying parameter, we obtain families of lines. Each family of lines admits orthogonal lines. Let us denote the original lines by the index 1. The line passing through point $P(x,y)$ admits the following orthogonality condition:

$$\cosh^2 \frac{y}{k} dx dx_1 + dy dy_1 = 0. \tag{4.1}$$

The lines verify

$$y_1 = y_1(x_1, p), \quad p = p(x_1, y_1), \tag{4.2}$$

where the parameter is denoted by p. By deriving this equation with respect to variable x_1, we obtain

$$\frac{dy_1}{dx_1} = f(x_1, p). \tag{4.3}$$

Using (4.2) and (4.3), we eliminate the parameter, thus we get

$$\frac{dy_1}{dx_1} = f[x_1, p(x_1, y_1)] = F(x_1, y_1). \tag{4.4}$$

From (4.1) and (4.4) it follows that the orthogonal lines verify

$$\cosh^2 \frac{y}{k} dx + F(x,y) dy = 0. \tag{4.5}$$

In the case of polar coordinates we use the formula

$$\rho_1 = \rho_1(\varphi_1, p).$$

Hence we get

$$\frac{d\rho_1}{d\varphi_1} = g(\varphi_1, p) = G(\rho_1, \varphi_1). \tag{4.6}$$

By applying the orthogonality condition

$$d\rho d\rho_1 + k^2 \sinh^2 \frac{\rho}{k} d\varphi d\varphi_1 = 0, \tag{4.7}$$

we get for the orthogonal lines

$$G(\rho, \varphi) d\rho + k^2 \sinh^2 \frac{\rho}{k} d\varphi = 0. \tag{4.8}$$

Now let us consider the distance along the line between points $P_1(0,b)$ and $P(x,y)$ represented in figure 4.

We denote the length of $\overline{PP_1}$ by d. The distance from the origin to P equals ρ. In figure 4 a right-angled triangle is formed, where the hypotenuse is equal to ρ and the other two sides are x and y. Equality (2.12) gives a formula considering these values.

Let P_3 be the foot of the perpendicular from P_1 to the line OP. Then two right-angled triangles, namely OP_3P_1 and P_1P_3P are formed.

In OP_3P_1 the length of the hypotenuse $\overline{OP_1}$ is denoted by b, while the legs $\overline{OP_3}$ and $\overline{P_1P_3}$ are denoted by ρ_0 and c. Therefore we can write

$$\cosh \tfrac{b}{k} = \cosh \tfrac{\rho_0}{k} \cosh \tfrac{c}{k}.$$

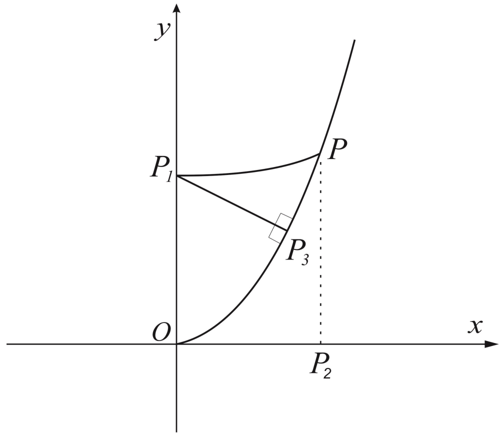

FIGURE 4

In triangle P_1P_3P the hypotenuse is d, while the value of the legs are $\rho - \rho_0$ and c, thus we get

$$\cosh\frac{d}{k} = \cosh\frac{\rho-\rho_0}{k}\cosh\frac{c}{k}.$$

If we eliminate c, we obtain

$$\cosh\frac{d}{k} = \cosh\frac{b}{k}\left(\cosh\frac{\rho}{k} - \tanh\frac{\rho_0}{k}\sinh\frac{\rho}{k}\right). \tag{4.9}$$

In triangle OP_3P_1 the angle between OP_1 and OP_3 is equal to $\frac{\pi}{2} - \varphi$. Hence, from (2.13) we can write

$$\tanh\frac{\rho_0}{k} = \sin\varphi\tanh\frac{b}{k}. \tag{4.10}$$

Applying (4.9) and (4.10), we get

$$\cosh\frac{d}{k} = \cosh\frac{b}{k}\cosh\frac{\rho}{k} - \sin\varphi\sinh\frac{b}{k}\sinh\frac{\rho}{k}. \tag{4.11}$$

If we use Cartesian coordinates, from (4.11), (2.12) and (2.11) it follows that

$$\cosh\frac{d}{k} = \cosh\frac{b}{k}\cosh\frac{x}{k}\cosh\frac{y}{k} - \sinh\frac{b}{k}\sinh\frac{y}{k}. \tag{4.12}$$

In the following sections we determine the orthogonal lines. During the analysis our choice of coordinates may vary depending on the form of calculations.

We will prove that the curvature of the orthogonal lines is constant. Moreover, any two orthogonal lines from the same family are parallel. We define parallelism in the following way: let us consider a geodesic and its two orthogonal lines passing through the line in two different points. If the distance between the two points of intersection is constant in the case of any geodesic, we say that the two orthogonal lines are parallel.

5. The orthogonal curves of the radial lines

Radial lines are lines which pass through a common point. We will consider four family of lines.

a) The first family consists of lines crossing the origin, which are determined by equation (3.1). Here we use

$$p = \tan\varphi$$

as parameter. In this case we can write

$$F(x, y) = \coth\frac{x}{k}\sinh\frac{y}{k}\cosh\frac{y}{k}.$$

Therefore, by using (4.5) we get

$$\frac{\cosh\frac{y}{k}}{\sinh\frac{x}{k}}d\left(\cosh\frac{x}{k}\cosh\frac{y}{k}\right) = 0.$$

The expression in bracket is constant. From equation (2.12) it follows that the points of an orthogonal curve are always at the same distance from the origin. Hence we obtain a circle with center O.

$$\cosh\frac{x}{k}\cosh\frac{y}{k} = \cosh\frac{R}{k}, \quad \rho = R, \tag{5.1}$$

where R denotes the radius of the circle.

If we use polar coordinates, from (5.1) and (2.2) it follows that

$$\frac{d\varphi}{ds} = \frac{1}{k\sinh\frac{R}{k}}.$$

From (2.9) we obtain the formula characterizing the curvature:

$$\frac{1}{r_g} = \frac{1}{k}\coth\frac{R}{k}. \tag{5.2}$$

b) Now we determine the orthogonal curves of lines crossing point $P_0(a, 0) \in Ox$. As the translation of the origin along the direction of the x-axis into point P_0 is a symmetry operation, we obtain circles with center P_0, which verify

$$\cosh\frac{x-a}{k}\cosh\frac{y}{k} = \cosh\frac{R}{k}. \tag{5.3}$$

The curvature of the orthogonal lines is determined by formula (5.2).

If we use polar coordinates, from (5.3), (2.10) and (2.12) it follows that the circles verify

$$\cosh\frac{a}{k}\cosh\frac{\rho}{k} - \cos\varphi\sinh\frac{a}{k}\sinh\frac{\rho}{k} = \cosh\frac{R}{k}.$$

c) The lines parallel to the x-axis admit the

$$p = \tanh\frac{b}{k}$$

parameter and satisfy equation (3.10).

As

$$F = -\sinh\frac{y}{k}\cosh\frac{y}{k},$$

from (4.5) it follows that

$$d\left(\ln\cosh\frac{y}{k}-\frac{x}{k}\right)=0.$$

The orthogonal curve crossing point $P(x_0,0)$ is called paracycloid, which verifies

$$\cosh\frac{y}{k}=e^{\frac{x-x_0}{k}}. \tag{5.4}$$

We can also determine the equation of the paracycloid by the following way. If $a>x_0$, the circle with center $P_0(a,0)$ passing through $P(x_0,0)$ has the radius $R=a-x_0$. If $a\longrightarrow\infty$, from (5.3) we obtain formula (5.4). Thus the paracycloid can be considered a semicircle with infinite radius.

d) The orthogonal curves of lines passing through point $P_1(0,b)$ are lines which cross or do not cross the x-axis. By the use of polar coordinates we obtain

$$\coth\frac{\rho}{k}=\coth\frac{b}{k}\sin\varphi+p\cos\varphi. \tag{5.5}$$

Here the parameters are given by

$$\coth\frac{a}{b}\ \text{ and }\quad \tanh\frac{x_m}{k}.$$

The orthogonal lines admit the following formula:

$$G(\rho,\varphi)=-\frac{k}{cos\varphi}\sinh^2\frac{\rho}{k}\left(\coth\frac{b}{k}-\sin\varphi\coth\frac{\rho}{k}\right).$$

Using (4.8), we get

$$-\frac{1}{cos\varphi}\left(\coth\frac{b}{k}-\sin\varphi\coth\frac{\rho}{k}\right)d\rho+kd\varphi=$$

$$=-\frac{k}{\cos\varphi\sinh\frac{b}{k}\sinh\frac{\rho}{k}}d\left(\cosh\frac{b}{k}\cosh\frac{\rho}{k}-\sin\varphi\sinh\frac{b}{k}\sinh\frac{\rho}{k}\right)=0.$$

Hence, by using (4.11) it follows that the orthogonal lines are circles with center $P_1(0,b)$, which verify the following equations:

$$\cosh\frac{b}{k}\cosh\frac{\rho}{k}-\sin\varphi\sinh\frac{b}{k}\sinh\frac{\rho}{k}=\cosh\frac{R}{k}, \tag{5.6}$$

$$\cosh\frac{b}{k}\cosh\frac{x}{k}\cosh\frac{y}{k}-\sinh\frac{b}{k}\sinh\frac{y}{k}=\cosh\frac{R}{k}. \tag{5.7}$$

We obtain the curvature by considering (2.2) and using the formulas below:

$$\sin\varphi=f(\rho),\quad f(\rho)=\coth\frac{b}{k}\coth\frac{\rho}{k}-\frac{\cosh\frac{R}{k}}{\sinh\frac{b}{k}\sinh\frac{\rho}{k}}. \tag{5.8}$$

Hence we obtain

$$\frac{d\rho}{ds}=\frac{\sinh\frac{b}{k}}{\sinh\frac{R}{k}}\sqrt{1-f^2}. \tag{5.9}$$

From (5.8) we get

$$\cos\varphi d\varphi=\sqrt{1-f^2}d\varphi=\frac{df}{d\rho}d\rho.$$

Thus by using (5.9) we obtain

$$\frac{d\varphi}{ds} = \frac{\sinh \frac{b}{k}}{\sinh \frac{R}{k}} \frac{df}{d\rho}. \tag{5.10}$$

The derivatives of the second kind are as follows:

$$\frac{d^2\rho}{ds^2} = -\frac{\sinh^2 \frac{b}{k}}{\sinh^2 \frac{R}{k}} f \frac{df}{d\rho}, \quad \frac{d^2\varphi}{ds^2} = \frac{\sinh^2 \frac{b}{k}}{\sinh^2 \frac{R}{k}} \sqrt{1-f^2} \frac{d^2 f}{d\rho^2}.$$

Hence, from (2.9) and (5.5) it follows that the curvature is given by formula (5.2).

If we consider two circles, the distance between the intersections with a geodesic is equal to the difference of the two radiuses, which is a constant value. This proves that the orthogonal lines determined above are parallel.

6. The orthogonal curves of lines not having common point

We will consider two different cases.

a) The first family consists of lines being parallel to the y-axis. The orthogonal lines are called hipercycloids having

$$y = b, \tag{6.1}$$

where b is a constant value. In the upper half-plane $b > 0$, while below the x-axis $b < 0$. The hipercycloids satisfy the orthogonality condition (4.1), because in the case of the geodesic satisfying $dx_1 = 0$ its orthogonal curve verifies $dy = 0$.

From (2.2) and (6.1) we obtain

$$\frac{dx}{ds} = \frac{1}{\cosh \frac{b}{k}}. \tag{6.2}$$

Using (6.1), (6.2) and (1.8), we get for the curvature

$$\frac{1}{r_g} = \frac{1}{k} \tanh \frac{b}{k}.$$

b) The lines of the second family do not cross the x-axis and have minimum point on the y-axis. By using (3.8), these lines verify

$$\tanh \frac{y}{k} = p \cosh \frac{x}{k}, \quad p = \tanh \frac{b}{k}, \tag{6.3}$$

where b can be either positive or negative value.

In this case

$$F(x,y) = \tanh \frac{x}{k} \sinh \frac{y}{k} \cosh \frac{y}{k}.$$

From the (4.5) orthogonality condition we get

$$\tanh \frac{x}{k} \sinh \frac{y}{k} dy + \cosh \frac{y}{k} dx = \frac{k}{\cosh \frac{x}{k}} d\left(\sinh \frac{x}{k} \cosh \frac{y}{k}\right) = 0.$$

Thus the orthogonal curve passing through point $P(x_0, 0)$ verifies

$$\sinh\frac{x}{k}\cosh\frac{y}{k} = \sinh\frac{x_0}{k}. \tag{6.4}$$

Figure 5 illustrates the geodesics determined by b and $-b$, and their orthogonal curves passing through points $P(x_0, 0)$ and $P'(-x_0, 0)$.

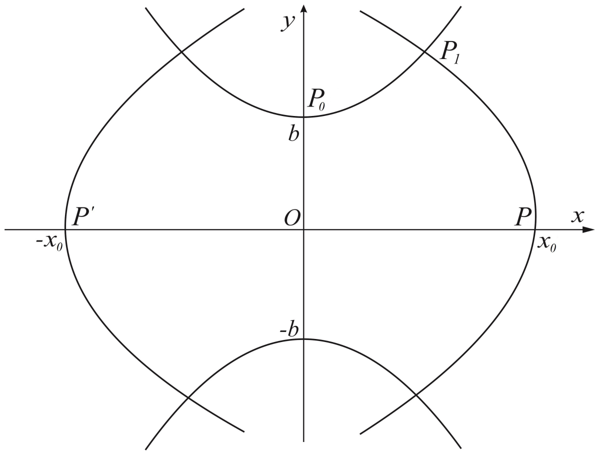

FIGURE 5

The tangent field of the orthogonal line admits

$$\tan\alpha = \frac{1}{\cosh\frac{y}{k}}\frac{dy}{dx} = -\sqrt{\coth^2\frac{x_0}{k}\coth^2\frac{y}{k} - 1}.$$

Thus as $y \longrightarrow \infty$, we obtain

$$\tan\alpha = -\frac{1}{\sinh\frac{x_0}{k}}.$$

We use polar coordinates in order to determine the curvature of the lines. From (6.4) and (2.10) we obtain

$$\cos\varphi\sinh\frac{\rho}{k} = \sinh\frac{x_0}{k}. \tag{6.5}$$

Also, from (2.2) and (6.5) it follows that

$$\frac{d\rho}{ds} = \sqrt{1 - \tanh^2\frac{x_0}{k}\coth^2\frac{\rho}{k}} = g(\rho).$$

Hence, by using equation (6.5) we obtain

$$\frac{d\varphi}{ds} = \frac{g(\rho)}{k}\cot\varphi\coth\frac{\rho}{k} = \frac{1}{k}\tanh\frac{x_0}{k}\frac{\cosh\frac{\rho}{k}}{\sinh^2\frac{\rho}{k}}.$$

The derivatives of the second kind are as follows:

$$\frac{d^2\rho}{ds^2} = \frac{1}{2}\frac{dg^2}{d\rho},$$

$$\frac{d^2\varphi}{ds^2} = \frac{\tanh\frac{x_0}{k}}{k^2\sinh\frac{\rho}{k}}\left(1 - 2\coth^2\frac{\rho}{k}\right)g.$$

From (2.9) we obtain for the curvature the following formula:

$$\frac{1}{r_g} = \frac{1}{k}\tanh\frac{x_0}{k}. \tag{6.6}$$

The orthogonal curves are parallel to the y-axis. We will prove this by determining the distance between the points $P_0(0, b)$ and $P_1(x_1, y_1)$, illustrated on figure 5. Point P_1 is the intersection point of the geodesic and its orthogonal curve. The coordinates are given by the formulas

$$\tanh\frac{y_1}{k} = \tanh\frac{b}{k}\cosh\frac{x_1}{k}, \quad \sinh\frac{x_1}{k}\cosh\frac{y_1}{k} = \sinh\frac{x_0}{k}.$$

Hence we obtain

$$\sinh\frac{y_1}{k} = \sinh\frac{b}{k}\cosh\frac{x_0}{k}, \quad \cosh\frac{x_1}{k} = \frac{\tanh\frac{y_1}{k}}{\tanh\frac{b}{k}}.$$

From (4.12) we get

$$\cosh\frac{d}{k} = \cosh\frac{b}{k}\cosh\frac{x_1}{k}\cosh\frac{y_1}{k} - \sinh\frac{b}{k}\sinh\frac{y_1}{k} = \cosh\frac{x_0}{k},$$

thus

$$d = x_0.$$

The distance of two orthogonal lines is given by the difference of the values x_0, which proves the parallelism of the orthogonal curves.

These orthogonal lines are the duals of the hipercycloids, fact which is illustrated also by the curvatures determined above. This family of lines was not considered in the past.

Note that any orthogonal line can be described by formula (1.11). In the case of hipercycloids the value of x_2, while in the case of (6.4) the value of x_1 is constant. Along the circles (5.1), (5.3) and (5.7) the following values are constant:

$$\cosh\frac{a}{k}x_0 - \sinh\frac{a}{k}x_1, \quad \cosh\frac{b}{k}x_0 - \sinh\frac{b}{k}x_2.$$

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 463–470

A unified local convergence for Chebyshev-Halley-type methods in Banach space under weak conditions

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Abstract. We present a unified local convergence analysis for Chebyshev-Halley-type methods in order to approximate a solution of a nonlinear equation in a Banach space setting. Our methods include the Chebyshev; Halley; super-Halley and other high order methods. The convergence ball and error estimates are given for these methods under the same conditions. Numerical examples are also provided in this study.

Mathematics Subject Classification (2010): 65D10, 65D99, 65G99, 47H17, 49M15.

Keywords: Chebyshev-Halley-type methods, Banach space, convergence ball, local convergence.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought in a form like (1.1) using mathematical modeling [2, 3, 4, 5, 11, 14, 15]. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. In particular, the practice of Numerical Functional Analysis for finding solution x^* of equation (1.1) is essentially connected to variants of Newton's method. This method converges quadratically to x^* if the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev-Halley-type methods [1, 3, 5, 7]–[16] require the evaluation of the second Fréchet-derivative, which is very expensive in general. However, there are integral equations, where the second Fréchet-derivative is diagonal by blocks and inexpensive [10]–[13] or for quadratic equations the second Fréchet-derivative is constant [4, 12]. Moreover, in some applications involving stiff systems [2], [5], [9], high order methods are usefull. That is why in a unified way we study the local convergence of Chebyshev-Halley-type methods(CHTM) defined for each $n = 0, 1, 2, \cdots$ by

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_n(I - \theta T_n)^{-1}\right]\Gamma_n F(x_n), \tag{1.2}$$

where x_0 is an initial point, I is the identity operator, $\Gamma_n = F'(x_n)^{-1}$, $T_n = \Gamma_n B(x_n)\Gamma_n F(x_n)$, B a bilinear operator and θ a real parameter. If $B(x_n) = F''(x_n)$, then: for $\theta = 0$ we obtain the Chebyshev method; for $\theta = \frac{1}{2}$ we obtain the Halley method, for $\theta = 1$ we obtain the super-Halley method [3], [5], [7]–[16] and for $\theta \in [0, 1]$ we obtain the method studied by Gutierrez and Hernandez [10], [11]. Other choices of operator B and parameter θ are possible [3]–[5]. The usual conditions for the semi-local convergence of these methods are $(\mathcal{C})$:

$(\mathcal{C}_1)$ There exists $\Gamma_0 = F'(x_0)^{-1}$ and $\|\Gamma_0\| \le \beta$;
$(\mathcal{C}_2)$ $\|\Gamma_0 F(x_0)\| \le \eta$;
$(\mathcal{C}_3)$ $\|F''(x)\| \le \beta_1$ for each $x \in D$;
$(\mathcal{C}_4)$ $\|F'''(x)\| \le \beta_2$ for each $x \in D$;
$(\mathcal{C}_5)$ $\|F'''(x) - F'''(y)\| \le \beta_3\|x - y\|$ for each $x, y \in D$.

The local convergence conditions are similar but x_0 is x^* in $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$. There is a plethora of local and semi-local convergence results under the $(\mathcal{C})$ conditions [1]–[16]. The conditions $(\mathcal{C}_4)$ and $(\mathcal{C}_5)$ restrict the applicability of these methods. That is why, in our study we assume conditions $(\mathcal{A})$:

$(\mathcal{A}_1)$ $F : D \to Y$ is Fréchet-differentiable and there exists $x^* \in D$ such that

$$F(x^*) = 0 \text{ and } F'(x^*)^{-1} \in L(Y, X);$$

$(\mathcal{A}_2)$ $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0\|x - x^*\|$ for each $x \in D$;
$(\mathcal{A}_3)$ $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L\|x - y\|$ for each $x, y \in D$;
$(\mathcal{A}_4)$ $\|F'(x^*)^{-1}F'(x)\| \le N$ for each $x \in D$
$(\mathcal{A}_5)$ $\|F'(x^*)^{-1}B(x)\| \le M$ for each $x \in D$.

Notice that the $(\mathcal{A})$ conditions are weaker than the $(\mathcal{C})$ conditions.

In the rest of this study, $U(w, q)$ and $\overline{U}(w, q)$ stand, respectively, for the open and closed ball in X with center $w \in X$ and of radius $q > 0$.

As a motivational example, let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Notice that $f'''(x)$ is unbounded on D. That is condition $(\mathcal{C}_4)$ is not satisfied. Hence, the results depending on $(\mathcal{C}_4)$ cannot apply in this case. However, we have $f'(x^*) = 3$ and $f(x^*) = 0$. That is, conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$, $(\mathcal{A}_4)$ are satisfied for $L_0 = L = 146.6629073$, $N = 101.5578008$ and if e.g, we choose $B = 0$, then we can set $M = 0$. Then condition $(\mathcal{A}_5)$ is also satisfied. Hence, the results of our Theorem 2.1 that follows can apply to solve equation $f(x) = 0$ using CHTM. Hence, the applicability of CHTM is expanded under the conditions $(\mathcal{A})$.

The paper is organized as follows: In Section 2 we present the local convergence of these methods. The numerical examples are given in the concluding Section 3.

2. Local convergence

We present the local convergence of method CHTM in this section. It is convenient for the local convergence of CHTM to introduce some functions and parameters. Define parameter r_s by

$$r_s = \frac{2}{2L_0 + |\theta| MN + \sqrt{(2L_0 + |\theta| MN)^2 - 4L_0^2}}. \tag{2.1}$$

Let

$$\varphi(t) = L_0^2 t^2 - (2L_0 + |\theta| MN)t + 1. \tag{2.2}$$

Notice that r_s is the smallest positive root of polynomial φ. Note also that

$$r_s \leq \frac{1}{L_0} \tag{2.3}$$

and $\varphi(t)$ is decreasing for all $t \in [0, \frac{2L_0 + |\theta| MN}{2L_0^2}]$.

Let us define function f on $[0, r_s)$ by

$$f(t) = \frac{1}{2}\left[L + \frac{MN^2}{(1 - L_0 t)^2 - |\theta| MNt}\right] \frac{t}{1 - L_0 t}. \tag{2.4}$$

Then $f(t)$ is increasing for all $t \in [0, r_s)$. This can be seen as follows:

$$f(t) = f_1(t) f_2(t)$$

where $f_1(t) = \frac{1}{2}\left[L + \frac{MN^2}{\varphi(t)}\right]$ and $f_2(t) = \frac{t}{1 - L_0 t}$ are increasing for all $t \in [0, r_s)$.

Define polynomial g by

$$\begin{aligned} g(t) &= L_0^2 (L + 2L_0) t^3 - [(2L_0 + |\theta| MN)L + 2L_0(2L_0 + |\theta| MN) + 2L_0^2] t^2 \\ &\quad + [L + MN^2 + 2(2L_0 + |\theta| MN) + 2L_0] t - 2. \end{aligned} \tag{2.5}$$

It follows from the definition of r_s and f that function f is well defined on $[0, r_s)$. We have that $g(0) = -2$ and $g(t) \to \infty$ as $t \to \infty$. Hence, polynomial g has roots in $(0, \infty)$. Denote by r_m the smallest such root. Set

$$r^* = \min\{r_s, r_m\}. \tag{2.6}$$

Then for

$$r \in [0, r^*), \tag{2.7}$$

we have that

$$g(r) < 0 \tag{2.8}$$

and

$$f(r) < 1. \tag{2.9}$$

Then, we can show the following local convergence result for method (1.2) under $(\mathcal{A})$ conditions

Theorem 2.1. *Suppose that the $(\mathcal{A})$ conditions and $\overline{U}(x^*, r) \subseteq D$, hold, where r is given by (2.1). Then, sequence $\{x_n\}$ generated by CHTM method (1.2) for some $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$.*

$$\|x_{n+1} - x^*\| \leq f(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|. \tag{2.10}$$

Proof. We shall use induction to show that estimates (2.10) hold and $x_{n+1} \in U(x^*, r)$ for each $n = 0, 1, 2, \cdots$. Using $(\mathcal{A}_2)$ and the hypothesis $x_0 \in U(x^*, r)$ we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.11}$$

It follows from (2.11) and the Banach Lemma on invertible operators [2, 5, 14] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r}. \tag{2.12}$$

We also have by $(\mathcal{A}_4)$, $(\mathcal{A}_5)$ and (2.12) that, since

$$\begin{aligned}
T_0 &= [F'(x_0)^{-1}F'(x^*)][F'(x^*)^{-1}F''(x_0)][F'(x_0)^{-1}F'(x^*)] \\
&\quad \times [F'(x^*)^{-1}\int_0^1 F'(x^* + \tau(x_0 - x^*))(x_0 - x^*)d\tau \\
\|\theta T_0\| &\leq |\theta|\|F'(x_0)^{-1}F'(x^*)\|^2\|F'(x^*)^{-1}F''(x_0)\| \\
&\quad \times \|F'(x^*)^{-1}\int_0^1 F'(x^* + \tau(x_0 - x^*))d\tau\|\|x_0 - x^*\| \\
&\leq \frac{|\theta|MN\|x_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)^2} \\
&\leq \frac{|\theta|MNr}{(1 - L_0 r)^2} < 1 \qquad (2.13)
\end{aligned}$$

by the choice of r and r_s.

It follows from (2.13) and the Banach lemma that $(I-\theta T_0)^{-1}$ exists and

$$\begin{aligned} \|(I-\theta T_0)^{-1}\| &\leq \frac{1}{1-\frac{|\theta| MN\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2}} \\ &\leq \frac{1}{1-\frac{|\theta|MNr}{(1-L_0 r)^2}}. \end{aligned} \tag{2.14}$$

It follows from CHTM for $n=0$ that x_1 is well defined. We shall show (2.10) holds for $n=0$ and $x_1 \in U(x^*, r)$. Using CHTM for $n=0$, we get the identity

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - F'(x_0)^{-1}F'(x_0) - \frac{1}{2}T_0(1-\theta T_0)^{-1}\Gamma_0 F(x_0) \\ &= -[F'(x_0)^{-1}F'(x^*)][F'(x^*)^{-1}\int_0^1 (F'(x^*+\tau(x_0-x^*)) \\ &\quad -F'(x_0))d\tau(x_0-x^*)] - \frac{1}{2}[T_n(1-\theta T_n)^{-1}][F'(x_0)^{-1}F'(x^*)] \\ &\quad \times[F'(x^*)^{-1}\int_0^1 F'(x^*+\tau(x_0-x^*))d\tau(x_0-x^*)]. \end{aligned} \tag{2.15}$$

Using $(\mathcal{A}_3)$, (2.4), (2.9), (2.12)- (2.15) we get in turn

$$\begin{aligned} \|x_1-x^*\| &\leq \frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)} \\ &\quad + \frac{1}{2}\frac{\frac{MN\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2}}{1-\frac{|\theta|MN\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2}}\frac{N\|x_0-x^*\|}{1-L_0\|x_0-x^*\|} \\ &\leq f(\|x_0-x^*\|)\|x_0-x^*\| \\ &\leq f(r)\|x_0-x^*\| < \|x_0-x^*\|, \end{aligned} \tag{2.16}$$

which shows (2.10) for $n=0$ and $x_1 \in U(x^*, r)$. The induction is completed, if we simply replace x_0 by x_k in the preceding estimates to obtain that

$$\begin{aligned} \|x_{k+1}-x^*\| &\leq f(\|x_k-x^*\|)\|x_k-x^*\| \\ &\leq f(r)\|x_k-x^*\| < \|x_k-x^*\|, \end{aligned} \tag{2.17}$$

which implies that (2.10) holds for each $k=0,1,2,\cdots$, $x_{k+1} \in U(x^*, r)$ for each $k=0,1,2,\cdots$, and from $\|x_{k+1}-x^*\| < \|x_k-x^*\|$ we deduce that $\lim_{k\to\infty} x_k = x^*$. □

Remark 2.2. (a) Condition $(\mathcal{A}_2)$ can be dropped, since this condition follows from $(\mathcal{A}_3)$. Notice, however that

$$L_0 \leq L \tag{2.18}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [2]–[6].

(b) In view of condition $(\mathcal{A}_2)$ and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}[F'(x)-F'(x^*)]+I\| \\ &\leq 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\| \\ &\leq 1+L_0\|x-x^*\|^p, \end{aligned}$$

condition $(\mathcal{A}_4)$ can be dropped and N can be replaced by

$$N(r) = 1 + L_0 r^p. \tag{2.19}$$

(c) It is worth noticing that it follows from the first term in (2.16) that r is such that

$$r < r_A = \frac{2}{2L_0 + L}. \tag{2.20}$$

The convergence ball of radius r_A was given by us in [2, 3, 5] for Newton's method under conditions $(\mathcal{A}_1)$- $(\mathcal{A}_3)$. Estimate (2.20) shows that the convergence ball of higher than two CHTM methods are smaller than the convergence ball of the quadratically convergent Newton's method. The convergence ball given by Rheinboldt [15] for Newton's method is

$$r_R = \frac{2}{3L} < r_A \tag{2.21}$$

if $L_0 < L$ and $\frac{r_R}{r_A} \to \frac{1}{3}$ as $\frac{L_0}{L} \to 0$. Hence, we do not expect r to be larger than r_A no matter how we choose θ, L_0, L, M and N.

(d) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2]– [5], [14, 15].

(e) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [2]– [5], [14, 15]:

$$F'(x) = T(F(x)),$$

where T is a known continuous operator. Since

$$F'(x^*) = T(F(x^*)) = T(0), \;\; F''(x^*) = F'(x^*)T'(F(x^*)) = T(0)T'(0),$$

we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $T(x) = x + 1$ and $x^* = 0$.

3. Numerical Examples

We present numerical examples where we compute the radii of the convergence balls.

Example 3.1. Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define F on D for $v = (x, y, z)^T$ by

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T. \tag{3.1}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)^T$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1 < L = M = N = e$. The values of r_s, r_m, r^*, r_A and r_R are given in Table 1.

Table 1

θ	r_s	r_m	r^*	r_R	r_A
0	0.5820	0.0636	0.0636	0.2453	0.3249
0.5	0.1495	0.0527	0.0527	0.2453	0.3249
1	0.0948	0.0448	0.0448	0.2453	0.3248

Example 3.2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.2}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = M = 15$ and $N = N(t) = 1 + 7.5t$. The values of r_s, r_m, r^*, r_A and r_R are given in Table 2.

Table 2

θ	r_s	r_m	r^*	r_R	r_A
0	0.1333	0.0018	0.0018	0.0444	0.0667
0.5	0.0128	0.0016	0.0016	0.0444	0.0667
1	0.0070	0.0014	0.0014	0.0444	0.0667

Example 3.3. Returning back to the motivational example at the introduction of this study, we have

Table 3

θ	r_s	r_m	r^*	r_R	r_A
0	0.0068	0.0045	0.0045	0.0045	0.0045

Note that, since $M = 0$ the value of r_s, r_m, r^*, r_A and r_R will not change for different values of θ.

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 471–483

Multiple Stackelberg variational responses

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Abstract. Contrary to the standard literature (where the Stackelberg response function is single-valued), we provide a whole class of functions to show that the Stackelberg variational response set may contain at least three elements.

Mathematics Subject Classification (2010): 91B02, 91B50, 91A90, 91A35.

Keywords: Stackelberg equilibria, Nash equilibria, Palais-Smale condition, Motreanu-Panagiotopoulos functional.

1. Introduction

The Stackelberg duopoly is a game in which the leader moves first and the follower moves sequentially. In the usual Nash competition, however, the two players are competing with each other at the same level. The Stackelberg model can be handled by the backward induction method, i.e., we find the best response for the follower (by considering the strategy action of the leader as a parameter) and then choose the best strategy of the leader.

We assume in the sequel that the strategies of both players are some sets $K_1, K_2 \subset \mathbb{R}^m$. Let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the payoff function of the *f*ollower player. The first step is to determine for every fixed $x_1 \in K_1$ the *Stackelberg equilibrium response set*, defined by

$$R_{SE}(x_1) = \{x_2 \in K_2 : f(x_1, y) - f(x_1, x_2) \geq 0,\ \forall y \in K_2\}.$$

Now, assuming that $R_{SE}(x_1) \neq \emptyset$ for every $x_1 \in K_1$, the concluding step (for the leader) is to minimize the map $x \mapsto l(x, r(x))$ on K_1 where r is a suitable selection of the set-valued map R_{SE}.

The main objective is to locate the elements of the Stackelberg equilibrium response set. In order to do that, we introduce a larger set by means of variational inequalities. For simplicity, we assume that the follower's payoff function $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ has the property that $f(x_1, \cdot)$ is a locally Lipschitz function for every $x_1 \in K_1$. Now, we introduce the so-called *Stackelberg variational response set* defined by

$$R_{SV}(x_1) = \left\{x_2 \in K_2 : f^0_{x_2}((x_1, x_2); y - x_2) \geq 0,\ \forall y \in K_2\right\},$$

where $f^0_{x_2}((x_1,x_2);v)$ is the generalized directional derivative of $f(x_1,\cdot)$ at the point $x_2 \in K_2$ in the direction $v \in \mathbb{R}^m$. It is clear that

$$R_{SE}(x_1) \subseteq R_{SV}(x_1).$$

Usually, the standard literature provides examples where the set $R_{SV}(x_1)$ is a singleton, see A. Kristály and Sz. Nagy [4], Sz. Nagy [7] and the monograph by A. Kristály, V. Rădulescu and Cs. Varga [5] for functions of class C^1. However, as expected, one can happen to have examples where this set contains several elements. In fact, this is precisely the aim of the paper to provide a whole class of functions with the latter property.

We focus our attention to a specific payoff function for the follower player; namely, we assume that $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is given by

$$f_\lambda(x_1,x_2) := f(x_1,x_2) = \frac{1}{2}\|x_2\|^2 - \lambda \tilde{f}(x_1,x_2) + \delta_{K_2}(x_2), \tag{1.1}$$

where $K_2 \subset \mathbb{R}^m$ is a non-empty, closed, non-compact set, $\lambda > 0$ is a parameter and $\tilde{f}(x_1,\cdot)$ is locally Lipschitz for every $x_1 \in \mathbb{R}^m$. As usual, δ_{K_2} denotes the indicator function of the set K_2.

Let $x_1 \in \mathbb{R}^m$ be arbitrarily fixed. On the locally Lipschitz function $\tilde{f}(x_1,\cdot)$ we assume:

$(H^1_{x_1})$ $\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1,x_2)\} = o(\|x_2\|)$ whenever $\|x_2\| \to 0$;
$(H^2_{x_1})$ $\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1,x_2)\} = o(\|x_2\|)$ whenever $\|x_2\| \to +\infty$;
$(H^3_{x_1})$ $\tilde{f}(x_1,0) = 0$ and there exists $\tilde{x}_2 \in K_2$ such that $\tilde{f}(x_1,\tilde{x}_2) > 0$.

Here, $o(\cdot)$ is the usual Landau symbol.

Remark 1.1. (a) Hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$ mean that $\partial_{x_2}\tilde{f}(x_1,\cdot)$ is *superlinear* at the origin and *sublinear* at infinity, respectively. Hypothesis $(H^3_{x_1})$ implies that $\tilde{f}(x_1,\cdot)$ is not identically zero.

(b) According to hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$, the number

$$\tilde{c} = \max_{x_2 \in \mathbb{R}^m \setminus \{0\}} \frac{\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1,x_2)\}}{\|x_2\|} \tag{1.2}$$

is well-defined, finite, and from the upper semicontinuity of $\partial_{x_2}\tilde{f}(x_1,\cdot)$ and hypothesis $(H^3_{x_1})$, we have $0 < \tilde{c} < \infty$.

(c) We also introduce the number

$$\tilde{\lambda} = \frac{1}{2} \inf_{\substack{\tilde{f}(x_1,x_2)>0 \\ x_2 \in K_2}} \frac{\|x_2\|^2}{\tilde{f}(x_1,x_2)}, \tag{1.3}$$

which is well-defined, finite and $0 < \tilde{\lambda} < \infty$. The discussion on this number is postponed to Proposition 4.2.

Note that the Stackelberg variational response set for the function f_λ in (1.1) is given by

$$R^\lambda_{SV}(x_1) = \left\{ x_2 \in K_2 : \langle x_2, y - x_2 \rangle + \lambda \tilde{f}^0_{x_2}((x_1,x_2); -y + x_2) \geq 0, \ \forall y \in K_2 \right\}.$$

The main theorem of our paper is the following.

Theorem 1.2. *Let $K_i \subset \mathbb{R}^m$ be two convex sets ($i = 1, 2$), and let $f_\lambda : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the follower payoff function of the form* (1.1) *such that $\tilde{f}(x_1, \cdot)$ is locally Lipschitz for every $x_1 \in K_1$. Assume that K_2 is closed and non-compact such that $0 \in K_2$. Fix $x_1 \in K_1$ and assume that the hypotheses $(H^i_{x_1})$ hold true, $i \in \{1, 2, 3\}$. Then the following statements hold:*

(a) $0 \in R^\lambda_{SV}(x_1)$ *for every* $\lambda > 0$;
(b) $R^\lambda_{SV}(x_1) = \{0\}$ *for every* $\lambda \in (0, \tilde{c}^{-1})$, *where* $\tilde{c}$ *is from* (1.2);
(c) $\mathrm{card}(R^\lambda_{SV}(x_1)) \geq 3$ *for every* $\lambda > \tilde{\lambda} > 0$, *where* $\tilde{\lambda}$ *is from* (1.3).

Remark 1.3. By the conclusions of Theorem 1.2 (b) and (c) it is clear that

$$\tilde{c}^{-1} < \tilde{\lambda}.$$

At this moment, we have no precise information what can be said about Stackelberg responses in the gap-interval $[\tilde{c}^{-1}, \tilde{\lambda}]$; in fact, this will be the subject of Section 5.

In the sequel we provide an application.

Example 1.4. Let $K_2 = [0, \infty)$ and $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(x_1, x_2) = (1 + |x_1|) \left(\min \left(8x_2^3, (x_2 + 3)^{\frac{3}{2}} \right) \right)_+ ,$$

where $s_+ = \max(s, 0)$. A simple calculation shows that

$$\partial_{x_2} \tilde{f}(x_1, x_2) = \begin{cases} \{0\}, & \text{if} \quad x_2 < 0; \\ \left\{24(1 + |x_1|)x_2^2\right\}, & \text{if} \quad x_2 \in [0, 1); \\ [3(1 + |x_1|), 24(1 + |x_1|)], & \text{if} \quad x_2 = 1; \\ \left\{ \frac{3}{2}(1 + |x_1|)(x_2 + 3)^{\frac{1}{2}} \right\}, & \text{if} \quad x_2 > 1. \end{cases}$$

Now, hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$ hold since

$$\lim_{x_2 \searrow 0} \frac{24(1 + |x_1|)x_2^2}{x_2} = \lim_{x_2 \to \infty} \frac{\frac{3}{2}(1 + |x_1|)(x_2 + 3)^{\frac{1}{2}}}{x_2} = 0.$$

Hypothesis $(H^3_{x_1})$ holds since

$$\tilde{f}(x_1, 0) = 0 < \tilde{f}(x_1, 1) = 8(1 + |x_1|).$$

Let $x_1 \in \mathbb{R}$ be fixed. We notice that $\tilde{c} = 24(1+|x_1|)$ and $\tilde{\lambda} = \frac{1}{16(1+|x_1|)}$. According to Theorem 1.2, only the zero solution is given for $\lambda \in (0, \frac{1}{24(1+|x_1|)})$, while for $\lambda > \frac{1}{16(1+|x_1|)}$ there are three solutions for the inclusion

$$x_2 \in \lambda \partial_{x_2} \tilde{f}(x_1, x_2), \ x_2 \geq 0, \tag{1.4}$$

which is equivalent to $x_2 \in R^\lambda_{SV}(x_1)$.

For λ large enough we solve the inclusion (1.4), obtaining that $R^\lambda_{SV}(x_1)$ contains exactly three elements; namely, $R^\lambda_{SV}(x_1) = \{0, x_2^\lambda, y_2^\lambda\}$ where

$$x_2^\lambda = \frac{9\lambda^2(1 + |x_1|)^2 + 3\lambda(1 + |x_1|)\sqrt{9\lambda^2(1 + |x_1|)^2 + 48}}{8}$$

and

$$y_2^\lambda = \frac{1}{24\lambda(1+|x_1|)}.$$

After a simple computation we conclude that the Stackelberg equilibrium response set is $R_{SE}^\lambda(x_1) = \{x_2^\lambda\}$ whenever λ is large. □

The paper has the following structure. In the next section we recall some notions and results from non-smooth analysis for Lipschitz functions and critical point theory for Motreanu-Panagiotopoulos functionals. In Section 3 the proof of Theorem 1.2 (a) and (b) is provided while Section 4 is devoted the proof of Theorem 1.2 (c). Finally, the last section is devoted to the study of the gap-interval.

2. Preliminaries

Let X be a real Banach space and $U \subset X$ an open subset.

Definition 2.1. (F.H. Clarke [3]) *A function $f : U \to \mathbb{R}$ is called* locally Lipschitz *if every point $x \in U$ possesses a neighborhood $N_x \subset U$ such that*

$$|f(x_1) - f(x_2)| \leq K\|x_1 - x_2\|, \quad \forall x_1, x_2 \in N_x,$$

for a constant $K > 0$ depending on N_x.

Definition 2.2. (F.H. Clarke [3]) *The* generalized directional derivative *of the locally Lipschitz function $f : U \to \mathbb{R}$ at the point $x \in U$ in the direction $v \in X$ is defined by*

$$f^0(x; v) = \limsup_{\substack{w \to x \\ t \searrow 0}} \frac{1}{t}(f(w + tv) - f(w)).$$

The following result presents some important properties of the generalized directional derivative.

Proposition 2.3. (D. Motreanu and P.D. Panagiotopoulos [6]) *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function. Then we have:*

(a) *For every $x \in U$ the function $f^0(x;\ \cdot) : X \to \mathbb{R}$ is positively homogeneous and subadditive (therefore convex) and satisfies*

$$|f^0(x; v)| \leq K\|v\|, \quad \forall v \in X. \tag{2.1}$$

Moreover, it is Lipschitz continuous on X with the Lipschitz constant K, where $K > 0$ is a Lipschitz constant of f near x.

(b) *$f^0(\cdot;\cdot) : U \times X \to \mathbb{R}$ is upper semicontinuous.*

(c) *$f^0(x; -v) = (-f)^0(x; v), \quad \forall x \in U, \ \forall v \in X.$*

Definition 2.4. *The* generalized gradient *of f at the point $x \in X$ is defined by*

$$\partial f(x) = \{x^* \in X^* :\ \langle x^*, v\rangle \leq f^0(x; v) \ \text{ for each } v \in X\}.$$

By using the Hahn-Banach theorem it follows that the set $\partial f(x) \neq \emptyset$ for every $x \in U$. Some important properties of the generalized gradient are collected below.

Proposition 2.5. (F.H. Clarke [3], D. Motreanu and P.D. Panagiotopoulos [6]) *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function. We have:*

(a) *For every* $x \in U$, $\partial f(x)$ *is a nonempty, weak***-compacts and convex subset of* $X^\star$ *which is bounded by the Lipschitz constant* $K > 0$ *of* f *near* x.
(b) *For every* $\lambda \in \mathbb{R}$ *and* $x \in U$ *one has* $\partial(\lambda f)(x) = \lambda \partial f(x)$.
(c) *If* $g : U \to \mathbb{R}$ *is another locally Lipschitz function then for every* $x \in U$, *one has* $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$.
(d) *For every* $x \in U$, $f^0(x; \cdot)$ *is the support function of* $\partial f(x)$, *i.e.,* $f^0(x; v) = \max\{\langle z,\ v\rangle :\ z \in \partial f(x)\},\ \forall\ v \in X$.
(e) (Upper semicontinuity) *The set-valued map* $\partial f : U \to 2^{X^\star}$ *is weakly***-closed, that is, if* $\{x_n\} \subset U$ *and* $\{z_n\} \subset X^*$ *are sequences such that* $x_n \to x$ *strongly in* X *and* $z_n \in \partial f(x_n)$ *with* $z_n \rightharpoonup z$ *weakly** *in* X^*, *then* $z \in \partial f(x)$. *In particular, if* X *is finite dimensional, then* ∂f *is upper semicontinuous.*
(f) (Lebourg's mean value theorem) *If* $x, y \in U$ *are two points such that* $[x, y] \subset U$ *then there exists a point* $z \in [x, y] \setminus \{x, y\}$ *such that for some* $z^* \in \partial f(z)$ *the following relation is satisfied:*

$$f(y) - f(x) = \langle z^*, y - x\rangle.$$

Let $E : X \to \mathbb{R}$ be a locally Lipschitz function and let $\zeta : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. Then, $I = E + \zeta$ is called *Motreanu-Panagiotopoulos-type functional*, see [6]. In particular, if E is of class C^1, the functional I is a *Szulkin-type functional*, see A. Szulkin [8].

Definition 2.6. (D. Motreanu and P.D. Panagiotopoulos [6, p.64]) *An element* $x \in X$ *is called a* critical point of $I = E + \zeta$ *if*

$$E^0(x; v - x) + \zeta(v) - \zeta(x) \geq 0 \quad \text{for all } v \in X. \tag{2.2}$$

The number $I(x)$ *is a* critical value *of* I.

Remark 2.7. We notice that an equivalent formulation for (2.2) is

$$0 \in \partial E(x) + \partial_C \zeta(x) \quad \text{in} \quad X^\star, \tag{2.3}$$

where ∂_C denotes the subdifferential in the sense of convex analysis.

Proposition 2.8. *Every local minimum point of* $I = E + \zeta$ *is a critical point of* I *in the sense of* (2.2).

Definition 2.9. (D. Motreanu and P.D. Panagiotopoulos [6, p.64]) *The functional* $I = E+\zeta$ *satisfies the* Palais-Smale condition at level $c \in \mathbb{R}$, (*shortly,* $(PS)_c$*-condition*) *if every sequence* $\{x_n\} \subset X$ *such that* $\lim_{n\to\infty} I(x_n) = c$ *and*

$$E^0(x_n; v - x_n) + \zeta(v) - \zeta(x_n) \geq -\varepsilon_n \|v - x_n\| \quad \textit{for all } v \in X,$$

where $\varepsilon_n \to 0$, *possesses a convergent subsequence.*

Remark 2.10. When $\zeta = 0$, $(PS)_c$-condition is equivalent to the $(PS)_c$-condition introduced by K.-C. Chang [2]. In particular, if E is of class C^1 and $\zeta = 0$, the $(PS)_c$-condition from Definition 2.9 reduces to the standard Palais-Smale condition.

Theorem 2.11. *Let* X *be a Banach space,* $I = E + \zeta : X \to \mathbb{R} \cup \{+\infty\}$ *a Motreanu-Panagiotopoulos-type functional which is bounded from below. If* $I = E + \zeta$ *satisfies the Palais-Smale condition at level* $c = \inf_{x\in X} I(x)$, *then* $c \in \mathbb{R}$ *is a critical value of* I.

We conclude this section by recalling a non-smooth version of the Mountain Pass theorem (initially established by A. Ambrosetti and P. Rabinowitz [1] for C^1 functionals):

Theorem 2.12. (D. Motreanu and P.D. Panagiotopoulos [6, p. 77]) *Let X be a Banach space, $I = E + \zeta : X \to \mathbb{R} \cup \{+\infty\}$ a Motreanu-Panagiotopoulos-type functional and we assume that*

(a) *$I(u) \geq \alpha$ for all $\|u\| = \rho$ with α, $\rho > 0$, and $I(0) = 0$;*
(b) *there is $e \in X$ with $\|e\| > \rho$ and $I(e) \leq 0$.*

If I satisfies the $(PS)_c$-condition (in the sense of Definition 2.9) for

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of I and $c \geq \alpha$.

3. Null Stackelberg response: proof of Theorem 1.2 (a) and (b)

Proof of Theorem 1.2 (a). The claim is equivalent to prove that

$$\tilde{f}^0_{x_2}((x_1, 0); -y) \geq 0, \ \forall y \in K_2. \tag{3.1}$$

By contradiction, we assume that there exists $y_0 \in K_2$ such that $\tilde{f}^0_{x_2}((x_1, 0); -y_0) < 0$. By Proposition 2.5 (d), we have that

$$0 > \tilde{f}^0_{x_2}((x_1, 0); -y_0) = \max\{\langle z, -y_0 \rangle : z \in \partial_{x_2} \tilde{f}(x_1, 0)\},$$

thus, with our assumption, it follows that

$$0 \notin \partial_{x_2} \tilde{f}(x_1, 0).$$

Since the set $\partial_{x_2} \tilde{f}(x_1, 0)$ is compact (see Proposition 2.5), we have that

$$\varepsilon_0 = \operatorname{dist}\left(0, \partial_{x_2} \tilde{f}(x_1, 0)\right) > 0.$$

The upper semicontinuity of $\partial_{x_2} \tilde{f}(x_1, \cdot)$ (see Proposition 2.5 (e)) implies that there exists $\eta_0 > 0$ such that for every $y \in B_{\mathbb{R}^m}(0, \eta_0)$, we have

$$\partial_{x_2} \tilde{f}(x_1, y) \subseteq \partial_{x_2} \tilde{f}(x_1, 0) + B_{\mathbb{R}^m}\left(0, \frac{\varepsilon_0}{2}\right).$$

If $\{x_n\} \subset \mathbb{R}^m$ is a sequence such that $\lim_{n \to \infty} x_n = 0$, for large enough $n \in \mathbb{N}$, we have that

$$z_n \in \partial_{x_2} \tilde{f}(x_1, 0) + B_{\mathbb{R}^m}\left(0, \frac{\varepsilon_0}{2}\right), \ \forall z_n \in \partial_{x_2} \tilde{f}(x_1, x_n).$$

In particular, for every large $n \in \mathbb{N}$, there exists $z_0^n \in \partial_{x_2} \tilde{f}(x_1, 0)$ such that

$$\|z_n - z_0^n\| \leq \frac{\varepsilon_0}{2}.$$

Consequently,

$$\|z_n\| \geq \|z_0^n\| - \|z_n - z_0^n\| \geq \operatorname{dist}\left(0, \partial_{x_2} \tilde{f}(x_1, 0)\right) - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2}.$$

Therefore,

$$\max\left\{\|z_n\| : z_n \in \partial_{x_2}\tilde{f}(x_1,x_n)\right\} \geq \frac{\varepsilon_0}{2}.$$

Since $\lim_{n\to\infty} x_n = 0$, by hypothesis $(H^1_{x_1})$ and the above estimate we have that

$$\begin{aligned} 0 &= \lim_{x_2\to 0}\frac{\max\{\|z\| : z\in\partial_{x_2}\tilde{f}(x_1,x_2)\}}{\|x_2\|} \\ &\geq \lim_{n\to\infty}\frac{\max\{\|z_n\| : z_n\in\partial_{x_2}\tilde{f}(x_1,x_n)\}}{\|x_n\|} \\ &\geq +\infty, \end{aligned}$$

a contradiction. This fact shows that the claim (3.1) holds true, which implies that

$$0 \in R^\lambda_{SV}(x_1) \text{ for every } \lambda > 0.$$

Proof of Theorem 1.2 (b). Let us fix $\lambda \in (0,\tilde{c}^{-1})$ where $\tilde{c}$ comes from relation (1.2) and let $x_2 \in R^\lambda_{SV}(x_1)$, i.e.,

$$\langle x_2, y-x_2\rangle + \lambda\tilde{f}^0_{x_2}((x_1,x_2);-y+x_2) \geq 0,\ \forall y\in K_2.$$

Since $0\in K_2$, we may choose $y=0$ in the above inequality, obtaining that

$$\|x_2\|^2 \leq \lambda\tilde{f}^0_{x_2}((x_1,x_2);x_2). \tag{3.2}$$

By Proposition 2.5 (d) and (1.2), it follows that

$$\begin{aligned} |\tilde{f}^0_{x_2}((x_1,x_2);x_2)| &= |\max\{\langle z,x_2\rangle : z\in\partial_{x_2}\tilde{f}(x_1,x_2)\}| \\ &\leq \max\{\|z\| : z\in\partial_{x_2}\tilde{f}(x_1,x_2)\}\cdot\|x_2\| \\ &\leq \tilde{c}\|x_2\|^2. \end{aligned}$$

The latter estimate and (3.2) gives that

$$\|x_2\|^2 \leq \lambda\tilde{c}\|x_2\|^2.$$

Since $\lambda\in(0,\tilde{c}^{-1})$, we necessarily have that $x_2 = 0$. Therefore, we have

$$R^\lambda_{SV}(x_1) = \{0\},\ \forall\lambda\in(0,\tilde{c}^{-1}).$$

4. Geometry of Stackelberg responses: proof of Theorem 1.2 (c)

Let $x_1 \in K_1$ be fixed.

Lemma 4.1. *Let $\lambda > 0$ be fixed. The functional $f_\lambda(x_1,\cdot)$ defined in* (1.1) *is bounded from below and coercive, i.e., $f_\lambda(x_1,x_2)\to+\infty$ whenever $\|x_2\|\to+\infty$. Moreover, $f_\lambda(x_1,\cdot)$ satisfies the Palais-Smale condition in the sense of Definition* 2.9.

Proof. According to hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$ and to the upper semicontinuity of $\partial_{x_2}\tilde{f}(x_1,\cdot)$ (see Proposition 2.5 (e)), for every $\varepsilon>0$ there exists $M_\varepsilon>0$ such that

$$\max\left\{\|z\| : z\in\partial_{x_2}\tilde{f}(x_1,x_2)\right\} \leq \frac{\varepsilon}{2}\|x_2\| + M_\varepsilon.$$

By Lebourg mean value theorem and from the fact that $\tilde{f}(x_1, 0) = 0$, it follows that for every $x_2 \in \mathbb{R}^m$,

$$|\tilde{f}(x_1, x_2)| = |\tilde{f}(x_1, x_2) - \tilde{f}(x_1, 0)| \le \frac{\varepsilon}{2}\|x_2\|^2 + M_\varepsilon\|x_2\|.$$

Consequently, if $\varepsilon < \lambda^{-1}$ we have that

$$\begin{aligned} f_\lambda(x_1, x_2) &= \frac{1}{2}\|x_2\|^2 - \lambda\tilde{f}(x_1, x_2) + \delta_{K_2}(x_2) \\ &\ge \frac{1}{2}(1 - \varepsilon\lambda)\|x_2\|^2 - \lambda M_\varepsilon\|x_2\|. \end{aligned}$$

This estimate shows that $f_\lambda(x_1, \cdot)$ is bounded from below and coercive.

Now, let $\{x_n\} \subset \mathbb{R}^m$ be a Palais-Smale sequence for $f_\lambda(x_1, \cdot)$, i.e.,

$$\lim_{n\to\infty} f_\lambda(x_1, x_n) = c \tag{4.1}$$

and for every $v \in \mathbb{R}^m$,

$$\langle x_n, v - x_n\rangle + \lambda\tilde{f}^0(x_n; -v + x_n) + \delta_{K_2}(v) - \delta_{K_2}(x_n) \ge -\varepsilon_n\|v - x_n\|,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Since $f_\lambda(x_1, \cdot)$ is coercive, relation (4.1) immediately implies that the sequence $\{x_n\} \subset \mathbb{R}^m$ should be bounded. Consequently, we can extract a convergent subsequence of it, which proves the validity of the Palais-Smale condition. □

Proposition 4.2. *The number $\tilde{\lambda}$ in* (1.3) *is well-defined and*

$$0 < \tilde{\lambda} < \infty.$$

Proof. Let $x_1 \in K_1$ be fixed. By Lebourg mean value theorem (see Proposition 2.5 (f)), we have that

$$\tilde{f}(x_1, x_2) = \tilde{f}(x_1, x_2) - \tilde{f}(x_1, 0) = \langle z_\theta, x_2\rangle$$

for some $z_\theta \in \partial_{x_2}\tilde{f}(x_1, \theta x_2)$ with $\theta \in (0, 1)$. Now, by hypothesis $(H^1_{x_1})$ it follows that for arbitrary $\varepsilon > 0$ there exists $\eta > 0$ such that if $x_2 \in K_2$ with $\|x_2\| < \eta$ then

$$|\tilde{f}(x_1, x_2)| \le \varepsilon\|x_2\|^2.$$

Consequently,

$$\lim_{\substack{x_2\to 0\\ x_2\in K_2}} \frac{\|x_2\|^2}{|\tilde{f}(x_1, x_2)|} = +\infty.$$

A similar reasoning as above shows that

$$\lim_{\substack{\|x_2\|\to\infty\\ x_2\in K_2}} \frac{\|x_2\|^2}{|\tilde{f}(x_1, x_2)|} = +\infty. \tag{4.2}$$

Indeed, by $(H^2_{x_1})$ we have that for arbitrary $\varepsilon > 0$ there exists $\eta > 0$ such that if $\|x_2\| > \eta$ then

$$\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1, x_2)\} \le \varepsilon\|x_2\|.$$

Let $x_\eta \in K_2$ be such that $\|x_\eta\| = \eta$. By Lebourg mean value theorem, for every $x_2 \in K_2$ with $\|x_2\| > \eta$, we have that

$$\tilde{f}(x_1, x_2) - \tilde{f}(x_1, x_\eta) = \langle z_\eta, x_2 - x_\eta \rangle$$

for some $z_\eta \in \partial_{x_2} \tilde{f}(x_1, x_2')$ with $x_2' \in K_2$ and $\|x_2'\| > \eta$. Consequently, we obtain for every $x_2 \in K_2$ with $\|x_2\| > \eta$ that

$$|\tilde{f}(x_1, x_2)| \leq |\tilde{f}(x_1, x_\eta)| + \varepsilon \|x_2\| \|x_2 - x_\eta\|,$$

which shows the validity of (4.2). This ends the proof of the fact that $0 < \tilde{\lambda} < \infty$.

We also notice that the above arguments show that instead of "inf" we can write "min" in (1.3). □

Proof of Theorem 1.2 (c). Let us fix $\lambda > \tilde{\lambda}$.

Step 1. (First response) According to property (a), one has $0 \in R^\lambda_{SV}(x_1)$, which is the first (trivial) response.

Step 2. (Second response) Combining Lemma 4.1 with Theorem 2.11, it follows that the Motreanu-Panagiotopoulos-type functional $f_\lambda(x_1, \cdot)$ achieves its infimum at a point $x_2^\lambda \in \mathbb{R}^m$ which is a critical point in the sense of Definition 2.6. Therefore,

$$f_\lambda(x_1, x_2^\lambda) = \inf_{x \in \mathbb{R}^m} f_\lambda(x_1, x)$$

and

$$0 \in x_2^\lambda - \lambda \partial_{x_2} \tilde{f}(x_1, x_2^\lambda) + \partial_C \delta_{K_2}(x_2^\lambda) \text{ in } \mathbb{R}^m.$$

In fact, the latter relation is nothing but $x_2^\lambda \in R^\lambda_{SV}(x_1)$, which is the second response. Note that in fact $x_2^\lambda \in K_2$; otherwise, $f_\lambda(x_1, x_2^\lambda)$ would be $+\infty$, a contradiction.

It remains to prove that $x_2^\lambda \neq 0$. Since $\lambda > \tilde{\lambda}$, by the definition of $\tilde{\lambda}$ it follows the existence of an element $y_0 \in K_2$ such that

$$\lambda > \frac{1}{2} \frac{\|y_0\|^2}{\tilde{f}(x_1, y_0)} > \tilde{\lambda}.$$

Therefore,

$$\begin{aligned} f_\lambda(x_1, x_2^\lambda) &= \inf_{x \in \mathbb{R}^m} f_\lambda(x_1, x) \\ &\leq f_\lambda(x_1, y_0) \\ &= \frac{1}{2}\|y_0\|^2 - \lambda \tilde{f}(x_1, y_0) + \delta_{K_2}(y_0) \\ &= \frac{1}{2}\|y_0\|^2 - \lambda \tilde{f}(x_1, y_0) \\ &< 0. \end{aligned}$$

Since $f_\lambda(x_1, 0) = 0$, we have that $x_2^\lambda \neq 0$.

Step 3. (Third response) By hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$ again, for every $\varepsilon \in (0, \frac{1}{\lambda})$ there exists $M_\varepsilon > 0$ such that

$$\max\{\|z\| : z \in \partial_{x_2} \tilde{f}(x_1, x_2)\} \leq \frac{\varepsilon}{2}\|x_2\| + M_\varepsilon \|x_2\|^2, \ \forall x_2 \in \mathbb{R}^m.$$

By Lebourg mean value theorem, one has that

$$\tilde{f}(x_1, x_2) \le \frac{\varepsilon}{2}\|x_2\|^2 + M_\varepsilon \|x_2\|^3, \ \forall x_2 \in \mathbb{R}^m.$$

Let

$$0 < \rho < \min\left\{\|x_2^\lambda\|, \frac{1}{2M_\varepsilon}\left(\frac{1}{\lambda} - \varepsilon\right)\right\}.$$

Then, for every $x_2 \in \mathbb{R}^m$ with the property $\|x_2\| = \rho$, we have

$$\begin{aligned} f_\lambda(x_1, x_2) &= \frac{1}{2}\|x_2\|^2 - \lambda \tilde{f}(x_1, x_2) + \delta_{K_2}(x_2) \\ &\ge \frac{1}{2}(1 - \varepsilon\lambda)\|x_2\|^2 - \lambda M_\varepsilon \|x_2\|^3 \\ &= \rho^2\left(\frac{1}{2}(1 - \varepsilon\lambda) - \lambda M_\varepsilon \rho\right) \\ &> 0. \end{aligned}$$

Therefore, by the non-smooth Mountain Pass theorem (see Theorem 2.12), it follows that the number

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f_\lambda(x_1, \gamma(t))$$

is a critical value for $f_\lambda(x_1, \cdot)$, where $\Gamma = \{\gamma \in C([0,1], \mathbb{R}^m) : \gamma(0) = 0, \gamma(1) = x_2^\lambda\}$, and

$$c_\lambda \ge \rho^2\left(\frac{1}{2}(1 - \varepsilon\lambda) - \lambda M_\varepsilon \rho\right) > 0.$$

Thus, if $y_2^\lambda \in K_2$ is the mountain pass-type critical point of $f_\lambda(x_1, \cdot)$ with $c_\lambda = f_\lambda(x_1, y_2^\lambda) > 0$, we clearly have that $y_2^\lambda \neq 0$ and $y_2^\lambda \neq x_2^\lambda$, which is the third response.

Summing up the above three steps, we conclude that

$$\{0, x_2^\lambda, y_2^\lambda\} \subset R_{SV}^\lambda(x_1), \ \forall \lambda > \tilde{\lambda}.$$

This ends the proof of Theorem 1.2. □

Remark 4.3. As we pointed out before, the Stackelberg variational response set reduces to the null strategy whenever the parameter is small enough. However, when the parameter is beyond a threshold value (see Theorem 1.2 (c)), there are three possible Stackelberg variational responses; in this case, the follower enters actively into the game in order to minimize his loss. More precisely, besides the null strategy (see Step 1), he can choose the global minimum-type solution/response (see Step 2); in this case, his loss function takes a negative value, i.e., he is in a winning position. In the case when the player chooses the mountain pass-type minimax response (see Step 3), his payoff function takes a positive value.

5. Remarks on the gap-interval

The subject of this section is twofold:

(a) to give a direct proof for the inequality $\tilde{c}^{-1} \leq \tilde{\lambda}$ whenever $K_2 = \mathbb{R}^m$ (the strict inequality $\tilde{c}^{-1} < \tilde{\lambda}$ can be proven e.g. when $m = 1$ and the payoff function $\tilde{f}$ is of class C^1);
(b) to provide an example in order to show that the gap-interval $[\tilde{c}^{-1}, \tilde{\lambda}]$ can be arbitrary small.

Proposition 5.1. *When $K_2 = \mathbb{R}^m$, we have $\tilde{c}^{-1} \leq \tilde{\lambda}$.*

Proof. As we already pointed out in the proof of Theorem 1.2, in the definition of $\tilde{\lambda}$ we can write minimum instead of infimum. Accordingly, let $\tilde{x}_2 \in K_2 = \mathbb{R}^m$ be the minimum point of the function $x_2 \mapsto \frac{\|x_2\|^2}{2\tilde{f}(x_1,x_2)}$ in the set

$$S = \{x_2 \in \mathbb{R}^m : \tilde{f}(x_1, x_2) > 0\},$$

i.e.,

$$\tilde{\lambda} = \frac{\|\tilde{x}_2\|^2}{2\tilde{f}(x_1, \tilde{x}_2)}.$$

Since S is open and $0 \notin S$, the element $\tilde{x}_2 \neq 0$ is a local minimum point, thus a critical point of the above locally Lipschitz function. Applying the rules of subdifferentiation, we obtain

$$0 \in \frac{2\tilde{x}_2\tilde{f}(x_1, \tilde{x}_2) - \|\tilde{x}_2\|^2 \partial_{x_2}\tilde{f}(x_1, \tilde{x}_2)}{\tilde{f}(x_1, \tilde{x}_2)^2},$$

i.e.,

$$\frac{2\tilde{f}(x_1, \tilde{x}_2)}{\|\tilde{x}_2\|^2}\tilde{x}_2 \in \partial_{x_2}\tilde{f}(x_1, \tilde{x}_2). \tag{5.1}$$

Therefore,

$$\begin{aligned} \tilde{c} &= \max_{x_2 \in \mathbb{R}^m \setminus \{0\}} \frac{\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1, x_2)\}}{\|x_2\|} \\ &\geq \frac{1}{\|\tilde{x}_2\|} \cdot \left\| \frac{2\tilde{f}(x_1, \tilde{x}_2)}{\|\tilde{x}_2\|^2}\tilde{x}_2 \right\| = \frac{2\tilde{f}(x_1, \tilde{x}_2)}{\|\tilde{x}_2\|^2} \\ &= \tilde{\lambda}^{-1}, \end{aligned}$$

which concludes the proof. □

Remark 5.2. In general, we have that $\tilde{c}^{-1} < \tilde{\lambda}$. Such a situation occurs e.g. when $m = 1$, $K_2 = [0, \infty)$ and the payoff function $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1 in the second variable.

Indeed, by contradiction, we assume that $\tilde{c}^{-1} = \tilde{\lambda}$. Let

$$\tilde{x}_2^0 = \inf\left\{\tilde{x}_2 > 0 : \tilde{\lambda} = \frac{\tilde{x}_2^2}{2\tilde{f}(x_1, \tilde{x}_2)}\right\},$$

and fix $y_0 \in (0, \tilde{x}_2^0)$. By the latter construction, one clearly has that

$$\tilde{\lambda} < \frac{y_0^2}{2\tilde{f}(x_1, y_0)}.$$

Since $\tilde{f}(x_1,\cdot)$ is of class C^1, it follows that $\partial_{x_2}\tilde{f}(x_1,x_2) = \{\tilde{f}'_{x_2}(x_1,x_2)\}$; thus by the definition of the number $\tilde{c}$ we obtain in particular that

$$\tilde{f}'_{x_2}(x_1,t) \le \tilde{c}t, \ \forall t > 0.$$

Thus, the above relations imply that

$$\begin{aligned} 0 &= 2\tilde{f}(x_1,\tilde{x}_2) - \tilde{\lambda}^{-1}\tilde{x}_2^2 \\ &= 2\tilde{f}(x_1,y_0) - \tilde{\lambda}^{-1}y_0^2 \\ &\quad +2\tilde{f}(x_1,\tilde{x}_2) - \tilde{\lambda}^{-1}\tilde{x}_2^2 - (2\tilde{f}(x_1,y_0) - \tilde{\lambda}^{-1}y_0^2) \\ &= 2\tilde{f}(x_1,y_0) - \tilde{\lambda}^{-1}y_0^2 + 2\int_{y_0}^{\tilde{x}_2} (\tilde{f}'_{x_2}(x_1,t) - \tilde{c}t)dt \\ &< 0, \end{aligned}$$

a contradiction, which proves the claim.

Proposition 5.3. *The gap-interval $[\tilde{c}^{-1},\tilde{\lambda}]$ can be arbitrarily small.*

Proof. For $\eta > 1$, let $\tilde{f} : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by

$$\tilde{f}(x_1,x_2) = (1+|x_1|)\int_0^{x_2} \min\{(s-1)_+,\eta-1\}ds,$$

and $K_2 = \mathbb{R}$. Note that $\tilde{f}(x_1,\cdot)$ is of class C^1 and

$$\partial_{x_2}\tilde{f}(x_1,x_2) = \{\tilde{f}'_{x_2}(x_1,x_2)\} = \{(1+|x_1|)\min\{(x_2-1)_+,\eta-1\}\}.$$

Consequently, on one hand, we have

$$\tilde{c} = \max_{x_2\in\mathbb{R}\setminus\{0\}} \frac{\max\{|z| : z\in\partial_{x_2}\tilde{f}(x_1,x_2)\}}{|x_2|} = (1+|x_1|)\frac{\eta-1}{\eta}.$$

On the other hand,

$$\tilde{\lambda} = \frac{1}{2}\inf_{\substack{\tilde{f}(x_1,x_2)>0 \\ x_2\in\mathbb{R}}} \frac{|x_2|^2}{\tilde{f}(x_1,x_2)} = \frac{1}{1+|x_1|}\cdot\frac{\eta^2}{(\eta-1)^2}.$$

We can see that $\tilde{c}^{-1} < \tilde{\lambda}$ and these numbers can be arbitrary close to each other whenever η is large enough. □

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Book reviews

Michael Harris, Mathematics without apologies: portrait of a problematic vocation, Princeton University Press, Princeton, NJ, 2015, xxii+438,
ISBN: 978-0-691-15423-7/hbk; 978-1-400-85202-4/ebook.

This is an unusual book written by an eminent professional mathematicians, a well known expert in number theory and algebraic geometry. It reflects author's opinions on mathematics, both pure and applied, and on mathematicians, as seen from the inside of the caste and from outside as well. The author combines his own experience as a mathematician with that obtained from various domains of the human knowledge - philosophy (Plato, Archimedes, Omar Khayam, hindustan philosophy, Witgenstein, Pascal, Russel Kant, Nietsche, etc), literature (a lot of quotations for Goethe's *Faust* and from Shakespeare), history, religion (the case of Pavel Florensky), cinematography and even pop music.

A large part of the book is devoted to the discussion whether mathematics is useful only by the prism of his applications (as considered Fourier) or as a creation of the human spirit, without any reference to practical applications. The conclusion is that pure mathematics, considered by the author a scientific discipline very close to art (an abstract one), deserves to be studied, developed and supported *per se*, without any reference to applications. And besides, some branches of pure mathematics turned to find unexpected applications, in spite of the fact that at the origins they were developed as pure theoretical achievements. As an example, the number theory, a highly theoretical discipline, found applications in cryptography which is basic in bank and e-commerce security. This contradicts Hardy's opinions, who at several occasions toasted "for pure mathematics, and to never find applications", with emphasis on number theory. In fact the title evokes G. H. Hardy's classic *A Mathematician's Apology*, Cambridge Univ. Press, Cambridge, 1940. The author discusses also the allegations blaming mathematicians involved in mathematical economy (another field where results in pure mathematics found deep applications) for the 2008 financial crisis.

The gallery of mathematicians presented in the book is dominated by two giants of the XX century - Alexander Grothendieck and Robert Langlands. Besides the visionary results of Grothendieck in algebraic geometry ("schemas" and "motifs"), the author discusses in several places ideas from *Récoltes te Sémailles*, a collection of reminiscences and reflections about mathematics, philosophy, politics and others, written by Grothendieck. Robert Langlands is best known by his long term "Langlands program", still in progress, the author himself being involved in it. The realization of some intermediary steps led Maxim Kontsevich to a Fields medal in 1998.

The book contains also five sessions entitled "How to Explain Number Theory at a Dinner Party", written as a dialogue between two imagined interlocutors at a dinner party, Performing Artist and Number Theorist, whose challenges and responses elaborate key ideas and themes.

Some practical advices on the career-shaping of a mathematician - the role of "charisma", the quality of publications, and, the last but not the least, the chance - are included as well.

It is difficult to present in a few lines the wealthy of information contained in this marvelous book, our strong advice being to read it (far from being an easy task) and benefit from author's erudition and his charming style of presentation.

P. T. Mocanu

Miroslav Bačák, Convex analysis and optimization in Hadamard spaces, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 22, viii + 185 pp, Walter de Gruyter, Berlin, 2014, ISBN 978-3-11-036103-2/hbk; 978-3-11-036162-9/ebook.

In recent years, concepts and traditional results from convex analysis have been extended to nonlinear settings. Due to their rich geometry, CAT(0) spaces (also known as spaces of non-positive curvature in the sense of Alexandrov) proved to be relevant in this context. The aim of this book is to present a systematic discussion of various topics in convex analysis in the setting of Hadamard spaces (that is, complete CAT(0) spaces). The book contains eight chapters which combine techniques from analysis, geometry, probability and optimization to study different problems in convex analysis.

The first chapter defines Hadamard spaces giving equivalent conditions, examples and construction methods. The second chapter introduces many convexity concepts, properties and results used throughout the book. Special attention is given, among others aspects, to barycenters and resolvents of convex lower semi-continuous functions defined on a Hadamard space. The next chapter deals with a notion of weak convergence defined in terms of asymptotic centers of bounded sequences which recovers the usual weak convergence in Hilbert spaces. Properties of nonexpansive mappings and gradient flows of convex lower semi-continuous functions are the central focus of the following two chapters. Chapter 6 is devoted to convex optimization algorithms used to study convex feasibility problems, to approximate fixed points of a nonexpansive mapping or to find a minimizer of a convex lower semi-continuous function as well as a finite sum thereof when the Hadamard space is in addition locally compact. The next chapter is concerned with random variables with values in Hadamard spaces. In the last chapter, the author gives an example of a Hadamard space, the so-called tree space constructed by L. Billera, S. Holmes and K. Vogtmann, which finds interesting applications in phylogenetics.

This book is written in a very lucid way and can be used both by students and researchers interested in analysis in Hadamard spaces. Each chapter contains a set of exercises and ends with detailed bibliographical remarks, where the author carefully refers to the sources of the presented results. Some comments and challenging questions are also included.

Adriana Nicolae

Francesco Altomare, Mirella Cappelletti Montano, Vita Leonessa, Ioan Raşa; Markov operators, positive semigroups and approximation processes, De Gruyter Studies in Mathematics, vol. 61, Walter de Gruyter, Berlin, 2014, xi+ 313 pp.
ISBN 978-3-11-037274-8/hbk; 978-3-11-036697-6/ebook.

Let $C(X), C(Y)$ be the Banach spaces (with respect to the uniform norm $\|\cdot\|_\infty$) of real- or complex-valued continuous functions on compact Hausdorff spaces X, Y, respectively. A positive linear operator $T : C(X) \to C(Y)$ is called a Markov operator if $T1_X = 1_Y$, where 1_Z denotes the function identically equal to 1 on Z. It follows $\|T\| = \|T1_X\|_\infty = 1$. As a special class of positive linear operators, the Markov operators inherit their properties. For reader's convenience, the authors present in the first chapter, *Positive linear operators and approximation problems*, the main notions, tools and results from the theory of linear operators – positive Radon measures, Choquet boundaries, Bauer simplices, Korovkin-type approximation, etc. Good sources for results of this kind are the book by F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications*, de Gruyter Studies in Mathematics, vol. 17, W. de Gruyter, Berlin, 1994, and the survey paper by F. Altomare, *Korovkin-type theorems and approximation by positive linear operators*, Surv. Approx. Theory vol. 5 (2010), 92-164.

The main theme of the book is the theory of Markov semigroups and the approximation processes which can be generated by a Markov operator acting on $C(K)$, where K is a compact convex subset of a (possibly infinite dimensional) Hausdorff locally convex space.

After a first contact with semigroups of Markov operators in Section 1.4 of the first chapter, their real study starts in Chapter 2, *C_0-semigroups of operators and linear evolution equations*, including a presentation of general properties of semigroups of operators on Banach spaces and their relations with Markov processes and multidimensional second-order differential operators.

Of particular interest in the theory of semigroups of Markov operators are the Bernstein-Schnabl operators associated with Markov operators, whose study begins in the third chapter. After giving several interpretations of these operators – probabilistic, via tensor products – several key examples are discussed: Bernstein-Schnabl operators on [0,1] (which turn to classical Bernstein polynomials), on Bauer simplices, associated with strictly elliptic differential operators, with tensor and convolution products, or with convex combinations of Markov operators. Then one discusses the approximation properties of this class of operators and the rate of convergence. A special attention is paid to their preservation properties – of Hölder and Lipschitz continuity, of convexity and of monotonicity.

Chapters 4, *Differential operators and Markov semigroups associated with Markov operators*, and 5, *Perturbed differential operators and modified Bernstein-Schnabl operators*, treat other main theme of the book. It is shown that "under suitable assumptions on T, the associated (abstract) differential operator is closable and its closure generates a Markov semigroup $(T(t))_{t\geq 0}$ on $C(K)$ which, in turn, is the transition semigroup of a suitable right-continuous Markov process with state space K" (from Introduction, page 2).

Two appendices, A1, *A classification of Markov operators on two dimensional compact convex subsets*, and A2, *Rate of convergence for the limit semigroup of Bernstein operators*, complete the main text.

The bibliography contains 210 items, including many papers by the authors of the book.

Based mainly on the original results of the authors, the book is very well written and contains a lot of interesting material. It can be viewed as an extension of the book by Altomare and Campity mentioned above, and can serve as a reference for researchers in various domain of approximation theory, functional analysis and probability theory. Taking into account the detailed presentation of the results, it can be also used by novices as an accessible introduction to this fertile area of investigation.

S. Cobzaş

Lawrence Craig Evans and Ronald F. Gariepy, Measure theory and fine properties of functions, 2nd revised ed., Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015, xi+309 pp., ISBN: 978-1-4822-4238-6/hbk.

This is a new revised edition of a very successful book (published by CRC Press in 1992) dealing with measure theory in $\mathbb{R}^n$ and some special properties of functions, usually omitted from books dealing with abstract measure theory, but which a working mathematician analyst must to know. Among these special topics we mention: Vitali's and Besicovitch's covering theorems, Hausdorff measures and capacities (for classifying classes of negligible sets for various fine properties of functions), Rademacher's Theorem on the a.e. differentiability of Lipschitz functions (a topic of great actual interest in connection with its extension to infinite dimensions), the area and coarea formulas (yielding change-of-variable rules for Lipschitz functions between $\mathbb{R}^n$ and $\mathbb{R}^m$), the Lebesgue-Besicovitch differentiation theorem, the precise structure of BV and Sobolev functions, Alexandrov's theorem on a.e. twice differentiability of convex functions, Whithney's extension theorem with applications to approximation of Sobolev and BV functions, etc. The book is clearly written with complete proofs, including all technicalities. One assumes that the reader is familiar with Lebesgue measure and abstract measure theory.

The new edition benefits from LaTeX retyping, yielding better cross-references, as well as of numerous improvements in notation, format and clarity of exposition. The bibliography has been updated and several new sections were added: on π-λ-Theorem (on the relations between σ-algebras and Dynkin systems), weak compactness criteria in L^1, the method of Young measures in the study weak convergence, etc.

Undoubtedly that this welcome updated and revised edition of a very popular book will continue to be of great interest for the community of mathematicians interested in mathematical analysis in $\mathbb{R}^n$.

Valeriu Anisiu

Bernardo Lafuerza Guillén, Panackal Harikrishnan; Probabilistic normed spaces, Imperial College Press, London 2014, World Scientific, London-Singapore-Hong Kong 2014, xi+220 pp, ISBN 978-1-78326-468-1/hbk; 978-1-78326-470-4/ebook.

Probabilistic metric (PM) spaces are spaces on which there is a "distance function" taking as values distribution functions - the "distance" between two points p, q is a distribution function (in the sense of probability theory) $F(p, q)$, whose value $F(p, q)(t)$ at $t \in \mathbb{R}$ can be interpreted as the probability that the distance between p and q be less than t. Probabilistic metric spaces were first considered by K. Menger in 1942, who made important contributions to the subject, followed almost immediately by A. Wald in 1943. A good presentation of results up to 1983 is given in the book by B. Schweizer and A. Sklar, *Probabilistic metric spaces*, North Holland, Amsterdam 1983 (reprinted and updated by Dover Publications, New York 2012).

Probabilistic normed (PN) spaces entered the stage only in 1962, introduced by Šerstnev and developed in a series of papers by him and other Russian mathematicians from the Probability School of the University of Kazan. After that the theory laid into lethargy until 1983 when Alsina, Schweizer and Sklar proposed a new approach to probabilistic normed spaces, which is more general and more adequate for developing a consistent theory. Menger PN spaces are particular cases of those defined by Alsina, Schweizer and Sklar, and Šerstnev PN spaces are particular cases of Menger PN spaces.

The first chapter, 1. *Preliminaries*, includes some background material from probability theory (distribution functions) and on copulas, triangular norms, probabilistic metric spaces.

The rest of the book is devoted to a systematic presentation of various aspects of the theory of PN spaces, which are well reflected by the headings of the chapters: 2. *Probabilistic normed spaces*; 3. *The topology of* PN *spaces*; 4. *Probabilistic norms and convergence*; 5. *Products and quotients of* PN *spaces*; 6. *$\mathcal{D}$-Boundedness and $\mathcal{D}$-compactness*; 7. *Normability*; 8. *Invariant and semi-invariant* PN *spaces*; 9. *Linear operators*.

Applications to functional equations are given in Chapter 10. *Stability of some functional equations in* PN *spaces*, while Chapter 11. *Menger's* 2*-probabilistic normed spaces*, presents a probabilistic version of 2-metric spaces introduced by S. Gähler, Mathematische Nachrichten (1964).

The book is clearly written and can be used as an introductory text to this area of research. Based on recent results, including authors' contributions, it is a good reference in the domain. Most of the chapters ends with a list of open problems, inviting the reader to further investigation and new developments in this active area of research.

S. Cobzaş