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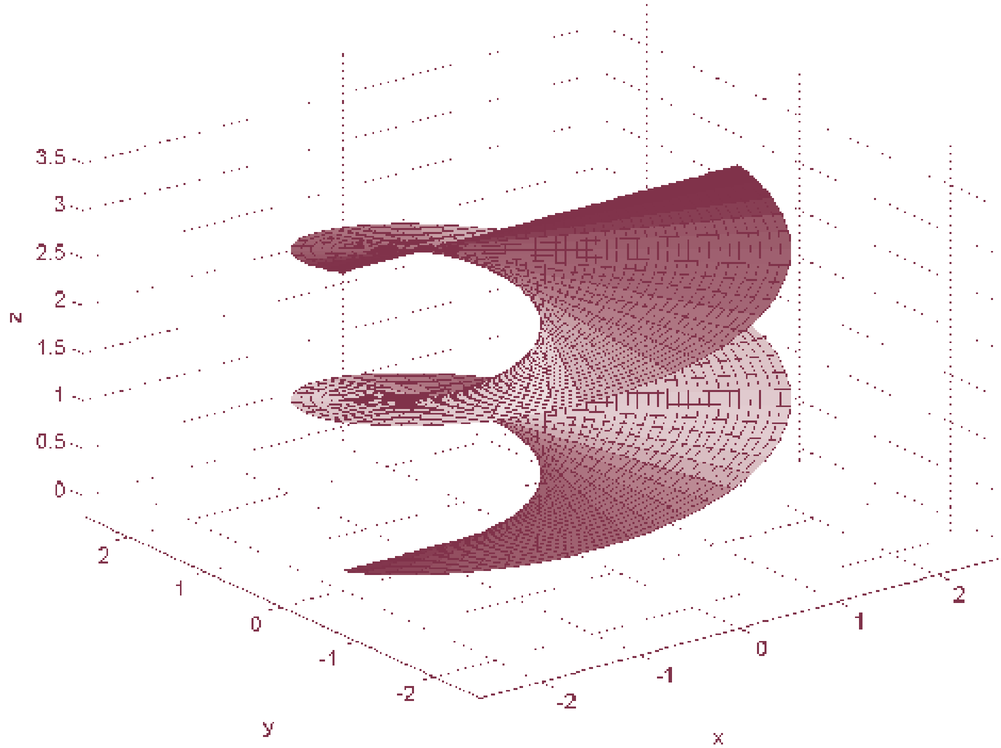


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**CONTENTS**

Marlène Frigon, Fixed point theorems for maps on cones in Fréchet spaces via the projective limit approach ........ 393
Saïd Abbas, Wafaa Albarakati, Mouffak Benchohra and Adrian Petruşel, Existence and Ulam stability results for Hadamard partial fractional integral inclusions via Picard operators ........ 409
Diana Otrocol, Hybrid differential equations with maxima via Picard operators theory ........ 421
Kazimierz Goebel, Remarks on fixed point sets ........ 429
Monica-Felicia Bota, Fixed point theorems for operators with a contractive iterate in $b$-metric spaces ........ 435
Radu Precup, Abstract method of upper and lower solutions and application to singular boundary value problems ........ 443
Alexandru-Darius Filip, Coupled fixed point theorems for Zamfirescu type operators in ordered generalized Kasahara spaces ........ 453
Nicolaie Lungu and Sorina Anamaria Ciplea, Ulam-Hyers stability of Black-Scholes equation ........ 467
Anca Oprea and Gabriela Petruşel, Coupled fixed point theorems for rational type contractions ........ 473
Sever S. Dragomir, Inequalities of Hermite-Hadamard type for $AH$-convex functions ........ 489
Abhijit Banerjee and Sujoy Majumder, Nonlinear differential polynomial sharing a nonzero polynomial with certain degree ........ 503
Book reviews ........ 531

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 4, 393–408

# Fixed point theorems for maps on cones in Fréchet spaces via the projective limit approach

Marlène Frigon

*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** We present fixed point results for admissibly compact maps on cones in Fréchet spaces. We first extend the Krasnosel'skiĭ fixed point theorem with order type cone-compression and cone-expansion conditions. Then, we extend the monotone iterative method to this context. Finally, we present fixed point results under a combination of the assumptions of the previous results. More precisely, we combine a cone-compressing or cone-extending condition only on one side of the boundary of an annulus with an assumption on the existence of an upper fixed point. In addition, we show that the usual monotonicity condition can be weaken.

**Mathematics Subject Classification (2010):** 47H10, 47H04.

**Keywords:** Fixed point, Fréchet space, cone, fixed point index, cone-compressing and cone-extending conditions, multivalued map, monotone iterative method.

## 1. Introduction

The classical Krasnosel'skiĭ fixed point theorem is very well known and useful, see [13, 14]. Assuming cone-compression and cone-expansion conditions on the boundary of two nested bounded, neighborhoods of the origin relative to a cone, it establishes the existence of nontrivial fixed points of maps on cones in Banach spaces. Two types of cone-compression and cone-expansion conditions were considered: one involving the norm and the other involving the order on the space induced by the cone. This result was extended to Fréchet spaces in [1, 2, 12] using the fact that a Fréchet space is the projective limit of a sequence of Banach spaces. All those generalizations rely on at least one cone-compression condition involving the norm of the values of maps on the relative boundary of suitable bounded, open sets in those Banach spaces.

On the other hand, the monotone iterative method is often applied to deduce the existence of fixed points of nondecreasing maps $f$ defined on closed intervals $[\alpha, \beta]$ in ordered Banach spaces, where $\alpha$ is a *lower fixed point of* $f$ (i.e. $\alpha \leq f(\alpha)$) and $\beta$ is an *upper fixed point of* $f$ (i.e. $f(\beta) \leq \beta$). The fixed points are obtained as the limits of iterative sequences. This method was introduced by Amann [3] for single-valued maps and extended to multivalued maps in [7].

In a series of papers, Cabada, Cid, Infante and their collaborators (see [4, 5, 6, 8, 10]) obtained many fixed point theorems on cones in Banach spaces by imposing cone-compression or cone-extension conditions on the boundary relative to a cone of only one bounded, neighborhood of the origin instead of two. The usual second condition was replaced by assuming that the map $f$ is nondecreasing (or nonincreasing) on a suitable shell and by assuming the existence of an upper fixed point (or a lower fixed point) instead of assuming the existence of both as in the monotone iterative method.

In this paper, we present fixed point results for maps on cones in Fréchet spaces. In section 3, we extend the Krasnosel'skiĭ fixed point theorem with order type cone-compression and cone-expansion conditions instead of norm-type conditions. Our results will rely on the fixed point index theory for multivalued mapping in cones obtained by Fitzpatrick and Petryshyn [9].

In section 4, we extend the monotone iterative method to Fréchet spaces. In addition, we show that the monotonicity condition can be dropped. In that case, the existence of a fixed point is still insured but some precision on its localization is lost.

Finally, in the last section, existence results are presented relying on one cone-compression or cone-expansion condition combined with one condition of the type upper fixed point or lower fixed point. It is not assumed that the cone is normal or solid. Also, a condition weaker than monotonicity is imposed. Therefore, even in the particular case where the Fréchet space is a Banach space, our results generalize theorems due to Cabada, Cid and Infante [6].

Using the fact that a Fréchet space is the projective limit of a sequence of Banach spaces, our results are presented for admissibly compact maps. This notion was introduced in [11]. It is worthwhile to mention that our results could have been presented for admissibly condensing maps or admissible maps satisfying a Leggett-William type condition as in [1]. We first present some preliminaries on the fixed point index for multivalued maps on closed, convex sets, then on Fréchet spaces, and finally on admissibly compact maps.

## 2. Preliminaries

### 2.1. Fixed point index

In all this text, $E$ denotes a Fréchet space endowed with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $X, Y$ be subsets of $E$ and $F : X \to Y$ a multivalued map with non-empty closed values. The map $F$ is *compact* if $F(X)$ is relatively compact in $Y$; it is *completely continuous* if $F(B)$ is relatively compact in $Y$ for every $B \subset X$ bounded. It is *upper semi-continuous* (u.s.c.) if $\{x \in X : F(x) \cap A \neq \emptyset\}$ is closed in $Y$ for every $A$ closed in $X$.

Let $C$ be a closed, convex set in $E$. For $U$ a nonempty, open set in $E$, we denote $U_C = U \cap C$, $\overline{U}_C = \overline{U} \cap C$ and $\partial_C U = \overline{U}_C \backslash U_C$ the boundary of $U$ in $C$.

In [9], Fitzpatrick and Petryshyn defined a fixed point index for upper semi-continuous, condensing, multivalued maps $F : \overline{U}_C \to C$ with nonempty, convex, compact values such that $F$ has no fixed point on $\partial_C U$. This fixed point index is denoted $i_C(F, U)$. Here is their Theorem 2.1 in the particular case of compact maps.

**Theorem 2.1** ([9])**.** *Let $F : \overline{U}_C \to C$ be a compact, u.s.c., multivalued map with nonempty, convex, compact values and such that $x \notin F(x)$ for all $x \in \partial_C U$. Then, the following statements hold:*

(1) *If $i_C(F, U) \neq 0$, then $F$ has a fixed point.*
(2) *If $x_0 \in U_C$, then $i_C(\{x_0\}, U) = 1$, where $\{x_0\}$ denotes the constant map.*
(3) *If $U = U_1 \cup U_2$, where $U_1$ and $U_2$ are disjoint open sets and are such that $x \notin F(x)$ if $x \in \partial_C U_1 \cup \partial_C U_2$, then*
$$i_C(F, U) = i_C(F, U_1) + i_C(F, U_2).$$
(4) *If $H : [0, 1] \times \overline{U}_C \to C$ is a compact, u.s.c., multivalued map with nonempty, convex, compact values and such that $x \notin H(t, x)$ for all $t \in [0, 1]$ and $x \in \partial_C U$, then*
$$i_C(H(1, \cdot), U) = i_C(H(0, \cdot), U).$$

By $K$, we denote a *cone* in $E$; that is a closed set such that, for every $x, y \in K$ and every $\lambda, \delta \geq 0$, $\lambda x + \delta y \in K$ and $K \cap (-K) = \{0\}$. A cone $K$ is called *normal* if, for every $n \in \mathbb{N}$, there exists $c_n \geq 1$ such that

$$\|x\|_n \leq c_n \|y\|_n \quad \text{for every } x, y \in K \text{ such that } y - x \in K.$$

Fitzpatrick and Petryshyn [9] obtained the following Krasnosel'skiĭ type fixed point result which relied on the previous theorem in the particular case where the closed, convex set is a cone. Using the fact that a Fréchet space is metrizable, they considered $d$ a metric on $E$ generating the same topology. For $r > 0$, let

$$B_d(x_0, r) = \{x \in E : d(x, x_0) < r\} \quad \text{and} \quad \overline{B_d(x_0, r)} = \{x \in E : d(x, x_0) \leq r\}.$$

Again, their theorem is stated for compact maps instead of condensing maps.

**Theorem 2.2** ([9])**.** *Let $r_1, r_2 \in (0, \infty)$, $r = \min\{r_1, r_2\}$ and $R = \max\{r_1, r_2\}$. Let $K$ be a cone in $E$ and $F : \overline{B_d(0, R)} \cap K \to K$ a compact, u.s.c., multivalued map with nonempty, convex, compact values satisfying the following conditions:*

(i) $\big(F(x) - x\big) \subset K$ *if $x \in \partial_K B_d(0, r_1)$;*
(ii) $\big(x - F(x)\big) \subset K$ *if $x \in \partial_K B_d(0, r_2)$;*
(iii) *there exists a continuous semi-norm $p$, non-vanishing on $K$, such that $(I - F)(\overline{B_d(0, r_1)} \cap K)$ is $p$-bounded.*

*Then, $F$ has a fixed point $x_0 \in \overline{B_d(0, R)} \backslash B_d(0, r)$.*

It could be difficult to apply this result to deduce the existence of solutions to differential or integral equations on unbounded intervals. Indeed, in general, the operator associated to the problem will not be compact on open sets. The problem is that open sets in Fréchet spaces are big.

Let us give an example. Let $C(\mathbb{R})$ be the space of continuous functions on the real line and, for $n \in \mathbb{N}$, the semi-norm

$$\|x\|_n = \max_{t\in[-n,n]} |x(t)|.$$

Endowed with the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$, $C(\mathbb{R})$ is a Fréchet space. Let $U \subset C(\mathbb{R})$ be a neighborhood of 0. Then, there exist $n_0 \in \mathbb{N}$ and $r > 0$ such that

$$\{x \in C(\mathbb{R}) : \|x\|_{n_0} < r\} \subset U.$$

Also, in this context, it could be more difficult to get non trivial fixed points. For example, let

$$B(0,r) = \{x \in C(\mathbb{R}) : |x(t)| < r \ \forall t \in \mathbb{R}\}.$$

From the previous remark, $B(0,r)$ has empty interior. Therefore, there exists a sequence $\{x_n\}$ in $C(\mathbb{R})$ such that $x_n \to 0$ and $\|x_n\|_n \geq r$ for every $n \in \mathbb{N}$.

### 2.2. Fréchet spaces and projective limits

For sake of completeness, we recall some notations and properties of Fréchet spaces presented in [11].

Let $E$ be a Fréchet space with the topology generated by a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. In what follows, we will always assume that the following condition is satisfied:

$$\|x\|_1 \leq \|x\|_2 \leq \cdots \quad \text{for every } x \in E. \tag{2.1}$$

For $\hat{x} \in E$, $r > 0$, $R = (r_1, r_2, \dots) \in (0,\infty)^{\mathbb{N}}$ and $n \in \mathbb{N}$, we denote

$$\begin{aligned}
B_n(\hat{x}, r) &= \{x \in E : \|x - \hat{x}\|_n < r\},\\
\overline{B_n(\hat{x}, r)} &= \{x \in E : \|x - \hat{x}\|_n \leq r\},\\
B(\hat{x}, R) &= \{x \in E : \|x - \hat{x}\|_n < r_n \ \forall n \in \mathbb{N}\},\\
\overline{B(\hat{x}, R)} &= \{x \in E : \|x - \hat{x}\|_n \leq r_n \ \forall n \in \mathbb{N}\}.
\end{aligned}$$

For $X \subset E$ and $n \in \mathbb{N}$, we denote by $\operatorname{diam}_n$, the $n$-diameter of $X$ induced by $\|\cdot\|_n$; that is,

$$\operatorname{diam}_n(X) = \sup\{\|x - y\|_n : x, y \in X\} \in [0,\infty) \cup \{\infty\}.$$

We say that $X$ is *bounded* if there exists $R \in (0,\infty)^{\mathbb{N}}$ such that $X \subset B(0,R)$; so, $\operatorname{diam}_n(X) < \infty$ for every $n \in \mathbb{N}$.

**Remark 2.3.** Observe that if $E$ is not a Banach space, then

(1) an open set in $E$ is never bounded;
(2) a bounded set in $E$ has empty interior.

The space $E$ is the projective limit of a sequence of Banach spaces $\{E_n\}$. Indeed, for each $n \in \mathbb{N}$, we write

$$x \sim_n y \quad \text{if and only if} \quad \|x - y\|_n = 0. \tag{2.2}$$

This defines an equivalence relation on $E$. We denote by $E_n$ the completion of the quotient space $E/{\sim_n}$ with respect to $\|\cdot\|_n$ (the norm on $E/{\sim_n}$ induced by $\|\cdot\|_n$ and its extension to $E_n$ are still denoted by $\|\cdot\|_n$). This construction defines a continuous map $\mu_n : E \to E_n$ such that

$$\mu_n(x) = [x]_n, \quad \text{(i.e. } \mu_n(x) = \mu_n(y) \quad \Longleftrightarrow \quad x \sim_n y).$$

Similarly, for every $m \geq n$, we can define an equivalence relation on $E_m$, still noted $\sim_n$, which defines a continuous map $\mu_{n,m} : E_m \to E_n$ since $E_m/{\sim_n}$ can be regarded as a subset of $E_n$. So, $E$ is the projective limit of $\{E_n\}$.

For each subset $X \subset E$ and each $n \in \mathbb{N}$, we set $X_n = \mu_n(X)$, and we denote $\overline{X}_n$, and $\partial_n X_n$, respectively the closure and the boundary of $X_n$ with respect to $\|\cdot\|_n$ in $E_n$.

The following lemma gives an important property of closed subsets of $E$.

**Lemma 2.4** ([11]). *Let $E$ be a Fréchet space endowed with a family of semi-norms satisfying* (2.1)*, and let $X$ be a closed subset of $E$. Then, for every sequence $\{z_n\}$ with $z_n \in \overline{X}_n$, such that for every $n \in \mathbb{N}$, $\{\mu_{n,m}(z_m)\}_{m\geq n}$ is a Cauchy sequence in $\overline{X}_n$, there exists $x \in X$ such that $\{\mu_{n,m}(z_m)\}_{m\geq n}$ converges to $\mu_n(x) \in X_n$ for every $n \in \mathbb{N}$.*

For every $n \in \mathbb{N}$, let $A(n) \subset E_n$. We define

$$\begin{aligned} \operatorname*{Lim}_{n\to\infty} A(n) = \{x \in E :\ & \exists N_0 \subset \mathbb{N} \text{ infinite and } z_n \in A(n) \text{ for } n \in N_0 \\ & \text{such that} \quad \forall\, n \in \mathbb{N}, \quad \mu_{n,m}(z_m) \to \mu_n(x) \\ & \text{as } m \to \infty \text{ with } m \in N_0 \text{ and } m \geq n\}. \end{aligned} \tag{2.3}$$

Notice that if $X$ is closed, then

$$X = \operatorname*{Lim}_{n\to\infty} \overline{X}_n.$$

Taking into account the fact that many applications in Fréchet spaces lead to look for solutions in a closed set with empty interior, the notion of pseudo-interior was introduced in [11].

**Definition 2.5.** Let $X$ be a subset of $E$. The *pseudo-interior* of $X$ is defined by

$$\text{pseudo-int}(X) = \{x \in X : \mu_n(x) \in \overline{X}_n \backslash \partial X_n \ \text{ for every } n \in \mathbb{N}\}.$$

The set $X$ is *pseudo-open* if $X = \text{pseudo-int}(X)$.

For $n \in \mathbb{N}$, let $C_n$ be a closed, convex set in $E_n$. In what follows, the topology in $C_n$ induced by $\|\cdot\|_n$ will play a key role. So, we introduce the following notation. Let $U$ be a nonempty pseudo-open set in $E$, we denote

$$U_{C_n} = U_n \cap C_n, \quad \overline{U}_{C_n} = \overline{U}_n \cap C_n \quad \text{and} \quad \partial_{C_n} U_n = \overline{U}_{C_n} \backslash U_{C_n} = \big(\overline{U}_n \backslash U_n\big) \cap C_n.$$

### 2.3. Admissibly compact maps

Here is the notion of admissibly compact maps introduced in [11].

**Definition 2.6.** Let $X \subset E$ and $C$ closed and convex in $E$. A map $f : X \to C$ is called *admissibly compact* if it satisfies the following properties for every $n \in \mathbb{N}$:

(i) The multivalued map $\widehat{F}_n : X_n \to \overline{C}_n$ defined by

$$\widehat{F}_n(\mu_n(x)) = \overline{\mathrm{co}}\Big(\mu_n\big(f(\{x\}_{n,X})\big)\Big),$$

admits an upper semi-continuous compact extension $F_n : \overline{X}_n \to \overline{C}_n$ with convex, compact values, where

$$\{x\}_{n,X} = \{y \in X : \mu_n(y) = \mu_n(x)\} = \mu_n^{-1}\big([x]_n\big) \cap X.$$

(ii) For every $\varepsilon > 0$, there exists $m \geq n$ such that, for every $x \in X$,

$$\mathrm{diam}_n\Big(f\big(\{x\}_{m,X}\big)\Big) < \varepsilon.$$

A map $f : X \to C$ is called *admissibly completely continuous* if it is admissibly compact on every bounded sets in $X$.

The following proposition will play a key role in the proof of the forthcoming fixed point theorems.

**Proposition 2.7.** *Let $X \subset E$ be closed, $C \subset E$ closed, convex, and $f : X \to C$ an admissibly compact map. Assume that there exists $N_0 \subset \mathbb{N}$ infinite such that, for every $n \in N_0$, there exists $z_n \in \overline{X}_n$ such that $z_n \in F_n(z_n)$. Then, $f$ has a fixed point.*

*Proof.* For $m \in N_0$, $F_m$ has a fixed point $z_m \in \overline{X}_m$. From the definition of $F_n$, one sees that

$$\mu_{n,m}(z_m) \in F_n(\mu_{n,m}(z_m)) \quad \text{for every } n \leq m.$$

Thus, without lost of generality, we can assume that $N_0 = \mathbb{N}$.

The compactness of $F_1$ implies that the sequence $\{\mu_{1,k}(z_k)\}_{k\geq 1}$ has a subsequence $\{\mu_{1,k}(z_k)\}_{k\in N_1}$ converging to some $x_1 \in \overline{X}_1$. It follows from the upper semi-continuity of $F_1$ that $x_1 \in F_1(x_1)$.

Similarly, the sequence $\{\mu_{2,k}(z_k)\}_{k\in N_1}$ has a subsequence $\{\mu_{2,k}(z_k)\}_{k\in N_2}$ converging to $x_2 \in \overline{X}_2$, with $x_2 \in F_2(x_2)$. The uniqueness of the limit implies that $\mu_{1,2}(x_2) = x_1$.

Repeating this argument gives, for every $n \in \mathbb{N}$, the existence of $x_n \in \overline{X}_n$ such that $x_n \in F_n(x_n)$ and $\mu_{n,m}(x_m) = x_n$ for every $m \geq n$. It follows from Lemma 2.4 that there exists $x \in X$ such that $\mu_n(x) \in F_n(\mu_n(x))$ for every $n \in \mathbb{N}$.

We have to show that $x = f(x)$. If this is false, there exist $n \in \mathbb{N}$ and $r > 0$ such that $\|x - f(x)\|_n = r$. Let $\varepsilon < r/2$. By Definition 2.6(ii), there exists $m \geq n$ such that $\mathrm{diam}_n\Big(f\big(\{x\}_{m,X}\big)\Big) < \varepsilon$. Observe that

$$\mathrm{diam}_n\Big(f\big(\{x\}_{m,X}\big)\Big) = \mathrm{diam}_n\Big(\mathrm{co}\Big(f\big(\{x\}_{m,X}\big)\Big)\Big).$$

On the other hand, since $\mu_m(x) \in F_m(\mu_m(x))$, there is $w \in \mathrm{co}\big(f\big(\{x\}_{m,X}\big)\big)$ such that $\|x - w\|_m < \varepsilon$. Thus,

$$\begin{aligned} r = \|x - f(x)\|_n &\leq \|x - w\|_n + \|w - f(x)\|_n \\ &< \|x - w\|_m + \mathrm{diam}_n\Big(\mathrm{co}\Big(f\big(\{x\}_{m,X}\big)\Big)\Big) < 2\varepsilon < r. \end{aligned}$$

Thus, $x = f(x)$. □

## 3. Krasnosel'skiĭ type fixed point results

In this section, we present Krasnosel'skiĭ type fixed point results with order-type cone-compressing and cone-extending conditions on the pseudo-boundary of bounded sets in $E$.

Let us first recall the following two fixed point results obtained in [12] for admissibly completely continuous maps in Fréchet spaces satisfying norm-type cone-compressing and cone-extending type conditions. Notice that, for $K$ a cone in $E$, one has that $\overline{K}_n$ is a cone in $E_n$ for every $n \in \mathbb{N}$.

**Theorem 3.1** ([12]). *Let $f : K \to K$ be an admissibly completely continuous map. Assume that there exist $U, V$ two bounded, pseudo-open subsets of $E$ satisfying the following conditions for every $n \in \mathbb{N}$:*

(i) $\|y\|_n \geq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} U_n$
*(resp.* $\|y\|_n \leq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} U_n$*);*
(ii) $\|y\|_n \leq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} V_n$
*(resp.* $\|y\|_n \geq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} V_n$*);*
(iii) $0 \in \overline{U}_n \backslash \partial_n U_n \subset \overline{U}_n \subset \overline{V}_n \backslash \partial_n V_n$ *for every* $n \in \mathbb{N}$.

*Then, there exists $x$ a fixed point of $f$ such that*

$$x \in \operatorname*{Lim}_{n\to\infty} A(n),$$

*where* $A(n) = \overline{K}_n \cap \overline{V_n \backslash U_n}$ *and* $\operatorname{Lim}_{n\to\infty} A(n)$ *is defined in* (2.3).

In the particular case where $U$ and $V$ are pseudo-balls, the previous result can be stated as follows.

**Corollary 3.2** ([12]). *Let $f : K \to K$ be an admissibly completely continuous map. Assume that there exist $\{r_{1,n}\}$ and $\{r_{2,n}\}$ nondecreasing sequences in $(0, \infty)$ such that, for every $n \in \mathbb{N}$,*

(i) $\|y\|_n \geq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} B_n(0, r_{1,n})$*;*
(ii) $\|y\|_n \leq \|x\|_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} B_n(0, r_{2,n})$*;*
(iii) $r_{1,n} \neq r_{2,n}$.

*Then, there exists $x$ a fixed point of $f$ such that*

$$x \in \operatorname*{Lim}_{n\to\infty} \overline{K}_n \cap \overline{B_n(0, R_n) \backslash B_n(0, r_n)},$$

*where* $R_n = \max\{r_{1,n}, r_{2,n}\}$ *and* $r_n = \min\{r_{1,n}, r_{2,n}\}$.

Analogous results can be obtained if the norm-type cone-compressing and cone-extending conditions are replaced by order-type conditions.

**Theorem 3.3.** *Let $f : K \to K$ be an admissibly completely continuous map. Assume that there exist $U, V$ two bounded, pseudo-open subsets of $E$ satisfying the following conditions for every $n \in \mathbb{N}$:*

(i) $\big(F_n(x) - x\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ *for all* $x \in \partial_{\overline{K}_n} U_n$
*(resp.* $\big(x - F_n(x)\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ *for all* $x \in \partial_{\overline{K}_n} U_n$*);*

(ii) $\big(x - F_n(x)\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ *for all* $x \in \partial_{\overline{K}_n} V_n$
*(resp.* $\big(F_n(x) - x\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ *for all* $x \in \partial_{\overline{K}_n} V_n$*);*

(iii) $0 \in \overline{U}_n \backslash \partial_n U_n \subset \overline{U}_n \subset \overline{V}_n \backslash \partial_n V_n$ *for every* $n \in \mathbb{N}$.

*Then, there exists* $x$ *a fixed point of* $f$ *such that*

$$x \in \operatorname*{Lim}_{n \to \infty} A(n),$$

*where* $A(n) = \overline{K}_n \cap \overline{V_n \backslash U_n}$.

*Proof.* For every $n \in \mathbb{N}$, we claim that

$$\exists z_n \in F_n(z_n) \quad \text{such that } z_n \in A(n). \tag{3.1}$$

If this is false, we define

$$H_n : [0,1] \times \overline{U}_{\overline{K}_n} \to \overline{K}_n \quad \text{by} \quad H_n(t,x) = tF_n(x).$$

For $x \in \partial_{\overline{K}_n} U_n$ and $t \in (0,1]$, $x \notin H_n(t,x)$. Otherwise,

$$\left(\frac{1}{t} - 1\right) x \in \big(F_n(x) - x\big) \cap \overline{K}_n,$$

which contradicts (i). It follows from (iii) and Theorem 2.1(2), (4) that

$$i_{\overline{K}_n}(F_n, U_n) = i_{\overline{K}_n}(0, U_n) = 1. \tag{3.2}$$

On the other hand, choose $\hat{u} \in \overline{K}_n$ such that

$$\|\hat{u}\|_n > \max\{\|x - y\|_n : x \in \overline{V}_{\overline{K}_n},\ y \in F_n(x)\}. \tag{3.3}$$

Such $\hat{u}$ exists since $V_n$ and $F_n(\overline{V}_{\overline{K}_n})$ are bounded. Let us define

$$\widehat{H}_n : [0,1] \times \overline{V}_{\overline{K}_n} \to \overline{K}_n \quad \text{by} \quad \widehat{H}_n(t,x) = t\hat{u} + F_n(x).$$

By (ii), $x \notin \widehat{H}_n(t,x)$ for all $t \in [0,1]$ and $x \in \partial_{\overline{K}_n} V_n$. It follows from (3.3) that $x \notin \widehat{H}_n(1,x)$ for every $x \in \overline{V}_{\overline{K}_n}$. Theorem 2.1(1), (4) implies that

$$i_{\overline{K}_n}(F_n, V_n) = i_{\overline{K}_n}(\widehat{H}_n(1,\cdot), V_n) = 0. \tag{3.4}$$

Combining (3.2) and (3.4) and applying Theorem 2.1(3) permit us to deduce that

$$i_{\overline{K}_n}(F_n, V_n \backslash \overline{U}_n) = i_{\overline{K}_n}(F_n, V_n) - i_{\overline{K}_n}(F_n, U_n) = -1.$$

Therefore, (3.1) holds.

The conclusion follows from Proposition 2.7. □

Here is a corollary of the previous theorem in the particular case where $U$ and $V$ are pseudo-balls.

**Corollary 3.4.** *Let* $f : K \to K$ *be an admissibly completely continuous map. Assume that there exist* $\{r_{1,n}\}$ *and* $\{r_{2,n}\}$ *nondecreasing sequences in* $(0,\infty)$ *such that, for every* $n \in \mathbb{N}$,

(i) $x - F_n(x) \subset \overline{K}_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} B_n(0, r_{1,n})$;

(ii) $F_n(x) - x \subset \overline{K}_n \quad \forall y \in F_n(x), \forall x \in \partial_{\overline{K}_n} B_n(0, r_{2,n})$.

*Then, there exists $x$ a fixed point of $f$ such that*

$$x \in \operatorname*{Lim}_{n\to\infty} \overline{K}_n \cap \overline{B_n(0,R_n)} \backslash B_n(0,r_n),$$

*where* $R_n = \max\{r_{1,n}, r_{2,n}\}$ *and* $r_n = \min\{r_{1,n}, r_{2,n}\}$.

## 4. Monotone iterative method in cones

Let $K$ be a cone in $E$ and $\overline{K}_n$ the associated cone in $E_n$ for every $n \in \mathbb{N}$. The cone $K$ defines the partial orderings in $E$ and in $E_n$ given by

$$\begin{aligned} &\text{for } x,y \in E, && x \preceq y && \text{if and only if} && y - x \in K, \\ &\text{for } n \in \mathbb{N} \text{ and } x,y \in E_n, && x \preceq_n y && \text{if and only if} && y - x \in \overline{K}_n. \end{aligned} \tag{4.1}$$

For $x, y \in E$ such that $x \preceq y$ (resp. $x,y \in E_n$ such that $x \preceq_n y$ for some $n \in \mathbb{N}$) we denote

$$\begin{aligned} [x,y] &= \{z \in E : x \preceq z \preceq y\} && (\text{resp.} \quad [x,y]_n = \{z \in E_n : x \preceq_n z \preceq_n y\}), \\ [x,\infty) &= \{z \in E : x \preceq z\} && (\text{resp.} \quad [x,\infty)_n = \{z \in E_n : x \preceq_n z\}). \end{aligned}$$

Arguing as in [3], the well-known monotone iterative method permits to get the following fixed point result in Fréchet space.

**Theorem 4.1.** *Let $\alpha \preceq \beta$ be in $E$ and $f : [\alpha,\beta] \to E$ a compact map. Assume the following conditions are satisfied:*

(i) $\alpha \preceq f(\alpha)$ *and* $f(\beta) \preceq \beta$;
(ii) $f$ *is* nondecreasing*; that is, for every $x, y \in [\alpha,\beta]$ such that $x \preceq y$, one has* $f(x) \preceq f(y)$.

*Then, $f$ has a fixed point and the iterative sequences $\{f^k(\alpha)\}$ and $\{f^k(\beta)\}$ converge respectively to the smallest and the greatest fixed point of $f$ in $[\alpha,\beta]$.*

For some $\alpha \in E$ (resp. $\beta \in E$) such that $\alpha \not\preceq f(\alpha)$ (resp. $f(\beta) \not\preceq \beta$), there could exist some $n \in \mathbb{N}$ such that $\mu_n(\alpha) \preceq_n \mu_n(f(\alpha))$ (resp. $\mu_n(f(\beta) \preceq_n \mu_n(\beta))$). This remark leads us to consider admissibly compact maps. Since they involve multivalued maps, different notions of monotonicity can be defined.

**Definition 4.2.** Let $Y$ be a space endowed with a partial order $\leq$, $X \subset Y$ and $T : X \to Y$ a multivalued map. Let $x^-, x^+ \in X$ and $y^-, y^+ \in Y$ be such that $x^- \leq x^+$ and $y^- \leq y^+$.

(i) The map $T$ is *right-nondecreasing on* $[x^-, x^+]$ *and in* $[y^-, y^+]$ if $y^- \in T(x^-)$ and, for every $x_1, x_2 \in X$ and every $y_1 \in T(x_1)$ such that

$$x^- \leq x_1 \leq x_2 \leq x^+ \quad \text{and} \quad y^- \leq y_1 \leq y^+,$$

there exists $y_2 \in T(x_2)$ such that $y_1 \leq y_2 \leq y^+$.

(ii) The map $T$ is *left-nondecreasing on* $[x^-, x^+]$ *and in* $[y^-, y^+]$ if $y^+ \in T(x^+)$ and, for every $x_1, x_2 \in X$ and every $y_2 \in T(x_2)$ such that

$$x^- \leq x_1 \leq x_2 \leq x^+ \quad \text{and} \quad y^- \leq y_2 \leq y^+,$$

there exists $y_1 \in T(x_1)$ such that $y^- \leq y_1 \leq y_2$.

Similarly, one can define that $T$ is *right-nonincreasing* (resp. *left-nonincreasing*) *on* $[x^-, x^+]$ *and in* $[y^-, y^+]$.

The following fixed point result concerns admissibly compact maps which are nondecreasing in the sense of the previous definition.

**Theorem 4.3.** *Let $X \subset E$ be closed and $f : X \to E$ an admissibly compact map. Assume the following conditions are satisfied:*

(i) *there exists $N_0 \subset \mathbb{N}$ infinite such that, for every $n \in N_0$, there exist $\alpha_n, \beta_n \in \overline{X}_n$ such that $\alpha_n \preceq_n \beta_n$ in $E_n$ and $[\alpha_n, \beta_n]_n \subset \overline{X}_n$;*
(ii) *for every $n \in N_0$, there exists $\xi_n \in F_n(\alpha_n) \cap [\alpha_n, \beta_n]_n$ (resp. $\zeta_n \in F_n(\beta_n) \cap [\alpha_n, \beta_n]_n$);*
(iii) *for every $n \in N_0$, $F_n$ is right-nondecreasing on $[\alpha_n, \beta_n]_n$ and in $[\xi_n, \beta_n]_n$ (resp. $F_n$ is left-nondecreasing on $[\alpha_n, \beta_n]_n$ and in $[\alpha_n, \zeta_n]_n$).*

*Then, $f$ has a fixed point*

$$x \in \operatorname*{Lim}_{\substack{n\to\infty \\ n\in N_0}} A(n),$$

*where*

$$A(n) = \Big\{ z \in [\alpha_n, \beta_n]_n : z = \lim_{k\to\infty} u_k \text{ with } u_{k+1} \in F_n(u_k) \text{ and } \alpha_n \preceq_n \xi_n = u_1 \preceq_n u_2 \preceq_n \cdots \preceq_n \beta_n \Big\},$$

$$\Big(\textit{resp.}\quad A(n) = \Big\{ z \in [\alpha_n, \beta_n]_n : z = \lim_{k\to\infty} v_k \text{ with } v_{k+1} \in F_n(v_k) \text{ and } \alpha_n \preceq_n \cdots \preceq_n v_2 \preceq_n v_1 = \zeta_n \preceq_n \beta_n \Big\}\Big).$$

*Proof.* For $n \in N_0$, $F_n : [\alpha_n, \beta_n]_n \to E_n$ is compact, u.s.c. with compact, convex values. From (i)-(iii), one can construct a sequence $\{u_k^n\}$ in $[\alpha_n, \beta_n]_n$ such that $u_1^n = \xi_n$, $u_{k+1}^n \in F_n(u_k^n)$ and $u_k^n \preceq_n u_{k+1}^n$ for every $k \in \mathbb{N}$. Arguing as in the proof of Theorem 3.4 in [7], one deduces that there exists $z_n = \lim_{k\to\infty} u_k^n \in A(n)$ such that $z_n \in F_n(z_n)$. The conclusion follows from Proposition 2.7. $\square$

Observe that assumption (iii) of the previous theorem implies that

$$F_n(x) \cap [\alpha_n, \beta_n]_n \neq \emptyset \quad \forall x \in [\alpha_n, \beta_n]_n, \quad \forall n \in N_0.$$

In fact, this is sufficient to insure that $f$ has a fixed point. However, we loose some precision on its localization.

**Theorem 4.4.** *Let $X \subset E$ be closed and $f : X \to E$ an admissibly compact map. Assume the following conditions are satisfied:*

(i) *there exists $N_0 \subset \mathbb{N}$ infinite such that, for every $n \in N_0$, there exist $\alpha_n, \beta_n \in \overline{X}_n$ such that $\alpha_n \preceq_n \beta_n$ in $E_n$ and $[\alpha_n, \beta_n]_n \subset \overline{X}_n$;*
(ii) *for every $n \in N_0$, $x \in [\alpha_n, \beta_n]_n$, there exists $u \in F_n(x) \cap [\alpha_n, \beta_n]_n$.*

*Then, $f$ has a fixed point*

$$x \in \operatorname*{Lim}_{\substack{n\to\infty \\ n\in N_0}} [\alpha_n, \beta_n]_n.$$

*Proof.* For $n \in N_0$, let us define $\widetilde{F}_n : [\alpha_n, \beta_n]_n \to [\alpha_n, \beta_n]_n$ by

$$\widetilde{F}_n(x) = F_n(x) \cap [\alpha_n, \beta_n]_n.$$

The assumptions imply that $\widetilde{F}_n$ is a compact, u.s.c., multivalued map with nonempty, compact, convex values and defined on a closed, convex subset of the Banach space $E_n$. The Kakutani fixed point theorem insures the existence of $z_n \in \widetilde{F}_n(z_n)$. The conclusion follows from Proposition 2.7. □

**Remark 4.5.** In the results of this section, one can replace the compactness assumption by the complete continuity if, in addition, we assume that $K$ is normal. Indeed, in a normal cone, an interval $[\alpha, \beta]$ (resp. $[\alpha_n, \beta_n]_n$) is bounded.

## 5. Fixed point results in cones with mixed type conditions

In this section, we present fixed point results relying on a combination of conditions imposed in the theorems obtained in the two previous sections. In particular, the existence of suitable pairs $(\alpha_n, \beta_n)$ is not assumed. More precisely, the assumption on the existence of a suitable $\{\alpha_n\}$ in Theorem 4.4 is removed and replaced by some conditions on the pseudo-boundary of a suitable pseudo-open set. As before, $\overline{K}_n$ is the cone in $E_n$ associated to a cone $K$ in $E$.

**Theorem 5.1.** *Let $\beta \in K$ and $f : [0, \beta] \to K$ an admissibly compact map. Assume the following conditions are satisfied:*

(i) *there exists $U$ a bounded, pseudo-open set in $E$ such that, for every $n \in \mathbb{N}$, $0 \in \overline{U}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} U_n \subset \overline{U}_{\overline{K}_n} \subset [0, \mu_n(\beta)]_n$;*

(ii) *the set*

$$N_0 = \{n \in \mathbb{N} : \forall x \in \partial_{\overline{K}_n} U_n,\ (F_n(x) - x) \cap \overline{K}_n = \emptyset \quad \text{or} \quad F_n(x) \cap [x, \mu_n(\beta)]_n \neq \emptyset\}$$

*is infinite;*

(iii) *for every $n \in N_0$ and every $\hat{x} \in \partial_{\overline{K}_n} U_n$ such that $F_n(\hat{x}) \cap [\hat{x}, \infty)_n \neq \emptyset$, one has that $F_n(x) \cap [\hat{x}, \mu_n(\beta)]_n \neq \emptyset$ for every $x \in [\hat{x}, \mu_n(\beta)]_n$.*

*Then, $f$ has a fixed point $x^*$ such that*

$$x^* \in \operatorname*{Lim}_{\substack{n \to \infty \\ n \in N_0}} A(n),$$

*where*

$$A(n) = \Big(\overline{U}_{\overline{K}_n}\Big) \cup \Big(\bigcup_{x \in \partial_{\overline{K}_n} U_n} [x, \mu_n(\beta)]_n\Big).$$

*Proof.* It follows from (i) that, for every $n \in N_0$ and every $x \in \partial_{\overline{K}_n} U_n$, one has $x \preceq_n \mu_n(\beta)$.

Let

$$N_1 = \Big\{n \in N_0 : \exists \alpha_n \in \partial_{\overline{K}_n} U_n \text{ such that } F_n(\alpha_n) \cap [\alpha_n, \infty)_n \neq \emptyset\}.$$

If $N_1$ is infinite, then the assumptions of Theorem 4.4 are satisfied with $\alpha_n$ and $\beta_n = \mu_n(\beta)$. Therefore, $f$ has a fixed point

$$x \in \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_1}} [\alpha_n, \mu_n(\beta)]_n \subset \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}} A(n).$$

On the other hand, if $N_1$ is empty or finite, then $N_2 = N_0\backslash N_1$ is infinite and, for every $n \in N_2$, $(F_n(z) - z) \cap \overline{K}_n = \emptyset$ for every $z \in \partial_{\overline{K}_n} U_n$. Arguing as in the proof of Theorem 3.3, one deduces that the fixed point index

$$i_{\overline{K}_n}(F_n, U_n) = 1 \quad \forall n \in N_2.$$

Hence, there exists $z_n \in U_{\overline{K}_n}$ such that $z_n \in F_n(z_n)$. Proposition 2.7 permits to conclude that $f$ has a fixed point

$$x \in \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_2}} U_{\overline{K}_n} \subset \operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}} A(n).$$

□

We obtain the following corollary by adding a monotonicity condition.

**Corollary 5.2.** *Let $\beta \in K$ and $f : [0, \beta] \to K$ an admissibly compact map satisfying conditions* (i) *and* (ii) *of Theorem* 5.1*. In addition, assume that*

(iii') *for every $n \in N_0$ and every $\hat{x} \in \partial_{\overline{K}_n} U_n$ such that $F_n(\hat{x}) \cap [\hat{x}, \infty)_n \neq \emptyset$, there exists $\hat{y} \in F_n(\hat{x})$ such that $F_n$ is right-nondecreasing on $[\hat{x}, \mu_n(\beta)]_n$ and in $[\hat{y}, \mu_n(\beta)]_n$.*

*Then, $f$ has a fixed point.*

In [6], Cabada, Cid and Infante consided completely continuous maps defined on a solid, normal cone in a Banach space and which are nondecreasing on $[0, \beta]\backslash B(0, r/c)$. Here is a corollary of Theorem 5.1 for admissibly completely continuous maps satisfying a monotonicity condition analogous to the condition imposed in [6].

**Corollary 5.3.** *Let $K$ be a normal cone and $f : K \to K$ an admissibly completely continuous map. Assume there exist $\beta \in K$ and $\{r_n\}$ a nondecreasing sequence in $(0, \infty)$ satisfying the following conditions:*

(i) $\overline{B_n(0, r_n)} \cap \overline{K}_n \subset [0, \mu_n(\beta)[_n$*;*

(ii) *the set*

$$N_0 = \{n \in \mathbb{N} : \forall x \in \partial_{\overline{K}_n} B_n(0, r_n),\ (F_n(x) - x) \cap \overline{K}_n = \emptyset \\ \text{or} \quad F_n(x) \cap [x, \mu_n(\beta)]_n \neq \emptyset\}$$

*is infinite;*

(iii) *for every $n \in N_0$ and every $\hat{x} \in \overline{K}_n \backslash B_n(0, r_n/c_n)$ such that $F_n(\hat{x}) \cap [0, \mu_n(\beta)]_n \neq \emptyset$, one has that $F_n$ is right-nondecreasing on $[\hat{x}, \mu_n(\beta)]_n$ and in $[\hat{y}, \mu_n(\beta)]_n$ for every $\hat{y} \in F_n(\hat{x}) \cap [0, \mu_n(\beta)]_n$.*

*Then, $f$ has a fixed point.*

*Proof.* Since $K$ is normal, $[0,\beta]$ is bounded and hence, $f:[0,\beta]\to K$ is admissibly compact. Moreover,

$$[\hat{x},\mu_n(\beta)]_n\subset \overline{K}_n\backslash B_n(0,r_n/c_n)\quad \forall \hat{x}\in\partial_{\overline{K}_n}B_n(0,r_n).$$

So, (iii) insures that condition (iii') of Corollary 5.2 is satisfied. □

Adding extra assumptions to Theorem 5.1 permits to obtain more precision on the localization of the fixed point.

**Theorem 5.4.** *Let $\beta\in K$, $X\subset K$ closed such that $[0,\beta]\subset X$ and let $f:X\to K$ be an admissibly compact map satisfying conditions* (i)-(iii) *of Theorem* 5.1. *In addition, assume that the following conditions are satisfied:*

(iv) *there exists $V$ a pseudo-open set in $E$ such that, for every $n\in N_0$,*

$$0\in\overline{V}_{\overline{K}_n}\backslash\partial_{\overline{K}_n}V_n\subset\overline{V}_{\overline{K}_n}\subset\overline{U}_{\overline{K}_n}\backslash\partial_{\overline{K}_n}U_n,$$

$$\Big(\textit{resp.}\quad 0\in\overline{U}_{\overline{K}_n}\backslash\partial_{\overline{K}_n}U_n\subset\overline{U}_{\overline{K}_n}\subset\overline{V}_{\overline{K}_n}\backslash\partial_{\overline{K}_n}V_n\subset\overline{V}_{\overline{K}_n}\subset\overline{X}_n\Big);$$

(v) *for every $n\in N_0$, the fixed point index*

$$i_{\overline{K}_n}(F_n,V_n)=0.$$

*Then, $f$ has a fixed point $x^*$ such that*

$$x^*\in\operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}}\widehat{A}(n),$$

*where*

$$\widehat{A}(n)=\Big(\overline{U}_{\overline{K}_n}\backslash V_{\overline{K}_n}\Big)\cup\Big(\bigcup_{x\in\partial_{\overline{K}_n}U_n}[x,\mu_n(\beta)]_n\Big),$$

$$\Big(\textit{resp.}\quad \widehat{A}(n)=\Big(\overline{V}_{\overline{K}_n}\backslash U_{\overline{K}_n}\Big)\cup\Big(\bigcup_{x\in\partial_{\overline{K}_n}U_n}[x,\mu_n(\beta)]_n\Big)\Big).$$

*Proof.* It follows from the proof of Theorem 5.1 that $f$ has a fixed point

$$x\in\operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}}\Big(\bigcup_{x\in\partial_{\overline{K}_n}U_n}[x,\mu_n(\beta)]_n\Big),$$

or, there exists $N_2\subset N_0$ infinite such that $i_{\overline{K}_n}(F_n,U_n)=1$ for every $n\in N_2$. Theorem 2.1(1), (3), and assumptions (iv) and (v) imply that, for every $n\in N_2$,

$$i_{\overline{K}_n}(F_n,U_n\backslash\overline{V}_n)=-1\quad(\text{resp. } i_{\overline{K}_n}(F_n,V_n\backslash\overline{U}_n)=-1).$$

So, there exists

$$z_n\in F_n(z_n)\cap U_{\overline{K}_n}\backslash\overline{V}_{\overline{K}_n}\quad(\text{resp. } z_n\in F_n(z_n)\cap V_{\overline{K}_n}\backslash\overline{U}_{\overline{K}_n}).$$

The conclusion follows from Proposition 2.7. □

**Remark 5.5.** The fixed point obtained in the previous theorem is non trivial if

$$0\notin\operatorname*{Lim}_{\substack{n\to\infty\\ n\in N_0}}\widehat{A}(n).$$

**Remark 5.6.** Even in the particular case where $E$ is a Banach space, Theorem 5.4 generalizes Theorem 2.3 in [6]. In particular, the cone is not assumed to be normal and solid, and no monotonicity condition is imposed on $f$.

**Corollary 5.7.** *Let $\beta \in K$, $X \subset K$ closed such that $[0, \beta] \subset X$ and let $f : X \to K$ be an admissibly compact map satisfying conditions* (i)-(iv) *of Theorem* 5.4. *In addition, assume that*

(v') *for every $n \in N_0$, $\big(x - F_n(x)\big) \cap \overline{K}_n \backslash \{0\} = \emptyset$ for every $x \in \partial_{\overline{K}_n} V_n$.*

*Then, $f$ has a fixed point $x^*$ such that*

$$x^* \in \operatorname*{Lim}_{\substack{n\to\infty \\ n\in N_0}} \widehat{A}(n),$$

*where $\widehat{A}(n)$ is defined in Theorem* 5.4.

*Proof.* Arguing as in the proof of Theorem 3.3, one can show that, for every $n \in N_0$, $F_n$ has a fixed point in $\partial_{\overline{K}_n} V_n$ or

$$i_{\overline{K}_n}(F_n, V_n) = 0.$$

The conclusion follows from Theorem 5.4. □

Condition (iii) in Theorem 5.1 insured that, for suitable $x$, there exists $y \in F_n(x)$ such that $y \leq \mu_n(\beta)$. In the next result, we assume the opposite inequality.

**Theorem 5.8.** *Let $\alpha \in K$, $X \subset K$ closed such that $[0, \alpha] \subset X$ and $f : X \to K$ an admissibly compact map. Assume the following conditions are satisfied:*

(i) *there exists $U$ a bounded pseudo-open set in $E$ such that, for every $n \in \mathbb{N}$, $0 \in \overline{U}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} U_n \subset \overline{U}_{\overline{K}_n} \subset [0, \mu_n(\alpha)]_n$;*

(ii) *the set*

$$N_0 = \{n \in \mathbb{N} : \forall x \in \partial_{\overline{K}_n} U_n,\ (x - F_n(x)) \cap \overline{K}_n = \emptyset \quad \text{or} \quad F_n(x) \subset [x, \infty)_n$$

*is infinite;*

(iii) *there exists $V$ a pseudo-open set in $E$ such that, for every $n \in N_0$,*

$$0 \in \overline{V}_n \backslash \partial_{\overline{K}_n} V_n \subset \overline{V}_{\overline{K}_n} \subset \overline{U}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} U_n,$$
$$\Big(\textit{resp.}\quad 0 \in \overline{U}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} U_n \subset \overline{U}_{\overline{K}_n} \subset \overline{V}_{\overline{K}_n} \backslash \partial_{\overline{K}_n} V_n \subset \overline{V}_{\overline{K}_n} \subset \overline{X}_n\Big);$$

(iv) *for every $n \in N_0$, the fixed point index*

$$i_{\overline{K}_n}(F_n, V_n) = 1.$$

*Then, $f$ has a fixed point $x^*$ such that*

$$x^* \in \operatorname*{Lim}_{\substack{n\to\infty \\ n\in N_0}} \widetilde{A}(n),$$

*where*

$$\widetilde{A}(n) = \Big(\overline{U}_{\overline{K}_n} \backslash V_{\overline{K}_n}\Big) \cup \Big( \bigcup_{x \in \partial_{\overline{K}_n} U_n} [x, \mu_n(\alpha)]_n \Big),$$

$$\Big(\textit{resp.} \quad \widetilde{A}(n) = \Big(\overline{V}_{\overline{K}_n} \backslash U_{\overline{K}_n}\Big) \cup \Big( \bigcup_{x \in \partial_{\overline{K}_n} U_n} [x, \mu_n(\alpha)]_n \Big)\Big).$$

*Proof.* It follows from (i) that, for every $n \in N_0$ and every $x \in \partial_{\overline{K}_n} U_n$, one has $x \preceq_n \mu_n(\alpha)$.

Let

$$N_1 = \Big\{ n \in N_0 : \exists x \in \partial_{\overline{K}_n} U_n \text{ such that } \big(x - F_n(x)\big) \cap \overline{K}_n \neq \emptyset \}.$$

If $N_1$ is infinite, then, for $z_n \in \partial_{\overline{K}_n} U_n$ such that there exists $u \in F_n(z_n)$ with $z_n - u \in \overline{K}_n$, one has $F_n(z_n) \subset [z_n, \infty)_n$. So, $u \preceq_n z_n \preceq_n u$. Thus, $z_n \in F_n(z_n)$ and $f$ has a fixed point

$$x \in \operatorname*{Lim}_{\substack{n \to \infty \\ n \in N_1}} \Big( \bigcup_{x \in \partial_{\overline{K}_n} U_n} [x, \mu_n(\alpha)]_n \Big) \subset \operatorname*{Lim}_{\substack{n \to \infty \\ n \in N_0}} \widetilde{A}(n).$$

On the other hand, if $N_1$ is empty or finite, then $N_2 = N_0 \backslash N_1$ is infinite and, for every $n \in N_2$, $(z - F_n(z)) \cap \overline{K}_n = \emptyset$ for every $z \in \partial_{\overline{K}_n} U_n$. Arguing as in the proof of Theorem 3.3, one deduces that the fixed point index

$$i_{\overline{K}_n}(F_n, U_n) = 0 \quad \forall n \in N_2.$$

Theorem 2.1(3), and assumptions (iii) and (iv) imply that, for every $n \in N_2$,

$$i_{\overline{K}_n}(F_n, U_n \backslash \overline{V}_n) = -1 \quad (\text{resp. } i_{\overline{K}_n}(F_n, V_n \backslash \overline{U}_n) = -1).$$

So, there exists

$$z_n \in F_n(z_n) \cap U_{\overline{K}_n} \backslash \overline{V}_{\overline{K}_n} \quad (\text{resp. } z_n \in F_n(z_n) \cap V_{\overline{K}_n} \backslash \overline{U}_{\overline{K}_n}).$$

The conclusion follows from Proposition 2.7. □

**Remark 5.9.** Even in the particular case where $E$ is a Banach space, Theorem 5.8 generalizes Theorem 2.5 in [6]. Again, the cone is not assumed to be normal and solid, and no monotonicity condition is imposed on $f$.

## References


[1] Agarwal, R.P., O'Regan, D., *Cone compression and expansion fixed point theorems in Fréchet spaces with applications*, J. Differential Equations, **171**(2001), 412-429.

[2] Agarwal, R.P., O'Regan, D., *Fixed points of cone compression and expansion multimaps defined on Fréchet spaces: the projective limit approach*, J. Appl. Math. Stoch. Anal., 2006, Art. ID 92375, 13 pp.

[3] Amann, H., *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Funct. Anal., **11**(1972), 346-384.

[4] Cabada, A., Cid, J.A., *Existence of a non-zero fixed point for nondecreasing operators proved via Krasnoselskiĭ's fixed point theorem*, Nonlinear Anal., **71**(2009), 2114-2118.

[5] Cabada, A., Cid, J.A., Infante, G., *New criteria for the existence of non-trivial fixed points in cones*, Fixed Point Theory Appl., **2013:125** (2013), 12 pp.

[6] Cabada, A., Cid, J.A., Infante, G., *A positive fixed point theorem with applications to systems of Hammerstein integral equations*, Bound. Value Probl., **2014:254** (2014), 10 pp.

[7] Chengbo, Z., Chen, Y., *Some fixed point theorems for multivalued maps in ordered Banach spaces and applications*, Int. J. Math. Math. Sci., **20**(2005), 3247-3259.

[8] Cid, J.A., Franco, D., Minhós, F., *Positive fixed points and fourth-order equations*, Bull. Lond. Math. Soc., **41**(2009), 72-78.

[9] Fitzpatrick, P.M., Petryshyn, M.V., *Fixed point theorems and the fixed point index for multivalued mappings in cones*, J. Lond. Math. Soc., **12**(1975/76), no. 2, 75-85.

[10] Franco, D., Infante, G., Perán, J., *A new criterion for the existence of multiple solutions in cones*, Proc. Roy. Soc. Edinburgh Sect. A, **142**(2012), 1043-1050.

[11] Frigon, M., *Fixed point results for compact maps on closed subsets of Fréchet spaces and applications to differential and integral equations*, Bull. Belg. Math. Soc. Simon Stevin, **9**(2002), 23-37.

[12] Frigon, M., O'Regan, D., *Fixed points of cone-compressing and cone-extending operators in Fréchet spaces*, Bull. Lond. Math. Soc., **35**(2003), 672-680.

[13] Krasnosel'skiĭ, M.A., *Fixed points of cone-compressing and cone-extending operators*, Dokl. Akad. Nauk SSSR, **135** (1960), 527-530 (Russian); English transl.: Soviet Math. Dokl., **1**(1960), 1285-1288.

[14] Petryshyn, W.V., *Existence of fixed points of positive $k$-set contractive maps as consequences of suitable boundary condition*, J. Lond. Math. Soc., **185**(1988), 503-512.



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# Existence and Ulam stability results for Hadamard partial fractional integral inclusions via Picard operators

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*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** In this paper, by using the weakly Picard operators theory, we investigate some existence and Ulam type stability results for a class of Hadamard partial fractional integral inclusions.

**Mathematics Subject Classification (2010):** 26A33, 34G20, 34A40, 45N05, 47H10.

**Keywords:** Hadamard fractional integral inclusion, multivalued weekly Picard operator, fixed point inclusion, Ulam-Hyers stability.

## 1. Introduction

The fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [16, 27, 38]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [1, 3, 4], Kilbas *et al.* [22], Miller and Ross [24], the papers of Abbas *et al.* [2, 5, 6, 7], Vityuk and Golushkov [40], and the references therein. In [10], Butzer *et al.* investigate properties of the Hadamard fractional integral and the derivative. In [11], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators and in [28], Pooseh *et al.* obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [29] and the references therein.

The stability of functional equations was originally raised by Ulam [39] in 1940 and Hyers [17] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [30] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs of [18, 19]. Bota-Boriceanu and Petruşel [9], Petru *et al.* [25, 26], and Rus [31, 32] discussed the Ulam-Hyers stability for operatorial equations and inclusions. Castro and Ramos [12], and Jung [21] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative are proposed by Wang *et al.* [41, 42]. Some stability results for fractional integral equation are obtained by Wei *et al.* [43]. More details from historical point of view, and recent developments of such stabilities are reported in [20, 31, 43].

The theory of Picard operators was introduced by Ioan A. Rus (see [33, 34, 35] and their references) to study problems related to fixed point theory. This abstract approach was used later on by many mathematicians as a very powerful method in the study of integral equations and inequalities, ordinary and partial differential equations (existence, uniqueness, differentiability of the solutions, ...), see [35] and the references therein. The theory of Picard operators is also a very powerful tool in the study of Ulam-Hyers stability of functional equations. We only have to define a fixed point equation from the functional equation we want to study, then if the defined operator is c-weakly Picard we also have immediately the Ulam-Hyers stability of the desired equation. Of course it is not always possible to transform a functional equation or a differential equation into a fixed point problem and actually this point shows a weakness of this theory. The uniform approach with Picard operators to the discuss of the stability problems of Ulam-Hyers type is due to Rus [32].

In [2, 5, 6], Abbas *et al.* studied some Ulam stabilities for functional fractional partial differential and integral inclusions via Picard operators. In this paper, we discuss the Ulam-Hyers and the Ulam-Hyers-Rassias stability for the following new class of fractional partial integral inclusions of the form

$$u(x,y)-\mu(x,y)\in({}^HI^r_\sigma F)(x,y,u(x,y));\ (x,y)\in J:=[1,a]\times[1,b], \tag{1.1}$$

where $a,b>1,\ \sigma=(1,1),\ F:J\times E\to\mathcal{P}(E)$ is a set-valued function with nonempty values in a (real or complex) separable Banach space $E,\ \mathcal{P}(E)$ is the family of all nonempty subsets of $E,\ {}^HI^r_\sigma F$ is the definite Hadamard integral for the set-valued function $F$ of order $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, and $\mu:J\to E$ is a given continuous function.

This paper initiates the existence of the solution and the Ulam stability via Picard operators for such new class of fractional integral inclusions.

## 2. Basic concepts and auxiliary results

Let $L^1(J)$ be the space of Bochner-integrable functions $u:J\to E$ with the norm

$$\|u\|_{L^1}=\int_1^a\int_1^b\|u(x,y)\|_E dydx,$$

where $\|\cdot\|_E$ denotes a complete norm on $E$. By $L^\infty(J)$ we denote the Banach space of measurable functions $u : J \to E$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(x,y)\|_E \leq c,\ a.e.\ (x,y) \in J\}.$$

As usual, by $\mathcal{C} := C(J)$ we denote the Banach space of all continuous functions from $J$ into $E$ with the norm $\|.\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(x,y)\in J} \|u(x,y)\|_E.$$

Let $(X,d)$ be a metric space induced from the normed space $(X, \|.\|)$. Denote
$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
$\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
$\mathcal{P}_{cp}(E) = \{Y \in \mathcal{P}(E) : Y\ \text{ compact}\}$ and
$\mathcal{P}_{cp,cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ compact and convex}\}$.

**Definition 2.1.** *A multivalued map $T : X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$, $T$ is called upper semi-continuous (u.s.c.) on $X$ if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $T(x_0)$, there exists an open neighborhood $N_0$ of $x_0$ such that $T(N_0) \subseteq N$. $T$ is lower semi-continuous (l.s.c.) if the set $\{t \in X : T(t) \cap B \neq \emptyset\}$ is open for any open set $B$ in $X$. $T$ is said to be completely continuous if $T(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. $T$ has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator $T$ will be denoted by $Fix(T)$. The graph of $T$ will be denoted by $Graph(T) := \{(u,v) \in X \times \mathcal{P}(X) : v \in T(u)\}$.*

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,\infty) \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b),\ d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a Hausdorff metric space.

Notice that $A : X \to X$ is a selection for $T : X \to \mathcal{P}(X)$ if $A(u) \in T(u)$; for each $u \in X$. For each $u \in \mathcal{C}$, define the set of selections of the multivalued $F : J \times \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$S_{F,u} = \{v :\in L^1(J) : v(x,y) \in F(x,y,u(x,y));\ (x,y) \in J\}.$$

**Definition 2.2.** *A multivalued map $G : J \to \mathcal{P}_{cl}(E)$, is said to be measurable if for every $v \in E$ the function $(x,y) \to d(v, G(x,y)) = \inf\{d(v,z) : z \in G(x,y)\}$ is measurable.*

In what follows we will give some basic definitions and results on Picard operator theory [35]. Let $(X,d)$ be a metric space and $A : X \to X$ be an operator. We denote by $F_A$ the set of the fixed points of $A$. We also denote $A^0 := 1_X,\ A^1 := A, \ldots, A^{n+1} := A^n \circ A;\ n \in \mathbb{N}$ the iterate operators of the operator $A$.

**Definition 2.3.** *The operator $A : X \to X$ is a Picard operator $(PO)$ if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;
(ii) *The sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to $x^*$ for all $x_0 \in X$.*

**Definition 2.4.** *The operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on $x$) is a fixed point of $A$.*

**Definition 2.5.** *If $A$ is weakly Picard operator then we consider the operator $A^\infty$ defined by*

$$A^\infty : X \to X;\ A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

**Remark 2.6.** *It is clear that $A^\infty(X) = F_A$.*

**Definition 2.7.** *Let $A$ be a weakly Picard operator and $c > 0$. The operator $A$ is c-weakly Picard operator if*

$$d(x, A^\infty(x)) \le c\ d(x, A(x));\ x \in X.$$

In the multivalued case we have the following concepts (see [36]).

**Definition 2.8.** *Let $(X, d)$ be a metric space, and $F : X \to \mathcal{P}_{cl}(X)$ be a multivalued operator. By definition, $F$ is a multivalued weakly Picard operator (MWPO), if for each $u \in X$ and each $v \in F(x)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that*

(i) $u_0 = u,\ u_1 = v$;
(ii) $u_{n+1} \in F(u_n)$, *for each* $n \in \mathbb{N}$;
(iii) *the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of $F$.*

**Remark 2.9.** *A sequence $(u_n)_{n\in\mathbb{N}}$ satisfying condition (i) and (ii) in the above Definition is called a sequence of successive approximations of $F$ starting from $(x, y) \in Graph(F)$.*

If $F : X \to \mathcal{P}_{cl}(X)$ is a (MWPO) then we define $F_1 : Graph(F) \to \mathcal{P}(Fix(F))$ by the formula $F_1(x, y) := \{u \in Fix(F)$ : there exists a sequence of successive approximations of $F$ starting from $(x, y)$ that converges to $u\}$.

**Definition 2.10.** *Let $(X, d)$ be a metric space and let $\Psi : [0, \infty) \to [0, \infty)$ be an increasing function which is continuous at 0 and $\Psi(0) = 0$. Then $F : X \to \mathcal{P}_{cl}(X)$ is said to be a multivalued $\Psi-$weakly Picard operator ($\Psi-MWPO$) if it is a multivalued weakly Picard operator and there exists a selection $A^\infty : Graph(F) \to Fix(F)$ of $F^\infty$ such that*

$$d(u, A^\infty(u, v)) \le \Psi(d(u, v));\ for\ all\ (u, v) \in Graph(F).$$

*If there exists $c > 0$ such that $\Psi(t) = ct$, for each $t \in [0.\infty)$, then $F$ is called a multivalued c-weakly Picard operator ($c - MWPO$).*

Let us recall the notion of comparison function.

**Definition 2.11.** *A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a comparison function (see [35]) if it is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0$.*

As a consequence, we have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and $\varphi$ is continuous at 0.

**Definition 2.12.** *A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a strict comparison function (see [35]) if it is strictly increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, for each $t > 0$.*

**Example 2.13.** *The mappings* $\varphi_1, \varphi_2 : [0, \infty) \to [0, \infty)$ *given by* $\varphi_1(t) = ct;\ c \in [0, 1)$, *and* $\varphi_2(t) = \frac{t}{1+t};\ t \in [0, \infty)$, *are strict comparison functions.*

**Definition 2.14.** *A multivalued operator* $N : X \to \mathcal{P}_{cl}(X)$ *is called*

a) $\gamma$*-Lipschitz if and only if there exists* $\gamma \geq 0$ *such that*

$$H_d(N(u), N(v)) \leq \gamma d(u, v);\ \textit{for each}\ \ u,\ v \in X,$$

b) *a multivalued* $\gamma-$*contraction if and only if it is* $\gamma$*-Lipschitz with* $\gamma \in [0, 1)$,

c) *a multivalued* $\varphi-$*contraction if and only if there exists a strict comparison function* $\varphi : [0, \infty) \to [0, \infty)$ *such that*

$$H_d(N(u), N(v)) \leq \varphi(d(u, v));\ \textit{for each}\ \ u,\ v \in X.$$

Now, we introduce notations and definitions concerning to partial Hadamard integral of fractional order.

**Definition 2.15.** [15, 22] *The Hadamard fractional integral of order* $q > 0$ *for a function* $g \in L^1([1, a], \mathbb{R})$, *is defined as*

$$({}^H I_1^r g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

*where* $\Gamma(\cdot)$ *is the Euler gamma function.*

**Definition 2.16.** *Let* $r_1,\ r_2 \geq 0,\ \ \sigma = (1, 1)$ *and* $r = (r_1, r_2)$. *For* $w \in L^1(J, \mathbb{R})$, *define the Hadamard partial fractional integral of order* $r$ *by the expression*

$$({}^H I_\sigma^r w)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{w(s, t)}{st} dtds.$$

**Definition 2.17.** *Let* $F : J \times E \to \mathcal{P}(E)$ *be a set-valued function with nonempty values in* $E$. $({}^H I_\sigma^r F)(x, y, u(x, y))$ *is the definite Hadamard integral for the set-valued functions* $F$ *of order* $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$ *which is defined as*

$${}^H I_\sigma^r F(x, y, u(x, y)) = \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t)}{st\Gamma(r_1)\Gamma(r_2)} dtds : f \in S_{F,u} \right\}.$$

**Remark 2.18.** *Solutions of the inclusion (*1.1*) are solutions of the fixed point inclusion* $u \in N(u)$ *where* $N : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ *is the multivalued operator defined by*

$$(Nu)(x, y) = \left\{\mu(x, y) + ({}^H I_\sigma^r f)(x, y) :\ f \in S_{F,u}\right\};\ (x, y) \in J.$$

Let us give the definition of Ulam-Hyers stability of the fixed point inclusion $u \in N(u)$, see for instance [2]. Let $\epsilon$ be a positive real number and $\Phi : J \to [0, \infty)$ be a continuous function.

**Definition 2.19.** *The fixed point inclusion* $u \in N(u)$ *is said to be Ulam-Hyers stable if there exists a real number* $c_N > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $u \in \mathcal{C}$ *of the inequality* $H_d(u(x, y), (Nu)(x, y)) \leq \epsilon;\ (x, y) \in J$, *there exists a solution* $v \in \mathcal{C}$ *of the inclusion* $u \in N(u)$ *with*

$$\|u(x, y) - v(x, y)\|_E \leq \epsilon c_N;\ (x, y) \in J.$$

**Definition 2.20.** *The fixed point inclusion $u \in N(u)$ is said to be generalized Ulam-Hyers stable if there exists an increasing function $\theta_N \in C([0,\infty),[0,\infty))$, $\theta_N(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y),(Nu)(x,y)) \leq \epsilon$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with*

$$\|u(x,y) - v(x,y)\|_E \leq \theta_N(\epsilon);\ (x,y) \in J.$$

**Definition 2.21.** *The fixed point inclusion $u \in N(u)$ is said to be Ulam-Hyers-Rassias stable with respect to $\Phi$ if there exists a real number $c_{N,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y),(Nu)(x,y)) \leq \epsilon\Phi(x,y)$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with*

$$\|u(x,y) - v(x,y)\|_E \leq \epsilon c_{N,\Phi}\Phi(x,y);\ (x,y) \in J.$$

**Definition 2.22.** *The fixed point inclusion $u \in N(u)$ is said to be generalized Ulam-Hyers-Rassias stable with respect to $\Phi$ if there exists a real number $c_{N,\Phi} > 0$ such that for each solution $u \in \mathcal{C}$ of the inequality $H_d(u(x,y),(Nu)(x,y)) \leq \Phi(x,y)$; $(x,y) \in J$, there exists a solution $v \in \mathcal{C}$ of the inclusion $u \in N(u)$ with*

$$\|u(x,y) - v(x,y)\|_E \leq c_{N,\Phi}\Phi(x,y);\ (x,y) \in J.$$

**Remark 2.23.** *It is clear that*

(i) *Definition* 2.19 $\Rightarrow$ *Definition* 2.20*,*
(ii) *Definition* 2.21 $\Rightarrow$ *Definition* 2.22*,*
(iii) *Definition* 2.21 *for* $\Phi(x,y) = 1 \ \Rightarrow$ *Definition* 2.19.

The following result, a generalization of Covitz-Nadler fixed point principle (see [14]), is known in the literature as Węgrzyk's fixed point theorem.

**Lemma 2.24.** [44] *Let $(X,d)$ be a complete metric space. If $A : X \to \mathcal{P}_{cl}(X)$ is a $\varphi-$contraction, then $Fix(A)$ is nonempty and for any $u_0 \in X$, there exists a sequence of successive approximations of $A$ starting from $u_0$ which converges to a fixed point of $A$.*

Also, the following result is known in the literature as Węgrzyk's theorem.

**Lemma 2.25.** [44] *Let $(X,d)$ be a Banach space. If an operator $A : X \to \mathcal{P}_{cl}(X)$ is a $\varphi-$contraction, then $A$ is a $(MWPO)$.*

Now we present an important characterization Lemma from the point of view of Ulam-Hyers stability.

**Lemma 2.26.** [26] *Let $(X,d)$ be a metric space. If $A : X \to \mathcal{P}_{cp}(X)$ is a $(\Psi-MWPO)$, then the fixed point inclusion $u \in A(u)$ is generalized Ulam-Hyers stable. In particular, if $A$ is $(c-MWPO)$, then the fixed point inclusion $u \in A(u)$ is Ulam-Hyers stable.*

Another Ulam-Hyers stability result, more efficient for applications, was proved in [23].

**Theorem 2.27.** [23] *Let $(X,d)$ be a complete metric space and $A : X \to \mathcal{P}_{cp}(X)$ be a multivalued $\varphi-$contraction. Then:*

$(i)$ Existence of the fixed point: *$A$ is a $(MWPO)$;*

$(ii)$ Ulam-Hyers stability for the fixed point inclusion: *If additionally $\varphi(ct) \leq c\varphi(t)$ for every $t \in [0, \infty)$ (where $c > 1$), then $A$ is a ($\Psi-MWPO$), with $\Psi(t) := t + \sum_{n=1}^{\infty} \varphi^n(t)$, for each $t \in [0, \infty)$;*

$(iii)$ Data dependence of the fixed point set: *Let $S : X \to \mathcal{P}_{cl}(X)$ be a multivalued $\varphi-$contraction and $\eta > 0$ be such that $H_d(S(x), A(x)) \leq \eta$, for each $x \in X$. Suppose that $\varphi(ct) \leq c\varphi(t)$ for every $t \in [0, \infty)$ (where $c > 1$). Then,*

$$H_d(Fix(S), Fix(F)) \leq \Psi(\eta).$$

## 3. Existence and Ulam-Hyers stability results

In this section, we present conditions for the existence and the Ulam stability of the Hadamard integral inclusion (1.1).

**Theorem 3.1.** *Assume that the multifunction $F : J \times E \to \mathcal{P}_{cp}(E)$ satisfies the following hypotheses:*

$(H_1)$ *$(x, y) \longmapsto F(x, y, u)$ is jointly measurable for each $u \in E$;*

$(H_2)$ *$u \longmapsto F(x, y, u)$ is lower semicontinuous for almost all $(x, y) \in J$;*

$(H_3)$ *There exists $p \in L^\infty(J, [0, \infty))$ and a strict comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that for each $(x, y) \in J$ and each $u, v \in E$, we have*

$$H_d(F(x, y, u(x, y), F(x, y, \overline{u})\| \leq p(x, y)\varphi(\|u - \overline{u}\|_E), \tag{3.1}$$

*and*

$$\frac{(\log a)^{r_1}(\log b)^{r_2}\|p\|_{L^\infty}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \leq 1; \tag{3.2}$$

$(H_4)$ *There exists an integrable function $q : [1, b] \to [0, \infty)$ such that for each $x \in [1, a]$ and $u \in E$, we have $F(x, y, u) \subset q(y)B(0, 1)$, a.e. $y \in [1, b]$, where $B(0, 1) = \{u \in E : \|u\|_E < 1\}$.*

*Then the following conclusions hold:*

$(a)$ *The integral inclusion (1.1) has least one solution and $N$ is a $(MWPO)$.*

$(b)$ *If additionally $\varphi(ct) \leq c\varphi(t)$ for every $t \in [0, \infty)$ (where $c > 1$), then the integral inclusion (1.1) is generalized Ulam-Hyers stable, and $N$ is a ($\Psi-MWPO$), with the function $\Psi$ defined by $\Psi(t) := t + \sum_{n=1}^{\infty} \varphi^n(t)$, for each $t \in [0, \infty)$. Moreover, in this case the continuous data dependence of the solution set of the integral inclusion (3.1) holds.*

**Remark 3.2.** *For each $u \in \mathcal{C}$, the set $S_{F,u}$ is nonempty since by $(H_1)$, $F$ has a measurable selection (see* [13], *Theorem III.6).*

*Proof.* We shall show that $N$ defined in Remark 2.18 satisfies the assumptions of Theorem 2.27. The proof will be given in two steps.

**Step 1.** $N(u) \in P_{cp}(\mathcal{C})$ *for each* $u \in \mathcal{C}$.

From the continuity of $\mu$ and Theorem 2 in Rybiński [37] we have that for each $u \in \mathcal{C}$

there exists $f \in S_{F,u}$, for all $(x,y) \in J$, such that $f(x,y)$ is integrable with respect to $y$ and continuous with respect to $x$. Then the function $v(x,y) = \mu(x,y) +^H I^r_\sigma f(x,y)$ has the property $v \in N(u)$. Moreover, from $(H_1)$ and $(H_4)$, via Theorem 8.6.3. in Aubin and Frankowska [8], we get that $N(u)$ is a compact set, for each $u \in \mathcal{C}$.

**Step 2.** $H_d(N(u), N(\overline{u})) \leq \varphi(\|u - \overline{u}\|_\infty)$ *for each* $u, \overline{u} \in \mathcal{C}$.

Let $u, \overline{u} \in \mathcal{C}$ and $h \in N(u)$. Then, there exists $f(x,y) \in F(x,y,u(x,y))$ such that for each $(x,y) \in J$, we have

$$h(x,y) = \mu(x,y) +^H I^r_\sigma f(x,y).$$

From $(H_3)$ it follows that

$$H_d(F(x,y,u(x,y)), F(x,y,\overline{u}(x,y))) \leq p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E).$$

Hence, there exists $w(x,y) \in F(x,y,\overline{u}(x,y)$ such that

$$\|f(x,y) - w(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E);\ (x,y) \in J.$$

Consider $U : J \to \mathcal{P}(E)$ given by

$$U(x,y) = \{w \in E : \|f(x,y) - w(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E)\}.$$

Since the multivalued operator $u(x,y) = U(x,y) \cap F(x,y,\overline{u}(x,y))$ is measurable (see Proposition III.4 in [13]), there exists a function $\overline{f}(x,y)$ which is a measurable selection for $u$. So, $\overline{f}(x,y) \in F(x,y,\overline{u}(x,y))$, and for each $(x,y) \in J$,

$$\|f(x,y) - \overline{f}(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E).$$

Let us define for each $(x,y) \in J$,

$$\overline{h}(x,y) = \mu(x,y) +^H I^r_\sigma \overline{f}(x,y).$$

Then for each $(x,y) \in J$, we have

$$\begin{aligned}
\|h(x,y) - \overline{h}(x,y)\|_E &\leq {}^H I^r_\sigma \|f(x,y) - \overline{f}(x,y)\|_E \\
&\leq {}^H I^r_\sigma (p(x,y)\varphi(\|u(x,y) - \overline{u}(x,y)\|_E)) \\
&\leq \|p\|_{L^\infty}\varphi(\|u - \overline{u}\|_\infty)\left(\int_1^x \int_1^y \frac{\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}}{st\Gamma(r_1)\Gamma(r_2)} dtds\right) \\
&\leq \frac{(\log a)^{r_1}(\log b)^{r_2}\|p\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}\varphi(\|u - \overline{u}\|_\infty).
\end{aligned}$$

Thus, by (3.2), we get

$$\|h - \overline{h}\|_\infty \leq \varphi(\|u - \overline{u}\|_\infty).$$

By an analogous relation, obtained by interchanging the roles of $u$ and $\overline{u}$, it follows that

$$H_d(N(u), N(\overline{u})) \leq \varphi(\|u - \overline{u}\|_\infty).$$

Hence, $N$ is a $\varphi-$contraction.

**(a)** By Lemma 2.24, $N$ has a fixed point witch is a solution of the inclusion (1.1) on $J$, and by [Theorem 2.27,(i)], $N$ is a $(MWPO)$.

**(b)** We will prove that the fixed point inclusion problem (1.1) is generalized Ulam-Hyers stable. Indeed, let $\epsilon > 0$ and $v \in \mathcal{C}$ for which there exists $u \in \mathcal{C}$ such that

$$u(x,y) \in \mu(x,y) + ({}^H I_\sigma^r F)(x,y,v(x,y)); \quad \text{if } (x,y) \in J,$$

and

$$\|u - v\|_\infty \leq \epsilon.$$

Then $H_d(v, N(v)) \leq \epsilon$. Moreover, by the above proof we have that $N$ is a multivalued $\varphi-$contraction and using [Theorem 2.27,(i)-(ii)], we obtain that $N$ is a is a $(\Psi - MWPO)$. Then, by Lemma 2.26 we obtain that the fixed point problem $u \in N(u)$ is generalized Ulam-Hyers stable. Thus, the integral inclusion (1.1) is generalized Ulam-Hyers stable.
Concerning the conclusion of the theorem, we apply [Theorem 2.27,(iii)].

## 4. An example

Let $E = l^1 = \left\{ w = (w_1, w_2, \ldots, w_n, \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty \right\}$, be the Banach space with norm

$$\|w\|_E = \sum_{n=1}^{\infty} |w_n|,$$

and consider the following partial functional fractional order integral inclusion of the form

$$u(x,y) \in \mu(x,y) + ({}^H I_\sigma^r F)(x,y,u(x,y)); \text{ a.e. } (x,y) \in [1,e] \times [1,e], \tag{4.1}$$

where $r = (r_1, r_2), \ r_1, r_2 \in (0, \infty)$,

$$u = (u_1, u_2, \ldots, u_n, \ldots), \ \mu(x,y) = (x + e^{-y}, 0, \ldots, 0, \ldots),$$

and

$$F(x,y,u(x,y))$$
$$= \{v \in C([1,e] \times [1,e], \mathbb{R}) : \|f_1(x,y,u(x,y))\|_E \leq \|v\|_E \leq \|f_2(x,y,u(x,y))\|_E\};$$

$(x,y) \in [1,e] \times [1,e]$, where $f_1, f_2 : [1,e] \times [1,e] \times E \to E$,

$$f_k = (f_{k,1}, f_{k,2}, \ldots, f_{k,n}, \ldots); \ k \in \{1,2\}, \ n \in \mathbb{N},$$

$$f_{1,n}(x,y,u_n(x,y)) = \frac{xy^2 u_n}{(1 + \|u_n\|_E)e^{10+x+y}}; \ n \in \mathbb{N},$$

and

$$f_{2,n}(x,y,u_n(x,y)) = \frac{xy^2 u_n}{e^{10+x+y}}; \ n \in \mathbb{N}.$$

We assume that $F$ is closed and convex valued. We can see that the solutions of the inclusion(4.1) are solutions of the fixed point inclusion $u \in A(u)$ where $A : C([1,e] \times [1,e], \mathbb{R}) \to \mathcal{P}(C([1,e] \times [1,e], \mathbb{R}))$ is the multifunction operator defined by

$$(Au)(x,y) = \left\{ \mu(x,y) + ({}^H I_\sigma^r f)(x,y); \ f \in S_{F,u} \right\}; \ (x,y) \in [1,e] \times [1,e].$$

For each $(x,y) \in [1,e] \times [1,e]$ and all $z_1, z_2 \in E$, we have

$$\|f_2(x,y,z_2) - f_1(x,y,z_1)\|_E \le xy^2 e^{-10-x-y}\|z_2 - z_1\|_E.$$

Thus, the hypotheses $(H_1) - (H_3)$ are satisfied with $p(x,y) = xy^2e^{-10-x-y}$. We shall show that condition (3.2) holds with $a = b = e$. Indeed, $\|p\|_{L^\infty} = e^{-9}$, $\Gamma(1+r_i) > \frac{1}{2}$; $i = 1,2$. A simple computation shows that

$$\zeta := \frac{(\log a)^{r_1}(\log b)^{r_2}\|p\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 4e^{-9} < 1.$$

The condition $(H_4)$ is satisfied with $q(y) = \frac{y^2e^{-10-y}}{\|F\|_{\mathcal{P}}}$; $y \in [1,e]$, where

$$\|F\|_{\mathcal{P}} = \sup\{\|f\|_{\mathcal{C}} : f \in S_{F,u}\}; \quad \text{for all}\, u \in \mathcal{C}.$$

Consequently, by Theorem 3.1 we concluded that:

($a$) The integral inclusion (4.1) has least one solution and $A$ is a $(MWPO)$.

($b$) The function $\varphi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) = \zeta t$ satisfies $\varphi(\zeta t) \le \zeta\varphi(t)$ for every $t \in [0,\infty)$. Then the integral inclusion (4.1) is generalized Ulam-Hyers stable, and $A$ is a ($\Psi-$MWPO), with the function $\Psi$ defined by $\Psi(t) := t + (1-\zeta t)^{-1}$, for each $t \in [0, \zeta^{-1})$. Moreover, the continuous data dependence of the solution set of the integral inclusion (3.1) holds.

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## References

[1] Abbas, S., and Benchohra, M., *Advanced Functional Evolution Equations and Inclusions*, Developments in Mathematics, 39, Springer, New York, 2015.

[2] Abbas, S., Benchohra, M., Henderson, J., *Ulam stability for partial fractional integral inclusions via Picard operators*, J. Frac. Calc. Appl., **5**(2014), 133-144.

[3] Abbas, S., Benchohra, M., N'Guérékata, G.M., *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[4] Abbas, S., Benchohra, M., N'Guérékata, G.M., *Topics in Fractional Differential Equations*, Developments in Mathematics, 27, Springer, New York, 2012.

[5] Abbas, S., Benchohra, M., Petruşel, A., *Ulam stabilities for the Darboux problem for partial fractional differential inclusions via Picard operators*, Electron. J. Qual. Theory Differ. Eq., **2014**, no. 51, 13 pp.

[6] Abbas, S., Benchohra, M., Sivasundaram, S., *Ulam stability for partial fractional differential inclusions with multiple delay and impulses via Picard operators*, J. Nonlinear Stud., **20**(2013), no. 4, 623-641.

[7] Abbas, S., Benchohra, M., Trujillo, J.J., *Upper and lower solutions method for partial fractional differential inclusions with not instantaneous impulses*, Prog. Frac. Diff. Appl., **1**(2015), no. 1, 11-22.

[8] Aubin, J.P., Frankowska, H., *Set-Valued Analysis,* Birkhauser, Basel, 1990.

[9] Bota-Boriceanu, M.F., Petruşel, A., *Ulam-Hyers stability for operatorial equations and inclusions*, Analele Univ. I. Cuza Iaşi, **57**(2011), 65-74.

[10] Butzer, P.L., Kilbas, A.A., Trujillo, J.J., *Fractional calculus in the mellin setting and Hadamard-type fractional integrals*, J. Math. Anal. Appl., **269**(2002), 1-27.

[11] Butzer, P.L., Kilbas, A.A., Trujillo, J.J., *Mellin transform analysis and integration by parts for hadamard-type fractional integrals*, J. Math. Anal. Appl., **270**(2002), 1-15.

[12] Castro, L.P., Ramos, A., *Hyers-Ulam-Rassias stability for a class of Volterra integral equations*, Banach J. Math. Anal., **3**(2009), 36-43.

[13] Castaing, C., Valadier, M., *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[14] Covitz, H., Nadler Jr., S.B., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., **8**(1970), 5-11.

[15] Hadamard, J., *Essai sur l'étude des fonctions données par leur développment de Taylor*, J. Pure Appl. Math., **4**(8)(1892), 101-186.

[16] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[17] Hyers, D.H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., **27**(1941), 222-224.

[18] Hyers, D.H., Isac, G., Rassias, Th.M., *Stability of Functional Equations in Several Variables*, Birkhäuser, 1998.

[19] Jung, S.M., *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[20] Jung, S.M., *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[21] Jung, S.M., *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory Appl., **2007**(2007), Article ID 57064, 9 pages.

[22] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[23] Lazăr, V.L., *Fixed point theory for multivalued $\varphi-$contractions*, Fixed Point Theory Appl., 2011, 2011:50, 12 pp.

[24] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[25] Petru, T.P., Bota, M.F., *Ulam-Hyers stabillity of operational inclusions in complete gauge spaces*, Fixed Point Theory, **13**(2012), 641-650.

[26] Petru, T.P., Petruşel, A., Yao, J.C., *Ulam-Hyers stability for operatorial equations and inclusions via nonself operators*, Taiwanese J. Math., **15**(2011), 2169-2193.

[27] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[28] Pooseh, S., Almeida, R., Torres, D., *Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative*, Numer. Funct. Anal. Optim., **33**(2012), no. 3, 301-319.

[29] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[30] Rassias, Th.M., *On the stability of linear mappings in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[31] Rus, I.A., *Ulam stability of ordinary differential equations*, Studia Univ. Babeş-Bolyai Math., **54**(2009), no. 4, 125-133.

[32] Rus, I.A., *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10**(2009), 305-320.

[33] Rus, I.A., *Fixed points, upper and lower fixed points: abstract Gronwall lemmas*, Carpathian J. Math., **20**(2004), 125-134.

[34] Rus, I.A., *Picard operators and applications*, Sci. Math. Jpn., **58**(2003), 191-219.

[35] Rus, I.A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[36] Rus, I.A., Petruşel, A., Sîtămărian, A., *Data dependence of the fixed points set of some multivalued weakly Picard operators*, Nonlinear Anal., **52**(2003), 1947-1959.

[37] Rybinski, L., *On Carathédory type selections*, Fund. Math., **125**(1985), 187-193.

[38] Tarasov, V.E., *Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg, 2010.

[39] Ulam, S.M., *A Collection of Mathematical Problems*, Interscience Publ., New York, 1968.

[40] Vityuk, A.N., Golushkov, A.V., *Existence of solutions of systems of partial differential equations of fractional order*, Nonlinear Oscil., **7**(2004), no. 3, 318-325.

[41] Wang, J., Lv, L., Zhou, Y., *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, Electron. J. Qual. Theory Differ. Eq., **2011**, no. 63, 10 pp.

[42] Wang, J., Lv, L., Zhou, Y., *New concepts and results in stability of fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., **17**(2012), 2530-2538.

[43] Wei, W., Li, X., Li, X., *New stability results for fractional integral equation*, Comput. Math. Appl., **64**(2012), 3468-3476.

[44] Węgrzyk, R., *Fixed point theorems for multifunctions and their applications to functional equations*, Dissertationes Math. (Rozprawy Mat.), **201**(1982), 28 pp.

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# Hybrid differential equations with maxima via Picard operators theory

Diana Otrocol

*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** The aim of this paper is to discuss some basic problems (existence and uniqueness, data dependence) of the Cauchy problem for a hybrid differential equation with maxima using weakly Picard operators technique.

**Mathematics Subject Classification (2010):** 47H10, 34K05.

**Keywords:** Differential equations with maxima, Cauchy problem, data dependence, weakly Picard operators.

## 1. Introduction

Recently, the interest in differential equations with "maxima" has increased exponentially. Such equations model real world problems whose present state depends significantly on its maximum value on a past time interval. For example, many problems in the control theory correspond to the maximal deviation of the regulated quantity. Some qualitative properties of the solutions of ordinary differential equations with "maxima" can be found in [1, 2, 5], [16, 17] and the references therein.

The main goal of the presented paper is to study a hybrid differential equation with maxima, using the theory of weakly Picard operators. The theory of Picard operators was introduced by I. A. Rus (see [12], [14] and their references) to study problems related to fixed point theory. This abstract approach is used by many mathematicians and it seemed to be a very useful and powerful method in the study of integral equations and inequalities, ordinary and partial differential equations (existence, uniqueness, differentiability of the solutions), etc.

In this paper we consider the following hybrid differential equation with maxima

$$x'(t) = f(t, x(t)) + g(t, \max_{a \leq \xi \leq t} x(\xi)), \tag{1.1}$$

with initial condition

$$x(a) = x_0, \tag{1.2}$$

where $t \in [a,b],\ a,b \in \mathbb{R}, x_0 \in \mathbb{R}^m,\ f,g : [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$.

We use the terminologies and notations from [12] and [14]. For the convenience of the reader we recall some of them.

Let $(X,d)$ be a metric space and $A : X \to X$ an operator. We denote by $A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A^n \circ A,\ \ n \in \mathbb{N}$, the iterate operators of the operator $A$. We also have:

$$P(X) := \{Y \subseteq X \mid Y \neq \phi\},$$
$$F_A := \{x \in X \mid A(x) = x\},$$
$$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}.$$

**Definition 1.1.** *Let $(X,d)$ be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to $x^*$, for all $x_0 \in X$.*

**Definition 1.2.** *Let $(X,d)$ be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on $x$) is a fixed point of $A$.*

**Definition 1.3.** *If $A$ is weakly Picard operator then we consider the operator $A^\infty$ defined by $A^\infty : X \to X,\ A^\infty(x) := \lim\limits_{n\to\infty} A^n(x)$.*

Obviously, $A^\infty(X) = F_A$. Moreover, if $A$ is a PO and we denote by $x^*$ its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

## 2. Existence and uniqueness

We prove the existence and uniqueness for the solution of the problem (1.1)-(1.2) using the Perov's Theorem as in [7]. For standard techniques, when it is used the Banach contraction principle, see [13], [9] and [10].

**Theorem 2.1.** *(Perov's fixed point theorem) Let $(X,d)$ with $d(x,y) \in \mathbb{R}^m$, be a complete generalized metric space and $A : X \to X$ an operator. We suppose that there exists a matrix $Q \in M_{m\times m}(\mathbb{R}_+)$, such that*

(i) *$d(A(x),A(y)) \le Qd(x,y)$, for all $x,y \in X$;*
(ii) *$Q^n \to 0$, as $n \to \infty$.*

*Then*

(a) *$F_A = \{x^*\}$,*
(b) *$A^n(x) \to x^*$, as $n \to \infty$ and*

$$d(A^n(x),x^*) \le (I-Q)^{-1}Q^n d(x_0,A(x_0)),\ \forall x_0,x \in X, \forall n \in \mathbb{N}^*;$$

(c) *$d(x,x^*) \le (I-Q)^{-1}d(x,A(x)),\ \forall x \in X$.*

We consider on $\mathbb{R}^m$ the following vectorial norm

$$|x| := \begin{pmatrix} |x_1| \\ \vdots \\ |x_m| \end{pmatrix}.$$

We have the following result:

**Theorem 2.2.** *We assume that:*

*(i)* $f, g \in C([a,b] \times \mathbb{R}^m, \mathbb{R}^m)$;

*(ii) there exist* $L_f$ *and* $L_g$ *nonnegative matrices such that*

$$\left|f(t,u^1) - f(t,u^2)\right| \leq L_f \left|u^1 - u^2\right|,$$
$$\left|g(t,v^1) - g(t,v^2)\right| \leq L_g \left|v^1 - v^2\right|,$$

$\forall\, t \in [a,b]$ *and* $u^1 = (u_1^1, \ldots, u_m^1)$, $u^2 = (u_1^2, \ldots, u_m^2)$, $v^1 = (v_1^1, \ldots, v_m^1)$, $v^2 = (v_1^2, \ldots, v_m^2) \in \mathbb{R}^m$;

*(iii) the matrix*

$$Q := (b-a)(L_f + L_g) \tag{2.1}$$

*is convergent to 0, i.e.* $Q^n \to 0$, *as* $n \to \infty$.

*Then, the problem* (1.1)*-*(1.2) *has a unique solution* $x^* \in C([a,b], \mathbb{R}^m)$.

*Proof.* We consider the generalized Banach space $X = (C([a,b], \mathbb{R}^m), \|\cdot\|)$ where $\|\cdot\|$ is the norm,

$$\|x\| := \begin{pmatrix} \max\limits_{a \leq t \leq b} |x_1(t)| \\ \vdots \\ \max\limits_{a \leq t \leq b} |x_m(t)| \end{pmatrix}. \tag{2.2}$$

The problem (1.1)-(1.2), $x \in C^1([a,b], \mathbb{R}^m)$ is equivalent with the following fixed point equation

$$x(t) = x_0 + \int_a^t f(s, x(s))ds + \int_a^t g(s, \max_{a \leq \xi \leq s} x(\xi))ds, \ t \in [a,b]. \tag{2.3}$$

We consider the operator $A : X \to X$, where

$$A(x)(t) = x_0 + \int_a^t f(s, x(s))ds + \int_a^t g(s, \max_{a \leq \xi \leq s} x(\xi))ds. \tag{2.4}$$

It is easy to see that if $x^* \in F_A$ then $x^*$ is a solution of (1.1)-(1.2).

Condition (ii) implies that

$$|A(x)(t)-A(y)(t)|$$
$$\leq \int_a^t |f(s,x(s))-f(s,y(s))|\,ds+\int_a^t \left|g(s,\max_{a\leq\xi\leq s}x(\xi))-g(s,\max_{a\leq\xi\leq s}y(\xi))\right|ds$$
$$\leq (b-a)L_f\begin{pmatrix}\max\limits_{a\leq s\leq b}|x_1(s)-y_1(s)|\\ \vdots\\ \max\limits_{a\leq s\leq b}|x_m(s)-y_m(s)|\end{pmatrix}$$
$$+(b-a)L_g\begin{pmatrix}\max\limits_{a\leq s\leq b}\left|\max\limits_{a\leq\xi\leq s}x_1(s)-\max\limits_{a\leq\xi\leq s}y_1(s)\right|\\ \vdots\\ \max\limits_{a\leq s\leq b}\left|\max\limits_{a\leq\xi\leq s}x_m(s)-\max\limits_{a\leq\xi\leq s}y_m(s)\right|\end{pmatrix}.$$

But

$$\max_{a\leq s\leq b}\left|\max_{a\leq\xi\leq s}x_i(s)-\max_{a\leq\xi\leq s}y_i(s)\right|\leq \max_{a\leq s\leq b}|x_i(s)-y_i(s)|\,.$$

So,

$$\|A(x)-A(y)\|\leq Q\,\|x-y\|\,.$$

Using (iii), we get that the operator $A:X\to X$ is a $Q$-contraction, so

$$F_A=(x_1^*,\ldots,x_m^*)=x^*$$

is the unique solution of (1.1)-(1.2). □

The equation (1.1) is equivalent with

$$x(t)=x(a)+\int_a^t f(s,x(s))ds+\int_a^t g(s,\max_{a\leq\xi\leq s}x(\xi))ds,\ t\in[a,b], \tag{2.5}$$

$x\in C([a,b],\mathbb{R}^m)$.

In what follows we consider the operator $B:X\to X$ defined by $B(x)(t):=$the right hand side of (2.5). For $x_0\in\mathbb{R}^m$, we consider

$$X_{x_0}:=\{x\in C([a,b],\mathbb{R}^m)|\ x(a)=x_0\}.$$

It is clear that

$$X=\underset{x_0\in\mathbb{R}^m}{\cup}X_{x_0}$$

is a partition of $X$. We have

**Lemma 2.3.** *We suppose that the condition $(C_1)$ is satisfied. Then*
*(a)* $A(X)\subset X_{x_0}$ *and* $A(X_{x_0})\subset X_{x_0}$;
*(b)* $A|_{X_{x_0}}=B|_{X_{x_0}}$.

**Remark 2.4.** From Theorem 2.2 we have that the operator $A$ is PO. Because $A|_{X_{x_0}} = B|_{X_{x_0}}$, $X := C([a,b],\mathbb{R}^m) = \underset{x_0\in\mathbb{R}^m}{\cup} X_{x_0}, X_{x_0} \in I(B)$ it follows that the operator $B$ is WPO and

$$F_B \cap X_{x_0} = \{x^*\},\ \forall x_0 \in \mathbb{R}^m,$$

where $x^*$ is the unique solution of the problem (1.1)-(1.2).

## 3. Data dependence: comparison results

Now we consider the operators $A$ and $B$ on the ordered Banach space $(C([a,b],\mathbb{R}^m), \|\cdot\|, \leq)$ where the order relation on $\mathbb{R}^m$ is given by: $x \leq y \Leftrightarrow x_i \leq y_i$, $i = \overline{1,m}$.

In order to establish the Čaplygin type inequalities we need the following abstract result.

**Lemma 3.1.** *(see [14]) Let $(X,d,\leq)$ be an ordered metric space and $A : X \to X$ an operator. Suppose that $A$ is increasing and WPO. Then the operator $A^\infty$ is increasing.*

We have the following result

**Theorem 3.2.** *Suppose that:*

*(a) the conditions of Theorem 2.2 are satisfied;*

*(b) $f(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m, g(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ are increasing, $\forall t \in [a,b]$.*

*Let $x^*$ be a solution of equation (1.1) and $y^*$ a solution of the inequality*

$$y'(t) \leq f(t,y(t)) + g(t, \max_{a\leq\xi\leq t} y(\xi)),\ t \in [a,b].$$

*Then $y^*(a) \leq x^*(a)$ implies that $y \leq x$.*

*Proof.* From Remark 2.4 we have that $B$ is WPO. On the other hand, from the condition (b) and Lemma 3.1 we get that the operator $B^\infty$ is increasing. If $x_0 \in \mathbb{R}^m$, then we denote by $\widetilde{x}_0$ the following function

$$\widetilde{x}_0 : [a,b] \to \mathbb{R}^m, \widetilde{x}_0(t) = x_0,\ \forall t \in [a,b].$$

Hence $y^* \leq B(y^*) \leq B^2(y^*) \leq \ldots \leq B^\infty(y^*) = B^\infty(\widetilde{y^*(a)}) \leq B^\infty(\widetilde{x^*(a)}) = x^*$. □

In order to study the monotony of the solution of the problem (1.1)-(1.2) with respect to $x_0, f, g$ we need the following result from WPOs theory.

**Lemma 3.3.** *(Abstract comparison lemma, [15]) Let $(X,d,\leq)$ be an ordered metric space and $A,B,C : X \to X$ be such that:*

*(i) the operator $A,B,C$ are WPOs;*

*(ii) $A \leq B \leq C$;*

*(iii) the operator $B$ is increasing.*

*Then $x \leq y \leq z$ imply that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.*

From this abstract result we obtain the following result:

**Theorem 3.4.** *Let $f^j, g^j \in C([a,b]\times\mathbb{R}^m, \mathbb{R}^m), j = \overline{1,3}$, and suppose that the conditions from Theorem 2.2 hold. Furthermore suppose that:*

*(i) $f^1 \le f^2 \le f^3, g^1 \le g^2 \le g^3$;*

*(ii) $f^2(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m, g^2(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ are increasing.*

*Let $x^{*j}$ be a solution of the equation*

$$x^{j\prime}(t) = f^j(t, x(t)) + g^j(t, \max_{a\le\xi\le t} x(\xi)),\ t \in [a,b] \text{ and } j = \overline{1,3}.$$

*Then $x^{*1}(a) \le x^{*2}(a) \le x^{*3}(a)$, implies $x^{*1} \le x^{*2} \le x^{*3}$, i.e. the unique solution of the problem (1.1)-(1.2) is increasing with respect to $x_0, f$ and $g$.*

*Proof.* From Remark 2.4, the operators $B_j, j = \overline{1,3}$, are WPOs. From the condition (ii) the operator $B_2$ is monotone increasing. From the condition (i) it follows that $B_1 \le B_2 \le B_3$. Let $\widetilde{x}^j(a) \in (C[a,b], \mathbb{R}^m)$ be defined by $\widetilde{x}^j(a) = x^j(a),\ \forall t \in [a,b]$. We notice that

$$\widetilde{x}^1(a)(t) \le \widetilde{x}^2(a)(t) \le \widetilde{x}^3(a)(t),\ \forall t \in [a,b].$$

From Lemma 3.3 we have that $B_1^\infty(\widetilde{x}^{*1}(a)) \le B_2^\infty(\widetilde{x}^{*2}(a)) \le B_3^\infty(\widetilde{x}^{*3}(a))$.
But $x^{*j} = B_j^\infty(\widetilde{x}^{*j}(a))$, so $x^{*1} \le x^{*2} \le x^{*3}$. □

## 4. Data dependence: continuity

In this section we prove the continuous dependence of the solution for equation (1.1) and suppose the conditions of Theorem 2.2 are satisfied.

**Theorem 4.1.** *Let $x_0^j, f^j, g^j, j = 1,2$ satisfy the conditions from Theorem 2.2. Furthermore we suppose there exist $\eta^1, \eta^2, \eta^3 \in \mathbb{R}^m_+$, such that*

*(i) $\left|x_0^j - x_0^j\right| \le \eta^1$;*

*(ii) $\left|f^1(t,u) - f^2(t,u)\right| \le \eta^2, \left|g^1(t,v) - g^2(t,v)\right| \le \eta^3,\ \forall t \in C[a,b], u, v \in \mathbb{R}^m$.*

*Then*

$$\left\|x^*(t; x_0^1, f^1, g^1) - x^*(t; x_0^2, f^2, g^2)\right\| \le (I-Q)^{-1}(\eta^1 + (b-a)(\eta^2 + \eta^3)),$$

*where $x^*(t; x_0^j, f^j, g^j)$ are the solutions of the problem (1.1)-(1.2) with respect to $x_0^j, f^j, g^j, j = 1,2$.*

*Proof.* Consider the operator $A_{x_0^j, f^j, g^j}, j = 1,2$. From Theorem 2.2 it follows that

$$\left\|A_{x_0^1,f^1,g^1}(x) - A_{x_0^1,f^1,g^1}(y)\right\| \le Q\left\|x-y\right\|, \forall x, y \in X.$$

Additionally

$$\left\|A_{x_0^1,f^1,g^1}(x) - A_{x_0^2,f^2,g^2}(x)\right\| \le \eta^1 + (b-a)(\eta^2 + \eta^3).$$

Then

$$\begin{aligned}
&\left\|x^*(t;x_0^1,f^1,g^1)-x^*(t;x_0^2,f^2,g^2)\right\| \\
&= \left\|A_{x_0^1,f^1,g^1}(x^*(t;x_0^1,f^1,g^1)) - A_{x_0^2,f^2,g^2}(x^*(t;x_0^2,f^2,g^2))\right\| \\
&\le \left\|A_{x_0^1,f^1,g^1}(x^*(t;x_0^1,f^1,g^1)) - A_{x_0^1,f^1,g^1}(x^*(t;x_0^2,f^2,g^2))\right\| \\
&\quad + \left\|A_{x_0^1,f^1,g^1}(x^*(t;x_0^2,f^2,g^2)) - A_{x_0^2,f^2,g^2}(x^*(t;x_0^2,f^2,g^2))\right\| \\
&\le Q\left\|x^*(t;x_0^1,f^1,g^1)-x^*(t;x_0^2,f^2,g^2)\right\| + \eta^1 + (b-a)(\eta^2+\eta^3).
\end{aligned}$$

Since $Q^n \to 0$ as $n \to \infty$, implies that $(I-Q)^{-1} \in M_{mm}(\mathbb{R}_+)$ and we finally obtain

$$\left\|x^*(t;x_0^1,f^1,g^1)-x^*(t;x_0^2,f^2,g^2)\right\| \le (I-Q)^{-1}(\eta^1 + (b-a)(\eta^2+\eta^3)). \qquad \square$$

## 5. Remarks

In this section we emphasize some special cases of (1.1).

Let $\tau > 0$ be a given number and we define the operator $G : C([-\tau,\infty),\mathbb{R}^m) \to \mathbb{R}^m$ such that for any function $x \in C([-\tau,\infty),\mathbb{R}^m)$ and any point $t \in \mathbb{R}_+$ there exists a point $\xi \in [t-\tau,t]$ such that $G(x)(t) = a(t)x(\xi)$ where $a \in C(\mathbb{R}_+,\mathbb{R})$.

Consider the nonlinear delay functional differential equation

$$x'(t) = f(t,x(t)) + g(t,G(x)(t)) \tag{5.1}$$

for $t \ge t_0$ with initial condition

$$x(t+t_0) = \varphi(t),\ t \in [-\tau,0],$$

where $x \in \mathbb{R}^m,\ f : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m,\ t_0 \in \mathbb{R}_+,\ \varphi : [-\tau,0] \to \mathbb{R}^m$.

Particular cases of (1.1):

(i) For $G(x)(t) = x(t-\tau),\ t \in \mathbb{R}_+$, then (5.1) reduces to a delay differential equation (see [6], [12], [14], [15]);

(ii) For $G(x)(t) = \max\limits_{s\in[t-\tau,t]} x(s),\ t \in \mathbb{R}_+$, then (5.1) reduces to a differential equation with maxima (see [16], [17], [9], [10], [1]);

(iii) For $G(x)(t) = \int_{t-\tau}^{t} x(s)ds,\ t \in \mathbb{R}_+, \tau > 0$, then (5.1) reduces to a differential equation with distributed delay (see [11], [4]);

(iv) For $g(t,G(x)(t)) = h(x)(t)$, where $h : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is an abstract Volterra operator, then (5.1) reduces to a differential equation with abstract Volterra operator (see [8]);

(v) If $x'(t) - f(t,x(t)) := \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right],\ G(x)(t) = x(t),\ t \ge t_0$, then (5.1) reduces to a quadratic differential equation (see [3]).

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## References


[1] Bainov, D.D., Hristova, S., *Differential equations with maxima*, Chapman & Hall/CRC Pure and Applied Mathematics, 2011.

[2] Bainov, D.D., Petrov, V.A., Proyteheva, V.S., *Existence and asymptotic behavior of nonoscillatory solutions of second order neutral differential equations with "maxima"*, J. Comput. Appl. Math., **83**(1997), no. 2, 237-249.

[3] Dhage, B.C., Lakshmikantham, V., *Basic results on hybrid differential equations*, Nonlinear Anal. Hybrid Syst., **4**(2010), 414-424.

[4] Dobriţoiu, M., Rus, I.A., Şerban, M.A., *An integral equation arising from infections diseases, via Picard operators*, Stud. Univ. Babeş-Bolyai Math., **52**(2007), no. 3, 81-94.

[5] Georgiev, L., Angelov, V.G., *On the existence and uniqueness of solutions for maximum equations*, Glas. Mat., **37**(2002), no. 2, 275-281.

[6] Hale, J., *Theory of functional differential equations*, Springer, 1977.

[7] Otrocol, D., *Systems of functional differential equations with maxima, of mixed type*, Electron. J. Qual. Theory Differ. Eq., **2014**(2014), no. 5, 1-9.

[8] Otrocol, D., Ilea, V.A., *Qualitative properties of a functional differential equation*, Electron. J. Qual. Theory Differ. Eq., **47**(2014), 1-8.

[9] Otrocol, D., Rus, I.A., *Functional-differential equations with "maxima" via weakly Picard operators theory*, Bull. Math. Soc. Sci. Math. Roumanie, **51(99)**(2008), no. 3, 253-261.

[10] Otrocol, D., Rus, I.A., *Functional-differential equations with maxima of mixed type argument*, Fixed Point Theory, **9**(2008), no. 1, 207-220.

[11] Precup, R., *Monotone techniques to the initial values problem for a delay integral equation from Biomathematics*, Stud. Univ. Babeş-Bolyai Math., **40**(1995), no. 2, 63-73.

[12] Rus, I.A., *Picard operators and applications*, Sci. Math. Jpn., **58**(2003), no. 1, 191-219.

[13] Rus, I.A., *Functional differential equations of mixed type, via weakly Picard operators*, Seminar on Fixed Point Theory Cluj-Napoca, **3**(2002), 335-346.

[14] Rus, I.A., *Generalized contractions and applications*, Cluj University Press, 2001.

[15] Rus, I.A., *Weakly Picard operators and applications*, Seminar on Fixed Point Theory, Cluj-Napoca, **2**(2001), 41-58.

[16] Stepanov, E., *On solvability of same boundary value problems for differential equations with "maxima"*, Topol. Methods Nonlinear Anal., **8**(1996), 315-326.

[17] Zhang, B.G., Zhang, G., *Qualitative properties of functional equations with "maxima"*, Rocky Mountain J. Math., **29**(1999), no. 1, 357-367.



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# Remarks on fixed point sets

Kazimierz Goebel

*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** This is a collection of examples illustrating various properties of fixed point sets under regularity assumptions on mappings.

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**Keywords:** Fixed point sets, convergence of iterates.

Working for many years on the fixed point theory I came across several original questions asked by the newcomers to the field and young researchers. Answering very often required constructions of various examples. Here, samples are recalled, answered and discussed. Presented examples concern geometrical and topological properties of the fixed point sets of mappings satisfying certain regularity conditions. If known, are formulated and shown with some modifications and novelties.

For the whole text, let $X$ denotes a Banach space with norm $\|\cdot\|$ and $C$ be a convex, closed and bounded subset of $X$. $H$ stands for Hilbert space. We shall deal with the classical situation of a mapping $T : C \to C$ satisfying the Lipschitz condition (lipschitzian mappings),

$$\|Tx - Ty\| \leq k \|x - y\| . \tag{1}$$

The fixed point set for $T$ is defined as,

$$FixT = [x \in C : x = Tx] .$$

If $T$ is a *contraction,* $k < 1$, $FixT$ consists of exactly one point, say $z$, and for any $x_0 \in C$, the sequence of iterates $x_n = T^n x_0$ converges, $\lim_{n\to\infty} x_n = z$.

If $T$ is *nonexpansive,* $k = 1$, the set $FixT$ may be empty unless the set $C$ catisfies some additional regularity condition. However, always *the minimal displacement* by $T$ satisfies,

$$d(T) = \inf [\|x - Tx\| : x \in C] = 0.$$

Even if $FixT \neq \emptyset$ the iterates $T^n x_0$, in general, do not converge. If the space $X$, or the set $C$, show some special conditions, it may have influence on the regularity of $FixT$. For example, if $X$ is strictly convex, then $FixT$ is convex. If $C$ is weakly compact and $T$ has a fixed point in every $T-$invarint closed convex subset of $C$, then $FixT$ is *a nonexpansive retract of* $C$. The last means that there exists a nonexpansive mapping

$R : C \to FixT$ such that $FixR = FixT = R(C)$. Without regularity conditions the fixed point sets can be very bizzare (see below and [1]).

If $k > 1, FixT$ is in many cases empty unless $C$ is compact. If $C$ is not compact, it may be that $d(T) > 0$.

The above standard facts in fixed point theory can be found, among other places, in books [2], [5], [6] see also [3].

The first to discuss is:

**Question 1.** Given a Banach space $X$ and a nonempty, closed set $F \subset X$. Is there a way, procedure, algorithm to find out whether $F$ is a fixed point set for a nonexpansive mapping $T : X \to X$ or $T : C \to C$ for a convex $C$ containing $F$?

The situation is clear if $X = H$ is a Hilbert space. The fixed point sets of nonexpansive mappings are convex. Since for any nonempty closed convex suset $C \subset H$ the nearest point retraction $P_C$: $X \to C$ assigning to each $x \in H$ the unique point

$P_C(x) \in C$ such that

$$\|x - P_C x\| = \min [\|x - y\| : y \in C]$$

is nonexpansive, convexity is the necessary and sufficient condition for a closed set to be of the form $C = FixT$ for nonexpansive $T$.

The nearest point projection $P$ is not the only one nonexpansive mapping having $C$ as the fixed point set. The reflection with respect to $P, S = 2P - I$ is also nonexpansive with $FixS = C$. There are more examples in [4].

In other, even very regular, spaces convexity is not enough.

**Example 1.** Let for $p \in (1, \infty)$, $l_3^p$ be the three dimensional space $\mathbb{R}^3$ furnished with the norm

$$\|x\|_p = \|(x_1, x_2, x_3)\|_p = (|x_1|^p + |x_2|^p + |x_3|^p)^{\frac{1}{p}} .$$

Let $K$ be the triangle with vertices at unit vectors, $e^1 = (0,0,1), e^2 = (0,1,0), e^3 = (0,0,1)$. There is the unique closed ball containing $K$ having minimal radius. Simple calculus shows that this is $B(z_p, r_p)$ centered at $z_p = (t_p, t_p, t_p)$ with

$$\begin{aligned} t_p &= \left(2^{\frac{1}{p-1}} + 1\right)^{-1} \\ r_p &= 2^{\frac{1}{p}} \left(2^{\frac{1}{p-1}} + 1\right)^{\frac{1-p}{p}} . \end{aligned}$$

For two extremal cases, $p = 1$ and $p = \infty$, we have $z_1 = \lim_{p \to 1} z_p = (0,0,0), r_1 = 1$ and $z_\infty = \lim_{p \to \infty} z_p = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), r_\infty = \frac{1}{2}$. Observe that $z_p \in K$ if and only if $p = 2$. Let $C \subset l_3^p$ be a closed convex set containing $K$ and such that $z_p \in C$. Assume that $T : C \to C$ is nonexpansive and such that $FixT = K$. For any $x \in K$ we have

$$\|Tz_p - Tx\| \leq \|z_p - x\| \leq r_p .$$

Thus $K \subset B(Tz_p, r_p)$. In view that the ball of radius $r_p$ containing $K$ is unique, it must be $z_p = Tz_p$. For $p \neq 2, z_p \notin K$ and thus, we have the contradiction $K \neq FixT$.

In general, no topological propertis can characterize whether a closed set $F$ is, or not is, of the form $F = FixT$ for nonexpansive $T$ . There may be the case that the set $C \subset X$ is not the fixed point set of of any nanexpansive mapping $T : X \to X$, but for certain larger space $Y, X \subset Y$ there is even an isometry $G : Y \to Y$ having having $FixG$ isometric to $F$.

**Example 2.** Let $X$ be an arbitrary space and let $F \subset X$ be nonempty and closed. Consider the space $Z = C \times c_0$ normed for $(x, y) \in Z, x \in X, y = (y_1, y_2, ...) \in c_0$, by

$$\|(x, y)\| = \max \left[\|x\|, \|y\|_{c_0}\right].$$

Since for any $x_1, x_2$ in $X$

$$|dist(x_1, F) - dist(x_2, F)| \leq \|x_1 - x_2\|,$$

the mapping $T : Z \to Z$, defined as

$$T(x, y) = T(x, (y_1, y_2, ...)) = (x, (dist(x, F), y_1, y_2, ...))$$

is not only nonexpansive but isometric. If $(x, y) = T(x, y)$, then

$$(y_1, y_2, ...) = (dist(x, F), y_1, y_2, ...)$$

which implies

$$dist(x, F) = y_1 = y_2 = ... = 0.$$

Thus, $Fix(T) \subset Z$ is

$$Fix(T) = F \times \{0\}$$

which is an isometric the copy of $F \subset X$.

Summing up, any closed subset of any Banach space can be considered as a fixed point set of an isometry.

Next two questions concern mappings which are lipschitzian with $k > 1$.

**Question 2.** Do there exist some characteristics of fixed point sets for Lipschitz mapping with constant $k > 1$? Do they depennd on the size of $k$ or the regularity of the space?

**Question 3.** The same as above with the additional assumption that $T$ has convergent iterates, for any $x$ there exists $T_\infty x = \lim_{n\to\infty} T^n x$.

The first answer is that there is no dependence on the size of $k$. If $C$ is a closed and convex set and $T : C \to C$ is $k-$lipschitzian, then for any $\alpha \in (0, 1)$ the mapping $T_\alpha = (1 - \alpha) I + \alpha T$ satisfies the Lipschitz condition wit a constant $k_\alpha \leq (1 - \alpha + \alpha k)$ and has $FixT_\alpha = FixT$. Since $\lim_{\alpha\to 0} k_\alpha = 1$ possible characteristics may not depend on the size of $k$.

The rest is answered by the following fact.

**Claim 1.** Let $C \subset X$ be a nonempty closed convex and bounded. Suppose $F \subset C$ is nonempty and closed. For any $\varepsilon > 0$ there exists a mapping $T : C \to C$ such that $T$ is $k-$lipschtzian with $k \leq 1 + \varepsilon, FixT = F$.
Moreover, for any $x \in C, T_\infty x = \lim_{n\to\infty} T^n x$ does exist.

*Proof.* Let $F$ and $C$ be as assumed. Fix a point $z \in F$ and $0 < \varepsilon < 1$. Define the mapping $T : C \to C$ as

$$Tx = x + \varepsilon \frac{dist\,(x, F)\,(z - x)}{2diamC}.$$

Observe that $FixT = F$. Now it is easy to see that

$$\|dist\,(x, F)\,(z - x) - dist\,(x, F)\,(z - y)\| \leq 2diamC\,\|x - y\|$$

and consequently,

$$\|Tx - Ty\| \leq (1 + \varepsilon)\,\|x - y\|\,.$$

Finally, for any $x \notin F$ the iterates $\{x, Tx, T^2x, ...\}$ form on the segment $[x, z]$ the sequence monotone in the way that for any $n = 0, 1, 2, ...,$ $T^{n+1}x \in [T^n x, z]$. It implies convergence. □

Additional information can be obtained that $T_\infty x = \lim_{n\to\infty} T^n x$ is the point in $F \cap [x, z]$ closest to $x$, $\|x - T^\infty x\| = dist\,(x, F \cap [x, z])$. Obviously, the above construction does not imply the continuity of $T_\infty$. The simpliest example of it is, if we consider $F$ consisting of only two distinct points.

The next question concerns retracts. Let us remain in our standartd assumptions for $F \subset C$. The set $F$ is said to be the retract (lipschitzian, nonexpansive) if there is a continuous (lipschitzian, nonexpansive) mapping $R : C \to F$ such that $R\,(C) = FixR = F$. Any such $R$ is said to be a retraction of $C$ onto $F$. For a given retract $F$ of $C$, there may be many retractions of various regularity. Retractions are characterized by the condition $R^2 = R$ which implies that $R\,(C)$ is the retract of $C$. For any $x \in C$ the iterates of each retraction $R$ stabilize after one step, $R^n x = Rx, n = 1, 2, 3, ...$ .

**Question 4.** What about mappings having iterates stabilized after $k$ steps, meaning $T^{k+1}x = T^k x$?

Under this condition for all $n \geq k$, we have $T^n = T^k$. In other notation, for any $x \in C$,

$$T^k x = T^{k+1}x = T^{k+2}x = ...\ .$$

Thus $T^{2k} = T^k$ and $R = T^k$ is a retraction. Hence the fixed point sets of such mappings are the retracts of $C$. There are simple example of such mappings.

**Example 3.** Let $B \subset X$ be the unit ball. It is the retract of $X$. The standart retraction $Q : X \to B$ is the radial projection mapping

$$Qx = \begin{cases} x & \text{if } \|x\| \leq 1 \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1. \end{cases}.$$

Let $S = 2Q - I$. Since $Q$ satisfies Lipschitz condition with constant $k \in [1, 2]$, so $S$ is also lipschitzian. More precisely,

$$Sx = \begin{cases} x & \text{if } \|x\| \leq 1 \\ \frac{2 - \|x\|}{\|x\|}x & \text{if } \|x\| > 1 \end{cases},$$

and

$$\|Sx\| = \begin{cases} \|x\| & \text{if } \|x\| \leq 1 \\ |\|x\| - 2| & \text{if } \|x\| > 1 \end{cases}.$$

For any $x$ we have $S^{n+1}x = S^n x$ begining from $n = Ent\frac{\|x\|}{2}$. Consequently for any $r > 1$, $S : rB \to rB$ for $n = Ent\frac{r}{2}, S^n$ is a lipschitzian retraction of $rB$ onto $B$.

It is easy to observe that there are more Lipschitz retractions of the whole space $X$ on $B$ other than $Q$ . Such are for example mappings of the form,

$$R_n = Q \circ S^n, n = 1, 2, \ldots .$$

In the most interesting case of the Hilbert space $H$, both mappings $Q$ and $S$ are nonexpansive. So, nonexpansive are all $R_n, n = 1, 2, \ldots$ .

There is an advice for interested reader to prove:

**Claim 2.** Suppose $F, C$, $F \subset C \subset H$ are closed, convex and bounded. Assume that $F$ (so, also $C$) has nonempty interior. Let $P = P_F : C \to F$ be the closest point projection and $S = 2P - I$. Then there exists $n \geq 1$ such that $S^n$ is a nonexpansive retraction of $C$ onto $F$.

The last is the following:

**Question 5.** What can be said about mappings similar to presented above if we only assume that for each individual point $x$ the sequence $T^n x$ stabilizes but not necessarily on on the same level?

The situation is unclear. This condition does not bring much of regularity. There are sets $F, C, F \subset C$ such that $F$ is not the retract of $C$ or even are disconnected which satisfy the following.

For any $\varepsilon > 0$ there is a $(1 + \varepsilon)$ −lipschitzian mapping $T : C \to F$ having $F = FixT$ and such that for each $x \in C, T^{k+1}x = T^k x$ for certain $k$ depending on $x, k = k(x)$ . It looks like it is unknown how to characterize such sets. Here we present only a few simple examples.

**Example 4.** Consider the plain $\mathbb{R}^2$ with standard Eucledean norm, or any Hilbert space. Let $B, S$ denote the unit ball and the unit sphere and let $P$ be the radial retraction on $B$. Then the mapping $T : B \to B$,

$$Tx = P((1 + \varepsilon)x), \tag{2}$$

is $(1 + \varepsilon)$ −lipschitzian, $FixT = S \cup \{0\}$ and for any $x \in B$ the iterates of $T$ stabilize. The level of stabilization $k = k(x)$ for $x \notin FixT$ depends only on the norm of $x$ and grows to infinity as $x \to 0$.

It is only a technicality to construct similar examples with $F = FixT = S \cup \frac{1}{2}S \cup \{0\}$ or $F = rB \cup S, 0 < r < 1$ and other modifications as,

**Example 5.** Again for the Euclidean plane $\mathbb{R}^2$ plain consider the square

$$K = [(x_1, x_2) : |x_1| \leq 1, |x_2| \leq 1].$$

Let $P = P_K$ be the closest point projection of $\mathbb{R}^2$ on $K$,

$$P(x_1, x_2) = (\alpha(x_1), \alpha(x_2)),$$

where

$$\alpha(t) = \begin{cases} -1 & \text{if} \quad t < -1 \\ t & \text{if} \quad -1 \leq t \leq 1 \\ 1 & \text{if} \quad t > 1 \end{cases} .$$

Let again the mapping $T : K \to K$ be defined by (2). Then $T$ fulfils required conditions and has the fixed point set consisting of nine points: the origin, positive and negative unit vectors and vertices of the square.

## References

[1] Bruck, R., Goebel, K., *Bizzare fixed point sets*, Fixed Point Theory and Applications (Proceedings of FPTA 1991, Halifax), World Scientific Publishing, Singapore/ New Jersey/ London/ Hong Kong, 1992, 67-70.

[2] Goebel, K., Kirk, W.A., *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.

[3] Goebel, K., Prus, S., *Shapes and sizes of the fixed point sets*, Nonlinear Analysis and Convex Analysis I, (Proceedings of NACA2011), Yokohama Publishers, Yokohama 2012, 75-97.

[4] Goebel, K., Sedłak, E., *Nonexpansive retracts in Hilbert space*, Annales UMCS, **63**(2009), 1-6.

[5] Kirk, W.A., Sims, B., *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht/ Boston/ London, 2001.

[6] Piasecki, L., *Classification of Lipschitz Mappings*, CRC Press, Boca Raton, FL, 2014.

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# Fixed point theorems for operators with a contractive iterate in $b$-metric spaces

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*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** We consider, in this paper, mappings with a contractive iterate at a point, which are not contractions, and prove some uniqueness and existence results in the case of $b-$metric spaces. A data dependence result and an Ulam-Hyers stability result are also proved.

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**Keywords:** Fixed point, $b$-metric space, contractive iterate, data dependence, Ulam-Hyers stability.

## 1. Introduction

The well known Banach contraction's principle states that in a complete metric space each contraction has a unique fixed point and the sequence of successive approximations converges to the fixed point. We consider, in this paper, mappings with a contractive iterate at a point, which are not contractions, and prove some uniqueness and existence results in the case of $b-$metric spaces. Some related results for the case of metric spaces can be found in [12, 4, 17, 19] The starting point of this theory is the article of V.M. Sehgal [22], where the author proves the following result:

**Theorem 1.1.** *Let $(X, d)$ be a complete metric space and $f : X \to X$ a continuous mapping satisfying the condition: there exists a $k < 1$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(f^{n(x)}(y), f^{n(x)}(x)) \leq kd(y, x).$$

*Then $f$ has a unique fixed point $u$ and $f^n(x_0) \to u$, for each $x_0) \in X$.*

We investigate mappings that are not necessary continuous and extend the previous result to the case of $b-$metric spaces. The data dependence of the fixed points is also considered. In the second part of the paper we prove an Ulam-Hyers stability result. For more results regarding this concepts see [8, 13, 20, 21].

## 2. Preliminaries

The $b$-metric space is a generalization of a usual metric space, which was introduced by Czerwik [15, 14]. In fact, such general setting of metric spaces were considered earlier, for example, by Bourbaki [11], Bakhtin [3], Heinonen [18]. Following these initial papers, $b$-metric spaces and related fixed point theorems have been investigated by a number of authors, see e.g. Boriceanu et al.[9], Bota [10], Aydi et al. [1, 2].

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. We recollect some essential definitions and fundamental results. We begin with the definition of a $b$-metric space.

**Definition 2.1.** *(Bakhtin [3], Czerwik [15]) Let $X$ be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a $b$-metric if the following conditions are satisfied:*

1. $d(x, y) = 0$ *if and only if* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$,

*for all $x, y, z \in X$. A pair $(X, d)$ is called a $b$-metric space.*

It is clear that a $b$-metric is a usual metric if we take $s = 1$. Hence, we conclude that the class of $b$-metric spaces is larger than the class of usual metric spaces. For more details and examples on $b$-metric spaces, see e.g. [3, 5, 11, 14, 15, 18].

For the sake of completeness we state the following examples, see [5, 6].

**Example 2.2.** Let $X$ be a set with the cardinal $card(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of $X$ such that $card(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then, the functional $d : X \times X \to [0, \infty)$ defined by:

$$d(x, y) := \begin{cases} 0, & x = y \\ 2s, & x, y \in X_1 \\ 1, & \text{otherwise.} \end{cases}$$

is a $b$-metric on $X$ with coefficient $s > 1$.

**Example 2.3.** The set $l^p(\mathbb{R})$ (with $0 < p < 1$), where

$$l^p(\mathbb{R}) := \left\{ (x_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) := \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p},$$

(where $x = (x_n), y = (y_n) \in l^p(\mathbb{R})$) is a $b$-metric space with coefficient $s = 2^{1/p} > 1$. Notice that the above result holds for the general case $l^p(X)$ with $0 < p < 1$, where $X$ is a Banach space.

**Example 2.4.** The space $L^p[0,1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional

$$d(x,y) := \left( \int_0^1 |x(t) - y(t)|^p dt \right)^{1/p}, \text{ for each } x, y \in L^p[0,1],$$

is a $b$-metric space. Notice that $s = 2^{1/p}$.

We will present now the notions of convergence, compactness, closedness and completeness in a $b$-metric space.

**Definition 2.5.** *Let $(X,d)$ be a b-metric space. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ is called:*

(a) *convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.*

(b) *Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to +\infty$.*

**Remark 2.6.** Notice that in a $b$-metric space $(X,d)$ the following assertions hold:

(i) a convergent sequence has a unique limit;

(ii) each convergent sequence is Cauchy;

(iii) $(X, \overset{d}{\to})$ is an $L$-space (see Fréchet [16], Blumenthal [7]);

(iv) in general, a $b$-metric is not continuous;

Taking into account of (iii), we have the following concepts.

**Definition 2.7.** *Let $(X,d)$ be a b-metric space. Then a subset $Y \subset X$ is called:*

*(i) closed if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ in $Y$ which converges to an element $x$, we have $x \in Y$;*

*(ii) compact if and only if for every sequence of elements of $Y$ there exists a subsequence that converges to an element of $Y$.*

**Definition 2.8.** *The b-metric space $(X,d)$ is complete if every Cauchy sequence in $X$ converges.*

**Lemma 2.9.** *(Czerwik [15]) Let $(X,d)$ be a b-metric space. Then and let $\{x_k\}_{k=0}^n \subset X$. Then $d(x_n, x_0) \le sd(x_0, x_1) + \ldots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^n d(x_{n-1}, x_n)$.*

## 3. Main results

In order to prove the first main result we need the following Lemma:

**Lemma 3.1.** *Let $(X,d)$ be a complete b-metric space with $s \ge 1$ and $f : X \to X$ a mapping which satisfies the condition: there exists an $a \in (0, \frac{1}{s})$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(f^{n(x)}(x), f^{n(x)}(y)) \le ad(x,y).$$

*Then for each $x \in X$, $r(x) = sup_n d(f^n(x), x)$ is finite.*

*Proof.* Let $x \in X$ and let $l(x) = max\{d(f^k(x), x), k = 1, 2, \ldots, n(x)\}$.

If $n \in \mathbb{N}$ there exists $k \geq 0$ such that

$$k \cdot n(x) < n \leq (k+1) \cdot n(x).$$

We have:

$$\begin{aligned} d(f^n(x), x) &\leq s[d(f^{n(x)}(f^{n-n(x)}(x)), f^{n(x)}(x)) + d(f^{n(x)}(x), x)] \\ &\leq s \cdot a \cdot d(f^{n-n(x)}(x), x) + s \cdot l(x) \\ &\leq s \cdot l(x) + a \cdot s^2 \cdot l(x) + a^2 \cdot s^3 \cdot l(x) + \ldots + a^k \cdot s^{k+1} \cdot l(x) \\ &= s \cdot l(x)[1 + s \cdot a + s^2 \cdot a^2 + \ldots + s^k \cdot a^k] \\ &= s \cdot l(x) \cdot \frac{1 - (s \cdot a)^{k+1}}{1 - s \cdot a} \leq s \cdot l(x) \cdot \frac{1}{1 - sa}. \end{aligned}$$

Hence $r(x) = sup_n d(f^n(x), x)$ is finite. □

The next result presents a fixed point theorem for a mapping with a contractive iterate. A data dependence result is also proved.

**Theorem 3.2.** *Let $(X, d)$ be a complete b-metric space with $s \geq 1$ and $f : X \to X$ a mapping which satisfies the condition: there exists an $a \in (0, \frac{1}{s})$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(f^{n(x)}(x), f^{n(x)}(y)) \leq ad(x, y).$$

*Then:*

(i) *$f$ has a unique fixed point $x^* \in X$ and $f^n(x_0) \to x^*$, for each $x_0 \in X$, as $n \to \infty$.*
*If, in addition, the b−metric is continuous we have:*
(ii) *$d(x_0, x^*) \leq sd(x_0, f^{n(x_0)}(x_0)) + \frac{s^2}{1-sa} r(x_0)$, for each $x_0 \in X$.*
(iii) *Let $g : X \to X$ such that there exists $\eta > 0$ with*

$$d(f^{n(x)}(x), g(x)) \leq \eta, \ \forall x \in X.$$

*Then*

$$d(x^*, y^*) \leq s \cdot \eta + \frac{s}{1 - sa} \cdot r(y^*),$$

*for all $y^* \in Fix(g)$.*

*Proof.* (i) Let $x_0 \in X$ be arbitrary. Let $m_0 = n(x_0)$, $x_1 = f^{m_0}(x_0)$ and inductively $m_i = n(x_i)$, $x_{i+1} = f^{m_i}(x_i)$. We show that the sequence $\{x_n\}$ is convergent. By routine calculation we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f^{m_{n-1}}(f^{m_n}(x_{n-1})), f^{m_{n-1}}(x_{n-1})) \\ &\leq a \cdot d(f^{m_n}(x_{n-1}), x_{n-1}) \leq \ldots \leq a^n \cdot d(f^{m_n}(x_0), x_0). \end{aligned}$$

Estimating $d(x_n, x_{n+p})$ we obtain

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s \cdot d(x_n, x_{n+1}) + s^2 \cdot d(x_{n+1}, x_{n+2}) + \ldots + s^{p-1} \cdot d(x_{n+p-1}, x_{n+p}) \\ &\leq s \cdot a^n \cdot d(f^{m_n}(x_0), x_0) + s^2 \cdot a^{n+1} \cdot d(f^{m_n}(x_0), x_0) + \ldots \\ &+ s^p \cdot a^{n+p-1} \cdot d(f^{m_n}(x_0), x_0) \\ &\leq s \cdot a^n \cdot r(x_0) + s^2 \cdot a^{n+1} \cdot r(x_0) + \ldots + s^p \cdot a^{n+p-1} r(x_0) \\ &= s \cdot a^n \cdot r(x_0)[1 + s \cdot a + \ldots + (s \cdot a)^{p-1}] \\ &= s \cdot a^n \cdot r(x_0) \cdot \frac{1-(sa)^p}{1-sa} \to 0, n \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is Cauchy. Let $x_n \to x^* \in X$. We want to show that $f(x^*) = x^*$. First we show that

$$f^{n(x^*)}(x_m) = y_m \to f^{n(x^*)}(x^*), \text{ as } m \to \infty.$$

We have

$$d(f^{n(x^*)}(x_m), f^{n(x^*)}(x^*)) \leq ad(x_m, x^*) \to 0, \text{as } m \to \infty.$$

On the other side we can write

$$d(f^{n(x^*)}(x^*), x^*) \leq s \cdot [d(f^{n(x^*)}(x^*), f^{n(x^*)}(x_i)) + d(f^{n(x^*)}(x_i), x^*)]$$

where for $i$ sufficiently large we have

$$d(f^{n(x^*)}(x^*), f^{n(x^*)}(x_i)) < \frac{\varepsilon}{3s}.$$

We also have that

$$\begin{aligned} d(f^{n(x^*)}(x_i), x_i) &= d(f^{n(x^*)}(f^{m_{i-1}}(x_{i-1})), f^{m_{i-1}}(x_{i-1})) \\ &= d(f^{m_{i-1}}(f^{n(x^*)}(x_{i-1})), f^{m_{i-1}}(x_{i-1})) \\ &\leq a \cdot d(f^{n(x^*)}(x_{i-1}), x_{i-1}) \leq a^i \cdot d(f^{n(x^*)}(x^*), x^*) < \frac{\varepsilon}{3s^2} \end{aligned}$$

for $i$ sufficiently large.
We also have

$$d(f^{n(x^*)}(x_i), x^*) \leq s \cdot [d(f^{n(x^*)}(x_i), x_i) + d(x_i, x^*)] < s\frac{\varepsilon}{3s^2} + s\frac{\varepsilon}{3s^2} = \frac{2\varepsilon}{3s}$$

Hence

$$d(f^{n(x^*)}(x_i), x^*) \leq s\left[s\frac{\varepsilon}{3s^2} + s\frac{\varepsilon}{3s^2}\right] + \frac{\varepsilon}{3s} = \varepsilon.$$

Thus $f^{n(x^*)}(x^*) = x^*$ which gives us the existence of a fixed point for $g = f^{n(x^*)}$.

In order to prove the uniqueness of the fixed point let us consider $x^*$ and $y^*$ two fixed points with $x^* \neq y^*$. We have

$$d(x^*, y^*) = d(g(x^*), g(y^*)) = d(f^{n(x^*)}(x^*), f^{n(x^*)}(y^*)) \leq a \cdot d(x^*, y^*),$$

which is a contradiction with $a \in (0, 1)$.

From the uniqueness of the fixed point and from $f^{n(x^*)} = x^*$ we can conclude that $x^*$ is a fixed point for $f$ too. Indeed we have

$$f(x^*) = f(f^{n(x^*)}(x^*)) = f^{n(x^*)}(f(x^*)),$$

so $f(x^*)$ is a fixed point for $f^{n(x^*)}$. But $f^{n(x^*)}$ has a unique fixed point $x^*$. Hence $f(x^*) = x^*$.

To show that $f^n(x_0) \to x^*$ let us consider the set

$$\rho_* = \max\{d(f^m(x_0), x^*) : m = 0, 1, 2, \ldots, (n(x^*) - 1)\}.$$

For $n \in \mathbb{N}$ sufficiently large we have: $n = r \cdot n(x^*) + q,\ 0 \leq q < n(u), r > 0$ and

$$\begin{aligned} d(f^n(x_0), x^*) &= d(f^{rn(x^*)+q}(x_0), f^{n(x^*)(x^*)}) \\ &\leq ad(f^{(r-1)n(x^*)+q}(x_0), x^*) \leq \ldots \\ &\leq a^r d(f^q(x_0), x^*) \leq a^r \rho_* \end{aligned}$$

Since $n \to \infty$ implies $r \to \infty$, we have $d(f^n(x_0), x^*) \to 0$, as $n \to \infty$. This establish the theorem.

(ii) In order to prove the second assertion we consider the following inequality obtained above:

$$d(x_n, x_{n+p}) \leq s \cdot a^n \cdot r(x_0) \cdot \frac{1 - (sa)^p}{1 - sa}.$$

Since the $b-$metric is continuous and letting $p \to \infty$ we obtain:

$$d(x_n, x^*) \leq \frac{sa^n}{1 - sa} \cdot r(x_0).$$

For $n = 1$ we have

$$d(x_1, x^*) = d(f^{n(x_0)}(x_0), x^*) \leq \frac{s}{1 - sa} r(x_0).$$

Taking into account the previous inequalities we have:

$$\begin{aligned} d(x_0, x^*) &\leq s(d(x_0, x_1) + d(x_1, x^*)) \\ &\leq sd(x_0, x_1) + \frac{s^2}{1 - sa} r(x_0) \\ &= s \cdot d(x_0, f^{n(x_0)}(x_0)) + \frac{s^2}{1 - sa} r(x_0) \end{aligned}$$

(iii) For the data dependence of the fixed points, using the result from (ii) for $x_0 = y^*$, we obtain:

$$\begin{aligned} d(x^*, y^*) &\leq sd(y^*, f^{n(y^*)}(y^*)) + \frac{s^2}{1 - sa} r(y^*) \\ &= s \cdot d(g(y^*), f^{n(y^*)}(y^*)) + \frac{s^2}{1 - sa} r(y^*) \\ &\leq s \cdot \eta + \frac{s^2}{1 - sa} r(y^*) \end{aligned}$$

□

In the second part of the paper is presented an Ulam-Hyers stability result. We begin with the definition of the Ulam-Hyers stability for a fixed point equation.

**Definition 3.3.** *Let $(X, d)$ be a $b$-metric space with $s \geq 1$ and $f : X \to X$ a mapping. The fixed point equation*

$$x = f(x), \ x \in X \tag{3.1}$$

*is called Ulam-Hyers stable if $\forall \varepsilon > 0$ and $\forall x \in X$ there exists $n(x) \in \mathbb{N}^*$ such that $\forall y^*$ a solution of the inequality*

$$d(y, f^{n(y)}(y)) \leq \varepsilon \tag{3.2}$$

*there exist $c > 0$ and $x^* \in X$ a solution of (3.1) such that*

$$d(y^*, x^*) \leq \varepsilon. \tag{3.3}$$

**Theorem 3.4.** *Let $(X, d)$ be a complete $b$-metric space with $s \geq 1$. Suppose that all the hypothesis of Theorem 3.2 hold.*

*Then the fixed point problem (3.1) is Ulam-Hyers stable.*

*Proof.* Let us estimate the following:

$$\begin{aligned} d(y^*, x^*) &\leq s(d(y^*, f^{n(y^*)}(y^*)) + d(f^{n(y^*)}(y^*), x^*) \\ &= s(\varepsilon + d(f^{n(y^*)}(y^*), f^{n(y^*)}(x^*))) \\ &\leq s\varepsilon + s \cdot a \cdot d(y^*, x^*) \end{aligned}$$

Hence:

$$d(y^*, x^*) \leq \frac{s\varepsilon}{1 - sa}$$

□

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## References

[1] Aydi, H., Bota, M.F., Karapınar, E., Mitrović, S., *A fixed point theorem for set-valued quasi-contractions in b-metric spaces*, Fixed Point Theory Appl., 2012, 2012:88.

[2] Aydi, H., Bota, M.F., Karapınar, E., Moradi, S., *A common fixed point for weak $\phi$-contractions on b-metric spaces*, Fixed Point Theory, **13**(2012), no. 2, 337-346.

[3] Bakhtin, I.A., *The contraction mapping principle in quasimetric spaces*, Funct. Anal., Unianowsk Gos. Ped. Inst., **30**(1989), 26-37.

[4] Barada, K., Rhoades, B., *Fixed point jtheorems for mappings with a contractive*, Pacific J. Math., **71**(1977), no. 2, 517-520.

[5] Berinde, V., *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory, Preprint **3**(1993), 3-9.

[6] Berinde, V., *Sequences of operators and fixed points in quasimetric spaces*, Stud. Univ. Babeş-Bolyai Math., **16**(1996), no. 4, 23-27.

[7] Blumenthal, L.M., *Theory and Applications of Distance Geometry*, Oxford, 1953.

[8] Bota-Boriceanu, M.F., Petruşel, A., *Ulam-Hyers stability for operatorial equations*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), **57**(2011), 65-74.

[9] Boriceanu, M.F., Petruşel, A., Rus, I.A., *Fixed point theorems for some multivalued generalized contractions in b-metric spaces*, Int. J. Math. Stat., **6**(2010), 65-76.

[10] Bota, M.F., *Dynamical Aspects in the Theory of Multivalued Operators*, Cluj University Press, 2010.

[11] Bourbaki, N., *Topologie Générale*, Herman, Paris, 1974.

[12] Ćirić, L., *On Sehgal's maps with a contractive iterate at a point*, Publ. Inst. Math. (N.S.), **33**(1983), no. 47, 59-62.

[13] Chiş-Novac, A., Precup, R., Rus, I.A., *Data dependence of fixed points for nonself generalized contractions*, Fixed Point Theory, **10**(2009), no. 1, 73-87.

[14] Czerwik, S., *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1**(1993), 5-11.

[15] Czerwik, S., *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Univ. Modena, **46**(1998), 263-276.

[16] Fréchet, M., *Les Espaces Abstraits*, Gauthier-Villars, Paris,1928.

[17] Guseman, Jr., L.F., *Fixed Point Theorems for Mappings with a Contractive Iterate at a Point*, Proc. Amer. Math. Soc., **26**(1970), no. 4, 615-618.

[18] Heinonen, J., *Lectures on Analysis on Metric Spaces*, Springer Berlin, 2001.

[19] Matkowski, J., *Fixed Point Theorems for Mappings with a Contractive Iterate at a Point*, Proc. Amer. Math. Soc., **62**(1977), no. 2, 344-348.

[20] Rus, I.A., *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9**(2008), no. 2, 541-559.

[21] Rus, I.A., Petruşel, A., Sîntămărian, A., *Data dependence of the fixed points set of some multivalued weakly Picard operators*, Nonlinear Anal., **52**(2003), 1947-1959.

[22] Sehgal, V.M., *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Soc., **23**(1969), 631-634.

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# Abstract method of upper and lower solutions and application to singular boundary value problems

Radu Precup

*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** The method of upper and lower solutions is presented for the fixed point problem associated to operators which are compositions of a linear operator and a nonlinear mapping. Spectral properties of the linear part together with growth and monotonicity properties of the nonlinear part are involved. An application to singular boundary value problems is included.

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## 1. Introduction

One of the most useful methods for solving nonlinear equations arising from mathematical modeling of real processes is the method of upper and lower solutions (see [1], [2], [5], [6], [8], [10], [12], [13], [14]). It consists in localizing solutions of an operator equation

$$u = Tu$$

in an order interval $[u_0, v_0]$, where $u_0$ is a *lower solution*, i.e.

$$u_0 \leq Tu_0,$$

$v_0$ is an *upper solution*, i.e.

$$v_0 \geq Tv_0,$$

and $u_0, v_0$ are *comparable* in the sense of order, that is $u_0 \leq v_0$. Thus a basic problem is to find comparable lower and upper solutions. In this paper we present such type of results for the abstract Hammerstein equation

$$u = ANu \tag{1.1}$$

in an ordered Banach space $X$. Here $A$ is a linear operator and $N$ is a nonlinear mapping from $X$ to $X$. Although the main motivation is in applications to real processes from science and engineering, a general abstract method is essential in order to understand unitarily particular results and to make clear the applicability of the method to a specific problem.

## 2. Main results

The first result guarantees that the solutions of an equation $u = A\Phi u$ simpler than (1.1), are upper (lower) solutions for (1.1) provided that $\Phi$ (respectively, $N$) dominates $N$ (respectively, $\Phi$).Throughout this paper we shall use the same symbol $\leq$ to denote the order relation in different ordered sets.

**Theorem 2.1.** *Let $X$ and $Y$ be two ordered sets, $N : X \to Y$ be any mapping and $A : Y \to X$ be an increasing operator. Assume that there are $D \subset X$ and $\Phi : D \to Y$ such that*

$$Nu \leq \Phi u \quad (\textit{respectively, } Nu \geq \Phi u) \tag{2.1}$$

*for all $u \in D$. Then any solution $u \in D$ of the equation*

$$u = A\Phi u, \tag{2.2}$$

*if there is one, is an upper (respectively, lower) solution of the equation $u = ANu$.*

*Proof.* Assume $v_0 \in D$ solves (2.2). Then, from (2.1) we have

$$Nv_0 \leq \Phi v_0$$

and since $A$ is increasing,

$$ANv_0 \leq A\Phi v_0 = v_0.$$

Hence $v_0$ is an upper solution. Similarly, if $Nu \geq \Phi u$ on $D$, then any solution of (2.2) is a lower solution of the equation $u = ANu$. □

If in Theorem 2.1 we add linearity, then we obtain the following result.

**Corollary 2.2.** *Let $X$, $Y$ be ordered linear spaces, $N : X \to Y$ any mapping and $A : Y \to X$ a linear increasing operator. Let $K_X$ be the cone of all elements $u$ of $X$ with $u \geq 0$. Assume there are $c \in \mathbf{R}_+$ and $w_0 \in Y$ such that*

$$Nu \leq cu + w_0 \tag{2.3}$$

*for all $u \in K_X \cap Y$. Then any solution $v_0 \in K_X$ of the equation*

$$u - cAu = Aw_0 \tag{2.4}$$

*is an upper solution of* (1.1). *If in addition,*

$$-N(-u) \leq cu + w_0 \tag{2.5}$$

*for all $u \in K_X \cap Y$, then $u_0 := -v_0$ is a lower solution of* (1.1).

*Proof.* In Theorem 2.1 take $D = K_X \cap Y$ and $\Phi u = cu + w_0$. For the second part of the corollary, take $D = (-K_X) \cap Y$, $\Phi u = cu - w_0$. □

Equation (2.4) suggests that more applicable results can be established if we take into account the spectral properties of $A$.

**Theorem 2.3.** *Let $X$ be a Banach space ordered by a normal cone $K$. Assume that $A : X \to X$ is a completely continuous linear operator whose non-zero eigenvalues are positive and that $A$ satisfies the weak maximum principle*

$$u - \alpha Au = Aw, \ w \in K \quad \textit{implies} \ \ u \in K \tag{2.6}$$

*for every $\alpha \in (-\infty, |A|^{-1})$. In addition assume that $N : X \to X$ is a continuous mapping such that*

$$Nu \le cu + w_0, \qquad N(-u) \ge -cu - w_0 \tag{2.7}$$

*for all $u \in K$ and some $0 < c < |A|^{-1}$, $w_0 \in K$, and there exists $a \in \mathbf{R}_+$ such that the operator*

$$Nu + au \quad \textit{is increasing on} \quad [-v_0, v_0],$$

*where $v_0$ is the (unique) solution of the equation $u - cAu = Aw_0$.*

*Then equation (1.1) has at least one solution. Moreover, if the set $\mathcal{S}_+$ $(\mathcal{S}_-)$ of all solutions $u \ge 0$ (respectively, $u \le 0$) is nonempty, then it has a maximal (respectively, minimal) element.*

*Proof.* First note that for any constant $\alpha < |A|^{-1}$, the operator $I - \alpha A$ is injective (equivalently, bijective, according to the Fredholm's alternative [4, p. 92]). Indeed, otherwise for some $u \in X \setminus \{0\}$ one has $u - \alpha Au = 0$. For $\alpha < 0$ this is impossible since all non-zero eigenvalues are assumed to be positive (here $1/\alpha$ is a non-zero eigenvalue). If $\alpha = 0$, this equality is obviously impossible. It remains to discuss the case $\alpha > 0$. Then $|u| = \alpha |Au| \le \alpha |A| |u|$, whence $\alpha \ge |A|^{-1}$, a contradiction. Thus our claim is proved.

Let $v_0$ be the unique solution of the equation $u - cAu = Aw_0$. From (2.6) one has $v_0 \ge 0$. Now, (2.7) guarantees both (2.3), (2.5). Thus, by Corollary 2.2, $v_0$ is an upper solution and $u_0 := -v_0$ is a lower solution. Let

$$N_a u = Nu + au.$$

The equation $u = ANu$ is equivalent to

$$u = (I + aA)^{-1} AN_a u.$$

Let

$$T_a = (I + aA)^{-1} AN_a.$$

Clearly $T_a$ is completely continuous on $[u_0, v_0]$. Also $T_a$ is increasing on $[u_0, v_0]$ since $N_a$ is increasing by our hypothesis and $(I + aA)^{-1} A$ is increasing as well. Indeed, if $w \in K$ and $u := (I + aA)^{-1} Aw$, then $u + aAu = Aw$ and by the weak maximum principle $u \in K$. Hence the linear operator $(I + aA)^{-1} A$ is increasing. In addition

$$T_a v_0 \le v_0. \tag{2.8}$$

To prove this denote $u := T_a v_0$. Then

$$\begin{aligned} u + aAu &= ANv_0 + aAv_0 = A(cv_0 + w_0 - h) + aAv_0 \\ &= cAv_0 + Aw_0 - Ah + aAv_0 = v_0 - Ah + aAv_0 \end{aligned}$$

where $h := cv_0 + w_0 - Nv_0 \in K$. Consequently

$$v_0 - u + aA(v_0 - u) = Ah$$

and by the weak maximum principle $v_0 - u \geq 0$ which proves (2.8). Similarly,

$$u_0 \leq T_a u_0.$$

Let $u^*, v^*$ be the minimal, respectively maximal solution in $[u_0, v_0]$ as guaranteed by the Monotone Iterative Principle (see [9] and [13]). One has

$$-v_0 \leq u^* \leq v^* \leq v_0.$$

We now show that if $w \in K$ solves $w = ANw$, then $w \leq v_0$. Indeed, from

$$w = ANw = A(cw + w_0 - h) = cAw + Aw_0 - Ah,$$

where $h := cw + w_0 - Nw \in K$, and

$$v_0 = cAv_0 + Aw_0, \tag{2.9}$$

by subtraction, we obtain

$$v_0 - w - cA(v_0 - w) = Ah.$$

Then by the weak maximum principle, $v_0 - w \geq 0$ and so $w \in [0, v_0]$. Consequently $w \leq v^*$. Hence $v^*$ is maximal in $\mathcal{S}_+$. Similarly, if $w \in -K$ and $w = ANw$, then $-v_0 \leq w$. Hence $u^*$ is minimal in $\mathcal{S}_-$. □

For our next theorem, an existence and localization result of a nonnegative nonzero solution, we assume that $X$ is a Hilbert space with inner product and norm $(.,.)$, $|.|$ ordered by a normal cone $K$, which is also a vector lattice with respect to the order relation introduced by $K$. Then any element $x \in X$ can be written as a difference of two elements $x^+, x^-$ of $K$, that is $x = x^+ - x^-$, where $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Thus for an element $x$ one has $x \geq 0$, if and only if $x^- = 0$. We also assume that

$$(x,\ y) \geq 0 \text{ for all } x, y \in K \text{ and } \left(x^+,\ x^-\right) = 0 \text{ for every } x \in X. \tag{2.10}$$

We also note that if $A : X \to X$ is a completely continuous positive (with $(Au,\ u) \geq 0$ for all $u \in X$) self-adjoint linear operator, then there exists $u_1 \in X$, $|u_1| = 1$, such that

$$|A| = (Au_1,\ u_1). \tag{2.11}$$

This follows from the characterization of the norm of self-adjoint linear operators:

$$|A| = \sup_{u \neq 0} \frac{|(Au,\ u)|}{|u|^2}.$$

Since $(x^+,\ x^-) = 0$ for every $x \in X$, we have $\left|u_1^+ + u_1^-\right| = \left|u_1^+ - u_1^-\right| = 1$. Also if $A(K) \subset K$, then

$$\begin{aligned} |A| &= \left(A\left(u_1^+ - u_1^-\right),\ u_1^+ - u_1^-\right) \\ &= \left(Au_1^+,\ u_1^+\right) + \left(Au_1^-,\ u_1^-\right) - 2\left(Au_1^+,\ u_1^-\right) \\ &\leq \left(Au_1^+,\ u_1^+\right) + \left(Au_1^-,\ u_1^-\right) + 2\left(Au_1^+,\ u_1^-\right) \\ &= \left(A\left(u_1^+ + u_1^-\right),\ u_1^+ + u_1^-\right) \leq |A| \left|u_1^+ + u_1^-\right|^2 \\ &= |A|. \end{aligned}$$

Hence in (2.11) we may assume that $u_1 \geq 0$ (otherwise, replace $u_1$ by $u_1^+ + u_1^-$).

**Theorem 2.4.** *Let $A : X \to X$ be a completely continuous positive self-adjoint linear operator such that weak maximum principle* (2.6) *holds, and let $N : X \to X$ be any continuous mapping such that $N(0) = 0$,*

$$Nu \leq cu + w_0 \tag{2.12}$$

*for all $u \in K$ and some $0 \leq c < |A|^{-1}$, $w_0 \geq u_1$, and*

$$N(\varepsilon u_1) \geq \varepsilon |A|^{-1} u_1 \tag{2.13}$$

*for all $\varepsilon \in [0, \varepsilon_0]$ and some $\varepsilon_0 > 0$. Here $u_1 \in K$, $|u_1| = 1$ and $(Au_1,\ u_1) = |A|$. In addition assume that there exists $a \in \mathbf{R}_+$ with*

$$Nu + au \quad \textit{increasing on} \quad [0, v_0],$$

*where $v_0$ is the* (*unique*) *solution of the equation $u - cAu = Aw_0$. Then equation* (1.1) *has a maximal solution in $K \setminus \{0\}$.*

*Proof.* First note that the non-zero eigenvalues of $A$ are positive since $A$ is positive. As above, the unique solution $v_0$ of the equation $u - cAu = Aw_0$ is an upper solution of the equation $u = ANu$. Since $N(0) = 0$, the null element is a solution, and so a lower solution. Now we apply the Monotone Iterative Principle to deduce the existence of a maximal fixed point $v^*$ in $[0, v_0]$ of the operator

$$T_a = (I + aA)^{-1} AN_a.$$

As in the proof of Theorem 2.3 we can show that $v^*$ is maximal in the set of all nonnegative solutions. To show that $v^* \neq 0$, we prove that $v^*$ is the maximal fixed point of $T_a$ in an order subinterval $[u_0, v_0] \subset [0, v_0]$ with $u_0 \neq 0$.

For any fixed $v \in X$ we consider the function

$$g(t) = \frac{(A(u_1 + tv),\ u_1 + tv)}{|u_1 + tv|^2},$$

which can be defined on a neighborhood of $t = 0$. This function attains its maximum $|A|$ at $t = 0$, so $g'(0) = 0$. Notice

$$g'(0) = 2\left[(Au_1,\ v) - |A|(u_1,\ v)\right].$$

Hence

$$u_1 = |A|^{-1} Au_1$$

(i.e., $|A|$ is the largest eigenvalue of $A$ and $u_1$ is an eigenvector). Let $u_0 = \varepsilon u_1$, where $0 < \varepsilon \leq \varepsilon_0$. Clearly

$$u_0 \geq 0, \quad u_0 \neq 0, \quad u_0 = |A|^{-1} A u_0.$$

Using (2.13), we deduce

$$\begin{aligned} u_0 &= |A|^{-1} A u_0 = A\left(|A|^{-1} u_0\right) \\ &\leq ANu_0. \end{aligned}$$

Thus $u_0$ is a lower solution of $u = ANu$. Also, from

$$v_0 = cAv_0 + Aw_0, \quad u_0 = |A|^{-1} A u_0,$$

we have

$$\begin{aligned} v_0 - u_0 &= cA(v_0 - u_0) + \left(c - |A|^{-1}\right) Au_0 + Aw_0. \\ &= cA(v_0 - u_0) + A\left[\left(c - |A|^{-1}\right) u_0 + w_0\right]. \end{aligned}$$

Since $w_0 \geq u_1$, we may write $w_0 = u_1 + h$, where $h = w_0 - u_1 \in K$. Then

$$v_0 - u_0 = cA(v_0 - u_0) + A\left[\left(\left(c - |A|^{-1}\right)\varepsilon + 1\right) u_1 + h\right].$$

Now we choose $\varepsilon > 0$ small enough so that $\left(c - |A|^{-1}\right)\varepsilon + 1 \geq 0$. Then

$$\left(\left(c - |A|^{-1}\right)\varepsilon + 1\right) u_1 \in K$$

and

$$\left(\left(c - |A|^{-1}\right)\varepsilon + 1\right) u_1 + h \in K$$

too, and by the maximum principle, $v_0 - u_0 \geq 0$. Next we apply the Monotone Iterative Principle to deduce the existence of a maximal fixed point in $[u_0, v_0]$ of $T_a$. Clearly it is equal to $v^*$. □

**Remark 2.5.** *Under the assumptions on $X$ from Theorem 2.4, the weak maximum principle holds for $A$ on $\left(-\infty, |A|^{-1}\right)$ if it holds on $(-\infty, 0]$.*

*Indeed, if (2.6) holds on $(-\infty, 0]$, then, in particular (take $\alpha = 0$ in (2.6)) $A(K) \subset K$. Furthermore, assume $\alpha \in \left(0, |A|^{-1}\right)$ and $u := v - \alpha Av \in K$. We have to show that $v \geq 0$, equivalently $v^- = 0$. Assume the contrary, i.e. $v^- \neq 0$. Then if we multiply by $v^-$ and we use (2.10), we obtain*

$$\begin{aligned} 0 &\leq (v^-, u) = (v^-, v) - \alpha(v^-, Av) \\ &= (v^-, v^+) - |v^-|^2 - \alpha(v^-, Av^+) + \alpha(v^-, Av^-) \\ &\leq -|v^-|^2 + \alpha(v^-, Av^-). \end{aligned}$$

*It follows that*

$$\alpha \geq \frac{|v^-|^2}{(v^-, Av^-)}.$$

*But* $(v^-,\ Av^-) \le |A|\,|v^-|^2$. *Then* $\alpha \ge \dfrac{1}{|A|}$, *a contradiction. Thus* $v^- = 0$.

## 3. Application to singular boundary value problems

We shall apply the above results to the boundary value problem for a singular second order differential equation

$$\begin{cases} -\frac{1}{p}(pu')' = q(t)\,f(u), & 0<t<1 \\ (pu')(0) = 0 \\ u(1) = 0 \end{cases} \tag{3.1}$$

where $p \in C[0,1] \cap C^1(0,1)$ with $p > 0$ on $(0,1)$, $\int_0^1 \frac{1}{p(t)}dt < \infty$ and $q \in L^\infty([0,1], \mathbf{R}_+)$. The equation is singular if $p$ is zero at $t = 0$ or/and $t = 1$. Such kind of problems are in connection with radial solutions to stationary diffusion and waves equations and arise from mathematical modelling of many processes in physics and biology [3], [7], [11].

By a solution of (3.1) we mean a function $u \in C[0,1] \cap C^1(0,1)$, with $pu' \in AC[0,1]$ which satisfies the differential equation for almost every $t \in (0,1)$.

Let $X = L_p^2[0,1]$ with inner product and norm

$$(u,v) = \int_0^1 p\ uv\ dt, \quad |u| = \left(\int_0^1 p\ u^2 dt\right)^{1/2}.$$

Clearly, $X$ is vector lattice ordered by the regular (hence, normal) cone $K$ of all nonnegative functions, with the additional property (2.10).

Denote $Lu = -\frac{1}{p}(pu')'$, where

$$\begin{aligned} D(L) &= \{u \in C[0,1] \cap C^1(0,1) :\ pu' \in AC[0,1], \\ Lu &\in L_p^2[0,1],\ (pu')(0) = u(1) = 0\}. \end{aligned}$$

It is easy to see that for every $h \in L_p^2[0,1]$ there is a unique $u \in D(L)$ with $Lu = h$, and

$$u(t) = \int_t^1 \frac{1}{p(s)} \int_0^s p(\tau)\,h(\tau)\,d\tau ds.$$

Let $A$ be the inverse of $L$, more exactly

$$A : L_p^2[0,1] \to L_p^2[0,1], \quad (Ah)(t) = \int_t^1 \frac{1}{p(s)} \int_0^s p(\tau)\,h(\tau)\,d\tau ds.$$

We note that $A$ has all the required properties, i.e., it is completely continuous, positive, self-adjoint (see e.g. [11]) and satisfies the weak maximum principle. To prove the last property, according to Remark 2.5, it is sufficient to show that (2.6) holds for $\alpha \le 0$. For $\alpha = 0$ this trivially holds as follows looking at the expression of $A$. Let $\alpha < 0$ and let $u - \alpha Au = Aw$ for some $w \in K$. Then

$$\begin{cases} -\frac{1}{p}(pu')' - \alpha u = w, & 0<t<1 \\ u(1) = (pu')(0) = 0. \end{cases} \tag{3.2}$$

Suppose that $u \notin K$. Then there would be an interval $[a,b]$, $0 \leq a < b \leq 1$ such that

$$\begin{aligned} u &< 0 \text{ in } (a,b), \quad u(b) = 0 \text{ and} \\ \text{either } a &= 0, \text{ or } 0 < a \text{ and } u(a) = 0. \end{aligned}$$

Then on $[a,b]$, one has $-\frac{1}{p}(pu')' \geq 0$, i.e. $(pu')' \leq 0$. Hence $pu'$ is decreasing on $[a,b]$ and since $(pu')(b) \geq 0$, we must have $pu' \geq 0$ on $[a,b]$. Then $u$ is increasing and since $u(b) = 0$, we have $u(a) < 0$. Hence $a = 0$ and $u < 0$ in $(0,b)$. Now integration from 0 to $b$ in (3.2) gives

$$-(pu')(b) - \alpha \int_0^b pudt = \int_0^b pwdt.$$

Since $\int_0^b pwdt \geq 0$ and $\alpha \int_0^b pudt > 0$, we deduce $(pu')(b) < 0$, a contradiction. Therefore $u \geq 0$ in $[0,1]$.

**Theorem 3.1.** *Assume $f \in C^1(\mathbf{R})$ and*

$$\limsup_{|x| \to \infty} \left| \frac{f(x)}{x} \right| < |A|^{-1} |q|_{L^\infty[0,1]}^{-1}. \tag{3.3}$$

*Then problem* (3.1) *has at least one solution. Moreover, if the set $\mathcal{S}_+$ $(\mathcal{S}_-)$ of all solutions $u \geq 0$ (respectively, $u \leq 0$) is nonempty, then it has a maximal (respectively, minimal) element.*

*Proof.* From (3.3) we can find a $c_0 \in \left(0, |A|^{-1} |q|_{L^\infty[0,1]}^{-1}\right)$ and a $\mu > 0$ such that

$$|f(x)| \leq c_0 |x| \quad \text{for} \quad |x| > \mu.$$

Next the continuity of $f$ on $[-\mu, \mu]$ guarantees the existence of a $c_1 > 0$ with

$$|f(x)| \leq c_1 \quad \text{on} \quad [-\mu, \mu].$$

Thus

$$|f(x)| \leq c_0 |x| + c_1 \quad \text{for all } x \in \mathbf{R}. \tag{3.4}$$

This implies that the mapping

$$N(u)(t) = q(t) f(u(t))$$

is well-defined and continuous from $L_p^2[0,1]$ to itself and

$$|Nu|_{L_p^2[0,1]} \leq |q|_{L^\infty[0,1]} \left(c_0 |u|_{L_p^2[0,1]} + c_1 |1|_{L_p^2[0,1]}\right).$$

On the other hand, if $u \in K = L_p^2([0,1], \mathbf{R}_+)$, then (3.4) guarantees

$$N(u) \leq cu + w_0 \quad \text{and} \quad -N(-u) \leq cu + w_0,$$

where $c = c_0 |q|_{L^\infty[0,1]} < |A|^{-1}$ and $w_0 = c_1 |q|_{L^\infty[0,1]}$.

If $v_0$ is the (unique) solution of the equation $u - cAu = Aw_0$, then $v_0 \in C([0,1], \mathbf{R}_+)$ and so $0 \leq v_0(t) \leq M$ for all $t \in [0,1]$ and some $M > 0$. Function $f$ being $C^1$, there is a number $a \in \mathbf{R}_+$ such that

$$|q|_{L^\infty[0,1]} f'(x) + a \geq 0 \quad \text{for all} \quad x \in [-M, M].$$

Consequently, $N(u) + au$ is an increasing operator on $[-v_0, v_0]$ and we may apply Theorem 2.3. □

Finally Theorem 2.4 yields the following result.

**Theorem 3.2.** *Assume* $q \equiv 1,\ f \in C^1(\mathbf{R}_+, \mathbf{R}),\ f(0) = 0$ *and*

$$\limsup_{x\to\infty} \frac{f(x)}{x} < |A|^{-1} < f'(0).$$

*Then problem* (3.1) *has a maximal positive solution.*

## References


[1] Agarwal, R.P., O'Regan, D., Precup, R., *Construction of upper and lower solutions with applications to singular boundary value problems*, J. Comput. Anal. Appl., **7**(2005), 205-221.

[2] Amman, H., *Fixed point equations and nonlinear eigenvalues problems in ordered Banach spaces,* SIAM Review, **18**(1976), 620-709.

[3] Anderson, N., Arthurs, A.M., *Analytic bounding functions for diffusion problems with Michaelis-Menten kinetics,* Bull. Math. Biol., **47**(1985), 145-153.

[4] Brezis, H., *Analyse fonctionnelle,* Masson, Paris, 1983.

[5] Cabada, A., *An overview of the lower and upper solutions method with nonlinear boundary value conditions*, Boundary Value Problems, **2011**(2011), Article ID 893753, 18 pages.

[6] Carl, S., Heikkilä, S., *Nonlinear Differential Equations in Ordered Spaces*, Chapman & Hall/CRC, Boca Raton, 2000.

[7] Chan, C.Y., Hon, Y.C., *Computational methods for generalized Emden Fowler models of neutral atoms,* Quart. Appl. Math., **46**(1988), 711-726.

[8] De Coster, C., Habets, P., *Two-Point Boundary Value Problems: Lower and Upper Solutions,* Elsevier, Amsterdam, 2006.

[9] Deimling, K., *Nonlinear Functional Analysis,* Springer, Berlin, 1985.

[10] Guo, D., Lakshmikantham, V., *Nonlinear Problems in Abstract Cones,* Academic Press, Boston, 1988.

[11] O'Regan, D., *Theory of Singular Boundary Value Problems,* World Scientific, Singapore, 1994.

[12] Precup, R., *Monotone technique to the initial values problem for a delay integral equation from biomathematics,* Studia Univ. Babeş-Bolyai Math., **40**(1995), no. 2, 63-73.

[13] Precup, R., *Methods in Nonlinear Integral Equations,* Kluwer, Dordrecht, 2002.

[14] Rus, I.A., *Fixed points, upper and lower fixed points: abstract Gronwall lemmas,* Carpathian J. Math., **20**(2004), 125-134.



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# Coupled fixed point theorems for Zamfirescu type operators in ordered generalized Kasahara spaces

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*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** In this paper we give some coupled fixed point theorems for Zamfirescu type operators in ordered generalized Kasahara spaces $(X, \to, d, \leq)$, where $d : X \times X \to \mathbb{R}^m_+$ is a premetric. An application concerning the existence and uniqueness of solutions for systems of functional-integral equations is also given.

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## 1. Introduction and preliminaries

Many coupled fixed point results were given in the context of complete generalized metric spaces, for generalized contraction mappings. If we carefully examine their proofs by the iteration method, we can see that in some cases, not all of the metric properties are essentials. We give here some coupled fixed point theorems and applications in a more general setting, the so called generalized Kasahara space.

We recall first the notion of $L$-space, given by M. Fréchet in [4].

**Definition 1.1.** *Let $X$ be a nonempty set. Let*

$$s(X) := \left\{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\right\}.$$

*Let $c(X)$ be a subset of $s(x)$ and $Lim : c(X) \to X$ be an operator. By definition the triple $(X, c(X), Lim)$ is called an $L$-space (denoted by $(X, \to)$) if the following conditions are satisfied:*

*(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.*

*(ii) if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and*

$$Lim(x_{n_i})_{i\in\mathbb{N}} = x.$$

**Remark 1.2.** For examples and more considerations on $L$-spaces, see I.A. Rus, A. Petruşel and G. Petruşel [10, pp.77-80].

The notion of generalized Kasahara space was introduced by I.A. Rus in [9] as follows:

**Definition 1.3.** *Let $(X, \to)$ be an L-space, $(G, +, \leq, \xrightarrow{G})$ be an L-space ordered semigroup with unity, $0$ be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is called a generalized Kasahara space if and only if the following compatibility condition between $\to$ and $d_G$ holds:*

$$\text{for all } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1}) < +\infty$$
$$\Rightarrow (x_n)_{n\in\mathbb{N}} \text{ is convergent in } (X, \to). \tag{1.1}$$

**Remark 1.4.** Notice that by the inequality with the symbol $+\infty$ in the compatibility condition (1.1), we understand that the series $\sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1})$ is bounded in $(G, \leq)$.

**Remark 1.5.** In the context of generalized Kasahara spaces, fixed point results for self generalized contractions were already given by S. Kasahara in [5], for the case when $G = \mathbb{R}_+ \cup \{+\infty\}$ and by I.A. Rus in [9], for the case when $G = \mathbb{R}^m_+$.

An example of generalized Kasahara space is the following one:

**Example 1.6 (I.A. Rus,** [9]**).** Let $\rho : X \times X \to \mathbb{R}^m_+$ be a generalized complete metric on a set $X$. Let $x_0 \in X$ and $\lambda \in \mathbb{R}^m_+$ with $\lambda \neq 0$. Let $d_\lambda : X \times X \to \mathbb{R}^m_+$ be defined by

$$d_\lambda(x, y) = \begin{cases} \rho(x, y) & , \text{ if } x \neq x_0 \text{ and } y \neq x_0, \\ \lambda & , \text{ if } x = x_0 \text{ or } y = x_0. \end{cases}$$

Then $(X, \xrightarrow{\rho}, d_\lambda)$ is a generalized Kasahara space.

We recall also a very useful tool which helps us to prove the uniqueness of the fixed point for operators defined on generalized Kasahara spaces.

**Lemma 1.7 (Kasahara's lemma** [5]**).** *Let $(X, \to, d_G)$ be a generalized Kasahara space. Then $d_G(x, y) = d_G(y, x) = 0$ implies $x = y$, for all $x, y \in X$.*

**Remark 1.8.** For more considerations on Kasahara spaces, see [3] and [9].

We introduce now the notion of ordered generalized Kasahara space.

**Definition 1.9.** *Let $(X, \to, d_G)$ be a generalized Kasahara space. Then $(X, \to, d_G, \leq)$ is an ordered generalized Kasahara space if and only if $(X, \leq)$ is a partially ordered set.*

**Example 1.10.** Let $X := C([a, b], \mathbb{R}^m) = \{x : [a, b] \to \mathbb{R}^m \mid x \text{ is continuous on } [a, b]\}$ be endowed with the partial order relation

$$x \leq_C y \Leftrightarrow x(t) \leq y(t) \Leftrightarrow x_i(t) \leq y_i(t), \text{ for all } t \in [a, b],\ i = \overline{1, m}.$$

We consider $\xrightarrow{\rho}$, the convergence structure induced by the Cebîşev norm

$$\rho : C([a, b], \mathbb{R}^m) \times C([a, b], \mathbb{R}^m) \to \mathbb{R}^m_+,$$

defined by

$$\rho(x,y)=\|x-y\|_C=\max_{t\in[a,b]}|x(t)-y(t)|=\begin{pmatrix}\max\limits_{t\in[a,b]}|x_1(t)-y_1(t)|\\ \vdots\\ \max\limits_{t\in[a,b]}|x_m(t)-y_m(t)|\end{pmatrix}.$$

Let $d: C([a,b],\mathbb{R}^m)\times C([a,b],\mathbb{R}^m)\to\mathbb{R}^m_+$, defined by

$$d(x,y)=\|x-y\|_C+\|(x-y)^p\|_C=\max_{t\in[a,b]}|x(t)-y(t)|+\max_{t\in[a,b]}|x(t)-y(t)|^p$$

$$=\begin{pmatrix}\max\limits_{t\in[a,b]}|x_1(t)-y_1(t)|+\max\limits_{t\in[a,b]}|x_1(t)-y_1(t)|^p\\ \vdots\\ \max\limits_{t\in[a,b]}|x_m(t)-y_m(t)|+\max\limits_{t\in[a,b]}|x_m(t)-y_m(t)|^p\end{pmatrix},$$

where $p\in\mathbb{N}$, $p\geq 2$.
Since $\rho(x,y)\leq d(x,y)$, for all $x,y\in C([a,b],\mathbb{R}^m)$ we get that $(C([a,b],\mathbb{R}^m),\overset{\rho}{\to},d,\leq_C)$ is an ordered generalized Kasahara space. (See also I.A. Rus, [9]).

Let $(X,\to,d_G,\leq)$ be an ordered generalized Kasahara space. Then we define

$$X_{\leq}:=\{(x_1,x_2)\in X\times X\mid x_1\leq x_2 \text{ or } x_2\leq x_1\}.$$

In the above setting, if $f: X\to X$ is an operator, then the Cartesian product of $f$ with itself is

$$f\times f: X\times X\to X\times X, \text{ given by } (f\times f)(x_1,x_2):=(f(x_1),f(x_2)).$$

In this paper, we consider the ordered generalized Kasahara space $(X,\to,d,\leq)$, where $d: X\times X\to\mathbb{R}^m_+$ is a premetric, i.e., $d(x,x)=0$, for all $x\in X$ and $d(x,y)\leq d(x,z)+d(z,y)$, for all $x,y,z\in X$.

We mention that if $\alpha,\beta\in\mathbb{R}^m$, $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_m)$, $\beta=(\beta_1,\beta_2,\ldots,\beta_m)$ and $c\in\mathbb{R}$ , then by $\alpha\leq\beta$ (respectively $\alpha<\beta$), we mean that $\alpha_i\leq\beta_i$ (respectively $\alpha_i<\beta_i$), for all $i=\overline{1,m}$ and by $\alpha\leq c$ we mean that $\alpha_i\leq c$, for all $i=\overline{1,m}$.

We denote by $\mathcal{M}_{m,m}(\mathbb{R}_+)$ the set of all $m\times m$ matrices with positive elements, by $O_m$ the zero $m\times m$ matrix and by $I_m$ the identity $m\times m$ matrix. If $A=(a_{ij})_{i,j=\overline{1,m}}$, $B=(b_{ij})_{i,j=\overline{1,m}}\in\mathcal{M}_{m,m}(\mathbb{R}_+)$, then by $A\leq B$ we understand $a_{ij}\leq b_{ij}$, for all $i,j=\overline{1,m}$. The symbol $A^\tau$ stands for the transpose of the matrix $A$. Notice also that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

A matrix $A\in\mathcal{M}_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if and only if $A^n\to O_m$ as $n\to\infty$ (see [10]). Regarding this class of matrices we have the following classical result in matrix analysis (see [1, Lemma 3.3.1, page 55], [11], [8, page 37], [13, page 12].

**Theorem 1.11.** *Let $A\in\mathcal{M}_{m,m}(\mathbb{R}_+)$. The following statements are equivalent:*

*(i) $A$ is convergent to zero;*
*(ii) $A^n\to O_m$ as $n\to\infty$;*

(*iii*) *the eigenvalues of $A$ lies in the open unit disc, i.e.,* $|\lambda| < 1$, *for all* $\lambda \in \mathbb{C}$ *with* $det(A - \lambda I_m) = 0$;

(*iv*) *the matrix* $I_m - A$ *is non-singular and*

$$(I_m - A)^{-1} = I_m + A + A^2 + \ldots + A^n + \ldots;$$

(*v*) *the matrix* $(I_m - A)$ *is non-singular and* $(I_m - A)^{-1}$ *has nonnegative elements;*

(*vi*) $A^n q \to 0 \in \mathbb{R}^m$ *and* $q^\tau A^n \to 0 \in \mathbb{R}^m$ *as* $n \to \infty$, *for all* $q \in \mathbb{R}^m$.

**Remark 1.12.** Some examples of matrices which converge to zero are:

a) any matrix $A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

b) any matrix $A := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

c) any matrix $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

We consider now the following particular matrix set:

$$\mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+) := \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1m} \\ 0 & a_{22} & a_{23} & \ldots & a_{2m} \\ 0 & 0 & a_{33} & \ldots & a_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & a_{mm} \end{pmatrix} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \;\middle|\; \max_{i=\overline{1,m}} a_{ii} < \frac{1}{2} \right\}.$$

**Lemma 1.13.** *Let* $A \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$. *Then the matrices* $A$ *and* $(I_m - A)^{-1}A$ *are convergent to zero.*

*Proof.* Since the eigenvalues of $A$ and $(I_m - A)^{-1}A$ are in the open unit disk, the conclusion follows from Theorem 1.11. □

**Remark 1.14.** For more considerations on matrices which converge to zero, see [6], [8] and [12].

Let $(X, \to)$ be an $L$-space and $f : X \to X$ be an operator. The following notations and notions will be needed in the sequel of this paper:

- $Fix(f) := \{x \in X \mid x = f(x)\}$ the set of all fixed points for $f$.
- $I(f) := \{Y \subset X \mid f(Y) \subset Y\}$ - the set of all invariant subsets of $X$ with respect to $f$.
- $Graph(f) := \{(x, y) \in X \times X \mid y = f(x)\}$ the graph of $f$. We say that $f$ has closed graph with respect to $\to$ or $Graph(f)$ is closed in $X \times X$ with respect to $\to$ if and only if for any sequences $(x_n)_{n\in\mathbb{N}} \subset X$, $(y_n)_{n\in\mathbb{N}} \subset X$ with $y_n = f(x_n)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$, $y_n \to y \in X$, as $n \to \infty$, we have that $y = f(x)$.
- A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called sequence of successive approximations for $f$ starting from a given point $x_0 \in X$ if $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Notice that $x_n = f^n(x_0)$, for all $n \in \mathbb{N}$.

## 2. Main results

Our first main result is the following one:

**Theorem 2.1.** *Let* $(X, \to, d, \leq)$ *be an ordered generalized Kasahara space, where* $d : X \times X \to \mathbb{R}^m_+$ *is a premetric, i.e.,* $d(x, x) = 0$ *and* $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$. *Let* $f : X \to X$ *be an operator. We assume that:*

*(i) for each* $(x, y) \in X_\leq$, *there exists* $z_{(x,y)} := z \in X$ *such that* $(x, z), (y, z) \in X_\leq$*;*
*(ii) for each* $(x, y) \in X_\leq$, *we have* $(x, f(x))$, $(y, f(y)) \in X_\leq$*;*
*(iii)* $X_\leq \in I(f \times f)$*;*
*(iv)* $f : (X, \to) \to (X, \to)$ *has closed graph;*
*(v)* $f$ *is a Zamfirescu type operator, i.e., at least one of the following conditions holds:*
*($v_1$) there exists* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *which converges to zero such that*

$$d(f(x), f(y)) \leq Ad(x, y), \text{ for all } (x, y) \in X_\leq$$

*($v_2$) there exists* $B \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ *such that*

$$d(f(x), f(y)) \leq B[d(x, f(x)) + d(y, f(y))], \text{ for all } (x, y) \in X_\leq$$

*($v_3$) there exists* $C \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ *such that*

$$d(f(x), f(y)) \leq C[d(x, f(y)) + d(y, f(x))], \text{ for all } (x, y) \in X_\leq$$

*(vi) there exists* $x_0 \in X$ *such that* $(x_0, f(x_0)) \in X_\leq$.

*Then* $f : (X, \to) \to (X, \to)$ *is a Picard operator.*

*Proof.* Let $x \in X$ be arbitrary.

Since $(x_0, f(x_0)) \in X_\leq$, by *(iii)* we have $(f(x_0), f^2(x_0)) \in X_\leq$.

If $f$ satisfies $(v_1)$ then

$$d(f(x_0), f^2(x_0)) \leq Ad(x_0, f(x_0)).$$

If $f$ satisfies $(v_2)$ then

$$d(f(x_0), f^2(x_0)) \leq B[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))],$$
$$\text{i.e., } d(f(x_0), f^2(x_0)) \leq (I_m - B)^{-1}Bd(x_0, f(x_0)).$$

If $f$ satisfies $(v_3)$ then

$$\begin{aligned} d(f(x_0), f^2(x_0)) &\leq C[d(x_0, f^2(x_0)) + d(f(x_0), f(x_0))] \\ &\leq C[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))], \\ \text{i.e., } d(f(x_0), f^2(x_0)) &\leq (I_m - C)^{-1}Cd(x_0, f(x_0)). \end{aligned}$$

Let $\Omega := \{A, (I_m - B)^{-1}B, (I_m - C)^{-1}C\}$. For any matrix $M \in \Omega$, we have $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and by Lemma 1.13, it follows that $M$ is a matrix that converges to zero. In addition, we have

$$d(f(x_0), f^2(x_0)) \leq Md(x_0, f(x_0)), \text{ for all } (x_0, f(x_0)) \in X_\leq \text{ and all } M \in \Omega.$$

Now, since $(f(x_0), f^2(x_0)) \in X_\leq$, by *(iii)* it follows that $(f^2(x_0), f^3(x_0)) \in X_\leq$.

If $f$ satisfies $(v_1)$ then

$$d(f^2(x_0), f^3(x_0)) \le Ad(f(x_0), f^2(x_0)) \le A^2 d(x_0, f(x_0)).$$

If $f$ satisfies $(v_2)$ then

$$\begin{aligned} d(f^2(x_0), f^3(x_0)) &\le B[d(f(x_0), f^2(x_0)) + d(f^2(x_0), f^3(x_0))], \\ \text{i.e., } d(f^2(x_0), f^3(x_0)) &\le (I_m - B)^{-1} B d(f(x_0), f^2(x_0)) \\ &\le [(I_m - B)^{-1} B]^2 d(x_0, f(x_0)). \end{aligned}$$

If $f$ satisfies $(v_3)$ then

$$\begin{aligned} d(f^2(x_0), f^3(x_0)) &\le C[d(f(x_0), f^3(x_0)) + d(f^2(x_0), f^2(x_0))] \\ &\le C[d(f(x_0), f^2(x_0)) + d(f^2(x_0), f^3(x_0))], \\ \text{i.e., } d(f^2(x_0), f^3(x_0)) &\le (I_m - C)^{-1} C d(f(x_0), f^2(x_0)) \\ &\le [(I_m - C)^{-1} C]^2 d(x_0, f(x_0)). \end{aligned}$$

In all three cases presented above, we conclude that

$$d(f^2(x_0), f^3(x_0)) \le M^2 d(x_0, f(x_0))$$

for all $(x_0, f(x_0)) \in X_\le$ and all $M \in \Omega$.

By induction, for $n \in \mathbb{N}$, we get

$$d(f^n(x_0), f^{n+1}(x_0)) \le M^n d(x_0, f(x_0))$$

for all $(x_0, f(x_0)) \in X_\le$ and all $M \in \Omega$.

Next, we obtain

$$\begin{aligned} \sum_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) &\le \sum_{n\in\mathbb{N}} M^n d(x_0, f(x_0)) \\ &= (I_m - M)^{-1} d(x_0, f(x_0)) < +\infty \end{aligned}$$

for all $(x_0, f(x_0)) \in X_\le$ and all $M \in \Omega$.

Since $(X, \to, d)$ is a generalized Kasahara space, we get that the sequence of successive approximations for $f$, starting from $x_0$, is convergent in $(X, \to)$. So, there exists $x^* \in X$ such that $f^n(x_0) \to x^*$ as $n \to \infty$. By $(iv)$ we get that $x^* \in Fix(f)$.

Notice also that:

• If $(x, x_0) \in X_\le$ then by $(iii)$ we have $(f^n(x), f^n(x_0)) \in X_\le$ and by $(ii)$ that $(x, f(x))$, $(y, f(y)) \in X_\le$.

If $f$ satisfies $(v_1)$ then

$$\begin{aligned} 0 &\le d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(x)) \\ &\le Ad(f^{n-1}(x), f^{n-1}(x_0)) + Ad(f^{n-1}(x_0), f^{n-1}(x)) \\ &\le \ldots \le A^n d(x, x_0) + A^n d(x_0, x) \overset{\mathbb{R}^m_+}{\to} 0 \text{ as } n \to \infty. \end{aligned}$$

If $f$ satisfies $(v_2)$ then

$$\begin{aligned}0 &\le d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(x))\\ &\le 2B[d(f^{n-1}(x), f^n(x)) + d(f^{n-1}(x_0), f^n(x_0))]\\ &\le 2B[(I_m - B)^{-1}B]^{n-1}[d(x, f(x)) + d(x_0, f(x_0))]\\ &\le 2(I_m + B + B^2 + \dots)B[(I_m - B)^{-1}B]^{n-1}[d(x, f(x)) + d(x_0, f(x_0))]\\ &= 2[(I_m - B)^{-1}B]^n[d(x, f(x)) + d(x_0, f(x_0))] \overset{\mathbb{R}^m_+}{\to} 0 \text{ as } n \to \infty.\end{aligned}$$

If $f$ satisfies $(v_3)$ then

$$\begin{aligned}0 &\le d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(x))\\ &\le 2C[d(f^{n-1}(x), f^n(x_0)) + d(f^{n-1}(x_0), f^n(x))]\\ &\le 2C[d(f^{n-1}(x), f^n(x)) + d(f^n(x), f^n(x_0))\\ &\qquad + d(f^{n-1}(x_0), f^n(x_0)) + d(f^n(x_0), f^n(x))],\\ \text{i.e., } 0 &\le d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(x))\\ &\le (I_m - 2C)^{-1}2C[d(f^{n-1}(x), f^n(x)) + d(f^{n-1}(x_0), f^n(x_0))]\\ &\le (I_m - 2C)^{-1}2C[(I_m - C)^{-1}C]^{n-1}[d(x, f(x)) + d(x_0, f(x_0))]\\ &\le (I_m - 2C)^{-1}2[(I_m - C)^{-1}C]^n[d(x, f(x)) + d(x_0, f(x_0))] \overset{\mathbb{R}^m_+}{\to} 0\\ &\quad \text{as } n \to \infty.\end{aligned}$$

In all three cases we get that $d(f^n(x), f^n(x_0)) = d(f^n(x_0), f^n(x)) = 0$. By Kasahara's lemma 1.7, it follows that $f^n(x) = f^n(x_0)$, for all $n \in \mathbb{N}$.

• If $(x, x_0) \notin X_\le$, then by $(i)$, there exists $z_{(x,x_0)} := z \in X$ such that $(x, z)$, $(x_0, z) \in X_\le$. Since $(x, z) \in X_\le$, by $(iii)$ we have $(f^n(x), f^n(z)) \in X_\le$ and by $(ii)$ that $(x, f(x))$, $(z, f(z)) \in X_\le$. In a similar way as presented above, we obtain $f^n(x) = f^n(z)$, for all $n \in \mathbb{N}$. On the other hand, since $(x_0, z) \in X_\le$ we get that $f^n(x_0) = f^n(z)$, for all $n \in \mathbb{N}$. Hence $f^n(x) = f^n(x_0) \to x^*$ as $n \to \infty$.

We show next the uniqueness of the fixed point $x^*$.

Let $y^* \in Fix(f)$ such that $y^* \neq x^*$.

If $(x^*, y^*) \in X_\le$, then by $(iii)$ we have $(f^n(x^*), f^n(y^*)) \in X_\le$ and by $(ii)$ that $(x^*, f(x^*))$, $(y^*, f(y^*)) \in X_\le$.

If $f$ satisfies $(v_1)$ then we have:

$$\begin{aligned}0 &\le d(f(x^*), f(y^*)) + d(f(y^*), f(x^*)) \le Ad(x^*, y^*) + Ad(y^*, x^*),\\ \text{i.e., } 0 &\le d(x^*, y^*) + d(y^*, x^*) \le (I_m - A)^{-1}0 = 0.\end{aligned}$$

If $f$ satisfies $(v_2)$ then we have:

$$\begin{aligned}0 &\le d(f(x^*), f(y^*)) + d(f(y^*), f(x^*)) \le 2B[d(x^*, f(x^*)) + d(y^*, f(y^*))],\\ \text{i.e., } 0 &\le d(x^*, y^*) + d(y^*, x^*) \le 2B[d(x^*, x^*) + d(y^*, y^*)] = 0.\end{aligned}$$

If $f$ satisfies $(v_3)$ then we have:

$$\begin{aligned}0 &\le d(f(x^*), f(y^*)) + d(f(y^*), f(x^*))\\ &\le 2C[d(x^*, f(y^*)) + d(y^*, f(x^*))] = 2C[d(x^*, y^*) + d(y^*, x^*)],\end{aligned}$$

i.e., $0 \le d(x^*, y^*) + d(y^*, x^*) \le (I_m - 2C)^{-1}0 = 0.$

So, in all three cases, we conclude that $d(x^*, y^*) = d(y^*, x^*) = 0$. By Kasahara's lemma 1.7, it follows that $x^* = y^*$.

If $(x^*, y^*) \notin X_{\le}$, then by $(i)$, there exists $z_{(x^*,y^*)} := z \in X$ such that $(x^*, z)$, $(y^*, z) \in X_{\le}$. Since $(x^*, z) \in X_{\le}$, by following the same way of proof as presented above, replacing $y^*$ with $z$, we get that $x^* = z$. On the other hand, since $(y^*, z) \in X_{\le}$, we get in a similar way that $y^* = z$. Hence $x^* = y^*$. □

In the sequel, we will apply the above result to the coupled fixed point problem generated by an operator.

Let $X$ be a nonempty set, endowed with a partial order relation denoted by $\le$. If we consider two arbitrary elements $z := (x, y)$, $w = (u, v)$ of $X \times X$, then, we can introduce a partial ordering relation on $X \times X$, denoted by $\preceq$ and defined as follows:

$$z \preceq w \text{ if and only if } (x \ge u \text{ and } y \le v).$$

**Theorem 2.2.** *Let $(X, \to, d, \le)$ be an ordered Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional, satisfying the following conditions: $d(x, x) = 0$, for all $x \in X$ and $d(x, y) \le d(x, z) + d(z, y)$, for all $x, y, z \in X$.*

*Let $S : X \times X \to X$ be an operator. We suppose that:*

*(i) for each $z = (x, y)$, $w = (u, v) \in X \times X$, which are not comparable with respect to the partial ordering $\preceq$ in $X \times X$, there exists $t := (t_1, t_2) \in X \times X$, which may depend on $(x, y)$ and $(u, v)$, such that $t$ is comparable with respect to the partial ordering $\preceq$, with both $z$ and $w$;*

*(ii) for each $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$, with $(x_1 \ge y_1$ and $x_2 \le y_2)$ or $(y_1 \ge x_1$ and $y_2 \le x_2)$ we have*

$$\left( \begin{cases} x_1 \ge S(x_1, x_2) \\ x_2 \le S(x_2, x_1) \end{cases} \text{or} \begin{cases} S(x_1, x_2) \ge x_1 \\ S(x_2, x_1) \le x_2 \end{cases} \right)$$

*and*

$$\left( \begin{cases} y_1 \ge S(y_1, y_2) \\ y_2 \le S(y_2, y_1) \end{cases} \text{or} \begin{cases} S(y_1, y_2) \ge y_1 \\ S(y_2, y_1) \le y_2 \end{cases} \right)$$

*(iii) for all $(x \ge u$ and $y \le v)$ or $(u \ge x$ and $v \le y)$, we have*

$$\begin{cases} S(x, y) \ge S(u, v) \\ S(y, x) \le S(v, u) \end{cases} \text{or} \begin{cases} S(u, v) \ge S(x, y) \\ S(v, u) \le S(y, x) \end{cases}$$

*i.e., $S$ has the generalized mixed monotone property;*

*(iv) $S : X \times X \to X$ has closed graph with respect to $\to$;*

*(v) at least one of the following conditions holds:*

$(v_1)$ *there exists* $k_1, k_2 \in \mathbb{R}_+$, $k_1 + k_2 < 1$ *such that*

$$d(S(x,y), S(u,v)) \leq k_1 d(x,u) + k_2 d(y,v)$$

$(v_2)$ *there exists* $k \in [0, \frac{1}{2}[$ *such that*

$$d(S(x,y), S(u,v)) \leq k[d(x, S(x,y)) + d(u, S(u,v))]$$

$(v_3)$ *there exists* $k \in [0, \frac{1}{2}[$ *such that*

$$d(S(x,y), S(u,v)) \leq k[d(x, S(u,v)) + d(u, S(x,y))]$$

$(vi)$ *there exists* $z_0 := (z_0^1, z_0^2) \in X \times X$ *such that*

$$\begin{cases} z_0^1 \geq S(z_0^1, z_0^2) \\ z_0^2 \leq S(z_0^2, z_0^1) \end{cases} \quad or \quad \begin{cases} S(z_0^1, z_0^2) \geq z_0^1 \\ S(z_0^2, z_0^1) \leq z_0^2 \end{cases}.$$

*Then there exists a unique element* $(x^*, y^*) \in X \times X$ *such that* $x^* = S(x^*, y^*)$ *and* $y^* = S(y^*, x^*)$ *and the sequence of successive approximations* $(S^n(w_0^1, w_0^2),\ S^n(w_0^2, w_0^1))$ *converges to* $(x^*, y^*)$ *as* $n \to \infty$, *for all* $w_0 = (w_0^1, w_0^2) \in X \times X$.

*Proof.* Let $Z := X \times X$ and consider $\preceq$, the partial order relation on $Z$, defined as follows: for all $z := (x,y)$, $w := (u,v) \in Z$, $z \preceq w$ if and only if ($x \geq u$ and $y \leq v$).

Let $Z_{\preceq} := \{(z,w) := ((x,y),(u,v)) \in Z \times Z \mid z \preceq w \text{ or } w \preceq z\}$.

Let $F : Z \to Z$ be an operator defined by

$$F(x,y) := \begin{pmatrix} S(x,y) \\ S(y,x) \end{pmatrix} = (S(x,y), S(y,x)).$$

We show that all of the assumptions of Theorem 2.1 are satisfied.

By $(i)$ and $(iv)$ it follows that the assumptions $(i)$ and $(iv)$ of Theorem 2.1 are satisfied.

By $(ii)$, since $x = (x_1, x_2) \in X \times X$ with

$$\begin{cases} x_1 \geq S(x_1, x_2) \\ x_2 \leq S(x_2, x_1) \end{cases} \quad \text{or} \quad \begin{cases} S(x_1, x_2) \geq x_1 \\ S(x_2, x_1) \leq x_2 \end{cases}$$

we have $(x_1, x_2) \preceq (S(x_1, x_2), S(x_2, x_1))$ and so, $x \preceq F(x)$. By a similar approach we get $F(x) \preceq x$. So, $(x, F(x)) \in Z_{\preceq}$. By following the same way of proof, we get $(y, F(y)) \in Z_{\preceq}$. Hence, the assumption $(ii)$ of Theorem 2.1 holds.

By $(iii)$, we have $Z_{\preceq} \in I(F \times F)$.

Indeed, let $z = (x,y)$, $w = (u,v) \in Z_{\preceq}$ be two arbitrary elements, where ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) such that

$$(1) \begin{cases} S(x,y) \geq S(u,v) \\ S(y,x) \leq S(v,u) \end{cases} \quad \text{or} \quad (2) \begin{cases} S(u,v) \geq S(x,y) \\ S(v,u) \leq S(y,x) \end{cases}$$

From (1) and (2) we have that $(S(x,y), S(y,x)) \preceq (S(u,v), S(v,u))$, i.e., $F(x,y) \preceq F(u,v)$ or $F(z) \preceq F(w)$. Similarly, we get $F(w) \preceq F(z)$. Hence, $(F(z), F(w)) \in Z_{\preceq}$, for all $(z,w) \in Z_{\preceq}$. So, $(F \times F)(Z_{\preceq}) \subset Z_{\preceq}$, i.e., $Z_{\preceq} \in I(F \times F)$. Thus, the assumption $(iii)$ holds.

By $(vi)$, since $(z_0^1, z_0^2) \in X \times X$ such that

$$\begin{cases} z_0^1 \geq S(z_0^1, z_0^2) \\ z_0^2 \leq S(z_0^2, z_0^1) \end{cases} \quad \text{or} \quad \begin{cases} S(z_0^1, z_0^2) \geq z_0^1 \\ S(z_0^2, z_0^1) \leq z_0^2 \end{cases}$$

we get that $(z_0^1, z_0^2) \preceq (S(z_0^1, z_0^2), S(z_0^2, z_0^1))$ and thus, $z_0 \preceq F(z_0)$. By a similar approach we get $F(z_0) \preceq z_0$. Hence, there exists $z_0 \in Z$ such that $(z_0, F(z_0)) \in Z_{\preceq}$, so, the assumption $(vi)$ of Theorem 2.1 holds.

Finally, we prove the assumption $(v)$ of Theorem 2.1.

Let $\tilde{d} : Z \times Z \to \mathbb{R}^2_+$, defined by $\tilde{d}((x,y),(u,v)) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix}$.

Since $(X, \to, d, \leq)$ is an ordered Kasahara space, it follows that $(X, \to, \tilde{d}, \leq)$ is an ordered generalized Kasahara space.

• If $(v_1)$ holds, then we have

$$\begin{aligned} \tilde{d}(F(x,y), F(u,v)) &= \tilde{d}((S(x,y), S(y,x)), (S(u,v), S(v,u))) \\ &= \begin{pmatrix} d(S(x,y), S(u,v)) \\ d(S(y,x), S(v,u)) \end{pmatrix} \leq \begin{pmatrix} k_1 d(x,u) + k_2 d(y,v) \\ k_1 d(y,v) + k_2 d(x,u) \end{pmatrix} \\ &= \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix} = A\tilde{d}((x,y),(u,v)). \end{aligned}$$

Since $k_1 + k_2 < 1$, we get that the matrix $A := \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix}$ is convergent to zero.

• If $(v_2)$ holds, then we have

$$\begin{aligned} \tilde{d}(F(x,y), F(u,v)) &= \tilde{d}((S(x,y), S(y,x)), (S(u,v), S(v,u))) \\ &= \begin{pmatrix} d(S(x,y), S(u,v)) \\ d(S(y,x), S(v,u)) \end{pmatrix} \leq \begin{pmatrix} k[d(x, S(x,y)) + d(u, S(u,v))] \\ k[d(y, S(y,x)) + d(v, S(v,u))] \end{pmatrix} \\ &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} d(x, S(x,y)) + d(u, S(u,v)) \\ d(y, S(y,x)) + d(v, S(v,u)) \end{pmatrix} \\ &= B[\tilde{d}((x,y), (S(x,y), S(y,x))) + \tilde{d}((u,v), (S(u,v), S(v,u)))] \\ &= B[\tilde{d}((x,y), F(x,y)) + \tilde{d}((u,v), F(u,v))]. \end{aligned}$$

Since $0 \leq k < \frac{1}{2}$, we get that the matrix $B := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \in \mathcal{M}^{\Delta}_{2,2}(\mathbb{R}_+)$.

• If $(v_3)$ holds, then we have

$$\begin{aligned} \tilde{d}(F(x,y), F(u,v)) &= \tilde{d}((S(x,y), S(y,x)), (S(u,v), S(v,u))) \\ &= \begin{pmatrix} d(S(x,y), S(u,v)) \\ d(S(y,x), S(v,u)) \end{pmatrix} \leq \begin{pmatrix} k[d(x, S(u,v)) + d(u, S(x,y))] \\ k[d(y, S(v,u)) + d(v, S(y,x))] \end{pmatrix} \\ &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} d(x, S(u,v)) + d(u, S(x,y)) \\ d(y, S(v,u)) + d(v, S(y,x)) \end{pmatrix} \\ &= C[\tilde{d}((x,y), (S(u,v), S(v,u))) + \tilde{d}((u,v), (S(x,y), S(y,x)))] \\ &= C[\tilde{d}((x,y), F(u,v)) + \tilde{d}((u,v), F(x,y))]. \end{aligned}$$

Since $0 \le k < \frac{1}{2}$, we get that the matrix $C := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \in \mathcal{M}^{\Delta}_{2,2}(\mathbb{R}_+)$.

We apply next Theorem 2.1 and the conclusion follows. □

## 3. Application

Let us consider the following system of functional-integral equations

$$(\mathbb{S}) \quad \begin{cases} x(t) = f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds) \\ y(t) = f(t, y(t), \int_a^b \Phi(t,s,y(s),x(s))ds) \end{cases}, \text{ for all } t \in [a,b] \subset \mathbb{R}_+.$$

By a solution of the system $(\mathbb{S})$ we understand a couple $(x,y) \in C[a,b] \times C[a,b]$, which satisfies the system for all $t \in [a,b] \subset \mathbb{R}_+$.

Let $X = C[a,b]$ be endowed with the partial order relation

$$x \le_C y \Leftrightarrow x(t) \le y(t), \text{ for all } t \in [a,b].$$

We consider $\xrightarrow{\rho}$, the convergence structure induced by the Cebîşev norm

$$\rho : C[a,b] \times C[a,b] \to \mathbb{R}_+,\ \rho(x,y) = \|x-y\|_C = \max_{t\in[a,b]} |x(t) - y(t)|.$$

Let $d : C[a,b] \times C[a,b] \to \mathbb{R}_+$, defined by

$$d(x,y) = \|(x-y)\|_C + \|(x-y)^2\|_C = \max_{t\in[a,b]} |x(t)-y(t)| + \max_{t\in[a,b]} (x(t)-y(t))^2.$$

Since $\rho(x,y) \le d(x,y)$, for all $x, y \in C[a,b]$ we get that $(C[a,b], \xrightarrow{\rho}, d, \le_C)$ is an ordered Kasahara space.

**Theorem 3.1.** *Let $\Phi : [a,b] \times [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ and $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ be two continuous mappings and consider the system $(\mathbb{S})$. We suppose that:*

$(i)$ *there exists $z_0 := (z_0^1, z_0^2) \in C[a,b] \times C[a,b]$ such that*

$$\begin{cases} z_0^1(t) \ge f(t, z_0^1(t), \int_a^b \Phi(t,s,z_0^1(t),z_0^2(t))ds) \\ z_0^2(t) \le f(t, z_0^2(t), \int_a^b \Phi(t,s,z_0^2(t),z_0^1(t))ds) \end{cases}$$

*or*

$$\begin{cases} z_0^1(t) \le f(t, z_0^1(t), \int_a^b \Phi(t,s,z_0^1(t),z_0^2(t))ds) \\ z_0^2(t) \ge f(t, z_0^2(t), \int_a^b \Phi(t,s,z_0^2(t),z_0^1(t))ds) \end{cases};$$

$(ii)$ *$f(t,\cdot,z)$ is increasing for all $t \in [a,b]$, $z \in \mathbb{R}$ and $\Phi(t,s,\cdot,z)$ is increasing, $\Phi(t,s,w,\cdot)$ is decreasing and $f(t,w,\cdot)$ is increasing for all $t,s \in [a,b]$, $w,z \in \mathbb{R}$, or, $f(t,\cdot,z)$ is decreasing for all $t \in [a,b]$, $z \in \mathbb{R}$ and $\Phi(t,s,\cdot,z)$ is decreasing, $\Phi(t,s,w,\cdot)$ is increasing and $f(t,w,\cdot)$ is decreasing for all $t,s \in [a,b]$, $w,z \in \mathbb{R}$*

$(iii)$ *there exists $k_1, k_2 \in [0, \frac{\sqrt{5}-1}{4}[$ such that*

$$|f(t,w_1,z_1) - f(t,w_2,z_2)| \le k_1|w_1 - f(t,w_1,z_1)| + k_2|w_2 - f(t,w_2,z_2)|$$

*for all $t \in [a,b]$ and $w_1, w_2, z_1, z_2 \in \mathbb{R}$.*

(iv) *for all* $x = (x_1, x_2)$, $y = (y_1, y_2) \in C[a,b] \times C[a,b]$, *with* ($x_1(t) \geq y_1(t)$ *and* $x_2(t) \leq y_2(t)$) *or* ($y_1(t) \geq x_1(t)$ *and* $y_2(t) \leq x_2(t)$) *we have*

$$\left(\begin{cases} x_1(t) \geq f(t, x_1(t), \int_a^b \Phi(t,s,x_1(t),x_2(t))ds) \\ x_2(t) \leq f(t, x_2(t), \int_a^b \Phi(t,s,x_2(t),x_1(t))ds) \end{cases} \quad or \right.$$

$$\left.\begin{cases} f(t, x_1(t), \int_a^b \Phi(t,s,x_1(t),x_2(t))ds) \geq x_1(t) \\ f(t, x_2(t), \int_a^b \Phi(t,s,x_2(t),x_1(t))ds) \leq x_2(t) \end{cases}\right)$$

*and*

$$\left(\begin{cases} y_1(t) \geq f(t, y_1(t), \int_a^b \Phi(t,s,y_1(t),y_2(t))ds) \\ y_2(t) \leq f(t, y_2(t), \int_a^b \Phi(t,s,y_2(t),y_1(t))ds) \end{cases} \quad or \right.$$

$$\left.\begin{cases} f(t, y_1(t), \int_a^b \Phi(t,s,y_1(t),y_2(t))ds) \geq y_1(t) \\ f(t, y_2(t), \int_a^b \Phi(t,s,y_2(t),y_1(t))ds) \leq y_2(t) \end{cases}\right)$$

*for all* $t \in [a,b]$.

*Then there exists a unique solution* $(x^*, y^*)$ *for the system* $(\mathbb{S})$.

*Proof.* Let us consider the operator $S : C[a,b] \times C[a,b] \to C[a,b]$, defined by

$$S(x,y)(t) := f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds).$$

Then the system $(\mathbb{S})$ is equivalent with $\begin{cases} x = S(x,y) \\ y = S(y,x) \end{cases}$.

Since $S(x,y)$ is a continuous operator on $(C[a,b] \times C[a,b], \overset{\rho}{\to})$, it follows that $Graph(S)$ is closed with respect to $\overset{\rho}{\to}$.

For all ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) we have

$$\begin{aligned} &|S(x,y)(t) - S(u,v)(t)| \\ &\quad = |f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds) - f(t, u(t), \int_a^b \Phi(t,s,u(s),v(s))ds)| \\ &\quad \overset{(iii)}{\leq} k_1 |x(t) - f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds)| \\ &\qquad + k_2 |u(t) - f(t, u(t), \int_a^b \Phi(t,s,u(s),v(s))ds)| \\ &\quad \leq k_1 \big(|x(t) - S(x,y)(t)| + |x(t) - S(x,y)(t)|^2\big) \\ &\qquad + k_2 \big(|u(t) - S(u,v)(t)| + |u(t) - S(u,v)(t)|^2\big). \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&|S(x,y)(t)-S(u,v)(t)|^2\\
&\quad\le\Big(k_1|x(t)-f(t,x(t),\int_a^b\Phi(t,s,x(s),y(s))ds)|\\
&\qquad\quad+k_2|u(t)-f(t,u(t),\int_a^b\Phi(t,s,u(s),v(s))ds)|\Big)^2\\
&\quad=\big(k_1|x(t)-S(x,y)(t)|+k_2|u(t)-S(u,v)(t)|\big)^2\\
&\quad\le 2\big(k_1^2|x(t)-S(x,y)(t)|^2+k_2^2|u(t)-S(u,v)(t)|^2\big)\\
&\quad\le 2k_1^2\big(|x(t)-S(x,y)(t)|+|x(t)-S(x,y)(t)|^2\big)\\
&\qquad\quad+2k_2^2\big(|u(t)-S(u,v)(t)|+|u(t)-S(u,v)(t)|^2\big).
\end{aligned}
$$

We get further that:

$$
\begin{aligned}
&|S(x,y)(t)-S(u,v)(t)|+|S(x,y)(t)-S(u,v)(t)|^2\\
&\quad\le(k_1+2k_1^2)\big(|x(t)-S(x,y)(t)|+|x(t)-S(x,y)(t)|^2\big)\\
&\qquad+(k_2+2k_2^2)\big(|u(t)-S(u,v)(t)|+|u(t)-S(u,v)(t)|^2\big).
\end{aligned}
$$

Hence, by taking the maximum over $t\in[a,b]$ we get

$$d(S(x,y),S(u,v))\le\mathcal{K}\big[d(x,S(x,y))+d(u,S(u,v))\big],$$

for all ($x\ge u$ and $y\le v$) or ($u\ge x$ and $v\le y$), where

$$\mathcal{K}:=\max\{k_1+2k_1^2,k_2+2k_2^2\}.$$

Since $k_1,k_2\in[0,\frac{\sqrt{5}-1}{4}[$, we get that $0\le\mathcal{K}<\frac{1}{2}$.

We see that all the assumptions of Theorem 2.2 are satisfied and the conclusion follows. □

**Remark 3.2.** Similar applications were given in [2] and [7].

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## References

[1] Allaire, G., *Numerical Linear Algebra*, Springer, New York, 2008.

[2] Bota, M.F., Petruşel, A., Petruşel, G., Samet, B., *Coupled fixed point theorems for single-valued operators in b-metric spaces*, Fixed Point Theory Appl., (2015), DOI 10.1186/s13663-015-0482-3.

[3] Filip, A.-D., *Fixed Point Theory in Kasahara Spaces*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2015.

[4] Fréchet, M., *Les espaces abstraits*, Gauthier-Villars, Paris, 1928.

[5] Kasahara, S., *On some generalizations of the Banach contraction theorem*, Publ. RIMS, Kyoto Univ., **12**(1976), 427-437.

[6] Perov, A.I., *On Cauchy problem for a system of ordinary differential equations*, Pviblizhen, Met. Reshen. Differ. Uvavn., **2**(1964), 115-134.

[7] Petruşel, A., Petruşel, G., Samet, B., Yao, J.-C., *Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to operator equation systems*, Fixed Point Theory, **17**(2016), no. 2, 457-476.

[8] Rus, I.A., *Principles and Applications of the Fixed Point Theory*, (Romanian), Ed. Dacia, 1979.

[9] Rus, I.A., *Kasahara spaces*, Sci. Math. Jpn., **72**(2010), no. 1, 101-110.

[10] Rus, I.A., Petruşel, A., Petruşel, G., *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008.

[11] Precup, R., *The role of the matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comput. Modelling, **49**(2009), 703-708.

[12] Turinici, M., *Finite dimensional vector contractions and their fixed points*, Studia Univ. Babeş-Bolyai Math., **35**(1990), no.1, 30-42.

[13] Varga, R.S., *Matrix Iterative Analysis*, Springer, Berlin, 2000.

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# Ulam-Hyers stability of Black-Scholes equation

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*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** The goal of this paper is to give a Ulam-Hyers stability result for Black-Scholes equation, in which the unknown function is the price of a derivative financial product. Our approach is based on Green function.

**Mathematics Subject Classification (2010):** 35L70, 45H10, 47H10.

**Keywords:** Black-Scholes equation, Ulam-Hyers stability, generalized Ulam-Hyers stability, derivative financial product, Green function.

## 1. Introduction

The Black-Scholes equation was introduced as a model for the financial mathematics ([1]). We will consider the following equation ([10], [11]):

$$\frac{\partial V(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) = F(s,t) \tag{1.1}$$

$$\Omega = \{(s,t) \mid s \in (s_1,s_2),\ t \in (T_1,T)\},\ V \in C^2(\Omega),$$

where $V(s,t)$ represents the price of the derivative financial product. The independent variables $(s,t)$ are the share price of the underlying assets and time, respectively. The constants $\sigma$ and $r$ are the volatility of the underlying asset and the risk-free interest rate, respectively. This equation is of the parabolic type and it can be considered as a diffusion equation. In what follows, we refer to this equation as BS equation. In this case we consider the conditions ([11]):

(i) Cauchy problem:

$$V(s,T) = \varphi(s), \tag{1.2}$$

$\varphi(s)$ is the pay-off function of a given derivative problem at $t = T$.

(ii) The boundary conditions (Darboux):

$$V(s_1,t) = b_1(t), \quad V(s_2,t) = b_2(t). \tag{1.3}$$

By an appropriate substitution ([11]), we obtain the equation:

$$\frac{\partial v(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 v(s,t)}{\partial s^2} + rs\frac{\partial v(s,t)}{\partial s} - rv(s,t) = h(s,t), \tag{1.4}$$

with

$$v(s,T) = f(s) \tag{1.5}$$

and homogeneous conditions:

$$v(s_1,t) = v(s_2,t) = 0, \tag{1.6}$$

$$\Omega = \{(s,t) \mid s \in (s_1,s_2),\ t \in (T_1,T)\},\ h \in C(\Omega,\mathbb{R}).$$

In what follows we consider the Cauchy-Darboux problem (1.4)+(1.5)+(1.6).
Here ([11])

$$\begin{aligned} h(s,t) = F(s,t) &+ \frac{s-s_1}{s_2-s_1}[r(b_2(t)-b_1(t)) + b_1'(t) - b_2'(t)] \\ &- b_1'(t) + rb_1(t) + rs\frac{b_1(t)-b_2(t)}{s_2-s_1} \end{aligned} \tag{1.7}$$

and

$$v(s,t) = \int_{s_1}^{s_2} G(s,t;\eta)f(\eta)d\eta + \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta)h(\eta,\tau)d\eta d\tau. \tag{1.8}$$

Further we study the problem of Ulam-Hyers stability of this equation, where the unknown function appears here as the price of financial derivatives.

We recall that this equation can be called Black-Merton-Scholes equation and it was a subject of the Nobel Prize in Economics in 1997.

## 2. Notions and definitions

In this section we will present some types of Ulam stability for the Black-Scholes equation.

In 1940, on a talk given at Wisconsin University, S.M. Ulam formulated the following problem: "*Under what conditions does there exist near every approximately homomorphism of a given metric group an homomorphism of the group?*" ([4], [8], [9], [12], [13], [20]). Generally, we say that a differential equation is stable in Ulam sense if for every approximate solution of the differential equation, there exists an exact solution near it. The goal of this paper is to give a stability result for Black-Scholes equation ([1], [11]).

It seems that the first paper on the Ulam-Hyers stability of partial differential equations was written by Prástaro and Rassias ([15]). For other results on the stability of differential equations and partial differential equations we refer to ([2], [3], [5], [6], [7], [14], [17], [19]).

Let $\varepsilon > 0$, $\varphi \in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\varphi(0) = 0$. We consider the following inequations:

$$\left|\frac{\partial u(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2} + rs\frac{\partial u(s,t)}{\partial s} - ru(s,t) - h(s,t)\right| \le \varepsilon,\ \forall\ (s,t) \in \Omega \tag{2.1}$$

$$\left|\frac{\partial u(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2} + rs\frac{\partial u(s,t)}{\partial s} - ru(s,t) - h(s,t,u)\right| \le \varepsilon,\ \forall\ (s,t) \in \Omega. \tag{2.2}$$

**Definition 2.1.** ([17], [18]) *The equation* (1.4) *is Ulam-Hyers stable if there exists a real number* $c_1$ *such that for each solution* $u$ *of* (2.1) *there exists a solution* $v$ *of* (1.4) *with*

$$|u(s,t) - v(s,t)| \le c_1 \cdot \varepsilon,\ \forall\ (s,t) \in \Omega. \tag{2.3}$$

**Definition 2.2.** *The equation* (1.4) *is generalized Ulam-Hyers stable if there exists* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi(0) = 0$, *continuous, such that for each solution* $u$ *of* (2.2) *there exists a solution* $v$ *of* (1.4) *with*

$$|u(s,t) - v(s,t)| \le \varphi(\varepsilon),\ \forall\ (s,t) \in \Omega. \tag{2.4}$$

**Remark 2.3.** A function $u$ is a solution of (2.1) if and only if there exists a function $g \in C(\Omega)$ such that

(i) $|g(s,t)| \le \varepsilon,\ \forall\ (s,t) \in \Omega$;

(ii) $\dfrac{\partial u(s,t)}{\partial t} + \dfrac{\sigma^2 s^2}{2} \dfrac{\partial^2 u(s,t)}{\partial s^2} + rs\dfrac{\partial u(s,t)}{\partial s} - ru(s,t) = h(s,t) + g(s,t)$.

**Remark 2.4.** A function $u$ is a solution of (2.2) if and only if there exists a function $g \in C(\Omega)$ such that

(i) $|g(s,t)| \le \varepsilon,\ \forall\ (s,t) \in \Omega$;

(ii) $\dfrac{\partial u(s,t)}{\partial t} + \dfrac{\sigma^2 s^2}{2} \dfrac{\partial^2 u}{\partial s^2} + rs\dfrac{\partial u(s,t)}{\partial s} - ru(s,t) = h(s,t) + g(s,t)$.

## 3. Ulam-Hyers stability of equation BS

Here we will present some results of Ulam-Hyers stability for the equation BS.

**Theorem 3.1.** *We suppose that:*

*(i)* $\Omega$ *is a bounded domain and* $G$ *is the Green function for the BS equation;*

*(ii)* $h \in C(\overline{\Omega})$, $f \in C(s_1, s_2)$;

*(iii)* $\displaystyle\int_t^T \int_{s_1}^{s_2} |G(s, t-\tau; \eta)| d\eta d\tau \le q < 1,\ \forall\ (s,t) \in \Omega.$

*Then:*

*(a) the problem* (1.5) + (1.6) *has a unique solution;*

*(b) the equation BS,* (1.5), *is Ulam-Hyers stable.*

*Proof.* (a) This is a well known result, consequence of Banach principle ([16]).

(b) Let $u$ be a solution of the inequation (2.1). Let $v$ be the unique solution of the problem (1.5)+(1.6). From Remark 2.3 and the condition (iii) we have that

$$|u(s,t) - v(s,t)| \le \left| \int_{s_1}^{s_2} G(s,t;\eta) f(\eta) d\eta + \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(\eta,\tau) d\eta d\tau \right.$$

$$+ \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) g(\eta,\tau) d\eta d\tau - \int_{s_1}^{s_2} G(s,t;\eta) f(\eta) d\tau$$

$$\left. - \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(\eta,\tau) d\eta d\tau \right|$$

$$\leq \int_t^T \int_{s_1}^{s_2} |G(s,t-\tau;\eta)| \cdot |g(\eta,\tau)| d\eta d\tau \leq q \cdot \varepsilon.$$

So, the equation (1.5) is Ulam-Hyers stable.

## 4. Generalized Ulam-Hyers stability of nonlinear BS equation

In this paragraph we will consider the nonlinear BS equation. Let $\Omega$ be the domain considered above.

In what follows, we consider the mixed problem (Cauchy-Darboux) ([11]):

$$\frac{\partial v(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2} \frac{\partial^2 v(s,t)}{\partial s^2} + rs\frac{\partial v(s,t)}{\partial s} - rv(s,t) = h(s,t,v), \tag{4.1}$$

$$\begin{aligned} &v(s,T) = f(s), \\ &v(s_1,t) = v(s_2,t) = 0. \end{aligned} \tag{4.2}$$

**Theorem 4.1.** *Let us consider the equation* (4.1) *and the inequation* (2.2). *Let $G$ be the Green function corresponding to BS equation.*

*We suppose that:*

*(i) $h \in C(\Omega)$ and there exists $l_h > 0$ with*

$$l_h \int_t^T \int_{s_1}^{s_2} |G(s,t-\tau;\eta)| d\eta d\tau \leq q < 1$$

*and a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|h(s,t,u) - h(s,t,v)| \leq l_h \varphi(|u-v|).$$

*Then*

*(a) the Cauchy-Darboux problem* (4.1) + (4.2) *has a unique solution $v$;*

*(b) if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(z) = z - \varphi(z)$, is strictly increasing and onto or bijective, the problem* (4.1) + (4.2) *is generalized Ulam-Hyers stable.*

*Proof.* (a) This result is a consequence of Banach theorem.

(b) Let $u$ be a solution of the inequality (2.2) and $v$ the unique solution of the problem (4.1)+(4.2). From the above conditions we have

$$|u(s,t) - v(s,t)| = \Bigg| \int_{s_1}^{s_2} G(s,t;\eta) f(\eta) d\eta + \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(s,t,u) d\eta d\tau$$

$$+ \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) g(\eta,\tau) d\eta d\tau - \int_{s_1}^{s_2} G(s,t;\eta) f(\eta) d\eta$$

$$- \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(\eta,\tau,v) d\eta d\tau \Bigg|$$

$$= \Bigg| \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) h(s,t,u) d\eta d\tau + \int_t^T \int_{s_1}^{s_2} G(s,t-\tau;\eta) g(\eta,\tau) d\eta d\tau$$

$$- \int_t^T \int_{s_1}^{s_2} G(s, t-\tau; \eta) h(s,t,v) d\eta d\tau \Bigg|$$

$$\leq \int_t^T \int_{s_1}^{s_2} |G(s, t-\tau; \eta)| \cdot |h(s,t,u) - h(s,t,v)| d\eta d\tau$$

$$+ \int_t^T \int_{s_1}^{s_2} |G(s, t-\tau; \eta)| \cdot |g(\eta, \tau)| d\eta d\tau$$

$$\leq \int_t^T \int_{s_1}^{s_2} |G(s, t-\tau; \eta)| l_h \varphi(|u-v|) d\eta d\tau + \int_t^T \int_{s_1}^{s_2} |G(s, t-\tau; \eta)| \cdot |g(\eta, \tau)| d\eta d\tau,$$

then we have

$$|u(s,t) - v(s,t)| \leq \varphi(|u(s,t) - v(s,t)|) + \frac{\varepsilon}{l_h}$$

and

$$\psi(|u(s,t) - v(s,t)|) \leq \frac{\varepsilon}{l_h},$$

therefore we have

$$|u(s,t) - v(s,t)| \leq \psi^{-1} \left( \frac{\varepsilon}{l_h} \right).$$

So the equation (4.1) is generalized Ulam-Hyers stable.

## References


[1] Black, F., and Scholes, M.S., *The pricing of options and corporate liabilities*, Journal of Political Economics, 81, 1973, 637-654.

[2] Brzdek, J., Popa, D., Xu, B., *The Hyers-Ulam stability of nonlinear recurrences*, J. Math. Anal. Appl., **335**(2007), 443-449.

[3] Câmpean, D.S., Popa, D., *Hyers-Ulam stability of Euler's equation*, Appl. Math. Lett., **24**(2011), 1539-1543.

[4] Hyers, D.H., *On the stability of the linear functional equation*, Proc. Math. Acad. Sci. USA, **27**(1941), 222-242.

[5] Hyers, D.H., Isac, G., Rassias, Th.M., *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[6] Jung, S.-M., *Hyers-Ulam stability of linear partial differential equations of first order*, Appl. Math. Lett., **22**(2009), 70-74.

[7] Lungu, N., Ciplea, S.A., *Ulam-Hyers-Rassias stability of pseudoparabolic partial differential equations*, Carpathian J. Math., **31**(2015), no. 2, 233-240.

[8] Lungu, N., Popa, D., *Hyers-Ulam stability of a first order partial differential equation*, J. Math. Anal. Appl., **385**(2012), 86-91.

[9] Lungu, N., Popa, D., *On the Ulam-Hyers stability of first order partial differential equations*, Carpathian J. Math., **28**(2012), no. 1, 77-82.

[10] Melnikov, M.Y., Melnikov, Y.A., *Construction of Green's functions for the Black-Scholes equation*, Electron. J. Differential Equations, **153**(2007), 1-14.

[11] Melnikov, Y.A., Melnikov, M.Y., *Green's Functions, Construction and Applications*, De Gruyter Studies in Mathematics, Berlin/Boston, **42**(2012).

[12] Popa, D., *Hyers-Ulam-Rassias stability of linear recurrence*, J. Math. Anal. Appl., **309**(2005), 591-597.

[13] Popa, D., *Hyers-Ulam stability of the linear recurrence with constant coefficients*, Adv. Difference Equ., **2**(2005), 101-107.

[14] Popa, D., Raşa, I., *On Hyers-Ulam stability of the linear differential equation*, J. Math. Anal. Appl., **381**(2011), 530-537.

[15] Prástaro, A., Rassias, Th.M., *Ulam stability in geometry PDE's*, Nonlinear Funct. Anal. Appl., **8**(2)(2003), 259-278.

[16] Rus, I.A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979.

[17] Rus, I.A., Lungu, N., *Ulam stability of a nonlinear hyperbolic partial differential equation*, Carpathian J. Math., **24**(2008), no. 3, 403-408.

[18] Rus, I.A., *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10**(2009), no. 2, 305-320.

[19] Rus, I.A., *Ulam stability of ordinary differential equations*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), no. 4, 125-133.

[20] Ulam, S.M., *A Collection of Mathematical Problems*, Interscience, New York, 1960.

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# Coupled fixed point theorems for rational type contractions

Anca Oprea and Gabriela Petruşel

*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** In this paper, we will consider the coupled fixed problem in $b$-metric space for single-valued operators satisfying a generalized contraction condition of rational type. First part of the paper concerns with some fixed point theorems, while the second part presents a study of the solution set of the coupled fixed point problem. More precisely, we will present some existence and uniqueness theorems for the coupled fixed point problem, as well as a qualitative study of it (data dependence of the coupled fixed point set, well-posedness, Ulam-Hyers stability and the limit shadowing property of the coupled fixed point problem) under some rational type contraction assumptions on the mapping.

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**Keywords:** Fixed point, ordered metric space, rational type contraction, coupled fixed point, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing property.

## 1. Introduction and preliminaries

The notion of b-metric spaces and discussion on the topological structure of it appeared in several papers, such as L.M. Blumenthal [2], S. Czerwik [6], N. Bourbaki [5], Heinonen [10].

On the other hand, the concept of coupled fixed point problem, was considered, for the first time, by Opoitsev in [14]-[15], but a very fruitful approach in this field was proposed by D. Guo, V. Lashmikantham [9] and T. Gnana Bhaskar and V. Lashmikantham [7]. Later on, many results related to this kind of problem appeared (see, for example [8], [13],. . . ).

Moreover, starting with the paper of Dass and Gupta [9], several extensions of the contraction principle considered the case of self mappings satisfying some rational type contraction assumptions, see, for example, [7].

Our aim is to consider both of the above research directions. More precisely, we will prove, using some adequate fixed point theorems for monotone rational contractions in ordered $b$-metric spaces, some coupled fixed point theorems for operators $T : X \times X \to X$ satisfying some rational type assumptions on comparable elements.

We shall recall some well known notions and definition of the b-metric spaces.

**Definition 1.1.** Let $X$ be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if the following axioms are satisfied:

1. if $x, y \in X$, then $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

A pair $(X, d)$ with the above properties is called a b-metric space.

Let $(X, \leq)$ be a partially ordered set and $d$ a metric on $X$. Notice that we can endow the product space $X \times X$ with the partial order $\leq_p$ given by

$$(x, y) \leq_p (u, v) \Leftrightarrow x \leq u, y \geq v.$$

**Definition 1.3.** Let $(X, \leq)$ be a partially ordered set and let $T : X \times X \to X$. We say that $T$ has the mixed monotone property if $T(\cdot, y)$ is monotone increasing for any $y \in X$ and $T(x, \cdot)$ is monotone decreasing for any $x \in X$.

**Lemma 1.4.** Let $(X, d)$ be a b-metric space. Then the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called:

i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case we write $\lim\limits_{n\to\infty} x_n = x$;

ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

If $(X, d)$ is a metric space and $T : X \times X \to X$ is an operator, then by definition, a coupled fixed point for $T$ is a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases} \qquad (P_1)$$

We will denote by $CFix(T)$ the coupled fixed point set for $T$.

The aim of this paper is to present, in the framework of complete ordered $b$-metric spaces, some existence and uniqueness theorems for the coupled fixed point problem, as well as, a qualitative study of this problem (data dependence of the coupled fixed point set, well-posedness, Ulam-Hyers stability and the limit shadowing property of the coupled fixed point problem) under some rational type contraction assumptions on the mapping. Our results extend and complement some theorems given in the recent literature, see e.g. [21], [22].

## 2. Fixed point theorems

In this part of the paper, we will present a fixed point theorems in ordered b-metric spaces for a single-valued operstor satisfying a rational type contraction condition.

**Theorem 2.1.** Let $(X, \leq)$ be a partially ordered set and $d : X \times X \to \mathbb{R}_+$ be a complete b-metric with constant $s \geq 1$. Let $f : X \to X$ be an operator which has closed graph with respect to $d$ and it is increasing with respect to " $\leq$ ". Suppose that there exists $\alpha, \beta \geq 0$ with $\alpha + \beta s < 1$ satisfying

$$d(f(x), f(y)) \leq \frac{\alpha \cdot d(y, f(y))[1 + d(x, f(x))]}{1 + d(x, y)} + \beta \cdot d(x, y), \tag{2.1}$$

for $x, y \in X$ with $x \leq y$ .

If there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$, then there exists $x^* \in X$ such that $x^* = f(x^*)$ and $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, as $n \to \infty$.

*Proof.* We have two cases:

**Case 1.** If $f(x_0) = x_0$, then $Fix(f) \neq \emptyset$.

**Case 2.** Suppose that $x_0 < f(x_0)$.

Using that $f$ is an increasing operator and by mathematical induction, we have

$$x_0 < f(x_0) \leq f^2(x_0) \leq \ldots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \ldots$$

By this method we get a sequence $(x_n)_{n\subset\mathbb{N}} \in X$ defined by

$$x_{n+1} = f(x_n) = f(f(x_{n-1})) = f^2(x_{n-1}) = \ldots = f^n(x_1) = f^{n+1}(x_0).$$

If there exists $n \geq 1$ such that $x_{n+1} = x_n$, then $f(x_n) = x_n$. So we get that $x_n$ is a fixed point of $f$, which implies $Fix(f) \neq \emptyset$.

Suppose that $x_{n+1} \neq x_n$ for $n \geq 0$.

Since $x_n \leq x_{n+1}$ for any $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n))$$

$$\leq \frac{\alpha \cdot d(x_n, f(x_n))[1 + d(x_{n-1}, f(x_{n-1}))]}{1 + d(x_{n-1}, x_n)} + \beta \cdot d(x_{n-1}, x_n)$$

$$= \frac{\alpha \cdot d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta \cdot d(x_{n-1}, x_n)$$

$$= \alpha \cdot d(x_n, x_{n+1}) + \beta \cdot d(x_{n-1}, x_n).$$

So we obtain

$$d(x_n, x_{n+1}) \leq \frac{\beta}{1-\alpha} \cdot d(x_{n-1}, x_n) \text{ for any } n \in \mathbb{N}.$$

Using mathematical induction we get that

$$d(x_n, x_{n+1}) \leq \frac{\beta}{1-\alpha} \cdot d(x_{n-1}, x_n) \leq \ldots \leq \left(\frac{\beta}{1-\alpha}\right)^n \cdot d(x_0, x_1)$$

or

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \left(\frac{\beta}{1-\alpha}\right)^n \cdot d(x_0, f(x_0)) \quad for \quad any \quad n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$. We will prove that $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = f^n(x_0)$ is a Cauchy sequence in $X$.

$$d(f^n(x_0), f^{n+p}(x_0)) \leq s \cdot d(f^n(x_0), f^{n+1}(x_0)) + s^2 \cdot d(f^{n+1}(x_0), f^{n+2}(x_0)) + \ldots$$
$$+ s^{p-1} \cdot d(f^{n+p-2}(x_0), f^{n+p-1}(x_0)) + s^{p-1} \cdot d(f^{n+p-1}(x_0), f^{n+p}(x_0)).$$

We denote

$$A = \frac{\beta}{1-\alpha}.$$

So we obtain

$$d(f^n(x_0), f^{n+p}(x_0)) \leq s \cdot A^n \cdot d(x_0, f(x_0)) + s^2 \cdot A^{n+1} \cdot d(x_0, f(x_0)) + \ldots$$
$$+ s^{p-1} \cdot A^{n+p-2} \cdot d(x_0, f(x_0)) + s^p \cdot A^{n+p-1} \cdot d(x_0, f(x_0))$$
$$= s \cdot A^n [1 + s \cdot A + \ldots + (s \cdot A)^{p-1}] \cdot d(x_0, f(x_0)) = s \cdot A^n \cdot \frac{1 - (s \cdot A)^p}{1 - s \cdot A} \cdot d(x_0, f(x_0)).$$

But $A = \frac{\beta}{1-\alpha} < \frac{1}{s}$, then we get that

$$d(f^n(x_0), f^{n+p}(x_0)) \leq s \cdot A^n \cdot \frac{1 - (s \cdot A)^p}{1 - s \cdot A} \cdot d(x_0, f(x_0)) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $(f^n(x_0))_{n \in \mathbb{N}}$ is a Cauchy sequence on $X$. We also know that $(X, d)$ is a complete b-metric space. So there exists $x^* \in X$ such that $(f^n(x_0))_{n \in \mathbb{N}} \to x^*$ as $n \to \infty$. Because $f$ has closed graph, then $x^* \in Fix(f)$, which implies $Fix(f) \neq \emptyset$.

Or $f$ is continuous, we have

$$f(x^*) = f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = x^*. \qquad \square$$

A uniqueness result concerning the fixed point equation is the following.

**Theorem 2.2.** Suppose that all the hypotheses of Theorem 2.1. take place. Additionally, suppose that the following condition holds: for all $x, y \in X$ there exists $z \in X$ such that $z \leq x$ and $z \leq y$.

Then $Fix(f) = \{x^*\}$.

*Proof.* Suppose that $x^*, y^* \in X$ are two fixed points of $f$. We have two cases:

**Case 1.** $x^*$ and $y^*$ are comparable. Suppose $x^* \leq y^*$(or $y^* \leq x^*$ is the same)

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \frac{\alpha \cdot d(y^*, f(y^*))[1 + d(x^*, f(x^*))]}{1 + d(x^*, y^*)} + \beta \cdot d(x^*, y^*)$$
$$= \beta \cdot d(x^*, y^*).$$

Since $\beta < 1$, this is only possible when $d(x^*, y^*) = 0$. This implies $x^* = y^*$, so $Fix(f) = \{x^*\}$.

**Case 2.** $x^*$ and $y^*$ are not comparable.

By our additional assumption, we have that there exists $z \in X$ with $z \leq x^*$ and $z \leq y^*$.

Since $z \leq x^*$, then $f^n(z) \leq f^n(x^*) = x^*$ for any $n \in \mathbb{N}$.

We obtain

$$d(f^n(z), x^*) = d(f^n(z), f^n(x^*)) \leq \frac{\alpha \cdot d(f^{n-1}(x^*), f^n(x^*))[1 + d(f^{n-1}(z), f^n(z)]}{1 + d(f^{n-1}(z), f^{n-1}(x^*))}$$
$$+ \beta \cdot d(f^{n-1}(z), f^{n-1}(x^*)) = \beta \cdot d(f^{n-1}(z), f^{n-1}(x^*)) = \beta \cdot d(f^{n-1}(z), x^*)$$

So we have

$$d(f^n(z), x^*) \leq \beta \cdot d(f^{n-1}(z), x^*) \leq \beta^2 \cdot d(f^{n-2}(z), x^*) \leq \ldots \leq \beta^n \cdot d(z, x^*)$$

and since $\beta < 1$, $\beta^n \to 0$ then we get that

$$\lim_{n \to \infty} d(f^n(z), x^*) = 0$$

This implies $\lim_{n\to\infty} f^n(z) = x^*$. Using a similar argument, we get that $\lim_{n\to\infty} f^n(z) = y^*$. Then $x^* = y^*$.

A global version of the previous result is the following:

**Theorem 2.3.** Let $(X, d)$ be a complete b- metric space with constant $s \geq 1$, $f : X \to X$ be an operator of $X$ with the following condition: there exists $\alpha, \beta \geq 0$ with $\max\{\alpha, \frac{\beta}{1-\alpha}\} < \frac{1}{s}$ such that

$$d(f(x), f(y)) \leq \frac{\alpha \cdot d(y, f(y))[1 + d(x, f(x))]}{1 + d(x, y)} + \beta \cdot d(x, y), \tag{2.2}$$

for $x, y \in X$. Then $f$ has a unique fixed point.

*Proof.* Let $x_0 \in X$ be arbitrary chosen. Using the same method as in previous proof, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$, which is a Cauchy sequence.

Since $(X, d)$ is a complete b-metric space, we get that there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Then, we have

$$\begin{aligned} d(x^*, f(x^*)) &\leq s \cdot d(x^*, f(x_n)) + s \cdot d(f(x_n), f(x^*)) \\ &\leq s \cdot d(x^*, f(x_n)) + s \cdot \frac{\alpha \cdot d(x^*, f(x^*))[1 + d(x_n, f(x_n))]}{1 + d(x_n, x^*)} + s \cdot \beta \cdot d(x_n, x^*) \\ &= s \cdot d(x^*, x_{n+1}) + s \cdot \frac{\alpha \cdot d(x^*, f(x^*))[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x^*)} + s \cdot \beta \cdot d(x_n, x^*). \end{aligned}$$

Thus, we obtain

$$d(x^*, f(x^*)) \left[\frac{1 + d(x_n, x^*) - s \cdot \alpha - s \cdot \alpha \cdot d(x_n, x_{n+1})}{1 + d(x_n, x^*)}\right] \leq s \cdot d(x^*, x_{n+1}) + s \cdot \beta \cdot d(x_n, x^*).$$

Letting $n \to \infty$ we have $d(x^*, f(x^*))(1 - s \cdot \alpha) \leq 0$. Thus $d(x^*, f(x^*)) = 0$, i.e., $x^* \in Fix(f)$.

We prove that $x^*$ is the unique fixed point of $f$. Suppose that $y^*$ is a fixed point of $f$, i.e. $f(y^*) = y^*$. Then

$$d(y^*, x^*) = d(f(y^*), f(x^*)) \leq \frac{\alpha \cdot d(x^*, f(x^*))[1 + d(y^*, f(y^*))]}{1 + d(x^*, y^*)} + \beta \cdot d(y^*, x^*)$$

Hence $d(y^*, x^*) \leq \beta \cdot d(y^*, x^*)$ and thus $y^* = x^*$.

Therefore $x^*$ is the unique fixed point of $f$.

## 3. Coupled fixed point theorems

In this section, using the fixed point theorems proved in Section 2, we will obtain some existence and uniqueness theorems for the coupled fixed point problem.

**Theorem 3.1.** Let $(X, \leq)$ be a partially ordered set and $d : X \times X \to \mathbb{R}_+$ be a complete b-metric on $X$ with constant $s \geq 1$. Let $T : X \times X \to X$ be an operator with closed graph (or in particular, it is continuous) which has the mixed monotone property on $X \times X$. Assume that the following conditions are satisfied:

i) Suppose that there exists $\alpha, \beta \geq 0$ with $\frac{\beta}{1-\alpha} < \frac{1}{s}$ such that

$$d(T(x,y),T(u,v)) + d(T(y,x),T(v,u)) \leq \frac{\alpha \cdot [d(u,T(u,v)) + d(v,T(v,u))][1 + d(x,T(x,y)) + d(y,T(y,x))]}{1 + d(x,u) + d(y,v)} + \beta \cdot [d(x,u) + d(y,v)], \tag{3.1}$$

for all $(x,y),(u,v) \in X \times X$ with $x \leq u, y \geq v$ ;

ii) there exists $x_0, y_0 \in X$ such that $x_0 \leq T(x_0,y_0), y_0 \geq T(y_0,x_0)$, i.e. $(x_0,y_0) \leq_p (T(x_0,y_0),T(y_0,x_0))$.

Then, the following conclusions hold:

a) there exists $(x^*,y^*) \in X \times X$ a solution of the coupled fixed point problem $(P_1)$, such that the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in $X$ defined by

$$\begin{cases} x_{n+1} = T(x_n,y_n), \\ y_{n+1} = T(y_n,x_n), \quad for \quad all \quad n \in \mathbb{N}. \end{cases}$$

have the property that $(x_n)_{n\in\mathbb{N}} \to x^*, (y_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$.

b) in particular, if $d$ is a continuous b-metric on $X$, then

$$d(x_n,x^*) + d(y_n,y^*) \leq \frac{s \cdot A^n}{1 - s \cdot A}[d(x_0,x_1) + d(y_0,y_1)]$$

where $A = \frac{2\beta}{1-2\alpha}$ and $\begin{cases} x_1 = T(x_0,y_0) \\ y_1 = T(y_0,x_0). \end{cases}$

*Proof.* By ii) we have that $z_0 = (x_0,y_0) \leq_p (T(x_0,y_0),T(y_0,x_0)) = (x_1,y_1) = z_1$. So we have $z_0 \leq_p z_1$.

If we consider $x_2 = T(x_1,y_1)$ and $y_2 = T(y_1,x_1)$, then we get $x_2 = T(x_1,y_1) = T^2(x_0,y_0)$ and $y_2 = T(y_1,x_1) = T^2(y_0,x_0)$. Using the mixed monotone property of $T$, we get

$$x_2 = T(x_1,y_1) \geq T(x_0,y_0) = x_1 \quad \text{implies} \quad x_1 \leq x_2$$
$$y_2 = T(y_1,x_1) \leq T(y_0,x_0) = y_1 \quad \text{implies} \quad y_1 \geq y_2$$

Hence $z_1 = (x_1,y_1) \leq_p (x_2,y_2) = z_2$.

By this approach we obtain the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in $X$ with

$$\begin{cases} x_{n+1} = T(x_n,y_n) \\ y_{n+1} = T(y_n,x_n) \end{cases}$$

and by mathematical induction we obtain $z_n = (x_n,y_n) \leq_p (x_{n+1},y_{n+1}) = z_{n+1}$, which implies $(z_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence in $(Z,\leq_p)$, where $Z = X \times X$.

Consider the metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$ , defined by

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v).$$

Then, $\widetilde{d}$ is a b-metric on $Z$ with the same constant $s \geq 1$ and if $(X,d)$ is complete, we have $(Z,\widetilde{d})$ is complete, too.

Let $F : Z \to Z$ be an operator defined by $F(x,y) = (T(x,y),T(y,x)), \ \forall (x,y) \in Z$.

We have $z_{n+1} = F(z_n)$, for $n \geq 0$ where $z_0 = (x_0, y_0)$. Using the mixed monotone property of $T$, then the operator $F$ is monotone increasing with respect to " $\leq_p$ " i.e. $(x, y), (u, v) \in Z$, with $(x, y) \leq_p (u, v) \Rightarrow F(x, y) \leq_p F(u, v)$.

Because $T$ has a closed graph (or, in particulat it is continuous on $X \times X$), then $F$ has a closed graph (or respectively is continuous on $Z$).

$F$ is a contraction in $(Z, \widetilde{d})$ on all comparable elements of $Z$. Let $z = (x, y) \leq_p (u, v) = w \in Z$, so we have

$$\widetilde{d}(F(z), F(w)) = \widetilde{d}((T(x, y), T(y, x)), (T(u, v), T(v, u))$$

$$= d(T(x, y), T(u, v)) + d(T(y, x), T(v, u))$$

$$\leq \frac{\alpha \cdot [d(u, T(u, v)) + d(v, T(v, u))][1 + d(x, T(x, y)) + d(y, T(y, x))]}{1 + d(x, u) + d(y, v)}$$

$$+\beta \cdot [d(x, u) + d(y, v)]$$

$$= \frac{\alpha \cdot \widetilde{d}(w, F(w))[1 + \widetilde{d}(z, F(z))]}{1 + \widetilde{d}(z, w)} + \beta \cdot \widetilde{d}(z, w).$$

The operator $F : Z \to Z$ has the following properties:

1) $F : Z \to Z$ has a closed graph;
2) $F : Z \to Z$ is increasing on $Z$;
3) there exist $z_0 = (x_0, y_0) \in Z$ such that $z_0 \leq_p F(z_0)$;
4) there exists $\alpha, \beta \geq 0$ with $\frac{\beta}{1-\alpha} < \frac{1}{s}$ such that

$$\widetilde{d}(F(z), F(w)) \leq \frac{\alpha \cdot \widetilde{d}(w, F(w))[1 + \widetilde{d}(z, F(z))]}{1 + \widetilde{d}(z, w)} + \beta \cdot \widetilde{d}(z, w).$$

We can apply the conclusion of the Theorem 2.1. and we get that $F$ has at least one fixed point. Hence, there exists $z^* \in Z$ with $F(z^*) = z^*$. Let $z^* = (x^*, y^*) \in Z$, so we have $F(x^*, y^*) = (x^*, y^*)$.

This implies

$$(T(x^*, y^*), T(y^*, x^*)) = (x^*, y^*) \Rightarrow \begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases}$$

and the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in $X$ defined by

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n) \end{cases} \qquad \text{for } n \in \mathbb{N}$$

have the property that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

We know that $z_{n+1} = F(z_n) = F(x_n, y_n)$ for $n \geq 0$. This yields to

$$\widetilde{d}(z_n, z_{n+1}) = \widetilde{d}(F(z_{n-1}), F(z_n))$$

$$= \widetilde{d}((T(x_{n-1}, y_{n-1}), T(y_{n-1}, x_{n-1})), (T(x_n, y_n), T(y_n, x_n)))$$

$$= d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) + d(T(y_{n-1}, x_{n-1}), T(y_n, x_n))$$

$$\leq \frac{\alpha[d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))][1 + d(x_{n-1}, T(x_{n-1}, y_{n-1})) + d(y_{n-1}, T(y_{n-1}, x_{n-1}))]}{1 + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}$$

$$+\beta[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]$$

$$= \frac{\alpha \cdot \widetilde{d}(z_n, F(z_n))[1+\widetilde{d}(z_{n-1}, F(z_{n-1}))]}{1+\widetilde{d}(z_{n-1}, z_n)} + \beta \cdot \widetilde{d}(z_{n-1}, z_n)$$

$$= \frac{\alpha \cdot \widetilde{d}(z_n, F(z_n))[1+\widetilde{d}(z_{n-1}, z_n)]}{1+\widetilde{d}(z_{n-1}, z_n)} + \beta \cdot \widetilde{d}(z_{n-1}, z_n) = \alpha \cdot \widetilde{d}(z_n, z_{n+1}) + \beta \cdot \widetilde{d}(z_{n-1}, z_n).$$

This yields to

$$\widetilde{d}(z_n, z_{n+1}) \leq \frac{\beta}{1-\alpha} \cdot \widetilde{d}(z_{n-1}, z_n) \leq \left(\frac{\beta}{1-\alpha}\right)^2 \cdot \widetilde{d}(z_{n-2}, z_{n-1})$$
$$\leq \ldots \leq \left(\frac{\beta}{1-\alpha}\right)^n \cdot \widetilde{d}(z_0, z_1)$$

where $\frac{\beta}{1-\alpha} < \frac{1}{s} < 1$.

We denote $A = \frac{\beta}{1-\alpha} < 1$. Moreover, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$\widetilde{d}(z_n, z_{n+p}) \leq s \cdot \widetilde{d}(z_n, z_{n+1}) + s^2 \cdot \widetilde{d}(z_{n+1}, z_{n+2}) + \ldots + +s^{p-1} \cdot \widetilde{d}(z_{n+p-1}, z_{n+p})$$
$$\leq s \cdot A^n \cdot \widetilde{d}(z_0, z_1) + s^2 \cdot A^{n+1} \cdot \widetilde{d}(z_0, z_1) + \ldots + s^{p-1} \cdot A^{n+p-1} \cdot \widetilde{d}(z_0, z_1)$$
$$\leq s \cdot A^n \cdot [1 + s \cdot A + \ldots + (s \cdot A)^{p-1}] \cdot \widetilde{d}(z_0, z_1)$$
$$= s \cdot A^n \cdot \frac{1-(s \cdot A)^{p-1}}{1 - s \cdot A} \cdot \widetilde{d}(z_0, z_1) \leq s \cdot A^n \cdot \frac{1}{1 - s \cdot A} \cdot \widetilde{d}(z_0, z_1).$$

If the b-metric is continuous, letting $p \to \infty$ we obtain

$$\widetilde{d}(z_n, z^*) \leq \frac{s \cdot A^n}{1 - s \cdot A} \cdot \widetilde{d}(z_0, z_1).$$

But $z_n = (x_n, y_n)$, so we get

$$\widetilde{d}((x_n, y_n), z^*) \leq \frac{s \cdot A^n}{1 - s \cdot A} \cdot \widetilde{d}((x_0, y_0), (x_1, y_1))$$

and, by definition of $\widetilde{d}$, we finally get

$$d(x_n, z^*) + d(y_n, z^*) \leq \frac{s \cdot A^n}{1 - s \cdot A} \cdot [d(x_0, x_1) + d(y_0, y_1)].$$ □

The following theorem gives the uniqueness of the coupled fixed point.

**Theorem 3.2.** Consider that we have the hypotheses of Theorem 3.1. and the following condition holds:

for all $(x, y), (u, v) \in X \times X$ there exists $(z, w) \in X \times X$ such that

$$(z, w) \leq_p (x, y) \text{ and } (z, w) \leq_p (u, v).$$

Then $CFix(T) = \{(x^*, y^*)\}$.

*Proof.* The operator $T$ verifies the hypotheses of Theorem 3.1. Hence there exists $(x^*, y^*) \in Z := X \times X$ such that

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases}$$

Let $(\overline{x}, \overline{y}) \in CFix(T)$ and $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, defined by

$$\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v),$$

where $Z = X \times X$.

We have two cases:

**Case 1.** $(x^*, y^*) \leq_p (\overline{x}, \overline{y})$, which implies

$$\begin{aligned}\widetilde{d}((x^*, y^*), (\overline{x}, \overline{y})) &= \widetilde{d}((T(x^*, y^*), T(y^*, x^*)), (T(\overline{x}, \overline{y}), T(\overline{y}, \overline{x})))\\ &= d(T(x^*, y^*), T(\overline{x}, \overline{y})) + d(T(y^*, x^*), T(\overline{y}, \overline{x}))\\ \leq \frac{\alpha \cdot [d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x}))][1 + d(x^*, T(x^*, y^*)) + d(y^*, T(y^*, x^*))]}{1 + d(x^*, \overline{x}) + d(y^*, \overline{y})}\\ +\beta \cdot [d(x^*, \overline{x}) + d(y^*, \overline{y})] = \beta \cdot [d(x^*, \overline{x}) + d(y^*, \overline{y})] = \beta \cdot \widetilde{d}((x^*, y^*), (\overline{x}, \overline{y}))\end{aligned}$$

This yields to

$$\widetilde{d}((x^*, y^*), (\overline{x}, \overline{y})) \leq \beta \cdot \widetilde{d}((x^*, y^*), (\overline{x}, \overline{y}))$$

or

$$(1 - \beta) \cdot \widetilde{d}((x^*, y^*), (\overline{x}, \overline{y})) \leq 0 \quad (\text{but} \quad 1 - \beta > 0)$$

Hence, we have

$$(x^*, y^*) = (\overline{x}, \overline{y}).$$

**Case 2.** $(x^*, y^*), (\overline{x}, \overline{y})$ are not comparable.

Let $F : Z \to Z$ be defined by $F(x, y) = (T(x, y), T(y, x)) \quad \forall (x, y) \in Z$. There exists $(z, w) \in Z$, such that $(z, w) \leq_p (x^*, y^*)$, implies $F^n(z, w) \leq_p F^n(x^*, y^*)$ because $F$ is an increasing operator and $(z, w) \leq_p (\overline{x}, \overline{y})$, implies $F^n(z, w) \leq_p F(\overline{x}, \overline{y})$, $F$ is an increasing operator.

We have

$$\begin{aligned}\widetilde{d}(F^n(z, w), (x^*, y^*)) &= \widetilde{d}(F^n(z, w), F^n(x^*, y^*)) = \widetilde{d}(F(F^{n-1}(z, w)), F(F^{n-1}(x^*, y^*)))\\ &\leq \frac{\alpha \cdot \widetilde{d}(F^{n-1}(x^*, y^*), F^n(x^*, y^*))[1 + \widetilde{d}(F^{n-1}(z, w), F^n(z, w))]}{1 + \widetilde{d}(F^{n-1}(z, w), F^{n-1}(x^*, y^*))}\\ &\quad +\beta \cdot \widetilde{d}(F^{n-1}(z, w), F^{n-1}(x^*, y^*))\\ &= \beta \cdot \widetilde{d}(F^{n-1}(z, w), F^{n-1}(x^*, y^*)).\end{aligned}$$

By mathematical induction we get

$$\begin{aligned}\widetilde{d}(F^n(z, w), F^n(x^*, y^*)) &\leq \beta \cdot \widetilde{d}(F^{n-1}(z, w), F^{n-1}(x^*, y^*))\\ &\leq \ldots \leq \beta^n \cdot \widetilde{d}((z, w), (x^*, y^*)) \to 0, \text{ as } n \to \infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty} F^n(z, w) = (x^*, y^*). \tag{3.2}$$

But, we also know,

$$(z, w) \leq_p (\overline{x}, \overline{y})$$

implies

$$F^n(z, w) \leq_p F^n(\overline{x}, \overline{y}) = (\overline{x}, \overline{y}).$$

Similarly, we obtain that

$$\widetilde{d}(F^n(z, w), (\overline{x}, \overline{y})) \leq \beta^n \cdot \widetilde{d}((z, w), (\overline{x}, \overline{y})) \to 0 \quad \text{as} \quad n \to \infty$$

Hence,

$$\lim_{n\to\infty} F^n(z, w) = (\overline{x}, \overline{y}). \tag{3.3}$$

By (3.1)+(3.2) we obtain that

$$(x^*, y^*) = (\overline{x}, \overline{y}). \qquad \square$$

A global version of the previous results is the following.

**Theorem 3.3.** Let $(X, d)$ be a complete b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator such that there exist $\alpha, \beta \geq 0$ with $\max\{\alpha, \frac{\beta}{1-\alpha}\} < \frac{1}{s}$ such that

$$d(T(x,y), T(u,v)) + d(T(y,x), T(v,u))$$
$$\leq \frac{\alpha \cdot [d(u, T(u,v)) + d(v, T(v,u))][1 + d(x, T(x,y)) + d(y, T(y,x))]}{1 + d(x,u) + d(y,v)}$$
$$+\beta \cdot [d(x,u) + d(y,v)] \quad \text{for} \quad (x,y), (u,v) \in X \times X.$$

Then, there exists an unique solution $(x^*, y^*) \in X \times X$ of the coupled fixed point problem $(P_1)$, and for any initial element $(x_0, y_0) \in X \times X$ the sequence $z_{n+1} = (x_{n+1}, y_{n+1}) = (T(x_n, y_n), T(y_n, x_n)) \in X \times X$ converges to $(x^*, y^*)$.

*Proof.* Let $Z = X \times X$ and the functional $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, such that

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v).$$

We know that $\widetilde{d}$ is a b-metric on $Z$ with the same constant $s \geq 1$. Moreover, if $(X, d)$ is a complete b-metric space, then $(Z, \widetilde{d})$ is a complete b-metric space too.

Consider the operator $F : Z \to Z$ defined by $F(x,y) = (T(x,y), T(y,x))$ for $(x,y) \in Z$.

Let $z = (x,y) \in Z$ and $w = (u,v) \in Z$.

We have

$$\widetilde{d}(F(z), F(w)) = \widetilde{d}((T(x,y), T(y,x)), (T(u,v), T(v,u)))$$
$$= d(T(x,y), T(u,v)) + d(T(y,x), T(v,u))$$
$$\leq \frac{\alpha \cdot [d(u, T(u,v)) + d(v, T(v,u))][1 + d(x, T(x,y)) + d(y, T(y,x))]}{1 + d(x,u) + d(y,v)}$$
$$+\beta \cdot [d(x,u) + d(y,v)]$$
$$= \frac{\alpha \cdot \widetilde{d}((u,v), (T(u,v), T(v,u)))[1 + \widetilde{d}((x,y), (T(x,y), T(y,x)))]}{1 + \widetilde{d}((x,y),(u,v))} + \beta \cdot \widetilde{d}((x,y),(u,v))$$
$$= \frac{\alpha \cdot \widetilde{d}(w, F(w))[1 + \widetilde{d}(z, F(z))]}{1 + \widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w).$$

Therefore

$$\widetilde{d}(F(z), F(w)) \leq \frac{\alpha \cdot \widetilde{d}(w, F(w))[1 + \widetilde{d}(z, F(z))]}{1 + \widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w).$$

From Theorem 2.3. we have that $Fix(F) = \{(x^*, y^*)\}$, so the coupled fixed point problem $(P_1)$ has a unique solution $(x^*, y^*) \in Z$.

An existence and uniqueness result for the fixed point of $T$ is given now.

**Theorem 3.4.** If we suppose that we have the hypotheses of Theorem 3.2., then for the unique coupled fixed point $(x^*, y^*)$ of $T$ we have that $x^* = y^*$ i.e. $T(x^*, x^*) = x^*$.

*Proof.* From Theorem 3.2., there exists an unique coupled fixed point of $T$, $(x^*, y^*) \in X \times X$.

We have two cases:

**Case 1.** If $x^*$ and $y^*$ are comparable, $x^* \leq y^*$.

Then we have

$$d(T(x,y),T(u,v)) + d(T(y,x),T(v,u))$$
$$\leq \frac{\alpha \cdot [d(u,T(u,v)) + d(v,T(v,u))][1 + d(x,T(x,u)) + d(y,T(y,x))]}{1 + d(x,u) + d(y,v)}$$
$$+\beta \cdot [d(x,u) + d(y,v)].$$

Let

$$x = v = x^* \quad \text{and} \quad y = u = y^*.$$

Thus we obtain

$$2 \cdot d(T(x^*,y^*),T(y^*,x^*))$$
$$\leq \frac{\alpha \cdot [d(y^*,T(y^*,x^*)) + d(x^*,T(x^*,y^*))][1 + d(x^*,T(x^*,y^*)) + d(y^*,T(y^*,x^*))]}{1 + 2d(x^*,y^*)}$$
$$+\beta \cdot 2 \cdot d(x^*,y^*).$$

This yields to

$$d(x^*,y^*) \leq \beta \cdot d(x^*,y^*).$$

So

$$(1-\beta) \cdot d(x^*,y^*) \leq 0,$$

follows that $x^* = y^*$.

**Case 2.** Suppose that $x^*$ and $y^*$ are not comparable.

Hence, there exists $z \in X$ such that $z \leq x^*$ and $z \leq y^*$. Thus, the following relations are satisfied:

$$(z,y^*) \leq_p (y^*,z),\ (z,y^*) \leq_p (x^*,y^*),\ (y^*,x^*) \leq_p (y^*,z).$$

Let $F : Z \to Z$ be defined by $F(x,y) = (T(x,y),T(y,x)) \quad \forall (x,y) \in Z$. Then,

$$d(x^*,y^*) = \frac{1}{2} \cdot \widetilde{d}((y^*,x^*),(x^*,y^*)) = \frac{1}{2} \cdot \widetilde{d}(F^n(y^*,x^*),F^n(x^*,y^*))$$
$$\leq \frac{s}{2} \cdot \widetilde{d}(F^n(y^*,x^*),F^n(y^*,z)) + \frac{s}{2} \cdot \widetilde{d}(F^n(y^*,z),F^n(x^*,y^*))$$
$$\leq \frac{s}{2} \cdot \widetilde{d}(F^n(y^*,x^*),F^n(y^*,z)) + \frac{s^2}{2} \cdot \widetilde{d}(F^n(y^*,z),F^n(z,y^*)) + \frac{s^2}{2} \cdot \widetilde{d}(F^n(z,y^*),F^n(x^*,y^*)).$$

But we know that

$$\widetilde{d}(F^n(y^*,x^*),F^n(y^*,z)) \leq \beta^n \cdot \widetilde{d}((y^*,x^*),(y^*,z)) = \beta^n \cdot d(x^*,z)$$
$$\widetilde{d}(F^n(y^*,z),F^n(z,y^*)) \leq \beta^n \cdot \widetilde{d}((y^*,z),(z,y^*)) = 2\beta^n \cdot d(y^*,z)$$
$$\widetilde{d}(F^n(z,y^*),F^n(x^*,y^*)) \leq \beta^n \cdot \widetilde{d}((z,y^*),(x^*,y^*)) = \beta^n \cdot d(z,x^*).$$

Using this assumptions, we get that

$$d(x^*,y^*) \leq \frac{s}{2} \cdot \beta^n \cdot d(x^*,z) + \frac{s^2}{2} \cdot \beta^n \cdot 2 \cdot d(y^*,z) + \frac{s^2}{2} \cdot \beta^n \cdot d(z,x^*)$$
$$= \frac{s}{2} \cdot \beta^n \cdot [(1+s)d(x^*,z) + 2 \cdot s \cdot d(y^*,z)] \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, we have that $x^* = T(x^*, x^*)$.

## 4. Properties of the coupled fixed point problem

This section presents data dependence, well-posedness, Ulam-Hyers stability and limit shadowing property for the coupled fixed point problem.

The following theorem is a data dependence result of a coupled fixed point problem.

**Theorem 4.1.** Let $(X, d)$ be a complete b-metric space with constant $s \geq 1$ and $T_i : X \times X \to X \quad (i \in \{1, 2\})$ be two operators which satisfy the following conditions:

i) there exist $\alpha, \beta \geq 0$ with $\max\{\alpha, \frac{\beta}{1-\alpha}\} < \frac{1}{s}$ such that

$$d(T_1(x,y), T_1(u,v)) + d(T_1(y,x), T_1(v,u))$$
$$\leq \frac{\alpha \cdot [d(u, T_1(u,v)) + d(v, T_1(v,u))][1 + d(x, T_1(x,y)) + d(y, T_1(y,x))]}{1 + d(x,u) + d(y,v)}$$
$$+\beta \cdot [d(x,u) + d(y,v)] \quad \text{all for} \quad (x,y), (u,v) \in X \times X;$$

ii) $CFix(T_2) \neq \emptyset$;

iii) there exists $\eta > 0$ such that $d(T_1(x,y), T_2(x,y)) \leq \eta$ for all $(x,y) \in X \times X$.

In the above conditions, if $(x^*, y^*) \in X \times X$ is the unique coupled fixed point for $T_1$, then $d(x^*, \overline{x}) + d(y^*, \overline{y}) \leq \frac{2s(1+\alpha)}{1-s\beta} \cdot \eta$, where $(\overline{x}, \overline{y}) \in CFix(T_2)$.

*Proof.* By Theorem 3.3, there exists $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_1(y^*, x^*). \end{cases}$$

Let $(\overline{x}, \overline{y}) \in CFix(T_2)$, i.e. $\begin{cases} \overline{x} = T_2(\overline{x}, \overline{y}) \\ \overline{y} = T_2(\overline{y}, \overline{x}). \end{cases}$

Consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, defined by

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v)$$

for $(x,y), (u,v) \in Z$, where $Z = X \times X$.

Consider two operators $F_i : Z \to Z$ defined by $F_i(x,y) = (T_i(x,y), T_i(y,x))$, for $(x,y) \in Z$, $i \in \{1,2\}$.

We denote by $z = (x^*, y^*) \in Z$, which means $F_1(z) = z$ and $w = (\overline{x}, \overline{y}) \in Z$, which means $F_2(w) = w$. Then,

$$\widetilde{d}(F_1(z), F_1(w)) = \frac{\alpha \cdot \widetilde{d}(w, F_1(w))[1 + \widetilde{d}(z, F_1(z))]}{1 + \widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)$$

$$= \frac{\alpha \cdot \widetilde{d}(w, F_1(w))}{1 + \widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w) \leq \alpha \cdot \widetilde{d}(w, F_1(w)) + \beta \cdot \widetilde{d}(z,w) \leq 2\alpha \cdot \eta + \beta \cdot \widetilde{d}(z,w).$$

Since

$$\widetilde{d}(z,w) = \widetilde{d}(F_1(z), F_2(w)) \leq s \cdot [\widetilde{d}(F_1(z), F_1(w)) + \widetilde{d}(F_1(w), F_2(w))]$$
$$\leq s \cdot [2\alpha \cdot \eta + \beta \cdot \widetilde{d}(z,w)] + 2s \cdot \eta,$$

we will obtain that $(1 - s\beta) \cdot \widetilde{d}(z, w) \leq 2s \cdot (1 + \alpha) \cdot \eta$.

Since $\max\{\alpha, \frac{\beta}{1-\alpha}\} < \frac{1}{s}$, we get that $1 - s\beta > 0$. Therefore $\widetilde{d}(z, w) \leq \frac{2s(1+\alpha)}{1-s\beta} \cdot \eta$ and by definition of the metric $\widetilde{d}$, we have

$$d(x^*, \overline{x}) + d(y^*, \overline{y}) \leq \frac{2s(1 + \alpha)}{1 - s\beta} \cdot \eta. \qquad \square$$

**Definition 4.2.** Let $(X, d)$ be a b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator. By definition, the coupled fixed point problem $(P_1)$ is said to be well-posed if:

i) $CFix(T) = \{(x^*, y^*)\}$;

ii) for any sequence $(x_n, y_n)_{n\in\mathbb{N}} \in X \times X$ for which $d(x_n, T(x_n, y_n)) \to 0$ and $d(y_n, T(y_n, x_n)) \to 0$ as $n \to \infty$, we have that $(x_n)_{n\in\mathbb{N}} \to x^*$ and $(y_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$.

**Theorem 4.3.** Assume that all the hypotheses of Theorem 3.3. take place. Then the coupled fixed problem $(P_1)$ is well-possed.

*Proof.* We denote by $Z = X \times X$. By Theorem 3.3. we have $CFix(T) = \{(x^*, y^*)\}$.

Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence on $Z$. We know that $d(x_n, T(x_n, y_n)) \to 0$ and $d(y_n, T(y_n, x_n)) \to 0$ as $n \to \infty$.

Consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, such that $\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v)$ for all $(x, y), (u, v) \in Z$.

Let $F : Z \to Z$ be an operator defined by $F(x, y) = (T(x, y), T(y, x))$ for all $(x, y) \in Z$. We know that $F(x^*, y^*) = (x^*, y^*)$, so we have

$$\widetilde{d}((x_n, y_n), (x^*, y^*)) = d(x_n, x^*) + d(y_n, y^*)$$
$$\leq s \cdot d(x_n, T(x_n, y_n)) + s \cdot d(T(x_n, y_n), T(x^*, y^*))$$
$$+s \cdot d(y_n, T(y_n, x_n)) + s \cdot d(T(y_n, x_n), T(y^*, x^*))$$
$$= s \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]$$
$$+s \cdot [d(T(x_n, y_n), T(x^*, y^*)) + d(T(y_n, x_n), T(y^*, x^*))]$$
$$\leq s \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]$$
$$+s \cdot \frac{\alpha \cdot [d(x^*, T(x^*, y^*)) + d(y^*, T(y^*, x^*))][1 + d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]}{1 + d(x_n, x^*) + d(y_n, y^*)}$$
$$+s \cdot \beta \cdot [1 + d(x_n, x^*) + d(y_n, y^*)]$$
$$\leq s \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))] + s \cdot \beta \cdot \widetilde{d}((x_n, y_n), (x^*, y^*)).$$

We obtain that

$$(1 - s\beta) \cdot \widetilde{d}((x_n, y_n), (x^*, y^*)) \leq s \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]$$

$$\widetilde{d}((x_n, y_n), (x^*, y^*)) \leq \frac{s}{1 - s\beta} \cdot [d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))] \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$.

**Definition 4.4.** Let $(X, d)$ be a b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator. Let $\tilde{d}$ any *b*-metric on $X \times X$ generated by $d$ By definition, the coupled fixed point problem $(P_1)$ is said to be Ulam-Hyers stable if there exists a function

$\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $(\overline{x}, \overline{y}) \in X \times X$ of the inequality

$$\tilde{d}((x, y), (T(x, y), T(y, x))) \le \varepsilon,$$

there exists a solution $(x^*, y^*) \in X \times X$ of the coupled fixed point problem $(P_1)$ such that

$$\tilde{d}((\overline{x}, \overline{y}), (x^*, y^*)) \le \psi(\varepsilon).$$

**Theorem 4.5.** Assume that all the hypotheses of Theorem 3.3. take place. Then the coupled fixed point problem $(P_1)$ is Ulam-Hyers stable.

*Proof.* Let $Z = X \times X$. By Theorem 3.3., we have $CFix(T) = \{(x^*, y^*)\}$. Let any $\varepsilon > 0$ and let $(\overline{x}, \overline{y}) \in Z$ such that $d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x})) \le \varepsilon$.

Consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$ given by

$$\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v),\ \forall (x, y), (u, v) \in Z$$

and $F : Z \to Z$ an operator defined by $F(x, y) = (T(x, y), T(y, x))$ for all $(x, y) \in Z$.

We have

$$\widetilde{d}((\overline{x}, \overline{y}), (x^*, y^*)) = d(\overline{x}, x^*) + d(\overline{y}, y^*) = d(\overline{x}, T(x^*, y^*)) + d(\overline{y}, T(y^*, x^*))$$
$$\le s \cdot [d(\overline{x}, T(\overline{x}, \overline{y})) + d(T(\overline{x}, \overline{y}), T(x^*, y^*))] + s \cdot [d(\overline{y}, T(\overline{y}, \overline{x})) + d(T(\overline{y}, \overline{x}), T(y^*, x^*))]$$
$$\le s \cdot [d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x}))]$$
$$+s \cdot \frac{\alpha \cdot [d(x^*, T(x^*, y^*)) + d(y^*, T(y^*, x^*))][1 + d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x}))]}{1 + d(\overline{x}, x^*) + d(\overline{y}, y^*)}$$
$$+s \cdot \beta \cdot [d(\overline{x}, x^*) + d(\overline{y}, y^*)].$$

Thus

$$\widetilde{d}((\overline{x}, \overline{y}), (x^*, y^*)) \le \frac{s}{1 - s\beta} \cdot [d(\overline{x}, T(\overline{x}, \overline{y})) + d(\overline{y}, T(\overline{y}, \overline{x}))] \le \frac{s}{1 - s\beta} \cdot \varepsilon.$$

Therefore the coupled fixed point problem $(P_1)$ is Ulam-Hyers stable, with a mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := ct$, where $c = \frac{s}{1-s\beta} > 0$.

**Definition 4.6.** Let $(X, d)$ be a b-metric space with constant $s \ge 1$ and $T : X \times X \to X$ be an operator. By definition, the coupled fixed point problem $(P_1)$ has the limit shadowing property, if for any sequence $(x_n, y_n)_{n\in\mathbb{N}} \in X \times X$ for which $d(x_{n+1}, T(x_n, y_n)) \to 0$ and respectively $d(y_{n+1}, T(y_n, x_n)) \to 0$ as $n \to \infty$, there exists a sequence $(T^n(x, y), T^n(y, x))_{n\in\mathbb{N}}$ such that $d(x_n, T^n(x, y)) \to 0$ and $d(y_n, T^n(y, x)) \to 0$ as $n \to \infty$.

**Theorem 4.7.** Assume that the hypotheses from Theorem 3.3. take place. Then the coupled fixed point problem $(P_1)$ for $T$ has the limit shadowing property.

*Proof.* By Theorem 3.3, we have $CFix(T) = \{(x^*, y^*)\}$ and for any initial point $(x, y) \in X \times X$ the sequence $z_{n+1} = (T^n(x, y), T^n(y, x)) \in X \times X$ converge to $(x^*, y^*)$ as $n \to \infty$.

Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence on $Z = X \times X$ such that $d(x_{n+1}, T(x_n, y_n)) \to 0$ and $d(y_{n+1}, T(y_n, x_n)) \to 0$ as $n \to \infty$.

We consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$, defined by

$$\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v) \text{ for all } (x, y), (u, v) \in Z.$$

Let $F : Z \to Z$ be an operator defined by $F(u,v) = (T(u,v), T(v,u))$ for all $(u,v) \in Z$. We know that $F(x^*, y^*) = (x^*, y^*)$. Then for every $(x,y) \in Z$ we have:

$$\widetilde{d}((x_{n+1}, y_{n+1}), (T^{n+1}(x,y), T^{n+1}(y,x)))$$
$$\leq s \cdot [\widetilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) + \widetilde{d}((x^*, y^*), (T^{n+1}(x,y), T^{n+1}(y,x)))])$$

But

$$\widetilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) \leq s \cdot [\widetilde{d}((x_{n+1}, y_{n+1}), F(x_n, y_n)) + \widetilde{d}(F(x_n, y_n), F(x^*, y^*))]$$

$$\leq s \cdot \widetilde{d}((x_{n+1}, y_{n+1}), F(x_n, y_n)) + s \cdot \frac{\alpha \cdot \widetilde{d}((x^*, y^*), F(x^*, y^*))[1 + \widetilde{d}((x_n, y_n), F(x_n, y_n))]}{1 + \widetilde{d}((x_n, y_n), (x^*, y^*))}$$
$$+ s \cdot \beta \cdot \widetilde{d}((x_n, y_n), (x^*, y^*))$$
$$= s \cdot [d(x_{n+1}, T(x_n, y_n)) + d(y_{n+1}, T(y_n, x_n))] + s \cdot \beta \cdot \widetilde{d}((x_n, y_n), (x^*, y^*)).$$

This yields to

$$\widetilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) \leq s \cdot [d(x_{n+1}, T(x_n, y_n)) + d(y_{n+1}, T(y_n, x_n))]$$
$$+ s \cdot \beta \cdot \{s \cdot [d(x_n, T(x_{n-1}, y_{n-1})) + d(y_n, T(y_{n-1}, x_{n-1}))] + s \cdot \beta \cdot \widetilde{d}((x_{n-1}, y_{n-1}), (x^*, y^*))\}.$$

Therefore,

$$\widetilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) \leq s \cdot [d(x_{n+1}, T(x_n, y_n)) + d(y_{n+1}, T(y_n, x_n))]$$
$$+ s \cdot (s \cdot \beta) \cdot [d(x_n, T(x_{n-1}, y_{n-1})) + d(y_n, T(y_{n-1}, x_{n-1}))] + (s \cdot \beta)^2 \cdot \widetilde{d}((x_{n-1}, y_{n-1}), (x^*, y^*))$$
$$\leq \ldots \leq (s \cdot \beta)^{n+1} \cdot \widetilde{d}((x_0, y_0), (x^*, y^*)) + s \cdot \left[\sum_{p=0}^{n} (s \cdot \beta)^{n-p} \cdot \widetilde{d}((x_{p+1}, y_{p+1}), F(x_p, y_p))\right].$$

From Cauchy's Lemma we have $\widetilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) \to 0$ as $n \to \infty$.

Thus $\widetilde{d}((x_{n+1}, y_{n+1}), (T^{n+1}(x,y), T^{n+1}(y,x))) \to 0$ as $n \to \infty$, so there exists a sequence $(T^n(x,y), T^n(y,x)) \in Z$ with

$$\widetilde{d}((x_n, y_n), (T^n(x,y), T^n(y,x))) = d(x_n, T^n(x,y)) + d(y_n, T^n(y,x)) \to 0 \quad \text{as} \quad n \to \infty.$$

## References


[1] Bakhtin, I.A., *The contraction mapping principle in almost metric spaces*, Funct. Anal., Unianowsk, Gos. Ped. Inst., **30**(1989), 26-37.

[2] Blumenthal, L.M., *Theory and Applications of Distance Geometry,* Oxford, 1953.

[3] Berinde, V., *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory, 1993, 3-9.

[4] Berinde, V., *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **74**(2011), 7347-7355.

[5] Bota, M.F., Petruşel, A., Petruşel, G., Samet, B., *Coupled fixed point theorems for singlevalued operators in b-metric spaces*, Fixed Point Theory Appl., 2015, 2015:231 doi:10.1186/s13663-015-0482-3.

[6] Bourbaki, N., *Topologie générale*, Herman, Paris, 1974.

[7] Cabrera, I., Harjani, J., Sadarangani, K., *A fixed point theorem for contractions of rational type in partially ordered metric spaces*, Ann. Univ. Ferrara, **59**(2013), 251-258.

[8] Czerwik, S., *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1**(1993), 5-11.

[9] Dass, B.K., Gupta, S., *An extension of Banach contraction principle through rational expressions*, Indian J. Pure Appl. Math., **6**(1975), 1455-1458.

[10] Gnana Bhaskar, T., Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(2006), 1379-1393.

[11] Guo, D., Cho, Y.J., Zhu, Z., *Partial Ordering Methods in Nonlinear Problems*, Nova Science Publishers Inc., Hauppauge, NY, 2004.

[12] Guo, D., Laksmikantham, V., *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., **11**(1987), 623-632.

[13] Heinonen, J., *Lectures on Analysis on Metric Spaces*, Springer Berlin, 2011.

[14] Jleli, M., Samet, B., *On positive solutions for a class of singular nonlinear fractional differential equations*, Boundary Value Problems, 2012, 2012:73, 11 pp.

[15] Kirk, W.A., Shahzad, N., *Fixed Point Theory in Distance Spaces*, Springer Heidelberg, 2014.

[16] Laksmikantham, V., Ćirić, L., *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70**(2009), 4341-4349.

[17] Opoitsev, V.I., *Dynamics of collective behavior. III. Heterogenic systems*, (Russian), Avtomat. i Telemekh., **36**(1975), 124-138.

[18] Opoitsev, V.I., Khurodze, T.A., *Nonlinear operatorsin spaces with a cone*, (Russian), Tbilis. Gos. Univ., 1984, 271.

[19] Petruşel, A., Rus, I.A., *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134**(2005), no. 2, 411-418.

[20] Petru, T.P., Petruşel, A., Yao, J.C., *Ulam-Hyers stability for operatorial equations and inclusions via nonself operators*, Taiwanese J. Math., **15**(2011), no. 5, 2195-2212.

[21] Ur Rahman, M., Sarwar, M., *Coupled fixed point theorem for rational contraction in dislocated quasi-metric space*, Palestine J. Math., **5**(2016), no. 2, 6-11.

[22] Singh Chouhan, V., Sharma, R., *Coupled fixed point theorems for rational contractions in partially ordered metric spaces*, Int. J. Modern Math. Sci., **12**(2014), no. 3, 165-174.

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# Inequalities of Hermite-Hadamard type for $AH$-convex functions

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*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** Some inequalities of Hermite-Hadamard type for $AH$-convex functions defined on convex subsets in real or complex linear spaces are given. The case for functions of one real variable is explored in depth. Applications for special means are provided as well.

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**Keywords:** Convex functions, integral inequalities, $AH$-Convex functions.

## 1. Introduction

The following inequality holds for any convex function $f$ defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a,b\in\mathbb{R},\, a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [41]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [41]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [1]-[19], [22]-[24], [25]-[34] and [35]-[44].

Let $X$ be a vector space over the real or complex number field $\mathbb{K}$ and $x,y\in X,\ x\neq y$. Define the segment

$$[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}.$$

We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t):=f[(1-t)x+ty],\ t\in[0,1].$$

Note that $f$ is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [20, p. 2], [21, p. 2])

$$f\left(\frac{x+y}{2}\right)\le\int_0^1 f[(1-t)x+ty]dt\le\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

Let $X$ be a linear space and $C$ a convex subset in $X$. A function $f:C\to\mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the convex set $C$ if the following inequality holds

$$f\left((1-\lambda)x+\lambda y\right)\le(\ge)\frac{1}{(1-\lambda)\frac{1}{f(x)}+\lambda\frac{1}{f(y)}}=\frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)} \tag{AH}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x\in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)}+\lambda\frac{1}{f(y)}\le(\ge)\frac{1}{f\left((1-\lambda)x+\lambda y\right)}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

Therefore we can state the following fact:

**Criterion 1.1.** *Let $X$ be a linear space and $C$ a convex subset in $X$. The function $f:C\to(0,\infty)$ is AH-convex (concave) on $C$ if and only if $\frac{1}{f}$ is concave (convex) on $C$ in the usual sense.*

If we apply the Hermite-Hadamard inequality (1.2) for the function $\frac{1}{f}$ then we state the following result:

**Proposition 1.2.** *Let $X$ be a linear space and $C$ a convex subset in $X$. If the function $f:C\to(0,\infty)$ is AH-convex (concave) on $C$, then*

$$\frac{f(x)+f(y)}{2f(x)f(y)}\le(\ge)\int_0^1\frac{d\lambda}{f\left((1-\lambda)x+\lambda y\right)}\le(\ge)\frac{1}{f\left(\frac{x+y}{2}\right)} \tag{1.3}$$

*for any $x,y\in C$.*

Motivated by the above results, in this paper we establish some new Hermite-Hadamard type inequalities for $AH$-convex (concave) functions, first in the general setting of linear spaces and then in the particular case of functions of a real variable. Some examples for special means are provided as well.

## 2. Some Hermite-Hadamard type inequalities

The following result holds:

**Theorem 2.1.** *Let $X$ be a linear space and $C$ a convex subset in $X$. If the function $f : C \to (0, \infty)$ is AH-convex (concave) on $C$, then for any $x, y \in C$ we have*

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \leq (\geq) \frac{G^2(f(x), f(y))}{L(f(x), f(y))}, \tag{2.1}$$

*where the Logarithmic mean of positive numbers $a, b$ is defined as*

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \\ a & \text{if } a = b, \end{cases}$$

*and the geometric mean is $G = \sqrt{ab}$.*

*Proof.* Let $x, y \in C$ with $x \neq y$. If $f : C \to (0, \infty)$ is $AH$-convex (concave) on $C$, then $\frac{1}{f}$ is concave (convex) on $C$. This implies that the function

$$\varphi_{x,y} : [0,1] \to (0, \infty), \varphi_{x,y}(t) = \frac{1}{f((1-\lambda)x + \lambda y)}$$

is concave (convex) on $[0,1]$ and therefore continuous on $(0,1)$ with $\varphi_{x,y}(0) = \frac{1}{f(x)}$ and $\varphi_{x,y}(1) = \frac{1}{f(y)}$. The function $[0,1] \ni t \mapsto f((1-t)x + ty)$ is continuous on $(0,1)$ and since $f(x), f(y) > 0$ are finite, then the Lebesgue integral $\int_0^1 f((1-t)x + ty)\, dt$ exists and by (AH) we have

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \leq (\geq) f(x) f(y) \int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)}. \tag{2.2}$$

If $f(y) = f(x)$, then

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)} = \frac{1}{f(y)}.$$

If $f(y) \neq f(x)$, then by changing the variable $u = \lambda(f(x) - f(y)) + f(y)$ we have

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)} = \frac{\ln f(x) - \ln f(y)}{f(x) - f(y)} = \frac{1}{L(f(x), f(y))}.$$

By the use of (2.2) we get the desired result (2.1). □

**Remark 2.2.** Using the following well known inequalities

$$H(a,b) \leq G(a,b) \leq L(a,b)$$

we have

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \leq \frac{G^2(f(x), f(y))}{L(f(x), f(y))} \leq G(f(x), f(y)) \tag{2.3}$$

for any $x, y \in C$, provided that $f : C \to (0, \infty)$ is $AH$-convex.

If $f : C \to (0, \infty)$ is $AH$-concave, then

$$\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda \geq \frac{G^2(f(x), f(y))}{L(f(x), f(y))} \geq \frac{G(f(x), f(y))}{L(f(x), f(y))} H(f(x), f(y)) \tag{2.4}$$

for any $x, y \in C$.

**Theorem 2.3.** *Let $X$ be a linear space and $C$ a convex subset in $X$. If the function $f : C \to (0, \infty)$ is AH-convex (concave) on $C$, then for any $x, y \in C$ we have*

$$f\left(\frac{x+y}{2}\right) \leq (\geq) \frac{\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y)\,d\lambda}{\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda}. \tag{2.5}$$

*Proof.* By the definition of $AH$-convexity (concavity) we have

$$f\left(\frac{u+v}{2}\right) \leq (\geq) \frac{2f(u) f(v)}{f(u) + f(v)} \tag{2.6}$$

for any $u, v \in C$.

Let $x, y \in C$ and $\lambda \in [0, 1]$. If we take in (2.6) $u = (1-\lambda)x + \lambda y$ and $v = \lambda x + (1-\lambda)y$, then we get

$$f\left(\frac{x+y}{2}\right) \leq (\geq) \frac{2f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y)}{f((1-\lambda)x + \lambda y) + f(\lambda x + (1-\lambda)y)},$$

which is equivalent to

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) [f((1-\lambda)x + \lambda y) + f(\lambda x + (1-\lambda)y)] \leq (\geq) f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y). \tag{2.7}$$

Integrating the inequality on $[0, 1]$ over $\lambda \in [0, 1]$ and taking into account that

$$\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda = \int_0^1 f(\lambda x + (1-\lambda)y)\,d\lambda$$

we deduce from (2.7) the desired result (2.5). □

**Remark 2.4.** By the Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\begin{aligned} &\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y)\,d\lambda \\ &\leq \left[\int_0^1 f^2((1-\lambda)x + \lambda y)\,d\lambda \int_0^1 f^2(\lambda x + (1-\lambda)y)\,d\lambda\right]^{1/2} \\ &= \int_0^1 f^2((1-\lambda)x + \lambda y)\,d\lambda \end{aligned} \tag{2.8}$$

for any $x, y \in C$.

If the function $f : C \to (0, \infty)$ is $AH$-convex on $C$, then we have

$$f\left(\frac{x+y}{2}\right) \le \frac{\int_0^1 f\left((1-\lambda)x + \lambda y\right) f\left(\lambda x + (1-\lambda)y\right) d\lambda}{\int_0^1 f\left((1-\lambda)x + \lambda y\right) d\lambda} \tag{2.9}$$
$$\le \frac{\int_0^1 f^2\left((1-\lambda)x + \lambda y\right) d\lambda}{\int_0^1 f\left((1-\lambda)x + \lambda y\right) d\lambda}.$$

If the function $\psi_{x,y}(t) = f\left((1-t)x + ty\right)$, for some given $x, y \in C$ with $x \ne y$, is monotonic nondecreasing on $[0,1]$, then $\chi_{x,y}(t) = f\left(tx + (1-t)y\right)$ is monotonic nonincreasing on $[0,1]$ and by Čebyšev's inequality for monotonic opposite functions we have

$$\int_0^1 f\left((1-\lambda)x + \lambda y\right) f\left(\lambda x + (1-\lambda)y\right) d\lambda$$
$$\le \int_0^1 f\left((1-\lambda)x + \lambda y\right) d\lambda \int_0^1 f\left(\lambda x + (1-\lambda)y\right) d\lambda$$
$$= \left(\int_0^1 f\left((1-\lambda)x + \lambda y\right) d\lambda\right)^2.$$

So, for some given $x, y \in C$ with $x \ne y$, $\psi_{x,y}(t) = f\left((1-t)x + ty\right)$ is monotonic nondecreasing (nonincreasing) on $[0,1]$ and if the function $f : C \to (0, \infty)$ is $AH$-convex on $C$, then we have

$$f\left(\frac{x+y}{2}\right) \le \frac{\int_0^1 f\left((1-\lambda)x + \lambda y\right) f\left(\lambda x + (1-\lambda)y\right) d\lambda}{\int_0^1 f\left((1-\lambda)x + \lambda y\right) d\lambda} \tag{2.10}$$
$$\le \int_0^1 f\left((1-\lambda)x + \lambda y\right) d\lambda.$$

If $(X, \|\cdot\|)$ is a normed space, then the function $g : X \to [0, \infty)$, $g(x) = \|x\|^p$, $p \ge 1$ is convex and then the function $f : C \subset X \to (0, \infty)$, $f(x) = \frac{1}{\|x\|^p}$ is $AH$-*concave* on any convex subset of $X$ which does not contain $\{0\}$.

Utilising (2.1) we have

$$\int_0^1 \frac{d\lambda}{\|(1-\lambda)x + \lambda y\|^p} \ge \frac{1}{L\left(\|x\|^p, \|y\|^p\right)}, \tag{2.11}$$

for any linearly independent $x, y \in X$ and $p \ge 1$.

Making use of (2.5) we also have

$$\int_0^1 \frac{d\lambda}{\|(1-\lambda)x + \lambda y\|^p} \tag{2.12}$$
$$\ge \left\|\frac{x+y}{2}\right\|^p \int_0^1 \frac{d\lambda}{\|(1-\lambda)x + \lambda y\|^p \|\lambda x + (1-\lambda)y\|^p}$$

for any linearly independent $x, y \in X$ and $p \ge 1$.

## 3. More results for scalar case

If the function $f$ is defined on an interval $I$ and $a, b \in I$ with $a < b$, then

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda = \frac{1}{b-a}\int_a^b f(t)\, dt$$

and the inequalities (1.3), (2.1) and (2.5) can be written as

$$\frac{f(a)+f(b)}{2f(a)f(b)} \le (\ge) \frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt \le (\ge) \frac{1}{f\left(\frac{a+b}{2}\right)}, \tag{3.1}$$

$$\frac{1}{b-a}\int_a^b f(t)\, dt \le (\ge) \frac{G^2(f(a), f(b))}{L(f(a), f(b))}, \tag{3.2}$$

and

$$f\left(\frac{a+b}{2}\right) \le (\ge) \frac{\int_a^b f(t) f(a+b-t)\, dt}{\int_a^b f(t)\, dt}, \tag{3.3}$$

respectively, where $f : I \to (0, \infty)$ is assumed to be $AH$-convex (concave) on $I$.

The following proposition holds:

**Proposition 3.1.** *Let $f : I \to (0, \infty)$ be AH-convex (concave) on $I$. Let $x, y \in \mathring{I}$, the interior of $I$, then there exists $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$ such that*

$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{\varphi(y)}{f(y)}(y-x) \tag{3.4}$$

*holds.*

*Proof.* Let $x, y \in \mathring{I}$. Since the function $\frac{1}{f}$ is concave (convex) then the lateral derivatives $f'_-(y), f'_+(y)$ exists for $y \in \mathring{I}$ and $\left(\frac{1}{f}\right)'_{-(+)}(y) = -\frac{f'_{-(+)}(y)}{f^2(y)}$.

Since $\frac{1}{f}$ is concave (convex) then we have the *gradient inequality*

$$\frac{1}{f(y)} - \frac{1}{f(x)} \ge (\le) \lambda(y)(y-x) = -\lambda(y)(x-y)$$

with $\lambda(y) \in \left[-\frac{f'_+(y)}{f^2(y)}, -\frac{f'_-(y)}{f^2(y)}\right]$, which is equivalent to

$$\frac{1}{f(y)} - \frac{1}{f(x)} \ge (\le) \frac{\varphi(y)}{f^2(y)}(x-y) \tag{3.5}$$

with $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$.

The inequality (3.5) can be also written as

$$1 - \frac{f(y)}{f(x)} \ge (\le) \frac{\varphi(y)}{f(y)}(x-y)$$

or as

$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{\varphi(y)}{f(y)}(y-x)$$

and the inequality (3.4) is proved. □

**Corollary 3.2.** *Let $f : I \to (0,\infty)$ be AH-convex (concave) on $I$. If $f$ is differentiable on $\mathring{I}$ then for any $x, y \in \mathring{I}$, we have*

$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{f'(y)}{f(y)}(y - x). \tag{3.6}$$

The following result also holds:

**Theorem 3.3.** *Let $f : I \to (0,\infty)$ be AH-convex (concave) on $I$. If $a, b \in I$ with $a < b$, then we have the inequality*

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \le (\ge) \left[\frac{b-s}{b-a} f(b) + \frac{s-a}{b-a} f(a)\right] f(s) \tag{3.7}$$

*for any $s \in [a,b]$.*

*In particular, we have*

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \le (\ge) f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{2} \tag{3.8}$$

*and*

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \le (\ge) f(a) f(b). \tag{3.9}$$

*Proof.* If the function $f : I \to (0,\infty)$ is $AH$-convex (concave) on $I$, then the function $f$ is differentiable almost everywhere on $I$ and we have the inequality

$$\frac{f(t)}{f(s)} - 1 \le (\ge) \frac{f'(t)}{f(t)}(t - s) \tag{3.10}$$

for every $s \in [a,b]$ and almost every $t \in [a,b]$.

Multiplying (3.10) by $f(t) > 0$ and integrating over $t \in [a,b]$ we have

$$\frac{1}{f(s)}\int_a^b f^2(t)\,dt - \int_a^b f(t)\,dt \le (\ge) \int_a^b f'(t)(t-s)\,dt. \tag{3.11}$$

Integrating by parts we have

$$\int_a^b f'(t)(t-s)\,dt = f(b)(b-s) + f(a)(s-a) - \int_a^b f(t)\,dt$$

and by (3.11) we get the desired result (3.7).

We observe that (3.8) follows by (3.7) for $s = \frac{a+b}{2}$ while (3.9) follows by (3.7) for either $s = a$ or $s = b$. □

**Remark 3.4.** By the Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2 \le \frac{1}{b-a}\int_a^b f^2(t)\,dt$$

and if we assume that $f : I \to (0,\infty)$ is $AH$-convex on $I$, then we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{1}{b-a}\int_a^b f^2(t)\,dt\right)^{1/2} \le \sqrt{f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{2}} \tag{3.12}$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{1}{b-a}\int_a^b f^2(t)\,dt\right)^{1/2} \le \sqrt{f(a)\,f(b)}. \tag{3.13}$$

The following result also holds:

**Theorem 3.5.** *Let $f : I \to (0,\infty)$ be AH-convex (concave) on $I$. If $a, b \in I$ with $a < b$, then we have the inequality*

$$\begin{aligned}&\int_a^b \ln f(t)\,dt + \frac{1}{f(s)}\int_a^b f(t)\,dt \\ &\le (\ge)\, b - a + (b-s)\ln f(b) + (s-a)\ln f(a)\end{aligned} \tag{3.14}$$

*for any $s \in [a,b]$.*

*In particular, we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \frac{1}{f\left(\frac{a+b}{2}\right)}\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge)\, 1 + \ln\sqrt{f(b)\,f(a)}\end{aligned} \tag{3.15}$$

*and*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \left[\frac{f(b)+f(a)}{2f(a)\,f(b)}\right]\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge)\, 1 + \ln\sqrt{f(b)\,f(a)}.\end{aligned} \tag{3.16}$$

*Proof.* Integrating the inequality (3.10) over $t \in [a,b]$ we have

$$\frac{1}{f(s)}\int_a^b f(t)\,dt - (b-a) \le (\ge) \int_a^b \frac{f'(t)}{f(t)}(t-s)\,dt. \tag{3.17}$$

Observe that

$$\begin{aligned}\int_a^b \frac{f'(t)}{f(t)}(t-s)\,dt &= \int_a^b (\ln f(t))'(t-s)\,dt \\ &= (t-s)\ln f(t)\big|_a^b - \int_a^b \ln f(t)\,dt \\ &= (b-s)\ln f(b) + (s-a)\ln f(a) - \int_a^b \ln f(t)\,dt\end{aligned}$$

and by (3.17) we get

$$\begin{aligned}&\frac{1}{f(s)}\int_a^b f(t)\,dt - (b-a) \\ &\le (\ge)\,(b-s)\ln f(b) + (s-a)\ln f(a) - \int_a^b \ln f(t)\,dt,\end{aligned}$$

which is equivalent to

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(s)}\int_a^b f(t)\,dt$$
$$\leq (\geq)\, b-a+(b-s)\ln f(b)+(s-a)\ln f(a)$$

for any $s\in[a,b]$.

If we take in (3.14) $s=\frac{a+b}{2}$ then we get the desired result (3.15).

If we take in (3.14) $s=a$ and $s=b$ we get

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(a)}\int_a^b f(t)\,dt \leq (\geq)\, b-a+(b-a)\ln f(b)$$

and

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(b)}\int_a^b f(t)\,dt \leq (\geq)\, b-a+(b-a)\ln f(a),$$

which by addition produces

$$2\int_a^b \ln f(t)\,dt + \frac{1}{f(a)}\int_a^b f(t)\,dt + \frac{1}{f(b)}\int_a^b f(t)\,dt$$
$$\leq (\geq)\, 2(b-a)+(b-a)\ln f(b)+(b-a)\ln f(a)$$

and then

$$\int_a^b \ln f(t)\,dt + \left[\frac{f(b)+f(a)}{2f(a)f(b)}\right]\int_a^b f(t)\,dt$$
$$\leq (\geq)\, b-a+(b-a)\ln\sqrt{f(b)f(a)},$$

which is equivalent to (3.16). □

**Remark 3.6.** We observe that

$$(b-s)\ln f(b)+(s-a)\ln f(a)=0$$

iff

$$s=\frac{b\ln f(b)-a\ln f(a)}{\ln f(b)-\ln f(a)}=\frac{L(f(a),f(b))}{L\left([f(a)]^a,[f(b)]^b\right)}.$$

If

$$s=\frac{L(f(a),f(b))}{L\left([f(a)]^a,[f(b)]^b\right)}\in I$$

then from (3.14) we have

$$\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \frac{1}{f\left(\frac{L(f(a),f(b))}{L\left([f(a)]^a,[f(b)]^b\right)}\right)}\frac{1}{b-a}\int_a^b f(t)\,dt \leq (\geq)\, 1. \tag{3.18}$$

Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$ two linearly independent vectors on $X$. Since the function $g : [0,1] \to (0,\infty)$, $g(t) = \|(1-t)x + ty\|^p$, $p \geq 1$ is convex on $[0,1]$, then the function $f : [0,1] \to (0,\infty)$, $g(t) = \frac{1}{\|(1-t)x+ty\|^p}$ is $AH$-concave on $[0,1]$.

Making use of the inequalities (3.8) and (3.9) we get

$$\left\|\frac{x+y}{2}\right\|^p \int_0^1 \frac{1}{\|(1-t)x+ty\|^{2p}} dt \geq \frac{\|x\|^p + \|y\|^p}{2\|x\|^p \|y\|^p} \tag{3.19}$$

and

$$\int_0^1 \frac{1}{\|(1-t)x+ty\|^{2p}} dt \geq \frac{1}{\|x\|^p \|y\|^p}. \tag{3.20}$$

## 4. Applications for special means

Let us recall the following means:

a) The *arithmetic mean*

$$A(a,b) := \frac{a+b}{2}, \quad a, b > 0,$$

b) The *geometric mean*

$$G(a,b) := \sqrt{ab}; \quad a, b \geq 0,$$

c) The *harmonic mean*

$$H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}; \quad a, b > 0,$$

d) The *identric mean*

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad b \neq a \\ a & \text{if} \quad b = a \end{cases}; \quad a, b > 0$$

e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if} \quad b \neq a \\ a & \text{if} \quad b = a \end{cases}; \quad a, b > 0$$

f) The $p-$*logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if} \quad b \neq a, \quad p \in \mathbb{R} \setminus \{-1, 0\} \\ a & \text{if} \quad b = a \end{cases}; \quad a, b > 0.$$

It is well known that, if $L_{-1} := L$ and $L_0 := I$, then the function $\mathbb{R} \ni p \mapsto L_p$ is monotonically strictly increasing. In particular, we have

$$H(a,b) \le G(a,b) \le L(a,b) \le I(a,b) \le A(a,b).$$

Consider the function

$$f(t) = t^p = \frac{1}{t^{-p}}$$

if $-p > 1$ or $-p < 0$, i.e. $p \in (-\infty,-1) \cup (0,\infty)$ then the function $f(t) = t^p, t > 0$ is $AH$-concave. If $p \in (-1,0)$ then the function $f(t) = t^p, t > 0$ is $AH$-convex.

Now, if we write the inequality (3.2) for the function $f(t) = t^p$ and $0 < a < b$ we get

$$\frac{1}{b-a}\int_a^b t^p dt \le (\ge) \frac{G^2(a^p,b^p)}{L(a^p,b^p)}, \tag{4.1}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty))$.

Now, observe that

$$\frac{1}{b-a}\int_a^b t^p dt = \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} = L_p^p(a,b),$$

$$\begin{aligned} L(a^p,b^p) &= \frac{b^p - a^p}{\ln b^p - \ln a^p} = \frac{b^p - a^p}{p(b-a)}\frac{b-a}{\ln b - \ln a} \\ &= L_{p-1}^{p-1}(a,b)\,L(a,b),\ p \in \mathbb{R}\setminus\{0,1\} \end{aligned}$$

and

$$G^2(a^p,b^p) = G^{2p}(a,b).$$

Then by (4.1) we get

$$L_p^p(a,b)\,L_{p-1}^{p-1}(a,b)\,L(a,b) \le (\ge)\, G^{2p}(a,b), \tag{4.2}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty)\setminus\{1\})$.

If we write the inequality (3.8) for the function $f(t) = t^p$ and $0 < a < b$ we get

$$\frac{1}{b-a}\int_a^b t^{2p} dt \le (\ge) \left(\frac{a+b}{2}\right)^p \frac{a^p + b^p}{2} \tag{4.3}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty))$.

Since

$$\frac{1}{b-a}\int_a^b t^{2p} dt = L_{2p}^{2p}(a,b),\ p \in \mathbb{R}\setminus\left\{-\frac{1}{2},0\right\},$$

$$\left(\frac{a+b}{2}\right)^p = A^p(a,b),\ \frac{a^p + b^p}{2} = A(a^p,b^p),$$

then by (4.3) we have

$$L_{2p}^{2p}(a,b) \le (\ge)\, A^p(a,b)\,A(a^p,b^p) \tag{4.4}$$

where $p \in (-1,0)\setminus\left\{-\frac{1}{2}\right\}\,(p \in (-\infty,-1) \cup (0,\infty))$.

Now consider the function $f(t) = \ln t$, $t > 1$. The function

$$g(t) := \frac{1}{\ln t},\ t > 1$$

is convex on $(1,\infty)$. If we apply the inequality (3.2) for the $AH$-concave function $f(t)=\ln t$, $t>1$ on $[a,b]\subset(1,\infty)$, then we get

$$\ln I(a,b)\geq\frac{G^{2}(\ln a,\ln b)}{L(\ln a,\ln b)}. \tag{4.5}$$

## References

[1] Alomari, M., Darus, M., *The Hadamard's inequality for s-convex function*, Int. J. Math. Anal., (Russian), **2**(2008), no. 13-16, 639-646.

[2] Alomari, M., Darus, M., *Hadamard-type inequalities for s-convex functions*, Int. Math. Forum, **3**(2008), no. 37-40, 1965-1975.

[3] Anastassiou, G.A., *Univariate Ostrowski inequalities,* revisited, Monatsh. Math., **135**(2002), no. 3, 175-189.

[4] Barnett, N.S., Cerone, P., Dragomir, S.S., Pinheiro, M.R., Sofo, A., *Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications*, Inequality Theory and Applications, vol. 2 (Chinju/Masan, 2001), 19-32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: RGMIA Res. Rep. Coll., **5**(2002), no. 2, Art. 1 `[Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf]`.

[5] Beckenbach, E.F., *Convex functions*, Bull. Amer. Math. Soc., **54**(1948), 439-460.

[6] Bombardelli, M., Varošanec, S., *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl., **58**(2009), no. 9, 1869-1877.

[7] Breckner, W.W., *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, (German), Publ. Inst. Math., Beograd, N.S., **23(37)**(1978), 13-20.

[8] Breckner, W.W., Orbán, G., *Continuity properties of rationally s-convex mappings with values in an ordered topological linear space*, Babeş-Bolyai Univ., Fac. of Math., Cluj-Napoca, 1978, viii+92 pp.

[9] Cerone, P., Dragomir, S.S., *Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou*, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press, New York, 135-200.

[10] Cerone, P., Dragomir, S.S., *New bounds for the three-point rule involving the Riemann-Stieltjes integrals*, in: Advances in Statistics Combinatorics and Related Areas, C. Gulati, et al.(Eds.), World Science Publishing, 2002, 53-62.

[11] Cerone, P., Dragomir, S.S., Roumeliotis, J., *Some Ostrowski type inequalities for n-time differentiable mappings and applications*, Demonstratio Math., **32**(1999), no. 2, 697-712.

[12] Cristescu, G., *Hadamard type inequalities for convolution of h-convex functions*, Ann. Tiberiu Popoviciu Semin. Funct. Eq. Approx. Convexity, **8**(2010), 3-11.

[13] Dragomir, S.S., *Ostrowski's inequality for monotonous mappings and applications*, J. KSIAM, **3**(1999), no. 1, 127-135.

[14] Dragomir, S.S., *The Ostrowski's integral inequality for Lipschitzian mappings and applications*, Comp. Math. Appl., **38**(1999), 33-37.

[15] Dragomir, S.S., *On the Ostrowski's inequality for Riemann-Stieltjes integral*, Korean J. Appl. Math., **7**(2000), 477-485.

[16] Dragomir, S.S., *On the Ostrowski's inequality for mappings of bounded variation and applications*, Math. Ineq. Appl., **4**(2001), no. 1, 33-40.

[17] Dragomir, S.S., *On the Ostrowski inequality for Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *where* $f$ *is of Hölder type and* $u$ *is of bounded variation and applications*, J. KSIAM, **5**(2001), no. 1, 35-45.

[18] Dragomir, S.S., *Ostrowski type inequalities for isotonic linear functionals*, J. Inequal. Pure Appl. Math., **3**(2002), no. 5, Art. 68.

[19] Dragomir, S.S., *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math., **3**(2002), no. 2, Article 31, 8 pp.

[20] Dragomir, S.S., *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math., **3**(2002), No. 2, Article 31.

[21] Dragomir, S.S., *An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math., **3**(2002), no. 3, Article 35.

[22] Dragomir, S.S., *An Ostrowski like inequality for convex functions and applications*, Revista Math. Complutense, **16**(2003), no. 2, 373-382.

[23] Dragomir, S.S., *Operator Inequalities of Ostrowski and Trapezoidal Type*, Springer Briefs in Mathematics, Springer, New York, 2012. x+112 pp.

[24] Dragomir, S.S., Cerone, P., Roumeliotis, J., Wang, S., *A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis*, Bull. Math. Soc. Sci. Math. Romania, **42(90)**(1999), no. 4, 301-314.

[25] Dragomir, S.S., Fitzpatrick, S., *The Hadamard inequalities for s-convex functions in the second sense*, Demonstratio Math., **32**(1999), no. 4, 687-696.

[26] Dragomir, S.S., Fitzpatrick, S., *The Jensen inequality for s-Breckner convex functions in linear spaces*, Demonstratio Math., **33**(2000), no. 1, 43-49.

[27] Dragomir, S.S., Mond, B., *On Hadamard's inequality for a class of functions of Godunova and Levin*, Indian J. Math., **39**(1997), no. 1, 1-9.

[28] Dragomir, S.S., Pearce, C.E.M., *On Jensen's inequality for a class of functions of Godunova and Levin*, Period. Math. Hungar., **33**(1996), no. 2, 93-100.

[29] Dragomir, S.S., Pearce, C.E.M., *Quasi-convex functions and Hadamard's inequality*, Bull. Austral. Math. Soc., **57**(1998), 377-385.

[30] Dragomir, S.S., Pečarić, J., Persson, L., *Some inequalities of Hadamard type*, Soochow J. Math., **21**(1995), no. 3, 335-341.

[31] Dragomir, S.S., Rassias, Th.M. (Eds.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.

[32] Dragomir, S.S., Wang, S., *A new inequality of Ostrowski's type in* $L_1-$*norm and applications to some special means and to some numerical quadrature rules*, Tamkang J. of Math., **28**(1997), 239-244.

[33] Dragomir, S.S., Wang, S., *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules*, Appl. Math. Lett., **11**(1998), 105-109.

[34] Dragomir, S.S., Wang, S., *A new inequality of Ostrowski's type in* $L_p-$*norm and applications to some special means and to some numerical quadrature rules*, Indian J. of Math., **40**(1998), no. 3, 245-304.

[35] El Farissi, A., *Simple proof and refeinment of Hermite-Hadamard inequality*, J. Math. Ineq., **4**(2010), no. 3, 365-369.

[36] Godunova, E.K., Levin, V.I., *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, (Russian) Numerical mathematics and mathematical physics, Moskov. Gos. Ped. Inst., Moscow, **166**(1985), 138-142.

[37] Hudzik, H., Maligranda, L., *Some remarks on s-convex functions*, Aequationes Math., **48**(1994), no. 1, 100-111.

[38] Kikianty, E., Dragomir, S.S., *Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space*, Math. Inequal. Appl. (in press).

[39] Kirmaci, U.S., Klaričić Bakula, M., Özdemir, M.E., Pečarić, J., *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput., **193**(2007), no. 1, 26-35.

[40] Latif, M.A., *On some inequalities for h-convex functions* Int. J. Math. Anal., (Russian), **4**(2010), no. 29-32, 1473-1482.

[41] Mitrinović, D.S., Lacković, I.B., *Hermite and convexity*, Aequationes Math., **28**(1985), 229-232.

[42] Mitrinović, D.S., Pečarić, J.E., *Note on a class of functions of Godunova and Levin*, C. R. Math. Rep. Acad. Sci. Canada, **12**(1990), no. 1, 33-36.

[43] Pearce, C.E.M., Rubinov, A.M., *P-functions, quasi-convex functions, and Hadamard-type inequalities*, J. Math. Anal. Appl., **240**(1999), no. 1, 92-104.

[44] Pečarić, J.E., Dragomir, S.S., *On an inequality of Godunova-Levin and some refinements of Jensen integral inequality*, Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1989), 263-268, Preprint, 89-6, Babeş-Bolyai Univ., Cluj-Napoca, 1989.

[45] Pečarić, J., Dragomir, S.S., *A generalization of Hadamard's inequality for isotonic linear functionals*, Radovi Mat., Sarajevo, **7**(1991), 103-107.

[46] Radulescu, M., Radulescu, S., Alexandrescu, P., *On the Godunova-Levin-Schur class of functions*, Math. Inequal. Appl., **12**(2009), no. 4, 853-862.

[47] Sarikaya, M.Z., Saglam, A., Yildirim, H., *On some Hadamard-type inequalities for h-convex functions*, J. Math. Inequal., **2**(2008), no. 3, 335-341.

[48] Set, E., Özdemir, M.E., Sarıkaya, M.Z., *New inequalities of Ostrowski's type for s-convex functions in the second sense with applications*, Facta Univ. Ser. Math. Inform., **27**(2012), no. 1, 67-82.

[49] Sarikaya, M.Z., Set, E., Özdemir, M.E., *On some new inequalities of Hadamard type involving h-convex functions*, Acta Math. Univ. Comenian., **79**(2010), no. 2, 265-272.

[50] Tunç, M., *Ostrowski-type inequalities via h-convex functions with applications to special means*, J. Inequal. Appl., 2013, 2013:326.

[51] Varošanec, S., *On h-convexity*, J. Math. Anal. Appl., **326**(2007), no. 1, 303-311.



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# Nonlinear differential polynomial sharing a nonzero polynomial with certain degree

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*Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary*

**Abstract.** In this paper we study the uniqueness problems of meromorphic functions when certain non-linear differential polynomial sharing a nonzero polynomial with certain degree. We obtain some results which will rectify, improve and generalize some recent results of C. Wu and J. Li [15] in a large extent. Our results will also improve and generalize some recent results due to Fang [5], Zhang-Zhang [24], Zhang [22], Xu et al. [16], Qi-Yang [14], Dou-Qi-Yang [4], Zhang-Xu [26] and Liu-Yang [13].

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## 1. Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let $f$ and $g$ be two non-constant meromorphic functions and let $a$ be a finite complex number. We say that $f$ and $g$ share $a$ CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $a$ IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that $f$ and $g$ share $\infty$ CM, if $1/f$ and $1/g$ share 0 CM, and we say that $f$ and $g$ share $\infty$ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [7]). For a non-constant meromorphic function $f$, we denote by $T(r,f)$ the Nevanlinna characteristic of $f$ and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set of finite linear measure.

A meromorphic function $a(z)$ is called a small function with respect to $f$, provided that $T(r,a)=S(r,f)$.

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty$, outside of a possible exceptional set of finite linear measure.

A finite value $z_0$ is said to be a fixed point of $f(z)$ if $f(z_0)=z_0$.

For the sake of simplicity we also use the notations $m^* := \chi_\mu m$, where $\chi_\mu=0$ if $\mu=0$, $\chi_\mu=1$ if $\mu\neq 0$.

In 1959, W.K. Hayman (see [7], Corollary of Theorem 9) proved the following theorem.

**Theorem A.** *Let $f$ be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f'=1$ has infinitely many solutions.*

*Theorem A* was extended by Chen [3] in the following manner:

**Theorem B.** *Let $f$ be a transcendental entire function, $n$, $k$ two positive integers with $n\geq k+1$. Then $(f^n)^{(k)}-1$ has infinitely many zeros.*

Latter Fang [5] obtained the following two uniqueness theorem corresponding to *Theorem B.*

**Theorem C.** *Letf and $g$ be two non-constant entire functions, and let $n$, $k$ be two positive integers with $n>2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$ or $f\equiv tg$ for a constant $t$ such that $t^n=1$.*

**Theorem D.** *Let $f$ and $g$ be two non-constant entire functions, and let $n$, $k$ be two positive integers with $n>2k+8$. If $(f^n(z)(f(z)-1))^{(k)}$ and $(g^n(z)(g(z)-1))^{(k)}$ share 1 CM, then $f(z)\equiv g(z)$.*

In 2008, improving the above results J. F. Zhang and X. Y. Zhang [24] obtained the following theorem:

**Theorem E.** *Let $f$ and $g$ be two non-constant entire functions and let $n$, $k$ be two positive integers with $n>5k+7$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 IM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $(-1)^k(c_1c_2)^n[nc]^{2k}=1$ or $f\equiv tg$ for a constant $t$ such that $t^n=1$.*

**Theorem F.** *Let $f$ and $g$ be two non-constant entire functions, and let $n$, $k$ be two positive integers with $n>5k+13$. If $(f^n(z)(f(z)-1))^{(k)}$ and $(g^n(z)(g(z)-1))^{(k)}$ share 1 IM, then $f(z)\equiv g(z)$.*

In 2008 Zhang [22] obtained similar type of result as mentioned in *Theorem E* in the the following way:

**Theorem G.** *Let $f$ and $g$ be two non-constant entire functions, and let $n$, $k$ be two positive integers with $n>2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share $z$ CM, then either*

(1) *$k=1$, $f(z)=c_1e^{cz^2}$, $g(z)=c_2e^{-cz^2}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $4(c_1c_2)^n(nc)^2=-1$ or*
(2) *$f\equiv tg$ for a constant $t$ such that $t^n=1$.*

Recently Xiao-Bin Zhang and Jun-Feng Xu [26] proved the following result for meromorphic function.

**Theorem H.** [26] *Let $f$ and $g$ be two non-constant meromorphic functions, and $a(z)$ $(\not\equiv 0, \infty)$ be a small function with respect to $f$. Let $n$, $k$ and $m$ be three positive integers with $n > 3k + m + 8$ and let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$ or $P(w) \equiv c_0$ where $a_0(\neq 0), a_1, \ldots, a_{m-1}, a_m(\neq 0), c_0(\neq 0)$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $a$ CM, then*

(I) *when $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$, one of the following three cases holds:*

(I1) *$f(z) \equiv tg(z)$ for a constant $t$ such that $t^d = 1$, where*

$$d = GCD(n+m, \ldots, n+m-i, \ldots, n),\ a_{m-i} \neq 0$$

*for some $i = 1, 2, \ldots, m$,*

(I2) *$f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \ldots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \ldots + a_0),$$

(I3) *$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv a^2$;*

(II) *when $P(w) \equiv c_0$, one of the following two cases holds:*

(II1) *$f \equiv tg$ for some constant $t$ such that $t^n = 1$,*

(II2) *$c_0^2 [f^n]^{(k)} [g^n]^{(k)} \equiv a^2$.*

Generalized results in the above directions for entire function were obtained by Qi-Yang [14] and Dou-Qi-Yang [4] in the following manner:

**Theorem I.** *Let $f$ and $g$ be two transcendental entire functions, and let $n$, $k$ and $m$ be three positive integers with $n > 2k + m^* + 4$, $\lambda$, $\mu$ be two constants such that $|\lambda| + |\mu| \neq 0$. If $[f^n(\lambda f^m + \mu)]^{(k)}$ and $[g^n(\lambda g^m + \mu)]^{(k)}$ share $z$ CM, then one of the following conclusions hold:*

(1) *If $\lambda\mu \neq 0$, then $f^d \equiv g^d$, where $d = gcd(n, m)$; in particular $f \equiv g$, when $d = 1$;*

(2) *If $\lambda\mu = 0$, then $f \equiv cg$, where $c$ is a constant satisfying $c^{n+m^*} = 1$, or $k = 1$ and $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$, for some constants $b_1$, $b_2$ and $b$ that satisfy $4(\lambda + \mu)^2 (b_1 b_2)^{n+m^*} [(n + m^*) b]^2 = -1$.*

**Theorem J.** *Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$ or $P(z) = C$, where $a_0, a_1, \ldots, a_{m-1}$, $a_m(\neq 0), C(\neq 0)$ are complex constants. Suppose that $f$ and $g$ be two transcendental entire functions, and let $n$, $k$ and $m$ be three positive integers with $n > 2k + m^{**} + 4$, where $m^{**} = 0$, if $P(z) \equiv C$, otherwise $m^{**} = m$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $z$ CM then the following conclusions hold:*

(i) *If $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$ is not a monomial, then $f(z) \equiv tg(z)$ for a constant $t$ such that $t^d = 1$, where $d = gcd(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \ldots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \ldots + a_0);$$

(ii) *If* $P(z) \equiv C$ *or* $P(z) = a_m z^m$ *then* $f \equiv tg$ *for some constant* $t$ *such that* $t^{n+m^{**}} = 1$, *or then* $f = b_1 e^{bz^2}$, $g = b_2 e^{-bz^2}$, *for three constants* $b_1$, $b_2$ *and* $b$ *that satisfy* $4a_m^2(b_1b_2)^{n+m}[(n+m)b]^2 = -1$ *or* $4C^2(b_1b_2)^n[nb]^2 = -1$.

In 2013, Liu and Yang [13] replaced the CM value sharing concept by IM fixed point sharing one in the above two theorems. They proved the following results:

**Theorem K.** *Let* $f$ *and* $g$ *be two transcendental entire functions, and let* $n$, $k$ *and* $m$ *be three positive integers with* $n > 5k + 4m^* + 7$, $\lambda$, $\mu$ *be two constants such that* $|\lambda| + |\mu| \neq 0$. *If* $[f^n(\lambda f^m + \mu)]^{(k)}$ *and* $[g^n(\lambda g^m + \mu)]^{(k)}$ *share* $z$ *IM, then the conclusion of* Theorem I *holds*

**Theorem L.** *Let* $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ *or* $P(\omega) = C$, *where* $a_0, a_1, \ldots, a_{m-1}$, $a_m(\neq 0), C(\neq 0)$ *are complex constants. Suppose that* $f$ *and* $g$ *be two transcendental entire functions, and let* $n$, $k$ *and* $m$ *be three positive integers with* $n > 5k + 4m^{**} + 7$. *If* $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share* $z$ *IM then the conclusion of* Theorem J *holds*

In this paper, we always use $P(\omega)$ denoting an arbitrary polynomial of degree $n$ as follows:

$$P(\omega) = a_n\omega^n + a_{n-1}\omega^{n-1} + \ldots + a_0 = a_n(\omega - c_{l_1})^{l_1}(\omega - c_{l_2})^{l_2}\ldots(\omega - c_{l_s})^{l_s}, \quad (1.1)$$

where $a_i (i = 0, 1, \ldots, n-1)$, $a_n \neq 0$ and $c_{l_j} (j = 1, 2, \ldots, s)$ are distinct finite complex numbers and $l_1, l_2, \ldots, l_s$, $s$, $n$ and $k$ are all positives integers satisfying

$$\sum_{i=1}^{s} l_i = n.$$

Also we let

$$l = \max\{l_1, l_2, \ldots, l_s\}$$

and from (1.1) we have

$$P(w) = (w - c_l)^l P_*(w),$$

where $P_*(w)$ is a polynomial in degree $n - l = r(say)$.

We also use $P_1(\omega_1)$ as an arbitrary non-zero polynomial defined by

$$P_1(\omega_1) = a_n \prod_{\substack{i=1 \\ l_i \neq l}}^{s} (\omega_1 + c_l - c_{l_i}) = b_m\omega_1^m + b_{m-1}\omega_1^{m-1} + \ldots + b_0, \quad (1.2)$$

where $\omega_1 = \omega - c_l$ and $m = n - l$.
Obviously

$$P(\omega) = \omega_1^l P_1(\omega_1).$$

If we observe the above theorems carefully we see that all the investigations were done on the basis of sharing of the expression of the particular form $h^nP(h)$ ($h = f$ or $g$). So it will be quiet natural to investigate all the results for the most standard form $P(h)$ instead of $h^nP(h)$ ($h = f$ or $g$).

Recently, C. Wu and J. Li [15] obtained the following results in this direction:

**Theorem M.** *Let $f$ and $g$ be two non-constant meromorphic functions, and let $n$, $k$ and $l$ be three positive integers satisfying $4lk + 12l > 4kn + 11n + 9k + 14$. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share 1 IM, then either $f = b_1e^{bz} + c$, $g = b_2e^{-bz} + c$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $b_1$, $b_2$, $b$ are three constants satisfying $(-1)^k(b_1b_2)^n(nb)^{2k} = 1$ and $R(\omega_1, \omega_2) = P(\omega_1) - P(\omega_2)$.*

**Theorem N.** *Let $f$ and $g$ be two non-constant meromorphic functions, and let $n$, $k$ and $l$ be three positive integers satisfying $kl + 6l > nk + 5n + 3k + 8$. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share 1 CM, then conclusion of* Theorem M *holds.*

**Theorem O.** *Let $f$ and $g$ be two non-constant entire functions, and let $n$, $k$ and $l$ be three positive integers satisfying $5l > 4n + 5k + 7$. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share 1 IM, then conclusion of* Theorem M *holds.*

**Theorem P.** *Let $f$ and $g$ be two non-constant entire functions, and let $n$, $k$ and $l$ be three positive integers satisfying $2l > n + 2k + 4$. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share 1 CM, then conclusion of* Theorem M *holds.*

**Remark 1.1.** *The results* [15] *are new and seem fine. However in the proof of Theorem 11* [15]*, one can easily point out a number of gaps.*

*We first consider p.* 299 *under the* **case 1.1.2** *fifth line from top.*
*The calculations are true only when $p_{j_0} > k$, : A question arises: When $p_{j_0} \leq k$ ? Actually the author did not consider this case.*

*In the same page the author used the inequality*

$$\overline{N}(r, \infty; f) \leq \sum_{j=1}^{s} \overline{N}(r, c_j; g) + \overline{N}(r, 0; g^{'}),$$

*which is not true for any arbitrary $k$ and the situation when*

$$[L(f)]^{(k)}[L(g)]^{(k)} \equiv 1.$$

**Remark 1.2.** The authors declare that *Lemma 11* in [15] can be obtained from [17]. But in [17] there is no such lemma. One can easily verify that the *lemma 11* in [15] is actually *Lemma 2.12* of [25]. Also one can easily observe that to prove *Lemma 2.12* in [25], *Lemma 2.8* plays a vital role [see p.8 last line in [25]]. But the following example shows that *Lemma 2.8* of [25] is invalid.

**Example 1.1.** Let $F = ze^z$, $G = \frac{1}{ze^z}$, then $F$ and $G$ share 1 and $-1$ and satisfy

$$N(r, 0; F) + N(r, \infty; F) = S(r, F)$$

and

$$N(r, 0; G) + N(r, \infty; G) = S(r, G).$$

It is clear that $F$ and $G$ share neither 0 nor $\infty$.

So the very existence of *Lemma 11* in [15] and proof of *Theorem 11*, where *Lemma 11* plays a vital role is questionable. In this paper we tackle the problem by obtaining the correct proof of *Lemma 11* as well as *Theorem 11.* We also observe that in *Theorems M* and *N* as $n = l+r$, the relation $4lk+12l > 4kn+11n+9k+14$ ($kl+6l >$

$nk+5n+3k+8$) ultimately produce $l>(4k+11)r+9k+14$ ($l>(k+5)r+3k+8$) which are very much stronger result in-comparison to the lower bound of $l$ obtained by the previous authors. In that sense in this paper we shall decrease the lower bound of $l$ to a large extent. Not only that our results will largely improve and generalize all the previous results mentioned earlier. To proceed further we require the following definition. In 2001 an idea of gradation of sharing of values was introduced in {[8], [9]} which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

**Definition 1.1.** [8, 9] Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that $f$, $g$ share the value $a$ with weight $k$.

The definition implies that if $f$, $g$ share a value $a$ with weight $k$ then $z_0$ is an $a$-point of $f$ with multiplicity $m$ ($\leq k$) if and only if it is an $a$-point of $g$ with multiplicity $m$ ($\leq k$) and $z_0$ is an $a$-point of $f$ with multiplicity $m$ ($>k$) if and only if it is an $a$-point of $g$ with multiplicity $n$ ($>k$), where $m$ is not necessarily equal to $n$.

We write $f$, $g$ share $(a,k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a,k)$, then $f$, $g$ share $(a,p)$ for any integer $p$, $0\leq p<k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a,0)$ or $(a,\infty)$ respectively.

The main results of the paper are as follows.

**Theorem 1.1.** *Let $f$ and $g$ be two transcendental meromorphic functions and $p(z)$ be a nonzero polynomial with $deg(p)\leq l-1$, where $n$, $k$, $r$ and $l$ be four positive integers with $n=l+r$ such that $l>3k+r+8$. Suppose $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $(p,2)$, where $P(\omega)$ be defined as in (1.1). Now*

- (I) *when $s\geq 2$ then one of the following three cases holds:*
  - (I1) *$f\equiv tg$ for a constant $t$ such that $t^d=1$, where $d=GCD(n,\ldots,n-i,\ldots,1)$, $a_{n-i}\neq 0$ for some $i\in\{1,2,\ldots,n-1\}$;*
  - (I2) *$f$ and $g$ satisfy the algebraic equation $R(f,g)\equiv 0$, where*

$$R(\omega_1,\omega_2)=(a_n\omega_1^n+a_{n-1}\omega_1^{n-1}+\ldots+a_1\omega_1)-(a_n\omega_2^n+a_{n-1}\omega_2^{n-1}+\ldots+a_1\omega_2);$$

  - (I3) *$[P(f)]^{(k)}[P(g)]^{(k)}\equiv p^2$;*
- (II) *when $s=1$ then one of the following two cases holds:*
  - (II1) *$f\equiv tg$ for some constant $t$ such that $t^n=1$,*
  - (II2) *if $p(z)$ is not a constant, then $f=c_1e^{cQ(z)}+c_l$, $g=c_2e^{-cQ(z)}+c_l$, where*

$$Q(z)=\int_0^z p(z)dz,$$

    *$c_1$, $c_2$ and $c$ are constants such that $b_i^2(c_1c_2)^{l+i}[(l+i)c]^2=-1$, if $p(z)$ is a nonzero constant $b$, then $f=c_3e^{cz}+c_l$, $g=c_4e^{-cz}+c_l$, where $c_3$, $c_4$ and $c$ are constants such that $(-1)^k b_i^2(c_3c_4)^{l+i}[(l+i)c]^{2k}=b^2$.*

*In particular when $l_i>k(i=1,2,\ldots,s)$ and*

$$\Theta(0;f)+\Theta(\infty;f)>\frac{n(3-s)-2ks+4k}{n+2k},$$

*then* $(I3)$ *does not hold.*

**Theorem 1.2.** *Let $f$ and $g$ be two transcendental meromorphic functions and $p(z)$ be a nonzero polynomial with $deg(p) \leq l-1$, where $n$, $k$, $r$ and $l$ be four positive integers with $n = l + r$ such that $l > 9k + 4r + 14$. Suppose $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $p(z)$ IM, where $P(\omega)$ be defined as in (1.1). Then the conclusion of Theorem 1.1 holds.*

**Remark 1.3.** Theorems 1.1 and 1.2 both hold for two non-constant meromorphic functions $f$ and $g$ when $p(z)$ is a non-zero constant.

**Remark 1.4.** When $l = n$, $c_l = 0$ from Theorem 1.1 we can easily get an improved version of *Theorem H.*

**Corollary 1.1.** *Let $f$ and $g$ be two transcendental entire functions and $p(z)$ be a nonzero polynomial with $deg(p) \leq l - 1$, where $n$, $k$, $r$ and $l$ be four positive integers with $n = l + r$ such that $l > 2k + r + 4$. Suppose $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $(p, 2)$, where $P(\omega)$ be defined as in (1.1). Then one of the following three cases holds:*

(1) *$f \equiv tg$ for a constant $t$ such that $t^d = 1$,where $d = GCD(n, \ldots, n - i, \ldots, 1)$, $a_{n-i} \neq 0$ for some $i \in \{1, 2, \ldots, n - 1\}$;*

(2) *$f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where*

$$R(\omega_1, \omega_2) = (a_n\omega_1^n + a_{n-1}\omega_1^{n-1} + \ldots + a_1\omega_1) - (a_n\omega_2^n + a_{n-1}\omega_2^{n-1} + \ldots + a_1\omega_2);$$

(3) *if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)} + c_l$, $g = c_2e^{-cQ(z)} + c_l$, where $Q(z) = \int_0^z p(z)dz$, $c_1$, $c_2$ and $c$ are constants such that $b_i^2(c_1c_2)^{l+i}[(l+i)c]^2 = -1$, if $p(z)$ is a nonzero constant $b$, then $f = c_3e^{cz} + c_l$, $g = c_4e^{-cz} + c_l$, where $c_3$, $c_4$ and $c$ are constants such that $(-1)^kb_i^2(c_3c_4)^{l+i}[(l + i)c]^{2k} = b^2$.*

**Corollary 1.2.** *Let $f$ and $g$ be two transcendental entire functions and $p(z)$ be a nonzero polynomial with $deg(p) \leq l - 1$, where $n$, $k$, $r$ and $l$ be four positive integers with $n = l + r$ such that $l > 5k + 4r + 7$. Let $P(\omega)$ be defined as in (1.1). If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $p(z)$ IM then the conclusion of Corollary 1.1 holds.*

**Remark 1.5.** Corollaries 1.1 and 1.2 both hold for two non-constant entire functions $f$ and $g$ when $p(z)$ is a non-zero constant.

**Remark 1.6.** When $l = n$, $c_l = 0$, from Corollary 1.1 and Corollary 1.2 we can easily get the improved version of *Theorems C, G* and *Theorem E* respectively.

**Remark 1.7.** When $l = n_1$, $n = n_1+1$ and $c_l = 0$, from Corollary 1.1 and Corollary 1.2 we can easily obtain the improved version of *Theorem D* and *Theorem F* respectively.

**Remark 1.8.** When $l = n_1$, $n = n_1 + m^*$ and $c_l = 0$, from Corollary 1.1, Lemmas 2.16 and 2.17 we can easily obtained the improvement of *Theorem I* where as from Corollary 1.2 we get the improved version of *Theorem K.*

**Remark 1.9.** When $l = n_1$, $n = n_1 + m^{**}$ and $c_l = 0$, from Corollary 1.1 and Corollary 1.2 we can easily get an improved version of *Theorem J* and *Theorem L* respectively.

We now explain some definitions and notations which are used in the paper.

**Definition 1.2.** [11] Let $p$ be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r,a;f\mid\geq p)$ ($\overline{N}(r,a;f\mid\geq p)$)denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not less than $p$.
(ii) $N(r,a;f\mid\leq p)$ ($\overline{N}(r,a;f\mid\leq p)$)denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not greater than $p$.

**Definition 1.3.** {11, cf.[19]} For $a\in\mathbb{C}\cup\{\infty\}$ and a positive integer $p$ we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+\ldots+\overline{N}(r,a;f\mid\geq p)$. Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

**Definition 1.4.** Let $a,b\in\mathbb{C}\cup\{\infty\}$. Let $p$ be a positive integer. We denote by $\overline{N}(r,a;f\mid\geq p\mid g=b)$ ($\overline{N}(r,a;f\mid\geq p\mid g\neq b)$) the reduced counting function of those $a$-points of $f$ with multiplicities $\geq p$, which are the $b$-points (not the $b$-points) of $g$.

**Definition 1.5.** {cf.[1], 2} Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share the value 1 IM. Let $z_0$ be a 1-point of $f$ with multiplicity $p$, a 1-point of $g$ with multiplicity $q$. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p=q=1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

**Definition 1.6.** {cf.[1], 2} Let $k$ be a positive integer. Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share the value 1 IM. Let $z_0$ be a 1-point of $f$ with multiplicity $p$, a 1-point of $g$ with multiplicity $q$. We denote by $\overline{N}_{f>k}(r,1;g)$ the reduced counting function of those 1-points of $f$ and $g$ such that $p>q=k$. $\overline{N}_{g>k}(r,1;f)$ is defined analogously.

**Definition 1.7.** [8, 9] Let $f$, $g$ share a value $a$ IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$.
Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

## 2. Lemmas

Let $F$ and $G$ be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by $H$ the function as follows:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{2.1}$$

**Lemma 2.1.** [17] *Let $f$ be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z),\ldots, a_0(z)$ be meromorphic functions such that $T(r,a_i(z))=S(r,f)$ for $i=0,1,2,\ldots,n$. Then*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0)=nT(r,f)+S(r,f).$$

**Lemma 2.2.** [23] *Let $f$ be a non-constant meromorphic function, and $p$, $k$ be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right)\leq T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}(r,0;f)+S(r,f),$$

$$N_p\left(r,0;f^{(k)}\right) \le k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

**Lemma 2.3.** [10] *If $N(r,0;f^{(k)}|f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f \neq 0) \le k\overline{N}(r,\infty;f) + N(r,0;f| < k) + k\overline{N}(r,0;f| \ge k) + S(r,f).$$

**Lemma 2.4.** *Let $f$ be a non-constant meromorphic function. Let $n$, $k$ and $l$ be three positive integers such that $l > k+2$ and $P(\omega)$ be defined as in (1.1), $a(z)(\not\equiv 0,\infty)$ be a small function with respect to $f$. Then $[P(f)]^{(k)} - a(z)$ has infinitely many zeros.*

*Proof.* Let us take $F = P(f)$.
In view of *Lemmas* 2.1, 2.2 and by the second theorem for small functions (see [18]) we get

$$\begin{aligned}
& nT(r,f) \\
= \;& T(r,F) + O(1) \\
\le \;& T(r,F^{(k)}) - \overline{N}(r,0;F^{(k)}) + N_{k+1}(r,0;F) + S(r,f) \\
\le \;& \overline{N}(r,0;F^{(k)}) + \overline{N}(r,\infty;F^{(k)}) + \overline{N}(r,a(z);F^{(k)}) - \overline{N}(r,0;F^{(k)}) + N_{k+1}(r,0;F) \\
& +(\varepsilon + o(1))\, T(r,f) \\
\le \;& \overline{N}(r,\infty;f) + (k+1)\,\overline{N}(r,c_l;f) + N(r,0;P(f)|f \neq c_l) + \overline{N}(r,a(z);F^{(k)}) \\
& +(\varepsilon + o(1))\, T(r,f) \\
\le \;& (n-l+k+2)\, T(r,f) + \overline{N}(r,a(z);F^{(k)}) + (\varepsilon + o(1))\, T(r,f),
\end{aligned}$$

for all $\varepsilon > 0$. Take $\varepsilon < 1$. Since $l > k+2$ from above one can easily say that $F^{(k)} - a(z)$ has infinitely many zeros. □

**Lemma 2.5.** ([20], Lemma 6) *If $H \equiv 0$, then $F$, $G$ share 1 CM. If further $F$, $G$ share $\infty$ IM then $F$, $G$ share $\infty$ CM.*

**Lemma 2.6.** [12] *Let $f_1$ and $f_2$ be two non-constant meromorphic functions satisfying $\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$ for $i = 1,2$. If $f_1^s f_2^t - 1$ is not identically zero for arbitrary integers $s$ and $t(|s|+|t| > 0)$, then for any positive $\varepsilon$, we have*

$$N_0(r,1;f_1,f_2) \le \varepsilon T(r) + S(r;f_1,f_2),$$

*where $N_0(r,1;f_1,f_2)$ denotes the reduced counting function related to the common 1-points of $f_1$ and $f_2$ and $T(r) = T(r,f_1) + T(r,f_2)$, $S(r;f_1,f_2) = o(T(r))$ as $r \longrightarrow \infty$ possibly outside a set of finite linear measure.*

**Lemma 2.7.** *Let $f$ and $g$ be two non-constant meromorphic functions. Let $n$, $k$ and $l$ be three positive integers such that $2l > n + 3k$. If $[P(f)]^{(k)} \equiv [P(g)]^{(k)}$, then $P(f) \equiv P(g)$, where $P(\omega)$ be defined as in (1.1).*

*Proof.* We have $[P(f)]^{(k)} \equiv [P(g)]^{(k)}$.
Integrating we get

$$[P(f)]^{(k-1)} \equiv [P(g)]^{(k-1)} + c_{k-1}.$$

If possible suppose $c_{k-1} \neq 0$.
Now in view of *Lemma* 2.2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned}
& n\,T(r,f) \\
&= T(r,P(f)) + O(1) \\
&\leq T(r,[P(f)]^{(k-1)}) - \overline{N}(r,0;[P(f)]^{(k-1)}) + N_k(r,0;P(f)) + S(r,f) \\
&\leq \overline{N}(r,0;[P(f)]^{(k-1)}) + \overline{N}(r,\infty;f) + \overline{N}(r,c_{k-1};[P(f)]^{(k-1)}) - \overline{N}(r,0;[P(f)]^{(k-1)}) \\
&+ N_k(r,0;P(f)) + S(r,f) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;[P(g)]^{(k-1)}) + N_k(r,0;P(f)) + S(r,f) \\
&\leq \overline{N}(r,\infty;f) + (k-1)\overline{N}(r,\infty;g) + N_k(r,0;P(g)) + N_k(r,0;P(f)) + S(r,f) \\
&\leq \overline{N}(r,\infty;f) + (k-1)\overline{N}(r,\infty;g) + k\overline{N}(r,c_l;g) + N(r,0;P(g)|g \neq c_l) + k\overline{N}(r,c_l;f) \\
&+ N(r,0;P(f)|f \neq c_l) + S(r,f) \\
&\leq (n-l+k+1)\,T(r,f) + (n-l+2k-1)\,T(r,g) + S(r,f) + S(r,g) \\
&\leq (2n-2l+3k)\,T(r) + S(r).
\end{aligned}$$

Similarly we get

$$n\,T(r,g) \leq (2n-2l+3k)\,T(r) + S(r).$$

Combining these we get

$$(2l-n-3k)\,T(r) \leq S(r),$$

which is a contradiction since $2l > n + 3k$.
Therefore $c_{k-1} = 0$ and so $[P(f)]^{(k-1)} \equiv [P(g)]^{(k-1)}$.
Proceeding in this way we obtain

$$[P(f)]^{'} \equiv [P(g)]^{'}.$$

Integrating we get

$$P(f) \equiv P(g) + c_0.$$

If possible suppose $c_0 \neq 0$. Now using the second fundamental theorem we get

$$\begin{aligned}
& nT(r,f) \\
=\;& T(r,P(f)) + O(1) \\
\leq\;& \overline{N}(r,0;P(f)) + \overline{N}(r,\infty;P(f)) + \overline{N}(r,c_0;P(f)) \\
\leq\;& \overline{N}(r,0;P(f)) + \overline{N}(r,\infty;f) + \overline{N}(r,0;P(g)) \\
\leq\;& (n-l+2)\,T(r,f) + (n-l+1)\,T(r,g) + S(r,f) \\
\leq\;& (2n-2l+3)\,T(r) + S(r).
\end{aligned}$$

Similarly we get

$$n\,T(r,g) \leq (2n-2l+3)\,T(r) + S(r).$$

Combining these we get

$$(2l - n - 3)\, T(r) \le S(r),$$

which is a contradiction since $2l > n + 3$.
Therefore $c_0 = 0$ and so

$$P(f) \equiv P(g).$$

This proves the Lemma. □

**Lemma 2.8.** *Let $f$, $g$ be two non-constant meromorphic functions. Let $n$, $k$ and $l$ be three positive integers such that $l > k + 2$ and $P(\omega)$ be defined as in (1.1). If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $\alpha$ IM, where $\alpha(\not\equiv 0, \infty)$ is a small function of $f$ and $g$, then $T(r, f) = O(T(r, g))$ and $T(r, g) = O(T(r, f))$.*

*Proof.* Let $F = P(f)$. By the second fundamental theorem for small functions {see [18]}, we have

$$T(r, F^{(k)}) \le \overline{N}(r, \infty; F^{(k)}) + \overline{N}(r, 0; F^{(k)}) + \overline{N}(r, \alpha; F^{(k)}) + (\varepsilon + o(1))T(r, F),$$

for all $\varepsilon > 0$.
Now in the view of *Lemmas* 2.1 and 2.2 for $p = 1$ and using above we get

$$\begin{aligned}
& n\, T(r, f) \\
\le\; & T(r, F^{(k)}) - \overline{N}(r, 0; F^{(k)}) + N_{k+1}(r, 0; P(f)) + (\varepsilon + o(1))T(r, f) \\
\le\; & \overline{N}(r, 0; F^{(k)}) + \overline{N}(r, \infty; f) + \overline{N}(r, \alpha; F^{(k)}) - \overline{N}(r, 0; F^{(k)}) + N_{k+1}(r, 0; P(f)) \\
& + (\varepsilon + o(1))T(r, f) \\
\le\; & \overline{N}(r, \infty; f) + \overline{N}(r, \alpha; [P(f)]^{(k)}) + (k + 1)\overline{N}(r, c_l; f) + N(r, 0; P(f) | f \ne c_l) \\
& + (\varepsilon + o(1))T(r, f) \\
\le\; & (n - l + k + 2)\, T(r, f) + \overline{N}(r, \alpha; [P(g)]^{(k)}) + (\varepsilon + o(1))T(r, f) \\
\le\; & (n - l + k + 2)\, T(r, f) + (k + 1)n\, T(r, g) + (\varepsilon + o(1))T(r, f),
\end{aligned}$$

i.e.,

$$(l - k - 2)\, T(r, f) \le (k + 1)n\, T(r, g) + (\varepsilon + o(1))T(r, f).$$

Since $l > k + 2$, take $\varepsilon < 1$ and we have $T(r, f) = O(T(r, g))$. Similarly we have $T(r, g) = O(T(r, f))$. This completes the proof of Lemma. □

**Lemma 2.9.** *Let $f$, $g$ be two non-constant meromorphic functions and let*

$$F = [P(f)]^{(k)}/\alpha(z),\; G = [P(g)]^{(k)}/\alpha(z),$$

*where $P(\omega)$ be defined as in (1.1), $\alpha(z)$ be a small function with respect to $f$, $g$ and $n$, $k$ and $l$ be positive integers such that $2l > n + 3k + 3$. Suppose $H \equiv 0$. Then one of the following holds:*

(i) $[P(f)]^{(k)}[P(g)]^{(k)} \equiv \alpha^2$;
(ii) *$f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = GCD(n, \dots, n - i, \dots, 1)$, $a_{n-i} \ne 0$ for some $i \in \{1, 2, \dots, n - 1\}$;*

(iii) *$f$ and $g$ satisfy the algebraic equation $R(f,g)\equiv 0$, where*

$$R(\omega_1,\omega_2)=(a_n\omega_1^n+a_{n-1}\omega_1^{n-1}+\ldots+a_1\omega_1)-(a_n\omega_2^n+a_{n-1}\omega_2^{n-1}+\ldots+a_1\omega_2).$$

*Proof.* Since $H\equiv 0$, by *Lemma* 2.5 we get $F$ and $G$ share 1 CM.
On integration we get

$$\frac{1}{F-1}\equiv\frac{bG+a-b}{G-1}, \tag{2.2}$$

where $a,b$ are constants and $a\neq 0$. We now consider the following cases.
**Case 1.** Let $b\neq 0$ and $a\neq b$.
If $b=-1$, then from (2.2) we have

$$F\equiv\frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f).$$

So in view of *Lemma* 2.2 and the second fundamental theorem we get

$$\begin{aligned}
&n\,T(r,g)\\
&=T(r,P(f))+O(1)\\
&\leq T(r,G)+N_{k+1}(r,0;P(g))-\overline{N}(r,0;G)\\
&\leq\overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,a+1;G)+N_{k+1}(r,0;P(g))-\overline{N}(r,0;G)+S(r,g)\\
&\leq\overline{N}(r,\infty;g)+(k+1)\overline{N}(r,c_l;g)+N(r,0;P(g)|g\neq c_l)+\overline{N}(r,\infty;f)+S(r,g)\\
&\leq\ T(r,f)+(n-l+k+2)\,T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set $I$ with infinite measure such that $T(r,f)\leq T(r,g)$ for $r\in I$.
So for $r\in I$ we have

$$(l-k-3)\,T(r,g)\leq S(r,g),$$

which is a contradiction since $l>k+3$.
If $b\neq -1$, from (2.2) we obtain that

$$F-(1+\frac{1}{b})\equiv\frac{-a}{b^2[G+\frac{a-b}{b}]}.$$

So

$$\overline{N}(r,\frac{(b-a)}{b};G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f).$$

Using *Lemma* 2.2 and the same argument as used in the case when $b=-1$ we can get a contradiction.
**Case 2.** Let $b\neq 0$ and $a=b$.
If $b=-1$, then from (2.2) we have

$$FG\equiv\alpha^2,$$

i.e.,

$$[P(f)]^{(k)}[P(g)]^{(k)}\equiv\alpha^2,$$

where $[P(f)]^k$ and $[P(g)]^k$ share $\alpha$ CM.
If $b \neq -1$, from (2.2) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F).$$

So in view of *Lemma* 2.2 and the second fundamental theorem we get

$$\begin{aligned}
& n\,T(r,g) \\
& \leq T(r,G) + N_{k+1}(r,0;P(g)) - \overline{N}(r,0;G) + S(r,g) \\
& \leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\frac{1}{1+b};G) + N_{k+1}(r,0;P(g)) - \overline{N}(r,0;G) + S(r,g) \\
& \leq \overline{N}(r,\infty;g) + N_{k+1}(r,0;P(g)) + \overline{N}(r,0;F) + S(r,g) \\
& \leq \overline{N}(r,\infty;g) + N_{k+1}(r,0;P(g)) + N_{k+1}(r,0;P(f)) + k\overline{N}(r,\infty;f) + S(r,f) + S(r,g) \\
& \leq (n-l+k+2)\,T(r,g) + (n-l+2k+1)\,T(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

So for $r \in I$ we have

$$(2l-n-3k-3)\,T(r,g) \leq S(r,g),$$

which is a contradiction since $2l > n+3k+3$.
**Case 3.** Let $b = 0$. From (2.2) we obtain

$$F \equiv \frac{G+a-1}{a}. \tag{2.3}$$

If $a \neq 1$ then from (2.3) we obtain

$$\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in *Case 2*. Therefore $a = 1$ and from (2.3) we obtain

$$F \equiv G,$$

i.e.,

$$[P(f)]^{(k)} \equiv [P(g)]^{(k)}.$$

Then by Lemma 2.7 we have

$$P(f) \equiv P(g). \tag{2.4}$$

Let $h = \frac{f}{g}$. If $h$ is a constant, by putting $f = hg$ in (2.4) we get

$$a_n g^{n-1}(h^n - 1) + a_{n-1} g^{n-2}(h^{n-1} - 1) + \ldots + a_1(h-1) \equiv 0,$$

which implies that $h^d = 1$, where $d = GCD(n, \ldots, n-i, \ldots, 1)$, $a_{n-i} \neq 0$ for some $i \in \{1, 2, \ldots, n-1\}$. Thus $f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = GCD(n, \ldots, n-i, \ldots, 1)$, $a_{n-i} \neq 0$ for some $i \in \{1, 2, \ldots, n-1\}$.
If $h$ is not constant then $f$ and $g$ satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = (a_n\omega_1^n + a_{n-1}\omega_1^{n-1} + \ldots + a_1\omega_1) - (a_n\omega_2^n + a_{n-1}\omega_2^{n-1} + \ldots + a_1\omega_2)$. □

**Lemma 2.10.** [6] *Let $f(z)$ be a non-constant entire function and let $k \geq 2$ be a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a \neq 0, b$ are constant.*

**Lemma 2.11.** [[7], Theorem 3.10] *Suppose that $f$ is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,\infty,f) + N(r,0;f) + N(r,0;f^{(k)}) = S(r,\frac{f^{'}}{f}),$$

*then $f = e^{az+b}$, where $a \neq 0$, $b$ are constants.*

**Lemma 2.12.** [[21], Theorem 1.24] *Let $f$ be a non-constant meromorphic function and let $k$ be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r,0;f^{(k)}) \leq N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

**Lemma 2.13.** *Let $f$, $g$ be two transcendental meromorphic functions and $p(z)$ be a non-zero polynomial with $deg(p) \leq n-1$, where $n$ and $k$ be two positive integers such that $n > \max\{2k, k+2\}$. Suppose $[f^n]^{(k)}[g^n]^{(k)} \equiv p^2$, where $[f^n]^{(k)} - p(z)$ and $[g^n]^{(k)} - p(z)$ share 0 CM. Now*

*(i) if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)}$, $g = c_2e^{-cQ(z)}$, where*

$$Q(z) = \int_0^z p(z)dz,$$

*$c_1$, $c_2$ and $c$ are constants such that $(nc)^2(c_1c_2)^n = -1$,*

*(ii) if $p(z)$ is a nonzero constant $b$, then $f = c_3e^{dz}$, $g = c_4e^{-dz}$, where $c_3$, $c_4$ and $d$ are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.*

*Proof.* Suppose

$$[f^n]^{(k)}[g^n]^{(k)} \equiv p^2. \tag{2.5}$$

We consider the following cases:

**Case 1.** Let $deg(p(z)) = l(\geq 1)$.

Let $z_0(p(z_0) \neq 0)$ be a zero of $f$ with multiplicity $q$. Note that $z_0$ is a zero of $[f^n]^{(k)}$ with multiplicity $nq-k$. Obviously $z_0$ will be a pole of $g$ with multiplicity $q_1$, say. Note that $z_0$ is a pole of $[g^n]^{(k)}$ with multiplicity $nq_1+k$ and so $nq-k = nq_1+k$. Now

$$nq - k = nq_1 + k$$

implies that

$$n(q-q_1) = 2k. \tag{2.6}$$

Since $n > 2k$, we get a contradiction from (2.6).

This shows that $z_0$ is a zero of $p(z)$ and so we have $N(r,0;f) = O(logr)$. Similarly we can prove that $N(r,0;g) = O(logr)$.

Thus in general we can take $N(r,0;f) + N(r,0;g) = O(logr)$.

We know that

$$N(r,\infty;[f^n]^{(k)}) = nN(r,\infty;f) + k\overline{N}(r,\infty;f).$$

Also by *Lemma* 2.12 we have

$$\begin{aligned} N(r,0;[g^n]^{(k)}) &\leq nN(r,0;g) + k\overline{N}(r,\infty;g) + S(r,g) \\ &\leq k\overline{N}(r,\infty;g) + O(logr) + S(r,g). \end{aligned}$$

From (2.5) we get

$$N(r,\infty;[f^n]^{(k)}) = N(r,0;[g^n]^{(k)}),$$

i.e.,

$$nN(r,\infty;f) + k\overline{N}(r,\infty;f) \leq k\overline{N}(r,\infty;g) + O(logr) + S(r,g). \tag{2.7}$$

Similarly we get

$$nN(r,\infty;g) + k\overline{N}(r,\infty;g) \leq k\overline{N}(r,\infty;f) + O(logr) + S(r,f). \tag{2.8}$$

Since $f$ and $g$ are transcendental, it follows that

$$S(r,f) + O(logr) = S(r,f), \quad S(r,g) + O(logr) = S(r,g).$$

Now combining (2.7) and (2.8) we get

$$N(r,\infty;f) + N(r,\infty;g) = S(r,f) + S(r,g).$$

By *Lemma* 2.8 we have $S(r,f) = S(r,g)$ and so we obtain

$$N(r,\infty;f) = S(r,f), \quad N(r,\infty;g) = S(r,g). \tag{2.9}$$

Let

$$F_1 = \frac{[f^n]^{(k)}}{p} \quad and \quad G_1 = \frac{[g^n]^{(k)}}{p}. \tag{2.10}$$

Note that $T(r,F_1) \leq n(k+1)T(r,f) + S(r,f)$ and so $T(r,F_1) = O(T(r,f))$. Also by *Lemma* 2.2, one can obtain $T(r,f) = O(T(r,F_1))$. Hence $S(r,F_1) = S(r,f)$. Similarly we get $S(r,G_1) = S(r,g)$. Hence we get $S(r,F_1) = S(r,G_1)$. From (2.5) we get

$$F_1G_1 \equiv 1. \tag{2.11}$$

If $F_1 \equiv cG_1$, where $c$ is a nonzero constant, then $F_1$ is a constant and so $f$ is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv cG_1$ and so in the view of (2.11) we see that $F_1$ and $G_1$ share $-1$ IM.

Now by *Lemma* 2.12 we have

$$N(r,0;F_1) \leq nN(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f) \leq S(r,F_1).$$

Similarly we have

$$N(r,0;G_1) \leq nN(r,0;g) + k\overline{N}(r,\infty;g) + S(r,g) \leq S(r,G_1).$$

Also we see that

$$N(r,\infty;F_1) = S(r,F_1), \quad N(r,\infty;G_1) = S(r,G_1).$$

It is clearly that $T(r,F_1) = T(r,G_1) + O(1)$. Let

$$f_1 = \frac{F_1}{G_1}.$$

and

$$f_2 = \frac{F_1 - 1}{G_1 - 1}.$$

Clearly $f_1$ is non-constant. If $f_2$ is a nonzero constant then $F_1$ and $G_1$ share $\infty$ CM and so from (2.11) we conclude that $F_1$ and $G_1$ have no poles.

Next we suppose that $f_2$ is non-constant. We see that

$$F_1 = \frac{f_1(1-f_2)}{f_1-f_2}, \quad G_1 = \frac{1-f_2}{f_1-f_2}.$$

Clearly

$$T(r,F_1) \leq 2[T(r,f_1)+T(r,f_2)] + O(1)$$

and

$$T(r,f_1)+T(r,f_2) \leq 4T(r,F_1) + O(1).$$

These give $S(r,F_1) = S(r;f_1,f_2)$. Also we note that

$$\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$$

for $i = 1,2$.

Next we suppose $\overline{N}(r,-1;F_1) \neq S(r,F_1)$, otherwise by the second fundamental theorem $F_1$ will be a constant.

Also we see that

$$\overline{N}(r,-1;F_1) \leq N_0(r,1;f_1,f_2).$$

Thus we have

$$T(r,f_1)+T(r,f_2) \leq 4\; N_0(r,1;f_1,f_2) + S(r,F_1).$$

Then by *Lemma* 2.6 there exist two mutually prime integers $s$ and $t(|s|+|t|>0)$ such that

$$f_1^s f_2^t \equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s \left[\frac{F_1-1}{G_1-1}\right]^t \equiv 1. \tag{2.12}$$

If either $s$ or $t$ is zero then we arrive at a contradiction and so $st \neq 0$.
We now consider following cases:

**Case (i).** Suppose $s>0$ and $t=-t_1$, where $t_1>0$. Then we have

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1-1}{G_1-1}\right]^{t_1}. \tag{2.13}$$

Let $z_1$ be a pole of $F_1$ of multiplicity $p$. Then from (2.11) we see that $z_1$ must be a zero of $G_1$ of multiplicity $p$. Now from (2.13) we get $2s = t_1$, which is impossible. Hence $F_1$ has no pole. Similarly we can prove that $G_1$ also has no poles.

**Case (ii).** Suppose either $s>0$ and $t>0$ or $s<0$ and $t<0$. Then from (2.13) one can easily prove that $F_1$ and $G_1$ have no poles.

Consequently from (2.11) we see that $F_1$ and $G_1$ have no zeros. So we deduce from (2.10) that both $f$ and $g$ have no pole.

Since $F_1$ and $G_1$ have no zeros and poles, we have

$$F_1 \equiv e^{\gamma_1} G_1,$$

i.e.,

$$[f^n]^{(k)} \equiv e^{\gamma_1}[g^n]^{(k)},$$

where $\gamma_1$ is a non-constant entire function. Then from (2.5) we get

$$[f^n]^{(k)} \equiv ce^{\frac{1}{2}\gamma_1}p(z), \quad [g^n]^{(k)} \equiv ce^{-\frac{1}{2}\gamma_1}p(z),$$

where $c$ is a nonzero constant. This shows that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 0 CM.
Also we deduce from (2.10) that both $f$ and $g$ are transcendental entire functions.
Since $N(r,0;f) = O(logr)$ and $N(r,0;g) = O(logr)$, so we can take

$$f(z) = h_1(z)e^{\alpha(z)}, \quad g(z) = h_2(z)e^{\beta(z)}, \tag{2.14}$$

where $h_1$ and $h_2$ are nonzero polynomials and $\alpha$, $\beta$ are two non-constant entire functions.
We deduce from (2.5) and (2.14) that either both $\alpha$ and $\beta$ are transcendental entire functions or both $\alpha$ and $\beta$ are polynomials.
We consider the following cases:
**Subcase 1.1:** Let $k \geq 2$.
First we suppose both $\alpha$ and $\beta$ are transcendental entire functions.
Let $\alpha_1 = \alpha' + \frac{h_1'}{h_1}$ and $\beta_1 = \beta' + \frac{h_2'}{h_2}$. Clearly both $\alpha_1$ and $\beta_1$ are transcendental entire functions.
Note that

$$S(r, n\alpha_1) = S(r, \frac{[f^n]'}{f^n}), \quad S(r, n\beta_1) = S(r, \frac{[g^n]'}{g^n}).$$

Moreover we see that

$$N(r,0;[f^n]^{(k)}) \leq N(r,0;p^2) = O(logr).$$

$$N(r,0;[g^n]^{(k)}) \leq N(r,0;p^2) = O(logr).$$

From these and using (2.14) we have

$$N(r,\infty;f^n) + N(r,0;f^n) + N(r,0;[f^n]^{(k)}) = S(r,n\alpha_1) = S(r, \frac{[f^n]'}{f^n}) \tag{2.15}$$

and

$$N(r,\infty;g^n) + N(r,0;g^n) + N(r,0;[g^n]^{(k)}) = S(r,n\beta_1) = S(r, \frac{[g^n]'}{g^n}). \tag{2.16}$$

Then from (2.15), (2.16) and Lemma 2.9 we must have

$$f = e^{az+b}, \quad g = e^{cz+d}, \tag{2.17}$$

where $a \neq 0$, $b$, $c \neq 0$ and $d$ are constants. But these types of $f$ and $g$ do not agree with the relation (2.5).
Next we suppose $\alpha$ and $\beta$ are both polynomials.
From (2.5) we get $\alpha + \beta \equiv C$ i.e., $\alpha' \equiv -\beta'$. Therefore $deg(\alpha) = deg(\beta)$.
We deduce from (2.14) that

$$[f^n]^{(k)} \equiv Ah_1^{n-k}[h_1^k(\alpha')^k + P_{k-1}(\alpha', h_1')]e^{n\alpha} \equiv p(z)e^{n\alpha}, \tag{2.18}$$

and

$$[g^n]^{(k)} = Bh_2^{n-k}[h_2^k(\beta^{'})^k + Q_{k-1}(\beta^{'}, h_2^{'})]e^{n\beta} \equiv p(z)e^{n\beta}, \tag{2.19}$$

where $A$, $B$ are nonzero constants, $P_{k-1}(\alpha^{'}, h_1^{'})$ and $Q_{k-1}(\beta^{'}, h_2^{'})$ are differential polynomials in $\alpha^{'}, h_1^{'}$ and $\beta^{'}, h_2^{'}$ respectively.
Since $deg(p) \leq n-1$, from (2.17) and (2.19) we conclude that both $h_1$ and $h_2$ are nonzero constant.
So we can rewrite $f$ and $g$ as follows:

$$f = e^{\gamma_2}, \quad g = e^{\delta_2}, \tag{2.20}$$

where $\gamma_2 + \delta_2 \equiv C$ and $deg(\gamma_2) = deg(\delta_2)$.
If $deg(\gamma_2) = deg(\delta_2) = 1$, then we again get a contradiction from (2.5).
Next we suppose $deg(\gamma_2) = deg(\delta_2) \geq 2$.
We deduce from (2.20) that

$$[f^n]^{(k)} = A_1[(\gamma_2^{'})^k + P_{k-1}(\gamma_2^{'})]e^{n\gamma_2}, \quad [g^n]^{(k)} = B_1[(\delta_2^{'})^k + Q_{k-1}(\delta_2^{'})]e^{n\delta_2},$$

where $A_1$, $B_1$ are nonzero constants, $P_{k-1}(\gamma_2^{'})$ and $Q_{k-1}(\delta_2^{'})$ are differential polynomials in $\gamma_2^{'}$ and $\delta_2^{'}$ of degree atmost $k-1$ respectively.
Since $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 0 CM, it follows that

$$[(\gamma_2^{'})^k + P_{k-1}(\gamma_2^{'})] \equiv D[(\delta_2^{'})^k + Q_{k-1}(\delta_2^{'})],$$

where $D$ is a nonzero constant, which is impossible as $k \geq 2$.

Actually $[(\gamma_2^{'})^k + P_{k-1}(\gamma_2^{'})]$ and $[(\delta_2^{'})^k + Q_{k-1}(\delta_2^{'})]$ contain the terms $(\gamma_2^{'})^k + K(\gamma_2^{'})^{k-2}\gamma_2^{''}$ and $(\delta_2^{'})^k + K(\delta_2^{'})^{k-2}\delta_2^{''}$ respectively, where $K$ is a suitably positive integer. But these two terms are not identical.
**Subcase 1.2:** Let $k = 1$.
Now from (2.5) we get

$$f^{n-1}f^{'}g^{n-1}g^{'} \equiv p_1^2, \tag{2.21}$$

where $p_1^2 = \frac{1}{n^2}p^2$.
First we suppose both $\alpha$ and $\beta$ are transcendental entire functions.
Let $h = fg$. Clearly $h$ is a transcendental entire function. Then from (2.21) we get

$$\left(\frac{g^{'}}{g} - \frac{1}{2}\frac{h^{'}}{h}\right)^2 \equiv \frac{1}{4}\left(\frac{h^{'}}{h}\right)^2 - h^{-n}p_1^2. \tag{2.22}$$

Let

$$\alpha_2 = \frac{g^{'}}{g} - \frac{1}{2}\frac{h^{'}}{h}.$$

From (2.22) we get

$$\alpha_2^2 \equiv \frac{1}{4}\left(\frac{h^{'}}{h}\right)^2 - h^{-n}p_1^2. \tag{2.23}$$

First we suppose $\alpha_2 \equiv 0$. Then we get $h^{-n}p_1^2 \equiv \frac{1}{4}\left(\frac{h'}{h}\right)^2$ and so $T(r,h) = S(r,h)$, which is impossible. Next we suppose that $\alpha_2 \not\equiv 0$. Differentiating (2.23) we get

$$2\alpha_2\alpha_2' \equiv \frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)' + n\,h'h^{-n-1}p_1^2 - 2h^{-n}p_1p_1'.$$

Applying (2.23) we obtain

$$h^{-n}\left(-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2\right) \equiv \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right). \tag{2.24}$$

First we suppose

$$-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2 \equiv 0.$$

Then there exist a non-zero constant $c$ such that $\alpha_2^2 \equiv ch^{-n}p_1^2$ and so from (2.23) we get

$$(c+1)h^{-n}p_1^2 \equiv \frac{1}{4}\left(\frac{h'}{h}\right)^2.$$

If $c = -1$, then $h$ will be a constant. If $c \neq -1$, then we have $T(r,h) = S(r,h)$, which is impossible. Next we suppose that

$$-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2 \not\equiv 0.$$

Then by (2.24) we have

$$\begin{aligned}
& n\,T(r,h) && (2.25)\\
= \; & n\,m(r,h) &&\\
\leq \; & m\left(r, h^n\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right)\right) + m\left(r, \frac{1}{\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right)}\right) + O(1) &&\\
\leq \; & T\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right)\right) + m\left(r, n\frac{h'}{h}p_1^2 - 2p_1p_1' + 2\frac{\alpha_2'}{\alpha_2}p_1^2\right) &&\\
\leq \; & \overline{N}(r,0;\alpha_2) + \overline{N}(r,\infty;\alpha_2) + S(r,h) + S(r,\alpha_2) &&\\
\leq \; & T(r,\alpha_2) + S(r,h). && (2.26)
\end{aligned}$$

From (2.23) we get

$$T(r,\alpha_2) \leq \frac{1}{2}n\,T(r,h) + S(r,h).$$

Now from (2.25) we get

$$\frac{1}{2}n\,T(r,h) \leq S(r,h),$$

which is impossible .
Thus $\alpha$ and $\beta$ are both polynomials. Also from (2.5) we can conclude that $\alpha(z)+\beta(z) \equiv C$ for a constant $C$ and so $\alpha^{'}(z) + \beta^{'}(z) \equiv 0$. We deduce from (2.5) that

$$[f^n]^{'} \equiv n[h_1^n\alpha^{'} + h_1^{n-1}h_1^{'}]e^{n\alpha} \equiv p(z)e^{n\alpha}, \tag{2.27}$$

and

$$[g^n]^{'} = n[h_2^n\beta^{'} + h_2^{n-1}h_2^{'}]e^{n\beta} \equiv p(z)e^{n\beta}. \tag{2.28}$$

Since $deg(p) \leq n-1$, from (2.27) and (2.28) we conclude that both $h_1$ and $h_2$ are nonzero constant.
So we can rewrite $f$ and $g$ as follows:

$$f = e^{\gamma_3}, \quad g = e^{\delta_3}. \tag{2.29}$$

Now from (2.5) we get

$$n^2\gamma_3^{'}\delta_3^{'}e^{n(\gamma_3+\delta_3)} \equiv p^2. \tag{2.30}$$

Also from (2.30) we can conclude that $\gamma_3(z) + \delta_3(z) \equiv C$ for a constant $C$ and so $\gamma_3^{'}(z) + \delta_3^{'}(z) \equiv 0$. Thus from (2.30) we get $n^2e^{nC}\gamma_3^{'}\delta_3^{'} \equiv p^2(z)$. By computation we get

$$\gamma_3^{'} = cp(z), \quad \delta_3^{'} = -cp(z). \tag{2.31}$$

Hence

$$\gamma_3 = cQ(z) + b_1, \quad \delta_3 = -cQ(z) + b_2, \tag{2.32}$$

where $Q(z) = \int_0^z p(z)dz$ and $b_1$, $b_2$ are constants. Finally we take $f$ and $g$ as

$$f(z) = c_1e^{cQ(z)}, \quad g(z) = c_2e^{-cQ(z)},$$

where $c_1$, $c_2$ and $c$ are constants such that $(nc)^2(c_1c_2)^n = -1$.
**Case 2.** Let $p(z)$ be a nonzero constant $b$. Since $n > 2k$, one can easily prove that $f$ and $g$ have no zeros. Now proceeding in the same way as done in the proof of the **Case 1** we get $f = e^{\alpha}$ and $g = e^{\beta}$, where $\alpha$ and $\beta$ are two non-constant entire functions.
We now consider the following two subcases:
**Subcase 2.1:** Let $k \geq 2$.
We see that

$$N(r, 0; [f^n]^{(k)}) = 0$$

and

$$f^n(z)[f^n(z)]^{(k)} \neq 0. \tag{2.33}$$

Similarly we have

$$g^n(z)[g^n(z)]^{(k)} \neq 0. \tag{2.34}$$

Then from (2.33), (2.34) and Lemma 2.10 we must have

$$f = e^{az+b}, \quad g = e^{cz+d}, \tag{2.35}$$

where $a \neq 0$, $b$, $c \neq 0$ and $d$ are constants.
**Subcase 2.1:** Let $k = 1$.
Considering **Subcase 1.2** one can easily get

$$f = e^{az+b}, \quad g = e^{cz+d}, \tag{2.36}$$

where $a \neq 0$, $b$, $c \neq 0$ and $d$ are constants.
Finally we can take $f$ and $g$ as

$$f = c_3 e^{dz}, \quad g = c_4 e^{-dz},$$

where $c_3$, $c_4$ and $d$ are nonzero constants such that $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.
This completes the proof of Lemma. □

**Lemma 2.14.** *Let $f$, $g$ be two transcendental meromorphic functions, $p(z)$ be a nonzero polynomial with $deg(p) \leq n-1$, where $n$ and $k$ be two positive integers such that $n > \max\{2k, k+2\}$.*
*Let $[(f-a)^n]^{(k)}$, $[(g-a)^n]^{(k)}$ share $p$ CM and $[(f-a)^n]^{(k)}[(g-a)^n]^{(k)} \equiv p^2$. Now*
*(i) if $p(z)$ is not a constant, then $f = c_1 e^{cQ(z)} + a$, $g = c_2 e^{-cQ(z)} + a$, where*

$$Q(z) = \int_0^z p(z)dz,$$

*$c_1$, $c_2$ and $c$ are constants such that $(nc)^2(c_1c_2)^n = -1$,*
*(ii) if $p(z)$ is a nonzero constant $b$, then $f = c_3 e^{dz} + a$, $g = c_4 e^{-dz} + a$, where $c_3$, $c_4$ and $d$ are constants such that $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.*

*Proof.* The Lemma follows from *Lemma* 2.13. □

**Lemma 2.15.** *Let $f$, $g$ be two transcendental entire functions and $P(\omega)$ be defined as in (1.1), $p(z)$ be a nonzero polynomial such that $deg(p) \leq l-1$, where $n$, $k$ and $l$ be three positive integers such that $2l > n+3k+3$. Suppose $[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2$. Then*
*(i) if $p(z)$ is not a constant, then $f = c_1 e^{cQ(z)} + c_l$, $g = c_2 e^{-cQ(z)} + c_l$, where*

$$Q(z) = \int_0^z p(z)dz,$$

*$c_1$, $c_2$ and $c$ are constants such that $(nc)^2(c_1c_2)^n = -1$,*
*(ii) if $p(z)$ is a nonzero constant $b$, then $f = c_3 e^{dz} + c_l$, $g = c_4 e^{-dz} + c_l$, where $c_3$, $c_4$ and $d$ are constants such that $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.*

*Proof.* Suppose

$$[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2. \tag{2.37}$$

Since $l > k$, we can take

$$f(z) - c_l = h(z)e^{\alpha(z)}, \tag{2.38}$$

where $h$ is a nonzero polynomial and $\alpha$ is a non-constant entire function.
Let $f_1 = f - c_l$, $g_1 = g - c_l$.
Clearly $P(f) = f_1^l P_1(f_1)$ and $P(g) = g_1^l P_1(g_1)$,
i.e.,

$$P(f) = f_1^l [b_m f_1^m + b_{m-1} f_1^{m-1} + \ldots + b_0]$$

and

$$P(g) = g_1^l [b_m g_1^m + b_{m-1} g_1^{m-1} + \ldots + b_0].$$

We now consider the following two cases:

**Case 1.** Let $s \geq 2$, where $s$ denotes the number of distinct zeros of $P(\omega) = 0$.
In this case $m \geq 1$ and so atleast two of $b_i$, where $i \in \{0, 1, \ldots, m\}$ are nonzero.
Since $f_1 = he^{\alpha}$, then by induction we get

$$(b_i f_1^{l+i})^{(k)} = t_i(\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)}, h, h^{'}, \ldots, h^{(k)}) e^{(l+i)\alpha}, \tag{2.39}$$

where $t_i(\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)}, h, h^{'}, \ldots, h^{(k)})$ $(i = 0, 1, 2, \ldots, m)$ are differential polynomials in
$\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)}, h, h^{'}, \ldots, h^{(k)}$.
Obviously

$$t_i(\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)}, h, h^{'}, \ldots, h^{(k)}) \not\equiv 0$$

and $[P(f)]^{(k)} \not\equiv 0$.
From (2.37) and (2.39) we obtain

$$\overline{N}(r, 0; t_m e^{m\alpha(z)} + t_{m-1} e^{(m-1)\alpha(z)} + \ldots + t_0) \leq N(r, 0; p^2) = S(r, f). \tag{2.40}$$

Since $\alpha$ is an entire function, we obtain $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2, \ldots, k$. Hence $T(r, t_i) = S(r, f)$ for $i = 0, 1, 2, \ldots, m$. So from (2.40) and using second fundamental theorem for small functions{see [18]}, we obtain

$$\begin{aligned}
& m\,T(r, f) \\
= \;& T(r, t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + S(r, f) \\
\leq \;& \overline{N}(r, 0; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + \overline{N}(r, 0; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha} + t_0) \\
& + \overline{N}(r, \infty; t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + (\varepsilon + o(1))\, T(r, f) \\
\leq \;& \overline{N}(r, 0; t_m e^{(m-1)\alpha} + \ldots + t_1) + (\varepsilon + o(1))\, T(, f) \\
\leq \;& (m-1) T(r, f) + (\varepsilon + o(1))\, T(r, f),
\end{aligned}$$

for all $\varepsilon > 0$. Take $\varepsilon < 1$ and we obtain a contradiction.
**Subcase 2.2:** Let $s = 1$.
In this case $l = n$. From (2.37) we get

$$[(f_1)^n]^{(k)} [(g_1)^n]^{(k)} \equiv p^2. \tag{2.41}$$

Finally Lemma follows from *Lemma* 2.14.
This completes the proof of the Lemma. □

**Lemma 2.16.** [14] *Let $f$ and $g$ be two non-constant entire functions and $\lambda$, $\mu$ be two constants such that $\lambda\mu \neq 0$. Let $n$, $m$ and $k$ be three positive integers such that $n > 2k+m$. If $[f^n(\lambda f^m + \mu)]^{(k)} \equiv [g^n(\lambda g^m + \mu)]^{(k)}$, then $f^d(z) \equiv g^d(z)$, $d = GCD(n, m)$.*

**Lemma 2.17.** [16] *Let $f$ and $g$ be two non-constant meromorphic functions, $k$, $n > 2k+1$ be two positive integers. If $[f^n]^{(k)} \equiv [g^n]^{(k)}$, then $f \equiv tg$ for a constant $t$ such that $t^n = 1$.*

**Lemma 2.18.** *Let $f$ and $g$ be two non-constant meromorphic functions and $a(z)(\not\equiv 0, \infty)$ be a small functions of $f$ and $g$. Let $n$, $k$ and $s \geq 2$ be three positive integers such that $n > 2ks + k$ and $P(\omega)$ be defined as in (1.1). If $l_i > k (i = 1, 2, \ldots, s)$ and $\Theta(0; f) + \Theta(\infty; f) > \frac{n(3-s) - 2ks + 4k}{n + 2k}$ then*

$$[P(f)]^{(k)} [P(g)]^{(k)} \not\equiv a^2,$$

*Proof.* First suppose that

$$[P(f)]^{(k)}[P(g)]^{(k)} \equiv a^2,$$

i.e.,

$$[(f-c_1)^{l_1}(f-c_2)^{l_2}\dots(f-c_s)^{l_s}]^{(k)}[(g-c_1)^{l_1}(g-c_2)^{l_2}\dots(g-c_s)^{l_s}]^{(k)} \equiv a^2. \quad (2.42)$$

Now by *Lemma* 2.8, we have

$$S(r,f) = S(r,g).$$

Now by the second fundamental theorem for $f$ and $g$ we get respectively

$$s\,T(r,f) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \sum_{i=1}^{s}\overline{N}(r,c_i;f) - \overline{N}_0(r,0;f') + S(r,f) \quad (2.43)$$

and

$$s\,T(r,g) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \sum_{i=1}^{s}\overline{N}(r,c_i;g) - \overline{N}_0(r,0;g') + S(r,g), \quad (2.44)$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function of those zeros of $f'$ which are not the zeros of $f$ and $f-c_i$, $i=1,2,\dots,s$ and $\overline{N}_0(r,0;g')$ can be similarly defined.

Let $z_1(a(z_1)\ne 0,\infty)$ be a zero of $f-c_i$ with multiplicity $q_i$, $i=1,2,\dots,s$. Obviously $z_1$ must be a pole of $g$ with multiplicity $r$. Then from (2.42) we get $l_iq_i-k = nr+k$. This gives $q_i \ge \frac{n+2k}{l_i}$ for $i=1,2,\dots,s$ and so we get

$$\overline{N}(r,c_i;f) \le \frac{l_i}{n+2k}\,N(r,c_i;f) \le \frac{l_i}{n+2k}\,T(r,f).$$

Clearly

$$\sum_{i=1}^{s}\overline{N}(r,c_i;f) \le \frac{n}{n+2k}\,T(r,f). \quad (2.45)$$

Similarly we have

$$\sum_{i=1}^{s}\overline{N}(r,c_i;g) \le \frac{n}{n+2k}\,T(r,g). \quad (2.46)$$

Then by (2.43) and (2.45) we get

$$\begin{aligned} &s\,T(r,f) \quad (2.47)\\ \le\ &\left(2+\frac{n}{n+2k}-\Theta(0;f)-\Theta(\infty;f)+\varepsilon\right)T(r,f)+S(r,f).\end{aligned}$$

Then from (2.47) we get

$$\left(s-2-\frac{n}{n+2k}+\Theta(0;f)+\Theta(\infty;f)-\varepsilon\right)T(r,f) \le S(r,f).$$

Since $\Theta(0;f)+\Theta(\infty;f) > \frac{n(3-s)-2ks+4k}{n+2k}$, we arrive at a contradiction.
This completes the proof. □

**Lemma 2.19.** [2] *Let $f$ and $g$ be two non-constant meromorphic functions sharing 1 IM. Then*

$$\begin{aligned}&\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}^{(2}_E(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)\\ \leq\ & N(r,1;g)-\overline{N}(r,1;g).\end{aligned}$$

**Lemma 2.20.** [2] *Let $f$, $g$ share 1 IM. Then*

$$\overline{N}_L(r,1;f)\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)$$

**Lemma 2.21.** [2] *Let $f$, $g$ share 1 IM. Then*

$$\begin{aligned}&(i)\quad \overline{N}_{f>1}(r,1;g)\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_0(r,0;f^{'})+S(r,f)\\ &(ii)\quad \overline{N}_{g>1}(r,1;f)\leq\overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_0(r,0;g^{'})+S(r,g).\end{aligned}$$

## 3. Proofs of the Theorems

*Proof of Theorem* 1.1. Let $F=\frac{[P(f)]^{(k)}}{p(z)}$ and $G=\frac{[P(g)]^{(k)}}{p(z)}$. Note that since $f$ and $g$ are transcendental meromorphic functions, $p(z)$ is a small function with respect to both $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$. Also $F$ and $G$ share $(1,2)$ except the zeros of $p(z)$.

**Case 1.** Let $H\not\equiv 0$.

From (2.1) it can be easily calculated that the possible poles of $H$ occur at (i) multiple zeros of $F$ and $G$, (ii) those 1 points of $F$ and $G$ whose multiplicities are different,(iii) those poles of $F$ and $G$, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.

Since $H$ has only simple poles we get

$$\begin{aligned}&N(r,\infty;H) \qquad (3.1)\\ \leq\ &\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)\\ &+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Let $z_0$ be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then $z_0$ is a simple zero of $G-1$ and a zero of $H$. So

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \qquad (3.2)$$

Now using (3.1) and (3.2) we get

$$\begin{aligned}&\overline{N}(r,1;F) \qquad (3.3)\\ \leq\ &N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2)\\ \leq\ &\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\ &+\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g).\end{aligned}$$

Now in view of *Lemma* 2.3 we get

$$\begin{aligned}&\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\,|\geq 2)+\overline{N}_*(r,1;F,G) \qquad (3.4)\\ \leq\ &N(r,0;G^{'}\mid G\neq 0)\leq\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,g),\end{aligned}$$

Hence using (3.3), (3.4), *Lemmas* 2.1 and 2.2 we get from the second fundamental theorem that

$$\begin{aligned}
& nT(r,f) \\
& \leq T(r,F) + N_{k+2}(r,0;P(f)) - N_2(r,0;F) + S(r,f) \\
& \leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;P(f)) - N_2(r,0;F) - N_0(r,0;F^{'}) \\
& \leq 2\overline{N}(r,\infty,F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;F) + N_{k+2}(r,0;P(f)) + \overline{N}(r,0;F| \geq 2) \\
& + \overline{N}(r,0;G| \geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F| \geq 2) + \overline{N}_0(r,0;G^{'}) - N_2(r,0;F) \\
& + S(r,f) + S(r,g) \\
& \leq 2\,\overline{N}(r,\infty;P(f)) + 2\,\overline{N}(r,\infty;P(g)) + N_{k+2}(r,0;P(f)) + N_2(r,0;G) \\
& + S(r,f) + S(r,g) \\
& \leq 2\,\overline{N}(r,\infty;P(f)) + (2+k)\,\overline{N}(r,\infty;P(g)) + N_{k+2}(r,0;P(f)) + N_{k+2}(r,0;P(g)) \\
& + S(r,f) + S(r,g) \\
& \leq 2\,\overline{N}(r,\infty;f) + (2+k)\,\overline{N}(r,\infty;g) + N_{k+2}(r,0;(f-c_l)^l P_*(f)) \qquad (3.5) \\
& + N_{k+2}(r,0;(g-c_l)^l P_*(g)) + S(r,f) + S(r,g) \\
& \leq 2\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) + (k+2)\{T(r,f)+T(r,g)\} + r\{T(r,f)+T(r,g)\} \\
& \leq (3k+2r+8)T(r) + S(r) \qquad (3.6)
\end{aligned}$$

In a similar way we can obtain

$$nT(r,g) \quad \leq \quad (3k+2r+8)T(r) + S(r). \qquad (3.7)$$

From (3.5) and (3.7) we get

$$(l-3k-r-8)\,T(r) \leq S(r),$$

which is a contradiction since $l > 3k+r+8$.

**Case 2.** Let $H \equiv 0$. Then the Theorem follows from *Lemmas* 2.9, 2.14 and 2.18. □

*Proof of Theorem* 1.2. In this case $F$ and $G$ share 1 IM.

**Case 1.** Let $H \not\equiv 0$. Here we see that

$$N_E^{1)}(r,1;F \mid= 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F) + S(r,G). \qquad (3.8)$$

Now using *Lemmas* 2.3, 2.19, 2.20, 2.21, (3.1) and (3.8) we get

$$\overline{N}(r,1;F) \leq N_E^{1)}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) \qquad (3.9)$$

$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{*}(r,1;F,G)$$
$$+\overline{N}_{L}(r,1;F)+\overline{N}_{L}(r,1;G)+\overline{N}_{E}^{(2}(r,1;F)$$
$$+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})+S(r,f)+S(r,g)$$
$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_{L}(r,1;F)$$
$$+2\overline{N}_{L}(r,1;G)+\overline{N}_{E}^{(2}(r,1;F)+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})+S(r,f)+S(r,g)$$
$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{F>1}(r,1;G)$$
$$+\overline{N}_{G>1}(r,1;F)+\overline{N}_{L}(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})$$
$$+S(r,f)+S(r,g)$$
$$\leq 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{2}(r,0;F)+\overline{N}(r,0;F)+N_{2}(r,0;G)+N(r,1;G)$$
$$-\overline{N}(r,1;G)+\overline{N}_{0}(r,0;G^{'})+\overline{N}_{0}(r,0;F^{'})+S(r,f)+S(r,g)$$
$$\leq 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{2}(r,0;F)+\overline{N}(r,0;F)+N_{2}(r,0;G)$$
$$+N(r,0;G^{'}|G\neq 0)+\overline{N}_{0}(r,0;F^{'})+S(r)$$
$$\leq 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{2}(r,0;F)+\overline{N}(r,0;F)+N_{2}(r,0;G)+\overline{N}(r,0;G)$$
$$+\overline{N}_{0}(r,0;F^{'})+S(r).$$

Hence using (3.9), *Lemmas* 2.1 and 2.2 we get from second fundamental theorem that

$$nT(r,f)$$
$$\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;P(f))-N_{2}(r,0;F)-N_{0}(r,0;F^{'})$$
$$\leq 4\overline{N}(r,\infty,P(f))+3\overline{N}(r,\infty;P(g))+N_{2}(r,0;F)+2\,\overline{N}(r,0;F)+N_{k+2}(r,0;P(f))$$
$$+N_{2}(r,0;G)+\overline{N}(r,0;G)-N_{2}(r,0;F)+S(r,f)+S(r,g)$$
$$\leq 4\overline{N}(r,\infty;P(f))+3\overline{N}(r,\infty;P(g))+N_{k+2}(r,0;P(f))+2\,\overline{N}(r,0;F)+N_{2}(r,0;G)$$
$$+\overline{N}(r,0;G)+S(r,f)+S(r,g)$$
$$\leq 4\overline{N}(r,\infty;P(f))+3\overline{N}(r,\infty;P(g))+N_{k+2}(r,0;P(f))+2\,k\overline{N}(r,\infty;P(f))$$
$$+2\,N_{k+1}(r,0;P(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;P(g))+k\overline{N}(r,\infty;g)$$
$$+N_{k+1}(r,0;P(g))+S(r,f)+S(r,g)$$
$$\leq (2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+(3k+3r+4)T(r,f)$$
$$+(2k+2r+3)T(r,g)+S(r,f)+S(r,g)$$
$$\leq (9k+5r+14T(r)+S(r). \tag{3.10}$$

In a similar way we can obtain

$$n\,T(r,g)\leq (9k+5r+14)T(r)+S(r). \tag{3.11}$$

Combining (3.10) and (3.11) we see that

$$(l-9k-4r-14)\,T(r)\leq S(r). \tag{3.12}$$

When $l>9k+4r+14$, (3.12) leads to a contradiction.

**Case 2.** Let $H \equiv 0$. Then the Theorem follows from *Lemmas* 2.9, 2.14 and 2.18. This completes the proof of the Theorem. □

*Proof of Corollary* 1.1 *and* 1.2. From Theorem 1.1 and 1.2 one can easily prove the corollaries. So we omit the details. □

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## References

[1] Alzahary, T.C., Yi, H.X., *Weighted value sharing and a question of I.Lahiri*, Complex Var. Theory Appl., **49**(2004), no. 15, 1063-1078.

[2] Banerjee, A., *Meromorphic functions sharing one value*, Int. J. Math. Math. Sci., **22**(2005), 3587-3598.

[3] Chen, H.H., *Yoshida functions and Picad values of integral functions and their derivatives*, Bull. Aust. Mat. Soc., **54**(1996), 373-381.

[4] Dou, J., Qi, X.G., Yang, L.Z., *Entire functions that share fixed points*, Bull. Malays. Math., **34**(2011), no. 2, 355-367.

[5] Fang, M.L., *Uniqueness and value-sharing of entire functions*, Comput. Math. Appl., **44**(2002), 823-831.

[6] Frank, G., *Eine Vermutung Von Hayman über Nullslellen meromorphic Funktion*, Math. Z., **149**(1976), 29-36.

[7] Hayman, W.K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[8] Lahiri, I., *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161**(2001), 193-206.

[9] Lahiri, I., *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46**(2001), 241-253.

[10] Lahiri, I., Dewan, S., *Value distribution of the product of a meromorphic function and its derivative*, Kodai Math. J., **26**(2003), 95-100.

[11] Lahiri, I., Sarkar, A., *Nonlinear differential polynomials sharing* 1*-points with weight two*, Chinese J. Contemp. Math., **25**(2004), no. 3, 325-334.

[12] Li, P., Yang, C.C., *On the characteristics of meromorphic functions that share three values CM*, J. Math. Anal. Appl., **220**(1998), 132-145.

[13] Liu, Y., Yang, L.Z., *Some Further Results On Uniqueness Of Entire Functions And Fixed Points*, Kyungpook. Math. J., **53**(2013), 371-383.

[14] Qi, X.G., Yang, L.Z., *Uniqueness of entire functions and fixed points*, An. Polon. Math., **97**(2010), 87-100.

[15] Wu, C., Li, J., *Uniqueness of meromorphic functions sharing a value*, Vietnam J. Math., **41**(2013), 289-302.

[16] Xu, J.F., Lü, F., Yi, H.X., *Fixed points and uniqueness of meromorphic functions*, Comput. Math. Appl., **59**(2010), 9-17.

[17] Yang, C.C., *On deficiencies of differential polynomials II*, Math. Z., **125**(1972), 107-112.

[18] Yamanoi, K., *The second main theorem for small functions and related problems*, Acta Math., **192**(2004), 225-294.

[19] Yi, H.X., *On characteristic function of a meromorphic function and its derivative*, Indian J. Math., **33**(1991), no. 2, 119-133.

[20] Yi, H.X., *Meromorphic functions that share one or two values II*, Kodai Math. J., **22**(1999), 264-272.

[21] Yang, C.C., Yi, H.X., *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[22] Zhang, J.L., *Uniqueness theorems for entire functions concerning fixed-points*, Comput. Math. Appl., **56**(2008), 3079-3087.

[23] Zhang, Q.C., *Meromorphic function that shares one small function with its derivative*, J. Inequal. Pure Appl. Math., **6**(2005), no. 4, Art. 116 [ONLINE *http://jipam.vu.edu.au/*].

[24] Zhang, J.F., Zhang, X.Y., *Uniqueness of entire finctions that share one value*, Comput. Math. Appl., **56**(2008), 3000-3014.

[25] Zhang, X., Xu, J., Yi, H., *Value sharing of meromorphic functions and Fang's problem* , arXiv: 1009.2132v1 [math.CV].

[26] Zhang, X.B., Xu, J.F., *Uniqueness of meromorphic functions sharing a small function and its applications*, Comput. Math. Appl., **61**(2011), 722-730.

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# Book reviews

**Antonio J. Guirao, Vicente Montesinos and Václav Zizler, Open problems in the geometry and analysis of Banach spaces.** Cham: Springer 2016, xii + 169 p., ISBN 978-3-319-33571-1/hbk; 978-3-319-33572-8/ebook.

This is a collection of 304 open research problems from Banach space theory and related areas (measure theory, vector measures, nonlinear analysis, best approximation and optimization).

The problems are grouped into seven chapters: 1. *Basic linear structure* (Schauder bases, approximation properties, weak Hilbert spaces, Daugavet property); 2. *Basic linear geometry* (Chebyshev sets, isometries, Banach-Mazur distance, rotund renormings); 3. *Biorthogonal systems* (Markushevich bases, Auerbach bases, weakly compactly generated Banach spaces); 4. *Differentiability and structure, renormings* (Asplund spaces, weak Asplund spaces, Gâteaux and Fréchet differentiability, Krein-Milman and Radon-Nikodým properties, norm-attaining functionals and operators); 5. *Nonlinear geometry* (Lipschitz-free spaces, Lipschitz homeomorphisms and Lipschitz quotients); 6. *Some more nonseparable problems* (Schauder basis in nonseparable setting, equilateral sets); 7. *Some applications* (fixed points, Riemann integrability of vector-valued functions).

As the authors point out in the Preface:

> Some of the problems are longstanding open problems, some are recent, some are more important, and some are only "local" problems. Some would require new ideas, and some may go only with a subtle combination of known facts.

The book is very well organized - every problem is preceded by an introductory part containing the notions and previous results necessary for its understanding, as well as references to significant papers or books containing partial solutions or related results. At the end there are a detailed index and a comprehensive table referring to the listed problems by subject (and a reference list, of course).

The second and the third named authors are coauthors of two impressive volumes:
M. Fabian, P. Habala, P. Hájek, V. Montesinos Santaluca, J. Pelant and V. Zizler, *Functional analysis and infinite-dimensional geometry.* CMS Books in Mathematics, 451 p., Springer, 2001, and
M. Fabian, P. Habala, P. Hájek, V. Montesinos Santaluca and V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis*, CMS Books in Mathematics, 820 p, Springer, 2011.

The present collection of problems is tightly connected with the two books mentioned above, being often used by the authors to upgrade and update information provided in these two references (as they confess in the Preface). All in all, the authors produced a marvelous piece of mathematical writing of great use for researchers in various fields of functional and mathematical analysis as well as for young graduate or PhD students.

S. Cobzaş

**Advanced Courses of Mathematical Analysis V Proceedings of the Fifth International School; (edited by Juan Carlos Navarro Pascual and El Amn Kaidi);** V International Course of Mathematical Analysis in Andalusia Universidad de Almería, Almería, Spain, 12 - 16 September 2011. ISBN: 978-981-4699-68-6 (hardcover), 978-981-4699-70-9 (ebook).

The courses of Mathematical Analysis in Andalusia started in 2002 at the University of Cádiz at the initiative of the late Professor Antonio Aizpuru. Their aim was to provide opportunities for different research groups in Andalusia working in various areas of Mathematical Analysis to share information about their research and to cooperate, and, at the same time, to introduce the young researchers to the most advanced research lines.

The project turned to be a great success, both concerning the conferences and the published volumes. The present volume is dedicated to the V International Course on Mathematical Analysis, carried out at the University of Almería, September 12–16, 2011, following the first one from 2002, the second (Granada 2004), the third (Huelva, 2007), and the fourth (Cádiz, 2009).

It contains the elaborated versions of four mincourses of three ours each and five plenary one-our presentations. Besides these plenary lectures the interested participants had the occasion to present their recent contributions, short communications or posters.

The minicourses are the following: B. Cascales, *Measurability and semicontinuity of multifunctions* (26 p), F. Cobos, *Introduction to interpolation theory* (22 p), L. Pick, *Optimality of function spaces in Sobolev embeddings* (69 p), and B. Russo, *Derivations and projections on Jordan triples: An introduction to nonassociative algebra, continuous cohomology, and quantum functional analysis* (10 p).

The one-our plenary lectures are dealing with topics as: weighted inequalities and extrapolation (J. Duoandikoetxea), Muckenhoupt-Wheeden Conjecture for Calderón-Zygmund operators (D. Cruz-Uribe, J. M Martell and C. Pérez), nonlinear partial differential equations and game theory (J D Rossi), the Radon-Nikodým theorem for vector measures and integral representation of operators on Banach function spaces (E. A. Sánchez Pérez), the Orlicz-Pettis theorem for multiplier convergent series (C. Swartz).

The volume contains papers of great interest, both for researchers in Functional Analysis, Operator Theory, Measure Theory as well as for young researchers and graduate students desiring to get a first-hand acquaintance with the last developments and open problems in various areas of Mathematical Analysis.

V. Anisiu