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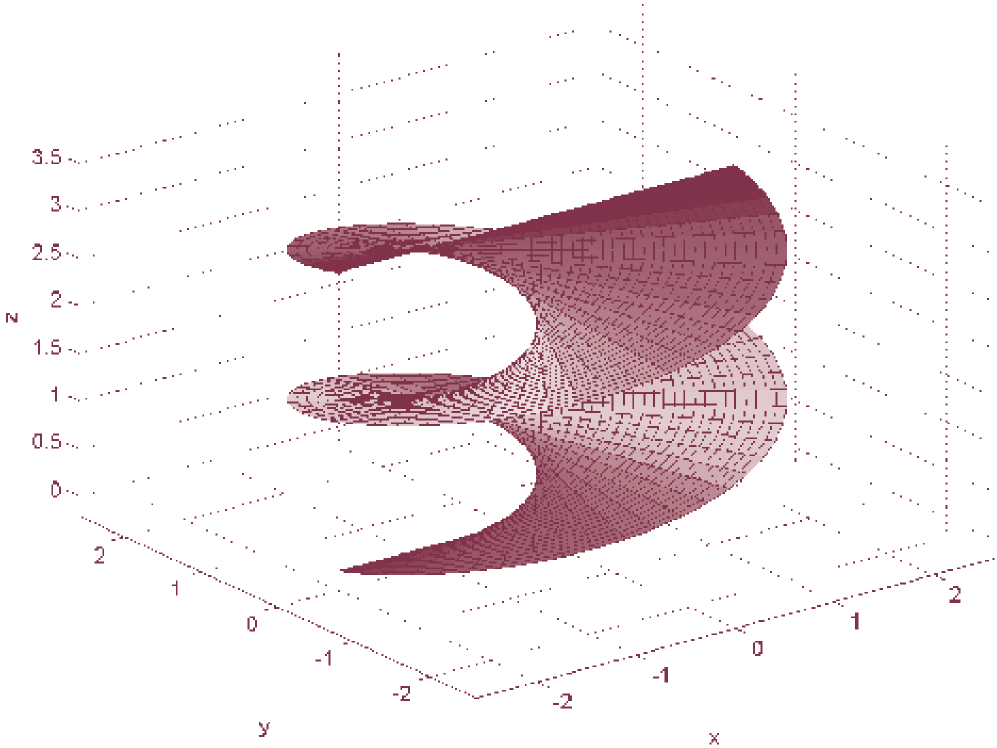

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Preface

During the period September 17-20, 2014, the third edition of the International Conference on Numerical Analysis and Approximation Theory (NAAT 2014) was held at Babeş-Bolyai University of Cluj-Napoca, Romania. Previous editions have taken place in July 2006 and September 2010.

This edition was dedicated to the memory of Dimitrie Dumitru Stancu (February 11, 1927 – April 17, 2014) eminent teacher and researcher of our university, honorary member of the Romanian Academy.

The aim of NAAT 2014 was to bring together specialists in Approximation, Optimization, Numerical Analysis, Statistics, Stochastic Processes and Wavelets. Also, the meeting promoted interactions between researches and Ph. D. students. The conference was attended by over 60 mathematicians coming from nine countries: Brazil, Germany, Italy, Netherlands, Romania, Serbia, Spain, Hungary and Turkey.

We are particularly indebted to our invited speakers: Francesco Altomare (Aldo Moro University, Bari, Italy), Lajos Gergó (Eötvös Loránd University, Budapest, Hungary), Gradimir V. Milovanović (Mathematical Institute of the Serbian Academy of Sciences and Arts, Serbia), Francisco-Javier Muñoz-Delgado (University of Jaen, Spain), Maria Neuss-Radu (University of Erlangen-Nürnberg, Germany), Iuliu Sorin Pop (University of Technology, Eindhoven, Netherlands), Björn Schmalfuss (Friedrich Schiller University, Jena, Germany).

This issue of the journal contains a selection of 18 refereed papers presented at the conference.

Finally we mention that this conference was partly funded by grant 69M/27.08.2014 of the National Agency for Scientific Research (ANCS).

The Organizing Committee

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Optimal cubic Lagrange interpolation: Extremal node systems with minimal Lebesgue constant

Heinz-Joachim Rack and Robert Vajda

Abstract. In the theory of interpolation of continuous functions by algebraic polynomials of degree at most $n-1 \geq 2$, the search for explicit analytic expressions of extremal node systems which lead to the minimal Lebesgue constant is still an intriguing topic in mathematics today [33]. The first non-trivial case $n-1=2$ (quadratic interpolation) has been completely resolved, even in two alternative fashions, see [25], [27]. In the present paper we proceed to completely resolve the cubic case ($n-1=3$) of optimal polynomial Lagrange interpolation on the unit interval $[-1,1]$. We will provide two explicit analytic expressions for the uncountable infinitely many extremal node systems $x_1^* < x_2^* < x_3^* < x_4^*$ in $[-1,1]$ which all lead to the (known) minimal Lebesgue constant of cubic Lagrange interpolation on $[-1,1]$. The descriptions of the extremal node systems (which need not be zero-symmetric) resemble the solutions for the quadratic case and incorporate two intrinsic constants expressed by radicals, of which one constant looks particularly intricate. Our results encompass earlier related work provided in [17], [23], [24], [29], [30] and are guided by symbolic computation.

Mathematics Subject Classification (2010): 05C35, 33F10, 41A05, 41A44, 65D05, 68W30.

Keywords: Constant, cubic, extremal, interpolation, Lagrange interpolation, Lebesgue constant, minimal, node, node system, optimal, point, polynomial, symbolic computation.

1. Introduction

Lagrange polynomial interpolation on node systems $x_1 < x_2 < \cdots < x_{n-1} < x_n$ in some interval $\mathbf{I}$ is a classical and feasible method to approximate (continuous) functions on $\mathbf{I}$ by algebraic polynomials of maximal degree $n-1$, see e.g. [9], [18], [21],

This paper was presented at the third edition of the International Conference on Numerical Analysis and Approximation Theory (NAAT 2014), Cluj-Napoca, Romania, September 17-20, 2014.

[26] or [28] for details. The goodness of this approximation method, as compared with the best possible approximation in Chebyshev's sense, is measured by means of the Lebesgue constants which can be viewed as operator norms or condition numbers, see also [33]. They depend, for a given n, solely on the chosen configuration of interpolation points, and Lebesgue's lemma suggests that we choose them in such a manner that the corresponding Lebesgue constant becomes least. Such extremal (or: optimal) node systems are in a sense the opposite to the equidistant interpolation points which may yield disastrous approximation results, see e.g. Runge's example in [18]. Although near-optimal node systems are known, and algorithms exist to numerically compute optimal (canonical) node systems, see [1] and [18], the search for explicit analytic formulas characterizing optimal node systems remains an intriguing and challenging topic in mathematics today. Here are three quotations in this regard, see also [6], [7], [13], [14], [17], [33]:

- *The nature of the optimal set X^* remains a mystery* [15, p. xlvii]
- *The problem of analytical description of the optimal matrix of nodes is considered by pure mathematicians as a great challenge.* [4]
- *To this day there is no explicit representation for the nth row of the optimal array, and in all likelihood there never will be.* [16]

It suffices to restrict the search for optimal node systems to the unit interval $\mathbf{I} = [-1, 1]$. The first non-trivial case $n - 1 = 2$ of quadratic Lagrange interpolation on $\mathbf{I}$ has been completely resolved: All optimal node systems $x_1^* < x_2^* < x_3^*$ in $\mathbf{I}$ can be explicitly described in two alternative fashions, see [25], [27]. In the present paper we proceed to completely resolve the cubic case $n-1 = 3$. Similarly to the quadratic case we will provide two alternative, but equivalent, explicit analytic (i.e., non-numeric) descriptions of all extremal node systems $x_1^* < x_2^* < x_3^* < x_4^*$ (with $-1 \leq x_1^*$ and $x_4^* \leq 1$) which, by their definition, lead to the (known) minimal Lebesgue constant of cubic Lagrange interpolation on $\mathbf{I}$. Amplifying Theorem 2 in [17] which states that optimal node systems are not unique, we will furthermore show that actually there exist, for each $n - 1 \geq 2$, uncountable infinitely many such node systems in $\mathbf{I}$. Generally, they are zero-asymmetrically distributed in $\mathbf{I}$; however, the subset of optimal zero-symmetric node systems deserves special attention: The investigation, in the cubic case, of optimal node systems $-x_4^* < -x_3^* < x_3^* < x_4^*$ produces two intrinsic constants (b and t, as defined below, of which b is particularly intricate) which are key in describing the general distribution of all optimal node systems $x_1^* < x_2^* < x_3^* < x_4^*$. These can be geometrically visualized by a 2D-region in the plane spanned by the two outer optimal nodes.

Historically, the first investigation into optimal node systems for the cubic case seems to be [29, p. 229], [30, Problem 6.43], where an analytic, but implicit (i.e., not by radicals) formula for the optimal zero-symmetric node systems as well as for the minimal Lebesgue constant has been provided. However, the reader of [29], [30] might be left with the impression that actually all optimal node systems had been determined in this way, since optimal zero-asymmetric ones are not mentioned there. A particular case of a zero-symmetric node system in $\mathbf{I}$ (for $n - 1 = 3$) is a canonical node distribution $-1 < -x_3 < x_3 < 1$. The unique node $x_3 = x_3^* = t$ which turns it to the optimal canonical node configuration has been explicitly determined in [23], [24], where also the minimal Lebesgue constant for the cubic Lagrange interpolation on $\mathbf{I}$ has been determined in an explicit form. Both quantities can be expressed, using Cardan's formula, by means of roots of certain cubic polynomials with integer coefficients.

The manuscript is organized as follows: After providing some necessary notations and definitions, we will consider, in an ascending order of generalization, first canonical node systems, then zero-symmetric node systems, and finally arbitrary (zero-symmetric and zero-asymmetric) node systems and will establish corresponding optimal node configurations. The proofs are postponed to section 6.

We hope that this paper, along with [25], [27] and our dedicated web repository `www.math.u-szeged.hu/~vajda/Leb/` will add to the dissemination of computer-aided optimal quadratic and cubic Lagrange interpolation and may stimulate research of the higher-degree polynomial cases, see Remark 7.5.

2. Definitions and basic theoretical background

Let $C(\mathbf{I})$ denote the Banach space of continuous real functions f on $\mathbf{I}$, equipped with the uniform norm:

$$||f|| = \max_{x\in\mathbf{I}} |f(x)|. \tag{2.1}$$

We wish to approximate f by an algebraic polynomial of degree at most $n-1$, where $n \geq 3$. An old idea, named after Lagrange, see [19], is to sample f at n distinct points in $\mathbf{I}$,

$$X_n:\ -1 \leq x_1 < x_2 < \ldots < x_{n-1} < x_n \leq 1, \tag{2.2}$$

and to construct an interpolating polynomial of degree at most $n-1$ as follows:

$$L_{n-1}(x) = L_{n-1}(f, X_n, x) = \sum_{j=1}^{n} f(x_j)\ell_{n-1,j}(X_n, x) \tag{2.3}$$

where

$$\ell_{n-1,j}(x) = \ell_{n-1,j}(X_n, x) = \prod_{i=1, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}, \tag{2.4}$$

so that

$$\ell_{n-1,j}(X_n, x_i) = \delta_{j,i} \quad \text{(Kronecker delta)} \tag{2.5}$$

and hence

$$L_{n-1}(x_i) = f(x_i), \quad 1 \leq i \leq n \ \text{(interpolatory condition)} \tag{2.6}$$

If $||f|| \leq 1$ then (2.3) implies that $|L_{n-1}(x)|$ can be estimated from above by

$$\sum_{j=1}^{n} |\ell_{n-1,j}(X_n, x)| = \lambda_n(x) = \lambda_n(X_n, x). \tag{2.7}$$

Definition 2.1. We call the x_i's in (2.2) the *interpolation nodes* and the grid X_n the *node system*, the unique L_{n-1} the *Lagrange interpolation polynomial*, the $\ell_{n-1,j}$'s (of exact degree $n-1$) the *Lagrange fundamental polynomials*, and λ_n the Lebesgue function (named after Lebesgue, see [34]).

Three properties of λ_n are summarized in the following statement, see [4], [17], [28, p. 95]:

Proposition 2.2.
i) *λ_n is a piecewise polynomial satisfying $\lambda_n(x) \geq 1$ with equality only if $x = x_i$ $(1 \leq i \leq n)$.*
ii) *λ_n has precisely one local maximum, which we will denote by $\mu_i = \mu_i(X_n)$, in each open sub-interval (x_i, x_{i+1}) of X_n $\quad (1 \leq i \leq n-1)$. The extremum point in (x_i, x_{i+1}), at which the maximum μ_i is attained, we will denote by $\xi_i = \xi_i(X_n)$ so that $\lambda_n(\xi_i) = \mu_i$ holds.*
iii) *λ_n is strictly decreasing and convex in $(-\infty, x_1)$ and strictly increasing and convex in (x_n, ∞).*

Definition 2.3. The largest value of λ_n in $\mathbf{I}$, denoted by $\Lambda_n = \Lambda_n(X_n)$, is called the *Lebesgue constant*:

$$\Lambda_n = \max_{x \in \mathbf{I}} \lambda_n(x). \tag{2.8}$$

The importance of Λ_n in interpolation theory stems from the following inequality ("Error Comparison Theorem") which can be viewed as a version of Lebesgue's lemma, but can also be proved directly [26, Theorem 4.1]:

$$||f - L_{n-1}|| \leq (1 + \Lambda_n)||f - P^*_{n-1}||, \tag{2.9}$$

where $f \in C(\mathbf{I})$, and P^*_{n-1} denotes the polynomial of best uniform approximation to f out of the linear space of all algebraic polynomials of degree at most $n-1$. Usually, P^*_{n-1} is much harder to determine than L_{n-1}, and of course there always holds $||f - P^*_{n-1}|| \leq ||f - L_{n-1}||$. The estimate (2.9), which is sharp for some f, tells us that a small Lebesgue constant implies that the approximation to f by the Lagrange interpolation polynomial is nearly as good as the best uniform approximation to f by means of P^*_{n-1}. Therefore, it is desirable to minimize Λ_n which can be achieved by a strategic placement of the interpolation nodes.
It is known [26, p. 100] that for each $n \geq 3$ there exists, in $\mathbf{I}$, an extremal (or optimal) node system $X_n = X^*_n : -1 \leq x^*_1 < x^*_2 < \cdots < x^*_{n-1} < x^*_n \leq 1$ such that there holds

$$\Lambda^*_n = \Lambda_n(X^*_n) \leq \Lambda_n = \Lambda_n(X_n) \textit{ for all possible choices of node systems } X_n. \tag{2.10}$$

Definition 2.4. The Lebesgue constant Λ^*_n is called *minimal*.

It is furthermore known that for a given $n \geq 3$ an extremal node system is not unique, see [17, Theorem 2], and there in particular exists an extremal node system which includes the endpoints of $\mathbf{I}$ as interpolation nodes, see [26, p. 100]. Obviously, all extremal node systems, for a given n, generate the same minimal Lebesgue constant. We take the opportunity to amplify [17, Theorem 2] in the following way:

Theorem 2.5. For each $n \geq 3$ there exist uncountable infinitely many optimal node systems $X^*_n : x^*_1 < x^*_2 < \cdots < x^*_{n-1} < x^*_n$ in $\mathbf{I}$ which all yield (2.10).

Definition 2.6. The construction of $L_{n-1}(f, X^*_n, x)$ is called *optimal* Lagrange polynomial interpolation on $\mathbf{I}$ since it furnishes, for a given n, the minimal interpolation error in the sense of (2.9).

Definition 2.7. A node system which includes the endpoints of **I** as interpolation nodes (that is, $x_1 = -1$ and $x_n = 1$) is called a *canonical* node system (abbreviated CNS).

In answering a conjecture which goes back to [3], it was proved in [8] and in [14] that the following deep result holds true:

Proposition 2.8. If a Lebesgue function corresponding to a CNS X_n satisfies the so-called equioscillation property

$$\mu_1 = \mu_2 = \ldots = \mu_{n-2} = \mu_{n-1}, \tag{2.11}$$

then X_n is an extremal node system, i.e. $X_n = X_n^*$ with $\Lambda_n(X_n^*) = \Lambda_n^*$.

Thus, the fulfillment of (2.11) is a sufficient condition for a CNS to be extremal. Actually, it was additionally proved that a CNS which satisfies (2.11) is unique and *zero-symmetric*, i.e., $x_i^* = -x_{n-i+1}^*$ for $1 \le i \le n$.
To the best of our knowledge, currently the optimal CNS $X_n = X_n^* : -1 = -x_n^* < -x_{n-1}^* < \cdots < x_{n-1}^* < x_n^* = 1$ and the associated minimal Lebesgue constant $\Lambda_n^* = \Lambda(X_n^*)$ are explicitly known only if $n = 3$ ($X_3^* : -1 < 0 < 1$ and $\Lambda_3^* = 1.25$, see [3], [25], [27] or [33]), or if $n = 4$ (see next section).
In the next three sections we will focus on cubic interpolation on **I** by the Lagrange interpolation polynomial L_3, i.e., we set $n = 4$.

3. The optimal canonical node system and the minimal Lebesgue constant for cubic Lagrange interpolation

We consider first optimal cubic Lagrange interpolation on a CNS. It must be zero-symmetric if it has to be extremal, so that the goal is to find, analytically and explicitly, the unique node $x_3 = x_3^* = t$ in $X_4 : -1 = -x_4 < -x_3 < x_3 < x_4 = 1$ which turns X_4 into the unique extremal node system $X_4^* : -1 < -t < t < 1$ and to determine the associated minimal Lebesgue constant $\Lambda_4^* = \Lambda_4(X_4^*)$ for cubic Lagrange interpolation on **I**. This problem, which is also addressed in [21, Example 2.5.3] and [22, Exercise 4.10], has been solved in [23], [24] by means of roots of certain cubic polynomials with integer coefficients: *The minimal Lebesgue constant, Λ_4^*, is the unique real root of the polynomial*

$$P_3^*(x) = -11 + 53x - 93x^2 + 43x^3. \tag{3.1}$$

This root can be explicitly expressed, with the aid of Cardan's formula, as

$$\Lambda_4^* = \frac{1}{129}\left(93 + \sqrt[3]{125172 + 11868\sqrt{69}} + \sqrt[3]{125172 - 11868\sqrt{69}}\right) \tag{3.2}$$

$$= 1.4229195732\ldots . \tag{3.3}$$

The square of the (unique positive) optimal node $x_3^* = t$ is the unique real root of the polynomial

$$Q_3^*(x) = -1 + 2x + 17x^2 + 25x^3. \tag{3.4}$$

Hence t itself can be explicitly expressed, with the aid of Cardan's formula, as

$$t = \frac{1}{5\sqrt{3}}\sqrt{-17+\sqrt[3]{\frac{14699+1725\sqrt{69}}{2}}+\sqrt[3]{\frac{14699-1725\sqrt{69}}{2}}} \quad (3.5)$$

$$= 0.4177913013\ldots, \quad (3.6)$$

so that

$$X_4^*: -1 < -t < t < 1, \quad \textit{with}\, t\ \textit{from}\ (3.5), \quad (3.7)$$

is the (unique and zero-symmetric) optimal CNS in **I**.

Furthermore, also the three extremum points $\xi_i = \xi_i^*$, with $\xi_1^* = -\xi_3^* < \xi_2^* = 0 < \xi_3^*$, of the cubic Lebesgue function $\lambda_4^*(x) = \lambda_4^*(X_4^*, x)$ are expressed in an analogous way in [24], but we only provide here the numerical value $\xi_3^* = 0.7331726239\ldots$, see Figure 1.

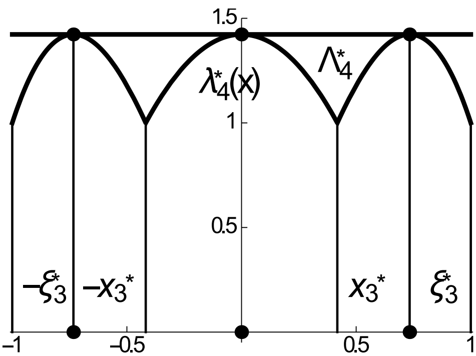

Figure 1

An analytical, but implicit, solution for t and Λ_4^* had been provided earlier in [29, p. 229], [30, Problem 6.43], see also [23]: t is the unique positive root of the polynomial

$$S_6(x) = Q_3^*(x^2) \quad (3.8)$$

and

$$\Lambda_4^* = \frac{1+t^2}{1-t^2}. \quad (3.9)$$

The polynomial (3.1) and the explicit expressions (3.2), (3.5) were not given there. Numerical values for t and/or Λ_4^* are given, for example, in [1], [2], [4], [11], [20], [21] and [33].

4. Optimal zero-symmetric node systems for cubic Lagrange interpolation

We consider next optimal cubic Lagrange interpolation on zero-symmetric node systems in **I**.

Problem 4.1. Find, analytically and explicitly, all optimal zero-symmetric node systems $X_4^* : -x_4^* < -x_3^* < x_3^* < x_4^*$ in $\mathbf{I}$!

Our solution to this problem will be expressed by means of the already deployed explicit constant t according to (3.5) and by means of a real parameter $\beta \in [1, b]$ where the right-hand endpoint $b > 1$ of that interval has still to be determined. We will first state our solution to Problem 4.1 and then turn to the task of determining the constant b.

Theorem 4.2. *All optimal zero-symmetric node systems for the cubic Lagrange interpolation on* $\mathbf{I}$ *are given by* $X_4^* : -x_4^* < -x_3^* < x_3^* < x_4^*$, *where*

$$-x_4^* = -\frac{1}{\beta},\ -x_3^* = -\frac{t}{\beta},\ x_3^* = \frac{t}{\beta},\ x_4^* = \frac{1}{\beta}. \tag{4.1}$$

Here, t *is given by* (3.5) *and* β *is an arbitrary number from the interval* $[1, b]$. *The right-hand endpoint* $b > 1$ *of that interval will be specified implicitly and numerically in Lemma* 4.3, *and explicitly in Lemma* 4.4.

Note that the choice $\beta = 1$ takes us back to the optimal CNS as given in (3.7). For the constant b in Theorem 4.2 we will now provide an implicit and an explicit analytical description. However, both of them are intricate.

Lemma 4.3. *The constant* b *in Theorem* 4.2 *can be expressed implicitly as the unique positive root of the following polynomial of degree 18 with integer coefficients:*

$$\begin{aligned} P_{18}(x) = &-121 + 220x - 1014x^2 + 1344x^3 + 3283x^4 - 5166x^5 + 4502x^6 \\ &+ 15692x^7 - 84178x^8 + 7868x^9 + 210676x^{10} - 25694x^{11} - 310732x^{12} \\ &+ 34154x^{13} + 255377x^{14} - 8450x^{15} - 124700x^{16} + 26875x^{18}. \end{aligned} \tag{4.2}$$

The numerical evaluation of this root by computer algebra systems yields

$$b = 1.0433133411\ldots\ . \tag{4.3}$$

However, the computer algebra systems we have checked do not render the real root b of P_{18} by means of radicals. But such an explicit representation of b is possible:

Lemma 4.4. *The constant* b *in Theorem* 4.2 *can be expressed explicitly in terms of radicals as follows:*

$$\begin{aligned} b = b(t) = &\left(\frac{t+t^3}{2+2t} - \sqrt{\left(\frac{-1}{27}\right)(1+(-1+t)t)^3 + \frac{(t+t^3)^2}{4(1+t)^2}}\right)^{1/3} \\ &+ \left(\frac{t+t^3}{2+2t} + \sqrt{\left(\frac{-1}{27}\right)(1+(-1+t)t)^3 + \frac{(t+t^3)^2}{4(1+t)^2}}\right)^{1/3}, \end{aligned} \tag{4.4}$$

where t *denotes the constant in* (3.5), *so that* b *still depends on* t. *Upon inserting the value* (3.5) *for* t, *one obtains in place of* (4.4) *a non-parametric explicit expression for* *b which, however, is quite intricate:*

$$
\begin{aligned}
b = &\Bigg(\bigg(58+\Big(\frac{1}{2}\big(14699-1725\sqrt{69}\big)\Big)^{1/3}+\Big(\frac{1}{2}\big(14699+1725\sqrt{69}\big)\Big)^{1/3}\bigg) \qquad (4.5)\\
&\Big/\Big(150+750\sqrt{\Big(6\Big/\Big(-34+2^{2/3}\big(14699-1725\sqrt{69}\big)^{1/3}+2^{2/3}\big(14699+1725\sqrt{69}\big)^{1/3}\Big)\Big)}\Big)\\
&-\sqrt{\Bigg(\bigg(58\ 2^{1/3}+\big(14699-1725\sqrt{69}\big)^{1/3}+\big(14699+1725\sqrt{69}\big)^{1/3}\bigg)^2\Big/}\\
&\Big(3750\ 2^{2/3}\Big(\sqrt{6}+30\Big/\Big(\sqrt{\Big(-34+2^{2/3}\big(14699-1725\sqrt{69}\big)^{1/3}+}\\
&\quad 2^{2/3}\big(14699+1725\sqrt{69}\big)^{1/3}\Big)\Big)\Big)^2\Bigg)-\\
&\frac{1}{91125000}\Big(116+2^{2/3}\big(14699-1725\sqrt{69}\big)^{1/3}+2^{2/3}\big(14699+1725\sqrt{69}\big)^{1/3}-\\
&5\sqrt{\Big(6\Big(-34+2^{2/3}\big(14699-1725\sqrt{69}\big)^{1/3}+}\\
&\quad 2^{2/3}\big(14699+1725\sqrt{69}\big)^{1/3}\Big)\Big)^3\Big)\Bigg)^{1/3}+\\
&\Bigg(\bigg(58+\big(\tfrac{1}{2}\big(14699-1725\sqrt{69}\big)\big)^{1/3}+\big(\tfrac{1}{2}\big(14699+1725\sqrt{69}\big)\big)^{1/3}\bigg)\Big/\\
&\Big(150+750\sqrt{\Big(6\Big/\Big(-34+2^{2/3}\big(14699-1725\sqrt{69}\big)^{1/3}+2^{2/3}\big(14699+1725\sqrt{69}\big)^{1/3}\Big)\Big)}\Big)+\\
&\sqrt{\Bigg(\bigg(58\ 2^{1/3}+\big(14699-1725\sqrt{69}\big)^{1/3}+\big(14699+1725\sqrt{69}\big)^{1/3}\bigg)^2\Big/}\\
&\Big(3750\ 2^{2/3}\Big(\sqrt{6}+30\Big/\Big(\sqrt{\Big(-34+2^{2/3}\big(14699-1725\sqrt{69}\big)^{1/3}+}\\
&\quad 2^{2/3}\big(14699+1725\sqrt{69}\big)^{1/3}\Big)\Big)\Big)^2\Bigg)-\\
&\frac{1}{91125000}\Big(116+2^{2/3}\big(14699-1725\sqrt{69}\big)^{1/3}+2^{2/3}\big(14699+1725\sqrt{69}\big)^{1/3}-\\
&5\sqrt{\Big(6\Big(-34+2^{2/3}\big(14699-1725\sqrt{69}\big)^{1/3}+\ 2^{2/3}\big(14699+1725\sqrt{69}\big)^{1/3}\Big)\Big)^3\Big)\Bigg)^{1/3}}
\end{aligned}
$$

The first few decimal digits of b read, more precisely as in (4.3), *as follows:*

$$b = 1.0433133411115929136310868319544936747984794694039995\ldots\ . \qquad (4.6)$$

To give an example, we consider the shortest sub-interval $[-x_4^*, x_4^*]$ of $\mathbf{I}$ which allows optimal cubic Lagrange interpolation. According to Theorem 4.2 we have to choose $\beta = b$, so that we get, as a kind of counterpart to the CNS,

Example 4.5. The configuration

$$-x_4^* = -\frac{1}{b},\ -x_3^* = -\frac{t}{b},\ x_3^* = \frac{t}{b},\ x_4^* = \frac{1}{b} \qquad (4.7)$$

is the unique optimal zero-symmetric node system in $\mathbf{I}$ with the shortest interval length $x_4^* - (-x_4^*) = \frac{2}{b}$. The corresponding numerical values are:

$$-0.9584848200\ldots < -0.4004466202\ldots < 0.4004466202\ldots < 0.9584848200\ldots \qquad (4.8)$$

and

$$\frac{2}{b} = 1.9169696400\ldots . \tag{4.9}$$

We note that (4.7) is also unique in having the property that the corresponding Lebesgue function $\lambda_4(x)$ equioscillates on **I** most, that is, five ($= n+1$) times, see Figure 2.

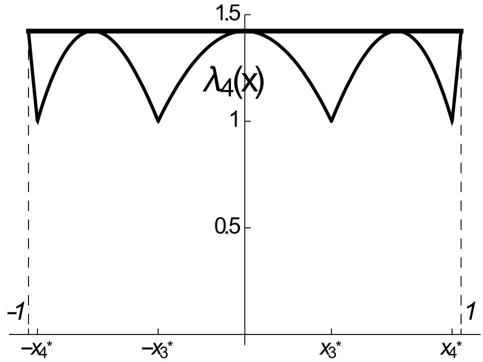

Figure 2

It is interesting to remark that the analogous configuration for the quadratic case ($n = 3$) is $-\frac{2\sqrt{2}}{3} < 0 < \frac{2\sqrt{2}}{3}$, and the corresponding Lebesgue function $\lambda_3(x)$ equioscillates on **I** most, that is, four ($= n+1$) times. The extremum points $-1 < -\frac{\sqrt{2}}{3} < \frac{\sqrt{2}}{3} < 1$ in **I** of that particular $\lambda_3(x)$ were considered by Bernstein in [3] and motivated him to state his famous equioscillation conjecture. We therefore call (4.7) the Bernstein-type node system (BNS).

An analytical, but implicit, solution similar to (4.1) had been provided earlier in [29, p. 229] (misprinted) and [30, Problem 6.43]: The optimal zero-symmetric nodes in **I** are $-a < -at < at < 1$, where $a \in [a_0, 1]$ and t is, as before, the unique positive root of the polynomial $S_6(x) = Q_3^*(x^2)$ and a_0 is the unique positive root of the (parameterized) polynomial

$$S_3(x) = -(t+1) + (t^3+1)x^2 + (t^3+t)x^3. \tag{4.10}$$

However, Tureckii did not express t and a_0 by radicals, and in [29, p. 229] the term $(t^3+1)x^2$ of $S_3(x)$ was misprinted as $(t^3+1)x$. We observe that the left-hand endpoint a_0 of the interval $[a_0, 1]$ coincides with $\frac{1}{b}$, where the constant b is given in Lemma 4.3 and 4.4. After all, we point out that optimal zero-asymmetric node systems are not mentioned in [29], [30], [31].

5. Optimal arbitrary node systems for cubic Lagrange interpolation

Finally we consider optimal cubic Lagrange interpolation on arbitrary (zero-symmetric and zero-asymmetric) node systems in **I**.

Problem 5.1. Find, analytically and explicitly, all optimal node systems $X_4^* : x_1^* < x_2^* < x_3^* < x_4^*$ in $\mathbf{I}$!

Our solution to this problem is twofold: Building on the already deployed constants b in (4.4) and t in (3.5), the first solution describes all four optimal nodes by means of two parameters, $\alpha \in [-b, -1]$ and $\beta \in [1, b]$, whereas the second equivalent solution gives only the selection range for the outer optimal nodes x_1^* and x_4^* and expresses the inner optimal nodes x_2^* and x_3^* as functions of them. The first solution is similar to the one given in [25], whereas the second solution is similar to the one given in [27], for the quadratic case.

Theorem 5.2. *All optimal node systems for the cubic Lagrange interpolation on* $\mathbf{I}$ *are given by* $X_4^* : x_1^* < x_2^* < x_3^* < x_4^*$, *where*

$$x_1^* = \frac{-2-\alpha-\beta}{-\alpha+\beta},\ x_2^* = \frac{-2t-\alpha-\beta}{-\alpha+\beta},\ x_3^* = \frac{2t-\alpha-\beta}{-\alpha+\beta},\ x_4^* = \frac{2-\alpha-\beta}{-\alpha+\beta}, \tag{5.1}$$

in which $\alpha \in [-b, -1]$ *and* $\beta \in [1, b]$ *are arbitrary numbers, and the constants* b *and* t *are defined in* (4.4) *respectively* (3.5).

Note that the choice $\alpha = -\beta$ takes us back to the zero-symmetric case (4.1). To illustrate Theorem 5.2, we give an analytic example:

Example 5.3. Choose $\alpha = -1.04 \in [-b, -1]$ and $\beta = 1.03 \in [1, b]$, say. According to (5.1) we get

$$X_4^* : x_1^* = -\frac{199}{207} < x_2^* = \frac{100}{207}\left(\frac{1}{100} - 2t\right) < x_3^* = \frac{100}{207}\left(\frac{1}{100} + 2t\right) < x_4^* = \frac{201}{207} \tag{5.2}$$

which is an optimal (zero-asymmetric) node system in $\mathbf{I}$. A little computation reveals that indeed

$$\max_{x \in \mathbf{I}} \lambda_4(X_4^*, x) = \Lambda_4^* = \frac{1}{129}\left(93 + \sqrt[3]{125172 + 11868\sqrt{69}} + \sqrt[3]{125172 - 11868\sqrt{69}}\right)$$

holds, e.g., $\lambda_4(X_4^*, \frac{1}{207}) = \Lambda_4^*$ and $\lambda_4'(X_4^*, x)|_{x=\frac{1}{207}} = 0$. The numerical values in (5.2) read, after inserting t from (3.6):

$$\begin{aligned} x_1^* = -0.9613526570\cdots < x_2^* = -0.3988321752\cdots < \\ < x_3^* = 0.4084940109\cdots < x_4^* = 0.9710144927\ldots. \end{aligned} \tag{5.3}$$

The corresponding Lebesgue function $\lambda_4(x) = \lambda_4(X_4^*, x)$ is depicted in Figure 3.

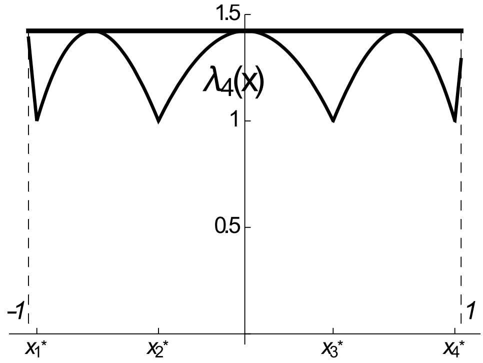

Figure 3

Our second solution to Problem 5.1 likewise builds on the deployed constants b in (4.4) and t in (3.5). The two parameters α and β and their selection ranges are now superseded by selection ranges for the two outer optimal nodes which, once fixed, uniquely determine the corresponding two inner optimal nodes.

Theorem 5.4. *(Alternative solution to Problem* 5.1*) All optimal node systems for the cubic Lagrange interpolation on* $\mathbf{I}$ *are given by* $X_4^* : x_1^* < x_2^* < x_3^* < x_4^*$*, where either*

$$-1 \le x_1^* \le -\frac{1}{b} = -0.9584848200\ldots \text{ and } \left(\frac{b-1}{b+1}\right) x_1^* + \frac{2}{b+1} \le x_4^* \le 1 \tag{5.4}$$

or

$$-\frac{1}{b} < x_1^* \le \frac{b-3}{b+1} = -0.9576047978\ldots \text{ and } \left(\frac{b+1}{b-1}\right) x_1^* + \frac{2}{b-1} \le x_4^* \le 1 \tag{5.5}$$

with

$$x_2^* = \left(\frac{1+t}{2}\right) x_1^* + \left(\frac{1-t}{2}\right) x_4^*, \tag{5.6}$$

and with

$$x_3^* = \left(\frac{1-t}{2}\right) x_1^* + \left(\frac{1+t}{2}\right) x_4^*. \tag{5.7}$$

When the outer optimal nodes x_1^* and x_4^* vary in their respective ranges (5.4) and (5.5), then also the ranges of the inner optimal nodes x_2^* and x_3^* are exhausted. That ranges are substantiated in the next statement.

Theorem 5.5. *The two inner optimal nodes* x_2^* *and* x_3^* *in Theorem* 5.4 *may vary within the ranges*

$$\frac{-1}{b+1}(2t+b-1) = -0.4301327291\ldots \le x_2^* \le \frac{-1}{b+1}(2t-b+1) = -0.3877375269\ldots \tag{5.8}$$

and

$$\frac{1}{b+1}(2t-b+1) = 0.3877375269\cdots \le x_3^* \le \frac{1}{b+1}(2t+b-1) = 0.4301327291\ldots \tag{5.9}$$

To illustrate Theorems 5.4 and 5.5 we give a numerical example which can be turned into an analytical one:

Example 5.6. Choose $x_1^* = -0.99$, say. Then one gets, according to (5.4), $0.9578167738\cdots \leq x_4^* \leq 1$. Choose now $x_4^* = 0.96$, say. This implies, according to (5.6) and (5.7), $x_2^* = -0.4223465188\ldots$ and $x_3^* = 0.3923465188\ldots$. Hence these four numerical values constitute a zero-asymmetric optimal node system for the cubic Lagrange interpolation on **I**. The associated Lebesgue function is depicted in Figure 4. It is readily verified that this specific optimal node system corresponds to (5.1) if we insert there $\alpha = -\frac{197}{195} = -1.0102564102\ldots$ and $\beta = \frac{203}{195} = 1.0410256410\ldots$. In particular, the two inner nodes can thus be expressed explicitly as $x_2^* = \frac{1}{200}(-195t-3)$ and $x_3^* = \frac{1}{200}(195t-3)$.

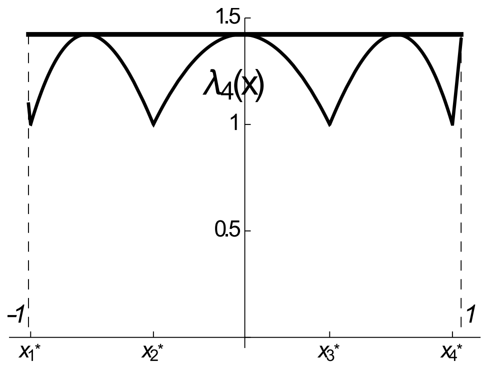

Figure 4

The somewhat curious description of the ranges of the two outer optimal nodes x_1^* and x_4^* in Theorem 5.4 can be given a geometric visualization by a 2D-quadrilateral whose one 1D-diagonal represents the zero-symmetric node systems (ZSNS) (4.1). The two endpoints of that diagonal are represented by the CNS (3.7) and by the BNS (4.7). The quadrilateral itself is not quite a square, rather two of its sides (the lower one and the right one) can be expressed as line segments of linear functions g (flat) and h (steep) of the variable x_1^*, see (5.4) and (5.5):

$$g(x_1^*) = \left(\frac{b-1}{b+1}\right) x_1^* + \frac{2}{b+1} = 0.0211976010\ldots x_1^* + 0.9788023989\ldots \tag{5.10}$$

and

$$h(x_1^*) = \left(\frac{b+1}{b-1}\right) x_1^* + \frac{2}{b-1} = 47.1751494729\ldots x_1^* + 46.1751494729\ldots. \tag{5.11}$$

The graphical representation is given in Figure 5.

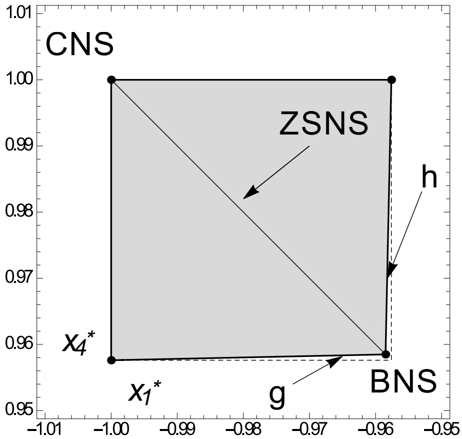

Figure 5

We are not aware of any published studies that provide explicit analytical solutions to the problem of determining all optimal node systems in the frame of optimal polynomial Lagrange interpolation on **I** with polynomials of degree ≥ 4 (i.e., $n \geq 5$), notwithstanding that this large project is vibrant, see [5], and also Remark 7.5 below. The results for the polynomial degree $= 3$ as obtained in [23], [24], and [30, Problem 6.43] do not cover zero-asymmetric node systems. Thus the present paper seems to be the first one where, for the polynomial degree $= 3$, *all* optimal node systems in **I** have been determined, explicitly and analytically. For the polynomial degree $= 2$ this has been achieved in [25] and in [27].

6. Proofs

6.1. Proof of Theorem 2.5

Proof. Let $\lambda_n^*(x) = \lambda_n^*(X_n^*, x)$ denote the Lebesgue function corresponding to the optimal CNS $X_n^* : -1 = -x_n^* = x_1^* < -x_{n-1}^* = x_2^* < \cdots < x_{n-1}^* < x_n^* = 1$ in **I**. We know that $\lambda_n^*(\pm 1) = 1$ and that $\lambda_n^*(x)$ is strictly decreasing resp. increasing in $(-\infty, -1)$ and $(1, \infty)$, see Proposition 2.2. As $\Lambda_n^* > 1$, there exist unique values $x = c_n < -1$ resp. $x = b_n > 1$ with the property $\lambda_n^*(c_n) = \lambda_n^*(b_n) = \Lambda_n^*$. Hence, $\max_{x\in[c_n,b_n]} \lambda_n^*(x) = \Lambda_n^*$ and in particular $\max_{x\in[\alpha,\beta]} \lambda_n^*(x) = \Lambda_n^*$ for any subinterval $[\alpha, \beta]$ of $[c_n, b_n]$ which covers **I**. Choose now an arbitrary $\alpha \in [c_n, -1]$ and an arbitrary $\beta \in [1, b_n]$ and consider the linear transformation

$$S(x) = \frac{1}{-\alpha + \beta}(2x - \alpha - \beta) \tag{6.1}$$

which maps $[\alpha, \beta]$ onto **I**, because $S(\alpha) = -1$ and $S(\beta) = 1$. In particular, the nodes of the optimal CNS X_n^* (which is contained in $[\alpha, \beta]$) will be mapped as follows:

$$
\begin{aligned}
S(-1) = y_1^* &= \frac{1}{-\alpha+\beta}(-2-\alpha-\beta) \\
S(-x_{n-1}^*) = y_2^* &= \frac{1}{-\alpha+\beta}(-2x_{n-1}^*-\alpha-\beta) \\
&\cdots \\
S(x_{n-1}^*) = y_{n-1}^* &= \frac{1}{-\alpha+\beta}(2x_{n-1}^*-\alpha-\beta) \\
S(1) = y_n^* &= \frac{1}{-\alpha+\beta}(2-\alpha-\beta). \qquad (6.2)
\end{aligned}
$$

As $S(x)$ has the positive slope $\frac{2}{-\alpha+\beta}$, these images are ordered ascendingly in $\mathbf{I}$:

$$Y_n^* = Y_{n,\alpha,\beta}^* : y_1^* < y_2^* \cdots < y_{n-1}^* < y_n^*. \qquad (6.3)$$

We proceed to show that Y_n^* is an extremal node system (with minimal Lebesgue constant) in $\mathbf{I}$. To this end, we observe that, for $y \in \mathbf{I}$ and $x \in [\alpha, \beta]$, we have

$$
\begin{aligned}
\lambda_n(Y_n^*, y) &= \sum_{j=1}^{n} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{|y-y_i^*|}{|y_j^*-y_i^*|} = \sum_{j=1}^{n} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{|S(x)-S(x_i^*)|}{|S(x_j^*)-S(x_i^*)|} = \\
&= \sum_{j=1}^{n} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{|\frac{1}{-\alpha+\beta}(2x-\alpha-\beta)-\frac{1}{-\alpha+\beta}(2x_i^*-\alpha-\beta)|}{|\frac{1}{-\alpha+\beta}(2x_j^*-\alpha-\beta)-\frac{1}{-\alpha+\beta}(2x_i^*-\alpha-\beta)|} = \sum_{j=1}^{n} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{|x-x_i^*|}{|x_j^*-x_i^*|} = \\
&= \lambda_n^*(x), \qquad (6.4)
\end{aligned}
$$

and hence

$$\max_{y\in\mathbf{I}} \lambda_n(Y_n^*, y) = \max_{x\in[\alpha,\beta]} \lambda_n^*(x) = \Lambda_n^*,$$

see also [21, Theorem 2.5.3] and [6, Problem 1, p. 22]. Thus, $Y_n^* = Y_{n,\alpha,\beta}^*$ is indeed an optimal node system in $\mathbf{I}$, and since α and β were chosen arbitrarily, there exist uncountable infinitely many such Y_n^*'s. □

6.2. Proof of Theorem 4.2

Proof. The statement is a corollary to Theorem 5.2: insert there $\alpha = -\beta$. □

6.3. Proof of Theorem 5.2

Proof. The Lebesgue function $\lambda_4^*(x) = \lambda_4^*(X_4^*, x)$ which corresponds to the optimal CNS X_4^* in (3.7) is readily found to be given by

$$
\begin{aligned}
\lambda_4^*(x) = |\ell_{31}(x)| + |\ell_{32}(x)| + |\ell_{33}(x)| + |\ell_{34}(x)| = \qquad (6.5) \\
= \quad & |(-t-x)(-1+x)(-t+x)/(2(-1-t)(-1+t))| + \\
& |(-1+x)(1+x)(-t+x)/(2(-1-t)(1-t)t)| + \\
& |(-1+x)(1+x)(t+x)/(2(-1+t)t(1+t))| + \\
& |(1+x)(-t+x)(t+x)/(2(1-t)(1+t))|,
\end{aligned}
$$

where t is defined in (3.5).

For $x \in (1, \infty)$, $\lambda_4^*(x)$ is strictly increasing and is representable there as

$$\lambda_4^*(x) = -\ell_{31}(x) + \ell_{32}(x) - \ell_{33}(x) + \ell_{34}(x) = \frac{x(1 - t + t^2 - x^2)}{(-1 + t)t}. \tag{6.6}$$

Let $x = b > 1$ denote the unique point on the x-axis where $\lambda_4^*(x)$ intercepts with the constant function $f(x) = \Lambda_4^*$, see (3.2). A numerical solution of the equation $\lambda_4^*(x) - \Lambda_4^* = 0$ is the value b given in (4.3). The expression of b as explicit analytical solution we will provide below (see the proof of Lemma 4.4). Similarly, for $x \in (-\infty, -1)$, $\lambda_4^*(x)$ is strictly decreasing and is representable there as

$$\lambda_4^*(x) = \ell_{31}(x) - \ell_{32}(x) + \ell_{33}(x) - \ell_{34}(x) = \frac{x(-1 + t - t^2 + x^2)}{(-1 + t)t}, \tag{6.7}$$

so that $x = -b < -1$ denotes the unique point on the x-axis where $\lambda_4^*(x)$ intercepts with the constant function $f(x) = \Lambda_4^*$. Thus we have, on the interval $[-b, b]$ as well as on any subinterval $[\alpha, \beta]$ thereof which covers $\mathbf{I}$, $\max_{x\in[-b,b]} \lambda_4^*(x) = \Lambda_4^* = \max_{x\in\mathbf{I}} \lambda_4^*(x)$. We now apply the linear transformation (6.1) to X_4^* in (3.7) with arbitrary $\alpha \in [-b, -1]$ and arbitrary $\beta \in [1, b]$, and thus get (5.1), after renaming y_i^* to x_i^*, $i = 1, 2, 3, 4$ (see the proof of Theorem 2.5). In this way we generate uncountable infinitely many extremal node systems $x_1^* < x_2^* < x_3^* < x_4^*$ in $\mathbf{I}$, and in fact we so obtain *all* extremal node systems in $\mathbf{I}$.
For let $X_4^0 : x_1^0 < x_2^0 < x_3^0 < x_4^0$ be an arbitrary extremal node system in $\mathbf{I}$. The linear transformation $T(x) = \frac{2x - x_1^0 - x_4^0}{-x_1^0 + x_4^0}$ maps X_4^0 onto $\mathbf{I}$ since $T(x_1^0) = -1$ and $T(x_4^0) = 1$. Since its slope $\frac{2}{-x_1^0 + x_4^0}$ is positive, the mapped values are ordered ascendingly, that is $X_{4,T}^0 : -1 = T(x_1^0) < T(x_2^0) < T(x_3^0) < T(x_4^0) = 1$, and hence $X_{4,T}^0$ is a CNS in $\mathbf{I}$. The Lebesgue constant corresponding to X_4^0 (which equals Λ_4^* as given in (3.2)) is identical with the one corresponding to $X_{4,T}^0$, see the proof of Theorem 2.5 or [21, Theorem 2.5.3] or [6, Problem 1, p. 22]. But this implies that $X_{4,T}^0$ is necessarily the unique CNS X_4^* on $\mathbf{I}$, as given in (3.7). In particular we thus have $T(x_2^0) = -t$ and $T(x_3^0) = t$, and this implies, by the definition of $T(x)$, $x_2^0 = \left(\frac{1+t}{2}\right) x_1^0 + \left(\frac{1-t}{2}\right) x_4^0$ and $x_3^0 = \left(\frac{1-t}{2}\right) x_1^0 + \left(\frac{1+t}{2}\right) x_4^0$, where t is from (3.5).
With this information at hand we proceed as follows: The linear transformation S from (6.1) with $\alpha = \alpha^0 = \frac{-2 - x_4^0 - x_1^0}{x_4^0 - x_1^0}$ and $\beta = \beta^0 = \frac{2 - x_4^0 - x_1^0}{x_4^0 - x_1^0}$ maps the optimal CNS X_4^* onto $x_1^0 < x_2^0 < x_3^0 < x_4^0$. Indeed, $S(-1) = x_1^0$ and $S(1) = x_4^0$ as is immediately verified. Furthermore we get $S(-t) = \left(\frac{1+t}{2}\right) x_1^0 + \left(\frac{1-t}{2}\right) x_4^0$ and $S(t) = \left(\frac{1-t}{2}\right) x_1^0 + \left(\frac{1+t}{2}\right) x_4^0$, and these values are identical with x_2^0 and x_3^0 as shown above.
It remains to show that $\alpha^0 \in [-b, -1]$ and $\beta^0 \in [1, b]$. We will prove it by symbolic computation employing the built-in language symbols **InterpolatingPolynomial** and **Resolve** in Mathematica® as well as Mathematica®-specific notation. Furthermore, we will use the fact that $\lambda_n(\pm 1) \le \Lambda_n^*$, see Proposition 2.2. Assume that the variable LF contains the Lebesgue function corresponding to the node system in (5.1). This can be defined e.g. in Mathematica® as follows:

LF=
Abs[InterpolatingPolynomial[
$\{\{x1[\alpha,\beta],1\},\{x2[\alpha,\beta],0\},\{x3[\alpha,\beta],0\},\{x4[\alpha,\beta],0\}\},x]]$ +
Abs[InterpolatingPolynomial[
$\{\{x1[\alpha,\beta],0\},\{x2[\alpha,\beta],1\},\{x3[\alpha,\beta],0\},\{x4[\alpha,\beta],0\}\},x]]$ +
Abs[InterpolatingPolynomial[
$\{\{x1[\alpha,\beta],0\},\{x2[\alpha,\beta],0\},\{x3[\alpha,\beta],1\},\{x4[\alpha,\beta],0\}\},x]]$ +
Abs[InterpolatingPolynomial[
$\{\{x1[\alpha,\beta],0\},\{x2[\alpha,\beta],0\},\{x3[\alpha,\beta],0\},\{x4[\alpha,\beta],1\}\},x]]$.

Denote the the specific parameter values α^0 and β^0 by

$$\alpha 0 = \frac{-2 - x4 - x1}{x4 - x1} \quad \beta 0 = \frac{2 - x4 - x1}{x4 - x1}. \tag{6.8}$$

Then one gets, with $\Lambda = \Lambda_4^*$ as given in (3.2) and (3.9) and b as given in (4.2) and (4.5),

[in:]

$$\begin{aligned}&\text{Resolve[ForAll}[\{x1, x4\}, (-1 \le x1 < x4 \le 1 \wedge \\ &(\text{LF}/.x \to -1) \le \Lambda \wedge (\text{LF}/.x \to 1) \le \Lambda /.\{\alpha \to \alpha 0, \beta \to \beta 0\}) \\ &\Rightarrow \\ &(-b \le \alpha 0 \le -1 \ \wedge\ 1 \le \beta 0 \le b)], \text{Reals}]\end{aligned} \tag{6.9}$$

[out:] True. □

6.4. Proof of Lemma 4.3

Proof. We want to determine the point $x = b > 1$ on the x-axis where $\lambda_4^*(x)$ intercepts with the constant function $f(x) = \Lambda_4^*$ (see the proof of Theorem 5.2). To this end, we employ a computer algebra system. The built-in language symbol **RootReduce** of Mathematica® immediately gives, in view of (3.1), (3.4) and (6.6), the claimed polynomial of degree 18:

[in:]

$$\begin{aligned}&\text{RootReduce}[x/.\text{Solve}[x(1 - t + t^2 - x^2)/((-1 + t)t) == \\ &\text{Root}[-11 + 53\#1 - 93\#1^2 + 43\#1^3\&, 1] \\ &/.t \to \text{Root}[-1 + 2\#1^2 + 17\#1^4 + 25\#1^6\&, 2], x, \text{Reals}]]\end{aligned} \tag{6.10}$$

[out:]

$$\begin{aligned}&\{\text{Root}[-121 + 220\#1 - 1014\#1^2 + 1344\#1^3 + 3283\#1^4 - 5166\#1^5 + \\ &4502\#1^6 + 15692\#1^7 - 84178\#1^8 + 7868\#1^9 + 210676\#1^{10} - \\ &25694\#1^{11} - 310732\#1^{12} + 34154\#1^{13} + 255377\#1^{14} - \\ &8450\#1^{15} - 124700\#1^{16} + 26875\#1^{18}\&, 2]\}.\end{aligned} \tag{6.11}$$

This polynomial P_{18} can also be deduced in a reverse way: If one employs in Mathematica® the built-in language symbol **FullSimplify** to the explicit expression (4.5), then one gets (6.11). □

6.5. Proof of Lemma 4.4

Proof. The equation

$$x(1-t+t^2-x^2)/((-1+t)t) = \frac{1+t^2}{1-t^2} \tag{6.12}$$

see (3.9) and (6.6), amounts to the following cubic algebraic equation in x:

$$(t+t^3)+(1+t^3)x+(-1-t)x^3 = 0. \tag{6.13}$$

Solving (6.13) with the aid of Cardan's formula yields the (parametric) solution $x = b = b(t)$ as given in (4.4). Inserting then into (4.4) the value for the constant t according to (3.5) and simplifying eventually gives (4.5). The said algebraic manipulations can be executed by pencil and paper, but it is more convenient to guide them by a computer algebra system. □

6.6. Proof of Theorem 5.4

Proof. To prove (5.4) and (5.5), we use quantifier elimination to eliminate the existentially quantified variables α and β from the parametric representation of $x_1^* = x_1^*(\alpha,\beta)$ and $x_4^* = x_4^*(\alpha,\beta)$, taking into account the range of α and β, see Theorem 5.2. This elimination can be executed by means of the built-in language symbol **Resolve** in Mathematica®:

[in :]

$$\text{Resolve[Exists[}\{\alpha,\beta\}, \big(b>1 \wedge -b \le \alpha \le -1 \wedge 1 \le \beta \le b \wedge$$
$$x_1^*(\beta-\alpha) == -2-\alpha-\beta \wedge x_4^*(\beta-\alpha) == 2-\alpha-\beta\big)], \{x_1^*, x_4^*\}, \text{Reals]}$$

[out :]

$$b>1 \wedge \Big((-1 \le x_1^* \le -\frac{1}{b} \wedge \frac{2-x_1^*+bx_1^*}{1+b} \le x_4^* \le 1) \vee$$
$$(-\frac{1}{b} < x_1^* \le \frac{-3+b}{1+b} \wedge \frac{2+x_1^*+bx_1^*}{-1+b} \le x_4^* \le 1)\Big), \tag{6.14}$$

which coincides with (5.4) and (5.5). To prove (5.6) and (5.7), consider the parametric representation of the optimal nodes $x_i^* = x_i^*(\alpha,\beta)$, $i = 1,2,3,4$ in (5.1). Computing now $\left(\frac{1+t}{2}\right)x_1^*(\alpha,\beta) + \left(\frac{1-t}{2}\right)x_4^*(\alpha,\beta)$ and $\left(\frac{1-t}{2}\right)x_1^*(\alpha,\beta) + \left(\frac{1+t}{2}\right)x_4^*(\alpha,\beta)$ gives immediately (5.6) and (5.7). □

6.7. Proof of Theorem 5.5

Proof. According to (5.7), the node $x_3^* = x_3^*(x_1^*, x_4^*) = \left(\frac{1-t}{2}\right)x_1^* + \left(\frac{1+t}{2}\right)x_4^*$ is a bivariate polynomial which is linear in both variables x_1^* and x_4^*. Hence $x_3^*(x_1^*, x_4^*)$ will attain its extreme values on the boundary of the ranges of x_1^* and x_4^*, so that it suffices to investigate the values of $x_3^*(x_1^*, x_4^*)$ at five points, see (5.4), (5.5) and also Figure 5:

$$x_3^*\left(-1, \frac{3-b}{b+1}\right) = \frac{2t-b+1}{b+1} = 0.3877375269\ldots \tag{6.15}$$

$$x_3^*(-1,1) = t = 0.4177913013\ldots \tag{6.16}$$

$$x_3^*\left(\frac{-1}{b}, \frac{1}{b}\right) = \frac{t}{b} = 0.4004466202\ldots \tag{6.17}$$

$$x_3^*\left(\frac{-1}{b},1\right)=\frac{-1+b+bt+t}{2b}=0.4298765508\ldots \tag{6.18}$$

$$x_3^*\left(\frac{b-3}{b-1},1\right)=\frac{2t+b-1}{b+1}=0.4301327291\ldots\,. \tag{6.19}$$

Obviously, (6.15) is the minimum and (6.19) is the maximum of $x_3^*=x_3^*(x_1^*,x_4^*)$ as claimed in (5.9). The verification of (5.8) follows similar lines and will be left to the reader. □

7. Concluding remarks

Remark 7.1. Let $n=4$. According to (4.7), the largest possible value for the first optimal interpolation node in **I** is $x_1^*=-\frac{1}{b}=-0.9584848200\ldots$, if we consider zero-symmetric node systems. But if we allow arbitrary node configurations, then we can get beyond this number: the largest possible value for the first optimal interpolation node in **I** is in fact $x_1^*=\frac{b-3}{b+1}=-0.9576047978\ldots$, see (5.5).

Remark 7.2. According to section 6.6, Schurer's description in [27, Theorem 1] of the optimal arbitrary node systems $X_3^*: x_1^*<x_2^*<x_3^*$ for quadratic Lagrange interpolation on **I** can be restated almost verbatim in the form as given in our Theorem 5.4, if we replace in (5.4) and (5.5) x_4^* by x_3^* and if we replace the constant $b=1.0433133411\ldots$ by the corresponding constant $b^\circ=\frac{3}{2\sqrt{2}}=1.0606601717\ldots$. For example, (5.4) would then read

$$-1\le x_1^*\le -\frac{1}{b^\circ}(\approx -0.9428)\wedge\left(\frac{b^\circ-1}{b^\circ+1}\right)x_1^*+\frac{2}{b^\circ+1}\le x_3^*\le 1 \tag{7.1}$$

which is identical with

$$-1\le x_1^*\le -\frac{2\sqrt{2}}{3}(\approx -0.9428)\wedge(17-12\sqrt{2})x_1^*+12\sqrt{2}-16\le x_3^*\le 1 \tag{7.2}$$

as given in [27]. Note that in the quadratic case we have $x_2^*=\frac{x_1^*+x_3^*}{2}$.

Remark 7.3. The proof of Theorem 5.4 rests on Theorem 5.2. We have also found an alternative, computer-aided proof of Theorem 5.4 which avoids Theorem 5.2. It uses quantifier elimination and can be compared with section 3.5 in [25]. However, to reduce computational complexity, this automated proof requires a reduction of the variables by means of

$$x_2^*=x_1^*+x_4^*-x_3^*. \tag{7.3}$$

This equation, which is of some interest in itself, follows readily from Theorem 5.2, but, to stay independent of that theorem, we had to establish an alternative new proof for (7.3).

Remark 7.4. Suppose one chooses $x_1^*=c$ (constant) and $x_4^*=d$ (constant) from the indicated ranges (5.4) respectively (5.5) in Theorem 5.4. Then one gets, in generalizing Example 5.6, $\alpha=\frac{-2-c-d}{d-c}$ and $\beta=\frac{2-c-d}{d-c}$, and hence $x_2^*=\frac{c+d+t(c-d)}{2}$ and $x_3^*=\frac{c+d+t(d-c)}{2}$, according to Theorem 5.2, or directly from (5.6) and (5.7).

Remark 7.5. The natural question arises: How to determine the minimal Lebesgue constant Λ_n^* and all optimal node systems X_n^* in $\mathbf{I}$ for the next cases $n = 5, 6, 7, \ldots$? We have achieved some progress for $n = 5$ and $n = 6$. For example, we have implicitly determined the Lebesgue constants Λ_5^* and Λ_6^* by symbolic computation as roots of certain high-degree polynomials with integer coefficients. To be more specific, for $n = 5$ (quartic case) we have obtained

$$\Lambda_5^* = 1.5594902098\ldots \tag{7.4}$$

as a root of a polynomial of degree 73 with integer coefficients. This polynomial has the contour

$$P_{73}^*(x) = \tag{7.5}$$

$$4919208440669185186769320586798345151056312259772478803766113281 25 + \ldots$$

$$+1415665151043813144584984996241786441414714228396379218167033600 4096\, x^{73}.$$

We intend to expose our findings in a separate manuscript, see also [32].

Remark 7.6. That the topic of optimal cubic Lagrange interpolation awakens interest in the reader may be deduced from the fact that the online-version of [23] on the publishers website has received more than 60 article views subject to charge, see `http://www.tandfonline.com/doi/abs/10.1080/0020739840150312`

Remark 7.7. The desire to precisely determine the values of interesting constants (here: b, t, Λ_4^*) is reflected on in [12, p. 79].

Remark 7.8. In [10, p. 70] it is erroneously claimed that an optimal node system in $\mathbf{I}$ must necessarily be a CNS.

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Estimates for the ratio of gamma functions by using higher order roots

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Abstract. It is the aim of this paper to give a systematically way for obtaining higher order roots estimates of the ratio $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$, as $x \to \infty$ and the Wallis ratio $\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$, as $n \to \infty$.

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1. Introduction

The factorial function $n! = 1\cdot 2\cdot 3\cdots n$ (defined for positive integers n), and its extension gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

(to the real and complex values z, excepting $-1, -2, -3, ...$) has a great importance in pure mathematics, as in applied mathematics and other branches of science, such as chemistry, statistical physics, or cuantum mechanics.

The ratio $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$ is strongly related to the Wallis sequence

$$P_n = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$$

and to other aspects in the theory of the gamma function, as for example Kershaw-Gautschi inequalities. For this reason, many mathematicians have been preocuppied by the approximation of this ratio. There exists a broad literature on this subject. In particular, many inequalities, sharp bounds for these functions, and accurate approximations have been published. See, e.g. the classical results from [2] and the recent

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article [3] and all references therein. A first result was stated by Kazarinoff [4, pp. 47-48 and pp. 65-67]:

$$\sqrt{n+\frac{1}{4}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{2}},$$

then this result was improved by Chu [3]:

$$\sqrt{n+\frac{1}{4}-\frac{1}{(4n-2)^2}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{4}+\frac{1}{16n-4}},$$

and then by Boyd [1] and Slavič [23] as:

$$\sqrt{n+\frac{1}{4}+\frac{1}{32n+32}} < \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \sqrt{n+\frac{1}{4}+\frac{1}{32n-\frac{64n-148}{8n+11}}}.$$

Motivated by these formulas, Mortici [5] proposed the following approximations family:

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt[2k]{P_k(n)}, \tag{1.1}$$

where $P_k(n)$ is a polynomial of kth order (the notation "$f(n) \approx g(n)$" means that the ratio $f(n)/g(n)$ tends to 1, as n approaches infinity). Mortici calculated in [5] the first approximations as $n \to \infty$:

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt[4]{n^2+\frac{1}{2}n+\frac{1}{8}}$$

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt[6]{n^3+\frac{3}{4}n^2+\frac{9}{32}n+\frac{5}{128}}$$

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt[8]{n^4+n^3+\frac{1}{2}n^2+\frac{1}{8}n}.$$

In [5, p. 427] it is shown that these approximations are increasingly accurate as the root order grows.

Mortici used an original method, however, this method doesn't allow us to determine the general formula of this approximation.

The aim of this paper is to give a systematically method for obtaining the approximations (1.1) for any order $2k$.

The method we propose is related to the theory of asymptotic series and it is inspired from a recent result of Chen and Lin [2].

2. The theoretical results

The asymptotic theory is a strong tool for improving and obtaining new approximation formulas.

Let $f:(0,\infty)\to\mathbb{R}$ be a function. We say that $\sum\limits_{k=1}^{\infty}\frac{\alpha_k}{x^k}$ is an asymptotic series expansion for $f(x)$ as $x\to\infty$, and denote

$$f(x)\sim\sum_{k=1}^{\infty}\frac{\alpha_k}{x^k}\qquad\text{as}\ \ x\to\infty,$$

if for all $m\in\mathbb{N}^*$

$$f(x)-\sum_{k=1}^{m}\frac{\alpha_k}{x^k}=\mathcal{O}\left(\frac{1}{x^{m+1}}\right)\qquad\text{as}\ \ x\to\infty.$$

For a positive function f we write

$$f(x)\sim\exp\left\{\sum_{k=1}^{m}\frac{\alpha_k}{x^k}\right\}\qquad\text{as}\ \ x\to\infty,$$

if for all $m\in\mathbb{N}^*$

$$\ln f(x)-\sum_{k=1}^{m}\frac{\alpha_k}{x^k}=\mathcal{O}\left(\frac{1}{x^{m+1}}\right)\qquad\text{as}\ \ x\to\infty.$$

Using the idea first presented by Chen and Lin in [2], we give the following theorem:

Theorem 2.1. *If the function f has the asymptotic expansion as $x\to\infty$:*

$$f(x)\sim\exp\left\{\sum_{k=1}^{\infty}\frac{\alpha_k}{x^k}\right\}\quad(x>0),$$

then

$$f(x)\sim\sqrt[r]{1+\sum_{j=1}^{\infty}\frac{b_j}{x^j}}\quad(r,x>0),$$

where

$$b_j=\sum_{k_1+2k_2+\ldots+jk_j=j}\frac{r^{k_1+k_2+\ldots+k_j}}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_j^{k_j}.$$

Proof. This proof is based on the ideas of Chen and Lin presented in [2]. We have

$$f(x)=\exp\left\{\sum_{k=1}^{m}\frac{\alpha_k}{x^k}+R_m(x)\right\},$$

where

$$R_m(x)=\mathcal{O}\left(\frac{1}{x^{m+1}}\right).$$

Thus

$$\begin{aligned}
[f(x)]^r &= e^{r\cdot R_m(x)} \cdot \exp\left\{\sum_{k=1}^{m} \frac{r\alpha_k}{x^k}\right\} \\
&= e^{r\cdot R_m(x)} \prod_{k=1}^{m} \left\{1 + \frac{r\alpha_k}{x^k} + \frac{1}{2!}\cdot\left(\frac{r\alpha_k}{x^k}\right)^2 + \ldots\right\} \\
&= e^{r\cdot R_m(x)} \sum_{k_1,k_2,..k_m=0}^{\infty} \frac{1}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\left(\frac{r\alpha_1}{x}\right)^{k_1}\cdot\left(\frac{r\alpha_2}{x^2}\right)^{k_2}\cdot\ldots\cdot\left(\frac{r\alpha_m}{x^m}\right)^{k_m} \\
&= e^{r\cdot R_m(x)} \sum_{k_1,k_2,..k_m=0}^{\infty} \frac{r^{k_1+k_2+\ldots+k_m}}{k_1!\cdot k_2!\cdot\ldots\cdot k_m!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_m^{k_m}\cdot\frac{1}{x^{k_1+2k_2+\ldots+mk_m}} \\
&= 1 + \sum_{j=1}^{\infty} \frac{b_j}{x^j}
\end{aligned}$$

where

$$b_j = \sum_{k_1+2k_2+\ldots+jk_j=j} \frac{r^{k_1+k_2+\ldots+k_j}}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_j^{k_j}$$

The proof is now completed.□

In [23], Slavič gave the following integral representation for every $x > 0$:

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\exp\left\{\sum_{k=1}^{n} \frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}} \cdot \int_0^{\infty}\left[\frac{\tanh t}{2t} - \sum_{k=1}^{n} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{k(2k)!}t^{2k-2}\right]e^{-4/x}dt\right\}$$

from which, a more accurate double inequality was established:

$$\sqrt{x}\exp\left(\sum_{k=1}^{2m} \frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}\right) < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\exp\left(\sum_{k=1}^{2l-1} \frac{\left(1-2^{-2k}\right)B_{2k}}{k(2k-1)x^{2k-1}}\right)$$

for $x > 0$. Here m and l are any natural numbers and B_{2k} for $k \in N$ are Bernoulli numbers defined by the generating function

$$\frac{t}{e^t-1} = \sum_{j=0}^{\infty} \frac{B_j}{j!}t^j \quad (|t| < 2\pi)\,.$$

The following asymptotic formula is presented in [23], as $x \to \infty$:

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\exp\left\{\sum_{j=1}^{\infty} \frac{\left(1-2^{-2j}\right)B_{2j}}{j(2j-1)x^{2j-1}}\right\},$$

which is equivalent to

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\exp\left\{\sum_{k=1}^{\infty}\frac{\left(2-2^{-k}\right)B_{k+1}}{k(k+1)x^k}\right\} \tag{2.1}$$

(in the last formula, the terms involving $B_{2j+1}=0$ were added, for sake of symmetry).

3. Approximations for $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$

By applying Theorem 1 to the function

$$f(x)=\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)} \quad (x>0). \tag{3.1}$$

with the coefficients of the asymptotic series

$$\alpha_k=\frac{\left(2-2^{-k}\right)B_{k+1}}{k(k+1)}, \tag{3.2}$$

see (2.1), and then replacing r by $2r$, we obtain:

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)}\right)^{2r} \sim 1+\sum_{j=1}^{\infty}\frac{b_j}{x^j},$$

where

$$b_j=\sum_{k_1+2k_2+\ldots+jk_j=j}\frac{(2r)^{k_1+k_2+\ldots+k_j}}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_j^{k_j}. \tag{3.3}$$

Then, we deduce that

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[2r]{x^r+b_1x^{r-1}+\ldots+b_{r-1}x+b_r}$$

where $b_1, b_2, \ldots b_r$ are given in (3.3). Concrete values are presented below:

$$r=1 \Rightarrow b_1=\frac{1}{4} \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt{x+\frac{1}{4}}$$

$$r=2 \Rightarrow b_1=\frac{1}{2}, b_2=\frac{1}{8} \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[4]{x^2+\frac{1}{2}x+\frac{1}{8}}$$

$$r=3 \Rightarrow b_1=\frac{3}{4}, b_2=\frac{9}{32}, b_3=\frac{5}{128} \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[6]{x^3+\frac{3}{4}x^2+\frac{9}{32}x+\frac{5}{128}}$$

$$r=4 \Rightarrow b_1=1, b_2=\frac{1}{2}, b_3=\frac{1}{8}, b_4=0 \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[8]{x^4+x^3+\frac{1}{2}x^2+\frac{1}{8}x}$$

$$\begin{aligned} r &= 5 \Rightarrow b_1=\frac{5}{4}, b_2=\frac{25}{32}, b_3=\frac{35}{128}, b_4=\frac{75}{2048}, b_5=\frac{3}{8192} \\ &\Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \approx \sqrt[10]{x^5+\frac{5}{4}x^4+\frac{25}{32}x^3+\frac{35}{128}x^2+\frac{75}{2048}x+\frac{3}{8192}} \end{aligned}$$

4. Approximations for Wallis ratio

Let us now apply once again Theorem 1 to the function f given by (3.1), with α_k given by (3.2). Now we replace r by $-2r$ to obtain:

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)}\right)^{-2r} \sim 1+\sum_{j=1}^{\infty}\frac{b'_j}{x^j}$$

where

$$b'_j = \sum_{k_1+2k_2+\ldots+jk_j=j} \frac{(-2r)^{k_1+k_2+\ldots+k_j}}{k_1!\cdot k_2!\cdot\ldots\cdot k_j!}\cdot\alpha_1^{k_1}\cdot\ldots\cdot\alpha_j^{k_j}. \tag{4.1}$$

Hence

$$\left(\frac{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}\right)^{2r} \sim 1+\sum_{j=1}^{\infty}\frac{b'_j}{x^j}$$

where $b_1, b_2, ...b_r$ are given in (4.1). Furthermore, we obtain:

$$\left(\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}\right)^{2r} \sim \frac{1}{x^r}+\sum_{j=1}^{\infty}\frac{b'_j}{x^{j+r}}$$

and therefore

$$\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} \sim \sqrt[2r]{\frac{1}{x^r}+\frac{b'_1}{x^{r+1}}+\frac{b'_2}{x^{r+2}}+\ldots}.$$

Using this result, we obtain the following asymptotic expansion for the Wallis sequence, using the relation:

$$P_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}. \tag{4.2}$$

We get

$$P_n \approx \frac{1}{\sqrt{\pi}}\sqrt[2r]{\frac{1}{n^r}+\frac{b'_1}{n^{r+1}}+\frac{b'_2}{n^{r+2}}+\ldots},$$

which is equivalent to

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt[2r]{1+\frac{b'_1}{n}+\frac{b'_2}{n^2}+\ldots}.$$

We present the following particular cases:

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt{1-\frac{1}{4n}}$$

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt[4]{1-\frac{1}{2n}+\frac{1}{8n^2}}$$

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt[6]{1-\frac{3}{4n}+\frac{9}{32n^2}-\frac{5}{128n^3}}$$

$$P_n \approx \frac{1}{\sqrt{n\pi}}\sqrt[8]{1-\frac{1}{n}+\frac{1}{2n^2}-\frac{1}{8n^3}}$$

$$P_n \approx \frac{1}{\sqrt{n\pi}} \sqrt[10]{1 - \frac{5}{4n} + \frac{25}{32n^2} - \frac{35}{128n^3} + \frac{75}{2048n^4} - \frac{3}{8192n^5}} := \delta_n. \tag{4.3}$$

5. Conclusions

Mortici's formula stated in [5]:

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt[8]{n^4 + n^3 + \frac{1}{2}n^2 + \frac{1}{8}n}$$

can be rewritten using (4.2) in the form

$$P_n \approx \frac{1}{\sqrt{\pi}\sqrt[8]{n^4 + n^3 + \frac{1}{2}n^2 + \frac{1}{8}n}} := \mu_n. \tag{5.1}$$

Our formula (4.3) gives results of the same order of accuracy with Mortici's formula (5.1). A comparison table is given below:

n	$\lvert P_n - \mu_n \rvert$	$\lvert P_n - \delta_n \rvert$
10	1.4655×10^{-10}	1.8666×10^{-10}
50	4.8252×10^{-15}	4.8432×10^{-15}
100	5.4202×10^{-17}	5.2798×10^{-17}
200	6.0379×10^{-19}	5.7940×10^{-19}
500	1.5718×10^{-21}	1.4948×10^{-21}
1000	1.7395×10^{-23}	1.6493×10^{-23}

The formula (4.3) can be equivalently written in terms of gamma function as follows:

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \approx \frac{1}{\sqrt{n}} \sqrt[10]{1 - \frac{5}{4n} + \frac{25}{32n^2} - \frac{35}{128n^3} + \frac{75}{2048n^4} - \frac{3}{8192n^5}}.$$

The associated function satisfies the following properties:

Theorem 5.1. *The function* $\varphi : [2, \infty) \to \mathbb{R}$, *defined by*

$$\begin{aligned} \varphi(x) &= \ln\Gamma\left(x+\frac{1}{2}\right) - \ln\Gamma(x+1) + \frac{1}{2}\ln x \\ &+ \frac{1}{10}\ln\left(1 - \frac{5}{4x} + \frac{25}{32x^2} - \frac{35}{128x^3} + \frac{75}{2048x^4} - \frac{3}{8192x^5}\right) \end{aligned}$$

is monotonically increasing and concave.

The proof of this theorem is now classical. The same method was used by Chen and Lin, or Mortici in some of their papers. See, *e.g.*, [2], [6]-[22]. We omit the proof for sake of simplicity.

As

$$\varphi(2) = \ln\frac{3}{4}\sqrt{\frac{\pi}{2}} + \frac{1}{10}\ln\frac{141\,141}{262\,144} := \tau$$

(numerically $\tau = -0.1238\cdots$) and $\lim_{x\to\infty} \varphi(x) = 0$, we deduce that

$$\tau \leq \varphi(x) < 0 \qquad (x \in \mathbb{R};\ x \geq 2).$$

By exponentiating this double inequality, we get the following result:

Theorem 5.2. *The following double inequality holds true, for every real number* $x \geq 2$:

$$\begin{aligned} & \frac{\beta}{\sqrt{x}} \sqrt[10]{1 - \frac{5}{4x} + \frac{25}{32x^2} - \frac{35}{128x^3} + \frac{75}{2048x^4} - \frac{3}{8192x^5}} \\ \leq & \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} \\ < & \frac{\alpha}{\sqrt{x}} \sqrt[10]{1 - \frac{5}{4x} + \frac{25}{32x^2} - \frac{35}{128x^3} + \frac{75}{2048x^4} - \frac{3}{8192x^5}}. \end{aligned}$$

The constants

$$\begin{aligned} \alpha &= 1.0000 \\ \beta &= e^{\tau} = \frac{3}{4}\sqrt{\frac{\pi}{2}} \cdot \sqrt[10]{\frac{141\,141}{262\,144}} = 0.8835\cdots \end{aligned}$$

are sharp.

Further studies on ratio of gamma functions are highly motivated since a deep knowledge of the quotient $\Gamma(x+a)/\Gamma(x+b)$ $(a, b \in \mathbb{R};\ x \to \infty)$ is required in many problems, such as the theory of Mellin-Barnes integrals, the theory of the generalized weighted mean values, or in the theory of hypergeometric functions.

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Weighted Ostrowski-Grüss type inequalities

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Abstract. Several inequalities of Ostrowski-Grüss-type availabe in the literature are generalized considering the weighted case of them. Involving the least concave majorant of the modulus of continuity we provide upper bounds of our inequalities.

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1. Introduction

Over the last decades, integral inequalities have attracted much attention because of their applications in statistical analysis and the theory of distributions. In this paper we improve the classical Ostrowski type inequality for weighted integrals and generalize some Ostrowski-Grüss type inequalities involving differentiable mappings. Also, applications to special weight functions are investigated.

For each $x \in [a,b]$ consider the linear functional

$$\mathcal{L}(f)(x) := f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right), f \in C[a,b].$$

If $f : [a,b] \to \mathbb{R}$ is differentiable with bounded derivative, then

$$\begin{aligned} |\mathcal{L}(f)(x)| &\leq \frac{1}{8}(b-a)(\Gamma-\gamma) &(1.1)\\ &\leq \frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma) &(1.2)\\ &\leq \frac{1}{4}(b-a)(\Gamma-\gamma), &(1.3)\end{aligned}$$

where $\gamma := \inf\{f'(x) | x \in [a,b]\}$ and $\Gamma := \sup\{f'(x) | x \in [a,b]\}$.

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The inequality (1.3) was proven by S.S. Dragomir and S. Wang [3] and it is known as the Ostrowski-Grüss-type inequality. This inequality was improved by M. Matić et al. [6], and we recall their result in (1.2). An improvement of this result was given by X.L. Cheng in [2], as shown in relation (1.1). He also proved that the constant 1/8 is best possible.

In [5] the authors introduced the linear functional $\mathcal{L}_c : C[a,b] \to \mathbb{R}$ defined by

$$\mathcal{L}_c(f)(x) := f(x) - \frac{1}{b-a}\int_a^b f(t)dt - c\cdot\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right), \text{ where } c \geq 0.$$

The following result gives bounds of the functional $\mathcal{L}_c$ involving differences of upper and lower bounds of first order derivatives.

Theorem 1.1. [5] *For all $x \in [a,b]$, $c \in [0,2]$ and $f \in C^1[a,b]$ we have*

$$\frac{1}{2(b-a)}\left[(x-a-u_c(x))^2\gamma - (x-b-u_c(x))^2\Gamma\right] \leq \mathcal{L}_c(f)(x) \tag{1.4}$$

$$\leq \frac{1}{2(b-a)}\left[(x-a-u_c(x))^2\Gamma - (x-b-u_c(x))^2\gamma\right],$$

where $u_c(x) := c\left(x - \frac{a+b}{2}\right)$.

Remark 1.2. a) From (1.4), with $c = 1$, inequality (1.1) follows which was established by X.L. Cheng in [2].

b) As a consequence of (1.4), for $c = 0$, the following inequality holds:

$$-\frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty \leq \frac{(x-a)^2\gamma-(b-x)^2\Gamma}{2(b-a)} \leq f(x) - \frac{1}{b-a}\int_a^b f(t)dt$$

$$\leq \frac{(x-a)^2\Gamma-(b-x)^2\gamma}{2(b-a)} \leq \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty.$$

This inequality improves the classical Ostrowski inequality presented by Anastassiou in [1] in the form

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty.$$

Weighted versions of (1.3), (1.2) and (1.1) were established by J. Roumeliotis in [9] and [10]. These results are given below.

Definition 1.3. *Let $w : (a,b) \to (0,\infty)$ be integrable, i.e., $\int_a^b w(t)dt < \infty$, then $m(\alpha,\beta) := \int_\alpha^\beta w(t)dt$ and $M(\alpha,\beta) := \int_\alpha^\beta tw(t)dt$ are the first moments, for $[\alpha,\beta] \subseteq [a,b]$. Define the mean of the interval $[\alpha,\beta]$ with respect to the weight function w as $\sigma(\alpha,\beta) := \frac{M(\alpha,\beta)}{m(\alpha,\beta)}$.*

The weighted variant of the functional $\mathcal{L}$ can be written in the following way:

$$\mathcal{L}_w(f)(x) := f(x) - \frac{1}{m(a,b)} \int_a^b f(t)w(t)dt - \frac{f(b)-f(a)}{b-a}(x - \sigma(a,b)).$$

Theorem 1.4. [9] *Let $f : [a,b] \to \mathbb{R}$ be differentiable with bounded derivative and let $w : (a,b) \to (0,\infty)$ be integrable. Then*

$$\begin{aligned}|\mathcal{L}_w(f)(x)| &\leq \frac{1}{2}(\Gamma - \gamma)\frac{\sqrt{b-a}}{m(a,b)}\left\{\int_a^b K^2(x,t)dt - \frac{m(a,b)^2(x-\sigma(a,b))^2}{b-a}\right\}^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Gamma - \gamma)(b-a),\end{aligned}$$

for all $x \in [a,b]$, where $K(x,t) = \begin{cases} \int_a^t w(u)du, a \leq t \leq x \\ \int_b^t w(u)du, x < t \leq b. \end{cases}$

Theorem 1.5. [10] *Let $f : [a,b] \to \mathbb{R}$ be differentiable with bounded derivative and let $w : (a,b) \to (0,\infty)$ be integrable. Then*

$$|\mathcal{L}_w(f)(x)| \leq \frac{\Gamma - \gamma}{m(a,b)} \int_x^{t^*} (t-x)w(t)dt, \tag{1.5}$$

for all $x \in [a,b]$, where $t^* \in [a,b]$ is unique and verifies

$$\frac{m(a,b)}{b-a}|x - \sigma(a,b)| = \begin{cases} m(t^*,b), a \leq x \leq \sigma(a,b) \\ m(a,t^*), \sigma(a,b) < x \leq b. \end{cases}$$

For each $x \in [a,b]$ consider the linear functional $\mathcal{L}_{w,c} : C[a,b] \to \mathbb{R}$ defined by

$$\mathcal{L}_{w,c}(f)(x) := f(x) - \frac{1}{m(a,b)} \int_a^b f(t)w(t)dt - c \cdot \frac{f(b)-f(a)}{b-a}(x - \sigma(a,b)),$$

where $c \geq 0$.

In this paper we propose the weighted analogue of (1.4). Also, new inequalities of $\mathcal{L}_{w,c}$ will be considered involving the least concave majorants of the first order moduli of continuity.

2. Generalized Ostrowski-Grüss type inequalities

In this section we will give the upper bounds of $\mathcal{L}_{w,c}$ involving differences of upper and lower bounds of first order derivatives. First, we need the following lemma.

Lemma 2.1. *For $c \in [0,1]$ and $x \in [0,1]$, there exists a unique $t^* = t^*(x) \in [a,b]$ satisfying*

$$u_{w,c}(x) = \begin{cases} \dfrac{1}{m(a,b)} \displaystyle\int_b^{t^*} w(u)du, \ a \leq x \leq \sigma(a,b) \\ \dfrac{1}{m(a,b)} \displaystyle\int_a^{t^*} w(u)du, \ \sigma(a,b) < x \leq b, \end{cases}$$

where $u_{w,c}(x) = \dfrac{c}{b-a}(x - \sigma(a,b))$.

Proof. Let us consider $a \le x \le \sigma(a,b)$ and
$$f(t) = \frac{1}{m(a,b)} \int_b^t w(u)du - u_{w,c}(x), t \in [x,b].$$
Since f is strictly increasing on (x,b), $f(b) = -u_{w,c}(x) \ge 0$, then to show that $t^* \in [x,b]$ exists, it will suffice to establish that $f(x) \le 0$, where
$$f(x) = \frac{1}{m(a,b)} \int_b^x w(u)du - u_{w,c}(x).$$
It follows
$$\begin{aligned} u_{w,c}(x) &= \frac{c}{b-a}\left(x - \sigma(a,b)\right) = \frac{c}{b-a}\left(x - \frac{M(a,b)}{m(a,b)}\right) \\ &= -\frac{c}{(b-a)m(a,b)} \int_a^b (t-x)w(t)dt \\ &= -\frac{c}{(b-a)m(a,b)} \left\{ \int_a^x (t-x)w(t)dt + \int_x^b (t-x)w(t)dt \right\} \\ &\ge -\frac{c}{(b-a)m(a,b)} \int_x^b (t-x)w(t)dt \ge -\frac{c}{(b-a)m(a,b)}(b-x) \int_x^b w(t)dt \\ &\ge -\frac{c}{m(a,b)} \int_x^b w(t)dt \ge \frac{1}{m(a,b)} \int_b^x w(t)dt. \end{aligned}$$
In a similar way for $\sigma(a,b) < x \le b$ follows that there exists a unique $t^* \in [a,x]$ such that $u_{w,c}(x) = \frac{1}{m(a,b)} \int_a^{t^*} w(u)du$. □

Denote
$$\mathcal{P}(x,t) = \begin{cases} \frac{1}{m(a,b)} \int_a^t w(u)du - u_{w,c}(x), a \le t < x, \\ \frac{1}{m(a,b)} \int_b^t w(u)du - u_{w,c}(x), x \le t \le b. \end{cases}$$
It is easy to verify that, for all $f \in C^1[a,b]$, $\mathcal{L}_{w,c}(f)(x) = \int_a^b \mathcal{P}(x,t)f'(t)dt$.

Theorem 2.2. *For all $x \in [a,b], c \in [0,1]$ and $f \in C^1[a,b]$ we have*
$$\begin{aligned} (1-c)(x-\sigma(a,b))\gamma + (\gamma - \Gamma)\nu(x,t^*) &\le \mathcal{L}_{w,c}(f)(x) \\ &\le (1-c)(x-\sigma(a,b))\Gamma + (\Gamma-\gamma)\nu(x,t^*), \end{aligned} \tag{2.1}$$
where $\nu(x,t^*) := \frac{1}{m(a,b)} \int_x^{t^*} (t-x)w(t)dt$.

Proof. If $a \le x \le \sigma(a,b)$, then
$$\mathcal{P}(x,t) \ge 0, \text{ for } t \in [a,x) \cup [t^*,b] \text{ and } \mathcal{P}(x,t) < 0, \text{ for } t \in [x,t^*).$$
Also, if $\sigma(a,b) < x \le b$, it follows
$$\mathcal{P}(x,t) \le 0, \text{ for } t \in [a,t^*] \cup [x,b] \text{ and } \mathcal{P}(x,t) > 0, \text{ for } t \in (t^*,x).$$

Let $a \le x \le \sigma(a,b)$. It follows

$$\mathcal{L}_{w,c}(f)(x) \le \Gamma \left(\int_a^x \mathcal{P}(x,t)dt + \int_{t^*}^b \mathcal{P}(x,t)dt \right) + \gamma \int_x^{t^*} \mathcal{P}(x,t)dt.$$

We have

$$\int_a^x \mathcal{P}(x,t)dt + \int_{t^*}^b \mathcal{P}(x,t)dt = \int_a^x \left(\frac{1}{m(a,b)} \int_a^t w(u)du - u_{w,c}(x) \right) dt$$
$$+ \int_{t^*}^b \left(\frac{1}{m(a,b)} \int_b^t w(u)du - u_{w,c}(x) \right) dt$$
$$= \frac{1}{m(a,b)} \left\{ x \int_a^x w(u)du - \int_a^x tw(t)dt - t^* \int_b^{t^*} w(u)du - \int_{t^*}^b tw(t)dt \right\}$$
$$- u_{w,c}(x)(x-a+b-t^*) = \frac{1}{m(a,b)} \left\{ x \int_a^x w(u)du - \int_a^x tw(t)dt - \int_{t^*}^b tw(t)dt \right\}$$
$$- u_{w,c}(x)(x-a+b) + t^* \left(u_{w,c}(x) - \frac{1}{m(a,b)} \int_b^{t^*} w(u)du \right)$$
$$= \frac{1}{m(a,b)} \left\{ x \int_a^x w(u)du - \int_a^x tw(t)dt - (x-a+b) \int_b^{t^*} w(u)du - \int_{t^*}^b tw(t)dt \right\}$$
$$= \frac{1}{m(a,b)} \left\{ \int_{t^*}^b (x-a+b-t)w(t)dt + \int_a^x (x-t)w(t)dt \right\}$$
$$= \frac{1}{m(a,b)} \left\{ (b-a) \int_{t^*}^b w(t)dt + \int_a^b (x-t)w(t)dt - \int_x^{t^*} (x-t)w(t)dt \right\}$$
$$= \frac{1}{m(a,b)} \left\{ -(b-a)m(a,b)u_{w,c}(x) + xm(a,b) - M(a,b) + \int_x^{t^*} (t-x)w(t)dt \right\}$$
$$= (1-c)(x-\sigma(a,b)) + \frac{1}{m(a,b)} \int_x^{t^*} (t-x)w(t)dt.$$

$$\int_x^{t^*} \mathcal{P}(x,y)dt = \int_x^{t^*} \left\{ \frac{1}{m(a,b)} \int_b^t w(u)du - u_{w,c}(x) \right\} dt$$
$$= \frac{1}{m(a,b)} \left\{ t^* \int_b^{t^*} w(u)du - x \int_b^x w(u)du - \int_x^{t^*} tw(t)dt \right\} - u_{w,c}(x)(t^* - x)$$
$$= t^* \left\{ \frac{1}{m(a,b)} \int_b^{t^*} w(u)du - u_{w,c}(x) \right\} - \frac{1}{m(a,b)} \left\{ x \int_b^x w(u)du + \int_x^{t^*} tw(t)dt \right\}$$
$$+ x \cdot \frac{1}{m(a,b)} \int_b^{t^*} w(u)du = -\frac{1}{m(a,b)} \int_x^{t^*} (t-x)w(t)dt = -\nu(x,t^*).$$

Therefore,

$$\begin{aligned}\mathcal{L}_{w,c}(f)(x) &\leq \Gamma\left[(1-c)(x-\sigma(a,b))+\nu(x,t^*)\right]-\gamma\nu(x,t^*) \\ &= (1-c)(x-\sigma(a,b))\Gamma+(\Gamma-\gamma)\nu(x,t^*).\end{aligned} \tag{2.2}$$

By similar reasoning it follows that (2.2) is also valid if $\sigma(a,b)\leq x\leq b$.

If we write (2.2) for $-f$ instead of f, we obtain

$$\mathcal{L}_{w,c}(f)(x)\geq(1-c)(x-\sigma(a,b))\gamma+(\gamma-\Gamma)\nu(x,t^*).$$ □

Corollary 2.3. *For all $x\in[a,b]$ and $f\in C^1[a,b]$, we obtain*

$$|\mathcal{L}_w(f)(x)|\leq\frac{\Gamma-\gamma}{m(a,b)}\int_x^{t^*}(t-x)w(t)dt. \tag{2.3}$$

Proof. The inequality (2.3) follows from (2.1) with $c=1$. □

Remark 2.4. The coefficient $\frac{1}{m(a,b)}\int_x^{t^*}(t-x)w(t)dt$ is sharp in the sense that it cannot be replaced by a smaller one. The inequality (2.3) holds for all $x\in[a,b]$. Let $x=\sigma(a,b)$ and

$$f(t)=\begin{cases}\Gamma(t-a),\ a\leq t<\sigma(a,b)\\ \Gamma\left(\sigma(a,b)-a\right)+\gamma\left(t-\sigma(a,b)\right),\ \sigma(a,b)\leq t\leq b.\end{cases}$$

If $x=\sigma(a,b)$, then $t^*=b$ and it follows

$$\mathcal{L}_w(f)(\sigma(a,b))=\frac{\Gamma-\gamma}{m(a,b)}\int_{\sigma(a,b)}^{b}\left(t-\sigma(a,b)\right)w(t)dt.$$

Therefore, in (2.3) equality holds.

Corollary 2.5. *For all $x\in[a,b]$ and $f\in C^1[a,b]$, the following inequality holds*

$$\begin{aligned}&-\frac{\|f'\|_\infty}{m(a,b)}\int_a^b|t-x|w(t)dt\leq(x-\sigma(a,b))\gamma+(\gamma-\Gamma)\nu(x,t^*) \\ &\leq f(x)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt \\ &\leq(x-\sigma(a,b))\Gamma+(\Gamma-\gamma)\nu(x,t^*)\leq\frac{\|f'\|_\infty}{m(a,b)}\int_a^b|t-x|w(t)dt.\end{aligned} \tag{2.4}$$

Proof. This result is a consequence of (2.1), for $c=0$. □

Remark 2.6. a) Inequality (2.3) was established by J. Roumeliotis [10].

b) Inequality (2.4) improves the classical weighted Ostrowski inequality

$$\left|f(x)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right|\leq\|f'\|_\infty\cdot\frac{1}{m(a,b)}\int_a^b|t-x|w(t)dt.$$

3. Ostrowski-Grüss-type inequalities in terms of the least concave majorant

The aim of this section is to extend the inequalities mentioned in the previous section, by using the least concave majorant of the modulus of continuity. This approach was inspired by a paper of Gavrea & Gavrea [4] who were the first to observe the possibility of using moduli in this context.

Proposition 3.1. *The linear functional* $\mathcal{L}_{w,c} : C[a,b] \to \mathbb{R}$ *satisfies*

i) $|\mathcal{L}_{w,c}(f)(x)| \leq 4\|f\|_\infty$, *for all* $f \in C[a,b]$.

ii) $|\mathcal{L}_{w,c}(f)(x)| \leq [(c-1)\,|x-\sigma(a,b)| + 2\nu(x,t^*)]\,\|f'\|_\infty$, *for all* $f \in C^1[a,b]$, *where* $c \in [0,1]$ *and* ν *is defined in Theorem* 2.2.

Proof. Inequality i) follows immediately from definition of $\mathcal{L}_{w,c}$. The second inequality is obtained after elementary calculations as follows:

$$|\mathcal{L}_{w,c}(f)(x)| \leq \int_a^b |\mathcal{P}(x,t)|\, f'(t)dt \leq \|f'\|_\infty \int_a^b |\mathcal{P}(x,t)|\, dt$$

If $a \leq x \leq \sigma(a,b)$, then

$$\begin{aligned}\int_a^b |\mathcal{P}(x,t)|\, dt &= \int_a^x \mathcal{P}(x,t)dt + \int_{t^*}^b \mathcal{P}(x,t)dt - \int_x^{t^*} \mathcal{P}(x,t)dt \\ &= (1-c)(x-\sigma(a,b)) + 2\nu(x,t^*).\end{aligned} \tag{3.1}$$

By similar reasoning, for $\sigma(a,b) < x \leq b$, it follows

$$\begin{aligned}\int_a^b |\mathcal{P}(x,t)|\, dt &= -\int_a^{t^*} \mathcal{P}(x,t)dt - \int_x^b \mathcal{P}(x,t)dt + \int_{t^*}^x \mathcal{P}(x,t)dt \\ &= (c-1)(x-\sigma(a,b)) + 2\nu(x,t^*).\end{aligned}$$

□

Theorem 3.2. *If* $f \in C[a,b]$, $c \in [0,1]$, *then*

$$|\mathcal{L}_{w,c}(f)(x)| \leq 2\tilde{\omega}\left(f; \frac{1}{2}\left[(c-1)\,|x-\sigma(a,b)| + 2\nu(x,t^*)\right]\right),$$

where ν *is defined in Theorem* 2.2 *and* $\tilde{\omega}$ *is the least concave majorant of the usual modulus of continuity.*

Proof. Taking an arbitrary $g \in C^1[a,b]$ and using Proposition 3.1 we obtain

$$\begin{aligned}|\mathcal{L}_{w,c}(f)(x)| &\leq |\mathcal{L}_{w,c}(f-g)(x)| + |\mathcal{L}_{w,c}(g)(x)| \\ &\leq 4\|f-g\|_\infty + [(c-1)\,|x-\sigma(a,b)| + 2\nu(x,t^*)]\,\|g'\|_\infty.\end{aligned}$$

Passing to the inf we arrive at

$$\begin{aligned}|\mathcal{L}_{w,c}(f)(x)| &\leq 4 \inf_{g\in C^1[a,b]} \left\{\|f-g\|_\infty + \frac{1}{4}\left[(c-1)\,|x-\sigma(a,b)| + 2\nu(x,t^*)\right]\|g'\|_\infty\right\} \\ &= 2\tilde{\omega}\left(f; \frac{1}{2}\left[(c-1)\,|x-\sigma(a,b)| + 2\nu(x,t^*)\right]\right),\end{aligned}$$

so the result follows as a consequence of the relation [11]:

$$\inf_{g\in C^1([a,b])}\left(\|f-g\|_\infty+\frac{t}{2}\|g'\|\right)=\frac{1}{2}\tilde{\omega}(f;t), t\geq 0. \qquad \square$$

Corollary 3.3. *For all $x\in[a,b]$ and $f\in C[a,b]$, we obtain*

$$|\mathcal{L}_w(f)(x)|\leq 2\tilde{\omega}\left(f;\frac{2}{m(a,b)}\int_x^{t^*}(t-x)w(t)dt\right).$$

Corollary 3.4. *For all $x\in[a,b]$ and $f\in C[a,b]$, the following inequality holds*

$$\left|f(x)-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right|$$

$$=2\tilde{\omega}\left(f;\frac{1}{2m(a,b)}\left(\int_x^a(t-x)w(t)dt+\int_x^b(t-x)w(t)dt\right)\right).$$

4. Numerical example

In this section the inequality (2.4) is evaluated for some specific weight functions.
1. Let the weight function w be the probability density function of the Beta distribution,

$$w_{p,q}(x)=\begin{cases}\dfrac{1}{B(p,q)}x^{p-1}(1-x)^{q-1},\ x\in[0,1],\\ 0,\ x\in\mathbb{R}\setminus[0,1],\end{cases}$$

where $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,\ p,q>0$.

Substituting $w_{p,q}$ in relation (2.4), it follows

$$\left(x-\frac{p}{p+q}\right)\gamma+(\gamma-\Gamma)\tilde{\nu}(x) \tag{4.1}$$

$$\leq f(x)-\int_a^b w_{p,q}(t)f(t)dt\leq\left(x-\frac{p}{p+q}\right)\Gamma+(\Gamma-\gamma)\tilde{\nu}(x),$$

where

$$\tilde{\nu}(x)=\begin{cases}\dfrac{p}{p+q}-x-B(x;p+1,q)+xB(x;p,q),\ 0\leq x\leq\dfrac{p}{p+q},\\ xB(x;p,q)-B(x;p+1,q),\ \dfrac{p}{p+q}<x\leq 1,\end{cases}$$

and $B(x;p,q)=\dfrac{1}{B(p,q)}\int_0^x t^{p-1}(1-t)^{q-1}dt,\ 0\leq x\leq 1$ is the incomplete Beta function.

In the below table , for $p=q=\dfrac{1}{2}$ and $f(t)=\dfrac{t^2}{2}, t\in[0,1]$ we calculate the left hand side and the right hand side of inequality (4.1):

Table 1. Error estimate of $E(x) = f(x) - \int_0^1 w_{p,q}(t)f(t)dt$

x	$\tilde{\nu}(x)$	l.h.s of (4.1)	r.h.s of (4.1)	$E(x)$
0	0.500000000000000	-0.500000000000000	0	-0.1875
0.1	0.406636443481054	-0.406636443481054	0.006636443481054	-0.1825
0.2	0.318514120706339	-0.318514120706339	0.018514120706339	-0.1675
0.3	0.233428745882118	-0.233428745882118	0.033428745882118	-0.1425
0.4	0.150335250602855	-0.150335250602855	0.050335250602855	-0.1075
0.5	0.068309886183791	-0.068309886183791	0.068309886183791	-0.0625
0.6	0.086241033753880	-0.086241033753880	0.186241033753880	-0.0075
0.7	0.102438865447664	-0.102438865447664	0.302438865447664	0.0575
0.8	0.113681356007205	-0.113681356007205	0.413681356007205	0.1325
0.9	0.111469208180188	-0.111469208180188	0.511469208180188	0.2175
1	0	0	0.500000000000000	0.3125

2. Let the weight w be the probability density function of the normal distribution

$$w_{m,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}},\ m, \sigma \in \mathbb{R}, \sigma > 0, x \in \mathbb{R}.$$

Then we have

$$m(a,b) = F(b) - F(a), \text{where } F \text{ is the cumulative distribution,}$$

$$\sigma(a,b) = m - \frac{\sigma}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{(b-m)^2}{2\sigma^2}} - e^{-\frac{(a-m)^2}{2\sigma^2}}}{F(b) - F(a)},$$

$$\nu(x,t^*) = \begin{cases} \frac{1}{F(b)-F(a)} \left[\frac{-\sigma}{\sqrt{2\pi}} \left(e^{-\frac{(b-m)^2}{2\sigma^2}} - e^{-\frac{(x-m)^2}{2\sigma^2}} \right) + (m-x)\left(F(b)-F(x)\right) \right], \\ a \le x \le \sigma(a,b), \\ \frac{1}{F(b)-F(a)} \left[\frac{-\sigma}{\sqrt{2\pi}} \left(e^{-\frac{(a-m)^2}{2\sigma^2}} - e^{-\frac{(x-m)^2}{2\sigma^2}} \right) + (m-x)\left(F(a)-F(x)\right) \right], \\ \sigma(a,b) < x \le b. \end{cases}$$

If we consider the probability density function of the standard normal distribution, namely

$$w_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

inequality (2.4) on the interval $[0,1]$ becomes

$$\left(x - \frac{1 - e^{-\frac{1}{2}}}{\phi(1)\sqrt{2\pi}} \right) \gamma + (\gamma - \Gamma)\tilde{\nu}(x)$$

$$\le f(x) - \frac{1}{\phi(1)} \int_a^b f(t) w_{0,1}(t) dt \le \left(x - \frac{1 - e^{-\frac{1}{2}}}{\phi(1)\sqrt{2\pi}} \right) \Gamma + (\Gamma - \gamma)\tilde{\nu}(x),$$

where

$$\tilde{\nu}(x)=\begin{cases}\dfrac{1}{\phi(1)}\left[\dfrac{-1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}}-e^{-\frac{x^2}{2}}\right)-x\left(\phi(1)-\phi(x)\right)\right], 0\le x\le \dfrac{1-e^{-\frac{1}{2}}}{\phi(1)\sqrt{2\pi}},\\ \dfrac{1}{\phi(1)}\left[\dfrac{-1}{\sqrt{2\pi}}\left(1-e^{-\frac{x^2}{2}}\right)+x\phi(x)\right], \dfrac{1-e^{-\frac{1}{2}}}{\phi(1)\sqrt{2\pi}}< x\le 1.\end{cases}$$

Here ϕ is Laplace's function and $\phi(1)=0.3413$.

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Spline and fractal spline interpolation

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Abstract. The classical methods of real data interpolation can be generalized by fractal interpolation. These fractal interpolation functions provide new methods of approximation of experimental data. This paper presents an application of these interpolation methods.

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1. Spline interpolation

Let $H^{m,2}[a,b]$, $m \in \mathbb{N}^*$ be the set of functions $f \in C^{m-1}[a,b]$ with $f^{(m-1)}$ absolutely continuous on $[a,b]$ and $f^{(m)} \in L^2[a,b]$, $\Lambda = \{\lambda_i \mid \lambda_i : H^{m,2}[a,b] \to \mathbb{R}, i = 1..., n\}$ a set of linear functionals, $y \in \mathbb{R}^n$ and

$$U = U_y = \{f \in H^{m,2}[a,b] | \lambda_i(f) = y_i, i = 1, ..., n\}.$$

Definition 1.1. *The problem that consists of determining the elements $s \in U$ such that*

$$\|s^{(m)}\|_2 = \inf_{u \in U} \|u^{(m)}\|_2$$

is called a polynomial spline interpolation problem.

For the solution of a spline interpolation problem we can give the following structural characterization theorem ([3]) in the most general case, when we have Birkhoff type functionals. The set of Birkhoff type functionals is given by:

$$\Lambda = \{\lambda_{ij} | \lambda_{ij} f = f^{(j)}(x_i), i = 1, ..., n, j \in I_i\},$$

for $I_i \subseteq \{0, ..., r_i\}$, $r_i \in \mathbb{N}$, $r_i < m$, and $x_i \in [a,b], i = 1, ..., k$.

Theorem 1.2. *Let Λ be a set of Birkhoff type functionals and let U be the corresponding interpolatory set. The functions $s \in U$ is a solution of the spline interpolation problem if and only if:*

1. $s^{(2m)}(x) = 0, \quad x \in [x_1, x_k] \setminus \{x_1, ...x_k\}$,

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2. $s^{(m)}(x) = 0, \quad x \in (a, x_1) \cup (x_k, b),$
3. $s^{(2m-\mu-1)}(x_i - 0) = s^{(2m-\mu-1)}(x_i + 0), \quad \mu \in \{0, 1, ..., m-1\} - I_i$ *for* $i = 1, ..., k.$

The characterization theorem states that the solution s of the polynomial spline interpolation problem is a polynomial of $2m - 1$ degree on each interior interval (x_i, x_{i+1}) and it is a polynomial of $m - 1$ degree on the intervals $[a, x_1)$ and $(x_k, b]$. Furthermore, the derivative of order $2m - \mu - 1$ is continuous in x_i if the value of the νth ordin derivative in x_i does not belong to Λ.

Definition 1.3. *The solution* s *of the polynomial spline interpolation problem is called a natural spline function of order* $2m - 1$.

When $\Lambda = \{\lambda_i | \lambda_i(f) = f(x_i), i = 1, ..., n\}$ is the set of Lagrange type functionals, with $x_i \in [a, b], i = 1, ..., n$ and $n \geq m$, then for every $f \in H^{m,2}[a, b]$ the interpolation spline function $S_L f$ exists and is unique.
The function $S_L f$ may be written in the form

$$S_L f = \sum_{k=1}^{n} s_k f(x_k),$$

where $s_k, k = 1, .., n$ are the fundamental interpolation spline functions. To determine these functions we can use the characterization theorem and we have

$$s_k(x) = \sum_{i=0}^{m-1} a_i^k x_i + \sum_{j=1}^{n} b_j^k (x - x_j)_+^{2m-1}, k = 1, ..., n,$$

with $a_i^k, i = 0, ..., m-1$ and $b_j^k, j = 1, ..., n$ obtained as the solution of the following systems:

$$s_k^p(\alpha) = 0, \; p = m, ..., 2m-1, \; \text{and } \alpha > x_n$$
$$s_k(x_\nu) = \delta_{k\nu}, \; \nu = 1, ..., n$$

for $k = 1, ..., n$.
We collect form the server of our university some data regarding the internet traffic. We process these data with Lagrange type cubic spline function, and we obtain the following figure:

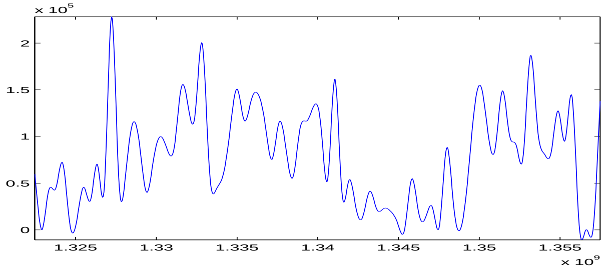

FIGURE 1. Spline interpolation: for internet traffic data

2. Fractal functions

Let (X, d) be a complete metric space, and $D(X)$ be the class of all non-empty closed bounded subsets of X. Then $(D(X), h)$ is a complete metric space with the Hausdorff metric: $h : D(X) \times D(X) \to R$

$$h(A, B) := \sup\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}$$

Let $E \subset X$, $p \geq 0$, $\epsilon > 0$, $|E|$ denote the diameter of the subset E, and define the Hausdorff p-dimensional measure of E:

$$\mathcal{H}^p(E) := \lim_{\epsilon\to 0} \mathcal{H}^p_\epsilon(E) = \sup_{\epsilon>0} \mathcal{H}^p_\epsilon(E),$$

where

$$\mathcal{H}^p_\epsilon(E) := \inf \left\{ \sum_{i=1}^{\infty} |E_i|^p, E \subset \cup_{i=1}^{\infty} E_i, |E_i| < \epsilon \right\}.$$

For each E there is a unique real number q, named the Hausdorff dimension of E, such that

$$\mathcal{H}^p(E) = \begin{cases} +\infty & if \quad 0 \leq p < q \\ 0 & if \quad q < p < \infty \end{cases}$$

B. Mandelbrot define fractal as the set of which Hausdorff dimension is noninteger.

The functions $f : I \to \mathbf{R}$, where I is a real closed interval, is named by M. F. Barnsley *fractal function* if the Hausdorff dimensions of their graphs are noninteger.

Let be N a natural number, $N > 1$, and let $w_i : \ X \to X : \ i \in \{1, ..., N\}$ be continuous functions. Then we call $\{X, w_i : i = 1, ..., N\}$ an *iterated function system (IFS).*

If, for some $0 \leq k < 1$ and all $i \in \{1, ..., N\}$,

$$d(w_i(x), w_i(x')) \leq kd(x, x'), \forall\, x, x' \in X,$$

then the IFS is named *hyperbolic.*

Define $W : \mathcal{D}(X) \to \mathcal{D}(X)$ by

$$W(A) := \cup_{i=1}^{N} w_i(A),$$

where $w_i(A) = \{w_i(x) : x \in A\}$.

W is a contraction mapping if the IFS is hyperbolic:

$$h(W(A), W(B)) \leq kh(A, B)\, \forall\, A, B \in \mathcal{D}(X).$$

Any set $G \in D(X)$ such that $W(G) = G$ is called an *attractor* for the IFS.

Theorem 2.1. (Hutchinson [4]) *Let $\{X, w_i\, i = 1, ..., N\}$ an hyperbolic IFS. There is a unique compact set $G \subset X$, such that $W(G) = G$, and*

$$G := \lim_{n\to\infty} W^n(E),\ E \in \mathcal{D}(X),\ W^0.$$

Let $\{(x_i, y_i) \in R^2, i = 0, 1, \cdots, N\}$ be given, and $I = [x_0, x_N]$. The functions $f : I \to R,$ which interpolate the data according to f$(x_i) = y_i$, $i = 0, 1, ..., N$, and whose graphs are attractors of IFS are *fractal interpolation functions* .

Let be $X = I \times [a,b]$ with Euclidean metric d, $I_n = [x_{n-1}, x_n]$ $u_n : I \to I_n$, $n \in \{1,2,...,N\}$, contractive homeomorphism such that

$$u_n(x_0) := x_{n-1},\ u_n(x_N) := x_n,\ \forall n \in \{1,\cdots,N\}.$$

$$|u_n(c_1) - u_n(c_2)| \leq l|c_1 - c_2|,\ c_1, c_2 \in I,\ 0 \leq l < 1$$

$v_n : X \to [a,b]$ continuous, with

$$v_n(x_0, y_0) := y_{n-1},\ v_n(x_N, y_N) := y_n,\ \forall n \in \{1,\cdots,N\}.$$

$$|v_n(c,d_1) - v_n(c,d_2)| \leq q|d_1 - d_2|,\ c \in I,\ d_1, d_2 \in [a,b], 0 \leq q < 1.$$

Let $w_n : X \to X$, $n \in \{1,2,...,N\}$

$$w_n(x,y) = (u_n(x), v_n(x,y)).$$

Than $\{X, w_n : n = 1,2,...,N\}$ is an IFS but may not be hyperbolic.

Theorem 2.2. (Barnsley [1]) *For the IFS $\{X, w_n : n = 1,2,...,N\}$ defined above, there is a metric d equivalent to the Euclidean metric, such that the IFS is hyperbolic with respect to d. The unique attractor G of the IFS is the graph of a continuous function $f : I \to R$ which interpolates the data set $\{(x_i, y_i) \in R^2, i = 0,1,\cdots,N\}$*

The following example given by Barnsley is used in many articles to give the iterated function system for the most widely studied fractal interpolation function.

Example 2.3. [1] Let $\{(x_i, y_i) \in R^2, i = 0,1,\cdots,N\}$, $N > 1$

$$w_n(x,y) = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix},$$

where $|d_n| < 1$ is given, a_n, c_n, e_n, f_n are real number such that

$$w_n(x_0, y_0) := (x_{n-1}, y_{n-1}),\ w_n(x_N, y_N) := (x_n, y_n)$$

From the above equations follows that

$$a_n = \frac{x_n - x_{n-1}}{x_N - x_0},$$

$$c_n = \frac{y_n - y_{n-1}}{x_N - x_0} - \frac{d_n(y_N - y_0)}{x_N - x_0},$$

$$e_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}$$

$$f_n = \frac{x_N y_{n-1} - x_0 y_n}{x_N - x_0} - \frac{d_n(x_N y_0 - x_0 y_N)}{x_N - x_0}.$$

w_n is a shear transformation: it maps lines parallel to the y-axis into the lines parallel to the y-axis, d_n is the vertical scaling factor.

Using the same data regarding the internet traffic, we construct the iterated function system, and we implement this in MatLab. The graph of the fractal interpolation function for these data is given in Figure 2.

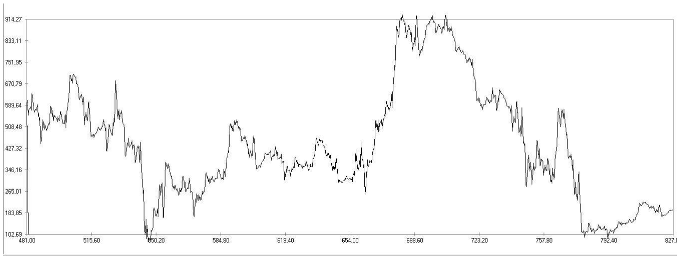

FIGURE 2. Fractal interpolation: for internet traffic data

3. Spline fractal interpolation functions

Let be

$$u_n(x) = a_n x + b_n$$

and

$$v_n(x, y) = \alpha_n y + q_n(x)$$

where a_n and b_n can be obtained from the relations for a_n, e_n, given in example and $-1 < \alpha_n < 1$.

The function f is a cubic spline fractal interpolation function, which interpolates the set of ordinates $y_0, y_1, ..., y_N$ with respect to the mesh $x_0 < x_1 < ... < x_N$ if f is of class $C^2[x_0, x_N]$, who satisfies the interpolation conditions $f(x_i) = y_i$, $i = 0, 1, ..., N$, and the graph of f is fixed point of the iterated function system $\{\mathbb{R}^2; \omega_n(x, y), n = 1, 2, ..., N\}$, where $\omega_n(x, y) = (u_n x, v_n(x, y))$, and the function $q_n(x)$ is a suitable cubic polynomial.

In order to construct the cubic spline fractal interpolation function for the internet traffic data, we use the algorithm used by Chand and Kapoor in ([2]), where the cubic spline functions are constructed by the moments $M_n = f''(x_n)$ for $n = 1, 2, ..., N$.

Let $G = \{f : I \to \mathbb{R},\ f \text{ is continuous},\ f(x_0) = y_0,\ f(x_N) = y_N\}$, and ρ be the sup-norm on G, then (G, ρ) is a complete metric space, and the fractal interpolation function is the unique fixed point of the Read-Bajraktarević operator T on (G, ρ) so that

$$Tf(x) \equiv v_n(u_n^{-1}(x), f(u_n^{-1}(x))) = f(x), n = 1, 2, ..., N. \tag{3.1}$$

In the algorithm mentioned above, Chand and Kapoor obtained from the properties of a fractal spline interpolation function, that the cubic spline fractal interpolation function in terms of the moments can be written as

$$\begin{aligned} f(u_n(x)) = a_n^2 \Bigg\{ & \alpha_n f(x) + \frac{(M_n - \alpha_n M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{n-1} - \alpha_n M_0)(x_N - x)^3}{6(x_N - x_0)} \\ & - \frac{(M_{n-1}\alpha_n M_0)(x_N - x_0)(x_N - x)}{6} - \frac{(M_N - \alpha_n M_N)(x_N - x_0)(x - x_0)}{6} \\ & + \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0\right) \frac{x_N - x}{x_N - x_0} + \left(\frac{y_n}{a_n^2} - \alpha_n y_N\right) \frac{x - x_0}{x_N - x_0} \Bigg\}, n = 1, 2, ..., N, \end{aligned}$$

also they give the system of equations from where the moments can be obtained:

$$A_n^* f'(x_0) + A_n M_0 + \mu M_{n-1} + 2M_n + \lambda M_{n+1} + B_n M_N + B_n^* f'(x_N)$$
$$= \frac{6[(y_{n+1}-y_n)/h_{n+1} - (y_n - y_{n-1})/h_n]}{h_n + h_{n+1}} - \frac{6(a_{n+1}\alpha_{n+1} - a_n\alpha_n)}{h_n + h_{n+1}} \frac{y_N - y_0}{x_N - x_0},$$

where

$$A_n^* = \frac{-6a_{n+1}\alpha_{n+1}}{h_n + h_{n+1}}, \; A_n = \frac{-(\alpha_n h_n + 2\alpha_{n+1}h_{n+1})}{h_n + h_{n+1}},$$

$$\alpha_n = \frac{6}{h_n + h_{n+1}}, \; \mu_n = 1 - \lambda_n,$$

$$B_n = \frac{-(2\alpha_n h_n + \alpha_{n+1}h_{n+1})}{h_n + h_{n+1}}, \; B_n^* = \frac{6a_n\alpha_n}{h_n + h_{n+1}}$$

for $n = 1, 2, ..., N-1$ and $x_n - x_{n-1} = h_n$ for $n = 1, 2, ..., N$.

Solving the systems of equations and using the boundary conditions where the values of the first derivative are prescribed at the endpoints of the interval $[x_0, x_N]$, we have the moments $M_n, n = 0, 1, ..., N$ which are used in the construction of the iterated function system given by the relations from ([2])

$$\{\mathbb{R}^2; \omega_n(x, y) = (u_n(x), v_n(x, y)), n = 1, 2, ..., N\}, \tag{3.2}$$

where $u_n(x) = a_n x + b_n$ and

$$v_n(x,y) = a_n^2 \left\{ \alpha_n f(x) + \frac{(M_n - \alpha_n M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{n-1} - \alpha_n M_0)(x_N - x)^3}{6(x_N - x_0)} \right.$$
$$- \frac{(M_{n-1}\alpha_n M_0)(x_N - x_0)(x_N - x)}{6} - \frac{(M_N - \alpha_n M_N)(x_N - x_0)(x - x_0)}{6}$$
$$\left. + \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0\right) \frac{x_N - x}{x_N - x_0} + \left(\frac{y_n}{a_n^2} - \alpha_n y_N\right) \frac{x - x_0}{x_N - x_0} \right\}, n = 1, 2, ..., N.$$

The graph of the cubic spline is the fixed point of the iterated function system given by (3.2).

We made a MatLab implementation of this algorithm. It can be used for arbitrary set of data, in the case of the data studied before, the cubic spline fractal interpolation function will have the form given in Figure 3.

The cubic spline interpolation function is an important tool in computer graphics, CAGD, differential equations and several engineering applications and it is of class C^2. Generally, affine fractal interpolation functions are nondifferentiable functions. Therefore the cubic spline fractal interpolation it seems to be a good method in data processing, because it has better properties and it can be used with all possible boundary conditions like in the case of classical splines.

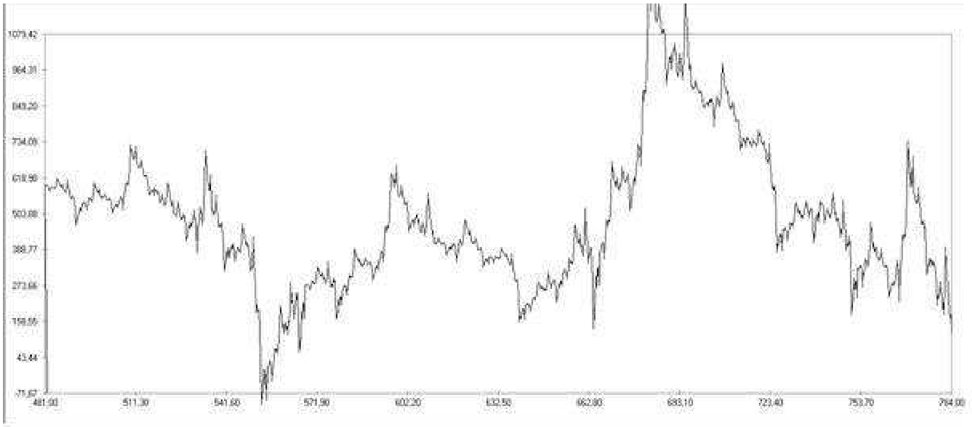

FIGURE 3. Spline fractal interpolation: for internet traffic data

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A direct approach for proving Wallis ratio estimates and an improvement of Zhang-Xu-Situ inequality

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Abstract. In time, inequalities about Wallis ratio and related functions were presented by many mathematicians. In this paper, we show how estimates on the Wallis ratio can be obtained using the asymptotic series. Finally, an improvement of an inequality due to X.-M. Zhang, T.-Q. Xu and L.-B. Situ [Geometric convexity of a function involving gamma function and application to inequality theory, J. Inequal. Pure Appl. Math. 8 (1) (2007) Art. 17, 9 pp.] is presented.

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1. Introduction and motivation

Wallis ratio

$$P_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}, \quad n = 1, 2, 3 \cdots$$

plays a main role in mathematics and other branches of science. This expression is closely related to the Euler gamma function defined for all real $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

We have:

$$P_n = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n+1)}. \tag{1.1}$$

For further details, we recommend the basic monograph [1]. Many mathematicians were preoccupied to give estimates for P_n and other expressions related to gamma function.We refer for example to the following recent titles: Chen and Qi [2]-[3], Hirschhorn [5], Lin, Deng and Chen [7], Mortici [9]-[13], Păltănea [25].

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Chronologically, we mention the following inequalities for every integer $n \geq 1$ due to Wallis [29]

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi n}}, \tag{1.2}$$

Kazarinoff [6]

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \tag{1.3}$$

Hirschhorn [5]

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{8}{3}}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{7}{3}}\right)}} \tag{1.4}$$

or Panaitopol [24]

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{1.5}$$

Chen and Qi [2] proposed the following inequality

$$\frac{1}{\sqrt{\pi\left(n+A\right)}} \leq P_n < \frac{1}{\sqrt{\pi\left(n+B\right)}}, \quad n \geq 1, \tag{1.6}$$

where the constants $A = \frac{4}{\pi} - 1$ and $B = \frac{1}{4}$ are sharp. Zhao [27] proved

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < P_n \leq \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{3}}\right)}} \tag{1.7}$$

and Zhang et al. [28] showed:

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} < P_n \leq \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}}. \tag{1.8}$$

The above estimates were obtained using a various of methods such as mean inequality, Jensen inequality, monotonicity of some sequences, or monotonicity and complete monotonicity of some functions. In this work, we exploit some inequalities obtained by truncation of certain asymptotic series.

As a new result, in the last section of this work we present an improvement of an inequality due to X.-M. Zhang, T.-Q. Xu and L.-B. Situ stated in [28].

2. The asymptotic series of P_n

The following inequalities were presented by Slavić [26], for every real $x > 0$ and integers m, $l \geq 1$:

$$\sqrt{x}\exp\left(a_m(x)\right) < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\exp\left(b_l(x)\right), \tag{2.1}$$

where

$$a_m(x) = \sum_{k=1}^{2m} \frac{\left(1 - 2^{-2k}\right) B_{2k}}{k\,(2k-1)\,x^{2k-1}}$$

and

$$b_l(x) = \sum_{k=1}^{2l-1} \frac{\left(1 - 2^{-2k}\right) B_{2k}}{k\,(2k-1)\,x^{2k-1}}.$$

Here B_j are the Bernoulli numbers given by the generating function

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}.$$

The first Bernoulli numbers are $B_0 = 1$, $B_1 = 1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, while $B_{2m+1} = 0$, for every integer $m \geq 1$. See, *e.g.* [1].

As a direct consequence of Slavić inequalities (2.1), the following asymptotic formula holds true as $x \to \infty$:

$$\frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right)} \sim \sqrt{x} \exp\left\{\sum_{k=1}^{\infty} \frac{\left(1 - 2^{-2k}\right) B_{2k}}{k\,(2k-1)\,x^{2k-1}}\right\}.$$

Using (1.1), we get:

$$P_n \sim \frac{1}{\sqrt{n\pi}} \exp\left\{-\sum_{k=1}^{\infty} \frac{\left(1 - 2^{-2k}\right) B_{2k}}{k\,(2k-1)\,n^{2k-1}}\right\}, \quad n \to \infty, \tag{2.2}$$

while inequalities (2.1) can be rewritten as

$$\frac{1}{\sqrt{n\pi}} \exp\left\{\alpha_l(n)\right\} < P_n < \frac{1}{\sqrt{n\pi}} \exp\left\{\beta_m(n)\right\}, \tag{2.3}$$

where $\alpha_l(n) = -b_l(n)$ and $\beta_m(n) = -a_m(n)$. The first truncations are the following:

$$\begin{aligned} \alpha_1(n) &= -\frac{1}{8n}, \\ \alpha_2(n) &= -\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5}, \ldots \end{aligned}$$

and

$$\begin{aligned} \beta_1(n) &= -\frac{1}{8n} + \frac{1}{192n^3}, \\ \beta_2(n) &= -\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5} + \frac{17}{14336n^7}, \ldots . \end{aligned}$$

As examples, we show complete arguments of our method for proving Kazarinoff's inequality and Panaitopol's inequality. All other inequalities on Wallis ratio presented in the first part of this work can be similarly proven, as we indicate in the next section.

3. Kazarinoff's inequality

We start with Kazarinoff's inequality (1.3). In his proof, Kazarinoff used the Legendre's formula for digamma function:

$$\psi(x) = -\gamma + \int_0^1 \frac{t^x - 1}{t-1} dt$$

($\gamma = 0.577215\cdots$ is the Euler-Mascheroni constant) and the inequality

$$[\ln \phi(t)]'' - \left\{[\ln \phi(t)]'\right\}^2 > 0,$$

where

$$\phi(t) = \int_0^1 \sin^t x \; dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{t+2}{2}\right)}.$$

Chen and Qi [2] rediscovered the right-hand side of Wallis' inequality using the monotonicity of the sequence

$$Q_n = \left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 - n,$$

with $\lim_{n\to\infty} Q_n = 1/4$.

Our idea for proving Kazarinoff's inequality using the asymptotic series (2.2) is to consider as many as necessary terms α_l and β_m such that

$$\begin{aligned} \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} &< \frac{1}{\sqrt{n\pi}} \exp\left(\alpha_l(n)\right) \\ &< P_n \\ &< \frac{1}{\sqrt{n\pi}} \exp\left(\beta_m(n)\right) < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \end{aligned} \tag{3.1}$$

Individual tryings we made showed that already first truncations α_1 and β_1 make inequalities (3.1) true, namely

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{1}{\sqrt{n\pi}} \exp\left(-\frac{1}{8n}\right) \tag{3.2}$$

and

$$\frac{1}{\sqrt{n\pi}} \exp\left(-\frac{1}{8n} + \frac{1}{192n^3}\right) < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{3.3}$$

By taking the logarithm, the inequalities (3.2)-(3.3) are equivalent to

$$-\frac{1}{8n} - \frac{1}{2}\ln n + \frac{1}{2}\ln\left(n+\frac{1}{2}\right) > 0$$

and

$$-\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{2}\ln n + \frac{1}{2}\ln\left(n+\frac{1}{4}\right) < 0,$$

for every integer $n \geq 1$. It suffices $f > 0$ and $g < 0$ on $[1, \infty)$, where

$$f(x) = -\frac{1}{8x} - \frac{1}{2}\ln x + \frac{1}{2}\ln\left(x + \frac{1}{2}\right)$$

and

$$g(x) = -\frac{1}{8x} + \frac{1}{192x^3} - \frac{1}{2}\ln x + \frac{1}{2}\ln\left(x + \frac{1}{4}\right).$$

As

$$f'(x) = -\frac{(2x-1)}{8(2x+1)x^2} < 0, \quad g'(x) = \frac{8x^2 - 4x - 1}{64x^4(4x+1)} > 0,$$

the function f is strictly decreasing and g is strictly increasing on $[1, \infty)$. But $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, so $f > 0$ and $g < 0$ on $[1, \infty)$ and our assertion is proved.

4. Panaitopol's inequality

Panaitopol [24] improved the left-hand side of Wallis' inequality as

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32n}\right)}} < P_n.$$

As above, we search a truncation of asymptotic series such that

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32n}\right)}} < \frac{1}{\sqrt{n\pi}} \exp\left(\alpha_2(n)\right) < P_n. \tag{4.1}$$

Remark that in this case, the second truncation should be selected:

$$\alpha_2(n) = -\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5}.$$

The first inequality (4.1) is equivalent to $h > 0$ on $[1, \infty)$, where

$$h(x) = -\frac{1}{8x} + \frac{1}{192x^3} - \frac{1}{640x^5} - \frac{1}{2}\ln x + \frac{1}{2}\ln\left(x + \frac{1}{4} + \frac{1}{32x}\right).$$

As

$$h'(x) = \frac{1}{8x^2(32x^2 + 4x + 1)} > 0,$$

the function h is strictly increasing.

Then $h(x) \geq h(1) = \frac{1}{2}\ln\frac{41}{32} - \frac{233}{1920} = 0.00256\cdots > 0$, for every $x \in [1, \infty)$ and the assertion follows.

5. Further examples

All the other inequalities presented in the first section can be similarly proven. More precisely, we reduced Zhao De Jun's inequality (1.7) to

$$\begin{aligned}\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} &< \frac{1}{\sqrt{n\pi}}\exp\left(\alpha_1(n)\right)\\ &< P_n\\ &< \frac{1}{\sqrt{n\pi}}\exp\left(\beta_1(n)\right) < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{3}}\right)}},\end{aligned}$$

Hirschhorn's inequality (1.4) to

$$\begin{aligned}\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{8}{3}}\right)}} &< \frac{1}{\sqrt{n\pi}}\exp\left(\alpha_2(n)\right)\\ &< P_n\\ &< \frac{1}{\sqrt{n\pi}}\exp\left(\beta_1(n)\right) < \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{7}{3}}\right)}}\end{aligned}$$

and Zhang et al. inequality (1.8) to

$$\begin{aligned}\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} &< \frac{1}{\sqrt{n\pi}}\exp\left(\alpha_1(n)\right)\\ &< P_n\\ &< \frac{1}{\sqrt{n\pi}}\exp\left(\beta_1(n)\right) < \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}}.\end{aligned}$$

The great advantage of the asymptotic series method we present here is that all computations are reduced to some elementary functions involving polynomial functions and logarithmic functions. In consequence, the monotonicity, or positivity, of such functions can be easily stated.

6. An improvement of Zhang-Xu-Situ inequality

In this section, motivated by Zhang-Xu-Situ inequality (1.8), we propose the following better approximation

$$P_n \approx \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}+\frac{1}{48n^2}-\frac{1}{2880n^3}}. \tag{6.1}$$

This approximation is obtained firstly by considering the following class of approximations

$$P_n \approx \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{an+\frac{b}{n}+\frac{c}{n^2}+\frac{d}{n^3}}, \tag{6.2}$$

where a, b, c, $d \in \mathbb{R}$ are any real parameters. In order to find the values of a, b, c, d that provide the most accurate approximation (6.2), we use a method first introduced by Mortici in [8]. This method was proven to be a strong tool for constructing asymptotic expansions, or for accelerating some convergences. See, *e.g.* [14]-[23].

Let us define the relative error sequence w_n by the following formulas for every integer $n \geq 1$:

$$P_n = \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{an+\frac{b}{n}+\frac{c}{n^2}+\frac{d}{n^3}} \exp w_n .$$

We consider an approximation (6.2) better when the speed of convergence of w_n to zero is higher. But w_n is faster convergent together to the difference $w_n - w_{n+1}$. Using a Maple software, we get

$$\begin{aligned} w_n - w_{n+1} \quad = \quad & \left(\frac{1}{8}a - \frac{1}{8}\right)\frac{1}{n^2} + \left(-\frac{5}{24}a - b + \frac{1}{8}\right)\frac{1}{n^3} \\ & + \left(\frac{19}{64}a + \frac{15}{8}b - \frac{3}{2}c - \frac{7}{64}\right)\frac{1}{n^4} \\ & + \left(-\frac{197}{480}a - \frac{35}{12}b + \frac{7}{2}c - 2d + \frac{3}{32}\right)\frac{1}{n^5} \\ & + \left(\frac{217}{384}a + \frac{815}{192}b - \frac{155}{24}c + \frac{45}{8}d - \frac{31}{384}\right)\frac{1}{n^6} \\ & + O\left(\frac{1}{n^7}\right). \end{aligned}$$

The fastest convergence is obtained when the first four coefficients vanish of n^{-k}, that is for the values

$$a = 1,\ b = -\frac{1}{12},\ c = \frac{1}{48},\ d = -\frac{1}{2880}.$$

Now the approximation (6.1) is completely justified. We are now in a position to improve the upper bound of Zhang-Xu-Situ inequality as follows.

Theorem 6.1. *The following inequality is valid, for every integer* $n \geq 1$:

$$P_n \leq \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}+\frac{1}{48n^2}-\frac{1}{2880n^3}}. \tag{6.3}$$

Proof. It suffices to show that

$$\frac{1}{\sqrt{n\pi}}\exp\left(\beta_2(n)\right) < \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}+\frac{1}{48n^2}-\frac{1}{2880n^3}},$$

or equivalently

$$\begin{aligned} & -\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5} + \frac{17}{14336n^7} + \frac{1}{2} \\ < & \left(n - \frac{1}{12n} + \frac{1}{48n^2} - \frac{1}{2880n^3}\right)\ln\left(1+\frac{1}{2n}\right). \end{aligned}$$

We have to prove that $\varphi < 0$ for every $x \in [1, \infty)$, where

$$\varphi(x) = \frac{-\frac{1}{8x} + \frac{1}{192x^3} - \frac{1}{640x^5} + \frac{17}{14336x^7} + \frac{1}{2}}{\left(x - \frac{1}{12x} + \frac{1}{48x^2} - \frac{1}{2880x^3}\right)} - \ln\left(1 + \frac{1}{2x}\right).$$

But

$$\varphi'(x) = \frac{P(x-1)}{56x\,(2x+1)\,(60x - 240x^2 + 2880x^4 - 1)^2} > 0,$$

with

$$\begin{aligned} P(x) \quad = \quad & 784\,374\,011 + 2985\,594\,595x + 4717\,628\,082x^2 \\ & +4035\,936\,400x^3 + 2026\,365\,656x^4 \\ & +598\,429\,920x^5 + 96\,371\,520x^6 + 6531\,840x^7, \end{aligned}$$

so the function φ is strictly increasing on $[1, \infty)$. But $\lim_{x\to\infty} \varphi(x) = 0$, so $\varphi(x) < 0$ for all real $x \geq 1$. The proof is now completed. □

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Construction and applications of Gaussian quadratures with nonclassical and exotic weight functions

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Abstract. In 1814 Carl Friedrich Gauß (1777–1855) developed his famous method of numerical integration which dramatically improves the earlier method of Isaac Newton (1643–1727) from 1676. Beside the some historical details in this survey, a formulation of this classical theory in modern terminology using theory of orthogonlity on real line, as well as the characterization, existence and uniqueness of these formulas, are presented. A special attention is devoted to the algorithms for constructing such quadrature formulas for nonclassical weight functions, their numerical stability and the corresponding software. Finally, some recent progress in this subject, as well as new important applications of these methods in several different directions (distributions in statistics and physics, summation of slowly convergent series, etc.) are presented.

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1. Introduction

Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n, $\mathcal{P}$ be the set of all algebraic polynomials, and $\mathrm{d}\mu$ be a finite positive Borel measure on the real line $\mathbb{R}$ such that its support $\mathrm{supp}(\mathrm{d}\mu)$ is an infinite set, and all its moments $\mu_k = \int_{\mathbb{R}} t^k \,\mathrm{d}\mu$, $k = 0, 1, \ldots$, exist and are finite.

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The n-point quadrature formula

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\mu(t) = \sum_{k=1}^{n} A_k f(\tau_k) + R_n(f), \tag{1.1}$$

which is exact on the set $\mathcal{P}_{2n-1}$ ($R_n(\mathcal{P}_{2n-1}) = 0$) is known as the *Gauss-Christofell quadrature formula* (cf. [10, p. 29], [14, p. 324]). It is a quadrature formula of the maximal algebraic degree of exactness $d_{\max} = 2n - 1$. First formula of this type

$$\int_0^1 f(t)\,\mathrm{d}t = \sum_{k=1}^{n} A_k f(\tau_k) + R_n(f), \tag{1.2}$$

was descovered by Carl Friedrich Gauss two centuries ago.

In this survey paper we give an account on this kind of quadrature rules and several their new applications. The paper is organized as follows. Starting with the famous idea of Gauss and some historical details, in Section 2 we give its formulation in modern terminology and a connection with orthogonal polynomials. Section 3 is devoted to constructive theory of orthogonal polynomials. Numerical construction of Gaussian quadratures with respect to strong non-classical weights and some exotic weight functions, as well as several applications of such rules in approximation theory, statistics, and summation of slowly convergent series are studied in Section 4. Special attention is paid to available software, which is based on recent progress in symbolic computation and variable precision arithmetic.

2. Two centuries of Gaussian rules

After Newton formula of numerical integration from 1676 (known as Newton-Cotes rules),

$$\int_a^b f(t)\,\mathrm{d}t \approx Q_n(f) = \sum_{k=1}^{n} A_k f(\tau_k), \tag{2.1}$$

obtained by an integration of the corresponding interpolation polynomal of $f(t)$ at n different fixed points (nodes), $\tau_1, \ldots, \tau_n$ (usually selected equidistantly on $[a, b]$), Gauss in 1814 developed his famous method [4][1], which dramatically improves the previous Newton method. While Newton-Cotes formula exact only for polynomials of degree at most $n - 1$, Gauss' question was what is the maximum degree of exactness that can be achieved in (2.1) (i.e., in (1.2) supposing that $[a, b] = [0, 1]$) if the nodes $\tau_1, \ldots, \tau_n$ are free.

Since in the quadrature sum

$$Q_n(f) = \sum_{k=1}^{n} A_k f(\tau_k)$$

there are $2n$ unknowns parameters: τ_k, A_k, $k = 1, \ldots, n$, Gauss started with the conjecture that the quadrature formula (1.2) could be exact for all algebraic polynomials of degree at most $2n - 1$. Starting from the work of Newton and Cotes and using only

[1]Gauss submitted his manuscript on September 16, 1814.

his own result on continued fractions associated with hypergeometric series, Gauss proved this result. It is interesting to mention that Gauss determined numerical values of quadrature parameters, the nodes τ_k and the weights A_k, $k = 1, \dots, n$, for all $n \le 7$, with almost 16 significant decimal digits[2]. This discovery was the most significant event of the 19th century in the field of numerical integration and perhaps in all of numerical analysis.

An elegant alternative derivation of these formulas was provided by Jacobi [13], and furher contributions by Mehler, Radau, Heine, etc. A significant generalization to arbitrary measures was given by Christoffel (see a nice survey of Gauss-Christoffel quadrature formulae written by Gautschi [6]). The error term and convergence were proved by Markov and Stieltjes, respectively. It was only in 1928 Uspensky gave the first proof for the convergence of Gaussian formula on unbounded intervals with the classical measures of Laguerre and Hermite.

As we mentioned in Section 1, these formulae with maximal degree of precision are known today as the Gauss-Christoffel quadrature formulae.

In modern terminology, the formulation of this classical theory can be given in the following form: *Let* $\mathrm{d}\mu(t)$ *is a positive measure on* $\mathbb{R}$ *with finite or unbounded support, for which all moments* $\mu_k = \int_{\mathbb{R}} t^k \, \mathrm{d}\mu(t)$ *exist and are finite, and* $\mu_0 > 0$. *Then, for each* $n \in \mathbb{N}$, *there exists the* n*-point Gauss-Christoffel quadrature formula* (1.1) *which is exact for all algebraic polynomials of degree* $\le 2n-1$, *i.e.,* $R_n(f) = 0$ *for each* $f \in \mathcal{P}_{2n-1}$.

The Gauss–Christoffel quadrature formula (1.1) can be characterized as an interpolatory formula for which its *node polynomial* $\omega_n(t) = \prod_{k=1}^{n}(t - \tau_k)$ is orthogonal to $\mathcal{P}_{n-1}$ with respect to the inner product defined by

$$(p, q) = \int_{\mathbb{R}} p(t)q(t) \, \mathrm{d}\mu(t) \quad (p, q \in \mathcal{P}). \tag{2.2}$$

Therefore, orthogonal polynomials play an important role in the analysis and construction of quadrature formulas of the maximal, or nearly maximal, algebraic degree of exactness (cf. [10], [14], [9], [19]). The inner product (2.2) gives rise to a unique system of monic orthogonal polynomials $\pi_k(\,\cdot\,) = \pi_k(\,\cdot\,; \mathrm{d}\mu)$, such that

$$\pi_k(t) \equiv \pi_k(\,\mathrm{d}\mu; t) = t^k + \text{terms of lower degree}, \quad k = 0, 1, \dots, \tag{2.3}$$

and

$$(\pi_k, \pi_n) = ||\pi_n||^2 \delta_{kn} = \begin{cases} 0, & n \ne k, \\ ||\pi_n||^2, & n = k. \end{cases}$$

The following theorem is due to Jacobi [13] (cf. [14, p. 297]).

Theorem 2.1. *Given a positive integer* m ($\le n$), *the quadrature formula* (1.1) *has degree of exactness* $d = n - 1 + m$ *if and only if the following conditions are satisfied:*

1° *Formula* (1.1) *is interpolatory;*

2° *The node polynomial* $\omega_n(t) = (t - \tau_1) \cdots (t - \tau_n)$ *satisfies*

$$(\forall p \in \mathcal{P}_{m-1}) \qquad (p, \omega_n) = \int_{\mathbb{R}} p(t)\omega_n(t) \, \mathrm{d}\mu(t) = 0.$$

[2]Otherwise, τ_k, $k = 1, \dots, n$, are zeros of the shifted Legendre polynomial $P_n(2x - 1)$.

According to this theorem, the n-point quadrature formula (1.1) with respect to the positive measure $\mathrm{d}\mu(t)$ has the maximal algebraic degree of exactness $2n-1$, i.e., $m=n$ is optimal ($\omega_n=\pi_n$). The higher m ($>n$) is impossible. Indeed, according to 2°, the case $m=n+1$ requires the orthogonality $(p,\omega_n)=0$ for all $p\in\mathcal{P}_n$, which is impossible when $p=\omega_n$.

The cases $m=n-1$ and $m=n-2$ lead to the Gauss-Radau (one of the endpoints a or b is included in the set of nodes) and Gauss-Lobatto formulas ($\tau_1=a$ and $\tau_n=b$), respectively.

2.1. *Fundamental three–term recurrence relation.* Because of the property $(tp,q)=(p,tq)$ of the inner product (2.2), the monic orthogonal polynomials (2.3) satisfy the three–term recurrence relation

$$\pi_{k+1}(t)=(t-\alpha_k)\pi_k(t)-\beta_k\pi_{k-1}(t),\quad k=0,1,2\ldots, \tag{2.4}$$

with $\pi_0(t)=1$ and $\pi_{-1}(t)=0$, where $(\alpha_k)=(\alpha_k(\mathrm{d}\mu))$ and $(\beta_k)=(\beta_k(\mathrm{d}\mu))$ are sequences of recursion coefficients which depend on the measure $\mathrm{d}\mu$. The coefficient β_0 may be arbitrary, but is conveniently defined by $\beta_0=\mu_0=\int_{\mathbb{R}}\mathrm{d}\mu(t)$.

There are many reasons way the coefficients α_k and β_k in the three–term recurrence relation (2.4) are fundamental quantities in the constructive theory of orthogonal polynomials (for details see [7]).

First, α_k and β_k provide a compact way of representing and easily calculating orthogonal polynomials, their derivatives, and their linear combinations, requiring only a linear array of parameters.

The same recursion coefficients α_k and β_k appear in the *Jacobi continued fraction associated with the measure* $\mathrm{d}\mu$,

$$F(z)=\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{z-t}\sim\frac{\beta_0}{z-\alpha_0-}\,\frac{\beta_1}{z-\alpha_1-}\cdots,$$

which is known as the *Stieltjes transform* of the measure $\mathrm{d}\mu$ (for details see [10, p. 15], [14, p. 114]). For the n-th *convergent* of this continued fraction, it is easy to see that

$$\frac{\beta_0}{z-\alpha_0-}\,\frac{\beta_1}{z-\alpha_1-}\cdots\frac{\beta_{n-1}}{z-\alpha_{n-1}}=\frac{\sigma_n(z)}{\pi_n(z)}, \tag{2.5}$$

where σ_n are the so–called *associated polynomials*, defined by

$$\sigma_k(z)=\int_{\mathbb{R}}\frac{\pi_k(z)-\pi_k(t)}{z-t}\,\mathrm{d}\mu(t),\quad k\ge 0.$$

The associated polynomials satisfy the same fundamental relation (2.4), i.e.,

$$\sigma_{k+1}(z)=(z-\alpha_k)\sigma_k(z)-\beta_k\sigma_{k-1}(z),\quad k\ge 0,$$

only with starting values $\sigma_0(z)=0$, $\sigma_{-1}(z)=-1$.

The function of the second kind,

$$\varrho_k(z)=\int_{\mathbb{R}}\frac{\pi_k(t)}{z-t}\,\mathrm{d}\mu(t),\quad k\ge 0,$$

where z is outside the spectrum of $\mathrm{d}\mu$, also satisfy the same three–term recurrence relation (2.4) and represent its *minimal solution*, normalized by $\varrho_{-1}(z)=1$, as observed by Gautschi in [5].

It is easy to see that the rational function (2.5) has simple poles at the zeros $z = \tau_{n,k}$, $k = 1, \ldots, n$, of the polynomial $\pi_n(t)$. If by $\lambda_{n,k}$ we denote the corresponding residues of $\sigma_n(z)/\pi_n(z)$ at these poles, i.e.,

$$\lambda_{n,k} = \lim_{z \to \tau_{n,k}} (z - \tau_{n,k}) \frac{\sigma_n(z)}{\pi_n(z)} = \frac{1}{\pi_n'(\tau_{n,k})} \int_{\mathbb{R}} \frac{\pi_n(t)}{t - \tau_{n,k}} \, \mathrm{d}\mu(t), \tag{2.6}$$

then for the continued fraction representation (2.5) we can get the following form

$$\frac{\sigma_n(z)}{\pi_n(z)} = \sum_{k=1}^{n} \frac{\lambda_{n,k}}{z - \tau_{n,k}} .$$

As we can see, the coefficients $\lambda_{n,k}$ are exactly the weight coefficients (Christoffel numbers) in the Gauss–Christoffel quadrature formula (1.1) and they can be expressed by the so–called Christoffel function $\lambda_n(\mathrm{d}\mu; t)$ (cf. [14, Chapters 2 & 5]) in the form

$$A_k = \lambda_n(\mathrm{d}\mu; \tau_k), \quad k = 1, \ldots, n,$$

and zeros of the polynomial $\pi_n(t)$ are the nodes of (1.1), i.e., $\tau_k = \tau_{n,k}$, $k = 1, \ldots, n$.

3. Constructive theory of orthogonal polynomials and quadratures

A classical approach in construction of Gauss-Christoffel quadrature rules is based on a computation of nodes by using Newton's method and then a direct application of some expressions derived from (2.6) for the weight coefficients (cf. Davis & Rabinowitz [3]).

However, a characterization of the Gaussian formula via an eigenvalue problem for one symmetric tridiagonal *Jacobi matrix*, of order n associated with the measure $\mathrm{d}\mu$,

$$J_n(\mathrm{d}\mu) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}, \tag{3.1}$$

has become the basis of current methods for generating Gaussian quadratures. The most popular of them is the *Golub-Welsch procedure*, obtained by a simplification of QR algorithm, so that beside all eigenvalues only the first components of the eigenvectors are computed [12].

Theorem 3.1. *The nodes τ_k in the Gauss-Christoffel quadrature rule* (1.1) *are eigenvalues of the Jacobi matrix $J_n(\mathrm{d}\mu)$ given by* (3.1). *The weight coefficients A_k are given by*

$$A_k = \lambda_{n,k} = \beta_0 v_{k,1}^2, \quad k = 1, \ldots, n,$$

where $\beta_0 = \mu_0 = \int_{\mathbb{R}} \mathrm{d}\mu(t)$ and $v_{k,1}$ is the first component of the normalized eigenvector $\mathbf{v}_k$ $(= [v_{k,1} \ \ldots \ v_{k,n}]^{\mathrm{T}})$ corresponding to the eigenvalue τ_k,

$$J_n(\mathrm{d}\mu)\mathbf{v}_k = \tau_k \mathbf{v}_k, \quad \mathbf{v}_k^{\mathrm{T}} \mathbf{v}_k = 1, \quad k = 1, \ldots, n.$$

Therefore, if we know recursive coefficients α_k and β_k in the fundamental three-term recurrence relation (2.4), the problem of construction Gaussian rules can be easily solved by the Golub-Welsch procedure. This procedure is implemented in several packages including the most known `ORTPOL` given by Gautschi [8].

Unfortunately, the recursion coefficients are known explicitly only for some narrow classes of orthogonal polynomials, e.g. they are known for the so-called *very classical* orthogonal polynomials (Jacobi, the generalized Laguerre, and Hermite polynomials). Orthogonal polynomials for which the recursion coefficients are not known we call *strongly non–classical polynomials.* For these, if we know how to compute the first n recursion coefficients α_k and β_k, $k = 0, 1, \ldots, n-1$, then we can compute all orthogonal polynomials of degree at most n by a straightforward application of the three–term recurrence relation (2.4), construct the corresponding Gauss–Christoffel quadratures for any number of nodes less than or equal to n, etc.

An important progress for strongly non–classical measures was given by Walter Gautschi. In [7] he started with an arbitrary positive measure $\mathrm{d}\mu(t)$, which is given explicitly or implicitly via moment information, and considered the actual (numerical) construction of orthogonal polynomials as a basic computational problem: *For a given measure* $\mathrm{d}\mu$ *and for given* $n \in \mathbb{N}$, *generate the first coefficients* $\alpha_k(\mathrm{d}\mu)$ *and* $\beta_k(\mathrm{d}\mu)$, $k = 0, 1, \ldots, n-1$. In about two dozen papers, Gautschi developed the so–called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, including effective algorithms for numerically generating orthogonal polynomials, a detailed stability analysis of such algorithms, the corresponding software implementation, etc. (cf. [8], [9], [10], [20], [21]).

Following [10] we mention here some basic facts in the constructive theory of orthogonal polynomials and Gaussian quadratures. We consider two tasks:

(a) *Construction of recursion coefficients* α_k, β_k, $k = 0, 1, \ldots, n-1$;

(b) *Construction of the Gauss-Christoffel quadrature* (1.1), i.e.,

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\mu(t) = \sum_{k=1}^{n} A_k f(\tau_k) + R_n(f). \tag{3.2}$$

The first construction (a) is, in fact, a map, in notation $\mathbf{K}_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, of the first $2n$ moments to $2n$ recursive coefficients,

$$\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_{2n-1}) \;\mapsto\; \boldsymbol{\rho} = (\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1}).$$

An important aspect in the numerical construction (a) is the sensitivity of this problem with respect to small perturbation in the data, i.e., perturbations in the first $2n$ moments μ_k, $k = 0, 1, \ldots, 2n-1$ (when we calculate coefficients for $k \le n-1$).

There is a simple algorithm, due to Chebyshev, which transforms the moments to desired recursion coefficients, but its viability is strictly dependent on the conditioning of this mapping. Usually it is severely ill conditioned so that these calculatios via moments, in finite precision on a computer, are quite ineffective, especially for measures on unbounded supports. The only salvation, in this case, is to either use symbolic computation, which however requires special resources and often is not possible, or else to use the explicit form of the measure. In the latter case, an appropriate discretization of the measure and subsequent approximation of the recursion coefficients is a viable alternative.

In his analysis, Gautschi introduced also another map $\mathbf{G}_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, as a map of moments into the parameters of the Gauss-Christoffel quadrature (3.2),

$$\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_{2n-1}) \ \mapsto \ \boldsymbol{\gamma} = (A_1, \ldots, A_n, \tau_1, \ldots, \tau_n),$$

and represented it as a *composition* of two maps

$$\mathbf{K}_n = \mathbf{H}_n \circ \mathbf{G}_n,$$

where $\mathbf{H}_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ maps the Gaussian parameters into the recursion coefficients, $\boldsymbol{\gamma} \ \to \ \boldsymbol{\rho}$.

The map $\mathbf{H}_n$, as well as its inverse map $\mathbf{H}_n^{-1}$, are generally *well-conditioned*, and the condition of $\mathbf{K}_n$ is more or less the same as the condition of $\mathbf{G}_n$. Notice that an implementation of the map $\mathbf{H}_n^{-1}$ can be done by the Golub-Welsch procedure.

The map $\mathbf{G}_n$ is usually *ill-conditioned*, i.e., its condition number is much larger than one, $\operatorname{cond} \mathbf{G}_n \gg 1$. If the condition number is of order 10^m, it roughly means a loss of m decimal digits in results when the input data are perturbed by one units in the last digit. For example, if the working precision is d decimal digits, e.g., $d = 16$ and the condition number is 10^{14}, then results will be accurate to about $16 - 14 = 2$ digits!

The (absolute) condition number of the map $\mathbf{G}_n$ is defined as a norm of the Fréchet derivative of this map,

$$(\operatorname{cond} \mathbf{G}_n)(\boldsymbol{\mu}) = \left\| \frac{\partial \mathbf{G}_n(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}} \right\|.$$

Otherwise, the Fréchet derivative is a linear transformation defined by the Jacobian matrix.

In order to determine $\operatorname{cond} \mathbf{G}_n$, Gautschi introduced the inverse map of $\mathbf{G}_n$ as $\mathbf{F}_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ $(\boldsymbol{\gamma} = (A_1, \ldots, A_n, \tau_1, \ldots, \tau_n) \to \boldsymbol{\mu})$, defined by

$$\mu_k = \sum_{\nu=1}^{n} A_\nu \tau_\nu^k, \quad k = 0, 1, \ldots, 2n-1. \tag{3.3}$$

In fact, (3.3) is a system of $2n$ non-linear equations obtained from (3.2) by taking $f(t) = t^k$, $k = 0, 1, \ldots, 2n-1$, for which the remainder term $R_n(f)$ is equal to zero. It is clear that

$$\frac{\partial \mathbf{G}_n(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}} = \begin{bmatrix} \dfrac{\partial \mu_0}{\partial A_1} & \cdots & \dfrac{\partial \mu_0}{\partial A_n} & \dfrac{\partial \mu_0}{\partial \tau_1} & \cdots & \dfrac{\partial \mu_0}{\partial \tau_n} \\ \vdots & & & & & \\ \dfrac{\partial \mu_{2n-1}}{\partial A_1} & \cdots & \dfrac{\partial \mu_{2n-1}}{\partial A_n} & \dfrac{\partial \mu_{2n-1}}{\partial x\tau_1} & \cdots & \dfrac{\partial \mu_{2n-1}}{\partial \tau_n} \end{bmatrix} = \mathbf{T}\boldsymbol{\Lambda},$$

where $\mathbf{\Lambda} = \mathrm{diag}(1, \ldots, 1, A_1, \ldots, A_n)$ and $\mathbf{T}$ is a confluent Vandermonde matrix

$$\mathbf{T} = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \tau_1 & \cdots & \tau_n & 1 & \cdots & 1 \\ \tau_1^2 & \cdots & \tau_n^2 & 2\tau_1 & \cdots & 2\tau_n \\ \vdots & & & & & \\ \tau_1^{2n-1} & \cdots & \tau_n^{2n-1} & (2n-1)\tau_1^{2n-1} & \cdots & (2n-1)\tau_n^{2n-1} \end{bmatrix}.$$

Since

$$\frac{\partial \mathbf{G}_n}{\partial \boldsymbol{\mu}} = \left(\frac{\partial \mathbf{F}_n}{\partial \boldsymbol{\gamma}}\right)^{-1} = \mathbf{\Lambda}^{-1}\mathbf{T}^{-1},$$

the following expression for calculating the condition number

$$(\mathrm{cond}\,\mathbf{G}_n)(\boldsymbol{\mu}) = \left\|\mathbf{\Lambda}^{-1}\mathbf{T}^{-1}\right\|$$

holds. Several estimates of $(\mathrm{cond}\,\mathbf{G}_n)(\boldsymbol{\mu})$ and examples for different measures can be found in [10]. As a rule, the conditional number grows exonentially fast with n (see Fig. 1).

Suppose that we have a numerical method for realizing the mapping $\mathbf{G}_n$ in an arithmetic with the working precision of d decimal digits. Then, the accuracy of results (here, the recursion coefficients α_k and β_k) depends on the working precision, but also on the condition number of this mapping. Roughly speaking, if we need the accuracy of ℓ decimal digits in results for each $k < n$, then the condition number $(\mathrm{cond}\,\mathbf{G}_n)$ must be less than 10^m, where $m = d - \ell$. For example, among the methods (A), (B), (C) (see Fig. 1), only the method (C) provides the required accuracy for a fixed $n = N$.

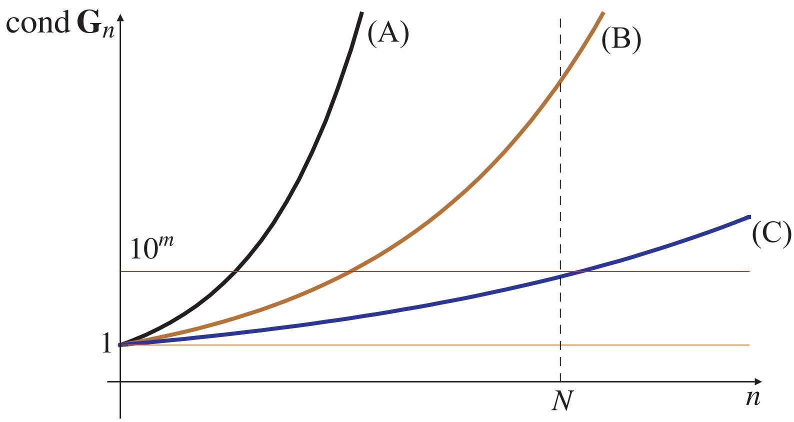

FIGURE 1. The condition number $\mathrm{cond}\,\mathbf{G}_n$ for three different methods of construction (A), (B) and (C)

There are three basic procedures for generating the recursion coefficients: (1) the *method of* (*modified*) *moments*, (2) the *discretized Stieltjes–Gautschi procedure*, (3) the *Lanczos algorithm*, and they play the central role in the constructive theory of orthogonal polynomials (cf. [7], [9], [10], [14]).

Recent progress in symbolic computation and variable-precision arithmetic now makes it possible to generate the recurrence coefficients α_k and β_k directly by using the original Chebyshev method of moments, but in a sufficiently high precision arithmetic, i.e., we should take the working precision to be $d = \ell + m$. Such an approach enables us to overcome the numerical instability!

Respectively symbolic/variable-precision software for orthogonal polynomials is available: Gautschi's package `SOPQ` in MATLAB and our MATHEMATICA package `OrthogonalPolynomials` (see [1] and [23]), which is downloadable from the web site `http://www.mi.sanu.ac.rs/~gvm/`.

All that is required is a procedure for symbolic calculation of the moments or their numerical calculation in variable-precision arithmetic. Details on applications this package to construction of recursive coefficients and parameters of Gaussian formulas will be done in the next section.

4. Construction of orthogonal polynomials and quadratures for some non-classical weights

4.1. Some distributions in physics

Bose-Einstein and Fermi-Dirac weights on $\mathbb{R}^+$ are defined by

$$\varepsilon(t) = \frac{t}{\mathrm{e}^t - 1} \quad \text{and} \quad \varphi(t) = \frac{1}{\mathrm{e}^t + 1}, \tag{4.1}$$

respectively. These functions and the corresponding quadratures are widely used in solid state physics, e.g., the total energy of thermal vibration of a crystal lattice can be expressed in the form $\int_0^{+\infty} f(t)\varepsilon(t)\,\mathrm{d}t$, where $f(t)$ is related to the phonon density of states. Integrals with $\varphi(t)$ are encountered in the dynamics of electrons in metals. Also, integrals of the previous type can be used for summation of slowly convergent series (see Section 5).

The moments of the functions (4.1) can be exactly calculated in terms of Riemann zeta function as

$$\mu_k(\varepsilon) = \int_0^{+\infty} \frac{t^{k+1}}{\mathrm{e}^t - 1}\,\mathrm{d}t = (k+1)!\zeta(k+2), \quad k \in \mathbb{N}_0,$$

and

$$\mu_k(\varphi) = \int_0^{+\infty} \frac{t^k}{\mathrm{e}^t + 1}\,\mathrm{d}t = \begin{cases} \log 2, & k = 0, \\ (1 - 2^{-k})k!\zeta(k+1), & k > 0, \end{cases}$$

respectively, and these moments are enough for constructing recursive coefficients in the corresponding three-term recurrence relations for orthogonal polynomials with respect to the weight functions (4.1).

For example, using our MATHEMATICA package `OrthogonalPolynomials` (see [1] and [23]) and executing the following commands (for Einstein's weight):

```
<< orthogonalPolynomials‘
mEin =Table[(k+1)! Zeta[k+2], {k,0,100}];
{alE, beE} = aChebyshevAlgorithm[mEin, WorkingPrecision->55];
{alE1,beE1}=aChebyshevAlgorithm[mEin, WorkingPrecision -> 80];
N[Max[Abs[alE/alE1-1], Abs[beE/beE1-1]],3]
```

we obtain the first 50 recurrence coefficients with the maximal relative error 3.31×10^{-21}, using the working precision of 55 decimal digits. Notice that for calculating this maximal relative error in recursive coefficients we have to compute them with some better precision (in this case we used 80 decimal digits).

Now, we can calculate Gaussian parameters (nodes and weights) for each $n \leq 50$. For example, for $n = 10$ we have:

```
PGQ[n_] :=aGaussianNodesWeights[n,alE,beE,WorkingPrecision->25,
          Precision->20]
{n10, w10} = N[PGQ[10],20]

  {{0.17127645878001723630, 0.89167285640716281560,
    2.1546962419952769267, 3.9409621944320753085,
    6.2730549781202005837, 9.2198332084047489872,
    12.896129024261770678, 17.492620202296984539,
    23.375068766890757875, 31.480929908705477946},
   {0.40175819838719705508, 0.61781515020685988777,
    0.43092384916712431584, 0.16018318534772922234,
    0.031116001568317075487, 0.0030029502799063140584,
    0.00013244003563186081692, 2.2807340153227672644*10^-6,
    1.1114755872888526597*10^-8,6.6895094339315858173*10^-12}}
```

For details see [11], [20], [23].

4.2. Exotic exponential weights on $\mathbb{R}^+$

In this subsection we mention only the weight function of the form $w(t) = w^{(\alpha,\beta)}(x) = \exp(-t^{-\alpha} - t^{\beta})$ on $\mathbb{R}^+$, with parameters $\alpha > 0$ and $\beta > 1$.

In a simpler case when $\alpha = \beta$, we can determine the moments in an analytic form as

$$\mu_k^{(\beta,\beta)} = \int_0^{+\infty} t^k w^{(\beta,\beta)}(t)\,\mathrm{d}t = \frac{2}{\beta} K_{(k+1)/\beta}(2), \quad k \in \mathbb{N}_0, \tag{4.2}$$

where $K_r(z)$ is the modified Bessel function of the second kind.

The general case $w(t) = w^{(\alpha,\beta)}(x)$, $\alpha \neq \beta$, can be solved by the the so-called Meijer G function

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) \equiv G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1,\ldots,a_n; a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m; b_{m+1},\ldots,b_q \end{matrix}\right)$$

$$= \frac{1}{2\pi \mathrm{i}} \int_L \frac{\prod_{\nu=1}^{m} \Gamma(b_\nu - s) \prod_{\nu=1}^{n} \Gamma(1 - a_\nu + s)}{\prod_{\nu=m+1}^{q} \Gamma(1 - b_\nu + s) \prod_{\nu=n+1}^{p} \Gamma(a_\nu - s)} z^s \,\mathrm{d}s.$$

For some specific values of α i β we have (see [15])

$$\mu_k^{(1,2)} = \frac{1}{2^{k+2}\sqrt{\pi}} G_{2,4}^{3,1}\left(\frac{1}{4}\,\middle|\, \begin{array}{c} -\,;- \\ -\frac{k+1}{2}, -\frac{k}{2}, 0; - \end{array}\right), \quad k \ge 0;$$

$$\mu_k^{(2,1)} = \frac{2^k}{\sqrt{\pi}} G_{2,4}^{3,1}\left(\frac{1}{4}\,\middle|\, \begin{array}{c} -\,;- \\ 0, \frac{k+1}{2}, \frac{k+2}{2}; - \end{array}\right), \quad k \ge 0;$$

$$\mu_k^{(1,3)} = \frac{1}{2\cdot 3^{k+3/2}\pi} G_{2,5}^{4,1}\left(\frac{1}{27}\,\middle|\, \begin{array}{c} -\,;- \\ -\frac{k+1}{3}, -\frac{k}{3}, -\frac{k-1}{3}, 0; - \end{array}\right), \quad k \ge 0.$$

As an example we take $\alpha = \beta = 2$. In order to generate quadratures, for example, for $m \le n = 100$, we need the first two hundred moments $\mu_k^{(2,2)}$, given by (4.2). Using the Mathematica package `OrthogonalPolynomials`, with the following commands

```
<< orthogonalPolynomials`
mom = Table[BesselK[(k+1)/2, 2], {k,0,200}];
{al,be} = aChebyshevAlgorithm[mom, WorkingPrecision -> 120];
{al1,be1} = aChebyshevAlgorithm[mom, WorkingPrecision -> 140];
N[Max[Abs[al/al1 - 1], Abs[be/be1 - 1]], 3]
```

we obtain the first 100 recursive coefficients with relative errors less than 2.21×10^{-23}. As we can see, the calculation of the recursive coefficients in this case is a very sensitive process, which here, in the worst case, causes a loss of about 98 decimal digits!

The corresponding Gaussian quadrature formulas have an application in integration of functions which can *increase exponentially* at the endpoints 0 and $+\infty$. For the so-called "truncated" Gaussian quadratures the stability and convergence with the order of the best polynomial approximation in suitable function spaces are proved in [15].

4.3. Some distribution in statistics

Following Stoyanov [26, §7.1] we give an example with the inverse Gaussian distribution (IG) with "easy" parameters, say $(1,1)$. Thus, we consider a random variable $\theta \sim \mathrm{IG}$, with density function

$$w_1(x) = \begin{cases} \dfrac{\mathrm{e}}{\sqrt{2\pi}}\, x^{-3/2} \exp\Big[-\dfrac{1}{2}\Big(x + \dfrac{1}{x}\Big)\Big], & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

In terms of the modified Bessel function of the second kind, we have its moments as

$$\mu_k^{(1)} = \int_0^{+\infty} x^k w_1(x)\,\mathrm{d}x = \mathrm{e}\sqrt{\frac{2}{\pi}}\, K_{k-1/2}(1), \quad k \in \mathbb{N}_0.$$

Now, taking `WorkingPrecision -> WP` (WP=50) in the package `OrthogonalPolynomials`, we can obtain the first 50 recurrence coefficients for orthogonal polynomials with respect to this weight function $w_1(x)$, with the maximal relative error 1.88×10^{-25}.

If we consider a power transformation of θ, i.e., θ^r for a real r, for example $r = 3$, the density function of the random variable $X = \theta^3$ is given by (see [26, §7.1])

$$w_3(x) := \begin{cases} \dfrac{\mathrm{e}}{3\sqrt{2\pi}}\, x^{-7/6} \exp\Big[-\dfrac{1}{2}\Big(x^{1/3} + \dfrac{1}{x^{1/3}}\Big)\Big], & \text{if } \ x > 0, \\ 0, & \text{if } \ x \le 0, \end{cases}$$

and its moments are

$$\mu_k^{(3)} = \int_0^{+\infty} x^k w_3(x)\,\mathrm{d}x = \mathrm{e}\sqrt{\frac{2}{\pi}}\, K_{3k-1/2}(1), \quad k \in \mathbb{N}_0.$$

Now, the construction problem is slightly better conditioned. Namely, in this case in order to obtain the first 50 recurrence coefficients with a similar maximal relative error (3.84×10^{-26}) we need only WP=35, i.e, 15 digits less!

Graphs of previous weight functions w_1 and w_3 are displayed in Fig. 2.

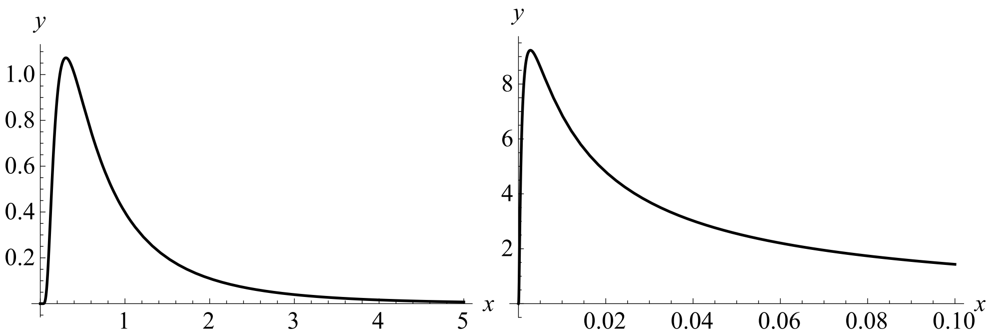

FIGURE 2. Graphs of $x \mapsto w_1(x)$ (left) and $\mapsto w_3(x)$ (right)

In order to calculate the following integral

$$\int_{\mathbb{R}} w_1(x) \cos x \,\mathrm{d}x = 0.538295818310337041115777\ldots,$$

we apply n-point Gaussian quadratures obtained for each $n \le 50$ by the following commands

```
<< orthogonalPolynomials`
 f[x_] := Cos[x]; exact = 0.538295818310337041115777;
 mom=Table[Exp[1] Sqrt[2/Pi] BesselK[k-1/2, 1], {k,0,99}];
 {alB,beB}=aChebyshevAlgorithm[mom, WorkingPrecision -> 50];
 PQ[n_] :=aGaussianNodesWeights[n, alB, beB,
           WorkingPrecision -> 25,Precision -> 20];
 ss = Table[N[PQ[n][[2]].f[PQ[n][[1]]], 20], {n,5,50,5}];
 err = Table[N[Abs[ss[[k]]/exact - 1], 3], {k, 1, 10}];
```

Gaussian approximations $Q_n(f; w_1)$ and the corresponding relative errors err(n) for $n = 5(5)50$ are presented in Table 1. Numbers in parenthesis indicate the decimal exponents.

TABLE 1. Gaussian approximations $Q_n(f; w_1)$ and relative errors for $f(x) = \cos x$

n	$Q_n(f; w_1)$	err(n)
5	0.54279156780936401515	8.35(−3)
10	0.53844179972070903368	2.71(−4)
15	0.53829287281685621212	5.47(−6)
20	0.53829574913263199781	1.29(−7)
25	0.53829582036400719491	3.82(−9)
30	0.53829581835353617306	8.03(−11)
35	0.53829581830877861990	2.90(−12)
40	0.53829581831030650190	5.67(−14)
45	0.53829581831033828714	2.31(−15)
50	0.53829581831033706428	4.30(−17)

In this subsection we also mention a few distribution for which, using the MATHEMATICA package `OrthogonalPolynomials`, we can get the recursion coefficients for $k \le n$ in a symbolic form, where n is a finite number. Assuming these expressions as hypothesis, in some cases we can prove the analytic expressions for recurrence coefficients.

First, we consider the Stieltjes-Wigert weight function

$$w(x) := \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} \exp\Big[-\dfrac{\log^2(x)}{2\sigma^2}\Big], & \text{if } \ x > 0, \\ 0, & \text{if } \ x \le 0, \end{cases}$$

for which the moments are given by

$$\mu_k = \int_0^{+\infty} x^k w(x)\, \mathrm{d}x = q^{k^2/2}, \quad k \in \mathbb{N}_0 \quad (q = \mathrm{e}^{\sigma^2}).$$

In this case, executing the following commands

```
<< orthogonalPolynomials`
mom = Table[q^(k^2/2), {k, 0, 39}];
{al, be} = aChebyshevAlgorithm[mom, Algorithm -> Symbolic]
```

we can obtain the first twenty coefficients in the three-term recurrence relation in an analytic form, and then prove that

$$\alpha_k = q^{k-1/2}\big(q^{k+1} + q^k - 1\big); \quad \beta_0 = 1, \ \beta_k = q^{3k-2}\big(q^k - 1\big), \quad k = 0, 1, \dots .$$

Similarly, for the weight function on $\mathbb{R}$ given by

$$w(x) = \frac{x^2 \mathrm{e}^{-\pi x}}{(1 - \mathrm{e}^{-\pi x})^2} = \Big(\frac{x}{2\sinh(\pi x/2)}\Big)^2 = \frac{1}{4}[w^A(x/2)]^2,$$

where $w^A(x)$ is the Abel weight on $\mathbb{R}$ (see [14, p. 159]), we can determine the moments in terms of Bernoulli numbers

$$\mu_k = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{k/2} 2^{k+2} \dfrac{B_{k+2}}{\pi}, & k \text{ is even}. \end{cases}$$

Using the package package `OrthogonalPolynomials`, for the corresponding sequence $\{\beta_k\}_{k\geq 0}$ we obtain (see [22])

$$\left\{ \frac{2}{3\pi}, \frac{4}{5}, \frac{72}{35}, \frac{80}{21}, \frac{200}{33}, \frac{1260}{143}, \frac{784}{65}, \frac{1344}{85}, \frac{6480}{323}, \frac{3300}{133}, \frac{4840}{161}, \frac{20592}{575}, \frac{9464}{225}, \dots \right\}.$$

After some experiments, we conjectured and proved that

$$\beta_0 = \mu_0 = \frac{2}{3\pi}, \quad \beta_k = \frac{k(k+1)^2(k+2)}{(2k+1)(2k+3)}, \quad k \in \mathbb{N}.$$

Finally, for the weight function on $\mathbb{R}$, given by

$$w(x) = x^2 \frac{e^{\pi x/2} + e^{-\pi x/2}}{(e^{\pi x/2} - e^{-\pi x/2})^2} = 2\cosh\frac{\pi x}{2}\Big(\frac{x}{2\sinh(\pi x/2)}\Big)^2,$$

we get the moments (cf. [22])

$$\mu_k = \begin{cases} 0, & k \text{ is odd}, \\ \dfrac{2^{k+3}}{\pi}(2^{k+2}-1)|B_{k+2}|, & k \text{ is even}. \end{cases}$$

In this case we have that

$$\beta_0 = \mu_0 = \frac{4}{\pi}, \quad \beta_k = \begin{cases} (k+1)^2, & k \text{ is odd}, \\ k(k+2), & k \text{ is even}. \end{cases}$$

5. Summation of slowly convergent series

There are many methods for fast summation of slowly convergent series. In this section we consider only the so-called *summation/integration* procedures. The basic idea in such procedures is to transform the sum to an integral with respect to some weight function on $\mathbb{R}$ (or $\mathbb{R}_+$), and then to approximate this integral by a finite quadrature sum,

$$\sum_{k=1}^{+\infty} (\pm 1)^k f(k) = \int_{\mathbb{R}} g(x) w(x)\,dx \approx \sum_{\nu=1}^{N} A_\nu g(x_\nu),$$

where the function g is connected with f in some way. Thus, these procedures need two steps:

(a) Methods of transformation $\sum \Rightarrow \int$;

(b) Construction of Gaussian quadratures

$$\int_{\mathbb{R}} g(x) w(x)\,dx = \sum_{\nu=1}^{N} A_\nu g(x_\nu) + R_n(f),$$

where w is a *non-classical* weight.

5.1. Laplace transformation method

In this subsection we mention only the basic idea of the *Laplace transform method.*

Suppose that the general term of series is expressible in terms of the Laplace transform, or its derivative, of a known function.

Let $f(s) = \int_0^{+\infty} \mathrm{e}^{-st} g(t)\,\mathrm{d}t$, $\operatorname{Re} s \geq 1$. Then

$$T = \sum_{k=1}^{+\infty} f(k) = \sum_{k=1}^{+\infty} \int_0^{+\infty} \mathrm{e}^{-kt} g(t)\,\mathrm{d}t = \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} \mathrm{e}^{-kt} \right) g(t)\,\mathrm{d}t,$$

i.e.,

$$T = \int_0^{+\infty} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\, g(t)\,\mathrm{d}t = \int_0^{+\infty} \frac{t}{\mathrm{e}^t - 1} \frac{g(t)}{t}\,\mathrm{d}t.$$

Thus, the *summation of series* is now transformed to an *integration problem* with respect to the Bose-Einstein weight function $\varepsilon(t) = t/(\mathrm{e}^t - 1)$ on $\mathbb{R}^+$, which is considered in Subsection 4.1.

Similarly, for "alternating" series, we have

$$S = \sum_{k=1}^{+\infty} (-1)^k f(k) = \int_0^{+\infty} \frac{1}{\mathrm{e}^t + 1} (-g(t))\,\mathrm{d}t, \tag{5.1}$$

where the Fermi-Dirac weight function on $\mathbb{R}^+$, $\varphi(t) = 1/(\mathrm{e}^t + 1)$, is appeared on the right-hand side in (5.1).

For details and examples see [11], [18], [22].

5.2. Hyperbolic weight functions and $\sum \Rightarrow \int$ transformation

In this subsection we consider an alternative summation/integration procedure for the series

$$T_{m,n} = \sum_{k=m}^{n} f(k) \quad \text{and} \quad S_{m,n} = \sum_{k=m}^{n} (-1)^k f(k), \tag{5.2}$$

where $m, n \in \mathbb{Z}$ $(m < n \leq +\infty)$ and the function f is holomorphic in the region

$$\bigl\{ z \in \mathbb{C} \;\big|\; \operatorname{Re} z \geq \alpha,\; m-1 < \alpha < m \bigr\}. \tag{5.3}$$

Our method of transformation "sum" to "integral" requires the indefinite integral F of f chosen so as to satisfy the following decay properties (see [16], [14]),

(C1) *F is a holomorphic function in the region* (5.3);

(C2) $\lim\limits_{|t|\to+\infty} \mathrm{e}^{-c|t|} F(x + \mathrm{i}t/\pi) = 0$, *uniformly for* $x \geq \alpha$;

(C3) $\lim\limits_{x\to+\infty} \int_{\mathbb{R}} \mathrm{e}^{-c|t|} \left| F(x + \mathrm{i}t/\pi) \right| \mathrm{d}t = 0$,

where $c = 2$ or $c = 1$, when we consider $T_{m,n}$ or $S_{n,m}$, respectively.

Let $m - 1 < \alpha < m$, $n < \beta < n+1$, $\delta > 0$, and

$$G = \Bigl\{ z \in \mathbb{C} \,:\, \alpha \leq \operatorname{Re} z \leq \beta,\; |\operatorname{Im} z| \leq \frac{\delta}{\pi} \Bigr\}.$$

Using contour integration of a product of functions $z \mapsto f(z)g(z)$ over the rectangle $\Gamma = \partial G$ in the complex plane, where $g(z) = \pi/\tan \pi z$ and $g(z) = \pi/\sin \pi z$, by Cauchy's residue theorem, we obtain

$$T_{m,n} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \frac{\pi}{\tan \pi z}\, \mathrm{d}z \quad \text{and} \quad S_{m,n} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \frac{\pi}{\sin \pi z}\, \mathrm{d}z.$$

After integration by parts, these formulas reduce to

$$T_{m,n} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \left(\frac{\pi}{\sin \pi z}\right)^2 F(z)\, \mathrm{d}z, \quad S_{m,n} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \left(\frac{\pi}{\sin \pi z}\right)^2 \cos \pi z\, F(z)\, \mathrm{d}z,$$

where F is an integral of f.

Finally, setting $\alpha = m - 1/2$, $\beta = n + 1/2$, and letting $\delta \to +\infty$, under conditions (C1), (C2), and (C3), the previous integrals over Γ reduce to the weighted integrals over $(0, +\infty)$, giving transformations

$$\sum_{k=m}^{+\infty} f(k) = \int_0^{+\infty} w_1(t) \Phi\left(m - \frac{1}{2}, \frac{t}{\pi}\right) \mathrm{d}t \tag{5.4}$$

and

$$\sum_{k=m}^{+\infty} (-1)^k f(k) = (-1)^m \int_0^{+\infty} w_2(t) \Psi\left(m - \frac{1}{2}, \frac{t}{\pi}\right) \mathrm{d}t, \tag{5.5}$$

where the weight functions are given by

$$w_1(t) = \frac{1}{\cosh^2 t} \quad \text{and} \quad w_2(t) = \frac{\sinh t}{\cosh^2 t}, \tag{5.6}$$

respectively. Here F is an integral of f, as well as

$$\Phi(x, y) = -\frac{1}{2}\left[F(x + \mathrm{i}y) + F(x - \mathrm{i}y)\right] = -\mathrm{Re}\, F(x + \mathrm{i}y)$$

and

$$\Psi(x, y) = \frac{1}{2\mathrm{i}}\left[F(x + \mathrm{i}y) - F(x - \mathrm{i}y)\right] = \mathrm{Im}\, F(x + \mathrm{i}y).$$

The second our task is a numerical construction of Gaussian quadratures with respect to the hyperbolic weights w_1 and w_2, given in (5.6),

$$\int_0^{+\infty} g(t) w_s(t)\, \mathrm{d}t = \sum_{\nu=1}^{N} A_{\nu,s}^N g(\tau_{\nu,s}^N) + R_{N,s}(g) \quad (s = 1, 2), \tag{5.7}$$

with weights $A_{\nu,s}^N$ and nodes $\tau_{\nu,s}^N$, $\nu = 1, \ldots, N$ $(s = 1, 2)$, which are exact for all $g \in \mathcal{P}_{2N-1}$.

The moments of the hyperbolic weights w_1 and w_2 can be expressed in explicit form (see [22])

$$\mu_k^{(1)} = \int_0^{+\infty} t^k w_1(t)\, \mathrm{d}t = \begin{cases} 1, & k = 0, \\ \log 2, & k = 1, \\ (2^{k-1} - 1)k!/4^{k-1}\zeta(k), & k \geq 2; \end{cases}$$

and

$$\mu_k^{(2)} = \int_0^{+\infty} t^k w_2(t)\,\mathrm{d}t = \begin{cases} 1, & k = 0, \\ k\left(\dfrac{\pi}{2}\right)^k |E_{k-1}|, & k\,(\text{odd}) \ge 1, \\ \dfrac{2k}{4^k}\left[\psi^{(k-1)}(\tfrac{1}{4}) - \psi^{(k-1)}(\tfrac{3}{4})\right], & k\,(\text{even}) \ge 2, \end{cases}$$

where $\zeta(k)$ is the Riemann zeta function, E_k are Euler's numbers, defined by the generating function

$$\frac{2}{\mathrm{e}^t + \mathrm{e}^{-t}} = \sum_{k=0}^{+\infty} E_k \frac{t^k}{k!},$$

and $\psi(z)$ is the so-called digamma function, i.e., the logarithmic derivative of the gamma function, $\psi(z) = \Gamma'(z)/\Gamma(z)$. MATHEMATICA evaluates derivatives $\psi^{(n)}(z)$ to arbitrary numerical precision, using the function `PolyGamma[n,z]`.

In order to construct Gaussian rules with respect to the weight $w_2(t)$ on $(0, +\infty)$ for $N \le 50$, we need the recursion coefficients α_k and β_k for $k \le N - 1 = 49$, i.e., the moments for $k \le 2N - 1 = 99$. Taking the `WorkingPrecision` to be 100, and executing the following commands:

```
<< orthogonalPolynomials`
   mom2=Join[{1},Table[If[OddQ[k],k(Pi/2)^k Abs[EulerE[k-1]],
       2k/4^k(PolyGamma[k-1,1/4]-PolyGamma[k-1,3/4])],{k,1,99}]];
   {al,be}=aChebyshevAlgorithm[mom2, WorkingPrecision -> 100];
```

we obtain the first 50 recursion coefficients α_k and β_k, with the relative errors less than 6.18×10^{-60}.

In construction the corresponding recursion coefficients for the weight $w_1(t)$ on $(0, +\infty)$ for $N \le 50$, the second line in the previous commands should be replaced by

```
mom1=Join[{1,Log[2]},Table[(2^(k-1)-1)k!/4^(k-1)Zeta[k],{k,2,99}]];
```

In this case, the first 50 recursion coefficients are obtained with slightly better accuracy (precisely, with the maximal relative error 3.65×10^{-63}).

These 50 recursive coefficients are enough for constructing Gaussian formulas (5.7) for each $N \le 50$ and $s = 1, 2$.

Example 5.1. Now we consider a typical slowly convergent series

$$T(p) = \sum_{k=1}^{+\infty} \frac{1}{k^{1/p}(k+1)}, \quad p \ge 1, \tag{5.8}$$

which can be also represented in the form, by extracting a finite number of terms,

$$T(p) = \sum_{k=1}^{m-1} \frac{1}{k^{1/p}(k+1)} + \sum_{k=m}^{+\infty} \frac{1}{k^{1/p}(k+1)}. \tag{5.9}$$

Then, we apply our integral transformation (5.4) to the second (infinity) series in (5.9). Thus, using Gaussian quadrature formula (5.7) with respect to the weight $w_1(t) =$

$1/\cosh^2 t$ on $\mathbb{R}_+$, we obtain

$$T(p) \approx Q_m^{(N)}(p) = \sum_{k=1}^{m-1} \frac{1}{k^{1/p}(k+1)} + \sum_{\nu=1}^{N} A_{\nu,1}^{N} \Phi_p(m-1/2, \tau_{\nu,1}^{N}/\pi), \tag{5.10}$$

with $\Phi_p(x,y) = -\frac{1}{2}\left[F_p(x+\mathrm{i}y) + F_p(x-\mathrm{i}y)\right]$, where $\tau_{\nu,1}^N$ and $A_{\nu,1}^N$ are nodes and weights of the N-point Gaussian rule (5.7) ($s=1$).

Taking different values for m, we can notice the change rate of convergence of this quadrature processes. Namely, the rapidly increasing of convergence of the summation process as m increases is due to the singularities (poles) of $\Phi_p(m-1/2, z/\pi)$ moving away from the real line (see [20] and [22]). Here, $f_p(z) = 1/(z^{1/p}(z+1))$ and

$$F_1(z) = \log(z) - \log(z+1), \qquad F_2(z) = 2\arctan\left(\sqrt{z}\right) - \pi,$$

$$F_3(z) = \frac{1}{2}\log\frac{z+1}{\left(\sqrt[3]{z}+1\right)^3} + \sqrt{3}\arctan\left(\frac{2\sqrt[3]{z}-1}{\sqrt{3}}\right) - \frac{\pi\sqrt{3}}{2}, \quad \text{etc.}$$

For $p=2$ the series $T(2)$ appears in a study of spirals (cf. [2]) and defines the well-known *Theodorus constant*,

$$T(2) = \sum_{k=1}^{+\infty} \frac{1}{\sqrt{k}(k+1)} = 1.8580\ldots.$$

The first 10^6 terms of $T(2)$ give the result $T(2) \approx 1.86$ (only 3-digit accuracy).

For larger values of p, the corresponding series $T(p)$ is slower. For example, for $p=6$ an accuracy with only 3 digits in $T(6)$ by a direct summation needs 10^{18} terms. However, our summation/integration formula (5.10) for $p=6$ and $m=10$ gives approximations $Q_{10}^{(N)}(6)$, $N = 5(10)45$, which are presented in Table 2. In each entry the first digit in error is underlined.

Table 2. Gaussian approximations $Q_{10}^{(N)}(6)$ for $N = 5(10)45$

N	$Q_{10}^{(N)}(6)$
5	$5.69941177638356\underline{3}0$
15	$5.6994117763835619667430485043356\underline{4}1$
25	$5.699411776383561966743048504335627732872048290\underline{0}3$
35	$5.69941177638356196674304850433562773287204829113580049\underline{9}7$
45	$5.699411776383561966743048504335627732872048291135800493867761\underline{0}6$

As we can see, this method is very efficient; the quadrature formula with only $N=45$ nodes gives more than 60 exact digits in the sum $T(6)$!

For details and other applications see [16], [17], and [22].

6. Construction of some perfectly symmetric cubature rules

Using the MATHEMATICA package `OrthogonalPolynomias` we can construct some perfectly symmetric two-dimensional cubature formulas in $D \subset \mathbb{R}^2$ with minimal number of nodes,

$$I(f) = \iint\limits_D w(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{N} A_i f(P_i) + R_N(f). \tag{6.1}$$

Such a cubature formula has the nodes of the form $(\pm x_j, \pm y_j)$ and $(\pm y_j, \pm x_j)$ with the same weights. Regarding [25] we call it as a "good" formula if all of its weights are positive.

For completely symmetric weight functions $w(x,-y) = w(-x,y) = w(x,y) \ge 0$, the typical domains D are the square with vertices $(\pm 1, \pm 1)$, the unit circle, and the entire plane $\mathbb{R}^2$.

We recall that a two-dimensional cubature rule of degree d integrates exactly all monomials $x^i y^j$, i.e., $R_N(x^i y^j) = 0$, for which $i + j \le d$.

In order to obtain "good" cubature rules for some weights $w(x,y)$ of degree $d \le 7$, Stroud and Secrest [28] used the nodes whose "generators" are of the form $(0,0)$, $(\alpha,0)$, (β,β). For rules of degree $d \ge 8$, it is necessary to include nodes whose "generators" are of the form (γ,δ). Each of them generates eight nodes of the form: $(\pm\gamma,\pm\delta)$, $(\pm\delta,\pm\gamma)$ with the same weight, while $(\alpha,0)$ and (β,β) generate only four nodes: $(\pm\alpha,0)$, $(0,\pm\alpha)$ and $(\pm\beta,\pm\beta)$, respectively. Of course, $(0,0)$ gives only one node $(0,0)$.

Following [25], the method of construction needs integrals

$$\begin{cases} I(x^{2k}) \ \text{ and } \ I(x^{2j}y^{2k}) \\ k = 0, 1 \ldots, [N/2]; \\ 1 \le j \le k; \ j + k = 2, \ldots, [N/2], \end{cases}$$

as well as the following "special moments", i.e., integrals of the form

$$\begin{aligned} \mu_{jk} &= I\big[(x^2-y^2)^2(x^2y^2)^j(x^2+y^2)^k\big] \\ &= \int_0^{+\infty}\!\!\int_0^{+\infty} w(x,y)(x^2-y^2)^2(x^2y^2)^j(x^2+y^2)^k \,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{6.2}$$

where $j \ge 1$, $k \ge 0$. The corresponding system of nonlinear equations $R_N(x^i y^j) = 0$, $i + j \le d$, can be separated in a few systems of the Gaussian type (3.3), which can be solved using the MATHEMATICA package `OrthogonalPolynomials`.

In this section we show only construction of cubature formulas (6.1) on $D = \mathbb{R}^2$, with respect to the complete symmetric weight function of the form $w(x,y) = w_\nu(x,y)$, $\nu = 1,2,3$, where

$$w_1(x,y) = \mathrm{e}^{-(x^2+y^2)}, \quad w_2(x,y) = \mathrm{e}^{-\sqrt{x^2+y^2}}, \quad w_3(x,y) = \mathrm{e}^{-(|x|+|y|)}.$$

The special moments for the previous weight functions w_1 and w_2 can be calculated in an analytic form as

$$\mu_{jk}^{(1)} = \frac{(2j+k+2)!\pi}{2^{4j+3}(j+1)}\binom{2j}{j} \qquad \text{and} \qquad \mu_{jk}^{(2)} = \frac{(4j+2k+5)!\pi}{2^{4j+2}(j+1)}\binom{2j}{j},$$

respectively. In the third case it can be expressed by the following integral

$$\mu_{jk}^{(3)} = \frac{(4j+2k+5)!}{2^{4j+k+1}}\int_0^1 \sqrt{z}\,(1-z)^{2j}(1+z)^k\,\mathrm{d}z,$$

or in terms of hypergeometric functions as

$$\mu_{jk}^{(3)} = \frac{(4j+2k+5)!}{2^{k-2}(2j+1)\binom{4(j+1)}{2(j+1)}}\left\{{}_2F_1\left(-\frac{1}{2},-k;2j+\frac{5}{2};-1\right)\right.$$
$$\left.+\frac{k-2j-1}{2(j+1)}\,{}_2F_1\left(\frac{1}{2},-k;2j+\frac{5}{2};-1\right)\right\}.$$

Example 6.1. In order to construct 44-point cubature formulas of degree $d = 15$ with respect to the weight $w(x,y) = w_3(x,y) = \exp(-|x|-|y|)$ on $\mathbb{R}^2$, we use the following generators:

$$(u_i,0),\ i=1,2,3,4;\quad (v_i,v_i),\ j=1,2,3;\quad (w_i,z_i),\ i=1,2.$$

Generator P_i	Weight A_i	Number of points
$(u_i,0)$	a_i	4
(v_i,v_i)	b_i	4
(w_i,z_i)	c_i	8

Then the corresponding system of equations is given by

$$\sum_i c_i(w_i^2-z_i^2)^2(w_i^2z_i^2)^j(w_i^2+z_i^2)^k = \frac{1}{8}\,\mu_{jk}$$

$$j=1,\ldots,\left[\frac{m-2}{2}\right];\quad k=0,1,\ldots,m-2j-2.$$

Using the Mathematica package `OrthogonalPolynomials` for the "generator nodes" in this 44-point cubature formula of degree $d = 15$ we obtain:

```
{{16.75517334835192,0}, {9.520295794790188,0},
{4.451284933071043,0}, {1.326612922551803,0},
{10.40246868263913,10.40246868263913},
{6.307197292644404,6.307197292644404},
{2.533316709591005,2.533316709591005},
{13.16709143114937,3.265192228507983},
{6.770241049738993,2.369872911188105}},
```

and for the corresponding weight coefficients the following values:

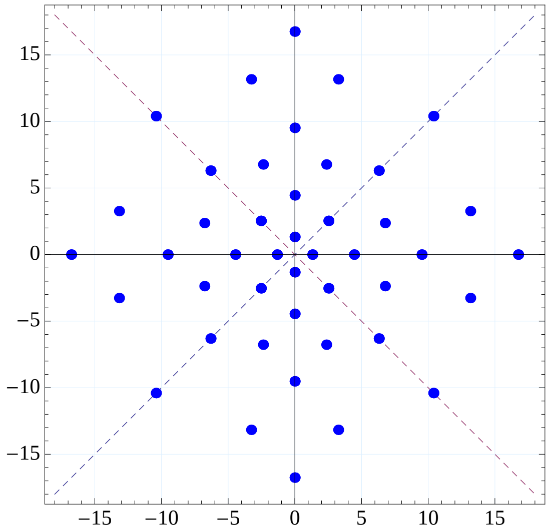

FIGURE 3. Distribution of nodes in 44-point cubature formula of degree $d = 15$ for the weight function w_3

```
{8.186694686950403*(10^-7), 0.0006529201474967032,
0.06663038092243385, 0.8569723144924805,
6.812119062461652*(10^-8), 0.00007773406088317548,
0.07219519187714604, 2.913841882561950*(10^-6),
0.001732372012567657}.
```

Finally, the distribution of nodes in this cubature formula ($N = 44$ and $d = 15$) is displayed in in Fig. 3.

Remark 6.2. Cubature formulas (6.1) with the exponential weights w_ν, $\nu = 1, 2, 3$, have been recently used in [24] and [27].

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On the unbounded divergence of interpolatory product quadrature rules on Jacobi nodes

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Abstract. This paper is devoted to prove the unbounded divergence on superdense sets, with respect to product quadrature formulas of interpolatory type on Jacobi nodes.

Mathematics Subject Classification (2010): 41A10, 41A55, 65D32.

Keywords: Product quadrature formulas, superdense sets, condensation of singularities.

1. Introduction

Let μ be the Lebesgue measure on the interval $[-1,1]$ of $\mathbb{R}$ and let denote by L_1 the Banach space of all measurable functions (equivalence classes of functions with respect to the equality $\mu-$a.e.) $g:[-1,1]\to\mathbb{R}$, such that $|g|$ is Lebesgue integrable on the interval $[-1,1]$, endowed with the norm $||g||_1=\int_{-1}^{1}|g(x)|dx, g\in L_1$. Analogously, L_∞ is the Banach space of all measurable functions (equivalence classes of functions with respect to the equality $\mu-$ a.e) $g:[-1,1]\to\mathbb{R}$, normed by $||g||_\infty=\text{ess sup}\,|g|$.

Given a nonnegative function $\rho\in L_\infty$ such that $\rho(x)>0$ μ -a.e. on [-1, 1], let consider, in accordance with [8], [9], the Banach space $(L_1^{(1/\rho)}, ||.||_1^{(1/\rho)})$, where $L_1^{(1/\rho)}$ is the set of all measurable functions (classes of functions) $g:[-1,1]\to\mathbb{R}$ for which $g/\rho\in L_1$ and $||g||^{(1/\rho)}=||g/\rho||_1$.

Further, let denote by $(C,||.||)$ the Banach space of all continuous functions $f:[-1,1]\to\mathbb{R}$, where $||.||$ stands for the uniform (supremum) norm, and let consider the Banach space $(C^s,||.||_s)$ of all functions $f:[-1,1]\to\mathbb{R}$, that are continuous together with their derivatives up to the order $s\geq 1$, endowed with the norm

$$||f||_s=\sum_{r=0}^{s-1}|f^{(r)}(0)|+||f^{(s)}||;$$

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we admit $f^{(0)} = f$ and $C^0 = C$.

For each integer $n \geq 1$, let denote by $x_n^k = \cos\ \theta_n^k, 1 \leq k \leq n, 0 < \theta_n^1 < \theta_n^2 < \ldots < \theta_n^n < \pi$, the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$, with $\alpha > -1$ and $\beta > -1$, referred to as Jacobi nodes.

We specify, also, the usual notations

$$(L_n f)(x) = \sum_{k=1}^{n} f(x_n^k) l_n^k(x),\ |x| \leq 1$$

and

$$\Lambda_n(x) = \sum_{k=1}^{n} |l_n^k(x)|;\ |x| \leq 1,\ n \geq 1,$$

denoting the Lagrange polynomials which interpolate a function $f\colon [-1,1] \to \mathbb{R}$ at the Jacobi nodes, and the Lebesgue functions associated to the Jacobi nodes, respectively.

In this paper, we deal with *product-quadrature formulas* of interpolatory type, as follows:

$$I(f;g) = I_n(f;g) + R_n(f;g), n \geq 1, f \in C, g \in L_1^{(1/\rho)}, \tag{1.1}$$

where

$$I : C \times L_1^{(1/\rho)} \longrightarrow \mathbb{R}\ I(f;g) = \int_{-1}^{1} f(x)g(x)dx, \tag{1.2}$$

and

$$I_n(f;g) = \int_{-1}^{1} (L_n f)(x)g(x)dx, n \geq 1; f \in C, g \in L_1^{(1/\rho)}. \tag{1.3}$$

Numerous papers have studied the convergence of the product quadrature formulas of type (1.1), involving Jacobi, Gauss-Kronrod or equidistant nodes and various functions $g \in L_1$ (i.e. $\rho(x) = 1,\ \forall\ x \in [-1,1]$), [1, Ch. 5], [3], [4], [5], [7], [8], [9]. Regarding the divergence of these formulas, I.H.Sloan and W.E.Smith, [9, Th.7, ii] proved the following statement in the case of *Jacobi nodes:*

If $\alpha > -1$, $\beta > -1$ and $\rho(x) = (1-x)^{\max\{0,(2\alpha+1)/4\}}(1+x)^{\max\{0,(2\beta+1)/4\}}$, then there exist a function $f_0 \in C$ and a function $g_0 \in L_1^{(1/\rho)}$ so that the sequence $I_n(f_0, g_0) : n \geq 1$ is not convergent to $I(f_0, g_0)$.

In fact, the divergence phenomenon holds on large subsets of $L_1^{(1/\rho)}$ and C, in topological sense. More exactly, the following assertion is a particular case of [6, Theorem 3.2]:

Suppose that $\mu\{x \in [-1,1] : \rho(x) > 0\} > 0$. Then, there exists a superdense set X_0 in the Banach space $L_1^{(1/\rho)}$ such that for every g in X_0 the subset of C consisting of all functions f for which the product integration rules (1.1) unboundedly diverge, namely

$$Y_0(g) = \left\{ f \in C : \sup\left\{ \left| \int_{-1}^{1} (L_n f)(x)g(x)dx \right| ; n \geq 1 \right\} = \infty \right\},$$

is superdense in the Banach space C.

We recall that a subset S of the topological space T is said to be *superdense* in T if it is residual (namely its complement is of first Baire category), uncountable and dense in T.

The aim of this paper is to highlight the phenomenon of double condensation of singularities for the product quadratures formulas (1.1) in the case of the Banach spaces $(C^s, ||.||_s)$, $s \geq 1$. If $\rho(x) = 1$, $\forall\, x \in [-1, 1]$, and $\alpha = \beta = 2$, this property was emphasized in [5, Th.3], for $s = 1$ and $s = 2$. In the next section, we point out the double superdense unbounded divergence of the formulas (1.1) for $\alpha > -1$, $\beta > -1$ and $s \geq 1$ satisfying the inequality $s < \alpha + 1/2$ or $s < \beta + 1/2$ and more general conditions regarding the function ρ.

In what follows, we denote by m, M, M_k, $k \geq 1$, some generic positive constants which are independent of any positive integer n and we use the notation $a_n \sim b_n$ if the sequences (a_n) and (b_n) satisfy the inequalities $0 < m \leq |a_n/b_n| \leq M$.

2. The unbounded divergence of the product quadrature formulas (1.1)

Let $T_n f : C^s \to (L_1^{(1/\rho)})^*$, be the continuous linear operators given by $T_n f : L_1^{(1/\rho)} \to \mathbb{R}, f \in C^s$ and $(T_n f)(g) = \int_{-1}^{1} g(x)(L_n f)(x)dx,\ g \in L_1^{(1/\rho)}$ $n \geq 1$, where $(L_1^{(1/\rho)})^*$ is the Banach space of all continuous linear functionals defined on $L_1^{(1/\rho)}$.

By standard reasoning, via the Theorem of Riesz concerning the representation of continuous linear functionals, we get:

$$||T_n|| = \sup\{||\rho L_n f||_\infty : f \in C^s, ||f||_s \leq 1\}. \tag{2.1}$$

Now, we are in the position to state the following divergence result:

Theorem 2.1. *Suppose that the integer $s \geq 0$ and the real numbers $A > 0$, $a \in (0, 1)$, $\alpha > -1$, $\beta > -1$ satisfy at least one of the following conditions:*

(i) $s < \alpha + 1/2$ and $\rho(x) \geq A$, for $x \in (a, 1)$;

(ii) $s < \alpha + 1/2$ and $\rho(1) > 0$;

(iii) $s < \beta + 1/2$ and $\rho(x) \geq A$, for $x \in (-1, -a)$;

(iv) $s < \beta + 1/2$ and $\rho(-1) > 0$.

Then, there exists a superdense set X_0 in the Banach space $L_1^{(1/\rho)}$, such that for every g in X_0 the subset of C^s consisting of all functions f for which the product integration rules (1.1) unboundedly diverge, namely

$$Y_0(g) = \left\{ f \in C^s : \sup\left\{ \left| \int_{-1}^{1} (L_n f)(x) g(x) dx \right| ; n \geq 1 \right\} = \infty \right\},$$

is superdense in the Banach space C^s.

Proof. For each integer $n \geq 2$, let us define the numbers $\delta_n^k, 1 \leq k \leq n$, and δ_n as follows: $3\delta_n^k = \min\{x_n^{k-1} - x_n^k, x_n^k - x_n^{k+1}\}$, $1 \leq k \leq n$, with $x_n^0 = 1, x_n^{n+1} = -1$, and $\delta_n = \max\{\delta_n^k, 1 \leq k \leq n\}$.

In analogy with [5, Th.2.3], we obtain:

$$||T_n|| \geq M_1\, \frac{\rho(\tau_n)}{(\delta_n)^{s+2}} \sum_{k=1}^{n} (\delta_n^k)^{2s+2} |l_n^k(\tau_n)|, \tag{2.2}$$

where τ_n is an arbitrary number of $[-1, 1]$.

For the beginning, let us suppose that the hypothesis (i) of this theorem is satisfied. The estimate sin $\theta_n^k \sim k/n$, [7], implies

$$\theta_n^k \sim k/n. \tag{2.3}$$

The relations $P_n^{(\alpha,\beta)}(x_n^1) = 0$ and $P_n^{(\alpha,\beta)}(1) \sim n^\alpha$, [10], lead to the existence of a point τ_n so that

$$\tau_n \in (x_n^1, 1);\ P_n^{(\alpha,\beta)}(\tau_n) = (1/2)P_n^{(\alpha,\beta)}(1) \sim n^\alpha. \tag{2.4}$$

Now, let us estimate δ_n^k, δ_n and $|l_n^k(\tau_n)|$.
The estimates $\theta_n^k - \theta_n^{k-1} \sim 1/n$, sin $\theta_n^k \sim k/n$ and $\theta \sim \theta_n^k$, if $\theta_n^{k-1} \le \theta \le \theta_n^k$, [7] , combined with $x_n^{k-1} - x_n^k = 2\sin(\theta_n^k - \theta_n^{k-1})/2 \sin(\theta_n^k + \theta_n^{k+1})/2$, yield:

$$\delta_n^k \sim k/n^2,\ 1 \le k \le n;\ \delta_n \sim 1/n. \tag{2.5}$$

The relation $\tau_n \in (x_n^1, 1)$ of (2.4), together with (2.3) and $x_n^1 \ge x_n^k$, $1 \le k \le n$, gives $|\tau_n - x_n^k| = \tau_n - x_n^k \le 1 - x_n^k = 2\sin^2(\theta_n^k/2) \sim k^2/n^2$, namely

$$|\tau_n - x_n^k| \le M_2 k^2/n^2, 1 \le k \le n. \tag{2.6}$$

Now, by combining the inequality (2.6) with the estimates (2.4) and $|(P_n^{(\alpha,\beta)}(x_n^k))'| \sim n^{\alpha+2}k^{-\alpha-3/2}$, if $0 < \theta_n^k < \pi/2$, [10], we get:

$$|l_n^k(\tau_n)| = |P_n^{(\alpha,\beta)}(\tau_n)||\tau_n - x_n^k|^{-1}|(P_n^{(\alpha,\beta)}(x_n^k))'|^{-1} \ge M_2 k^{\alpha-1/2}. \tag{2.7}$$

Further, the relation (2.3) with $k = 1$, together with (2.4), implies $\tau_n \in (a, 1)$, for n sufficiently large, which leads to:

$$\rho(\tau_n) \ge A > 0. \tag{2.8}$$

Finally, the relations (2.2), (2.4), (2.7) and (2.8) provide the inequality

$$||T_n|| \ge M_4 n^{\alpha+1/2-s}, \tag{2.9}$$

for n sufficiently large. Secondly, if the condition (ii) is fulfilled, we proceed in a similar manner, taking τ_n =1 in (2.2) and obtaining the unboundedness of the set of norms $\{||T_n|| : n \ge 1\}$ from an analogous inequality of (2.9). Also, it is easily seen that the hypotheses (iii) and (iv) lead to an inequality of type (2.9), namely:

$$||T_n|| \ge M_4 n^{\beta+1/2-s}, \tag{2.10}$$

for n sufficiently large.

To complete the proof, we apply, in a standard manner, firstly the principle of condensation of singularities, [2,Th.5.4], and the relations (2.9) and (2.10), in order to conclude that the set of unbounded divergence of the family $\{T_n : n \ge 1\}$ is superdense in the Banach spaces $(C^s, ||.||_s)$ and secondly, based on this result, the principle of double condensation of singularities, [2, Th.5.2], to provide the conclusion of this theorem. □

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The generalization of Mastroianni operators using the Durrmeyer's method

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Abstract. In the present paper, we define a sequence of Durrmeyer's type operators associated with Mastroianni operators and introduce a new operator.

Mathematics Subject Classification (2010): 41A25.

Keywords: Mastroianni operator, operator of Durrmeyer type, approximation properties.

1. Introduction

In [5], [6] G. Mastroianni defined and studied a general class of linear positive approximation operators, which was generalized by O. Agratini, B. Della Vecchia [1]. In brief we recall this construction.

Taking $[0,\infty) := \mathbb{R}_+$, we consider the next spaces of functions:
$B(\mathbb{R}_+) = \{f : \mathbb{R}_+ \longrightarrow \mathbb{R} | (\exists) M_f > 0 : |f(x)| \leq M_f\}$, a normed space with the uniform norm $\|f\|_B = \sup\{|f(x)| : x \in \mathbb{R}_+\}$;
$B_\rho(\mathbb{R}_+) = \left\{f : \mathbb{R}_+ \longrightarrow \mathbb{R} | \, |f(x)| \leq N_f \rho(x),\, N_f > 0,\, \rho(x) = 1 + x^2\right\}$, a normed space with the norm $\|f\|_\rho = \sup\left\{\dfrac{|f(x)|}{\rho(x)} : x \geq 0\right\} = \sup\left\{\dfrac{|f(x)|}{1+x^2} : x \geq 0\right\}$;
$C_\rho(\mathbb{R}_+) = \{f \in B_\rho(\mathbb{R}_+) | f \text{ continuous function}\}$;
$C^*_\rho(\mathbb{R}_+) = \left\{f \in C_\rho(\mathbb{R}_+) | (\exists) \lim\limits_{x\to\infty} \dfrac{|f(x)|}{1+x^2} < \infty\right\}$.
The space $C^*_\rho(\mathbb{R}_+)$ endowed with the norm $\|f\|_\rho$ is a Banach space.
In our estimations we use the first modulus of continuity on a finite interval $[0,b]$, $b > 0$, $\omega_{[0,b]}(f;\delta) = \sup\{|f(x+h) - f(x)| : 0 < h \leq \delta,\, x \in [0,b]\}$ and the Peetre's K-functional defined as

$$K_2(f;\delta) = \inf\left\{\|f-g\|_B + \delta\|g''\|_B : g \in W^2_\infty\right\},\ \delta > 0,$$

where $W^2_\infty = \{g \in C_B(\mathbb{R}_+) : g',\, g'' \in C_B(\mathbb{R}_+)\}$.

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It is known (see [9] p.177, th. 2.4) that, there exists a positive constant C such that $K_2(f;\delta) \leq C\omega_2\left(f;\sqrt{\delta}\right)$, where

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} \left\{|f(x+2h)-2f(x+h)+f(x)|\right\}.$$

Let $(\Phi_n)_{n\geq 1}$ be a sequence of real functions defined on $[0,\infty) := \mathbb{R}_+$ which are infinitely differentiable on $\mathbb{R}_+$ and satisfy the conditions:

(i) $\Phi_n(0) = 1$, $n \in \mathbb{N}$;

(ii) for every $n \in \mathbb{N}$, $x \in \mathbb{R}_+$ and $k \in \mathbb{N}\cup\{0\} := \mathbb{N}_0$,

$$(-1)^k \Phi_n^{(k)}(x) \geq 0; \tag{1.1}$$

(iii) for each $(n,k) \in \mathbb{N}\times\mathbb{N}_0$ there exists a number $p(n,k) \in \mathbb{N}$ and a function $\alpha_{n,k} \in \mathbb{R}^{\mathbb{R}}$ such that $\Phi_n^{(i+k)}(x) = (-1)^k \Phi_{p(n,k)}^{(i)}(x)\alpha_{n,k}(x)$, $i \in \mathbb{N}_0$, $x \in \mathbb{R}_+$ and

(iv) $\lim\limits_{n\to\infty} \dfrac{n}{p(n,k)} = \lim\limits_{n\to\infty} \dfrac{\alpha_{n,k}(x)}{n^k} = 1.$

Remark 1.1. There is easy to see that

$$\lim_{n\to\infty} \frac{\Phi_n^{(k)}(0)}{n^k} = \lim_{n\to\infty} \frac{(-1)^k \alpha_{n,k}(0)}{n^k} = (-1)^k. \tag{1.2}$$

The Mastroianni operators $M_n : C_B(\mathbb{R}_+) \longrightarrow C(\mathbb{R}_+)$ are defined by the following formula

$$M_n(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n}\right) \tag{1.3}$$

with the basis functions,

$$m_{n,k}(x) = \frac{(-x)^k \Phi_n^{(k)}(x)}{k!}. \tag{1.4}$$

For these operators and for the test functions $e_r(x) = x^r$, $r = 0,\ 1,\ 2$ the following results were obtained [5]:

$$\begin{aligned} M_n(e_0;x) &= \Phi_n(0), \\ M_n(e_1;x) &= -\frac{\Phi_n'(0)}{n}x, \\ M_n(e_2;x) &= \frac{\Phi_n''(0)x^2 - \Phi_n'(0)x}{n^2}. \end{aligned} \tag{1.5}$$

In terms of the hypergeometric and confluent hypergeometric functions, recent results, about Durrmeyer type operators [2], [3], [4], [8], have considered in the definition of the basis functions, the family's functions:

$$\Phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0,\ x \geq 0 \\ (1+cx)^{-\frac{n}{c}}, & c \in \mathbb{N},\ x \geq 0 \end{cases}$$

For these functions we have

$$\Phi_{n,c}^{(k+1)}(x) = -n\Phi_{n+c,c}^{(k)}(x),\ n > \max\{0,-c\}$$

respectively

$$\Phi_{n,c}^{(i+k)}(x) = (-1)^k n_{[k,-c]}\Phi_{n+kc,c}^{(i)}(x)$$

where $n_{[k,-c]} = n(n+c)(n+2c)\cdots(n+\overline{k-1}c)$ is the factorial power of order k of n with the increment $-c$ and $n_{[0,-c]} = 1$.

So, the conditions (iii)-(iv) are true, for $p(n,k) = n + kc$ and $\alpha_{n,k}(x) = n_{[k,-c]}$.

In the next section we propose a Mastroianni–Durrmeyer operator, when the sequence of functions $(\Phi_n)_{n\geq 1}$ satisfy the conditions (i)-(iv) and other supplementary conditions, is non-nominated.

2. Main results

Let $(\Phi_n)_{n\geq 1}$ be the sequence of functions which satisfy the conditions (i)-(iv) and the next supplementary conditions, for any $n \in \mathbb{N}$ and $r,\ k \in \mathbb{N}_0$:

(v) $\lim\limits_{x\to\infty} x^r \Phi_n^{(k)}(x) = 0$

(vi) $(\exists) J_{n,k,r} := \int\limits_0^\infty x^r \Phi_n^{(k)}(x)dx < \infty$, $(\exists) J_{n,0,0} := \int\limits_0^\infty \Phi_n(x)dx \neq 0$.

We define the operators of Durrmeyer type associated with Mastroianni operators (1.3)-(1.4) for each real value function $f \in \mathbb{R}^{\mathbb{R}}$ for which the series exists:

$$DM_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{\int\limits_0^\infty m_{n,k}(t)f(t)dt}{\int\limits_0^\infty m_{n,k}(t)dt} = \int_0^\infty K_n(t,x)f(t)dt \qquad (2.1)$$

with the kernel

$$K_n(t,x) = \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x)m_{n,k}(t),\ I_{n,0,0} = J_{n,0,0} = \int_0^\infty \Phi_n(t)dt \neq 0. \qquad (2.2)$$

Lemma 2.1. *The next identity is true for any $n \in \mathbb{N}$ and $r,\ k \in \mathbb{N}_0$*

$$I_{n,k,r} = \frac{(r+1)_k}{k!} I_{n,0,r},$$

where

$$I_{n,k,r} := \int_0^\infty t^r m_{n,k}(t)dt = \frac{(-1)^k}{k!} J_{n,k,r+k}$$

and

$$(n)_k = n(n+1)(n+2)\cdots(n+k-1) = n_{[k,-1]},\ (n)_0 = 1$$

is the Pochhammer symbol or the factorial power of order k of n and the increment -1. So, $(1)_k = k!$, $(2)_k = (k+1)!$.

The proof suppose an easy computation, so we have omitted them. We remark that

$I_{n,0,r} = J_{n,0,r} = \int\limits_0^\infty t^r \Phi_n(t)dt$, $r \geq 0$, (the moments of the r-th order reported to Φ_n)

$I_{n,k,0} = \int\limits_0^\infty m_{n,k}(t)dt = \frac{(-1)^k}{k!} J_{n,k,k},$
$J_{n,k,k} = (-1)^k k! J_{n,0,0},\ k \geq 0,$
$I_{n,k,0} = I_{n,0,0} = J_{n,0,0} = \int\limits_0^\infty \Phi_n(t)dt,$
$I_{n,k,r} = \int\limits_0^\infty t^r m_{n,k}(t)dt = \frac{(-1)^k}{k!} \int\limits_0^\infty t^{r+k} \Phi_n^{(k)}(t)dt = \frac{(r+1)_k}{k!} \int\limits_0^\infty t^r \Phi_n(t)dt$
$= \frac{(r+1)_k}{k!} J_{n,0,r} = \frac{(r+1)_k}{k!} I_{n,0,r}$, (the moments of the r-th order reported to $m_{n,k}$).

Lemma 2.2. *The moments of the operators $DM_n(f;x)$ are given for $e_r(x) = x^r$, $r \in \mathbb{N}_0$ as*

$$DM_n\left(e_r;x\right) = \frac{I_{n,0,r}}{I_{n,0,0}} \sum_{k=0}^{\infty} \frac{(r+1)_k}{k!} m_{n,k}(x). \tag{2.3}$$

Further, we have

$$\begin{aligned} DM_n\left(e_0;x\right) &= 1, \\ DM_n\left(e_1;x\right) &= \frac{I_{n,0,1}}{I_{n,0,0}}\left(1 - x\Phi_n'(0)\right), \\ DM_n\left(e_2;x\right) &= \frac{I_{n,0,2}}{2I_{n,0,0}}\left(x^2\Phi_n''(0) - 4x\Phi_n'(0) + 2\right), \\ DM_n\left((e_1 - xe_0)^2;x\right) &= x^2\left(\frac{I_{n,0,2}}{2I_{n,0,0}}\Phi_n''(0) + 2\frac{I_{n,0,1}}{I_{n,0,0}}\Phi_n'(0) + 1\right) \\ &\quad -2x\left(\frac{I_{n,0,2}}{I_{n,0,0}}\Phi_n'(0) + \frac{I_{n,0,1}}{I_{n,0,0}}\right) + \frac{I_{n,0,2}}{I_{n,0,0}}. \end{aligned} \tag{2.4}$$

Proof. Using Lemma 2.1 we obtain

$$DM_n\left(e_r;x\right) = \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,r} = \frac{I_{n,0,r}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(r+1)_k}{k!}.$$

$$DM_n\left(e_0;x\right) = \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,0} = \frac{I_{n,0,0}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(1)_k}{k!} = 1,$$

$$\begin{aligned} DM_n\left(e_1;x\right) &= \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,1} = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(2)_k}{k!} \\ &= \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(k+1)!}{k!} = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x)(k+1) \\ &= \frac{nI_{n,0,1}}{I_{n,0,0}}\left(M_n\left(e_1;x\right) + \frac{1}{n}\right) = \frac{nI_{n,0,1}}{I_{n,0,0}}\left(-\frac{\Phi_n'(0)}{n}x + \frac{1}{n}\right) \end{aligned}$$

$$\begin{aligned} DM_n(e_2;x) &= \frac{1}{I_{n,0,0}}\sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,2} = \frac{I_{n,0,2}}{I_{n,0,0}}\sum_{k=0}^{\infty} m_{n,k}(x)\frac{(3)_k}{k!} \\ &= \frac{I_{n,0,2}}{2I_{n,0,0}}\sum_{k=0}^{\infty} m_{n,k}(x)\frac{(k+2)!}{k!} \\ &= \frac{n^2 I_{n,0,2}}{2I_{n,0,0}}\left(M_n(e_2;x)+\frac{3}{n}M_n(e_1;x)+\frac{2}{n^2}\right) \\ &= \frac{n^2 I_{n,0,2}}{2I_{n,0,0}}\left(\frac{\Phi_n''(0)x^2-\Phi_n'(0)x}{n^2}-\frac{3\Phi_n'(0)x}{n^2}+\frac{2}{n^2}\right). \end{aligned}$$

Because $DM_n\left((e_1-xe_0)^2;x\right) = DM_n(e_2;x) - 2xDM_n(e_1;x) + x^2 DM_n(e_0;x)$ is easy to obtain the relation of enunciation. □

Lemma 2.3. *Let*

$$\overline{DM_n}(f;x) = DM_n(f;x) - f\left(\frac{nI_{n,0,1}}{I_{n,0,0}}\left(\frac{1}{n}-\frac{\Phi_n'(0)}{n}x\right)\right) + f(x). \tag{2.5}$$

The following assertions hold:

$$\begin{aligned} \overline{DM_n}(e_0;x) &= 1, \\ \overline{DM_n}(e_1;x) &= x, \\ \overline{DM_n}(e_1-xe_0;x) &= 0. \end{aligned}$$

The proof suppose an easy computation, so we have omitted them.

Further, we consider the next conventions:

$$|DM_n(e_1-xe_0;x)| = \left|\frac{I_{n,0,1}}{I_{n,0,0}}(1-x\Phi_n'(0))-x\right| := \lambda_n(x), \tag{2.6}$$

$$\begin{aligned} DM_n\left((e_1-xe_0)^2;x\right) &= x^2\left(\frac{I_{n,0,2}}{2I_{n,0,0}}\Phi_n''(0)+2\frac{I_{n,0,1}}{I_{n,0,0}}\Phi_n'(0)+1\right) \\ &\quad -2x\left(\frac{I_{n,0,2}}{I_{n,0,0}}\Phi_n'(0)+\frac{I_{n,0,1}}{I_{n,0,0}}\right)+\frac{I_{n,0,2}}{I_{n,0,0}} \\ &= \beta_n(x). \end{aligned} \tag{2.7}$$

From (1.1) we have $\Phi_n(x)\ge 0$, $\Phi_n'(x)\le 0$, $\Phi_n''(x)\ge 0$, $x\in\mathbb{R}_+$ and so

$$\beta_n(x) \le x^2\left(\frac{I_{n,0,2}}{I_{n,0,0}}\Phi_n''(0)+1\right)-2x\frac{I_{n,0,2}}{I_{n,0,0}}\Phi_n'(0)+\frac{I_{n,0,2}}{I_{n,0,0}} := \eta_n(x). \tag{2.8}$$

Because DM_n is a linear positive operator, using the Cauchy-Schwarz's inequality we have

$$\begin{aligned} \lambda_n(x) &= |DM_n(e_1-xe_0;x)| \le DM_n(|e_1-xe_0|;x) \\ &\le \sqrt{DM_n((e_1-xe_0)^2;x)} = \sqrt{\beta_n(x)}. \end{aligned}$$

Let

$$\gamma_n(x) = \beta_n(x)+\lambda_n^2(x) \le 2\beta_n(x). \tag{2.9}$$

Lemma 2.4. *For every $x \in \mathbb{R}_+$ and $f'' \in C_B(\mathbb{R}_+)$ we have*

$$\left|\overline{DM_n}(f;x) - f(x)\right| \leq \frac{\|f''\|_B}{2}\gamma_n(x).$$

Proof. Using Taylor's expansion

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du$$

we obtain with Lemma 2.3 that

$$\overline{DM_n}(f;x) - f(x) = \overline{DM_n}\left(\int_x^t (t-u)f''(u)du; x\right).$$

Because $\left|\int_x^t (t-u)f''(u)du\right| \leq \|f''\|_B \dfrac{(t-x)^2}{2}$ using Lemma 2.2 we get

$$\left|\overline{DM_n}(f;x) - f(x)\right| \leq DM_n\left(\int_x^t (t-u)f''(u)du; x\right)$$

$$- \int_x^{\frac{I_{n,0,2}}{I_{n,0,0}}(1-x\Phi_n'(0))} \left(\frac{I_{n,0,1}}{I_{n,0,0}}(1-x\Phi_n'(0)) - u\right) f''(u)du$$

$$\leq \frac{\|f''\|_B}{2} DM_n\left((t-x)^2; x\right) + \frac{\|f''\|_B}{2}\left(\frac{I_{n,0,1}}{I_{n,0,0}}(1-x\Phi_n'(0)) - x\right)^2$$

$$\leq \frac{\|f''\|_B}{2}\left(\beta_n(x) + \lambda_n^2(x)\right). \qquad \square$$

Theorem 2.5. *For every $x \in \mathbb{R}_+$ and $f \in C_B(\mathbb{R}_+)$ the operators* (2.1)-(2.2) *satisfy the following relations*

(i) *If* $\lim\limits_{n\to\infty} \dfrac{n^r I_{n,0,r}}{r! I_{n,0,0}} = 1$, $r = 0, 1, 2$, *then* $\lim\limits_{n\to\infty} DM_n(f;x) = f(x)$,

(ii) $|DM_n(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{\beta_n(x)}\right)$,

(iii) $|DM_n(f;x) - f(x)| \leq 2C\omega_2\left(f, \sqrt{\gamma_n(x)}\right) + \omega\left(f, \lambda_n(x)\right)$.

with $\lambda_n(x)$, $\beta_n(x)$, $\eta_n(x)$, $\gamma_n(x)$ *defined as* (2.6), (2.7), (2.8), (2.9).

Proof. (i) Because $\lim\limits_{n\to\infty} \dfrac{\Phi_n^{(k)}(0)}{n^k} = \lim\limits_{n\to\infty} \dfrac{(-1)^k \alpha_{n,k}(0)}{n^k} = (-1)^k$ and $\lim\limits_{n\to\infty} \dfrac{n^r I_{n,0,r}}{r! I_{n,0,0}} = 1$, $r = 0, 1, 2$ using Lemma 2.2 we have $\lim\limits_{n\to\infty} DM_n\left(e_r; x\right) = e_r(x)$, $r = 0, 1, 2$ and the Bohmann-Korovkin assure the conclusion (i) of the theorem.

(ii) Using a result of O. Shisha, B. Mond [7] with the modulus of continuity of f we obtain a quantitative estimation of the remainder of the approximation formula. Indeed,

$$|DM_n(f;x) - f(x)| \leq \left(1 + \delta_n^{-1}(x)\sqrt{DM_n\left((e_1 - xe_0)^2; x\right)}\right)\omega\left(f, \delta_n(x)\right).$$

Taking

$$\delta_n(x) = \sqrt{\beta_n(x)} = \sqrt{DM_n\left((e_1 - xe_0)^2; x\right)}$$

$$= \left\{\frac{I_{n,0,2}}{2I_{n,0,0}}\left(x^2\Phi_n''(0) - 4x\Phi_n'(0) + 2\right) - 2x\frac{I_{n,0,1}}{I_{n,0,0}}\left(1 - x\Phi_n'(0)\right) + x^2\right\}^{\frac{1}{2}}$$

the proof of (ii) is completed.

(iii) From (2.5) we obtain for $g \in W_\infty^2$

$$\begin{aligned}
&|DM_n(f;x) - f(x)| \\
&\leq \left|\overline{DM_n}(f-g;x) - (f-g)(x) + \overline{DM_n}(g;x) - g(x)\right| \\
&+\left|f\left(\frac{nI_{n,0,1}}{I_{n,0,0}}\left(\frac{1}{n} - \frac{\Phi_n'(0)}{n}x\right)\right) - f(x)\right| \\
&\leq 2\|f-g\|_B + \frac{\|g''\|_B\,\gamma_n(x)}{2} \\
&+\left|f\left(\frac{nI_{n,0,1}}{I_{n,0,0}}\left(\frac{1}{n} - \frac{\Phi_n'(0)}{n}x\right)\right) - f(x)\right| \\
&\leq 2\|f-g\|_B + \frac{\|g''\|_B\,\gamma_n(x)}{2} + \omega\left(f, \lambda_n(x)\right).
\end{aligned}$$

Taking infimum over $g \in W_\infty^2$ on the right hand side, we get

$$\begin{aligned}
|DM_n(f;x) - f(x)| &\leq 2K_2\left(f, \gamma_n(x)\right) + \omega\left(f, \lambda_n(x)\right) \\
&\leq 2C\omega_2\left(f, \sqrt{\gamma_n(x)}\right) + \omega\left(f, \lambda_n(x)\right).
\end{aligned}$$

□

Theorem 2.6. *Let $f \in C_\rho(\mathbb{R}_+)$ and $\omega_{[0,b+1]}(f;\delta)$ be its modulus of continuity on the finite interval $[0, b+1]$, $b > 0$. Then*

$$\|DM_n(f) - f\|_{C[0,b]} \leq 3N_f\eta_n(b)(1+b)^2 + 2\omega_{[0,b+1]}\left(f, \sqrt{\eta_n(b)}\right),$$

with $\eta_n(x)$ defined as (2.8)*.*

Proof. Let $x \in \mathbb{R}_+$ and $t > b+1$ Because $f \in C_\rho(\mathbb{R}_+)$ using the growth condition of f since $t - x > 1$ we have

$$\begin{aligned}
|f(t) - f(x)| &\leq N_f(2 + t^2 + x^2) \leq N_f(2 + (t - x + x)^2 + x^2) \\
&\leq 3N_f(t-x)^2(1+b)^2.
\end{aligned}$$

For $x \in \mathbb{R}_+$, $\delta > 0$ and $t < b+1$ we have

$$|f(t) - f(x)| \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}\left(f, \delta\right).$$

So,

$$|f(t)-f(x)| \leq 3N_f(t-x)^2(1+b)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta).$$

and with (2.8) we obtain

$$\begin{aligned}|DM_n(f;x)-f(x)| &\leq 3N_f DM_n\left((e_1-xe_0)^2;x\right)(1+b)^2\\ &+\left(1+\frac{DM_n\left(|e_1-xe_0|\,;x\right)}{\delta}\right)\omega_{[0,b+1]}(f,\delta)\\ &\leq 3N_f DM_n\left((e_1-xe_0)^2;x\right)(1+b)^2\\ &+\left(1+\frac{\sqrt{DM_n\left((e_1-xe_0)^2;x\right)}}{\delta}\right)\omega_{[0,b+1]}(f,\delta)\\ &\leq 3N_f\eta_n(b)(1+b)^2+\left(1+\frac{\sqrt{\eta_n(b)}}{\delta}\right)\omega_{[0,b+1]}(f,\delta)\\ &\leq 3N_f\eta_n(b)(1+b)^2+2\omega_{[0,b+1]}\left(f,\sqrt{\eta_n(b)}\right).\end{aligned}$$

□

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A Voronovskaya-type theorem for a certain nonlinear Bernstein operators

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Abstract. The present paper concerns with the nonlinear Bernstein operators $NB_n f$ of the form

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \quad , \quad 0 \le x \le 1 \quad , \quad n \in \mathbb{N},$$

acting on bounded functions on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions. As a continuation of the very recent paper of the authors [11], we estimate the rate of convergence by modulus of continuity and provide a Voronovskaya-type formula for these operators. We note that our results are strict extensions of the classical ones, namely, the results dealing with the linear Bernstein polynomials.

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1. Introduction

We consider the problem of approximating a given real-valued function f, defined on $[0,1]$, by means of a sequence of nonlinear Bernstein operators $(NB_n f)$. Operators like positive linear, convolution, moment and sampling operators play an important role in several branches of Mathematics, for instance in reconstruction of signals and images, in Fourier analysis, operator theory, probability theory and approximation theory.

In this paper, we deal with nonlinear Bernstein operators generated by the classical Bernstein operators. These operators considered in the papers [1], [4] and [11], in which other kinds of convergence properties are studied.

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Let f be a function defined on the interval $[0,1]$ and let $\mathbb{N} := \{1,2,...\}$. The classical Bernstein operators $B_n f$ applied to f are defined as

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N}, \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. These polynomials were introduced by Bernstein [7] in 1912 to give the first constructive proof of the Weierstrass approximation theorem. Some properties of the polynomials (1.1) can be found in Lorentz [14].

We now state a brief and technical explanation of the relation between approximation by linear and nonlinear operators. Approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [15] and widely developed in [5] (and the references contained therein). In [15], the assumption of linearity of the singular integral operators was replaced by an assumption of a Lipschitz condition for the kernel function $K_\lambda(t,u)$ with respect to the second variable. Especially, nonlinear integral operators of type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t))\, dt, \qquad x \in (a,b),$$

and its special cases were studied by Bardaro-Karsli and Vinti [2], [3] and Karsli [10], [12] in some Lebesgue spaces.

Very recently, by using the techniques due to Musielak [15], Karsli-Tiryaki and Altin [11] considered the following type nonlinear counterpart of the well-known Bernstein operators;

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N}, \tag{1.2}$$

acting on bounded functions f on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions. They proved some existence and approximation theorems for the nonlinear Bernstein operators. In particular, they obtain some pointwise convergence for the nonlinear sequence of Bernstein operators (1.2) to some discontinuity point of the first kind of f, as $n \to \infty$.

As a continuation of the very recent paper of the authors [11], we estimate a Voronovskaya-type formula for this class nonlinear Bernstein operators on the interval $[0,1]$. Let us note that such kind of results for a general class of discrete operators were studied by Bardaro and Mantellini [4].

An outline of the paper is as follows: The next section contains basic definitions and notations.

In Section 3, the main approximation results of this study are given. They are dealing with some approximation theorems for nonlinear Bernstein operators (1.2) and rate of convergence by modulus of continuity. Also we give a Voronovskaya-type formula for this class nonlinear Bernstein operators on the interval $[0,1]$.

In Section 4, we give some certain results which are necessary to prove the main result.

The final section, that is Section 5, concerns with the proof of the main results presented in Section 3.

2. Preliminaries

In this section, we recall the following structural assumptions according to [11], which will be fundamental in proving our convergence theorems.

In the following we will denote by $C(I)$ the space of all uniformly continuous and bounded functions $f : I \to \mathbb{R}$, endowed with the norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is non-decreasing, continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$, and $\lim\limits_{u\to\infty} \psi(u) = +\infty$.

We now introduce a sequence of functions. Let $\{P_{n,k}\}_{n\in\mathbb{N}}$ be a sequence of functions $P_{n,k} : [0,1] \,\mathrm{x}\, \mathbb{R} \to \mathbb{R}$ defined by

$$P_{n,k}(t,u) = p_{n,k}(t) H_n(u) \tag{2.1}$$

for every $t \in [0,1], u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{n,k}(t)$ is the Bernstein basis.

Throughout the paper we assume that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing sequence such that $\lim\limits_{n\to\infty} \mu(n) = \infty$.

First of all we assume that the following conditions hold:

a) $H_n : \mathbb{R} \to \mathbb{R}$ is such that

$$|H_n(u) - H_n(v)| \le \psi(|u-v|), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$, for every $n \in \mathbb{N}$. That is, H_n satisfies a (L-ψ) Lipschitz condition.

b) We now set

$$K_n(x,u) := \begin{cases} \sum\limits_{k\le nu} p_{n,k}(x), & 0 < u \le 1 \\ 0, & u = 0 \end{cases} \tag{2.2}$$

and from (2.2) one can write

$$\lambda_n(x,t) := \int\limits_0^t d_u K_n(x,u).$$

Similar approach and some particular examples can be found in [6], [9], [11], [13] and [16].

c) Denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Assume that for n sufficiently large

$$\sup_u |r_n(u)| \leq \frac{1}{\mu(n)},$$

holds.

3. Convergence Results

Let X be the set of all bounded Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$.

We will consider the following type nonlinear Bernstein operators,

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right)$$

defined for every $f \in X$ for which $NB_n f$ is well-defined, where $P_{n,k}(x,u)$ satisfies (2.1) for every $x \in [0,1]$, $u \in \mathbb{R}$.

Definition 3.1. *Let $f \in C[a,b]$ and $\delta > 0$ be given. Then the modulus of continuity is given by;*

$$\omega_\psi(f;\delta) = \sup_{|t-x| \leq \delta\ ,\ t,x \in [a,b]} \psi\left(|f(t) - f(x)|\right). \tag{3.1}$$

Definition 3.2. *We will say that the sequence $(P_{n,k})_{n \in \mathbb{N}}$ is $\psi-$singular if the following assumptions are satisfied;*

(P.1) For every $x \in I$ and $\delta > 0$ there holds

$$\psi\left(\sum_{\left|\frac{k}{n}-x\right| \geq \delta} \left|\frac{k}{n} - x\right| p_{n,k}(x)\right) = o\left(n^{-1}\right)\ ,\ (n \to \infty).$$

(P.2) For every $u \in \mathbb{R}$ and for every $x \in I$ we have

$$\lim_{n \to \infty} n \left[\sum_{k=0}^{n} P_{n,k}(x,u) - u\right] = 0.$$

We are now ready to establish the main results of this study:

Theorem 3.3. *Let $f : I \to \mathbb{R}$, $f \in C(I)$ and suppose that a kernel satisfies (a), (b) and (c). Then*

$$\|NB_n f - f\|_\infty \to 0 \text{ as } n \to \infty$$

where $I = [0,1]$ and $\psi \in \Psi$.

Theorem 3.4. *If $f(x)$ is continuous and $\omega_\psi(f;\delta)$ the modulus of continuity of $f(x)$ given in (3.1), then*

$$|NB_n f(x) - f(x)| \leq \psi(\epsilon) + \frac{5}{4}\omega_\psi(f;\delta) + \frac{1}{\mu(n)}$$

where $\delta = n^{-\frac{1}{2}}$.

Theorem 3.5. *Let $f \in L_1[0,1]$ be a function such that $f'(x)$ exists at a point $x \in (0,1)$. Let us assume that the sequence $(P_{n,k})_{n\in\mathbb{N}}$ is $\psi-$singular and*

$$\limsup_{n\to\infty} n\ \psi\left(M_1\left(p_{n,k},x\right)\right) = l_1(x) \in \mathbb{R}, \tag{3.2}$$

where M_1 is the first order absolute moment of Bernstein polynomials given in Lemma 4.3. Then,

$$\limsup_{n\to\infty} n\ \left|\left(NB_n f\right)(x) - f(x)\right| \leq M\ l_1(x),$$

where $M > 0$ be a sufficiently large integer.

4. Auxilary Results

In this section we give certain results, which are necessary to prove our theorems.

Lemma 4.1. *$\omega_\psi(f;\delta)$ has the following properties,*
i) $\omega_\psi(f;\delta) \geq 0$,
ii) If $\delta_1 \leq \delta_2$, then $\omega_\psi(f;\delta_1) \leq \omega_\psi(f;\delta_2)$,
iii) Let $m \in \mathbb{N}$, then $\omega_\psi(f;m\delta) \leq m\ \omega_\psi(f;\delta)$,
iv) Let $\lambda \in \mathbb{R}^+$, then $\omega_\psi(f;\lambda\delta) \leq (\lambda+1)\,\omega_\psi(f;\delta)$,
v) $\lim\limits_{\delta\to 0^+} \omega_\psi(f;\delta) = 0$,
vi) $\psi(|f(t) - f(x)|) \leq \omega_\psi(f;|t-x|)$,
vii) $\psi(|f(t) - f(x)|) \leq \left(\frac{|t-x|}{\delta} + 1\right)\omega_\psi(f;\delta)$, and they can be proven as similar with the classical ones.

Lemma 4.2. *It is well known that for $(B_n t^s)(x)$, $s = 0,1,2$, one has*

$$(B_n 1)(x) = 1, (B_n t)(x) = x, (B_n t^2)(x) = x^2 + \frac{x(1-x)}{n}.$$

For proof of this Lemma see [14].

By direct calculation, we find the following equalities:

$$(B_n\,(t-x)^2)(x) = \frac{x(1-x)}{n} \quad , \quad (B_n\,(t-x))(x) = 0.$$

Lemma 4.3. *The first order absolute moment for Bernstein polynomial is defined as*

$$M_1\left(p_{n,k},x\right) = \sum_{k=0}^{n} \left|\left(\frac{k}{n} - x\right)\right| p_{n,k}(x)$$

and

$$M_1\left(p_{n,k},x\right) \leq \left(\frac{2x(1-x)}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

which can be found [8].

5. Proof of the Theorems

Proof of Theorem 3.3. We evaluate $\|NB_n f - f\|_\infty$. We have

$$\begin{aligned}
|NB_n f(x) - f(x)| &= \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x)\right| \\
&= \left|\sum_{k=0}^{n} \left\{H_n\left(f\left(\frac{k}{n}\right)\right) - f(x)\right\} p_{n,k}(x)\right| \\
&\leq \sum_{k=0}^{n} \left|H_n\left(f\left(\frac{k}{n}\right)\right) - H_n(f(x))\right| p_{n,k}(x) \\
&\quad + \sum_{k=0}^{n} |H_n(f(x)) - f(x)|\, p_{n,k}(x) \\
&= I_{n,1}(x) + I_{n,2}(x).
\end{aligned}$$

First we consider $I_{n,2}(x)$. From (c) we have

$$\begin{aligned}
I_{n,2}(x) &= \sum_{k=0}^{n} |H_n(f(x)) - f(x)|\, p_{n,k}(x) \\
&\leq \frac{1}{\mu(n)}.
\end{aligned}$$

Next we consider $I_{n,1}(x)$

$$\begin{aligned}
I_{n,1}(x) &= \sum_{k=0}^{n} \left|H_n\left(f\left(\frac{k}{n}\right)\right) - H_n(f(x))\right| p_{n,k}(x) \\
&\leq \sum_{k=0}^{n} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\
&= \int_0^1 \psi(|f(t) - f(x)|)\, d_t(K_n(x,t)) \\
&= \int_{|t-x|\leq\delta} \psi(|f(t) - f(x)|)\, d_t(K_n(x,t)) \\
&\quad + \int_{|t-x|>\delta} \psi(|f(t) - f(x)|)\, d_t(K_n(x,t)) \\
&\leq \psi(\epsilon) + \psi(2\|f\|_\infty)\ \epsilon
\end{aligned}$$

holds true, since ψ is non-decreasing and concave function. Finally we have

$$|NB_n f(x) - f(x)| \leq \psi(\epsilon) + \psi(2\|f\|_\infty)\ \epsilon + \frac{1}{\mu(n)}$$

and so, since $\frac{1}{\mu(n)} \to 0$ when $n \to \infty$, we obtain

$$\limsup_{n\to\infty} |NB_n f(x) - f(x)| \leq \psi(\epsilon) + \psi\left(2\|f\|_\infty\right)\ \epsilon.$$

Hence the assertion follows, $\epsilon > 0$ being arbitrary.

Proof of Theorem 3.4. We can write the difference as in the previous theorem

$$\begin{aligned} |NB_n f(x) - f(x)| &= \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x)\right| \\ &\leq I_{n,1}(x) + I_{n,2}(x) \end{aligned}$$

where

$$\begin{aligned} I_{n,2}(x) &= \sum_{k=0}^{n} |H_n(f(x)) - f(x)|\, p_{n,k}(x) \\ &\leq \frac{1}{\mu(n)}. \end{aligned}$$

First we consider $I_{n,1}(x)$. If we think $I_{n,1}(x)$ as two sum,

$$\begin{aligned} I_{n,1}(x) &\leq \sum_{\left|\frac{k}{n}-x\right|\leq\delta} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &\quad + \sum_{\left|\frac{k}{n}-x\right|>\delta} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &= \psi(\epsilon) + I_{n,1,2}(x). \end{aligned}$$

Now we will consider $I_{n,1,2}(x)$. Taking into account that $\omega_\psi(f;\delta)$ is the modulus of continuity

$$\begin{aligned} I_{n,1,2}(x) &= \sum_{\left|\frac{k}{n}-x\right|>\delta} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &\leq \omega_\psi(f;\delta) \sum_{\left|\frac{k}{n}-x\right|>\delta} \left(\frac{\left|\frac{k}{n}-x\right|}{\delta} + 1\right) p_{n,k}(x) \\ &\leq \omega_\psi(f;\delta) \left\{1 + \delta^{-1} \sum_{\left|\frac{k}{n}-x\right|>\delta} \left|\frac{k}{n} - x\right| p_{n,k}(x)\right\} \\ &\leq \omega_\psi(f;\delta) \left\{1 + \delta^{-2} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 p_{n,k}(x)\right\} \\ &\leq \omega_\psi(f;\delta) \left\{1 + \left(4n\delta^2\right)^{-1}\right\}. \end{aligned}$$

In conclusion;

$$\begin{aligned}|NB_n f(x) - f(x)| &\leq \psi(\epsilon) + \omega_\psi(f;\delta)\left\{1 + \left(4n\delta^2\right)^{-1}\right\} + \frac{1}{\mu(n)} \\ &\leq \psi(\epsilon) + \frac{5}{4}\omega_\psi(f;\delta) + \frac{1}{\mu(n)}\end{aligned}$$

where $\delta = n^{-\frac{1}{2}}$.

Proof of Theorem 3.5. Since f is differentiable at the point x, then there exists a bounded function h such that $\lim_{y\to 0} h(y) = 0$. By Taylor's formula we have

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right) h\left(\frac{k}{n} - x\right).$$

Now we can write

$$\begin{aligned}n\ |(NB_n f)(x) - f(x)| &= n\left|\sum_{k=0}^{n}\left\{H_n\left(f\left(\frac{k}{n}\right)\right) - f(x)\right\} p_{n,k}(x)\right| \\ &\leq n\sum_{k=0}^{n}\psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &\quad + n\left|\sum_{k=0}^{n}\left\{H_n(f(x)) - f(x)\right\} p_{n,k}(x)\right| \\ &= I_1(x) + I_2(x).\end{aligned}$$

By assumption (P.2), $I_2(x)$ tends to zero. We can estimate the first term in the following way:

Let $M > 0$ be an integer such that $|f'(x)| + \left|h\left(\frac{k}{n} - x\right)\right| \leq M$. Using subadditivity of the function $\psi(x)$, $x \geq 0$, we have

$$\begin{aligned}I_1(x) &= n\sum_{k=0}^{n}\psi\left(\left|\left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right) h\left(\frac{k}{n} - x\right)\right|\right) p_{n,k}(x) \\ &\leq n\sum_{k=0}^{n}\psi\left(\left|\left(\frac{k}{n} - x\right)\right|\left[|f'(x)| + \left|h\left(\frac{k}{n} - x\right)\right|\right]\right) p_{n,k}(x) \\ &\leq n\left\{\sum_{k=0}^{n}\psi\left(M\left|\left(\frac{k}{n} - x\right)\right|\right) p_{n,k}(x)\right\} \\ &\leq n\ M\left\{\sum_{k=0}^{n}\psi\left(\left|\left(\frac{k}{n} - x\right)\right|\right) p_{n,k}(x)\right\}\end{aligned}$$

In virtue of Jensen's Inequality, we can write

$$\begin{aligned}I_1(x) &\leq n\ M\ \psi\left(\sum_{k=0}^{n}\left|\left(\frac{k}{n} - x\right)\right| p_{n,k}(x)\right) \\ &= n\ M\ \psi\left(M_1\left(p_{n,k}, x\right)\right).\end{aligned}$$

In view of (3.2), one has

$$\limsup_{n\to\infty} n \ \left|(NB_n f)(x) - f(x)\right| \leq M \ l_1(x).$$

This completes the proof of the theorem.

As a corollary of the Theorem 3.5 we have:

Corollary 5.1. *Let $f \in L_1[0,1]$ be a function such that $f'(x)$ exists at a point $x \in (0,1)$. Let us assume that the sequence $(P_{n,k})_{n\in\mathbb{N}}$ is $\psi-$singular satisfies (3.2) and let $\psi(x) = x^\gamma$ where $0 < \gamma \leq 1$. Then*

$$\limsup_{n\to\infty} n \ \left|(NB_n f)(x) - f(x)\right| \leq l_1(x) \left|f'(x)\right|^\gamma.$$

We note that to prove the above Corollary we can also use the following inequality;

$$\psi(|a|\,|b|) \leq \psi(|a|)\,\psi(|b|),$$

(*see* [4]).

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On convergence of a kind of complex nonlinear Bernstein operators

Harun Karsli and Esra Unal

Abstract. The present article deals with the approximation properties and Voronovskaja type results with quantitative estimates for a certain class of complex nonlinear Bernstein operators $(NB_n f)$ of the form

$$(NB_n f)(z) = \sum_{k=0}^{n} p_{k,n}(z) G_n \left(f\left(\frac{k}{n}\right)\right), \quad |z| \leq 1$$

attached to analytic functions on compact disks.

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Keywords: Nonlinear Bernstein operators, Lipschitz condition, Voronovskaja-type result, compact disks.

1. Introduction

Approximation properties of complex Bernstein polynomials were initially studied by Lorentz [6]. Recently S. G. Gal has done a commendable work in this direction and he compiled the important papers in his recent book [2]. Concerning the convergence of the Bernstein polynomials in the complex plane, Bernstein proved that if $f : G \to \mathbb{C}$ is analytic in the open set $G \subseteq \mathbb{C}$ with $\overline{D}_1 \subset G$ where $\overline{D}_1 = \{z \in \mathbb{C} : \ |z| \leq 1\}$ then the complex Bernstein polynomials

$$(B_n f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f\left(\frac{k}{n}\right)$$

converge uniformly to f in $\overline{D}_1$. In the present paper we study the rate of approximation of analytic functions and give a Voronovskaja type result for the nonlinear

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complex Bernstein operator $(NB_n f)$. Nonlinear Bernstein operator of complex variable is defined as

$$(NB_n f)(z) = \sum_{k=0}^{n} p_{k,n}(z) G_n\left(f\left(\frac{k}{n}\right)\right) \quad (1.1)$$

where $G_n : \mathbb{C} \to \mathbb{C}$ satisfies the Hölder condition i.e,

$$|G_n(u) - G_n(v)| \leq R|u - v|^{\gamma}$$

for every $n \in \mathbb{N}$, $0 < \gamma \leq 1$ and suitable constant $R > 0$ and

$$\lim_{n\to\infty} [G_n(u) - u] = 0 \quad (1.2)$$

for every $u \in \overline{D}_1$ where $\overline{D}_1 = \{z \in \mathbb{C} : \ |z| \leq 1\}$.

2. Convergence Results

We will consider the following nonlinear version of complex Bernstein operator,

$$(NB_n f)(z) = \sum_{k=0}^{n} p_{k,n}(z) G_n\left(f\left(\frac{k}{n}\right)\right), \quad |z| \leq 1$$

defined for every $f \in \overline{D}_1$ for which $(NB_n f)$ is well-defined, where

$$D_1 = \{z \in \mathbb{C} : \ |z| < 1\}$$

The real case of above operator (1.1) and some of its properties can be found in [5].

We are now ready to establish the main results of this study:

Theorem 2.1. *Suppose that $f : D_1 \to \mathbb{C}$is analytic in D_1, that is*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

for all $z \in D_1$. For all $|z| \leq 1$ and $n \in \mathbb{N}$, we have

$$|(NB_n f)(z) - f(z)| \leq R\left(\frac{3}{n}C(f)\right)^{\gamma},$$

where $0 < C(f) = \sum\limits_{k=2}^{\infty} k(k-1)|c_k| < \infty$.

Theorem 2.2. *Suppose that $f : D_1 \to \mathbb{C}$ is analytic in D_1. We can write*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

for all $z \in D_1$.The following Voronovskaja-type result in the closed unit disk holds,

$$\left|(NB_n f)(z) - f(z) - \frac{z(1-z)}{n} f''(z)\right| \leq R\left(\frac{|z(1-z)|}{2n}\frac{10}{n}M(f)\right)^{\gamma}$$

for all $n \in \mathbb{N}$, $z \in \overline{D}_1$, where $0 < M(f) = \sum\limits_{k=3}^{\infty} k(k-1)(k-2)^2 c_k < \infty$ and $0 < \gamma \leq 1$.

The linear counterpart of Theorem 2.2 is given by Gal [4]. Notice that our theorems contain appropriate result of Gal [4] as a special case.

3. Auxiliary Result

In this section we give a certain result, which is necessary to prove our theorems.

Lemma 3.1. (Lorentz [7, p. 40, Theorem 4]) *For polynomials* $P_n(z) = \sum_{k=0}^{n} a_k z^k$ *with complex coefficients on the disk* $|z| \leq 1$ *we put*

$$\|P_n\|_1 = \max_{|z|\leq 1} |P_n(z)| \,.$$

Then

$$\|P_n'\| \leq n \, \|P_n\| \,.$$

4. Proof of the Theorems

Proof of Theorem 2.1. We consider

$$\begin{aligned} |(NB_n f)(z) - f(z)| &= \left| \sum_{k=0}^{n} p_{k,n}(z) \, G_n\left(f\left(\frac{k}{n}\right)\right) - f(z) \sum_{k=0}^{n} p_{k,n}(z) \right| \\ &= \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - f(z) \right\} \right| \end{aligned}$$

$$\begin{aligned} &= \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) + G_n(f(z)) - f(z) \right\} \right| \\ &\leq \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) \right\} \right| + \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n(f(z)) - f(z) \right\} \right| \end{aligned}$$

the last term in the last inequality goes to zero because of (1.2). Then we will estimate the first sum $I_1 = \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) \right\} \right|$

$$\begin{aligned} I_1 &= \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) \right\} \right| \\ &\leq \sum_{k=0}^{n} |p_{k,n}(z)| \left| G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(f(z)) \right| . \end{aligned}$$

By using Hölder condition $0 < \gamma \leq 1$,

$$\leq R \sum_{k=0}^{n} |p_{k,n}(z)| \left| f\left(\frac{k}{n}\right) - f(z) \right|^{\gamma}$$

if we use Hölder inequality then we have

$$\leq R \left(\sum_{k=0}^{n} |p_{k,n}(z)| \left| f\left(\frac{k}{n}\right) - f(z) \right| \right)^{\gamma} .$$

Denoting $e_k(z) = z^k$, $k = 0, 1, ...$ and $\pi_{k,n}(z) = B_n(e_k)(z)$, we evidently have

$$(B_n f)(z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$$

and by using this representation we get

$$= R\left(\sum_{k=0}^{\infty} |c_k| \, |\pi_{k,n}(z) - e_k(z)|\right)^{\gamma}.$$

So that we need an estimate for

$$|\pi_{k,n}(z) - e_k(z)|.$$

For this purpose we use the recurrence proved for the real variable case in Andrica [1]. It is valid for complex variable as well in [2] and [3]:

$$\pi_{k+1,n}(z) = \frac{z(1-z)}{n}\pi'_{k,n}(z) + z\pi_{k,n}(z)$$

for all $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $k = 0, 1, ...$

From this recurrence we easily obtain that degree $(\pi_{k,n}(z)) = k$. Also, by replacing k with $k-1$, we get

$$\pi_{k,n}(z) - z^k = \frac{z(1-z)}{n}[\pi_{k-1,n}(z) - z^{k-1}]' + \frac{(k-1)z^{k-1}(1-z)}{n} + z[\pi_{k-1,n}(z) - z^{k-1}]$$

which by Bernstein's inequality for complex polynomials where $|z| \leq r \leq 1$ gives

$$\begin{aligned} |\pi_{k,n}(z) - e_k(z)| &\leq (k-1)\frac{1+r}{n}\left\|\pi_{k-1,n}(z) - e_{k-1}(z)\right\|_1 \\ &\quad + \frac{r^{k-1}(1+r)(k-1)}{n} + r\left|\pi_{k-1,n}(z) - e_{k-1}(z)\right|. \end{aligned}$$

As a conclusion, for all $|z| \leq 1$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned} |(NB_n f)(z) - f(z)| &\leq R\left(\sum_{k=0}^{\infty} |c_k| \, |\pi_{k,n}(z) - e_k(z)|\right)^{\gamma} \\ &\leq R\left(\frac{r(1+r)(1+2r)}{2n}\sum_{k=2}^{\infty} k(k-1)\,|c_k|\right)^{\gamma}. \end{aligned}$$

Since $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is absolutely and uniformly convergent in $|z| \leq 1$, then one has $f''(z) = \sum_{k=2}^{\infty} k(k-1)c_k z^{k-2}$. Note that $\sum_{k=2}^{\infty} k(k-1)c_k z^{k-2}$ is also absolutely convergent for $|z| \leq 1$, which implies $\sum_{k=2}^{\infty} k(k-1)\,|c_k| < \infty$.

Proof of Theorem 2.2. Denoting $h(z) := f(z) + \frac{z(1-z)}{n} f''(z)$

$$\begin{aligned}
&\left| (NB_n f)(z) - f(z) - \frac{z(1-z)}{n} f''(z) \right| \\
= &\left| \sum_{k=0}^{n} p_{k,n}(z) G_n\left(f\left(\frac{k}{n}\right)\right) - f(z) - \frac{z(1-z)}{n} f''(z) \right| \\
= &\left| \sum_{k=0}^{n} p_{k,n}(z) G_n\left(f\left(\frac{k}{n}\right)\right) - h(z) \right| \\
\leq &\left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(h(z)) \right\} \right| \\
&+ \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n(h(z)) - h(z) \right\} \right|
\end{aligned}$$

it is clearly seen that the last term in the last inequality goes to zero because of (1.2). Then we will estimate $I_1 = \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(h(z)) \right\} \right|$

$$\begin{aligned}
I_1 &= \left| \sum_{k=0}^{n} p_{k,n}(z) \left\{ G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(h(z)) \right\} \right| \\
&\leq \sum_{k=0}^{n} |p_{k,n}(z)| \left| G_n\left(f\left(\frac{k}{n}\right)\right) - G_n(h(z)) \right|
\end{aligned}$$

by using Hölder condition $0 < \gamma \leq 1$

$$\leq \sum_{k=0}^{n} |p_{k,n}(z)| R \left| f\left(\frac{k}{n}\right) - h(z) \right|^{\gamma}.$$

Substituting $h(z)$ and using the Hölder inequality,

$$\leq R \left(\sum_{k=0}^{n} |p_{k,n}(z)| \left| f\left(\frac{k}{n}\right) - f(z) - \frac{z(1-z)}{n} f''(z) \right| \right)^{\gamma}$$

Denoting $e_k(z) = z^k$, $k = 0, 1, ...$ and $\pi_{k,n}(z) = B_n(e_k)(z)$, we can write

$$(B_n f)(z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$$

which immediately implies

$$\begin{aligned}
&\left| (NB_n f)(z) - f(z) - \frac{z(1-z)}{n} f''(z) \right| \\
\leq \; & R \left(\sum_{k=0}^{\infty} |c_k| \left| \pi_{k,n}(z) - e_k(z) - \frac{z^{k-1}(1-z)k(k-1)}{2n} \right| \right)^{\gamma}
\end{aligned}$$

for all $z \in \overline{D}_1$, $n \in \mathbb{N}$.

In what follows, we will use the recurrence obtained in the proof of Theorem 1.1.2 [2].

$$\pi_{k+1,n}(z) = \frac{z(1-z)}{n}\pi'_{k,n}(z) + z\pi_{k,n}(z)$$

for all $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $k = 0, 1, ...$

If we denote

$$E_{k,n}(z) = \pi_{k,n}(z) - e_k(z) - \frac{z^{k-1}(1-z)k(k-1)}{2n}$$

then it is clear that $E_{k,n}(z)$ is a polynomial of degree$\leq k$ and by a simple calculation and the use of the above recurrence we obtain the following relationship

$$\begin{aligned} E_{k,n}(z) &= \frac{z(1-z)}{n}E'_{k,n}(z) + zE_{k-1,n}(z) \\ &+\frac{z^{k-2}(1-z)(k-1)(k-2)}{2n^2}\left[(k-2) - z(k-1)\right] \end{aligned}$$

for all $k \geq 2$, $n \in \mathbb{N}$ and $z \in \overline{D}_1$.

According to Bernstein's inequality $\left\|E'_{k-1,n}(z)\right\| \leq (k-1)\left\|E_{k-1,n}(z)\right\|$ the above relationship implies for all $|z| \leq 1$, $k \geq 2, n \in \mathbb{N}$ that

$$\begin{aligned} |E_{k,n}(z)| &\leq \frac{|z|\,|1-z|}{2n}\left[2\left\|E'_{k-1,n}(z)\right\|\right] + |E_{k-1,n}(z)| \\ &+\frac{|z|\,|1-z|}{2n}\frac{|z|^{k-3}(k-1)(k-2)}{n}(2k-3) \\ &\leq |E_{k-1,n}(z)| + \frac{|z|\,|1-z|}{2n}\left[2(k-1)\left\|\pi_{k-1,n}(z) - e_{k-1}(z)\right\|\right. \\ &\left.+2(k-1)\left\|\frac{(k-1)(k-2)\left[e_{k-2}(z) - e_{k-1}(z)\right]}{2n}\right\| + \frac{2k(k-1)(k-2)}{n}\right] \end{aligned}$$

where $\|.\|$ denotes the uniform norm in $C(\overline{D}_1)$.

It follows

$$\left\|\pi_{k,n}(z) - e_k(z)\right\| \leq \frac{3}{n}k(k-1)$$

and as a consequence, we get

$$\begin{aligned} |E_{k,n}(z)| &\leq |E_{k-1,n}(z)| + \frac{|z|\,|1-z|}{2n}\left[2(k-1)\frac{3(k-1)(k-2)}{n}\right. \\ &\left.+2(k-1)\left\|\frac{(k-1)(k-2)\left[e_{k-2}(z) - e_{k-1}(z)\right]}{2n}\right\| + \frac{2k(k-1)(k-2)}{n}\right] \end{aligned}$$

which by simple calculation implies

$$|E_{k,n}(z)| \leq |E_{k-1,n}(z)| + \frac{|z|\,|1-z|}{2n}\frac{10}{n}k(k-1)(k-2).$$

Since $E_{0,n}(z) = E_{1,n}(z) = E_{2,n}(z) = 0$, for any $z \in \mathbb{C}$ it follows that from the last inequality for $k = 3, 4, ...$ we easily obtain, step by step the following

$$|E_{k,n}(z)| \leq \frac{|z|\,|1-z|}{2n}\frac{10}{n}\sum_{j=3}^{k} j(j-1)(j-2) \leq \frac{|z|\,|1-z|}{2n}\frac{10}{n}k(k-1)(k-2)^2$$

In conclusion

$$\left|(NB_n f)(z) - f(z) - \frac{z(1-z)}{n}f''(z)\right| \leq R\left(\sum_{k=3}^{\infty}|c_k|\,|E_{k,n}(z)|\right)^{\gamma}$$

$$\leq R\left(\frac{|z|\,|1-z|}{2n}\frac{10}{n}\sum_{k=3}^{\infty}|c_k|\,k(k-1)(k-2)^2\right)^{\gamma}.$$

Note that since $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$ and the series is absolutely convergent in $\overline{D}_1$, it easily follows that $\sum_{k=3}^{\infty} |c_k|\,k(k-1)(k-2)^2 < \infty$.

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On convergence of nonlinear singular integral operators with non-isotropic kernels

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Abstract. Here we give some approximation theorems concerning pointwise convergence and rate of pointwise convergence of nonlinear singular integral operators with non-isotropic kernels of the form

$$T_{w,\lambda}(f)(s) = \int\limits_{\mathbb{R}^n} K_w\left(|s-t|_\lambda, f(t)\right) dt,$$

where the kernel function satisfies Lipschitz condition and some singularity assumptions. Here Λ is a non-empty set of indices, 0 is an accumulation point of Λ and $|s-t|_\lambda$ denotes the non-isotropic distance between the points $s, t \in \mathbb{R}^n$.

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1. Introduction

The theory of approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [12] and widely developed in [4]. He considers nonlinear integral operators, replacing linearity assumption by Lipschitz condition for kernel functions generating the operators and satisfying suitable singularity assumptions. After this discovery, several matematicians have undertaken the program of extending approximation by nonlinear integral operators in many ways and to several settings, including modular function spaces, pointwise and uniform convergence of operators, Korovkin type theorems, abstract function spaces, sampling series and so on. Especially, this kind of operators were extensively studied by C. Bardaro, J. Musielak and G. Vinti [5],[6] in connection with the modular space. Operators of type

$$T_w(f)(x) = \int_a^b K_w\left(x-t, f(t)\right) dt, x \in (a,b) \tag{1.1}$$

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and its special cases were studied by Swiderski and Wachnicki [15], Karsli [9], [10] and Karsli-Ibikli [11] in some Lebesgue spaces. Such developments delineated a theory which is nowadays referred to as the theory of approximation by nonlinear integral operators.

The kernel of the operator of type (1.1) depends on Euclidean distance, so it holds the properties of isotropy. As an extension of the isotropic distance, non-isotropy was defined by Besov and Nikolsky 1975 in [7] . It is useful to mention that, non-isotropy were studied on linear singular integral operators [2], [3], [16], and on potential theory [8], [14]. In this paper we assume that the kernel of the operator depends on non-isotropic distance.

The present paper concerns with pointwise convergence of families of nonlinear singular integral operators $T_{w,\lambda}(f)(s)$ of the form

$$T_{w,\lambda}(f)(s) = \int\limits_{\mathbb{R}^n} K_w\left(|s-t|_\lambda, f(t)\right) dt. \tag{1.2}$$

The convergence of the family of operators of type 1.2 is proved for some points with the suitable assumptions and properties. The next section contains some definitions, notations, assumptions and properties.

2. Preminilaries

Definition 2.1. [7] Let λ_1, λ_2, ...,λ_n be positive real numbers. The non-isotropic λ-distance of $x \in \mathbb{R}^n$ to the origin is

$$|x|_\lambda = \left(|x_1|^{\frac{1}{\lambda_1}} + |x_2|^{\frac{1}{\lambda_2}} + ... + |x_n|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}}$$

where

$$\begin{aligned} x &= (x_1, x_2, ..., x_n), \\ \lambda &= (\lambda_1, \lambda_2, ..., \lambda_n), \\ |\lambda| &= \lambda_1 + \lambda_2 + ... + \lambda_n. \end{aligned}$$

In the case $\lambda_k = \frac{1}{2}(k = 1, ...n)$ we have the well-known Euclidean norm. Note also that for any $t > 0$

$$\left(\left|t^{\lambda_1}x_1\right|^{\frac{1}{\lambda_1}} + \left|t^{\lambda_2}x_2\right|^{\frac{1}{\lambda_2}} + ... + \left|t^{\lambda_n}x_n\right|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}} = t^{\frac{|\lambda|}{n}} |x|_\lambda .$$

It means that non-isotropic λ-distance has the homogeneity of the degree $\frac{|\lambda|}{n}$, and also λ-distance has the following properties,

a) $|x|_\lambda = 0 \Leftrightarrow x = 0,$

b) $\left|p^\lambda x\right|_\lambda = p^{\frac{|\lambda|}{n}} |x|_\lambda ,$

c) $|x+y|_\lambda \leq 2^{(1+\frac{1}{\lambda_{\min}})\frac{|\lambda|}{n}} (|x|_\lambda + |y|_\lambda).$

Definition 2.2. [13] Let $g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function and f be a one-variable function, defined almost everywhere on $[0, \infty)$. g is called radial function, if there is a representation such that

$$g(x_1, x_2, ..., x_n) = f\left(\sqrt{(x_1)^2 + (x_2)^2 + ... + (x_n)^2}\right).$$

Definition 2.3. [1] Let $f \in L_p(\mathbb{R}^n)$ $1 \le p \le +\infty$ be a function and a point $x \in \mathbb{R}^n$ is called (λ, p)-Lebesgue point of the function f, if

$$\lim_{h\to 0}\left(\frac{1}{h^{2|\lambda|}}\int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}}\le h} |f(x-t) - f(x)|^p\, dt\right)^{\frac{1}{p}} = 0\ .$$

Definition 2.4. [1] λ-spherical coordinates are given by

$$\begin{aligned}
x_1 &= (u\cos\theta_1)^{2\lambda_1},\\
x_2 &= (u\sin\theta_1\cos\theta_2)^{2\lambda_2},\\
&\vdots\\
x_{n-1} &= (u\sin\theta_1\sin\theta_1 ... \sin\theta_{n-2}\cos\theta_{n-1})^{2\lambda_{n-1}},\\
x_n &= (u\sin\theta_1\sin\theta_1 ... \sin\theta_{n-2})^{2\lambda_n}
\end{aligned}$$

where

$$0 \le \theta_1, \theta_2, ..., \theta_{n-2} \le \pi, 0 \le \theta_{n-1} \le 2\pi, u \ge 0.$$

Denoting the Jacobian of this transformation by $J_\lambda(u, \theta_1, \theta_2, ..., \theta_{n-1})$, we obtain

$$J_\lambda(u, \theta_1, \theta_2, ..., \theta_{n-1}) = u^{2|\lambda|-1}\Omega_\lambda(\theta),$$

where

$$\Omega_\lambda(\theta) = 2^n\lambda_1...\lambda_n\prod_{j=1}^{n-1}(\cos\theta_j)^{2\lambda_j-1}(\sin\theta_j)^{\sum\limits_{k=j}^{j+1}2\lambda_k-1}$$

and we can easily see that

$$\omega_{\lambda,n-1} = \int\limits_{S^{n-1}}\Omega_\lambda(\theta)d\theta$$

is finite, where $S^{n-1}$is the unit sphere in $\mathbb{R}^n$.

Definition 2.5. $K_w\left(|.|_\lambda, .\right)$ belongs to λ-class, if the following conditions are satisfied

(A) There exists a summable function $L_w(|.|_\lambda) : \mathbb{R}^n \to \mathbb{R}$ such that

$$|K_w(|t|_\lambda, u) - K_w(|t|_\lambda, v)| \le L_w(|t|_\lambda)\,|u - v|$$

for any $w \in W$, $t \in \mathbb{R}^n$, $u, v \in \mathbb{R}$,

(B) For every $t \in \mathbb{R}^n$ and $u \in \mathbb{R}$

$$\lim_{w\to 0}\left|\int\limits_{\mathbb{R}^n} K_w(|t|_\lambda, u)\, dt - u\right| = 0.$$

(C) $L_w(|t|_\lambda) = w^{-|\lambda|}L\left(\left|\frac{t}{w^\lambda}\right|_\lambda\right)$ for any $w \in W$, $t \in \mathbb{R}^n$,

(D) $\lim\limits_{w\to 0} R_q^{\delta}(w)=0$, where

$$R_q^{\delta}(w)=\begin{cases}\left(\int\limits_{|t|_{\lambda}^{\frac{n}{2|\lambda|}}>\delta}(L_w(|t|_{\lambda}))^q dt\right)^{\frac{1}{q}} & 1\le q<+\infty\\ \underset{|t|_{\lambda}^{\frac{n}{2|\lambda|}}>\delta}{ess\sup} L_w(|t|_{\lambda}) & q=+\infty.\end{cases}$$

3. Convergence results

Theorem 3.1. *Let $f\in L_p(\mathbb{R}^n)$, $1\le p\le +\infty$. Assume that the kernel of the family of operators of type* 1.2 *is in λ-class, then*

$$|T_{w,\lambda}(f)(s)-f(s)|\to 0 \quad as\ w\to 0$$

when s is the continuity point of the function f.
Proof. We can easily observe that

$$\int\limits_{\mathbb{R}^n} K_w\left(|s-t|_{\lambda}, f(s)\right)dt=\int\limits_{\mathbb{R}^n} K_w\left(|t|_{\lambda}, f(s+t)\right)dt$$

then

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^n} K_w\left(|t|_{\lambda}, f(s+t)\right)dt-f(s)\right| &= \left|\int\limits_{\mathbb{R}^n} K_w\left(|t|_{\lambda}, f(s+t)\right)dt-\int\limits_{\mathbb{R}^n} K_w\left(|t|_{\lambda}, f(s)\right)dt\right.\\ &\quad \left.+\int\limits_{\mathbb{R}^n} K_w\left(|t|_{\lambda}, f(s)\right)dt-f(s)\right|\\ &\le \int\limits_{\mathbb{R}^n}\left|K_w\left(|t|_{\lambda}, f(s+t)\right)dt-K_w\left(|t|_{\lambda}, f(s)\right)\right|dt\\ &\quad +\left|\int\limits_{\mathbb{R}^n} K_w\left(|t|_{\lambda}, f(s)\right)dt-f(s)\right|\\ &= I_1+I_2.\end{aligned}$$

from the condition (B) of the λ-class $I_2\to 0$ whenever $w\to 0$.

Now we will prove $I_1\to 0$ whenever $w\to 0$. From the condition (A) of the λ-class, there exists a function which is in $L_q(\mathbb{R}^n)$ with

$$I_1\le\int\limits_{\mathbb{R}^n} L_w(|t|_{\lambda})\,|f(s+t)-f(s)|\,dt.$$

Due to the continuity of the function f at $t=s, \forall\varepsilon>0$ we can find a $\delta>0$ such that

$$|f(s+t)-f(s)|<\varepsilon \quad \text{whenever} \quad |t|_{\lambda}^{\frac{n}{2|\lambda|}}\le\delta$$

$$I_1 \leq \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} \leq \delta} L_w(|t|_\lambda) \, |f(s+t) - f(s)| \, dt + \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w(|t|_\lambda) \, |f(s+t) - f(s)| \, dt$$
$$= J_1 + J_2$$

from the property of continuity of the function $f(x)$ and the summability of the function $L_w(|t|_\lambda)$, $J_1 \to 0$ whenever $w \to 0$, and

$$J_2 = \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w(|t|_\lambda) \, |f(s+t) - f(s)| \, dt$$
$$\leq R_q^\delta(w) \left(\int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} (|f(s+t) - f(s)|)^p dt \right)^{\frac{1}{p}}$$

from the condition (D), $J_2 \to 0$ whenever $w \to 0$.

Theorem 3.2. *Let $f \in L_p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$. Assume that the kernel of the family of operators of type* 1.2 *is in λ-class, then*

$$|T_{w,\lambda}(f)(s) - f(s)| \to 0 \quad \text{as } w \to 0$$

when s is the (λ, p)-Lebesgue point of the function f.

Proof. We can easily observe that

$$\int\limits_{\mathbb{R}^n} K_w\left(|s-t|_\lambda, f(s)\right) dt = \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt$$

then

$$\left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt - f(s) \right| = \left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt - \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt \right.$$
$$\left. + \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt - f(s) \right|$$
$$\leq \int\limits_{\mathbb{R}^n} \left| K_w\left(|t|_\lambda, f(s+t)\right) dt - K_w\left(|t|_\lambda, f(s)\right) \right| dt$$
$$+ \left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt - f(s) \right|$$
$$= I_1 + I_2$$

from the condition (B) of the λ-class $I_2 \to 0$ whenever $w \to 0$.

Now we will prove $I_1 \to 0$ whenever $w \to 0$. From the condition (A) of the λ-class there exists a function which is in $L_q(\mathbb{R}^n)$ with

$$I_1 \leq \int\limits_{\mathbb{R}^n} L_w(|t|_\lambda)\,|f(s+t) - f(s)|\,dt.$$

We give some inequalities to prove this part.

Define a function,

$$\psi_\lambda(x) = \underset{|t|_\lambda \geq |x|_\lambda}{ess\sup} \int\limits_{\mathbb{R}^n} L_w(|t|_\lambda)dt.$$

It is seen that $\psi_\lambda(x)$ is radial function of the non-isotropic distance, so

$$\psi_\lambda(x) = \psi_\lambda(|x|).$$

If $|x|_\lambda^{\frac{n}{2|\lambda|}} = r$, then $\psi_\lambda(r) = \psi_\lambda(x)$. So, from the definition of the function $\psi_\lambda(x)$, it is a monotone decreasing function with respect to r. Hence,

$$\begin{aligned}
\int\limits_{\frac{r^{\frac{1}{2}}}{2} < |t|_\lambda^{\frac{n}{2|\lambda|}} < r^{\frac{1}{2}}} \psi_\lambda(t)dt &\geq \psi_\lambda(r^{\frac{1}{2}}) \int\limits_{\frac{r^{\frac{1}{2}}}{2} < |t|_\lambda^{\frac{n}{2|\lambda|}} < r^{\frac{1}{2}}} dt \\
&= \psi_\lambda(r^{\frac{1}{2}}) \int\limits_{\frac{r^{\frac{1}{2}}}{2}}^{r^{\frac{1}{2}}} \int\limits_{S^{n-1}} \Omega_\lambda(\theta)\rho^{2|\lambda|-1} d\theta d\rho \\
&= \psi_\lambda(r^{\frac{1}{2}}) \left(\frac{r^{|\lambda|}}{2\,|\lambda|} - \frac{r^{|\lambda|}}{2\,|\lambda|\,2^{2|\lambda|}} \right) \omega_{\lambda,n-1} \\
&= \psi_\lambda(r^{\frac{1}{2}}) r^{|\lambda|} \left(\frac{1}{2\,|\lambda|} - \frac{1}{2\,|\lambda|\,2^{2|\lambda|}} \right) \omega_{\lambda,n-1}.
\end{aligned}$$

Since $\psi_\lambda(r^{\frac{1}{2}}) r^{|\lambda|} \to 0$ whenever $r \to 0$ and $r \to \infty$, so we can find a constant $A > 0$ such that

$$\psi_\lambda(r^{\frac{1}{2}}) r^{|\lambda|} \leq A \qquad 0 < r < \infty$$

and to obtain the second inequality, we use the property of (λ, p)-Lebesgue point,

$$\lim_{h \to 0} \left(\frac{1}{h^{2|\lambda|}} \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} \leq h} \left\{ \int\limits_{S^{n-1}} \left| f(s + (\theta\rho)^{2|\lambda|}) - f(s) \right|^p \Omega_\lambda(\theta) d\theta \right\} \rho^{2|\lambda|-1} d\rho \right)^{\frac{1}{p}}$$

$$g_\lambda(\rho) := \int\limits_{S^{n-1}} \left| f((s + \theta\rho)^{2|\lambda|}) - f(s) \right|^p \Omega_\lambda(\theta) d\theta$$

$$\lim_{h\to 0}\left(\frac{1}{h^{2|\lambda|}}\int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}}\le h} g_\lambda(\rho)\rho^{2|\lambda|-1}d\rho\right)^{\frac{1}{p}}=0$$

$$G_\lambda(\rho):=\int\limits_0^\rho g_\lambda(\eta)\eta^{2|\lambda|-1}d\eta$$

so it is obvious that

$$G_\lambda(\rho)\le\varepsilon^p\rho^{2|\lambda|}$$

then

$$\begin{aligned}
I_1 &\le \int\limits_{\mathbb{R}^n} L_w(|t|_\lambda)\,|f(s+t)-f(s)|\,dt\\
&= \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}}\le\delta} L_w(|t|_\lambda)\,|f(s+t)-f(s)|\,dt+\int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} L_w(|t|_\lambda)\,|f(s+t)-f(s)|\,dt\\
&= J_1+J_2.
\end{aligned}$$

$$\begin{aligned}
J_1 &= \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}}\le\delta} w^{-|\lambda|}L\left(\left|\frac{t}{w^\lambda}\right|_\lambda\right)|f(s+t)-f(s)|\,dt\\
&= \left(\int\limits_0^\delta\left\{\int\limits_{S^{n-1}}\left|f(s+(\theta\rho)^{2|\lambda|})-f(s)\right|^p\Omega_\lambda(\theta)d\theta\right\}\rho^{2|\lambda|-1}w^{-|\lambda|}\psi_\lambda\left(\frac{\rho}{w^{\frac12}}\right)d\rho\right)^{\frac1p}\\
&= \left(\int\limits_0^\delta g_\lambda(\rho)\rho^{2|\lambda|-1}w^{-|\lambda|}\psi_\lambda\left(\frac{\rho}{w^{\frac12}}\right)d\rho\right)^{\frac1p}.
\end{aligned}$$

Applying integration by parts,

$$\begin{aligned}
&= \left(G_\lambda(\rho)w^{-|\lambda|}\psi_\lambda\left(\frac{\rho}{w^{\frac12}}\right)|_0^\delta-\int\limits_0^\delta G_\lambda(\rho)d\left(w^{-|\lambda|}\psi_\lambda\left(\frac{\rho}{w^{\frac12}}\right)\right)\right)^{\frac1p}\\
&\le \left(\varepsilon^p\rho^{2|\lambda|}w^{-|\lambda|}\psi_\lambda\left(\frac{\rho}{w^{\frac12}}\right)|_0^\delta-\int\limits_0^{\frac{\delta}{w^{\frac12}}} G_\lambda(w^{\frac12}u)w^{-|\lambda|}d\left(\psi_\lambda(u)\right)\right)^{\frac1p}\\
&= \left(\varepsilon^p\left(\frac{\delta^2}{w}\right)\psi_\lambda\left(\frac{\delta}{w^{\frac12}}\right)-\int\limits_0^{\frac{\delta}{w^{\frac12}}}\varepsilon^p w^{|\lambda|}u^{2|\lambda|}w^{-|\lambda|}d\left(\psi_\lambda(u)\right)\right)^{\frac1p}
\end{aligned}$$

$$\leq \varepsilon \left(A - \int_0^\infty u^{2|\lambda|} d\left(\psi_\lambda(u)\right) \right)^{\frac{1}{p}}.$$

Using integration by parts to calculate the last integral,

$$\begin{aligned} -\int_0^\infty u^{2|\lambda|} d\left(\psi_\lambda(u)\right) &= \lim_{r\to\infty} -r^{|\lambda|}\psi_\lambda(r) + 2|\lambda| \int_0^\infty u^{2|\lambda|-1}\psi_\lambda(r)du \\ &= \frac{2|\lambda|}{\omega_{\lambda,n-1}} \int_{\mathbb{R}^n} \psi_\lambda(x)dx < B. \end{aligned}$$

Here B is a constant, hence, $J_1 < \varepsilon B$. Now we investigate J_2 whenever $w \to 0$,

$$\begin{aligned} J_2 &= \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w(|t|_\lambda) \left| f(s+t) - f(s) \right| dt \\ &\leq \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} \left| f(s+t)) \right| L_w(|t|_\lambda) dt + \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} \left| f(s) \right| L_w(|t|_\lambda) dt. \end{aligned}$$

Let ψ_δ is the characteristic function of the set $\left\{ t \in \mathbb{R}^n : |t|^{\frac{n}{2|\lambda|}} > \delta \right\}$, then

$$\begin{aligned} &= \int_{\mathbb{R}^n} \left| f(s+t) \right| L_w(|t|_\lambda)\psi_\delta dt + \int_{\mathbb{R}^n} \left| f(s) \right| L_w(|t|_\lambda)\psi_\delta dt \\ &= \|f\|_p \left\| \psi_\delta L_w\left(|t|_\lambda\right) \right\|_q + \left| f(s) \right| \left\| \psi_\delta L_w\left(|t|_\lambda\right) \right\|_1. \end{aligned}$$

In view of the conditions (D),$\left\| \psi_\delta L_w\left(|t|_\lambda\right) \right\|_q$ goes to zero whenever $w \to 0$. From (C) of λ-class, one has

$$\begin{aligned} \left\| \psi_\delta L_w\left(|t|_\lambda\right) \right\|_1 = \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w\left(|t|_\lambda\right) dt &= \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} w^{-|\lambda|} L\left(\left| \frac{t}{w^\lambda} \right|_\lambda \right) dt \\ = \int_{|tw^\lambda|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L(|t|_\lambda) dt &= \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \frac{\delta}{\sqrt{w}}} L(|t|_\lambda) dt \end{aligned}$$

and this part also goes to zero whenever $w \to 0$.

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Adaptive algorithm for polyhedral approximation of 3D solids

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Abstract. In this paper we discuss theoretical foundations of developing general methods for volume-based approximation of three-dimensional solids. We construct an iterative method that can be used for approximation of regular subsets of $\mathbb{R}^d$ ($d \in \mathbb{N}$) in particular $\mathbb{R}^3$. We will define solid meshes and investigate the connection between solid meshes, regular sets and polyhedra. First the general description of the method will be given. The main idea of our algorithm is a kind of space partitioning with increasing atomic σ-algebra sequences. In every step one atom will be divided into two nonempty atoms. We define a volume-based distance metric and we give sufficient conditions for the convergence and monotonicity of the method. We show a possible application, a polyhedral approximation (or approximate convex decomposition) of triangular meshes.

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1. Introduction

Approximation of $2D$ or $3D$ subsets is useful many times. The main application is mesh simplification, i.e. we would like to approximate a mesh having n triangles with another mesh having $n' << n$ triangles. Mesh simplification algorithms sometimes use volume-based error metric [1], but the most common metric is the Hausdorff-distance, it can be found in many papers e.g. in [2]. A volume-based metric allows to approximate meshes which can not be realized to generate meshes which have a geometric realization. We investigated some topological properties of triangular meshes. There are some common topological error, e.g. holes, dangling faces, isolated faces, etc, a good summary of this topic can be read in [5]. Mesh repairing algorithms like [4] or [2] deal with changing the topological properties of meshes. Result of our method is

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a common finite polyhedron, which always has an obvious geometric realization. The discussed algorithm can be applied to recover topological properties, if we define properly the measure of an object having topological errors. Another intensively studied related topic is the approximate convex decomposition, see e.g. [16]. It can be used in physical simulation e.g. collision detection, fracture simulation [12] and obviously finite element methods. Similar approaches exist for volume-like approximations, for example spatial decomposition and mesh generation. Common spatial decomposition algorithms like octree and k-d trees can be found in [9] and comparative analysis of some algorithms in [22]. There are some useful information about mesh generation in [19]. During our work we studied the above topics, and it became clear, that the methods are very similar to each other, but there is no general theory published yet that connects these subjects. This was our motivation to give a theoretical foundation, which can describe most of the volume-based approximation methods. Reader will notice, that our concept is similar to the construction of the measure and integral theory, where it is needed we refer to the literature.

2. Solids

The subject of our study is the solid mesh. The solids can be triangular meshes, polyhedra or solid geometries, e.g. a solid cube. The solid can be determined by its vertices and faces, as we discussed in [13]. We will define the solid as a special subset of $\mathbb{R}^3$. General theory of solids can be found in [10], [11],[20], including the concept of regularity, which plays a central role, as we will see further. We draw up some requirements before defining the solid. Let us consider an $S \subset \mathbb{R}^d$ set. If S is a solid, it should meet the following expectations:

- S does not contain dangling faces, edges, isolated points or gaps
- S has volume and surface area
- S cannot be arbitrary large
- S cannot be decomposed into some parts satisfying the first 3 conditions.

As we will see, the requirements above can be translated as regularity, measurability, boundedness and connectedness. The last two criterions are obvious. In the followings, some properties of regularity will be detailed.

Let S be an arbitrary set. Denote $S^c, \overline{S}, \mathrm{int}S, \mathrm{ext}S, \partial S$ the complement, closure, interior, exterior and boundary of S, respectively. Let μ be the common Lebesgue-measure in $\mathbb{R}^d$, and denote $\mathbb{B}^d$ the d dimensional unit ball with respect to the Euclidean norm denoted by $\|.\|$, moreover let $B_r(x)$ be the ball with radius r centered at x. Topological subjects concerned with the notions of this paper can be found in [15].

Definition 2.1. *Let S be an arbitrary set. We define the regularized of S as*

$$S^* := \overline{intS}.$$

Definition 2.2. *S is said to be regular if $S = S^*$.*

Corollary 2.3. *If $S \subset \mathbb{R}^d$ and $S = S^*$ then*

$$\partial S = \partial \overline{intS} = \partial intS.$$

Therefore

$$S = S^* = \overline{intS} = intS \cup \partial intS = intS \cup \partial S.$$

Moreover $\mathbb{R}^d = intS \cup \partial S \cup extS = S \cup extS$, *consequently*

$$S^c = extS.$$

Lemma 2.4. *If $S = S^*$ then $x \in \partial S$ if and only if*

$$\forall r > 0 \ : \ (B_r(x) \setminus \{x\}) \cap intS \neq \emptyset \quad \textit{and} \quad (B_r(x) \setminus \{x\}) \cap extS \neq \emptyset.$$

Proof. Since x is a boundary point, then each $B_r(x)$ satisfies $B_r(x) \cap S \neq \emptyset$ and $B_r(x) \cap S^c \neq \emptyset$. Notice, that int$S$ does not contain isolated points and S^* contains all of its limit points. Therefore $x \in \partial S$ if and only if

$$\forall r > 0 \ : \ (B_r(x) \setminus \{x\}) \cap S \neq \emptyset \quad \text{and} \quad (B_r(x) \setminus \{x\}) \cap S^c \neq \emptyset.$$

As a consequence of Corollary 2.3 $S^c = \text{ext}S$, i.e.

$$(B_r(x) \setminus \{x\}) \cap S^c \neq \emptyset \ \Leftrightarrow \ (B_r(x) \setminus \{x\}) \cap \text{ext}S \neq \emptyset. \tag{2.1}$$

On the other hand $x \in \partial S = \partial\text{int}S$, thus $B_r(x) \cap \text{int}S \neq \emptyset$ for all $r > 0$, and $x \in \partial S$ implies $(B_r(x) \setminus \{x\}) \cap \text{int}S \neq 0$, hence

$$x \in \partial S \ \Rightarrow \ (B_r(x) \setminus \{x\}) \cap \text{int}S \neq \emptyset. \tag{2.2}$$

In the opposite direction $(B_r(x) \setminus \{x\}) \cap \text{int}S \neq 0$ is sufficient to fulfill $(B_r(x) \setminus \{x\}) \cap S \neq 0$, thus

$$(B_r(x) \setminus \{x\}) \cap \text{int}S \neq \emptyset \ \Rightarrow \ (B_r(x) \setminus \{x\}) \cap S \neq \emptyset. \tag{2.3}$$

(2.1),(2.2),(2.3) are sufficient to satisfy the statement. □

We need a measure such that the concept of volume can be interpreted. Most of our theorems can be proved supposing that μ is an outer measure, for simplicity we suppose that μ is the Lebesgue-measure.

Corollary 2.5. *The closed (and opened) sets of $\mathbb{R}^d$ are Lebesgue-measurable.*

It is known, that any open ball in $\mathbb{R}^d$ has a positive Lebesgue-measure, because for all $x \in \mathbb{R}^d$

$$\mu(B_r(x)) = \frac{\pi^{d/2}}{\Gamma(d/2+1)} r^d > 0 \tag{2.4}$$

is the volume of the d-dimensional ball with radius r.

Corollary 2.6. *An open set $S \subset \mathbb{R}^d$ is empty if and only if $\mu(S) = 0$.*

If S is regular and $S \neq \emptyset$ then int$S \neq 0$, since int$S = \emptyset$ implies $S = S^* = \overline{\text{int}S} = \overline{\emptyset} = \emptyset$.

Corollary 2.7. *A regular set $S \subset \mathbb{R}^d$ is empty if and only if $\mu(S) = 0$.*

We deal with regular, connected and bounded sets. It is easy to see, that there exists an isomorphism between every bounded subset of $\mathbb{R}^d$ and the subsets of the unit ball, so it is enough to consider the subsets of $\mathbb{B}^d$ without loss of generality.

Definition 2.8. *Let us define the following set:*

$$\Omega^d := \{S \subset \mathbb{B}^d \mid S \ \textit{connected, regular, and} \ \mu(\partial S) = 0\}.$$

If $S \in \Omega^d$ then we say S is a d-dimensional solid.

A commonly convex polyhedron is defined to be the finite intersection of half-spaces. Then a polyhedron can be defined as the finite union of convex polyhedra. By definition we have two important corollary with respect to polyhedra, see e.g. [14], [6], [3].

Corollary 2.9. *If B is a polyhedron, then B is a solid.*

Corollary 2.10. *Let B be a convex polyhedron, and H be a half-space, such that $B' := B \cap H$, $B'' := B \cap H^c$ with $intB' \neq \emptyset$, $intB'' \neq \emptyset$.Then $\overline{B'}$ and $\overline{B''}$ are convex polyhedra.*

3. Distance, approximation problem

In this section we use some notes from integral theory. The connection between measure and integral is studied in [8], [23]. The following concept of distance can be found in [8].

Definition 3.1. *Let us define the following function $\rho : 2^{\mathbb{R}^d} \times 2^{\mathbb{R}^d} \to \mathbb{R}_0^+$ which defines the distance between subsets A and B of $\mathbb{R}^d$*

$$\rho(A, B) := \mu(A \triangle B) \quad (A, B \subset \mathbb{R}^d)$$

where $A \triangle B := (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference.

We can see that if $\mu(A \triangle B) = 0$ then A and B may differ only on a set of measure zero. Denote χ_S the indicator function of some set S. If A, B are measurable and bounded then χ_A, χ_B are integrable. Therefore we can express the measure of set operations by integrals.

$$\mu(A \cap B) = \int_{\mathbb{R}^d} \chi_A \chi_B \, d\mu \tag{3.1}$$

$$\mu(A^c) = \mu(\mathbb{R}^d \setminus A) = \int_{\mathbb{R}^d} 1 - \chi_A \, d\mu \tag{3.2}$$

$$\mu(A \cup B) = \int_{\mathbb{R}^d} \chi_A + \chi_B - \chi_A \chi_B \, d\mu. \tag{3.3}$$

The distance can be reformulated as

$$\rho(A, B) = \int_{\mathbb{R}^d} \chi_A + \chi_B - 2\chi_A \chi_B \, d\mu.$$

There are connections between measure and integral. The range of χ is $\{0, 1\}$ then $\chi \equiv \chi^p$ for all $1 \leq p < +\infty$, consequently

$$\rho(A, B) = \int_{\mathbb{R}^d} \chi_A^2 + \chi_B^2 - 2\chi_A \chi_B \, d\mu = \int_{\mathbb{R}^d} (\chi_A - \chi_B)^2 \, d\mu$$
$$= \int_{\mathbb{R}^d} |\chi_A - \chi_B| \, d\mu = \int_{\mathbb{R}^d} |\chi_A - \chi_B|^p \, d\mu =: \|\chi_A - \chi_B\|_{L_\mu^p}^p. \tag{3.4}$$

As we see, $\rho(A, B)$ equals to the p-th power of the common L_μ^p-norm of the measurable function $\chi_A - \chi_B$. This implies, that ρ is a pseudo-metric, since ρ is non-negative, triangle inequality holds and $\rho(A, A) = 0$, but $\rho(A, B) = 0$ does not imply $A = B$, because A and B may differ on a set of measure zero. Now we show, that A and B can not differ in the case $\rho(A, B) = 0$ if they are regular sets.

Theorem 3.2. *Let $A, B \subset \mathbb{R}^d$ be regular sets. Then $\rho(A, B) = 0$ if and only if $A = B$.*

Proof. If $A = B$ then obviously $A \triangle B = \emptyset$, thus $\rho(A, B) = \mu(A \triangle B) = 0$. To verify the opposite direction, let us suppose $\rho(A, B) = 0$. Since A, B are regular sets, we can express the symmetric difference this way:

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cap B^c) \cup (B \cap A^c)$$

$$= ((\mathrm{int}A \cup \partial A) \cap \mathrm{ext}B) \cup ((\mathrm{int}B \cup \partial B) \cap \mathrm{ext}A)$$

$$(\mathrm{int}A \cap \mathrm{ext}B) \cup (\partial A \cap \mathrm{ext}B) \cup (\mathrm{int}B \cap \mathrm{ext}A) \cup (\partial B \cap \mathrm{ext}A).$$

All four members of the last term are disjoint. Using properties of the measure we get

$$\rho(A, B) = \mu(\mathrm{int}A \cap \mathrm{ext}B) + \mu(\mathrm{int}B \cap \mathrm{ext}A)$$

$$+\mu(\partial A \cap \mathrm{ext}B) + \mu(\partial B \cap \mathrm{ext}A) = 0.$$

It can be seen $\rho(A, B) = 0$ if and only if every member has zero Lebesgue-measure. Notice that $\mathrm{int}A$ and $\mathrm{ext}B$ are open sets, therefore $\mathrm{int}A \cap \mathrm{ext}B$ is open, so they have zero measure if and only if they are empty. Since $\mathrm{ext}A \cap \mathrm{int}B$ is open, the followings are necessarily true:

$$\mathrm{int}A \cap \mathrm{ext}B = \emptyset$$

$$\mathrm{ext}A \cap \mathrm{int}B = \emptyset.$$

Let us suppose, that $\partial A \cap \mathrm{ext}B \neq \emptyset$. Then exists $x \in \mathbb{R}^d$ such for some $r > 0$ $B_r(x) \subset \mathrm{ext}B$, moreover as a consequence of Lemma 2.4

$$(B_r(x) \setminus \{x\}) \cap \mathrm{int}A \neq \emptyset.$$

Thus

$$(B_r(x) \setminus \{x\}) \cap \mathrm{int}A \cap \mathrm{ext}B \neq \emptyset,$$

i.e. $\mathrm{int}A \cap \mathrm{ext}B \neq \emptyset$, which is a contradiction. Consequently $\partial A \cap \mathrm{ext}B$ is empty, similarly $\partial B \cap \mathrm{ext}A$ is empty, as well. To sum up, $\rho(A, B) = 0$ implies $A \triangle B = \emptyset$, which proves the statement. □

As a consequence of the preceding theorem ρ is a metric on the regular subsets of $\mathbb{R}^d$. As Ω^d contains only regular sets, we get the following result.

Corollary 3.3. *ρ is a metric on Ω^d.*

The abstract approximation problem is the following:

Given $S \subset \mathbb{R}^d$ a solid and $\varepsilon > 0$ a positive real. We are looking for a solid $\tilde{S} \subset \mathbb{R}^d$ which has ,,better properties" than S in some sense, and $\mu(S, \tilde{S}) < \varepsilon$.

In the next sections we will discuss our construction to give a possible solution for the problem above. Here ,,better properties" refers to e.g. topological correctness, simplicity, convexity, etc. depending on the objective.

4. Construction

Consider $\mathcal{B} \subset 2^{\mathbb{B}^d}$ to be a finite atomic σ-algebra on $\mathbb{B}^d$, i.e.

$$\begin{aligned} \mathcal{B} &:= \sigma(B_i \subset \mathbb{B}^d \ : \ i = 0, 1, \dots, n) \quad (n \in \mathbb{N}) \\ B_i \cap B_j &= \emptyset \ (i \neq j), \quad \cup_{i=0}^n B_j = \mathbb{B}^d. \end{aligned} \tag{4.1}$$

Then the functions contained in $L_\mu(\mathcal{B})$ are integrable, and by definition they are constant on an arbitrary B_i atom. Thus $L_\mu(\mathcal{B}) \subset L_\mu(\mathbb{B}^d)$, moreover $L_\mu(\mathcal{B})$ is a finite dimensional subspace in $L_\mu(\mathbb{B}^d)$. Denote by $\chi(X)$ the set of indicator functions defined on subsets of some set system X. Obviously $\chi(\mathcal{B}) \subset L_\mu(\mathcal{B})$. Moreover, if $S \subset \mathbb{B}^d$ then $\chi_S \in \chi(\mathbb{B}^d)$, i.e. a bijection can be defined between the subsets of $\mathbb{B}^d$ and the functions from $\chi(\mathbb{B}^d)$.

Let us define the following functions:

$$\phi_i := \frac{1}{\sqrt{\mu(B_i)}} \chi_{B_i}. \tag{4.2}$$

It is easy to see, that $\{\phi_i\}_{i=0}^n$ forms an orthonormed system under the common inner product:

$$\langle \phi_i, \phi_j \rangle := \int_{\mathbb{B}^d} \phi_i \phi_j d\mu = \frac{1}{\sqrt{\mu(B_i)\mu(B_j)}} \int_{\mathbb{B}^d} \chi_{B_i} \chi_{B_j} d\mu.$$

If $i \neq j$ then $B_i \cap B_j = \emptyset$ implies that the inner product is zero, and obviously

$$\langle \phi_i, \phi_i \rangle := \frac{1}{\sqrt{\mu(B_i)^2}} \int_{\mathbb{B}^d} \chi_{B_i}^2 d\mu = \frac{1}{\mu(B_i)} \int_{B_i} d\mu = 1.$$

Consequently $\{\phi_i\}_{i=0}^n$ is an orthonormed system on a finite dimensional subspace of the Hilbert-space $L_\mu(\mathbb{B}^d)$. By Riesz projection theorem the best approximation in $L_\mu(\mathcal{B})$ of an arbitrary function $f \in L_\mu(\mathbb{B}^d)$ can be expressed by the Fourier-series with respect to $\mathcal{B}$, see e.g. [7],[17]:

$$\mathcal{F}^{\mathcal{B}} f := \sum_{i=0}^n \langle f, \phi_i \rangle \phi_i. \tag{4.3}$$

Let be given $S \subset \mathbb{B}^d$. Then we get the following formula for the i-th Fourier-coefficient of χ_S:

$$\begin{aligned} \langle \chi_S, \phi_i \rangle &= \frac{1}{\sqrt{\mu(B_i)}} \langle \chi_S, \chi_{B_i} \rangle = \frac{1}{\sqrt{\mu(B_i)}} \int_{\mathbb{B}^d} \chi_S \chi_{B_i} \, d\mu \\ &= \frac{1}{\sqrt{\mu(B_i)}} \int_{B_i \cap S} \, d\mu = \frac{1}{\sqrt{\mu(B_i)}} \mu(B_i \cap S). \end{aligned}$$

Thus the Fourier-series of χ_S is

$$\mathcal{F}^{\mathcal{B}} \chi_S := \sum_{i=0}^n \frac{1}{\sqrt{\mu(B_i)}} \mu(B_i \cap S) \phi_i = \sum_{i=0}^n \frac{\mu(B_i \cap S)}{\mu(B_i)} \chi_{B_i} = \sum_{i=0}^n b_i \chi_{B_i} \tag{4.4}$$

where

$$b_i := \frac{\mu(B_i \cap S)}{\mu(B_i)}. \tag{4.5}$$

$\mathcal{F}^{\mathcal{B}}\chi_S$ is also the best approximation of χ_S on the subspace spanned by $\mathcal{B}$. The main problem is, that $\mathcal{F}^{\mathcal{B}}\chi_S$ is generally not an element of $\chi(\mathbb{B}^d)$, therefore we can not assign a set from $\mathbb{B}^d$ to the resulting function. To solve this problem, we will introduce the following operator.

Definition 4.1. *Let $f \in L_\mu(\mathbb{B}^d)$ be an arbitrary function and $\alpha \in (0,1)$. Define the operator $\mathcal{X}_\alpha : L_\mu(\mathbb{B}^d) \to \chi(\mathbb{B}^d)$ as follows*

$$\mathcal{X}_\alpha f(x) := \begin{cases} 0, & f(x) \le \alpha \\ 1, & f(x) > \alpha. \end{cases}$$

$\mathcal{X}_\alpha f$ is an approximation of f by an indicator function, and the resulting function depends on a real parameter $\alpha \in (0,1)$. Later the importance of the value of α will be explained.

Corollary 4.2. *Let $S \subset \mathbb{B}^d$ be an arbitrary set, $\alpha \in (0,1)$.*

$$\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}}\chi_S \in \chi(\mathcal{B})$$

in other words

$$\exists \tilde{S} \subset \mathbb{B}^d \ : \ \chi_{\tilde{S}} = \mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}}\chi_S$$

namely

$$\tilde{S} := \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}}\chi_S = 1\}.$$

In our approach the set $\tilde{S}$ is the approximation of S with respect to the system $\mathcal{B}$ and α.

Our strategy is similar to Schipp's construction in [21]. We would like to construct a sequence of increasing σ-algebras to refine the approximation. Let us suppose, we have a σ-algebra $\mathcal{B}_n$ and a set S to be approximated. Now we can draw up the approximation with respect to $\mathcal{B}_n$ as $S_n := \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n}\chi_S = 1\}$. We need to check if $\rho(S_n, S) < \varepsilon$. If not, then we have to construct a larger σ-algebra $\mathcal{B}_{n+1}$, compute the $(n+1)$-th approximation and its distance. Repeat this process while $\rho(S_n, S) \ge \varepsilon$. It seems to be easy, but we have to work out some conditions for refinement to ensure the convergence. Denote $[a..b]$ the interval of natural numbers between a and b, i.e. $[a..b] := [a,b] \cap \mathbb{N}$.

Let us consider the following atomic decomposition sequence of the unit ball

$$\begin{aligned} &\mathcal{B}_n = \sigma(B_0, B_1, \dots, B_n) \\ &B_i \cap B_j = \emptyset \quad (i \neq j), \quad \bigcup_{i=0}^{n} B_i = \mathbb{B}^d. \end{aligned} \tag{4.6}$$

By definition, the n-th σ-algebra is generated by exactly $n+1$ subsets, then $\mathcal{B}_{n+1}$ can be obtained only by splitting a generator element of $\mathcal{B}_n$, for details see [18]:

$$\exists k \in [0..n] \ : \ \mathcal{B}_{n+1} = \sigma(B_0, \dots, B_{k-1}, B_k', B_k'', B_{k+1}, \dots, B_n)$$

such that

$$B_k' \neq \emptyset, \quad B_k'' \neq \emptyset, \quad B_k' \cup B_k'' = B_k, \quad B_k' \cap B_k'' = \emptyset.$$

Then the indicator function of the n-th approximation can be defined by

$$\chi_{S_n} := \mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n} \chi_S. \tag{4.7}$$

Denote diam(A) the diameter of some set $A \subset \mathbb{B}^d$, i.e.

$$\operatorname{diam}(A) := \sup_{x,y\in A} \|x-y\|. \tag{4.8}$$

We give a sufficient condition for the convergence of S_n.

Theorem 4.3. *Let $(\mathcal{B}_n)$ be an increasing sequence of σ-algebra, where*

$$\mathcal{B}_n = \sigma(B_0^{(n)}, B_1^{(n)}, \dots, B_n^{(n)})$$

and consider $S \in \Omega^d$ to be a solid, an arbitrary $\alpha \in (0,1)$ and let

$$S_n = \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n} \chi_S = 1\}.$$

If the diameter of all the generator atoms converge to zero, i.e.

$$\lim_{n\to\infty} \max_{k\in[0..n]} \operatorname{diam}(B_k^{(n)}) = 0$$

then

$$\lim_{n\to\infty} S_n = S.$$

Proof. Since S is solid, $\lim_{n\to\infty} S_n = S$ if and only if $\lim_{n\to\infty} \rho(S_n, S) = 0$ as a consequence of Theorem 3.2, and

$$\rho(S_n, S) = \int_{\mathbb{B}^d} |\chi_{S_n} - \chi_S| \, d\mu$$

$$= \int_{\text{int}S} |\chi_{S_n} - \chi_S| \, d\mu + \int_{\text{ext}S} |\chi_{S_n} - \chi_S| \, d\mu + \int_{\partial S} |\chi_{S_n} - \chi_S| \, d\mu. \tag{4.9}$$

Notice that any function in $\chi(\mathbb{B}^d)$ can be dominated by $\chi_{\mathbb{B}^d}$, which is integrable. Using this fact the third integral equals to zero, because $\mu(\partial S) = 0$.

We know by Lebesgue's density theorem, that for almost every $x \in S$

$$\lim_{r\to 0} \frac{\mu(B_r(x) \cap S)}{\mu(B_r(x))} = 1. \tag{4.10}$$

This implies that

$$\forall x \in \text{int}S \ : \ \lim_{r\to 0} \frac{\mu(B_r(x) \cap S)}{\mu(B_r(x))} = 1$$

$$\forall x \in \text{ext}S \ : \ \lim_{r\to 0} \frac{\mu(B_r(x) \cap S)}{\mu(B_r(x))} = 0.$$

If $x \in \text{int}S$ then exists $r > 0$ such $B_r(x) \subset \text{int}S$. On the other hand since $\mathcal{B}_n$ is atomic, then for all $n \in \mathbb{N}$ exists a unique k such $x \in B_k^{(n)}$. Since diam$(B_k^{(n)})$ tends to zero, we get $B_k^{(n)} \subset B_r(x)$ for sufficiently large n, i.e. pointwise convergence is true for any $x \in \text{int}S$. Similarly, pointwise convergence can be proved in the case $x \in \text{ext}S$. Now using dominant convergence theorem we have

$$\lim \int_{\text{int}S} |\chi_{S_n} - \chi_S| \, d\mu = \int_{\text{int}S} \lim |\chi_{S_n} - \chi_S| \, d\mu = 0$$

$$\lim \int_{\text{ext}S} |\chi_{S_n} - \chi_S| \, d\mu = \int_{\text{ext}S} \lim |\chi_{S_n} - \chi_S| \, d\mu = 0$$

to sum up, all the three integrals in (4.9) equals to 0 as $n \to \infty$, therefore $\rho(S_n, S) = 0$, which proves the theorem. □

We should not require that the maximal diameter of the decomposition tends to zero, it would be too expensive in applications. To avoid this problem and to simplify the procedure, we have the following idea. If a Fourier-coefficient of an atom is exactly 0 or 1, it is unnecessary to split. Accurately, if every Fourier-coefficients tends to 0 or 1, then the preceding theorem is automatically satisfied.

Lemma 4.4. *Let be*

$$I_n := \{i \in [0..n] \mid b_i^{(n)} > \alpha\}$$

and

$$J_n := [0..n] \setminus I_n.$$

Then

$$S_n = \bigcup_{i \in I_n} B_i^{(n)}$$

moreover

$$\rho(S, S_n) = \sum_{i \in J_n} \mu(B_i^{(n)}) b_i^{(n)} + \sum_{i \in I_n} \mu(B_i^{(n)})(1 - b_i^{(n)}).$$

Proof. By definition

$$S_n = \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n} \chi_S = 1\} = \left\{ \mathcal{X}_\alpha \sum_{i=0}^{n} b_i^{(n)} \chi_{B_i} = 1 \right\}$$
$$= \left\{ \sum_{i \in I_n} \chi_{B_i} = 1 \right\} = \bigcup_{i \in I_n} B_i^{(n)}.$$

Using the formula above, we can write

$$\begin{aligned}
\rho(S, S_n) &= \mu(S \triangle S_n) = \mu(S \cup S_n) - \mu(S \cap S_n) \\
&= \mu(S \cup \bigcup_{i \in I_n} B_i^{(n)}) - \mu(S \cap \bigcup_{i \in I_n} B_i^{(n)}) \\
&= \mu(S \cup \bigcup_{i \in I_n} (B_i^{(n)} \cap S^c)) - \mu(\bigcup_{i \in I_n} (B_i^{(n)} \cap S)) \\
&= \mu(S) - \mu(\bigcup_{i \in I_n} (B_i^{(n)} \cap S) + \mu(\bigcup_{i \in I_n} (B_i^{(n)} \cap S^c)) \\
&= \mu(\bigcup_{i \in J_n} (B_i^{(n)} \cap S)) + \mu(\bigcup_{i \in I_n} (B_i^{(n)} \cap S^c)) \\
&= \sum_{i \in J_n} \mu(B_i^{(n)} \cap S) + \sum_{i \in I_n} \mu(B_i^{(n)}) - \mu(B_i^{(n)} \cap S) \\
&= \sum_{i \in J_n} \mu(B_i^{(n)}) b_i^{(n)} + \sum_{i \in I_n} \mu(B_i^{(n)})(1 - b_i^{(n)}).
\end{aligned}$$

□

We can estimate the distance, if we define the relevant indices.

Definition 4.5. *Let be*

$$\Delta_n := \{i \in [0..n] \mid b_i^{(n)} \in (0,1)\}.$$

An i index and the $B_i^{(n)}$ generator atom are said to be relevant if $i \in \Delta_n$.

Δ_n contains all the indices of atoms which have Fourier-coefficients not exactly equal to 0 or 1. Notice, that if $b_j^{(n)} = 0$ then $j \in J_n$ for all $\alpha \in (0,1)$. Consequently in the first sum there is a multiplication with $b_j^{(n)} = 0$, moreover $I_n \cap J_n = \emptyset$ implies $j \notin I_n$ therefore j can be left from both sums. Similarly, if $b_i^{(n)} = 1$ then we multiply with $(1 - b_i^{(n)}) = 0$ in the second sum and $i \notin J_n$, therefore i can be left, as well. Because of this we have the following corollary.

Corollary 4.6.

$$\begin{aligned}\rho(S, S_n) &= \sum_{i \in J_n \cap \Delta_n} \mu(B_i^{(n)}) b_i^{(n)} + \sum_{i \in I_n \cap \Delta_n} \mu(B_i^{(n)})(1 - b_i^{(n)}) \\ &\le \sum_{i \in J_n \cap \Delta_n} \mu(B_i^{(n)}) + \sum_{i \in I_n \cap \Delta_n} \mu(B_i^{(n)}) = \sum_{i \in \Delta_n} \mu(B_i^{(n)}).\end{aligned}$$

We are ready to state a stronger convergence theorem.

Theorem 4.7. *Let $(\mathcal{B}_n)$ be an increasing sequence of σ-algebra where*

$$\mathcal{B}_n = \sigma(B_0^{(n)}, B_1^{(n)}, \ldots, B_n^{(n)})$$

and consider a solid $S \in \Omega^d$, an arbitrary $\alpha \in (0,1)$ and let

$$S_n = \{\mathcal{X}_\alpha \mathcal{F}^{\mathcal{B}_n} \chi_S = 1\}.$$

If the diameter of all the relevant generator atoms converge to zero, i.e.

$$\lim_{n\to\infty} \max_{k \in \Delta_n} \operatorname{diam}(B_k^{(n)}) = 0$$

then

$$\lim_{n\to\infty} S_n = S.$$

Proof. If $b_i^{(n)} = 0$ then $\mu(B_i^{(n)} \cap S) = \mu(B_i^{(n)})$. Since atoms and S are regular sets, we have $B_i^{(n)} \cap S = B_i^{(n)}$, therefore $\chi_{B_i^{(n)}}(x) = \chi_S(x)$ for all $x \in B_i^{(n)}$. The same result is true if $b_i^{(n)} = 1$. Consequently,

$$\begin{aligned}\rho(S_n, S) = \int_{\mathbb{B}^d} |\chi_{S_n} - \chi_S| \, d\mu &= \sum_{i=0}^{n} \int_{B_i^{(n)}} |\chi_{S_n} - \chi_S| \, d\mu \\ &= \sum_{i \in \Delta_n} \int_{B_i^{(n)}} |\chi_{S_n} - \chi_S| \, d\mu.\end{aligned}$$

Using the same idea as in Theorem 4.3 we can prove that the remaining integrals tend to zero as $n \to \infty$. ☐

We have convergence theorems, finally we would like to investigate the monotonicity of the convergence. The following theorem gives a monotonicity condition of the approximation method. We will show, that it is closely related to the value of α.

Theorem 4.8. (S_n) *is an improving approximation sequence, i.e.*

$$\rho(S_{n+1}, S) \leq \rho(S_n, S)$$

for all $n \in \mathbb{N}$ *if and only if* $\alpha = \frac{1}{2}$.

Proof. Let us suppose, that $B_k^{(n)} =: B \in \mathcal{B}_n$ is divided into two disjoint non-empty $B', B'' \in \mathcal{B}_{n+1}$ sets. We have to show, that the sequence $\delta_n := \rho(S, S_n) - \rho(S, S_{n+1})$ is non-negative, i.e. $\delta_n \geq 0$ $(n \in \mathbb{N})$. We have 6 possible outcomes with respect to the relationship of the Fourier-coefficients b, b', b'' and α, viz. it depends on the Fourier-coefficients if B is a subset of S_n and if B' or B'' or both B', B'' are subsets of S_{n+1}.

1. $b > \alpha,\ b' > \alpha,\ b'' > \alpha$
   Then
$$\delta_n = \mu(B \cap S) - (\mu(B' \cap S) + \mu(B'' \cap S)) = 0.$$
2. $b \leq \alpha,\ b' \leq \alpha,\ b'' \leq \alpha$
   Similarly,
$$\delta_n = \mu(B \cap S^c) - (\mu(B' \cap S^c) + \mu(B'' \cap S^c)) = 0.$$
3. $b > \alpha,\ b' \leq \alpha,\ b'' \leq \alpha$
   This implies $\mu(B' \cap S) \leq \alpha\mu(B')$ s $\mu(B'' \cap S) \leq \alpha\mu(B'')$, therefore
$$\mu(B \cap S) = \mu(B' \cap S) + \mu(B'' \cap S) \leq \alpha(\mu(B') + \mu(B'')) = \alpha\mu(B)$$
   consequently $b \leq \alpha$, and it is a contradiction. This case cannot be realized.
4. $b \leq \alpha,\ b' > \alpha,\ b'' > \alpha$
   Similarly to the preceding case, this case is also impossible.
5. $b > \alpha,\ b' > \alpha,\ b'' \leq \alpha$
   Under the assumptions
$$\begin{aligned}\delta_n &= \mu(B \cap S^c) - (\mu(B' \cap S^c) + \mu(B'' \cap S)) \\ &= (\mu(B \cap S^c) - \mu(B' \cap S^c)) - \mu(B'' \cap S) \\ &= \mu(B'' \cap S^c) - \mu(B'' \cap S) \\ &= \mu(B'') - 2\mu(B'' \cap S) \geq 0,\end{aligned}$$
   if and only if
$$\mu(B'') \geq 2\mu(B'' \cap S)$$
$$\frac{1}{2} \geq b''.$$
   This holds for arbitrary B'' only if $\alpha \leq \frac{1}{2}$.
6. $b \leq \alpha,\ b' \leq \alpha,\ b'' > \alpha$
   In this case, similarly to the preceding we get
$$\delta_n = \mu(B'') - 2\mu(B'' \cap S^c)) \geq 0,$$
   if and only if
$$\mu(B'') \geq 2\mu(B'' \cap S^c)$$

$$\frac{1}{2} \geq \frac{\mu(B'' \cap S^c)}{\mu(B'')} = \frac{\mu(B'') - \mu(B'' \cap S)}{\mu(B'')} = 1 - b''$$

$$b'' \geq \frac{1}{2}.$$

It is true in general case only if $\alpha \geq \frac{1}{2}$.

In summary, 3. and 4. are impossible, 1. and 2. implies $\delta_n = 0$, in the cases 5. and 6. $\delta_n \geq 0$ is satisfied only if $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, respectively. Consequently $\delta_n \geq 0$ for all possible outcomes if and only if $\alpha = \frac{1}{2}$. □

To guarantee the monotonicity of the convergence let us redefine the n-th approximation for $\alpha = \frac{1}{2}$

$$S_n := \{\mathcal{X}_{\frac{1}{2}} \mathcal{F}^{\mathcal{B}_n} \chi_S = 1\}. \tag{4.11}$$

Applying the above idea, we can give a general schema to develop volume-based approximation methods:

1. Let be given a solid $S \in \Omega^d$, moreover let $B_0^{(0)} \supseteq \mathbb{B}^d$ be an arbitrary superset of the unit ball, which is the unique generator set of the $\mathcal{B}_0$ σ-algebra.
2. **Choose** an index k, where $0 < b_k^{(n)} < 1$, and **divide** the set $B_k^{(n)}$ into two non-empty disjoint sets. In this way we obtain the algebra $\mathcal{B}_{n+1}$.
3. Compute the new Fourier-coefficients b'_k, b''_k as well as $\rho(S_n, S)$. While $\rho(S_n, S) \geq \varepsilon$ for some given $\varepsilon > 0$ tolerance, go back to step 2.
4. If $\rho(S_n, S) < \varepsilon$ we are done, we could define ∂S if it is needed.

It can be seen, there are two important questions unanswered, namely: how could we choose the k index and how could we divide the $B_k^{(n)}$ atom such that the assumptions of our convergence theorems are satisfied. Let us define the following functions.

Definition 4.9. *The $\mathcal{C} : 2^{2^{\mathbb{B}^d}} \to \mathbb{N}$ type function is said to be a choosing function if*

$$\forall n \in \mathbb{N} \ \exists! k \in [0..n] \ : \ \mathcal{C}(\mathcal{B}_n) = k.$$

Moreover $\mathcal{D} : 2^{\mathbb{B}^d} \to 2^{\mathbb{B}^d} \times 2^{\mathbb{B}^d}$ is said to be a dividing function if

$$\forall B \in 2^{\mathbb{B}^d} \ \exists! B', B'' \in 2^{\mathbb{B}^d} \ : \ B' \neq \emptyset, \ B'' \neq \emptyset$$

$$\mathcal{D}(B) = (B', B''), \ B' \cap B'' = \emptyset, \ B' \cup B'' = B.$$

With our new notations we can draw up the approximation schema more precisely.

VolumeBasedApproximation$(S, B_0^{(0)}, \varepsilon, \mathcal{C}, \mathcal{D})$
1. $n := 0$
2. $\mathcal{B}_n := \sigma(B_0^{(n)}, \dots, B_n^{(n)})$
3. $k := \mathcal{C}(\mathcal{B}_n)$
4. $S_n := \{\mathcal{X}_{\frac{1}{2}} \mathcal{F}^{\mathcal{B}_n} \chi_S = 1\}$
5. **if** $\rho(S_n, S) \geq \varepsilon$ **then**
6. $\quad \mathcal{B}_{n+1} := \sigma(B_0^{(n)}, \dots, B_{k-1}^{(n)}, \mathcal{D}(B_k^{(n)}), B_{k+1}^{(n)}, \dots, B_n^{(n)})$
7. $\quad n := n + 1$
8. $\quad$ **goto** 3.
9. $\quad$ **else stop**

(4.12)

To develop a volume-based approximation algorithm we need only to define exactly the choosing function and the dividing function. As we can see $\mathcal{C}, \mathcal{D}, B_0^{(0)}, \varepsilon$ are free parameters, the result can be affected by all of them. In the next section we give a simple example for an approximation algorithm.

5. Application

We applied our method successfully to give approximation of three-dimensional triangular meshes. Let $d := 3$, $B_0^{(0)} \supset \mathbb{B}^3$ is the cube of side 2 centered at the origin, $\varepsilon > 0$ be an arbitrary real. Denote

$$\mathcal{C}_0(\mathcal{B}_n) := \left\{ k \in \Delta_n \mid \mu(B_k^{(n)}) = \max_{j \in \Delta_n} \mu(B_j^{(n)}) \right\}. \tag{5.1}$$

Let be $a, b \in \mathbb{B}^3$ and

$$H_{a,b} := \left\{ x \in \mathbb{R}^3 \mid \langle x - \tfrac{a+b}{2}, b - a \rangle \leq 0 \right\} \tag{5.2}$$

is a half-space, the points below the plane determined by the midpoint of a and b and the normal vector in the direction of $b - a$. Then we can define the following function

$$\begin{aligned} \mathcal{D}_0(B) := \{(B', B'') \mid B' = B \cap H_{a,b},\ B'' = B \setminus B' \\ \|a - b\| = \operatorname{diam}(B)\}. \end{aligned} \tag{5.3}$$

It is obvious, that $|\mathcal{C}_0| \geq 1$ and $|\mathcal{D}_0| \geq 1$, therefore $\mathcal{C}_0$ and $\mathcal{D}_0$ are not functions in generally. It can be thought, that we can give some extra conditions to obtain functions, e.g. let k be the minimal index which satisfies (5.1), and let a have the smallest x, y, z coordinate value satisfying (5.3). Let us denote the functions describing these additional conditions with $\mathcal{C}_1$ and $\mathcal{D}_1$ respectively. For instance, we chose the maximal indices for k and for a using that a finite convex polyhedron's diameter can be spanned by only two of its vertices, and vertices can be indexed. Here we used Corollary 2.10 i.e. a convex polyhedron divided a by plane results two convex polyhedra. As a consequence of the definition of $\mathcal{C}_1$ and $\mathcal{D}_1$, it is easy to see that

$$\mathcal{C} := \mathcal{C}_1 \circ \mathcal{C}_0 \tag{5.4}$$

is a choosing function,

$$\mathcal{D} := \mathcal{D}_1 \circ \mathcal{D}_0 \tag{5.5}$$

is a dividing function.

Theorem 5.1. *Let $B_0^{(0)} \supset \mathbb{B}^3$, $S \subset \Omega^3$ and $\varepsilon > 0$, and let $\mathcal{C}, \mathcal{D}$ be defined by (5.4), (5.5), (5.1), (5.3). Then the algorithm defined by (4.12) is convergent.*

Proof. In terms of Theorem 4.7 it is enough to prove, that the definitions of $\mathcal{C}$ and $\mathcal{D}$ ensure that

$$\lim_{n\to\infty} \max_{k\in\Delta_n} \operatorname{diam}(B_k^{(n)}) = 0.$$

Let us suppose, that in the n-th step of the iteration the k-th atom has maximal diameter, i.e. $r := \max_{k\in\Delta_n} \operatorname{diam}(B_k^{(n)})$, furthermore let us suppose $\mathcal{C}(\mathcal{B}_n) = k$. Corollary 2.10 implies that $B_k^{(n)}$ is a convex polyhedron for all $n \in \mathbb{N}, k \in \Delta_n$ determined by a finite number of vertices. Because of this there are finite number of vertex pairs $(a_0, b_0), \dots, (a_{n_1}, b_{n_1})$ which satisfy $\|a_i - b_i\| \geq \frac{r}{2}$. Let us suppose, that we apply a $\mathcal{D}$ function to $B_k^{(n)}$. By definition we choose a vertex pair (a_i, b_i), and after the dividing operation there are no atoms with diameter determined by (a_i, b_i), since the a_i and b_i vertices are assigned to different ones. Moreover, it is impossible, that after the operation new vertex pairs were formed, whose distance is greater than $\frac{r}{2}$. Therefore the new atoms generated from $B_k^{(n)}$ can have at least one less vertex pairs, which satisfy $\|a_i - b_i\| \geq \frac{r}{2}$. Consequently, after a finite number of iterations all the atoms obtained from $B_k^{(n)}$ will have the diameter strictly less than $\frac{r}{2}$, i.e.

$$\exists N_1 \in \mathbb{N}\ \forall k_1 \in \Delta_{N_1} \ : \ B_k^{(n)} \cap B_{k_1}^{(N_1)} \neq \emptyset \Rightarrow \operatorname{diam}(B_{k_1}^{(N_1)}) < \frac{r}{2}.$$

On the other hand, we have finite number of atoms, $n+1$ in the n-th iteration, so there are only a finite number of atoms $B_0, \dots, B_{n_2}$, which satisfy $\operatorname{diam}(B_i) \geq \frac{r}{2}$. By definition of $\mathcal{C}$, if $n_2 > 0$ and $n > N_1$ we need to choose an atom for which $B_i \cap B_k^{(n)} = \emptyset$, since $\operatorname{diam}(B_i) \geq \frac{r}{2} > \operatorname{diam}(B_{k_1}^{(N_1)})$ for all $k_1 \in \Delta_{N_1}$. Due to the above explanation, after finitely many iterations all the atoms obtained from $B_0, \dots, B_{n_2}$ will have the diameter strictly less than $\frac{r}{2}$, therefore *all* the atoms will have this property, as well. In other words

$$\exists N_2 \in \mathbb{N} \ : \ \max_{k_2\in\Delta_{N_2}} \operatorname{diam}(B_{k_2}^{(N_2)}) < \frac{r}{2}.$$

The above idea can be applied arbitrary many times. Using the fact that $r \leq 1$, we find $\left(\frac{r}{2}\right)^n \to 0$ as $n \to \infty$. □

6. Results, Future Work

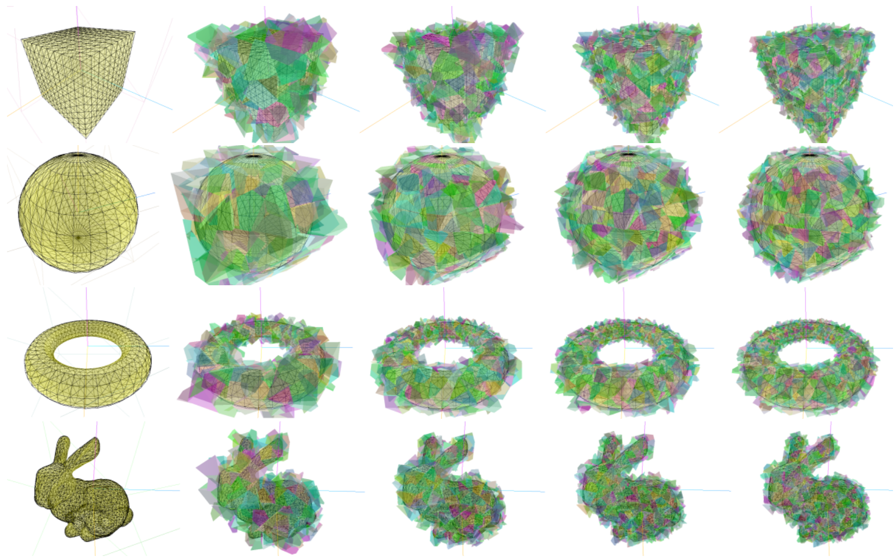

Figure 1. Test models were Cube, Sphere, Torus and Bunny. From left to right we can see the test model, and the resulting approximation, S_n after $n = 1000, 2000, 3000, 4000$ iterations, respectively.

TABLE 1. Results for our 4 test models. After the model name the number of vertices and faces are shown. The table contains the $d^*(S_n)$ maximal diameter and the $\rho^*(S_n, S)$ error estimation for iteration number n.

Cube (602, 1200)			Sphere (482, 960)		
n	$d^*(S_n)$	$\rho^*(S_n, S)$	n	$d^*(S_n)$	$\rho^*(S_n, S)$
1000	0,39	1,98	1000	0,43	3,16
2000	0,27	1,35	2000	0,31	2,31
3000	0,22	1,07	3000	0,26	1,90
4000	0,19	0,90	4000	0,22	1,63
Torus (576, 1152)			Bunny (2503, 4968)		
n	$d^*(S_n)$	$\rho^*(S_n, S)$	n	$d^*(S_n)$	$\rho^*(S_n, S)$
1000	0,31	1,14	1000	0,30	0,97
2000	0,22	0,84	2000	0,27	0,71
3000	0,18	0,69	3000	0,17	0,58
4000	0,16	0,60	4000	0,15	0,50

The algorithm obtained from our approximation schema was implemented by choosing and dividing functions defined in the last section. Some tests were performed on triangular meshes as we can see on Figure 1. We show 4 models: cube, sphere, torus, and the Stanford Bunny (see [24]) from top to bottom, respectively. From left to right we can see 5 level of approximation, the test model and the resulting S_n after $n = 0, 1000, 2000, 3000, 4000$ iterations, respectively. The most important properties of the approximations were indicated in Table 6. These tables contain the maximal diameters

$$d^*(S_n) := \max_{k \in \Delta_n} \operatorname{diam}(B_k^{(n)}) \tag{6.1}$$

and the estimation of the error, defined as

$$\rho^*(S_n, S) := \sum_{i \in \Delta_n} \mu(B_i^{(n)}) \tag{6.2}$$

where $\rho^*(S_n, S) \geq \rho(S_n, S)$ according to Corollary 4.6.

Our future plans are to work out some new choosing and dividing functions, to show that our method contains in particular most of the space partitioning methods as Octree, spatial decomposition, approximate or exact convex decomposition, etc. We are working on some strategies for choosing and dividing, that can have a dramatic effect on rate of convergence. We are designing an effective data structure for the decomposition, in addition we try to increase the efficiency of the implementation. The source code of the preparing software package will be publicly available in the future.

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Discrete operators associated with the Durrmeyer operator

Marius Birou

Abstract. In [3] the author constructed discrete operators associated with certain integral operators using a probabilistic approach. In this article we obtain positive linear operators of discrete type associated with the classical Durrmeyer operator with the aid of some quadrature formulas with positive coefficients. Using Gaussian quadratures we get operators which preserve the moments of the classical Durrmeyer operator up to a given order. Another class of discrete operators is obtained by using the quadratures generated by some positive linear operators. We study the convergence of the new operators and compare them with the Durrmeyer operator. Also, we present some problems of optimality and give numerical examples.

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Keywords: Positive linear operators, quadrature.

1. Introduction

If f is an integrable function on $[0,1]$, then the classical Durrmeyer operator is defined by

$$M_n f(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,\ x\in[0,1], \tag{1.1}$$

$$p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}.$$

This operator was introduced by Durrmeyer in [1].

We observe that the Durrmeyer operator is an integral operator. It uses some integrals as information about the approximated function. In practice it is hard to obtain these integrals. It is known that the Riemann integrals can be approximated by quadrature formulas.

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Using the quadratures

$$(n+1)\int_0^1 p_{n,k}(t)f(t)dt \approx \sum_{j=0}^{m} A_{j,m}^{n,k} f(t_{j,m}^{n,k}),\ \ k=0,1,...,n \tag{1.2}$$

with positive coefficients, i.e. $A_{j,m}^{n,k} \geq 0$, $j=0,...,m$, $k=0,...,n$, we get the associated discrete operator

$$D_{n,m}f(x) = \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{m} A_{j,m}^{n,k} f(t_{j,m}^{n,k}),\ \ x \in [0,1]. \tag{1.3}$$

The operator $D_{n,m}$ uses the values of the function f at the nodes $t_{j,m}^{n,k}$ as information about the approximated function.

Discrete operators associated with some positive operators using a probabilistic approach were obtained in [3].

The remainder terms of the quadratures (1.2) are given by

$$R_{n,m}^{k}(f) = (n+1)\int_0^1 p_{n,k}(t)f(t)dt - \sum_{j=0}^{m} A_{j,m}^{n,k} f(t_{j,m}^{n,k}),\ \ k=0,1,...,n.$$

If the quadrature formulas have the degree of exactness r, i.e.

$$\begin{aligned} R_{n,m}^{k}(e_i) &= 0,\ \ i=0,...,r \\ R_{n,m}^{k}(e_{r+1}) &\neq 0 \end{aligned}$$

where $e_i(x) = x^i$, then the associated discrete operator has the same images of the monomials e_i, $i=0,...,r$ as the original operator, i.e.

$$M_n e_i = D_{n,m} e_i,\ \ i=0,...,r.$$

The next result follows by using the well known Korovkin theorem and taking into account the uniform convergence of the Durrmeyer operators on the test functions e_i, $i=0,1,2$.

Theorem 1.1. *If the quadratures (*1.2*) have the degree of exactness at least two, then the sequence* $(D_{n,m}f)_{n\geq 1}$ *converges uniformly to the function* f*, for every* $f \in C[0,1]$.

Using the Shisha-Mond result from [2] we have the following estimations of the errors for the Durrmeyer operator and the associated discrete operator respectively:

$$|M_n f(x) - f(x)| \leq \left(1 + \frac{2(n-3)x(1-x)+2}{(n+2)(n+3)\delta^2}\right)\omega(f,\delta),\ \ x \in [0,1], \tag{1.4}$$

$$\begin{aligned} |D_{n,m}f(x) - f(x)| \leq |f(x)|\,|D_{n,m}e_0(x) - e_0(x)| + \\ \left(D_{n,m}e_0(x) + \frac{1}{\delta^2}D_{n,m}(e_2 - 2xe_1 + x^2 e_0)(x)\right)\omega(f,\delta),\ \ x \in [0,1], \end{aligned} \tag{1.5}$$

where $\delta > 0$ and $\omega(f,\cdot)$ is the modulus of continuity i.e.,

$$\omega(f,\delta) = \sup\{|f(x+h) - f(x)| : x, x+h \in [0,1], 0 \leq h \leq \delta\}.$$

We observe that if the quadrature formulas have the degree of exactness at least two then the associated discrete operator has the same approximation order as the Durrmeyer operator.

Also, we get the estimation

$$|M_n f(x) - D_{n,m} f(x)| \leq \max_{k\in\{0,\dots,n\}} |R^k_{n,m}(f)|, \; x \in [0,1]. \tag{1.6}$$

In this article we obtain discrete operators associated with the Durrmeyer operator generated by quadratures of Gauss type and by quadratures obtained using positive linear operators.

2. Discrete operators generated by Gaussian quadratures

We have to approximate the integrals

$$\int_0^1 t^k(1-t)^{n-k} f(t)dt, \; k = 0,1,\dots,n.$$

Using the substitution $t = (1+u)/2$ we get

$$\frac{1}{2^{n+1}} \int_{-1}^1 (1+u)^k(1-u)^{n-k} f\left(\frac{1+u}{2}\right) du, \; k = 0,1,\dots,n. \tag{2.1}$$

To approximate the integrals from (2.1) we can use the Gauss Jacobi quadratures

$$\int_{-1}^1 (1+u)^\alpha(1-u)^\beta g(u)\, du \approx \sum_{j=0}^m B_{j,m} g(u_{j,m}). \tag{2.2}$$

We consider two cases.

2.1. The first case

We take

$$\alpha = k, \; \beta = n-k, \; g(u) = f\left(\frac{1+u}{2}\right).$$

The quadrature formulas (2.2) become

$$\int_{-1}^1 (1+u)^k(1-u)^{n-k} g(u)\, du \approx \sum_{j=0}^m B^{n,k}_{j,m} g(u^{n,k}_{j,m}), \; k = 0,1,\dots,n. \tag{2.3}$$

The nodes $u^{n,k}_{j,m}$, $j = 0,\dots,m$, $k = 0,\dots,n$ are the roots of the Jacobi orthogonal polynomial of degree $m+1$:

$$J^{(k,n-k)}_{m+1}(u) = \frac{1}{2^{m+1}(m+1)!} \frac{1}{\rho(u)} \frac{d^{m+1}}{du^{m+1}} \left[\rho(u)(u^2-1)^{m+1}\right], \; u \in [-1,1]$$

where

$$\rho(u) = (1+u)^k(1-u)^{n-k}.$$

Using [5, Th. 11.5.3], we get the coefficients

$$B_{j,m}^{n,k} = \frac{2^n(2m+n+2)(m+k)!(m+n-k)!}{(m+1)!(m+n+1)!J_m^{(k,n-k)}(u_{j,m}^{n,k})\frac{d}{du}\left[J_{m+1}^{(k,n-k)}(u)\right]_{u=u_{j,m}^{n,k}}}$$

for $j=0,...,m$ and $k=0,...,n$.

The associated discrete operator is

$$D_{n,m}^{GJ}f(x) = \frac{n+1}{2^{n+1}}\sum_{k=0}^{n} p_{n,k}(x)\binom{n}{k}\sum_{j=0}^{m} B_{j,m}^{n,k} f\left(\frac{1+u_{j,m}^{n,k}}{2}\right), \ x\in[0,1].$$

We have

$$D_{n,m}^{GJ}e_i = M_n e_i, \ i=0,...,2m+1.$$

For $m=0$ we get a Stancu operator [4]

$$D_{n,0}^{GJ}f(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+1}{n+2}\right), \ x\in[0,1].$$

This operator was associated with the Durrmeyer operator in [3].

Theorem 2.1. *For every $m\in\mathbb{N}$ and for every $f\in C[0,1]$ we have that the sequence $(D_{n,m}^{GJ}f)_{n\geq 1}$ converges uniformly to the function f.*

Proof. If $m=0$ then the convergence follows from [4].

For $m\geq 1$ the quadrature formulas (2.3) have the degree of exactness at least three. Using the Theorem 1.1 we get the conclusion. □

2.2. The second case

We take

$$\alpha=\beta=0, \ g_{k,n}(u) = (1+u)^k(1-u)^{n-k} f\left(\frac{1+u}{2}\right)$$

and we use the Gauss Legendre quadratures

$$\int_{-1}^{1} g_{k,n}(u)du \approx \sum_{j=0}^{m} B_{j,m} g_{k,n}(u_{j,m}), \ k=0,1,...,n. \tag{2.4}$$

The discrete operator is

$$D_{n,m}^{GL}f(x) =$$

$$\frac{n+1}{2^{n+1}}\sum_{k=0}^{n} p_{n,k}(x)\binom{n}{k}\sum_{j=0}^{m} B_{j,m}(1+u_{j,m})^k(1-u_{j,m})^{n-k} f\left(\frac{1+u_{j,m}}{2}\right), \ x\in[0,1],$$

where the nodes $u_{j,m}$, $j=0,...,m$ are roots of the Legendre orthogonal polynomial $J_{m+1}^{(0,0)}(u)$ and the coefficients are given by (see [5, Th. 11.6.2])

$$B_{j,m} = \frac{2}{(m+1)J_m^{(0,0)}(u_{j,m})\frac{d}{du}\left[J_{m+1}^{(0,0)}(u)\right]_{u=u_{j,m}}}$$

for $j=0,...,m$ and $k=0,...,n$.

Theorem 2.2. *If* $m \in \mathbb{N}$ *and*

$$m \geq \frac{n+1}{2}, \tag{2.5}$$

then the sequence $(D_{n,m}^{GL} f)_{n \geq 1}$ *converges to the function* f*, for every* $f \in C[0,1]$.

Proof. The quadrature formulas (2.4) have the degree of exactness $2m+1$. From the inequality (2.5) we get that the degree of exactness of the quadrature formulas are at least $n+2$. It follows that exists $r \in \mathbb{N}$, $r \geq 2$ such that

$$D_{n,m}^{GL} e_i = M_n e_i,\ i = 0, ..., r,$$

The convergence of the operators is assured by the Korovkin theorem taking into account the convergence of the Durrmeyer operators. □

For $m = n$ we get

$$D_{n,n}^{GL} e_i = M_n e_i,\ i = 0, ..., n+1.$$

We observe that the operator $D_{n,0}^{GJ}$ preserves the moments of the Durrmeyer operator up to order one while the operator $D_{n,n}^{GL}$ keeps the moments up to order $n+1$. Both operators use the same amount of information about the approximated function ($n+1$ evaluations of the function).

We approximate the function $f : [0,1] \to \mathbb{R}$, $f(x) = Exp(x^2)$ using the associated discrete operators $D_{n,0}^{GJ}$ and $D_{n,n}^{GL}$ for $n = 5$.

Operator	$\max_{x\in[0,1]} \lvert D_{n,m}f(x) - M_n f(x)\rvert$	$\max_{x\in[0,1]} \lvert D_{n,m}f(x) - f(x)\rvert$
$D_{n,0}^{GJ}$	$7.5 \cdot 10^{-2}$	$6.5 \cdot 10^{-1}$
$D_{n,n}^{GL}$	$4 \cdot 10^{-5}$	$5.8 \cdot 10^{-1}$

3. Discrete operators generated by quadratures obtained using positive linear operators

We consider the linear positive operators $L_n : C[0,1] \to C[0,1]$, $n \geq 1$ of the form

$$L_n f(t) = \sum_{j=0}^{n} w_{n,j}(t) f(t_{j,n}),\ f \in C[0,1],\ t \in [0,1],$$

where $w_{n,j} \in C[0,1]$, $w_{n,j} \geq 0$, and the corresponding approximation formula

$$f(t) = L_n f(t) + R_n f(t).$$

For $k = 0, ..., n$ we get the quadrature formulas

$$(n+1) \int_0^1 p_{n,k}(t) f(t) dt = \sum_{j=0}^{n} A_j^{n,k} f(t_{j,n}) + (n+1) \int_0^1 p_{n,k}(t) R_n f(t) dt,$$

where

$$A_j^{n,k} = (n+1) \int_0^1 p_{n,k}(t) w_{n,j}(t) dt,\ j,k = 0, ..., n.$$

We get the following associated discrete operator

$$D_n^{PL} f(x) = \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{n} A_j^{n,k} f(t_{j,n}),\ x \in [0,1].$$

Theorem 3.1. *If the sequence $(L_n f)_{n\geq 1}$ converges uniformly to the function $f \in C[0,1]$, then the sequence $\left(D_n^{PL} f\right)_{n\geq 1}$ also converges uniformly to the function f and*

$$|D_n^{PL} f(x) - M_n f(x)| \leq \sup_{x\in[0,1]} |R_n f(x)|,\ x \in [0,1].$$

Proof. For every $x \in [0,1]$ we have

$$\left|D_n^{PL} f(x) - M_n f(x)\right| \leq (n+1) \int_0^1 p_{n,k}(t)\, |R_n f(t)|\, dt \leq \sup_{x\in[0,1]} |R_n f(x)|\,.$$

The convergence of the associated operators follows from the inequality

$$\left|D_n^{PL} f(x) - f(x)\right| \leq \left|D_n^{PL} f(x) - M_n f(x)\right| + |M_n f(x) - f(x)|\,. \qquad \square$$

We present two examples.

Example 3.2. For $0 \leq \alpha \leq \beta$ the Bernstein-Stancu operator is given by (see [4])

$$P_n^{\alpha,\beta} f(t) = \sum_{j=0}^{n} p_{n,j}(t) f\left(\frac{j+\alpha}{n+\beta}\right),\ t \in [0,1].$$

We get the associated discrete operator

$$D_n^{BS} f(x) = \frac{1}{2n+1} \sum_{k=0}^{n} \binom{n}{k} p_{n,k}(x) \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{2n}{k+j}} f\left(\frac{j+\alpha}{n+\beta}\right),\ x \in [0,1].$$

For the case of the Bernstein operator ($\alpha = \beta = 0$) we have that the associated operator preserves the moments of the Durrmeyer operator up to the order 1, i.e.

$$D_n^{B} e_i = M_n e_i,\ i = 0,1.$$

Example 3.3. Let the sequence of divisions of the interval $[0,1]$ with the norm tending to 0

$$\Delta_n :\ 0 = t_{0,n} < t_{1,n} < ... < t_{n,n} = 1.$$

The spline linear operator is defined by

$$S_{n,1} f(t) = \sum_{j=0}^{n} s_j(t) f(t_{j,n}),\ t \in [0,1],$$

with the cardinal functions

$$s_0(t) = \begin{cases} \frac{t_{1,n}-t}{t_{1,n}-t_{0,n}},\ t \in [t_{0,n}, t_{1,n}] \\ 0,\ t \notin [t_{0,n}, t_{1,n}] \end{cases}$$

$$s_j(t) = \begin{cases} \frac{t-t_{j-1,n}}{t_{j,n}-t_{j-1,n}}, \ t \in [t_{j-1,n}, t_{j,n}) \\ \frac{t_{j+1,n}-t}{t_{j+1,n}-t_{j,n}}, \ t \in [t_{j,n}, t_{j+1,n}] \\ 0, \ t \notin [t_{j-1,n}, t_{j+1,n}] \end{cases}, \ j = 1, ..., n-1$$

$$s_n(t) = \begin{cases} \frac{t-t_{n-1,n}}{t_{n,n}-t_{n-1,n}}, \ t \in [t_{n-1,n}, t_{n,n}] \\ 0, \ t \notin [t_{n-1,n}, t_{n,n}] \end{cases}$$

The associated discrete operator is

$$D_n^{SL} f(x) = \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{n} A_j^{n,k} f(t_{j,n}), \ x \in [0,1], \tag{3.1}$$

where

$$A_j^{n,k} = \int_0^1 p_{n,k}(t) s_j(t) dt, \ j,k = 0, ..., n.$$

We have

$$D_n^{SL} e_i = M_n e_i, \ i = 0, 1.$$

Next we will show an optimal property of the operator (3.1). It is known the connection between the spline functions and the optimal quadrature in sense of Sard.

We consider that the quadratures

$$\int_0^1 p_{n,k}(t) f(t) dt = \sum_{j=0}^{n} A_j^{n,k} f(t_{j,n}) + R_{n,k}(f), \ k = 0, ..., n \tag{3.2}$$

have the degree of exactness at least 0.

Let the space of functions

$$H^{1,2}[0,1] = \{g | g \in C[0,1], \ g \ \textit{absolutely continuous}, \ g' \in L^2[0,1]\}.$$

If $f \in H^{1,2}[0,1]$ then the remainders of the quadrature formulas (3.2) can be written in the form

$$R_{n,k}(f) = \int_0^1 K_{n,k}(t) f'(t) dt, \ k = 0, ..., n$$

with

$$K_{n,k}(t) = R_{n,k}[(x-t)_+^0] = (x-t)_+^0 - \sum_{j=0}^{n} A_j^{n,k} (t_{j,n} - t)_+^0,$$

where

$$z_+ = \begin{cases} z, \ z \geq 0 \\ 0, \ z < 0 \end{cases}$$

We have the estimation

$$|R_{n,k}(f)| \leq \left(\int_0^1 K_{n,k}^2(t) dt \right)^{1/2} \left(\int_0^1 f'^2(t) dt \right)^{1/2}.$$

The quadratures (3.2) with the coefficients $A_j^{n,k}$ chosen such that we get

$$\inf_{A_j^{n,k}} \int_0^1 K_{n,k}^2(t) dt$$

are named optimal in sense of Sard (see [5, p. 261]). We obtain these quadratures by integration of the spline linear interpolation formula, i.e.

$$(n+1)\int_0^1 p_{n,k}(t)f(t)dt$$
$$= (n+1)\int_0^1 p_{n,k}(t)S_{n,1}f(t)dt + (n+1)\int_0^1 p_{n,k}(t)R_nf(t)dt.$$

We approximate the function $f:[0,1]\to\mathbb{R}$, $f(x)=Exp(x^2)$ using the associated discrete operators D_n^B and D_n^{SL} with equidistant nodes for $n=5$. Both operators use $n+1$ evaluations of the approximated function.

Operator	$\max_{x\in[0,1]} \lvert D_nf(x)-M_nf(x)\rvert$	$\max_{x\in[0,1]} \lvert D_nf(x)-f(x)\rvert$
D_n^B	$1.1\cdot 10^{-1}$	$5\cdot 10^{-1}$
D_n^{SL}	$3.5\cdot 10^{-2}$	$5.5\cdot 10^{-1}$

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On global smoothness preservation by Bernstein-type operators

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Abstract. We study global smoothness preservation of a function f by sequences of Bernstein-type operators with respect to a certain modulus of continuity of order two.

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Keywords: Global smoothness preservation, second order modulus of continuity, simultaneous approximation, Bernstein-type operators.

1. Introduction

Let $(L_n)_{n\in\mathbb{N}}$ a sequence of linear positive operators of approximation on $C[0,1]$-the space of continuous real-valued function on $[0,1]$. If global smoothness of a continuous function f is expressed by a Lipschitz condition with some modulus of continuity, it is of interest if L_nf verify same condition. Further, if the sequence of operators present simultaneous approximation property, namely for $f \in C^r[0,1]$ the sequence $(L_nf)^{(r)}$ converges to the $f^{(r)}$ uniformly on $[0,1]$, it is also important to study if is preserved global smoothness of the derivatives.

The preservation of global smoothness properties by the Bernstein operators

$$B_n(f,x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) P_{n,j}(x),\ f \in C[0,1],\ x \in [0,1],$$

$$P_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j},$$

were studied in [6], [7], [4], [2], [5], [3]. In [10], D.-X. Zhou showed that the Lipschitz classes with respect to the second order modulus

$$\omega_2(f,t) = \sup\left\{|f(x-h) - 2f(x) + f(x+h)| : \ x \pm h \in [0,1],\ 0 < h \le t\right\}$$

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are not preserved by the Bernstein operators. He introduced the following modulus of smoothness of order two

$$\begin{aligned}\widetilde{\omega}_2(f,t) &= \sup\{|f(x+h_1+h_2)-f(x+h_1)-f(x+h_2)+f(x)| : \\ &\qquad x, x+h_1+h_2 \in [0,1], \, h_1,h_2>0, \, h_1+h_2 \le 2t\} \qquad (1.1)\end{aligned}$$

and proved:
Theorem A. *Let* $f \in C[0,1]$, $n \in \mathbb{N}$, $M > 0$ *and* $0 < \alpha \le 1$.
If

$$\widetilde{\omega}_2(f,t) \le Mt^{\alpha}, \, 0 < t \le \frac{1}{2},$$

then

$$\widetilde{\omega}_2(B_n f,t) \le Mt^{\alpha}, \, 0 < t \le \frac{1}{2}.$$

We consider the Bernstein-type operators

$$L_n(f,x) = \sum_{j=0}^{n} P_{n,j}(x) F_{n,j}(f), \, f \in C[0,1], \, x \in [0,1], \qquad (1.2)$$

where $P_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}$ and $F_{n,j} : C[0,1] \longrightarrow \mathbb{R}$, $j = \overline{0,n}$, are linear positive functionals.

In the next section we study simultaneous global smoothness preservation in terms of modulus of continuity ω_2^* introduced by Păltănea [8], [1], [9], defined for $f \in C[0,1]$ and $t > 0$ by

$$\begin{aligned}\omega_2^*(f,t) &= \sup\{|\Delta(f;u,y,v)| : \, u,v \in [0,1], \, u \ne v, \\ &\qquad u \le y \le v, \, v-u \le 2t\}, \qquad (1.3)\end{aligned}$$

where

$$\Delta(f;u,y,v) = \frac{v-y}{v-u} f(u) + \frac{y-u}{v-u} f(v) - f(y).$$

2. Main result

Firstly, we present two auxiliary results.

Lemma 2.1. *The derivative of* r*-th order of the Bernstein-type polynomial* $L_n f$, $r \in \mathbb{N}$, $n \ge r$, *has the expression*

$$(L_n f)^{(r)}(x) = \sum_{j=0}^{n-r} P_{n-r,j}(x) G_{n,j,r}(f), \qquad (2.1)$$

with

$$G_{n,j,r}(f) = (n)_r \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} F_{n,j+i}(f), \qquad (2.2)$$

where $(n)_r = n(n-1)\cdots(n-r+1)$ *is the Pochhammer symbol.*

Proof. We prove the formula by induction with regard to r. For $r=1$ we have

$$\begin{aligned}(L_nf)'(x) &= -nF_{n,0}(f)(1-x)^{n-1}\\ &\quad+\sum_{j=1}^{n-1}\binom{n}{j}[x^j(1-x)^{n-j}]'F_{n,j}(f)+nF_{n,n}(f)x^{n-1}\\ &= n\sum_{j=1}^{n}\binom{n-1}{j-1}x^{j-1}(1-x)^{n-j}F_{n,j}(f)\\ &\quad-n\sum_{j=0}^{n-1}\binom{n-1}{j}x^j(1-x)^{n-1-j}F_{n,j}(f)\\ &= n\sum_{j=0}^{n-1}P_{n-1,j}(x)\left[F_{n,j+1}(f)-F_{n,j}(f)\right].\end{aligned}$$

Suppose now that the formula (2.1) is true for $r\in\mathbb{N}$, $r\geq 2$ and prove it for $r+1$, $n\geq r+1$.

$$\begin{aligned}(L_nf)^{(r+1)}(x) &= (n-r)\sum_{j=0}^{n-r-1}\binom{n-r-1}{j}x^j(1-x)^{n-r-1-j}\left[G_{n,j+1,r}(f)-G_{n,j,r}(f)\right]\\ &= \sum_{j=0}^{n-r-1}P_{n-r-1,j}(x)G_{n,j,r+1}(f).\end{aligned}$$ □

In [10] is given the following representation of the Bernstein operators

$$B_n(f,x+ty)=\sum_{k+l=0}^{n}P_{n,k,l}(x,y)\sum_{m=0}^{l}P_{l,m}(t)f\left(\frac{k+m}{n}\right),$$

$0\leq x\leq 1$, $y>0$, $x+y\leq 1$ and $0\leq t\leq 1$, where

$$P_{n,k,l}(x,y)=\frac{n!}{k!l!\,(n-k-l)!}x^ky^l(1-x-y)^{n-k-l}.$$

Similarly is obtained:

Lemma 2.2. *Let $f\in C[0,1]$, $r\in\mathbb{N}\cup\{0\}$, $0\leq u<v\leq 1$, $\lambda\in[0,1]$. Then for $n\geq r+1$ we have*

$$\begin{aligned}&(L_nf)^{(r)}((1-\lambda)u+\lambda v)\\ &=\sum_{k+l=0}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}(\lambda)G_{n,k+m,r}(f),\end{aligned}\tag{2.3}$$

where $G_{n,j,r}(f)$, $0\leq j\leq n-r$, $r\in\mathbb{N}$ is defined in (2.2) and $G_{n,j,0}(f)=F_{n,j}(f)$, $0\leq j\leq n$. We agree that $(L_nf)^{(0)}=L_nf$.

Sketch of proof. We have

$$\begin{aligned}
&(L_nf)^{(r)}\left((1-\lambda)u+\lambda v\right)=(L_nf)^{(r)}\left(u+\lambda(v-u)\right)\\
&=\sum_{j=0}^{n-r}P_{n-r,j}(u+\lambda(v-u))G_{n,j,r}(f)\\
&=\sum_{j=0}^{n-r}\binom{n-r}{j}\sum_{k=0}^{j}\binom{j}{k}u^k\lambda^{j-k}(v-u)^{j-k}\cdot\\
&\cdot\sum_{p=0}^{n-r-j}\binom{n-r-j}{p}(1-v)^{n-r-j-p}(1-\lambda)^p(v-u)^pG_{n,j,r}(f)\\
&=\sum_{j=0}^{n-r}\sum_{k=0}^{j}\sum_{p=0}^{n-r-j}P_{n-r,k,j-k+p}(u,v-u)P_{j-k+p,j-k}(\lambda)G_{n,j,r}(f)
\end{aligned}$$

It makes the change of index $j-k+p=l$ and reverses the order of summations, afterwards it changes the index $j-k=m$ and is obtained (2.3).

Theorem 2.3. *Let $r\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, L_n defined by (1.2), $n\geq r+1$, with $F_{n,j}:C[0,1]\longrightarrow\mathbb{R}$, $j=\overline{0,n}$ linear positive functionals such that*

$$\left|\Delta\left(G_{n,\bullet,r}(f);j_1,j_2,j_3\right)\right|\leq\omega_2^*\left(f^{(r)},\frac{j_3-j_1}{2n}\right)\tag{2.4}$$

hold $(\forall)f\in C^r[0,1]$, $(\forall)j_1,\,j_2,\,j_3\in\mathbb{N}:\ 0\leq j_1\leq j_2\leq j_3\leq n-r$, $j_1\neq j_3$, where $G_{n,j,r}(f)$, $0\leq j\leq n-r$, $r\in\mathbb{N}$ is defined in (2.2) and $G_{n,j,0}(f)=F_{n,j}(f)$, $0\leq j\leq n$. Let $f\in C^r[0,1]$, $M>0$, $\alpha\in(0,1]$.
If

$$\omega_2^*\left(f^{(r)},t\right)\leq Mt^\alpha,\ 0<t\leq\frac{1}{2},$$

then

$$\omega_2^*\left((L_nf)^{(r)},t\right)\leq Mt^\alpha,\ 0<t\leq\frac{1}{2}.$$

Proof. Let $t\in\left(0,\frac{1}{2}\right]$. Let $u,v\in[0,1]$, $u\neq v$, $u\leq y\leq v$, $v-u\leq 2t$. We use the representation (2.3). We have:

$$\begin{aligned}
\left|\Delta\left((L_nf)^{(r)};u,y,v\right)\right|&=\left|\frac{v-y}{v-u}(L_nf)^{(r)}(u)+\frac{y-u}{v-u}(L_nf)^{(r)}(v)-(L_nf)^{(r)}(y)\right|\\
&\leq\sum_{k+l=0}^{n-r}P_{n-r,k,l}(u,v-u)\cdot\\
\cdot\left|\frac{v-y}{v-u}G_{n,k,r}(f)\right.&\left.+\frac{y-u}{v-u}G_{n,k+l,r}(f)-\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)G_{n,k+m,r}(f)\right|\\
&=\sum_{\substack{k+l=0\\l\neq0}}^{n-r}P_{n-r,k,l}(u,v-u)\cdot
\end{aligned}$$

$$\cdot\left|\frac{v-y}{v-u}G_{n,k,r}(f)+\frac{y-u}{v-u}G_{n,k+l,r}(f)-\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)G_{n,k+m,r}(f)\right|$$

$$\leq \sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)\cdot$$

$$\cdot\left|\left(1-\frac{m}{l}\right)G_{n,k,r}(f)+\frac{m}{l}G_{n,k+l,r}(f)-G_{n,k+m,r}(f)\right|$$

$$=\sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)\left|\Delta\left(G_{n,\bullet,r}(f);k,k+m,k+l\right)\right|$$

$$\leq \sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)\omega_2^*\left(f^{(r)},\frac{l}{2n}\right)$$

$$\leq \sum_{\substack{k+l=0\\ l\neq 0}}^{n-r}P_{n-r,k,l}(u,v-u)\sum_{m=0}^{l}P_{l,m}\left(\frac{y-u}{v-u}\right)M\left(\frac{l}{2n}\right)^{\alpha}$$

$$=\frac{M}{2^{\alpha}}\left(\frac{n-r}{n}\right)^{\alpha}\sum_{k+l=0}^{n-r}P_{n-r,k,l}(u,v-u)\left(\frac{l}{n-r}\right)^{\alpha}$$

$$\leq \frac{M}{2^{\alpha}}\left(\sum_{k+l=0}^{n-r}P_{n-r,k,l}(u,v-u)\frac{l}{n-r}\right)^{\alpha}$$

$$=\frac{M}{2^{\alpha}}\left(v-u\right)^{\alpha}\leq Mt^{\alpha}.$$

Hence $\omega_2^*\left((L_nf)^{(r)},t\right)\leq Mt^{\alpha}$. □

3. Applications

3.1. Global smoothness preservation by the Stancu operators

For $0\leq a\leq b$, the Stancu-Bernstein operators are given by

$$S_n^{(a,b)}(f,x)=\sum_{j=0}^{n}f\left(\frac{j+a}{n+b}\right)P_{n,j}(x),\ f\in C[0,1],\ x\in[0,1].$$

So

$$G_{n,j,0}(f)=F_{n,j}(f)=f\left(\frac{j+a}{n+b}\right),\ 1\leq j\leq n$$

and for $r\in\mathbb{N}$, $n\geq r+1$, $1\leq j\leq n-r$

$$\begin{aligned}G_{n,j,r}(f) &= (n)_r\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}f\left(\frac{j+a}{n+b}+\frac{i}{n+b}\right)\\ &= (n)_r\Delta^r_{\frac{1}{n+b}}f\left(\frac{j+a}{n+b}\right),\end{aligned}$$

where $\Delta_h^r f(x)$ is the forward difference of order r with step h of f at x.

Let $r \in \mathbb{N}_0$, $j_1, j_2, j_3 \in \mathbb{N} : 0 \le j_1 \le j_2 \le j_3 \le n-r$, $j_1 \ne j_3$. For $r = 0$, $f \in C[0,1]$ we have:

$$\begin{aligned}|\Delta(F_{n,\cdot}(f); j_1, j_2, j_3)| &= \left| \frac{j_3 - j_2}{j_3 - j_1} F_{n,j_1}(f) + \frac{j_2 - j_1}{j_3 - j_1} F_{n,j_3}(f) - F_{n,j_2}(f) \right| \\ &= \left| \frac{j_3 - j_2}{j_3 - j_1} f\left(\frac{j_1 + a}{n+b}\right) + \frac{j_2 - j_1}{j_3 - j_1} f\left(\frac{j_3 + a}{n+b}\right) - f\left(\frac{j_2 + a}{n+b}\right) \right| \\ &= \left| \Delta\left(f; \frac{j_1 + a}{n+b}, \frac{j_2 + a}{n+b}, \frac{j_3 + a}{n+b}\right) \right| \\ &\le \omega_2^*\left(f, \frac{j_3 - j_1}{2(n+b)}\right) \le \omega_2^*\left(f, \frac{j_3 - j_1}{2n}\right).\end{aligned}$$

For $r \ge 1$, $f \in C^r[0,1]$ we have

$$\begin{aligned}&|\Delta(G_{n,\cdot,r}(f); j_1, j_2, j_3)| \\ &= (n)_r \left| \frac{j_3 - j_2}{j_3 - j_1} \Delta^r_{\frac{1}{n+b}} f\left(\frac{j_1 + a}{n+b}\right) + \frac{j_2 - j_1}{j_3 - j_1} \Delta^r_{\frac{1}{n+b}} f\left(\frac{j_3 + a}{n+b}\right) - \Delta^r_{\frac{1}{n+b}} f\left(\frac{j_2 + a}{n+b}\right) \right| \\ &\le (n)_r \int_0^{\frac{1}{n+b}} \cdots \int_0^{\frac{1}{n+b}} |\Delta(f^{(r)}; \frac{j_1 + a}{n+b} + u_1 + \cdots + u_r, \\ &\quad \frac{j_2 + a}{n+b} + u_1 + \cdots + u_r, \frac{j_3 + a}{n+b} + u_1 + \cdots + u_r)| du_r \cdots du_1 \\ &\le \frac{(n)_r}{(n+b)^r} \omega_2^*\left(f^{(r)}, \frac{j_3 - j_1}{2(n+b)}\right) < \omega_2^*\left(f^{(r)}, \frac{j_3 - j_1}{2n}\right).\end{aligned}$$

We used that $\Delta_h^r f(x) = \int_0^h \cdots \int_0^h f^{(r)}(x + u_1 + \cdots u_r)\, du_r \cdots du_1$.

Thus we obtain:

Theorem 3.1. *Let $f \in C^r[0,1]$, $r \in \mathbb{N}_0$, $n \ge r+1$, $M > 0$, $\alpha \in (0,1]$. If*

$$\omega_2^*\left(f^{(r)}, t\right) \le M t^\alpha,\ 0 < t \le \frac{1}{2},$$

then

$$\omega_2^*\left((S_n^{(a,b)} f)^{(r)}, t\right) \le M t^\alpha,\ 0 < t \le \frac{1}{2}.$$

In particular, when $b = 0$ we have the results for the Bernstein operator.

3.2. Global smoothness preservation by the Kantorovich operators

The Kantorovich operators are defined by

$$M_n(f, x) = \sum_{j=0}^n \left((n+1) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(u) du \right) P_{n,j}(x),\ f \in L_1[0,1],\ x \in [0,1].$$

So

$$F_{n,j}(f) = (n+1) \int\limits_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(u)du = \int\limits_0^1 f\left(\frac{s+j}{n+1}\right) ds$$

and for $r \in \mathbb{N}$, $n \geq r+1$, $1 \leq j \leq n-r$

$$\begin{aligned} G_{n,j,r}(f) &= (n)_r \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \int\limits_0^1 f\left(\frac{s+j}{n+1} + \frac{i}{n+1}\right) ds \\ &= (n)_r \int\limits_0^1 \Delta^r_{\frac{1}{n+1}} f\left(\frac{s+j}{n+1}\right) ds \end{aligned}$$

Let $r \in \mathbb{N}_0$, $j_1, j_2, j_3 \in \mathbb{N} : 0 \leq j_1 \leq j_2 \leq j_3 \leq n-r$, $j_1 \neq j_3$. For $r = 0$, $f \in C[0,1]$ we have:

$$\begin{aligned} &|\Delta\left(F_{n,\cdot}(f); j_1, j_2, j_3\right)| \\ &= \left|\frac{j_3-j_2}{j_3-j_1} F_{n,j_1}(f) + \frac{j_2-j_1}{j_3-j_1} F_{n,j_3}(f) - F_{n,j_2}(f)\right| \\ &= \left|\int\limits_0^1 \left[\frac{j_3-j_2}{j_3-j_1} f\left(\frac{s+j_1}{n+1}\right) + \frac{j_2-j_1}{j_3-j_1} f\left(\frac{s+j_3}{n+1}\right) - f\left(\frac{s+j_2}{n+1}\right)\right] ds\right| \\ &\leq \int\limits_0^1 \left|\Delta\left(f; \frac{s+j_1}{n+1}, \frac{s+j_2}{n+1}, \frac{s+j_3}{n+1}\right)\right| ds \\ &\leq \omega_2^*\left(f, \frac{j_3-j_1}{2(n+1)}\right) < \omega_2^*\left(f, \frac{j_3-j_1}{2n}\right). \end{aligned}$$

For $r \geq 1$, $f \in C^r[0,1]$ we have

$$\begin{aligned} &|\Delta\left(G_{n,\cdot,r}(f); j_1, j_2, j_3\right)| \\ &\leq (n)_r \int\limits_0^1 \Big|\frac{j_3-j_2}{j_3-j_1} \Delta^r_{\frac{1}{n+1}} f\left(\frac{s+j_1}{n+1}\right) + \frac{j_2-j_1}{j_3-j_1} \Delta^r_{\frac{1}{n+1}} f\left(\frac{s+j_3}{n+1}\right) \\ &\quad -\Delta^r_{\frac{1}{n+1}} f\left(\frac{s+j_2}{n+1}\right)\Big| ds \\ &\leq (n)_r \int\limits_0^1 \int\limits_0^{\frac{1}{n+1}} \cdots \int\limits_0^{\frac{1}{n+1}} |\Delta(f^{(r)}; \frac{s+j_1}{n+1} + u_1 + \cdots + u_r, \\ &\quad \frac{s+j_2}{n+1} + u_1 + \cdots + u_r, \frac{s+j_3}{n+1} + u_1 + \cdots + u_r)| du_r \cdots du_1 ds \\ &\leq \frac{(n)_r}{(n+1)^r} \omega_2^*\left(f^{(r)}, \frac{j_3-j_1}{2(n+1)}\right) < \omega_2^*\left(f^{(r)}, \frac{j_3-j_1}{2n}\right). \end{aligned}$$

So we obtain:

Theorem 3.2. *Let* $f \in C^r[0,1]$, $r \in \mathbb{N}_0$, $M > 0$, $\alpha \in (0,1]$.
If

$$\omega_2^*\left(f^{(r)}, t\right) \le Mt^\alpha,\ 0 < t \le \frac{1}{2},$$

then

$$\omega_2^*\left((M_n f)^{(r)}, t\right) \le Mt^\alpha,\ 0 < t \le \frac{1}{2}.$$

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Statistical aspects on the usage of some dermatological creams with metallic nanoparticles

Ioana Chiorean

Abstract. The main purpose of the present paper is to emphasize, by statistical studies, the anti-inflammatory effect of some new prepared nanomaterials on skin diseases (psoriasis). These new materials are based on silver and gold nanoparticles and natural compounds extracted from native plants of *Adoxaceae* family (European Cranberry Bush - *Viburnum opulus L.*, European black Elderberry - *Sambucus nigra L.* and *Cornus mas*), which grow in our country and possess a known anti-inflammatory activity mainly due to their high content of anthocyanins and other polyphenols.

Mathematics Subject Classification (2010): 62P10.

Keywords: Statistics, metallic nanomaterials, medical effects.

1. Introduction

In the 21-st century, the scientists from different domain of activity, all over the world, find more and more attractive the idea of "back to nature". Mainly in Medicine, many researchers try to discover new products, based on natural extract, to cure all kind of diseases, because it is known that the antocyanins obtained from plants are organic colorants which have good results in this. In order to get a better penetration of these products into the human body, the nanotechnology became "a great help", by new metallic (gold/silver) nanomaterials. For the moment, few studies have been made directly on human subjects, due to the fact that the toxicity of these metallic nanoproducts has to be carefully determined. For each case, and for each material, *in vitro* and *in vivo* studies have to be performed. Only then, if the results indicate the non-citotoxicity of the material, they may be used on humans.

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In paper [1], [2], the effect of the natural extract made from the different fruits of *Adoxaceae* family found in Romania has been studied. We are talking about *Călin* (*Viburnum opulus L.*), *Soc* (*Sambucus nigra, L.*) and *Corn* (*Cornus mas*).

The natural extracts of these fruits have been functionalized with metallic ions of gold/silver (more precisely the metallic ions have been reduced in the presence of natural extracts) and some new nanomaterials have been obtained. Their physical characteristics have been studied in [1] and [2].

In vitro and *in vivo* studies have been made (see also [1] and [2]). It has been found that our products are non-toxic. Their benefits on skin diseases have been also emphasized. Some dermatological creams have been produced with our nanomaterials and have been tested on a sample of patients with psoriatic lesions.

The histograms of the subjects' skin thickness presented an involution of diseases after treatment, fact that indicates, from medical point of view, the very good anti-inflammatory effect of all our nanomaterials. But, in order to state a final conclusion, the data has to be statistically interpreted.

2. Statistical studies

The data have been collected from a total sample of 45 subjects, all aged between 35-63 years old, with clinical diagnosis of psoriasis. Patients enrolled in the study signed an informed consent forms. The study has been approved by the Ethical Committee of the University of Medicine and Pharmacy "Iuliu Haţieganu" Cluj-Napoca, Romania. All subjects have been submitted to an ultrasonographic evaluation, images from different areas of the skin being taken and analyzed in the Dermavision software. The ultrasound evaluation allowed the acquirement of cross-sectional images of the skin up to a depth of 2.5 cm as well as the assessment of the echogenicity variation, by comparing the number of pixels (with different echogenicity levels) before and after therapy.

2.1. Study for the cream with gold nanoparticles functionalized with *Călin* (*Viburnum opulus L.*)

The pictures in Figure 1 indicates the skin lesions before and after treatment with this cream.

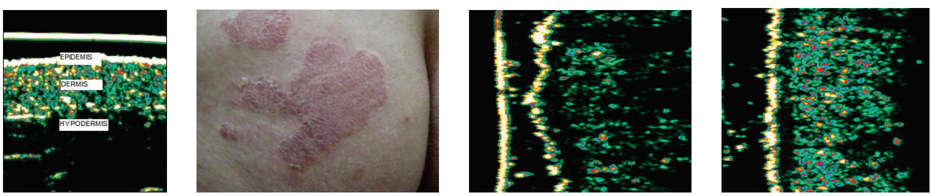

Figure 1. Anti-inflammatory effect on skin lesions; (a): normal skin, (b) psoriasis vulgaris, (c): histogram before treatment, (d) histogram after treatment

The cream has been used by 8 patients have used the cream, for two weeks, twice a day.

According with [1], the mean value of our data is $M = 3437.75$ before treatment, and $M = 983$, after treatment, which indicates that the cream has a very good anti-inflammatory effect. From statistical point of view, the two tailed Student test has been used. The correlation coefficient is $R = 0.874$ and $p = 0.005 < 0.05$, which means that this result is statistically significant.

2.2. Study for the cream with silver nanoparticles functionalized with *Călin* (*Viburnum opulus L.*)

In this case, 7 patients have used the cream, for two weeks, twice a day. The graphic in Figure 2 indicates the variations of psoriatic lesions skin thickness before and after the treatment with this cream.

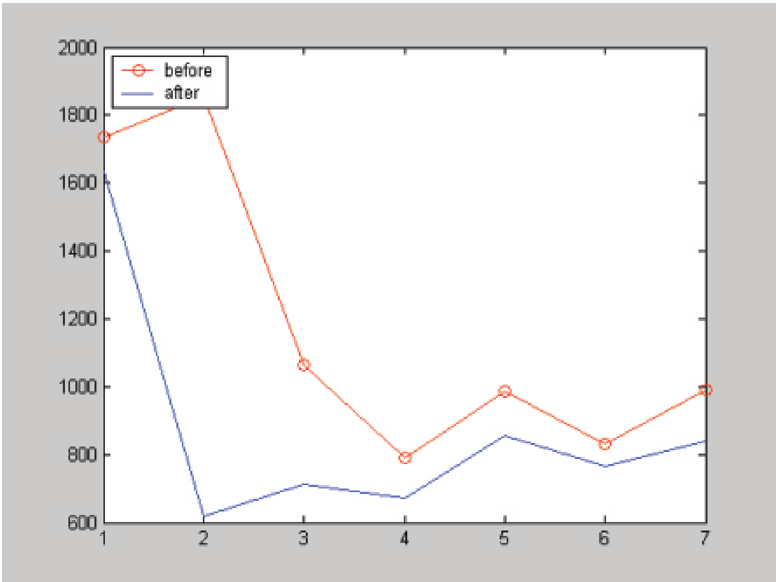

Figure 2. The skin thickness (in μm) before and after treatment with silver nanoparticles functionalized with *Călin* (*Viburnum opulus L.*) cream

In this situation, in spite of the fact that the values indicate a decrease in the thickness of the skin lesions, the data are not correlated, so the sample is too small to be statistical significant.

2.3. Study for the cream with gold nanoparticles functionalized with *Soc* (*Sambucus nigra, L.*)

In this case, also 7 patients have used the cream, for two weeks, twice a day.

Again, as in [1], in this situation, in spite of the fact that the values indicate a decrease in the thickness of the skin lesions, the data are not correlated, so the sample is too small to be statistical significant.

2.4. Study for the cream with silver nanoparticles functionalized with *Soc* (*Sambucus nigra, L.*)

In this case, the 7 patients have used also the cream, for two weeks, twice a day. The graphic in Figure 3 indicates the variations of psoriatic lesions skin thickness before and after the treatment with this cream.

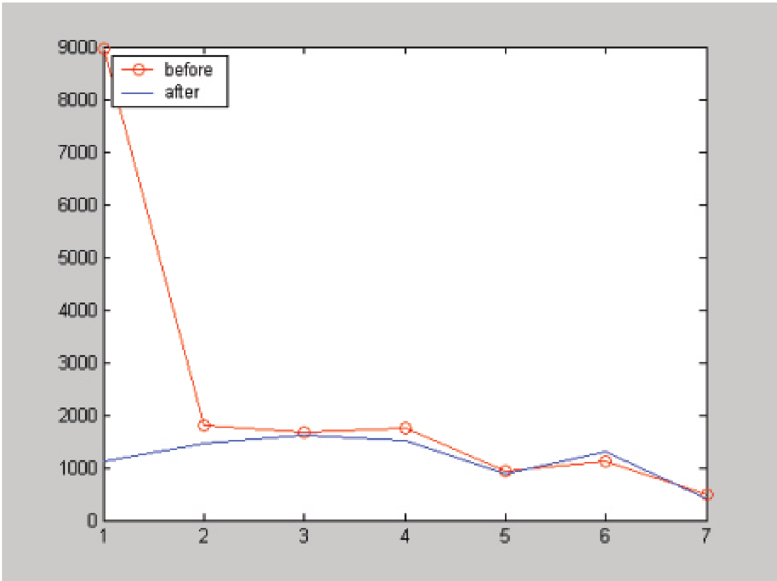

Figure 3. The skin thickness (in μm) before and after treatment with silver nanoparticles functionalized with *Soc* (*Sambucus nigra, L.*) cream

We note that we are in a situation similar with the previous two.

2.5. Study for the cream with gold nanoparticles functionalized with *Corn* (*Cornus mas*)

In this case, 8 patients have used the cream, for two weeks, twice a day. The graphic in Figure 4 indicates the variations of psoriatic lesions skin thickness before and after the treatment with this cream.

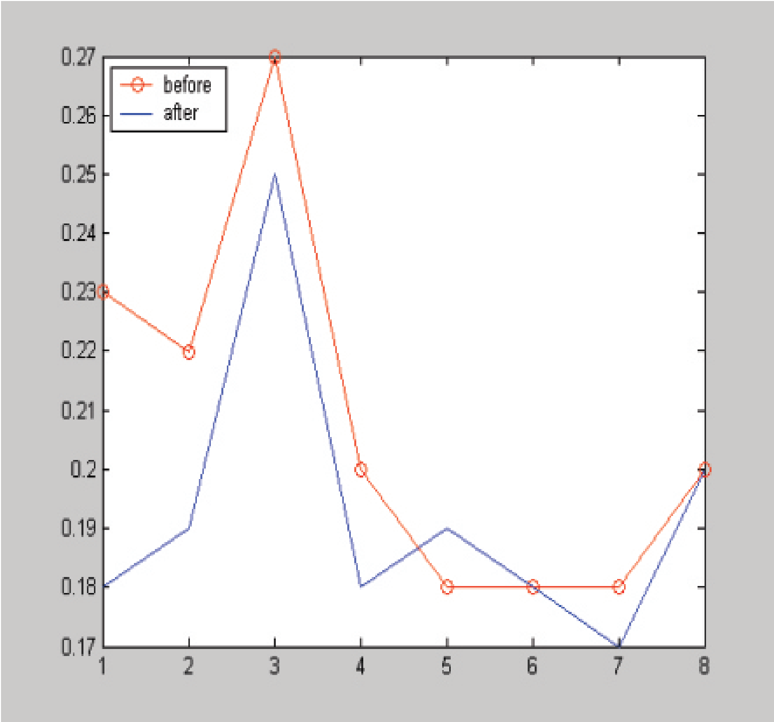

Figure 4. The skin thickness (in μm) before and after treatment with gold nanoparticles functionalized with *Corn* (*Cornus mas*) cream

From statistical point of view, the two tailed Student test has also been used. The mean value is $M = 0.2075$ before treatment, and $M = 0.1925$, after treatment. The diminishing of this value is small, but it is statistically significant, correlation coefficient being $R = 0.79$ and $p = 0.01 < 0.05$.

2.6. Study for the cream with silver nanoparticles functionalized with *Corn* (*Cornus mas*)

In this case, also 8 patients have used the cream, for two weeks, twice o day. The graphic in Figure 5 indicates the variations of psoriatic lesions skin thickness before and after the treatment with this cream.

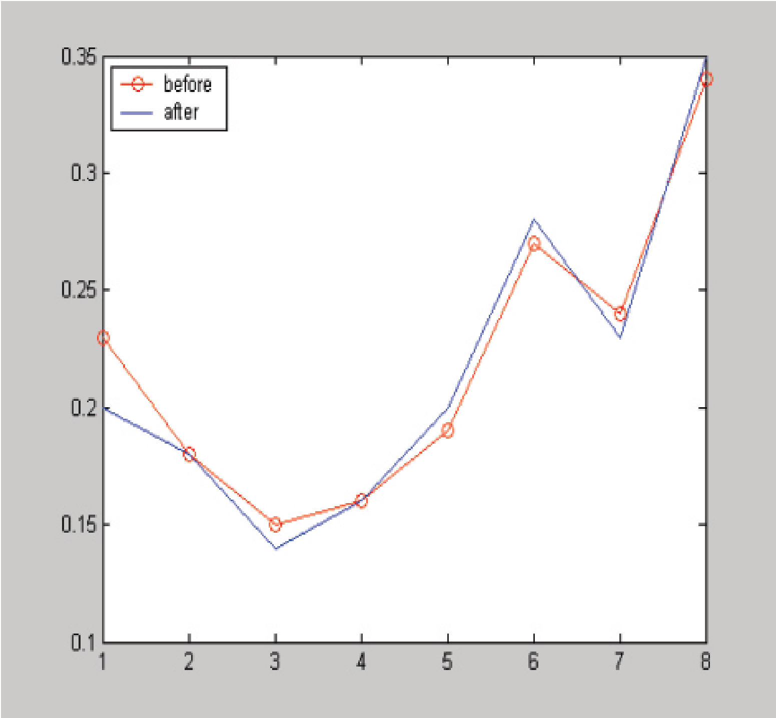

Figure 5. The skin thickness (in μm) before and after treatment with silver nanoparticles functionalized with *Corn* (*Cornus mas*) cream

The statistical study has revealed a very good correlation with $R = 0.98$ and $p < 0.00001$. The mean value of data before treatment has been $M = 0.22$, and $M = 0.2175$ after treatment, which indicates only a very slightly modification in the thickness of the skin. This permits us to state that, in spite of the good statistical results, this cream has not a very good anti-inflammatory effect.

Conclusion. Comparing from mathematically point of view our six creams, we may state that the cream with **gold nanoparticles functionalized with Călin** (*Viburnum opulus L.*) has the best anti-inflammatory effect. The second one is **the cream with gold nanoparticles functionalized with Corn** (*Cornus mas*) and the third is **the cream with silver nanoparticles functionalized with Corn** (*Cornus mas*). All the other may indicate some medical diminishing of the psoriatic skin lesions thickness, but they are not statistically significant.

Acknowledgements. This work was elaborated in the frame project no. 147/2012, through the program "Partnerships in priority areas - PN II", developed with the support of ANCS, CNDI - UEFISCDI.

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Finding initial solutions for a class of nonlinear BVP

Maria Gabriela Trîmbiţaş and Radu T. Trîmbiţaş

Abstract. The purpose of these paper is to solve a nonlinear boundary value problem having the origin in fluid mechanics. The equation has in general several solutions and the main difficulty is to find starting solutions. We follow a mixed symbolic-numeric approach.

Mathematics Subject Classification (2010): 65L10, 76R99.

Keywords: Fluid mechanics, boundary value problem, starting solution.

1. Introduction

1.1. The origin of the problem

Consider the steady stagnation-point flow of a viscous and incompressible fluid over a continuously stretching/shrinking sheet in its own plane.

Assumptions [4]:

- free stream velocity $u_e(x) = ax$, stretching/shrinking sheet velocity $u_w(x) = b(x + c) + u_{slip}(x)$, where $b > 0$ is the stretching rate and $b < 0$ is the shrinking rate,$a > 0$, $-c$ is the location of the stretching origin and is $u_{slip}(x)$ the second-order velocity slip
- x-axis is measured along the stretching surface and the y-axis is perpendicular to it

The basic equations are [1, 3]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{1.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \tag{1.3}$$

This paper was presented at the third edition of the International Conference on Numerical Analysis and Approximation Theory (NAAT 2014), Cluj-Napoca, Romania, September 17-20, 2014.

where u and v and are the velocity components along the x- and y-axes, p is the pressure, ν is the kinematic viscosity and ρ is the density.

For the previous equations, we have the following boundary conditions

$$\begin{cases} v = v_0,\ u = u_w(x) = b(x+c) + u_{slip}(x) & \text{at } y = 0 \\ u = u_e(x) = ax & \text{as } y \to \infty \end{cases} \tag{1.4}$$

where v_0 is the mass flux velocity $v_0 < 0$ with for suction and $v_0 > 0$ for injection.

[5] and [1] give

$$u_{slip}(x) = A\frac{\partial u}{\partial y} + B\frac{\partial^2 u}{\partial y^2},\ A \in \mathbb{R}, B < 0. \tag{1.5}$$

Following [4], we assume that Eqs. (1.1) to (1.3) subject to the boundary conditions (1.4) admit the similarity solution

$$u = axf'(\eta) + bcg(\eta),\ v = \sqrt{a\nu}f(\eta),\ \eta = y\sqrt{\frac{a}{\nu}}. \tag{1.6}$$

where prime ($'$) denotes differentiation with respect to η.

Using Eq. (1.3) and the boundary conditions (1.5), we obtain the following expression for the pressure p

$$p = p_0 - \rho\frac{u_e(x)^2}{2} - \mu\frac{v^2}{2} + \mu\frac{\partial v}{\partial y} \tag{1.7}$$

where p_0 is the stagnation pressure. Thus, we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u_e\frac{\partial u_e}{\partial x} \tag{1.8}$$

1.2. The boundary value problem

Substituting (1.6) and (1.8) into Eq. (1.2), we get the system of ODE

$$f''' + ff'' + f'^2 - 1 = 0 \tag{1.9}$$

$$g'' + fg' - f'g = 0 \tag{1.10}$$

subject to the boundary conditions

$$f(0) = s,\ f'(0) = \alpha + \beta f''(0) + \gamma f'''(0),\ f'(\infty) = 1 \tag{1.11}$$

$$g(0) = 1,\ g'(\infty) = 0. \tag{1.12}$$

Here $s = -v_0/\sqrt{a\nu}$ is the mass flux velocity with $s > 0$ for suction and $s < 0$ for injection, respectively, $\alpha > 0$ is the stretching parameter and $\alpha < 0$ is the shrinking parameter, respectively, β is the first order velocity slip parameter with $0 < \beta = a\sqrt{c/\nu}$ and γ is the second order velocity slip parameter with $\gamma = Bc/\nu < 0$.

2. The symbolic solution

Once f is determined, the second equation (1.10) is linear in g; so, we focus our attention to the solution of (1.9) + (1.11).

The next Maple code shows the result returned by Maple procedure `dsolve` for (1.9).

```
> ecd1:=diff(f(x),x$3)+f(x)*diff(f(x),x$2)+diff(f(x),x)^2-1=0;
```

$$ecd1 := \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) + f(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + \left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right)^2 - 1 = 0$$

```
> sed:=dsolve(ecd1,f(x));
```

$$f(x) = \frac{1}{6}\Big(\big(\_C1^4\_C3 - 2\_C1^3\_C3x + \_C1^2\_C3x^2$$
$$+ 2\_C1^2\_C2\_C3 - 4\_C1\_C2\_C3x + 2\_C2\_C3x^2 - 6\_C1^2\_C3$$
$$+12\_C1\_C3x - 6\_C3x^2\big)\,\mathrm{hypergeom}\left(\left[\frac{7}{4} - \frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right],\right.$$
$$\left.\left[\frac{5}{2}\right], \frac{1}{2}(\_C1 - x)^2\right) \Big/ \left((\_C1\_C3 - \_C3x)\,\mathrm{hypergeom}\left(\left[\frac{3}{4}\right.\right.\right.$$
$$\left.\left. -\frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(\_C1 - x^2)\right) + \mathrm{hypergeom}\left(\left[\frac{1}{4}\right.\right.$$
$$\left.\left. -\frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right], \left[\frac{1}{2}\right], \frac{1}{2}(\_C1 - x^2)\right) + \frac{1}{6}\Big(\big(3\_C1^3$$
$$-3\_C1^2x + 6\_C1\_C2 - 6\_C2x - 6\_C1 + 6x\big)\,\mathrm{hypergeom}\left(\left[\frac{5}{4}\right.\right.$$
$$\left.\left.\left. -\frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(\_C1 - x^2)\right)\right) \Big/ \big((\_C1\_C3$$
$$\_C3x)\,\mathrm{hypergeom}\left(\left[\frac{3}{4} - \frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(\_C1 - x^2)\right)$$
$$+ \mathrm{hypergeom}\left(\left[\frac{1}{4} - \frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right], \left[\frac{1}{2}\right], \frac{1}{2}(\_C1 - x^2)\right)$$
$$+ \frac{1}{6}\Big(\big(6\_C1^2\_C3 - 12\_C1\_C3x + 6\_C3x^2 - 12\_C3\big)$$
$$\mathrm{hypergeom}\left(\left[\frac{3}{4} - \frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(\_C1 - x^2)\right)$$
$$+ (6\_C1 - 6x)\,\mathrm{hypergeom}\left(\left[\frac{1}{4} - \frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right], \left[\frac{1}{2}\right],\right.$$
$$\left.\frac{1}{2}(\_C1 - x^2)\right) \Big/ \big((\_C1\_C3 - \_C3x)$$
$$\mathrm{hypergeom}\left(\left[\frac{3}{4} - \frac{1}{4}\_C2 - \frac{1}{8}\_C1^2\right], \left[\frac{3}{2}\right], \frac{1}{2}(\_C1 - x^2)\right) +$$

$$\text{hypergeom}\left(\left[\frac{3}{4}-\frac{1}{4}\_C2-\frac{1}{8}\_C1^2\right],\left[\frac{3}{2}\right],\frac{1}{2}\left(\_C1-x^2\right)\right)\right)$$

Here `hypergeom(a,b,z)` means the confluent hypergeometric function ${}_1\mathrm{F}_1(a,b,z)$ (see [2])

$${}_1\mathrm{F}_1(a,b,z)=\sum_{s=0}^{\infty}\frac{(a)_s}{(b)_2 s!}z^s=1+\frac{a}{b}z+\frac{a(a+1)}{2b(b+1)}z^2+\cdots$$

The solution depends on three free constants $\_C1, \_C2, \_C3$. (1.11) leads us to a nonlinear system in unknowns $\_C1, \_C2, \_C3$:

$$\begin{aligned} F(0)&=s\\ F'(0)&=\alpha+\beta F''(0)+\gamma F'''(0),\\ F'(\infty)&=1 \end{aligned} \tag{2.1}$$

3. The numerical solution

Our approach consists of:

- Replace ${}_1\mathrm{F}_1(a,b,z)$ in the expression of F by an approximation and solve the system
- Use the solution as a starting function for a numerical method for the BVP (1.9)+(1.10)+(1.11)+1.12).

When z is close to 0 we use a Taylor expansion with a small number of terms

$$\begin{aligned} {}_1\mathrm{F}_1(a,b,z)&\approx\sum_{s=0}^{n}\frac{(a)_s}{(b)_2 s!}z^s\\ &=1+\frac{a}{b}z+\frac{a(a+1)}{2b(b+1)}z^2+\cdots+\frac{a(a+1)\cdots(a+n-1)}{n!b(b+1)\dots(b+n-1)}z^n \end{aligned}$$

If z is large,

$${}_1\mathrm{F}_1(a,b,z)\approx\frac{e^z z^{a-b}\Gamma(b)}{\Gamma(a)}.$$

(see [2]).

With hypergeometric function replaced by previous approximation, Maple procedure `solve` returns the approximate solution of (2.1) in terms of s, α, β, γ. This approach allows us to obtain informations about the number of solutions and their behavior.

Finally, we solve BVP (1.9)+(1.10)+(1.11)+1.12) using the MATLAB function `bvp4c` and the approximation of F as the starting function (initial solution).

3.1. Numerical examples

We consider two numerical examples.

- For $s = -1$, $\alpha = 2$, $\beta = 0.5$, $\gamma = -1$, Maple returns two solutions

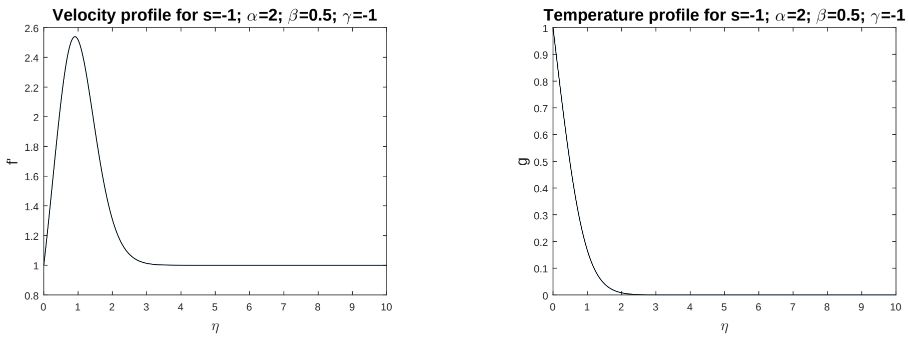
$$.986091543197571, -3.48618826578288, \quad -.848486949052433,$$
$$2.29686508693243, -5.63779461378456, \quad -1.16744311525857$$

Using these coefficients for the MATLAB starting solution, we obtain the graph in Figure 1.

- For $s = -1$, $\alpha = 2$, $\beta = 0.5$, $\gamma = -1$, Maple returns four solutions

$$2.17791394594416, -5.37165457796903, \quad -1.18215998400672,$$
$$-.497833460462187, -3.12391907717788, \quad .959680066577520,$$
$$-.654766323087723, -3.21435946892491, \quad 1.04272129118383,$$
$$-1.13575318603863, -3.64496764979845, \quad 1.16765563652685$$

Using these coefficients for the MATLAB starting solution, we obtain the graph in Figure 2.



(a) Velocity (b) Temperature

FIGURE 1. Velocity (left) and temperature profile for the first example

We give in the sequel the Maple code for the computation of initial solutions

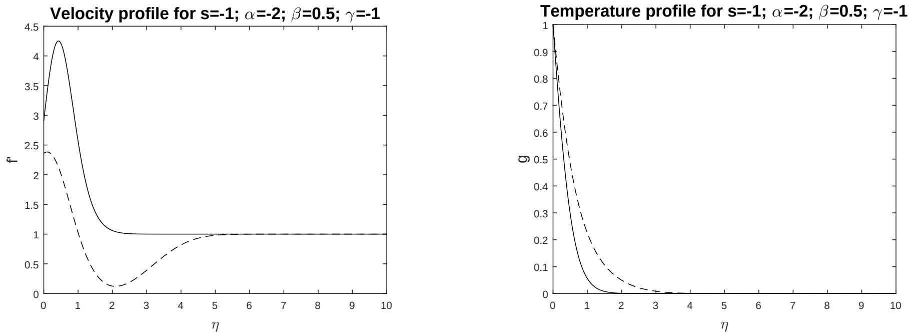
```
> restart;
> Digits:=15:
```

A Taylor expansion of the hypergeometric function

```
> hyge:=proc(a,b,z,n::integer:=2)
> add(z^k*factor(expand(pochhammer(op(a),k)))/
> (k!*factor(expand(pochhammer(op(b),k)))),k=0..n);
> end:
```

Equations and boundary conditions

```
> ecd1:=diff(f(x),x$3)+f(x)*diff(f(x),x$2)+diff(f(x),x)^2-1=0:
```



(a) Velocity (b) Temperature

FIGURE 2. Velocity (left) and temperature profile for the second example

```
>  ic:=f(0)=s,
>  D(f)(0)=alpha+beta*D[1,1](f)(0)+Gamma*D[1,1,1](f)(0),
>  D(f)(10)=1:

>  sed:=dsolve(ecd1,f(x)):
```

This is the ”analytic” solution!

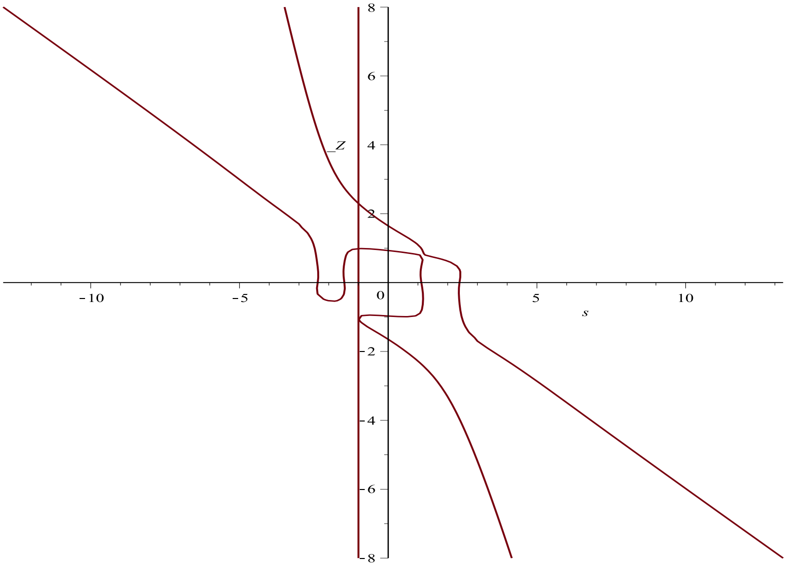
```
>  F:=unapply(rhs(%),x):
```

Fh is an approximation of *F*.

```
>  Fh:=unapply(subs(hypergeom=hyge,F(x)),x):

>  Fh:=unapply(simplify(Fh(x)),x):
```

The nonlinear system to be solved

```
>  ec1:=Fh(0)=s:

>  ec2:=eval(diff(Fh(x),x),x=0)=alpha+beta*eval(diff(Fh(x),x,x),x=0)
>  +Gamma*eval(diff(Fh(x),x$3),x=0):

>  ec3:=limit(diff(Fh(x),x),x=infinity)=1:

>  with(plots):

>  ss:=solve([ec1,ec2,ec3],[_C1,_C2,_C3]):

>  vss:=eval(ss,[alpha=2, beta=0.5, Gamma=-1]):

>  assign(%);

>  implicitplot([op(_C1),s=-1],s=-15..15,_Z=-8..8,gridrefine=2,
crossingrefine=2);
```

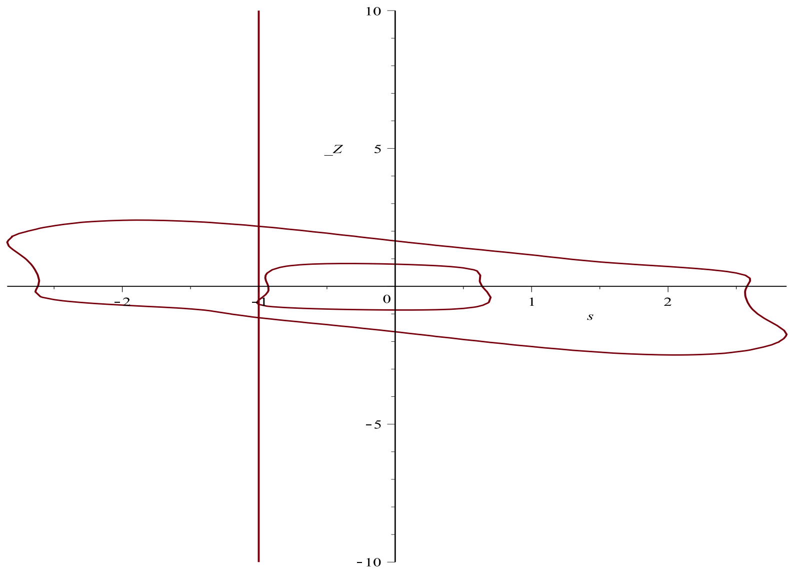



```
> unassign('_C1','_C2','_C3');

> vss1:=eval(ss,[s=-1,alpha=2, beta=0.5, Gamma=-1]):

> lv:=evalf(allvalues(vss1)):

> la:=map(op,[lv]):

> la2:=remove(has,la,I);
```

$la2 :=$

$[[\_C1 = .986091543197571, \_C2 = -3.48618826578288, \_C3 = -.848486949052433],$
$[\_C1 = 2.29686508693243, \_C2 = -5.63779461378456, \_C3 = -1.16744311525857]]$

```
> unassign('_C1','_C2','_C3');

> vssa:=eval(ss,[alpha=-2, beta=0.5, Gamma=-1]):

> assign(%);

> implicitplot([op(_C1),s=-1],s=-5..5,_Z=-10..10,gridrefine=2,
crossingrefine=2);
```




```
> unassign('_C1','_C2','_C3');
> vss1:=eval(ss,[s=-1,alpha=-2, beta=0.5, Gamma=-1]):
> lv:=evalf(allvalues(vss1)):
> la:=map(op,[lv]):
> la2:=remove(has,la,I);
```

$$la2 := [[\_C1 = 2.17791394594416, \_C2 = -5.37165457796903, \_C3 = -1.18215998400672], [\_C1 = -.497833460462187, \_C2 = -3.12391907717788, \_C3 = .959680066577520], [\_C1 = -.654766323087723, \_C2 = -3.21435946892491, \_C3 = 1.04272129118383], [\_C1 = -1.13575318603863, \_C2 = -3.64496764979845, \_C3 = 1.16765563652685]]$$

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A convergence result for a contact problem with adhesion

Anca Farcaş

Abstract. We prove a convergence result for a system coupling two integral equations with a history-dependent variational inequality. More exactly, we consider the variational formulation of a quasistatic contact problem with adhesion. Then we prove the dependence of the weak solution with respect to the data. The proof is based on arguments of variational inequalities, Fréchet spaces and Gronwall inequalities.

Mathematics Subject Classification (2010): 74G25, 74G30, 74M15, 74505.

Keywords: Viscoplastic material, frictionless contact, unilateral constraint, weak solution, finite element, numerical simulations.

1. Introduction

The aim of this paper is to present a convergence result associated to a contact problem with adhesion. It is known that the processes of contact involving adhesion overcome in many industrial settings, when different parts are glued together. For this reason a lot of studies have been developed so the literature concerning this area is in a continuous expansion. According to [2] if we want to model a process in which bonding is not present and debonding may take place, an adhesion process is needed in order to describe the contact. Such models containing adhesion can be found in [1, 3, 5, 6, 9, 10].

The present paper represents a continuation of the paper [14] which covers the modelling and the variational analysis of a contact problem with adhesion and surface memory effects within the infinitesimal strain theory. Taking note of that, the present paper aims to prove a convergence result associated to the problem approached in [14].

The paper is structured as follows. Second Section presents the notations we have made and some short preliminary material. In Section 3 we describe the model

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and we list the assumptions on the data as well as the variational formulation of the problem as it was given in [14]. Finally in Section 4 we state and prove our main convergence result.

2. Notation and Preliminaries

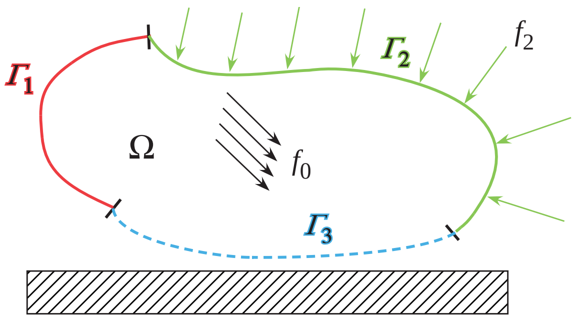

FIGURE 1. The physical setting; Γ_3 is the contact surface

We start this section by presenting the physical setting of the contact process we analyzed throughout the paper. We continue then with some important notation we also shall use throughout this paper. For further details we refer the reader to [2, 4, 7, 8]. Everywhere in this paper we use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+$ to denote the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. For a given $r \in \mathbb{R}$ we denote by r^+ its positive part, i.e. $r^+ = max\,\{r, 0\}$. Also Ω is a bounded domain with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\,(\Gamma_1) > 0$. Standard notation are used for the Lebesgue and Sobolev spaces associated to Ω and Γ and moreover we use the spaces

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\}$$

and

$$Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ji} \,\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator and it is given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \qquad \forall\, \boldsymbol{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality. Moreover, the below mentioned sets are used in the proof of our result.

$$U = \{\, \boldsymbol{v} \in V : v_\nu \leq g \quad \text{a.e. on} \quad \Gamma_3 \,\},$$

$$Z = \left\{\, \omega \in L^2(\Gamma_3) \,:\, 0 \leq \omega \leq 1 \quad \text{a.e. on} \ \ \Gamma_3 \,\right\},$$

where g is a positive constant. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. Their corresponding inner product and norm are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij} \ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used.

In this paper we assume that the material's behavior follows a viscoelastic constitutive law with long memory of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \quad \text{in } \Omega, \tag{2.1}$$

where, here and below, $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress field, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor and $t \in \mathbb{R}_+$ represents the time variable. Also, $\mathcal{A}$ and $\mathcal{B}$ represent the elasticity operator and the relaxation tensor, respectively, and are assumed to verify the following conditions.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\qquad \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \, \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \ \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{2.2}$$

$$\mathcal{B} \in C(\mathbb{R}_+; \mathbf{Q}_\infty). \tag{2.3}$$

The contribution of the bonding to the normal traction, $\sigma_\nu^A(t)$, satisfies

$$\sigma_\nu^A(t) = \gamma_\nu \beta^2(t) \widetilde{R}(u_\nu(t)) \quad \text{on } \Gamma_3, \tag{2.4}$$

where $\widetilde{R}$ is the truncation function given by

$$\widetilde{R}(s) = \left\{\begin{array}{ll} L & \text{if } s < -L \\ -s & \text{if } -L \leq s \leq 0 \\ 0 & \text{if } s > 0 \end{array}\right. \tag{2.5}$$

We follow [5, 6, 11] and assume that the bonding field satisfies the unilateral constraint

$$0 \leq \beta(t) \leq 1 \qquad \text{on} \ \ \Gamma_3. \tag{2.6}$$

Moreover, its evolution is governed by the differential equation

$$\dot{\beta}(t) = -\big(\gamma_\nu \beta(t)[R(u_\nu(t))]^2 - \varepsilon_a\big)^+ \quad \text{on} \ \ \Gamma_3 \tag{2.7}$$

in which ε_a represents the Dupré energy and R is the truncation operator given by

$$R(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \leq s \leq L, \\ L & \text{if } s > L. \end{cases} \tag{2.8}$$

In order to complete the differential equation we give the initial condition

$$\beta(0) = \beta_0 \quad \text{on} \quad \Gamma_3 \tag{2.9}$$

and we assume that the adhesion coefficient, γ_ν, the Dupré energy ε_a, and initial bonding field, β_0, satisfy the conditions

$$\gamma_\nu \in L^\infty(\Gamma_3), \quad \gamma_\nu \geq 0, \quad \varepsilon_a \in L^\infty(\Gamma_3), \quad \varepsilon_a \geq 0, \tag{2.10}$$

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \quad \text{a.e. on} \quad \Gamma_3. \tag{2.11}$$

Note that here and below $L > 0$ is the characteristic length of the bond, beyond which it stretches without offering any additional resistance (see, e.g., [9]). More details on this condition can be found in [11] and references therein. According to [14] when all the adhesive bonds are inactive, or broken, the motion is frictionless. Thus, the tangential traction depends on the intensity of adhesion and on the tangential displacement, but only up to the bond length L, that is

$$-\boldsymbol{\sigma}_\tau(t) = p_\tau(\beta(t))\boldsymbol{R}^*(\boldsymbol{u}_\tau(t)) \quad \text{on } \Gamma_3. \tag{2.12}$$

The truncation operator $\boldsymbol{R}^*$ is given by

$$\boldsymbol{R}^*(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } \|\boldsymbol{v}\| \leq L \\ \dfrac{L}{\|\boldsymbol{v}\|}\boldsymbol{v} & \text{if } \|\boldsymbol{v}\| \geq L. \end{cases} \tag{2.13}$$

The function p_ν will be used later in the paper. It satisfies

$$\left\{ \begin{array}{l} \text{(a) } p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_\nu > 0 \text{ such that} \\ \qquad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leq L_\nu |r_1 - r_2| \\ \qquad\quad \forall\, r_1,\, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))(r_1 - r_2) \geq 0 \\ \qquad \forall\, r_1,\, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \\ \qquad \text{for any } r \in \mathbb{R}. \\ \text{(e) } p_\nu(\boldsymbol{x}, r) = 0 \text{ for all } r \leq 0, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{array} \right. \tag{2.14}$$

Next we will briefly present some of the other notation that are used in the paper during the proofs of the main result.

We use the Riesz representation Theorem to define the operator $P : V \to V$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu \, da \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{2.15}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \qquad \forall\, \boldsymbol{v} \in V. \tag{2.16}$$

We assume that the densities of body forces and surface tractions have regularity

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{2.17}$$

We also consider b a surface memory function which verifies

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)), \quad b(t, \boldsymbol{x}) \geq 0 \quad \text{for all } t \in \mathbb{R}_+, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{2.18}$$

Finally, we consider the functional $j : Z \times V \times V \to \mathbb{R}$ defined by

$$j(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \Big[p_\tau(\beta(t)) \boldsymbol{R}^*(\boldsymbol{u}_\tau(t)) \cdot \boldsymbol{v}_\tau - \gamma_\nu \beta^2(t) \widetilde{R}(u_\nu(t)) v_\nu \Big]\, da \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \beta \in Z. \tag{2.19}$$

3. The model

We start this section by presenting the problem statement as it was given in [14].
Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *and an adhesion field* $\beta : \Gamma_3 \times \mathbb{R}_+ \to [0, 1]$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{3.4}$$

$$-\boldsymbol{\sigma}_\tau(t) = p_\tau(\beta(t))\boldsymbol{R}^*(\boldsymbol{u}_\tau(t)) \quad \text{on} \quad \Gamma_3, \tag{3.5}$$

$$\dot{\beta}(t) = -\big(\gamma_\nu \beta(t)[R(u_\nu(t))]^2 - \varepsilon_a\big)^+ \quad \text{on} \quad \Gamma_3, \tag{3.6}$$

for all $t \in \mathbb{R}_+$*, there exists* $\xi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\left.\begin{aligned}
&u_\nu(t) \le g,\ \sigma_\nu(t) + p_\nu(u_\nu(t)) + \xi(t) - \gamma_\nu \beta^2(t)\widetilde{R}(u_\nu(t)) \le 0,\\
&(u_\nu(t) - g)[\sigma_\nu(t) + p_\nu(u_\nu(t)) + \xi(t) - \gamma_\nu \beta^2(t)\widetilde{R}(u_\nu(t))] = 0,\\
&0 \le \xi(t) \le \int_0^t b(t-s)\, u_\nu^+(s)\, ds,\\
&\xi(t) = 0 \ \text{ if } u_\nu(t) < 0,\\
&\xi(t) = \int_0^t b(t-s)\, u_\nu^+(s)\, ds \text{ if } u_\nu(t) > 0
\end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{3.7}$$

for all $t \in \mathbb{R}_+$ *and, moreover,*

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{on} \ \ \Omega. \tag{3.8}$$

$$\beta(0) = \beta_0 \qquad \text{on} \ \ \Gamma_3, \tag{3.9}$$

We recall that (3.7) describes a condition with unilateral constraint. We assume that at a given moment t there is penetration which did not reach the bound g, i.e. $0 < u_\nu(t) < g$. Then, (3.7) yields

$$-\sigma_\nu(t) = p_\nu(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds. \tag{3.10}$$

This equality shows that at the moment t, the reaction of the foundation depends both on the current value of the penetration (represented by the term $p_\nu(u_\nu(t))$) and on the history of the penetration (represented by the integral term in (3.10)). A contact condition with unilateral constraint, normal compliance and surface memory effects was used in [12] and [13]. Assume now that at a given moment t there is separation between the body and the foundation, i.e. $u_\nu(t) < 0$. Then, (3.7) shows that

$$\sigma_\nu(t) = \gamma_\nu \beta^2(t)\widetilde{R}(u_\nu(t)), \tag{3.11}$$

which means that the reaction of the foundation is nonnegative and depends on the adhesion coefficient, on the square of intensity of adhesion and on the normal displacement, but as it does not exceed the bound length L. Once it exceeds it the normal traction remains constant and $|\sigma_\nu(t)| \le \gamma_\nu L$.

The unique weak solvability of this problem was proved in [14]. Further on, we present its variational formulation.

Problem $\mathcal{P}^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to U$*, a stress field* $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ *and a bonding field* $\beta : \mathbb{R}_+ \to Z$ *such that for all* $t \in \mathbb{R}_+$ *we have*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \tag{3.12}$$

$$\begin{aligned}
&(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))\\
&+\Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds, v_\nu^+ - u_\nu^+(t)\Big)_{L^2(\Gamma_3)} \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall \boldsymbol{v} \in U,
\end{aligned} \tag{3.13}$$

$$\beta(t) = \beta_0 - \int_0^t \big(\gamma_\nu \beta(s)[R(u_\nu(s))]^2 - \varepsilon_a\big)^+ ds. \tag{3.14}$$

In the next section we will present the main result of this paper namely the continuous dependence of the weak solution with respect to the data.

4. Dependence on the data

For each $\rho > 0$ let $\mathcal{B}_\rho$, b_ρ, $\boldsymbol{f}_{0\rho}$, $\boldsymbol{f}_{2\rho}$, $\beta_{0\rho}$ represent perturbations of $\mathcal{B}$, b, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ β_0, respectively, which satisfy conditions (2.3), (2.18), (2.17) and (2.9), respectively. In other words, let

$$\mathcal{B}_\rho \to \mathcal{B} \quad \text{in } C(\mathbb{R}_+; \mathbf{Q}_\infty) \quad \text{as} \quad \rho \to 0, \tag{4.1}$$

$$b_\rho \to b \quad \text{in } \ C(\mathbb{R}_+; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0, \tag{4.2}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in } \ C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0, \tag{4.3}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in } \ C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0, \tag{4.4}$$

$$\beta_{0\rho} \to \beta_0 \quad \text{in } \quad C(\mathbb{R}_+; L^2(\Gamma_3)) \quad \text{as} \quad \rho \to 0. \tag{4.5}$$

Moreover, there exists

$$\begin{cases} F : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \alpha \in \mathbb{R}_+ \text{ s. t.} \\ \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \le F(\rho)(|r| + \alpha) \\ \qquad \forall\, r \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \ \text{for each } \rho > 0. \\ \text{(b) } F(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \end{cases} \tag{4.6}$$

So, the perturbed variational problem is as follows.

Problem ${\mathcal{P}_\rho}^V$. *Find a displacement field* $\boldsymbol{u}_\rho : \mathbb{R}_+ \to U$, *a stress field* $\boldsymbol{\sigma}_\rho : \mathbb{R}_+ \to Q$ *and a bonding field* $\beta_\rho : \mathbb{R}_+ \to Z$ *such that for all* $t \in \mathbb{R}_+$ *we have*

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \int_0^t \mathcal{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))ds, \tag{4.7}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \\ &+ j(\beta_\rho(t), \boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t)) + \Big(\int_0^t b_\rho(t-s)\, u_{\rho\nu}^+(s)\, ds, v_\nu^+ - u_{\rho\nu}^+(t)\Big)_{L^2(\Gamma_3)} \\ &\ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \quad \forall\, \boldsymbol{v} \in U, \end{aligned} \tag{4.8}$$

$$\beta_\rho(t) = \beta_0 - \int_0^t \big(\gamma_\nu \beta_\rho(s)[R(u_\rho \nu(s))]^2 - \varepsilon_a\big)^+ ds. \tag{4.9}$$

Theorem 4.1. *Under the assumptions* (4.1)–(4.5), *the solution* $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \beta_\rho)$ *of Problem* $\mathcal{P}_\rho^V$ *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ *of Problem* $\mathcal{P}^V$,

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad in \quad C(\mathbb{R}_+; U)$$
$$\boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad in \quad C(\mathbb{R}_+; Q)$$
$$\beta_\rho \to \beta \quad in \quad C(\mathbb{R}_+; Z)$$

as $\rho \to 0$.

Proof. Let $n \in \mathbb{N}$ and $t \in [0,n]$. We put $\boldsymbol{v} = \boldsymbol{u}_\rho(t)$ in $\mathcal{P}^V$ and $\boldsymbol{v} = \boldsymbol{u}(t)$ in ${\mathcal{P}_\rho}^V$ then we combine the variational problem $\mathcal{P}^V$ with the perturbed variational problem ${\mathcal{P}_\rho}^V$ and we get

$$
\begin{aligned}
&(P\boldsymbol{u} - P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)) \qquad (4.10)\\
&\qquad + j(\beta_\rho(t), \boldsymbol{u}_\rho(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))\\
&+ \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds - \int_0^t b_\rho(t-s)\, u_{\rho\nu}^+(s)\, ds, u_{\rho\nu}^+(t) - u_\nu(t)^+ \Big)_{L^2(\Gamma_3)}\\
&\qquad + (\boldsymbol{f}_\rho(t) - f(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V +\\
&+ \Big(\int_0^t \mathcal{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s)) ds - \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q\\
&\geq \Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q
\end{aligned}
$$

Using (2.2) we deduce that

$$
\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q \geq m_{\mathcal{A}} \|\boldsymbol{u}_\rho - \boldsymbol{u}\|_V^2 \qquad (4.11)
$$

In addition

$$
\begin{aligned}
&\Big(\int_0^t \mathcal{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s)) ds - \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q \qquad (4.12)\\
&\qquad \leq \Big[\Theta_{\rho n} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\, ds + \omega_{\rho n} \int_0^t \|\boldsymbol{u}(s)\|\, ds \Big] \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V,\\
&\Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds - \int_0^t b_\rho(t-s)\, u_{\rho\nu}^+(s)\, ds, u_{\rho\nu}^+(t) - u_\nu(t)^+ \Big)_{L^2(\Gamma_3)} \qquad (4.13)\\
&\qquad \leq \Big[\Theta_{\rho n}^b \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\, ds + \omega_{\rho n}^b \int_0^t \|\boldsymbol{u}(s)\|\, ds \Big] \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V,
\end{aligned}
$$

and

$$
(\boldsymbol{f}_\rho(t) - f(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \leq \delta_{\rho n} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V, \qquad (4.14)
$$

where

$$
\Theta_{\rho n} = c_0^2 \max_{r \in [0,n]} \|\mathcal{B}_\rho(r)\|_Q, \qquad (4.15)
$$

$$
\omega_{\rho n} = c_0^2 \max_{r \in [0,n]} \|\mathcal{B}_\rho(r) - \mathcal{B}(r)\|_Q, \qquad (4.16)
$$

$$
\Theta_{\rho n}^b = c_0^2 \max_{r \in [0,n]} \|b_\rho(r)\|_{L^\infty(\Gamma_3)}, \qquad (4.17)
$$

$$
\omega_{\rho n}^b = c_0^2 \max_{r \in [0,n]} \|b_\rho(r) - b(r)\|_{L^\infty(\Gamma_3)}, \qquad (4.18)
$$

$$\delta_{\rho n} = \max_{r\in[0,n]} \|\boldsymbol{f}_\rho(r) - \boldsymbol{f}(r)\|_V. \tag{4.19}$$

Moreover,

$$\begin{aligned}(P\boldsymbol{u} - P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \\ \leq F(\rho)\big(c_0^2\|\boldsymbol{u}(t)\|_V + c_0\,\alpha\, meas(\Gamma_3)^{1/2}\big)\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.\end{aligned} \tag{4.20}$$

From [14] we have that

$$j(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq c\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V.$$

Analogous, in our context we have

$$j(\beta, \boldsymbol{u}, \boldsymbol{u}_\rho - \boldsymbol{u}) + j(\beta_\rho, \boldsymbol{u}_\rho, \boldsymbol{u} - \boldsymbol{u}_\rho) \leq c\|\beta - \beta_\rho\|_{L^2(\Gamma_3)}\|\boldsymbol{u} - \boldsymbol{u}_\rho\|_V. \tag{4.21}$$

Note that c is a constant which does not depend on t and whose values can change from line to line. From (4.11) – (4.21) we deduce that

$$\begin{aligned}m_A\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2 &\leq \delta_{\rho n}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\ &+F(\rho)\big(c_0^2\|\boldsymbol{u}(t)\|_V + c_0\,\alpha\, meas(\Gamma_3)^{1/2}\big)\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\ &+\Big[\Theta_{\rho n}^b \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\,ds + \omega_{\rho n}^b \int_0^t \|\boldsymbol{u}(s)\|\,ds\Big]\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\ &+\Big[\Theta_{\rho n} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\,ds + \omega_{\rho n} \int_0^t \|\boldsymbol{u}(s)\|\,ds\Big]\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\ &+c\|\beta - \beta_\rho\|_{L^2(\Gamma_3)}\|\boldsymbol{u}(t) - \boldsymbol{u}_\rho(t)\|_V\end{aligned} \tag{4.22}$$

Consequently

$$\begin{aligned}&m_A\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2 \\ &\leq \Big[\delta_{\rho n} + F(\rho)\big(c_0^2\|\boldsymbol{u}(t)\|_V + c_0\,\alpha\, meas(\Gamma_3)^{1/2}\big) + (\omega_{\rho n}^b + \omega_{\rho n})\int_0^t \|\boldsymbol{u}(s)\|\,ds\Big] \\ &\qquad +(\Theta_{\rho n}^b + \Theta_{\rho n})\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|\,ds + c\|\beta - \beta_\rho\|_{L^2(\Gamma_3)}\end{aligned} \tag{4.23}$$

Next, we denote

$$\xi_{n,n} = \frac{1}{m_A}\max\Big\{1,\ c_0^2\|\boldsymbol{u}(t)\|_V + c_0\,\alpha\, meas(\Gamma_3)^{1/2},\ \int_0^t \|\boldsymbol{u}(s)\|\,ds\Big\}. \tag{4.24}$$

Once again, from [2] we have that

$$\|\beta - \beta_\rho\|_{L^2(\Gamma_3)} \leq c\int_0^t \|\boldsymbol{u}(s) - \boldsymbol{u}_\rho(s)\|_V\,ds \tag{4.25}$$

So we have that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \tag{4.26}$$

$$\leq \Big[\delta_{\rho n} + F(\rho) + \omega^b_{\rho n} + \omega_{\rho n}\Big]\xi_{n,n} + \frac{\Theta^b_{\rho n} + \Theta_{\rho n} + c}{m_A}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds.$$

We know that $((\Theta_{\rho n})_\rho, (\Theta^b_{\rho n})_\rho)$ are bounded sequences so we can conclude that there exists $\zeta_n > 0$ which only depends on n and it is independent of ρ such that

$$0 \leq \frac{\Theta^b_{\rho n} + \Theta_{\rho n} + c}{m_A} \leq \zeta_n, \text{ for all } \rho \geq 0.$$

We deduce that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \tag{4.27}$$

$$\leq \Big[\delta_{\rho n} + F(\rho) + \omega^b_{\rho n} + \omega_{\rho n}\Big]\xi_{n,n} + \zeta_n \int_0^n \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \, ds.$$

Using the Gronwall inequality we get that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq \Big[F(\rho) + \delta_{\rho n} + \omega^b_{\rho n} + \omega_{\rho n}\Big]\xi_{n,n} e^{\xi_n \rho n}. \tag{4.28}$$

Now using the fact that $F(\rho) \to 0, \omega_{\rho n} \to 0, \delta_{\rho n} \to 0, \omega^b_{\rho n} \to 0$ we deduce that

$$\max_{t\in[0,n]} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \to 0 \text{ for } \rho \to 0. \tag{4.29}$$

In conclusion, we have that

$$\max_{t\in[0,n]} \|\beta(t) - \beta_\rho(t)\|_{L^2(\Gamma_3)} \to 0 \text{ for } \rho \to 0. \tag{4.30}$$

In the same time

$$\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_\rho(t)\|_Q \leq L_{\mathcal{A}}\|\boldsymbol{u}(t) - \boldsymbol{u}_\rho(t)\|_V + \tag{4.31}$$

$$+\Theta_{\rho n}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds + \omega_{\rho n}\int_0^n \|\boldsymbol{u}(s)\|_V \, ds.$$

Taking into account (4.29) and the fact that $((\Theta_{\rho n})_\rho)$ is bounded and $\omega_{\rho n} \to 0$ we get

$$\max_{t\in[0,n]} \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_\rho(t)\|_Q \to 0.$$ □

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Fully developed mixed convection through a vertical porous channel with an anisotropic permeability: case of heat flux

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Abstract. The effect of anisotropy on the steady fully developed mixed convection flow in a vertical porous channel is analytically studied. The side walls of the channel are prescribed by a constant heat flux and the flow at the entrance is upward, so that natural convection aids the forced flow. It is shown that the anisotropy parameter has a significant effect of the flow and heat transfer characteristics. We extend the study in [9] with the case of opposing flow by using Computer Algebra software.

Mathematics Subject Classification (2010): 76S99, 76R99, 68W30.

Keywords: Fluid mechanics, porous media, computer algebra.

1. Introduction

This work is an extension of the paper [9], which considers all the cases of assisting and oposing flow.

Convective heat transfer in a saturated porous medium has attracted considerable interest in recent years, due to its frequent occurrence in industrial and technological applications. Examples of these applications include geothermal reservoirs, thermal insulation, enhanced oil recovery, drying of porous solids, packed-bed catalytic reactors, volcanic eruption, electronic circuits, and many others, see, for example the books [13], [10], [15], [18, 19], [12], [11], and [17].

The aim of this paper is to study the effects of anisotropy on the fully developed mixed convection flow through a vertical channel filled with a porous medium. Such studies were performed by [16], [14], [8], [7], [4], [5], [6], [20], etc. It was found by these authors that the effect of the anisotropy ratio parameter on the flow characteristics was significant.

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We have followed here also the papers by [2], and [3].

2. The basic equations

We consider the problem of steady fully developed flow in a vertical porous channel bounded by two parallel walls at a distance L, which are maintained at uniform and equal wall heat fluxes q_w (Figure 1). The channel has a rectangular cross-section infinitely long in the z-direction. The porous medium is assumed to be anisotropic in permeability with its principal axes of the porous matrix denoted by K_1 and K_2.

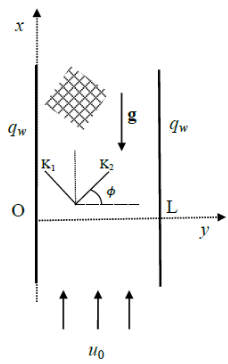


FIGURE 1. The geometry of the problem

The anisotropy of the porous medium is then characterized by the anisotropy ratio $K^* = \frac{K_1}{K_2}$ and the orientation angle ϕ, defined as the angle between the horizontal and K_2. It is also assumed that the flow is uniform with the characteristic velocity at the entrance of the channel denoted by u_0.

Under these assumptions, along with the Boussinesq approximation, the basic equations governing the steady conservation of mass, momentum (Brinkman-Darcy's law) and energy can be written as follows ([6])

$$\nabla \cdot \mathbf{v} = 0 \tag{2.1}$$

$$\mathbf{v} = \frac{\overline{\overline{K}}}{\mu}\left(-\nabla p - \mu \nabla^2 \mathbf{v} + \rho\left[1 - \beta\left(T - T_0\right)\right]\mathbf{g}\right) \tag{2.2}$$

$$\nabla\left(vT\right) = \alpha_m \nabla^2 T \tag{2.3}$$

Here $\overline{\overline{K}}$ is the symmetrical second-order permeability tensor, which is defined as

$$\overline{\overline{K}} = \begin{bmatrix} K_1 \cos^2\phi + K_2 \sin^2\phi & (K_1 - K_2)\sin\phi\cos\phi \\ (K_1 - K_2)\sin\phi\cos\phi & K_1\cos^2\phi + K_2\sin^2\phi \end{bmatrix} \tag{2.4}$$

Equation (2.1) can be also written in the form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.5}$$

Assuming that the flow is fully developed it results in that $v = 0$ and $u = u(y)$. Thus, the governing equations (2.2) and (2.3) can be written in reduced form as follow

$$\mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \frac{a\mu}{K_1} u + \rho g \beta (T - T_0) = \frac{\partial p}{\partial x} \tag{2.6}$$

$$\frac{\partial p}{\partial x} = 0 \tag{2.7}$$

$$u \frac{\partial T}{\partial x} = \alpha_m \frac{\partial^2 T}{\partial y^2}, \tag{2.8}$$

where the constant a is given by

$$a = \cos^2 \phi + K^* \sin^2 \phi. \tag{2.9}$$

The boundary conditions for Eqs. (2.6) and (2.8) are

$$u(0) = u(L) = 0, \quad \left.\frac{\partial T}{\partial y}\right|_{y=0} = \frac{q}{k}, \quad \left.\frac{\partial T}{\partial y}\right|_{y=L} = -\frac{q}{k} \tag{2.10}$$

along with the mass flux ([2]) and thermal ([3]) conditions

$$\int_0^L u \mathrm{d}\, y = Q_0, \quad T_0 = \frac{1}{L} \int_0^L T \mathrm{d}\, y, \tag{2.11}$$

where Q_0 is the mass flux across the channel and T_0 is the mean temperature in the horizontal direction of the channel, and is chosen as the reference temperature.

3. The dimensionless equations

Introducing the dimensionless variables defined as

$$\begin{gathered} X = \frac{\alpha_m x}{u_0 L^2}, \; Y = \frac{y}{L}, \; U(Y) = \frac{u}{u_0} \\ \theta(X,Y) = \frac{T - T_0}{\frac{q_w L}{K}}, \; P(X) = \frac{\alpha_m p}{\mu u_0^2 L}, \end{gathered} \tag{3.1}$$

equations (2.6) and (2.7) become

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\, y^2} - \zeta^2 U + \lambda \theta + \gamma = 0 \tag{3.2}$$

$$U \frac{\partial \theta}{\partial X} = \frac{\partial^2 \theta}{\partial Y^2} \tag{3.3}$$

Here ζ is the *anisotropic parameter*, γ is the *constant pressure gradient parameter* and λ is the *mixed convection parameter*, which are defined as

$$\zeta^2 = \sqrt{\frac{a}{\mathrm{Da}}}, \; \gamma = -\frac{\partial P}{\partial X} = -\frac{\partial P}{\partial x}, \; \lambda = \frac{Gr}{\mathrm{Re}} \tag{3.4}$$

where $\mathrm{Da} = K_1/L$ is the Darcy number, $\mathrm{Gr} = g\beta(q_w \frac{L}{k})\frac{L^3}{v^2}$ is the Grashof number based on the heat flux q_w and $\mathrm{Re} = u_0 L/w$ is the Reynolds number.

The boundary conditions (2.10) become

$$U(0) = U(1) = 0,\ \left.\frac{\partial\theta}{\partial Y}\right|_{Y=0} = -1,\ \left.\frac{\partial\theta}{\partial Y}\right|_{Y=1} = 1 \tag{3.5}$$

and the mass flux and thermal conditions (2.11) reduces to

$$\int_0^1 U\mathrm{d}Y = 1,\ \int_0^1 \theta\mathrm{d}Y = 0 \tag{3.6}$$

where we took $Q_0 = u_0 L$.

By integrating Eq. (3.3) over the channel cross-section, and making use the boundary conditions (3.5) for θ, it can be shown that

$$\frac{\partial\theta}{\partial X} = 2 \tag{3.7}$$

Thus, Eq. (3.3) becomes

$$\frac{\partial^2\theta}{\partial Y^2} = 2U \tag{3.8}$$

Note that Al-Hadrhrami et al. [1] have considered

$$\theta(y) = \overline{\theta}(y) + 2x$$

Thus, it results that

$$\frac{\partial^2\overline{\theta}}{\partial Y^2} = 2U \tag{3.9}$$

If we integrate this equation twice, we get

$$\bar{\theta}(Y) = 2\int\left(\int U(Y)\mathrm{d}\,y + C_5\right)\mathrm{d}\,y + C_6$$

where C_5 and C_6 are constants of integration. Thus, we can obtain θ from U.

The physical quantities of interest are the *wall skin friction* or *wall shear stress coefficient* Cf and the *Nusselt number* Nu, which are defined as

$$C_f = \frac{\tau_w}{\rho u_0^2},\quad Nu = \frac{qL}{k(T_w - T_0)} \tag{3.10}$$

where

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0,L}$$

Using (2.11) and (3.8) we get

$$\mathrm{Re}\,C_f = \left.\frac{\partial U}{\partial Y}\right|_{Y=0,1},\qquad Nu = \left.\frac{1}{\theta}\right|_{Y=0,1}.$$

4. Analysis and modeling

In order to obtain U, we consider Eq. (3.2) and differentiate it twice, by taking into account Eq. (3.8). We obtain the following equations

$$\frac{\mathrm{d}^4U}{dY^4} - \zeta^2\frac{d^2U}{dY^2} + 2\lambda U = 0 \quad (4.1)$$

$$\frac{d^2\theta}{dY^2} = 2U(Y) \quad (4.2)$$

with boundary conditions

$$\begin{aligned} &U(0) = 0,\ U(1) = 0 \\ &\theta'(0) = -1,\ \theta'(1) = 1 \\ &\int_0^1 \theta(Y)\mathrm{d}Y = 0. \end{aligned} \quad (4.3)$$

The form of the solutions of (4.1)+(4.2)+(4.3) depends on the roots of the characteristic equations

$$r^4 - \zeta^2 r^2 + 2\lambda = 0.$$

Formally, these roots are

$$\pm\frac{1}{2}\sqrt{2\,\zeta^2 \pm 2\,\sqrt{\zeta^4 - 8\,\lambda}}.$$

Let

$$\Delta = \zeta^4 - 8\lambda.$$

According to the sign of Δ and λ, there will be several cases to be considered.

1. $\lambda > 0$ (assisting flow)
   (a) $\Delta < 0$, four complex roots
   (b) $\Delta = 0$, two double real roots, $\pm\frac{\sqrt{2}}{2}\zeta$
   (c) $\Delta > 0$, four real roots
2. $\lambda = 0$, the roots are $0, 0, \zeta, -\zeta$
3. $\lambda < 0$, (opposing flow) two real and two complex roots

For the solution of (4.1)+(4.2)+(4.3) and solutions plot we used the computer algebra system Maple.

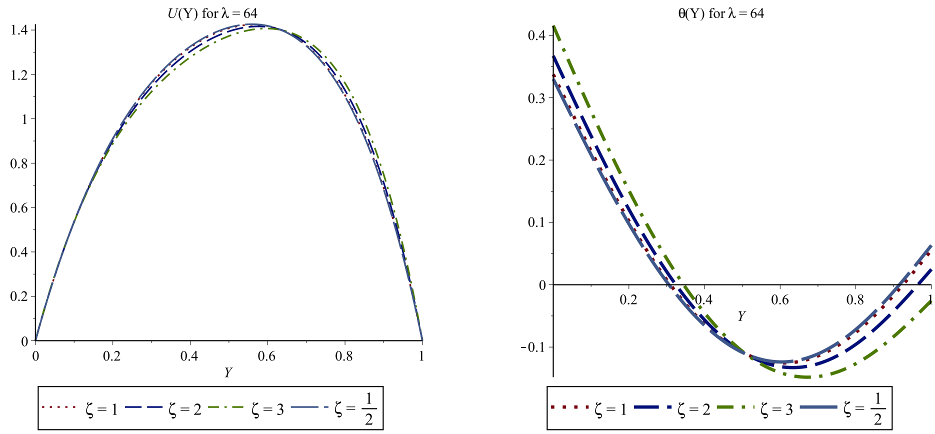
5. Results and discussion

We present several graphs which illustrate the influence of λ and ζ on the velocity and temperature profile.

It can be seen from Figs. 2(a), 3(a) and 4(a) that the effect of an increasing anisotropic parameter ζ leads to a decrease of the dimensionless fluid velocity next to left wall of the channel and to an increase of the dimensionless velocity profiles near the right wall of the channel, for all three cases considered $\Delta < 0$, $\Delta = 0$ and $\Delta < 0$. This is true for opposing flow for small values of λ (see Figures 6 and 7).

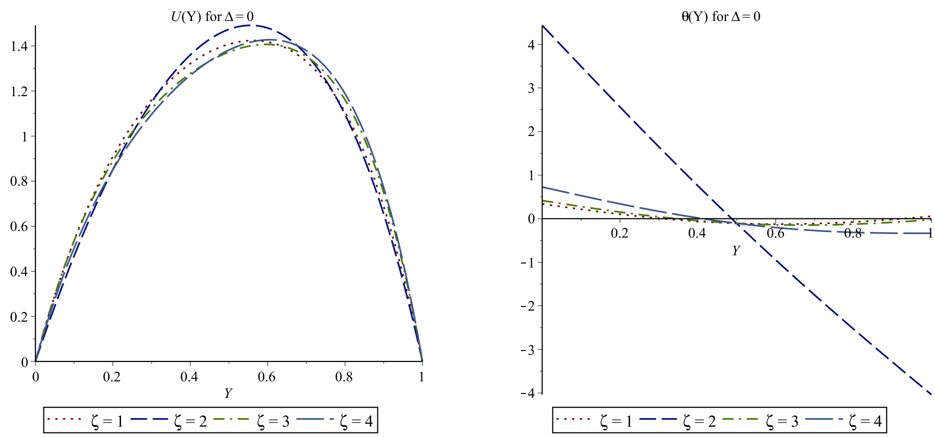
However, Figs. 2(b) and 4(b) shows that in these cases of Δ the dimensionless temperature increases with the increase of the anisotropic parameter ζ.

It should also be mentioned that the dimensionless velocity and temperature profiles illustrated in Figs. 2(a) to 5(b) resemble the same shapes as in the paper [3].



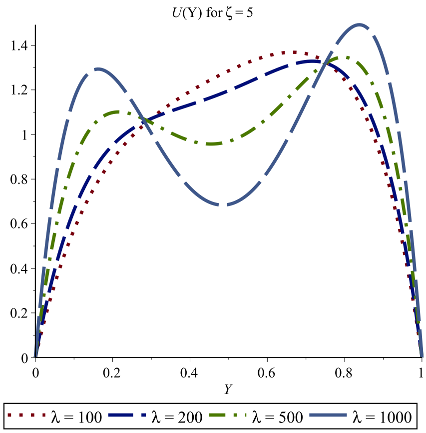
(a) Velocity (b) Temperature

FIGURE 2. Velocity (left) and temperature profile for $\lambda = 64$ and various values of ζ when $\Delta < 0$

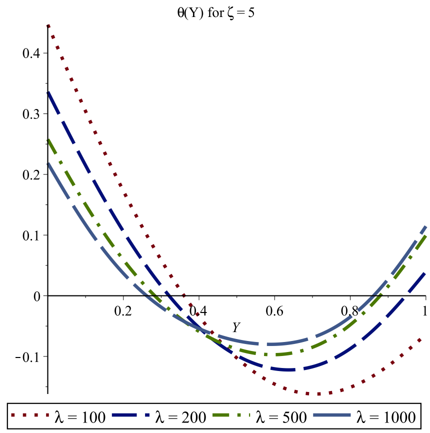


(a) Velocity (b) Temperature

FIGURE 3. Velocity (left) and temperature profile for $\Delta = 0$ and various values of ζ

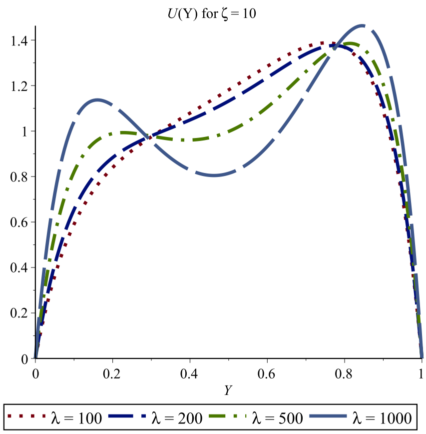


(a) Velocity

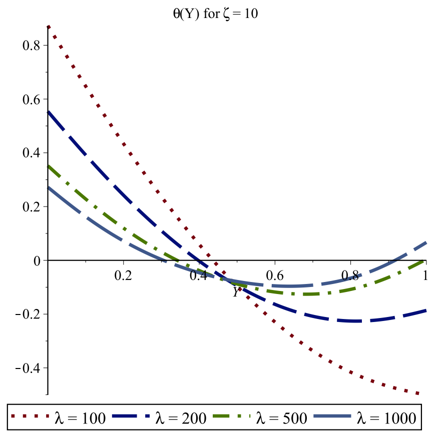


(b) Temperature

FIGURE 4. Velocity (left) and temperature profile for $\zeta = 5$ and various values of λ when $\Delta < 0$



(a) Velocity

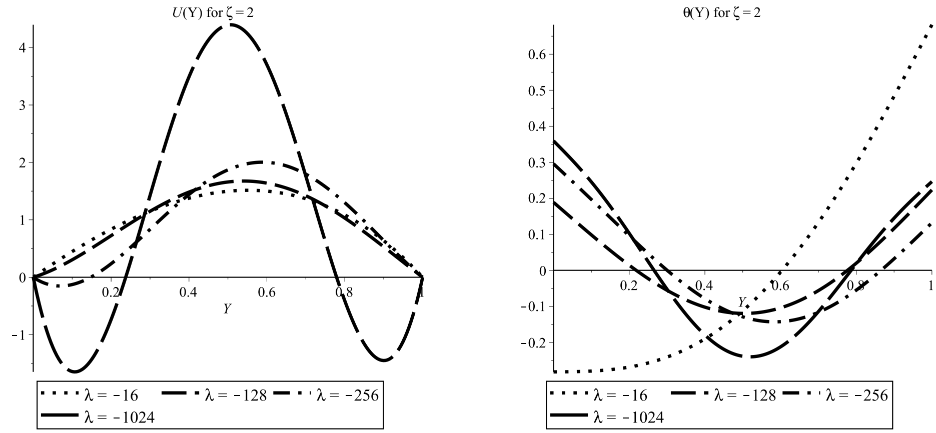


(b) Temperature

FIGURE 5. Velocity (left) and temperature profile for $\zeta = 10$ and various values of λ when $\Delta > 0$

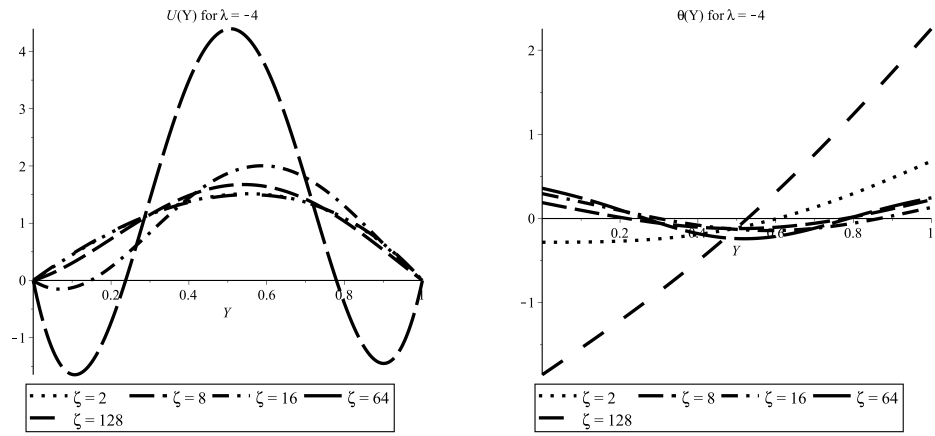
6. Conclusions

The paper presents an analytical study of the fully developed assisting mixed convection flow through a porous channel with an anisotropic permeability when the walls of the channels are kept at constant heat fluxes. The Brinkman-Darcy model has been used.



(a) Velocity (b) Temperature

FIGURE 6. Velocity (left) and temperature profile for $\zeta = 2$ and various values of $\lambda < 0$ (opposing flow)



(a) Velocity (b) Temperature

FIGURE 7. Velocity (left) and temperature profile for $\lambda = -2$ and various values of ζ (opposing flow)

The following conclusions can be drawn:

1. The effect of anisotropy on the dimensionless velocity profiles is substantial, especially for large values of the mixed convection parameter λ;
2. The effect of anisotropy is less important for the dimensionless temperature profiles when the mixed convection parameter λ increases.

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Book reviews

Alexander B. Kharazishvili, Set Theoretical Aspects of Real Analysis, CRC Press, Taylor & Francis Group, Boca Raton 2015, xxii + 433 pp, ISBN: 13: 978-1-4822-4201-0.

As it is well known the Axiom of Choice (**AC**) plays a fundamental role in mathematics, many fundamental results depending on it, or even being equivalent to it. At the same time the acceptance of **AC**, i.e. working within **ZFC** (Zermelo-Frenkel set theory + **AC**), leads to counterintuitive and paradoxical results, the most intriguing being the Banach-Tarski paradox. On the other side, the restriction to **ZF** axioms also leads to paradoxical situations as, for instance, the possibility to represent the set **R** of real numbers as a countable union of countable sets (see page 11 of the book). For these reasons the mathematicians tried to understand to what extent some results depend essentially on **AC**, or on some weaker variants: Countable Choice (**CC**), Dependent Choice (**DC**), Product Countable Choice (**PCC**), Continuum Hypothesis (**CH**), Martin's Axiom (**MA**). Some of the results equivalent to **AC** are: (1) Tychonov's theorem on the compactness of the product of compact spaces; (2) Kuratowski-Zorn lemma on the existence of maximal chains in partially ordered sets; (3) the total ordering of cardinal numbers; (3) the existence of a Hamel basis in any vector space. Other results, as (i) the existence of nonmeasurable subsets of **R**; (ii) the existence of subsets of $\mathbb{R}$ not having the Baire property; (iii) the existence of a Hamel basis in **R** (considered as a vector space over **Q**); the Hahn-Banach theorem, cannot be proved within **ZF** and need some uncountable forms of the **AC**. As it is known, Solovay constructed a mathematical theory based on **ZF** plus the existence of a nonmeasurable Lebesgue set, taken as an axiom.

The book contains a detailed presentation of various aspects relating the foundation of mathematics with some fundamental results in real analysis, measure theory, set theory and topology. Among the topics included in the book we mention: measurability properties of sets and functions, the existence of nonmeasurable sets (Vitali sets, Berstein sets) and nonmeasurable functions and functions having some pathological properties (e.g. the Sierpinski-Zygmund function), measurability and continuity (Luzin-type results), the existence of nonmeasurable additive functions, measurability properties of well-orderings, etc.

The exposition is completed with five appendices containing brief but thorough presentations of various topics, that are essential for the understanding of the main text: A1. *The axioms of set theory*; A2. *The Axiom of Choice and the Continuum*

Hypothesis; A3. *Martin's Axiom and its consequences in real analysis*; A4. ω_1-*dense subsets of the real line*; A5. *The beginning of the descriptive set theory.* These appendices make the book fairly self-contained, preventing the reader to browse through specialized volumes.

The main text is completed with Exercises containing further results, ranging from routine to very difficult. For these ones, marked by stars, some hints are supplied. The bibliography at the end of the book counts 279 items.

The author is a well known expert in the area with numerous journal contributions and 6 books (3 in Russian published at Tbilisi, his home university, and other 3 in English published with various international editors), treating various aspects of the interconnections between set theory, measure theory and real analysis.

The book, containing a lot of results of interest to a broad spectrum of readers, in fact to every mathematician involved in research or teaching, is written in a didactic manner with clear proofs and many examples and comments. It can be used as a reference text or as a complementary text for courses on real analysis and measure theory.

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