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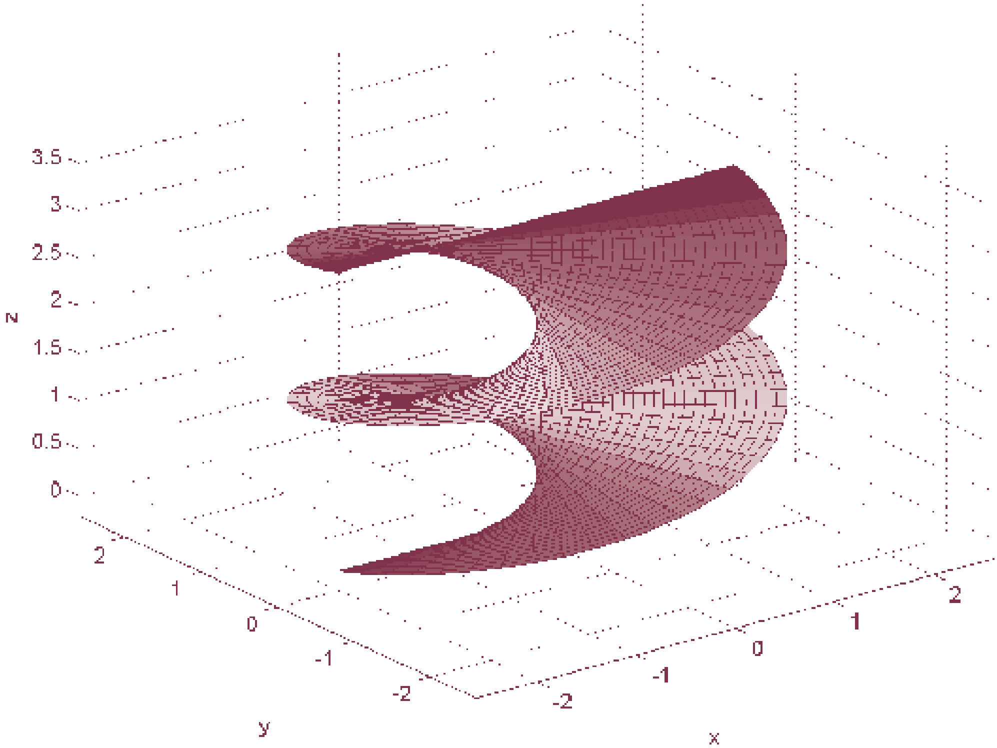

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MONTH **MARCH**
ISSUE **1**

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 3–15

New fractional estimates of Hermite-Hadamard inequalities and applications to means

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Abstract. The main objective of this paper is to obtain some new fractional estimates of Hermite-Hadamard type inequalities via h-convex functions. A new fractional integral identity for three times differentiable function is established. This result plays an important role in the development of new results. Several new special cases are also discussed. Some applications to means of real numbers are also discussed.

Mathematics Subject Classification (2010): 26A33, 26D15, 26A51.

Keywords: Convex functions, h-convex, fractional integrals, Hermite-Hadamard inequality.

1. Introduction and preliminaries

Throughout the sequel of the paper, let set of real numbers be denoted by $\mathbb{R}$, $I = [a, b] \subset \mathbb{R}$ be the real interval and I° be the interior of I unless otherwise specified.

Definition 1.1. *A function $f : I \to \mathbb{R}$ is said to be classical convex function, if*

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in I, t \in [0,1]. \tag{1.1}$$

In recent years numerous generalizations of classical convex functions have been proposed, see [1, 2, 3, 4, 5, 6, 11, 19]. Varosanec [19] investigated a new class of convex functions which she named as h-convex functions. This class is unifying one and it includes some other classes of convex functions, such as, s-Breckner convex functions [1], s-Godunova-Levin-Dragomir convex functions [3], Godunova-Levin functions [6] and P-functions [5].
The h-convexity is defined as:

Definition 1.2. [19] *Let $h : [0,1] \to \mathbb{R}$ be a non-negative function. A non-negative function $f : I \to \mathbb{R}$ is said to be h-convex function, if*

$$f((1-t)x + ty) \leq h(1-t)f(x) + h(t)f(y), \quad \forall x, y \in I, t \in [0,1]. \tag{1.2}$$

For different suitable choices of function $h(.)$ one can have other classes of convex functions.
Every one is familiar with the fact that theory of convex functions has a close relation with theory of inequalities. In fact many classical inequalities are derived using convexity property. Thus these facts inspired a number of researchers to investigate both theories. Consequently several new generalizations of classical inequalities have been obtained via different generalizations of convex functions, see [3, 4, 5, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24].
Nowadays fractional calculus is a vibrant area of research in mathematics. The history of fractional calculus started with the letter of L'Hospital to Leibniz on 30th September 1695 in which he enquired Leibniz about the notation he used in his publications for n-th order derivative of the linear function $f(x) = x$, $\frac{\mathrm{D}^n x}{\mathrm{D}x^n}$. L'Hospital asked a question to Leibniz that what would happen if $n = \frac{1}{2}$. Leibniz's replied: "An apparent paradox, from which one day useful consequences will be drawn." With this the study of fractional calculus had begun. Several applications of fractional calculus have been found till now. For some useful information on fractional calculus and its applications, see [7, 8, 9]. A recent approach of obtaining fractional version of classical integral inequalities has also attracted researchers. For example, see [11, 12, 15, 19, 22]. The motivation of this article is to establish some new fractional estimates of Hermite-Hadamard type inequalities via h-convex functions. Some special cases which can be derived from our main results are also discussed. In the end some application to special means of real numbers are also discussed.
We now recall some preliminary concepts which are widely used throughout the paper.

Definition 1.3. [9] *Let $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a,$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1}\mathrm{d}x,$$

is the Gamma function.

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2F_1(x,y;c;z) = \frac{1}{\mathrm{B}(y,c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\mathrm{d}t$$

for $|z| < 1, c > y > 0$.

Recall that

1. For arbitrary $a, b \in \mathbb{R} \setminus \{0\}$ and $a \neq b$, $L(b,a) = \dfrac{b-a}{\log b - \log a}$, is the logarithmic mean.
2. For arbitrary $a, b \in \mathbb{R}$ and $a \neq b$, $A(a,b) = \dfrac{a+b}{2}$, is the arithmetic mean.
3. The extended logarithmic mean L_p of two positive numbers a, b is given for $a = b$ by $L_p(a,a) = a$ and for $a \neq b$ by

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0, \\ \frac{b-a}{\log b - \log a}, & p = -1, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0. \end{cases}$$

2. Main results

To prove our main results, we need following auxiliary result.

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a,b]$, then*

$$L_f(a,b;n;\alpha) = \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}$$

$$\times \int_0^1 (1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) + f'''\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right] \mathrm{d}t,$$

where

$$\begin{aligned} L_f(a,b;n;\alpha) = {} & \frac{(n+1)^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-} f(a) + J^\alpha_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+} f(b)\right] \\ & - \frac{(b-a)^2}{(n+1)^3(\alpha+1)(\alpha+2)}\left[f''\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)\right. \\ & \left. + f''\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)\right] + \frac{b-a}{(n+1)^2(\alpha+1)}\left[f'\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)\right. \\ & \left. + f'\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)\right] - \frac{1}{n+1}\left[f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)\right. \\ & \left. + f\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)\right]. \end{aligned}$$

Proof. Let

$$I \triangleq \int_0^1 (1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]\mathrm{d}t$$

$$= -\int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t + \int_0^1 (1-t)^{\alpha} f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t$$

$$= -I_1 + I_2. \tag{2.1}$$

Integrating I_1 on $[0,1]$ yields

$$\begin{aligned} I_1 &\triangleq \int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t \\ &= \frac{n+1}{b-a}f''\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right) - \frac{(n+1)^2(\alpha+2)}{(b-a)^2}f'\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right) \\ &+ \frac{(n+1)^3(\alpha+1)(\alpha+2)}{(b-a)^3}f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right) \\ &- \frac{(n+1)^{\alpha+3}\Gamma(\alpha+3)}{(b-a)^{\alpha+3}}J^{\alpha}_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-}f(a). \end{aligned} \tag{2.2}$$

Similarly, integrating I_2 on $[0,1]$, we have

$$\begin{aligned} I_2 &\triangleq \int_0^1 (1-t)^{\alpha+2} f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t \\ &= -\frac{n+1}{b-a}f''\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right) - \frac{(n+1)^2(\alpha+2)}{(b-a)^2}f'\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right) \\ &- \frac{(n+1)^3(\alpha+1)(\alpha+2)}{(b-a)^3}f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right) \\ &+ \frac{(n+1)^{\alpha+3}\Gamma(\alpha+3)}{(b-a)^{\alpha+3}}J^{\alpha}_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+}f(b). \end{aligned} \tag{2.3}$$

Summation of (2.2), (2.3) and (2.1) and then multiplying both sides by

$$\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}$$

completes the proof. □

Note that for $n=1$ and $\alpha=1$ in Lemma 2.1, we have previously Lemma [24]. If $n=1$ in Lemma 2.1, then, we have Lemma 3.1 [14].

Theorem 2.2. *Let $f : I \to \mathbb{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a,b]$ and $|f'''|$ is h-convex function, then*

$$|L_f(a,b;n;\alpha)| \le \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\Psi(n;h;t)\left[|f'''(a)|+|f'''(b)|\right],$$

where

$$\Psi(h;n;t)=\int_0^1(1-t)^{\alpha+2}\left[h\left(\frac{n+t}{n+1}\right)+h\left(\frac{1-t}{n+1}\right)\right]\mathrm{d}t.$$

Proof. Using Lemma 2.1 and the given hypothesis, we have

$$\begin{aligned}
&|L_f(a,b;n;\alpha)|\\
&=\Bigg|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\\
&\times\int_0^1(1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]\mathrm{d}t\Bigg|\\
&\le\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\\
&\times\left\{\left|\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t\right|\right.\\
&\left.+\left|\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t\right|\right\}\\
&\le\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|\mathrm{d}t\\
&+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|\mathrm{d}t\\
&\le\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left[h\left(\frac{n+t}{n+1}\right)|f'''(a)|+h\left(\frac{1-t}{n+1}\right)|f'''(b)|\right]\mathrm{d}t\\
&+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left[h\left(\frac{1-t}{n+1}\right)|f'''(a)|+h\left(\frac{n+t}{n+1}\right)|f'''(b)|\right]\mathrm{d}t\\
&=\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\\
&\times\left(\int_0^1(1-t)^{\alpha+2}\left[h\left(\frac{n+t}{n+1}\right)+h\left(\frac{1-t}{n+1}\right)\right]\mathrm{d}t\right)\left[|f'''(a)|+|f'''(b)|\right].
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a,b]$ and $|f'''|^q$ is h-convex function where $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, then*

$$|L_f(a,b;n;\alpha)| \le \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left\{\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t\right\}|f'''(a)|^q+\left\{\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left\{\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right\}|f'''(a)|^q+\left\{\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right].$$

Proof. Using given hypothesis, Lemma 2.1 and the Hölder's inequality, we have

$$|L_f(a,b;n;\alpha)|$$
$$=\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\right.$$
$$\left.\times\int_0^1(1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]\mathrm{d}t\right|$$
$$\le\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t\right|$$
$$+\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t\right|$$
$$\le\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|\mathrm{d}t$$
$$+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|\mathrm{d}t$$
$$\le\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1(1-t)^{p(\alpha+2)}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1(1-t)^{p(\alpha+2)}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left\{\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t\right\}|f'''(a)|^q+\left\{\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left\{\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right\}|f'''(a)|^q+\left\{\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Theorem 2.4. *Let $f: I \to \mathbb{R}$ be three times differentiable function on the interior I° of I. If $f''' \in L[a,b]$ and $|f'''|^q$ is h-convex function where $q > 1$, then*

$$|L_f(a,b;n;\alpha)|$$
$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1(1-t)^{\alpha+2}\left\{h\left(\frac{n+t}{n+1}\right)|f'''(a)|^q+h\left(\frac{1-t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1(1-t)^{\alpha+2}\left\{h\left(\frac{1-t}{n+1}\right)|f'''(a)|^q+h\left(\frac{n+t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].$$

Proof. Using given hypothesis, Lemma 2.1 and power mean inequality, we have

$$|L_f(a,b;n;\alpha)|$$
$$=\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\right.$$
$$\left.\times\int_0^1(1-t)^{\alpha+2}\left[-f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]\mathrm{d}t\right|$$
$$\leq\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t\right|$$
$$+\left|\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\int_0^1(1-t)^{\alpha+2}f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\mathrm{d}t\right|$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1(1-t)^{\alpha+2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1(1-t)^{\alpha+2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-t)^{\alpha+2}\left|f'''\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1(1-t)^{\alpha+2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-t)^{\alpha+2}\left\{h\left(\frac{n+t}{n+1}\right)|f'''(a)|^q+h\left(\frac{1-t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\int_0^1(1-t)^{\alpha+2}\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-t)^{\alpha+2}\left\{h\left(\frac{1-t}{n+1}\right)|f'''(a)|^q+h\left(\frac{n+t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1(1-t)^{\alpha+2}\left\{h\left(\frac{n+t}{n+1}\right)|f'''(a)|^q+h\left(\frac{1-t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1(1-t)^{\alpha+2}\left\{h\left(\frac{1-t}{n+1}\right)|f'''(a)|^q+h\left(\frac{n+t}{n+1}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

We now discuss some special cases of the results proved in previous section.

I. If $h(t)=t^s$ in Theorem 2.2, then, we have result for s-Breckner convex function.

Corollary 2.5. *Under the assumptions of Theorem 2.2, if $|f'''|$ is s-Breckner convex function, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{(b-a)^3}{(n+1)^{s+4}(\alpha+1)(\alpha+2)}\Psi(n;s;t)\left[|f'''(a)|+|f'''(b)|\right],$$

where

$$\begin{aligned}\Psi(n;s;t) &= \int_0^1 (1-t)^{\alpha+2}\left[(n+t)^s+(1-t)^s\right]\mathrm{d}t \\ &= \frac{n^s}{\alpha+3}\,{}_2F_1\left[1,-s;\alpha+4;-\frac{1}{n}\right]+\frac{1}{\alpha+s+3}.\end{aligned}$$

II. If $h(t)=t^{-s}$ in Theorem 2.2, then, we have result for s-Godunova-Levin-Dragomir function.

Corollary 2.6. *Under the assumptions of Theorem 2.2, if $|f'''|$ is s-Godunova-Levin-Dragomir function, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{(b-a)^3}{(n+1)^{4-s}(\alpha+1)(\alpha+2)}\Psi(n;-s;t)\left[|f'''(a)|+|f'''(b)|\right],$$

where

$$\begin{aligned}\Psi(n;-s;t) &= \int_0^1 (1-t)^{\alpha+2}\left[(n+t)^{-s}+(1-t)^{-s}\right]\mathrm{d}t \\ &= \frac{1}{n^s(\alpha+3)}\,{}_2F_1\left[1,s;\alpha+4;-\frac{1}{n}\right]+\frac{1}{\alpha-s+3}.\end{aligned}$$

III. If $h(t)=1$ in Theorem 2.2, then, we have result for P-function.

Corollary 2.7. *Under the assumptions of Theorem 2.2, if $|f'''|$ is P-function, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{2(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)(\alpha+3)}\left[|f'''(a)|+|f'''(b)|\right].$$

IV. If $h(t)=t^s$ in Theorem 2.3, then, we have result for s-Breckner convex function.

Corollary 2.8. *Under the assumptions of Theorem 2.3, if $|f'''|^q$ is s-Breckner convex function, then*

$$\begin{aligned}&|L_f(a,b;n;\alpha)| \\ &\leq \frac{(b-a)^3}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}} \\ &\times\left[\left(\left\{\frac{(1+n)^{1+s}-n^{1+s}}{1+s}\right\}|f'''(a)|^q+\left\{\frac{1}{1+s}\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\left\{\frac{1}{1+s}\right\}|f'''(a)|^q+\left\{\frac{(1+n)^{1+s}-n^{1+s}}{1+s}\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

V. If $h(t) = t^{-s}$ in Theorem 2.3, then, we have result for s-Godunova-Levin-Dragomir convex function.

Corollary 2.9. *Under the assumptions of Theorem 2.3, if $|f'''|^q$ is s-Godunova-Levin-Dragomir convex function, then*

$$\begin{aligned}
&|L_f(a,b;n;\alpha)| \\
&\leq \frac{(b-a)^3}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}} \\
&\times\left[\left(\left\{\frac{(n(1+n))^{-s}(-n^s(1+n)+n(1+n)^s)}{s-1}\right\}|f'''(a)|^q+\left\{\frac{1}{1-s}\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left\{\frac{1}{1-s}\right\}|f'''(a)|^q+\left\{\frac{(n(1+n))^{-s}(-n^s(1+n)+n(1+n)^s)}{s-1}\right\}|f'''(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

VI. If $h(t) = 1$ in Theorem 2.3, then, we have result for P-function.

Corollary 2.10. *Under the assumptions of Theorem 2.3, if $|f'''|^q$ is P-function, then*

$$|L_f(a,b;n;\alpha)| \leq \frac{2(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)}\left(\frac{1}{p(\alpha+2)+1}\right)^{\frac{1}{p}}\left[|f'''(a)|^q+|f'''(b)|^q\right]^{\frac{1}{q}}.$$

VII. If $h(t) = t^s$ in Theorem 2.4, then, we have result for s-Breckner convex function.

Corollary 2.11. *Under the assumptions of Theorem 2.4, if $h(t) = t^s$, then, we have result for s-Breckner convex function.*

$$\begin{aligned}
&|L_f(a,b;n;\alpha)| \\
&\leq \frac{(b-a)^3}{(n+1)^{4+\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\left\{\left(\frac{n^s\,{}_2F_1\left[1,-s;\alpha+4;-\frac{1}{n}\right]}{\alpha+3}\right)|f'''(a)|^q+\left(\frac{1}{\alpha+s+3}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left\{\left(\frac{1}{\alpha+s+3}\right)|f'''(a)|^q+\left(\frac{n^s\,{}_2F_1\left[1,-s;\alpha+4;-\frac{1}{n}\right]}{\alpha+3}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].
\end{aligned}$$

VIII. If $h(t) = t^{-s}$ in Theorem 2.4, then, we have result for s-Godunova-Levin-Dragomir convex function.

Corollary 2.12. *Under the assumptions of Theorem 2.4, if $h(t) = t^s$, then, we have result for s-Godunova-Levin-Dragomir convex function.*

$$\begin{aligned}
&|L_f(a,b;n;\alpha)| \\
&\leq \frac{(b-a)^3}{(n+1)^{4-\frac{s}{q}}(\alpha+1)(\alpha+2)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\left\{\left(\frac{{}_2F_1\left[1,2;\alpha+4;\frac{1}{n}\right]}{n^s(\alpha+3)}\right)|f'''(a)|^q+\left(\frac{1}{\alpha-s+3}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left\{\left(\frac{1}{\alpha-s+3}\right)|f'''(a)|^q+\left(\frac{{}_2F_1\left[1,2;\alpha+4;\frac{1}{n}\right]}{n^s(\alpha+3)}\right)|f'''(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].
\end{aligned}$$

IX. If $h(t) = 1$ in Theorem 2.4, then, we have result for P-function.

Corollary 2.13. *Under the assumptions of Theorem 2.4, if $h(t) = t^s$, then, we have result for p-function.*

$$\begin{aligned}
&|L_f(a,b;n;\alpha)| \\
&\leq \frac{2(b-a)^3}{(n+1)^4(\alpha+1)(\alpha+2)(\alpha+3)}\left[|f'''(a)|^q+|f'''(b)|^q\right].
\end{aligned}$$

3. Applications

In this section, we present some applications to means of real numbers.

Proposition 3.1. *For some $s \in (0,1)$, $0 \leq a < b$, then*

$$\begin{aligned}
&\left|L(a,b)-\frac{s(s-1)(b-a)^3}{24}A^{s-2}(a,b)-A^s(a,b)\right| \\
&\leq \frac{s(s-1)(s-2)(b-a)^3}{192}\left[|a|^{s-3}+|b|^{s-3}\right].
\end{aligned}$$

Proof. The assertion directly follows from Theorem 2.2 applying for $h(t) = t^s$, $f : [0,1] \to [0,1]$, $f(x) = x^s$ and $\alpha = 1, n = 1$. □

Proposition 3.2. *For some $s \in (0,1)$, $0 \leq a < b$ and $\frac{1}{p}+\frac{1}{q} = 1$, $1 < q < \infty$, then*

$$\begin{aligned}
&\left|L(a,b)-\frac{s(s-1)(b-a)^3}{24}A^{s-2}(a,b)-A^s(a,b)\right| \\
&\leq \frac{s(s-1)(s-2)(b-a)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{3}{4}|a|^{q(s-3)}+\frac{1}{2}|b|^{q(s-3)}\right)^{\frac{1}{q}}+\left(\frac{1}{2}|a|^{q(s-3)}+\frac{3}{4}|b|^{q(s-3)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. The assertion directly follows from Theorem 2.3 applying for $h(t) = t^s$, $f : [0,1] \to [0,1]$, $f(x) = x^s$ and $\alpha = 1, n = 1$. □

Proposition 3.3. *For some $s \in (0,1)$, $0 \leq a < b$ and $q > 1$, then*

$$\left|L(a,b) - \frac{s(s-1)(b-a)^3}{24} A^{s-2}(a,b) - A^s(a,b)\right|$$
$$\leq \frac{s(s-1)(s-2)(b-a)^3}{384}\left(\frac{4}{5}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\frac{3}{2}|a|^{q(s-3)} + |b|^{q(s-3)}\right)^{\frac{1}{q}} + \left(|a|^{q(s-3)} + \frac{3}{2}|b|^{q(s-3)}\right)^{\frac{1}{q}}\right].$$

Proof. The assertion directly follows from Theorem 2.4 applying for $h(t) = t^s$, $f : [0,1] \to [0,1]$, $f(x) = x^s$ and $\alpha = 1, n = 1$. □

Acknowledgements. The authors would like to thank anonymous referees for their comments and suggestions. The authors are grateful to Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research and academic environment.

References

[1] Breckner, W.W., *Stetigkeitsaussagen für eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen*, Publ. Inst. Math., **23**(1978), 13-20.

[2] Cristescu, G., *Improved integral inequalities for products of convex functions*, J. Ineq. Pure Appl. Math., **6**(2)(2005).

[3] Dragomir, S.S., *Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces*, RGMIA Research Report Collection, **16**(2013), Article 72.

[4] Dragomir, S.S., Pearce, C.E.M., *Selected topics on Hermite-Hadamard inequalities and applications*, Victoria University, Australia, 2000.

[5] Dragomir, S.S., Pečarić, J., Persson, L.E., *Some inequalities of Hadamard type*, Soochow J. Math., **21**(1995), 335-341.

[6] Godunova, E.K., Levin, V.I., *Neravenstva dlja funkcii sirokogo klassa soderzascego vypuklye monotonnye i nekotorye drugie vidy funkii*, Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. MGPI Moskva, (1985), 138142 (in Russian).

[7] Katugampola, U.N., *A new approach to generalized fractional derivatives*, Bulletin of Math. Anal. Appl., **6**(4)(2014), 1-15.

[8] Katugampola, U.N., *Mellin transforms of generalized fractional integrals and derivatives*, Appl. Math. Comput., **257**(2015), 566-580.

[9] Kilbas, A., Srivastava, H.M., Trujillo, J.J., *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, Netherlands, 2006.

[10] Khattri, S.K., *Three proofs of the inequality $e < \left(1 + \frac{1}{n}\right)^{n+0.5}$*, Amer. Math. Monthly, **117**(3)(2010), 273-277.

[11] Noor, M.A., Awan, M.U., *Some integral inequalities for two kinds of convexities via fractional integrals*, Trans. J. Math. Mech. **5**(2)(2013), 129-136.

[12] Noor, M.A., Cristescu, G., Awan, M.U., *Generalized fractional Hermite-Hadamard inequalities for twice differentiable s-convex functions*, Filomat, **29**(4)(2015), 807-815.

[13] Noor, M.A., Noor, K.I., Awan, M.U., *Hermite-Hadamard inequalities for relative semi-convex functions and applications*, Filomat, **28**(2)(2014), 221-230.

[14] Noor, M.A., Noor, K.I., Awan, M.U., *Fractional Hermite-Hadmard inequalities for convex functions and applications*, Tbilisi J. Math., **8**(2)(2015), 103-113.

[15] Ozdemir, M.E., Kavurmaci, H., Yildiz, C., *Fractional integral inequalities via s-convex functions*, available online at: arXiv:1201.4915v1, (2012).

[16] Park, J., *Some inequalities of Hermite-Hadamard type via differentiable are (s, m)-convex mappings*, Far East J. Math. Sci., **52**(2)(2011), 209-221.

[17] Park, J., *On the left side inequality of Hermite-Hadamard inequality for differentiable (α, m)-convex mappings*, Far East J. Math. Sci., **58**(2011)(2), 179-191.

[18] Park, J., *Hermite-Hadamard-like type inequalities for n-times differentiable functions which are m-convex and s-convex in the second sense*, Int. J. Math. Anal., **8**(25)(2014), 1187-1200.

[19] Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N., *HermiteHadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comp. Modelling, **57**(2013), 2403-2407.

[20] Shuang, Y., Yin, H.P., Qi, F., *HermiteHadamard type integral inequalities for geometric-arithmetically s-convex functions*, Analysis, 33(2013), 197208.

[21] Varosanec, S., *On h-convexity*, J. Math. Anal. Appl., 326(2007), 303-311.

[22] Wang, J., Li, X., Feckan, M., Zhou, Y., *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Appl. Anal., 2012, http://dxdoi.org/10.1080/00036811.2012.727986.

[23] Xi, B.Y., Qi, F., *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Func. Spaces Appl., 2012(2012), Article ID 980438.

[24] Xi, B.Y., Wang, S.H., Qi, F., *Some inequalities of Hermite-Hadamard type for functions whose third derivatives are P-convex*, Appl. Math., **3**(2012), 1898-1902.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 17–26

Generalized g-fractional calculus and iterative methods

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Abstract. We approximated solutions of some iterative methods on a generalized Banach space setting in [5]. Earlier studies such as [7-12] the operator involved is Fréchet-differentiable. In [5] we assumed that the operator is only continuous. This way we extended the applicability of these methods to include generalized fractional calculus and problems from other areas. In the present study applications include generalized g-fractional calculus. Fractional calculus is very important for its applications in many applied sciences.

Mathematics Subject Classification (2010): 26A33, 65G99, 47J25.

Keywords: Generalized Banach space, semilocal convergence, g-fractional calculus.

1. Introduction

Many problems in Computational sciences can be formulated as an operator equation using Mathematical Modelling [8, 10, 13, 14, 15]. The fixed points of these operators can rarely be found in closed form. That is why most solution methods are usually iterative.

The semilocal convergence is, based on the information around an initial point, to give conditions ensuring the convergence of the method.

We presented a semilocal convergence analysis for some iterative methods on a generalized Banach space setting in [5] to approximate fixed point or a zero of an operator. A generalized norm is defined to be an operator from a linear space into a partially order Banach space (to be precised in section 2). Earlier studies such as [7-12] for Newton's method have shown that a more precise convergence analysis is obtained when compared to the real norm theory. However, the main assumption is that the operator involved is Fréchet-differentiable. This hypothesis limits the applicability of Newton's method. In [5] study we only assumed the continuity of the operator. This may be expanded the applicability of these methods.

The rest of the paper is organized as follows: section 2 contains the basic concepts on generalized Banach spaces and the semilocal convergence analysis of these methods. Finally, in the concluding section 3, we present special cases and applications in generalized g-fractional calculus.

2. Generalized Banach spaces

We present some standard concepts that are needed in what follows to make the paper as self contained as possible. More details on generalized Banach spaces can be found in [5-12], and the references there in.

Definition 2.1. *A generalized Banach space is a triplet $(X, E, /\cdot/)$ such that*

(i) X is a linear space over $\mathbb{R}(\mathbb{C})$.

(ii) $E = (E, K, \|\cdot\|)$ is a partially ordered Banach space, i.e.

(ii_1) $(E, \|\cdot\|)$ is a real Banach space,

(ii_2) E is partially ordered by a closed convex cone K,

(iii_3) The norm $\|\cdot\|$ is monotone on K.

(iii) The operator $/\cdot/ : X \to K$ satisfies

$/x/ = 0 \Leftrightarrow x = 0$, $/\theta x/ = |\theta| \, /x/$,

$/x + y/ \le /x/ + /y/$ for each $x, y \in X$, $\theta \in \mathbb{R}(\mathbb{C})$.

(iv) X is a Banach space with respect to the induced norm $\|\cdot\|_i := \|\cdot\| \cdot /\cdot/$.

Remark 2.2. The operator $/\cdot/$ is called a generalized norm. In view of (iii) and (ii_3) $\|\cdot\|_i$, is a real norm. In the rest of this paper all topological concepts will be understood with respect to this norm.

Let $L\left(X^j, Y\right)$ stand for the space of j-linear symmetric and bounded operators from X^j to Y, where X and Y are Banach spaces. For X, Y partially ordered $L_+\left(X^j, Y\right)$ stands for the subset of monotone operators P such that

$$0 \le a_i \le b_i \Rightarrow P(a_1, ..., a_j) \le P(b_1, ..., b_j).$$

Definition 2.3. *The set of bounds for an operator $Q \in L(X, X)$ on a generalized Banach space $(X, E, /\cdot/)$ is defined to be:*

$$B(Q) := \{P \in L_+(E, E),\ /Qx/ \le P\,/x/ \text{ for each } x \in X\}.$$

Let $D \subset X$ and $T : D \to D$ be an operator. If $x_0 \in D$ the sequence $\{x_n\}$ given by

$$x_{n+1} := T(x_n) = T^{n+1}(x_0)$$

is well defined. We write in case of convergence

$$T^{\infty}(x_0) := \lim (T^n(x_0)) = \lim_{n\to\infty} x_n.$$

Let $(X, (E, K, \|\cdot\|), /\cdot/)$ and Y be generalized Banach spaces, $D \subset X$ an open subset, $G : D \to Y$ a continuous operator and $A(\cdot) : D \to L(X, Y)$. A zero of operator G is to be determined by a method starting at a point $x_0 \in D$. The results are presented for an operator $F = JG$, where $J \in L(Y, X)$. The iterates are determined through a fixed point problem:

$$x_{n+1} = x_n + y_n, \quad A(x_n)\, y_n + F(x_n) = 0 \tag{2.1}$$
$$\Leftrightarrow y_n = T(y_n) := (I - A(x_n))\, y_n - F(x_n).$$

Let $U(x_0, r)$ stand for the ball defined by

$$U(x_0, r) := \{x \in X : /x - x_0/ \leq r\}$$

for some $r \in K$.

Next, we state the semilocal convergence analysis of method (2.1) using the preceding notation.

Theorem 2.4. [5] *Let $F : D \subset X$, $A(\cdot) : D \to L(X, Y)$ and $x_0 \in D$ be as defined previously. Suppose:*

(H_1) There exists an operator $M \in B(I - A(x))$ for each $x \in D$.

(H_2) There exists an operator $N \in L_+(E, E)$ satisfying for each $x, y \in D$

$$/F(y) - F(x) - A(x)(y - x)/ \leq N /y - x/.$$

(H_3) There exists a solution $r \in K$ of

$$R_0(t) := (M + N)\, t + /F(x_0)/ \leq t.$$

(H_4) $U(x_0, r) \subseteq D$.

(H_5) $(M + N)^k r \to 0$ as $k \to \infty$.

Then, the following hold:

(C_1) The sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n + T_n^{\infty}(0), \quad T_n(y) := (I - A(x_n))\, y - F(x_n) \tag{2.2}$$

is well defined, remains in $U(x_0, r)$ for each $n = 0, 1, 2, \ldots$ and converges to the unique zero of operator F in $U(x_0, r)$.

(C_2) An apriori bound is given by the null-sequence $\{r_n\}$ defined by $r_0 := r$ and for each $n = 1, 2, \ldots$

$$r_n = P_n^{\infty}(0), \quad P_n(t) = Mt + Nr_{n-1}.$$

(C_3) An aposteriori bound is given by the sequence $\{s_n\}$ defined by

$$s_n := R_n^{\infty}(0), \quad R_n(t) = (M + N)\, t + Na_{n-1},$$

$$b_n := /x_n - x_0/ \leq r - r_n \leq r,$$

where

$$a_{n-1} := /x_n - x_{n-1}/ \text{ for each } n = 1, 2, \ldots$$

Remark 2.5. The results obtained in earlier studies such as [7-12] require that operator F (i.e. G) is Fréchet-differentiable. This assumption limits the applicability of the earlier results. In the present study we only require that F is a continuous operator. Hence, we have extended the applicability of these methods to include classes of operators that are only continuous.

Example 2.6. The j-dimensional space $\mathbb{R}^j$ is a classical example of a generalized Banach space. The generalized norm is defined by componentwise absolute values. Then, as ordered Banach space we set $E = \mathbb{R}^j$ with componentwise ordering with e.g. the maximum norm. A bound for a linear operator (a matrix) is given by the corresponding matrix with absolute values. Similarly, we can define the "N" operators. Let $E = \mathbb{R}$. That is we consider the case of a real normed space with norm denoted by $\|\cdot\|$. Let us see how the conditions of Theorem 2.4 look like.

Theorem 2.7. *(H_1) $\|I - A(x)\| \leq M$ for some $M \geq 0$.*

(H_2) $\|F(y) - F(x) - A(x)(y - x)\| \leq N \|y - x\|$ for some $N \geq 0$.

(H_3) $M + N < 1$,

$$r = \frac{\|F(x_0)\|}{1 - (M + N)}. \tag{2.3}$$

(H_4) $U(x_0, r) \subseteq D$.

(H_5) $(M + N)^k r \to 0$ as $k \to \infty$, where r is given by (2.3).

Then, the conclusions of Theorem 2.4 hold.

3. Applications to g-fractional calculus

We apply Theorem 2.7 in this section. Here basic concepts and facts come from [4]. We need

Definition 3.1. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = m$, $\lceil \cdot \rceil$ the ceiling of the number. Here $g \in AC([a, b])$ (absolutely continuous functions) and g is strictly increasing.*
Let $G : [a, b] \to \mathbb{R}$ such that $\left(G \circ g^{-1}\right)^{(m)} \circ g \in L_\infty([a, b])$.

We define the left generalized g-fractional derivative of G of order α as follows:

$$\left(D^\alpha_{a+;g} G\right)(x) := \frac{1}{\Gamma(m - \alpha)} \int_a^x (g(x) - g(t))^{m-\alpha-1} g'(t) \left(G \circ g^{-1}\right)^{(m)} (g(t))\, dt, \tag{3.1}$$

$a \leq x \leq b$, where Γ is the gamma function.

We also define the right generalized g-fractional derivative of G of order α as follows:

$$\left(D^\alpha_{b-;g} G\right)(x) := \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left(G \circ g^{-1}\right)^{(m)} (g(t))\, dt, \tag{3.2}$$

$a \leq x \leq b$.

Both $\left(D^\alpha_{a+;g} G\right), \left(D^\alpha_{b-;g} G\right) \in C([a, b])$.

(I) Let $a < a^* < b$. In particular we have that $\left(D^\alpha_{a+;g} G\right) \in C([a^*, b])$. We notice that

$$\left|\left(D^\alpha_{a+;g} G\right)(x)\right| \leq \frac{\left\|\left(G \circ g^{-1}\right)^{(m)} \circ g\right\|_{\infty,[a,b]}}{\Gamma(m - \alpha)} \left(\int_a^x (g(x) - g(t))^{m-\alpha-1} g'(t)\, dt\right) \tag{3.3}$$

$$= \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(m-\alpha)} \frac{(g(x)-g(a))^{m-\alpha}}{(m-\alpha)}$$

$$= \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(m-\alpha+1)} (g(x)-g(a))^{m-\alpha}, \quad \forall\, x \in [a,b].$$

We have proved that

$$\left| \left(D_{a+;g}^{\alpha} G\right)(x) \right| \leq \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(m-\alpha+1)} (g(x)-g(a))^{m-\alpha}$$

$$\leq \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(m-\alpha+1)} (g(b)-g(a))^{m-\alpha} < \infty, \ \forall\, x \in [a,b], \tag{3.4}$$

in particular true $\forall\, x \in [a^*, b]$.
We obtain that

$$\left(D_{a+;g}^{\alpha} G\right)(a) = 0. \tag{3.5}$$

Therefore there exist $x_1, x_2 \in [a^*, b]$ such that $D_{a+;g}^{\alpha} G(x_1) = \min D_{a+;g}^{\alpha} G(x)$, and $D_{a+;g}^{\alpha} G(x_2) = \max D_{a+;g}^{\alpha} G(x)$, for $x \in [a^*, b]$.
We assume that

$$D_{a+;g}^{\alpha} G(x_1) > 0. \tag{3.6}$$

(i.e. $D_{a+;g}^{\alpha} G(x) > 0$, $\forall\, x \in [a^*, b]$).
Furthermore

$$\left\| D_{a+;g}^{\alpha} G \right\|_{\infty,[a^*,b]} = D_{a+;g}^{\alpha} G(x_2). \tag{3.7}$$

Here it is

$$J(x) = mx, \ m \neq 0. \tag{3.8}$$

The equation

$$JG(x) = 0, \quad x \in [a^*, b], \tag{3.9}$$

has the same set of solutions as the equation

$$F(x) := \frac{JG(x)}{2D_{a+;g}^{\alpha} G(x_2)} = 0, \quad x \in [a^*, b]. \tag{3.10}$$

Notice that

$$D_{a+;g}^{\alpha} \left(\frac{G(x)}{2D_{a+;g}^{\alpha} G(x_2)} \right) = \frac{D_{a+;g}^{\alpha} G(x)}{2D_{a+;g}^{\alpha} G(x_2)} \leq \frac{1}{2} < 1, \quad \forall\, x \in [a^*, b]. \tag{3.11}$$

We call

$$A(x) := \frac{D_{a+;g}^{\alpha} G(x)}{2D_{a+;g}^{\alpha} G(x_2)}, \quad \forall\, x \in [a^*, b]. \tag{3.12}$$

We notice that

$$0 < \frac{D_{a+;g}^{\alpha} G(x_1)}{2D_{a+;g}^{\alpha} G(x_2)} \leq A(x) \leq \frac{1}{2}. \tag{3.13}$$

Hence it holds

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{D_{a+;g}^{\alpha} G(x_1)}{2D_{a+;g}^{\alpha} G(x_2)} =: \gamma_0, \quad \forall\, x \in [a^*, b]. \tag{3.14}$$

Clearly $\gamma_0 \in (0,1)$.
We have proved that

$$|1 - A(x)| \leq \gamma_0 \in (0,1), \quad \forall\, x \in [a^*, b]. \tag{3.15}$$

Next we assume that $F(x)$ is a contraction over $[a^*, b]$, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall\, x, y \in [a^*, b], \tag{3.16}$$

and $0 < \lambda < \frac{1}{2}$.
Equivalently we have

$$|JG(x) - JG(y)| \leq 2\lambda \left(D^{\alpha}_{a+;g} G(x_2)\right) |x - y|, \quad \forall\, x, y \in [a^*, b]. \tag{3.17}$$

We observe that

$$|F(y) - F(x) - A(x)(y - x)| \leq |F(y) - F(x)| + |A(x)| |y - x|$$

$$\leq \lambda |y - x| + |A(x)| |y - x| = (\lambda + |A(x)|) |y - x| =: (\xi_1), \quad \forall\, x, y \in [a^*, b]. \tag{3.18}$$

Hence by (3.4), $\forall\, x \in [a^*, b]$ we get that

$$|A(x)| = \frac{\left|D^{\alpha}_{a+;g} G(x)\right|}{2D^{\alpha}_{a+;g} G(x_2)} \leq \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)} < \infty. \tag{3.19}$$

Consequently we observe

$$(\xi_1) \leq \left(\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)} \right) |y - x|, \tag{3.20}$$

$\forall\, x, y \in [a^*, b]$.
Call

$$0 < \gamma_1 := \lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)}, \tag{3.21}$$

choosing $(g(b) - g(a))$ small enough we can make $\gamma_1 \in (0,1)$.
We proved that

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \quad \text{where } \gamma_1 \in (0,1), \ \forall\, x, y \in [a^*, b]. \tag{3.22}$$

Next we call and need

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{D^{\alpha}_{a+;g} G(x_1)}{2D^{\alpha}_{a+;g} G(x_2)} + \lambda$$

$$+ \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)} < 1, \tag{3.23}$$

equivalently we find,

$$\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{a+;g} G(x_2)} < \frac{D^{\alpha}_{a+;g} G(x_1)}{2D^{\alpha}_{a+;g} G(x_2)}, \tag{3.24}$$

equivalently,

$$2\lambda D^{\alpha}_{a+;g}G\left(x_2\right)+\frac{\left(g\left(b\right)-g\left(a\right)\right)^{m-\alpha}}{\Gamma\left(m-\alpha+1\right)}\left\|\left(G\circ g^{-1}\right)^{(m)}\circ g\right\|_{\infty,[a,b]}<D^{\alpha}_{a+;g}G\left(x_1\right), \quad (3.25)$$

which is possible for small λ, $(g(b)-g(a))$.
That is $\gamma\in(0,1)$. Hence equation (3.9) can be solved with our presented iterative algorithms.

Conclusion 3.2. (for (I))

Our presented earlier semilocal results, see Theorem 2.7, can apply in the above generalized fractional setting for $g(x)=x$ for each $x\in[a,b]$ since the following inequalities have been fulfilled:

$$\|1-A\|_{\infty}\leq\gamma_0, \quad (3.26)$$

and

$$|F(y)-F(x)-A(x)(y-x)|\leq\gamma_1|y-x|, \quad (3.27)$$

where $\gamma_0,\gamma_1\in(0,1)$, furthermore it holds

$$\gamma=\gamma_0+\gamma_1\in(0,1), \quad (3.28)$$

for all $x,y\in[a^*,b]$, where $a<a^*<b$.

The specific functions $A(x)$, $F(x)$ have been described above, see (3.12) and (3.10), respectively.

(II) Let $a<b^*<b$. In particular we have that $\left(D^{\alpha}_{b-;g}G\right)\in C([a,b^*])$. We notice that

$$\left|\left(D^{\alpha}_{b-;g}G\right)(x)\right|\leq\frac{\left\|\left(G\circ g^{-1}\right)^{(m)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(m-\alpha)}\left(\int_x^b\left(g(t)-g(x)\right)^{m-\alpha-1}g'(t)\,dt\right) \quad (3.29)$$

$$=\frac{\left\|\left(G\circ g^{-1}\right)^{(m)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(m-\alpha+1)}\left(g(b)-g(x)\right)^{m-\alpha}$$

$$\leq\frac{\left\|\left(G\circ g^{-1}\right)^{(m)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(m-\alpha+1)}\left(g(b)-g(a)\right)^{m-\alpha}<\infty,\quad\forall\,x\in[a,b], \quad (3.30)$$

in particular true $\forall\,x\in[a,b^*]$.
We obtain that

$$\left(D^{\alpha}_{b-;g}G\right)(b)=0. \quad (3.31)$$

Therefore there exist $x_1,x_2\in[a,b^*]$ such that $D^{\alpha}_{b-;g}G(x_1)=\min D^{\alpha}_{b-;g}G(x)$, and $D^{\alpha}_{b-;g}G(x_2)=\max D^{\alpha}_{b-;g}G(x)$, for $x\in[a,b^*]$.
We assume that

$$D^{\alpha}_{b-;g}G(x_1)>0. \quad (3.32)$$

(i.e. $D^{\alpha}_{b-;g}G(x)>0$, $\forall\,x\in[a,b^*]$).
Furthermore

$$\left\|D^{\alpha}_{b-;g}G\right\|_{\infty,[a,b^*]}=D^{\alpha}_{b-;g}G(x_2). \quad (3.33)$$

Here it is

$$J(x) = mx, \; m \neq 0. \tag{3.34}$$

The equation

$$JG(x) = 0, \quad x \in [a, b^*], \tag{3.35}$$

has the same set of solutions as the equation

$$F(x) := \frac{JG(x)}{2D^{\alpha}_{b-;g}G(x_2)} = 0, \quad x \in [a, b^*]. \tag{3.36}$$

Notice that

$$D^{\alpha}_{b-;g}\left(\frac{G(x)}{2D^{\alpha}_{b-;g}G(x_2)}\right) = \frac{D^{\alpha}_{b-;g}G(x)}{2D^{\alpha}_{b-;g}G(x_2)} \leq \frac{1}{2} < 1, \quad \forall\, x \in [a, b^*]. \tag{3.37}$$

We call

$$A(x) := \frac{D^{\alpha}_{b-;g}G(x)}{2D^{\alpha}_{b-;g}G(x_2)}, \quad \forall\, x \in [a, b^*]. \tag{3.38}$$

We notice that

$$0 < \frac{D^{\alpha}_{b-;g}G(x_1)}{2D^{\alpha}_{b-;g}G(x_2)} \leq A(x) \leq \frac{1}{2}. \tag{3.39}$$

Hence it holds

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{D^{\alpha}_{b-;g}G(x_1)}{2D^{\alpha}_{b-;g}G(x_2)} =: \gamma_0, \quad \forall\, x \in [a, b^*]. \tag{3.40}$$

Clearly $\gamma_0 \in (0, 1)$.
We have proved that

$$|1 - A(x)| \leq \gamma_0 \in (0, 1), \quad \forall\, x \in [a, b^*]. \tag{3.41}$$

Next we assume that $F(x)$ is a contraction over $[a, b^*]$, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall\, x, y \in [a, b^*], \tag{3.42}$$

and $0 < \lambda < \frac{1}{2}$.
Equivalently we have

$$|JG(x) - JG(y)| \leq 2\lambda \left(D^{\alpha}_{b-;g}G(x_2)\right) |x - y|, \quad \forall\, x, y \in [a, b^*]. \tag{3.43}$$

We observe that

$$|F(y) - F(x) - A(x)(y - x)| \leq |F(y) - F(x)| + |A(x)||y - x| \leq$$

$$\lambda |y - x| + |A(x)||y - x| = (\lambda + |A(x)|)|y - x| =: (\xi_2), \quad \forall\, x, y \in [a, b^*]. \tag{3.44}$$

Hence by (3.30), $\forall\, x \in [a, b^*]$ we get that

$$|A(x)| = \frac{\left|D^{\alpha}_{b-;g}G(x)\right|}{2D^{\alpha}_{b-;g}G(x_2)} \leq \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m - \alpha + 1)} \frac{\left\|\left(G \circ g^{-1}\right)^{(m)} \circ g\right\|_{\infty,[a,b]}}{D^{\alpha}_{b-;g}G(x_2)} < \infty. \tag{3.45}$$

Consequently we observe

$$(\xi_2) \leq \left(\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{b-;g} G(x_2)} \right) |y - x|, \tag{3.46}$$

$\forall\, x, y \in [a, b^*]$.
Call

$$0 < \gamma_1 := \lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{b-;g} G(x_2)}, \tag{3.47}$$

choosing $(g(b) - g(a))$ small enough we can make $\gamma_1 \in (0, 1)$.
We proved that

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \quad \text{where } \gamma_1 \in (0,1), \ \forall\, x, y \in [a, b^*]. \tag{3.48}$$

Next we call and need

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{D^{\alpha}_{b-;g} G(x_1)}{2 D^{\alpha}_{b-;g} G(x_2)} + \lambda$$

$$+ \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{b-;g} G(x_2)} < 1, \tag{3.49}$$

equivalently we find,

$$\lambda + \frac{(g(b) - g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]}}{D^{\alpha}_{b-;g} G(x_2)} < \frac{D^{\alpha}_{b-;g} G(x_1)}{2 D^{\alpha}_{b-;g} G(x_2)}, \tag{3.50}$$

equivalently,

$$2\lambda D^{\alpha}_{b-;g} G(x_2) + \frac{(g(b) - g(a))^{m-\alpha}}{\Gamma(m-\alpha+1)} \left\| \left(G \circ g^{-1}\right)^{(m)} \circ g \right\|_{\infty,[a,b]} < D^{\alpha}_{b-;g} G(x_1), \tag{3.51}$$

which is possible for small λ, $(g(b) - g(a))$.
That is $\gamma \in (0, 1)$. Hence equation (3.35) can be solved with our presented iterative algorithms.

Conclusion 3.3. (for (II))

Our presented earlier semilocal iterative methods, see Theorem 2.7, can apply in the above generalized fractional setting for $g(x) = x$ for each $x \in [a, b]$ since the following inequalities have been fulfilled:

$$\|1 - A\|_{\infty} \leq \gamma_0, \tag{3.52}$$

and

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \tag{3.53}$$

where $\gamma_0, \gamma_1 \in (0, 1)$, furthermore it holds

$$\gamma = \gamma_0 + \gamma_1 \in (0, 1), \tag{3.54}$$

for all $x, y \in [a, b^*]$, where $a < b^* < b$.

The specific functions $A(x)$, $F(x)$ have been described above, see (3.38) and (3.36), respectively.

References

[1] Amat, S., Busquier, S., Plaza, S., *Chaotic dynamics of a third-order Newton-type method*, J. Math. Anal. Appl., **366**(2010), no. 1, 164-174.

[2] Anastassiou, G., *Fractional Differentiation Inequalities*, Springer, New York, 2009.

[3] Anastassiou, G., *Fractional representation formulae and right fractional inequalities*, Mathematical and Computer Modelling, **54**(2011), no. 11-12, 3098-3115.

[4] Anastassiou, G., *Advanced fractional Taylor's formula*, J. Comput. Anal. Appl., **21**(7)(2016), 1185-1204.

[5] Anastassiou, G., Argyros, I.K., *Convergence for iterative methods on Banach spaces of convergence structure with applications in fractional calculus*, SeMA, DOI 10.1007/s40324-015-0044-y.

[6] Argyros, I.K., *Newton-like methods in partially ordered linear spaces*, J. Approx. Th. Appl., **9**(1993), no. 1, 1-10.

[7] Argyros, I.K., *Results on controlling the residuals of perturbed Newton-like methods on Banach spaces with a convergence structure*, Southwest J. Pure Appl. Math., **1**(1995), 32-38.

[8] Argyros, I.K., *Convergence and Applications of Newton-type iterations*, Springer-Verlag, New York, 2008.

[9] Ezquerro, J.A., Gutierrez, J.M., Hernandez, M.A., Romero, N., Rubio, M.J., *The Newton method: From Newton to Kantorovich* (Spanish), Gac. R. Soc. Mat. Esp., **13**(2010), 53-76.

[10] Magrenan, A.A., *Different anomalies in a Surrutt family of iterative root finding methods*, Appl. Math. Comput., **233**(2014), 29-38.

[11] Magrenan, A.A., *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput., **248**(2014), 215-224.

[12] Meyer, P.W., *Newton's method in generalized Banach spaces*, Numer. Func. Anal. Optimiz., **9**(1987), no. 3 and 4, 244-259.

[13] Mukeen, S., Habibullah, G.M., *k-Fractional integrals and Application*, Int. J. Contemp. Math. Sciences, **7**(2012), no. 2, 89-94.

[14] Potra, F.A., Ptak, V., *Nondiscrete induction and iterative processes*, Pitman Publ., London, 1984.

[15] Zekisarikaya, M., Karaca, A., *On the Riemann-Liouville fractional integral and applications*, Intern. J. Stat. and Math., **1**(2014), no. 3, 33-43.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 27–36

On a certain subclass of analytic univalent function defined by using Komatu integral operator

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Abstract. In this paper a certain class of analytic univalent functions in the open unit disk is defined. Some interesting results including inclusion relations argument properties and the effect of the certain integral operator to the elements of this class are investigated.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic function, Komatu operator, strongly starlike function, strongly convex function.

1. Introduction

Let A denote the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disk $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Also let S denotes the subclass of A consisting of univalent functions in U. A function $f \in A$ is said to be starlike of order γ $(0 \leq \gamma < 1)$ in U if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \gamma.$$

We denote by $S^*(\gamma)$, the class of all such functions. A function $f \in A$ is said to be convex of order γ $(0 \leq \gamma < 1)$ in U if

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma.$$

Let $K(\gamma)$ denote the class of all those functions $f \in A$ which are convex of order γ in U. We have

$$f \in K(\gamma) \quad \text{if and only if} \quad zf'(z) \in S^*(\gamma).$$

Recently, Komatu [4] has introduced a certain integral operator L_a^λ $(a > 0,\ \lambda > 0)$

$$L_a^\lambda f(z) = \frac{a^\lambda}{\Gamma(\lambda)} \int_0^1 t^{a-2} \left(\log \frac{1}{t}\right)^{\lambda-1} f(zt)dt, \quad z \in U,\ a > 0,\ \lambda > 0. \tag{1.1}$$

Thus, if $f \in A$ is of the form $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, it is easily seen form (1.1) that

$$L_a^\lambda f(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1}\right)^\lambda a_n z^n, \quad a > 0,\ \lambda > 0.$$

According to the above series expansion for L_a^λ one can define L_a^λ for all real λ. Using the above relation, it is easy to verify that

$$z(L_a^{\lambda+1} f(z))' = aL_a^\lambda f(z) - (a-1)L_a^{\lambda+1} f(z), \quad a > 0,\ \lambda \geq 0. \tag{1.2}$$

We note that

(i) For $a = 1$, $\lambda = k$ (k is any integer number), the multiplier transformation $L_1^\lambda = I^k$, was studied by Flet [2] and Sălăgean [9];

(ii) For $a = 1$, $\lambda = -k$ ($k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$), the differential operator $L_1^{-k} = D^k$, was studied by Sălăgean [9];

(iii) For $a = 2$, $\lambda = k$ (k is any integer number), the operator $L_2^k = L^k$, was studied by Uralegddi and Somantha [10];

(iv) For $a = 2$, the multiplier transformation $L_2^\lambda = I^\lambda$, was studied by Jung et all [3].

If $f \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \eta\right)\right| < \frac{\pi}{2}\beta, \quad z \in U,\ 0 \leq \eta < 1,\ 0 < \beta \leq 1,$$

then f is said to be strongly starlike of order β and type η in U. If $f \in A$ satisfies

$$\left|\arg\left(\frac{1 + zf''(z)}{f'(z)} - \eta\right)\right| < \frac{\pi}{2}\beta, \quad z \in U,\ 0 \leq \eta < 1,\ 0 < \beta \leq 1,$$

then f is said to be strongly convex of order β and type η in U. We denote by $S^*(\beta, \eta)$ and $K(\beta, \eta)$, respectively, the subclasses of a consisting of all strongly starlike and strongly convex of order β and type η in U. We also note that $S^*(1, \eta) = S^*(\eta)$ and $K(1, \eta) = K(\eta)$. We shall use $S^*(\beta)$ and $K(\beta)$ to denote $S^*(\beta, 0)$ and $K(\beta, 0)$, respectively, which are the classes of univalent starlike and univalent convex functions of order β $(0 \leq \beta < 1)$.

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic in U and satisfy the condition $\operatorname{Re} p(z) > 0$. For two functions f and g, analytic in U, we say that the function f is subordinate to g, and write $f(z) \prec g(z)$, if there exists a Schwarz function w in U, such that $f(z) = g(w(z))$.

For $a > 0$, let $S^\lambda(a, \eta, h)$ be the class of functions $f \in A$ satisfying the condition

$$\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda f(z)} - \eta\right) \prec h(z), \quad 0 \leq \eta < 1,\ h \in \mathcal{P}.$$

For simplicity we write

$$S^\lambda\left(a, \eta, \frac{1+Az}{1+Bz}\right) = S^\lambda(a, \eta, A, B), \quad -1 \leq B < A \leq 1.$$

2. Preliminaries

Lemma 2.1. [1] *For $\beta, \gamma \in \mathbb{C}$ let h be convex univalent in U with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \gamma) > 0$, if p is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

implies that $p(z) \prec h(z)$.

Lemma 2.2. [5] *Let h be convex univalent in U and w be analytic in U with $\operatorname{Re} w(z) > 0$. If p is analytic in U and $p(0) = h(0)$, and*

$$p(z) + zw(z)p'(z) \prec h(z),$$

then $p(z) \prec h(z)$.

Lemma 2.3. [7] *Let p be analytic in U with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in U$. Suppose that there exists a point $z_0 \in U$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha, \quad |z| < |z_0|,$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha, \quad 0 < \alpha \leq 1,$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right), \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\alpha,$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right), \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia.$$

Lemma 2.4. [8] *The function*

$$(1-z)^\gamma \equiv \exp(\gamma \log(1-z)), \quad \gamma \neq 0,$$

is univalent if only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.

Lemma 2.5. [6] *Let q be analytic in U and let Θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)),\ \ h(z) = \Theta(q(z)) + Q(z),$$

and suppose that

(1) *Q is starlike; either*

(2) *h is convex;*

(3) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re} \left(\dfrac{\Theta'(q(z)}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)} \right) > 0.$

If p is analytic in U with $p(0) = q(0)$ and $p(U) \subset \mathbb{D}$, and

$$\Theta(p(z)) + zp'(z)\phi(p(z)) \prec \Theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

3. Main results

Theorem 3.1. *$S^{\lambda}(a, \eta, h) \subset S^{\lambda+1}(a, \eta, h)$, where*

$$\operatorname{Re}\ ((1-\eta)h(z) + \eta + (a-1)) > 0.$$

Proof. Suppose that $f \in S^{\lambda}(a, \eta, h)$, set

$$p(z) = \frac{1}{1-\eta} \left(\frac{z(L_a^{\lambda+1} f(z))'}{L_a^{\lambda+1} f(z)} - \eta \right), \quad z \in U,\ 0 \leq \eta < 1,$$

where p is analytic function with $p(0) = 1$. By using the equation

$$z(L_a^{\lambda+1} f(z))' = aL_a^{\lambda} f(z) - (a-1)L_a^{\lambda+1} f(z), \quad a > 0,\ \lambda > 0, \tag{3.1}$$

we have

$$(a-1) + \eta + (1-\eta)p(z) = (a-1) + \frac{z(L_a^{\lambda+1} f(z))'}{L_a^{\lambda+1} f(z)}. \tag{3.2}$$

Hence from (3.1) and (3.2) we have

$$(a-1) + \eta + (1-\eta)p(z) = a\frac{L_a^{\lambda} f(z)}{L_a^{\lambda+1} f(z)}. \tag{3.3}$$

Differentiating logarithmically derivatives in both sides of (3.3) and using (3.1) we have

$$\frac{1}{1-\eta} \left(\frac{z(L_a^{\lambda} f(z))'}{L_a^{\lambda} f(z)} - \eta \right) = \frac{zp'(z)}{(a-1) + \eta + (1-\eta)p(z)} + p(z),\ 0 \leq \eta < 1,\ z \in U.$$

Since $\operatorname{Re}\ ((1-\eta)h(z) + \eta + (a-1)) > 0$, applying Lemma 2.1, it follows that $p(z) \prec h(z)$, that is

$$\frac{1}{1-\eta} \left(\frac{z(L_a^{\lambda+1} f(z))'}{L_a^{\lambda+1} f(z)} - \eta \right) \prec h(z),$$

and $f \in S^{\lambda+1}(a, \eta, h)$. □

Taking $h(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 3.1, we have the following Corollary:

Corollary 3.2. *The inclusion relation* $S^{\lambda}(a,\eta,A,B) \subset S^{\lambda+1}(a,\eta,A,B)$ *holds for any* $a > 0$.

Letting $a = 1$, $\lambda = 0$ and $h(z) = (\frac{1+z}{1-z})^{\beta}$ in Theorem 3.1 and using $S^{\lambda-1}(a,\eta,h) \subset S^{\lambda}(a,\eta,h)$ we have the following inclusion relation:

Corollary 3.3. $K(\beta,\eta) \subset S^{*}(\beta,\eta)$.

Theorem 3.4. *Let* $0 < \rho < 1$, $\gamma \neq 1$ *and* $a \ge 1$ *be a real number satisfying either* $|2a\gamma\rho - 1| \le 1$ *or* $|2a\gamma\rho + 1| \le 1$. *If* $f \in A$ *satisfies the condition*

$$\mathrm{Re}\left(1 + \frac{L_a^{\lambda} f(z)}{L_a^{\lambda+1} f(z)}\right) > 1 - \rho, \quad z \in U, \tag{3.4}$$

then

$$\left(z^{a-1} L_a^{\lambda+1} f(z)\right)^{\gamma} \prec q_1(z) = \frac{1}{(1-z)^{2a\gamma\rho}},$$

where q_1 *is the best dominant.*

Proof. Denoting $p(z) = (z^{a-1}L_a^{\lambda+1}f(z))^{\gamma}$, it follows that

$$\frac{zp'(z)}{p(z)} = \gamma a \frac{L_a^{\lambda} f(z)}{L_a^{\lambda+1} f(z)}. \tag{3.5}$$

Combing (3.4) and (3.5), we find that

$$1 + \frac{zp'(z)}{a\gamma p(z)} \prec \frac{1 + (2\rho - 1)z}{1 - z}, \tag{3.6}$$

and if we set $\Theta(w) = 1$, $\phi(w) = \frac{1}{\gamma a w}$, and $q_1(z) = \frac{1}{(1-z)^{2a\gamma\rho}}$, then by the assumption of the theorem and making use of Lemma 2.5, we know that q_1 is univalent in U. It follows that

$$Q(z) = zq_1'(z)\phi(q_1(z)) = \frac{2\rho z}{1 - z},$$

and

$$h(z) = \Theta(q_1(z)) + Q(z) = \frac{1 + (2\rho - 1)z}{1 - z}.$$

If we consider D such that

$$q(U) = \left\{w : |w^{\frac{1}{\xi}} - 1| < |w^{\frac{1}{\xi}}|,\ \xi = 2\gamma\rho a\right\} \subset D,$$

then it is easy to check that the conditions (i) and (ii) of Lemma 2.5 hold true. Thus, the desired result of Theorem 3.4 follows from (3.6). □

Theorem 3.5. *Let* h *be convex univalent function in* U *and*

$$\mathrm{Re}\ (c + \eta + (1 - \eta)h(z)) > 0, \quad z \in U.$$

If $f \in A$ *satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(L_a^{\lambda} f(z))'}{L_a^{\lambda} f(z)} - \eta\right) \prec h(z), \quad 0 \le \eta < 1,$$

then

$$\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda F_c(f)(z))'}{L_a^\lambda F_c(f)(z)}-\eta\right)\prec h(z), \quad 0\le\eta<1,$$

where F_c is the integral operator defined by

$$F_c(f)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{3.7}$$

Proof. From (3.7), we have

$$z(L_a^\lambda F_c(f)(z))'=(c+1)L_a^\lambda f(z)-cL_a^\lambda F_c(f)(z). \tag{3.8}$$

Let

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda F_c(f)(z))'}{L_a^\lambda F_c(f)(z)}-\eta\right), \tag{3.9}$$

where p is analytic function with $p(0)=1$. Then, using (3.8) we get

$$c+\eta+(1-\eta)p(z)=(c+1)\frac{L_a^\lambda f(z)}{L_a^\lambda F_c(f)(z)}. \tag{3.10}$$

Differentiating logarithmically in both sides of (3.10) and multiplying by z, we have

$$p(z)+\frac{zp'(z)}{c+\eta+(1-\eta)p(z)}=\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda f(z)}-\eta\right).$$

Since Re $(c+\eta+(1-\eta)p(z))>0$ thus by Lemma 2.1, we have

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(L_a^\lambda F_c(f)(z))'}{L_a^\lambda F_c(f)(z)}-\eta\right)\prec h(z).$$

□

Letting $h(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in the Theorem 3.5, we have the following Corollary.

Corollary 3.6. *If $f\in S^\lambda(a,\eta,A,B)$, then $F_c(f)\in S^\lambda(a,\eta,A,B)$, where $F_c(f)$ is the integral operator defined by* (3.7)*.*

Theorem 3.7. *Let $f\in A$, $0<\delta\le 1$, $a\ge 1$ and $0\le\gamma<1$. If*

$$\left|\arg\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

for some $g\in S^\lambda(a,\eta,A,B)$. Then

$$\left|\arg\left(\frac{z(L_a^{\lambda+1} f(z))'}{L_a^{\lambda+1} g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha.$$

where α $(0<\alpha\le 1)$ is the solution of the equation

$$\delta=\begin{cases}\alpha+\frac{2}{\pi}+\dfrac{\alpha\cos\frac{\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B}+\eta+(a-1)+\alpha\sin\frac{\pi}{2}t_1}, & B\neq-1,\\ \alpha, & B=-1,\end{cases} \tag{3.11}$$

and

$$t_1 = \frac{2}{\pi} \arcsin\left(\frac{(1-\eta)(A-B)}{(1-\eta)(1-AB)+(\eta+a-1)(1-B^2)}\right). \tag{3.12}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(L_a^{\lambda+1}f(z))'}{L_a^{\lambda+1}g(z)} - \gamma\right).$$

Using (1.2), it is easy to see that

$$((1-\gamma)p(z)+\gamma)\, L_a^{\lambda+1}g(z) = aL_a^{\lambda}f(z) - (a-1)L_a^{\lambda+1}f(z). \tag{3.13}$$

Differentiating (3.13) and multiplying by z, we obtain

$$\begin{aligned}(1-\gamma)zp'(z)L_a^{\lambda+1}g(z) &+ ((1-\gamma)p(z)+\gamma)\, z(L_a^{\lambda+1}g(z))' \\ &= az(L_a^{\lambda}f(z))' - (a-1)z(L_a^{\lambda+1}f(z))'.\end{aligned} \tag{3.14}$$

Since $g \in S^{\lambda}(a,\eta,A,B)$, by Theorem 3.1, we have $g \in S^{\lambda+1}(a,\eta,A,B)$. Let

$$q(z) = \frac{1}{1-\gamma}\left(\frac{z(L_a^{\lambda+1}g(z))'}{L_a^{\lambda+1}g(z)} - \eta\right).$$

Then by using (1.2) once again, we have

$$q(z)(1-\eta)+\eta+(a-1) = a\frac{L_a^{\lambda}g(z))'}{L_a^{\lambda+1}g(z)}. \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$\frac{zp'(z)}{q(z)(1-\eta)+\eta+(a-1)} + p(z) = \frac{1}{1-\gamma}\left(\frac{z(L_a^{\lambda+1}f(z))'}{L_a^{\lambda+1}g(z)} - \eta\right).$$

Since $q(z) \prec \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, we have

$$\left|q(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}, \quad z \in U,\ B \ne -1, \tag{3.16}$$

and

$$\frac{1-A}{2} \le \mathrm{Re}\ q(z), \quad z \in U,\ B \ne -1. \tag{3.17}$$

Therefore, from (3.16) and (3.17), for $B \ne -1$, we obtain

$$\left|q(z)(1-\eta)+\eta+(a-1) - \frac{(1-\eta)(1-AB)}{1-B^2} - \eta - (a-1)\right| < \frac{(1-\eta)(A-B)}{1-B^2}.$$

For $B \ne -1$, we have

$$\mathrm{Re}\ (q(z)(1-\eta)+\eta+(a-1)) > \frac{(1-\eta)(1-A)}{2} + \eta + (a-1).$$

Let

$$q(z)(1-\eta)+\eta+(a-1) = r\exp\left(i\frac{\Phi}{2}\right),$$

where

$$\frac{(1-\eta)(1-A)}{1-B} + \eta + (a-1) < r < \frac{(1-\eta)(1+A)}{1+B} + \eta + (a-1), \quad -t_1 < \Phi < t_1,$$

and t_1 is given by (3.12), and

$$\frac{(1-\eta)(1-A)}{2}+\eta+(a-1)<r<\infty.$$

We note that p is analytic in U with $p(0)=1$, so by applying the assumption and Lemma 2.2 with

$$w(z)=\frac{1}{q(z)(1-\eta)+\eta+(a-1)},$$

we have Re $w(z)>0$. Set

$$Q(z)=\frac{1}{1-\gamma}\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda g(z)}-\gamma\right),\quad 0\le\gamma<1.$$

At first, suppose that $p(z_0)^{\frac{1}{\alpha}}=ia(a>0)$. For $B\neq -1$ we have

$$\begin{aligned}
\arg\ Q(z_0) &= \arg\left(\frac{z_0p'(z_0)}{q(z_0)(1-\eta)+\eta+(a-1)}+p(z_0)\right)\\
&= \frac{\pi}{2}\alpha+\arg\left(1+ik\alpha\left(r\exp\left(\frac{i\pi}{2}\Phi\right)\right)^{-1}\right)\\
&= \frac{\pi}{2}\alpha+\arg\left(1+\frac{ik\alpha}{r}\left(\exp\left(\frac{-i\pi}{2}\Phi\right)\right)\right)\\
&\ge \frac{\pi}{2}\alpha+\arctan\left(\frac{k\alpha\sin\frac{\pi}{2}(1-\Phi)}{r+k\alpha\cos\frac{i\pi}{2}(1-\Phi)}\right)\\
&\ge \frac{\pi}{2}\alpha+\arctan\left(\frac{\alpha\cos\frac{\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B}+\eta+(a-1)+\alpha\sin\frac{\pi}{2}t_1}\right)\\
&= \frac{\pi}{2}\delta,
\end{aligned}$$

where δ and t_1 are given by (3.11) and (3.12), respectively.

Similarly, for the case $B=-1$, we have

$$\arg\ Q(z)=\arg\left(\frac{z_0p'(z_0)}{q(z_0)(1-\eta)+\eta+(a-1)}+p(z_0)\right)\ge\frac{\pi}{2}\alpha.$$

These results obviously contradict the assumption.
Next, suppose that $p(z_0)^{\frac{1}{\alpha}}=-ia$ $(a>0)$, $B=-1$ and $z_0\in U$. Applying the same

method we have

$$\begin{aligned}\arg\ Q(z_0) &= \arg\left(\frac{z_0p'(z_0)}{q(z_0)(1-\eta)+\eta+(a-1)}+p(z_0)\right)\\ &= \frac{-\pi}{2}\alpha+\arg\left(1-ik\alpha\left(r\exp\left(\frac{i\pi}{2}\Phi\right)\right)^{-1}\right)\\ &\leq \frac{-\pi}{2}\alpha-\arctan\left(\frac{k\alpha\sin\frac{\pi}{2}(1-\Phi)}{r+k\alpha\cos\frac{i\pi}{2}(1-\Phi)}\right)\\ &\leq \frac{-\pi}{2}\alpha-\arctan\left(\frac{\alpha\cos\frac{i\pi}{2}t_1}{\frac{(1-\eta)(1+A)}{1+B}+\eta+(a-1)+\alpha\sin\frac{\pi}{2}t_1}\right)\\ &= \frac{-\pi}{2}\delta,\end{aligned}$$

where δ and t_1 are given by (3.11) and (3.12) respectively.
Similarly, for the case $B=-1$, we have

$$\arg\ Q(z)=\arg\left(\frac{z_0p'(z_0)}{q(z_0)(1-\eta)+\eta+(a-1)}+p(z_0)\right)\leq\frac{-\pi}{2}\alpha,$$

which contradicts the assumption of Theorem 3.7. Therefore, the proof of Theorem 3.7 is completed. □

Theorem 3.8. *Let $f\in A$, $0<\delta\leq 1$, $a\geq 1$, $0\leq\gamma<1$ and* $\mathrm{Re}\ (c+\eta(1-\eta)h(z))>0$. *If*

$$\left|\arg\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

for some $g\in S^\lambda(a,\eta,A,B)$. Then

$$\left|\arg\left(\frac{z(L_a^{\lambda+1}F_c(f)(z))'}{L_a^{\lambda+1}F_c(g)(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where F_c is defined by (3.8)*, and α $(0<\alpha\leq 1)$ is the solution of the equation given by* (3.11)*.*

Proof. Let

$$p(z)=\frac{1}{1-\gamma}\left(\frac{z(L_a^\lambda F_c(f)(z))'}{L_a^\lambda F_c(g)(z)}-\gamma\right).$$

Since $g\in S^\lambda(a,\eta,A,B)$, so Theorem 3.5 implies that $F_c(g)\in S^\lambda(a,\eta,A,B)$. Using (3.9) we have

$$((1-\gamma)p(z)+\gamma)L_a^\lambda F_c(g)(z)=z(L_a^\lambda F_c(f)(z))'.$$

Now, by a simple calculation, we get

$$(1-\gamma)zp(z)'+((1-\gamma)p(z)+\gamma)\left(c+\eta+(1-\eta)q(z)\right)=(c+1)\frac{z(L_a^\lambda f(z))'}{L_a^\lambda F_c(g)(z)},$$

where

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(L_a^\lambda F_c(g)(z))'}{L_a^\lambda F_c(g)(z)} - \eta\right).$$

Hence we have

$$\frac{1}{1-\gamma}\left(\frac{z(L_a^\lambda f(z))'}{L_a^\lambda g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{q(z)(1-\eta)+\eta+c}.$$

The remaining part of the proof in Theorem 3.8 is similar to that Theorem 3.7 and so we omit it. □

References

[1] Eenigenburg, P., Miller, S.S., Read, M.O., *On a Briot-Bouquet differetial subordination*, Gen. Inequal., **13**(1983), no. 3, 339-348.

[2] Flett, T.M., *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl., **38**(1972), 746-765.

[3] Jung, I.B., Kim, C., Srivastava, H.M., *The Hardy space of analytic functions associated with certain one-parameter families of integral operator*, J. Math. Anal. Appl., **176**(1993), 138-147.

[4] Komatu, Y., *On analytic prolongation of family of operators*, Mathematica (Cluj), **55**(1990), no. 32, 141-145.

[5] Miller, S.S., Mocanu, P.T., *Differential subordintions and univalent functions*, Michigan Math. J., **32**(1985), 157-172.

[6] Miller, S.S., Mocanu, P.T., *On some classes of first order differential subordinations*, Michigan Math. J., **32**(1985), 185-195.

[7] Nunokawa, M., *On the order of strangly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci., **69**(1993), 234-237.

[8] Robertson, M.S., *Certian class of starlike function*, Michigan Math. J., **32**(1985), 135-140.

[9] Sălăgean, G.S., *Subclasses of univalent functions*, Springer-Verlag Heideberg, Lecture Notes in Math., **1013**(1983), 362-372.

[10] Uralegadi, B.A., Somantha, C., *Certian classes of univalent function, In current topic in analytic function theory*, H.M. Srivastava and S. Owa (eds.), World Scientific, Singapore., 1992, 371-374.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 37–44

Faber polynomial coefficient bounds for a subclass of bi-univalent functions

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Abstract. In this work, considering a general subclass of bi-univalent functions and using the Faber polynomials, we obtain coefficient expansions for functions in this class. In certain cases, our estimates improve some of those existing coefficient bounds.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic and univalent functions, bi-univalent functions, Faber polynomials.

1. Introduction

Let A denote the class of functions f which are analytic in the open unit disk $U = \{z : |z| < 1\}$ with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let S be the subclass of A consisting of the form (1.1) which are also univalent in U and let P be the class of functions $\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n$ that are analytic in U and satisfy the condition $\mathrm{Re}(\varphi(z)) > 0$ in U. By the Caratheodory's lemma (e.g., see [11]) we have $|\varphi_n| \leq 2$.

The Koebe one-quarter theorem [11] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w \ - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. For a brief history and interesting examples in the class Σ, see [27].

Lewin [20] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Netanyahu [22] showed that $max\,|a_2| = \frac{4}{3}$ if $f \in \Sigma$. Subsequently, Brannan and Clunie [7] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [8] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([5], [10], [13], [18], [19], [21], [24], [27], [28], [29]).

The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in geometric function theory. Grunsky [14] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalency of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [25] gave a differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (Schaeffer-Spencer [26]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([6], [9], [15], [16], [17]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \backslash \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, ...\}$) is still an open problem.

For $f(z)$ and $F(z)$ analytic in U, we say that f subordinate to F, written $f \prec F$, if there exists a Schwarz function $u(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ with $|u(z)| < 1$ in U, such that $f(z) = F(u(z))$. For the Schwarz function $u(z)$ we note that $|c_n| < 1$. (e.g. see Duren [11]).

A function $f \in \Sigma$ is said to be $B_\Sigma(\mu, \lambda, \varphi)$, $\lambda \geq 1$ and $\mu \geq 0$, if the following subordination hold

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \varphi(z) \tag{1.2}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec \varphi(w) \tag{1.3}$$

where $g(w) = f^{-1}(w)$.

In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $B_\Sigma(\mu, \lambda, \varphi)$ as well as providing estimates for the initial coefficients of these functions.

2. Main results

Using the Faber polynomial expansion of functions $f \in A$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [3],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)\, w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!\,(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!\,(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!\,(n-5)!} a_2^{n-5} \left[a_5 + (-n+2)\, a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!\,(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5)\, a_3 a_4\right] \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{2.1}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ [4]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2} K_1^{-2} &= -a_2, \\ \frac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4} K_3^{-4} &= -\left(5a_2^3 - 5a_2 a_3 + a_4\right). \end{aligned} \tag{2.2}$$

In general, for any $p \in \mathbb{N}$, an expansion of K_n^p is as, [3],

$$K_n^p = p a_n + \frac{p(p-1)}{2} E_n^2 + \frac{p!}{(p-3)!3!} E_n^3 + ... + \frac{p!}{(p-n)!n!} E_n^n, \tag{2.3}$$

where $E_n^p = E_n^p(a_2, a_3, ...)$ and by [1],

$$E_n^m(a_1, a_2, ..., a_n) = \sum_{m=1}^{\infty} \frac{m!\,(a_1)^{\mu_1} ... (a_n)^{\mu_n}}{\mu_1! ... \mu_n!}, \tag{2.4}$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$\begin{aligned} \mu_1 + \mu_2 + \ ... \ + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \ ... \ + n\mu_n &= n. \end{aligned} \tag{2.5}$$

Evidently, $E_n^n(a_1, a_2, ..., a_n) = a_1^n$, [2].

Theorem 2.1. *For* $\lambda \geq 1$ *and* $\mu \geq 0$, *let* $f \in \mathrm{B}_{\Sigma}(\mu, \lambda, \varphi)$. *If* $a_m = 0$; $2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{2}{\mu + (n-1)\lambda}; \qquad n \geq 4 \tag{2.6}$$

Proof. Let functions f given by (1.1). We have

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = 1 + \sum_{n=2}^{\infty} F_{n-1}\left(a_2, a_3, ..., a_n\right) a_n z^{n-1}, \tag{2.7}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = 1 + \sum_{n=1}^{\infty} F_{n-1}\left(A_2, A_3, ..., A_n\right) a_n w^{n-1}$$

where

$$\begin{aligned} F_1 &= (\mu+\lambda)a_2, \\ F_2 &= (\mu+2\lambda)\left[\frac{\mu-1}{2}a_2^2 + a_3\right], \\ F_3 &= (\mu+3\lambda)\left[\frac{(\mu-1)(\mu-2)}{3!}a_2^3 + (\mu-1)\,a_2a_3 + a_4\right], \end{aligned} \tag{2.8}$$

In general, (see [9]).

On the other hand, the inequalities (1.2) and (1.3) imply the existence of two positive real part functions $u(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $v(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$ where $\mathrm{Re}\,u(z) > 0$ and $\mathrm{Re}\,v(w) > 0$ in P so that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = \varphi(u(z)) \tag{2.9}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = \varphi(v(w)) \tag{2.10}$$

where

$$\varphi(u(z)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \varphi_k E_n^k\left(c_1, c_2, ..., c_n\right) z^n, \tag{2.11}$$

and

$$\varphi(v(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \varphi_k E_n^k\left(d_1, d_2, ..., d_n\right) w^n. \tag{2.12}$$

Comparing the corresponding coefficients of (2.9) and (2.11) yields

$$[\mu + (n-1)\lambda]\, a_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^k\left(c_1, c_2, ..., c_{n-1}\right), \; n \geq 2 \tag{2.13}$$

and similarly, from (2.10) and (2.12) we obtain

$$[\mu + (n-1)\lambda]\, b_n = \sum_{k=1}^{n-1} \varphi_k E_{n-1}^{k}\left(d_1, d_2, ..., d_{n-1}\right), \quad n \geq 2. \tag{2.14}$$

Note that for $a_m = 0$; $2 \leq m \leq n-1$ we have $b_n = -a_n$ and so

$$\begin{aligned} [\mu + (n-1)\lambda]\, a_n &= \varphi_1 c_{n-1} \\ -[\mu + (n-1)\lambda]\, a_n &= \varphi_1 d_{n-1} \end{aligned}$$

Now taking the absolute values of either of the above two equations and using the facts that $|\varphi_1| \leq 2$, $|c_{n-1}| \leq 1$and $|d_{n-1}| \leq 1$, we obtain

$$|a_n| \leq \frac{|\varphi_1 c_{n-1}|}{\mu + (n-1)\lambda} = \frac{|\varphi_1 d_{n-1}|}{\mu + (n-1)\lambda} \leq \frac{2}{\mu + (n-1)\lambda}. \tag{2.15}$$

□

Theorem 2.2. *Let* $f \in \mathrm{B}_\Sigma(\mu, \lambda, \varphi)$, $\lambda \geq 1$ *and* $\mu \geq 0$. *Then*

$$\begin{aligned} &(i) \quad |a_2| \leq \min\left\{\frac{2}{\mu+\lambda}, \sqrt{\frac{8}{(\mu+2\lambda)(\mu+1)}}\right\} \\ &(ii) \quad |a_3| \leq \min\left\{\frac{4}{(\mu+\lambda)^2} + \frac{2}{\mu+2\lambda}, \frac{8}{(\mu+2\lambda)(\mu+1)} + \frac{2}{\mu+2\lambda}\right\} \end{aligned} \tag{2.16}$$

Proof. Replacing n by 2 and 3 in (2.13) and (2.14), respectively, we find that

$$(\mu+\lambda)a_2 = \varphi_1 c_1, \tag{2.17}$$

$$(\mu+2\lambda)\left[\frac{\mu-1}{2}a_2^2 + a_3\right] = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{2.18}$$

$$-(\mu++\lambda)a_2 = \varphi_1 d_1, \tag{2.19}$$

$$(\mu+2\lambda)\left[\frac{\mu+3}{2}a_2^2 - a_3\right] = \varphi_1 d_2 + \varphi_2 d_1^2 \tag{2.20}$$

From (2.17) or (2.19) we obtain

$$|a_2| \leq \frac{|\varphi_1 c_1|}{\mu+\lambda} = \frac{|\varphi_1 d_1|}{\mu+\lambda} \leq \frac{2}{\mu+\lambda}. \tag{2.21}$$

Adding (2.18) to (2.20) implies

$$(\mu+2\lambda)(\mu+1)\, a_2^2 = \varphi_1 (c_2 + d_2) + \varphi_2\left(c_1^2 + d_1^2\right)$$

or, equivalently,

$$|a_2| \leq \sqrt{\frac{8}{(\mu+2\lambda)(\mu+1)}}. \tag{2.22}$$

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.20) from (2.18). We thus get

$$2(\mu+2\lambda)\left(a_3 - a_2^2\right) = \varphi_1 (c_2 - d_2) + \varphi_2\left(c_1^2 - d_1^2\right) \tag{2.23}$$

or

$$|a_3| = |a_2|^2 + \frac{|\varphi_1(c_2 - d_2)|}{2(\mu+2\lambda)} \leq |a_2|^2 + \frac{2}{\mu+2\lambda} \tag{2.24}$$

Upon substituting the value of a_2^2 from (2.21) and (2.22) into (2.24), it follows that

$$|a_3| \leq \frac{4}{(\mu+\lambda)^2} + \frac{2}{\mu+2\lambda}$$

and

$$|a_3| \leq \frac{8}{(\mu+2\lambda)(\mu+1)} + \frac{2}{\mu+2\lambda}.$$

□

If we put $\lambda = 1$ in Theorem 2.2, we obtain the following consequence.

Corollary 2.3. *Let* $f \in \mathrm{B}_\Sigma(\mu,\varphi)$, $\mu \geq 0$. *Then*

$$|a_2| \leq \frac{2}{\mu+1}$$

and

$$|a_3| \leq \frac{4}{(\mu+1)^2} + \frac{2}{\mu+2}$$

Remark 2.4. The above estimates for $|a_2|$ and $|a_3|$ show that Corollary 2.3 is an improvement of the estimates given in Prema and Keerthi ([23], Theorem 3.2) and Bulut ([9], Corollary 3).

If we put $\mu = 1$ in Theorem 2.2, we obtain the following consequence.

Corollary 2.5. *Let* $f \in \mathrm{B}_\Sigma(\lambda,\varphi)$, $\lambda \geq 1$. *Then*

$$|a_2| \leq \frac{2}{\lambda+1}$$

and

$$|a_3| \leq \frac{4}{(\lambda+1)^2} + \frac{2}{1+2\lambda}$$

Remark 2.6. The above estimates for $|a_2|$ and $|a_3|$ show that Corollary 2.5 is an improvement of the estimates given in Bulut ([9], Corollary 2).

References

[1] Airault, H., *Symmetric sums associated to the factorization of Grunsky coefficients*, in Conference, Groups and Symmetries, Montreal, Canada, 2007.

[2] Airault, H., *Remarks on Faber polynomials*, Int. Math. Forum, **3**(2008), no. 9, 449-456.

[3] Airault, H., Bouali, H., *Differential calculus on the Faber polynomials*, Bulletin des Sciences Mathematiques, **130**(2006), no. 3, 179-222.

[4] Airault, H., Ren, J., *An algebra of differential operators and generating functions on the set of univalent functions*, Bulletin des Sciences Mathematiques, **126**(2002), no. 5, 343-367.

[5] Altinkaya, Ş., Yalcin, S., *Initial coefficient bounds for a general class of bi-univalent functions*, International Journal of Analysis, Article ID 867871, (2014), 4 pp.

[6] Altinkaya, Ş., Yalcin, S., *Coefficient bounds for a subclass of bi-univalent functions*, TWMS Journal of Pure and Applied Mathematics, **6**(2015), no. 2, 180-185.

[7] Brannan, D.A., Clunie, J., *Aspects of contemporary complex analysis*, Proceedings of the NATO Advanced Study Instute Held at University of Durham, New York: Academic Press, 1979.

[8] Brannan, D.A., Taha, T.S., *On some classes of bi-univalent functions*, Stud. Univ. Babeş-Bolyai Math., **31**(1986), no. 2, 70-77.

[9] Bulut, S., *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions*, C.R. Acad. Sci. Paris, Ser. I, **352**(2014), no. 2, 479-484.

[10] Crişan, O., *Coefficient estimates certain subclasses of bi-univalent functions*, Gen. Math. Notes, **16**(2013), no. 2, 93-1002.

[11] Duren, P.L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, **259**(1983).

[12] Faber, G., *Über polynomische entwickelungen*, Math. Ann., **57**(1903), no. 3, 1569-1573.

[13] Frasin, B.A., Aouf, M.K., *New subclasses of bi-univalent functions,* Applied Mathematics Letters, **24**(2011), 1569-1573.

[14] Grunsky, H., *Koeffizientenbedingungen für schlict abbildende meromorphe funktionen*, Math. Zeit., **45**(1939), 29-61.

[15] Hamidi, S.G., Halim, S.A., Jahangiri, J.M., *Faber polynomial coefficient estimates for meromorphic bi-starlike functions*, International Journal of Mathematics and Mathematical Sciences, 2013, Article ID 498159, 4 p.

[16] Hamidi, S.G., Jahangiri, J.M., *Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C.R. Acad. Sci. Paris, Ser.I, **352**(2014), no. 1, 17-20.

[17] Jahangiri, J.M., Hamidi, S.G., *Coefficient estimates for certain classes of bi-univalent functions*, International Journal of Mathematics and Mathematical Sciences, 2013, Article ID 190560, 4 p.

[18] Keerthi, B.S., Raja, B., *Coefficient inequality for certain new subclasses of analytic bi-univalent functions*, Theoretical Mathematics and Applications, **3**(2013), no. 1, 1-10.

[19] Li, X.F., Wang, A.P., *Two new subclasses of bi-univalent functions*, Internat. Math. Forum, **7**(2012), 1495-1504.

[20] Lewin, M., *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., **18**(1967), 63-68.

[21] Magesh, N., Yamini, J., *Coefficient bounds for a certain subclass of bi-univalent functions*, International Mathematical Forum, **8**(2013), no. 27, 1337-1344.

[22] Netanyahu, E., *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Archive for Rational Mechanics and Analysis, **32**(1969), 100-112.

[23] Prema, S., Keerthi, B.S., *Coefficient bounds for certain subclasses of analytic functions*, J. Math. Anal., **4**(2013), no. 1, 22-27.

[24] Porwal, S., Darus, M., *On a new subclass of bi-univalent functions*, J. Egypt. Math. Soc., **21**(2013), no. 3, 190-193.

[25] Schiffer, M., *A method of variation within the family of simple functions*, Proc. London Math. Soc., **44**(1938), no. 2, 432-449.

[26] Schaeffer, A.C., Spencer, D.C., *The coefficients of schlict functions*, Duke Math. J., **10**(1943), 611-635.

[27] Srivastava, H.M., Mishra, A.K., Gochhayat, P., *Certain subclasses of analytic and bi-univalent functions,* Applied Mathematics Letters, **23**(2010), no. 10, 1188-1192.

[28] Srivastava, H.M., Bulut, S., Çağlar, M., Yağmur, N., *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat, **27**(2013), no. 5, 831-842.

[29] Xu, Q.H., Gui, Y.C., Srivastava, H.M., *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Applied Mathematics Letters, **25**(2012), 990-994.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 45–52

Coefficient inequality for subclass of analytic univalent functions related to simple logistic activation functions

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Abstract. The author investigates the relationship between unified subclasses of analytic univalent functions and simple logistic activation function to determine the initial Taylor series coefficients alongside classical Fekete-Szegő problem.

Mathematics Subject Classification (2010): 30C45, 33E99.

Keywords: Analytic, univalent, sigmoid, logistic activation, Fekete-Szegő, Sălăgean operator, coefficient inequalities.

1. Introduction and preliminaries

The theory of special functions are significantly important to scientist and engineers with mathematical calculations. Though not with any specific definition but its applications extends to physics, computer etc. In the recent time, theory of special function has been overshadowed by other fields such as real analysis, functional analysis, differential equation, algebra and topology.

There are various special functions but we shall concern ourselves with one of the activation function popularly known as sigmoid function or simple logistic function. By activation function, we meant an information process inspired by the same way biological nervous system (such as brain) process information. This composed of large number of highly interconnected processing element, that is neurons, working as a unit to solve or process a specific task. It also learns by examples, can not be programmed to solve a specific task. Sigmoid function (simple logistic activation function) has a gradient descendent learning algorithm, its evaluation could be done in several ways (even by truncated series expansion).

The simple logistic activation function is given as

$$L(z) = \frac{1}{1 + e^{-z}} \tag{1.1}$$

which is differentiable, it outputs real number between 0 and 1, it maps a very large input domain to a small range of outputs, it never loses information because it is one-to-one function and it increases monotonically. It is evidently clear from the aforementioned that sigmoid function is a great tool in geometric function theory.
As usual we denote by A the class of function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ (z \in U) \tag{1.2}$$

which are analytic in the open unit disk $U = \{z : |z| < 1, z \in C\}$ with normalization $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of univalent functions. For two functions f and φ analytic in the open unit disk, we say that f is subordinate to φ written as $f \prec \varphi$ in U or $f(z) \prec \varphi(z)$ if there exist Schwarz function $\omega(z)$ analytic in U with $w(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = \varphi(\omega(z)), z \in U$. It is clear from the Schwarz lemma that $f(z) \prec \varphi(z), (z \in U)$ which implies that $f(0) = \varphi(0)$ and $f(U) \subset \varphi(U)$. Suppose that φ is univalent in U then $f(z) \prec \varphi(z)$ if and only if $f(0) = \varphi(0)$ and $f(U) \subset \varphi(U)$.
Lemma A. [7] *If a function $p \in P$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \ (z \in U)$$

then $|p_k| \leq 2, k \in N$ where P is the family of all functions analytic in U for which $p(0) = 1$ and $Re(p(z)) > 0,\ (z \in U)$.
Let $\phi(z)$ be an analytic univalent function with positive real part in U and $\phi(U)$ be symmetric with respect to the real axis, starlike with respect to $\phi(0) = 1$ and $\Phi'(0) > 0$. Ma and Minda [6] gave unified representation of various subclasses of starlike and convex functions using the classes $S^*(\phi)$ and $C(\phi)$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$ respectively, which includes several well-known classes as special case.
Take for example, if

$$\phi(z) = \frac{1 + Az}{1 + Bz},\ (-1 \leq B < A \leq 1)$$

the class $S^*(\phi)$ reduces to the class $S^*[A, B]$ introduces by Janowski in [4].
In 1933, Fekete and Szegő [3] proved that

$$|a_2^2 - \mu a_3| \leq \begin{cases} 4\mu - 3, & \mu \geq 1 \\ 1 + \exp^{-\frac{2\mu}{1-\mu}}, & 0 \leq \mu \leq 1 \\ 3 - 4\mu & \mu \leq 0 \end{cases}$$

holds for function $f \in S$ and the result is sharp. The problem of finding the sharp bounds for the non-linear functional $|a_3 - \mu a_2^2|$ of any compact family of functions is popularly known as the Fekete-Szegő problem. Several known authors at different time have applied the classical Fekete-Szegő to various classes to obtain various sharp bounds the likes of Keogh and Merkes in 1969 [5] obtained the sharp upper bound of the Fekete-Szegő functional $|a_2^2 - \mu a_3|$ for some subclasses of univalent function S

(see also [1,11,12,14,15]). The Hadamard product (or convolution) of $f(z)$ given by (1.2) and

$$\varphi(z) = z + \sum_{k=2}^{\infty} \varphi_k z^k$$

is defined by

$$(f * \varphi)(z) = z + \sum_{k=2}^{\infty} a_k \varphi_k z^k = (\varphi * f)(z)$$

Therefore, $D^n(f * \varphi)(z) = D(D^{n-1}(f * \varphi)(z)) = z + \sum_{k=2}^{\infty} k^n a_k \varphi_k z^k$ where D^n is the well known Sălăgean derivative operator[13] defined as

$$D^0 f(z) = f(z), \;\; D^1 f(z) = D(f(z)) = zf'(z), \ldots,$$

$$D^n f(z) = D(D^{n-1} f(z)) = z + \sum_{k=2}^{\infty} k^n a_k z^k$$

Recently, Murugusundaramoorthy et al [8] also applied the Hadamard product to discuss a new class of functions denoted by $M_{g,h}(\phi)$ see for detail in [8].
Our major focus in this work is to investigate the simple logistic sigmoid activation function as related to the unified subclass of starlike and convex functions $M_{n,g}^{\alpha,h}(b, \Phi_{k,m})$ to determine the initial Taylor series coefficients and discuss its Fekete-Szegő functional.
For the purpose of our intention we recall the following:
Lemma B. [2] *Let L be a Sigmoid function defined in (1.1) and*

$$\Phi_{k,m} = 2L(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left[\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m \right]^k$$

then $\Phi_{k,m} \in P, |z| < 1$ where $\Phi_{k,m}$ is a modified sigmoid function.
Lemma C. [2] *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left[\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m \right]^k$$

then

$$|\Phi_{k,m}| < 2.$$

Lemma D. [2] *If $\Phi_{k,m} \in P$ is starlike then f is a normalized univalent function of the form (1.2).*
Taking $k = 1$, Joseph-Fadipe et al [2] proved that
Remark A. Let

$$\Phi(z) = 1 + \sum_{m=1}^{\infty} C_m z^m$$

where $C_m = \frac{(-1)(-1)^m}{2m!}$ $|C_m| \leq 2, m = 1, 2, 3, \ldots$ this result is sharp for each m (see also [10]).

Definition 1. For $b \in C$. Let the class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ denote the subclass of A consisting of functions f of the form (1.2), and

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k, \quad h(z) = z + \sum_{k=2}^{\infty} h_k z^k,$$

$$g_k > 0,\ h_k > 0,\ g_k - h_k > 0$$

satisfying the following subordination condition

$$1 + \frac{1}{b}\left[(1-\alpha)\frac{D^n(f*g)(z)}{D^n(f*h)(z)} + \alpha\frac{(D^n(f*g)(z))'}{(D^n(f*h)(z))'} - 1\right] \prec \Phi_{k,m}(z)$$

where $\alpha \geq 0, n \in N_0$, $\Phi_{k,m}$ is a simple logistic sigmoid activation function and D^n is the Sălăgean derivative operator [13].
We state here that we are not assuming $\Phi_{k,m}(U)$ in Definition 1 to be symmetric with respect to the real axis and starlike with respect to $\Phi_{k,m}(0) = 1$. To show that class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ is non empty, let us consider the function $f(z) = \frac{z}{1-z}$. We assume

$$\gamma(z) = 1 + \frac{1}{b}\left[(1-\alpha))\frac{D^n(f*g)(z)}{D^n(f*h)(z)} + \alpha\frac{[D^n(f*g)(z)]'}{[D^n(f*h)(z)]'} - 1\right],$$

we have

$$\gamma(z) = 1 + \frac{2^n}{b}(1-\alpha)(g_2 - h_2)z + \ldots.$$

Clearly $\gamma(0) = 1$ and

$$\gamma'(0) = \frac{2^n}{b}(1-\alpha)(g_2 - h_2) > 0,$$

hence

$$f(z) = \frac{z}{1+z} \in M_{n,h}^{\alpha,g}(b, \Phi_{k,m}).$$

Remark B. With various special choices of functions $g, h, \Phi_{k,m}, b$ and the real number α, the class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ reduces to several known classes and lead to other new classes.
Examples. 1. Suppose $\Phi_{k,m}(z) = \phi(z)$, then the class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ reduces to the class $M_{n,h}^{\alpha,g}(b, \phi)$
2. If $\Phi_{k,m} = \phi, n = 0, \alpha = 0$, the class $M_{0,h}^{0,g}(b, \Phi_{k,m}) = M_{g,h}(b, \phi)$ and if $b = 1$ in Example 2 the class reduces to class $M_{g,h}(\phi)$ studied in [8].
3. Furthermore, if we put $g(z) = \frac{z}{(1-z)^2}, h(z) = \frac{z}{1-z}$ then the class

$$M_{n,h}^{\alpha,g}(b, \Phi_{k,m}) = M_{n,\frac{z}{1-z}}^{\alpha,\frac{z}{(1-z)^2}}(b, \Phi_{k,m}),$$

we can continue to generate many classes with various special choices of the functions and parameters involved.
Suppose we let

$$\Phi_{k,m}(z) = \frac{\sqrt{1 \pm z^2} + z}{\sqrt{1 \pm z^2}},$$

then the class $M_{n,h}^{\alpha,g}(b, \Phi_{k,m})$ becomes $M_{n,h}^{\alpha,g}(b, \frac{\sqrt{1\pm z^2}+z}{\sqrt{1\pm z^2}})$.

2. Main result

Theorem 2.1. *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} [\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m]^k$$

where $\Phi_{k,m} \in A$ is a modified logistic sigmoid activation function and $\Phi'_{k,m}(0) > 0$. If $f(z)$ given by (1.2) belongs to the class $M_{g,h}^{\alpha,n}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$, $k \geq 2$ then

$$|a_2| \leq \frac{b}{2^{n+1}(1+\alpha)(g_2 - h_2)}$$

$$|a_3| \leq \frac{b^2(1+3\alpha)h_2}{2^2 \times 3^n(1+2\alpha)(1+\alpha)^2(g_2-h_2)(g_3-h_3)}$$

$$|a_4| \leq \frac{6^n b^3(h_2g_3 + h_3g_2 - 2h_2h_3)}{2^{3n+3} \cdot 3^n(g_4 - h_4)}$$

$$\times \left[\frac{(1+5\alpha)h_2}{(1+2\alpha)(1+\alpha)^3(g_3-h_3)(g_2-h_2)^2} - \frac{2^n \times 3^{n-1}}{6^n b^2(1+3\alpha)(h_2g_3+h_3g_2-2h_2h_3)} \right].$$

Proof. If $f \in M_{g,h}^{\alpha,n}(b, \Phi_{k,m})$, then

$$1 + \frac{1}{b}\left[(1-\alpha)\frac{D^n(f*g)(z)}{D^n(f*h)(z)} + \alpha\frac{[D^n(f*g)(z)]'}{[D^n(f*h)(z)]'} - 1\right] = \Phi_{k,m}(z) \tag{2.1}$$

A computation shows that

$$\frac{D^n(f*g)(z)}{D^n(f*h)(z)} = 1 + 2^n a_2(g_2-h_2)z + [2^{2n}(h_2^2 - g_2h_2) + 3^n a_3(g_3-h_3)]z^2 + \tag{2.2}$$

$$[4^n a_4(g_4-h_4) + 6^n a_2a_3(2h_2h_3 - h_3g_2 - h_2g_3)]z^3 + \ldots$$

$$\frac{[D^n(f*g)(z)]'}{[D^n(f*h)(z)]'} = 1+2^{n+1}a_2(g_2-h_2)z+[2^{2n+2}(h_2^2-g_2h_2)+3^{n+1}a_3(g_3-h_3)]z^2+ \tag{2.3}$$

$$[4^{n+1}a_4(g_4-h_4) + 6^{n+1}a_2a_3(2h_2h_3 - h_3g_2 - h_2g_3)]z^3 + \ldots$$

and Taylor series expansion of $\Phi_{k,m}$ is given as

$$\Phi_{k,m}(z) = 1 + \frac{1}{2}z - \frac{1}{24z^3} + \frac{1}{240z^5} - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - \ldots \tag{2.4}$$

From (2.1), (2.2), (2.3) and (2.4) we have

$$2^{n+1}(1+\alpha)(g_2-h_2)a_2 = b \tag{2.5}$$

$$3^n(1+2\alpha)(g_3-h_3)a_3 = \frac{b^2(1+3\alpha)h_2}{4(1+\alpha)^2(g_2-h_2)} \tag{2.6}$$

and

$$4^n(1+3\alpha)(g_4-h_4)a_4 = 6^n(1+5\alpha)(h_3g_2+h_2g_3-2h_2h_3)a_2a_3 - \frac{b}{24} \tag{2.7}$$

Equations (2.5), (2.6) and (2.7) give the desired results of Theorem 2.1.

Theorem 2.2. *Let* $\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$ *where* $\Phi_{k,m} \in A$ *is a modified logistic sigmoid activation function and* $\Phi'_{k,m}(0) > 0$. *If* $f(z)$ *given by (1.2) belongs to the class* $M_{g,h}^{\alpha,n}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$ *and* $\mu \in R$, $k \geq 2$ *then*

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{2^2(1+\alpha)^2(g_2 - h_2)(g_3 - h_3)} \left[\frac{(1+3\alpha)h_2}{3^n(1+2\alpha)} - \frac{\mu(g_3 - h_3)}{2^n(g_2 - h_2)}\right]. \tag{2.8}$$

Proof. A simple computation from (2.5) and (2.6) gives the desire result of Theorem 2.2.

Corollary 2.3. *Let* $\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$ *where* $\Phi_{k,m} \in A$ *is a modified logistic sigmoid activation function and* $\Phi'_{k,m}(0) > 0$. *If* $f(z)$ *given by (1.2) belongs to the class* $M_{0,h}^{\alpha,g}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$, $k \geq 2$ *then*

$$|a_2| \leq \frac{b}{2(1+\alpha)(g_2 - h_2)}$$

$$|a_3| \leq \frac{b^2(1+3\alpha)h_2}{4(1+2\alpha)(1+\alpha)^2(g_2 - h_2)(g_3 - h_3)}$$

$$|a_4| \leq \frac{b^3(h_2g_3 + h_3g_2 - 2h_2h_3)}{8(g_4 - h_4)} \left[\frac{(1+5\alpha)h_2}{(1+2\alpha)(1+\alpha)^3(g_3 - h_3)(g_2 - h_2)^2} - \frac{1}{3b^2(1+3\alpha)(h_2g_3 + h_3g_2 - 2h_2h_3)}\right].$$

Corollary 2.4. *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$$

where $\Phi_{k,m} \in A$ *is a modified logistic sigmoid activation function and* $\Phi'_{k,m}(0) > 0$. *If* $f(z)$ *given by (1.2) belongs to the class* $M_{0,h}^{\alpha,0}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$ *and* $\mu \in R$, $k \geq 2$ *then*

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{4(1+\alpha)^2(g_2 - h_2)(g_3 - h_3)} \left[\frac{(1+3\alpha)h_2}{(1+2\alpha)} - \frac{\mu(g_3 - h_3)}{2^n(g_2 - h_2)}\right].$$

Furthermore, suppose we put $\alpha = 0$ in Corollaries 2.3 and 2.4 we have respectively the following

Corollary 2.5. *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$$

where $\Phi_{k,m} \in A$ *is a modified logistic sigmoid activation function and* $\Phi'_{k,m}(0) > 0$. *If* $f(z)$ *given by (1.2) belongs to the class* $M_{0,h}^{0,g}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$, $k \geq 2$ *then*

$$|a_2| \leq \frac{b}{2(g_2 - h_2)}$$

$$|a_3| \leq \frac{b^2h_2}{4(g_2 - h_2)(g_3 - h_3)}$$

$$|a_4| \leq \frac{b^3(h_2g_3 + h_3g_2 - 2h_2h_3)}{8(g_4 - h_4)} \left[\frac{h_2}{(g_3 - h_3)(g_2 - h_2)^2} - \frac{1}{3b^2(h_2g_3 + h_3g_2 - 2h_2h_3)} \right].$$

Corollary 2.6. *Let*

$$\Phi_{k,m}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} (\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m)^k$$

where $\Phi_{k,m} \in A$ *is a modified logistic sigmoid activation function and* $\Phi'_{k,m}(0) > 0$*. If* $f(z)$ *given by (1.2) belongs to the class* $M_{0,h}^{0;g}(b, \Phi_{k,m}), g_k > 0, h_k > 0, g_k - h_k > 0$ *and* $\mu \in R$, $k \geq 2$ *then*

$$|a_3 - \mu a_2^2| \leq \frac{b^2}{4(g_2 - h_2)(g_3 - h_3)} \left[h_2 - \frac{\mu(g_3 - h_3)}{2^n(g_2 - h_2)} \right].$$

Concluding, with various special choices of α, n, b and other parameters involved, many interesting coefficient bounds and Fekete-Szegő inequalities could be obtained.

Acknowledgment. The author wish to thank the referees for their useful suggestions.

References

[1] El-Ashwah, R.M., Aouf, M.K., Hassan, A.H., *Fekete-Szegő problem for a new class of analytic functions with complex order defined by certain differential operator*, Stud. Univ. Babeş-Bolyai Math., **9**(2014), no. 1, 25-36.

[2] Fadipe-Joseph, O.A., Oladipo, A.T., Ezeafulukwe, U.A., *Modified sigmoid function in univalent function theory*, Internat. J. Math. Sci. Engr. Appl., **7**(2013), no. 7, 313-317.

[3] Fekete, M., Szegő, G., *Eine bemerkung uber ungerade schlichte funktionen*, J. Lond. Math. Soc., **8**(1933), 85-89.

[4] Janowski, W., *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Polon. Math., **23**(1970/1971), 159-177.

[5] Keogh, F.R., Merkes, E.P., *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc., **20**(1969), 8-12.

[6] Ma, W.C., Minda, D.A., *A unified treatment of some special classes of univalent functions*, in Proceedings of the Conference on Complex Analysis (Tianjin), 157-169, Conf. Proc. Lecture Notes Anal., I Int. press Cambridge, MA.

[7] Miller, S.S., Mocanu, P.T., *Differential subordinations*, Monographs and Text Books in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[8] Murugusundaramoorthy, G., Kavitha, S., Rosy, T., *On the Fekete-Szegő problem for some subclasses of analytic functions defined by convolution*, Proc. Pakistan Acad. Sci., **44**(2), no. 4, 249-254.

[9] Oladipo, A.T., Fadipe-Joseph, O.A., *Iterated integral transforms of activated sigmoid function as related to Caratheodory family*, Far East J. Appl. Math. (to appear).

[10] Olatunji, S.O., Gbolagade, A.M., Anake, T., Fadipe-Joseph, O.A., *Sigmoid function in the space of Univalent function of Bazilevic type*, Sci. Magna J., **9**(2013), no. 3, 43-51.

[11] Ravichandran, V., Gangadharam, A., Darus, M., *Fekete-Szegő inequality for certain class of Bazilevic functions*, Far East J. Math. Sci., **15**(2004), no. 2, 171-180.

[12] Ravichandran, V., Darus, M., Khan, M.H., Subramanian, K.G., *Fekete-Szegő inequality for certain class of analytic functions*, Aust. J. Math. Anal. Appl., **1**(2004), no. 2, Art. 4, 7 pp.

[13] Sălăgean, G.S., *Subclasses of univalent functions*, Lecture Notes in Math., Springer-Verlag, **1013**(1983), 362-372.

[14] Kumar, S.S., Kumar, V., *Fekete Szegő problem for a class of analytic functions*, Stud. Univ. Babeş-Bolyai Math., **58**(2013), no. 2, 181-188.

[15] Srivastava, H.M., Mishra, A.K., Das, M.K., *The Fekete-Szegő problem for a subclass of close-to-convex functions*, Complex Variables Theory Appl., **44**(2001), no. 2, 145-163.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 53–62

Certain sufficient conditions for parabolic starlike and uniformly close-to-convex functions

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Abstract. In the present paper, we study certain differential subordinations and obtain sufficient conditions for parabolic starlikeness and uniformly close-to-convexity of analytic functions.

Mathematics Subject Classification (2010): 30C80, 30C45.

Keywords: Analytic function, univalent function, parabolic starlike function, uniformly close-to-convex function, differential subordination.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{E} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore, Taylor's series expansion of $f \in \mathcal{A}$, is given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \le |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called a dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant

is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{E}. \tag{1.2}$$

The class of parabolic starlike functions is denoted by $\mathcal{S_P}$. A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|, z \in \mathbb{E}, \tag{1.3}$$

for some $g \in \mathcal{S_P}$. Let UCC denote the class of all such functions. Note that the function $g(z) \equiv z \in \mathcal{S_P}$. Therefore, for $g(z) \equiv z$, condition (1.3) becomes:

$$\Re\left(f'(z)\right) > |f'(z) - 1|, z \in \mathbb{E}. \tag{1.4}$$

Define the parabolic domain Ω as under:

$$\Omega = \{u + iv : u > \sqrt{(u-1)^2 + v^2}\}.$$

Note that the conditions (1.2) and (1.4) are equivalent to the condition that $\dfrac{zf'(z)}{f(z)}$ and $f'(z)$ take values in the parabolic domain Ω respectively.
Ronning [8] and Ma and Minda [4] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \tag{1.5}$$

maps the unit disk $\mathbb{E}$ onto the parabolic domain Ω. Therefore, the condition (1.2) is equivalent to

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \prec q(z), z \in \mathbb{E}, \tag{1.6}$$

and condition (1.4) is same as

$$\Re\left(f'(z)\right) \prec q(z), z \in \mathbb{E}, \tag{1.7}$$

where q(z) is given by (1.5).
It has always been a matter of interest for the researchers to find sufficient conditions for uniformly starlike and close-to-convex functions. The operators $f'(z), \dfrac{zf'(z)}{f(z)}, 1 + \dfrac{zf''(z)}{f'(z)}$ have played an important role in the theory of univalent functions. Various classes involving the combinations of above differential operators have been introduced in literature by different authors. For $f \in \mathcal{A}$, define differential operator $J(\alpha; f)$ as follows:

$$J(\alpha; f)(z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right), \alpha \in \mathbb{R}.$$

In 1973, Miller et al. [5] studied the class $\mathcal{M}_\alpha$ (known as the class of α-convex functions) defined as follows:

$$\mathcal{M}_\alpha = \{f \in \mathcal{A} : \Re[J(\alpha; f)(z)] > 0, z \in \mathbb{E}\}.$$

They proved that if $f \in \mathcal{M}_\alpha$, then f is starlike in $\mathbb{E}$. In 1976, Lewandowski et al. [3] proved that if $f \in \mathcal{A}$ satisfies the condition

$$\Re \frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{E},$$

then f is starlike in $\mathbb{E}$. Further, Silverman [9] defined the class $\mathcal{G}_b$ by taking quotient of operators $1+\dfrac{zf''(z)}{f'(z)}$ and $\dfrac{zf'(z)}{f(z)}$:

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, z \in \mathbb{E} \right\}.$$

The class $\mathcal{G}_b$ had been studied by Tuneski ([7], [12]). For $f \in \mathcal{A}$, define differential operator $I(\alpha; f)$ as follows:

$$I(\alpha; f)(z) = (1-\alpha)f'(z) + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right), \alpha \in \mathbb{R}.$$

Let $\mathcal{H}_\alpha(\beta)$ be the class of normalized analytic functions defined in $\mathbb{E}$ which satisfy the condition

$$\Re[I(\alpha; f)(z)] > \beta, z \in \mathbb{E},$$

where α and β are pre-assigned real numbers. The class $\mathcal{H}_\alpha(0)$ was introduced and studied by Al-Amiri and Reade [1] in 1975. They proved that the members of $\mathcal{H}_\alpha(0)$ are univalent for $\alpha \le 0$. In 2005, Singh et al. [11] studied the class $\mathcal{H}_\alpha(\alpha)$ and proved that the functions in $\mathcal{H}_\alpha(\alpha)$ are univalent for $0 < \alpha < 1$. Recently, the class $\mathcal{H}_\alpha(\beta)$ has been studied by Singh et al. [10]. They established that members of $\mathcal{H}_\alpha(\beta)$ are univalent for $\alpha \le \beta < 1$. In the present paper, we use the technique of differential subordination to study differential operators $I(\alpha; f)(z)$ and $J(\alpha; f)(z)$ and we obtain certain sufficient conditions for uniformly close-to-convex and parabolic starlike functions in terms of differential subordinations involving the operators $I(\alpha; f)(z)$ and $J(\alpha; f)(z)$. To prove our main results, we shall use the following lemma of Miller and Mocanu [6].

Lemma 1.1. *Let q be a univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q(z)$ and suppose that either*
(i) h is convex, or
(ii) Q is starlike.
In addition, assume that
(iii) $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ for all z in $\mathbb{E}$. If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. Main result

Theorem 2.1. *If $f \in \mathcal{A}$, satisfies the differential subordination*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec(1-\alpha)\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+\alpha\left\{1+\frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}, z\in\mathbb{E}, \tag{2.1}$$

for $0<\alpha\leq 1$, *then*

$$f'(z)\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z\in\mathbb{E}\ i.e.\ f\in UCC.$$

Proof. Let us define the function θ and ϕ as follows:

$$\theta(w)=(1-\alpha)w+\alpha$$

and

$$\phi(w)=\frac{\alpha}{w}.$$

Obviously, the function θ and ϕ are analytic in domain $\mathbb{D}=\mathbb{C}\setminus\{0\}$ and $\phi(w)\neq 0$ in $\mathbb{D}$. Define the functions Q and h as follows:

$$Q(z)=zq'(z)\phi(q(z))=\frac{\alpha zq'(z)}{q(z)}$$

and

$$h(z)=\theta(q(z))+Q(z)=(1-\alpha)q(z)+\alpha\left(1+\frac{zq'(z)}{q(z)}\right).$$

Further, select the functions $p(z)=f'(z), f\in\mathcal{A}$ and $q(z)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, we obtain (2.1) reduces to

$$(1-\alpha)p(z)+\alpha\left(1+\frac{zp'(z)}{p(z)}\right)\prec(1-\alpha)q(z)+\alpha\left(1+\frac{zq'(z)}{q(z)}\right)=h(z). \tag{2.2}$$

Now,

$$Q(z)=\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} \tag{2.3}$$

and

$$\frac{zQ'(z)}{Q(z)}=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}-\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}. \tag{2.4}$$

It can easily be verified that $\Re\dfrac{zQ'(z)}{Q(z)} > 0$ in $\mathbb{E}$ and hence Q is starlike in $\mathbb{E}$.
Also we have

$$h(z) = (1-\alpha)\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\} + \alpha\left\{1+\frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} + \left(\frac{1-\alpha}{\alpha}\right)\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}.$$

For $0<\alpha\leq 1$, we have $\Re\dfrac{zh'(z)}{Q(z)} > 0$.
The proof, now, follows from (2.2) by the use of Lemma 1.1. □

Theorem 2.2. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \alpha\left\{\frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}, z\in\mathbb{E}, \tag{2.5}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f\in S_{\mathcal{P}}.$$

Proof. Let us define the function θ and ϕ as follows:

$$\theta(w) = w$$

and

$$\phi(w) = \frac{\alpha}{w}.$$

Obviously, the function θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C}\setminus\{0\}$ and $\phi(w)\neq 0$ in $\mathbb{D}$. Define Q and h as under:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{\alpha zq'(z)}{q(z)}.$$

On writing $p(z) = \dfrac{zf'(z)}{f(z)}$, $f \in \mathcal{A}$ and $q(z) = 1+\dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, (2.5) becomes

$$p(z) + \frac{\alpha z p'(z)}{p(z)} \prec q(z) + \frac{\alpha z q'(z)}{q(z)}. \tag{2.6}$$

Here Q is given by (2.3) and $\frac{zQ'(z)}{Q(z)}$ is given by (2.4). It can easily be verified that $\Re\dfrac{zQ'(z)}{Q(z)} > 0$ in $\mathbb{E}$ and hence Q is starlike in $\mathbb{E}$. Further

$$h(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \alpha\left\{\frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}$$

and therefore, we have

$$\frac{zh'(z)}{Q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$
$$+\left(\frac{1}{\alpha}\right)\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}.$$

Since $\alpha > 0$, therefore, we have $\Re\dfrac{zh'(z)}{Q(z)} > 0$.
Thus, the proof follows from (2.6) by the use of Lemma 1.1. □

3. Deductions

Setting $\alpha = 1$ in Theorem 2.1, we get:

Corollary 3.1. *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf''(z)}{f'(z)} \prec \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z \in \mathbb{E},$$

then

$$f'(z) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in UCC.$$

Writing $\alpha = \frac{1}{2}$ in Theorem 2.1, we obtain:

Corollary 3.2. *Let $f \in \mathcal{A}$ satisfy the differential subordination*

$$f'(z) + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} = F(z), \tag{3.1}$$

then

$$f'(z) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in UCC.$$

Remark 3.3. In 2011, Billing et al. [2] proved the following result:
If $f \in \mathcal{A}$ satisfies the condition

$$\left|f'(z) + \frac{zf''(z)}{f'(z)} - 1\right| < \frac{5}{6}, z \in \mathbb{E}, \tag{3.2}$$

then $f \in UCC$.

Note that, Corollary 3.2 is a particular case of Theorem 2.1 corresponding to the above result (given by (3.2)). For comparison, we plot the image of unit disk under the function F(z) given by (3.1) and this image is given by light shaded portion of Figure 3.1. We notice that, by virtue of Corollary 3.2 the differential operator $f'(z)+\dfrac{zf''(z)}{f'(z)}$ takes values in the whole shaded portion of the Figure 3.1 to conclude that $f \in UCC$, whereas by (3.2) the same operator can take values only in a disk of radius 5/6 centered at 1 (shown by dark portion of Figure 3.1) to conclude the same result. Thus, the region for variability of operator $f'(z) + \dfrac{zf''(z)}{f'(z)}$ is extended largely in Corollary 3.2.

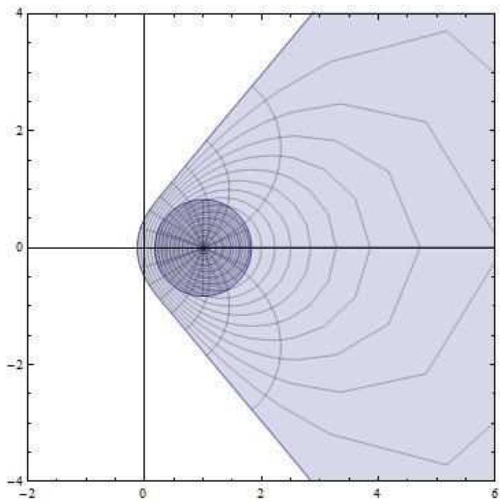

Figure 3.1

Taking $\alpha = 1$ in Theorem 2.2, we have the following result.

Corollary 3.4. *Suppose that $f \in \mathcal{A}$ satisfies*

$$\frac{zf''(z)}{f'(z)} \prec \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} = G(z), \tag{3.3}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \ \ i.e.\ f \in S_{\mathcal{P}}.$$

Remark 3.5. In 2011, Billing et al. [2] also proved the following result which gives the parabolic starlikeness for the functions belonging to the class $\mathcal{A}$:
If $f \in \mathcal{A}$ satisfies the differential inequality

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{5}{6},\ z \in \mathbb{E}, \tag{3.4}$$

then $f \in S_{\mathcal{P}}$.

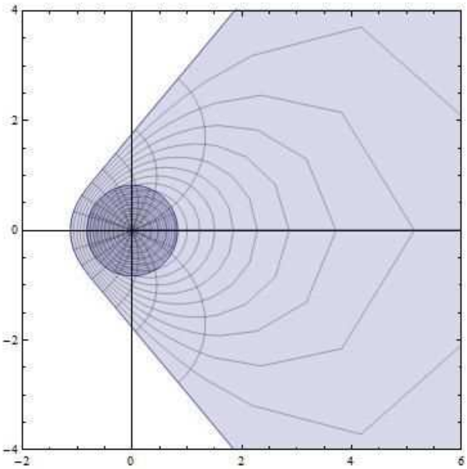

Figure 3.2

Clearly, Corollary 3.4 is a particular case of Theorem 2.2 corresponding to the above result given by (3.4). For comparison, we plot the image of unit disk under the function G(z) given by (3.3) and this image is shown in the light shaded portion of Figure 3.2. In the light of Corollary 3.4, the differential operator $\frac{zf''(z)}{f'(z)}$ takes values in the whole shaded portion of the Figure 3.2 to conclude that $f \in S_{\mathcal{P}}$, but (3.4) indicates that for the same conclusion, operator $\frac{zf''(z)}{f'(z)}$ can take values only in the disk of radius 5/6 centered at origin and this portion is shown by dark portion of Fig 3.2. Thus, the region for variability of operator $\dfrac{zf''(z)}{f'(z)}$ has been extended largely.

On writing $\alpha = \frac{1}{2}$ in Theorem 2.2, we get:

Corollary 3.6. *If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{4}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in S_{\mathcal{P}}.$$

References

[1] Al-Amiri, H.S., Reade, M.O., *On a linear combination of some expressions in the theory of univalent functions*, Monatshefto für Mathematik, **80**(1975), 257-264.

[2] Billing, S.S., Gupta, S., Dhaliwal, S.S., *Certain differential inequalities involving multivalent functions*, Int. J. Open Problems Complex Analysis, **3**(2011), no. 2, 43-49.

[3] Lewandowski, Z., Miller, S.S., Zlotkiewicz, E., *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc., **56**(1976), 111-117.

[4] Ma, W.C., Minda, D., *Uniformly convex functions*, Ann. Polon. Math., **57**(1992), no. 2, 165-175.

[5] Miller, S.S., Mocanu, P.T., Reade, M.O., *All α convex functions are univalent and starlike*, Proc. Amer. Math. Soc., **37**(1973), 553-554.

[6] Miller, S.S., Mocanu, P.T., *Differential subordination: Theory and applications*, Marcel Dekker, New York and Basel, 2000.

[7] Obradović, M., Tuneski, N., *On the starlike criteria defined by Silverman*, Zeszyty Nauk. Politech. Rzeszowskiej.Mat., **181**(2000), no. 24, 59-64.

[8] Ronning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993), no. 1, 189-196.

[9] Silverman, H., *Convex and starlike criteria*, Int. J. Math. and Math. Sci., **22**(1999), no. 1, 75-79.

[10] Singh, S., Gupta, S., Singh, S., *On a problem of univalence of functions satisfying a differential inequality*, Math. Inequal. Appl., **10**(2007), no. 1, 95-98.

[11] Singh, V., Singh, S., Gupta, S., *A problem in the theory of univalent functions*, Integral Transforms and Special Functions, **16**(2005), no. 2, 179-186.

[12] Tuneski, N., *On the quotient of the representations of convexity and starlikeness*, Math. Nachr., **248-249**(2003), no. 1, 200-203.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 63–76

Sufficient conditions for Janowski starlike functions

Kanika Sharma and V. Ravichandran

Abstract. Let p be an analytic function defined on the open unit disc $\mathbb{D}$ with $p(0) = 1$. The conditions on C, D, E, F are derived for $p(z)$ to be subordinate to $(1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$ when $C(z)z^2p''(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0$ or $C(z)p^2(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0$ or $|D(z)zp'(z) + E(z)p(z) + F(z)| < M$, $(M > 0)$, where C, D, E, F are complex-valued functions. Sufficient conditions for confluent (Kummer) hypergeometric function, generalized and normalized Bessel function of the first kind of complex order and integral operator to be subordinate to $(1 + Az)/(1 + Bz)$ are obtained as applications. Few more applications are discussed.

Mathematics Subject Classification (2010): 30C80, 30C45.

Keywords: Convex and starlike functions, differential subordination, confluent hypergeometric function, Bessel function.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For a fixed positive integer n, let $\mathcal{H}[a, n]$ be the subset of $\mathcal{H}$ consisting of functions p of the form $p(z) = a + p_nz^n + p_{n+1}z^{n+1} + \cdots$. Let $\mathcal{A}_n$ denote the class of analytic functions in $\mathbb{D}$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_kz^k,$$

and let $\mathcal{A} := \mathcal{A}_1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent (one-to-one) functions. For $-1 \leq B < A \leq 1$, the class $\mathcal{S}^*[A, B]$ defined by

$$\mathcal{S}^*[A, B] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

is the class of Janowski starlike functions [9]. For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the usual class of starlike functions of order β;

$$\mathcal{S}^*[1 - \beta, 0] := \mathcal{S}^*_\beta = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta\} \text{ and}$$

$$\mathcal{S}^*[\beta, -\beta] := \mathcal{S}^*[\beta] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta|zf'(z)/f(z) + 1|\}.$$

These classes have been studied, for example, in [2, 3, 14, 16]. The class $\mathcal{S}^* := \mathcal{S}^*(0)$ is the class of starlike functions. Recently, the authors have investigated the sufficient conditions for a function to belong to various subclasses of $\mathcal{S}^*[A, B]$ in [20, 19, 15]. A function $f \in \mathcal{A}$ is said to be close-to-convex of order β [13, 8] if $\operatorname{Re}(zf'(z)/g(z)) > \beta$ for some $g \in \mathcal{S}^*$. More results regarding these classes can be found in [7, 10].

In Theorem 2.1 of this paper, we investigate the conditions on C, D, E, F so that

$$C(z)z^2p''(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0$$

implies that $p(z) \prec (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$, where C, D, E, F are complex-valued functions. Miller and Mocanu [11] have obtained the linear integral operators that preserve analytic function with positive real part. We extend this result by investigating the sufficient conditions for integral operator to be subordinate to $(1 + Az)/(1 + Bz)$ by applying Theorem 2.1. We also apply Theorem 2.1 to obtain sufficient conditions for generalized and normalized Bessel function of the first kind of complex order and confluent (Kummer) hypergeometric function to be subordinate to $(1 + Az)/(1 + Bz)$. For $A = 1, B = -1$, all these applications get reduced to some well-known results. As an application, we also get some conditions on functions $f \in \mathcal{A}$, $g \in \mathcal{H}[1, 1]$ so that their product $fg \in \mathcal{S}^*[A, B]$. Section 3 deals with the problem of finding conditions on C, D, E, F so that $C(z)p^2(z)+D(z)zp'(z)+E(z)p(z)+F(z) = 0$ or $|D(z)zp'(z)+E(z)p(z)+F(z)| < M$, $(M > 0)$ implies that $p(z) \prec (1+Az)/(1+Bz)$.

Let Q be the class of functions q that are analytic and injective in $\overline{\mathbb{D}} \setminus R(q)$, where

$$R(q) := \{y \in \partial\mathbb{D} : \lim_{z\to y} q(z) = \infty\},$$

and are such that $q'(y) \neq 0$ for $y \in \partial\mathbb{D} \setminus R(q)$. The following results are required in our investigation.

Lemma 1.1. [13, Theorem 2.2d, p.24] *Let $p \in \mathcal{H}[a, n]$ and $q \in Q$ with $p(z) \not\equiv a$ and $q(0) = a$. If $p \not\prec q$, then there points $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D} \setminus R(q)$ and an $m \geq n \geq 1$ such that $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{D})$,*
(i) $p(z_0) = q(\zeta_0)$,
(ii) $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$,
(iii) $\operatorname{Re}((z_0p''(z_0)/p'(z_0)) + 1) \geq m \operatorname{Re}((z_0q''(z_0)/q'(z_0)) + 1)$.

Lemma 1.2. [13, Theorem 2.3i, p.35] *Let $\Omega \subset \mathbb{C}$ and suppose that $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfies the condition $\psi(i\rho, \sigma, \mu + i\nu; z) \notin \Omega$ whenever ρ, σ, μ and ν are real numbers, $\sigma \leq -n(1 + \rho^2)/2$, $\mu + \sigma \leq 0$. If $p \in \mathcal{H}[1, n]$ and $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$.*

2. Main results

Theorem 2.1. *Let n be a positive integer, $-1 \leq B < A \leq 1, C(z) = C \geq 0$. Suppose that the functions $D, E, F : \mathbb{D} \to \mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z) \geq C$,
(ii) Either $\operatorname{Re} E(z) > 0$ *and* $\operatorname{Re} F(z) > 0$ *or more generally,*

$$(A - B)(\operatorname{Re} D(z) - C)n + (1 + A)(1 + B) \operatorname{Re} E(z) + (1 + B)^2 \operatorname{Re} F(z) > 0,$$

(iii) $((AB-1)\operatorname{Im}E(z)-(B^2-1)\operatorname{Im}F(z))^2$

$$< \big((A-B)(\operatorname{Re}D(z)-C)n+(1+A)(1+B)\operatorname{Re}E(z)+(1+B)^2\operatorname{Re}F(z)\big)$$
$$\big((A-B)(\operatorname{Re}D(z)-C)n-(1-A)(1-B)\operatorname{Re}E(z)-(1-B)^2\operatorname{Re}F(z)\big).$$

If $p\in\mathcal{H}[1,n]$, $(1+B)p(z)\neq(1+A)$ *and satisfy*

$$Cz^2p''(z)+D(z)zp'(z)+E(z)p(z)+F(z)=0, \tag{2.1}$$

then $p(z)\prec(1+Az)/(1+Bz)$.

Proof. For $p\in\mathcal{H}[1,n]$, define the function $q:\mathbb{D}\to\mathbb{C}$ by

$$q(z)=\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)}. \tag{2.2}$$

Then q is analytic in $\mathbb{D}$ and $q(0)=1$. A simple computation shows that

$$p(z)=\frac{(1-A)+(1+A)q(z)}{(1-B)+(1+B)q(z)}, \tag{2.3}$$

$$p'(z)=\frac{2(A-B)q'(z)}{((1-B)+(1+B)q(z))^2} \tag{2.4}$$

and

$$p''(z)=\frac{2(A-B)((1-B)+(1+B)q(z))q''(z)-4(A-B)(1+B)(q'(z))^2}{((1-B)+(1+B)q(z))^3}. \tag{2.5}$$

Using (2.3), (2.4) and (2.5) in (2.1), a calculation shows that q satisfies the following equation

$$\begin{aligned} Cz^2q''(z)-&\frac{2C(1+B)}{(1-B)+(1+B)q(z)}(zq'(z))^2+D(z)zq'(z)\\ &+\frac{E(z)((1-A)+(1+A)q(z))((1-B)+(1+B)q(z))}{2(A-B)}\\ &+\frac{F(z)((1-B)+(1+B)q(z))^2}{2(A-B)}=0. \end{aligned} \tag{2.6}$$

Let $\psi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ be defined by

$$\begin{aligned} \psi(r,s,t;z)=Ct-&\frac{2C(1+B)}{(1-B)+(1+B)r}s^2+D(z)s\\ &+\frac{E(z)((1-A)+(1+A)r)((1-B)+(1+B)r)}{2(A-B)}\\ &+\frac{F(z)((1-B)+(1+B)r)^2}{2(A-B)}. \end{aligned} \tag{2.7}$$

Then the condition (2.6) is equivalent to $\psi(q(z),zq'(z),z^2q''(z);z)\in\Omega=\{0\}$.
To show that $\operatorname{Re}q(z)>0$ for $z\in\mathbb{D}$, from Lemma 1.2, it is sufficient to prove that

$\operatorname{Re}\psi(i\rho, \sigma, \mu + i\nu; z) < 0$ in $\mathbb{D}$ for any real ρ, σ, μ and ν satisfying $\sigma \leq -n(1+\rho^2)/2$, $\mu + \sigma \leq 0$. For $z \in \mathbb{D}$, it follows from (2.7) that

$$\begin{aligned}\operatorname{Re}\psi(i\rho, \sigma, \mu + i\nu; z) = C\mu &- \frac{2C(1-B^2)\sigma^2}{(1-B)^2 + (1+B)^2\rho^2} + \sigma \operatorname{Re} D(z) \\ &+ \frac{\operatorname{Re} E(z)\Big((1-A)(1-B) - (1+A)(1+B)\rho^2\Big)}{2(A-B)} \\ &+ \frac{((1-B)^2 - (1+B)^2\rho^2)\operatorname{Re} F(z)}{2(A-B)} \\ &+ \frac{(B^2-1)\rho \operatorname{Im} F(z)}{A-B} + \frac{(AB-1)\rho \operatorname{Im} E(z)}{A-B}.\end{aligned} \tag{2.8}$$

Using conditions $\operatorname{Re} D(z) \geq C \geq 0$, $\mu + \sigma \leq 0$ and $\sigma \leq -n(1+\rho^2)/2$, we get

$$C\mu + \sigma \operatorname{Re} D(z) \leq -C\sigma + \sigma \operatorname{Re} D(z) \leq -n(1+\rho^2)(\operatorname{Re} D(z) - C)/2$$

and

$$-\frac{2C(1-B^2)\sigma^2}{(1-B)^2 + (1+B)^2\rho^2} \leq 0.$$

Thus from (2.8), we have

$$\begin{aligned}\operatorname{Re}\psi(i\rho, \sigma, \mu + i\nu; z) \leq \frac{-n}{2}&(1+\rho^2)(\operatorname{Re} D(z) - C) + \frac{(AB-1)\rho \operatorname{Im} E(z)}{A-B} \\ &+ \frac{\operatorname{Re} E(z)\Big((1-A)(1-B) - (1+A)(1+B)\rho^2\Big)}{2(A-B)} \\ &+ \frac{((1-B)^2 - (1+B)^2\rho^2)\operatorname{Re} F(z)}{2(A-B)} \\ &+ \frac{(B^2-1)\rho \operatorname{Im} F(z)}{A-B} =: a\rho^2 + b\rho + c,\end{aligned}$$

where

$$\begin{aligned}a &= -\frac{1}{2(A-B)}\big((A-B)(\operatorname{Re} D(z) - C)n + (1+A)(1+B)\operatorname{Re} E(z) \\ &\quad + (1+B)^2 \operatorname{Re} F(z)\big), \\ b &= -\frac{1}{2(A-B)}\big(2(AB-1)\operatorname{Im} E(z) - 2(B^2-1)\operatorname{Im} F(z)\big), \\ c &= -\frac{1}{2(A-B)}\big((A-B)(\operatorname{Re} D(z) - C)n - (1-A)(1-B)\operatorname{Re} E(z) \\ &\quad - (1-B)^2 \operatorname{Re} F(z)\big).\end{aligned}$$

In view of the conditions (ii) and (iii) of Theorem 2.1, we see that $a < 0$ and $b^2 - 4ac < 0$ respectively. So, $\operatorname{Re}\psi(i\rho, \sigma, \mu + i\nu; z) < 0$ in $\mathbb{D}$. Hence by Lemma 1.2, we deduce that $\operatorname{Re} q(z) > 0$, that is, by using (2.2), we get

$$\frac{-(1-A) + (1-B)p(z)}{(1+A) - (1+B)p(z)} \prec \frac{1+z}{1-z}.$$

Therefore, there exist an analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)} = \frac{1+w(z)}{1-w(z)}$$

which gives that $p(z) = (1+Aw)/(1+Bw)$ and thus, $p(z) \prec (1+Az)/(1+Bz)$. □

By taking $A = 1$ and $B = -1$ in Theorem 2.1, we get the following result.

Corollary 2.2. *Let n be a positive integer, $C(z) = C \geq 0$. Suppose that the functions $D, E, F : \mathbb{D} \to \mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z) \geq C$,
(ii) $(\operatorname{Im} E(z))^2 < \big((\operatorname{Re} D(z) - C)n\big)\big((\operatorname{Re} D(z) - C)n - 2\operatorname{Re} F(z)\big)$.
If $p \in \mathcal{H}[1, n]$ and satisfy $Cz^2p''(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0$, then

$$\operatorname{Re} p(z) > 0.$$

By taking $C(z) = 0$ and $F(z) = 0$ in Theorem 2.1, we get the following result for first order differential subordination.

Corollary 2.3. *Let n be a positive integer, $-1 \leq B < A \leq 1$. Suppose that the functions $D, E : \mathbb{D} \to \mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z) \geq 0$,
(ii) $\operatorname{Re} E(z) > (-n(A-B)\operatorname{Re} D(z))/((1+A)(1+B))$,
(iii) $((AB-1)\operatorname{Im} E(z))^2 < \big((A-B)n\operatorname{Re} D(z) + (1+A)(1+B)\operatorname{Re} E(z)\big)$

$$\big((A-B)n\operatorname{Re} D(z) - (1-A)(1-B)\operatorname{Re} E(z)\big).$$

If $p \in \mathcal{H}[1, n]$, $(1+B)p(z) \neq (1+A)$ and satisfy $D(z)zp'(z) + E(z)p(z) = 0$, then $p(z) \prec (1+Az)/(1+Bz)$.

Next, we study the confluent (Kummer) hypergeometric function $\Phi(a, c; z)$ given by

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}, \tag{2.9}$$

where $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \cdots$, and $(\lambda)_n$ denotes the Pochhammer symbol given by $(\lambda)_0 = 1$, $(\lambda)_n = \lambda(\lambda+1)_{n-1}$. The function $\Phi \in \mathcal{H}[1, 1]$ is a solution of the differential equation

$$z\Phi''(a, c; z) + (c - z)\Phi'(a, c; z) - a\Phi(a, c; z) = 0 \tag{2.10}$$

introduced by Kummer in 1837 [21]. The function $\Phi(a, c; z)$ satisfies the following recursive relation

$$c\Phi'(a; c; z) = a\Phi(a+1; c+1; z).$$

When $\operatorname{Re} c > \operatorname{Re} a > 0$, Φ can be expressed in the integral form

$$\Phi(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}dt.$$

There has been several works [1, 11, 17, 18] studying geometric properties of the function $\Phi(a;c;z)$, such as on its close-to-convexity, starlikeness and convexity. By the use of Theorem 2.1, we obtain the following sufficient conditions for

$$\Phi(a,c;z) \prec (1+Az)/(1+Bz).$$

Corollary 2.4. *Let n be a positive integer and $-1 \leq B < A \leq 1$.*
If $(1+B)\Phi(a,c;z) \neq (1+A)$ and $a, c \in \mathbb{R}$ satisfy
(i) $(A-B)(c-2)-(1+A)(1+B)|a| > 0$,
(ii) $(a-1)^2B-(1+a)^2A < 0$,
(iii) $a^2(A-B)(AB-1)^2+2a(A+B)(AB-1)^2+(A-B)(AB(AB+4c^2-8c+2)+1) < 0$,
then $\Phi(a;c;z) \prec (1+Az)/(1+Bz)$.

Proof. To begin with, note that in view of (2.10), the function $\Phi(a,c;z)$ satisfies (2.1) with $C(z)=1$, $D(z)=c-z$, $E(z)=-az$ and $F(z)=0$. Since by the given condition (i), $c>2$, we get $\operatorname{Re} D(z) = c-x > C$ for $z \in \mathbb{D}$. The given condition (i) yields

$$\begin{aligned}(A-B)(\operatorname{Re} D(z)-C)n+(1+A)(1+B)\operatorname{Re} E(z)+(1+B)^2\operatorname{Re} F(z)\\ > (A-B)(c-2)-(1+A)(1+B)ax\\ > (A-B)(c-2)-(1+A)(1+B)|a| > 0.\end{aligned}$$

For $z = x+iy \in \mathbb{D}$, we have

$$\begin{aligned}&((AB-1)\operatorname{Im} E(z)-(B^2-1)\operatorname{Im} F(z))^2-\big((A-B)(\operatorname{Re} D(z)-C)n\\ &+(1+A)(1+B)\operatorname{Re} E(z)+(1+B)^2\operatorname{Re} F(z)\big)\big((A-B)\\ &(\operatorname{Re} D(z)-C)n-(1-A)(1-B)\operatorname{Re} E(z)-(1-B)^2\operatorname{Re} F(z)\big)\\ &=(AB-1)^2a^2y^2-((A-B)(c-x-1)-(1+A)(1+B)ax)\\ &((A-B)(c-x-1)+(1-A)(1-B)ax)\\ &<(AB-1)^2a^2(1-x^2)-((A-B)(c-x-1)\\ &-(1+A)(1+B)ax)((A-B)(c-x-1)\\ &+(1-A)(1-B)ax) =: G(x) = px^2+qx+r,\end{aligned}$$

where

$$p=(A-B)\left((a-1)^2B-(a+1)^2A\right),$$
$$q=2(c-1)(A-B)((a+1)A+(a-1)B)$$

and

$$r=a^2(AB-1)^2-(c-1)^2(A-B)^2.$$

Using (ii) and (iii), we get $p<0$ and $q^2-4pr<0$ respectively. So, $G(x)<0$ and thus, all the conditions of the Theorem 2.1 are satisfied.
Hence, $\Phi(a;c;z) \prec (1+Az)/(1+Bz)$. □

Remark 2.5. Taking $A=1$ and $B=-1$ in Corollary 2.4, we get the following well known result:

If $a, c \in \mathbb{R}$ such that $c > 1+\sqrt{1+a^2}$, then $\operatorname{Re}\Phi(a;c;z) > 0$.

Miller and Mocanu [11] have obtained the linear integral operators I such that $I[\mathcal{P}_n] \subseteq \mathcal{P}_n$, where $\mathcal{P}_n = \{f \in \mathcal{H}[1,n] : \operatorname{Re} f(z) > 0 \text{ for } z \in \mathbb{D}\}$. We extend this result by investigating the sufficient conditions for $I[f](z) \prec (1+Az)/(1+Bz)$ for $f \in \mathcal{P}_n$.

Corollary 2.6. *Let $\gamma \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Re}\gamma \geq 0$, n be a positive integer, $-1 \leq B < A \leq 1$. Suppose that ϕ, $\psi \in \mathcal{H}[1,n]$ such that $\phi(z) \neq 0$ and $\psi(z) \neq 0$ for $z \in \mathbb{D}$. Define the integral operator I as*

$$I[f](z) = \frac{\gamma}{z^\gamma \phi(z)} \int_0^z f(t) t^{\gamma-1} \psi(t) dt.$$

If for $f \in \mathcal{P}_n$, the following conditions hold:

$$\operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right) \geq 0, \tag{2.11}$$

$$\begin{aligned} n(A-B)\operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right) + (1+A)(1+B)\operatorname{Re}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\psi(z)}\right) \\ - (1+B)^2 \operatorname{Re} f(z) > 0, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\left((AB-1)\operatorname{Im}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\psi(z)}\right) + (B^2-1)\operatorname{Im} f(z)\right)^2 \\ &< \left(n(A-B)\operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right) + (1+A)(1+B)\operatorname{Re}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\psi(z)}\right)\right. \\ &\quad \left. - (1+B)^2 \operatorname{Re} f(z)\right)\left(n(A-B)\operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right)\right. \\ &\quad \left. - (1-A)(1-B)\operatorname{Re}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\psi(z)}\right) + (1-B)^2 \operatorname{Re} f(z)\right), \end{aligned} \tag{2.13}$$

then $I[f](z) \prec (1+Az)/(1+Bz)$.

Proof. Suppose that the function $f \in \mathcal{P}_n$ satisfy (2.11)– (2.13). Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{\gamma}{z^\gamma \phi(z)} \int_0^z f(t) t^{\gamma-1} \psi(t) dt. \tag{2.14}$$

Result [13, Lemma 1.2c, p. 11] together with some calculations show that p is well defined and $p \in \mathcal{H}[1,n]$. On differentiating (2.14), we see that p satisfies the differential equation

$$D(z) z p'(z) + E(z) p(z) - f(z) = 0,$$

where $D(z) = \phi(z)/\gamma\psi(z)$ and $E(z) = (\gamma\phi(z) + z\phi'(z))/\gamma\psi(z)$. It is easy to verify that (2.11), (2.12) and (2.13) respectively shows that the conditions (i), (ii) and (iii) of Theorem 2.1 are satisfied with $C = 0$, $F(z) = -f(z)$. Therefore, by Theorem 2.1, it follows that $p(z) \prec (1+Az)/(1+Bz)$. □

Taking $A = 1$ and $B = -1$ in Corollary 2.6, we get the following result.

Corollary 2.7. *Let* $\gamma \in \mathbb{C} \setminus \{0\}$ *such that* $\operatorname{Re}\gamma \geq 0$, n *be a positive integer. Suppose that* ϕ, $\psi \in \mathcal{H}[1,n]$ *such that* $\phi(z) \neq 0$ *and* $\psi(z) \neq 0$ *for* $z \in \mathbb{D}$. *Define the integral operator* I *as*

$$I[f](z) = \frac{\gamma}{z^{\gamma}\phi(z)} \int_0^z f(t)t^{\gamma-1}\psi(t)dt.$$

If for $f \in \mathcal{P}_n$, *the following conditions hold:*

$$\left(\operatorname{Im}\left(\frac{\gamma\phi(z) + z\phi'(z)}{\gamma\psi(z)}\right)\right)^2 < \left(n \operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right)\right)\left(n \operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right) + 2\operatorname{Re} f(z)\right), \quad (2.15)$$

then $\operatorname{Re}(I[f](z)) > 0$.

Remark 2.8. [13, Lemma 4.2a, p. 202] proves that if

$$\left|\operatorname{Im}\left(\frac{\gamma\phi(z) + z\phi'(z)}{\gamma\psi(z)}\right)\right| \leq \left(n \operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right)\right),$$

then $I[\mathcal{P}_n] \subset \mathcal{P}_n$. Since for any $f \in \mathcal{P}_n$, we have

$$\left(n \operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right)\right) < \left(n \operatorname{Re}\left(\frac{\phi(z)}{\gamma\psi(z)}\right) + 2\operatorname{Re} f(z)\right).$$

Therefore, Corollary 2.7 can be regarded as one of the particular case of [13, Lemma 4.2a, p. 202].

Next, we study the generalized and normalized Bessel function of the first kind of order p, $u_p(z) = u_{p,b,c}(z)$ given by the power series

$$u_p(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n} \frac{z^n}{n!},$$

where $b, p, c \in \mathbb{C}$ such that $k = p + (b+1)/2$ and $k \neq 0, -1, -2, \cdots$. The function $u_p \in \mathcal{H}[1,1]$ is a solution of the differential equation

$$4z^2 u_p''(z) + 4kzu_p'(z) + czu_p(z) = 0. \quad (2.16)$$

The function $u_p(z)$ also satisfy the following recursive relation

$$4ku_p(z) = cu_{p+1}(z),$$

which is useful for studying its various geometric properties. More results regarding this function can be found in [6, 5, 4]. By the use of Theorem 2.1, we obtain the following sufficient conditions for $u_p(z) \prec (1+Az)/(1+Bz)$.

Corollary 2.9. *Suppose that* $-1 \leq B < A \leq 1$ *and* $(1+B)u_p(z) \neq 1 + A$. *If* $b, p, c \in \mathbb{R}$ *satisfy the following conditions*
(i) $4(A-B)(k-1) - (1+A)(1+B)|c| > 0$,
(ii) $c^2 < AB((2-AB)c^2 - 64(k-1)^2)$,
then $u_p(z) \prec (1+Az)/(1+Bz)$.

Proof. In view of (2.16), the function $u_p(z)$ satisfies (2.1) with $C(z) = 4$, $D(z) = 4k$, $E(z) = cz$ and $F(z) = 0$. Since by the given condition (i), $k > 1$, we get $\operatorname{Re} D(z) = 4k > C$. The given condition (i) yields

$$(A-B)(\operatorname{Re} D(z) - C)n + (1+A)(1+B)\operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z) > 4(A-B)(k-1) - (1+A)(1+B)|c| > 0.$$

For $z = x + iy \in \mathbb{D}$, we have

$$\begin{aligned}
&((AB-1)\operatorname{Im} E(z) - (B^2-1)\operatorname{Im} F(z))^2 - \big((A-B)(\operatorname{Re} D(z) - C)n \\
&+(1+A)(1+B)\operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z)\big)\big((A-B) \\
&(\operatorname{Re} D(z) - C)n - (1-A)(1-B)\operatorname{Re} E(z) - (1-B)^2 \operatorname{Re} F(z)\big) \\
&= (AB-1)^2 c^2 y^2 - (4(A-B)(k-1) + (1+A)(1+B)cx) \\
&(4(A-B)(k-1) - (1-A)(1-B)cx) \\
&< (AB-1)^2 c^2 (1-x^2) - (4(A-B)(k-1) \\
&+(1+A)(1+B)cx)(4(A-B)(k-1) - (1-A)(1-B)cx) \\
&=: H(x) = px^2 + qx + r,
\end{aligned}$$

where

$$p = -(A-B)^2 c^2, \ q = -8c(k-1)\left(A^2 - B^2\right)$$

and

$$r = c^2(AB-1)^2 - 16(k-1)^2(A-B)^2.$$

From the hypothesis, we obtain $p < 0$ and

$$q^2 - 4pr = 4c^2(A-B)^2\left(AB\left(c^2(AB-2) + 64(k-1)^2\right) + c^2\right) < 0.$$

So, $H(x) < 0$. Therefore, by applying Theorem 2.1, we conclude that

$$u_p(z) \prec (1+Az)/(1+Bz).$$ □

Remark 2.10. If $A = 1$ and $B = -1$, then Corollary 2.9 reduces to [4, Theorem 2.2, p. 29]. So, Corollary 2.9 generalises [4, Theorem 2.2, p. 29].

The following result gives the sufficient conditions for functions $h \in \mathcal{A}_n$ to belong to the class of Janowski starlike functions.

Corollary 2.11. *Let n be a positive integer, $-1 \le B < A \le 1, C(z) = C \ge 0$. Suppose that the functions $D, E, F : \mathbb{D} \to \mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z) \ge C$,
(ii) Either $\operatorname{Re} E(z) > 0$ *and* $\operatorname{Re} F(z) > 0$, *or more generally,*

$$(A-B)(\operatorname{Re} D(z) - C)n + (1+A)(1+B)\operatorname{Re} E(z) + (1+B)^2 \operatorname{Re} F(z) > 0,$$

(iii) $((AB-1)\operatorname{Im} E(z) - (B^2-1)\operatorname{Im} F(z))^2$

$$\begin{aligned}
&< \big((A-B)(\operatorname{Re} D(z) - C)n + (1+A)(1+B)\operatorname{Re} E(z) \\
&+(1+B)^2 \operatorname{Re} F(z)\big)\big((A-B)(\operatorname{Re} D(z) - C)n \\
&-(1-A)(1-B)\operatorname{Re} E(z) - (1-B)^2 \operatorname{Re} F(z)\big).
\end{aligned}$$

If $h \in \mathcal{A}_n$*,* $(1+B)zh'(z)/h(z) \neq (1+A)$ *and satisfy*

$$Cz^3\Big(2\Big(\frac{h'(z)}{h(z)}\Big)^3 - \frac{3h'(z)h''(z)}{h^2(z)} + \frac{h'''(z)}{h(z)}\Big) + (2C + D(z))$$
$$z^2\Big(\frac{h''(z)}{h(z)} - \Big(\frac{h'(z)}{h(z)}\Big)^2\Big) + (D(z)+E(z))\frac{zh'(z)}{h(z)} + F(z) = 0,$$

then $h \in \mathcal{S}^*[A, B]$*.*

Proof. Let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = zh'(z)/h(z)$. Then p is analytic in $\mathbb{D}$ with $p(0) = 1$. A calculation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{zh''(z)}{h'(z)} - \frac{zh'(z)}{h(z)}.$$

The result now follows from Theorem 2.1. □

We obtain our next application by taking $n = 1$, $C(z) = F(z) = 0$, $D(z) = 1$, $h(z) = f(z)g(z)$ with $f \in \mathcal{A}$, $g \in \mathcal{H}[1, 1]$ and $E(z) = -1 - zh''(z)/h'(z) + zh'(z)/h(z)$ in Corollary 2.11.

Corollary 2.12. *Let* $-1 \leq B < A \leq 1$*. Suppose that the functions* $f \in \mathcal{A}$*,* $g \in \mathcal{H}[1, 1]$ *and* $K(z) = zf'(z)/f(z) + zg'(z)/g(z) - ((2zf'g' + zf''g + zg''f)/(f'g + g'f))$ *satisfy*
(i) $(1 + B)z(f'(z)g(z) + g'(z)f(z)) \neq (1 + A)f(z)g(z)$,
(ii) $\operatorname{Re} K(z) > 1 - (A - B)/((1 + A)(1 + B))$,
(iii) $((AB - 1)(\operatorname{Im} K(z) - 1))^2 < \big((A - B) + (1 + A)(1 + B)(\operatorname{Re} K(z) - 1)\big)$

$$\big((A - B) - (1 - A)(1 - B)(\operatorname{Re} K(z) - 1)\big)$$

then $fg \in \mathcal{S}^*[A, B]$*.*

3. Two more sufficient conditions for Janowski starlikeness

For $p \in \mathcal{H}[1, n]$, Miller and Mocanu [13, Example 2.4m, p. 43] obtained the conditions on C, D, E, F so that

$$\operatorname{Re}(C(z)p^2(z) + D(z)zp'(z) + E(z)p(z) + F(z)) > 0 \Rightarrow \operatorname{Re} p(z) > 0, z \in \mathbb{D}.$$

In contrast to the above result, in this section, for $-1 \leq B < A \leq 1$, we investigate conditions on C, D, E, F so that for $z \in \mathbb{D}$,

$$C(z)p^2(z) + D(z)zp'(z) + E(z)p(z) + F(z) = 0 \Rightarrow p(z) \prec (1 + Az)/(1 + Bz)$$

and then give an application.

Theorem 3.1. *Let* n *be a positive integer,* $-1 \leq B < A \leq 1$*. Suppose that the functions* $C, D, E, F : \mathbb{D} \to \mathbb{C}$ *satisfy*
(i) $\operatorname{Re} D(z) \geq 0$,
(ii) $(A-B)n \operatorname{Re} D(z)+(1+A)(1+B) \operatorname{Re} E(z)+(1+B)^2 \operatorname{Re} F(z)+(1+A)^2 \operatorname{Re} C(z) > 0$,
(iii) $((1 - AB) \operatorname{Im} E(z) + (1 - B^2) \operatorname{Im} F(z) + (1 - A^2) \operatorname{Im} C(z))^2$

$$< \Big((A - B)n \operatorname{Re} D(z) + (1 + A)(1 + B) \operatorname{Re} E(z) + (1 + B)^2 \operatorname{Re} F(z)$$

$$+(1+A)^2\operatorname{Re}C(z)\Big)\Big((A-B)n\operatorname{Re}D(z)$$
$$-(1-A)(1-B)\operatorname{Re}E(z)-(1-B)^2\operatorname{Re}F(z)-(1-A)^2\operatorname{Re}C(z)\Big).$$

If $p\in\mathcal{H}[1,n]$, $(1+B)p(z)\neq(1+A)$ *and satisfy*

$$C(z)p^2(z)+D(z)zp'(z)+E(z)p(z)+F(z)=0, \tag{3.1}$$

then $p(z)\prec(1+Az)/(1+Bz)$.

Proof. For $p\in\mathcal{H}[1,n]$, define the function $q:\mathbb{D}\to\mathbb{C}$ by

$$q(z)=\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)}. \tag{3.2}$$

Then q is analytic in $\mathbb{D}$ and $q(0)=1$. Proceeding as in Theorem 2.1, the differential equation (3.1) takes the following form

$$\begin{aligned} D(z)zq'(z)&+\frac{C(z)((1-A)+(1+A)q(z))^2}{2(A-B)}\\ &+\frac{E(z)((1-A)+(1+A)q(z))((1-B)+(1+B)q(z))}{2(A-B)}\\ &+\frac{F(z)((1-B)+(1+B)q(z))^2}{2(A-B)}=0.\end{aligned} \tag{3.3}$$

Let $\psi:\mathbb{C}^2\times\mathbb{D}\to\mathbb{C}$ be defined by

$$\begin{aligned}\psi(r,s;z)=D(z)s&+\frac{C(z)((1-A)+(1+A)r)^2}{2(A-B)}\\ &+\frac{E(z)((1-A)+(1+A)r)((1-B)+(1+B)r)}{2(A-B)}\\ &+\frac{F(z)((1-B)+(1+B)r)^2}{2(A-B)}.\end{aligned} \tag{3.4}$$

It follows from (3.3) that $\psi(q(z),zq'(z);z)\in\Omega=\{0\}$. Now to ensure that $\operatorname{Re}q(z)>0$ for $z\in\mathbb{D}$, from Lemma 1.2, it is enough to establish that $\operatorname{Re}\psi(i\rho,\sigma;z)<0$ in $\mathbb{D}$ for any real ρ, σ, satisfying $\sigma\leq-n(1+\rho^2)/2$. For $z\in\mathbb{D}$ in (3.4), a computation using condition (i) yields that

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma;z)&=\sigma\operatorname{Re}D(z)+\frac{((1-A)^2-(1+A)^2\rho^2)\operatorname{Re}C(z)}{2(A-B)}\\ &+\frac{(A^2-1)\rho\operatorname{Im}C(z)}{A-B}+\frac{(AB-1)\rho\operatorname{Im}E(z)}{A-B}\\ &+\frac{\operatorname{Re}E(z)\Big((1-A)(1-B)-(1+A)(1+B)\rho^2\Big)}{2(A-B)}\\ &+\frac{((1-B)^2-(1+B)^2\rho^2)\operatorname{Re}F(z)}{2(A-B)}+\frac{(B^2-1)\rho\operatorname{Im}F(z)}{A-B}\\ &\leq-\frac{n}{2}(1+\rho^2)\operatorname{Re}D(z)+\frac{((1-A)^2-(1+A)^2\rho^2)\operatorname{Re}C(z)}{2(A-B)}\end{aligned}$$

$$+\frac{\operatorname{Re} E(z)\Big((1-A)(1-B)-(1+A)(1+B)\rho^2\Big)}{2(A-B)}$$
$$+\frac{((1-B)^2-(1+B)^2\rho^2)\operatorname{Re} F(z)}{2(A-B)}+\frac{(B^2-1)\rho\operatorname{Im} F(z)}{A-B}$$
$$+\frac{(A^2-1)\rho\operatorname{Im} C(z)}{A-B}+\frac{(AB-1)\rho\operatorname{Im} E(z)}{A-B}=: a\rho^2+b\rho+c, \tag{3.5}$$

where

$$a=-\frac{1}{2(A-B)}\big((A-B)n\operatorname{Re} D(z)+(1+A)(1+B)\operatorname{Re} E(z)$$
$$+(1+B)^2\operatorname{Re} F(z)+(1+A)^2\operatorname{Re} C(z)\big),$$
$$b=-\frac{1}{(A-B)}\big((1-AB)\operatorname{Im} E(z)+(1-B^2)\operatorname{Im} F(z)+(1-A^2)\operatorname{Im} C(z)\big),$$
$$c=-\frac{1}{2(A-B)}\big((A-B)n\operatorname{Re} D(z)-(1-A)(1-B)\operatorname{Re} E(z)$$
$$-(1-B)^2\operatorname{Re} F(z)-(1-A)^2\operatorname{Re} C(z)\big).$$

In view of the conditions (ii) and (iii), we see that $a<0$ and $b^2-4ac<0$ respectively. So, $\operatorname{Re}\psi(i\rho,\sigma,\mu+i\nu;z)<0$ in $\mathbb{D}$. Hence, by Lemma 1.2, we deduce that $\operatorname{Re} q(z)>0$, that is, by using (3.2), we get

$$\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)}\prec\frac{1+z}{1-z}.$$

Therefore, there exist an analytic function w in $\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$ such that

$$\frac{-(1-A)+(1-B)p(z)}{(1+A)-(1+B)p(z)}=\frac{1+w(z)}{1-w(z)}$$

which gives that $p(z)=(1+Aw)/(1+Bw)$ and thus, $p(z)\prec(1+Az)/(1+Bz)$. □

The next result follows by taking $p(z)=zf'(z)/f(z)$ in Theorem 3.1.

Corollary 3.2. *Let n be a positive integer, $-1\le B<A\le 1, C(z)=C\ge 0$. Suppose that the functions $D,E,F:\mathbb{D}\to\mathbb{C}$ satisfy*
(i) $\operatorname{Re} D(z)\ge 0$,
(ii) $(A-B)n\operatorname{Re} D(z)+(1+A)(1+B)\operatorname{Re} E(z)+(1+B)^2\operatorname{Re} F(z)+(1+A)^2\operatorname{Re} C(z)>0$,
(iii) $((1-AB)\operatorname{Im} E(z)+(1-B^2)\operatorname{Im} F(z)+(1-A^2)\operatorname{Im} C(z))^2$

$$<\Big((A-B)n\operatorname{Re} D(z)+(1+A)(1+B)\operatorname{Re} E(z)+(1+B)^2\operatorname{Re} F(z)$$
$$+(1+A)^2\operatorname{Re} C(z)\Big)\Big((A-B)n\operatorname{Re} D(z)-(1-A)(1-B)\operatorname{Re} E(z)$$
$$-(1-B)^2\operatorname{Re} F(z)-(1-A)^2\operatorname{Re} C(z)\Big).$$

If $f\in\mathcal{A}_n$, $(1+B)zf'(z)/f(z)\ne(1+A)$ and satisfy

$$C(z)\Big(\frac{zf'(z)}{f(z)}\Big)^2+D(z)\Big(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\Big)\frac{zf'(z)}{f(z)}+E(z)\Big(\frac{zf'(z)}{f(z)}\Big)+F(z)=0,$$

then $f \in \mathcal{S}^*[A,B]$.

We close this section by finding conditions on D, E, F so that $p(z) \prec (1+Az)/(1+Bz)$ when $|D(z)zp'(z) + E(z)p(z) + F(z)| < M$, $(M > 0)$.

Theorem 3.3. *Let n be a positive integer, $M > 0$ and $-1 \le B < A \le 1$. Suppose that the functions $D, E, F : \mathbb{D} \to \mathbb{C}$ satisfy*

$$n(A-B)|D(z)| - (1+|A|)(1+|B|)|E(z)| \ge (1+|B|)^2(M+|F(z)|). \tag{3.6}$$

If $p \in \mathcal{H}[1,n]$ and satisfy

$$|D(z)zp'(z) + E(z)p(z) + F(z)| < M, \tag{3.7}$$

then $p(z) \prec (1+Az)/(1+Bz)$.

Proof. In view of condition (3.7), we must have $|E(0) + F(0)| < M$. Suppose that $G(z) = D(z)zp'(z) + E(z)p(z) + F(z)$. If we assume that p is not subordinate to $(1+Az)/(1+Bz) =: q(z)$, then by Lemma 1.1, there exist points $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D}$ and an $m \ge n$ such that

$$p(z_0) = q(\zeta_0) = \frac{1+A\zeta_0}{1+B\zeta_0} \tag{3.8}$$

and

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = \frac{m(A-B)\zeta_0}{(1+B\zeta_0)^2}. \tag{3.9}$$

Using (3.8), (3.9) , (3.6) and the fact that $|\zeta_0| = 1$ and $m \ge n$, we get

$$|G(z_0)| = \frac{1}{|1+B\zeta_0|^2}|m(A-B)\zeta_0 D(z_0) + (1+A\zeta_0)(1+B\zeta_0)E(z_0) + (1+B\zeta_0)^2 F(z_0)|$$

$$\ge \frac{1}{(1+|B|)^2}\big(m(A-B)|D(z_0)| - |1+A\zeta_0||1+B\zeta_0||E(z_0)| - |1+B\zeta_0|^2|F(z_0)|\big)$$

$$\ge \frac{1}{(1+|B|)^2}\big(n(A-B)|D(z_0)| - (1+|A|)(1+|B|)|E(z_0)| - (|1+|B|)^2|F(z_0)|\big) \ge M.$$

Since this contradicts (3.7), we get our required result. □

Acknowledgement. The work presented here is supported by a grant from University of Delhi.

References

[1] Acharya, A.P., *Univalence criteria for analytic funtions and applications to hypergeometric functions*, Ph.D Diss., University of Würzburg, 1997.

[2] Ali, R.M., Ravichandran, V., Seenivasagan, N., *Sufficient conditions for Janowski starlikeness*, Int. J. Math. Math. Sci., **2007**, Art. ID 62925, 7 pp.

[3] Ali, R.M., Chandrashekar, R., Ravichandran, V., *Janowski starlikeness for a class of analytic functions*, Appl. Math. Lett., **24**(2011), no. 4, 501-505.

[4] Baricz, Á., *Generalized Bessel Functions of the First Kind*, Lecture Notes in Mathematics, 1994, Springer, Berlin, 2010.

[5] Baricz, Á., *Applications of the admissible functions method for some differential equations*, Pure Math. Appl., **13**(2002), no. 4, 433-440.

[6] Baricz, Á., *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen, **73**(2008), no. 1-2, 155-178.

[7] Cho, N.E., Kwon, O.S., Ravichandran, V., *Coefficient, distortion and growth inequalities for certain close-to-convex functions*, J. Inequal. Appl., **2011**, 2011:100, 7 pp.

[8] Goodman, A.W., *Univalent Functions. Vol. II*, Mariner, Tampa, FL, 1983.

[9] Janowski, W., *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Polon. Math., **23**(1970/1971), 159-177.

[10] Lee, S.K., Ravichandran, V., Supramaniam, S., *Close-to-convexity and starlikeness of analytic functions*, Tamkang J. Math., **46**(2015), no. 2, 111-119.

[11] Miller, S.S., Mocanu, P.T., *Differential subordinations and inequalities in the complex plane*, J. Diff. Eq., **67**(1987), no. 2, 199-211.

[12] Miller, S.S., Mocanu, P.T., *The theory and applications of second-order differential subordinations*, Studia Univ. Babeş-Bolyai Math., **34**(1989), no. 4, 3-33.

[13] Miller, S.S., Mocanu, P.T., *Differential Subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[14] Polatoğlu, Y., Bolcal, M., *Some radius problem for certain families of analytic functions*, Turkish J. Math., **24**(2000), no. 4, 401-412.

[15] Ravichandran, V., Sharma, K., *Sufficient conditions for starlikeness*, J. Korean Math. Soc., **52**(2015), no. 4, 727-749.

[16] Robertson, M.S., *Certain classes of starlike functions*, Michigan Math. J., **32**(1985), no. 2, 135-140.

[17] Ruscheweyh, St., Singh, V., *On the order of starlikeness of hypergeometric functions*, J. Math. Anal. Appl., **113**(1986), no. 1, 1-11.

[18] Ali, R.M., Mondal, S.R., Ravichandran, V., *On the Janowski convexity and starlikeness of the confluent hypergeometric function*, Bull. Belg. Math. Soc. Simon Stevin, **22**(2015), no. 2, 227-250.

[19] Sharma, K., Ravichandran, V., *Applications of subordination theory to starlike functions*, Bull. Iranian Math. Soc. (to appear).

[20] Sharma, K., Jain, N.K., Ravichandran, V., *Starlike functions associated with a cardioid*, Afr. Mat. (2015), available online DOI: 10.1007/s13370-015-0387-7.

[21] Temme, N.M., *Special Functions*, Wiley, New York, 1996.

[22] Tepper, D.E., *On the radius of convexity and boundary distortion of Schlicht functions*, Trans. Amer. Math. Soc., **150**(1970), 519-528.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 77–85

Explicit limit cycles of a cubic polynomial differential systems

Ahmed Bendjeddou and Rachid Boukoucha

Abstract. In this paper, we determine sufficient conditions for a cubic polynomial differential systems of the form

$$\begin{cases} x' = x + ax^3 + bx^2y + cxy^2 + ny^3 \\ y' = y + sx^3 + ux^2y + vxy^2 + wy^3 \end{cases}$$

where a, b, c, n, s, u, v, w are real constants, to possess an algebraic, non-algebraic limit cycles, explicitly given. Concrete examples exhibiting the applicability of our result is introduced.

Mathematics Subject Classification (2010): 34A05, 34C05, 34CO7, 34C25.

Keywords: Planar polynomial differential system, algebraic limit cycle, non-algebraic limit cycle.

1. Introduction

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of the form

$$\begin{cases} x' = \frac{dx}{dt} = P(x,y) \\ y' = \frac{dy}{dt} = Q(x,y) \end{cases} \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are coprime polynomials and we denote by $n = \max\{\deg P, \deg Q\}$ and we say that n is the degree of system (1.1). A limit cycle of system (1.1) is an isolated periodic solution in the set of all periodic solution of system (1.1) see [4, 6, 10], and it is said to be algebraic if it is contained in the zero level set of a polynomial function, see for example [1, 2, 8]. We usually only ask for the number of such limit cycles, but their location as orbits of the system is also an interesting problem. And an even more difficult problem is to give an explicit expression of them. We are able to solve this last problem for a given system of the form (1.1). Until recently, the only limit cycles known in an explicit way were algebraic. In [3, 5, 7] examples of explicit limit cycles which are not algebraic are given. For instance, the limit cycle appearing in van der Pol's system is not algebraic as it is proved in [9].

In this paper, we determine sufficient conditions for a planar systems of the form

$$\begin{cases} x' = x + ax^3 + bx^2y + cxy^2 + ny^3 \\ y' = y + sx^3 + ux^2y + vxy^2 + wy^3 \end{cases} \tag{1.2}$$

where a, b, c, n, s, u, v and w are real constants, to possess an explicit algebraic, non-algebraic limit cycles. Concrete examples exhibiting the applicability of our result is introduced.

We define the trigonometric functions

$$\begin{aligned} f(\theta) &= 3a + c + u + 3w + 4(a - w)(\cos 2\theta) + 2(b + n + s + v)(\sin 2\theta) \\ &\quad + (a - c - u + w)(\cos 4\theta) + (b - n + s - v)(\sin 4\theta) \end{aligned}$$

$$\begin{aligned} g(\theta) &= 3s - 3n - b + v + 4(n + s)(\cos 2\theta) + 2(u - c - a + w)(\sin 2\theta) \\ &\quad + (c - a + u - w)(\sin 4\theta) + (b - n + s - v)(\cos 4\theta) \end{aligned}$$

2. Main result

Our main result is contained in the following theorem.

Theorem 2.1. *Consider a multi-parameter cubic polynomial differential system (*1.2*), then the following statements hold.*

H1) if

$$3a + c + u + 3w + 4|a - w| + 2|b + n + s + v| + |a - c - u + w| + |b - n + s - v| < 0,$$

$$3s - 3n - b + v + 4|n + s| + 2|u - c - a + w| + |c - a + u - w| + |b - n + s - v| < 0,$$

*then system (*1.2*) has limit cycle explicitly given in polar coordinates (r, θ), by*

$$r(\theta, r_*) = \exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)} d\mu\right) \sqrt{r_*^2 + 16 \int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega}$$

where a, b, c, n, s, u, v , w are real constants, and

$$r_* = 4\sqrt{\frac{\exp\left(2\int_0^{2\pi} \frac{f(\mu)}{g(\mu)} d\mu\right)}{1 - \exp\left(2\int_0^{2\pi} \frac{f(\mu)}{g(\mu)} d\mu\right)} \int_0^{2\pi} \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega}$$

H2) if $f(\theta)$, and $g(\theta)$ are not constant functions for all $\theta \in \mathbb{R}$, then this limit cycle is non algebraic limit cycle.

Moreover, this limit cycle is a stable hyperbolic limit cycle.

H3) if $f(\theta) = \lambda, g(\theta) = \beta$ are constant functions for all $\theta \in \mathbb{R}$ where $\lambda, \beta \in \mathbb{R}_-^*$, then this limit cycle is algebraic limit cycle given by $r_*^2 = \frac{-8}{\lambda}$ i e: $x^2 + y^2 = \frac{-8}{\lambda}$ is the circle.

In short, since it is well known that the polynomial differential systems of degree 1 have no limit cycles, it remains the following open question:
Open question. Are there or not polynomial differential systems of degree 2 exhibiting explicit non-algebraic limit cycles.

Proof. In order to prove our results we write the polynomial differential system (1.2) in polar coordinates (r,θ), defined by $x = r\cos\theta$, and $y = r\sin\theta$, then system becomes

$$\begin{cases} r' = r + f(\theta)\, r^3 \\ \theta' = g(\theta)\, r^2 \end{cases} \tag{2.1}$$

where $\theta' = \frac{d\theta}{dt}$, $r' = \frac{dr}{dt}$.
According to

$$3s - 3n - b + v + 4|n + s| + 2|u - c - a + w| + |c - a + u - w| + |b - n + s - v| < 0$$

hence $g(\theta) < 0$ for all $\theta \in \mathbb{R}$, then θ' is negative for all t, which means that the orbits $(r(t), \theta(t))$ of system (2.1) have the opposite orientation with respect to those $(x(t), y(t))$ of system (1.2).

Taking as new independent variable the coordinate θ, this differential system writes

$$\frac{dr}{d\theta} = \frac{f(\theta)}{g(\theta)} r + \frac{8}{g(\theta)} \frac{1}{r} \tag{2.2}$$

which is a Bernoulli equation.

By introducing the standard change of variables $\rho = r^2$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = \frac{16}{g(\theta)} + \frac{2f(\theta)}{g(\theta)} \rho \tag{2.3}$$

The general solution of linear equation (2.3) is

$$\rho(\theta) = \exp\left(\int_0^\theta \frac{2f(\mu)}{g(\mu)} d\mu\right)\left(k + 16\int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega\right) \tag{2.4}$$

where $k \in \mathbb{R}$

Then the general solution of Bernoulli equation (2.2) is

$$r(\theta) = \exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)} d\mu\right)\sqrt{k + 16\int_0^\theta \left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)}\right) d\omega} \tag{2.5}$$

where $k \in \mathbb{R}$

Notice that system (1.2) has a periodic orbit if and only if equation (2.5) has a strictly positive 2π periodic solution.

It is easy to check that the solution $r(\theta;r_0)$ of the differential equation (2.2) such that $r(0,r_0)=r_0$ is

$$r(\theta;r_0)=\exp\left(\int_0^{\theta}\frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{r_0^2+16\int_0^{\theta}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega} \tag{2.6}$$

where $r_0=r(0)$.
A periodic solution of system (2.1) must satisfy the condition $r(2\pi,r_0)=r(0,r_0)$, which leads to a unique value $r_0=r_*$, given by

$$r_*=4\sqrt{\frac{\exp\left(\int_0^{2\pi}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{1-\exp\left(\int_0^{2\pi}\frac{2f(\mu)}{g(\mu)}d\mu\right)}\int_0^{2\pi}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega} \tag{2.7}$$

Since

$$3a+c+u+3w+4|a-w|+2|b+n+s+v|+|a-c-u+w|+|b-n+s-v|<0$$

and

$$3s-3n-b+v+4|n+s|+2|u-c-a+w|+|c-a+u-w|+|b-n+s-v|<0$$

we have $f(\mu)<0$, $g(\mu)<0$ for all $\mu\in[0,2\pi]$ hence $r_*>0$.
Injecting this value of r_* in (2.6), we get the candidate solution

$$r(\theta,r_*)=4\exp\left(\int_0^{\theta}\frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{\begin{array}{r}\frac{\exp\left(\int_0^{2\pi}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{1-\exp\left(\int_0^{2\pi}\frac{2f(\mu)}{g(\mu)}d\mu\right)}\int_0^{2\pi}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\\+\int_0^{\theta}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\end{array}}$$

So, if $r(\theta;r_*)>0$ for all $\theta\in\mathbb{R}$, we shall have $r(\theta;r_*)>0$ would be periodic orbit, and consequently a limit cycle. In what follows it is proved that $r(\theta;r_*)>0$ for all $\theta\in\mathbb{R}$. Indeed

$$r(\theta,r_*)=4\exp\left(\int_0^{\theta}\frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{\begin{array}{r}\frac{\exp\left(\int_0^{2\pi}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{1-\exp\left(\int_0^{2\pi}\frac{2f(\mu)}{g(\mu)}d\mu\right)}\int_0^{2\pi}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\\+\int_0^{\theta}\left(\frac{\exp\left(-\int_0^{\omega}\frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\end{array}}$$

$$> 4\exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{\begin{array}{l}\displaystyle\int_0^{2\pi}\left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)}d\mu\right)}{-g(\omega)}\right)d\omega \\ \displaystyle+\int_0^{\theta}\left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\end{array}}$$

$$= 4\exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{\int_{2\pi}^{\theta}\left(\frac{e^{-\int_0^\omega \frac{2f(\mu)}{g(\mu)}d\mu}}{g(\omega)}\right)d\omega}$$

$$= 4\exp\left(\int_0^\theta \frac{f(\mu)}{g(\mu)}d\mu\right)\sqrt{\int_{\theta}^{2\pi}\left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)}d\mu\right)}{-g(\omega)}\right)d\omega} > 0$$

because $f(\mu) < 0, g(\mu) < 0$ for all $\mu \in \mathbb{R}$, hence $\frac{f(\mu)}{g(\mu)} > 0$ for all $\mu \in \mathbb{R}$
Consequently, this is a limit cycle for the differential system (1.2).
This completes the proof of statement H1 of Theorem 2.1.

If $f(\theta)$ and $g(\theta)$ are not constant functions for all $\theta \in \mathbb{R}$, the curve $(r(\theta)\cos\theta, r(\theta)\sin(\theta))$ in the (x, y) plane with

$$r(\theta; r_*)^2 = \exp\left(\int_0^\theta \frac{2f(\mu)}{g(\mu)}d\mu\right)\left(r_*^2 + 16\int_0^\theta\left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\right)$$

is not algebraic. More precisely, in Cartesian coordinates $r(\theta; r_*)^2 = x^2 + y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$, the curve defined by this limit cycle is

$$f(x, y) = x^2 + y^2 - \exp\left(\int_0^{\arctan\left(\frac{y}{x}\right)} \frac{2f(\mu)}{g(\mu)}d\mu\right)$$

$$\times\left(r_*^2 + 16\int_0^{\arctan\left(\frac{y}{x}\right)}\left(\frac{\exp\left(-\int_0^\omega \frac{2f(\mu)}{g(\mu)}d\mu\right)}{g(\omega)}\right)d\omega\right) = 0.$$

But there is no integer n for which both $\frac{\partial^{(n)} f}{\partial x^n}$ and $\frac{\partial^{(n)} f}{\partial y^n}$ vanish identically.
To be convinced by this fact, one has to compute for example $\frac{\partial f}{\partial x}$, that is

$$\frac{\partial f}{\partial x}(x,y) = 2x + \frac{y \exp\left(\frac{2f\left(\arctan\left(\frac{y}{x}\right)\right)}{g\left(\arctan\left(\frac{y}{x}\right)\right)}\right) \exp\left(\int_0^{\arctan\left(\frac{y}{x}\right)} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{x^2+y^2} r_*^2$$
$$+16 \frac{y \exp\left(\frac{2f\left(\arctan\left(\frac{y}{x}\right)\right)}{g\left(\arctan\left(\frac{y}{x}\right)\right)}\right) \exp\left(\int_0^{\arctan\left(\frac{y}{x}\right)} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{x^2+y^2}$$
$$\times \int_0^{\arctan\left(\frac{y}{x}\right)} \left(\frac{\exp\left(-\int_0^{\omega} \frac{2f(\mu)}{g(\mu)} d\mu\right)}{g(\omega)} \right) d\omega + \frac{16y}{(x^2+y^2)\, g\left(\arctan\left(\frac{y}{x}\right)\right)}$$

Since $f(x,y)$ appears again, it will remains in any order of derivation, therefore the curve $f(x,y) = 0$ is non-algebraic and the limit cycle will also be non-algebraic.

In order to prove the hyperbolicity of the limit cycle notice that the Poincaré return map is $\Pi(\rho_0) = \rho(2\pi, \rho_0)$, for more details see [5, section 1.6].

An easy computation shows that

$$\left.\frac{dr(2\pi; r_0)}{dr_0}\right|_{r_0=r_*} = \exp\left(\int_0^{2\pi} \frac{2f(\mu)}{g(\mu)} d\mu\right) > 1.$$

because $f(\mu)\, g(\mu) > 0$ for all $\mu \in \mathbb{R}$

Therefore the limit cycle of the differential equation (2.2) is unstable and hyperbolic. Consequently, this is a stable and hyperbolic limit cycle for the differential system (1.2). This completes the proof of statement H2 of Theorem 2.1.

Suppose now that $f(\theta) = \lambda$, $g(\theta) = \beta$ are constant functions for all $\theta \in \mathbb{R}$.

According to

$$3a + c + u + 3w + 4|a - w| + 2|b + n + s + v| + |a - c - u + w| + |b - n + s - v| < 0$$

$$3s - 3n - b + v + 4|n + s| + 2|u - c - a + w| + |c - a + u - w| + |b - n + s - v| < 0$$

hence $f(\theta) = \lambda < 0$, $g(\theta) = \beta < 0$ for all $\theta \in \mathbb{R}$. then

$$r_* = \sqrt{\frac{16}{\beta} \frac{\exp\left(2\int_0^{2\pi} \frac{\lambda}{\beta} d\mu\right)}{1 - \exp\left(2\int_0^{2\pi} \frac{\lambda}{\beta} d\mu\right)} \left(\int_0^{2\pi} \left(\exp\left(-2\int_0^{\omega} \frac{\lambda}{\beta} d\mu\right)\right) d\omega\right)} = \sqrt{-\frac{8}{\lambda}} > 0,$$

Injecting this value of r_* in (2.6), we get the solution

$$r(\theta, r_*) = \exp\left(\frac{\lambda}{\beta}\theta\right) \sqrt{\frac{-8}{\lambda} + \frac{16}{\beta} \int_0^{\theta} \left(\exp\left(-2\int_0^{\omega} \frac{\lambda}{\beta} d\mu\right)\right) d\omega}$$

$$r(\theta, r_*) = \sqrt{-\frac{8}{\lambda}} > 0,$$

for all $\theta \in \mathbb{R}$

In Cartesian coordinates

$$r(\theta; r_*)^2 = x^2 + y^2 = -\frac{8}{\lambda}$$

this limit cycle is algebraic (is the circle).
This completes the proof of statement H3 of Theorem 2.1. □

The following examples are given to illustrate our result.

Example 2.2. If we take $a = s = w = -1, b = 2, c = -2, n = 1$, and $u = v = 0$ then system (1.2) reads

$$\begin{cases} x' = x - x^3 + 2x^2y - 2xy^2 + y^3 \\ y' = y - x^3 - y^3 \end{cases}$$

equivalent to

$$\begin{cases} x' = x + (y - x)\left(x^2 - xy + y^2\right) \\ y' = y - (y + x)\left(x^2 - xy + y^2\right) \end{cases}$$

has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is,

$$r(\theta, r_*) = e^{\theta}\sqrt{r_*^2 - 4\int_0^{\theta}\left(\frac{e^{-2\omega}}{2 - \sin 2\omega}\right)d\omega}$$

where $\theta \in \mathbb{R}$, with $f(\theta) = g(\theta) = -8 + 4(\sin 2\theta)$, and the intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = \sqrt{\frac{2e^{4\pi}}{e^{4\pi} - 1}\int_0^{2\pi}\left(\frac{2}{2 - \sin 2\omega}e^{-2\omega}\right)d\omega} \simeq 1.1912$$

Moreover

$$\left.\frac{dr(2\pi; r_0)}{dr_0}\right|_{r_0 = r_*} = e^{4\pi} > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

Is the results presented by Jaume Llibre and Benterki Rebiha in [3] .

Example 2.3. If we take $a = s = w = -2, b = 5, c = -5, n = 2$, and $u = v = 1$ then system (1.2) reads

$$\begin{cases} x' = x - 2x^3 + 5x^2y - 5xy^2 + 2y^3 \\ y' = y - 2x^3 + x^2y + xy^2 - 2y^3 \end{cases}$$

has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is,

$$r(\theta, r_*) = \exp(\theta)\sqrt{r_*^2 + 4\int_0^{\theta}\left(\frac{\exp(-2\omega)}{-4 + 3(\sin 2\omega)}\right)d\omega}$$

where $\theta \in \mathbb{R}$, with $f(\theta) = -16 + 12(\sin 2\theta), g(\theta) = -16 + 12(\sin 2\theta)$, and the intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = 4\sqrt{\frac{\exp(4\pi)}{1 - \exp(4\pi)}\left(\int_0^{2\pi}\left(\frac{\exp(-2\omega)}{-16 + 12(\sin 2\omega)}\right)d\omega\right)} \simeq 1.0010$$

Moreover

$$\left.\frac{dr\left(2\pi; r_0\right)}{dr_0}\right|_{r_0=r_*} = e^{4\pi} > 1.$$

This limit cycle is a stable hyperbolic limit cycle.

Example 2.4. If we take $a = c = u = s = v = w = -1$ and $b = n = 1$, the system (1.2) reads

$$\begin{cases} x' = x - x^3 + x^2y - xy^2 + y^3 \\ y' = y - x^3 - x^2y - xy^2 - y^3 \end{cases}$$

in polar coordinates (r, θ) we obtained $f(\theta) = \lambda = -8$, $g(\theta) = \beta = -8$, and $r_* = \sqrt{\frac{-8}{\lambda}} = 1$ hence

$$r\left(\theta, r_*\right) = r\left(\theta, 1\right) = \exp\left(\int_0^\theta d\mu\right)\sqrt{1 + 16\int_0^\theta \left(\frac{\exp\left(-\int_0^\omega 2d\mu\right)}{-8}\right) d\omega} = 1$$

for all $\theta \in \mathbb{R}$.

The system has a algebraic limit cycle whose expression in Cartesian coordinates (x, y) becomes

$$r\left(\theta; r_*\right)^2 = x^2 + y^2 = 1$$

this limit cycle is the circle.

Example 2.5. If we take $a = c = u = w = -\frac{1}{2}, b = n = \frac{1}{4}$, and $s = v = -\frac{1}{4}$, the system (1.2) reads

$$\begin{cases} x' = x - \frac{1}{2}x^3 + \frac{1}{4}x^2y - \frac{1}{2}xy^2 + \frac{1}{4}y^3 \\ y' = y - \frac{1}{4}x^3 - \frac{1}{2}x^2y - \frac{1}{4}xy^2 - \frac{1}{2}y^3 \end{cases}$$

in polar coordinates (r, θ) we obtained $f(\theta) = \lambda = -4, g(\theta) = \beta = -2$ and $r_* = \sqrt{\frac{-8}{\lambda}} = \sqrt{2}$ hence

$$r\left(\theta, r_*\right) = r\left(\theta, \sqrt{2}\right) = \exp\left(\int_0^\theta 2ds\right)\sqrt{2 + 16\int_0^\theta \left(\frac{\exp\left(-\int_0^\omega 4ds\right)}{-2}\right) d\omega} = \sqrt{2}$$

for all $\theta \in \mathbb{R}$.

The system has a algebraic limit cycle whose expression in Cartesian coordinates (x, y) becomes

$$r\left(\theta; r_*\right)^2 = x^2 + y^2 = 2$$

this limit cycle is the circle.

References

[1] Bendjeddou, A., Cheurfa, R., *On the exact limit cycle for some class of planar differential systems,* Nonlinear Diff. Eq. Appl., **14**(2007), 491-498.

[2] Bendjeddou, A., Cheurfa, R., *Cubic and quartic planar differential systems with exact algebraic limit cycles,* Elect. J. Diff. Eq., **15**(2011), 1-12.

[3] Benterki, R., Llibre, J., *Polynomial differential systems with explicit non-algebraic limit cycles,* Elect. J. Diff. Eq., **78**(2012), 1-6.

[4] Dumortier, F., Llibre, J., Artés, J., *Qualitative Theory of Planar Differential Systems,* Springer, Berlin, 2006.

[5] Gasull, A., Giacomini, H., Torregrosa, J., *Explicit non-algebraic limit cycles for polynomial systems*, J. Comput. Appl. Math., **200**(2007), 448-457.

[6] Giacominiy, H., Llibre, J., Viano, M., *On the nonexistence existence and uniquencess of limit cycles,* Nonlinearity, **9**(1996), 501–516.

[7] Giné, J., Grau, M., *Coexistence of algebraic and non-algebraic limit cycles, explicitly given, using Riccati equations*, Nonlinearity, **19**(2006), 1939-1950.

[8] Llibre, J., Zhao, Y., *Algebraic Limit Cycles in Polynomial Systems of Differential Equations,* J. Phys. A, Math. Theor., **40**(2007), 14207-14222.

[9] Odani, K., *The limit cycle of the van der Pol equation is not algebraic,* J. Diff. Eq., **115**(1995), 146-152.

[10] Perko, L., *Differential equations and dynamical systems*, Third Edition, Texts in Applied Mathematics, 7, Springer-Verlag, New York, 2001.

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Fixed point theorems for a system of operator equations with applications

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Abstract. The purpose of this paper is to present some existence and uniqueness theorems for a general system of operator equations. The abstract result generalizes some existence results obtained in [V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 7347-7355] for the case of coupled fixed point problem. We also provide an application to a system of integro-differential equations.

Mathematics Subject Classification (2010): 47H10, 54H25, 34B15.

Keywords: Fixed point, contractive condition, metric space, coupled fixed point, integral equation.

1. Introduction

The classical Banach contraction principle is remarkable in its simplicity and it is perhaps the most widely applied fixed point theorem in all analysis. This is because the contractive condition on the operator is easy to test and it requires only the structure of a complete metric space for its setting (see S. Banach [1]). This principle is also a very useful tool in nonlinear analysis with many applications to operatorial equations, fractal theory, optimization theory and other topics. Several authors have been dedicated to the improvement and generalization of this principle (see [3], [6], [4], [5], etc.)

The purpose of this paper is to present some existence and uniqueness results which will extend and generalize some theorems obtained by V. Berinde in [2] for the case of coupled fixed point problems. We also provide an application to an integral equation system. For related results see also [7].

This work was possible due to the financial support of the Sectorial Operational Program for Human Resources Development 2007 − 2013, co-financed by the European Social Fund, under the project number POSDRU/159/1.5/S/132400 with the title ,,Young successful researchers- professional development in a international and interdisciplinary enviroment".

2. Main results

The first result is an existence and uniqueness result which generalizes Theorem 3 presented by V. Berinde in [2].

Theorem 2.1. *Let X be a nonempty set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let T_1, $T_2 : X \times X \to X$ be two operators for which there exists a constant $k \in [0, 1)$ such that*

$$d(T_1(x, y), T_1(u, v)) + d(T_2(x, y), T_2(u, v)) \leq k(d(x, u) + d(y, v)),$$

for all (x, y), $(u, v) \in X \times X$.
Then we have the following conclusions:
(i) there exists a unique element $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*) \end{cases}$$

(ii) the sequence $(T_1^n(x, y), T_2^n(x, y))_{n\in\mathbb{N}}$ converges to (x^*, y^*) as $n \to \infty$

$$\begin{aligned} T_1^{n+1}(x, y) \quad &: \quad = T_1^n(T_1(x, y), T_2(x, y)) \\ T_2^{n+1}(x, y) \quad &: \quad = T_2^n(T_1(x, y), T_2(x, y)) \end{aligned}$$

for all $n \in \mathbb{N}$.
(iii) we have the following estimation

$$\begin{aligned} d(T_1^n(x_0, y_0), x^*) &\leq \frac{k^n}{1-k} d(x_0, T_1(x_0, y_0)) \\ d(T_2^n(x_0, y_0), y^*) &\leq \frac{k^n}{1-k} d(y_0, T_2(x_0, y_0)) \end{aligned}$$

(iv) let $F_1, F_2 : X \times X \to X$ be two operators such that, there exist $\epsilon_1, \epsilon_2 > 0$ with

$$\begin{aligned} d(T_1(x, y), F_1(x, y)) &\leq \epsilon_1 \\ d(T_2(x, y), F_2(x, y)) &\leq \epsilon_2 \end{aligned}$$

for all $(x, y) \in X \times X$. If $(a^*, b^*) \in X \times X$ is such that

$$\begin{cases} a^* = F_1(a^*, b^*) \\ b^* = F_2(a^*, b^*) \end{cases}$$

then

$$d(x^*, a^*) + d(y^*, b^*) \leq \frac{\epsilon_1 + \epsilon_2}{1-k}$$

Proof. (i)- (ii)
We define $T : X \times X \to X \times X$ by

$$T(x, y) = (T_1(x, y), T_2(x, y)).$$

Lets denote $Z := X \times X$ and $d^* : Z \times Z \to \mathbb{R}_+$

$$d^*((x, y), (u, v)) := \frac{1}{2}(d(x, u) + d(y, v))$$

for all $(x, y), (u, v) \in X \times X$.

Then we have

$$d^*(T(x,y),T(u,v)) = \frac{d(T_1(x,y),T_1(u,v)) + d(T_2(x,y),T_2(u,v))}{2}.$$

Then we denote $(x,y) := z$, $(u,v) := w$ we get that

$$d^*(T(z),T(w)) \leq k \cdot d^*(z,w)$$

for every $z,w \in X \times X$.
Hence we obtained Banach's contraction condition. Applying Banach's contraction fixed point theorem we get that there exists a unique element $(x^*,y^*) := z^* \in X \times X$ such that

$$z^* = T(z^*)$$

and it is equivalent with

$$\begin{cases} x^* = T_1(x^*,y^*) \\ y^* = T_2(x^*,y^*) \end{cases}.$$

For each $z \in X \times X$, we have that $T^n(z) \to z^*$ as $n \to \infty$ where

$$\begin{aligned} T^0(z) &: = z, T^1(z) = T(x,y) = (T_1(x,y),T_2(x,y)) \\ T^2(z) &= T(T_1(x,y),T_2(x,y)) = (T_1^2(x,y),T_2^2(x,y)) \end{aligned}$$

and in generally

$$\begin{aligned} T_1^{n+1}(x,y) &: = T_1^n(T_1(x,y),T_2(x,y)) \\ T_2^{n+1}(x,y) &: = T_2^n(T_1(x,y),T_2(x,y)). \end{aligned}$$

We get that $T^n(z) = (T_1^n(z),T_2^n(z)) \to z^* = (x^*,y^*)$ as n$\to \infty$, for all $z = (x,y) \in X \times X$.
So for all $(x,y) \in X \times X$ we have that

$$\begin{aligned} T_1^n(x,y) &\to x^* \text{ as } n \to \infty \\ T_2^n(x,y) &\to y^* \text{ as } n \to \infty. \end{aligned}$$

(iii) We apply Banach's contraction principle and we have successively

$$\begin{aligned} d(T_1^n(x_0,y_0),x^*) &\leq \frac{k^n}{1-k} d(x_0,T_1(x_0,y_0)) \\ d(T_2^n(x_0,y_0),y^*) &\leq \frac{k^n}{1-k} d(x_0,T_2(x_0,y_0)) \end{aligned}$$

(iv) Let us consider $F : X \times X \to X \times X$ given by $F(x,y) = (F_1(x,y),F_2(x,y))$ and

$$\begin{aligned} d^*(T(x,y),F(x,y)) &= d^*((T_1(x,y),T_2(x,y)),(F_1(x,y),F_2(x,y))) \\ &= \frac{d(T_1(x,y),F_1(x,y)) + d(T_2(x,y),F_2(x,y))}{2} \leq \epsilon, \end{aligned}$$

where $\epsilon := \frac{\epsilon_1+\epsilon_2}{2}$.
Then, by the data dependence theorem related to Banachs contraction principle we get that

$$d(x^*,a^*) + d(y^*,b^*) \leq \frac{\epsilon_1+\epsilon_2}{1-k}.$$

Hence we get that
$$d^*((x^*,y^*),(a^*,b^*)) \le \frac{\epsilon}{1-k}. \qquad \square$$
An existence and uniqueness result, similar to Theorem 2.1, is the following theorem.

Theorem 2.2. *Let X be a nonempty set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $T_1,T_2 : X\times X \to X$ be two operators for which there exists a constant $k\in[0,1)$ such that, for each $(x,y),(u,v)\in X\times X$, we have*
$$\max\{d(T_1(x,y),T_2(u,v)), d(T_2(x,y),T_2(u,v))\} \le k\cdot\max\{d(x,u),d(y,v)\}.$$
Then we have the following conclusions:
(i) there exists a unique element $(x^*,y^*)\in X\times X$ such that
$$\begin{cases} x^* = T_1(x^*,y^*)\\ y^* = T_2(x^*,y^*)\end{cases}$$
(ii) the sequence $(T_1^n(x,y),T_2^n(x,y))_{n\in\mathbb{N}}$ converges to (x^*,y^*) as $n\to\infty$
$$\begin{aligned} T_1^{n+1}(x,y) &:= T_1^n(T_1(x,y),T_2(x,y))\\ T_2^{n+1}(x,y) &:= T_2^n(T_1(x,y),T_2(x,y))\end{aligned}$$
for all $n\in\mathbb{N}$.
(iii) we have the following estimation
$$\begin{aligned} d(T_1^n(x_0,y_0),x^*) &\le \frac{k^n}{1-k}d(x_0,T_1(x_0,y_0))\\ d(T_2^n(x_0,y_0),y^*) &\le \frac{k^n}{1-k}d(y_0,T_2(x_0,y_0))\end{aligned}$$
(iv) let $F_1,F_2 : X\times X\to X$ be two operators such that, there exist $\epsilon_1,\epsilon_2>0$ with
$$\begin{aligned} d(T_1(x,y),F_1(x,y)) &\le \epsilon_1\\ d(T_2(x,y),F_2(x,y)) &\le \epsilon_2\end{aligned}$$
for all $(x,y)\in X\times X$. If $(a^*,b^*)\in X\times X$ is such that
$$\begin{cases} a^* = F_1(a^*,b^*)\\ b^* = F_2(a^*,b^*)\end{cases}$$
then
$$\max\{d(x^*,a^*),d(y^*,b^*)\} \le \frac{\max\{\epsilon_1,\epsilon_2\}}{1-k}.$$

Proof. (i)- (ii)
We define $T : X\times X\to X\times X$ by
$$T(x,y) = (T_1(x,y),T_2(x,y)).$$
Lets denote $Z := X\times X$ and $d_* : Z\times Z\to\mathbb{R}_+$
$$d_*((x,y),(u,v)) := \frac{1}{2}\max\{d(x,u),d(y,v)\}$$
for all $(x,y),(u,v)\in X\times X$.

Then we have

$$d_*(T(x,y),T(u,v)) = \frac{1}{2}\max\{d(T_1(x,y),T_1(u,v)),d(T_2(x,y),T_2(x,y),T_2(u,v))\}$$

If we denote $(x,y) := z$, $(u,v) := w$ we get that

$$d_*(T(z),T(w)) \leq k \cdot \max\{d(x,u),d(y,v)\} = k \cdot d_*(z,w)$$

for every $z,w \in X \times X$.
Hence we obtained Banach's type contraction condition. By Banach's contraction fixed point theorem we get that there exists a unique element $(x^*,y^*) := z^* \in X \times X$ such that

$$z^* = T(z^*)$$

and it is equivalent with

$$\begin{cases} x^* = T_1(x^*,y^*) \\ y^* = T_2(x^*,y^*) \end{cases}.$$

For each $z \in X \times X$, we have that $T^n(z) \to z^*$ as $n \to \infty$ where

$$\begin{aligned} T^0(z) \quad &: \quad = z, T^1(z) = T(x,y) = (T_1(x,y),T_2(x,y)) \\ T^2(z) \quad &= \quad T(T_1(x,y),T_2(x,y)) = (T_1^2(x,y),T_2^2(x,y)) \end{aligned}$$

and in generally

$$\begin{aligned} T_1^{n+1}(x,y) \quad &: \quad = T_1^n(T_1(x,y),T_2(x,y)) \\ T_2^{n+1}(x,y) \quad &: \quad = T_2^n(T_1(x,y),T_2(x,y)). \end{aligned}$$

We get that $T^n(z) = (T_1^n(z),T_2^n(z)) \to z^* = (x^*,y^*)$ as n$\to \infty$, for all $z = (x,y) \in X \times X$.
So, for all $(x,y) \in X \times X$ we have that

$$\begin{aligned} T_1^n(x,y) \quad &\to \quad x^* \text{ as } n \to \infty \\ T_2^n(x,y) \quad &\to \quad y^* \text{ as } n \to \infty. \end{aligned}$$

(iii) We apply Banach's contraction principle and we have successively

$$\begin{aligned} d(T_1^n(x_0,y_0),x^*) \quad &\leq \quad \frac{k^n}{1-k}d(x_0,T_1(x_0,y_0)) \\ d(T_2^n(x_0,y_0),y^*) \quad &\leq \quad \frac{k^n}{1-k}d(x_0,T_2(x_0,y_0)) \end{aligned}$$

(iv) Let us consider $F : X \times X \to X \times X$ given by

$$F(x,y) = (F_1(x,y),F_2(x,y))$$

and

$$\begin{aligned} d_*(T(x,y),F(x,y)) \quad &= \quad d_*((T_1(x,y),T_2(x,y)),(F_1(x,y),F_2(x,y))) \\ &= \quad \frac{1}{2}\max\{d(T_1(x,y),F_1(x,y)),d(T_2(x,y),F_2(x,y))\} \leq \epsilon \end{aligned}$$

where $\epsilon := \frac{\max\{\epsilon_1,\epsilon_2\}}{2}$.

Then, applying the abstract data dependence theorem related to Banachs contraction principle we get that

$$\max\{d(x^*,a^*),d(y^*,b^*)\} \leq \frac{\max\{\epsilon_1,\epsilon_2\}}{1-k}.$$

We obtain that

$$d_*((x^*,y^*),(a^*,b^*)) \leq \frac{\epsilon}{1-k}.$$

□

3. An application

In this section, we will consider the following problem:

$$\begin{cases} x(t) = \int\limits_a^b K(s,t,x(s),y(s))ds \\ y''(t) = f(s,x(s),y(s)),\ y(a)=0\ y(b)=0 \end{cases} \tag{3.1}$$

This problem is equivalent to

$$\begin{cases} x(t) = \int\limits_a^b K(s,t,x(s),y(s))ds \\ y(t) = -\int\limits_a^b G(t,s)f(s,x(s),y(s))ds, \end{cases}$$

where $G:[a,b]\times[a,b]\to\mathbb{R}$ is given by

$$G(t,s) := \begin{cases} \frac{(s-a)(b-t)}{b-a}, & \text{if } s\leq t \\ \frac{(t-a)(b-s)}{b-a}, & \text{if } s\geq t \end{cases}.$$

Assumption (*) Suppose that $K:[a,b]^2\times\mathbb{R}^2\to\mathbb{R}$ and $f:[a,b]^2\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions and satisfy the following Lipschitz conditions

$$\begin{aligned} |K(t,s,u_1,v_1)-K(t,s,u_2,v_2)| &\leq \alpha|u_1-u_2|+\beta|v_1-v_2| \\ |f(s,u_1,v_1)-f(s,u_2,v_2)| &\leq \gamma|u_1-u_2|+\delta|v_1-v_2|, \end{aligned}$$

for every $t,s\in[a,b]$ and $u_1,v_1,u_2,v_2\in\mathbb{R}$, where $\alpha,\beta,\gamma,\delta>0$ such that

$$\max\left\{(\alpha(b-a)+\gamma\frac{(b-a)^2}{8}),\left(\beta(b-a)+\delta\frac{(b-a)^2}{8}\right)\right\}<1.$$

Let $X=(C[a,b],\|\cdot\|_C)$ be the Banach space of continuous functions endowed with the norm

$$\|x\|_c := \max_{t\in[a,b]}|x(t)|.$$

We define the following operators

$$T_1,T_2: X\times X\to X,\ (x,y)\to T_1(x,y) \text{ and } (x,y)\to T_2(x,y),$$

where

$$T_1(x,y)(t) = \int_a^b K(s,t,x(s),y(s))ds$$

$$T_2(x,y)(t) = -\int_a^b G(t,s)f(s,x(s),y(s))ds.$$

An existence and uniqueness result for the system (3.1) is the following theorem.

Theorem 3.1. *Consider the problem (*3.1*) with* $K, f : [a,b]^2 \times \mathbb{R}^2 \to \mathbb{R}$ *and suppose that Assumption (*) is satisfied. Then there exists a unique solution* (x^*, y^*) *of the problem (*3.1*).*

Proof. We verify that T_1 and T_2 satisfy the hypotheses of Theorem 2.1. Indeed, for every $t \in [a,b]$, we have

$$\begin{aligned}
|T_1(x,y)(t) - T_1(u,v)(t)| &= \left| \int_a^b K(s,t,x(s),y(s))ds - \int_a^b K(s,t,u(s),v(s))ds \right| \\
&\leq \int_a^b |K(s,t,x(s),y(s)) - K(s,t,u(s),v(s))|\, ds \\
&\leq \alpha \int_a^b |x(s)-u(s)|\, ds + \beta \int_a^b |y(s)-v(s)|\, ds \\
&\leq \alpha \|x-u\|_C (b-a) + \beta \|y-v\|_C (b-a).
\end{aligned}$$

Taking the $\max\limits_{t\in[a,b]}$ in the above relation we get that

$$\|T_1(x,y) - T_1(u,v)\|_C \leq \alpha(b-a)\|x-u\|_C + \beta(b-a)\|y-v\|_C .$$

On the other hand, for every $t \in [a,b]$, we have

$$\begin{aligned}
|T_2(x,y)(t) - T_2(u,v)(t)| &= \left| -\int_a^b G(t,s)f(s,x(s),y(s))ds + \int_a^b G(t,s)f(s,u(s),v(s))ds \right| \\
&\leq \int_a^b G(t,s)\, |f(s,u(s),v(s)) - f(s,x(s),y(s))|\, ds \\
&\leq \gamma \int_a^b G(t,s)\, |u(s)-x(s)|\, ds + \delta \int_a^b G(t,s)\, |v(s)-y(s)|\, ds \\
&\leq \gamma \|u-x\|_C \int_a^b G(t,s)ds + \delta \|v-y\|_C \int_a^b G(t,s)ds.
\end{aligned}$$

Taking the $\max\limits_{t\in[a,b]}$ in the above relation we obtain

$$\|T_2(x,y) - T_2(u,v)\|_C \leq \gamma \frac{(b-a)^2}{8} \|u-x\|_C + \delta \frac{(b-a)^2}{8} \|v-y\|_C .$$

Hence we get that

$$\|T_1(x,y)-T_1(u,v)\|_C+\|T_2(x,y)-T_2(u,v)\|_C$$
$$\leq [\alpha(b-a)+\gamma\frac{(b-a)^2}{8}]\,\|x-u\|_C+[\beta(b-a)+\delta\frac{(b-a)^2}{8}]\,\|y-v\|_C$$
$$\leq \max\left\{(\alpha(b-a)+\gamma\frac{(b-a)^2}{8}),(\beta(b-a)+\delta\frac{(b-a)^2}{8})\right\}(\|x-u\|_C+\|y-v\|_C).$$

Since the hypothesis of Theorem 2.1 is satisfied we get that the problem (3.1) has a unique solution on I. □

References

[1] Banach, S., *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math., **3**(1922), 133-181.

[2] Berinde, V., *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces,* Nonlinear Anal., **74**(2011) 7347-7355.

[3] Krasnoselskii, M.A., Zabreiko, P.P., *Geometrical Methods of Nonlinear Analysis*, Springer Verlag Berlin, 1984.

[4] Lakshmikantham, V., Ciric, L., *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70**(2009), 4341-4349.

[5] Rus, I.A., *Principles and Applications of Fixed Point Theory (in Romanian)*, Dacia, Cluj-Napoca, 1979.

[6] Rus, I.A., Petruşel, A., Petruşel G., *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008.

[7] Urs, C.L., *Contributions to the study of the coupled fixed point problem*, Ph. D. Thesis, 2013.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 95–108

Compact hypersurfaces in a locally symmetric manifold

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Abstract. Let M be an n-dimensional compact hypersurface in a locally symmetric manifold N^{n+1}. Denote by S and H the squared norm of the second fundamental form and the mean curvature of M. Let $|\Phi|^2$ be the nonnegative C^2-function on M defined by $|\Phi|^2 = S - nH^2$. In this paper, we prove that if M is oriented and has constant mean curvature and $|\Phi|$ satisfies $P_{n,H,\delta}(|\Phi|) \geq 0$, then (1) $|\Phi|^2 = 0$, (i) $H = 0$ and M is totally geodesic in N^{n+1}, (ii) $H \neq 0$ and M is totally umbilical in the unit sphere $S^{n+1}(1)$; or (2) $|\Phi|^2 = B_H$ if and only if (i) $H = 0$ and M is a Clifford torus, (ii) $H \neq 0$, $n \geq 3$, and M is an $H(r)$-torus with $r^2 < (n-1)/n$, (iii) $H \neq 0$, $n = 2$, and M is an $H(r)$-torus with $0 < r < 1$, $r^2 \neq \frac{1}{2}$. If M has constant normalized scalar curvature R, $\bar{R} = R - 1 \geq 0$, $\tilde{R} = R - \delta$ and S satisfies $\varphi_{n,\bar{R},\tilde{R},\delta}(S) \geq 0$, then (1) M is totally umbilical in $S^{n+1}(1)$; or (2) M is a product $S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $r = \sqrt{\frac{n-2}{n(R+1)}}$, where $P_{n,H,\delta}(x)$ and $\varphi_{n,\bar{R},\tilde{R},\delta}(x)$ are defined by (1.7) and (1.10).

Mathematics Subject Classification (2010): 53B20, 53A10.

Keywords: Locally symmetric, Riemannian manifolds, hypersurfaces, totally umbilical.

1. Introduction

If the ambient manifolds possess very nice symmetry, for example, the sphere, many important results had been obtained in the investigation of the minimal hypersurfaces and hypersurfaces with constant mean curvature or constant scalar curvature. One can see [1], [3], [5], [8], [9], [16] and [19]. For minimal hypersurfaces in a unit sphere, Simons [16], Chern-do Carmo-Kobayashi [5] and Lawson [8] obtained the following famous integral inequality and rigidity result:

Theorem 1.1. ([5, 8, 16]) *Let M be an n-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$. Then*

$$\int_M (S-n)S dv \geq 0. \tag{1.1}$$

In particular, if

$$0 \le S \le n,$$

then $S = 0$ *and* M *is totally geodesic, or* $S \le n$ *and* M *is the Clifford torus*

$$M_{m,n-m} = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right),$$

where S *is the squared norm of the second fundamental form of* M.

In the case of closed hypersurfaces with constant mean curvature H, H. Alencar and M. do Carmo [1] obtained the following integral inequality

$$\int_M |\Phi|^2\left\{n(1+H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - |\Phi|^2\right\}dv \le 0, \tag{1.2}$$

where $|\Phi|^2$ is a nonnegative C^2-function on M defined by $|\Phi|^2 = S - nH^2$.

In order to represent our theorem, we need some notation (one can see [1]). An $H(r)$-torus in $S^{n+1}(1)$ is the product immersion $S^{n-1}(r) \times S^1(\sqrt{1-r^2}) \hookrightarrow R^n \times R^2$, where $S^{n-1}(r) \subset R^n, S^1(\sqrt{1-r^2}) \subset R^2, 0 < r < 1$, are the standard immersions. In some orientation, $H(r)$-torus has principal curvatures given by

$$\lambda_1 = \cdots = \lambda_{n-1} = \sqrt{1-r^2}/r,\ \lambda_n = -r/\sqrt{1-r^2}.$$

For each $H \ge 0$, set

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(1+H^2),$$

and let B_H be the square of the positive root of $P_H(x) = 0$. By using (1.2), Alencar and do Carmo [1] also proved the following result:

Theorem 1.2. ([1]) *Let* M *be a closed and oriented hypersurface in a unit sphere* $S^{n+1}(1)$ *with constant mean curvature* H. *Assume that* $|\Phi|^2 \le B_H$, *then*

(1) *either* $|\Phi|^2 = 0$, M *is totally umbilical; or* $|\Phi|^2 = B_H$.

(2) $|\Phi|^2 = B_H$ *if and only if*

(*i*) $H = 0$ *and* M *is a Clifford torus;*

(*ii*) $H \ne 0$, $n \ge 3$, *and* M *is an* $H(r)$*-torus with* $r^2 < (n-1)/n$;

(*iii*) $H \ne 0$, $n = 2$, *and* M *is an* $H(r)$*-torus with* $0 < r < 1$, $r^2 \ne \frac{1}{2}$.

We should note that Zhong [19] also obtained the following important result:

Theorem 1.3. ([19]) *Let* M *be a closed hypersurface in a unit sphere* $S^{n+1}(1)$ *with constant mean curvature* H. *Then*

(1) *if* $S < 2\sqrt{n-1}$, M *is a small hypersphere* $S^n(r)$ *of radius* $r = \sqrt{\frac{n}{n+S}}$;

(2) *if* $S = 2\sqrt{n-1}$, M *is either a small hypersphere* $S^n(r_0)$ *or an* $H(r)$*-torus* $S^1(r) \times S^{n-1}(t)$, *where* $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ *and* $t^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.

In the case of hypersurfaces with constant scalar curvature, H. Li [9] obtained the following integral inequality

$$\int_M (S - n\bar{R})[n + 2(n-1)\bar{R} - \frac{n-2}{n}S - \frac{n-2}{n}\sqrt{(S + n(n-1)\bar{R})(S - n\bar{R})}]dv \leq 0, \tag{1.3}$$

and the following important result:

Theorem 1.4. ([9]) *Let M be an n-dimensional ($n \geq 3$) compact hypersurface in a unit sphere $S^{n+1}(1)$ with constant normalized scalar curvature R and $\bar{R} = R - 1 \geq 0$. If*

$$n\bar{R} \leq S \leq \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}, \tag{1.4}$$

then either $S = n\bar{R}$ and M is totally umbilical, or

$$S = \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}$$

and M is a product

$$S^1(\sqrt{1-r^2}) \times S^{n-1}(r), \ r = \sqrt{\frac{n-2}{n(R+1)}}.$$

Recently, many researchers begin to study the ambient manifolds which do not possess symmetry in general, for example, the locally symmetric manifolds and the pinched Riemannian manifolds. One can see [6], [7], [12-15] and [17].

Let N^{n+1} denote the locally symmetric manifold whose sectional curvature K_N satisfies the following condition

$$1/2 < \delta \leq K_N \leq 1,$$

at all points $x \in M$. If M is a compact minimal hypersurface in N^{n+1}, Hineva and Belchev [7], Chen [2], Shui and Wu[15], obtained the following important rigidity theorems:

Theorem 1.5. ([7]) *Let M be an n-dimensional compact minimal hypersurface in a locally symmetric manifold N^{n+1}. If*

$$S \leq \frac{(2\delta - 1)n}{n-1}, \tag{1.5}$$

then S is constant.

Theorem 1.6. ([2], [15]) *Let M be an n-dimensional compact minimal hypersurface in a locally symmetric manifold N^{n+1}. If*

$$S \leq (2\delta - 1)n, \tag{1.6}$$

then

(1) *$S = 0$, M is totally geodesic and locally symmetric; or*

(2) *$S = n$, M is a product $V^m(\frac{n}{m}) \times V^{n-m}(\frac{n}{n-m})$, $m = 1, 2, \cdots, n-1$, where $V^r(c)$ denotes the r-demensional Riemannian manifold with constant sectional curvature c.*

By making use of the generalized maximal principle duo to Omori [11] and Yau [18], the author [13] and [14] obtained the following:

Theorem 1.7. ([13]) *Let M be an n-dimensional complete hypersurface with constant mean curvature H in N^{n+1}. Assume that the sectional curvature $K_{n+1in+1i}$ of N^{n+1} at point x of M satisfies $\sum_i \lambda_i K_{n+1in+1i} = nH$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the principal curvatures at point x of M, then*

(1) *if $S < 2\sqrt{n-1}(2\delta - 1)$, M is totally umbilical;*

(2) *if $S = 2\sqrt{n-1}(2\delta - 1)$, $(n \geq 3)$, M is locally a piece of an $H(r)$-torus $S^1(r) \times S^{n-1}(t)$, where $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $t^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

Theorem 1.8. ([14]) *Let M be an n-dimensional complete hypersurface with constant mean curvature H in N^{n+1}. Assume that the sectional curvature $K_{n+1in+1i}$ of N^{n+1} at point x of M satisfies $\sum_i \lambda_i K_{n+1in+1i} = nH$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the principal curvatures at point x of M, then*

(1) *if $S < D(n, H)$, then M is totally umbilical;*

(2) *if $S = D(n, H)$, then*

(i) *$H = 0$ and M is locally a piece of a Clifford torus;*

(ii) *$H \neq 0$, $n \geq 3$ and M is locally a piece of an $H(r)$-torus with $r^2 < (n-1)/n$;*

(iii) *$H \neq 0$, $n = 2$ and M is locally a piece of an $H(r)$-torus with $r^2 \neq 1/2$, $0 < r < 1$, where*

$$D(n, H) = (2\delta - 1)n + \frac{n^3 H^2}{2(n-1)} - \frac{(n-2)nH}{2(n-1)}[n^2 H^2 + 4(n-1)(2\delta - 1)]^{1/2}.$$

Remark 1.1. We should note that in theorem 1.7 and theorem 1.8, the condition $\sum_i \lambda_i K_{n+1in+1i} = nH$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the principal curvatures at point x of M, is needed. But if $\delta = 1$, N^{n+1} is a unit sphere $S^{n+1}(1)$, then $K_{n+1in+1i} = 1$, $\sum_i \lambda_i K_{n+1in+1i} = nH$, by theorem 1.7 and theorem 1.8, we obtain some important results for complete hypersurfaces in a unit sphere $S^{n+1}(1)$ (see [13] and [14]).

In this paper, we shall study the compact hypersurfaces with constant mean curvature and constant scalar curvature in a locally symmetric manifold N^{n+1}. In order to present our theorems, we denote by H the mean curvature of M and S the squared norm of the second fundamental form of M. We define a polynomial $P_{n,H,\delta}(x)$ by

$$P_{n,H,\delta}(x) = \left(\frac{5\delta - 3}{2} + H^2\right) nx^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} Hx^3 - x^4 - \frac{1}{2}(1-\delta)n^2 H^2. \quad (1.7)$$

We shall prove the following:

Main Theorem 1.1. *Let M be an n-dimensional compact and oriented hypersurface with constant mean curvature in a locally symmetric manifold N^{n+1}. Then*

$$\int_M \left\{ \left(\frac{5\delta - 3}{2} + H^2\right) n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2 H^2 \right\} dv \leq 0. \quad (1.8)$$

In particular, if $|\Phi|$ satisfies

$$P_{n,H,\delta}(|\Phi|) \geq 0, \quad (1.9)$$

then either

(1) $|\Phi|^2 = 0$ *and*
(*i*) $H = 0$ *and* M *is totally geodesic in* N^{n+1},
(*ii*) $H \neq 0$ *and* M *is totally umbilical in the unit sphere* $S^{n+1}(1)$; *or*
(2) $|\Phi|^2 = B_H$, *if and only if*
(*i*) $H = 0$ *and* M *is a Clifford torus.*
(*ii*) $H \neq 0$, $n \geq 3$, *and* M *is an* $H(r)$*-torus with* $r^2 < (n-1)/n$.
(*iii*) $H \neq 0$, $n = 2$, *and* M *is an* $H(r)$*-torus with* $0 < r < 1$, $r^2 \neq \frac{1}{2}$.

We also define a function $\varphi_{n,\bar{R},\tilde{R},\delta}(x)$ by

$$\begin{aligned}\varphi_{n,\bar{R},\tilde{R},\delta}(x) =& \frac{n-1}{n}(x - n\bar{R})[\frac{5\delta-3}{2}n + 2(n-1)\bar{R} - \frac{n-2}{n}x \\ &- \frac{n-2}{n}\sqrt{(x + n(n-1)\tilde{R})(x - n\bar{R})}] \\ &- \frac{1}{2}(n-1)(1-\delta)\left[(5\delta-3)n + 2(n-1)\bar{R} + n\tilde{R} + \frac{3n-2}{n(n-1)}x\right].\end{aligned} \tag{1.10}$$

We shall prove the following:

Main Theorem 1.2. *Let* M *be an* n*-dimensional compact hypersurface in a locally symmetric manifold* N^{n+1} *with constant normalized scalar curvature* R *and* $\bar{R} = R - 1 \geq 0$, $\tilde{R} = R - \delta$. *Then*

$$\begin{aligned}\int_M \Big\{ &\frac{n-1}{n}(S - n\bar{R})\Big[\frac{5\delta-3}{2}n + 2(n-1)\bar{R} - \frac{n-2}{n}S \\ &- \frac{n-2}{n}\sqrt{(S + n(n-1)\tilde{R})(S - n\bar{R})}\Big] \\ &- \frac{1}{2}(n-1)(1-\delta)\Big[(5\delta-3)n + 2(n-1)\bar{R} + n\tilde{R} + \frac{3n-2}{n(n-1)}S\Big]\Big\}dv \leq 0.\end{aligned} \tag{1.11}$$

In particular, if S *satisfies*

$$\varphi_{n,\bar{R},\tilde{R},\delta}(S) \geq 0, \tag{1.12}$$

then either
(1) $S = n\bar{R}$ *and* M *is totally umbilical in the unit sphere* $S^{n+1}(1)$; *or*
(2) $S = \frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\}$ *and* M *is a product*

$$S^1(\sqrt{1-r^2}) \times S^{n-1}(r),\ r = \sqrt{\frac{n-2}{n(R+1)}}.$$

Remark 1.2. If $\delta = 1$, that is, N^{n+1} is the unit sphere $S^{n+1}(1)$, (1.8) reduces to (1.1) if $H = 0$ and (1.2). Main theorem 1.1 reduces to the theorem 1.1, if $H = 0$, of Simons, Chern-do Carmo-Kobayashi and Lawson [16, 5, 8] and theorem 1.2 of Alencar and do Carmo [1]. We should note that (1.11) reduces to (1.3) and Main theorem 1.2 reduces to the theorem 1.4 of H. Li [9].

2. Preliminaries

Let N^{n+1} be the locally symmetric manifold with sectional curvature K_N satisfies the condition $1/2 < \delta \leq K_N \leq 1$ at all points $x \in M$, M be the compact oriented hypersurface in N^{n+1}. Let $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ be a local frame of orthonormal vector fields in N^{n+1} such that, restricted to M the vectors $\{e_1, e_2, \ldots, e_n\}$ are tangent to M, the vector e_{n+1} is normal to M. We shall make use of the following convention on the ranges of indies:

$$1 \leq i, j, k, \ldots \leq n, \qquad 1 \leq A, B, C, \ldots \leq n+1.$$

Let $\{\omega_{ij}\}$ be the connection 1-form of M, $h = \{h_{ij}\}$ be the second fundamental form of M. The squared norm of h is denoted by $S = \sum_{i,j=1}^{n}(h_{ij})^2$. Let $\{K_{ABCD}\}$ and $\{R_{ijkl}\}$ be the components of the curvature tensors of N^{n+1} and M, respectively. Since N^{n+1} is a locally symmetric manifold, we have

$$K_{ABCD,E} = 0. \tag{2.1}$$

Let $\{h_{ijk}\}$ and $\{h_{ijkl}\}$ be the covariant derivative of $\{h_{ij}\}$ and $\{h_{ijk}\}$, respectively. We call $\xi = \frac{1}{n}\sum_{i=1}^{n} h_{ii}e_{n+1}$ the mean curvature vector of M. The mean curvature of M is given by $H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}$.

It is well known that for an arbitrary hypersurface M of N^{n+1}, we have

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.2}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + (1/2)\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.3}$$

$$R_{ijkl} = K_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.4}$$

$$n(n-1)R = \sum_{i,j} K_{ijij} + n^2H^2 - S, \tag{2.5}$$

where R is the normalized scalar curvature of M.

The Codazzi equation and Ricci identities are

$$h_{ijk} - h_{ikj} = K_{n+1ikj} = -K_{n+1ijk}, \tag{2.6}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}. \tag{2.7}$$

Let f be a smooth function on M. The first, second covariant derivatives f_i, f_{ij} and Laplacian of f are defined by

$$df = \sum_i f_i\theta_i, \quad \sum_j f_{ij}\theta_j = df_i + \sum_j f_j\theta_{ji}, \quad \Delta f = \sum_i f_{ii}. \tag{2.8}$$

We introduce an operator $\Box$ due to Cheng-Yau [4] by

$$\Box f = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{2.9}$$

Since M is compact, the operator $\Box$ is self-adjoint (see [4]) if and only if

$$\int_M (\Box f)g\,dv = \int_M f(\Box g)\,dv, \tag{2.10}$$

where f and g are smooth functions on M.

Setting $f = nH$ in (2.9), from (2.5), we have

$$\begin{aligned}\square(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \sum_i (nH)(nH)_{ii} - \sum_{i,j} h_{ij}(nH)_{ij} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH_i)^2 - \sum_{i,j} h_{ij}(nH)_{ij} \\ &= \frac{1}{2}n(n-1)\Delta R - \frac{1}{2}\Delta(\sum_{i,j} K_{ijij}) + \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}(nH)_{ij}.\end{aligned} \tag{2.11}$$

The Laplacian Δh_{ij} of the second fundamental form h of M is defined by $\Delta h_{ij} = \sum_{k=1}^{n} h_{ijkk}$. From Chern-do Carmo-Kobayashi [5], by a simple and direct calculation, we have

$$\begin{aligned}\Delta h_{ij} =& nHK_{n+1in+1j} - \sum_k K_{n+1kn+1k}h_{ij} + nH\sum_k h_{ik}h_{kj} \\ & - Sh_{ij} + \sum_{k,l}\{K_{lkik}h_{lj} + K_{lkjk}h_{li} + 2K_{lijk}h_{lk}\} + (nH)_{ij}.\end{aligned} \tag{2.12}$$

Choose a local frame of orthonormal vector fields $\{e_i\}$ such that at arbitrary point x of M

$$h_{ij} = \lambda_i\delta_{ij}, \tag{2.13}$$

then at point x we have

$$\begin{aligned}\sum_{i,j} h_{ij}\Delta h_{ij} =& nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i K_{n+1in+1i} \\ & + \sum_{i,j}(\lambda_i - \lambda_j)^2 K_{ijij} - S^2 + nH\sum_i \lambda_i^3 + \sum_i \lambda_i(nH)_{ii}.\end{aligned} \tag{2.14}$$

The following result due to Okumura [10], Alencar and do Carmo [1] will be very important to us.

Lemma 2.1. ([10], [1]) *Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$, and $\sum_i \mu_i^2 = \beta^2$, where $\beta = const. \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.15}$$

and equality holds in the right-hand (left-hand) side if and only if $(n-1)$ of the μ_i's are non-positive and equal ($(n-1)$ of the μ_i's are nonnegative and equal).

3. Proof of Main Theorem 1.1

Proof. We suppose that the mean curvature of M is constant and put a nonnegative C^2-function $|\Phi|^2$ by

$$|\Phi|^2 = S - nH^2, \tag{3.1}$$

then M is totally umbilical if and only if $|\Phi|^2 = 0$. Since $\frac{1}{2} < \delta \le K_N \le 1$, we have

$$\begin{aligned} nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i K_{n+1in+1i} & \\ = -\frac{1}{2}\sum_{i,j}(\lambda_i-\lambda_j)^2 K_{n+1in+1i} - \frac{1}{2}S\sum_i K_{n+1in+1i} + \frac{n}{2}\sum_i \lambda_i^2 K_{n+1in+1i} & \\ \ge -\frac{1}{2}\sum_{i,j}(\lambda_i-\lambda_j)^2 - \frac{1}{2}Sn + \frac{n}{2}\delta\sum_i \lambda_i^2 & \\ = -\frac{1}{2}[nS - 2n^2H^2 + nS] - \frac{1}{2}nS + \frac{\delta}{2}nS & \\ = -\frac{n}{2}(3-\delta)|\Phi|^2 - \frac{n^2}{2}(1-\delta)H^2, & \end{aligned} \tag{3.2}$$

$$\sum_{i,j}(\lambda_i-\lambda_j)^2 K_{ijij} \ge \delta\sum_{i,j}(\lambda_i-\lambda_j)^2 = 2n\delta(S-nH^2) = 2n\delta|\Phi|^2. \tag{3.3}$$

Since $\sum_i (H-\lambda_i) = 0$, $\sum_i (H-\lambda_i)^2 = S - nH^2 = |\Phi|^2$, by Lemma 2.1, we have

$$\left|\sum(H-\lambda_i)^3\right| \le \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3.$$

Thus

$$\begin{aligned} nH\sum_i \lambda_i^3 &= 3nH^2S - 2n^2H^4 - nH\sum_i(H-\lambda_i)^3 \\ &\ge 3nH^2(|\Phi|^2+nH^2) - 2n^2H^4 - n|H|\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3 \\ &= 3nH^2|\Phi|^2 + n^2H^4 - n|H|\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3. \end{aligned} \tag{3.4}$$

From (3.1)-(3.4), (2.14) and $H = const.$, we have

$$\sum_{i,j} h_{ij}\Delta h_{ij} \ge \left(\frac{5\delta-3}{2} + H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2. \tag{3.5}$$

Therefore, we have

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij} \\ &\ge \left(\frac{5\delta-3}{2} + H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2. \end{aligned} \tag{3.6}$$

Since M is compact and oriented, we can choose an orientation for M such that $H \ge 0$. From (3.6), we have

$$\int_M \left\{\left(\frac{5\delta-3}{2} + H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2\right\} dv \le 0. \tag{3.7}$$

From (1.9) and (3.7), we have

$$\left(\frac{5\delta-3}{2}+H^2\right)n|\Phi|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3-|\Phi|^4-\frac{1}{2}(1-\delta)n^2H^2=0. \tag{3.8}$$

(1) If $|\Phi|^2=0$, from (3.8), we have $(1-\delta)n^2H^2=0$. This implies that $H=0$ and M is totally geodesic in N^{n+1}, or $\delta=1$, that is, N^{n+1} is the unit sphere $S^{n+1}(1)$ and M is totally umbilical in $S^{n+1}(1)$.

(2) If $|\Phi|^2\neq 0$, we have the equalities in (3.2)-(3.7) hold. Therefore, we have for any i,j,k

$$h_{ijk}=0. \tag{3.9}$$

Putting

$$\vartheta=-\sum_{i,j,k}h_{ij}(K_{n+1kikj}+K_{n+1ijkk}).$$

From (2.17) of Chern-do Carmo-Kobayashi [5], we have

$$K_{n+1ijk,l}=K_{n+1ijkl}-K_{n+1in+1k}h_{jl}-K_{n+1ijn+1}h_{kl}+\sum_m K_{mijk}h_{ml}, \tag{3.10}$$

where $K_{n+1ijk,l}$ is the restriction to M of the covariant derivative $K_{ABCD,E}$ of K_{ABCD} as a curvature tensor of N^{n+1}. Since we suppose that N^{n+1} is a locally symmetric one, we have $K_{ABCD,E}=0$. From (3.10), we obtain that

$$K_{n+1ijkl}=K_{n+1in+1k}h_{jl}+K_{n+1ijn+1}h_{kl}-\sum_m K_{mijk}h_{ml}. \tag{3.11}$$

From (3.11) and the equalities of (3.2) and (3.3), we have

$$\begin{aligned}\vartheta=&nH\sum_i\lambda_iK_{n+1in+1i}-S\sum_k K_{n+1kn+1k}\\&+\sum_{i,j,k,m}h_{ij}(h_{mj}K_{mkik}+h_{mk}K_{mijk})\\=&-\frac{n}{2}(3-\delta)|\Phi|^2-\frac{n^2}{2}(1-\delta)H^2+\frac{1}{2}\sum_{i,k}(\lambda_i-\lambda_k)^2K_{ikik}\\=&\frac{3}{2}n(\delta-1)|\Phi|^2-\frac{1}{2}(1-\delta)n^2H^2\leq\frac{3}{2}n(\delta-1)|\Phi|^2.\end{aligned} \tag{3.12}$$

On the other hand, we define the globally vector field ϖ by

$$\varpi=\sum_{i,j,k}(h_{ik}K_{n+1jij}+h_{ij}K_{n+1ijk})e_k.$$

The divergence of ϖ can be written by

$$\mathtt{div}\varpi=\sum_{i,j,k}\nabla_k(h_{ik}K_{n+1jij}+h_{ij}K_{n+1ijk}).$$

From (3.9), we obtain that

$$\vartheta=\sum_{i,j,k}(h_{ikk}K_{n+1jij}+h_{ijk}K_{n+1ijk})-\mathtt{div}\varpi=-\mathtt{div}\varpi. \tag{3.13}$$

From (3.12) and (3.13), we have $\mathtt{div}\varpi \geq \frac{3}{2}n(1-\delta)|\Phi|^2 \geq 0$. Since M is compact, by the Green's divergence theorem, we have $\int_M \frac{3}{2}n(1-\delta)|\Phi|^2 = 0$. Since we suppose that $|\Phi|^2 \neq 0$, we have $\delta = 1$. We infer that N^{n+1} is the unit sphere $S^{n+1}(1)$ and (3.8) reduces to

$$|\Phi|^2\{n(1+H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - |\Phi|^2\} = 0. \tag{3.14}$$

From (3.14), we can get $|\Phi|^2 = B_H$. Therefore, from theorem 1.2 of Alencar and do Carmo [1], we know that Main theorem 1.1 is true. This completes the proof of Main Theorem 1.1. □

4. Proof of Main Theorem 1.2

In this section, we shall suppose that the normalized scalar curvature R of M is constant. We first need the following Lemma:

Lemma 4.1. *Let M be an n-dimensional hypersurface in a locally symmetric manifold N^{n+1} with constant normalized scalar curvature R and $\bar{R} = R - 1 \geq 0$. Then*

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2. \tag{4.1}$$

Proof. Taking the covariant derivative of (2.5), and using the fact $K_{ABCD,E} = 0$ and $R = const.$, we get

$$n^2 H H_k = \sum_{i,j} h_{ij}h_{ijk}.$$

It follows that

$$\sum_k n^4 H^2 (H_k)^2 = \sum_k \left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 \leq \left(\sum_{i,j} h_{ij}^2\right)\sum_{i,j,k} h_{ijk}^2, \tag{4.2}$$

that is

$$n^4 H^2 |\nabla H|^2 \leq S\sum_{i,j,k} h_{ijk}^2. \tag{4.3}$$

On the other hand, from (2.5), $K_{ijij} \leq 1$ and $R-1 \geq 0$, we have $n^2H^2 - S \geq 0$. From (4.3), we know that lemma 4.1 follows.

Now we shall prove Main theorem 1.2. From (2.1), (2.11), (2.13) and (3.6), we have

$$\begin{aligned}\Box(nH) \geq &\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 \\ &+ \left(\frac{5\delta-3}{2} + H^2\right)n|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 - |\Phi|^4 - \frac{1}{2}(1-\delta)n^2H^2.\end{aligned} \tag{4.4}$$

Putting $\bar{R} = R - 1$, $\tilde{R} = R - \delta$, by (2.5), we know that

$$\frac{1}{n^2}[n(n-1)\bar{R} + S] \leq H^2 \leq \frac{1}{n^2}[n(n-1)\tilde{R} + S], \tag{4.5}$$

$$\frac{n-1}{n}(S-n\tilde{R}) \le |\Phi|^2 = S-nH^2 \le \frac{n-1}{n}(S-n\bar{R}). \tag{4.6}$$

By (2.5), (4.5) and (4.6), we get from (4.4)

$$\begin{aligned}
\square(nH) \ge & \frac{n-1}{n}(S-n\tilde{R})\left[\frac{5\delta-3}{2}n+(n-1)\bar{R}+\frac{1}{n}S\right] \\
& -\frac{n-1}{n}(S-n\bar{R})\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})} \\
& -\left[\frac{n-1}{n}(S-n\bar{R})\right]^2-\frac{1}{2}(1-\delta)(S+n(n-1)\tilde{R}) \\
= & \frac{n-1}{n}(S-n\bar{R})\left[\frac{5\delta-3}{2}n+(n-1)\bar{R}+\frac{1}{n}S\right] \\
& +\frac{n-1}{n}(n\bar{R}-n\tilde{R})\left[\frac{5\delta-3}{2}n+(n-1)\bar{R}+\frac{1}{n}S\right] \\
& -\frac{n-1}{n}(S-n\bar{R})\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})} \\
& -\left[\frac{n-1}{n}(S-n\bar{R})\right]^2-\frac{1}{2}(1-\delta)(S+n(n-1)\tilde{R}) \\
= & \frac{n-1}{n}(S-n\bar{R})\left[\frac{5\delta-3}{2}n+2(n-1)\bar{R}-\frac{n-2}{n}S\right. \\
& \left.-\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})}\right] \\
& -\frac{1}{2}(n-1)(1-\delta)\left[(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S\right],
\end{aligned} \tag{4.7}$$

where $\bar{R}-\tilde{R}=\delta-1$ is used. Since M is compact, from (2.10), we have

$$\int_M \square(nH)dv = 0.$$

By (4.7), we get

$$\begin{aligned}
\int_M \Bigg\{ & \frac{n-1}{n}(S-n\bar{R})\Big[\frac{5\delta-3}{2}n+2(n-1)\bar{R}-\frac{n-2}{n}S \\
& -\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})}\Big] \\
& -\frac{1}{2}(n-1)(1-\delta)\Big[(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S\Big]\Bigg\}dv \le 0.
\end{aligned} \tag{4.8}$$

From (1.12) and (4.8), we have

$$\begin{aligned}&\frac{n-1}{n}(S-n\bar{R})\Big[\frac{5\delta-3}{2}n+2(n-1)\bar{R}-\frac{n-2}{n}S\\&\quad-\frac{n-2}{n}\sqrt{(S+n(n-1)\tilde{R})(S-n\bar{R})}\Big]\\&\quad-\frac{1}{2}(n-1)(1-\delta)\Big[(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S\Big]=0.\end{aligned}\tag{4.9}$$

(1) If $S=n\bar{R}$, from (4.9), we have

$$-\frac{1}{2}(n-1)(1-\delta)\left[(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S\right]=0.\tag{4.10}$$

Since $S=n\bar{R}$ and $\tilde{R}=\bar{R}+1-\delta$, from $\delta>\frac{1}{2}$ and $\bar{R}\geq 0$, we have

$$(5\delta-3)n+2(n-1)\bar{R}+n\tilde{R}+\frac{3n-2}{n(n-1)}S=2(2\delta-1)n+\frac{3n-2}{n-1}n\bar{R}>0.$$

Thus, from (4.10), we have $\delta=1$, that is, N^{n+1} is the unit sphere $S^{n+1}(1)$ and $S=n\bar{R}$. From (4.6), we have $|\Phi|^2=0$ and M is totally umbilical in $S^{n+1}(1)$.

(2) If $S\neq n\bar{R}$, we know that the equalities in (4.8), (4.7), (4.4), (3.2) and (3.3) hold. We infer that

$$K_{ijij}=\delta,\quad \sum_{i,j,k}h_{ijk}^2=n^2|\nabla H|^{\cdot}\tag{4.11}$$

From (4.9), we have $S=const.$, (2.5) and (4.11) imply that $H=const.$ and $h_{ijk}=0$, for any i,j,k. Putting

$$\vartheta=-\sum_{i,j,k}h_{ij}(K_{n+1kikj}+K_{n+1ijkk}),$$

and making use of the same assertion as in the proof of Main theorem 1.1, we conclude that $\delta=1$, that is, N^{n+1} is the unit sphere $S^{n+1}(1)$, and (4.9) reduces to

$$\begin{aligned}&\frac{n-1}{n}(S-n\bar{R})[n+2(n-1)\bar{R}-\frac{n-2}{n}S\\&\quad-\frac{n-2}{n}\sqrt{(S+n(n-1)\bar{R})(S-n\bar{R})}]=0.\end{aligned}\tag{4.12}$$

From (4.12), we have

$$S=\frac{n}{(n-2)(n\bar{R}+2)}\{n(n-1)\bar{R}^2+4(n-1)\bar{R}+n\}.$$

Therefore, from theorem 1.4 of H. Li [9], we know that Main theorem 1.2 is true. This completes the proof of Main Theorem 1.2. □

References

[1] Alencar, H., do Carmo, M., *Hypersurfaces with constant mean curvature in spheres*, Proc. of the Amer. Math. Soc., **120**(1994), 1223-1229.

[2] Chen, Q., *Minimal hypersurfaces of a locally symmetric space*, Chinese Science Bulletin, **38**(1993), 1057-1059.

[3] Cheng, Q.M., Shu, S.C., Suh, I.J., *Compact hypersurfaces in a unit sphere*, Proc. Royal Soc. Edinburgh, **135A**(2005), 1129-1137.

[4] Cheng, S.Y., Yau, S.T., *Hypersurfaces with constant scalar curvature*, Math. Ann., **225**(1977), 195-204.

[5] Chern, S.S., Do Carmo, M., Kobayashi, S., *Minimal submanifolds of a sphere with second fundamental form of constant length*, in Functional Analysis and Related Fields (F. Brower, ed.), Springer-Verlag, Berlin, 1970, 59-75.

[6] Fontenele, F., *Submanifolds with parallel mean curvature vector in pinched Riemannian manifolds*, Pacific J. of Math., **177**(1997), 47-70.

[7] Hlineva, S., Belchev, E., *On the minimal hypersurfaces of a locally symmetric manifold*, Lecture Notes in Math., **1481**(1990), 1-4.

[8] Lawson, H.B., *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math., **89**(1969), 187-197.

[9] Li, H., *Hypersurfaces with constant scalar curvature in space forms*, Math. Ann., **305**(1996), 665-672.

[10] Okumura, M., *Hypersurfaces and a pinching problem on the second fundamental tensor*, Amer. J. Math., **96**(1974), 207-213.

[11] Omor, H., *Isometric immersion of Riemmanian manifolds*, J. Math. Soc. Japan, **19**(1967), 205-214.

[12] Shiohama, K., Xu, H.W., *A general rigidity theorem for complete submanifolds*, Nagoya Math. J., **150**(1998), 105-134.

[13] Shu, S.C., Liu, S.Y., *Complete Hypersurfaces with Constant Mean Curvature in Locally Symmetric Manifold (I)*, Chinese Ann. of Math., **25A**(2004), 99-104.

[14] Shu, S.C., Liu, S.Y., *Complete Hypersurfaces with Constant Mean Curvature in Locally Symmetric Manifold (II)*, Advances in Math. (Chinese), **33**(2004), 563-569.

[15] Shui, N.X., Wu, G.Q., *Minimal hypersurfaces of a locally symmetric manifold*, Chinese Ann. of Math., **16A**(1995), 687-691.

[16] Simons, J., *Minimal varieties in Riemannian manifolds*, Ann. of Math., **88**(1968), 62-105.

[17] Xu, H.W., *On closed minimal submanifolds in pinched Riemannian manifolds*, Trans. Amer. Math. Soc., **347**(1995), 1743-1751.

[18] Yau, S.T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math., **28**(1975), 201-228.

[19] Zhong, H., *Hypersurfaces in a sphere with constant mean curvature*, Pro. of the Amer. Math. Soc., **125**(1997), 1193-1196.

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Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 1, 109–116

Convergence of the Neumann series for a Helmholtz-type equation

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Abstract. We pursue a constructive solution to the Robin problem of a Helmholtz-type equation in the form of a single layer potential. This representation method leads to a boundary integral equation. We study the problem on a bounded planar domain of class C^2. We prove the convergence of the Neumann series of iterations of the layer potential operators to the solution of the boundary integral equation. This study is inspired by several recent papers which cover the iteration techniques. In [7], [8], [9], D. Medkova obtained results regarding the successive approximation method for Neumann, Robin and transmission problems.

Mathematics Subject Classification (2010): 76D10, 35J05, 81Q05, 65N38.

Keywords: Helmholtz equation, Robin problem, single layer potential, integral equation method, successive approximation.

1. Introduction

The study of the Helmholtz equation has received broad attention in [1], [2], [3]. The equation is connected to several physical phenomena. The general form of the equation is

$$\Delta u + \lambda^2 u = 0$$

with $Im\ \lambda > 0$. In this paper we study the Robin problem for the Helmholtz equation in a bounded planar domain $D \subset \mathbb{R}^2$ of class C^2. We present an iteration technique which is suited to be used for a numerical computation of the solution of the Robin problem. The technique is based on the Neumann series of iterations of the layer potential operators. In the past the technique was studied by W.L. Wendland (see [11], [12]). More recently the Neumann series were used by D. Medkova for several problems associated with the Stokes system, including Robin and transmission problems, in the papers [7], [8], [9].

In general, the boundary value problems associated with the Helmholtz equation are not uniquely solvable when coupled with the general condition $Im\ \lambda > 0$.

The values of λ for which the Helmholtz equation is not uniquely solvable are called irregular frequencies (see also [2], section 2.1). In this paper we restrict the study of the equation to the case $Re\ \lambda = 0$. In this particular case the equation is

$$\Delta u - k^2 u = 0, \tag{1.1}$$

with $k > 0$. This equation is also known as the Klein-Gordon equation. It is connected to quantum mechanics. We consider the Robin boundary condition

$$\frac{\partial u}{\partial \nu} + \alpha u = g, \tag{1.2}$$

where ν is the outward unit normal vector of D, $\alpha > 0$ is a constant and $g \in C(\partial D, \mathbb{R}^2)$.

We pursue the solution $u \in C^2(D, \mathbb{R}^2) \cap C^1(\overline{D}, \mathbb{R}^2)$ of the boundary value problem (1.1),(1.2) in the form of a single layer potential

$$u(x) = \int_{\partial D} E(x, y)h(y)\, dy,$$

where $E(x, y)$ is the fundamental solution of the equation (1.1) and $h \in C(\partial D, \mathbb{R}^2)$ is a boundary function called density. The function u defined above as a single layer potential solves equation (1.1).

The fundamental solution of the Helmholtz equation in $\mathbb{R}^2$ is given by

$$E(x, y) = \frac{1}{2\pi} K_0(k|x - y|) = \frac{i}{4} H_0^{(1)}(k|x - y|),$$

where K is the modified Bessel function of the second kind and $H^{(1)}$ is the Hankel function of the first kind.

We will require to assume that the domains D have smooth boundaries because several proofs in this paper will use Green's formula and the compactness of the layer potential operators which are true for smooth domains. There are several established properties regarding the layer potentials on smooth domains (see [1], [2]). We simply state the following well known facts. In the sequel we will assume that the bounded domain $D \subset \mathbb{R}^2$ is of class C^2.

Definition 1.1. *For $h \in C(\partial D, \mathbb{C}^2)$ define the single layer potential S with density h by*

$$Sh(x) = \int_{\partial D} E(x, y)h(y)\, dy,\ x \in \mathbb{R}^2 \setminus \partial D,$$

and the double layer potential D with density h by

$$Dh(x) = \int_{\partial D} \frac{\partial E(x, y)}{\partial \nu} h(y)\, dy,\ x \in \mathbb{R}^2 \setminus \partial D.$$

Lemma 1.2. *The single layer potential operator $S : C(\partial D, \mathbb{C}^2) \to C(\partial D, \mathbb{C}^2)$ is given by*

$$Sh(x) = \int_{\partial D} E(x, y)h(y)\, dy = \lim_{z \to x} \int_{\partial D} E(z, y)h(y)\, dy,\ x \in \partial D.$$

The double layer potential operator $K : C(\partial D, \mathbb{C}^2) \to C(\partial D, \mathbb{C}^2)$ *is given by*

$$Kh(x) = \int_{\partial D} \frac{\partial E(x,y)}{\partial \nu} h(y)\, dy = \lim_{z \to x,\, z \in D} Dh(z) + \frac{1}{2} h(x),\ x \in \partial D.$$

The single layer potential operator satisfies

$$\frac{\partial Sh(x)}{\partial \nu} = \frac{1}{2} h(x) + K' h(x),\ x \in \partial D,$$

where K' *is the adjoint operator of* K.

The equalities above are called limiting relations. They relate the values of the layer potentials in the domain with the boundary values.

Lemma 1.3. *The operators* S, K *and* K' *are compact.*

Furthermore there are several other properties of the layer potential operators. If we define the operators $S,\ K,\ K'$ on the space $L^2(\partial D)$, then $S,\ K,\ K'$ are bounded.

2. Convergence of the Neumann series

Consider a solution of the problem (1.1),(1.2) in the form of a single layer potential $u = Sh$ with $h \in C(\partial D, \mathbb{R}^2)$. Since the single layer potential satisfies the limiting relations in Lemma 1.2, the Robin boundary condition (1.2) becomes

$$\frac{1}{2} h(x) + K' h(x) + \alpha Sh(x) = g(x),\ x \in \partial D. \tag{2.1}$$

This is a boundary integral equation. The invertibility of the operator

$$\frac{1}{2} I + K' + \alpha S$$

and the solvability of the equation were proved in [4]. The proof of the invertibility uses the Fredholm theory. Since the operators S and K' are compact, the operator $I/2 + K' + \alpha S$ has index 0. One can use Green's formula to prove the injectivity of the operator, from which it follows that the operator $I/2 + K' + \alpha S$ is invertible.

We will prove the convergence of a series of iterations of the layer potential operators to the solution of the boundary integral equation (2.1). The series is called Neumann series. In this way we give a constructive solution to the boundary value problem (1.1),(1.2).

In the proof of the convergence we will use the following lemmas which were proved for the Stokes system by D.Medkova in [7], [8], [10]. We prove the lemmas corresponding to the layer potential operators associated with the Klein-Gordon equation. They are instrumental in finding a range for the spectrum of the operator $I/2 + K' + \alpha S$.

Lemma 2.1. *Denote* $\|S\|_{L^2(\partial D, \mathbb{C}^2)} = M$. *Let* $h \in C(\partial D, \mathbb{C}^2)$. *Then*

$$\int_{\partial D} |Sh|^2\, dy \le M \int_{\partial D} h \cdot S\overline{h}\, dy.$$

Proof. For $f, g \in L^2(\partial D, \mathbb{C}^2)$ define

$$\langle f, g\rangle = \int_{\partial D} f \cdot S\overline{g}\, dy.$$

The integral operator S has a symmetric kernel $E(x, y)$. Therefore the product defined before is conjugate symmetric. Since the kernel $E(x, y)$ is positive, we deduce that the product $\langle\cdot,\cdot\rangle$ is positive definite. Then $\langle\cdot,\cdot\rangle$ is an inner product on $L^2(\partial D, \mathbb{C}^2)$. Holder's inequality gives

$$\int_{\partial D} |Sh|^2\, dy = \left(\sup_{f\in L^2, \|f\|=1} |\langle f, h\rangle|\right)^2.$$

From the Schwartz inequality we deduce

$$\int_{\partial D} |Sh|^2\, dy \le \langle h, h\rangle \sup_{f\in L^2, \|f\|=1} \langle f, f\rangle,$$

from which we get

$$\int_{\partial D} |Sh|^2\, dy \le \langle h, h\rangle \sup_{f\in L^2, \|f\|=1} \left(\int_{\partial D} |Sf|^2\, dy\right)^{1/2}.$$

This means

$$\int_{\partial D} |Sh|^2\, dy \le \|S\|_{L^2(\partial D, \mathbb{C}^2)} \int_{\partial D} h \cdot S\overline{h}\, dy.$$

The lemma is proved. □

Lemma 2.2. *Let $h \in C(\partial D, \mathbb{C}^2)$. Then*

$$\int_{\partial D} \overline{Sh} \cdot \left(\frac{\partial Sh}{\partial \nu} + \alpha Sh\right) dy = \int_D \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy + \int_{\partial D} \alpha |Sh|^2\, dy.$$

Proof. If we apply Green's formula

$$\int_G (\psi\Delta\varphi + \nabla\varphi \cdot \nabla\psi)\, dy = \int_{\partial G} \psi \frac{\partial \varphi}{\partial \nu} dy$$

for the vector components of $\overline{Sh}$ and Sh on the domain D, we obtain

$$\int_D \left(\overline{Sh} \cdot \Delta Sh + \nabla\overline{Sh} \cdot \nabla Sh\right) dy = \int_{\partial D} \overline{Sh} \cdot \frac{\partial Sh}{\partial \nu} dy.$$

The equality implies

$$\int_D \left(\overline{Sh} \cdot k^2 Sh + |\nabla Sh|^2\right) dy = \int_{\partial D} \overline{Sh} \cdot \frac{\partial Sh}{\partial \nu} dy,$$

and therefore

$$\int_{\partial D} \overline{Sh} \cdot \left(\frac{\partial Sh}{\partial \nu} + \alpha Sh\right) dy = \int_D \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy + \int_{\partial D} \alpha |Sh|^2\, dy. \quad \square$$

Lemma 2.3. *Let $h \in C(\partial D, \mathbb{C}^2)$. Then*

$$\int_{\mathbb{R}^2\setminus\partial D} \left(k^2|Sh|^2 + |\nabla Sh|^2\right) dy = \int_{\partial D} h \cdot \overline{Sh}\, dy.$$

Proof. From Green's formula we have

$$\int_D \left(\overline{Sh} \cdot k^2 Sh + |\nabla Sh|^2\right) dy = \int_{\partial D} \overline{Sh} \cdot \frac{\partial Sh}{\partial \nu} dy,$$

Using the limiting relations in Lemma 1.2, we get

$$\int_D \left(k^2 |Sh|^2 + |\nabla Sh|^2\right) dy = \int_{\partial D} \overline{Sh} \cdot (h/2 + K'h) dy. \tag{2.2}$$

If we apply Green's formula on the expanding domains $D^c \cap B(0,r)$ that converge to D^c and we use the Sommerfeld condition (see also [2], [10])

$$\frac{\partial u}{\partial |x|}(x) + ku(x) = o\left(|x|^{-1/2}\right),$$

then we deduce

$$\int_{D^c} \left(k^2 |Sh|^2 + |\nabla Sh|^2\right) dy = \int_{\partial D} \overline{Sh} \cdot (h/2 - K'h) dy. \tag{2.3}$$

From (2.2) and (2.3) we get

$$\int_{\mathbb{R}^2 \setminus \partial D} \left(k^2 |Sh|^2 + |\nabla Sh|^2\right) dy = \int_{\partial D} h \cdot \overline{Sh}\, dy. \qquad \square$$

The following theorem gives a range for the spectrum of the operator $I/2 + K' + \alpha S$. It will be used to find a suitable norm on $C(\partial D, \mathbb{C}^2)$, in order to prove the convergence of the Neumann series.

Theorem 2.4. *The spectrum σ of the operator*

$$\frac{1}{2} I + K' + \alpha S \;:\; C(\partial D, \mathbb{C}^2) \to\; C(\partial D, \mathbb{C}^2)$$

satisfies

$$\sigma(I/2 + K' + \alpha S) \subset (0, 1 + M\alpha].$$

Proof. Suppose λ is a complex eigenvalue of the operator $I/2 + K' + \alpha S$ with the corresponding eigenvector $h \in C(\partial D, \mathbb{C}^2)$. Then

$$\lambda \int_{\partial D} h \cdot S\overline{h}\, dy = \int_{\partial D} S\overline{h} \cdot (I/2 + K' + \alpha S)\, h\, dy,$$

from which it follows

$$\lambda \int_{\partial D} h \cdot S\overline{h}\, dy = \int_{\partial D} S\overline{h} \left(\frac{\partial Sh}{\partial \nu} + \alpha Sh\right) dy.$$

We showed in Lemma 2.3 that

$$\int_{\partial D} h \cdot S\overline{h}\, dy = \int_{\mathbb{R}^2 \setminus \partial D} \left(k^2 |Sh|^2 + |\nabla Sh|^2\right) dy \geq 0.$$

Assume that $\int_{\partial D} h \cdot S\overline{h}\, dy = 0$. Then $Sh \equiv 0$ and therefore

$$(I/2 + K' + \alpha S)h = \frac{\partial Sh}{\partial \nu} + \alpha Sh = 0.$$

From the invertibility of the operator $I/2+K'+\alpha S$ we deduce $h=0$, which is a contradiction. Therefore

$$\int_{\partial D} h\cdot S\overline{h}\,dy>0.$$

In Lemma 2.2 we proved that

$$\int_{\partial D} S\overline{h}\left(\frac{\partial Sh}{\partial\nu}+\alpha Sh\right)\,dy\geq 0.$$

It follows that $\lambda\geq 0$ and, since the operator $I/2+K'+\alpha S$ is invertible, we obtain $\lambda>0$, which proves the first part of the estimate of the range of

$$\sigma\left(\frac{1}{2}I+K'+\alpha S\right).$$

If we use Lemmas 2.1, 2.2 and 2.3, then we successively deduce

$$\lambda=\frac{\int_{\partial D} S\overline{h}\left(\frac{\partial Sh}{\partial\nu}+\alpha Sh\right)\,dy}{\int_{\partial D} h\cdot S\overline{h}\,\,dy},$$

$$\lambda=\frac{\int_{D}\left(k^2|Sh|^2+|\nabla Sh|^2\right)dy+\int_{\partial D}\alpha|Sh|^2\,dy}{\int_{\mathbb{R}^2\setminus\partial D}\left(k^2|Sh|^2+|\nabla Sh|^2\right)dy},$$

$$\lambda\leq 1+\frac{\int_{\partial D}\alpha|Sh|^2\,dy}{\int_{\mathbb{R}^2\setminus\partial D}\left(k^2|Sh|^2+|\nabla Sh|^2\right)dy},$$

$$\lambda\leq 1+\frac{\int_{\partial D}\alpha|Sh|^2\,dy}{\int_{\partial D} h\cdot S\overline{h}\,dy}\leq 1+M\alpha.$$

The theorem is proved. □

Theorem 2.5. *Let* $g\in C(\partial D,\mathbb{R}^2)$ *and* $0<c<2/(1+M\alpha)$. *Define the operator* $T=I-c(I/2+K'+\alpha S)$. *Then the series*

$$\sum_{j=0}^{\infty} c\,T^j g \tag{2.4}$$

converges in $C(\partial D,\mathbb{R}^2)$ *to the solution of the boundary integral equation*

$$\frac{1}{2}h+K'h+\alpha Sh=g.$$

Remark 2.6. The series (2.4) is called Neumann series. We will use the spectrum of the operator $I/2+K'+\alpha S$ to prove the convergence. It is well known (see [8]) that if $\|T\|<1$, then

$$\sum_{j=0}^{\infty} T^j=(I-T)^{-1}.$$

We need the following lemma about the relation between the eigenvalues and the norms in a complex Banach space. We state the lemma without proof. The lemma can be found in [8].

Lemma 2.7. *Let X be a complex Banach space and B the set of the norms on X that are equivalent to the original norm. Suppose A is a bounded linear operator in X and $r(A)$ is the spectral radius of A. Then*

$$r(A) = \inf_{\|\cdot\| \in B} \|A\|.$$

Proof. (proof of theorem 2.5) From Theorem 2.4 we have

$$\sigma(I/2 + K' + \alpha S) \subset (0, 1 + M\alpha].$$

Using the definitions that we made, $T = I - c(I/2 + K' + \alpha S)$ and

$$c \in \left(0, \frac{2}{1 + M\alpha}\right),$$

we obtain $\sigma(T) \subset (-1, 1)$ and therefore $r(T) < 1$.

From Lemma 2.7 we deduce that there is an equivalent norm $\|\cdot\|_*$ on $C(\partial D, \mathbb{C}^2)$, such that $\|T\|_* < 1$. It follows that the Neumann series

$$\sum_{j=0}^{\infty} c\, T^j g$$

converges to

$$c(I - T)^{-1} g = \left(\frac{1}{2} I + K' + \alpha S\right)^{-1} g = h,$$

which ends the proof. □

Acknowledgments. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PN-II-ID-PCE-2011-3-0994.

References

[1] Colton, D., Kress, R., *Inverse acoustic and electromagnetic scattering theory*, Springer, 2012.

[2] Hsiao, G.C., Wendland, W.L., *Boundary integral equations*, Springer, 2008.

[3] Jones, D.S., *Acoustic and electromagnetic waves*, Clarendon Press, Oxford University Press, 1986.

[4] Kryven, M., *Integral equation approach for the numerical solution of a Robin problem for the Klein-Gordon equation in a doubly connected domain*, arXiv preprint arXiv:1401.6957, 2014.

[5] Lanzani, L., Shen, Z., *On the Robin Boundary Condition for Laplace's Equation in Lipschitz Domains*, Comm. Partial Differential Equations, **29**(2004), 91-109.

[6] Medkova, D., *Convergence of the Neumann series in BEM for the Neumann problem of the Stokes system*, Acta Appl. Math., **116**(2011), 281-304.

[7] Medkova, D., *Integral representation of a solution of the Neumann problem for the Stokes system*, Numer. Algorithms, **54**(2010), 459-484.

[8] Medkova, D., *The third problem for the Stokes system in bounded domain*, preprint, 2010.

[9] Medkova, D., *Transmission problem for the Laplace equation and the integral equation method*, J. Math. Anal. Appl., **387**(2012), 837-843.

[10] Medkova, D., Varnhorn, W., *Boundary value problems for the Stokes equations with jumps in open sets*, Appl. Anal., **87**(2008), 829-849.

[11] Steinbach, O., Wendland, W.L., *On C. Neumann's method for second-order elliptic systems in domains with non-smooth boundaries*, J. Math. Anal. Appl., **262**(2001), 733-748.

[12] Wendland, W.L., *On Neumann's method for the exterior Neumann problem for the Helmholtz equation*, J. Math. Anal. Appl., 1977.

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Book reviews

René L. Schilling, Mass und Integral - Eine Einführung für Bachelor-Studenten, x+172 pp, Walter de Gruyter, Berlin/Bsoton, 2015, ISBN: 978-3-11-034814-9/pbk; eISBN(PDF): 978-3-11-035064-7; eISBN(EPUB): 978-3-11-038332-4.

This is a text of a course on measure theory and integration for students in mathematics and physics. The main goal of the book is to present the basic properties of the Lebesgue measure and integral needed in higher analysis, probability theory and mathematical physics.

The presentation is based on an abstract approach – σ-algebras, Dynkin systems, monotone classes. The measure is introduced using the Carathéodori extension theorem from semi-rings of sets, allowing a quick definition of Lebesgue measure on $\mathbb{R}$. The Lebesgue measure and integral on $\mathbb{R}^d$ are introduced as a product measure and integral, via the Fubini-Tonelli theorem.

The integrals of positive measurable functions f are defined as suprema of the integrals of measurable positive step functions majorized by f, and for real and extended real-valued measurable functions in the usual way, writing them as differences of positive measurable functions. The Lebesgue criterium of Riemann integrability is proved as well.

The convergence theorems (monotone convergence theorem, Lebesgue dominated convergence theorem, Egorov's theorem) are applied to the study of integrals with parameters – continuity and differentiability. The basic properties of L^p-spaces – Riesz-Fischer completeness theorem, Riesz theorem on the convergence of sequences of functions in L^p $(\|f_n - f\|_p \to 0 \iff \|f_n\|_p \to \|f\|_p$, provided $f_n(x) \to f(x)$ a.e.), Jensen inequality – are presented in detail. Applications are given to the convolution of functions and measures and to Fourier transform (Riemann-Lebesgue lemma, Wiener algebra, Plancherel's theorem).

The Lebesgue-Nikodým theorem is applied to the change of coordinate formula for integrals. The book ends with the study of functional analytic properties of the spaces L^p and $C(T)$ (for T a local compact metric space). One proves the density of some classes of functions in L^p, Riesz representation theorems for the duals of L^p and $C(T)$, and one studies the weak convergence of measures (an important topic in stochastic analysis).

Some additional questions are discussed in an Appendix: the existence of non-measurable sets, the integration of complex-valued functions, separability of $C(T)$, regularity of measures. Also, the exercises included at the end of each chapter complete the main text with further results and examples.

The book is very well organized, with clearly written conditions in all theorems, succeeding to present by a cleaver choice of the included topics, in a relatively small number of pages, some basic results of measure theory and integration, with emphasis on Lebesgue measure and integral in $\mathbb{R}^d$ and applications. For further results and applications, author's book, R. L. Schilling, *Measures, Integrals and Martingales*, Cambridge University Press, Cambridge 2001 (3rd printing), is highly recommended.

Hannelore Lisei

Daniel Alpay, An advanced complex analysis problem book. Topological vector spaces, functional analysis, and Hilbert spaces of analytic functions, Birkhäuser/Springer, New York, NY 2015, ix+525 p., ISBN 978-3-319-16058-0/pbk.

Usually in a first course on Complex Analysis analytic functions are considered as individuals, not as elements of some Hilbert, Banach or Fréchet spaces. Also some topological notions are introduced intuitively, without any rigorous topological foundation. Here one can mention the definition of the Riemann sphere $\widehat{\mathbb{C}}$ simply as $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the introduction of the uniform convergence on compact sets via Morera's theorem, proofs of the Riemann mapping theorem without appealing to compactness arguments.

The aim of this book is to fill in this gap, i.e. to get students familiar with some notions of functional analysis in the context of spaces of analytic functions, based on the unifying idea of reproducing kernel Hilbert space. By an adequate choice of the reproducing kernel one obtains the basic spaces of analytic functions: the Bargmann-Segal-Fock space, the Bergman space and the Hardy space. Besides the analytic description a geometric geometric one is considered as well.

The problems in the book are labeled *Exercise*, for which solutions are given, or *Question* or *Problem*, left without solutions or with solutions given in a previous book of the author:

[CAPB] D. Alpay, *A Complex Analysis Problem Book*, Birkhäuser/Springer Basel AG, Basel, 2011.

The first chapter of the book, 1. *Algebraic prerequisites*, contains some results on sets, functions, groups, matrices. The second one, 2. *Analytic functions*, contains some elements of complex analysis, a more detailed presentation being given in [CAPB].

The presentation of topological and functional analytic aspects is done in the second part of the book, II. *Topology and Functional Analysis*, having the chapters: 3. *Topological spaces*, 4. *Normed spaces* (Banach and Hilbert spaces, operators - bounded and unbounded), 5. *Locally convex topological vector spaces* (countably normed and Fréchet spaces, topologies on spaces of analytic functions and their duals, normal families), 6. *Some functional analysis* (Fourier transform, Stieltjes integral, density results in L_2-spaces). The third part, III. *Hilbert Spaces of Analytic Functions*, contains the chapters 7. *Reproducing kernel Hilbert spaces*, 8. *Hardy spaces*, 9. *de Branges-Rovnyak Spaces*, 10. *Bergman spaces* , and 11. *Fock spaces.*

Written by an expert in the area, the book is dedicated to beginning graduate students aiming a specialization in complex analysis. Teachers of complex analysis will find some supplementary material here and those of functional analysis a source of concrete examples. The presentation is restricted to one variable but, as the author promises in the Prologue to the book, a volume dedicated to several variables is in preparation.

S. Cobzaş

Marek Jarnicki and Petter Pflug, Continuous nowhere differentiable functions. The monsters of analysis, Springer Monographs in Mathematics, Springer - Cham, Heidelberg, New York, Dordrecht, London, 2015, xii+299 p., ISBN 978-3-319-12669-2/hbk; 978-3-319-12670-8/ebook.

After Newton put the basis of the differential calculus and applied it to the study the physical world, there was a general belief between mathematicians that a continuous function must be differentiable excepting a finite number of points. The famous mathematician and physicist A.-M. Ampère even published a proof of this result (based on some intuitively justified geometric reasonings on the behavior of curves) which was generally accepted by the mathematical community and included in almost every calculus book of that time. So the presentation by Karl Weierstrass in 1872 in front of the Königliche Akademie der Wissenschaften, Berlin, of his famous nowhere differentiable continuous function $\sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$, $x \in \mathbb{R}$, came as a great surprise, not very pleasant for some of them. Emile Picard said that if Newton had known about such functions he would never create calculus, and Ch. Hermite wrote to Stieltjes: "Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions continues qui nont point de dérivées.". H. Poincaré – who was the first to call such functions monsters – claimed that the functions were an arrogant distraction, and of little use to the subject: "They are invented on purpose to show that our ancestors reasoning was at fault, and we shall never get anything more out of them."

But, finally, the mathematicians had to accept the existence of these functions and to reconsider some results based on the idea of differentiability of continuous functions as well as the idea of proof – replace the geometric intuitive reasonings by rigorous analytic ones, as this was done by Weierstrass in his proof of nondifferentiability. These comments and others are nicely presented in the introductory chapter of the book, 1. *Introduction: A Historical Journey.*

The present book present in a rigorous and systematic way results related to this kind of functions, starting with classical and ending with some very recent and some open problems.

The first part I. *Classical results,* contains results obtained from the middle of the nineteenth century up to about 1950. Although the proofs are based on complicated arguments, they are accessible to undergraduate students.

Part II. *Topological methods*, shows that these strange functions not even that do exist, but they are in big quantities, in the sense of Baire category. For instance, the set of nowhere differentiable continuous functions are of second Baire category (and so contain a dense G_δ-set) in the space $C[a, b]$ (Banach-Mazurkiewicz-Jarnik and Saks theorems). The same is true about the set of functions $f \in A(\mathbb{D})$ (holomorphic in the open unit disk $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$ – the disk algebra) such that $f|_{\partial\mathbb{D}}$ is nowhere differentiable.

Part III. *Modern approach*, requires some more advanced tools from analysis, as measure theory and Fourier transform. The last chapter of this part (Chapter 12) is concerned with the existence of (closed) linear spaces of such functions (properties called lineability and spaceability - see the next review).

B. Riemann claimed that the function $\mathbf{R}(x) = \sum_{n=1}^{\infty} n^{-2}\sin(\pi n^2 x)$, $x \in \mathbb{R}$, is also nowhere differentiable, a result that turned out to be false. The difficult problem of the points of differentiability or nondifferentiability of this function, that preoccupied many famous mathematicians as, e.g., G. H. Hardy, is treated in detail in the fourth part of the book, having only one chapter, 13. *Riemann function.*

For reader's convenience 9 appendices, dealing with topics as Fourier transform, harmonic and holomorphic functions, Poisson summation formula, etc, are included.

By bringing together results scattered in various publications, some of them hardly to find or/and hardly to read (I mean old papers), presenting them in a unitary and rigorous way (using a modern language and style) with pertinent historical comments, the authors have done a great service to the mathematical community. The book presents interest for all mathematicians, but also for people (engineers, physicists, etc) having a basic background in calculus, interested in the evolution of some fascinating problems in this area, simply to formulate, but hardly to solve. Undergraduate, graduate students and teachers will find an accessible source of interesting examples, and possible be attracted by some hard problems remained unsolved till now.

S. Cobzaş

Richard M. Aron, Luis Bernal-González, Daniel M. Pellegrino and Juan B. Seoane Sepúlveda; Lineability. The search for linearity in mathematics, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015, xix+308 p, ISBN: 978-1-4822-9909-0/hbk;
978-1-4822-9910-6/ebook.

For a long time many mathematicians (including great names like A. M. Ampère) believed that any continuous function must be differentiable on a large subset of its domain of definition. So was a great shock when K. Weierstrass presented in 1872 his famous continuous and nowhere differentiable function: $\sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$, where $0 < a < 1$, b is an odd integer and $ab > 1+2\pi/2$. Although such functions were devised before by B. Bolzano (1830), B. Riemann (1861), H. Hankel (1870), Weierstrass was the first who published such a result. In spite of the natural assumption that the existence of these "pathological" (or "strange") functions is an exception, it turned out that they form large sets in the sense of Baire category. S. Banach (1931) and S. Mazurkiewicz (1936) proved that the set $\mathcal{ND}[a, b]$ of nowhere differentiable functions is of second Baire category (and so dense) in the space $C[a, b]$. The situations is the same with the set $(S)(I)$ of C^∞-functions with are nowhere real-analytic on the interval $I \subset \mathbb{R}$ (called singular functions). A classical example is that of the function

$f(x) = \exp(-1/x^2)$ which is of class C^∞ but no analytic at 0. Du Bois-Raymond (1876) constructed a function in $(S)(I)$ and H. Salzmann and K. Zeller (1955) proved that the set of singular functions is of second Baire category in $C^\infty(I)$.

The purpose of the present monograph is to analyze the situations when a class of functions in a given space contains a linear subspace. More exactly, a subset M of a topological vector space X is called:

- μ-lineable if $M \cup \{0\}$ contains a vector subspace of dimension μ;
- μ-spaceable if $M \cup \{0\}$ contains a closed vector subspace of dimension μ;
- μ-dense 0-lineable if $M \cup \{0\}$ contains a dense vector subspace of dimension μ.

Here μ is a cardinal number. If $\mu \geq \aleph_0$, then the set M is called simply lineable, spaceable, or dense-lineable. The first who proved the existence of such a subspace was V. I. Gurariy (1966): the set $\mathcal{ND}[0,1]$ contains an infinite dimensional vector space. Although some scattered results in this area were obtained in the last third of the preceding century, a systematic study of these problems started at the beginning of the current millennium. The survey paper by the last three authors, Bull. Amer. Math. Soc. 51 (2014), 71–130, can be considered as a forerunner of the present volume.

The authors examine the existence of linear subspaces in various classes of functions, as reflected by the headings of the chapters: 1. *Real analysis*, 2. *Complex analysis*, 3. *Sequence spaces, measure theory and integration*, 4. *Universality, hypercyclicity and chaos*, 5. *Zeros of polynomials in Banach spaces.* Other situations (divergent Fourier series, norm attaining functionals, etc) are discussed in Chapter 6. *Miscellaneous.*

The book is fairly self-contained - each chapter starts with a section, *What one needs to know*, and the first chapter (unnumbered), *Preliminary notions and tools*, contains also some additional notions and results used throughout the book.

Written by four experts in the area, whose substantial contributions are included in the book, most of them obtained in this millennium, the present monograph is addressed to postgraduate, but also to young or senior researchers wanting to enter the subject. Mathematicians interested in analysis, understood in a broad sense, will find a lot of interesting results collected in it.

Valeriu Anisiu

Valeriu Soltan; Lectures on Convex Sets, World Scientific Publishing Co. Pte. Ltd., Singapore 2015, x+405 pp, ISBN 978-9814656689; ISBN 978-9814656696.

The present book is devoted to a systematic study of algebraic and topological properties of convex subsets of the Euclidean space $\mathbb{R}^n$. As it is known these objects form the background of various mathematical disciplines, as convex geometry and operation research.

After two preliminary chapters 0. *Preliminaries*, and 1. *The affine structure of* $\mathbb{R}^n$, the study of convex sets starts in the second chapter with some algebraic and topological properties (relative interior, closure and relative boundary). Convex hulls, convex cones and conic hulls are treated in Chapters 3 and 4. Chapter 5 is concerned with some important topics in optimization and operation research – recession cones, normal cones and barrier cones. The separation and support properties of convex sets are discussed in Chapter 6. *Extreme points*, extreme faces and representations of convex sets in terms of extreme points are discussed in Chapter 7, while Chapter 8 is concerned with the exposed structure of convex sets and representation theorems (Straszewicz, Klee, Soltan). In the last chapter of the book, Chapter 9. *Polyhedra*, the results obtained in the previous chapters are applied to the study of this important class of convex sets.

The book is very well written. Each chapter ends with a section of notes and comments, and a set of exercises, with solutions given at the end of the book, completing the main text. Carefully done drawings illustrates the main notions introduced throughout the book.

The book is written at undergraduate level or entry-level graduate courses on geometry and convexity, the prerequisites being undergraduate courses on linear algebra, analysis and elementary topology. In spite of its relatively elementary level, the book contains many important results in finite dimensional convexity, necessary in many other mathematical areas. By the detailed and rigorous presentation of the material it can be recommended for self-study as well.

Nicolae Popovici

Arthur Benjamin, Gary Chartrand and Ping Zhang; The Fascinating World of Graph Theory, Princeton University Press, Princeton NJ, 2015, xii+322 p., ISBN 978-0-691-16381-9/hbk; 978-1-4008-5200-0/ebook).

The book is designed to introduce the field of Graph Theory to a broad audience and to also serve as an introductory textbook. Although the content is traditional for an undergraduate course, the way of presentation is not: the authors manage to motivate all topics with interesting applications, historical problems and discussion of concepts from an intuitive point of view.

After a funny prologue, chapter one deals exclusively with games, puzzles and problems that may be modeled using graphs. The basic notions of graphs and multigraphs are introduced and the way they capture the situations from reality are explained.

Graph classification is the topic of the second chapter. The basic idea of isomorphism is introduced and the reconstruction problem is the first unsolved problem discussed.

Chapter three introduces basic notions revolving around connectivity and distance. Both vertex and edge cuts are discussed along with several interesting applications.

Chapter four introduces trees and their basic properties. Among important topics we mention Cayley's formula for labeled trees and minimal spanning trees.

Graph traversal is treated in chapters five and six. There is a nice discussion of both Euler tours and the Chinese Postman Problem, both of which are edge traversals, and Hamilton cycles, which is edge traversal. Chapter six concludes with a discussion of the Traveling Salesman Problem.

Chapters seven and eight deal with graph decompositions, that is, partitions of the edge set of a graph. The latter concludes with Instant Insanity puzzle.

The ninth chapter treats orientations of graphs with an emphasis on tournaments and concludes with a nice application of tournaments for voting schemes.

Drawing graphs is the subject of Chapter ten. This is a well-presented standard material on topological graph theory.

The last two chapters deal with colouring: vertex colouring (Chapter 11) and edge colouring (Chapter 12).

The good integration of biographical sketches, human aspects and mathematics serves a good pedagogy. From the total of more than 300 pages, 50 is of gradual and well-chosen exercises A remarkable feature of this book is the constant effort to reveal the beauty of Mathematics in general and Graph Theory in particular.

Drawbacks of this works are little emphasizes on algorithms, the lack of mention to NP-Completeness, and lack of proofs for some important theorems.

Intended audience: undergraduates in Mathematics and Computer Science, professors which would like to use this as a textbook, anyone interested in discrete mathematics - its results, history and evolution.

Radu Trîmbiţaş

The Best Writing on Mathematics 2015, Edited by Mircea Pitici, Princeton University Press 2016, xxvi+376 pp., ISBN: 978-0-691-16965-1.

This is the sixth volume in a series edited by M. Pitici and published with PUP (2010, 2011, 2012, 2013, 2014). As the other volumes it contains a collection of 27 essays (a greater number than in the previous volumes), first published in 2014, dealing with various topics of mathematics and its applications. In a consistent Introduction (12 pages) the author explains the reasons for writing these volumes and the need for the popularization of mathematics: "That is why each volume should be seen in conjunction with the others, part of a serialized enterprise meant to facilitate the access to and exchange of ideas concerning diverse aspects of the mathematical experience.". This introductory part contains a brief survey on the writings on mathematics - both printed and online sources.

The volume contains more contributions, in comparison with previous ones, dealing with mathematical games and puzzles – *How puzzles made us human* (P. Mutalik), *Let the game continue* (C. Mulcahyand and D. Richards), *Challenging magic squares for magicians* (A. T. Benjamin and E. J. Brown), *Candy Crush's puzzling mathematics* (T. Walsh), *A prehistory of Nim* (L. Rougetet).

Some papers discuss philosophical and foundation aspects of mathematics – *Gödel, Gentzen, Goodstein: The magic sound of a G-string* (J. von Plato), *A guide for the perplexed: What mathematicians need to know to understand philosophers of mathematics* (M. Balaguer), *Writing about Math for the perplexed and the traumatized* (S. Strogatz).

In the paper *Synthetic biology, real mathematics* (by D. Mackenzie) applications to biology are discussed.

The future of high school mathematics and an analyze of the gap between Chinese and US students is presented in two papers. The relations between mathematics and art are discussed in two papers, one on Albrecht Dürer's painting and the other one on the quaternion group as a group of symmetry, while the beauty in mathematics is discussed in a paper by C. Cellucci.

Other contributions are dealing with geometry, the pigeonhole principle, chaos and billiard, big data manipulations, the Ontario lottery retailer scandal.

Dealing with topics of general interest – as history and philosophy, teaching, the occurrence of mathematics in everyday life, etc, – presented in an attractive and accessible manner, the books appeals to a large audience, including mathematicians of all levels of instruction, but also to anyone interested in the development of science and its applications.

Horia F. Pop