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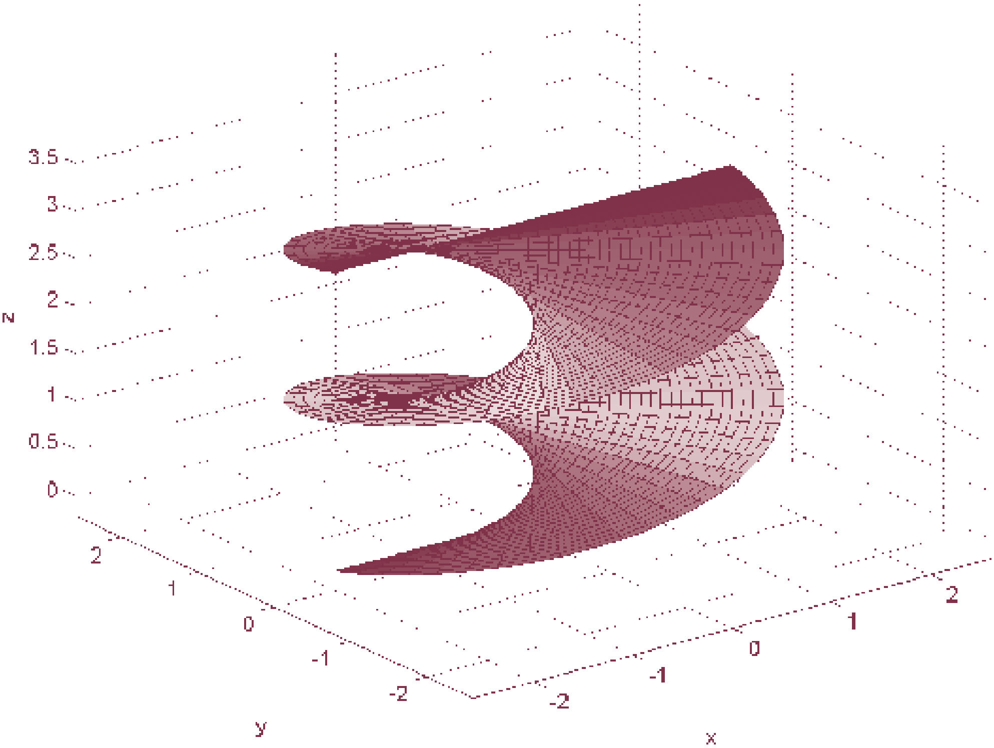

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Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
Telefon: 0264 405300

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CSABA VARGA – In Memoriam

Alexandru Kristály

Abstract. This note is devoted to present the scientific work of Professor Csaba Varga (1959-2021), who had contributions in Calculus of Variations and its applications in the theory of Partial Differential Equations and Finsler Geometry.

Mathematics Subject Classification (2010): 35A15, 35B38, 58J05, 58J60.

Keywords: Csaba Varga, critical points, Finsler geometry.

1. Introduction

Csaba György Varga passed away on 16 August 2021, after a long illness period. He was 62 years old.

Csaba was born on 5 February 1959 in Gyulakuta (Fântânele, Romania). He finished his university studies in 1983 at the Faculty of Mathematics of the Babeş-Bolyai University, Cluj-Napoca.

After being a highschool teacher for seven years in Bistriţa-Năsăud (Romania), he started his academic career in 1990. According to him, after "seven years of darkness", he had the opportunity to restart to work again in advanced mathematics together with his former students M. Crainic and G. Farkas. In that time, they learned and investigated together algebraic topology, Ljusternik-Schnirelmann category, density and condensation problems, see the early papers [28, 30, 31, 32, 33].

These papers have proved to be influential in the coming years when Csaba has got in contact with D. Motreanu. They started together to explore topological and variational phenomena in the context of elliptic problems. Due to this fruitful collaboration, Csaba defended his doctoral dissertation in 1996, entitled *Topological Methods in Optimizations*, under the supervision of J. Kolumbán. The central theme of his doctoral thesis is the non-smooth critical point theory (for locally Lipschitz functions) with applications in the theory of differential inclusions.

In the sequel, I invite the reader on a quick tour of Csaba's mathematical interests and contributions, placing them in the main research directions of *critical point theory* and *Finsler geometry.*

2. Critical point theory: from smooth to nonsmooth

From the mid of the 20th century, variational principles have been subject to relevant developments, when – among others – the modern critical point theory appeared. To be more precise, let X be a real Banach space, $E : X \to \mathbb{R}$ be a differentiable function; $x_0 \in X$ is said to be a *critical point* of E, if the derivative of E at x_0 vanishes, i.e., $dE(x_0) = 0$. This class of problems includes important chapters from modern mathematics:

- *weak solutions* of elliptic PDEs and related problems (weak solutions of differential equations are critical points of the energy functional associated to the original equation);
- *geodesic lines* in Riemannian/Finsler manifolds (these geometric objects occur as the critical points of the natural energy functionals defined on the space of curves with further particular properties).

A basic tool to guarantee critical points of energy functionals is the celebrated *Mountain Pass Theorem*, developed by A. Ambrosetti and P. Rabinowitz [3]. The proof of this result is based on a *deformation lemma*, which requires the existence of a suitable gradient vector field, coming from the high regularity of the functional. The Mountain Pass Theorem is applied to solve various elliptic problems; for simplicity, we consider the model problem

$$\begin{cases} -\Delta u(x) = f(u(x)) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary, Δ is the Laplace operator, while $f : \mathbb{R} \to \mathbb{R}$ is a continuous functions verifying certain growth conditions at the origin and at infinity. In such cases, we associate to problem (P) its natural energy functional $E : W_0^{1,2}(\Omega) \to \mathbb{R}$, defined by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega \int_0^{u(x)} f(t)dtdx,\ u \in W_0^{1,2}(\Omega),$$

where $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ is the usual Sobolev space. If f has appropriate properties, it follows that E is a C^2-class functional and

$$dE(u) = 0 \Longleftrightarrow u \text{ is a weak solution of } (P).$$

A highly nontrivial problem occurs when E is *not* differentiable, which requires a deep analysis; in this framework, Csaba has some relevant contributions, which are presented roughly in the next two subsections.

2.1. Critical points for locally Lipschitz functionals

In the early eighties, K.-C. Chang [7] proposed to study the problem (P) whenever f is not necessarily continuous, being only locally essentially bounded. Such phenomena arise in mathematical physics, engineering, etc.

Since in the new situation the nonlinear term f is only locally essentially bounded, it is possible to have the unlikely situation that problem (P) has only the zero solution, in spite of the fact that one could expect the presence of nontrivial solutions from practical point of view. For this reason, one usually substitutes the value $f(t)$ by the interval $[\underline{f}(t), \overline{f}(t)]$, where

$$\underline{f}(t) = \lim_{\delta \to 0^+} \text{essinf}_{|s-t|<\delta} f(s), \quad \overline{f}(t) = \lim_{\delta \to 0^+} \text{esssup}_{|s-t|<\delta} f(s),$$

while $\text{essinf}_A f = \sup\{a \in \mathbb{R} : f(x) \geq a \text{ for a.e. } x \in A\}$ and $\text{esssup}_A f = -\text{essinf}_A(-f)$, $A \neq \emptyset$. In this way, instead of problem (P) we consider the *differential inclusion*

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{DI}$$

where $F(t) = \displaystyle\int_0^t f(s)ds$ is a *locally Lipschitz* function[1], whose Clarke subgradient is

$$\partial F(t) = [\underline{f}(t), \overline{f}(t)], \ t \in \mathbb{R}.$$

The energy functional $E : W_0^{1,2}(\Omega) \to \mathbb{R}$ associated to (DI) is not of class C^1, being only locally Lipschitz on the Sobolev space $W_0^{1,2}(\Omega)$, while its critical point in the sense of Chang, i.e., $0 \in \partial E(u)$, is a solution of the differential inclusion (DI).

In general, if $E : X \to \mathbb{R}$ is a locally Lipschitz function in a given Banach space X, its *Clarke subgradient* at $u \in X$ is defined by

$$\partial E(u) = \{\xi \in X^* : E^o(u; v) \geq \langle \xi, v \rangle, \ \forall v \in X\},$$

see F. H. Clarke [8], where X^* is the dual of X, $\langle \cdot, \cdot \rangle$ is the duality mapping, and

$$E^o(u; v) = \limsup_{w \to v, t \to 0^+} \frac{E(w + tv) - E(w)}{t}$$

stands for the *Clarke directional derivative* of E at the point $u \in X$ and direction $v \in X$.

In a joint work with D. Motreanu, Csaba provided the *first extension of the celebrated Mountain Pass Theorem to locally Lipschitz functions*, see [27]. Moreover, they provided the so-called 'zero altitude' version of the result, which was new even in the smooth setting. The main tool they used is a *non-smooth deformation lemma*, where the key idea is the introduction of the so-called *pseuso-gradient vector field* for locally Lipschitz functions. Their non-smooth deformation lemma implies further non-smooth minimax results (saddle point, linking theorems).

The results from [27] has several applications and extensions, see e.g. C.O. Alves and J.A. Santos [1], or C.O. Alves, R.C. Duarte and M.A.S. Souto [2]. Moreover, various applications of the non-smooth Mountain Pass Theorem have been developed, both in the theory of *differential inclusions* and *hemivariational inequalities*. Moreover, spectacular arguments were provided not only in *bounded domains*, but also on *unbounded domains*. While in the former case Sobolev compactness is expected, in the latter case – in order to regain some sort of compactness – either certain coercivity or symmetric structures are required on unbounded domains. Such an approach is the so-called *principle of symmetric criticality* (both for smooth and nonsmooth

[1] The function $F : X \to \mathbb{R}$ is *locally Lipschitz*, if for every $x \in X$ there exist a neighborhood U and a constant $K_x > 0$ such that $|f(u) - f(v)| \leq K_x \|u - v\|$ for every $u, v \in U$, see F. H. Clarke [8].

functionals); over the years, Csaba became a worldwide expert of this principle, most of his results in this direction being influential in the literature, see e.g. [23, 24].

2.2. Critical points for continuous and set-valued functions

In the early nineties, M. Degiovanni and M. Marzocchi [11] have developed the theory of critical points for continuous functionals, by introducing the so-called *weak slope* of a continuous function $E : X \to \mathbb{R}$ defined on a metric space X. A point $u \in X$ is a *critical point* of E if its weak slope vanishes at u. In addition, if the functional E is of class C^1, the weak slope coincides with the norm of the usual differential of E.

Being an expert of the critical point theory for locally Lipschitz functions, Csaba obtained several important results also in the context of weak slopes. More precisely, Csaba and his co-authors obtained quantitative versions of the *deformation lemma* (without using pseudo gradient vector fields, which is not defined in such non-smooth settings), minimax results, see e.g. [22].

In addition, inspired by the work of M. Frigon [13], Csaba and his co-authors provided quantitative deformation lemmas and minimax results for *set-valued maps*, see [21]. The Mountain Pass Theorem for set-valued maps from [21] has a central place in the monograph of Y. Jabri [15].

3. Finsler geometry: from synthetic aspects to PDEs

In general, Finsler geometry is viewed as an extension of Riemannian geometry. S.-S. Chern claimed that Finsler geometry is just Riemannian geometry without the quadratic restriction. In certain sense, Chern's statement is confirmed, since many classical results can be easily extended from Riemannian to Finsler structures, as Hopf-Rinow, Cartan-Hadamard and Bonnet-Myers theorems, Rauch and Bishop-Gromov comparison principles, see D. Bao, S.-S. Chern and Z. Shen [4]. In spite of these facts, deep differences appear between the two geometries. Csaba was also extremely motivated to identify such nontrivial differences. In the sequel, we focus to the following two topics, both of them being his favorite research directions:

- *Busemann inequalities* and the existence of 'orthogonal' geodesic segments between Finsler submanifolds;
- *Sobolev spaces over Finsler manifolds* and their applications in the theory of PDEs.

To be more precise, let us give some basic notions from Finsler geometry. Let M be an $n(\geq 2)$-dimensional differentiable manifold and its tangent bundle $TM = \bigcup_{x\in M} T_xM$. The pair (M, F) is called a *Finsler manifold*, if the continuous function $F : TM \to [0, \infty)$ verifies the assumptions:

(a) $F \in C^\infty(TM \setminus \{0\})$;
(b) $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda \geq 0$ and $(x, y) \in TM$;
(c) $g_{ij}(x, y) = [\frac{1}{2}F^2]_{y^iy^j}(x, y)$ is positive definite for every $(x, y) \in TM \setminus \{0\}$, where $F(x, y) = F(y^i \frac{\partial}{\partial x^i}|_x)$.

(M, F) is *reversible*, if instead of (b) one has:

(b') $F(x, \lambda y) = |\lambda|F(x, y)$ for every $\lambda \in \mathbb{R}$ and $(x, y) \in TM$.

Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler setting; either the metric compatibility or the torsion-free property fails for a generic Finsler connection. Among these objects, the Chern connection has appropriate properties to provide qualitative results on Finsler manifolds, see D. Bao, S.-S. Chern and Z. Shen [4]. By means of this connection, one can introduce Jacobi fields, geodesics, flag curvature (replacing the sectional curvature), etc.

3.1. Busemann inequalities and 'orthogonal' geodesics on Finsler manifolds

In the forties, parallel to A.D. Alexandrov's theory, H. Busemann [5] developed a synthetic geometry on non-smooth metric spaces. Among others, H. Busemann elaborated axiomatically the theory of non-positively curved metric spaces, where no differential structure is needed. This notion of non-positive curvature requires that in small geodesic triangles the length of a side is at least the twice of the geodesic distance of the midpoints of the other two sides, see H. Busemann [5, p. 237]; if this property is valid in every small geodesic triangle, the space is called *Busemann NPC space*. By making a connection between smooth and synthetic objects, H. Busemann proved that a Riemannian manifold (M, g) is a Busemann NPC space if and only if its sectional curvature is non-positive. At the same time, he formulated the open question for non-Riemannian manifolds asking if non-positively curved Finsler manifolds are Busemann NPC space. It turned out that the picture for non-Riemannian Finsler spaces is totally different with respect to Riemannian manifolds. Indeed, P. Kelly and E. Straus [16] proved that a convex closed planar domain endowed with the standard Hilbert distance (providing a Finsler structure with constant flag curvature -1) is a Busemann NPC space if and only if the curve is an ellipse, thus the geometry reduces to the Riemannian one. After this result, nothing relevant happened till the early 2000s concerning Busemann's question on Finsler manifolds.

In 2003, Csaba and his co-authors proved in [17] that *non-positively curved Berwald manifolds*[2] *are Busemann NPC spaces*. In this way, Berwald manifolds became the first non-Riemannian Finsler spaces where H. Busemann's original question has been affirmatively answered. This result has been extended to further synthetic properties in [19], where the authors conjectured that non-positively curved Berwald manifolds are the largest Finsler objects which are Busemann NPC spaces. This question has been confirmed recently by S. Ivanov and A. Lytchak [14].

Since Busemann's inequality can be reformulated in terms of convexity, several applications can be found of the main results of [17, 19] by treating optimization problems, as Weber-type transportation phenomena on curved spaces; the reader may consult the monograph [20] for further applications in Economics and Geometry, written by Csaba and his co-authors.

Another important aspect of Finsler manifolds is to determine the number of geodesic segments perpendicular to certain submanifolds. Since the notion of perpendicularity as well as the behavior of the energy functional defined on the space of

[2]Special Finsler structures, where the coefficients of the Chern connection are not directional-dependent.

curves are delicate issues on Finsler manifolds, a comprehensive study of this problem was completed by Csaba and his co-authors in [18]; the most challenging part of the proof is the validity of the Palais-Smale compactness condition of the energy functional defined on the space of curves. The result from [18] has been extended by E. Caponio, M.Á. Javaloyes and A. Masiello [6] to stationary spacetimes over Finsler structures.

3.2. Sobolev spaces versus Finsler manifolds

Within the class of reversible Finsler manifolds (including in particular the class of Riemannian manifolds), the synthetic notion of Sobolev spaces on metric measure spaces and the analytic notion of Sobolev spaces coincide. However, the case when the Finsler manifold is *not* reversible (modeling e.g. Randers spaces, the Matsumoto mountain slope metric, or the Finsler-Poincaré ball), it turns out that surprising phenomena arise, which was described in the paper [12] of Csaba and his co-authors. To be more precise, let

$$W^{1,2}(M,F,\mathsf{m})=\left\{u\in W^{1,2}_{\mathrm{loc}}(M):\int_M F^{*2}(x,Du(x))\mathrm{d}\mathsf{m}(x)<+\infty\right\},$$

and $W_0^{1,2}(M,F,\mathsf{m})$ be the closure of $C_0^\infty(M)$ with respect to the (asymmetric) norm

$$\|u\|_F=\left(\int_M F^{*2}(x,Du(x))\mathrm{d}\mathsf{m}(x)+\int_M u^2(x)\mathrm{d}\mathsf{m}(x)\right)^{1/2}, \tag{3.1}$$

where m is the usual measure on (M,F). Let

$$r_F=\sup_{x\in M}\sup_{y\in T_xM\setminus\{0\}}\frac{F(x,y)}{F(x,-y)}$$

be the *reversibility constant* on (M,F). Clearly, $r_F\geq 1$ and $r_F=1$ if and only if (M,F) is reversible. Let

$$F_s(x,y)=\left(\frac{F^2(x,y)+F^2(x,-y)}{2}\right)^{1/2},\ (x,y)\in TM.$$

It is clear that (M,F_s) is a reversible Finsler manifold, F_s being the *symmetrized* Finsler metric associated with F.

In [12] the authors proved that if $r_F<+\infty$, then $(W_0^{1,2}(M,F,\mathsf{m}),\|\cdot\|_{F_s})$ is a reflexive Banach space, while the norm $\|\cdot\|_{F_s}$ and the asymmetric norm $\|\cdot\|_F$ are equivalent; in particular,

$$\left(\frac{1+r_F^2}{2}\right)^{-1/2}\|u\|_F\leq\|u\|_{F_s}\leq\left(\frac{1+r_F^{-2}}{2}\right)^{-1/2}\|u\|_F,\ \forall u\in W_0^{1,2}(M,F,\mathsf{m}).$$

A more surprising fact – which shows the genuine difference between Riemannian and Finsler geometry – is that the authors of [12] constructed a function u on the Finsler-Poincaré ball (having the reversibility constant $+\infty$) such that $u\in W_0^{1,2}(M,F,\mathsf{m})$ but $-u\notin W_0^{1,2}(M,F,\mathsf{m})$. In this way, the Sobolev space over a non-compact Finsler manifold (M,F) with $r_F=+\infty$ need not be even a vector space.

Csaba was also interested to study elliptic PDEs involving the Finsler-Laplace operator. Such kind of problems were discussed in [25], where the authors established Hardy-type inequalities on Finsler manifolds with some applications. Further results of Csaba and his co-authors, involving elliptic operators on different domains can be found in [10, 26, 29].

4. Concluding part

Csaba's most important contributions to applied mathematics have been published in internationally recognized journals such as *Calculus of Variations and Partial Differential Equations*, *Nonlinear Differential Equations and Applications*, *Discrete and Continuous Dynamical Systems-A*, *Advances in Differential Equations*, *Nonlinear Analysis Real World Applications*, etc. A summary of these results has been published in two monographs by *Cambridge University Press* in 2010 (see [20]) and *Springer* in 2021 (see [9]).

Csaba was invited to various research institutes and universities, as *Universita di Perugia, Eötvös Lóránd University, Alfréd Rényi Institute of Mathematics, Universita di Catania, Technical University of Athens*, etc. He collaborated with dozens of national and international mathematicians, resulting joint publications. He has more than 90 research papers, being cited in prestigious journals such as *Mathematische Annalen, Journal of Functional Analysis, Journal of Differential Equations* and others.

In addition to his scientific achievements, one of Csaba's greatest merits lies in discovering and educating young mathematical talents. Many of his former students became world-renowned mathematicians, working at prestigious European and American universities such as *Humboldt University, Utrecht University, Virginia Polytechnic Institute and State University.* As a doctoral supervisor, he advised numerous students, who became outstanding researchers and lecturers at the Babeş-Bolyai University and Sapientia University of Transylvania.

Csaba's absence remains an unfilled void in our soul.

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Alexandru Kristály
Babeş-Bolyai University,
Department of Economics,
58-60, T. Mihali Street,
Cluj-Napoca, Romania
e-mail: `alexandru.kristaly@ubbcluj.ro`

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Generalized de Jonquières divisors on generic curves

Gavril Farkas

To the memory of Csaba Varga (1959-2021)

Abstract. The classical de Jonquières and MacDonald formulas describe the virtual number of divisors with prescribed multiplicities in a linear system on an algebraic curve. We discuss the enumerative validity of the de Jonquières formulas for a general curve of genus g.

Mathematics Subject Classification (2010): 14H10, 14H51.

Keywords: Algebraic curves, de Jonquières divisors, moduli space of curves.

1. Introduction

De Jonquières' formula [11] is concerned with the following classical enumerative question: Given a suitably general (singular) plane curve of Γ degree d and geometric genus g, how many plane curves of given degree meet Γ in n_i unspecified points with contact order a_i, for $i = 1, \ldots, e$? De Jonquières using an ingenious recursive argument (later considerably simplified by Torelli [24] and then slightly generalized by Allen [1]) showed that the number in question equals

$$\frac{\left[a_1^{n_1} a_2^{n_2} \cdots a_e^{n_e}\right]}{n_1!\, n_2!\, \cdots n_e!}, \qquad \text{where in general we define the quantity}$$

$$[a_1 \cdots a_e] = a_1 \cdots a_e \ \frac{g!}{(g-e-1)!} \left(\frac{a_1 \cdots a_e}{g-e} - \sum_{i=1}^{e} \frac{a_1 \cdots \widehat{a_i} \cdots a_e}{g-e+1} + \cdots + (-1)^e \frac{1}{g} \right). \tag{1.1}$$

The formula (1.1) recovers many well known formulas in the theory of algebraic curves, for instance the number $2^{g-1}(2^g-1)$ of odd theta characteristics on a smooth curve of genus g, or the Plücker formula for the total number of ramification points on a linear series on a curve. The original proofs [11], [24] of the de Jonquières formula

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use an induction on the multiplicities a_i coupled with the *Brill-Cayley correspondence principle*. For a historic perspective on the de Jonquières formula we refer to Zeuthen's treatise [26, 136], or if one prefers English, the books of Coolidge [8, Book 3, Chapter 3.3] or Baker [5, pages 35-45]. De Jonquières' formula has been rediscovered by MacDonald [21] and Vainsencher [25] and a summary of their work, reinterpreting this number as a fundamental class of a modified diagonal on the symmetric product of a smooth curve can be found in the book [3].

In order to formulate the problem in modern terms, let C be a smooth curve of genus g and we fix a linear series $\ell = (L, V) \in G^r_d(C)$. For a partition $\mu = (a_1, \ldots, a_e)$ of d, we define the de Jonquières cycle $DJ_\mu(C, \ell)$ to be the locus of divisors of the type $a_1 \cdot x_1 + \cdots + a_e \cdot x_e$ lying in the linear system ℓ. Observe that $DJ_\mu(C, \ell)$ can be realized as the rank r degeneracy locus of the evaluation morphism of vector bundles

$$\chi \colon V \otimes \mathcal{O}_{C^e} \longrightarrow J_\mu(L)$$

over the product C^e, where the fibre of the vector bundle $J_\mu(L)$ over a point $(x_1, \ldots, x_e)$ equals the d-dimensional vector space $L_{|a_1 \cdot x_1 + \cdots + a_e \cdot x_e}$. Accordingly, the virtual dimension of $DJ_\mu(C, \ell)$ equals $e - d + r$. In the case $e = d - r$, this number equals zero and one expects ℓ to contain finitely many divisors with multiplicities prescribed by the partition μ. As pointed out in [3, page 359], the virtual class of this degeneracy locus can be realized via the Porteous formula as the coefficient of the monomial $t_1 t_2 \cdots t_e$ in the polynomial

$$\bigl(1 + a_1^2 t_1 + \cdots + a_e^2 t_e\bigr)^g \bigl(1 + a_1 t_1 + \cdots + a_e t_e\bigr)^{d-r-g}.$$

It is straightforward to see that this is simply a convenient way to repackage compactly the information contained in the formula (1.1). For instance, we obtain that a linear system $\ell \in G^r_d(C)$ is expected to contain precisely

$$2^r \left(\binom{d-r}{r} + g \binom{d-r-1}{r-1} + \binom{g}{2} \binom{d-r-2}{r-2} + \cdots \right)$$

divisors containing r double points, that is, of the type

$$2 \cdot x_1 + \cdots + 2 \cdot x_r + x_{r+1} + \cdots + x_{d-r}$$

and so on. Here we use the convention that $\binom{m}{-h} = 0$ when $h > 0$.

More generally, we consider a positive partition $\mu = (a_1, \ldots, a_e)$ and set

$$|\mu| := a_1 + \cdots + a_e \text{ and } \ell(\mu) := e.$$

For $0 \le f \le |\mu|$ we define the generalized de Jonquières (secant) locus

$$\begin{aligned} DJ^f_\mu(C, \ell) &:= Z_{|\mu| - f}(\chi) \\ &= \Bigl\{(x_1, \ldots, x_e) \in C^e : \dim \bigl|V(-a_1 \cdot x_1 - \cdots - a_e \cdot x_e)\bigr| \ge r - |\mu| + f\Bigr\}. \end{aligned}$$

Being a degeneracy locus, each component of $DJ^f_\mu(C, \ell)$ has dimension at least

$$e - f(r + 1 - |\mu| + f).$$

If $\mu = (1^e)$, then using the notations of [9] or [15], we observe that $DJ^f_\mu(\ell) = V^{e-f}_e(\ell)$ can be identified with the variety of e-secant $(e - f - 1)$-planes to the embedded curve $C \overset{|V|}{\hookrightarrow} \mathbb{P}^r$. Moreover, if $|\mu| = d$, then $DJ_\mu^{d-r}(C,\ell) = DJ_\mu(C,\ell)$ is the locus of de Jonquières divisors in the linear series ℓ. De Jonquières loci have been used to study the geometry of the moduli spaces of curves or that of strata of holomorphic differentials [4]. The class of effective divisors on $\overline{\mathcal{M}}_g$ involving de Jonquières conditions have been computed in [10], [16], [17], or [22].

The question of how to interpret the de Jonquières count when a curve $C \subseteq \mathbb{P}^r$ acquires singularities has been treated both in classical and modern times. The problem we address in this note on the other hand is the enumerative validity of the de Jonquières count when C is a general curve in moduli. We treat this problem variationally and consider de Jonquières cycles associated to all linear systems $\ell \in G^r_d(C)$, that is, we set up the correspondence:

$$\Sigma_\mu^f(C) := \Big\{\big(\ell, x_1, \ldots, x_e\big) : (x_1,\ldots,x_e) \in DJ_\mu^f(C,\ell)\Big\} \tag{1.2}$$

$$G^r_d(C) \xleftarrow{\ \pi_1\ } \Sigma_\mu^f(C) \xrightarrow{\ \pi_2\ } C^e$$

The main result of this paper is then summarized as follows:

Theorem 1.1. *Let C be a general curve of genus g and we fix a partition*

$$\mu = (a_1, \ldots, a_e),$$

as well as positive integers d, r and f with $\rho(g,r,d) \geq 0$ and $|\mu| - r \leq f \leq |\mu|$. Then each irreducible component of $\Sigma_\mu^f(C)$ has dimension $\rho(g,r,d) + e - f(r+1-|\mu|+f)$. Accordingly, if

$$\rho(g,r,d) + e - f(r+1-|\mu|+f) < 0,$$

then $DJ_\mu^f(C,\ell) = \emptyset$ for every linear series $\ell \in G^r_d(C)$.

This result generalizes [15, Theorem 0.1] to the case of an arbitrary partition μ, the result in *loc.cit.* corresponding to the case when $\mu = (1^e)$. It also generalizes Ungureanu's results [23, Theorem 1.5] corresponding to the case when $|\mu| = d = \deg(\ell)$, asserting that if C is a general curve, no linear series $\ell \in G^r_d(C)$ possesses a de Jonquières divisor of length $e < d - r$. Observe that the case $f = |\mu| - r$ in Theorem 1.1 can be obviously reduced to the classical de Jonquières case, by extending the partition μ to $\mu' = (\mu, 1^{d-|\mu|})$ of the degree d of the curve in question.

We now discuss several cases in which Theorem 1.1 applies. The first case beyond the classical de Jonquières situation treated for instance (under some restrictive assumptions) in [23] is when $f = |\mu| + 1 - r$, when the residual linear series $\big|V(-a_1 \cdot x_1 - \cdots - a_e \cdot x_e)\big|$ is a pencil, which can be formulated as saying that under the map $\varphi_\ell \colon C \to \mathbb{P}^r$ induced by the linear series ℓ, the $(a_i - 1)$-st osculating planes to C at the points x_i span a codimension two plane, that is,

$$\big\langle a_1 \cdot x_1, \ldots, a_e \cdot x_e\big\rangle \cong \mathbb{P}^{r-2}. \tag{1.3}$$

Tangential secants. Let us consider the case $a_1 = 2$ and $a_2 = \cdots = a_e = 1$ and $f = 1$, in which case the condition (1.3) translates into saying that $\langle 2 \cdot x_1, x_2, \ldots, x_e\rangle \cong \mathbb{P}^{e-1}$, that is, the tangent line to C at the point x_1 lies in the $(e-1)$-plane spanned by the points $x_1, \ldots, x_e$. Following classical terminology, we say that $\langle x_1, \ldots, x_e\rangle$ is a *tangential* $(e+1)$-*secant* to C. Theorem 1.1 can be formulated in this case as follows:

Corollary 1.2. *We fix positive integer* g, r, d *and* e *such that* $2e < r+1-\rho(g,r,d)$. *For a general curve* C *of genus* g*, no linear seris* $\ell \in G^r_d(C)$ *carries a tangential* $(e+1)$*-secant.*

Note that every space curve $C \subseteq \mathbb{P}^3$ of degree d and genus g is expected to have finitely many tangential *trisecants* and their number

$$T(d,g) = 2(d-2)(d-3) + 2g(d-6),$$

which can derived from the de Jonquières formula, has been first computed by Salmon and Zeuthen [26, 64], see also [3, page 364]. It is an interesting result of Kaji [18], valid to a large extent even in positive characteristic, that an *arbitrary* smooth space curve $C \subseteq \mathbb{P}^3$ cannot have infinitely many tangential trisecants, see also [7] for various extensions of this result. For space curves, our Corollary 1.2 reduces to the Brill-Noether Theorem, but already for curves $C \subseteq \mathbb{P}^4$ it goes beyond that and it states that when $\rho(g,r,d) = 0$ a general such curve has no tangential trisecants.

Multiple tangents. Passing now to the case of tangent planes, that is, when $a_1 = \cdots = a_e = 2$, we look at $(2e-2)$-planes in $\mathbb{P}^r$ that are tangent to C at e points, that is,

$$\big\langle 2 \cdot x_1, \ldots, 2 \cdot x_e\big\rangle \cong \mathbb{P}^{2e-2}.$$

We call such a configuration an *degenerate* e*-tangent* to $C \subseteq \mathbb{P}^r$. With this terminology, Theorem 1.1 takes the following form:

Corollary 1.3. *Fix positive integers* g, r, d, e *with* $\rho(g,r,d) \geq 0$ *and*

$$3e < r + 2 - \rho(g,r,d).$$

Then a general curve C *of genus* g *has no linear series* $\ell \in G^r_d(C)$ *with degenerate* e*-tangents.*

The simplest case where Corollary 1.3 applies is when $e = 2, r = 5$. It says that for a general curve C of genus g, no embedded curve $\varphi_\ell \colon C \to \mathbb{P}^5$ of degree d with $\rho(g,r,d) = 0$ has a pair of coplanar tangent lines.

Another immediate application of Theorem 1.1 is when again $a_1 = \cdots = a_e = 2$ but this time $f = 2e - r > 0$, hence

$$\big\langle 2 \cdot x_1, \ldots, 2 \cdot x_e\big\rangle \cong \mathbb{P}^{r-1}.$$

In other words, the points $x_1, \ldots, x_e$ span a *tangent hyperplane*. We find the following result:

Corollary 1.4. *Fix integers* $g \geq 1$, $r \geq 3$ *and* d *such that* $\rho(g,r,d) \geq 0$ *and* $e \geq r+1$. *Then for a general curve* C *of genus* g *the locus of linear systems* $\ell \in G^r_d(C)$ *such that* $\varphi_\ell \colon C \hookrightarrow \mathbb{P}^r$ *admits an* e*-secant tangent hyperplane is equal to* $\rho(g,r,d) + r - e$.

In particular, for $e = r + 1$ specializes to the known result [23], that for a Brill-Noether general curve $C \subseteq \mathbb{P}^r$ no hyperplane can be tangent at more than r points.

Flex lines and bitangents. A general smooth space curve $C \subseteq \mathbb{P}^3$ is expected to possess no bitangent or flex lines lines, that is, no de Jonquières divisors of length two corresponding to the partitions $\mu = (2, 2)$ and $\mu = (3, 1)$ respectively. We consider the problem more generally for curves $C \subseteq \mathbb{P}^r$ and our result in this case lends a sharp form to this expectation.

Corollary 1.5. *Fix positive integers $g \geq 1$, $r \geq 3$ and d with $\rho(g, r, d) \geq 0$ and a_1, a_2 such that*

$$a_1 + a_2 > \frac{\rho(g, r, d) + 2r}{r - 1}.$$

Then for a general curve C of genus g, no degree d embedding $\varphi_\ell \colon C \hookrightarrow \mathbb{P}^r$ possesses a secant line meeting the image of C with multiplicities a_1 and a_2 at the points of secancy.

For instance when $r = 3$, $e = 2$ and $|\mu| = 4$, Corollary 1.5 implies that when $\rho(g, 3, d) \leq 1$, for a general curve C of genus g no embedding $\varphi_\ell \colon C \hookrightarrow \mathbb{P}^3$ of degree d possesses either a bitangent or a flex line.

The last application of Theorem 1.1 is to the case when the partition μ is of length one.

Corollary 1.6. *We fix positive integers g, r, d and a such that $2a > \rho(g, r, d) - 1 + 2r$. Then a general curve C of genus g carries no linear series $\ell \in G^r_d(C)$ having a point $x \in C$ with $\ell(-a \cdot x) \in G^1_{d-a}(C)$.*

Specializing even further to the case $d = 2g - 2$ and $r = g - 1$ in which case ℓ necessarily equals the canonical linear series $|\omega_C|$, via the Riemann-Roch Theorem Corollary 1.6 can be reformulated as stating that for a general curve of genus g, if $a \geq g - 1$ we have that

$$h^0\big(C, \mathcal{O}_C(a \cdot x)\big) \leq a + 2 - g,$$

for each point $x \in C$. When $a = g - 1$ we obtain that C carries no pencil of degree $g - 1$ totally ramified at a point, which is a well-known result. The locus of curves $[C] \in \mathcal{M}_g$ having such a pencil has been studied by Diaz [12], who also computed the class of its compactification in $\overline{\mathcal{M}}_g$.

2. Generalized de Jonquières divisors on flag curves

We fix a smooth curve C of genus g and we denote by $G^r_d(C)$ the variety of linear systems of type g^r_d on C, that is, pairs $\ell = (L, V)$, where $L \in \mathrm{Pic}^d(C)$ and $V \subseteq H^0(C, L)$ is an $(r+1)$-dimensional subspace of sections. Recall that when C is a general curve of genus g, then $G^r_d(C)$ is a smooth variety of dimension equal to the *Brill-Noether number* $\rho(g, r, d) = g - (r + 1)(g - d + r)$. Our proof of Theorem 1.1 is by degeneration and we will use throughout the theory of limit linear series. We begin by quickly recalling the notation for vanishing and ramification sequences of linear series on curves largely following [13] and [14].

If $\ell = (L, V) \in G^r_d(C)$ is a linear series, the *ramification sequence* of ℓ at a point $q \in C$

$$\alpha^\ell(q) : 0 \le \alpha^\ell_0(q) \le \cdots \le \alpha^\ell_r(q) \le d - r$$

is obtained from the *vanishing sequence*

$$a^\ell(q) : 0 \le a^\ell_0(q) < \cdots < a^\ell_r(q) \le d$$

by setting $\alpha^\ell_i(q) := a^\ell_i(q) - i$, for $i = 0, \ldots, r$. In case the underlying line bundle L is clear from the context, we write $\alpha^V(q) = \alpha^\ell(q)$ and $a^V(q) = a^\ell(q)$. The *ramification weight* of q with respect to ℓ is defined as the quantity

$$\mathrm{wt}^\ell(q) := \sum_{i=0}^{r} \alpha^\ell_i(q).$$

We denote by

$$\rho(\ell, q) := \rho(g, r, d) - \mathrm{wt}^\ell(q)$$

the *adjusted Brill-Noether number* of ℓ with respect to q. We recall also the *Plücker formula*

$$\sum_{q \in C} \alpha^\ell(q) = (r+1)d + (r+1)r(g-1), \tag{2.1}$$

measuring the total ramification of ℓ. Incidentally, assuming that ℓ has only simple ramification points, that is, points with ramification sequence at most $(0, \ldots, 0, 1)$, then (2.1) is an instance of the de Jonquières formula (1.1) applied to the linear series ℓ and to the partition $\mu = (r+1, 1^{d-r-1})$ of d).

Following Eisenbud-Harris [13, page 364], let us recall that a *limit linear series* on a curve X of compact type consists of a collection

$$\ell = \big\{(L_C, V_C) \in G^r_d(C) : C \text{ is a component of } X\big\},$$

satisfying a compatibility condition on the vanishing sequences at the nodes of X in terms of the vanishing sequences of the aspects on the two (smooth) components of X on which each node of X lies. We denote by $\overline{G}^r_d(X)$ the variety of limit linear series of type g^r_d on X. More generally, if $q \in X_{\mathrm{req}}$ is a smooth point and

$$\alpha = \big(0 \le \alpha_0 \le \cdots \le \alpha_r \le d - r\big)$$

is a *Schubert index*, we denote by $\overline{G}^r_d\big(X, (q, \alpha)\big)$ the variety of limit linear series $\ell \in \overline{G}^r_d(X)$ satisfying the condition $\alpha^\ell(q) \ge \alpha$. From basic principles it follows that each component has dimension at least $\rho\big(g, r, d, \alpha\big) = \rho(g, r, d) - \mathrm{wt}(q)$. Eisenbud and Harris offer in [14, Theorem 1.1] sufficient conditions ensuring when the equality

$$\dim \overline{G}^r_d\big(X, (q, \alpha)\big) = \rho(g, r, d) - \mathrm{wt}(\alpha) \tag{2.2}$$

holds, which we will make an essential use of in the course of proving Theorem 1.1. In case a pointed curve $[X, q]$ satisfies the condition (2.2) for each $r, d \ge 1$ such that $\rho(g, r, d) \ge 0$ and for each choice of a Schubert index α, we say that $[X, q]$ verifies the strong Brill-Noether Theorem.

Having fixed a positive partition $\mu = (a_1, \ldots, a_e)$, a positive integer f with $|\mu| - r \le f \le |\mu|$ and a smooth curve C, we have defined in the Introduction the subvariety $\Sigma_\mu^f(C) \subseteq G^r_d(C) \times C^e$. Due to its determinantal structure, each irreducible component of $\Sigma_\mu^f(C)$ has dimension at least

$$\dim G^r_d(C) + e - f(r+1-|\mu|+f) \geq \rho(g,r,d) + e - f(r+1-|\mu|+f).$$

From this fact we obtain that once one shows that for a general curve C of genus g each irreducible component of $\Sigma_\mu^f(C)$ has dimension *at most* $\rho(g,r,d)+e-f(r+1-|\mu|+f)$, it will also follow that $\Sigma_\mu^f(C)$ is in fact equidimensional of this dimension.

Assume we are in a situation when $\Sigma_\mu^f(C)$ in nonempty for a general (and therefore for an arbitrary) smooth curve C of genus g.

2.1. Universal de Jonquières divisors on curves of compact type.

The proof of Theorem 1.1 relies, like several other proofs involving limit linear series, on degenerating a smooth curve of genus g to a flag curve consisting of a rational spine and g smooth elliptic tails. It is known [13] and [14] that such curves satisfy the Brill-Noether Theorem *independently* of the position of the g points of attachment on the rational spine. One has however to deal with the serious complication that, under this degeneration, although one has a good understanding of the aspects of the limit linear series on the flag curve, a priori there is no control on the position of the e marked points lying in the support of a generalized de Jonquières divisor. For the combinatorial argument required to prove Theorem 1.1 it is however essential to ensure that one can always find such a flag curve degeneration of a generic curve of genus g in which these e marked points specialize to a subcurve of the flag curve having relatively small arithmetic genus. To make sure this is possible, we employ a strategy already used in [15], which relies on considering *all* flag curves of genus g at once and using certain basic facts about the geometry of the (rational) parameter space of such a curves.

We set some further notation. Let $j\colon \overline{\mathcal{M}}_{0,g} \to \overline{\mathcal{M}}_g$ the map assigning to a stable rational pointed curve $[R, p_1, \ldots, p_g] \in \overline{\mathcal{M}}_{0,g}$ *fixed* smooth elliptic tails $E_1, \ldots, E_g$ at the marked points $p_1, \ldots, p_g$. We denote the resulting compact type curve by

$$X := R \cup_{p_1} E_1 \cup \ldots \cup_{p_g} E_g,$$

that is, $p_a(X) = g$ and let $p_R\colon X \to R$ be the map contracting each elliptic component E_i to the point p_i. We introduce the universal n-pointed curve $\overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1}$ of genus g and denote by $\pi\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ the morphism forgetting the $(n+1)$-st marked point. For $e \geq 1$, we write $\pi_e\colon \overline{\mathcal{C}}^e_{g,n} \to \overline{\mathcal{M}}_{g,n}$ for the e-fold fibre product of $\overline{\mathcal{C}}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$. We finally introduce the map

$$\chi\colon \overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{C}}^e_g \to \overline{\mathcal{C}}^e_{0,g}, \tag{2.3}$$

which collapses the fixed elliptic *tails* $E_1, \ldots, E_g$ and projects the corresponding marked points onto the rational *spine* R. With the notation introduced above, we thus have

$$\chi\Big([R, p_1, \ldots, p_g], (x_1, \ldots, x_e)\Big) = \Big([R, p_1, \ldots, p_g],\ p_R(x_1), \ldots, p_R(x_e)\Big),$$

where $x_1, \ldots, x_e \in X$.

Let $\overline{\mathfrak{DJ}} \subseteq \overline{\mathcal{C}}_g^e$ be the closure of the locus of generalized de Jonquières divisors on smooth curves of genus g, that is, of the following determinantal variety

$$\mathfrak{DJ} := \Big\{[C, x_1, \ldots, x_e] : [C] \in \mathcal{M}_g,\ x_i \in C,\ \exists \ell = (L, V) \in G^r_d(C) \text{ such that}$$
$$\dim \big|V(-a_1 \cdot x_1 - \cdots - a_e \cdot x_e)\big| \geq r - |\mu| + f\Big\}.$$

Since we assume that $\Sigma^f_\mu(C) \neq \emptyset$ for a general curve $[C] \in \mathcal{M}_g$, we have that

$$\pi_e\big(\overline{\mathfrak{DJ}}\big) = \overline{\mathcal{M}}_g,$$

where recall that $\pi_e \colon \overline{\mathcal{C}}_g^e \to \overline{\mathcal{M}}_g$. Next, we define the locus

$$\mathcal{U} := \chi\Big(\overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathfrak{DJ}}\Big) \subseteq \overline{\mathcal{C}}^e_{0,g}. \tag{2.4}$$

We use the commutativity of the following diagram, where the horizontal upper arrow is induced via the *stabilization* isomorphism $\overline{\mathcal{C}}_{g,n} \cong \overline{\mathcal{M}}_{g,n+1}$, see [20, page 175] by taking fibre products

$$\begin{array}{ccc} \overline{\mathcal{C}}^e_{0,g} & \longrightarrow & \overline{\mathcal{C}}^e_g \\ \Big\downarrow{\scriptstyle \pi_e} & & \Big\downarrow{\scriptstyle \pi_e} \\ \overline{\mathcal{M}}_{0,g} & \xrightarrow{\ j\ } & \overline{\mathcal{M}}_g \end{array}$$

in order to conclude that $\pi_e(\mathcal{U}) = \overline{\mathcal{M}}_{0,g}$. We denote by $e - m$ the generic fibre dimension of the map $\pi_{e|\mathcal{U}} \colon \mathcal{U} \to \overline{\mathcal{M}}_{0,g}$. Thus $0 \leq m \leq e$ and

$$\dim\big(\mathcal{U} \cap \pi_e^{-1}[R, p_1, \ldots, p_g]\big) = e - m,$$

for a general stable curve $[R, p_1, \ldots, p_g] \in \overline{\mathcal{M}}_{0,g}$.

We introduce the birational map

$$\vartheta \colon \overline{\mathcal{C}}^e_{0,g} \to \overline{\mathcal{M}}_{0,4}^{g-3+e} \cong (\mathbb{P}^1)^{g-3+e}$$

whose components are the forgetful morphisms $\pi_i \colon \overline{\mathcal{M}}_{0,g+e} \to \overline{\mathcal{M}}_{0,4}$ which for $i = 4, \ldots, g + e$ only retain the marked points labelled by $1, 2, 3$ and i respectively. Fixing for instance the first three marked points as usual $p_1 = 0$, $p_2 = 1$ and $p_3 = \infty \in \mathbb{P}^1$, by slightly abusing notation we can think of ϑ as the map assigning

$$\big([R, p_1, \ldots, p_g], x_1, \ldots, x_e\big) \stackrel{\vartheta}{\mapsto} \big(p_4, \ldots, p_g, x_1, \ldots, x_e\big) \in (\mathbb{P}^1)^{g-3+e}.$$

Using essentially only the elementary fact that the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ is ample, we then establish in [15, Proposition 2.2], that depending on whether $\vartheta(\mathcal{U}) \subseteq (\mathbb{P}^1)^{g-3+e}$ intersects the small diagonal $(x_1 = \cdots = x_e)$ in $(\mathbb{P}^1)^{g-3+e}$ or not, one of the following three possibilities occur:

- There exists a point $(p_4, \ldots, p_g, x_1, \ldots, x_e) \in \vartheta(\mathcal{U})$ with $x_1 = \cdots = x_e$ and at least $g - m - 3$ of the points $p_4, \ldots, p_g$ are mutually distinct.
- There exists a point $(p_4, \ldots, p_g, x_1, \ldots, x_e) \in \vartheta(\mathcal{U})$ such that at least $g - m$ of the points $p_4, \ldots, p_g$ are equal to a point $r \in \mathbb{P}^1 \setminus \{x_1, \ldots, x_e\}$.
- There exists a point $(p_4, \ldots, p_g, x_1, \ldots, x_e) \in \vartheta(\mathcal{U})$ such that $e - 1$ of the marked points $x_1, \ldots, x_e$ are equal and at least $g - m$ of the points $p_4, \ldots, p_g$ are equal to 0.

Investigating the fibres of the map ϑ in each of these cases we find the following, see [15]:

Proposition 2.1. *Keeping the notation above, if* $\dim(\mathcal{U}) = g-3+e-m$, *there exists a point*

$$\big([R, p_1, \ldots, p_g], x_1, \ldots, x_e\big) \in \overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathfrak{DJ}},$$

such that on the flag curve $X = R \cup_{p_1} E_1 \cup \ldots \cup_{p_g} E_g$ *the limiting de Jonquières divisor* $(x_1, \ldots, x_e)$ *satisfies either (i)* $x_1 = \cdots = x_e \in R \smallsetminus \{p_1, \ldots, p_g\}$, *or else, (ii)* $x_1, \ldots, x_e$ *all lie on a connected subcurve* $Y \subseteq X$ *of genus at most* m *and with* $\big|Y \cap \overline{(X \smallsetminus Y)}\big| \leq 1$.

2.2. The proof of Theorem 1.1

We fix a partition $\mu = (a_1, \ldots, a_e)$ and a positive integer $f \geq |\mu| - r$. We assume that the variety $\Sigma^f_\mu(C) \subseteq G^r_d(C) \times C^e$ is not empty for every smooth curve C of genus g. Keeping the notation above, we denote by $e-m$ the fibre dimension of the surjective morphism $\pi_e \colon \mathcal{U} \to \overline{\mathcal{M}}_{0,e}$. Recall that we defined $\overline{\mathfrak{DJ}} \subseteq \overline{\mathcal{C}}^e_g$ to be the closure of the universal locus of de Jonquières divisors and we assume that $e-n$ is the generic fibre dimension of the surjective morphism

$$\pi_{e|\overline{\mathfrak{DJ}}} \colon \overline{\mathfrak{DJ}} \to \overline{\mathcal{M}}_g.$$

When specializing to the subvariety of flag curves via the map $j \colon \overline{\mathcal{M}}_{0,g} \hookrightarrow \overline{\mathcal{M}}_g$ the fibre dimension of π_e can only go up, we have that $m \leq n$. We now apply Proposition 2.1 and let $X = R \cup E_1 \cup \ldots \cup E_g$ be the corresponding flag curve of genus g as above, where for $i = 1, \ldots, g$ we denote by $p_i \in R$ the node corresponding to the intersection of the *spine* R (which may itself well be reducible) with the subtree of X ending in the elliptic *tail* E_i. We denote by $Y \subseteq X$ the connected subcurve of X onto which the marked points $x_1, \ldots, x_e$ (limiting a generalized de Jonquières divisor) specialize. According to Proposition 2.1 there are two possibilities:

(i) $p_a(Y) = m \leq \min\{e, g\}$, or
(ii) $x_1 = \cdots = x_e \in R \smallsetminus \{p_1, \ldots, p_g\}$.

We first treat case (i). Let $Y' := \overline{X \smallsetminus Y}$ be the subcurve of X complementary to Y and set $\{p\} := Y \cap Y'$. When $m = g$, then set $Y := X$ and $Y' = \emptyset$ and we let $p \in X$ be a general (smooth) point. The divisor $a_1 \cdot x_1 + \cdots + a_e \cdot x_e$ is a limit of generalized de Jonquières divisors on smooth curves of genus g neighboring the genus g curve of compact type X. Applying the formalism of stable reduction, we can find a flat family of nodal curves of genus g

$$\varphi \colon \mathcal{X} \to (T, t_0)$$

over a smooth pointed curve, together with sections $s_1, \ldots, s_e \colon T \to \mathcal{X}$ such that:

(**1**) The generic fibre $\varphi^{-1}(t) = X_t$ is a smooth curve of genus g, whereas the central fibre

$$\widetilde{X} := \varphi^{-1}(0)$$

is stably equivalent to X, that is, it is a curve of arithmetic genus g obtained from X by possibly attaching chains of smooth rational curves at the singularities of X.

(**2**) $s_i(0) = x_i \in \widetilde{X}_{\mathrm{reg}}$ for all $i = 1, \ldots, e$.
(**3**) There exists a line bundle L_η of related degree d defined on the complement of the central fibre $X_\eta = \mathcal{X} \setminus \varphi^{-1}(0)$, and a subvector bundle $V_\eta \subseteq \varphi_*(L_\eta)$ of rank $r+1$, such that for $t \neq 0$, setting $L_t = L_{\eta|X_t} \in \mathrm{Pic}(X_t)$ and $V_t = V_{\eta|t} \subseteq H^0(X_t, L_t)$, we have that

$$\Big((L_t, V_t), s_1(0), \ldots, s_e(t)\Big) \in \Sigma_\mu^f(X_t),$$

that is, dim $\big|V_t(-a_1 \cdot s_1(t) - \cdots - a_e \cdot s_e(t))\big| \geq r - |\mu| + f$.

We shall denote by $\widetilde{Y} \subseteq \widetilde{X}$ the inverse image of Y under the contraction morphism $\widetilde{X} \to X$. Then set $\widetilde{Y}' := \overline{\widetilde{X} \setminus \widetilde{Y}}$ and we still denote by p the point of intersection of $\widetilde{Y}$ and $\widetilde{Y}'$.

Since when forming the family $\mathcal{X} \to T$ we allow us the possibility of a further base change and that of resolving the resulting singularities, we may furthermore assume that the flag curve $\widetilde{X}$ carries a (refined) limit linear series

$$\ell = \Big\{\ell_Z = (L_Z, V_Z) : Z \text{ is a component of } \widetilde{X}\Big\} \in \overline{G}^r_d\big(\widetilde{X}\big)$$

obtained following the procedure described by Eisenbud and Harris [13] as a limit of the linear series (L_t, V_t). Furthermore, the sublinear series described in (3) induce a limit linear series

$$\ell' = \Big\{\ell'_Z = (L_Z(-D_Z), V'_Z) : Z \text{ is a component of } \widetilde{X}\Big\} \in \overline{G}^{r-|\mu|+f}_{d-|\mu|}\big(\widetilde{X}\big),$$

where D_Z is an effective divisor on Z supported on the union of the points $s_1(0), \ldots, s_e(0)$ that happen to lie on Z and the point of intersection $Z \cap \overline{\widetilde{X} \setminus Z}$ (which is a smooth point of Z), and $V'_Z \subseteq H^0(Z, L'_Z)$ is respectively a subspace of sections of dimension $r + 1 - |\mu| + f$.

Note that p is a smooth point of both subcurves $\widetilde{Y}$ and $\widetilde{Y}'$ of $\widetilde{X}$, therefore it is a smooth point of a unique irreducible component of $\widetilde{Y}$, respectively of a unique irreducible component of $\widetilde{Y}'$. We consider the respective aspects of ℓ and slightly abusing notation, we denote by

$$a^{\ell_{\widetilde{Y}}}(p) = \big(a_0 < \cdots < a_r\big)$$

the sequence obtained by ordering the vanishing orders at p of the sections corresponding to the irreducible component of $\widetilde{Y}$ containing p. Similarly, we let

$$a^{\ell_{\widetilde{Y}'}}(p) = \big(b_0 < \cdots < b_r\big)$$

be the sequence obtained by ordering the vanishing orders at p of the sections contained in the aspect of ℓ corresponding to the component of $\widetilde{Y}'$ containing p. Note that $a_i + b_{r-i} = d$ for $i = 0, \ldots, r$. Furthermore, by ordering the vanishing orders at p of the aspect of ℓ' corresponding to the component of $\widetilde{Y}$ containing p, we obtain the sequence

$$a^{\ell'_{\widetilde{Y}}}(p) = \big(a_{i_0} < \cdots < a_{i_{r-|\mu|+f}}\big).$$

Clearly, this is a subsequence of $a^{\ell_{\widetilde{Y}}}(p)$. The entries in the complementary subsequence can be ordered as well and we denote this subsequence by

$$\big(a_{j_0} < a_{j_1} < \cdots < a_{j_{|\mu|-f-1}}\big).$$

Note that

$$\big\{a_{i_0}, \ldots, a_{i_{r-|\mu|+f}}\big\} \cup \big\{a_{j_0}, \ldots, a_{j_{|\mu|-f-1}}\big\} = \big\{a_0, \ldots, a_r\big\}.$$

While the entries in the sequence $(a_{j_0} < \cdots < a_{j_{|\mu|-f-1}})$ corresponding to vanishing orders of sections of a linear series on a single irreducible component of $\widetilde{Y}$, using the procedure described in [15, Lemma 2.1], one can construct a sublimit linear series $\ell^{\sharp}_{\widetilde{Y}} \in \overline{G}^{|\mu|-f-1}_d(\widetilde{Y})$ of $\ell_{\widetilde{Y}}$ such that its vanishing sequence $a^{\ell^{\sharp}_{\widetilde{Y}}}(p)$ equals precisely $\big(a_{j_0} < \cdots < a_{|\mu|-f-1}\big)$.

We first assume $\widetilde{Y}' \neq \emptyset$. The point $p \in \widetilde{Y}$ is a smooth point and lies on one of its rational component. In particular the genus m pointed curve $[\widetilde{Y}, p]$ verifies the strong Brill-Noether Theorem, that is, both varieties $\overline{G}^{r-|\mu|+f}_{d-|\mu|}\big(\widetilde{Y}, (p, \alpha^{\ell'_{\widetilde{Y}}}(p))\big)$ and $\overline{G}^{|\mu|-f-1}_d\big(\widetilde{Y}, (p, \alpha^{\ell^{\sharp}_{\widetilde{Y}}(p)})\big)$ have the expected dimension given by the corresponding adjusted Brill-Noether numbers, in particular these numbers must be non-negative, cf. [14, Theorem 1.1]. We thus obtain the following two inequalities by writing this for out for the limit linear series $\ell_{\widetilde{Y}}$ and $\ell_{\widetilde{Y}'}$, respectively:

$$\dim \overline{G}^{|\mu|-f-1}_d\Big(\widetilde{Y}, \big(p, \alpha^{\ell^{\sharp}_{\widetilde{Y}}(p)}\big)\Big) = \rho\big(\ell^{\sharp}_{\widetilde{Y}}, p\big) \tag{2.5}$$

$$= \rho\big(m, |\mu| - f - 1, d\big) - a_{j_0} - \cdots - a_{j_{|\mu|-f-1}} + \binom{|\mu| - f}{2} \geq 0,$$

as well as

$$\dim \overline{G}^{r-|\mu|+f}_{d-|\mu|}\Big(\widetilde{Y}, \big(p, \alpha^{\ell'_{\widetilde{Y}}}(p)\big)\Big) = \rho(\ell'_{\widetilde{Y}}, p) \tag{2.6}$$

$$= \rho\big(m, r - |\mu| + f, d - |\mu|\big) - a_{i_0} - \cdots - a_{r-|\mu|+f} + \binom{r + 1 - |\mu| + f}{2} \geq 0.$$

The same considerations can be applied to the complementary subcurve $\widetilde{Y}'$ of $\widetilde{X}$. The point of attachment p lies on a rational component component of $\widetilde{Y}'$, therefore the strong Brill-Noether inequality holds for $\ell_{Y'}$ as well, and we obtain:

$$\dim \overline{G}^r_d\Big(\widetilde{Y}', (p, \alpha^{\ell_{\widetilde{Y}'}}(p))\Big) = \rho(\ell_{\widetilde{Y}'}, p) = \rho(g - m, r, d) - \big(b_0 + \cdots + b_r\big) + \binom{r+1}{2} \geq 0. \tag{2.7}$$

We add the inequalities (2.5), (2.6) and (2.7) together and use the fact that $(\ell_{\widetilde{Y}}, \ell_{\widetilde{Y}'})$ form a refined limit linear series, therefore the vanishing orders of $\ell'_{\widetilde{Y}}$, $\ell^{\sharp}_{\widetilde{Y}}$ and those of $\ell_{\widetilde{Y}'}$ respectively add up, that is,

$$\sum_{k=0}^{r} b_k + \sum_{k=0}^{r-|\mu|+f} a_{i_k} + \sum_{k=0}^{|\mu|-f-1} a_{j_k} = \sum_{k=0}^{r}\big(a_k + b_{r-k}\big) = (r+1)d.$$

We obtain the following estimate:

$$\begin{aligned}0 \le \rho(g-m,r,d)&+\rho\big(m,r-|\mu|+f,d-|\mu|\big)+\rho\big(m,|\mu|-f-1,d\big)\\ &-(r+1)d+\binom{r+1}{2}+\binom{r+1-|\mu|+f}{2}+\binom{|\mu|-f}{2}\\ =\rho(g,r,d)-f(r+1-|\mu|+f)+m&\le \rho(g,r,d)-f(r+1-|\mu|+f)+e,\end{aligned}$$

which is precisely the second half of Theorem 1.1. Note that in the last inequality, the assumption $m \le e$ guaranteed by Proposition 2.1 is absolutely essential.

In the case $m=g$, when necessarily $e \ge g$ and $\widetilde{Y}=\widetilde{X}$, we proceed along similar lines. We add together inequalities (2.5) and (2.6) to obtain:

$$\begin{aligned}&\rho(g,r,d)+e-f(r+1-|\mu|+f)\\ &=\Bigg(\rho\big(g,r-|\mu|+f,d-|\mu|\big)-\sum_{k=0}^{r-|\mu|+f}a_{i_k}+\binom{r+1-|\mu|+f}{2}\Bigg)\\ &\quad+\Bigg(\rho\big(g,|\mu|-f-1,d\big)-\sum_{k=0}^{e-f-1}a_{j_k}+\binom{e-f}{2}\Bigg)\\ &\quad+\sum_{k=0}^{r-|\mu|+f}a_{i_k}+\sum_{k=0}^{|\mu|-f-1}a_{j_k}-\binom{r+1}{2}+e-g\\ &=\dim \overline{G}^{r-|\mu|+f}_{d-|\mu|}\Big(\widetilde{Y},\big(p,\alpha^{\ell'_{\widetilde{Y}}}(p)\big)\Big)+\dim \overline{G}^{|\mu|-f-1}_{d}\Big(\widetilde{Y},\big(p,\alpha^{\ell^{\sharp}_{\widetilde{Y}}(p)}\big)\Big)\\ &\quad+\sum_{k=0}^{r-|\mu|+f}a_{i_k}+\sum_{k=0}^{|\mu|-f-1}a_{j_k}-\binom{r+1}{2}+e-g\ge 0,\end{aligned}$$

since

$$\sum_{k=0}^{r-|\mu|+f}a_{i_k}+\sum_{k=0}^{|\mu|-f-1}a_{j_k}=\sum_{k=0}^{r}a_k\ge\binom{r+1}{2}$$

and, as explained, $e \ge g$.

Assume finally we are in the case (ii), that is, when $x_1=\cdots=x_e\in R\setminus\{p_1,\ldots,p_g\}$. Keeping the previous notation, we observe that the limit linear series $\ell\in\overline{G}^r_d(\widetilde{X})$ has vanishing sequence at x_1

$$a^{\ell}(x_1)\ge\Big(0,1,\ldots,|\mu|-f-1,|\mu|,|\mu|+1,\ldots,r+f-1,r+f\Big),$$

therefore $\mathrm{wt}^{\ell}(x_1)\ge f(r+1-|\mu|+f)$. Taking into account that $[\widetilde{X},q]$ satisfies the strong Brill-Noether Theorem, cf. [14, Theorem 1.1], Theorem 1.1, we obtain the inequality

$$\begin{aligned}0&\le\dim\overline{G}^r_d\Big(\widetilde{X},\big(x_1,\alpha^{\ell}(x_1)\big)\Big)\\ &\le\rho(g,r,d)-f(r+1-|\mu|+f)\\ &\le\rho(g,r,d)+e-f(r+1-|\mu|+f).\end{aligned}$$

This concludes the proof that the assumption $\Sigma_\mu^f(C) \neq \emptyset$ for a general curve of genus g implies that $\rho(g,r,d) + e - f(r+1-|\mu|+f) \geq 0$.

We come now to the dimensionality statement for the variety

$$\Sigma_\mu^f(C) \subseteq G_d^r(C) \times C^e,$$

when C is a general curve of genus g. Recalling from the Introduction that $\pi_2\colon \Sigma_\mu^f(C) \to C^e$ is the natural projection, with our notation we have $\dim \pi_2\big(\Sigma_\mu^f(C)\big) = e - n \leq e - m$, where $e-n$ has been defined as the minimal fibre dimension of the surjection $\overline{\mathfrak{DJ}} \to \overline{\mathcal{M}}_g$. We now estimate the fibre dimension of π_2 over a general point $(y_1, \ldots, y_e) \in \pi_2(\Sigma_\mu^f)$. To that end, we specialize once more to the locus of flag curves. For an e-pointed curve $[X, x_1, \ldots, x_e]$ of compact type, where the marked points are pairwise distinct smooth points of X, we denote by $\Sigma_\mu^f(X, x_1, \ldots, x_e)$ the subvariety of $\overline{G}_d^r(X)$ consisting of limit linear series

$$\ell = \Big\{\ell_Z = (\ell_Z, V_Z) : Z \text{ is a component of } X\Big\} \in \overline{G}_d^r(X)$$

possessing a sublimit linear series of the form

$$\ell' = \Big\{\ell'_Z = \big(L_Z(-D_Z), V'_Z\big) : Z \text{ is a component of } X\Big\} \in \overline{G}_{d-|\mu|}^{r-|\mu|+f}(X),$$

where $\mathrm{supp}(D_Z) = Z \cap \big(\overline{(X \smallsetminus Z)} \cup \{x_1, \ldots, x_e\}\big)$. As already explained, via Proposition 2.1 we may consider a further degeneration to a flag curve $[\widetilde{X}, x_1, \ldots, x_e]$, where $\widetilde{X} = \widetilde{Y} \cup \widetilde{Y}'$ with $\widetilde{Y} \cap \widetilde{Y}' = \{p\}$ satisfies the conditions (1)-(3). Recall that $x_1, \ldots, x_e \in \widetilde{Y}_{\mathrm{req}} \smallsetminus \{p\}$. It follows that for the generic fibre dimension of π_2 the following inequality holds:

$$\dim \pi_2^{-1}(y_1, \ldots, y_e) \leq \dim \Sigma_\mu^f\big(\widetilde{X}, x_1, \ldots, x_e\big).$$

Furthermore, the dimension of $\Sigma_\mu^f\big(\widetilde{X}, x_1, \ldots, x_e\big)$ cannot exceed the dimension of the space of triples $\big(\ell'_{\widetilde{Y}}, \ell^\sharp_{\widetilde{Y}}, \ell_{\widetilde{Y}'}\big)$ described earlier, which as explained, via the estimates (2.5), (2.6) and (2.7) equals

$$\begin{aligned}\dim \overline{G}_{d-|\mu|}^{r-|\mu|+f}\Big(\widetilde{Y}, \big(p, \alpha^{\ell'_{\widetilde{Y}}}(p)\big)\Big) &+ \dim \overline{G}_d^{|\mu|-f-1}\Big(\widetilde{Y}, \big(p, \alpha^{\ell^\sharp_{\widetilde{Y}}(p)}\big)\Big) \\ &+ \dim \overline{G}_d^r\Big(\widetilde{Y}', (p, \alpha^{\ell_{\widetilde{Y}'}}(p))\Big) \\ &= \rho(g,r,d) - f(r+1-|\mu|+f) + m.\end{aligned}$$

It follows that

$$\begin{aligned}\dim \Sigma_\mu^f(C) &\leq \dim \pi_2\big(\Sigma_\mu^f(C)\big) + \dim \Sigma_\mu^f\big(\widetilde{X}, x_1, \ldots, x_e\big) \\ &\leq e - n + m + \rho(g,r,d) - f(r+1-|\mu|+f) \\ &\leq e - f(r+1-|\mu|+f),\end{aligned}$$

since, as explained, $m \leq n$. This brings the proof of Theorem 1.1 to an end. □

Remark 2.2. A natural extension of Theorem 1.1 could be to consider the transversality of curves $C \subseteq \mathbb{P}^r$ with respect to non-linear spaces. For instance, staying at the level of space curves, it is expected that a general curve $C \subseteq \mathbb{P}^3$ has finitely many 8-secant conics (but no 9-secant conics), finitely many 12-secant twisted cubics (but not 13-secant twisted cubics) and so on. The smooth curves confirming this expectation have been recently characterized as those for which the blow-up of $\mathbb{P}^r$ along C yields a threefold with big and nef anticanonical divisor, see [6]. The (virtual) number of 8-secant conics to $C \subseteq \mathbb{P}^3$ has been computed by Katz [18] as an iteration of multiple point formulas. It would be interesting to have a study of the enumerative validity of this and other similar formulas mirroring Theorem 1.1. In this case however more subtle phenomena, related to the (Strong) Maximal Rank Conjecture [2, Conjecture 5.1], must come into play and which go beyond the Brill-Noether genericity of the curve in question. It is for instance clear that whenever $C \subseteq \mathbb{P}^3$ lies on a quadric there is a positive dimensional family of 8-secant conics, so at the very least these curves will have to be excluded, probably other as well.

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Gavril Farkas
Humboldt-Universität zu Berlin,
Institut Für Mathematik,
Unter den Linden 6,
10099 Berlin, Germany
e-mail: `farkas@math.hu-berlin.de`

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Generalized versus classical normal derivative

Lucas Fresse and Viorica V. Motreanu

Dedicated to the memory of Professor Csaba Varga

Abstract. Given a bounded domain with Lipschitz boundary, the general Green formula permits to justify that the weak solutions of a Neumann elliptic problem satisfy the Neumann boundary condition in a weak sense. The formula involves a generalized normal derivative. We prove a general result which establishes that the generalized normal derivative of an operator coincides with the classical one, provided that the operator is continuous. This result allows to deduce that, under usual regularity assumptions, the weak solutions of a Neumann problem satisfy the Neumann boundary condition in the classical sense. This information is necessary in particular for applying the strong maximum principle.

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1. Introduction and statement of the result

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with Lipschitz boundary. Then, it is a consequence of Rademacher Theorem that the outward unit normal $n(x)$ is defined almost everywhere on the boundary $\partial\Omega$ (endowed with the Hausdorff measure H^{N-1}). The normal derivative of a function $u \in C^1(\overline{\Omega})$ is then $\frac{\partial u}{\partial n} = \nabla u \cdot n$ on $\partial\Omega$.

The nonsmooth Green formula ([6], [2]) asserts that

$$\int_\Omega (\operatorname{div} a)\phi \, dx + \int_\Omega a \cdot \nabla\phi \, dx = \int_{\partial\Omega} \gamma_n(a)\gamma(\phi) \, dH^{N-1}$$

for all $\phi \in W^{1,p}(\Omega)$ and all a belonging to

$$V^{p'}(\Omega, \operatorname{div}) = \{a \in L^{p'}(\Omega, \mathbb{R}^N) : \operatorname{div} a \in L^{p'}(\Omega)\}.$$

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Here $p \in (1, +\infty)$ and $p' := \frac{p}{p-1}$ is its Hölder conjugate. The formula involves the classical trace operator $\gamma : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\partial\Omega)$ (see, e.g., [3], [5]) and the generalized normal derivative $\gamma_n : V^{p'}(\Omega, \operatorname{div}) \to W^{-\frac{1}{p'},p'}(\partial\Omega)$ introduced in [6] and [2].

If $\phi \in C^1(\overline{\Omega})$, then due to the classical Green formula we have $\gamma(\phi) = \phi|_{\partial\Omega}$. In fact, it is well known that the equality $\gamma(\phi) = \phi|_{\partial\Omega}$ holds whenever $\phi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ (see, e.g, [3]).

Similarly, if $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$, then we have $\gamma_n(a) = a \cdot n$. Our main result ensures that this equality holds more generally:

Theorem 1.1. *Let $\gamma_n : V^q(\Omega, \operatorname{div}) \to W^{-\frac{1}{q},q}(\partial\Omega)$ (with $q \in (1, +\infty)$) be the generalized normal derivative. Then, for all $a \in V^q(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$, we have $\gamma_n(a) = a \cdot n$.*

As far as we know, there was no proof of this result in the literature.

This result can be applied to Neumann elliptic boundary value problems driven by the p-Laplacian (or a more general nonlinear operator) for showing that a weak solution $u \in W^{1,p}(\Omega)$ (which belongs in fact to $C^1(\overline{\Omega})$ due to nonlinear regularity theory) satisfies the classical Neumann boundary condition $\frac{\partial u}{\partial n} = 0$. Without the result stated in Theorem 1.1, we can just say that $\gamma_n(|\nabla u|^{p-2}\nabla u) = 0$. The latter equality can be viewed as a Neumann boundary condition in a weak sense. However, it is a key point that for applying the strong maximum principle [9] to u (in order to show for instance that a nonnegative, nontrivial solution is positive on $\overline{\Omega}$), it is necessary to know that the strong Neumann condition $\frac{\partial u}{\partial n} = 0$ holds (the weak one is not sufficient).

The rest of the paper is organized as follows. In Section 2, we present the background on trace operator, generalized normal derivative, and Green formulas. In Section 3, we give the proof of Theorem 1.1. In Section 4, we present the application to Neumann and, more generally, Steklov boundary value problems.

2. Green formulas

In this section, we recall the generalized normal derivative operator defined in [6] and [2]. This operator permits to obtain a nonlinear Green formula, which is crucial for relating weak solutions of quasilinear elliptic problems and their boundary conditions.

Before stating the main definition and the general Green formula (Theorem 2.2), we review other versions of the Green formula involving relatively regular functions and operators.

Recall that $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with Lipschitz boundary $\partial\Omega$. This regularity of the domain implies that we have the $(N-1)$-dimensional Hausdorff measure H^{N-1} on $\partial\Omega$, and the outward unit normal $n(\cdot)$ is defined H^{N-1}-almost everywhere on $\partial\Omega$.

The *classical Green formula* states as follows : if $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$ and $v \in C^1(\overline{\Omega})(:= C^1(\overline{\Omega}, \mathbb{R}))$, then

$$\int_\Omega (\operatorname{div} a)v\,dx + \int_\Omega a \cdot \nabla v\,dx = \int_{\partial\Omega} (a \cdot n)v\,dH^{N-1} \tag{2.1}$$

where $\operatorname{div} a = \sum_{i=1}^{N} \frac{\partial a_i}{\partial x_i}$ and $\nabla v = (\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N})$, while "$\cdot$" stands for the scalar product in $\mathbb{R}^N$. For a first generalization of the Green formula, we take v in the Sobolev space $W^{1,p}(\Omega)$ $(p > 1)$ instead of being of class C^1. To this end, the notion of trace is needed:

Theorem 2.1 (see [3, §4.3] **and** [5, §1.5]**).** *There is a unique bounded linear operator* $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega, H^{N-1})$ *which extends the operator* $C^\infty(\overline{\Omega}) \to C(\partial\Omega)$, $v \mapsto v|_{\partial\Omega}$.

Moreover, we have the following properties:

(a) $\gamma(v) = v|_{\partial\Omega}$ *whenever* $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.

(b) *(Green formula) If* $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$ *and* $v \in W^{1,p}(\Omega)$, *then*

$$\int_\Omega (\operatorname{div} a) v \, dx + \int_\Omega a \cdot \nabla v \, dx = \int_{\partial\Omega} (a \cdot n) \gamma(v) \, dH^{N-1}. \tag{2.2}$$

(c) $\ker \gamma = W_0^{1,p}(\Omega)$ *and* $\operatorname{Im} \gamma = W^{\frac{1}{p'},p}(\partial\Omega)$.

In particular, in view of Theorem 2.1 (a)–(b), the Green formula (2.1) remains valid if we assume that $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ instead of $v \in C^1(\overline{\Omega})$.

The final stage of the discussion is to replace the assumption that $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$ by a more general one. To this end, for $q > 1$, we define

$$V^q(\Omega, \operatorname{div}) = \{a \in L^q(\Omega, \mathbb{R}^N) : \operatorname{div} a \in L^q(\Omega)\}$$

which is a Banach space for the norm

$$\|a\|_{V^q(\Omega,\operatorname{div})} = \left(\|a\|^q_{L^q(\Omega,\mathbb{R}^N)} + \|\operatorname{div} a\|^q_{L^q(\Omega)} \right)^{\frac{1}{q}}.$$

This requires the definition of a new operator which extends $a \mapsto a \cdot n$ to the space $V^{p'}(\Omega, \operatorname{div})$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p.

Theorem 2.2 ([6, 2]**).** *There is a unique bounded linear operator*

$$\gamma_n : V^{p'}(\Omega, \operatorname{div}) \to W^{-\frac{1}{p'},p'}(\partial\Omega) = W^{\frac{1}{p'},p}(\partial\Omega)^*$$

which extends the operator $C^\infty(\overline{\Omega}, \mathbb{R}^N) \to L^\infty(\partial\Omega, H^{N-1})$, $a \mapsto a \cdot n$.

Moreover, we have the following properties:

(a) *(Green formula) If* $a \in V^{p'}(\Omega, \operatorname{div})$ *and* $v \in W^{1,p}(\Omega)$, *then*

$$\int_\Omega (\operatorname{div} a) v \, dx + \int_\Omega a \cdot \nabla v \, dx = \langle \gamma_n(a), \gamma(v) \rangle_{W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p'},p}(\partial\Omega)}. \tag{2.3}$$

(b) $\operatorname{Im} \gamma_n = W^{-\frac{1}{p'},p'}(\partial\Omega)$.

Remark 2.3. Due to (2.2), (2.3), and the surjectivity of the trace operator $\gamma : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\partial\Omega)$, we have immediately that $\gamma_n(a) = a \cdot n$ whenever $a \in C^1(\overline{\Omega}, \mathbb{R}^N)$.

Example 2.4. (a) If $p = 2$, $u \in W^{1,2}(\Omega)$ is such that $\Delta u := \operatorname{div}(\nabla u) \in L^2(\Omega)$, then the Green formula (2.3) reads as

$$\int_\Omega (\Delta u) v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx = \langle \gamma_n(\nabla u), \gamma(v) \rangle_{W^{-\frac{1}{2},2}(\partial\Omega), W^{\frac{1}{2},2}(\partial\Omega)}.$$

(b) If $p > 1$ is arbitrary and letting $a = |\nabla u|^{p-2}\nabla u$ for $u \in W^{1,p}(\Omega)$, then the Green formula (2.3) becomes

$$\int_\Omega (\Delta_p u)v\,dx + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx = \langle \frac{\partial u}{\partial n_p}, \gamma(v)\rangle_{W^{-\frac{1}{p'},p'}(\partial\Omega),W^{\frac{1}{p'},p}(\partial\Omega)}$$

provided that $\Delta_p u \in L^{p'}(\Omega)$, where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and we denote $\frac{\partial u}{\partial n_p} := \gamma_n(|\nabla u|^{p-2}\nabla u)$. In the case $p \geq 2$, if $u \in C^2(\overline{\Omega})$ then $|\nabla u|^{p-2}\nabla u \in C^1(\overline{\Omega}, \mathbb{R}^N)$ and we get $\frac{\partial u}{\partial n_p} = |\nabla u|^{p-2}\nabla u \cdot n$ (see Remark 2.3). If, moreover, $p = 2$, then $\frac{\partial u}{\partial n_2} = \nabla u \cdot n = \frac{\partial u}{\partial n}$. Thus $\frac{\partial u}{\partial n_p}$ can be seen as a generalized normal derivative.

3. Proof of Theorem 1.1

The proof splits into several steps.

Lemma 3.1. *Let $a \in V^q(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$. Assume that a is the restriction of $a' \in V^q(\Omega', \operatorname{div}) \cap C(\overline{\Omega'}, \mathbb{R}^N)$ for a bounded domain $\Omega' \subset \mathbb{R}^N$ with $\overline{\Omega} \subset \Omega'$. Then, the equality $\gamma_n(a) = a \cdot n$ holds.*

In the following proof, whenever $\rho \in C_c^\infty(\mathbb{R}^N)$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we consider the convolution

$$\rho * h : \mathbb{R}^N \to \mathbb{R},\ x \mapsto \int_{\mathbb{R}^N} \rho(x-y)h(y)\,dy.$$

If $h \in L^q(\Omega')$ then we set $\rho * h = \rho * \bar{h}$ where $\bar{h} \in L^q(\mathbb{R}^N)$ is the extension by zero of h.

Proof of Lemma 3.1. Consider a regularizing sequence $(\rho_k)_{k\geq 1}$, that is,

$$\rho_k \in C_c^\infty(\mathbb{R}^N),\quad \operatorname{supp}\rho_k \subset B(0, \tfrac{1}{k}),\quad \int_{\mathbb{R}^N} \rho_k\,dx = 1,\quad \rho_k \geq 0 \text{ in } \mathbb{R}^N.$$

Choose $k_0 \geq 1$ such that

$$\overline{\Omega + B(0, \tfrac{1}{k_0})} \subset \Omega'. \tag{3.1}$$

Write $a = (a_1, \ldots, a_N)$ and $a' = (a'_1, \ldots, a'_N)$, so that $a_i = a'_i|_{\overline{\Omega}}$ for all $i \in \{1, \ldots, N\}$. Then we set

$$v_k = \rho_k * a' = (\rho_k * a'_1, \ldots, \rho_k * a'_N).$$

Thus, $v_k \in C^\infty(\mathbb{R}^N, \mathbb{R}^N) \cap L^q(\mathbb{R}^N, \mathbb{R}^N)$ (see [1, Théorème IV.15 and Proposition IV.20]) and we have that

$$v_k \to a' \text{ in } L^q(\Omega', \mathbb{R}^N) \text{ as } k \to \infty \tag{3.2}$$

(see [1, Théorème IV.22]) and moreover

$$v_k \to a \quad \text{uniformly on } \overline{\Omega} \text{ as } k \to \infty \tag{3.3}$$

(see [1, proof of Proposition IV.21]).

Since $\operatorname{div} a' \in L^q(\Omega')$, we have also that $\rho_k * \operatorname{div} a' \in C^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ and

$$\rho_k * \operatorname{div} a' \to \operatorname{div} a' \text{ in } L^q(\Omega') \text{ as } k \to \infty. \tag{3.4}$$

We claim that

$$\operatorname{div} v_k = \rho_k * \operatorname{div} a' \quad \text{in } \Omega, \text{ for all } k \geq k_0. \tag{3.5}$$

The functions on the left- and the right-hand side of (3.5) belong to $C^\infty(\mathbb{R}^N)$, but since we do not know that the partial derivatives $\frac{\partial a'_i}{\partial x_i}$ are defined almost everywhere (though it is the case for $\operatorname{div} a' \in L^q(\Omega')$), we will show (3.5) by reasoning in distributions. So let $\varphi \in C_c^\infty(\Omega)$. We compute

$$\begin{aligned}\langle \operatorname{div} v_k, \varphi\rangle = \sum_{i=1}^N \Big\langle \frac{\partial(\rho_k * a'_i)}{\partial x_i}, \varphi \Big\rangle &= -\sum_{i=1}^N \int_{\mathbb{R}^N} (\rho_k * a'_i) \frac{\partial \varphi}{\partial x_i}\, dx \\ &= -\sum_{i=1}^N \int_{\Omega'} a'_i \Big(\check{\rho}_k * \frac{\partial \varphi}{\partial x_i}\Big)\, dx \\ &= -\sum_{i=1}^N \int_{\Omega'} a'_i \frac{\partial(\check{\rho}_k * \varphi)}{\partial x_i}\, dx,\end{aligned}$$

where we denote $\check{\rho}_k(x) = \rho_k(-x)$ and use [1, Propositions IV.16 and IV.20]. Since $\rho_k \in C_c^\infty(B(0, \frac{1}{k}))$, $\varphi \in C_c^\infty(\Omega)$, and due to (3.1) and the fact that $k \geq k_0$, we have $\check{\rho}_k * \varphi \in C_c^\infty(\Omega')$ (see [1, Proposition IV.18]). Hence

$$\begin{aligned}\langle \operatorname{div} v_k, \varphi\rangle &= \sum_{i=1}^N \Big\langle \frac{\partial a'_i}{\partial x_i}, \check{\rho}_k * \varphi \Big\rangle = \langle \operatorname{div} a', \check{\rho}_k * \varphi\rangle \\ &= \int_{\Omega'} (\operatorname{div} a')(\check{\rho}_k * \varphi)\, dx \quad (\text{since } \operatorname{div} a' \in L^q(\Omega')) \\ &= \int_{\mathbb{R}^N} (\rho_k * \operatorname{div} a')\varphi\, dx \quad (\text{by [1, Proposition IV.16]}) \\ &= \langle \rho_k * \operatorname{div} a', \varphi\rangle.\end{aligned}$$

This establishes (3.5).

We have $v_k \in V^q(\Omega, \operatorname{div})$ because $v_k \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Formulas (3.2), (3.4), and (3.5) imply that

$$v_k \to a \quad \text{in } V^q(\Omega, \operatorname{div}).$$

Due to the continuity of the operator

$$\gamma_n : V^q(\Omega, \operatorname{div}) \to W^{-1/q,q}(\partial\Omega)$$

we have

$$\gamma_n(v_k) \to \gamma_n(a) \quad \text{in } W^{-1/q,q}(\partial\Omega) \quad \text{as } k \to \infty. \tag{3.6}$$

Since $v_k \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$, we have

$$\gamma_n(v_k) = v_k \cdot n \quad \text{on } \partial\Omega \quad \text{for all } k \tag{3.7}$$

(by definition of γ_n; see Theorem 2.2). By virtue of (3.3), we have

$$v_k \cdot n \to a \cdot n \quad \text{in } L^\infty(\partial\Omega, H^{N-1}) \text{ as } k \to \infty.$$

The continuity of the embeddings $L^\infty(\partial\Omega, H^{N-1}) \hookrightarrow L^q(\partial\Omega, H^{N-1}) \hookrightarrow W^{-1/q,q}(\partial\Omega)$ now implies that

$$v_k \cdot n \to a \cdot n \quad \text{in } W^{-1/q,q}(\partial\Omega) \ \text{ as } k \to \infty.$$

Combining this with (3.6) and (3.7), we conclude that

$$\gamma_n(a) = a \cdot n \quad \text{on } \partial\Omega.$$

The proof of the lemma is complete. □

Lemma 3.2. *There is an open covering*

$$\partial\Omega = \bigcup_{i=1}^m \Gamma_i,$$

a family of vectors $(\nu_i)_{i=1}^m \subset \mathbb{R}^N$ *and a constant* $\delta > 0$ *such that, for every* $i \in \{1, \dots, m\}$,

$$U_i(\delta_1, \delta_2) := \{x + t\nu_i : x \in \Gamma_i,\ t \in (-\delta_1, \delta_2)\}$$

is an open subset of $\mathbb{R}^N$ *for all* $\delta_1, \delta_2 \in (0, \delta]$ *and the following inclusions hold:*

$$U_i' := \{x + t\nu_i : x \in \Gamma_i,\ t \in (0, \delta)\} \subset \Omega,$$
$$U_i'' := \{x + t\nu_i : x \in \Gamma_i,\ t \in (-\delta, 0)\} \subset \mathbb{R}^N \setminus \overline{\Omega}.$$

Proof. Fix $x \in \partial\Omega$. Since Ω is assumed to have Lipschitz boundary, there is an open neighborhood $V \subset \mathbb{R}^N$ of x and a Lipschitz map $\chi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that (up to rotating and relabeling the axes)

$$\begin{aligned} V \cap \Omega &= \{(y_1, \dots, y_N) \in V : \chi(y_1, \dots, y_{N-1}) < y_N\}, \\ V \cap \partial\Omega &= \{(y_1, \dots, y_N) \in V : \chi(y_1, \dots, y_{N-1}) = y_N\} \end{aligned}$$

(see [3, §4.2]). There is $\delta > 0$ and an open neighborhood $W \subset V$ of x such that

$$\bigcup_{y \in W} B(y, \delta) \subset V,$$

where $B(y, \delta)$ stands for the open ball of radius δ with respect to the norm

$$(y_1, \dots, y_N) \mapsto \max_{1 \le i \le N} |y_i|.$$

Let $\Gamma_x = \Gamma = W \cap \partial\Omega$ and, given $\delta_1, \delta_2 \in (0, \delta]$, let

$$U(\delta_1, \delta_2) = \{y + t\nu : y \in \Gamma,\ t \in (-\delta_1, \delta_2)\}$$

where $\nu = (0, \dots, 0, 1)$. Note that we have equivalently

$$\begin{aligned} U(\delta_1, \delta_2) = \{&(y_1, \dots, y_N) \in \mathbb{R}^N : (y_1, \dots, y_{N-1}, \chi(y_1, \dots, y_{N-1})) \in W, \\ &y_N - \chi(y_1, \dots, y_{N-1}) \in (-\delta_1, \delta_2)\}, \end{aligned}$$

which shows that $U(\delta_1, \delta_2)$ is open. Moreover, for all $y = (y_1, \dots, y_N) \in \Gamma$ and $t \in (-\delta, \delta)$, we have $y + t\nu = (y_1, \dots, y_{N-1}, y_N + t) \in B(y, \delta) \subset V$ and

$$\chi(y_1, \dots, y_{N-1}) = y_N \begin{cases} < y_N + t & \text{if } t \in (0, \delta), \\ > y_N + t & \text{if } t \in (-\delta, 0), \end{cases}$$

whence

$$U' := \{y + t\nu : y \in \Gamma,\ t \in (0,\delta)\} \subset \Omega,$$
$$U'' := \{y + t\nu : y \in \Gamma,\ t \in (-\delta,0)\} \subset \mathbb{R}^N \setminus \overline{\Omega}.$$

By doing the same construction for every $x \in \partial\Omega$ and extracting a finite subcovering from the open covering $\bigcup_{x\in\partial\Omega} \Gamma_x = \partial\Omega$ so obtained, we get a family of open subsets/vectors satisfying the conditions stated in the lemma. □

Lemma 3.2 yields an open neighborhood $U := \bigcup_{i=1}^m U_i$ of the boundary $\partial\Omega$, where $U_i := U_i(\delta,\delta)$. Since $\partial\Omega$ is compact, we can find a relatively compact, open neighborhood V of $\partial\Omega$ such that $\overline{V} \subset U$. Let $U_0 := \Omega \setminus \overline{V}$. Then we have an open covering

$$\overline{\Omega} \subset \bigcup_{i=0}^m U_i.$$

Let $(\theta_i)_{i=0}^m$ be a partition of unity relative to this open covering, i.e.,

- $\theta_i \in C_c^\infty(U_i)$ and $0 \le \theta_i \le 1$ for all $i \in \{0,1,\dots,m\}$,
- $\theta_0 + \theta_1 + \ldots + \theta_m = 1$ in $\overline{\Omega}$

(see [1, Lemme IX.3]).

Lemma 3.3. *Let $a \in V^q(\Omega,\operatorname{div}) \cap C(\overline{\Omega},\mathbb{R}^N)$. For every $i \in \{0,\dots,m\}$, let $a_i = \theta_i a$ for θ_i as above, so that $a = a_0 + a_1 + \ldots + a_m$. Then:*

(a) *$a_i \in V^q(\Omega,\operatorname{div}) \cap C(\overline{\Omega},\mathbb{R}^N)$ and $\operatorname{supp} a_i \subset U_i$ for all i.*
(b) *In particular $\operatorname{supp} a_0 \subset \Omega$ and we have $\gamma_n(a_0) = a_0 \cdot n = 0$.*
(c) *If $\gamma_n(a_i) = a_i \cdot n$ for all $i \in \{1,\dots,m\}$, then $\gamma_n(a) = a \cdot n$.*

Proof. (a) Since $a_i = \theta_i a$ with $\theta_i \in C_c^\infty(U_i)$ and $a \in C(\overline{\Omega},\mathbb{R}^N)$, we get $a_i \in C(\overline{\Omega},\mathbb{R}^N)$ and $\operatorname{supp} a_i \subset U_i$. Moreover, we have

$$\operatorname{div} a_i = \theta_i \operatorname{div} a + a \cdot \nabla\theta_i$$

with $\operatorname{div} a \in L^q(\Omega)$, $a \in C(\overline{\Omega},\mathbb{R}^N)$, and $\theta_i \in C_c^\infty(U_i)$, whence $\operatorname{div} a_i \in L^q(\Omega)$ and, therefore, $a_i \in V^q(\Omega,\operatorname{div})$ for all $i \in \{0,\dots,m\}$. This shows (a).

(b) In particular, we get $\operatorname{supp} a_0 \subset U_0 \subset \Omega$. This guarantees that, if a_0' denotes the extension by zero of a_0, we have $a_0' \in V^q(\mathbb{R}^N,\operatorname{div}) \cap C(\mathbb{R}^N)$, and by Lemma 3.1 we deduce that $\gamma_n(a_0) = a_0 \cdot n = 0$ on $\partial\Omega$.

(c) Since γ_n is linear and $\gamma_n(a_0) = 0$, we have $\gamma_n(a) = \gamma_n(a_1) + \ldots + \gamma_n(a_m)$. On the other hand, since $a_0 \cdot n = 0$, we have $a \cdot n = a_1 \cdot n + \ldots + a_m \cdot n$. Part (c) of the lemma ensues. □

Lemma 3.4. *Let an open subset $\Gamma \subset \partial\Omega$, a vector $\nu_0 \in \mathbb{R}^N$, and a constant $\delta > 0$ such that*

$$U(\delta_1,\delta_2) := \{x + t\nu_0 : x \in \Gamma,\ t \in (-\delta_1,\delta_2)\}$$

is an open subset of $\mathbb{R}^N$ for all $\delta_1,\delta_2 \in (0,\delta]$, and

$$U' := \{x + t\nu_0 : x \in \Gamma,\ t \in (0,\delta)\} \subset \Omega,$$
$$U'' := \{x + t\nu_0 : x \in \Gamma,\ t \in (-\delta,0)\} \subset \mathbb{R}^N \setminus \overline{\Omega}.$$

Let $a \in V^q(\Omega, \mathrm{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ and suppose that $\operatorname{supp} a \subset U := U(\delta, \delta)$. Then, there is a sequence $(v_k)_{k\geq 1} \subset V^q(\Omega, \mathrm{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ satisfying the following properties:

(a) *$v_k \to a$ in $V^q(\Omega, \mathrm{div})$;*
(b) *$v_k \to a$ uniformly on $\overline{\Omega}$;*
(c) *for every $k \geq 1$, v_k is the restriction of $v'_k \in V^q(\Omega_k, \mathrm{div}) \cap C(\overline{\Omega_k}, \mathbb{R}^N)$ for a bounded domain $\Omega_k \subset \mathbb{R}^N$ with $\overline{\Omega} \subset \Omega_k$.*

In particular, by virtue of Lemma 3.1, we have $\gamma_n(v_k) = v_k \cdot n$ for all $k \geq 1$ and finally $\gamma_n(a) = a \cdot n$.

Proof. The final conclusion of the lemma can be justified as follows: on the basis of (c) we can apply Lemma 3.1 which yields $\gamma_n(v_k) = v_k \cdot n$ for all $k \geq 1$. Then, on the one hand, due to (a) and the continuity of γ_n, we have $\gamma_n(v_k) \to \gamma_n(a)$ in $W^{-\frac{1}{q},q}(\partial\Omega)$ as $k \to \infty$. On the other hand, due to (b), we have $v_k \cdot n \to a \cdot n$ in $L^q(\partial\Omega, H^{N-1}) \subset W^{-\frac{1}{q},q}(\partial\Omega)$. Altogether, this yields $\gamma_n(a) = a \cdot n$ as asserted.

Let us now show the rest of the lemma. Let $\epsilon \in (0, \delta)$ small so that

$$\operatorname{supp} a \subset W_\epsilon := \{x + t\nu_0 : x \in \Gamma,\ t \in (-\delta + \epsilon, \delta - \epsilon)\}.$$

Let $U_\epsilon = U(\epsilon, \delta - \epsilon) = \{x + t\nu_0 : x \in \Gamma,\ t \in (-\epsilon, \delta - \epsilon)\}$ and $V_\epsilon = \{x \in \mathbb{R}^N : x + \epsilon\nu_0 \notin \operatorname{supp} a\}$. The union $\Omega_\epsilon := U_\epsilon \cup V_\epsilon$ is then an open subset which contains $\overline{\Omega}$. The latter property can be shown as follows. Let $x \in \overline{\Omega}$ and assume that $x + \epsilon\nu_0 \in \operatorname{supp} a$ (otherwise, we get immediately $x \in V_\epsilon \subset \Omega_\epsilon$). Due to the inclusion $\operatorname{supp} a \subset W_\epsilon$, there are $x' \in \Gamma$ and $t \in (-\delta + \epsilon, \delta - \epsilon)$ such that $x + \epsilon\nu_0 = x' + t\nu_0$, hence $x = x' + (t - \epsilon)\nu_0$. Moreover, since $x \in \overline{\Omega}$, we must have $t - \epsilon \geq 0$. Hence $t - \epsilon \in [0, \delta - 2\epsilon) \subset (-\epsilon, \delta - \epsilon)$ and therefore $x \in U_\epsilon \subset \Omega_\epsilon$.

Now we define $v'_\epsilon \in C(\Omega_\epsilon, \mathbb{R}^N)$ by

$$v'_\epsilon(x) = \begin{cases} a(x + \epsilon\nu_0) & \text{if } x \in U_\epsilon, \\ 0 & \text{if } x \in V_\epsilon. \end{cases}$$

If $x \in U_\epsilon$ then $x + \epsilon\nu_0 \in U' \subset \Omega$ hence $a(x + \epsilon\nu_0)$ is well defined. If $x \in U_\epsilon \cap V_\epsilon$ then $x + \epsilon\nu_0 \notin \operatorname{supp} a$ (due to the definition of V_ϵ), thus $a(x + \epsilon\nu_0) = 0$. This shows that v'_ϵ is well defined and continuous (since $a \in C(\overline{\Omega}, \mathbb{R}^N)$).

Moreover, we have

$$\begin{aligned} \operatorname{div} v'_\epsilon(x) &= \begin{cases} \operatorname{div} a(x + \epsilon\nu_0) & \text{for a.e. } x \in U_\epsilon \\ 0 & \text{for } x \in V_\epsilon \end{cases} \\ &= \overline{\operatorname{div} a}(x + \epsilon\nu_0) \end{aligned} \tag{3.8}$$

where $\overline{\operatorname{div} a} \in L^q(\mathbb{R}^N)$ is the extension by zero of $\operatorname{div} a$. Indeed, if $x \in U_\epsilon$, we have $\operatorname{div} v'_\epsilon(x) = \operatorname{div} a(x + \epsilon\nu_0) = \overline{\operatorname{div} a}(x + \epsilon\nu_0)$. If $x \in V_\epsilon$, then $x + \epsilon\nu_0 \notin \operatorname{supp} a$, hence either we have $x + \epsilon\nu_0 \in \Omega \setminus \operatorname{supp} a$ in which case $\overline{\operatorname{div} a}(x + \epsilon\nu_0) = \operatorname{div} a(x + \epsilon\nu_0) = 0 = \operatorname{div} v'_\epsilon(x)$, or we have $x + \epsilon\nu_0 \notin \Omega$ in which case $\overline{\operatorname{div} a}(x + \epsilon\nu_0) = 0 = \operatorname{div} v'_\epsilon(x)$ (by definition of $\overline{\operatorname{div} a}$). This shows (3.8).

Since $\overline{\operatorname{div} a} \in L^q(\mathbb{R}^N)$ it follows that $\operatorname{div} v'_\epsilon \in L^q(\Omega_\epsilon)$. We define $v_\epsilon := v'_\epsilon|_{\overline{\Omega}}$. Then

$$v_\epsilon \in V^q(\Omega, \mathrm{div}) \cap C(\overline{\Omega}, \mathbb{R}^N).$$

Moreover, v_ϵ satisfies condition (c) of the statement. In addition, in view of (3.8) we can apply [1, Lemme IV.4] which yields

$$\operatorname{div} v_\epsilon \to \operatorname{div} a \quad \text{in } L^q(\Omega) \quad \text{as } \epsilon \to 0.$$

This will show condition (a) of the statement once we will have shown condition (b).

Let $\varepsilon > 0$. Since a is continuous on $\overline{\Omega}$ which is compact, it is uniformly continuous, hence there is $\alpha > 0$ such that

$$(x, y \in \overline{\Omega} \quad \text{and} \quad |x - y| < \alpha) \quad \Longrightarrow \quad |a(x) - a(y)| < \varepsilon,$$

where $|\cdot| : (x_1, \ldots, x_N) \in \mathbb{R}^N \mapsto \max_{1 \le i \le N} |x_i|$ is the infinite norm. Assume ϵ small enough so that $\epsilon \in (0, \alpha)$. For $x \in \overline{\Omega} \cap U_\epsilon$, we deduce that

$$|v_\epsilon(x) - a(x)| = |a(x + \epsilon\nu_0) - a(x)| \le \varepsilon.$$

Now let $x \in \overline{\Omega} \cap V_\epsilon$. If $x \notin \operatorname{supp} a$, then we have

$$|v_\epsilon(x) - a(x)| = 0.$$

If $x \in \operatorname{supp} a$, knowing that $\operatorname{supp} a \subset W_\epsilon$, by definition of W_ϵ we have that $x + \epsilon\nu_0 \in U' \subset \Omega$ (since $x \in U \cap \overline{\Omega}$) and $x + \epsilon\nu_0 \notin \operatorname{supp} a$ (since $x \in V_\epsilon$), hence

$$|v_\epsilon(x) - a(x)| = |0 - a(x)| = |a(x + \epsilon\nu_0) - a(x)| \le \varepsilon.$$

Finally we have shown

$$\|v_\epsilon - a\|_\infty \le \varepsilon.$$

This establishes the convergence

$$v_\epsilon \to a \quad \text{in } C(\overline{\Omega}, \mathbb{R}^N) \quad \text{as } \epsilon \to 0.$$

We obtain condition (b) of the statement. The proof of the lemma is therefore complete. ☐

Theorem 1.1 follows from the above lemmas. Specifically, Lemma 3.3 shows that it is sufficient to deal with elements a as those considered in Lemma 3.4. Then, the result follows from Lemma 3.4.

Remark 3.5. (a) Our proof of Theorem 1.1 relies on ideas used in [6] for showing that $C^\infty(\overline{\Omega}, \mathbb{R}^N)$ is dense in $V^q(\Omega, \operatorname{div})$.

(b) Theorem 1.1 could be already deduced from Lemma 3.1 if one can show that every element in $V^q(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ admits an extension to $V^q(\Omega', \operatorname{div}) \cap C(\overline{\Omega'}, \mathbb{R}^N)$ for some larger domain $\Omega' \supset \overline{\Omega}$. We have no indication whether this general extension property holds.

4. Boundary conditions for weak solutions of elliptic Neumann and Steklov problems

In this section, we first consider a Neumann problem involving the Carathéodory functions

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{and} \quad f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

(i.e., $a(\cdot, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $a(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$, and similarly for f). Let p^* be the Sobolev critical exponent given by $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ otherwise. In what follows, we assume:

Assumption 4.1. There are constants $r \in (p, p^*)$ and $a_1, a_2, a_3, c_1 \in (0, +\infty)$ such that

$$\begin{aligned} |a(x,s,\xi)| &\leq a_1(|\xi|^{p-1} + |s|^{r/p'} + 1), && (4.1)\\ a(x,s,\xi) \cdot \xi &\geq a_2|\xi|^p - a_3(|s|^r + 1), && (4.2)\\ |f(x,s,\xi)| &\leq c_1(|\xi|^{p-1} + |s|^{r-1} + 1) && (4.3) \end{aligned}$$

for a.e. $x \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

Parts (4.1) and (4.3) of this assumption guarantee that:

$$\begin{aligned} u \in W^{1,p}(\Omega) &\implies a(x,u,\nabla u) \in L^{p'}(\Omega, \mathbb{R}^N) \text{ and } f(x,u,\nabla u) \in L^{r'}(\Omega)\\ &\implies a(x,u,\nabla u), f(x,u,\nabla u) \in W^{1,p}(\Omega)^* \end{aligned}$$

so that the following definition makes sense.

Definition 4.2. A *weak solution* of the Neumann problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f(x,u,\nabla u) & \text{in } \Omega,\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

is a function $u \in W^{1,p}(\Omega)$ such that the equality

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla v \, dx = \int_\Omega f(x,u,\nabla u)\, v \, dx \tag{4.5}$$

holds for all $v \in W^{1,p}(\Omega)$.

For the moment, the boundary condition "$\frac{\partial u}{\partial n} = 0$" in problem (4.4) is just a notation, in the sense that the normal derivative is a priori not defined for elements in $W^{1,p}(\Omega)$. However, in Proposition 4.6, by using Theorem 1.1, we will show that the boundary condition is satisfied in the classical sense, under suitable regularity conditions on the operator a and the boundary $\partial\Omega$.

In the following lemma, we show that weak solutions to problem (4.4) satisfy a Neumann-type boundary condition in the "weak" sense.

Lemma 4.3. *Assume that $u \in W^{1,p}(\Omega)$ is a weak solution of problem* (4.4). *Then:*

(a) $u \in L^\infty(\Omega)$.
(b) $a(x,u,\nabla u) \in V^{p'}(\Omega, \operatorname{div})$ *and u satisfies the weak Neumann condition*

$$\gamma_n(a(x,u,\nabla u)) = 0 \quad \text{in} \quad W^{-\frac{1}{p'},p'}(\partial\Omega).$$

Proof. Part (a) can be shown by Moser iteration technique; see [4].

(b) First we note that part (a) combined with (4.3) ensures that

$$f(x,u,\nabla u) \in L^{p'}(\Omega).$$

Taking any smooth function $v \in C_c^\infty(\Omega)$ as test function, and using the definition of the divergence (as a distribution) and the fact that u is a weak solution of problem (4.4) gives

$$\int_\Omega -\mathrm{div}\, a(x,u,\nabla u)\, v\, dx = \int_\Omega a(x,u,\nabla u)\cdot \nabla v\, dx = \int_\Omega f(x,u,\nabla u) v\, dx.$$

This implies that $-\mathrm{div}\, a(x,u,\nabla u) = f(x,u,\nabla u) \in L^{p'}(\Omega)$, and yields in particular

$$a(x,u,\nabla u) \in V^{p'}(\Omega, \mathrm{div})$$

so that $\gamma_n(a(x,u,\nabla u))$ is a well-defined element of $W^{-\frac{1}{p'},p'}(\partial\Omega)$. Now taking an arbitrary $v \in W^{1,p}(\Omega)$ as test function in (4.5) and using the Green formula (2.3), we get

$$\begin{aligned} & \langle \gamma_n(a(x,u,\nabla u)), \gamma(v)\rangle \\ = \; & \int_\Omega \mathrm{div}\, a(x,u,\nabla u)\, v\, dx + \int_\Omega a(x,u,\nabla u)\cdot\nabla v\, dx \\ = \; & -\int_\Omega f(x,u,\nabla u)\, v\, dx + \int_\Omega f(x,u,\nabla u)\, v\, dx \; = \; 0 \end{aligned}$$

(here, for making the notation easier, we have dropped the reference to the pair $(W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p'},p}(\partial\Omega))$ in the duality brackets $\langle\cdot,\cdot\rangle$). Since $v \in W^{1,p}(\Omega)$ is arbitrary and the trace map $\gamma : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\partial\Omega)$ is surjective (see Theorem 2.1), it follows that

$$\gamma_n(a(x,u,\nabla u)) = 0 \quad \text{in} \quad W^{-\frac{1}{p'},p'}(\partial\Omega),$$

which concludes the proof. □

In order to apply the regularity theory and relate the generalized normal derivative with the classical one, in Proposition 4.6 and Corollary 4.8 below we assume that the domain Ω has $C^{1,\gamma}$ boundary $\partial\Omega$, for some $\gamma \in (0,1)$, and we also need to strengthen the hypothesis on a.

Assumption 4.4. (a) $a : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is continuous and its restriction to $\overline{\Omega}\times\mathbb{R}\times(\mathbb{R}^N\setminus\{0\}) \to \mathbb{R}$ is of class C^1. Moreover, a is of the form

$$a(x,s,\xi) = \alpha(x,s,\xi)\xi$$

with $\alpha : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N \to (0,+\infty)$.

(b) There are constants $\mu,\nu \in (0,1)$, $R \in [0,+\infty)$, a nonincreasing map $\kappa_1 : [0,+\infty) \to (0,+\infty)$ and a nondecreasing map $\kappa_2 : [0,+\infty) \to (0,+\infty)$ such that

$$\begin{aligned} a'_\xi(x,s,\xi)\eta\cdot\eta &\geq \kappa_1(|s|)(R+|\xi|)^{p-2}|\eta|^2, \\ \|a'_\xi(x,s,\xi)\| &\leq \kappa_2(|s|)(R+|\xi|)^{p-2}, \\ |a(x,s,\eta)-a(y,t,\eta)| &\leq \kappa_2(|s|+|t|)(|x-y|^\mu + |s-t|^\nu)(1+|\eta|)^{p-2}|\eta| \end{aligned}$$

for all $x,y \in \overline{\Omega}$, $s,t\in\mathbb{R}$, $\xi,\eta\in\mathbb{R}^N$, $\xi\neq 0$. Here $a'_\xi(x,s,\cdot)$ denotes the differential of the map $a(x,s,\cdot)$ and $\|\cdot\|$ denotes the norm in the space of linear endomorphisms of $\mathbb{R}^N$.

Example 4.5. For $p > 1$, the mapping $a : (x, s, \xi) \mapsto |\xi|^{p-2}\xi$ satisfies Assumption 4.4 with the map α given by

$$\alpha : (x, s, \xi) \mapsto \begin{cases} |\xi|^{p-2} & \text{if } \xi \neq 0, \\ 1 & \text{if } \xi = 0. \end{cases}$$

(Note that Assumption 4.4 does not require α to be continuous.) This mapping corresponds to the p-Laplacian operator $\Delta_p : u \in W^{1,p}(\Omega) \mapsto \operatorname{div} a(x, u, \nabla u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

Under Assumptions 4.1 and 4.4, we have:

Proposition 4.6. *Let $u \in W^{1,p}(\Omega)$ be a weak solution of problem* (4.4)*. Then:*

(a) $u \in C^{1,\lambda}(\overline{\Omega})$ *for some* $\lambda \in (0, 1)$.

(b) $a(x, u, \nabla u) \in V^{p'}(\Omega, \operatorname{div}) \cap C(\overline{\Omega}, \mathbb{R}^N)$ *and u satisfies the classical Neumann condition* $\frac{\partial u}{\partial n} = 0$ *on* $\partial\Omega$.

Proof. Part (a) follows from nonlinear regularity theory [7]. The first claim of Part (b) then follows from Lemma 4.3 and the continuity of a in Assumption 4.4. Then, Theorem 1.1 combined with Lemma 4.3 yields

$$a(x, u, \nabla u) \cdot n = \gamma_n(a(x, u, \nabla u)) = 0 \quad \text{on } \partial\Omega.$$

By Assumption 4.4, we have that $a(x, u, \nabla u) = \alpha(x, u, \nabla u)\nabla u$ with $\alpha(x, u, \nabla u) \in (0, +\infty)$, whence finally

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Note that this equality holds everywhere on $\partial\Omega$. Theorem 1.1 and Lemma 4.3 yield an equality almost everywhere, but in the present proposition due to the regularity assumption on the domain, the outward unit normal n is defined everywhere on $\partial\Omega$ so that the equality makes sense and holds everywhere by continuity. □

We strengthen our assumption in order to apply the strong maximum principle:

Assumption 4.7. (a) The mapping $a(x, s, \xi) = a(x, \xi)$ is independent of the variable s. Moreover, there are constants $d_1, d_2, d_3, \delta \in (0, +\infty)$ such that

$$a'_\xi(x, \xi)\eta \cdot \eta \geq d_1|\xi|^{p-2}|\eta|^2,$$
$$\|a'_\xi(x, \xi)\| \leq d_2|\xi|^{p-2},$$
$$|\xi| < \delta \ \Rightarrow \ \|a'_x(x, \xi)\| \leq d_3|\xi|^{p-1}$$

for all $x \in \overline{\Omega}$, $\xi, \eta \in \mathbb{R}^N$, $\xi \neq 0$.

(b) There is a constant $c > 0$ such that $f(x, s, \xi) \geq -cs^{p-1}$ for a.e. $x \in \Omega$, all $s \in [0, \delta)$, $\xi \in \mathbb{R}^N$.

Under Assumptions 4.1, 4.4, 4.7, we have:

Corollary 4.8. *Let $u \in C^{1,\lambda}(\overline{\Omega})$ be a weak solution of problem* (4.4), *as in Proposition* 4.6. *Assume that $u \geq 0$ on $\overline{\Omega}$ and $u \not\equiv 0$. Then we have $u > 0$ on $\overline{\Omega}$.*

Proof. By Assumption 4.7 (b), (4.3), and $u \in C^1(\overline{\Omega})$, we find $\tilde{c} > 0$ with

$$\operatorname{div} a(x, \nabla u) \le \tilde{c} u^{p-1} \quad \text{in } \Omega.$$

This combined with Assumption 4.7 allows us to invoke the strong maximum principle [8, Theorem 8.27], which yields $u > 0$ on Ω and

$$\forall x \in \partial\Omega,\ u(x) = 0 \ \Rightarrow \ \frac{\partial u}{\partial n}(x) < 0.$$

Since we know that $\frac{\partial u}{\partial n}(x) = 0$ for all $x \in \partial\Omega$ (by Proposition 4.6), we get $u(x) > 0$ for all $x \in \partial\Omega$. Whence $u > 0$ on $\overline{\Omega}$ as asserted. □

Finally we consider more general (Steklov-type) boundary conditions. Let $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition

$$|g(x, s)| \le c_2(|s|^{\sigma-1} + 1) \quad \text{for a.e. } x \in \partial\Omega, \text{ all } s \in \mathbb{R}, \tag{4.6}$$

for a constant $c_2 > 0$ and some $\sigma \in (1, \frac{(N-1)p}{N-p})$ if $p < N$ and an arbitrary $\sigma \in (1, +\infty)$ if $p \ge N$. Given a, f satisfying respectively (4.1) and (4.3) in Assumption 4.1, we say that $u \in W^{1,p}(\Omega)$ is a weak solution of the problem

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = f(x, u, \nabla u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} = g(x, u) & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

if the equality

$$\begin{aligned} \int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx \quad &= \quad \int_\Omega f(x, u, \nabla u)\, v \, dx \\ &\quad + \int_{\partial\Omega} g(x, \gamma(u))\gamma(v)\, dH^{N-1} \end{aligned} \tag{4.8}$$

holds for all $v \in W^{1,p}(\Omega)$ (there is a continuous embedding $W^{\frac{1}{p'},p}(\partial\Omega) \subset L^\sigma(\partial\Omega, H^{N-1})$, so the definition makes sense).

Proposition 4.9. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (4.7) such that $a(x, u, \nabla u) \in C(\overline{\Omega}, \mathbb{R}^N)$. Then,*

$$a(x, u, \nabla u) \cdot n = g(x, u) \quad \text{on } \partial\Omega.$$

In particular, if $a(x, u, \nabla u) = |\nabla u|^{p-2}\nabla u$, then $|\nabla u|^{p-2}\frac{\partial u}{\partial n} = g(x, u)$ on $\partial\Omega$.

Proof. Arguing as in the proof of Lemma 4.3, one has $\operatorname{div} a(x, u, \nabla u) = -f(x, u, \nabla u) \in L^{p'}(\Omega)$ hence $a(x, u, \nabla u) \in V^{p'}(\Omega, \operatorname{div})$. For every $v \in W^{1,p}(\Omega)$, by virtue of Theorem 2.2 and formula (4.8), we get

$$\begin{aligned} \langle \gamma_n(a(x, u, \nabla u)), \gamma(v) \rangle \quad &= \quad \int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx \\ &\quad - \int_\Omega f(x, u, \nabla u)\, v \, dx \\ &= \quad \int_{\partial\Omega} g(x, u)\gamma(v)\, dH^{N-1}, \end{aligned}$$

and Theorem 2.1 (c) yields $\gamma_n(a(x,u,\nabla u)) = g(x,u)$ in $W^{-\frac{1}{p'},p'}(\partial\Omega)$. On the other hand, Theorem 1.1 implies that $\gamma_n(a(x,u,\nabla u)) = a(x,u,\nabla u)\cdot n$ on $\partial\Omega$. The conclusion follows. □

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Lucas Fresse
Université de Lorraine, Institut Élie Cartan,
54506 Vandoeuvre-lès-Nancy, France
e-mail: `lucas.fresse@univ-lorraine.fr`

Viorica V. Motreanu
Lycée Varoquaux, 10 rue Jean Moulin,
54510 Tomblaine, France
e-mail: `vmotreanu@gmail.com`

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On a singular elliptic problem with variable exponent

Francesca Faraci

Dedicated to the memory of Professor Csaba Varga

Abstract. In the present note we study a semilinear elliptic Dirichlet problem involving a singular term with variable exponent of the following type

$$\begin{cases} -\Delta u = \frac{f(x)}{u^{\gamma(x)}}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{$\mathcal{P}$}$$

Existence and uniqueness results are proved when $f \geq 0$.

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1. Introduction

In the present note we consider the following semilinear singular elliptic problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^{\gamma(x)}}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{$\mathcal{P}$}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N > 2)$ with smooth boundary, $f \in L^p(\Omega)$ $(p > \frac{N}{2})$ is a nonnegative function and $\gamma \in C^1(\overline{\Omega})$ is positive. Singular nonlinear problems were introduced by Fulks and Maybee [10] as a mathematical model for describing the heat conduction in an electric medium and received a considerable attention after the seminal paper of Crandall, Rabinowitz and Tartar [8]. There is a wide literature dealing with singular term of the type $u^{-\gamma}$ (i.e. $\gamma(x) = const.$) when $0 < \gamma < 1$. In such a case one can associate to the problem an energy functional which, although not continuously Gâteaux differentiable, is strictly convex. Its global minimum turns out to be the unique (weak) solution of $(\mathcal{P})$ and variational methods

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apply (see for example [9, 11, 17] where the singular term is perturbed by suitable nonlinearities). When $\gamma \geq 1$ such kind of problems are less investigated. Notice in fact that the energy functional (when $\gamma > 1$) in general is not defined on the whole space $H_0^1(\Omega)$. However, one may still prove existence results in the framework of variational setting by constructing suitable approximation sequences or employing techniques from non smooth analysis (see for instance [3, 4, 5, 6, 13, 15, 16]).

As far as we know, the variable exponent case has been treated recently in [7]. Using Schauder's fixed point theorem, the authors prove the existence of an increasing sequence of solutions of non-singular approximating problems which converges to a weak solution of $(\mathcal{P})$ in the natural energy space $H_0^1(\Omega)$ or to a function of $H_{loc}^1(\Omega)$ according to the behaviour of γ on the boundary of Ω.

In the present note we will complete the result of [7] showing the uniqueness of the solution of $(\mathcal{P})$. For general variable exponent we don't expect to have solutions in $H_0^1(\Omega)$ (notice that in [2], where the uniqueness issue is addressed, the authors assume the solutions to be in $H_0^1(\Omega)$). As in [4], a weak solution is meant in the following sense:

Definition 1.1. A weak solution of $(\mathcal{P})$ is a function $u \in H_{loc}^1(\Omega)$ such that $u > 0$ in Ω, $(u-\varepsilon)^+ \in H_0^1(\Omega)$ for every $\varepsilon > 0$,

$$\frac{f(x)}{u^{\gamma(x)}} \in L_{loc}^1(\Omega),$$

and

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega \frac{f(x)}{u^{\gamma(x)}} \varphi \, dx \quad \text{for all } \varphi \in C_c^1(\Omega).$$

Our result reads as follows:

Theorem 1.2. *Assume that $f \in L^p(\Omega)$ $(p > \frac{N}{2})$ is a nonnegative function and $\gamma \in C^1(\overline{\Omega})$ is a positive function. Then, problem $(\mathcal{P})$ has a unique weak solution.*

2. Proof of Theorem 1.2

Existence of solution of $(\mathcal{P})$. The existence of a solution has been already proved in [7]. We propose here a slightly different approach which is purely variational and does not make use of the Schauder fixed point theorem. Denote by $g : \Omega \times (0, +\infty) \to \mathbb{R}$ and $g_n : \Omega \times \mathbb{R} \to \mathbb{R}$ the functions

$$g(x,t) = \frac{f(x)}{t^{\gamma(x)}}, \qquad \text{and}$$

$$g_n(x,t) = g(x, t^+ + \frac{1}{n}) \qquad \text{for every } n \in \mathbb{N}^+.$$

For every $n \in \mathbb{N}^+$, g_n is a Carathéodory function and if $G_n : \Omega \times \mathbb{R} \to \mathbb{R}$ is its primitive, i.e.

$$G_n(x,t) = \int_0^t g_n(x,s) ds,$$

the following inequalities hold:

$$0 < g_n(x,t) \le f(x) n^{\|\gamma\|_\infty}$$
$$|G_n(x,t)| \le f(x) n^{\|\gamma\|_\infty} |t|$$

Denote by $\mathscr{E}_n : H_0^1(\Omega) \to \mathbb{R}$ the functional

$$\mathscr{E}_n(u) = \frac{1}{2}\|u\|^2 - \int_\Omega G_n(x,u(x))$$

which is well defined, coercive, sequentially weakly lower semicontinuous. Let u_n be its global minimum.

Since the functional $\mathscr{E}_n$ is of class $C^1(H_0^1(\Omega))$ with derivative at u given by

$$\mathscr{E}_n'(u)(\varphi) = \int_\Omega \nabla u \nabla \varphi - \int_\Omega g_n(x,u)\varphi \qquad \text{for every } \varphi \in H_0^1(\Omega)$$

u_n turns out to be a weak solution of

$$\begin{cases} -\Delta u = g_n(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{P}_n$}$$

Thus, in particular,

$$\int_\Omega \nabla u_n \nabla \varphi = \int_\Omega g_n(x,u_n)\varphi \qquad \text{for all } \varphi \in H_0^1(\Omega). \tag{2.1}$$

Testing the above equality with $\varphi = u_n^-$ we obtain at once that $u_n \ge 0$. By classical regularity results, $u_n \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and by the strong maximum principle, $u_n > 0$ in Ω. Moreover, since the function $g_n(x,\cdot)$ is decreasing, in a standard way one can prove that u_n is the unique solution to $(\mathcal{P}_n)$.

As in [8], let $n > m$ and denote by $w = u_n - u_m$. Then $w \in C_0^1(\overline{\Omega})$ and

$$-\Delta w = \frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f(x)}{\left(u_m + \frac{1}{m}\right)^{\gamma(x)}}.$$

Using $w^- \in H_0^1(\Omega)$ as test function in the above equality, we deduce that

$$-\|w^-\|^2 = \int_{\{x\in\Omega \,:\, u_n < u_m\}} \left(\frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f(x)}{\left(u_m + \frac{1}{m}\right)^{\gamma(x)}} \right) w^- \ge 0,$$

which implies $w^- = 0$, i.e. $u_n(x) \ge u_m(x)$ for every $x \in \overline{\Omega}$.

Put now $z = u_m + \frac{1}{m} - \left(u_n + \frac{1}{n}\right)$. Then, $z \in C^1(\overline{\Omega})$ and $z^- \in H_0^1(\Omega)$ (recall that $n > m$) so, using z^- as test function in

$$-\Delta z = \frac{f(x)}{\left(u_m + \frac{1}{m}\right)^{\gamma(x)}} - \frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}},$$

we obtain

$$-\|z^-\|^2 = \int_{\{x\in\Omega \,:\, u_m + \frac{1}{m} < u_n + \frac{1}{n}\}} \left(\frac{f(x)}{\left(u_m + \frac{1}{m}\right)^{\gamma(x)}} - \frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \right) z^- \ge 0,$$

which implies $z^- = 0$, i.e. $u_n(x) + \frac{1}{n} \leq u_m(x) + \frac{1}{m}$ for every $x \in \overline{\Omega}$. In conclusion, if $n > m$ then

$$0 \leq u_n(x) - u_m(x) \leq \frac{1}{m} - \frac{1}{n} \text{ for all } x \in \overline{\Omega}.$$

Hence, there exists $u \in C^0(\overline{\Omega})$ such that $u_n \rightrightarrows u$ in $\overline{\Omega}$ and

$$u_n \leq u \leq u_n + \frac{1}{n} \text{ for every } n \in \mathbb{N}. \tag{2.2}$$

Let us prove that u is a solution of $(\mathcal{P})$. It is clear that $u > 0$ in Ω. Moreover if $K \subset \Omega$ is a compact set, then, for suitable constants $c_0, c_1, c_2 > 0$,

$$u(x) \geq c_0 \text{ for all } x \in K,$$

$$0 \leq \frac{f(x)}{u(x)^{\gamma(x)}} \leq c_1 f(x) \text{ and } 0 \leq \frac{f(x)}{u_n(x)^{\gamma(x)}} \leq c_2 f(x) \text{ for all } x \in K,$$

thus in particular, $\frac{f(x)}{u^{\gamma(x)}}$ is in $L^1_{\mathrm{loc}}(\Omega)$.

Let δ be a positive number and denote

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}.$$

We distinguish two cases. Assume that $\|\gamma\|_{L^\infty(\Omega_\delta)} \leq 1$. Following [7], the sequence $\{u_n\}$ is bounded in $H^1_0(\Omega)$ (for completeness we give the details). For a suitable constant c we obtain

$$\begin{aligned} \|u_n\|^2 &= \int_{\Omega_\delta} \frac{f(x)}{(u_n + \frac{1}{n})^{\gamma(x)}} u_n + \int_{\Omega \setminus \Omega_\delta} \frac{f(x)}{(u_n + \frac{1}{n})^{\gamma(x)}} u_n \\ &\leq \int_{\Omega_\delta} f(x) u_n^{1-\gamma(x)} + c \int_{\Omega \setminus \Omega_\delta} f(x) u_n \\ &\leq \int_\Omega f(x)(1 + (1+c)u_n) = \|f\|_1 + \mathcal{S}(1+c)\|f\|_{\frac{2N}{N+2}} \|u_n\|, \end{aligned}$$

being $\mathcal{S}$ the embedding constant of $H^1_0(\Omega) \hookrightarrow L^{2^\star}(\Omega)$.

Thus, u turns out to be also the limit in the weak topology of $H^1_0(\Omega)$ of $\{u_n\}$. Being $u \in H^1_0(\Omega)$, for every $\varepsilon > 0$, $(u - \varepsilon)^+ \in H^1_0(\Omega)$. Let $\varphi \in C^1_c(\Omega)$ and denote by c_1 the positive constant such that $u_n \geq c_1$ on suppφ. Since

$$g_n(x, u_n(x))\varphi(x) \to \frac{f(x)}{u(x)^{\gamma(x)}} \varphi(x) \text{ for all } \ x \in \Omega$$

and

$$0 \leq g_n(x, u_n(x))\varphi(x) \leq \frac{f(x)}{c_1^{\gamma(x)}} \varphi(x) \in L^1(\Omega),$$

passing to the limit in

$$\int_\Omega \nabla u_n \nabla \varphi = \int_\Omega g_n(x, u_n)\varphi \text{ for all } n \in \mathbb{N}$$

we obtain

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \frac{f(x)}{u(x)^{\gamma(x)}} \varphi,$$

as we claimed.

Otherwise, $\|\gamma\|_{L^\infty(\Omega_\delta)} > 1$. Set $\gamma^* = \|\gamma\|_{L^\infty(\Omega_\delta)}$. In this case we prove that $\left\{u_n^{\frac{\gamma^*+1}{2}}\right\}$ is bounded in $H_0^1(\Omega)$. Since

$$\int_\Omega \nabla u_n \nabla u_n^{\gamma^*} = \gamma^* \int_\Omega u_n^{\gamma^*-1}|\nabla u_n|^2 = \gamma^* \left(\frac{2}{\gamma^*+1}\right)^2 \int_\Omega \left|\nabla u_n^{\frac{\gamma^*+1}{2}}\right|^2,$$

using $u_n^{\gamma^*}$ as test function in (2.1), we obtain

$$\begin{aligned}\int_\Omega |\nabla u_n^{\frac{\gamma^*+1}{2}}|^2 &= \frac{4\gamma^*}{(\gamma^*+1)^2}\int_\Omega g_n(x,u_n)u_n^{\gamma^*} \\ &\leq \frac{4\gamma^*}{(\gamma^*+1)^2}\left(\int_{\Omega_\delta} f(x)u_n^{\gamma^*-\gamma(x)} + c_0\int_{\Omega\setminus\Omega_\delta} f(x)u_n^{\gamma^*}\right) \\ &\leq \frac{4\gamma^*}{(\gamma^*+1)^2}\left(\int_\Omega f(x)(1+(1+c_0)u_n^{\gamma^*})\right) \\ &= \frac{4\gamma^*}{(\gamma^*+1)^2}\|f\|_1 + \frac{4\gamma^*}{(\gamma^*+1)^2}(1+c_0)\int_\Omega f(x)u_n^{\gamma^*}.\end{aligned}$$

By the assumption, $f \in L^{\frac{N(\gamma^*+1)}{N+2\gamma^*}}$ and applying Hölder inequality we obtain

$$\begin{aligned}\int_\Omega f(x)u_n(x)^{\gamma^*} &\leq \left(\int_\Omega f(x)^{\frac{N(\gamma^*+1)}{N+2\gamma^*}}\right)^{\frac{N+2\gamma^*}{N(\gamma^*+1)}}\left(\int_\Omega u_n(x)^{\frac{N(\gamma^*+1)}{N-2}}\right)^{\frac{N(\gamma^*+1)}{(N-2)\gamma^*}} \\ &\leq \|f\|_{\frac{N(\gamma^*+1)}{N+2\gamma^*}}\|u_n^{\frac{\gamma^*+1}{2}}\|_{2^*}^{\frac{\gamma^*+1}{2\gamma^*}} \leq \|f\|_{\frac{N(\gamma^*+1)}{N+2\gamma^*}}(1+\mathcal{S}\|u_n^{\frac{\gamma^*+1}{2}}\|).\end{aligned}$$

Thus, for suitable constants one has

$$\|u_n^{\frac{\gamma^*+1}{2}}\|^2 \leq c_1 + c_2\|u_n^{\frac{\gamma^*+1}{2}}\|,$$

that is our claim. Thus, $u^{\frac{\gamma^*+1}{2}} \in H_0^1(\Omega)$ and from [5, Theorem 1.3], it follows that $(u-\varepsilon)^+ \in H_0^1(\Omega)$ for every $\varepsilon > 0$.

Moreover, if $K \subset \Omega$ is a compact set, there exists a constant $c > 0$ such that $u_n^{\gamma^*-1} \geq c$ uniformly on K. Since

$$c\int_K |\nabla u_n|^2 \leq \int_K u_n^{\gamma^*-1}|\nabla u_n|^2 = \frac{4}{(\gamma^*+1)^2}\int_K |\nabla u_n^{\frac{\gamma^*+1}{2}}|^2 \leq const,$$

we deduce at once that $\{u_n\}$ is bounded in $H^1_{\mathrm{loc}}(\Omega)$, thus $u \in H^1_{\mathrm{loc}}(\Omega)$. We conclude as above.

Uniqueness of solution of $(\mathcal{P})$.

In order to prove the uniqueness of the solution we follow [6] and prove that inequality (2.2) holds for every solution u of $(\mathcal{P})$.

Let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a solution of $(\mathcal{P})$, $n \in \mathbb{N}^+$ and u_n be the solution of $(\mathcal{P}_n)$. Let us prove that $u_n \leq u \leq u_n + \frac{1}{n}$. We first prove that $u \leq u_n + \frac{1}{n}$.

Fix a sequence $\{\varphi_k\} \subset C_c^1(\Omega)$ converging in $H_0^1(\Omega)$ to $\left(u - u_n - \frac{1}{n}\right)^+$ and let $\tilde{\varphi}_k = \min\{\varphi_k, \left(u - u_n - \frac{1}{n}\right)^+\}$. Thus, $\{\tilde{\varphi}_k\} \subset C_c^1(\Omega)$ still converges in $H_0^1(\Omega)$ to $\left(u - u_n - \frac{1}{n}\right)^+$ and $\text{supp}\tilde{\varphi}_k \subseteq \text{supp}\left(u - u_n - \frac{1}{n}\right)^+ \subseteq \text{supp}(u - \frac{1}{n})^+$. Then,

$$\int_\Omega \nabla u \nabla \tilde{\varphi}_k = \int_\Omega \frac{f(x)}{u^{\gamma(x)}} \tilde{\varphi}_k.$$

Since u is $H^1(\text{supp}\left(u - \frac{1}{n}\right)^+)$, passing to the limit one has also that

$$\int_\Omega \nabla u \nabla \tilde{\varphi}_k \to \int_\Omega \nabla u \nabla \left(u - u_n - \frac{1}{n}\right)^+.$$

From the definition of $\tilde{\varphi}_k$ and Fatou lemma, one also has

$$\int_\Omega \frac{f(x)}{u^{\gamma(x)}} \tilde{\varphi}_k \to \int_\Omega \frac{f(x)}{u^{\gamma(x)}} \left(u - u_n - \frac{1}{n}\right)^+.$$

Combining the above outcomes,

$$\int_\Omega \nabla u \nabla \left(u - u_n - \frac{1}{n}\right)^+ = \int_\Omega \frac{f(x)}{u^{\gamma(x)}} \left(u - u_n - \frac{1}{n}\right)^+.$$

Since u_n is a solution of $(\mathcal{P}_n)$,

$$\int_\Omega \nabla u_n \nabla \left(u - u_n - \frac{1}{n}\right)^+ = \int_\Omega \frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \left(u - u_n - \frac{1}{n}\right)^+,$$

and subtracting one has

$$\left\|\left(u - u_n - \frac{1}{n}\right)^+\right\|^2 = \int_\Omega f(x) \left(\frac{1}{u^{\gamma(x)}} - \frac{1}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}}\right) \left(u - u_n - \frac{1}{n}\right)^+ \le 0,$$

which implies the claim.

Let us prove now that $u \ge u_n$. Let $\varepsilon \le \frac{1}{n}$. Put $\psi_\varepsilon = (u_n - u - \varepsilon)^+$ for every $n \in \mathbb{N}$. Notice that ψ_ε has compact support since $u_n \le \varepsilon$ in a neighborhood of the boundary. Thus,

$$\int_\Omega \nabla u \nabla (u_n - u - \varepsilon)^+ = \int_\Omega \frac{f(x)}{u^{\gamma(x)}} (u_n - u - \varepsilon)^+,$$

and

$$\int_\Omega \nabla u_n \nabla (u_n - u - \varepsilon)^+ = \int_\Omega \frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} (u_n - u - \varepsilon)^+.$$

Subtracting,

$$\left\|(u_n - u - \varepsilon)^+\right\|^2 = \int_\Omega f(x) \left(\frac{1}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} - \frac{1}{u^{\gamma(x)}}\right) (u_n - u - \varepsilon)^+ \le 0,$$

which implies $u_n \le u + \varepsilon$. Letting $\varepsilon \to 0$ we obtain the desired inequality.

The proof of uniqueness follows at once: let u, v be solutons of $(\mathcal{P})$. Then, for every $n \in \mathbb{N}^+$ one has

$$u \leq v + \frac{1}{n},$$

which implies, passing to the limit that $u \leq v$. Analogously we get the converse inequality. □

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Francesca Faraci
Dipartimento di Matematica e Informatica,
Università degli Studi di Catania
e-mail: `ffaraci@dmi.unict.it`

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Existence results for Dirichlet double phase differential inclusions

Nicuşor Costea and Shengda Zeng

Dedicated to the memory of Professor Csaba Varga

Abstract. In this paper we consider a class of double phase differential inclusions of the type

$$\begin{cases} -\text{div } \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \in \partial_C^2 f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is a bounded domain with Lipschitz boundary, $f(x,t)$ is measurable w.r.t. the first variable on Ω and locally Lipschitz w.r.t. the second variable and $\partial_C^2 f(x,\cdot)$ stands for the Clarke subdifferential of $t \mapsto f(x,t)$. The variational formulation of the problem gives rise to a so-called hemivariational inequality and the corresponding energy functional is not differentiable, but only locally Lipschitz. We use nonsmooth critical point theory to prove the existence of at least one weak solution, provided the $\partial_C^2 f(x,\cdot)$ satisfies an appropriate growth condition.

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1. Introduction and main results

In this paper we are interested in a class of boundary value problems of the following type:

$$(P): \quad \begin{cases} -\text{div } \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \in \partial_C^2 f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is a bounded domain with Lipschitz boundary and $\partial_C^2 f(x,t)$ stands for the Clarke subdifferential of the locally Lipschitz mapping $t \mapsto f(x,t)$.

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The presence of the double phase operator in the left-hand side requires that weak solutions of problem (P) to be sought in the Musielak-Orlicz-Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$ (see Section 2.2), while the presence of the Clarke subdifferential in the right-hand gives rise to a hemivariational inequality. More precisely, we say that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a *weak solution* of (P) if it satisfies the following *hemivariational inequality*

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u \cdot \nabla v + \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v\right] dx \leq \int_\Omega f^0(x,u;v)dx,$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Here and hereafter, $f^0(x,\cdot;\cdot)$ denotes the *generalized directional derivative* of f (see Section 2.3).

The conditions, which guarantee the existence of weak solutions for problem (P), and the main results of the paper are listed as follows.

(H_1) $1 < p < q < +\infty$ and $0 \leq \mu(\cdot) \in L^1(\Omega)$.

(f_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that:

(i). $x \mapsto f(x,t)$ is measurable on Ω for all $t \in \mathbb{R}$;

(ii). $t \mapsto f(x,t)$ is locally Lipschitz for a.a. $x \in \Omega$;

(iii). $f(x,0) = 0$ for a.a. $x \in \Omega$.

(f_2) There exist $r \in (1,p^*)$, $\alpha \in L^{\frac{r}{r-1}}(\Omega)$ and $k > 0$ such that

$$|\zeta| \leq \alpha(x) + k|t|^{r-1}, \text{ for a.a. } x \in \Omega \text{ all } t \in \mathbb{R} \text{ and all } \zeta \in \partial_C^2 f(x,t),$$

where p^* is the *critical exponent* corresponding to p, i.e.,

$$p^* := \begin{cases} \frac{Np}{N-p}, & \text{if } p < N, \\ +\infty, & \text{otherwise.} \end{cases}$$

The first existence result is devoted to the case when the exponent controlling the growth of $\partial_C f(x,\cdot)$ is sufficiently small. The proof relies on the fact that in this case the associated energy functional is coercive. More precisely, we have the following existence result.

Theorem 1.1. *Assume (H_1), (f_1) and (f_2) hold. Then for any $r \in (1,p)$ the problem (P) possesses at least one nontrivial weak solution.*

If the exponent controlling the growth of $t \mapsto \partial_C f(x,t)$ is "large", i.e., $r \in (q,p^*)$, then the energy functional is no longer coercive. In this case we use the Ekeland variational principle to prove the existence of at least one weak solution by replacing (f_2) with the slightly more restrictive condition (f_2') and assume in addition condition (f_3), listed below:

(f_2') There exist $r \in (1,p^*)$, and $k > 0$ such that

$$|\zeta| \leq k|t|^{r-1}, \text{ for a.a. } x \in \Omega \text{ all } t \in \mathbb{R} \text{ and all } \zeta \in \partial_C^2 f(x,t).$$

(f_3) There exist a nonempty open subset $\omega \subset \Omega$ and $\delta, K > 0, s \in (1,p)$ such that

$$f(x,t) \geq Kt^s, \quad \text{whenever } (x,t) \in \omega \times (0,\delta].$$

Theorem 1.2. *Assume (H_1), (f_1), (f_2') and (f_3) hold. Then for any $r \in (q,p^*)$ the problem (P) possesses at least one nontrivial weak solution.*

We point out the fact that Theorem 1.2 is new even in the "smooth case", i.e., $f(x,\cdot) \in C^1(\mathbb{R})$ on the one hand due to the fact that we do not impose the condition $p < N$ and we allow $\mu \in L^1(\Omega)$ and, on the other hand, due to the fact that we do not impose an Ambrosetti-Rabinowitz type condition.

2. Preliminaries

2.1. Generalized N-functions and Musielak-Orlicz spaces

In this subsection we recall some definitions and basic properties of generalized Orlicz spaces also referred to as Musielak-Orlicz spaces. For more details and connections see, e.g., [2, 7, 10].

Definition 2.1. A continuous and convex function $\varphi : \mathbb{R} \to [0,\infty)$ is called *N-function* if it satisfies the following conditions:

(i). $\varphi(t) = 0$ if and only if $t = 0$;
(ii). $\varphi(-t) = \varphi(t)$ for all $t \in \mathbb{R}$;
(iii). $\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$.

Definition 2.2. Assume $\Omega \subset \mathbb{R}^N$ is a bounded domain. An application $\Phi : \Omega \times \mathbb{R} \to [0,\infty)$ is called *generalized N-function* if $x \mapsto \Phi(x,t)$ is measurabe for all $t \in \mathbb{R}$ and $t \mapsto \Phi(x,t)$ is an N-function for a.a. $x \in \Omega$.

Note that if Φ is a generalized N-function, then the corresponding *Young conjugate function*, $\tilde{\Phi} : \Omega \times \mathbb{R} \to [0,\infty)$, defined by

$$\tilde{\Phi}(x,s) := \sup_{t\geq 0} \{st - \Phi(x,t)\},$$

is also a generalized N-function.

Definition 2.3. A generalized N-function Φ is said to satisfy the *Δ_2-condition* if there exist a constant $k > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\Phi(x,2t) \leq k\Phi(x,t) + h(x) \text{ for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Let Φ_1, Φ_2 be two generalized N-functions. We say that Φ_1 *dominates* Φ_2, denoted $\Phi_1 \succeq \Phi_2$, if there exist two constants $K, L > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\Phi_2(x,t) \leq K\Phi_1(x,Lt) + h(x), \text{ for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

The functions Φ_1, Φ_2 are called *equivalent*, denoted $\Phi_1 \simeq \Phi_2$, if $\Phi_1 \succeq \Phi_2$ and $\Phi_2 \succeq \Phi_1$.

For a generalized N-function Φ, the *modular* $\varrho_\Phi : L^0(\Omega) \to \mathbb{R}$ is the functional given by

$$\varrho_\Phi(u) := \int_\Omega \Phi(x,|u|)\, dx,$$

where by $L^0(\Omega)$ we denote the set of measurable functions defined on Ω. We consider the following classes of functions:

(i). The *Musielak-Orlicz class* $K^\Phi(\Omega)$ defined by

$$K^\Phi(\Omega) := \left\{u \in L^0(\Omega) : \ \varrho_\Phi(u) < \infty\right\};$$

(ii). The *Musielak-Orlicz space* $L^{\Phi}(\Omega)$ is the linear space generated by $K^{\Phi}(\Omega)$. Note that $K^{\Phi}(\Omega) \subseteq L^{\Phi}(\Omega)$ and equality occurs if and only if $K^{\Phi}(\Omega)$ is a linear space, or equivalently Φ satisfies the Δ_2-condition.

The mapping $\|\cdot\|_{\Phi} : L^{\Phi}(\Omega \to [0,\infty)$ defined by

$$\|u\|_{\Phi} := \inf\left\{\beta > 0 : \; \varrho_{\Phi}\left(\frac{u}{\beta}\right) \leq 1\right\}$$

defines a norm (the so-called *Luxemburg norm*).

The following proposition highlights some useful properties of the Musielak-Orlicz spaces.

Proposition 2.4. *Let Φ, Ψ be two generalized N-functions. Then the following assertions hold:*

(i). *The Musielak-Orlicz space $(L^{\Phi}(\Omega), \|\cdot\|_{\Phi})$ is a Banach space;*

(ii). *If $\Phi \succeq \Psi$, then $L^{\Phi}(\Omega) \hookrightarrow L^{\Psi}(\Omega)$;*

(iii). $\varrho_{\Phi}(u) < 1$*(resp.* $\varrho_{\Phi}(u) = 1; \varrho_{\Phi}(u) > 1$) *if and only if* $\|u\|_{\Phi} < 1$ *(resp.* $\|u\|_{\Phi} = 1; \|u\|_{\Phi} > 1$)*;*

(iv). *The following Hölder-type inequality holds*

$$\int_{\Omega} |uv| \, dx \leq 2\|u\|_{\Phi}\|v\|_{\tilde{\Phi}}, \; \textit{for all } u \in L^{\Phi}(\Omega), v \in L^{\tilde{\Phi}}(\Omega).$$

For a generalized N-function Φ the corresponding *Musielak-Orlicz-Sobolev space* $W^{1,\Phi}(\Omega)$ is defined by

$$W^{1,\Phi}(\Omega) := \left\{u \in L^{\Phi}(\Omega) : \; |\nabla u| \in L^{\Phi}(\Omega)\right\}.$$

By a slight abuse, henceforth we denote $\|\nabla u\|_{\Phi}$ instead of $\||\nabla u|\|_{\Phi}$. Obviously the mapping $\|\cdot\|_{1,\Phi} : W^{1,\Phi}(\Omega) \to [0,\infty)$

$$\|u\|_{1,\Phi} := \|u\|_{\Phi} + \|\nabla u\|_{\Phi}$$

defines a norm.

The Musielak-Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ is defined as completion of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ w.r.t. the norm $\|\cdot\|_{1,\Phi}$.

Proposition 2.5 (Musielak [10]). *Assume Φ is a generalized N-function such that*

$$\inf_{x\in\Omega} \Phi(x,1) > 0. \tag{2.1}$$

Then $(W^{1,\Phi}(\Omega), \|\cdot\|_{1,\Phi})$ and $(W_0^{1,\Phi}(\Omega), \|\cdot\|_{1,\Phi})$ are Banach spaces. Furthermore, if $L^{\Phi}(\Omega)$ is reflexive, then $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are also reflexive.

2.2. The double phase space

Throughout this section we consider the particular case of the double-phase space, required to study problem (P). Note that if (H_1) holds, then the *double phase* function $\mathcal{H} : \Omega \times \mathbb{R} \to [0,\infty)$ given by

$$\mathcal{H}(x,t) := |t|^p + \mu(x)|t|^q$$

is a generalized N-function satisfying (2.1). Simple computations yield

$$\mathcal{H}(x,2t) \leq 2^q\mathcal{H}(x,t), \text{ for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R},$$

i.e., $\mathcal{H}$ satisfies the Δ_2-condition. Moreover, according to Colasuonno & Squassina [4, Proposition 2.14] the space $(L^{\mathcal{H}}(\Omega), \|\cdot\|_{\mathcal{H}})$ is uniformly convex. Consequently, Proposition 2.5 ensures that $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive. Moreover, if (H_1) holds, then the following Poincaré-type inequality holds

$$\|u\|_{\mathcal{H}} \leq C\|\nabla u\|_{\mathcal{H}}, \text{ for all } u \in W_0^{1,\mathcal{H}}(\Omega),$$

for some positive constant C independent of u. Thus, on the space $W_0^{1,\mathcal{H}}(\Omega)$ we can use the equivalent norm

$$\|u\| := \|\nabla u\|_{\mathcal{H}}.$$

We introduce next the space

$$L^q_\mu(\Omega) := \left\{ u \in L^0(\Omega) : \int_\Omega \mu(x)|u|^q \, dx < \infty \right\},$$

endowed with the seminorm

$$|u|_{q,\mu} := \left(\int_\Omega \mu(x)|u|^q \, dx \right)^{1/q}.$$

The definition of the Luxemburg norm together with the fact that $\mathcal{H}(x,t)$ is a generalized N-function which satisfies the Δ_2-condition. So, the following estimates hold:

$$\|u\|^q_{\mathcal{H}} \leq \int_\Omega [|u|^p + \mu(x)|u|^q] dx \leq \|u\|^p_{\mathcal{H}}, \ \forall u \in L^{\mathcal{H}}(\Omega), \text{ with } \|u\|_{\mathcal{H}} < 1, \tag{2.2}$$

and

$$\|u\|^p_{\mathcal{H}} \leq \int_\Omega [|u|^p + \mu(x)|u|^q] dx \leq \|u\|^q_{\mathcal{H}}, \ \forall u \in L^{\mathcal{H}}(\Omega), \text{ with } \|u\|_{\mathcal{H}} > 1. \tag{2.3}$$

The following proposition highlights some embedding results that will play a crucial role throughout the subsequent sections.

Proposition 2.6 (Colasuonno & Squassina [4]**).** *Assume (H_1) holds. The following statements are true:*

- (i). *The embedding $L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_\mu(\Omega)$ is continuous;*
- (ii). *If $\mu \in L^\infty(\Omega)$, then the embedding $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous;*
- (iii). *If $p \leq N$, then the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in (1, p^*)$;*
- (iv). *If $p > N$, then the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in (1, +\infty)$.*

2.3. Locally Lipschitz functionals

We recall that a functional $\phi : X \to \mathbb{R}$, with X being a Banach space, is said to be *locally Lipschitz* if, for every $u \in X$ there exists a neighborhood V of u and a positive constant L, which depends on the neighborhood V, such that

$$|\phi(w) - \phi(v)| \leq L\|w - v\|, \ \forall v, w \in V.$$

Definition 2.7. Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function. The *generalized directional derivative* of ϕ at $u \in X$ in the direction $v \in X$, denoted $\phi^0(u; v)$, is defined by

$$\phi^0(u; v) := \limsup_{\substack{w \to u \\ t \downarrow 0}} \frac{\phi(w + tv) - \phi(w)}{t}. \tag{2.4}$$

The following result points out some important properties of generalized directional derivatives that will be used in the sequel. For the proof one can consult Clarke [3].

Proposition 2.8. *Let $\phi, \rho : X \to \mathbb{R}$ be two locally Lipschitz functions. Then we have*

(*i*). *for each fixed $u \in X$, the function $v \mapsto \phi^0(u; v)$ is finite, subadditive and satisfies*

$$|\phi^0(u; v)| \le L\|v\|,$$

where $L > 0$ is the Lipschitz constant near the point u;

(*ii*). *the function $(u, v) \mapsto \phi^0(u; v)$ is upper semicontinuous;*

(*iii*). $\phi^0(u; -v) = (-\phi)^0(u; v)$, *for all $u, v \in X$;*

(*iv*). $\phi^0(u; \mu v) = \mu \phi^0(u; v)$, *for all $u, v \in X$ and all $\mu > 0$;*

(*v*). $(\phi + \rho)^0(u; v) \le \phi^0(u; v) + \rho^0(u; v)$, *for all $u, v \in X$.*

Definition 2.9. The *Clarke subdifferential* of a locally Lipschitz function $\phi : X \to \mathbb{R}$ at a point $u \in X$, denoted $\partial_C \phi(u)$, is the subset of X^* defined by

$$\partial_C \phi(u) := \left\{ \xi \in X^* : \ \phi^0(u; v) \ge \langle \xi, v \rangle, \ \forall v \in X \right\}. \tag{2.5}$$

We point out the fact that if ϕ is convex, then the Clarke subdifferential $\partial_C \phi$ coincides with the subdifferential of ϕ in the sense of Convex Analysis. Although is no longer monotone, for each $u \in X$ the generalized gradient $\partial_C \phi(u)$ is a nonempty, convex and weak*-compact subset of X^* (see, e.g., Clarke [3, Proposition 2.1.2]). Furthermore, if $\phi \in C^1(X, \mathbb{R})$, then $\partial_C \phi(u) = \{\phi'(u)\}$.

Theorem 2.10 (Lebourg's Mean Value Theorem [8]**).** *Let U be an open subset of a Banach space X and u, v be two points of U such that the line segment*

$$[u, v] := \{(1 - t)u + tv : 0 \le t \le 1\} \subset U.$$

If $\phi : U \to \mathbb{R}$ is a locally Lipschitz function, then there exist $t \in (0, 1)$ and $\zeta \in \partial_C \phi(u + t(v - u))$ such that

$$\phi(v) - \phi(u) = \langle \zeta, v - u \rangle.$$

Definition 2.11. We say that $u \in X$ is a *critical point* for the locally Lipschitz functional $\phi : X \to \mathbb{R}$ if $0 \in \partial_C \phi(u)$.

Remark 2.12. The point $u \in X$ is critical for ϕ if and only if $\phi^0(u; v) \ge 0, \ \forall v \in X$. Furthermore, any local extremum of ϕ is in fact a critical point.

We close this subsection by recalling the well-known Ekeland variational principle (see, e.g., [6]) which will play a key role in the proof of Theorem 1.2.

Theorem 2.13. *Let (Y, d) be a complete metric space and let $\varphi : Y \to (-\infty, \infty]$ be a proper, lower semicontinuous and bounded from below functional. Then for any $\varepsilon, \lambda > 0$ and any $v \in Y$ satisfying $\varphi(v) \le \inf_Y \varphi + \varepsilon$ there exists $u \in Y$ such that:*

(i). $\varphi(u) \leq \varphi(v)$;
(ii). $d(v, u) \leq \frac{1}{\lambda}$;
(iii). $-\varepsilon\lambda d(u, w) \leq \varphi(w) - \varphi(u)$, *for all* $w \in Y$.

3. Proof of the main results

Define the functionals $I : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ and $F : L^r(\Omega) \to \mathbb{R}$ by

$$I(u) := \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q\right] dx \text{ for all } u \in W_0^{1,\mathcal{H}}(\Omega)$$

and

$$F(w) := \int_\Omega f(x, w)dx \text{ for all } w \in L^r(\Omega)$$

respectively. Then $I \in C^1(W_0^{1,\mathcal{H}}(\Omega), \mathbb{R})$ with its derivative given by

$$\langle I'(u), v\rangle = \int_\Omega \left[|\nabla u|^{p-2}\nabla u \cdot \nabla v + \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v\right] dx, \tag{3.1}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$.

Due to the Aubin-Clarke Theorem (see, e.g., [3, Theorem 2.7.5]) F is locally Lipschitz and

$$\partial_C F(w) \subseteq \int_\Omega \partial_C^2 f(x, w)dx, \forall w \in L^r(\Omega),$$

in the sense that for any $\xi \in \partial_C F(w)$ there exists $\zeta \in L^{\frac{r}{r-1}}(\Omega)$ such that

$$\begin{cases} \langle \xi, z\rangle = \displaystyle\int_\Omega \zeta(x)z(x)dx, & \forall z \in L^r(\Omega), \\ \zeta(x) \in \partial_C^2 f(x, w(x)), & \text{for a.a. } x \in \Omega. \end{cases} \tag{3.2}$$

On the other hand, Proposition 2.6 ensures that for any $r \in (1, p^*)$ the embedding operator $i : W_0^{1,\mathcal{H}}(\Omega) \to L^r(\Omega)$ is compact and its adjoint operator, $i^* : L^{\frac{r}{r-1}}(\Omega) \to (W_0^{1,\mathcal{H}}(\Omega))^*$, is also compact. Consequently, the *energy functional* associated to problem (P), $E : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$

$$E(u) := I(u) - F(i(u)), \tag{3.3}$$

is well defined, weakly lower semicontinuous and locally Lipschitz. Moreover, basic subdifferential calculus (see, e.g., Carl, Le & Motreanu [1, Propositions 2.173, 2.174 & Corollary 2.180] ensures that

$$\partial_C E(u) \subseteq I'(u) - i^*\partial_C F(i(u)), \ \forall u \in W_0^{1,\mathcal{H}}(\Omega).$$

Henceforth, for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ and any $\zeta \in L^{\frac{r}{r-1}}(\Omega)$ we simply write u and ζ instead of $i(u)$ and $i^*(\zeta)$, respectivelly.

Lemma 3.1. *If* $r \in (1, p^*)$, *then any critical point of* E *(in the sense of Definition* 2.11*) is a weak solution for problem* (P).

Proof. Let $u \in W_0^{1,\mathcal{H}}(\Omega)$ be a critical point of E. Then $0 \in \partial_C E(u)$, or equivalently

$$I'(u) \in \partial_C F(u).$$

Keeping in mind (3.2), there exists $\zeta \in L^{\frac{r}{r-1}}(\Omega)$ such that $\zeta \in \partial_C^2 f(x,u)$ a.a. in Ω and

$$\langle I'(u), v\rangle = \int_\Omega \zeta v dx, \ \forall v \in W_0^{1,\mathcal{H}}(\Omega).$$

Now, using the definition of the Clarke subdifferential and (3.1) we get that

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u \cdot \nabla v + \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v\right] dx \leq \int_\Omega f^0(x,u;v)dx,$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$, i.e., u is indeed a weak solution for problem (P). □

Proof of Theorem 1.1. Let $u \in W_0^{1,\mathcal{H}}(\Omega)$ be fixed. Simple computations show that

$$\frac{1}{q}\varrho_{\mathcal{H}}(|\nabla u|) \leq I(u) \leq \frac{1}{p}\varrho_{\mathcal{H}}(|\nabla u|),$$

which combined with (2.2)-(2.3) leads to the following inequalities:

$$\frac{1}{q}\|u\|^q \leq I(u) \leq \frac{1}{p}\|u\|^p, \quad \text{if } \|u\| < 1, \tag{3.4}$$

and

$$\frac{1}{q}\|u\|^p \leq I(u) \leq \frac{1}{p}\|u\|^q, \quad \text{if } \|u\| > 1, \tag{3.5}$$

respectivelly. On the other hand, using Lebourg's Mean Value Theorem and the compact embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ we get

$$\begin{aligned} |F(u)| &= \left|\int_\Omega f(x,u)dx\right| \leq \int_\Omega |f(x,u) - f(x,0)|dx \leq \int_\Omega |\zeta||u|dx \\ &\leq \int_\Omega (\alpha(x) + k|u|^{r-1})|u|dx \leq \|\alpha\|_{\frac{r}{r-1}}\|u\|_r - k\|u\|_r^r \\ &\leq C_0\|\alpha\|_{\frac{r}{r-1}}\|u\| + C_1\|u\|^r, \end{aligned}$$

for some suitable constants $C_0, C_1 > 0$. Thus, for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|u\| > 1$ one has

$$E(u) = I(u) - F(u) \geq \frac{1}{q}\|u\|^p - C_0\|\alpha\|_{\frac{r}{r-1}}\|u\| - C_1\|u\|^r \to \infty, \text{ as } \|u\| \to \infty,$$

i.e., the energy functional E is coercive. Since E is also weakly lower semicontinuous, The Direct Method in the Calculus of Variations (see, e.g., [5, Theorem 1.7]) ensures the existence of a global minimizer u_0 of E, i.e.,

$$E(u_0) = \inf_{u \in W_0^{1,\mathcal{H}}(\Omega)} E(u).$$

The conclusion follows now from Remark 2.12 and Lemma 3.1. □

Lemma 3.2. *Assume* (H_1), (f_1) *and* (f_2') *hold. If* $r \in (q, p^*)$, *then there exist* $\rho \in (0,1)$ *and* $\gamma > 0$ *such that*

$$\inf_{u \in \partial B_\rho(0)} E(u) \geq \gamma,$$

where $\partial B_\rho(0) := \{u \in W_0^{1,\mathcal{H}}(\Omega) : \ \|u\| = \rho\}$.

Proof. Since (f_2') is in fact condition (f_2) with $\alpha \equiv 0$, it follows that there exists $C_1 > 0$ such that

$$|F(u)| \leq C_1 \|u\|^r, \ \forall u \in W_0^{1,\mathcal{H}}(\Omega).$$

Thus, for fixed $\rho \in \left(0, \min\left\{1, (qC_1)^{\frac{1}{q-r}}\right\}\right)$ and any $u \in \partial B_\rho(0)$ one has

$$E(u) \geq \frac{1}{q}\|u\|^q - C_1\|u\|^r = \frac{1}{q}\rho^q\left(1 - qC_1\rho^{r-q}\right).$$

The choice of ρ implies that $\gamma := \frac{1}{q}\rho^q\left(1 - qC_1\rho^{r-q}\right) > 0$, thus completing the proof. □

Lemma 3.3. *Assume* (H_1), (f_1) *and* (f_3) *hold. If* $r \in (q, p^*)$, *then there exist* $w_0 \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ *and* $t_0 \in (0,1)$ *such that*

$$E(tw_0) < 0, \ \forall t \in (0, t_0).$$

Proof. Let $x_0 \in \omega$ be fixed and choose $R > 0$ such that $\bar{B}_R(x_0) \subset \omega$. Then there exists $w_0 \in C_0^\infty(\omega)$ such that

$$\begin{cases} w_0(x) = 1, & \text{in } B_R(x_0), \\ 0 \leq w_0(x) \leq 1, & \text{on } \omega \setminus \bar{B}_R(x_0). \end{cases}$$

Obviously $w_0 \in W_0^{1,\mathcal{H}}(\Omega)$ and $\|w_0\| > 0$. Then, for any $0 < t < \min\{1, \delta, \|w_0\|^{-1}\}$ the following estimates hold

$$F(tw_0) = \int_\Omega f(x, tw_0(x))dx = \int_\omega f(x, tw_0(x))dx \geq \int_\omega Kt^s dx = K\text{meas}(\omega)t^s,$$

and

$$\begin{aligned} E(tw_0) &= I(tw_0) - F(tw_0) \leq \frac{1}{p}\|tw_0\|^p - K\text{meas}(\omega)t^s \\ &= K\text{meas}(\omega)t^s\left[\frac{\|w_0\|^p t^{p-s}}{pK\text{meas}(\omega)} - 1\right], \end{aligned}$$

which shows $E(tw_0) < 0$ for all $t \in (0, t_0)$ with

$$t_0 := \min\left\{1, \delta, \|w_0\|^{-1}, \left(\frac{pK\text{meas}(\omega)}{\|w_0\|^p}\right)^{\frac{1}{p-s}}\right\}.$$

□

Proof of Theorem 1.2. Lemmas 3.2 and 3.3 ensure that there exists $\rho \in (0,1)$ such that

$$\inf_{\bar{B}_\rho(0)} E < 0 < \inf_{\partial B_\rho(0)} E.$$

Let $\{w_n\} \subset \bar{B}_\rho(u)$ be a minimizing sequence for $E|_{\bar{B}_\rho(0)}$, i.e., $E(w_n) \to \inf_{\bar{B}_\rho(0)} E,$ as $n \to \infty$. Passing, if necessary, to a subsequence we may assume that

$$E(w_n) < \inf_{\bar{B}_\rho(0)} E + \frac{1}{n}, \ \forall n \geq 1. \tag{3.6}$$

Applying Ekeland's variational principle with $\varepsilon := \frac{1}{n}$ and $\lambda := \sqrt{n}$ we get that there exists $\{u_n\} \subset \bar{B}_\rho(0)$ such that

$$E(u_n) \leq E(w_n), \ \forall n \geq 1, \tag{3.7}$$

and

$$-\frac{1}{\sqrt{n}}\|v - u_n\| \leq E(v) - E(u_n), \ \forall v \in \bar{B}_\rho(0). \tag{3.8}$$

The sequence $\{u_n\}$ is clearly bounded, hence there exists $u \in \bar{B}_\rho(0)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \rightharpoonup u \text{ in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad u_{n_k} \to u \text{ in } L^r(\Omega).$$

For any $t \in (0,1)$ the element $v_t := u_{n_k} + t(u - u_{n_k})$ lies in $\bar{B}_\rho(0)$ and using (3.8) we have

$$-\frac{t}{\sqrt{n}}\|u - u_{n_k}\| \leq E(u_{n_k} + t(u - u_{n_k})) - E(u_{n_k}).$$

Dividing the last relation by $t > 0$ then taking the lim sup as $t \searrow 0$ we obtain

$$\begin{aligned} -\frac{1}{\sqrt{n}} \leq & \limsup_{t \searrow 0} \left[\frac{I(u_{n_k} + t(u - u_{n_k})) - I(u_{n_k})}{t} \right. \\ & \left. + \frac{(-F)(u_{n_k} + t(u - u_{n_k})) - (-F)(u_{n_k})}{t} \right] \\ \leq & \langle I'(u_{n_k}), u - u_{n_k} \rangle + (-F)^0(u_{n_k}; u - u_{n_k}), \end{aligned}$$

which can be rewritten as

$$\langle I'(u_{n_k}), u_{n_k} - u \rangle \leq \frac{1}{\sqrt{n}} + F^0(u_{n_k}; u_{n_k} - u), \ \forall n \geq 1.$$

Taking the lim sup as $n \to \infty$ and using Proposition 2.8 we have

$$\limsup_{n \to \infty} \langle I'(u_{n_k}, u_{n_k} - u) \rangle \leq F^0(u; 0) = 0.$$

Keeping in mind that I' of type $(S)_+$ (see, e.g., [9, Proposition 3.1]) we infer that

$$u_{n_k} \to u \text{ in } W_0^{1,\mathcal{H}}(\Omega).$$

But, due to (3.6) and (3.7), we conclude

$$E(u) = \lim_{n \to \infty} E(u_{n_k}) = \inf_{\bar{B}_\rho(0)} E < 0,$$

which shows that u is a nonzero local minimizer of E, and, according to Remark 2.12 a nontrivial critical point. □

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Nicuşor Costea
Department of Mathematics and Computer Science,
Politehnica University of Bucharest,
313 Splaiul Independenţei, 060042 Bucharest, Romania
e-mail: nicusorcostea@yahoo.com

Shengda Zeng
Guangxi Colleges and Universities Key Laboratory of Complex,
System Optimization and Big Data Processing,
Yulin Normal University,
Yulin 537000, Guangxi, P.R. China
and
Jagiellonian University in Krakow,
Faculty of Mathematics and Computer Science,
ul. Lojasiewicza 6, 30-348 Krakow, Poland
e-mail: zengshengda@163.com

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On eigenvalue problems governed by the (p,q)-Laplacian

Luminiţa Barbu and Gheorghe Moroşanu

Dedicated to the memory of Professor Csaba Varga

Abstract. This is a survey on recent results, mostly of the authors, regarding eigenvalue problems governed by the $(p,q)-$Laplacian and related open problems.

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Keywords: Eigenvalue problem, (p,q)-Laplacian, Sobolev space, Nehari manifold, variational methods, Lagrange multipliers.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. For $\theta \in (1,\infty)$, consider in Ω the θ-Laplace operator $\Delta_\theta u = \text{ div } (\mid \nabla u \mid^{\theta-2} \nabla u)$. Obviously, Δ_2 is the classic Laplacian Δ. There are many applications involving such kind of operators, including the so called two phase problems. For example, the operator $\big(\Delta + c\Delta_\theta\big)$, $c > 0$, $\theta \in (1,\infty)$, has applications in Born-Infeld theory for electrostatic fields (see Bonheure, Colasuonno & Fortunato [16], Fortunato, Orsina & Pisani [26]). We also refer to Benci et al. [14] and Benci, Fortunato & Pisani [15] for more general applications to quantum physics. Two phase equations arise also in other parts of mathematical physics as reaction diffusion equations (see Cherfils & Il'yasov [18]) and nonlinear elasticity theory (see Marcellini [35] and Zhikov [45]). In fact, the literature related to this subject is vast and daily increasing.

For $p,\ q \in (1,\infty)$, define $\mathcal{A}_{pq} := \Delta_p + \Delta_q$, which is usually called $(p,q)-$Laplacian. We assumes that $p \neq q$, because for $p = q$ $\mathcal{A}_{pq} = 2\Delta_p$ and this case is not relevant for our discussion here. Notice that the operator introduced above $\big(\Delta + c\Delta_\theta\big)$ with $c = 1$ is a $(2,\theta)-$Laplacian. The restriction to the case $c = 1$ does not affect the generality.

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In what follows we recall some facts concerning the classic eigenvalue problem for $-\Delta_p$, $p \in (1, \infty)$, under the Dirichlet boundary condition

$$\begin{cases} -\Delta_p u = \lambda \mid u \mid^{p-2} u \ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

A real number λ is called an *eigenvalue* of problem (1.1) if this problem admits a nontrivial weak solution, i.e. there exists $u_\lambda \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \mid \nabla u_\lambda \mid^{p-2} \nabla u_\lambda \cdot \nabla w \, dx = \lambda \int_\Omega \mid u_\lambda \mid^{p-2} u_\lambda w \, dx \ \forall \ w \in W_0^{1,p}(\Omega). \tag{1.2}$$

The nontrivial solutions u_λ of problem (1.1) are called *eigenfunctions* corresponding to the eigenvalue λ, and (λ, u_λ) are called *eigenpairs* of problem (1.1).

A standard method to show the existence of an increasing sequence of eigenvalues for problem (1.1),

$$0 < \lambda_1^D < \lambda_2^D \leq \lambda_3^D \leq \cdots \to \infty, \tag{1.3}$$

relies on the Ljusternik-Schnirelmann principle and on the concept of Krasnosel'skĭi genus. There are also other methods to prove the existence of such a sequence (see García-Azorero & Peral [28], Drábek & Robinson [23]). It is still not known whether this sequence includes all eigenvalues of problem (1.1), except for the well-known particular case $p = 2$.

On the other hand, it is well-known that $-\Delta_p$ with the Dirichlet boundary condition admits a lowest positive eigenvalue λ_1 (called *principal eigenvalue*), which is simple, and there exists a corresponding eigenfunction which is positive in Ω (see Lindqvist [34], Lê [33] and the references therein). Note also that the properties of the next lowest eigenvalue λ_2 have been investigated by Anane & Tsouli in [2], who proved that λ_2 has a variational characterization similar to that corresponding to the linear case $p = 2$.

Similar situations can be reported in the case of Neumann, Robin or Steklov boundary conditions.

2. Eigenvalue problems governed by the $(p, q)-$Laplacian

In this section we shall present some recent results on eigenvalue problems involving the $(p, q)-$Laplacian with various boundary conditions. More precisely, these results contain information regarding the corresponding eigenvalue sets. As seen below, the fact that the differential operator $\mathcal{A}_{pq}$ is *non-homogeneous* (i.e., $p \neq q$) implies that the eigenvalue sets are intervals or contain intervals. Throughout this section we will assume that $p, q \in (1, \infty)$, $p \neq q$, and introduce the following notations:

$$\begin{aligned} W &:= W^{1,\max\{p,q\}}(\Omega), \\ \frac{\partial u}{\partial \nu_{pq}} &:= \left(\mid \nabla u \mid^{p-2} + \mid \nabla u \mid^{q-2} \right) \frac{\partial u}{\partial \nu}, \end{aligned} \tag{2.1}$$

where ν is the outward unit normal to $\partial\Omega$.

2.1. The case of Dirichlet, Neumann, Robin or Steklov boundary conditions

Let us begin with the case of the *Dirichlet boundary condition.* Specifically, we consider the problem

$$\begin{cases} -\mathcal{A}_{pq}u = \lambda \mid u \mid^{p-2} u \ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

The definitions of eigenvalues, eigenfunctions and eigenpairs for problem (2.2) are similar to those corresponding to problem (1.1), the only differences being the following: the left hand side of equation (1.2) is replaced by

$$\int_\Omega \Big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} \Big) \nabla u_\lambda \cdot \nabla w \ dx,$$

and the Sobolev space in which the weak solution is sought is now $W_0^{1,\max\{p,q\}}(\Omega)$.

The existence of eigenvalues for this problem in the case when the right hand side of equation $(2.2)_1$ is of the form $\lambda m_p(x) \mid u \mid^{p-2} u$ in Ω, where $m_p \in L^\infty(\Omega)$ such that the Lebesgue measure of $\{x \in \Omega;\ m_p(x) > 0\}$ is positive, was studied by Tanaka in [42]. Using the Mountain Pass Theorem, Tanaka was able to obtain the full eigenvalue set ([42, Theorem 1, Theorem 2]). In the particular case $m_p \equiv 1$, Tanaka's result is the following:

Theorem 2.1. *If $p, q \in (1, \infty)$, $p \neq q$, then the set of eigenvalues of problem* (2.2) *is precisely (λ_1^D, ∞), where λ_1^D denotes the first eigenvalue of the negative Dirichlet $p-$Laplacian, more exactly*

$$\lambda_1^D := \inf \left\{ \frac{\int_\Omega \mid \nabla u \mid^p dx}{\int_\Omega \mid u \mid^p dx},\ u \in W_0^{1,p}(\Omega) \right\}. \tag{2.3}$$

Notice that the eigenvalue set of $-\mathcal{A}_{pq}$ with Dirichlet boundary condition has been completely determined, being an interval independent of q.

Next, let us consider the case of a generalized *Neumann boundary condition.* More precisely, consider the eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq}u = \lambda \mid u \mid^{q-2} u \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

The solution u of problem (2.4) is understood in a weak sense, as an element of the Sobolev space W satisfying equation $(2.4)_1$ in the sense of distributions and $(2.4)_2$ in the sense of traces. The scalar $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2.4) if there exists $u_\lambda \in W \setminus \{0\}$ such that for all $w \in W$ we have

$$\int_\Omega \Big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} \Big) \nabla u_\lambda \cdot \nabla w \ dx = \lambda \int_\Omega \mid u_\lambda \mid^{q-2} u_\lambda w \ dx. \tag{2.5}$$

Problem (2.4) was investigated by Mihăilescu [36, Theorem 1.1] (for $q = 2$, $p \in (2, \infty)$), Fărcăşeanu, Mihăilescu & Stancu-Dumitru [24, Theorem 1.1] (for $q = 2$, $p \in (1, 2)$), Mihăilescu & Moroşanu [37, Theorem 1.1] (for $q \in (2, \infty), p \in (1, \infty)$, $p \neq q$) and Barbu & Moroşanu [7, Theorem 1] (for $q \in (1, 2)$, $p \in (1, \infty)$, $p \neq q$).

To investigate such a problem, one can use techniques based on minimization arguments, which will be briefly described in what follows.

To begin with, let us choose $w = u_\lambda$ in (2.5). Clearly, we see that the eigenvalues of problem (2.4) cannot be negative. It is also obvious that $\lambda_0 = 0$ is an eigenvalue of this problem with the corresponding eigenfunctions given by the nonzero constant functions.

Now, if we assume that $\lambda > 0$ is an eigenvalue of problem (2.4) and choose $w \equiv 1$ in (2.5) we obtain that every eigenfunction u_λ corresponding to λ necessarily belong to the set

$$\mathcal{C}_{Ne} := \Big\{ u \in W;\ \int_\Omega \mid u \mid^{q-2} u\ dx = 0 \Big\}. \tag{2.6}$$

This is a symmetric cone. Moreover, $\mathcal{C}_{Ne}$ is a weakly closed subset of W and $\mathcal{C}_{Ne} \setminus \{0\} \neq \emptyset$ (see [6, Section 2]).

Next, we shall briefly describe the method we can use to solve the eigenvalue problem (2.4).

For $\lambda > 0$ consider the C^1 functional $\mathcal{J}_\lambda : W \to \mathbb{R}$, defined as

$$\mathcal{J}_\lambda(u) = \frac{1}{p} \int_\Omega \mid \nabla u \mid^p \ dx + \frac{1}{q} \int_\Omega \mid \nabla u \mid^q \ dx - \frac{\lambda}{q} \int_\Omega \mid u \mid^q \ dx. \tag{2.7}$$

This functional is often called the *energy functional* associated to problem (2.4). Clearly, λ is an eigenvalue of problem (2.4) if and only if there exists a critical point $u_\lambda \in W \setminus \{0\}$ of $\mathcal{J}_\lambda$, i. e. $\mathcal{J}_\lambda'(u_\lambda) = 0$.

Define

$$\widetilde{\lambda}^{Ne} := \inf_{w \in \mathcal{C}_{Ne} \setminus \{0\}} \frac{\int_\Omega \mid \nabla w \mid^q \ dx}{\int_\Omega \mid w \mid^q \ dx}. \tag{2.8}$$

Since $\widetilde{\lambda}^{Ne} = \lambda_1^{Ne_q}$ for $q > p$ and $\widetilde{\lambda}^{Ne} \geq \lambda_1^{Ne_q}$ for $q < p$, it follows that $\widetilde{\lambda}^{Ne} > 0$ (we have denoted by $\lambda_1^{Ne_q}$ the first positive eigenvalue of the negative Neumann $q-$Laplace operator).

Also, one can easily check that there is no eigenvalue of problem (2.4) in the set $(-\infty, \widetilde{\lambda}^{Ne}] \setminus \{0\}$. So, from now on we shall consider that λ is arbitrary but fixed in the interval $(\widetilde{\lambda}^{Ne}, \infty)$.

We distinguish two cases related to p and q:

Case 1: $1 < q < p$. In this case, as $\lambda > \widetilde{\lambda}^{Ne}$, the functional $\mathcal{J}_\lambda$ is coercive on $\mathcal{C}_{Ne} \subset W = W^{1,p}(\Omega)$, i.e.,

$$\lim_{\|u\|_{W^{1,p}(\Omega)} \to \infty, u \in \mathcal{C}_{Ne}} \mathcal{J}_\lambda(u) = \infty.$$

In particular, there exists $u_* \in \mathcal{C}_{Ne} \setminus \{0\}$ where $\mathcal{J}_\lambda$ attains its minimal value over $\mathcal{C}_{Ne}$,

$$J_\lambda(u_*) = \inf_{w \in \mathcal{C}_{Ne} \setminus \{0\}} \mathcal{J}_\lambda(w) \neq 0$$

(see [7, Lemma 6]).

Case 2: $1 < p < q$. Under this assumption, the functional $\mathcal{J}_\lambda$ is no longer coercive and may be unbounded below on $W = W^{1,q}(\Omega)$. So, we consider the restriction of functional $\mathcal{J}_\lambda$ to the Nehari type manifold (see [41]):

$$\mathcal{N}_\lambda = \{v \in \mathcal{C}_{Ne} \setminus \{0\}; \langle \mathcal{J}_\lambda'(v), v\rangle = 0\}.$$

We observe that

$$\mathcal{J}_\lambda(u) = \frac{q-p}{qp} \int_\Omega |\nabla u|^p \; dx > 0 \;\forall\; u \in \mathcal{N}_\lambda.$$

Moreover, any possible eigenfunction corresponding to λ belongs to $\mathcal{N}_\lambda$.

In addition, since $\lambda > \widetilde{\lambda}^{Ne}$, we can easily check that $\mathcal{N}_\lambda \neq \emptyset$.

In this case we have the following result (see [6, Case 2, Steps 1-4] and [7, Lemma 6]):

If $1 < p < q$ and $\lambda > \widetilde{\lambda}^{Ne}$, then there exists $u_* \in \mathcal{N}_\lambda$ where $\mathcal{J}_\lambda$ attains its minimal value over $\mathcal{N}_\lambda$,

$$m_\lambda := \inf_{w \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0.$$

Using the above preliminary results and applying the Lagrange Multipliers Rule in the case $q \geq 2$ and, respectively, an approximation technique in the case $1 < q < 2$, one can show that in fact the minimizer u_* of functional $\mathcal{J}_\lambda$ over $\mathcal{C}_{Ne}$ if $q < p$ and, respectively, over $\mathcal{N}_\lambda$ if $q > p$, is a global minimizer of $\mathcal{J}_\lambda$ over the whole W, i.e. u_* is an eigenfunction of problem (2.4) corresponding to the eigenvalue $\lambda > \widetilde{\lambda}^{Ne}$.

Thus, we have the following important result which provides the full spectrum of the eigenvalue problem (2.4):

Theorem 2.2. *Assume that* $p, q \in (1, \infty)$, $p \neq q$. *Then the set of eigenvalues of problem* (2.4) *is precisely* $\{0\} \cup (\widetilde{\lambda}^{Ne}, \infty)$, *where* $\widetilde{\lambda}^{Ne}$ *is the positive constant defined by* (2.8).

Now, consider the eigenvalue problem for the *Steklov* $(p, q)-$*Laplacian*, namely

$$\begin{cases} \mathcal{A}_{pq} u = 0 \;\; \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} = \lambda \mid u \mid^{q-2} u \text{ on } \partial\Omega. \end{cases} \tag{2.9}$$

Using an approach similar to that used before for the Neumann $(p, q)-$Laplacian, one can determine the full spectrum of the eigenvalue problem (2.9). More exactly, if we denote

$$\mathcal{C}_S := \Big\{ u \in W; \int_{\partial\Omega} \mid u_\lambda \mid^{q-2} u_\lambda \; d\sigma = 0 \Big\}, \tag{2.10}$$

$$\widetilde{\lambda}^S := \inf_{w \in \mathcal{C}_S \setminus \{0\}} \frac{\int_\Omega \mid \nabla w \mid^q \; dx}{\int_{\partial\Omega} \mid w \mid^q \; d\sigma}, \tag{2.11}$$

we have the following result

Theorem 2.3. *Assume that* $p, q \in (1, \infty)$, $p \neq q$. *Then the set of eigenvalues of problem* (2.9) *is precisely* $\{0\} \cup (\widetilde{\lambda}_S, \infty)$, *where* $\widetilde{\lambda}_S$ *is the positive constant defined by* (2.11).

This theorem was proved by Costea & Moroşanu [19, Theorem 3.1] in the case $p \in (1, \infty)$, $q \in [2, \infty)$, $p \neq q$ and later by Barbu & Moroşanu [7, Theorem 1] in the case $p \in (1, \infty)$, $q \in (1, 2)$, $p \neq q$.

Next, we pay attention to equation $(2.4)_1$ with a generalized *Robin boundary condition.* More precisely, we consider the following eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq}u = \lambda \mid u \mid^{q-2} u \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \beta \mid u \mid^{q-2} u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.12}$$

where β is a positive constant.

The eigenvalue problem (2.12) was studied by Gyulov & Moroşanu [30], who found an interval of eigenvalues for this problem. In order to state the main result in [30], we define

$$\begin{aligned} \widetilde{\lambda}^R &:= \inf_{w\in W\setminus\{0\}} \frac{\int_\Omega \mid \nabla w \mid^q \ dx + \beta \int_{\partial\Omega} \mid \nabla w \mid^q \ d\sigma}{\int_\Omega \mid w \mid^q \ dx}, \\ \lambda_0 &:= \beta \frac{\mid \partial\Omega \mid_{N-1}}{\mid \Omega \mid_N}, \end{aligned} \tag{2.13}$$

where $| \cdot |_N$ and $| \cdot |_{N-1}$ denote the Lebesgue measures of the two sets. Obviously, the constant $\widetilde{\lambda}_R$ coincides with the first eigenvalue of the Robin $q-$Laplace operator (see Lê [33]) in the case $q > p$ and is greater than or equal to that if $q < p$, so it is positive.

The results concerning the spectrum of problem (2.12) can be summarized as follows:

Theorem 2.4. *Assume that $p, q \in (1, \infty)$, $p \neq q$ and β is a positive constant. Then $\widetilde{\lambda}^R < \lambda_0$ and any $\lambda \in (\widetilde{\lambda}_R, \lambda_0)$ is an eigenvalue of problem* (2.12). *Moreover, the problem* (2.12) *has no nontrivial solution for $\lambda \in (-\infty, \widetilde{\lambda}^R]$.*

Note that this theorem does not say whether there are eigenvalues of problem (2.12) in the interval $[\lambda_0, \infty)$. On the other hand, we know that there exists a sequence of eigenvalues of problem (2.12) which converges to ∞ (see [5]). However, the full spectrum of problem (2.12) is still not completely known.

We also mention the paper by Papageorgiou, Vetro & Vetro [38] where an eigenvalue problem more general than (2.12) is considered in the case $1 < p < q$. Here the operator $\mathcal{A}_{pq}$ is perturbed with an indefinite and unbounded potential, $\zeta \in L^s(\Omega)$, $s < N/q$ if $q \leq N$ and $s = 1$ if $q > N$. The constant β is replaced by a function $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta \geq 0$, $\beta \not\equiv 0$ such that

$$\int_\Omega \zeta \ dx + \int_{\partial\Omega} \beta \ d\sigma > 0. \tag{2.14}$$

By arguing as in [30], the authors obtain a result similar to Theorem 2.4 (see [38, Theorem 1]).

Finally, let us consider the Steklov like eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq}u + \rho_1(x) \mid u \mid^{p-2} u + \rho_2(x) \mid u \mid^{q-2} u = 0, \ \ x \in \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \gamma_1(x) \mid u \mid^{p-2} u + \gamma_2(x) \mid u \mid^{q-2} u = \lambda \mid u \mid^{q-2} u, \ \ x \in \partial\Omega. \end{cases} \tag{2.15}$$

Assume that the following hypotheses are fulfilled:

$(h_{\rho_1\gamma_1})$ $\rho_1 \in L^\infty(\Omega)$ and $\gamma_1 \in L^\infty(\partial\Omega)$, $\rho_1,\ \gamma_1$ are nonnegative functions such that

$$\int_\Omega \rho_1\ dx + \int_{\partial\Omega} \gamma_1\ d\sigma > 0; \tag{2.16}$$

$(h_{\rho_2\gamma_2})$ $\rho_2 \in L^\infty(\Omega)$, $\gamma_2 \in L^\infty(\partial\Omega)$ and ρ_2 is a nonnegative function.

It is worth pointing out that the potential function γ_2 is allowed to be sign changing.

As usual, a scalar $\lambda \in \mathbb{R}$ is said to be an eigenvalue of the problem (2.15) if there exists $u_\lambda \in W \setminus \{0\}$ such that for all $w \in W$

$$\begin{aligned} &\int_\Omega \big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} \big) \nabla u_\lambda \cdot \nabla w\ dx \\ &\quad + \int_\Omega \big(\rho_1 \mid u_\lambda \mid^{p-2} + \rho_2 \mid u_\lambda \mid^{q-2} \big) u_\lambda w\ dx \\ &\quad + \int_{\partial\Omega} \big(\gamma_1 \mid u_\lambda \mid^{p-2} + \gamma_2 \mid u_\lambda \mid^{q-2} \big) u_\lambda w\ d\sigma = \lambda \int_{\partial\Omega} \mid u_\lambda \mid^{q-2} u_\lambda w\ d\sigma. \end{aligned} \tag{2.17}$$

The function u_λ is called an eigenfunction of the problem (2.15) (corresponding to the eigenvalue λ).

Define

$$\widetilde{\lambda}^{SR} := \inf_{w \in W \setminus \{0\}} \frac{\int_\Omega \big(\mid \nabla w \mid^q + \rho_2 \mid w \mid^q \big) dx + \int_{\partial\Omega} \gamma_2 \mid w \mid^q\ d\sigma}{\int_{\partial\Omega} \mid w \mid^q\ d\sigma}. \tag{2.18}$$

Problem (2.15) was studied by Barbu & Moroşanu [11]. Let us recall the main result on its eigenvalue set:

Theorem 2.5. ([11, Theorem 1]) *Assume that $p, q \in (1, \infty)$, $p \neq q$ and assumptions $(h_{\rho_i\gamma_i})$, $i = 1, 2$, are fulfilled. Then the set of eigenvalues of problem* (2.15) *is precisely* $(\widetilde{\lambda}^{SR}, \infty)$.

Note that if $\gamma_1 \equiv 0$ and $\gamma_2 \equiv$ const. > 0, then we have a Steklov-Robin boundary condition. The arguments we have used in the mentioned paper can easily be adapted to the following eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq} u + \rho_1(x) \mid u \mid^{p-2} u + \rho_2(x) \mid u \mid^{q-2} u = \lambda \mid u \mid^{q-2} u, & x \in \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \gamma_1(x) \mid u \mid^{p-2} u + \gamma_2(x) \mid u \mid^{q-2} u = 0, & x \in \partial\Omega, \end{cases} \tag{2.19}$$

under similar assumptions for the functions $\rho_i,\ \gamma_i,\ i = 1, 2$. While in the previous works [30] and [38] only subsets of the corresponding spectra were found, in this case the presence of the potential functions $\rho_i,\ \gamma_i$ satisfying assumptions $(h_{\rho_i\gamma_i})$, $i = 1, 2$, allows the full description of the spectrum.

2.2. The case of parametric boundary conditions

Consider the following eigenvalue problem

$$\begin{cases} -\mathcal{A}_{pq} u = \lambda\alpha(x) \mid u \mid^{r-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} = \lambda\beta(x) \mid u \mid^{r-2} u & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

under the following hypotheses

(h_{pqr}) $p, q, r \in (1, \infty),\ p \neq q$;

$(h_{\alpha\beta})$ $\alpha \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying

$$\int_\Omega \alpha\ dx + \int_{\partial\Omega} \beta\ d\sigma > 0. \tag{2.21}$$

Such eigenvalue problems were discussed for the first time by Von Below & François [43] (see also François [27]) who considered the linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda \beta u & \text{on } \partial\Omega. \end{cases}$$

They call it a *dynamical eigenvalue problem* since it can be derived from the study of the heat equation with dynamical boundary conditions. Also, the motivation behind problem (2.20) comes from the study of a double phase parabolic equation (see Arora & Shmarev [3], Huang [31], Marcellini [35] and the references therein) under a dynamical boundary condition. The existence theory for such parabolic problems relies on the spectral theory of associated elliptic problems with the parameter λ both in the equation and the boundary condition.

The eigenvalues and eigenfunctions of problem (2.20) can be defined as before. All eigenfunctions of problem (2.20) belong to the set

$$\mathcal{C}_r := \Big\{u \in W;\ \int_\Omega \alpha \mid u \mid^{r-2} u\ dx + \int_{\partial\Omega} \beta \mid u \mid^{r-2} u\ d\sigma = 0\Big\}. \tag{2.22}$$

In the case $r = q$, define

$$\widetilde{\lambda} := \inf_{w \in \mathcal{C}_q \setminus \{0\}} \frac{\int_\Omega \mid \nabla w \mid^q\ dx}{\int_\Omega \alpha \mid w \mid^q\ dx + \int_{\partial\Omega} \beta \mid w \mid^q\ d\sigma}. \tag{2.23}$$

If $r \neq q$ we assume, without any loss of generality, that $1 < p < q$ and for $r \in (p, q)$ define

$$\lambda_* := \inf_{v \in \mathcal{C}_r \setminus \mathcal{Z}_r} \Gamma \frac{K_q(v)^{1-\gamma} K_p(v)^\gamma}{\mathcal{K}_r(v)},\ \ \lambda^* := \frac{r}{q^{1-\gamma} p^\gamma} \lambda_*, \tag{2.24}$$

where

$$\begin{aligned} \mathcal{Z}_r &:= \{v \in W;\ \int_\Omega \alpha \mid v \mid^r dx + \int_{\partial\Omega} \beta \mid v \mid^r d\sigma = 0\}, \\ K_p(u) &:= \int_\Omega \mid \nabla u \mid^p dx,\ K_q(u) := \int_\Omega \mid \nabla u \mid^q dx, \\ \mathcal{K}_r(u) &:= \int_\Omega \alpha \mid u \mid^r dx + \int_{\partial\Omega} \beta \mid u \mid^r d\sigma\ \forall\ u \in W = W^{1,q}(\Omega), \\ \gamma &:= \frac{q-r}{q-p},\ \Gamma := \frac{q-p}{(r-p)^{1-\gamma}(q-r)^\gamma}. \end{aligned} \tag{2.25}$$

In the case $r = q$ we have obtained the following result:

Theorem 2.6. ([7, Theorem 1]) *Assume that $p, q \in (1, \infty),\ p \neq q,\ r = q$ and $(h_{\alpha\beta})$ holds. Then $\widetilde{\lambda} > 0$ and the set of eigenvalues of problem* (2.20) *(with $r = q$) is precisely $\{0\} \cup (\widetilde{\lambda}, \infty)$, where $\widetilde{\lambda}$ is the constant defined by* (2.23)*.*

Note that problem (2.20) in the case $q = 2$ and $p \in (1,\infty)$, $p \neq 2$, has been previously studied by Abreu & Madeira[1].

In the case $r \notin \{p,q\}$, we have the following result:

Theorem 2.7. ([8, Theorem 1.1], [10, Theorem 1]) *Suppose that assumption* $(h_{\alpha\beta})$ *holds.*
(a) *If either* $(1 < r < p < q < \infty)$ *or* $(1 < q < p < r < \infty$ *and* $r \in \Big(1, \frac{q(N-1)}{N-q}\Big)$ *if* $1 < q < N)$, *then the set of eigenvalues of problem* (2.20) *is* $[0,\infty)$.
(b) *If* $1 < p < r < q < \infty$, *with* $r < \frac{q(N-1)}{N-q}$ *if* $q < N$, *then* $0 < \lambda_* < \lambda^*$ *and for* $\lambda \in \{0\} \cup [\lambda^*,\infty)$ *there exists a weak solution* $u_\lambda \in W^{1,p}(\Omega) \setminus \{0\}$ *to problem* (2.20). *For any* $\lambda \in (-\infty,\lambda_*) \setminus \{0\}$ *problem* (2.20) *has only the trivial solution. Moreover, the constants* λ_*, λ^* *can be expressed as follows*

$$\lambda_* = \inf_{v \in \mathcal{C}_r \setminus \mathcal{Z}_r} \frac{K_p(v) + K_q(v)}{\mathcal{K}_r(v)}, \ \lambda^* = \inf_{v \in \mathcal{C}_r \setminus \mathcal{Z}} \frac{\frac{1}{p}K_p(v) + \frac{1}{q}K_q(v)}{\frac{1}{r}\mathcal{K}_r(v)}. \tag{2.26}$$

Thus, we were able to find the full eigenvalue sets in two of the three possible cases. The difficult case is $r \in (p,q)$, for which the eigenvalue set is not completely known.

Now, let us pay attention to the following eigenvalue problem governed by the $(p,q,r)-$Laplacian, which is defined by $\mathcal{A}_{pqr}u := \Delta_p u + \Delta_q u + \Delta_r u$,

$$\begin{cases} -\mathcal{A}_{pqr} = \lambda\alpha(x) \mid u \mid^{r-2} u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{pqr}} = \lambda\beta(x) \mid u \mid^{r-2} u \ \text{ on } \partial\Omega, \end{cases} \tag{2.27}$$

under the assumption $(h_{\alpha\beta})$ above and

$(h_{pqr})'$ $p,q,r \in (1,+\infty)$, $q < p$, $r \notin \{p,q\}$.

In the boundary condition $(2.27)_2$, $\frac{\partial u}{\partial \nu_{pqr}}$ denotes the conormal derivative corresponding to the differential operator $\mathcal{A}_{pqr}$, i.e.,

$$\frac{\partial u}{\partial \nu_{pqr}} := \Big(\sum_{\alpha \in \{p,q,r\}} \mid \nabla u \mid^{\alpha-2} \Big) \frac{\partial u}{\partial \nu}.$$

where ν is the outward unit normal to $\partial\Omega$.

Such a triple-phase eigenvalue problem is motivated by some models arising in mathematical physics. More exactly, let us consider the operator

$$Qu := -\text{div}\Big(\frac{\nabla u}{\sqrt{1- \mid \nabla u \mid^2}} \Big).$$

This operator occurs in the electrostatic Born-Infeld equation (see [16]), in string theory, in particular in the study of D-branes (see, e.g., [29]), and in classical relativity, where Q represents the mean curvature operator in Lorent-Minkowski space (see, e.g., [12] and [17]). A second order approximation of Q is $\mathcal{B} := -\triangle u - \triangle_4 u - \frac{3}{2}\triangle_6 u$, which is a negative $(2,4,6)$-Laplacian (see [40]), with the coefficient $-3/2$ instead of -1.

In fact, one can consider a more general eigenvalue problem, with

$$\mathcal{B}u := \Delta_p u + \rho_q \Delta_q u + \rho_r \Delta_r u, \ \ \rho_q, \, \rho_r > 0,$$

instead of $\mathcal{A}_{pqr}$, and with

$$\frac{\partial u}{\partial \nu_{\mathcal{B}}} := \Big(\sum_{\alpha \in \{p,q,r\}} \rho_\alpha \mid \nabla u \mid^{\alpha-2} \Big) \frac{\partial u}{\partial \nu}, \ \rho_p = 1,$$

instead of $\frac{\partial u}{\partial \nu_{pqr}}$ (see [9, Section 4]).

Under assumption $(h_{pqr})'$, the appropriate Sobolev space for problem (2.27) is $\widetilde{W} := W^{1,\max\{p,r\}}(\Omega)$. One can define the eigenvalues of problem (2.27) as follows: $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2.27) if there exists $u_\lambda \in \widetilde{W} \setminus \{0\}$ such that

$$\begin{aligned} &\int_\Omega \Big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} + \mid \nabla u_\lambda \mid^{r-2} \Big) \nabla u_\lambda \cdot \nabla w \; dx \\ &\quad = \lambda \Big(\int_\Omega a \mid u_\lambda \mid^{r-2} u_\lambda w \; dx + \int_{\partial\Omega} b \mid u_\lambda \mid^{r-2} u_\lambda w \; d\sigma \Big) \ \forall \ w \in \widetilde{W}. \end{aligned} \tag{2.28}$$

If u_λ is an eigenfunction corresponding to a positive eigenvalue λ then necessarily u_λ belongs to the set

$$\mathcal{C} := \Big\{ u \in \widetilde{W}; \ \int_\Omega \alpha \mid u \mid^{r-2} u \; dx + \int_{\partial\Omega} \beta \mid u \mid^{r-2} u \; d\sigma = 0 \Big\}. \tag{2.29}$$

Let us introduce the notations

$$\begin{aligned} K_\alpha(u) &:= \int_\Omega \mid \nabla u \mid^\alpha dx, \ \alpha \in \{p,q,r\}, \\ k_r(u) &:= \int_\Omega \alpha \mid u \mid^r dx + \int_{\partial\Omega} \beta \mid u \mid^r d\sigma \ \forall \ u \in W, \\ \mathcal{Z} &:= \{ v \in W; \ k_r(v) = 0 \}. \end{aligned} \tag{2.30}$$

Define

$$\Lambda_r := \inf_{v \in \mathcal{C} \setminus \mathcal{Z}} \frac{K_r(v)}{k_r(v)}. \tag{2.31}$$

For $r \in (q,p)$ denote

$$\begin{aligned} \Lambda_* &:= \inf_{v \in \mathcal{C} \setminus \mathcal{Z}} \left(\Gamma \frac{K_p(v)^{1-\gamma} K_q(v)^\gamma}{k_r(v)} + \frac{K_r(v)}{k_r(v)} \right), \\ \Lambda^* &:= \inf_{v \in \mathcal{C} \setminus \mathcal{Z}} \left(\Gamma \frac{r}{p^{1-\gamma} q^\gamma} \frac{K_p(v)^{1-\gamma} K_q(v)^\gamma}{k_r(v)} + \frac{K_r(v)}{k_r(v)} \right), \\ \gamma &:= \frac{p-r}{p-q}, \ \Gamma := \frac{p-q}{(r-q)^{1-\gamma}(p-r)^\gamma}. \end{aligned} \tag{2.32}$$

The main result concerning problem (2.27) is the following:

Theorem 2.8. (see [9, Theorems 1.1 and 1.2]) *Assume that (h'_{pqr}) and $(h_{\alpha\beta})$ above are fulfilled. If $r \notin (q,p)$, then $\Lambda_r > 0$ and the set of eigenvalues of problem (2.27) is precisely $\{0\} \cup (\Lambda_r, \infty)$, where Λ_r is the constant defined by (2.31). Otherwise, if $r \in (q,p)$, and $r < q(N-1)/(N-q)$ if $q < N$, then $0 < \Lambda_* < \Lambda^*$, every $\lambda \in \{0\} \cup [\Lambda^*, \infty)$*

is an eigenvalue of problem (2.27)*, and for any* $\lambda \in (-\infty, \Lambda_*) \setminus \{0\}$ *problem* (2.27) *has only the trivial solution.*

It would be nice to see whether some of the above result could be extended to the case in which operator $\mathcal{A}_{pq}$ is replaced by the operator $\mathcal{Q}_{pq} := \mathcal{Q}_p + \mathcal{Q}_q$, where for $\theta \in (1, \infty)$ we have denoted by $\mathcal{Q}_\theta$ the operator defined as follows

$$\mathcal{Q}_\theta u := \operatorname{div} \Big(F^{\theta-1}(\nabla u) F_\xi(\nabla u) \Big), \tag{2.33}$$

where F is a positive, one-homogeneous, convex function on $\mathbb{R}^N$ and F_ξ denotes the gradient of F.

If we assume that $F \in C^2(\mathbb{R}^N \setminus \{0\})$ and the Hessian matrix of F^p, $\big(F^p_{\xi_i\xi_j}(\xi)\big)_{i,j}$, is positive definite on $\mathbb{R}^N \setminus \{0\}$, then operator $\mathcal{Q}_\theta$ is elliptic. This operator is a natural generalization of Δ_θ which can be obtained from $\mathcal{Q}_\theta$ if F is the Euclidean norm. A typical example of F satisfying the above conditions is the l_r-norm (denoted by $\| \cdot \|_r$),

$$F(\xi) := \Big(\sum_{i=1}^{N} \mid \xi_i \mid^r \Big)^{1/r}, \ r \in (1, \infty),$$

for which the operator $\mathcal{Q}_\theta$ has the form

$$\Delta_{r\theta}(u) := \operatorname{div} \big(\| \nabla u \|_r^{\theta - r} \nabla^r u \big),$$

where

$$\nabla^r u := \Bigg(\Big| \frac{\partial u}{\partial x_1} \Big|^{r-2} \frac{\partial u}{\partial x_1}, \cdots, \Big| \frac{\partial u}{\partial x_N} \Big|^{r-2} \frac{\partial u}{\partial x_N} \Bigg).$$

Note that $\Delta_{r\theta}$ is a nonlinear operator unless $\theta = r = 2$ when it reduces to the usual Laplacian. An important special case is $r = \theta$, when $\Delta_{\theta\theta}$ is the so-called pseudo $\theta-$Laplacian.

The operator defined in (2.33) is often called anisotropic p-Laplacian or Finsler p-Laplacian. There exist many papers dedicated to the study of its eigenvalues, for different boundary conditions (Dirichlet, Neumann, Robin or Steklov). See, e.g., [13], [20], [21], [22], [25], [32], [44] and references therein.

As an example, let us consider the eigenvalue problem

$$\begin{cases} -\mathcal{Q}_p u = \lambda \alpha(x) \mid u \mid^{q-2} u \ \text{ in } \Omega, \\ F^{p-1}(\nabla u) \nabla_\xi F(\nabla u) \cdot \nu = \lambda \beta(x) \mid u \mid^{q-2} u \text{ on } \partial\Omega. \end{cases} \tag{2.34}$$

As usual, a real number λ is an eigenvalue of problem (2.34) if there exists $u_\lambda \in W^{1,p} \setminus \{0\}$ such that for all $w \in W^{1,p}(\Omega)$

$$\begin{aligned} &\int_\Omega F(\nabla u_\lambda)^{p-1} \nabla_\xi F(\nabla u_\lambda) \cdot \nabla w \, dx \\ &= \lambda \Big(\int_\Omega \alpha \mid u_\lambda \mid^{q-2} u_\lambda w \, dx + \int_{\partial\Omega} \beta \mid u_\lambda \mid^{q-2} u_\lambda w \, d\sigma \Big). \end{aligned} \tag{2.35}$$

The following result holds for problem (2.34).

Theorem 2.9. ([4, Theorem 1.2]) *Assume that* $q \in (1, \infty)$, $p \in \left(\frac{Nq}{N+q-1}, \infty\right)$, $p \neq q$, *and* $(h_{\alpha\beta})$ *are fulfilled. Then the set of eigenvalues of problem* (2.34) *is* $[0, \infty)$.

We expect that many of the above results will be extended to eigenvalue problems governed by the operator $\mathcal{Q}_{pq}$.

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Luminiţa Barbu
"Ovidius" University,
Faculty of Mathematics and Computer Science,
124 Mamaia Blvd, 900527 Constanţa, Romania
e-mail: `lbarbu@univ-ovidius.ro`

Gheorghe Moroşanu
"Babeş-Bolyai" University,
Faculty of Mathematics and Computer Science,
1 Mihail Kogălniceanu Street,
400084 Cluj-Napoca, Romania
e-mail: `morosanu@math.ubbcluj.ro`

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Quasilinear differential inclusions driven by degenerated p-Laplacian with weight

Dumitru Motreanu

Dedicated to the memory of Professor Csaba Varga

Abstract. The main result of the paper provides the existence of a solution to a quasilinear inclusion problem with Dirichlet boundary condition which exhibits a term with full dependence on the solution and its gradient (convection term) and is driven by the degenerated p-Laplacian with weight. The multivalued term in the differential inclusion is in form of the generalized gradient of a locally Lipschitz function expressed through the primitive of a locally essentially bounded function, which makes the problem to be of a hemivariational inequality type. The novelty of our result is that we are able to simultaneously handle three major features: degenerated leading operator, convection term and discontinuous nonlinearity. Results of independent interest regard certain nonlinear operators associated to the differential inclusion.

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1. Introduction

The aim of this paper is to study the quasilinear differential inclusion

$$\begin{cases} -\Delta_p^a u \in f(x,u,\nabla u) + [\underline{g}(u), \overline{g}(u)] & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, for $N \geq 1$, with a Lipschitz boundary $\partial\Omega$. Here $-\Delta_p^a$ denotes the (negative) degenerated p-Laplacian with the positive weight $a \in L^1_{\text{loc}}(\Omega)$ (see Section 3 for the precise definition). In the right-hand side of equation (1.1) there is the convection term $f(x,u,\nabla u)$, i.e., it depends on the solution u and its gradient ∇u, which is described by a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, that is,

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$f(\cdot, s, \xi)$ is measurable on Ω for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$. The multivalued term in (1.1) is expressed by means of a function $g \in L^\infty_{\text{loc}}(\mathbb{R})$ for which we set

$$\underline{g}(t) = \lim_{\delta \to 0} \operatorname{essinf}_{|\tau - t| < \delta} g(\tau), \ \forall\, t \in \mathbb{R}, \tag{1.2}$$

(i.e., the essential infimum of g at t) and

$$\overline{g}(t) = \lim_{\delta \to 0} \operatorname{esssup}_{|\tau - t| < \delta} g(\tau), \ \forall\, t \in \mathbb{R} \tag{1.3}$$

(i.e., the essential supremum of g at t). Since $g \in L^\infty_{\text{loc}}(\mathbb{R})$, it is clear that the expressions in (1.2) and (1.3) are well defined. If the function g is continuous, then the interval $[\underline{g}(u(x)), \overline{g}(u(x))]$ collapses to the singleton $g(u(x))$. Consequently, in this case (1.1) reduces to the quasilinear Dirichlet equation

$$\begin{cases} -\Delta_p^a u = f(x, u, \nabla(u)) + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The multivalued term $[\underline{g}(u), \overline{g}(u)]$ in (1.1) is actually the generalized gradient of a locally Lipschitz function that will be explicitly identified in Section 4. This fact qualifies problem (1.1) as a hemivariational inequality, which is of a special type due to the degenerated operator in the principal part and the presence of the convection term inducing a full gradient dependence. Besides their substantial mathematical interest in passing from convex nonsmooth potentials to nonconvex nonsmooth potentials, the hemivariational inequalities represent a powerful tool to model phenomena with various contact laws in mechanics and engineering. For theoretical developments in the study of hemivariational inequality based on nonsmooth variational methods, we refer to [4, 6, 7, 11, 12, 14, 15]. Nonvariational techniques, such as theoretic operator methods and sub-supersolution, have also been implementing in the nonsmooth multivalued setting of hemivariational inequalities, for instance, in [1, 8, 9, 10, 13, 16]. Problems (1.1) and (1.4) do not have variational structure due to the presence of the convection term, so the variational methods are not applicable. To overcome this difficulty we are going to apply in Section 5 the main theorem for multivalued pseudomonotone operators. Notice that if $f = 0$, problem (1.1) becomes a nonsmooth variational problem with discontinuous nonlinearities extending statements in [1, 2, 12]) to the case where the driving operator is degenerated exhibiting weights. In the situation of (1.4) with $f = 0$, we have a quasilinear elliptic equation that can be treated by using the smooth critical point theory. If $g = 0$, problems (1.1) and (1.4) extend previous statements to formulations involving degenerated operators with weights (see [10]).

The most significant contribution of the paper is to resolve problem (1.1) (and implicitly (1.4)) that incorporates in the same statement three challenging aspects: degenerated leading operator, convection term and discontinuous nonlinearity. This main result is stated as Theorem 5.1. It is the first available result encompassing the three relevant features mentioned before. The solutions to problems (1.1) and (1.4) are sought in a suitable Sobolev space $W_0^{1,p}(a, \Omega)$ that corresponds to the positive weight $a \in L^1_{\text{loc}}(\Omega)$ as discussed in Section 2. By a (weak) solution to problem (1.1) we mean any $u \in W_0^{1,p}(a, \Omega)$ for which it holds $f(x, u, \nabla u)$, $\underline{g}(u)$, $\overline{g}(u) \in L^{p/(p-1)}(\Omega)$

and

$$\begin{aligned}&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx-\int_\Omega f(x,u,\nabla u)v\,dx\\&\geq\int_\Omega \min\{\underline{g}(u(x))v(x),\overline{g}(u(x))v(x)\}\,dx\ \text{ for all } v\in W_0^{1,p}(a,\Omega).\end{aligned}\tag{1.5}$$

Replacing $v\in W_0^{1,p}(a,\Omega)$ with $-v$ it is seen that (1.5) is equivalent to

$$\begin{aligned}&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx-\int_\Omega f(x,u,\nabla u)v\,dx\\&\leq\int_\Omega \max\{\underline{g}(u(x))v(x),\overline{g}(u(x))v(x)\}\,\mathrm{d}x\ \text{ for all } v\in W_0^{1,p}(a,\Omega).\end{aligned}\tag{1.6}$$

As it is apparent from (1.5) (or (1.6)), (1.2) and (1.3), for the Dirichlet equation (1.4) the usual notion of weak solution is retrieved. Indeed, if $g:\mathbb{R}\to\mathbb{R}$ is continuous, then the interval $[\underline{g}(u),\overline{g}(u)]$ reduces to the singleton $g(u)$, thus $u\in W_0^{1,p}(\Omega)$ is a (weak) solution to equation (1.4) provided $f(x,u,\nabla u)$, $g(u)\in L^{p/(p-1)}(\Omega)$ and

$$\begin{aligned}&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx-\int_\Omega f(x,u,\nabla u)v\,dx\\&=\int_\Omega g(u(x))v(x)\,\mathrm{d}x \text{ for all } v\in W_0^{1,p}(a,\Omega).\end{aligned}\tag{1.7}$$

In addition to the existence result, the paper contains propositions of independent interest establishing properties of certain nonlinear operators associated to problem (1.1).

The rest of the paper is structured as follows. Section 2 collects needed preliminaries regarding multivalued pseudomonotone operators and nonsmooth analysis. Section 3 focuses on the degenerated p-Laplacian with weight driving (1.1). Section 4 investigates nonlinear operators related to problem (1.1). Section 5 presents our main result and its proof.

2. Prerequisites on multivalued pseudomonotone operators and nonsmooth analysis

This section provides necessary mathematical background for our results on problem (1.1), in particular (1.4).

We start by briefly reviewing the multivalued pseudomonotone operators. More details can be found in [1, 11, 17]. Let X be a reflexive Banach space with the norm $\|\cdot\|$, its dual X^* and the duality pairing $\langle\cdot,\cdot\rangle$ between X and X^*. The norm convergence in X and X^* is denoted by $\to$, while the weak convergence by $\rightharpoonup$. A multivalued map $A:X\to 2^{X^*}$ is called bounded if it maps bounded sets into bounded sets. It is said to be coercive if there is a function $\psi:\mathbb{R}_+\to\mathbb{R}$ with $\psi(t)\to+\infty$ as $t\to+\infty$ such that

$$\langle u^*,u-u_0\rangle\geq\psi(\|u\|)\|u\|$$

for all $u^* \in A(u)$ and a fixed element $u_0 \in X$. A multivalued map $A : X \to 2^{X^*}$ is called pseudomonotone if

(i) for each $v \in X$, the set $Av \subset X^*$ is nonempty, bounded, closed and convex;
(ii) A is upper semicontinuous from each finite dimensional subspace of X to X^* endowed with the weak topology;
(iii) for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ with

$$u_n \rightharpoonup u \text{ in } X,\ u_n^* \in Au_n \text{ for all } n \text{ and } \limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \le 0,$$

and for each $v \in X$, there exists $u^*(v) \in Au$ such that

$$\langle u^*(v), u - v\rangle \le \liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle.$$

We recall the main theorem for pseudomonotone operators (see, e.g., [1, Theorem 2.125]).

Theorem 2.1. *Let X be a reflexive Banach space, let $A : X \to 2^{X^*}$ be a pseudomonotone, bounded and coercive operator, and let $\eta \in X^*$. Then there exists at least a $u \in X$ with $\eta \in Au$.*

Next we outline some basic elements of nonsmooth analysis related to locally Lipschitz functions. An extensive study of this topic is available in [2, 3]). A function $\Phi : X \to \mathbb{R}$ on a Banach space X is called locally Lipschitz if for every $u \in X$ there is a neighborhood U of u in X and a constant $L_u > 0$ such that

$$|\Phi(v) - \Phi(w)| \le L_u\|v - w\|, \quad \forall\ v, w \in U.$$

The generalized directional derivative of a locally Lipschitz function $\Phi : X \to \mathbb{R}$ at $u \in X$ in the direction $v \in X$ is defined as

$$\Phi^0(u; v) := \limsup_{w\to u,\ t\to 0^+} \frac{1}{t}(\Phi(w + tv) - \Phi(w))$$

and the generalized gradient of Φ at $u \in X$ is the subset of the dual space X^* given by

$$\partial\Phi(u) := \left\{u^* \in X^* \ :\ \langle u^*, v\rangle \le \Phi^0(u; v), \quad \forall\, v \in X\right\}.$$

A continuous and convex function $\Phi : X \to \mathbb{R}$ is locally Lipschitz and its generalized gradient $\partial\Phi : X \to 2^{X^*}$ coincides with the subdifferential of Φ in the sense of convex analysis. As another important example, if $\Phi : X \to \mathbb{R}$ is continuously differentiable, the generalized gradient of Φ is just the differential $D\Phi$ of Φ.

The preceding notions of subdifferentiability theory for locally Lipschitz functions are needed to handle the multivalued term $[\underline{g}(u), \overline{g}(u)]$ in problem (1.1). Given $g : \mathbb{R} \to \mathbb{R}$ satisfying $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, we introduce

$$G(t) = \int_0^t g(t)\,\mathrm{d}t \quad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

The function $G : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and one can show that the generalized gradient $\partial G(t)$ of G at any $t \in \mathbb{R}$ is the compact interval

$$\partial G(t) = [\underline{g}(t), \overline{g}(t)], \tag{2.2}$$

where $\underline{g}(t)$ and $\overline{g}(t)$ are the functions in (1.2) and (1.3), respectively (see, e.g., [3, Example 2.2.5]).

3. The degenerated p-Laplacian with weight

Here we provide basic facts on the underlying space and driving operator in problem (1.1). An extensive related material can be found in [5]. The notation $|\cdot|$ will stand for the absolute value and Euclidean norm.
We assume the following hypothesis formulated in [5, p. 26] on the weight $a \in L^1_{\text{loc}}(\Omega)$:

$$\text{(H1)} \qquad a^{-s} \in L^1(\Omega) \ \text{ for some } s \in \left(\frac{N}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right).$$

Given a real number $p \in (1, +\infty)$, a positive function $a \in L^1_{\text{loc}}(\Omega)$ satisfying condition (H1), and a bounded domain $\Omega \subset \mathbb{R}^N$ of Lebesgue measure $|\Omega|$, with a Lipschitz boundary $\partial\Omega$, we introduce the weighted space

$$W^{1,p}(a, \Omega) := \{u \in L^p(\Omega) : \int_\Omega a(x)|\nabla u(x)|^p dx < \infty\}, \tag{3.1}$$

which is a Banach space endowed with the norm

$$\|u\|_{W^{1,p}(a,\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall\, u \in W^{1,p}(a, \Omega).$$

Noticing that $C^\infty_c(\Omega) \subset W^{1,p}(a, \Omega)$, the space $W^{1,p}_0(a, \Omega)$ is defined to be the closure of $C^\infty_c(\Omega)$ in $W^{1,p}(a, \Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(a,\Omega)}$. Hence $W^{1,p}_0(a, \Omega)$ is a separable Banach space. The dual space of $W^{1,p}_0(a, \Omega)$ is denoted $W^{1,p}_0(a, \Omega)^*$.
With the number s in hypothesis (H1) we set

$$p_s = \frac{ps}{s+1}. \tag{3.2}$$

By hypothesis (H1) it holds $s \geq 1/(p-1)$. From (3.2) it follows that $p_s \geq 1$, $p_s < p$ and $p_s/(p - p_s) = s$. Then Hölder's inequality and hypothesis (H1) yield

$$\int_\Omega |\nabla u(x)|^{p_s} dx = \int_\Omega (a(x)^{\frac{p_s}{p}} |\nabla u(x)|^{p_s}) a(x)^{-\frac{p_s}{p}} dx$$
$$\leq \|a^{-s}\|^{\frac{1}{s+1}}_{L^1(\Omega)} \|u\|^{p_s}, \quad \forall\, u \in W^{1,p}(a, \Omega).$$

This implies that $W^{1,p}_0(a, \Omega)$ is continuously embedded into the classical (unweighted) Sobolev space $W^{1,p_s}_0(\Omega)$,

$$W^{1,p}_0(a, \Omega) \hookrightarrow W^{1,p_s}_0(\Omega). \tag{3.3}$$

In view of the Rellich-Kondrachov embedding theorem there is the compact embedding

$$W^{1,p_s}_0(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

The preceding assertion is true because with the critical exponent p_s^* (corresponding to p_s), that is,

$$p_s^* := \begin{cases} \frac{Np_s}{N-p_s} & \text{if } N > p_s, \\ +\infty & \text{if } N \le p_s, \end{cases}$$

the assumption $s > N/p$ in (H1) implies $p_s^* > p$. Consequently, due to (3.3), there is the compact embedding

$$W_0^{1,p}(a,\Omega) \hookrightarrow\hookrightarrow L^p(\Omega). \tag{3.4}$$

Thanks to (3.4) we can conclude that

$$\|u\| := \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall\, u \in W_0^{1,p}(a,\Omega), \tag{3.5}$$

is an equivalent norm on $W_0^{1,p}(a,\Omega)$. The norm on $W_0^{1,p}(a,\Omega)$ introduced in (3.5) will be used throughout the rest of the paper.
By assumption (H1) it is known that $a^{-s} \in L^1(\Omega)$ when $s \ge 1/(p-1)$. This gives $a^{-\frac{1}{p-1}} \in L^1(\Omega)$ by noting that

$$\begin{aligned} \int_\Omega a(x)^{-\frac{1}{p-1}} dx &= \int_{\{a(x)<1\}} a(x)^{-\frac{1}{p-1}} dx + \int_{\{a(x)\ge 1\}} a(x)^{-\frac{1}{p-1}} dx \\ &\le \int_\Omega a(x)^{-s} dx + |\Omega| < \infty. \end{aligned}$$

Then [5, Theorem 1.3]) ensures that the space $W_0^{1,p}(a,\Omega)$ is uniformly convex. In particular, $W_0^{1,p}(a,\Omega)$ is a reflexive space.
The (negative) degenerated p-Laplacian with the positive weight $a \in L^1_{\mathrm{loc}}(\Omega)$ is the nonlinear operator $-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ given by

$$\langle -\Delta_p^a u, v\rangle := \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \ \forall\, u,v \in W_0^{1,p}(a,\Omega). \tag{3.6}$$

The operator $-\Delta_p^a$ is well defined as seen through Hölder's inequality that

$$\begin{aligned} &\left|\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) dx\right| \\ &\le \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega a(x)|\nabla v(x)|^p dx\right)^{\frac{1}{p}} < \infty \end{aligned} \tag{3.7}$$

for all $u, v \in W_0^{1,p}(a,\Omega)$. The positive number

$$\lambda_1 := \inf_{u\in W_0^{1,p}(a,\Omega),\ u\neq 0} \frac{\int_\Omega a(x)|\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{3.8}$$

is the first eigenvalue of $-\Delta_p^a$ (refer to [5, Lemma 3.1]). The following proposition addresses essential properties of the operator $-\Delta_p^a$ introduced in (3.6).

Proposition 3.1. *Assume that condition (H1) for the weight $a \in L^1_{\mathrm{loc}}(\Omega)$ positive almost everywhere is satisfied. Then the (negative) degenerated p-Laplacian $-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ has the properties:*

(a) *The operator $-\Delta_p^a$ is bounded.*

(b) *The operator $-\Delta_p^a$ is strictly monotone, that is,*

$$\langle -\Delta_p^a u - (-\Delta_p^a v), u - v\rangle > 0 \tag{3.9}$$

for all $u, v \in W_0^{1,p}(a,\Omega)$ with $u \neq v$. Moreover, it holds

$$\langle -\Delta_p^a(u) - (-\Delta_p^a v), u - v\rangle \geq (\|u\| - \|v\|)(\|u\|^{p-1} - \|v\|^{p-1}) \tag{3.10}$$

for all $u, v \in W_0^{1,p}(a,\Omega)$.

(c) *The operator $-\Delta_p^a$ has the S_+ property, that is, any sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and*

$$\limsup_{n\to\infty} \langle -\Delta_p^a u_n, u_n - u\rangle \leq 0 \tag{3.11}$$

fulfills $u_n \to u$ in $W_0^{1,p}(a,\Omega)$.

(d) *The operator $-\Delta_p^a$ is continuous.*

Proof. (a) It turns out from (3.7) that if $\|u\| \leq M$, then

$$\| - \Delta_p^a u\|_{W_0^{1,p}(a,\Omega)^*} \leq M^{p-1},$$

so $-\Delta_p^a$ is a bounded operator.

(b) Let us first prove (3.10). Given $u, v \in W_0^{1,p}(a,\Omega)$, by Hölder's inequality and (3.5) we find that

$$\begin{aligned}
&\langle -\Delta_p^a u - (-\Delta_p^a v), u - v\rangle \\
&= \int_\Omega a(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla(u-v)dx \\
&\geq \int_\Omega a(x)|\nabla u|^p dx + \int_\Omega a(x)|\nabla v|^p dx \\
&- \int_\Omega (a(x)^{\frac{p-1}{p}}|\nabla u|^{\frac{p-1}{p}})(a(x)^{\frac{1}{p}}|\nabla v|)dx - \int_\Omega (a(x)^{\frac{p-1}{p}}|\nabla v|^{\frac{p-1}{p}})(a(x)^{\frac{1}{p}}|\nabla u|)dx \\
&\geq \|u\|^p + \|v\|^p - \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega a(x)|\nabla v|^p dx\right)^{\frac{1}{p}} \\
&- \left(\int_\Omega a(x)|\nabla v|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}} \\
&= \|u\|^p + \|v\|^p - \|u\|^{p-1}\|v\| - \|v\|^{p-1}\|u\| \\
&= (\|u\| - \|v\|)(\|u\|^{p-1} - \|v\|^{p-1}).
\end{aligned}$$

Therefore (3.10) holds true.

From (3.10) we note that $\langle -\Delta_p^a u - (-\Delta_p^a v), u - v\rangle \geq 0$ whenever $u, v \in W_0^{1,p}(a,\Omega)$. Suppose that

$$\langle -\Delta_p^a u - (-\Delta_p^a v), u - v\rangle = 0 \tag{3.12}$$

for some $u, v \in W_0^{1,p}(a,\Omega)$. By (3.10) we have $\|u\| = \|v\|$. Furthermore, (3.10) and (3.12) imply

$$0 = \langle -\Delta_p^a u - (-\Delta_p^a)(\frac{1}{2}(u+v)), \frac{1}{2}(u-v)\rangle$$

$$+\langle(-\Delta_p^a)(\frac{1}{2}(u+v)) - (-\Delta_p^a v), \frac{1}{2}(u-v)\rangle.$$

Again by (3.10), this leads to

$$\|u\| = \|v\| = \|\frac{1}{2}(u+v)\|.$$

The space $W_0^{1,p}(a,\Omega)$ being uniformly convex, it is strictly convex. Consequently, the equality above ensures that $u = v$, thus (3.9) is proven.

(c) Consider a sequence $\{u_n\}$ in $W_0^{1,p}(a,\Omega)$ complying with the conditions required in the statement. From $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and (3.11), we derive

$$\limsup_{n\to\infty}\langle -\Delta_p^a u_n - (-\Delta_p^a u), u_n - u\rangle \leq 0. \tag{3.13}$$

Then (3.9) and (3.13) yield

$$\lim_{n\to\infty}\langle -\Delta_p^a u_n - (-\Delta_p^a u), u_n - u\rangle = 0.$$

Since the right-hand side of inequality (3.10) is nonnegative, we infer from the preceding equality and (3.10) that $\lim_{n\to\infty}\|u_n\| = \|u\|$. We deduce that $u_n \to u$ in $W_0^{1,p}(a,\Omega)$ because the space $W_0^{1,p}(a,\Omega)$ is uniformly convex (see Section 2), thus reaching the desired conclusion.

(d) We now check the continuity of the operator

$$-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*.$$

To this end, let $u_n \to u$ in $W_0^{1,p}(a,\Omega)$. Using the Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\langle -\Delta_p^a u_n - (-\Delta_p^a)u, v\rangle\right| \\
&= \left|\int_\Omega (a(x)^{\frac{p-1}{p}}(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v))(a(x)^{\frac{1}{p}}\nabla v)dx\right| \\
&\leq \left(\int_\Omega a(x)\left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\|v\|
\end{aligned}$$

for all $v \in W_0^{1,p}(a,\Omega)$. This amounts to saying that

$$\begin{aligned}
&\left\|-\Delta_p^a u_n - (-\Delta_p^a u)\right\|_{W_0^{1,p}(a,\Omega)^*} \\
&\leq \left(\int_\Omega a(x)\left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}.
\end{aligned} \tag{3.14}$$

The strong convergence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$, in conjunction with the compact embedding (3.4), shows that $u_n \to u$ in $L^p(\Omega)$, so along a relabeled subsequence one has $u_n(x) \to u(x)$ almost everywhere in Ω. In addition, the strong convergence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$ provides that $a(x)^{\frac{1}{p}}|\nabla u_n| \to a(x)^{\frac{1}{p}}|\nabla u|$ in $L^p(\Omega)$, which permits to find an $h \in L^p_+(\Omega)$ such that

$$a(x)^{\frac{1}{p}}|\nabla u_n(x)| \le h(x) \quad \text{for a.e. } x \in \Omega.$$

Through a well-known convexity inequality, this reflects in

$$a(x)\left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}} \le 2^{\frac{1}{p-1}}(h(x)^p + a(x)|\nabla u(x)|^p) =: q(x),$$

with $q \in L^1(\Omega)$. We have checked that we are allowed to apply the Lebesgue's dominated convergence theorem to the integral in (3.14). We infer that $-\Delta_p^a u_n \to -\Delta_p^a u$ in $W_0^{1,p}(a,\Omega)^*$, which completes the proof. □

4. Nemytskii type and multivalued operators associated to problem (1.1)

In order to simplify the notation, we pose $p' := p/(p-1)$. We assume that the nonlinearity $f(x,t,\xi)$ satisfies the growth condition:

(H2) There exist $\sigma \in L^{p'}(\Omega)$ and constants $b_1 \ge 0$ and $b_2 \ge 0$ such that

$$|f(x,t,\xi)| \le \sigma(x) + b_1|t|^{p-1} + b_2 a(x)^{\frac{1}{p'}}|\xi|^{p-1} \text{ for a.e. } x \in \Omega,\ \forall\, t \in \mathbb{R},\ \forall\, \xi \in \mathbb{R}^N.$$

Consider the weighted space

$$L^p(a,\Omega,\mathbb{R}^N) := \{w : \Omega \to \mathbb{R}^N \text{ measurable} : \int_\Omega a(x)|w(x)|^p dx < \infty\}, \tag{4.1}$$

which is a Banach space endowed with the norm

$$\|w\|_{L^p(a,\Omega,\mathbb{R}^N)} := \left(\int_\Omega a(x)|w(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall\, w \in L^p(a,\Omega,\mathbb{R}^N).$$

The multiplication operator $M_a : L^p(a,\Omega,\mathbb{R}^N) \to L^p(\Omega,\mathbb{R}^N)$ defined by

$$M_a(w) := a^{\frac{1}{p}} w, \quad \forall\, w \in L^p(a,\Omega,\mathbb{R}^N), \tag{4.2}$$

is an isometry, i.e.,

$$\|w\|_{L^p(a,\Omega,\mathbb{R}^N)} = \|M_a(w)\|_{L^p(\Omega,\mathbb{R}^N)}, \quad \forall\, w \in L^p(a,\Omega,\mathbb{R}^N).$$

Lemma 4.1. *Assume that conditions (H1) and (H2) are satisfied. Then the Nemytskii type operator* $N_f : L^p(\Omega) \times L^p(a,\Omega,\mathbb{R}^N) \to L^{p'}(\Omega)$ *given by*

$$N_f(u,w) := f(\cdot,u,w), \quad \forall\, (u,w) \in L^p(\Omega) \times L^p(a,\Omega,\mathbb{R}^N), \tag{4.3}$$

is well defined, bounded and continuous.

Proof. Given $(u,w) \in L^p(\Omega)\times L^p(a,\Omega,\mathbb{R}^N)$, we have from (4.3), hypothesis (H2) and a well-known convexity inequality that

$$\int_\Omega |N_f(u,w)|^{p'}dx \le \int_\Omega \left(\sigma(x) + b_1|u|^{p-1} + b_2 a(x)^{\frac{1}{p'}}|w|^{p-1}\right)^{p'}dx$$

$$\le C\left(\|\sigma\|^{p'}_{L^{p'}(\Omega)} + \|u\|^p_{L^{p'}(\Omega)} + \|w\|^p_{L^p(a,\Omega,\mathbb{R}^N)}\right),$$

with a constant $C > 0$. It follows that the map N_f is well defined and bounded. For proving the continuity of the mapping N_f, we introduce the Carathéodory function $F : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ by

$$F(x,t,\xi) := f(x,t,a(x)^{-\frac{1}{p}}\xi), \quad \forall\ (x,t,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^N.$$

Based on hypothesis (H2), we obtain the estimate

$$|F(x,t,\xi)| = |f(x,t,a(x)^{-\frac{1}{p}}\xi)|$$

$$\le \sigma(x) + b_1|t|^{p-1} + b_2 a(x)^{\frac{1}{p'}}(a(x)^{-\frac{1}{p}}|\xi|)^{p-1}$$

$$= \sigma(x) + b_1|t|^{p-1} + b_2|\xi|^{p-1} \text{ for a.e. } x\in\Omega,\ \forall\ t\in\mathbb{R},\ \forall\ \xi\in\mathbb{R}^{\mathrm{N}}.$$

This estimate guarantees that Krasnoselkii's theorem can be applied to F ensuring that the Nemytskii operator $N_F : L^p(\Omega)\times L^p(\Omega,\mathbb{R}^N) \to L^{p'}(\Omega)$ given by

$$N_F(u,w) := F(\cdot,u,w), \quad \forall\ (u,w)\in L^p(\Omega)\times L^p(\Omega,\mathbb{R}^N),$$

is continuous. From (4.2) and (4.3) we note

$$N_F(u,M_a(w)) = N_f(u,w), \quad \forall\ (u,w)\in L^p(\Omega)\times L^p(\Omega,\mathbb{R}^N).$$

Hence N_f is a composition of continuous mappings, whence its continuity. □

Proposition 4.2. *Assume that conditions (H1) and (H2) are satisfied. Then the Nemytskii type operator* $\mathcal{N}_f : W_0^{1,p}(a,\Omega) \to L^{p'}(\Omega)$ *given by*

$$\mathcal{N}_f(u) := N_f(u,\nabla u), \quad \forall\ u\in W_0^{1,p}(a,\Omega), \tag{4.4}$$

is well defined, bounded and continuous.

Proof. If $u\in W_0^{1,p}(a,\Omega)$, by embedding (3.4) we have that $u\in L^p(\Omega)$ and by (3.1) and (4.1) that $\nabla u\in L^p(a,\Omega,\mathbb{R}^N)$. Therefore the definition of $\mathcal{N}_f(u)$ in (4.4) makes sense. The boundedness and continuity of the mapping $u\in W_0^{1,p}(a,\Omega)\mapsto(u,\nabla u)\in L^p(\Omega)\times L^p(a,\Omega,\mathbb{R}^N)$ follow directly from (3.4) and

$$\|u\| = \|\nabla u\|_{L^p(a,\Omega,\mathbb{R}^N)}, \quad \forall\ u\in W_0^{1,p}(a,\Omega)$$

(refer to (3.5)). Taking into account (4.4) and Lemma 4.1, the desired conclusion is achieved. □

Next we focus on the multivalued term in problem (1.1). To this end we formulate the assumption:

(H3) The function $g : \mathbb{R} \to \mathbb{R}$ satisfies $g \in L^{\infty}_{\rm loc}(\mathbb{R})$ and there exists a constant $c > 0$ such that

$$\max\{|\underline{g}(t)|, |\overline{g}(t)\})| \le c(1 + |t|^{p-1}) \ \text{ for a.e. } t \in \mathbb{R},$$

with $\underline{g}$ and $\overline{g}$ in (1.2) and (1.3), respectively. If g is continuous, the above condition reduces to

$$|g(t)| \le c(1 + |t|^{p-1}) \ \text{ for all } t \in \mathbb{R}.$$

The function $G : \mathbb{R} \to \mathbb{R}$ in (2.1) corresponding to $g \in L^{\infty}_{\rm loc}(\mathbb{R})$ is locally Lipschitz. Then, by Lebourg's mean value theorem (see [3, Theorem 2.3.7]) and hypothesis (H3), the functional $\Phi : L^p(\Omega) \to \mathbb{R}$ given by

$$\Phi(v) = \int_\Omega G(v(x))dx \ \text{ for all } v \in L^p(\Omega) \tag{4.5}$$

is Lipschitz continuous on the bounded subsets of $L^p(\Omega)$, thus locally Lipschitz. The generalized gradient $\partial\Phi(u)$ is a nonempty, closed and convex subset of $L^{p'}(\Omega)$ for every $u \in L^p(\Omega)$. Therefore the multivalued mapping $\partial\Phi : L^p(\Omega) \to 2^{L^{p'}(\Omega)}$ is well defined. Since $W^{1,p}_0(a,\Omega)$ is continuously and densely embedded in $L^p(\Omega)$, it can be regarded as a multivalued mapping $\partial\Phi : W^{1,p}_0(a,\Omega) \to 2^{L^{p'}(\Omega)}$ (see [3, p. 47]).

Proposition 4.3. *Assume that conditions (H1) and (H3) are satisfied. Then the multivalued mapping $\partial\Phi : W^{1,p}_0(a,\Omega) \to 2^{L^{p'}(\Omega)}$ is bounded. Moreover, it is sequentially weakly upper semicontinuous in the following sense: if the sequences $\{u_n\} \subset W^{1,p}_0(a,\Omega)$ and $\{\zeta_n\} \subset L^{p'}(\Omega)$ satisfy $u_n \rightharpoonup u$ in $W^{1,p}_0(a,\Omega)$ for some $u \in W^{1,p}_0(a,\Omega)$ and $\zeta_n \in \partial\Phi(u_n)$ for all n, then along a relabeled subsequence one has $\zeta_n \rightharpoonup \zeta$ in $L^{p'}(\Omega)$ with some $\zeta \in \partial\Phi(u)$.*

Proof. Let $u \in W^{1,p}_0(a,\Omega)$ and $w \in \partial\Phi(u)$. By applying the Aubin-Clarke theorem (see [3, Theorem 2.7.5]), we derive from (4.5) and (2.2) that

$$w(x) \in \partial G(u(x)) = [\underline{g}(u(x)), \overline{g}(u(x))] \quad \text{for a.e. } x \in \Omega. \tag{4.6}$$

Then (4.6), (3.8), and hypothesis (H3) yield

$$\begin{aligned}\|w\|^{p'}_{L^{p'}(\Omega)} &\le \int_\Omega (\max\{|\underline{g}(u(x))|, |\overline{g}(u(x)|\})^{p'} dx \\ &\le c^{p'} \int_\Omega (1 + |u(x)|^{p-1})^{p'} dx \\ &\le 2^{\frac{1}{p-1}} c^{p'} (|\Omega| + \|u\|^p_{L^p(\Omega)}) \le 2^{\frac{1}{p-1}} c^{p'} (|\Omega| + \lambda_1^{-1}\|u\|^p).\end{aligned}$$

Hence the multivalued mapping $\partial\Phi$ is bounded.
For the second part of the statement, let $\{u_n\} \subset W^{1,p}_0(a,\Omega)$ and $\{\zeta_n\} \subset L^{p'}(\Omega)$ be sequences satisfying $u_n \rightharpoonup u$ in $W^{1,p}_0(a,\Omega)$ with a $u \in W^{1,p}_0(a,\Omega)$ and $\zeta_n \in \partial\Phi(u_n)$ for all n. The compact embedding (3.4) renders $u_n \to u$ in $L^p(\Omega)$. As known from the first part, the sequence $\{\zeta_n\}$ is bounded in $L^{p'}(\Omega)$, whence due to the reflexivity we have along a relabeled subsequence $\zeta_n \rightharpoonup \zeta$ in $L^{p'}(\Omega)$ with some $\zeta \in L^{p'}(\Omega)$. The fact that the multifunction $\partial\Phi$ is weak*-closed (see [3, Proposition 2.1.5]) implies that $\zeta \in \partial\Phi(u)$, which completes the proof. □

5. Existence of solutions to problem (1.1)

In order to prove the solvability of problem (1.1), a new hypothesis linking (H2) and (H3) is needed:
(H4) There holds

$$b_1\lambda_1^{-1} + (b_2 + c)\lambda_1^{-\frac{1}{p}} < 1,$$

where the constants b_1 and b_2 enter (H2), and c is the constant in (H3).

Our existence result on problems (1.1) and (1.4) is as follows.

Theorem 5.1. *Assume that conditions (H1)-(H4) hold. Then problem* (1.1) *admits at least one solution. In particular, if the function g is continuous, then a solution to problem* (1.4) *exists.*

Proof. The proof is conducted by applying Theorem 2.1. Towards this, we introduce the multivalued operator $A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$ by

$$Au := -\Delta_p^a u - \mathcal{N}_f(u) - \partial\Phi(u) \quad \text{ for all } u \in W_0^{1,p}(a,\Omega). \tag{5.1}$$

Since one has $L^{p'}(\Omega) \subset W_0^{1,p}(a,\Omega)^*$, the multifunction A in (5.1) is well defined. We verify that all the hypotheses of Theorem 2.1 are fulfilled.

Proposition 3.1 (a) ensures that $-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^*$ is a bounded operator. By Proposition 4.2 we get that $\mathcal{N}_f : W_0^{1,p}(a,\Omega) \to L^{p'}(\Omega) \subset W_0^{1,p}(a,\Omega)^*$ is bounded, while by virtue of Proposition 4.3 we know that the multivalued mapping $\partial\Phi : W_0^{1,p}(a,\Omega) \to 2^{L^{p'}(\Omega)}$ is bounded. In view of (5.1), we infer that the multivalued operator $A : W_0^{1,p}(\Omega) \to 2^{W^{-1,p'}(\Omega)}$ is bounded.

The next step in the proof is to show that the multivalued operator

$$A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$$

is pseudomonotone. In line with this, let sequences

$$\{u_n\} \subset W_0^{1,p}(a,\Omega) \text{ and } \{u_n^*\} \subset W_0^{1,p}(a,\Omega)^*$$

satisfy $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$, $u_n^* \in Au_n$ for all n, and

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \leq 0. \tag{5.2}$$

Take an arbitrary subsequence of $\{u_n\}$ still denoted $\{u_n\}$ and the corresponding subsequence of $\{\zeta_n\}$. According to (5.1) it holds

$$\zeta_n \in \partial\Phi(u_n), \quad \forall\, n, \tag{5.3}$$

with

$$u_n^* = -\Delta_p^a u_n - \mathcal{N}_f(u_n) - \zeta_n. \tag{5.4}$$

Exploiting the fact that the values of $\mathcal{N}_f$ belong to $L^{p'}(\Omega)$, we have

$$|\langle \mathcal{N}_f(u_n), u_n - u\rangle| \leq \|\mathcal{N}_f(u_n)\|_{L^{p'}(\Omega)} \|u_n - u\|_{L^p(\Omega)}.$$

Due to the compact embeddings of $W_0^{1,p}(a,\Omega)$ into $L^p(\Omega)$ and the boundedness of $\{\mathcal{N}_f(u_n)\}$ in $L^{p'}(\Omega)$, the above estimate entails

$$\lim_{n\to\infty} \langle \mathcal{N}_f(u_n), u_n - u\rangle = 0. \tag{5.5}$$

Then it stems from (5.2), (5.4) and (5.5) that

$$\limsup_{n\to\infty} \langle -\Delta_p^a u_n - \zeta_n, u_n - u\rangle \le 0. \tag{5.6}$$

Based on hypothesis (H3) we can invoke Proposition 4.3 that provides a subsequence of $\{\zeta_n\}$ (so, a fortiori, a subsequence of $\{u_n\}$) along which $\zeta_n \rightharpoonup \zeta$ in $L^{p'}(\Omega)$, with some $\zeta \in \partial\Phi(u)$, whereas $u_n \to u$ in $L^p(\Omega)$. Using that the values of the multifunction $\partial\Phi$ are in $L^{p'}(\Omega)$, along the relabeled subsequence we obtain

$$\lim_{n\to\infty} \langle \zeta_n, u_n - u\rangle = 0. \tag{5.7}$$

Combining (5.6) and (5.7) results in (3.11). This enables us to apply Proposition 3.1 (c). Hence, up to a subsequence, $u_n \to u$ in $W_0^{1,p}(\Omega)$. Actually, the preceding reasoning shows that every subsequence of $\{u_n\}$ contains a subsequence strongly converging to u in $W_0^{1,p}(\Omega)$, which ensures for the entire sequence that $u_n \to u$ in $W_0^{1,p}(\Omega)$. By the continuity of the mappings

$$-\Delta_p^a : W_0^{1,p}(a,\Omega) \to W_0^{1,p}(a,\Omega)^* \text{ and } \mathcal{N}_f : W_0^{1,p}(a,\Omega) \to L^{p'}(\Omega) \subset W_0^{1,p}(a,\Omega)^*$$

(see Proposition 3.1 (d) and Proposition 4.2) we have $-\Delta_p^a u_n \to -\Delta_p^a u$ in $W_0^{1,p}(a,\Omega)^*$ and $\mathcal{N}_f(u_n) \to \mathcal{N}_f(u)$ in $W_0^{1,p}(a,\Omega)^*$.

Let $v \in W_0^{1,p}(a,\Omega)$. From (5.3) and (5.4), in conjunction with the preceding comments, we note

$$\begin{aligned}
&\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle \qquad (5.8)\\
&= \liminf_{n\to\infty} \langle -\Delta_p^a u_n - \mathcal{N}_f(u_n) - \zeta_n, u_n - v\rangle\\
&= \langle -\Delta_p^a u - \mathcal{N}_f(u), u - v\rangle + \liminf_{n\to\infty} \langle -\zeta_n, u_n - v\rangle\\
&= \langle -\Delta_p^a u - \mathcal{N}_f(u), u - v\rangle - \limsup_{n\to\infty} \langle \zeta_n, u_n - v\rangle.\\
&\ge \langle -\Delta_p^a u - \mathcal{N}_f(u), u - v\rangle - \max_{\zeta\in\partial\Phi(u)} \langle \zeta, u - v\rangle.
\end{aligned}$$

Recall that the set $\partial\Phi(u)$ in weak*-compact in $W_0^{1,p}(a,\Omega)^*$ (refer to [3, Proposition 2.1.2]), so there exists $\zeta(v) \in \partial\Phi(u)$ for which it holds

$$\max_{\zeta\in\partial\Phi(u)} \langle \zeta, u - v\rangle = \langle \zeta(v), u - v\rangle.$$

On the basis of (5.8), this confirms that the multivalued operator

$$A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$$

is pseudomonotone.

We now turn to show that $A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$ defined in (5.1) is coercive. From (4.6) and by applying the Aubin-Clarke theorem (see [3, Theorem 2.7.5]) to (4.5), which is possible thanks to hypothesis (H3), we find that

$$\langle \zeta, v\rangle = \int_\Omega \zeta(x)v(x)\,dx \le \int_\Omega |\zeta(x)||v(x)|\,dx \tag{5.9}$$

$$\le \int_\Omega c(1+|v(x)|^{p-1})|v(x)|\,dx$$

$$\le c\lambda_1^{-\frac{1}{p}}\|v\|^p + c_0\|v\|$$

whenever $v \in W_0^{1,p}(a,\Omega)$ and $\zeta \in \partial\Phi(v)$, with a positive constant c_0.
Let $v \in W_0^{1,p}(a,\Omega)$ and $v^* \in Av$. Due to (5.1), we can write

$$v^* = -\Delta_p^a v - \mathcal{N}_f(v) - \zeta,$$

with $\zeta \in \partial\Phi(v)$. Then, by (3.4), hypothesis (H2), Hölder's inequality, (5.9), and (3.8), it turns out

$$\begin{aligned}\langle v^*, v\rangle &= \langle -\Delta_p^a v - \mathcal{N}_f(v) - \zeta, v\rangle\\ &\ge \|v\|^p - \|\sigma\|_{L^{p'}(\Omega)}\|v\|_{L^p(\Omega)} - b_1\|v\|^p_{L^p(\Omega)} - b_2\|v\|^{p-1}\|v\|_{L^p(\Omega)}\\ &- c\lambda_1^{-\frac{1}{p}}\|v\|^p - c_0\|v\|\\ &\ge (1 - b_1\lambda_1^{-1} - (b_2+c)\lambda_1^{-\frac{1}{p}})\|v\|^p - (\|\sigma\|_{L^{p'}(\Omega)}\lambda_1^{-1} + c_0)\|v\|.\end{aligned}$$

Hypothesis (H4) postulates that $1 - b_1\lambda_1^{-1} - (b_2+c)\lambda_1^{-\frac{1}{p}} > 0$. Therefore, owing to $p > 1$, the function $\psi : \mathbb{R}_+ \to \mathbb{R}$ given by

$$\psi(t) = (1 - b_1\lambda_1^{-1} - (b_2+c)\lambda_1^{-\frac{1}{p}})t^{p-1} - \|\sigma\|_{L^{p'}(\Omega)}\lambda_1^{-1} - c_0, \quad \forall\, t \in \mathbb{R}_+,$$

satisfies $\psi(t) \to +\infty$ as $t \to +\infty$. Furthermore, it holds $\langle v^*, v\rangle \ge \psi(\|v\|)\|v\|$ for all $v \in W_0^{1,p}(a,\Omega)$ and $v^* \in Av$. This means that the multivalued operator

$$A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$$

is coercive (with $u_0 = 0$ in the definition of coerciveness in Section 2).
Since the multivalued operator $A : W_0^{1,p}(a,\Omega) \to 2^{W_0^{1,p}(a,\Omega)^*}$ defined in (5.1) is pseudomonotone, bounded and coercive, Theorem 2.1 is applicable, which provides (choosing $\eta = 0$ in the statement of Theorem 2.1) the existence of a $u \in W_0^{1,p}(a,\Omega)$ solving the equation $Au = 0$, or equivalently

$$\langle -\Delta_p^a u - \mathcal{N}_f(u) - \zeta, v\rangle = 0, \quad \forall\, v \in W_0^{1,p}(a,\Omega),$$

with some $\zeta \in \partial\Phi(u)$. Inserting the expressions of the operators $-\Delta_p^a$ and $\mathcal{N}_f$, and for $\partial\Phi$ refering to (4.6) with $w = \zeta$, we get (1.5) (equivalently, (1.6)). The fact that $\underline{g}(u),\ \overline{g}(u) \in L^{p'}(\Omega)$ follows from hypothesis (H3) and $u \in L^p(\Omega)$. We conclude that $u \in W_0^{1,p}(a,\Omega)$ is a solution of problem (1.1). If the function $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ is continuous, we have that $\underline{g}(u(x)) = \overline{g}(u(x))$ almost everywhere in Ω, so (1.5) (equivalently, (1.6)) becomes (1.7). The proof is thus complete. $\square$

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Dumitru Motreanu
Département de Mathématiques,
Université de Perpignan,
66860 Perpignan, France
e-mail: `motreanu@univ-perp.fr`

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Multiple solutions for eigenvalue problems involving the (p, q)–Laplacian

Patrizia Pucci

Dedicated to the memory of Professor Csaba Varga with high feelings of admiration for his notable contributions in Mathematics and great affection

Abstract. This paper is devoted to a subject that Professor Csaba Varga suggested during his frequent visits to the University of Perugia and in my regular stays at the "Babeş-Bolyai" University. More specifically, continuing the work started in [7] jointly with Professor Varga, here we establish the existence of two nontrivial (weak) solutions of some one parameter eigenvalue (p, q)–Laplacian problems under homogeneous Dirichlet boundary conditions in bounded domains of $\mathbb{R}^N$.

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1. Introduction

The paper concerns certain nonlinear eigenvalue homogeneous Dirichlet boundary condition problems in bounded domains Ω of $\mathbb{R}^N$, involving the (p, q)–Laplacian. Hence the subject is strongly connected with the paper [7], we wrote jointly with Professor Csaba Varga during his frequent visits to the University of Perugia and in my regular stays at the "Babeş-Bolyai" University. More specifically, continuing the work started in [7] for problems involving a general elliptic operator in divergence form with p growth, we extend the existence theorems of two nontrivial (weak) solutions of [7] to eigenvalue (p, q)–Laplacian problems. More specifically, we consider for a nonnegative real parameter λ the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda\{a(x)|u|^{q-2}u + f(x,u)\} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_\lambda)$$

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in a bounded domain Ω of $\mathbb{R}^N$ and we assume for simplicity that the exponents p, q are such that $1 < p < q < N$. The operator $\Delta_\wp$, with $\wp \in \{p, q\}$, appearing in problem $(\mathscr{P}_\lambda)$, is the well known $\wp$*–Laplacian*, which is defined as

$$\Delta_\wp \varphi = \operatorname{div}(|\nabla \varphi|_H^{\wp-2} \nabla \varphi) \quad \text{for all } \varphi \in C^2(\mathbb{R}^N).$$

Throughout the paper, *the weight a in* $(\mathscr{P}_\lambda)$ *is required to be positive a.e. in* Ω *and of class* $L^\alpha(\Omega)$*, with* $\alpha > N/q$. The nonlinear perturbation $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which satisfies the natural growth conditions $(\mathscr{F})$ from (a) to (c) given in Section 3, with part (c) of $(\mathscr{F})$ fairly technical, due to the complexity in handling the nonhomogeneous (p, q)– Laplacian.

In Section 4 we find up the exact intervals of λ's for which problem $(\mathscr{P}_\lambda)$ admits only the trivial solution and for which $(\mathscr{P}_\lambda)$ has at least two nontrivial solutions. More precisely, following the strategies introduced in [7], we prove the existence theorems for problem $(\mathscr{P}_\lambda)$, using as a crucial tool Theorem 2.1 of [7], which is a differentiable version and a variant of Theorem 3.4 in [1] due to Arcoya and Carmona.

For further previous contributions in the subject, beside [7], we mention the papers [11, 13] due to Varga, the latter related articles [6, 19], written at the University of Perugia. For noteworthy comments and an extensive bibliography as well as for applications of the well known three critical points theorems we refer to the monumental monograph [12] of Kristály, Rădulescu and Varga.

In Section 5 we treat the different nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(x, u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

for which the technical assumption $(\mathscr{F})$–(c) is replaced by the more direct transparent request $(\mathscr{F})$–(c'), which is much easier to verify. This straight approach first started in [7] and we show here that the technique can be extended to cover the nonhomogeneous case of the (p, q)–Laplacian as well.

The importance of studying problems involving operators with non standard growth conditions, or (p, q) operators, begins independently with the pioneering papers of Zhikov in 1986 and Marcellini in 1991. The (p, q) operators were introduced in order to describe the behavior of highly anisotropic materials, that is, materials whose properties change drastically from point to point. Since then, increasing attention has been focused on the study of existence, regularity and qualitative properties of the solutions of problems of this type. For a detailed historical presentation and for a wide list of contributions on the subject we refer to the recent paper [17] due to Mingione and Rădulescu, editors of the Special Issue *New developments in non–uniformly elliptic and nonstandard growth problems.*

Concerning PDEs applications, the (p, q)–Laplacian $\Delta_p + \Delta_q$ arises from the study of general reaction–diffusion equations with nonhomogeneous diffusion and transport aspects. These nonhomogeneous operators have applications in biophysics, plasma physics and chemical reactions, with double phase features, where the function u corresponds to the concentration term, and the differential operator represents the diffusion coefficient. For further details we mention [14] as well as [17] and references therein. Different eigenvalue problems for the (p, q)–Laplacian have been extensively studied in recent years. In the context of Dirichlet boundary conditions we refer to the papers [4] by Bobkov and Tanaka, [8] by Colasuonno and Squassina, [14] by Marano and Mosconi, [15, 16] by Marano, Mosconi and Papageorgiou, to the recent paper [20] due to Tanaka and finally to the references therein.

For (p, q)–Laplacian eigenvalue problems under various boundary conditions (Robin, Steklov, etc.) we quote the recent papers [2] by Barbu and Moroşanu and [18] by Papageorgiou, Qin and Rădulescu, as well as their wide bibliography.

Let us end the comments by noting that the results of this note can be extended to the equations of problems $(\mathscr{P}_\lambda)$ and $(\mathcal{P}_\lambda)$ under Robin boundary conditions, as obtained in [7] via a delicate consequence of the three critical points Theorem 2.1 of [7].

2. Preliminaries and auxiliary results for $(\mathscr{P}_\lambda)$

In this section we introduce the main notation and assumptions for $(\mathscr{P}_\lambda)$. Throughout the paper, $\cdot$ denotes the Euclidean inner product and $|\cdot|$ the corresponding Euclidean norm in any space $\mathbb{R}^n$, $n = 1, 2, \dots$.

Let $1 < p < q < N$ and let $\mathscr{A}_{p,q} : \mathbb{R}^N \to \mathbb{R}$ be the potential

$$\mathscr{A}_{p,q}(\xi) = \frac{1}{p}|\xi|^p + \frac{1}{q}|\xi|^q \quad \text{of} \quad \mathbf{A}_{p,q}(\xi) = |\xi|^{p-2}\xi + |\xi|^{q-2}\xi. \tag{2.1}$$

Then both $\mathscr{A}_{p,q}$ and $\mathbf{A}_{p,q}$ are continuous in $\mathbb{R}^N$, $\mathscr{A}_{p,q}$ is even and strictly convex in $\mathbb{R}^N$. Clearly, $\mathbf{A}_{p,q}(\xi) \cdot \xi \geq \mathscr{A}_{p,q}(\xi)$ for all $\xi \in \mathbb{R}^N$.

Lemma 3 of [10] can also be generalized in this framework and we use the proof of Lemma 2.4 of [7], adopting the notation in (2.1).

Lemma 2.1. *Let ξ, $(\xi_n)_n$ be in $\mathbb{R}^N$ such that*

$$(\mathbf{A}_{p,q}(\xi_n) - \mathbf{A}_{p,q}(\xi)) \cdot (\xi_n - \xi) \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Then $(\xi_n)_n$ converges to ξ.

Proof. First we assert that $(\xi_n)_n$ is bounded. Otherwise, up to a subsequence, still denoted by $(\xi_n)_n$, we would have $|\xi_n| \to \infty$. Hence, as $n \to \infty$

$$(\mathbf{A}_{p,q}(\xi_n) - \mathbf{A}_{p,q}(\xi)) \cdot (\xi_n - \xi) \sim \mathbf{A}_{p,q}(\xi_n)\xi_n = |\xi_n|^p + |\xi_n|^q \to \infty,$$

which is impossible by (2.2). Therefore, $(\xi_n)_n$ is bounded and possesses a subsequence, still denoted by $(\xi_n)_n$, which converges to some $\eta \in \mathbb{R}^N$. Thus $(\mathbf{A}_{p,q}(\eta) - \mathbf{A}_{p,q}(\xi)) \cdot (\eta - \xi) = 0$ by (2.2) and the strict convexity of $\mathscr{A}_{p,q}$ implies that $\eta = \xi$. This also shows that actually the entire sequence $(\xi_n)_n$ converges to ξ. $\square$

Since $1 < p < q < N$, the natural solution space of $(\mathscr{P}_\lambda)$ is the separable uniformly convex Sobolev space $W_0^{1,q}(\Omega)$, endowed with the usual norm $\|u\| = \|\nabla u\|_q$, being Ω a bounded domain of $\mathbb{R}^N$. From here on, any Lebesgue space $L^\wp(\Omega)$, $\wp \geq 1$, is equipped with the canonical norm $\|\cdot\|_\wp$, while $\wp'$ is the conjugate exponent of $\wp$. It is clear that $W^{-1,q'}(\Omega)$ is the dual space of $W_0^{1,q}(\Omega)$ and that $q^\star = Nq/(N-q)$ is the Sobolev critical exponent of $W_0^{1,q}(\Omega)$.

Lemma 2.2. *Let $\mathscr{A}_{p,q}$ be as in (2.1). Then the functional*

$$\Phi_{p,q}(u)=\int_\Omega \mathscr{A}_{p,q}(\nabla u(x))dx=\frac{\|\nabla u\|_p^p}{p}+\frac{\|u\|^q}{q},\qquad \Phi_{p,q}:W_0^{1,q}(\Omega)\to\mathbb{R},$$

is convex, weakly lower semicontinuous and of class $C^1(W_0^{1,q}(\Omega))$.

Moreover, $\Phi'_{p,q}:W_0^{1,q}(\Omega)\to W^{-1,q'}(\Omega)$ verifies the $(\mathscr{S}_+)$ condition, i.e., for every sequence $(u_n)_n\subset W_0^{1,q}(\Omega)$ such that $u_n\rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$ and

$$\limsup_{n\to\infty}\int_\Omega \mathbf{A}_{p,q}(\nabla u_n)\cdot(\nabla u_n-\nabla u)dx\le 0, \tag{2.3}$$

then $u_n\to u$ strongly in $W_0^{1,q}(\Omega)$.

Proof. A simple calculation shows that the functional $\Phi_{p,q}$ is convex and of class $C^1(W_0^{1,q}(\Omega))$. Hence, in particular $\Phi_{p,q}$ is weakly lower semicontinuous in $W_0^{1,q}(\Omega)$ by Corollary 3.9 of [5].

Let $(u_n)_n$ be a sequence in $W_0^{1,q}(\Omega)$ as in the statement. Then

$$\Phi_{p,q}(u)\le\liminf_n \Phi_{p,q}(u_n),$$

since $\Phi_{p,q}$ is weakly lower semicontinuous on $W_0^{1,q}(\Omega)$.

We claim that $\int_\Omega \mathbf{A}_{p,q}(\nabla u)\cdot(\nabla u_n-\nabla u)dx\to 0$ as $n\to\infty$. Indeed, since $u_n\rightharpoonup u$ in $W_0^{1,q}(\Omega)$ as $n\to\infty$, in particular $\nabla u_n\rightharpoonup\nabla u$ in $[L^q(\Omega)]^N$ and $\nabla u_n\rightharpoonup\nabla u$ in $[L^p(\Omega)]^N$ as $n\to\infty$. Moreover, (2.1) implies that $|\mathbf{A}_{p,q}(\nabla u)|\le|\nabla u|^{p-1}+|\nabla u|^{q-1}$, with clearly $|\nabla u|^{p-1}\in L^{p'}(\Omega)$ and $|\nabla u|^{q-1}\in L^{q'}(\Omega)$. This gives at once that

$$\int_\Omega \mathbf{A}_{p,q}(\nabla u)\cdot(\nabla u_n-\nabla u)dx=\int_\Omega|\nabla u|^{p-2}\nabla u\cdot(\nabla u_n-\nabla u)dx \\ +\int_\Omega|\nabla u|^{q-2}\nabla u\cdot(\nabla u_n-\nabla u)dx$$

tends to 0 as $n\to\infty$, as claimed.

Therefore, by convexity and (2.3) we get that

$$0\le\limsup_{n\to\infty}\int_\Omega(\mathbf{A}_{p,q}(\nabla u_n)-\mathbf{A}_{p,q}(\nabla u))\cdot(\nabla u_n-\nabla u)dx\le 0.$$

In other words,

$$\lim_{n\to\infty}\int_\Omega(\mathbf{A}_{p,q}(\nabla u_n)-\mathbf{A}_{p,q}(\nabla u))\cdot(\nabla u_n-\nabla u)dx=0,$$

that is the sequence $n\mapsto(\mathbf{A}_{p,q}(\nabla u_n)-\mathbf{A}_{p,q}(\nabla u))\cdot(\nabla u_n-\nabla u)\ge 0$ converges to 0 in $L^1(\Omega)$. Hence, up to a subsequence, still denoted in the same way,

$$(\mathbf{A}_{p,q}(\nabla u_n)-\mathbf{A}_{p,q}(\nabla u))\cdot(\nabla u_n-\nabla u)\to 0\quad\text{a.e. in }\Omega.$$

Lemma 2.1 gives that also $\nabla u_n\to\nabla u$ a.e. in Ω. In particular, the Brézis–Lieb theorem gives as $n\to\infty$

$$\|\nabla u\|_p^p=\|\nabla u_n\|_p^p-\|\nabla u_n-\nabla u\|_p^p+o(1),$$
$$\|\nabla u\|_q^q=\|\nabla u_n\|_q^q-\|\nabla u_n-\nabla u\|_q^q+o(1),$$

and (2.3) holds in the stronger form

$$\lim_{n\to\infty}\int_\Omega \mathbf{A}_{p,q}(\nabla u_n)\cdot(\nabla u_n-\nabla u)dx=0.$$

Consequently, the combination of the above facts implies that as $n\to\infty$

$$\begin{aligned}o(1)&=\int_\Omega \mathbf{A}_{p,q}(\nabla u_n)\cdot(\nabla u_n-\nabla u)dx\\&=\|\nabla u_n\|_p^p-\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u dx\\&\qquad+\|\nabla u_n\|_q^q-\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u dx\\&=\|\nabla u\|_p^p+\|\nabla u_n-\nabla u\|_p^p-\|\nabla u\|_p^p\\&\qquad+\|\nabla u\|_q^q+\|\nabla u_n-\nabla u\|_q^q-\|\nabla u\|_q^q+o(1)\\&=\|\nabla u_n-\nabla u\|_p^p+\|\nabla u_n-\nabla u\|_q^q+o(1),\end{aligned}$$

since

$$|\nabla u_n|^{p-2}\nabla u_n\rightharpoonup|\nabla u|^{p-2}\nabla u \text{ in } [L^{p'}(\Omega)]^N$$

and similarly

$$|\nabla u_n|^{q-2}\nabla u_n\rightharpoonup|\nabla u|^{q-2}\nabla u \text{ in } [L^{q'}(\Omega)]^N.$$

In particular, $\|\nabla u_n-\nabla u\|_q=o(1)$ as $n\to\infty$, that is $u_n\to u$ strongly in $W_0^{1,q}(\Omega)$, as required. □

3. Formulation of the problem $(\mathscr{P}_\lambda)$

The assumptions on the coefficient a make it a good Lebesgue weight. Thus, throughout the paper, for brevity in notation, we denote by $L^\wp(\Omega;a)$, $\wp\geq 1$, the weighted $\wp$–Lebesgue space equipped with the norm

$$\|u\|_{\wp,a}=\left(\int_\Omega a(x)|u(x)|^\wp dx\right)^{1/\wp}.$$

In this section, we study $(\mathscr{P}_\lambda)$, so that $1<p<q<N$, the set Ω is a bounded domain of $\mathbb{R}^N$, and the natural solution space for $(\mathscr{P}_\lambda)$ is $W_0^{1,q}(\Omega)$. Before introducing the main structural assumptions on f, let us recall some basic properties, following somehow [7].

Since $a\in L^\alpha(\Omega)$ and $\alpha>N/q$, the embedding $W_0^{1,q}\hookrightarrow\hookrightarrow L^{\alpha' q}(\Omega)$ is compact. Moreover, $L^{\alpha' q}(\Omega)\hookrightarrow L^q(\Omega;a)$ is continuous, being by the Hölder inequality $\|u\|_{q,a}^q\leq\|a\|_\alpha\|u\|_{\alpha' q}^q$ for all $u\in L^{\alpha' q}(\Omega)$. Hence, also the embedding $W_0^{1,q}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega;a)$ is compact.

Let λ_1 be *the first eigenvalue* of the problem

$$-\Delta_q u=\lambda\, a(x)|u|^{q-2}u$$

in $W_0^{1,q}(\Omega)$, that is λ_1 is defined by the Rayleigh quotient

$$\lambda_1 = \inf_{\substack{u\in W_0^{1,q}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^q dx}{\int_\Omega a(x)|u|^q dx} = \inf_{\substack{u\in W_0^{1,q}(\Omega)\\ u\neq 0}} \frac{\|u\|^q}{\|u\|_{q,a}^q}. \tag{3.1}$$

By Proposition 3.1 of [9], the infimum in (3.1) is achieved and $\lambda_1 > 0$. Denote by $u_1 \in W_0^{1,q}(\Omega)$ the normalized eigenfunction corresponding to λ_1, that is $\|u_1\|_{q,a} = 1$ and $\|u_1\|^q = \lambda_1$. In particular,

$$\lambda_1 \|u\|_{q,a}^q \le \|u\|^q \quad \text{for every } u \in W_0^{1,q}(\Omega). \tag{3.2}$$

On f we assume the next condition.

($\mathscr{F}$) *Let $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function in $\mathbb{R}$, $f \not\equiv 0$, satisfying the following properties.*

(a) *There exist two measurable functions f_0, f_1 on Ω and a real exponent $m \in (1,q)$, such that $0 \le f_0 \le C_f a$, $0 \le f_1 \le C_f a$ a.e. in Ω for some appropriate constant $C_f > 0$, and*

$$|f(x,s)| \le f_0(x) + f_1(x)|s|^{m-1} \quad \textit{for a.a. } x\in\Omega \textit{ and all } s\in\mathbb{R}.$$

(b) *There exists $\gamma \in (q, q^\star/\alpha')$ such that $\limsup_{s\to 0} \dfrac{|f(x,s)|}{a(x)|s|^{\gamma-1}} < \infty$, uniformly a.e. in Ω.*

(c) $\displaystyle\int_\Omega F(x,u_1(x))dx \ge \frac{1}{q'} + \frac{q'}{p\lambda_1}\|\nabla u_1\|_p^p$, *where u_1 is the first normalized eigenfunction defined above, $F(x,s) = \displaystyle\int_0^s f(x,t)dt$ and q' is the Hölder conjugate of q.*

Note that, in the literature, $a \in L^\infty(\Omega)$ in the more familiar and standard setting of the p–Laplacian, so that the exponent γ in ($\mathscr{F}$)–(b) belongs to the open interval $(p, p^\star)$. For further comments on p–growth problems, we refer to [7].

As shown in [7], conditions ($\mathscr{F}$)–(a) and (b) imply that $f(x,0) = 0$ for a.a. $x \in \Omega$, that by the L'Hôpital rule

$$\limsup_{s\to 0} \frac{|F(x,s)|}{a(x)|s|^\gamma} < \infty \quad \text{uniformly a.e. in } \Omega, \tag{3.3}$$

and finally that

$$S_f = \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|f(x,s)|}{a(x)|s|^{q-1}} \in \mathbb{R}^+ \tag{3.4}$$

is positive and finite by ($\mathscr{F}$)–(b) and the fact that $\gamma > q$. Moreover, $|f(x,s)|/a(x)|s|^{m-1} \le 2C_f|s|^{m-q}$ for a.a. $x \in \Omega$ and all s, with $|s| \ge 1$, by ($\mathscr{F}$)–(a). Thus,

$$\lim_{s\to\infty} \frac{|f(x,s)|}{a(x)|s|^{q-1}} = 0 \quad \text{uniformly a.e. in } \Omega,$$

since $1 < m < q$ by ($\mathscr{F}$)–(a).

Hence *the positive number*

$$\lambda_\star = \frac{\lambda_1}{1+S_f} \tag{3.5}$$

is well defined. Furthermore, by (3.4)

$$\operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|F(x,s)|}{a(x)|s|^q} = \frac{S_f}{q}. \tag{3.6}$$

The main result of the section is proved by using the underlying energy functional J_λ associated to the variational problem $(\mathscr{P}_\lambda)$. For later purposes, we write J_λ in the form

$$\begin{gathered} J_\lambda(u) = \Phi_{p,q}(u) + \lambda\Psi(u), \\ \Psi(u) = -\mathcal{H}(u), \quad \mathcal{H}(u) = \mathcal{H}_1(u) + \mathcal{H}_2(u), \\ \mathcal{H}_1(u) = \frac{1}{q}\|u\|_{q,a}^q, \quad \mathcal{H}_2(u) = \int_\Omega F(x,u(x))dx. \end{gathered} \tag{3.7}$$

Thanks to Lemma 2.2, $(\mathscr{F})$–(a) and (b) it is easy to see that the functional J_λ is well defined in $W_0^{1,q}(\Omega)$ and of class $C^1(W_0^{1,q}(\Omega))$. Furthermore,

$$\begin{aligned} \langle J_\lambda'(u), \varphi\rangle = &\int_\Omega \mathbf{A}_{p,q}(\nabla u(x)) \cdot \nabla\varphi(x)dx \\ &- \lambda \int_\Omega \left\{a(x)|u(x)|^{q-2}u(x) + f(x,u(x))\right\} \varphi(x)dx, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,q}(\Omega)$ and its dual space $W^{-1,q'}(\Omega)$. Therefore, the critical points $u \in W_0^{1,q}(\Omega)$ of the functional J_λ are exactly the (weak) solutions of problem $(\mathscr{P}_\lambda)$.

By convenience, for every $r \in (\inf_{u\in W_0^{1,q}(\Omega)} \Psi(u), \sup_{u\in W_0^{1,q}(\Omega)} \Psi(u))$ let us introduce the two functions

$$\varphi_1(r) = \inf_{u\in\Psi^{-1}(I_r)} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi_{p,q}(v) - \Phi_{p,q}(u)}{\Psi(u) - r}, \quad I_r = (-\infty, r), \tag{3.8}$$

$$\varphi_2(r) = \sup_{u\in\Psi^{-1}(I^r)} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi_{p,q}(v) - \Phi_{p,q}(u)}{\Psi(u) - r}, \quad I^r = (r, \infty). \tag{3.9}$$

If $\Psi(v) < 0$ at some $v \in W_0^{1,q}(\Omega)$, then *the crucial positive number*

$$\lambda^\star = \varphi_1(0) = \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi_{p,q}(u)}{\Psi(u)}, \qquad I_0 = (-\infty, 0), \tag{3.10}$$

is well defined.

The proof of the next result, as well as the proof on the main existence theorem for $(\mathscr{P}_\lambda)$, is where we use the technical assumption $(\mathscr{F})$–(c).

Lemma 3.1. *If $(\mathscr{F})$–(a), (b) and (c) hold, then $\Psi^{-1}(I_0)$ is non–empty and moreover $\lambda_\star \le \lambda^\star < \lambda_1$.*

Proof. From $(\mathscr{F})$–(c) and (3.7) it follows in particular that that $\Psi(u_1) < 0$, since

$$\mathcal{H}(u_1) > \frac{1}{q}, \quad \text{i.e. } u_1 \in \Psi^{-1}(I_0).$$

Hence, $\lambda^\star$ is well defined. Again by $(\mathscr{F})$–(c) and (3.7)

$$\begin{aligned}\lambda^\star = \varphi_1(0) &= \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi_{p,q}(u)}{\Psi(u)}\\ &\le \frac{\Phi_{p,q}(u_1)}{\mathcal{H}(u_1)} = \frac{\|\nabla u_1\|_p^p/p + \|\nabla u_1\|_q^q/q}{\|u_1\|_{q,a}^q/q + \int_\Omega F(x,u_1(x))dx}\\ &\le \frac{\|\nabla u_1\|_p^p/p + \|\nabla u_1\|_q^q/q}{1/q + 1/q' + q'\|\nabla u_1\|_p^p/p\lambda_1}\\ &< \frac{\|\nabla u_1\|_p^p/p}{q'\|\nabla u_1\|_p^p/p\lambda_1} + \frac{\|u_1\|^q}{q} = \lambda_1,\end{aligned}$$

as required. Finally, by (3.7), (3.6) and (3.2), for all $u \in W_0^{1,q}(\Omega)$, with $u \ne 0$, we have

$$\frac{\Phi_{p,q}(u)}{|\Psi(u)|} \ge \frac{\|u\|^q/q}{(1+S_f)\|u\|_{q,a}^q/q} \ge \frac{\lambda_1}{1+S_f} = \lambda_\star.$$

Hence, in particular $\lambda^\star \ge \lambda_\star$ by (3.10). □

Lemma 3.2. *If $(\mathscr{F})$–(a) holds, then the operators*

$$\mathcal{H}_1', \quad \mathcal{H}_2', \quad \Psi' : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$$

are compact and $\mathcal{H}_1$, $\mathcal{H}_2$, Ψ are sequentially weakly continuous in $W_0^{1,q}(\Omega)$.

The proof is mutatis mutandis the same as the proof of the similar Lemma 3.2 of [7] and so we omit it here.

Lemma 3.3. *If $(\mathscr{F})$–(a) holds, then the functional $J_\lambda = \Phi_{p,q} + \lambda\Psi$ is coercive in $W_0^{1,q}(\Omega)$ for every $\lambda \in I$, $I = (-\infty, \lambda_1)$.*

Proof. Clearly, $(\mathscr{F})$–(a) implies that

$$|F(x,s)| \le f_0(x)|s| + f_1(x)|s|^m/m \le f_0(x) + (f_0(x) + f_1(x)/m)|s|^m \tag{3.11}$$

for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$.

Fix $\lambda \in (-\infty, \lambda_1)$ and $u \in W_0^{1,q}(\Omega)$. Then, (3.2), (3.7), (3.11) and the Hölder inequality give

$$\begin{aligned}J_\lambda(u) &\ge \frac{1}{q}\|u\|^q - \frac{\lambda}{q}\|u\|_{q,a}^q - |\lambda|\int_\Omega |F(x,u)|dx\\ &\ge \frac{1}{q}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^q - |\lambda|\cdot\|f_0\|_1 - |\lambda|\cdot\|f_0 + f_1/m\|_\alpha\|u\|_{\alpha'm}^m\\ &\ge \frac{1}{q}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^q - |\lambda|C_1 - |\lambda|\,C_2\|u\|^m,\end{aligned}$$

where $C_1 = \|f_0\|_1$ and $C_2 = c_{\alpha'm}^m\|f_0 + f_1/m\|_\alpha$, where $c_{\alpha'm}$ denotes the Sobolev constant of the compact embedding $W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^{\alpha'm}(\Omega)$. Clearly $C_1 < \infty$, since

$f_0 \in L^\alpha(\Omega) \subset L^1(\Omega)$ by $(\mathscr{F})$–(a), being $\alpha > N/q > 1$ and Ω bounded. This shows the assertion, since $1 < m < q$ by $(\mathscr{F})$–(a). □

4. Main result for $(\mathscr{P}_\lambda)$

Following the strategies proposed in [7], here we prove the main theorem for the (p, q) problem $\mathscr{P}_\lambda$.

Theorem 4.1. *Let $\mathscr{A}_{p,q}$ be as in* (2.1) *and let $\lambda_\star$ and $\lambda^\star$ be as defined in* (3.5) *and in* (3.10), *respectively. Assume that $(\mathscr{F})$–(a) and (b) hold.*

(i) If $\lambda \in [0, \lambda_\star]$, then $(\mathscr{P}_\lambda)$ has only the trivial solution.

(ii) If also $(\mathscr{F})$–(c) holds, then problem $(\mathscr{P}_\lambda)$ admits at least two nontrivial solutions for every $\lambda \in (\lambda^\star, \lambda_1)$, where $\lambda^\star = \varphi_1(0) < \lambda_1$ by Lemma 3.1.

Proof. (i) Let $u \in W_0^{1,q}(\Omega)$ be a nontrivial solution of $(\mathscr{P}_\lambda)$ for some $\lambda \geq 0$. Then,

$$\int_\Omega \mathbf{A}_{p,q}(\nabla u) \cdot \nabla\varphi \, dx = \lambda \int_\Omega \{a(x)|u|^{q-2}u + f(x, u)\}\varphi \, dx$$

for all $\varphi \in W_0^{1,q}(\Omega)$. Take $\varphi = u$ and by (2.1), (3.2), (3.4) and (3.7)

$$\begin{aligned}
\lambda_1\|u\|^q < \lambda_1 \int_\Omega \mathbf{A}_{p,q}(\nabla u)\nabla u \, dx &= \lambda_1\lambda \int_\Omega \{a(x)|u|^q + f(x, u)u\}dx \\
&= \lambda_1\lambda \left(\|u\|_{q,a}^q + \int_\Omega \frac{f(x, u)}{a(x)|u|^{q-1}} a(x)|u|^q dx \right) \\
&\leq \lambda_1\lambda \left(1 + S_f\right) \|u\|_{q,a}^q \leq \lambda \left(1 + S_f\right) \|u\|^q.
\end{aligned}$$

Therefore $\lambda > \lambda_\star$ by (3.5), as required.

(ii) By (2.1) the functional $\Phi_{p,q}$ is convex. Moreover, $\Phi_{p,q}$ is weakly lower semicontinuous and $\Phi'_{p,q}$ verifies condition $(\mathscr{S}_+)$ in $W_0^{1,q}(\Omega)$, as already proved in Lemma 2.2. Furthermore, $\Psi' : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ is compact and Ψ is sequentially weakly continuous in $W_0^{1,q}(\Omega)$ by Lemma 3.2. Moreover, the functional J_λ is coercive for every $\lambda \in I$, where $I = (-\infty, \lambda_1)$, thanks to Lemma 3.3.

We claim that $\Psi(W_0^{1,q}(\Omega)) \supset \mathbb{R}_0^- = (-\infty, 0]$. Indeed, $\Psi(0) = 0$ and $(\mathscr{F})$–(a) and (3.11) imply that

$$|F(x, s)| \leq f_0(x) + (1 + 1/m)C_f a(x)|s|^m$$

for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$. Hence, the Hölder inequality gives

$$\begin{aligned}
\Psi(u) &\leq -\frac{1}{q}\|u\|_{q,a}^q + \int_\Omega |F(x, u)|dx \\
&\leq -\frac{1}{q}\|u\|_{q,a}^q + \|f_0\|_1 + 2C_f \int_\Omega a(x)|u|^m dx \\
&\leq -\frac{1}{q}\|u\|_{q,a}^q + \|f_0\|_1 + 2C_f\|a\|_1^{(q-m)/q}\|u\|_{q,a}^m,
\end{aligned}$$

since $a \in L^1(\Omega)$, being $\alpha > N/q > 1$ and Ω bounded. Therefore,

$$\lim_{\substack{\|u\|_{q,a}\to\infty \\ u\in W_0^{1,q}(\Omega)}} \Psi(u) = -\infty,$$

thanks to the restriction $1 < m < q$ in assumption $(\mathscr{F})$–(a). Hence, the claim follows by the continuity of Ψ in $W_0^{1,q}(\Omega)$ and by (3.2).

Thus, $(\inf_{W_0^{1,q}(\Omega)} \Psi, \sup_{W_0^{1,q}(\Omega)} \Psi) \supset \mathbb{R}_0^-$. By (3.8) for every $u \in \Psi^{-1}(I_0)$ we have

$$\varphi_1(r) \le \frac{\Phi_{p,q}(u)}{r - \Psi(u)} \quad \text{for all } r \in (\Psi(u), 0),$$

so that

$$\limsup_{r\to 0^-} \varphi_1(r) \le -\frac{\Phi_{p,q}(u)}{\Psi(u)} \quad \text{for all } u \in \Psi^{-1}(I_0).$$

In other words, by (3.10)

$$\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0) = \lambda^\star. \tag{4.1}$$

From $(\mathscr{F})$–(a) and (b), that is (3.3) and (3.4), it follows the existence of a positive real number $\kappa > 0$ such that

$$|F(x,s)| \le \kappa\, a(x)|s|^\gamma \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}. \tag{4.2}$$

To this aim, denoting by ℓ_0 the limit number in (3.3), there exists $\delta > 0$ such that $|F(x,s)| \le (\ell_0 + 1)a(x)|s|^\gamma$ for a.a. $x \in \Omega$ and all s, with $|s| < \delta$. Fix s, with $|s| \ge \delta$, then by (3.6) for a.a. $x \in \Omega$

$$|F(x,s)| \le \frac{S_f}{q}|s|^{q-\gamma}a(x)|s|^\gamma \le \frac{S_f\delta^{q-\gamma}}{q}a(x)|s|^\gamma,$$

being $\gamma > q$ by $(\mathscr{F})$–(b). Hence, $\kappa = \max\{\ell_0 + 1, S_f\delta^{q-\gamma}/q\}$ and (4.2) holds.
We note in passing that the embedding $W_0^{1,q}(\Omega) \hookrightarrow L^\gamma(\Omega; a)$ is continuous. Indeed, by the Hölder inequality, with $1/\wp + 1/\alpha + \gamma/q^\star = 1$, where $\wp$ is the crucial exponent

$$\wp = \frac{\alpha' q^\star}{q^\star - \gamma\alpha'} > 1,$$

being $\gamma \in (q, q^\star/\alpha')$, as assumed in $(\mathscr{F})$–(b), we have

$$\int_\Omega a(x)|u|^\gamma dx \le |\Omega|^{1/\wp}\|a\|_\alpha \|u\|_{q^\star}^\gamma \le \tilde{C}\|u\|^\gamma, \tag{4.3}$$

where $\tilde{C} = c_{q^\star}^\gamma |\Omega|^{1/\wp}\|a\|_\alpha$ and $c_{q^\star}$ is the Sobolev constant for the continuous embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^\star}(\Omega)$.

Hence, by (3.2), (3.7), (4.2) and (4.3) for every $u \in W_0^{1,q}(\Omega)$, we get

$$|\Psi(u)| \le \frac{1}{q\lambda_1}\|u\|^q + C_\gamma\|u\|^\gamma, \tag{4.4}$$

where $C_\gamma = \tilde{C}\,\kappa$. Therefore, given $r < 0$ and $v \in \Psi^{-1}(r)$ we have by (2.1)

$$r = \Psi(v) \ge -\frac{1}{q\lambda_1}\|v\|^q - C_\gamma\|v\|^\gamma \ge -\frac{1}{\lambda_1}\Phi_{p,q}(v) - \ell\,\Phi_{p,q}(v)^{\gamma/q}, \tag{4.5}$$

where $\ell = C_\gamma q^{\gamma/q}$.

Since the functional $\Phi_{p,q}$ is bounded below, coercive and lower semicontinuous on the reflexive Banach space $W_0^{1,q}(\Omega)$, it is easy to see that $\Phi_{p,q}$ is also coercive on the sequentially weakly closed non–empty set $\Psi^{-1}(r)$ thanks to Lemma 3.2. Therefore, by Theorem 6.1.1 of [3], there exists an element $u_r \in \Psi^{-1}(r)$ such that

$$\Phi_{p,q}(u_r) = \inf_{v \in \Psi^{-1}(r)} \Phi_{p,q}(v).$$

By (3.9), we get

$$\varphi_2(r) \geq -\frac{1}{r} \inf_{v \in \Psi^{-1}(r)} \Phi_{p,q}(v) = \frac{\Phi_{p,q}(u_r)}{|r|},$$

being $u \equiv 0 \in \Psi^{-1}(I^r)$. From (4.5) we obtain

$$\begin{aligned} 1 &\leq \frac{1}{\lambda_1} \cdot \frac{\Phi_{p,q}(u_r)}{|r|} + \ell |r|^{\gamma/p-1} \left(\frac{\Phi_{p,q}(u_r)}{|r|} \right)^{\gamma/q} \\ &\leq \frac{\varphi_2(r)}{\lambda_1} + \ell |r|^{\gamma/q-1} \varphi_2(r)^{\gamma/q}. \end{aligned} \tag{4.6}$$

There are now two possibilities to be considered. Either φ_2 is locally bounded at 0^-, so that the above inequality shows at once that

$$\liminf_{r \to 0^-} \varphi_2(r) \geq \lambda_1, \tag{4.7}$$

being $\gamma > q$ by $(\mathscr{F})$–(b), or $\limsup_{r \to 0^-} \varphi_2(r) = \infty$. In both cases, (4.1) and Lemma 3.1 yield that

$$\limsup_{r \to 0^-} \varphi_1(r) \leq \lambda^\star < \lambda_1 \leq \limsup_{r \to 0^-} \varphi_2(r).$$

Hence, for all integers $n \geq n^\star = 1 + [2/(\lambda_1 - \lambda^\star)]$ there exists a number $r_n < 0$ so close to 0^- that $\varphi_1(r_n) < \lambda^\star + 1/n < \lambda_1 - 1/n < \varphi_2(r_n)$. In particular,

$$[\lambda^\star + 1/n, \lambda_1 - 1/n] \subset (\varphi_1(r_n), \varphi_2(r_n)) = (\varphi_1(r_n), \varphi_2(r_n)) \cap I \tag{4.8}$$

for all $n \geq n^\star$, where $I = (-\infty, \lambda_1)$ is the interval of λ's on which J_λ is coercive in $W_0^{1,q}(\Omega)$ by Lemma 3.3. Therefore, since all the assumptions of Theorem 2.1, Part (a) of (ii) of [7] are satisfied and $u \equiv 0$ is a critical point of J_λ, problem $(\mathscr{P}_\lambda)$ admits at least two nontrivial solutions for all $\lambda \in (\varphi_1(r_n), \varphi_2(r_n))$ and all $n \geq n^\star$. In conclusion, problem $(\mathscr{P}_\lambda)$ admits at least two nontrivial solutions for all $\lambda \in (\lambda^\star, \lambda_1)$, since

$$(\lambda^\star, \lambda_1) = \bigcup_{n=n^\star}^{\infty} [\lambda^\star + 1/n, \lambda_1 - 1/n] \subset \bigcup_{n=n^\star}^{\infty} (\varphi_1(r_n), \varphi_2(r_n))$$

by (4.8). □

5. The nonlinear eigenvalue problem $(\mathcal{P}_\lambda)$

In this last section we treat the different somehow simpler nonlinear eigenvalue problem $(\mathcal{P}_\lambda)$, for which the involved assumption $(\mathscr{F})$–(c) is replaced by a more direct transparent request, which is much easier to verify.

To this aim, let us denote by

$$B_0 = \{x \in \mathbb{R}^N : |x - x_0| \le r_0\}$$

the closed ball of $\mathbb{R}^N$ centered at a point $x_0 \in \mathbb{R}^N$ and of radius $r_0 > 0$. As in the previous paper [7], for the somehow simpler problem $(\mathcal{P}_\lambda)$, the *ad hoc* hypothesis $(\mathscr{F})$–(c) is replaced by the less stringent condition

$(\mathscr{F})$–(c') *Assume that there exist* $x_0 \in \Omega$, $s_0 \in \mathbb{R}$ *and* $r_0 > 0$ *so small that* $B_0 \subset \Omega$ *and*

$$\operatorname*{ess\,inf}_{B_0} F(x, |s_0|) = \mu_0 > 0, \qquad \operatorname*{ess\,sup}_{B_0} \max_{|t| \le |s_0|} |F(x,t)| = M_0 < \infty.$$

Clearly, when f does not depend on x, condition $(\mathscr{F})$–(c') simply reduces to the request that $F(s_0) > 0$ at a point $s_0 \in \mathbb{R}$, as first assumed in [11] by Kristály, Lisei and Varga. In this new setting, we derive the next result which improves the main theorem of [11] and extends Corollary 3.6 of [7] to the (p,q)–Laplacian case.

Theorem 5.1. *Let* $\mathscr{A}_{p,q}$ *be as in* (2.1) *and let* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *satisfy conditions* $(\mathscr{F})$–(a) *and* (b).

- (i) *If* $\lambda \in [0, \ell_\star]$, *where* $\ell_\star = \lambda_1/S_f$, *then problem* $(\mathcal{P}_\lambda)$ *has only the trivial solution.*
- (ii) *If furthermore* f *verifies* $(\mathscr{F})$–(c'), *then there exists* $\ell^\star \ge \ell_\star$ *such that* $(\mathcal{P}_\lambda)$ *admits at least two nontrivial solutions for all* $\lambda \in (\ell^\star, \infty)$.

Proof. Using the notation of (2.1) and Lemma 2.2, the energy functional $\mathcal{J}_\lambda$, associated to problem $(\mathcal{P}_\lambda)$, is given by $\mathcal{J}_\lambda = \Phi_{p,q} + \lambda\Psi_2$, where $\Phi_{p,q}$ is defined in Lemma 2.2 and

$$\Psi_2(u) = -\int_\Omega F(x, u(x))dx \quad \text{for all } u \in W_0^{1,q}(\Omega).$$

First, note that $\mathcal{J}_\lambda$ is coercive in $W_0^{1,q}(\Omega)$ for every $\lambda \in \mathbb{R}$. Indeed, as shown in the proof of Lemma 3.3, by (2.1) for all $u \in W_0^{1,q}(\Omega)$

$$\mathcal{J}_\lambda(u) \ge \frac{1}{q}\|u\|^q - |\lambda| \int_\Omega |F(x,u)|dx \ge \frac{1}{q}\|u\|^q - |\lambda| C_1 - |\lambda|\, C_2 \|u\|^m,$$

where $C_1 = \|f_0\|_1$, $C_2 = c_{\alpha' m}^m \|f_0 + f_1/m\|_\alpha$ and $c_{\alpha' m}$ denotes as before the Sobolev constant of the compact embedding $W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^{\alpha' m}(\Omega)$. This shows the claim, since $1 < m < q$ by $(\mathscr{F})$–(a). Hence, here $I = \mathbb{R}$.

(i) This part of the statement is proved using the same argument produced for the proof of Theorem 4.1–(i). Let $u \in W_0^{1,q}(\Omega)$ be a nontrivial solution of $(\mathcal{P}_\lambda)$ for some $\lambda \ge 0$. Then, by (2.1) and (3.4)

$$\begin{aligned}\lambda_1 \|u\|^q &< \lambda_1 \int_\Omega \mathbf{A}_{p,q}(\nabla u) \cdot \nabla u dx = \lambda_1 \lambda \int_\Omega f(x,u) u dx \le \lambda_1 \lambda S_f \|u\|_{q,a}^q \\ &\le \lambda S_f \|u\|^q\end{aligned}$$

thanks to (3.2). Thus, if u is a nontrivial (weak) solution of $(\mathcal{P}_\lambda)$, then necessarily $\lambda > \ell_\star = \lambda_1/S_f$, as required.

(ii) The proof of this part is again strongly based on an application of Theorem 2.1, Part (a) of (ii) of [7] and the fact that $u \equiv 0$ is a critical point of $\mathcal{J}_\lambda$. The new key functions φ_1 and φ_2 are now given by

$$\begin{aligned}\varphi_1(r) &= \inf_{u\in\Psi_2^{-1}(I_r)} \frac{\inf\limits_{v\in\Psi_2^{-1}(r)} \Phi_{p,q}(v)-\Phi_{p,q}(u)}{\Psi_2(u)-r}, \quad I_r=(-\infty,r),\\ \varphi_2(r) &= \sup_{u\in\Psi_2^{-1}(I^r)} \frac{\inf\limits_{v\in\Psi_2^{-1}(r)} \Phi_{p,q}(v)-\Phi_{p,q}(u)}{\Psi_2(u)-r}, \quad I^r=(r,\infty).\end{aligned} \tag{5.1}$$

We first show that there exists $u_0 \in W_0^{1,q}(\Omega)$ such that $\Psi_2(u_0) < 0$, so that the crucial number

$$\ell^\star = \varphi_1(0) = \inf_{u\in\Psi_2^{-1}(I_0)} -\frac{\Phi_{p,q}(u)}{\Psi_2(u)}, \quad I_0 = (-\infty, 0), \tag{5.2}$$

is well defined. To this aim, take $\sigma \in (0,1)$ and put

$$B = \{x \in \mathbb{R}^N : |x-x_0| \le \sigma r_0\}, \quad B_1 = \left\{x \in \mathbb{R}^N : |x-x_0| \le r_1\right\},$$

where $r_1 = (1+\sigma)r_0/2$. Hence,

$$B \subset B_1 \subset B_0 \subset \Omega.$$

Clearly, $F(x,0)=0$ a.e. in Ω, so that $s_0 \neq 0$ in $(\mathscr{F})$–(c'). Put $v_0 = |s_0|\chi_{B_1}$ in Ω and fix ε, with $0<\varepsilon<(1-\sigma)r_0/2$. Denote by ρ_ε the convolution kernel of fixed radius ε and define

$$u_0 = \rho_\varepsilon * v_0 \quad \text{in } \Omega.$$

Hence, $u_0 \equiv |s_0|$ in B, $0 \le u_0 \le |s_0|$ in Ω, $u_0 \in C_c^\infty(\Omega)$ and $\operatorname{supp} u_0 \subset B_0$. Therefore, $u_0 \in W_0^{1,q}(\Omega)$. By $(\mathscr{F})$–(c'),

$$\begin{aligned}\Psi_2(u_0) &= -\int_B F(x,|s_0|)dx - \int_{B_0\setminus B} F(x,u_0(x))dx \le M_0\int_{B_0\setminus B} dx - \mu_0\int_B dx\\ &\le \omega_N r_0^N\left[M_0(1-\sigma^N) - \mu_0\sigma^N\right],\end{aligned}$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$. Then, taking $\sigma \in (0,1)$ so close to 1^- that $\sigma^N > M_0/(\mu_0+M_0)$, we get that $\Psi_2(u_0)<0$, as claimed.

Furthermore, by (3.6) and (3.2), for all $u \in W_0^{1,q}(\Omega)$, with $u \not\equiv 0$, we easily obtain that

$$\frac{\Phi_{p,q}(u)}{|\Psi_2(u)|} \ge \frac{\|u\|^q/q}{S_f\|u\|_{q,a}^q/q} \ge \frac{\lambda_1}{S_f} = \ell_\star.$$

Hence, $\ell^\star \ge \ell_\star$ by (5.2).

In particular, for all $u \in \Psi_2^{-1}(I_0)$, we have by (5.1)

$$\varphi_1(r) \le \frac{\Phi_{p,q}(u)}{r-\Psi_2(u)} \quad \text{for all } r \in (\Psi_2(u), 0).$$

Hence, (4.1) holds in the form $\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0) = \ell^\star$, where now $\varphi_1(0)$ is given by (5.1) and (5.2). Also in this new setting (4.2) and (4.3) are still valid and (4.4) simply reduces to

$$|\Psi_2(u)| \le C_\gamma \|u\|^\gamma \quad \text{for all } u \in W_0^{1,q}(\Omega),$$

with the same constant $C_\gamma > 0$. Taking $r < 0$ and $v \in \Psi_2^{-1}(r)$, we get

$$r = \Psi_2(v) \ge -C_\gamma \|v\|^\gamma \ge -C_\gamma (q\Phi_{p,q}(v))^{\gamma/q}.$$

Therefore, by (5.1), since $u \equiv 0 \in \Psi_2^{-1}(I^r)$,

$$\varphi_2(r) \ge \frac{1}{|r|} \inf_{v\in\Psi_2^{-1}(r)} \Phi_{p,q}(v) \ge \kappa |r|^{q/\gamma - 1},$$

where $\kappa = C_\gamma^{-q/\gamma}/q$. This implies that $\lim_{r\to 0^-} \varphi_2(r) = \infty$, being $\gamma > q$ by $(\mathscr{F})$–(b). In conclusion, we have proved that

$$\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0) = \ell^\star < \lim_{r\to 0^-} \varphi_2(r) = \infty. \tag{5.3}$$

This shows that for all integers $n \ge n^\star = 2 + [\ell^\star]$ there exists $r_n < 0$ so close to 0^- that $\varphi_1(r_n) < \ell^\star + 1/n < n < \varphi_2(r_n)$. Hence, since all the assumptions of Theorem 2.1, Part (a) of (ii) of [7] are satisfied and $u \equiv 0$ a critical point of $\mathcal{J}_\lambda$, problem $(\mathcal{P}_\lambda)$ admits at least two nontrivial solutions for all

$$\lambda \in \bigcup_{n=n^\star}^{\infty} (\varphi_1(r_n), \varphi_2(r_n)) \supset \bigcup_{n=n^\star}^{\infty} [\ell^\star + 1/n, n] = (\ell^\star, \infty),$$

since here $I = \mathbb{R}$ is the interval of λ's in which the main functional $\mathcal{J}_\lambda$ is coercive in $W_0^{1,q}(\Omega)$. □

It is apparent from the main definitions (3.5), (3.10), Theorem 5.1 and (5.2) that $0 < \lambda_\star < \ell_\star \le \ell^\star \le \lambda^\star$. Hence, Theorem 5.1 provides also the useful information that $0 < \lambda_\star < \lambda^\star$.

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Patrizia Pucci
Università degli Studi di Perugia,
Dipartimento di Matematica e Informatica,
Via L. Vanvitelli, 1, 06123 Perugia, Italy
e-mail: `patrizia.pucci@unipg.it`

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Monotonicity with respect to p of the best constants associated with Sobolev immersions of type $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ when $q \in \{1, p, \infty\}$

Mihai Mihăilescu and Denisa Stancu-Dumitru

In memory of our good friend and collaborator Prof. Csaba Varga

Abstract. The goal of this paper is to collect some known results on the monotonicity with respect to p of the best constants associated with Sobolev immersions of type $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ when $q \in \{1, p, \infty\}$. More precisely, letting

$$\lambda(p, q; \Omega) := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \| \, |\nabla u|_D \, \|_{L^p(\Omega)} \|u\|_{L^q(\Omega)}^{-1} ,$$

we recall some monotonicity results related with the following functions

$$\begin{aligned} (1, \infty) \ni p &\mapsto |\Omega|^{p-1} \lambda(p, 1; \Omega)^p , \\ (1, \infty) \ni p &\mapsto \lambda(p, p; \Omega)^p , \\ (D, \infty) \ni p &\mapsto \lambda(p, \infty; \Omega)^p , \end{aligned}$$

when $\Omega \subset \mathbb{R}^D$ is a given open, bounded and convex set with smooth boundary.

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1. Introduction

1.1. Goal of the paper

For each open and bounded set $\Omega \subset \mathbb{R}^D$ ($D \geq 1$) the following continuous Sobolev immersions hold true $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, (see, e.g. H. Brezis [9, pp. 284-285 & 212-213]) for each $p \in [1, \infty)$ and each q that satisfies the following restrictions

$$q \in \begin{cases} \left[1, \dfrac{Dp}{D-p}\right], & \text{if } p \in [1, D) \ \& \ D \geq 2 , \\ [1, \infty), & \text{if } p = D \geq 2 , \\ [1, \infty], & \text{if } p \in (D, \infty) \ \& \ D \geq 2 \quad \text{or} \quad p \in [1, \infty] \ \& \ D = 1 . \end{cases}$$

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It follows that, for each Ω, p and q as above there exists a constant $c(p,q;\Omega) > 0$ such that

$$c(p,q;\Omega)\|u\|_{L^q(\Omega)} \leq \|\, |\nabla u|_D \,\|_{L^p(\Omega)}, \quad \forall\, u \in W_0^{1,p}(\Omega)\,.$$

Let $\lambda(p,q;\Omega)$ be the best constant in the above inequality, namely

$$\lambda(p,q;\Omega) := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\, |\nabla u|_D \,\|_{L^p(\Omega)}}{\|u\|_{L^q(\Omega)}}\,. \tag{1.1}$$

The goal of this paper is to recall certain results concerning some monotonicity properties of $\lambda(p,q;\Omega)$ with respect to p when $q \in \{1, p, \infty\}$ and Ω are fixed. More precisely, we will present some monotonicity results related with the following functions

$$(1,\infty) \ni p \mapsto |\Omega|^{p-1}\lambda(p,1;\Omega)^p\,, \tag{1.2}$$

$$(1,\infty) \ni p \mapsto \lambda(p,p;\Omega)^p\,, \tag{1.3}$$

$$(D,\infty) \ni p \mapsto \lambda(p,\infty;\Omega)^p\,, \tag{1.4}$$

when $\Omega \subset \mathbb{R}^D$ is a given open, bounded and convex set with smooth boundary.

1.2. Notations

For each positive integer $D \geq 2$ denot by $|\cdot|_D$ the Euclidean norm on $\mathbb{R}^D$. For each subset $\Omega \subset \mathbb{R}^D$, let $\partial\Omega$ be its boundary and denote by $|\partial\Omega|$ and $|\Omega|$, the $(D-1)$-dimensional Lebesgue perimeter of $\partial\Omega$ and the D-dimensional Lebesgue volume of Ω, respectively. Next, for each positive integer $D \geq 1$ define

$$\mathbb{P}^D \;:=\; \{\Omega \subset \mathbb{R}^D : \Omega \text{ is an open, bounded, convex set with smooth boundary } \partial\Omega\}\,,$$

and for each $\Omega \in \mathbb{P}^D$ let δ_Ω be the distance function to the boundary of Ω, i.e.

$$\delta_\Omega(x) := \inf_{y\in\partial\Omega} |x-y|_D, \quad \forall\, x \in \Omega\,.$$

Denote by R_Ω the inradius of Ω (that is the radius of the largest ball which can be inscribed in Ω, or, $R_\Omega = \|\delta_\Omega\|_{L^\infty(\Omega)}$). Further, let $\delta : \mathbb{P}^D \to [0,\infty)$ denote the average integral of δ_Ω, that is

$$\delta(\Omega) := \frac{1}{|\Omega|}\int_\Omega \delta_\Omega(x)\,dx\,,$$

and let $h : \mathbb{P}^D \to [0,\infty)$ denote the Cheeger constant of Ω, that is

$$h(\Omega) := \inf_{\omega\subset\Omega} \frac{|\partial\omega|}{|\omega|}\,, \tag{1.5}$$

where the quotient $\frac{|\partial\omega|}{|\omega|}$ is taken among all smooth subdomains $\omega \subset \Omega$. We recall that $h(\Omega)$ also has the equivalent definition

$$h(\Omega) := \inf_{u\in W_0^{1,1}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_D dx}{\displaystyle\int_\Omega |u| dx}\,, \tag{1.6}$$

and, consequently, by relation (1.1) with $p = q = 1$ we have $h(\Omega) = \lambda(1,1;\Omega)$.

1.3. A simple observation regarding the monotonicity of the functions (1.2), (1.3) and (1.4)

A simple application of Hölder's inequality leads to the following monotonicity results regarding functions (1.2), (1.3) and (1.4) when $\Omega \in \mathbb{P}^D$ is an arbitrary but fixed set (see, C. Enache and the first author of this paper [14, relation (2.1)], P. Lindqvist [27, Theorem 3.2], G. Ercole & G.A. Pereira [16, Lemma 3.1])

$$h(\Omega) \leq |\Omega|^{(p-1)/p}\lambda(p,1;\Omega) \leq |\Omega|^{(q-1)/q}\lambda(q,1;\Omega), \quad \forall\, 1 < p < q < \infty\,,$$

$$h(\Omega) \leq p\lambda(p,p;\Omega) \leq q\lambda(q,q;\Omega), \quad \forall\, 1 < p < q < \infty\,,$$

$$|\Omega|^{-1/p}\lambda(p,\infty;\Omega) \leq |\Omega|^{-1/q}\lambda(q,\infty;\Omega), \quad \forall\, D < p < q < \infty\,.$$

However, these results cannot offer any direct information in relation with the monotonicity of the functions (1.2), (1.3) and (1.4).

2. Monotonicity of the function $(1,\infty) \ni p \mapsto |\Omega|^{p-1}\lambda(p,1;\Omega)^p$

2.1. A connection with the p-torsion problem

By relation (1.1) with $q = 1$ we have that

$$\lambda(p,1;\Omega)^p := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\,|\nabla u|_D\,\|^p_{L^p(\Omega)}}{\|u\|^p_{L^1(\Omega)}}, \quad \forall\, p \in (1,\infty)\,.$$

It follows that

$$|\Omega|^{p-1}\lambda(p,1;\Omega)^p = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{|\Omega|^{-1}\displaystyle\int_\Omega |\nabla u|_D^p\,dx}{\left(|\Omega|^{-1}\displaystyle\int_\Omega |u|\,dx\right)^p}, \quad \forall\, p \in (1,\infty)\,.$$

It is standard to check that for each $p \in (1,\infty)$ there exists a nonnegative minimizer of $|\Omega|^{p-1}\lambda(p,1;\Omega)^p$ in $W_0^{1,p}(\Omega)\setminus\{0\}$. Moreover, it is well-known (see, e.g. L. Brasco [7, pp. 320-321]) that if $u_p \in W_0^{1,p}(\Omega)\setminus\{0\}$ is a nonnegative minimizer of $|\Omega|^{p-1}\lambda(p,1;\Omega)^p$ then

$$v_p(x) := \left(\int_\Omega |\nabla u_p(y)|_D^p\,dy\right)^{-1/(p-1)}\left(\int_\Omega u_p(y)\,dy\right)^{1/(p-1)} u_p(x)\,,$$

gives the unique (weak) solution of the *p-torsion problem*, namely

$$\begin{cases} -\Delta_p v = 1, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Delta_p v := \operatorname{div}(|\nabla v|_D^{p-2}\nabla v)$ stands for the p-Laplace operator. Conversely, if v_p is the unique (weak) solution of problem (2.1) then it is a positive minimizer of $|\Omega|^{p-1}\lambda(p,1;\Omega)^p$.

On the other hand, we recall that the *p-torsional rigidity* on Ω is defined as follows

$$T_p(\Omega) := \int_\Omega v_p\,dx,$$

and it has the following variational characterization (see, e.g., F. Della Pietra, N. Gavitone, & S. Guarino Lo Bianco [13, relations (18) and (19)])

$$T_p(\Omega)^{p-1} = \sup_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega |u|\, dx\right)^p}{\int_\Omega |\nabla u|_D^p\, dx}.$$

Consequently, we can relate function (1.2) with the p-torsional rigidity by the following formula

$$|\Omega|^{p-1}\lambda(p,1;\Omega)^p = |\Omega|^{p-1}T_p(\Omega)^{1-p}. \tag{2.2}$$

2.2. The case of a ball

In the particular case when $\Omega = B_R$ (that is a ball of radius R, centered at the origin) $v_p \in W_0^{1,p}(B_R)$, the unique solution of problem (2.1), can be explicitly computed (see, B. Kawohl [24, relation (3.8)]),

$$v_p(x) = \frac{D(p-1)}{p}\left[\left(\frac{R}{D}\right)^{\frac{p}{p-1}} - \left(\frac{|x|_D}{D}\right)^{\frac{p}{p-1}}\right], \quad \forall\, x \in B_R.$$

Therefore

$$T_p(B_R) = \int_{B_R} v_p\, dx = \frac{\omega_D}{D^{\frac{p}{p-1}}\left(D + \frac{p}{p-1}\right)} R^{D+\frac{p}{p-1}}, \tag{2.3}$$

where $\omega_D = |\partial B_1|$ (that is the area of the unit ball in $\mathbb{R}^D$), and, by relation (2.2), we get

$$|B_R|^{p-1}\lambda(p,1;B_R)^p = |B_R|^{p-1}T_p(B_R)^{1-p} = D\left(D + \frac{p}{p-1}\right)^{p-1} R^{-p},$$

where in the last relation we used the fact that $|B_R| = \dfrac{\omega_D R^D}{D}$. Consequently, in this particular case our problem reduces to the analysis of the monotonicity of the function

$$(1,\infty) \ni p \mapsto D\left(D + \frac{p}{p-1}\right)^{p-1} R^{-p}.$$

2.3. The case when $\Omega \in \mathbb{P}^D$ is a general set

In the general case explicit formulas for the quantity $|\Omega|^{p-1}\lambda(p,1;\Omega)^p$ are not available in the literature and, consequently, the analysis of the monotonicity of the function given in relation (1.2), i.e.

$$(1,\infty) \ni p \mapsto |\Omega|^{p-1}\lambda(p,1;\Omega)^p,$$

is not trivial. However, a hint regarding the possible monotonicity of the above function can be easily obtained by recalling the following asymptotic formula (see L.E. Payne & G.A. Philippin [32])

$$\lim_{p\to\infty}\int_\Omega v_p\, dx = \int_\Omega \delta_\Omega\, dx. \tag{2.4}$$

Note that, actually, v_p converges uniformly over $\overline{\Omega}$ to δ_Ω, as $p \to \infty$ (see, T. Bhattacharya, E. DiBenedetto, & J.J. Manfredi [2] and B. Kawohl [24]). Combining (2.4) with (2.2) we deduce that

$$\lim_{p\to\infty} |\Omega|^{\frac{p-1}{p}} \lambda(p,1;\Omega) = \delta(\Omega)^{-1}\,,$$

which further implies

$$\lim_{p\to\infty} |\Omega|^{p-1}\lambda(p,1;\Omega)^p = \begin{cases} +\infty & \text{if } \delta(\Omega) < 1,\\ 0 & \text{if } \delta(\Omega) > 1\,. \end{cases}$$

Consequently, if the map given in relation (1.2) has a certain monotonicity then it should be increasing if $\delta(\Omega) < 1$ and decreasing if $\delta(\Omega) > 1$. The precise result concerning the monotonicity of function (1.2) was obtained by C. Enache and the authors of this paper in [14] and [15]. More exactly, by [14, Theorem 2] and [15, Remark 2] we have the following result.

Theorem 2.1. *For each $D \geq 1$ there exists a constant $T \in [(2D)^{-1}, 1]$ such that for each set $\Omega \in \mathbb{P}^D$ with $\delta(\Omega) \leq T$ the map given in relation (1.2) is increasing. Moreover, for any $s > T$ there exists a set $\Omega \in \mathbb{P}^D$, with $\delta(\Omega) = s$, for which the map given in relation (1.2) is not monotone.*

Remark 1. We note that, actually, we can give a better lower bound for the constant T from the above theorem. Indeed, by [10, Proposition 6.1] we have that

$$\delta(\Omega) \geq \frac{R_\Omega}{D+1}, \quad \forall\, \Omega \in \mathbb{P}^D\,.$$

It follows that for each $\Omega \in \mathbb{P}^D$ with $\delta(\Omega) < (D+1)^{-1}$ we have $R_\Omega < 1$ which by [14, Lemma 1] implies that

$$T(p;\Omega) < T(q;\Omega), \quad \forall\, 1 < p < q < \infty,\ \forall\, \Omega \in \mathbb{P}^D \text{ with } \delta(\Omega) < (D+1)^{-1}\,.$$

This observation combined with the proof of [14, Proposition 1] implies that $T \geq (D+1)^{-1}$. Consequently, in the conclusion of Theorem 2.1 we have $T \in [(D+1)^{-1}, 1]$ which improves the older bounds for T, namely $T \in (0, 1]$ (obtained in [14, Theorem 2]) and $T \in [(2D)^{-1}, 1]$ (obtained in [15, Remark 2]).

2.3.1. Open problems related to the monotonicity of function (1.2).

Problem 1. Note that by [14, Proposition 2] we have that for each ball B_R with $R > D + 1$ (and consequently $\delta(B_R) > 1$) the map given in relation (1.2) is not monotone. Consequently, for each real number $s > 1$ a set $\Omega \in \mathbb{P}^D$ with $\delta(\Omega) = s$ for which the map given in relation (1.2) is not monotone could be chosen to be a ball. However, in general, the question if for any set $\Omega \in \mathbb{P}^D$ with $\delta(\Omega) > T$ the map given in relation (1.2) is not monotone is open.

Problem 2. Another open problem related with the result from Theorem 2.1 is the following: if $D \geq 2$ does the number T given by Theorem 2.1 satisfy $T = 1$ or can the situation $T < 1$ occur? Moreover, if the case $T < 1$ holds true, then does T depend on D (the dimension of the Euclidean space) or not?

2.4. An alternative variational characterization for $\lambda(p,1;\Omega)$ on sets with small $\delta(\Omega)$

The monotonicity result from Theorem 2.1 allow us to obtain an alternative variational characterization of the constant $\lambda(p,1;\Omega)$ on domains $\Omega\in\mathbb{P}^D$ with $\delta(\Omega)\le T$ (where T is the constant given by Theorem 2.1). More precisely, if for any $\Omega\in\mathbb{P}^D$ and each $p\in(1,\infty)$ let us define

$$\Lambda(p,1;\Omega):=\inf_{v\in X_0\setminus\{0\}}\frac{|\Omega|^{-1}\displaystyle\int_\Omega(\exp(|\nabla v|_D^p)-1)\,dx}{\exp\left(\left(|\Omega|^{-1}\displaystyle\int_\Omega|v|\,dx\right)^p\right)-1}\,, \tag{2.5}$$

where $X_0:=W^{1,\infty}(\Omega)\cap\left(\cap_{q>1}W_0^{1,q}(\Omega)\right)$. Then by [14, Theorem 3] we have the following result:

Theorem 2.2. *Let $D\ge1$ be an integer and $\Omega\in\mathbb{P}^D$ be a set. If $\|\delta_\Omega\|_{L^\infty(\Omega)}\le1$, then $\Lambda(p,1;\Omega)>0$, for all $p\in(1,\infty)$, while if $\delta(\Omega)>1$, then $\Lambda(p,1;\Omega)=0$, for all $p\in(1,\infty)$. Moreover, if $\delta(\Omega)\le T$, where T is the constant given by Theorem 2.1, then $\lambda(p,1;\Omega)=|\Omega|^{\frac{1-p}{p}}\Lambda(p,1;\Omega)^{1/p}$, for all $p\in(1,\infty)$.*

Remark 2. Note that the fact that $\|\delta_\Omega\|_{L^\infty(\Omega)}\le1$ implies $\delta(\Omega)\le1$. However, the fact that for any $\Omega\in\mathbb{P}^D$ with $\delta(\Omega)\le1$ it holds $\Lambda(p,1;\Omega)>0$, for all $p\in(1,\infty)$ is an open problem. This problem would be solved for instance if one can show that $T=1$.

2.5. Monotonicity of the p-torsional rigidity

Another monotonicity result that can be related with the above discussion is that of the function

$$(1,\infty)\ni p\to T_p(\Omega)\,, \tag{2.6}$$

when $\Omega\in\mathbb{P}^D$ is given. Note that by relation (2.2) this is equivalent with the monotonicity of the map

$$(1,\infty)\ni p\to\lambda(p,1;\Omega)^{-p/(p-1)}\,.$$

2.5.1. The case of a ball. The discussion of the particular case when Ω is a ball, say $\Omega=B_R$ consists in the investigation of the monotonicity of the function given by relation (2.3), namely

$$(1,\infty)\ni p\mapsto\frac{\omega_D}{D^{\frac{p}{p-1}}\left(D+\dfrac{p}{p-1}\right)}R^{D+\frac{p}{p-1}}\,.$$

By [15, Theorem 3] we have the following result.

Theorem 2.3. (a) *If $R\ge De^{\frac{1}{D+1}}$ then $(1,\infty)\ni p\to T_p(B_R)$ is decreasing on the entire interval $(1,\infty)$.*

(b) *If $R\in(D,De^{\frac{1}{D+1}})$ then $(1,\infty)\ni p\to T_p(B_R)$ is decreasing on $\left(1,\frac{1-D\ln\left(\frac{R}{D}\right)}{1-(D+1)\ln\left(\frac{R}{D}\right)}\right)$ and increasing on $\left(\frac{1-D\ln\left(\frac{R}{D}\right)}{1-(D+1)\ln\left(\frac{R}{D}\right)},\infty\right)$.*

(c) *If* $R \leq D$ *then* $(1,\infty) \ni p \to T_p(B_R)$ *is increasing on the entire interval* $(1,\infty)$.

2.5.2. The case when $\Omega \in \mathbb{P}^D$ is a general set. The analysis of the monotonicity of the map given by relation (2.6) on a general set $\Omega \in \mathbb{P}^D$ is, as in the case of the function (1.2), more difficult since we do not have explicit formulas for $T_p(\Omega)$. However, a hint can be given in this case if we take into account an asymptotic formula which can be found in one of the papers by H. Bueno & G. Ercole [11, Theorem 2, relation (16)] or H. Bueno, G. Ercole, & S. S. Macedo [12, relation (1.10)]), namely

$$\lim_{p\to 1^+} T_p(\Omega)^{1-p} = h(\Omega)\,, \tag{2.7}$$

where $h(\Omega)$ stands for the Cheeger constant of Ω given by relations (1.5) and (1.6). It follows that

$$\lim_{p\to 1^+} T_p(\Omega) = \begin{cases} 0, & \text{if } h(\Omega) > 1, \\ \infty, & \text{if } h(\Omega) < 1. \end{cases} \tag{2.8}$$

Consequently, if the function given in relation (2.6) has a certain monotonicity then it should be increasing if $h(\Omega) > 1$ and decreasing if $h(\Omega) < 1$. However, the analysis from [15] shows that it is more useful to work with the quotient $\frac{|\partial\Omega|}{|\Omega|}$ than with the Cheeger constant, $h(\Omega)$, when we analyse the monotonicity of the p-torsional rigidity with respect to $p \in (1,\infty)$. According to relation (1.5) that fact is not unexpected even if $h(\Omega) \leq \frac{|\partial\Omega|}{|\Omega|}$. Note that in the particular case when $\Omega = B_R$ the result from Theorem 2.3 is consistent with the above discussion since it is well-known that $\frac{|\partial B_R|}{|B_R|} = h(B_R) = \frac{D}{R}$ and then we observe that function $(1,\infty) \ni p \to T_p(B_R)$ is increasing if $h(B_R) = \frac{D}{R} > 1$ and decreasing if $h(B_R) = \frac{D}{R} \leq e^{-1/(D+1)} < 1$.

The general result concerning the monotonicity of function (2.6) was obtained by C. Enache and the authors of this paper in [15, Theorem 2]. We recall this result below.

Theorem 2.4. *Assume* $D \geq 2$. *Then there exist two real numbers* $A_1 \in \left[\frac{1}{2}, e^{\frac{-1}{D+1}}\right]$ *and* $A_2 \in [1, D]$ *such that*

(i) for each $\Omega \in \mathbb{P}^D$ *with* $\frac{|\partial\Omega|}{|\Omega|} \leq A_1$ *the map given in relation* (2.6) *is decreasing on the entire interval* $(1,\infty)$*;*
(ii) for each $\Omega \in \mathbb{P}^D$ *with* $\frac{|\partial\Omega|}{|\Omega|} \geq A_2$ *the map given in relation* (2.6) *is increasing on the entire interval* $(1,\infty)$*;*
(iii) for each real number $s \in (A_1, A_2)$ *there exists* $\Omega \in \mathbb{P}^D$ *with* $\frac{|\partial\Omega|}{|\Omega|} = s$ *such that the map given in relation* (2.6) *is not monotone on* $(1,\infty)$.

2.5.3. Open problems related to the monotonicity of function (2.6).

Problem 1. Note that by Theorem 2.3 we have that for each ball B_R with $R \in (D, De^{\frac{1}{D+1}})$ the map given in relation (2.6) is not monotone. Consequently, for each real number $s \in (e^{\frac{-1}{D+1}}, 1)$ a set $\Omega \in \mathbb{P}^D$ with $\frac{|\partial\Omega|}{|\Omega|} = s$ for which the map given in relation (2.6) is not monotone could be chosen to be a ball. However, in general, the question if for any set $\Omega \in \mathbb{P}^D$ with $\frac{|\partial\Omega|}{|\Omega|} \in (A_1, A_2)$ the map given in relation (2.6) is not monotone is open.

Problem 2. Another open problem related with the result from Theorem 2.3 is the following: do the numbers A_1 and A_2 given by Theorem 2.3 satisfy $A_1 = e^{\frac{-1}{D+1}}$ and $A_2 = 1$ or can the situations $A_1 < e^{\frac{-1}{D+1}}$ and $A_2 > 1$ occur? Further, if the case $A_2 > 1$ holds true, then does A_2 depend on D (the dimension of the Euclidean space) or not?

3. Monotonicity of the function $(1,\infty) \ni p \mapsto \lambda(p,p;\Omega)^p$

3.1. A connection with the eigenvalue problem of the p-Laplace operator

By relation (1.1) with $q = p$ we have that

$$\begin{aligned}\lambda(p,p;\Omega)^p \quad &:= \quad \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\| \, |\nabla u|_D \, \|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}} \\ &= \quad \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_D^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}, \quad \forall\, p \in (1,\infty)\,.\end{aligned}$$

It is well-known that this minimization problem is related to the eigenvalue problem for the p-Laplace operator, namely

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

in the sense that $\lambda(p,p;\Omega)^p$ represents the lowest eigenvalue of the problem (3.1), also known as the *principal frequency* of the p-Laplace operator (see, e.g. P. Lindqvist [26]). (We recall that by an eigenvalue of problem (3.1) we understand a parameter λ for which the problem possesses a nontrivial (weak) solution.)

3.2. The case $D = 1$

In the particular case when $D = 1$, if $\Omega \in \mathbb{P}^1$ then there exists $a,\ b \in \mathbb{R}$ with $a < b$ such that $\Omega = (a,b)$. It is well-known (see, e.g. P. Lindqvist [28]) that

$$\lambda(p,p;(a,b))^p = (p-1)\left(\frac{2}{b-a}\right)^p \left(\frac{\pi/p}{\sin(\pi/p)}\right)^p, \quad \forall\, p \in (1,\infty)\,.$$

Consequently, in this particular case our problem reduces to the analysis of the monotonicity of the function

$$(1,\infty) \ni p \mapsto (p-1)\left(\frac{2}{b-a}\right)^p \left(\frac{\pi/p}{\sin(\pi/p)}\right)^p .$$

The corresponding investigation was carried on by R. Kajikiya, M. Tanaka, & S. Tanaka in [23, Theorem 1.1]. More precisely, they proved the following result.

Theorem 3.1. *If $\frac{b-a}{2} \le 1$ then the map $p \mapsto \lambda(p,p;(a,b))^p$ is increasing on the entire interval $(1,\infty)$. If $\frac{b-a}{2} > 1$ then there exists $p^\star = p^\star\left(\frac{b-a}{2}\right) \in (1,\infty)$ such that $p \mapsto \lambda(p,p;(a,b))^p$ is increasing on $(1,p^\star)$ and decreasing on $(p^\star,\infty)$.*

3.3. The case $D \geq 2$

In the general case, when $D \geq 2$, there is no explicit formula of $\lambda(p,p;\Omega)^p$ when $p \in (1,\infty) \setminus \{2\}$, not even on simple domains such as balls or squares. This fact makes the study of the monotonicity of the map given in relation (1.3), i.e.

$$(1,\infty) \ni p \mapsto \lambda(p,p;\Omega)^p ,$$

more complicated. However, a hint regarding its monotonicity comes from the following asymptotic formula due to P. Juutinen, P. Lindqvist, & J. J. Manfredi [21, Lemma 1.5] and N. Fukagai, M. Ito, & K. Narukawa [18, Corollaries 3.2 and 4.5]

$$\lim_{p\to\infty} \lambda(p,p;\Omega) = R_\Omega^{-1} ,$$

which yields

$$\lim_{p\to\infty} \lambda(p,p;\Omega)^p = \begin{cases} +\infty & \text{if } R_\Omega < 1, \\ 0 & \text{if } R_\Omega > 1 . \end{cases}$$

Consequently, if the map given in relation (1.3) has a certain monotonicity then it should be increasing if $R_\Omega < 1$ and decreasing if $R_\Omega > 1$.

A first result concerning the monotonicity of the map (1.3) when $D \geq 2$ can be found in a paper by V. Bobkov & M. Tanaka, namely [3, Proposition 9], where the following theorem was proved.

Theorem 3.2. *Assume that $D \geq 2$ is an integer and $\Omega \subset \mathbb{R}^D$ is a domain satisfying*

$$B_r \subset \Omega \subset B_R ,$$

where r, $R \in (1,e)$ are two real numbers such that

$$\max\{1, e \ln R\} < r \leq R < e ,$$

and B_r, B_R stand for two balls having radii r and R, respectively. Then the map given by relation (1.3) is not monotone on $(1,\infty)$.

This result was complemented and improved by M. Bocea and the first author of this paper in [5, Theorem 1]. The precise result is formulated in the following theorem.

Theorem 3.3. *Let $D \geq 2$ be a given integer. Then there exists a real number $M \in [e^{-1}, 1]$ such that for each $\Omega \in \mathbb{P}^D$ with $R_\Omega \leq M$ the map given by relation (1.3) is increasing on the entire interval $(1,\infty)$. Moreover, for each $s > M$ there exists a domain $\Omega \in \mathbb{P}^D$ with $R_\Omega = s$ for which the map given by relation (1.3) is not monotone on the entire interval $(1,\infty)$.*

3.3.1. Open problems related to the monotonicity of function (1.3).

Problem 1. The following open problem can be formulated in relation with the above result: if $D \geq 2$ and M is the number given by Theorem 3.3 is it true that for all $\Omega \in \mathbb{P}^D$ with $R_\Omega > M$ the map given by relation (1.3) is not monotone on the entire interval $(1,\infty)$? In [5, Proposition 1 & Theorem 1] the authors proved the existence of such kind of domains by using as main argument a result due to R. Kajikiya [22, Proposition 2.3] (see also L. Brasco [8, Theorem 1.1] for a similar result). Moreover, the first author of this paper complemented the result by proving in [30, Theorem 1 (d)] that when Ω is a ball, say B_R, with the radius strictly larger than 1, ($R > 1$), then the map given by relation (1.3) is not monotone on the entire interval $(1, \infty)$. (That time the main argument was based on some estimates of the principal frequency due to J. Benedikt & P. Drábek [1, Theorem 2].) Excepting these particular investigations the general case is an open problem.

Problem 2. Another open problem related with the result from Theorem 3.3 is the following: if $D \geq 2$ does the number M given by Theorem 3.3 satisfy $M = 1$ or can the situation $M < 1$ occur? Moreover, if the case $M < 1$ holds true, then does M depend on D (the dimension of the Euclidean space) or not?

3.3.2. Monotonicity results for similar eigenvalue problems. In this section we recall certain papers where similar results with those formulated in Theorem 3.3 can be found.

Monotonicity results for variational eigenvalues of the Dirichlet p-Laplace operator. Since $\lambda(p, p; \Omega)^p$ represents the lowest eigenvalue of the problem (3.1) it is natural to ask if similar results hold true for other eigenvalues of the problem. In that context, firstly, we need to recall the well known fact that the description of the entire set of eigenvalues of problem (3.1), when $p \neq 2$, is still an open question. However, for the sequence of variational eigenvalues produced by using the Ljusternik-Schnirelman theory (see, e.g. P. Lindqvist [29] or A. Lê [25] for the description of the set of variational eigenvalues of problem (3.1)) similar results as those given in Theorem 3.3 were obtained by the first author of this paper in [30, Theorem 1].

Monotonicity results for the principal frequency on an annulus. A similar result with those from Theorem 3.3 was obtained when Ω is an annulus (i.e., Ω is the difference of two concentric balls), and consequently $\Omega \notin \mathbb{P}^D$, by A. Grecu and the first author of this paper in [19, Theorem 1].

Monotonicity of the first positive eigenvalue of the Neumann p-Laplace operator. Similar investigations with those from Theorem 3.3 were considered in the context of the first positive eigenvalue of the p-Laplace operator under the homogeneous Neumann boundary condition by the first author of this paper in collaboration with J. D. Rossi in [31, Theorem 1.1].

Monotonicity results for the principal frequency of the anisotropic p-Laplace operator. M. Bocea in collaboration with the authors of this paper discussed the monotonicity of the principal frequency of the anisotropic p-Laplace operator in [6, Theorem 1].

3.4. An alternative variational characterization for $\lambda(p, p; \Omega)$ on sets with small inradius

We note that combining the monotonicity result from Theorem 3.3 with those obtained by M. Bocea and the first author of this paper in [4, Theorem 2] we deduce that for each set $\Omega \in \mathbb{P}^D$ with $R_\Omega \in (0, M]$, where M is given by Theorem 3.3, we have the following alternative variational characterization of $\lambda(p,p;\Omega)^p$ when $p\in(1,\infty)$, namely

$$\lambda(p,p;\Omega)^p=\inf_{u\in X_0\setminus\{0\}}\frac{\displaystyle\int_\Omega \Phi_p(|\nabla u|)\,dx}{\displaystyle\int_\Omega \Phi_p(|u|)\,dx},$$

where, $X_0:=W^{1,\infty}(\Omega)\cap\left(\cap_{q>1}W_0^{1,q}(\Omega)\right)$ and $\Phi_p(t)$ can be taken to be either one of the functions $t\mapsto\sinh(|t|^p)$, $t\mapsto\cosh(|t|^p)-1$, or $t\mapsto\exp(|t|^p)-1$. It is interesting that this variational characterization fails to hold true when $\Omega\in\mathbb{P}^D$ with $R_\Omega\in(1,\infty)$ since in that case the above infimum vanishes (see, [4, Theorem 2]).

4. Monotonicity of the function $(D,\infty)\ni p\mapsto\lambda(p,\infty;\Omega)^p$

4.1. A connection between $\lambda(p,\infty;\Omega)^p$ and an eigenvalue problem

By relation (1.1) with $q=\infty$ we have that

$$\begin{aligned}\lambda(p,\infty;\Omega)^p &:= \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\|\,|\nabla u|_D\,\|^p_{L^p(\Omega)}}{\|u\|^p_{L^\infty(\Omega)}}\\ &= \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega|\nabla u|_D^p\,dx}{\|u\|^p_{L^\infty(\Omega)}},\quad \forall\,p\in(D,\infty)\,.\end{aligned}$$

Note that $\lambda(p,\infty;\Omega)^p$ is also known as the *best constant in Morrey's inequality*, that is the largest constant $C>0$ for which the following inequality holds true

$$C\|u\|^p_{L^\infty(\Omega)}\le\int_\Omega|\nabla u(x)|_D^p\,dx,\quad\forall\,u\in W_0^{1,p}(\Omega)\,.$$

It is well known (see, e.g. G. Ercole & G. A. Pereira [16, Theorem 2.5] or R. Hynd & E. Lindgren [20]) that for each $p\in(D,\infty)$ there exists a nonnegative minimizer of $\lambda(p,\infty;\Omega)^p$, say $u_p\in W_0^{1,p}(\Omega)$, such that

$$\|u_p\|_{L^\infty(\Omega)}=1\quad\text{and}\quad\|\,|\nabla u_p|_D\,\|^p_{L^p(\Omega)}=\lambda(p,\infty;\Omega)^p\,.$$

Moreover, there exists a unique point $x_p\in\Omega$ such that

$$u_p(x_p)=\|u_p\|_{L^\infty(\Omega)}=1\,,$$

and the following equation is satisfied in the sense of distributions

$$\begin{cases}-\operatorname{div}(|\nabla u_p(x)|_D^{p-2}\nabla u_p(x))=\lambda(p,\infty;\Omega)^p|u_p(x_p)|^{p-2}u_p(x_p)\delta_{x_p}(x), & \text{if } x\in\Omega,\\ u_p(x)=0, & \text{if } x\in\partial\Omega,\end{cases}$$

where by δ_{x_p} the Dirac mass concentrated at x_p was denoted.

4.1.1. The case of a ball. In the particular case when $\Omega = B_R$ (that is a ball of radius R, centered at the origin) then

$$\lambda(p,\infty;B_R)^p = \frac{Dv_D}{R^{p-D}}\left(\frac{p-D}{p-1}\right)^{p-1}, \tag{4.1}$$

where $v_D = |B_1|$ denotes the volume of the unit ball in $\mathbb{R}^D$, (see, e.g., [16, relation (1.9)]). Consequently, in this particular case our problem reduces to the analysis of the monotonicity of the function

$$(D,\infty) \ni p \mapsto \frac{Dv_D}{R^{p-D}}\left(\frac{p-D}{p-1}\right)^{p-1}.$$

The precise result in this case is the following (see [17, Theorem 1.2])

Theorem 4.1. *For every integer $D \geq 1$ if $R \leq 1$ then the map $p \mapsto \lambda(p,\infty;B_R)^p$ is increasing on the entire interval (D,∞), while, if $R > 1$ then the map $p \mapsto \lambda(p,\infty;B_R)^p$ is not monotone on (D,∞).*

4.1.2. The case when $\Omega \in \mathbb{P}^D$ is a general set. In the general case an explicit formula for the quantity $\lambda(p,\infty;\Omega)^p$ is not available in the literature and, consequently, the analysis of the monotonicity of the map given in relation (1.4), i.e.

$$(1,\infty) \ni p \mapsto \lambda(p,\infty;\Omega)^p,$$

is more complicated. However, a hint regarding its monotonicity comes from the following asymptotic formula (see, e.g. [16, Theorem 3.2])

$$\lim_{p\to\infty} \lambda(p,\infty;\Omega) = R_\Omega^{-1},$$

where $R_\Omega = \|\delta_\Omega\|_{L^\infty(\Omega)}$ denotes the inradius of Ω, which yields

$$\lim_{p\to\infty} \lambda(p,\infty;\Omega)^p = \begin{cases} +\infty & \text{if } R_\Omega < 1, \\ 0 & \text{if } R_\Omega > 1. \end{cases}$$

Consequently, if the map given in relation (1.4) has a certain monotonicity then it should be increasing if $R_\Omega < 1$ and decreasing if $R_\Omega > 1$.
On the other hand, by [16, Corollary 2.7] for each $\Omega \in \mathbb{P}^D$ and each $p \in (D,\infty)$ the following inequalities hold

$$\lambda\left(p,\infty;B_{\sqrt[D]{|\Omega|/v_D}}\right)^p \leq \lambda(p,\infty;\Omega)^p \leq \lambda\left(p,\infty;B_{R_\Omega}\right)^p. \tag{4.2}$$

Combining relations (4.1) and (4.2) we deduce that

$$\lim_{p\to D^+} \lambda(p,\infty;\Omega) = 0, \quad \forall\, \Omega \in \mathbb{P}^D.$$

The above pieces of information show that the map given in relation (1.4) is not monotone on (D,∞) for any set $\Omega \in \mathbb{P}^D$ with $R_\Omega > 1$. The general result on the monotonicity of the map given in relation (1.4) was obtained by M. Fărcăşeanu in collaboration with the first author of this paper in [17, Theorem 1.2] and is given in the following theorem.

Theorem 4.2. *For every integer $D \geq 2$ there exists $L \in [e^{-1}, 1]$ such that for each $\Omega \in \mathbb{P}^D$ with $R_\Omega \in (0, L]$ the map given in relation (1.4) is increasing on (D, ∞) while for each $\Omega \in \mathbb{P}^D$ with $R_\Omega > 1$ the map given in relation (1.4) is not monotone on (D, ∞).*

4.1.3. An open problem related to the monotonicity of function (1.4). The following open problem can be formulated in relation with the above result: if $D \geq 2$ does the number L given by Theorem 4.2 satisfy $L = 1$ or can the situation $L < 1$ occur? Moreover, if the case $L < 1$ holds true, then does L depend on D (the dimension of the Euclidean space) or not?

4.2. An alternative variational characterization for $\lambda(p, \infty; \Omega)$ on sets with small inradius

The monotonicity results from Theorems 4.1 and 4.2 allow us to obtain an alternative variational characterization of the constant $\lambda(p, \infty; \Omega)$ on domains $\Omega \in \mathbb{P}^D$ with $R_\Omega \leq L$ (where L is the constant given by Theorem 4.2). More precisely, if for any $\Omega \in \mathbb{P}^D$ and each $p \in (1, \infty)$ we define

$$\Lambda(p, \infty; \Omega) := \inf_{u \in X_0 \setminus \{0\}} \frac{\displaystyle\int_\Omega (\exp(|\nabla u|_D^p) - 1)\, dx}{\exp\left(\|u\|_{L^\infty(\Omega)}^p\right) - 1}, \tag{4.3}$$

where $X_0 := W^{1,\infty}(\Omega) \cap \left(\cap_{q>1} W_0^{1,q}(\Omega)\right)$, then by [17, Theorem 1.3] we have the following result.

Theorem 4.3. *Let $D \geq 1$ be an integer and $\Omega \in \mathbb{P}^D$ be a set. If $R_\Omega < 1$, then $\Lambda(p, \infty; \Omega) > 0$, for all $p \in (D, \infty)$, while if $R_\Omega > 1$, then $\Lambda(p, \infty; \Omega) = 0$, for all $p \in (D, \infty)$. Moreover, if $R_\Omega \leq L$, with L the constant given by Theorem 4.2, then $\Lambda(p, \infty; \Omega) = \lambda(p, \infty; \Omega)^p$, for all $p \in (D, \infty)$. In the particular case when $\Omega = B_R$ (i.e., Ω is a ball of radius R from $\mathbb{R}^D$) then $\Lambda(p, \infty; B_R) = 0$, for all $p \in (D, \infty)$ if $R > 1$ and $\Lambda(p, \infty; B_R) = \lambda(p, \infty; B_R)^p$, for all $p \in (D, \infty)$ if $R \in (0, 1]$.*

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Mihai Mihăilescu
(Corresponding author)
Department of Mathematics, University of Craiova,
200585 Craiova, Romania
and
“Gheorghe Mihoc – Caius Iacob” Institute of Mathematical Statistics
and Applied Mathematics of the Romanian Academy,
050711 Bucharest, Romania
e-mail: `mmihailes@yahoo.com`

Denisa Stancu-Dumitru
Department of Mathematics and Computer Sciences,
University Politehnica of Bucharest,
060042 Bucharest, Romania
and
Research Group of the Project PN-III-P1-1.1-TE-2021-1539,
The Research Institute of the University of Bucharest – ICUB,
University of Bucharest, 050663 Bucharest, Romania
e-mail: `denisa.stancu@yahoo.com`

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Multiplicity theorems involving functions with non-convex range

Biagio Ricceri

Dedicated to the memory of Professor Csaba Varga, with nostalgia

Abstract. Here is a sample of the results proved in this paper: Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, let $\rho > 0$ and let $\omega : [0, \rho[\to [0, +\infty[$ be a continuous increasing function such that

$$\lim_{\xi \to \rho^-} \int_0^\xi \omega(x)dx = +\infty.$$

Consider $C^0([0,1]) \times C^0([0,1])$ endowed with the norm

$$\|(\alpha, \beta)\| = \int_0^1 |\alpha(t)|dt + \int_0^1 |\beta(t)|dt.$$

Then, the following assertions are equivalent:
(a) the restriction of f to $\left[-\frac{\sqrt{\rho}}{2}, \frac{\sqrt{\rho}}{2}\right]$ is not constant;
(b) for every convex set $S \subseteq C^0([0,1]) \times C^0([0,1])$ dense in $C^0([0,1]) \times C^0([0,1])$, there exists $(\alpha, \beta) \in S$ such that the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2 dt\right) u'' = \beta(t) f(u) + \alpha(t) \text{ in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2 dt < \rho \end{cases}$$

has at least two classical solutions.

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1. Introduction

Let H be a real Hilbert space. A very classical result of Efimov and Stechkin ([3]) states that if X is a non-convex sequentially weakly closed subset of H, then there exists $y_0 \in H$ such that the restriction to X of the function $x \to \|x - y_0\|$ has at least two global minima. A more precise version of such a result was obtained by I.G. Tsar'kov in [10]. Actually, he proved that any convex set dense in H contains a point y_0 with the above property.

In the present paper, as a by product of a more general result, we get the following:

Theorem 1.1. *Let $X \subset H$ be a non-convex sequentially weakly closed set and let $u_0 \in \mathrm{conv}(X) \setminus X$.*

Then, if we put

$$\delta := \mathrm{dist}(u_0, X)$$

and, for each $r > 0$,

$$\rho_r := \sup_{\|y\|<r} ((\mathrm{dist}(u_0 + y, X))^2 - \|y\|^2),$$

for every convex set $S \subseteq H$ dense in H, for every bounded sequentially weakly lower semicontinuous function $\varphi : X \to \mathbf{R}$ and for every r satisfying

$$r > \frac{\rho_r - \delta^2 + \sup_X \varphi - \inf_X \varphi}{2\delta},$$

there exists $y_0 \in S$, with $\|y_0 - u_0\| < r$, such that the function $x \to \|x - y_0\|^2 + \varphi(x)$ has at least two global minima in X.

So, with respect to the Efimov-Stechkin-Tsar'kov result, Theorem 1.1 gives us two remarkable additional informations: a precise localization of the point y_0 and the validity of the conclusion not only for the function $x \to \|x-y_0\|^2$, but also for suitable perturbations of it.

Let us recall the most famous open problem in this area: if X is a subset of H such that, for each $y \in H$, the restriction of the function $x \to \|x - y\|$ to X has a unique global minimum, is it true that the set X is convex? So, Efimov-Stechkin's result provides an affirmative answer when X is sequentially weakly closed. However, it is a quite common feeling that the answer, in general, should be negative ([1], [2], [5], [8]). In the light of Theorem 1.1, we posit the following problem:

Problem 1.1. Let X be a subset of H for which there exists a bounded sequentially weakly lower semicontinuous function $\varphi : X \to \mathbf{R}$ such that, for each $y \in H$, the function $x \to \|x - y\|^2 + \varphi(x)$ has a unique global minimum in X. Then, must X be convex?

What allows us to reach the advances presented in Theorem 1.1 is our particular approach which is entirely based on the minimax theorem established in [9]. So, also the present paper can be regarded as a further ring of the chain of applications and consequences of that minimax theorem.

2. Results

In the sequel, X is a topological space and E is real normed space, with topological dual E^*.

For each $S \subseteq E^*$, we denote by $\mathcal{A}(X,S)$ (resp. $\mathcal{A}_s(X,S)$) the class of all pairs (I,ψ), with $I : X \to \mathbf{R}$ and $\psi : X \to E$, such that, for each $\eta \in S$ and each $s \in \mathbf{R}$, the set

$$\{x \in X : I(x) + \eta(\psi(x)) \leq s\}$$

is closed and compact (resp. sequentially closed and sequentially compact).

Let us start establishing the following useful proposition. E' denotes the algebraic dual of E.

Proposition 2.1. *Let $I : X \to \mathbf{R}$, let $\psi : X \to E$ and let $x_1, \dots, x_n \in X$, $\lambda_1, \dots, \lambda_n \in [0,1]$, with $\sum_{i=1}^{n} \lambda_i = 1$.*

Then, one has

$$\sup_{\eta \in E'} \inf_{x \in X} \left(I(x) + \eta \left(\psi(x) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \right) \leq \max_{1 \leq i \leq n} I(x_i).$$

Proof. Fix $\eta \in E'$. Clearly, for some $j' \in \{1, \dots, n\}$, we have

$$\eta \left(\psi(x_{j'}) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \leq 0. \tag{2.1}$$

Indeed, if not, we would have

$$\eta(\psi(x_j)) > \sum_{i=1}^{n} \lambda_i \eta(\psi(x_i))$$

for each $j \in \{1, \dots, n\}$. So, multiplying by λ_j and summing, we would obtain

$$\sum_{j=1}^{n} \lambda_j \eta(\psi(x_j)) > \sum_{i=1}^{n} \lambda_i \eta(\psi(x_i)),$$

a contradiction. In view of (2.1), we have

$$\begin{aligned} \inf_{x \in X} \left(I(x) + \eta \left(\psi(x) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \right) &\leq I(x_{j'}) + \eta \left(\psi(x_{j'}) - \sum_{i=1}^{n} \lambda_i \psi(x_i) \right) \\ &\leq I(x_{j'}) \leq \max_{1 \leq i \leq n} I(x_i) \end{aligned}$$

and so we get the conclusion due to the arbitrariness of η. □

Our main result is as follows:

Theorem 2.1. *Let $I : X \to \mathbf{R}$, let $\psi : X \to E$, let $S \subseteq E^*$ be a convex set dense in E^* and let $u_0 \in E$.*

Then, for every bounded function $\varphi : X \to \mathbf{R}$ such that $(I+\varphi,\psi) \in \mathcal{A}(X,S)$ and for every r satisfying

$$\sup_X \varphi - \inf_X \varphi < \inf_{x\in X}(I(x) + \|\psi(x)-u_0\|r) - \sup_{\|\eta\|_{E^*}<r} \inf_{x\in X}(I(x)+\eta(\psi(x)-u_0)), \quad (2.2)$$

there exists $\tilde{\eta} \in S$, with $\|\tilde{\eta}\|_{E^*} < r$, such that the function $I + \tilde{\eta}\circ\psi + \varphi$ has at least two global minima in X.

Proof. Consider the function $g : X \times E^* \to \mathbf{R}$ defined by

$$g(x,\eta) = I(x) + \eta(\psi(x) - u_0)$$

for all $(x,\eta) \in X \times E^*$. Let B_r denote the open ball in E^*, of radius r, centered at 0. Clearly, for each $x \in X$, we have

$$\sup_{\eta\in B_r} \eta(\psi(x)-u_0)) = \|\psi(x)-u_0\|r. \quad (2.3)$$

Then, from (2.2) and (2.3), it follows

$$\sup_X \varphi - \inf_X \varphi < \inf_X \sup_{B_r} g - \sup_{B_r} \inf_X g. \quad (2.4)$$

Now, consider the function $f : X \times (S\cap B_r) \to \mathbf{R}$ defined by

$$f(x,\eta) = g(x,\eta) + \varphi(x)$$

for all $(x,\eta) \in X \times (S\cap B_r)$. Since S is dense in E^*, the set $S \cap B_r$ is dense in B_r. Hence, since $g(x,\cdot)$ is continuous, we obtain

$$\inf_X \sup_{S\cap B_r} g = \inf_X \sup_{B_r} g. \quad (2.5)$$

Then, taking (2.4) and (2.5) into account, we have

$$\begin{aligned}\sup_{S\cap B_r} \inf_X f \le \sup_{B_r}\inf_X f \le \sup_{B_r}\inf_X g + \sup_X \varphi < \inf_X \sup_{B_r} g + \inf_X \varphi \\ \le \inf_{x\in X}\left(\sup_{\eta\in S\cap B_r} g(x,\eta) + \varphi(x)\right) = \inf_X \sup_{S\cap B_r} f. \quad (2.6)\end{aligned}$$

Now, since $(I+\varphi,\psi) \in \mathcal{A}(X,S)$ and f is concave in $S\cap B_r$, we can apply Theorem 1.1 of [9]. Therefore, since (by (2.6)) $\sup_{S\cap B_r}\inf_X f < \inf_X \sup_{S\cap B_r} f$, there exists of $\tilde{\eta} \in S \cap B_r$ such that the function $f(\cdot,\tilde{\eta})$ has at least two global minima in X which, of course, are global minima of the function $I + \tilde{\eta}\circ\psi + \varphi$. □

If we renounce to the very detailed informations contained in its conclusion, we can state Theorem 2.1 in an extremely simplified form.

Theorem 2.2. *Let $I : X \to \mathbf{R}$, let $\psi : X \to E$ and let $S \subset E^*$ be a convex set weakly-star dense in E^*. Assume that $\psi(X)$ is not convex and that $(I,\psi) \in \mathcal{A}(X,S)$.*

Then, there exists $\tilde{\eta} \in S$ such that the function $I + \tilde{\eta}\circ\psi$ has at least two global minima in X.

Proof. Fix $u_0 \in \mathrm{conv}(\psi(X)) \setminus \psi(X)$ and consider the function $g : X \times E^* \to \mathbf{R}$ defined by

$$g(x,\eta) = I(x) + \eta(\psi(x) - u_0)$$

for all $(x, \eta) \in X \times E^*$. By Proposition 2.1, we know that

$$\sup_{E^*} \inf_X g < +\infty.$$

On the other hand, for each $x \in X$, since $\psi(x) \neq u_0$, we have

$$\sup_{\eta \in E^*} \eta(\psi(x) - u_0) = +\infty.$$

Hence, since S is weakly-star dense in E^* and $g(x, \cdot)$ is weakly-star continuous, we have

$$\sup_{\eta \in S} g(x, \eta) = +\infty.$$

Therefore

$$\sup_S \inf_X g < \inf_X \sup_S g. \tag{2.7}$$

Now, taken into account that $(I, \psi) \in \mathcal{A}(X, S)$, we can apply Theorem 1.1 of [9] to $g_{|X \times S}$. So, in view of (2.7), there exists $\tilde{\eta} \in S$ such that the function $g(\cdot, \tilde{\eta})$ (and so $I + \tilde{\eta} \circ \psi$) has at least two global minima in X, as claimed. □

The next result is a sequential version of Theorem 1.1 of [9].

Theorem 2.3. *Let X be a topological space, E a topological vector space, $Y \subseteq E$ a non-empty separable convex set and $f : X \times Y \to \mathbf{R}$ a function satisfying the following conditions:*
(a) for each $y \in Y$, the function $f(\cdot, y)$ is sequentially lower semicontinuous, sequentially inf-compact and has a unique global minimum in X;
(b) for each $x \in X$, the function $f(x, \cdot)$ is continuous and quasi-concave.
Then, one has

$$\sup_Y \inf_X f = \inf_X \sup_Y f.$$

Proof. The pattern of the proof is the same as that of Theorem 1.1 of [9]. We limit ourselves to stress the needed changes. First, for every $n \in \mathbf{N}$, one proves the result when $E = \mathbf{R}^n$ and $Y = S_n := \{(\lambda_1, \ldots, \lambda_n) \in ([0, +\infty[)^n : \lambda_1 + \ldots + \lambda_n = 1\}$. In this connection, the proof agrees exactly with that of Lemma 2.1 of [9], with the only difference of using the sequential version of Theorem 1.A of [9] instead of such a result itself (see Remark 2.1 of [9]). Next, we fix a sequence $\{x_n\}$ dense in Y. For each $n \in \mathbf{N}$, set

$$P_n = \operatorname{conv}(\{x_1, \ldots, x_n\}).$$

Consider the function $\eta : S_n \to P$ defined by

$$\eta(\lambda_1, \ldots, \lambda_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n$$

for all $(\lambda_1, \ldots, \lambda_n) \in S_n$. Plainly, the function $(x, \lambda_1, \ldots, \lambda_n) \to f(x, \eta(\lambda_1, \ldots, \lambda_n))$ satisfies in $X \times S_n$ the assumptions of Theorem A, and so, by the case previously proved, we have

$$\sup_{(\lambda_1, \ldots, \lambda_n) \in S_n} \inf_{x \in X} f(x, \eta(\lambda_1, \ldots, \lambda_n)) = \inf_{x \in X} \sup_{(\lambda_1, \ldots, \lambda_n) \in S_n} f(x, \eta(\lambda_1, \ldots, \lambda_n)).$$

Since $\eta(S_n) = P_n$, we then have

$$\sup_{P_n} \inf_X f = \inf_X \sup_{P_n} f.$$

Now, set

$$D = \bigcup_{n \in \mathbf{N}} P_n.$$

In view of Proposition 2.2 of [9], we have

$$\sup_D \inf_X f = \inf_X \sup_D f.$$

Finally, by continuity and density, we have

$$\sup_{y \in D} f(x,y) = \sup_{y \in Y} f(x,y)$$

for all $x \in X$, and so

$$\inf_X \sup_Y f = \inf_X \sup_D f = \sup_D \inf_X f \leq \sup_Y \inf_X f \leq \inf_X \sup_Y f$$

and the proof is complete. □

Reasoning as in the proof of Theorem 2.1 and using Theorem 2.3, we get

Theorem 2.4. *Let the assumptions of Theorem 2.1 be satisfied. In addition, assume that E^* is separable.*

Then, the conclusion of Theorem 2.1 holds with $\mathcal{A}_s(X,S)$ instead of $\mathcal{A}(X,S)$.

Analogously, the sequential version of Theorem 2.2 is as follows:

Theorem 2.5. *Let $I : X \to \mathbf{R}$, let $\psi : X \to E$ and let $S \subset E^*$ be a convex set weakly-star separable and weakly-star dense in E^*. Assume that $\psi(X)$ is not convex and that $(I,\psi) \in \mathcal{A}_s(X,S)$.*

Then, there exists $\tilde{\eta} \in S$ such that the function $I + \tilde{\eta} \circ \psi$ has at least two global minima in X.

Here is a consequence of Theorem 2.1:

Theorem 2.6. *Let E be a Hilbert space, let $\psi : X \to E$ be a weakly continuous function and let $S \subseteq E$ be a convex set dense in E. Assume that $\psi(X)$ is not convex and that the function $\|\psi(\cdot)\|$ is inf-compact. Let $u_0 \in \operatorname{conv}(\psi(X)) \setminus \psi(X)$.*

Then, for every bounded function $\varphi : X \to \mathbf{R}$ such that $\|\psi(\cdot)\|^2 + \varphi(\cdot)$ is lower semicontinuous and for every r satisfying

$$r > \frac{\sup_{\|y\|<r}((\operatorname{dist}(u_0+y,\psi(X)))^2 - \|y\|^2) - (\operatorname{dist}(u_0,\psi(X)))^2 + \sup_X \varphi - \inf_X \varphi}{2\operatorname{dist}(u_0,\psi(X))}, \tag{2.8}$$

there exists $\tilde{y} \in S$, with $\|\tilde{y} - u_0\| < r$, such that the function $\|\psi(\cdot) - \tilde{y}\|^2 + \varphi(\cdot)$ has at least two global minima in X.

Proof. First, we observe that the set $\psi(X)$ is sequentially weakly closed (and so norm closed). Indeed, let $\{x_n\}$ be a sequence in X such that $\{\psi(x_n)\}$ converges weakly to $y \in E$. So, in particular, $\{\psi(x_n)\}$ is bounded and hence, since $\|\psi(\cdot)\|$ is inf-compact, there exists a compact set $K \subseteq X$ such that $x_n \in K$ for all $n \in \mathbf{N}$. Since ψ is weakly continuous, the set $\psi(K)$ is weakly compact and hence weakly closed. Therefore, $y \in \psi(K)$, as claimed. This remark ensures that $\mathrm{dist}(u_0, \psi(X)) > 0$. Now, we apply Theorem 2.1 identifying E with E^* and taking

$$I(x) = \frac{1}{2}\|\psi(x) - u_0\|^2$$

for all $x \in X$. Of course, we have

$$I(x) + \langle \psi(x) - u_0, y \rangle = \frac{1}{2}(\|\psi(x) - u_0 + y\|^2 - \|y\|^2) \tag{2.9}$$

for all $y \in E$. In view of (2.8) and (2.9), we have

$$\begin{aligned} \frac{1}{2}(\sup_X \varphi - \inf_X \varphi) &< \frac{1}{2}(\mathrm{dist}(u_0, \psi(X)))^2 + r\mathrm{dist}(u_0, \psi(X)) \\ &\quad - \frac{1}{2} \sup_{\|y\|<r} ((\mathrm{dist}(u_0 - y, \psi(X)))^2 - \|y\|^2) \\ &\le \inf_{x\in X}(I(x) + \|\psi(x) - u_0\|r) - \frac{1}{2} \sup_{\|y\|<r} ((\mathrm{dist}(u_0 - y, \psi(X)))^2 - \|y\|^2) \\ &= \inf_{x\in X}(I(x) + \|\psi(x) - u_0\|r) - \sup_{\|y\|<r} \inf_{x\in X}(I(x) + \langle \psi(x) - u_0, y\rangle). \end{aligned} \tag{2.10}$$

Let us show that $(I + \frac{1}{2}\varphi, \psi) \in \mathcal{A}(X, E)$. So, fix $y \in E$. Since ψ is weakly continuous, $\langle \psi(\cdot), v\rangle$ is continuous in X for all $v \in E$. Observing that

$$I(x) + \frac{1}{2}\varphi(x) + \langle \psi(x), y\rangle = \frac{1}{2}(\|\psi(x)\|^2 + \varphi(x)) + \langle \psi(x), y - u_0\rangle + \frac{1}{2}\|u_0\|^2,$$

we infer that $I(\cdot) + \frac{1}{2}\varphi(\cdot) + \langle \psi(\cdot), y\rangle$ is lower semicontinuous since $\|\psi(\cdot)\|^2 + \varphi(\cdot)$ is so by assumption. Now, let $s \in \mathbf{R}$. We readily have

$$\begin{aligned} &\left\{x \in X : I(x) + \frac{1}{2}\varphi(x) + \langle \psi(x), y\rangle \le s\right\} \\ &\subseteq \left\{x \in X : \|\psi(x)\|^2 - 2\|y - u_0\|\|\psi(x)\| \le 2s - \inf_X \varphi\right\}. \end{aligned} \tag{2.11}$$

Since $\|\psi(\cdot)\|$ is inf-compact, the set in the right-hand side of (2.11) is compact and hence so is the set in left-hand right, as claimed. Since the set $u_0 - S$ is convex and dense in E, in view of (2.10), Theorem 2.1 ensures the existence of $\tilde{v} \in u_0 - S$, with $\|\tilde{v}\| < r$, such that the function $I(\cdot) + \langle \psi(\cdot), \tilde{v}\rangle + \frac{1}{2}\varphi(\cdot)$ has at least two global minima in X. Consequently, since

$$I(x) + \langle \psi(x), \tilde{v}\rangle + \frac{1}{2}\varphi(x) = \frac{1}{2}(\|\psi(x) + \tilde{v} - u_0\|^2 + \varphi(x)) - \frac{1}{2}(\|u_0\|^2 - \|\tilde{v} - u_0\|^2),$$

if we put

$$\tilde{y} := u_0 - \tilde{v},$$

we have $\tilde{y} \in S$, $\|\tilde{y} - u_0\| < r$ and the function $\|\psi(\cdot) - \tilde{y}\|^2 + \varphi(\cdot)$ has at least two global minima in X. The proof is complete. □

Remark 2.1. Of course, Theorem 1.1 is an immediate corollary of Theorem 2.6: take $E = H$, consider X equipped with the relative weak topology, take $\psi(x) = x$ and observe that if $\varphi : X \to \mathbf{R}$ is sequentially weakly lower semicontinuous, then $\|\cdot\|^2+\varphi(\cdot)$ is weakly lower semicontinuous in view of the Eberlein-Smulyan theorem.

Here is an application of Theorem 2.2. An operator T between two Banach spaces F_1, F_2 is said to be sequentially weakly continuous if, for every sequence $\{x_n\}$ in F_1 weakly convergent to $x \in F_1$, the sequence $\{T(x_n)\}$ converges weakly to $T(x)$ in F_2.

Theorem 2.7. *Let V be a reflexive real Banach space, let $x_0 \in V$, let $r > 0$, let X be the open ball in V, of radius r, centered at x_0, let $\gamma : [0, r[\to \mathbf{R}$, with $\lim_{\xi\to r^-} \gamma(\xi) = +\infty$, let $I : X \to \mathbf{R}$ and $\psi : X \to E$ be two Gâteaux differentiable functions. Moreover, assume that I is sequentially weakly lower semicontinous, that ψ is sequentially weakly continuous, that $\psi(X)$ is bounded and non-convex, and that*

$$\gamma(\|x - x_0\|) \leq I(x)$$

for all $x \in X$.

Then, for every convex set $S \subseteq E^*$ weakly-star dense in E^*, there exists $\tilde{\eta} \in S$ such that the equation

$$I'(x) + (\tilde{\eta} \circ \psi)'(x) = 0$$

has at least two solutions in X.

Proof. We apply Theorem 2.2 considering X equipped with the relative weak topology. Let $\eta \in E^*$. Since $\psi(X)$ is bounded, we have $c := \inf_{x\in X} \eta(\psi(x)) > -\infty$. Let $s \in \mathbf{R}$. We have

$$\begin{aligned}\{x \in X : I(x) + \eta(\psi(x)) \leq s\} &\subseteq \{x \in X : I(x) \leq s - c\} \\ &\subseteq \{x \in X : \gamma(\|x - x_0\|) \leq s - c\}. \end{aligned} \tag{2.12}$$

Since $\lim_{\xi\to r^-} \gamma(t) = +\infty$, there is $\delta \in]0, r[$, such that $\gamma(\xi) > s - c$ for all $\xi \in]\delta, r[$. Consequently, from (2.12), we obtain

$$\{x \in X : I(x) + \eta(\psi(x)) \leq s\} \subseteq \{x \in V : \|x - x_0\| \leq \delta\}. \tag{2.13}$$

From the assumptions, it follows that the function $I + \eta \circ \psi$ is sequentially weakly lower semicontinuous in X. Hence, from (2.13), since $\delta < r$ and V is reflexive, we infer that the set $\{x \in X : I(x) + \eta(\psi(x)) \leq s\}$ is sequentially weakly compact and hence weakly compact, by the Eberlein-Smulyan theorem. In other words, $(I, \psi) \in \mathcal{A}(X, E^*)$. Therefore, we can apply Theorem 2.2. Accordingly, there exists $\tilde{\eta} \in S$ such that the function $I + \tilde{\eta} \circ \psi$ has at least two global minima in X which are critical points of it since X is open. □

Here is an application of Theorem 1.1:

Theorem 2.8. *Let H be a Hilbert space and let $I, J : H \to \mathbf{R}$ be two C^1 functionals with compact derivative such that $2I - J^2$ is bounded. Moreover, assume that $J(0) \neq 0$ and that there is $\hat{x} \in H$ such that $J(-\hat{x}) = -J(\hat{x})$.*

Then, for every convex set $S \subseteq H \times \mathbf{R}$ dense in $H \times \mathbf{R}$ and for every r satisfying

$$r > \frac{\|\hat{x}\|^2 + |J(\hat{x})|^2 - \inf_{x\in H}(\|x\|^2 + |J(x)|^2) + \sup_H(2I - J^2) - \inf_X(2I - J^2)}{2\inf_{x\in H} \sqrt{\|x\|^2 + |J(x)|^2}},$$

there exists $(y_0, \mu_0) \in S$, *with* $\|y_0\|^2 + |\mu_0|^2 < r^2$, *such that the equation*

$$x + I'(x) + \mu_0 J'(x) = y_0$$

has at least three solutions.

Proof. We consider the Hilbert space $E := H \times \mathbf{R}$ with the scalar product

$$\langle (x, \lambda), (y, \mu) \rangle_E = \langle x, y \rangle + \lambda\mu$$

for all $(x, \lambda), (y, \mu) \in E$. Take

$$X = \{(x, \lambda) \in E : \lambda = J(x)\}.$$

Since J' is compact, the functional J turns out to be sequentially weakly continuous ([11], Corollary 41.9). So, the set X is sequentially weakly closed. Moreover, notice that $(0, 0) \notin X$, while the antipodal points $(\hat{x}, J(\hat{x}))$ and $-(\hat{x}, J(\hat{x}))$ lie in X. So, $(0, 0) \in \operatorname{conv}(X)$. Now, with the notations of Theorem 1.1, taking, of course, $u_0 = (0, 0)$, we have

$$\delta = \inf_{x \in X} \sqrt{\|x\|^2 + |J(x)|^2}$$

and

$$\rho_r = \sup_{\|y\|^2 + |\mu|^2 < r^2} \inf_{x \in X} (\|x\|^2 + |J(x)|^2 - 2\langle (x, J(x)), (y, \mu) \rangle_E).$$

Then, from Proposition 2.1, we infer that

$$\rho_r \le \|\hat{x}\|^2 + |J(\hat{x})|^2.$$

Now, consider the function $\varphi : X \to \mathbf{R}$ defined by

$$\varphi(x, \lambda) = 2I(x) - \lambda^2$$

for all $(x, \lambda) \in X$. Notice that φ is sequentially weakly continuous and r satisfies the inequality of Theorem 1.1. Consequently, there exists $(y_0, \mu_0) \in S$ such that the functional

$$(x, \lambda) \to \|(x, \lambda)\|_E^2 - 2\langle (x, \lambda), (y_0, \mu_0) \rangle_E + 2I(x) - \lambda^2$$

has at least two global minima in X. Of course, if $(x, \lambda) \in X$, we have

$$\|(x, \lambda)\|_E^2 - 2\langle (x, \lambda), (y_0, \mu_0) \rangle_E + 2I(x) - \lambda^2$$
$$= \|x\|^2 + J^2(x) - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x) - J^2(x).$$

In other words, the functional

$$x \to \|x\|^2 - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x)$$

has two global minima in H. Since the functional

$$x \to -2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x)$$

has a compact derivative, a well know result ([11], Example 38.25) ensures that the functional

$$x \to \|x\|^2 - 2\langle x, y_0 \rangle - 2\mu_0 J(x) + 2I(x)$$

has the Palais-Smale property and so, by Corollary 1 of [6], it possesses at least three critical points. The proof is complete. □

Remark 2.2. In Theorem 2.8, apart from being C^1 with compact derivative, the truly essential assumption on J is, of course, that its graph is not convex. This amounts to say that J is not affine. The current assumptions are made to simplify the constants appearing in the conclusion. Actually, from the proof of Theorem 2.8, the following can be obtained:

Theorem 2.9. *Let H be a Hilbert space and let $I, J : H \to \mathbf{R}$ be two C^1 functionals with compact derivative such that $2I - J^2$ is bounded. Moreover, assume that J is not affine.*

Then, for every convex set $S \subseteq H \times \mathbf{R}$ dense in $H \times \mathbf{R}$, there exists $(y_0, \lambda_0) \in S$ such that the equation

$$x + I'(x) + \lambda_0 J'(x) = y_0$$

has at least three solutions.

Remark 2.3. For $I = 0$, the conclusion of Theorem 2.9 can be obtained from Theorem 4 of [7] (see also [4]) provided that, for some $r \in \mathbf{R}$, the set $J^{-1}(r)$ is not convex. Therefore, for instance, the fact that, for any non-constant bounded C^1 function $J : \mathbf{R} \to \mathbf{R}$, there are $a, b \in \mathbf{R}$ such that the equation

$$x + aJ'(x) = b$$

has at least three solutions, follows, in any case, from Theorem 2.9, while it follows from Theorem 4 of [7] only if J is not monotone.

We conclude presenting an application of Theorem 2.7 to a class of Kirchhoff-type problems.

Theorem 2.10. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, let $\rho > 0$ and let $\omega : [0, \rho[\to [0, +\infty[$ be a continuous increasing function such that*

$$\lim_{\xi \to \rho^-} \int_0^\xi \omega(x)dx = +\infty.$$

Consider $C^0([0,1]) \times C^0([0,1])$ endowed with the norm

$$\|(\alpha, \beta)\| = \int_0^1 |\alpha(t)|dt + \int_0^1 |\beta(t)|dt.$$

Then, the following assertions are equivalent:
(a) the restriction of f to $\left[-\frac{\sqrt{\rho}}{2}, \frac{\sqrt{\rho}}{2}\right]$ is not constant;
(b) for every convex set $S \subseteq C^0([0,1]) \times C^0([0,1])$ dense in $C^0([0,1]) \times C^0([0,1])$, there exists $(\alpha, \beta) \in S$ such that the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2 dt\right) u'' = \beta(t)f(u) + \alpha(t) \text{ in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2 dt < \rho \end{cases}$$

has at least two classical solutions.

Proof. Consider the Sobolev space $H^1_0(]0,1[)$ with the usual scalar product

$$\langle u,v\rangle = \int_0^1 u'(t)v'(t)dt.$$

Let $B_{\sqrt{\rho}}$ be the open ball in $H^1_0(]0,1[$, of radius $\sqrt{\rho}$, centered at 0. Let $g:[0,1]\times\mathbf{R}\to\mathbf{R}$ be a continuous function. Consider the functionals $I, J_g : B_{\sqrt{\rho}} \to \mathbf{R}$ defined by

$$I(u) = \frac{1}{2}\tilde{\omega}\left(\int_0^1 |u'(t)|^2 dt\right),$$

$$J_g(u) = \int_0^1 \tilde{g}(t,u(t))dt$$

for all $u \in B_{\sqrt{\rho}}$, where $\tilde{\omega}(\xi) = \int_0^\xi \omega(x)dx$, $\tilde{g}(t,\xi) = \int_0^\xi g(t,x)dx$. By classical results, taking into account that if $\omega(x) = 0$ then $x = 0$, it follows that the classical solutions of the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2 dt\right) u'' = g(t,u) \text{ in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2 dt < \rho \end{cases}$$

are exactly the critical points in $B_{\sqrt{\rho}}$ of the functional $I - J_g$.
Let us prove that $(a) \to (b)$. We are going to apply Theorem 2.7 taking $V = H^1_0(]0,1[)$, $x_0 = 0$, $r = \sqrt{\rho}$, I as above, $\gamma(\xi) = \frac{1}{2}\tilde{\omega}(\xi^2)$, $E = C^0([0,1]) \times C^0([0,1])$ and $\psi : B_{\sqrt{\rho}} \to E$ defined by

$$\psi(u)(\cdot) = (u(\cdot), \tilde{f}(u(\cdot)))$$

for all $u \in B_{\sqrt{\rho}}$, where $\tilde{f}(\xi) = \int_0^\xi f(x)dx$. Clearly, the functional I is continuous and strictly convex (and so weakly lower semicontinuous), while the operator ψ is Gâteaux differentiable and sequentially weakly continuous due to the compact embedding of $H^1_0(]0,1[)$ into $C^0([0,1])$. Recall that

$$\max_{[0,1]} |u| \le \frac{1}{2}\sqrt{\int_0^1 |u'(t)|^2 dt}$$

for all $u \in H^1_0(]0,1[)$. As a consequence, the set $\psi\left(B_{\sqrt{\rho}}\right)$ is bounded and, in view of (a), non-convex. Hence, each assumption of Theorem 2.7 is satisfied. Now, consider the operator $T : E \to E^*$ defined by

$$T(\alpha,\beta)(u,v) = \int_0^1 \alpha(t)u(t)dt + \int_0^1 \beta(t)v(t)dt$$

for all $(\alpha,\beta),(u,v) \in E$. Of course, T is linear and the linear subspace $T(E)$ is total over E. Hence, $T(E)$ is weakly-star dense in E^*. Moreover, notice that T is continuous with respect to the weak-star topology of E^*. Indeed, let $\{(\alpha_n,\beta_n)\}$ be a sequence in E converging to some $(\alpha,\beta) \in E$. Fix $(u,v) \in E$. We have to show that

$$\lim_{n\to\infty} T(\alpha_n,\beta_n)(u,v) = T(\alpha,\beta)(u,v). \tag{2.14}$$

Notice that

$$\lim_{n\to\infty}\left(\int_0^1 |\alpha_n(t)-\alpha(t)|dt+\int_0^1 |\beta_n(t)-\beta(t)|dt\right)=0. \tag{2.15}$$

On the other hand, we have

$$|T(\alpha_n,\beta_n)(u,v)-T(\alpha,\beta)(u,v)|=\left|\int_0^1(\alpha_n(t)-\alpha(t))u(t)dt+\int_0^1(\beta_n(t)-\beta(t))v(t)dt\right|$$

$$\leq\left(\int_0^1 |\alpha_n(t)-\alpha(t)|dt+\int_0^1 |\beta_n(t)-\beta(t)|dt\right)\max\left\{\max_{[0,1]}|u|,\max_{[0,1]}|v|\right\}$$

and hence (2.14) follows in view of (2.15).
Finally, fix a convex set $S\subseteq C^0([0,1])\times C^0([0,1])$ dense in $C^0([0,1])\times C^0([0,1])$. Then, by the kind of continuity of T just now proved, the convex set $T(-S)$ is weakly-star dense in E^* and hence, thanks to Theorem 2.7, there exists $(\alpha_0,\beta_0)\in -S$ such that, if we put

$$g(t,\xi)=\alpha_0(t)+\beta_0(t)f(\xi),$$

the functional $I-J_g$ has at least two critical points in $B_{\sqrt{\rho}}$ which are the claimed solutions of the problem in (b), with $\alpha=-\alpha_0$ and $\beta=-\beta_0$.

Now, let us prove that $(b)\to(a)$. Assume that the restriction of f to $\left[-\frac{\sqrt{\rho}}{2},\frac{\sqrt{\rho}}{2}\right]$ is constant. Let c be such a value. So, the classical solutions of the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2dt\right)u''=c\beta(t)+\alpha(t) \text{ in } [0,1] \\ u(0)=u(1)=0 \\ \int_0^1 |u'(t)|^2dt<\rho \end{cases}$$

are the critical points in $B_{\sqrt{\rho}}$ of the functional

$$u\to\frac{1}{2}\tilde{\omega}\left(\int_0^1 |u'(t)|^2dt\right)-\int_0^1(c\alpha(t)+\beta(t))u(t)dt.$$

But, since ω is increasing and non-negative, this functional is strictly convex and so it possesses a unique critical point. The proof is complete. □

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Biagio Ricceri
Department of Mathematics and Informatics,
University of Catania,
Viale A. Doria 6,
95125 Catania, Italy
e-mail: `ricceri@dmi.unict.it`

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Multiple solution for a fourth-order nonlinear eigenvalue problem with singular and sublinear potential

Csaba Farkas, Ildikó Ilona Mezei and Zsuzsánna-Tímea Nagy

Dedicated to the memory of Professor Csaba Varga

Abstract. Let (M, g) be a Cartan-Hadamard manifold. For certain positive numbers μ and λ, we establish the multiplicity of solutions to the problem

$$\Delta_g^2 u - \Delta_g u + u = \mu \frac{u}{d_g(x_0, x)^4} + \lambda\alpha(x)f(u), \quad \text{in } M,$$

where $x_0 \in M$, while $f : \mathbb{R} \to \mathbb{R}$ is continuous function, superlinear at zero and sublinear at infinity.

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1. Introduction

The biharmonic non-linear Schrödinger equation

$$i\partial_t\psi + a\Delta^2\psi + b\Delta\psi + c|\psi|^{2w}\psi = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d,$$

where $a, w > 0$ and $b, c \in \mathbb{R}$, $c \neq 0$ has been introduced by Karpman and Shagalov [13]. The problem, because of its physical applications, has received much attention in recent years. After a Lyapunov-Schmidt type reduction, i.e., a separation of variables the previous problem reduces to a fourth-order elliptic equation. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for such problems have been extensively studied in the literature over the last decades, see for instance [4, 5, 9, 16] and reference therein.

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Similarly, in recent years singular fourth order equations have been widely studied because of their wide application to physical models such as non-Newtonian fluids, see for instance [1, 3, 12, 11, 6, 17, 18].

Most of the aforementioned papers provide existence and multiplicity results by employing different techniques as variational methods, genus theory, the Nehari manifold etc.

As far as we know, no result is available in the literature concerning singular foruth order Schrödinger systems on non-compact Riemannian manifolds. Motivated by this fact, the purpose of the present paper is to provide multiplicity results in the case of the singular foruth order Schrödinger system in such a non-compact setting. Since this problem is very general, we shall restrict our study to Hadamard manifolds (simply connected, complete Riemannian manifolds with non-positive sectional curvature).

To be more precise, let (M, g) be a d-dimensional Hadamard manifold, with $d \geq 5$ and we shall consider the following problem

$$\begin{cases} \Delta_g^2 u - \Delta_g u + u = \mu \frac{u}{d_g(x_0,x)^4} + \lambda\alpha(x) f(u), & \text{in } M \\ u \in W_g^{2,2}(M) \end{cases} \tag{$\mathscr{P}_{\lambda,\mu}$}$$

where f is a given function, while λ and μ are positive constants, and $\alpha \in L^1(M) \cap L^\infty(M)$. On the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ we assume that

(f_1) is superlinear at zero, i.e. $\lim\limits_{s \to 0} \dfrac{f(s)}{s} = 0$,

(f_2) is sublinear at infinity, i.e., $\lim\limits_{s \to \infty} \dfrac{f(s)}{s} = 0$,

(f_3) denoting by $F(s) = \displaystyle\int_0^s f(t)\,\mathrm{d}t$, finally we assume that $\sup\limits_{s \in \mathbb{R}} F(s) > 0$.

Our main result reads as follows:

Theorem 1.1. *Let (M, g) be a d-dimensional Hadamard manifold, with $d \geq 5$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous function which satisfies (f_1), (f_2) and (f_3) and $\alpha \in L^1(M) \cap L^\infty(M)$ be a non-zero, non-negative function which depends on $d_g(x_0, \cdot)$ and satisfies $\sup\limits_{R>0} \operatorname*{essinf}\limits_{d_g(x_0,x) \leq R} \alpha(x) > 0$. Then for every $\mu \in \left[0, \frac{d^2(d-4)^2}{16}\right)$ there exist an open interval $I_\mu \subset (0, +\infty)$ and a real number $\sigma_\mu > 0$ such that for every $\lambda \in I_\mu$ the problem $(\mathscr{P}_{\lambda,\mu})$ has at least two distinct nontrivial weak solutions in $W_g^{2,2}(M)$ whose $W_g^{2,2}$-norms are less than σ_μ.*

The proof of Theorem 1.1 is based on a three critical point result of Bonanno [2] (which is actually a refinement of a general principle of Ricceri [20, 19]), combined with a compact embedding result(see Farkas, Kristály and Mester [8]) combined with variational arguments.

2. Preliminaries

Let (M,g) be a complete non-compact Riemannian manifold with $\dim M = d$. Let T_xM be the tangent space at $x \in M$, $TM = \bigcup_{x\in M} T_xM$ be the tangent bundle, and $d_g : M \times M \to [0,+\infty)$ be the distance function associated to the Riemannian metric g. Let $B_g(x,\rho) = \{y \in M : d_g(x,y) < \rho\}$ be the open metric ball with center x and radius $\rho > 0$; if dv_g is the canonical volume element on (M,g), the volume of a bounded open set $\Omega \subset M$ is $\mathrm{Vol}_g(\Omega) = \int_\Omega \mathrm{d}v_g = \mathcal{H}^d(\Omega)$. If $\mathrm{d}\sigma_g$ denotes the $(d-1)$-dimensional Riemannian measure induced on $\partial\Omega$ by g, then

$$\mathrm{Area}_g(\partial\Omega) = \int_{\partial\Omega} \mathrm{d}\sigma_g = \mathcal{H}^{d-1}(\partial\Omega)$$

stands for the area of $\partial\Omega$ with respect to the metric g. Hereafter, $\mathcal{H}^l$ denotes the l-dimensional Hausdorff measure.

Let $p > 1$. The norm of $L^p(M)$ is given by

$$\|u\|_p = \left(\int_M |u|^p \mathrm{d}v_g\right)^{1/p}.$$

Let $u : M \to \mathbb{R}$ be a function of class C^1. If (x^i) denotes the local coordinate system on a coordinate neighbourhood of $x \in M$, and the local components of the differential of u are denoted by $u_i = \frac{\partial u}{\partial x_i}$, then the local components of the gradient $\nabla_g u$ are $u^i = g^{ij}u_j$. Here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$. In particular, for every $x_0 \in M$ one has the eikonal equation

$$|\nabla_g d_g(x_0,\cdot)| = 1 \text{ a.e. on } M. \tag{2.1}$$

When no confusion arises, if $X, Y \in T_xM$, we simply write $|X|$ and $\langle X,Y\rangle$ instead of the norm $|X|_x$ and inner product $g_x(X,Y) = \langle X,Y\rangle_x$, respectively.

The $L^p(M)$ norm of $\nabla_g u : M \to TM$ is given by

$$\|\nabla_g u\|_p = \left(\int_M |\nabla_g u|^p \mathrm{d}v_g\right)^{\frac{1}{p}}.$$

The space $W_g^{2,2}(M)$ is the completion of $C_0^\infty(M)$ with respect to the norm

$$\|u\|^2_{W_g^{1,2}(M)} = \|u\|_2^p + \|\nabla_g u\|_2^2 + \|\Delta_g u\|_2^2.$$

Let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$, and let $\mathcal{O}_G^x = \{\xi x : \xi \in G\}$ be the orbit of the element $x \in M$. The action of G on $W_g^{2,2}(M)$ is defined by

$$(\xi u)(x) = u(\xi^{-1}x) \text{ for all } x \in M,\ \xi \in G,\ u \in W_g^{1,p}(M), \tag{2.2}$$

where $\xi^{-1} : M \to M$ is the inverse of the isometry ξ. We say that a continuous action of a group G on a complete Riemannian manifold M is *coercive* (see Tintarev [22, Definition 7.10.8] or Skrzypczak and Tintarev [21, Definition 1.2]) if for every $t > 0$, the set

$$\mathscr{O}_t = \{x \in M : \mathrm{diam}\mathcal{O}_G^x \le t\}$$

is bounded.

Let $m(y,\rho)$ be the maximal number of mutually disjoint geodesic balls with radius ρ on $\mathcal{O}_G^y$

$$m(y,\rho) = \sup\{n \in \mathbb{N} : \exists \xi_1,\dots,\xi_n \in G : B_g(\xi_i y,\rho) \cap B_g(\xi_j y,\rho) = \emptyset, \forall i \neq j\}$$

We also define

$$W_{g,G}^{2,2}(M) = \{u \in W_g^{2,2}(M) : \xi u = u \text{ for all } \xi \in G\}$$

be the subspace of G-invariant functions of $W_g^{2,2}(M)$.

Theorem 2.1 ([8], Theorem 1.1). *Let (M,g) be a d-dimensional Hadamard manifold, and let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix_M}(G) \neq \emptyset$. Then the following statements are equivalent:*

(i) *G is coercive;*
(ii) $\mathrm{Fix}_M(G)$ *is a singleton*;
(iii) *$m(y,\rho) \to \infty$ as $d_g(x_0,y) \to \infty$.*

Moreover, from any of the above statements it follows that the embedding $W_{g,G}^{2,2}(M) \subset W_{g,G}^{1,2}(M) \hookrightarrow L^q(M)$ is compact for every $2 \leq q < 2^{\#} = \frac{2d}{d-4}$ if $1 < p < d$.

In order to prove Theorem 1.1, we recall an abstract tool, which is the following critical point result of Bonanno [2] (which is actually a refinement of a general principle of Ricceri [20, 19]):

Theorem 2.2 ([2], Theorem 2.1). *Let X be a separable and reflexive real Banach space, and let $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals, such that $\Phi(u) \geq 0$ for every $u \in X$. Assume that there exist $u_0, u_1 \in X$ and $\rho > 0$ such that*

(1) $\Phi(u_0) = J(u_0) = 0$,
(2) $\rho < \Phi(u_1)$,
(3) $\displaystyle\sup_{\Phi(u)<\rho} J(u) < \rho \frac{J(u_1)}{\Phi(u_1)}$.

Further, put

$$\overline{a} = \zeta\rho \left(\rho \frac{J(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<\rho} J(u)\right)^{-1}, \quad \textit{where } \zeta > 1,$$

and assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(4) $\displaystyle\lim_{\|u\|\to\infty} (\Phi(u) - \lambda J(u)) = +\infty$, *for all $\lambda \in [0,\overline{a}]$.*

Then there exists an open interval $\Lambda \subset [0,\overline{a}]$ and a number $\mu > 0$ such that for each $\lambda \in \Lambda$, the equation $\Phi'(u) - \lambda J'(u) = 0$ admits at least three solutions in X having norm less than μ.

We conclude this section by stating the Rellich inequality: if (M,g) is a Hadamard manifold with $\dim M = d \geq 5$, then we have the following inequality (see for instance [15])

$$\int_M (\Delta_g u)^2 \,\mathrm{d}v_g \geq \frac{d^2(d-4)^2}{16} \int_M \frac{u^2}{d_g^4(x_0,x)} \,\mathrm{d}v_g, \ \forall u \in W_g^{2,2}(M). \tag{2.3}$$

where the constant is $\frac{d^2(d-4)^2}{16}$ sharp, but are never achieved

3. Proof of the main result

As in usual case we associate the energy functional with the problem $(\mathscr{P}_{\lambda,\mu})$, $E_{\lambda,\mu} : M \to \mathbb{R}$,

$$E_{\lambda,\mu}(u) = \int_M (\Delta_g u)^2 + |\nabla_g u|^2 + u^2 \, \mathrm{d}v_g \\ - \mu \int_M \frac{u^2}{d_g(x_0,x)^4} \, \mathrm{d}v_g - \lambda \int_M \alpha(x) F(u) \, \mathrm{d}v_g.$$

Based on the assumption of the continuous function f, a standard argument shows that $E_{\lambda,\mu} : W_g^{2,2}(M) \to \mathbb{R}$ is of class C^1 and its critical points are exactly the weak solutions of the studied problem. Therefore, it is enough to show the existence of multiple critical points of $E_{\lambda,\mu}$. For further use, let us denote by

$$\Phi_{\mu,0}(u) = \int_M (\Delta_g u)^2 + |\nabla_g u|^2 + u^2 \, \mathrm{d}v_g - \mu \int_M \frac{u^2}{d_g(x_0,x)^4} \, \mathrm{d}v_g$$

and

$$J_0(u) = \int_M \alpha(x) F(u) \, \mathrm{d}v_g.$$

Having in our mind the compactness result, see Theorem 2.1, we restrict the energy functional to the space $W_{g,G}^{2,2}(M)$. For simplicity, in the following we denote

$$\mathcal{E}_{\lambda,\mu} = E_{\lambda,\mu}|_{W_{g,G}^{2,2}(M)}, \quad \Phi_\mu = \Phi_{\mu,0}|_{W_{g,G}^{2,2}(M)}, \quad \text{and } J = J_0|_{W_{g,G}^{2,2}(M)}.$$

Lemma 3.1. *Let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ with $\mathrm{Fix}_M(G) = \{x_0\}$. Then $E_{\lambda,\mu}$ is G-invariant.*

Proof of Lemma 3.1. Let $u \in W_g^{2,2}(M)$ and $\sigma \in G$ be arbitrarily fixed. Since $\sigma : M \to M$ is an isometry on M, by (2.2), for every $x \in M$ we have

$$\nabla_g(\sigma u)(x) = D\sigma_{\sigma^{-1}(x)} \nabla_g u(\sigma^{-1}(x)),$$

where $D\sigma_{\sigma^{-1}(x)} : T_{\sigma^{-1}(x)}M \to T_x M$ denotes the differential of σ at the point $\sigma^{-1}(x)$. Note that the (signed) Jacobian determinant of σ is 1 and $D\sigma_{\sigma^{-1}(x)}$ preserves inner products. Therefore, by using the latter facts, relation (2.2) and a change of variables $y = \sigma^{-1}(x)$, it turns out that

$$\int_M \left(|\nabla_g(\sigma u)(x)|_x^2 + |(\sigma u)(x)|^2\right) \mathrm{d}v_g(x) \\ = \int_M \left(|\nabla_g u(\sigma^{-1}(x))|_{\sigma^{-1}(x)}^2 + |u(\sigma^{-1}(x))|^2\right) \mathrm{d}v_g(x) \\ = \int_M \left(|\nabla_g u(y)|_y^2 + |u(y)|^2\right) \mathrm{d}v_g(y),$$

We claim that

$$\Delta_g\left((\sigma \circ u)(x)\right) = \Delta_g u(\sigma^{-1}(x)).$$

To prove this claim, we choose an arbitrary test function φ, then we consider the following integral

$$\begin{aligned}
&\int_M \Delta_g\left((\sigma\circ u)(x)\right)\varphi(\sigma^{-1}(x))\mathrm{d}v_g(x)\\
&= -\int_M \langle D\sigma_{\sigma^{-1}(x)}\nabla_g u(\sigma^{-1}(x)), D\sigma_{\sigma^{-1}(x)}\varphi(\sigma^{-1}(x))\rangle \mathrm{d}v_g(x)\\
&= -\int_M \langle \nabla_g u(\sigma^{-1}(x)), \varphi(\sigma^{-1}(x))\rangle \mathrm{d}v_g(x)\\
&= -\int_M \langle \nabla_g u(y), \varphi(y)\rangle \mathrm{d}v_g(y)\\
&= \int_M \Delta_g u(y)\varphi(y)\mathrm{d}v_g(y)\\
&= \int_M \Delta_g u(\sigma^{-1}(x))\varphi(\sigma^{-1}(x))\mathrm{d}v_g(x),
\end{aligned}$$

the arbitrariness of the function φ proves the claim. Finally, since $\sigma\in G$ and $\alpha\in L^1(M)\cap L^\infty(M)$ is a non-zero, non-negative function which depends on $d_g(x_0,\cdot)$ and $\mathrm{Fix}_M(G)=\{x_0\}$, it turns out that for every $u\in W^{2,2}_{g,G}(M)$, we have $J_0(\sigma u)=J_0(u)$, which concludes the proof. □

The principle of symmetric criticality of Palais (see Kristály, Rădulescu and Varga [14, Theorem 1.50]) and the previous Lemma imply that the critical points of $\mathcal{E}_{\lambda,\mu}=E_{\lambda,\mu}|_{W^{2,2}_{g,G}(M)}$ are also critical points of the original functional $E_{\lambda,\mu}$. Therefore, it is enough to find critical points of $\mathcal{E}_{\lambda,\mu}$.

Lemma 3.2. *For every $\mu\in\left[0,\frac{d^2(d-4)^2}{16}\right)$ and $\lambda\in\mathbb{R}_+$, the functional $\mathcal{E}_{\lambda,\mu}$ is sequentially weakly lower semicontinuous on $W^{2,2}_{g,G}(M)$.*

Proof. First we prove that the functional Φ_μ is sequentially weakly lower semicontinuous on $W^{2,2}_g(M)$. To this end, we consider $u,v\in W^{2,2}_g(M)$ and $t\in[0,1]$, and thus

$$\begin{aligned}
\Phi_\mu(tu+(1-t)v) &= \int_M \left(\Delta_g(tu+(1-t)v)\right)^2 \mathrm{d}v_g + \int_M |\nabla_g(tu+(1-t)v)|^2\,\mathrm{d}v_g\\
&\quad + \int_M (tu+(1-t)v)^2\,\mathrm{d}v_g - \mu\int_M \frac{(tu+(1-t)v)^2}{d_g^4(x_0,x)}\mathrm{d}v_g\\
&\le \int_M \left(\Delta_g(tu+(1-t)v)\right)^2 \mathrm{d}v_g + \int_M t|\nabla_g u|^2+(1-t)|\nabla_g v|^2\,\mathrm{d}v_g\\
&\quad + \int_M tu^2+(1-t)v^2\,\mathrm{d}v_g - \mu\int_M \frac{(tu+(1-t)v)^2}{d_g^4(x_0,x)}\mathrm{d}v_g.
\end{aligned}$$

Now, using the following identity

$$(ta+(1-t)b)^2 = ta^2+(1-t)b^2-t(1-t)(a-b)^2,$$

we get that

$$\Phi_\mu(tu+(1-t)v) \leq t\Phi_\mu(u) + (1-t)\Phi_\mu(v) - t(1-t)\left(\int_M (\Delta_g(u-v))^2 \, dv_g - \mu \int_M \frac{(u-v)^2}{d_g^4(x_0,x)} \, dv_g\right).$$

Using the Rellich inequality (2.3) (see also Kristály and Repovs [15]), one has that

$$\int_M (\Delta_g(u-v))^2 \, dv_g - \mu \int_M \frac{(u-v)^2}{d_g^4(x_0,x)} \, dv_g \geq 0,$$

for every $u, v \in W_g^{2,2}(M)$, thus

$$\Phi_\mu(tu+(1-t)v) \leq t\Phi_\mu(u) + (1-t)\Phi_\mu(v).$$

Thus Φ_μ is positive and convex, therefore is sequentially weakly lower semicontinous.

It remains to prove that J is sequentially weakly continuous. To this end, consider a sequence $\{u_k\}_k$ in $W_{g,G}^{2,2}(M)$ which converges weakly to $u \in W_{g,G}^{2,2}(M)$, and suppose that

$$J(u_k) \not\to J(u_k) \quad \text{as } k \to \infty.$$

Thus, there exist $\varepsilon > 0$ and a subsequence of $\{u_n\}_n$, denoted again by $\{u_n\}_n$, such that $u_n \to u$ in $L^\infty(M)$ and

$$0 < \varepsilon \leq |J(u_k) - J(u)|, \quad \text{for every } k \in \mathbb{N}.$$

Thus, by the mean value theorem, there exists $\theta_k \in (0,1)$ such that

$$\begin{aligned} 0 < \varepsilon &\leq |\langle J'(u + \theta_k(u_k - u)), u_k - u\rangle| \\ &\leq \int_M \alpha(x)|f(u + \theta_k(u_k - u))| \cdot |u_k - u| \, dv_g. \end{aligned}$$

Using the assumptions (f_1), (f_2) and the Hölder inequality the last term tends to 0, which provides a contradiction. □

Lemma 3.3. *For every $\mu \in \left[0, \frac{d^2(d-4)^2}{16}\right)$ and $\lambda \in \mathbb{R}_+$, the functional $\mathcal{E}_{\lambda,\mu}$ is coercive and satisfies the Palais-Smale condition.*

Proof. First we prove that the functional $\mathcal{E}_{\lambda,\mu}$ is coercive. Let us fix $\mu \in \left[0, \frac{n^2(n-4)^2}{16}\right)$ and $\lambda \in \mathbb{R}_+$. We denote $\overline{\mu} = \frac{n^2(n-4)^2}{16}$. By the (f_1) and (f_2) for every $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0,1)$ such that

$$|f(s)| \leq \varepsilon |s| \ \text{ for all} |s| \leq \delta_\varepsilon \ \text{ and } |s| \geq \delta_\varepsilon^{-1}.$$

Since $f \in C(\mathbb{R}, \mathbb{R})$, there also exists a number $M_\varepsilon > 0$ such that

$$\frac{|f(s)|}{|s|^q} \leq M_\varepsilon \ \text{ for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where $q \in (0,1)$. Therefore

$$|f(s)| \leq \varepsilon|s| + M_\varepsilon |s|^q, \quad \text{for all } s \in \mathbb{R}. \tag{3.1}$$

Thus, for every $u \in W^{2,2}_{g,G}(M)$ we have

$$\begin{aligned}\mathcal{E}_{\lambda,\mu} &\geq \frac{1}{2}\left(1-\frac{\mu}{\overline{\mu}}\right)\|u\|^2 - \lambda\int_M \alpha(x)|F(u)|\,\mathrm{d}v_g \\ &\geq \frac{1}{2}\left(1-\frac{\mu}{\overline{\mu}}\right)\|u\|^2 - \frac{1}{2}\lambda\|\alpha\|_\infty\varepsilon\|u\|^2 - \frac{\lambda M_\varepsilon C}{q+1}\|u\|^{q+1}.\end{aligned}$$

If $\|u\| \to \infty$ we conclude that $\mathcal{E}_{\lambda,\mu}(u) \to \infty$ as well, i.e. $\mathcal{E}_{\lambda,\mu}$ is coercive. Now, let $\{u_k\}_k$ be a sequence in $W^{2,2}_{g,G}(M)$ such that $\{\mathcal{E}_{\lambda,\mu}(u_k)\}_k$ is bounded and $\|\mathcal{E}'_{\lambda,\mu}(u_k)\|_* \to 0$. Since $\mathcal{E}_{\lambda,\mu}$ is coercive, the sequence $\{u_k\}_k$ is bounded in $W^{2,2}_{g,G}(M)$. Therefore, up to a subsequence, $u_k \rightharpoonup u$ weakly in $W^{2,2}_{g,G}(M)$ for some $u \in W^{2,2}_{g,G}(M)$.
Hence, due to Theorem Theorem 2.1, it follows that $u_k \to u$ strongly in $L^p(M)$.
In particular, we have that

$$\mathcal{E}'_{\lambda,\mu}(u)(u-u_k) \to 0 \quad \text{and} \quad \mathcal{E}'_{\lambda,\mu}(u_k)(u-u_k) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.2}$$

On the one hand, it is easy to verify that

$$\begin{aligned}\left(1-\frac{\mu}{\overline{\mu}}\right)\|u_k-u\|^2 \leq &\|u_k-u\|^2 - \mu\int_M \frac{(u_k-u)^2}{d_g^4(x_0,x)}\,\mathrm{d}v_g \\ =&\mathcal{E}'_{\lambda,\mu}(u)(u-u_k) + \mathcal{E}'_{\lambda,\mu}(u_k)(u-u_k) \\ &+ \lambda\int_M \alpha(x)[f(u_k)-f(u)](u_k(x)-u(x))\,\mathrm{d}v_g.\end{aligned}$$

On the other hand, by means of (f_1) and (f_2) one has that

$$\int_M \alpha(x)[f(u_k)-f(u)](u_k(x)-u(x))\,\mathrm{d}v_g \to 0,$$

as $k \to \infty$. Thus we proved that $\|u_k - u\| \to 0$, which proves the claim. □

Lemma 3.4. *For every* $\mu \in \left[0, \frac{d^2(d-4)^2}{16}\right)$

$$\lim_{\rho\to 0^+} \frac{\sup\{J(u):\ \Phi_\mu(u) < \rho\}}{\rho} = 0.$$

Proof. Fix $\mu \in [0,\overline{\mu})$. Using again (f_1), for every $\varepsilon > 0$ there exists $\delta > 0$

$$|f(s)| < \frac{\varepsilon}{4}\left(1-\frac{\mu}{\overline{\mu}}\right)\|\alpha\|_\infty^{-1}\kappa_2^{-2}|s| \text{ for all } |s| < \delta.$$

For fixed $p > 2$, one has the following inequality

$$|F(s)| \leq \frac{\varepsilon}{4}\left(1-\frac{\mu}{\overline{\mu}}\right)\|\alpha\|_\infty^{-1}\kappa_2^{-2}|s| + c(\varepsilon)|s|^p \quad \text{for all } s \in \mathbb{R}.$$

For $\rho > 0$ define the sets

$$S^1_\rho = \{u : \Phi_\mu(u) < \rho\};\quad S^2_\rho = \{u : (1-\mu/\overline{\mu})\|u\| < 2\rho\}.$$

Using the Rellich inequality, we have that $S^1_\rho \subseteq S^2_\rho$. Moreover, for every $u \in S^2_\rho$ we have that

$$J(u) = \int_M \alpha(x)F(u)\,\mathrm{d}v_g \leq \frac{\varepsilon}{2}\rho + c\rho^{\frac{p}{2}}.$$

Thus there exists $\rho(\varepsilon) > 0$ such that for every $0 < \rho < \rho(\varepsilon)$

$$0 \le \frac{\sup\limits_{u\in S_\rho^1} J(u)}{\rho} \le \frac{\sup\limits_{u\in S_\rho^2} J(u)}{\rho} \le \frac{\varepsilon}{2} + c'\rho^{\frac{p-2}{2}} < \varepsilon,$$

which completes the proof. □

Proof of Theorem 1.1. Fix $\mu \in [0, \overline{\mu})$. We recall that $\sup\limits_{R>0} \operatorname*{essinf}\limits_{d_g(x_0,x)\le R} \alpha(x) > 0$, thus we choose an $R_0 > 0$ such that $\alpha_{R_0} := \operatorname*{essinf}\limits_{d_g(x_0,x)\le R_0} \alpha(x) > 0$.

From the assumption (f_3) there exists $s_0 > 0$ such that $F(s_0) > 0$. Let $u_\varepsilon \in W^{2,2}_{g,G}(M)$ such that $u_\varepsilon(x) = s_0$ for any $x \in B_g(x_0, \varepsilon R_0)$, $u_\varepsilon(x) = 0$ for any $M \setminus B_g(x_0, R_0)$, and $\|u_\varepsilon\|_\infty \le |s_0|$. We also have

$$\begin{aligned} J(u_\varepsilon) &\ge \alpha_{R_0} F(s_0)\mathrm{Vol}_g(B_g(x_0, \varepsilon R_0)) \\ &\quad - \|\alpha\|_\infty \max_{|t|\le|s_0|} |F(t)|\mathrm{Vol}_g\left(B_g(x_0, R_0) \setminus B_g(x_0, \varepsilon R_0)\right), \end{aligned}$$

For ε close enough to 1, the right-hand side of the last inequality becomes strictly positive; choose such a number, say ε_0. Now, taking into account Lemma 3.4, one can fix a small number $\rho = \rho(\varepsilon_0)$ such that

$$2\rho < \left(1 - \frac{\mu}{\overline{\mu}}\right) \|u\|^2,$$

$$\frac{\sup\{J(u) : \Phi_\mu(u) < \rho\}}{\rho} < \frac{2J(u_{\varepsilon_0})}{\|u_{\varepsilon_0}\|^2}.$$

In Theorem 2.2 we choose $u_1 = u_{\varepsilon_0}$ and $u_0 = 0$, and observe that the hypotheses (2) and (3) are satisfied. We define

$$\overline{a} = \frac{1+\rho}{\frac{J(u_{\varepsilon_0})}{\Phi(u_{\varepsilon_0})} - \frac{\sup\{J(u):\Phi_\mu(u)\le\rho\}}{\rho}}.$$

Taking into account Lemmas, 3.2 and 3.3, all the assumptions of Theorem 2.2 are verified. Thus there exists an open interval $I_\mu \subset [0, \overline{a}]$ and a number $\sigma_\mu > 0$ such that for each $\lambda \in I_\mu$, the equation $\mathcal{E}'_{\lambda,\mu}(u) = \Phi'_\mu(u) - \lambda J'(u)$ admits at least three solutions in $W^{2,2}_{g,G}(M)$ having $W^{2,2}_g(M)$-norms less than σ_μ. This concludes the proof. □

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Csaba Farkas
Sapientia Hungarian University of Transylvania,
Faculty of Technical and Human Sciences,
2, Sighişoarei Street,
540485 Tg. Mureş, Romania
e-mail: farkascs@ms.sapientia.ro & farkas.csaba2008@gmail.com

Ildikó Ilona Mezei
Babeş-Bolyai University,
Faculty of Mathematics and Computer Sciences,
1, Kogălniceanu Street,
400084 Cluj-Napoca, Romania
e-mail: ildiko.mezei@ubbcluj.ro

Zsuzsánna-Tímea Nagy
Sapientia Hungarian University of Transylvania,
Faculty of Technical and Human Sciences,
2, Sighişoarei Street,
540485 Tg. Mureş, Romania
e-mail: nagy.zsuzsa@ms.sapientia.ro

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Fuzzy differential subordinations connected with convolution

Sheza M. El-Deeb and Alina Alb Lupaş

Abstract. The object of the present paper is to obtain several fuzzy differential subordinations associated with Linear operator

$$\mathcal{D}^m_{n,\delta,g} f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\, c^n(\delta)\right]^m a_j b_j z^j .$$

Using the operator $\mathcal{D}^m_{n,\delta,g}$, we also introduce a class $\mathcal{H}^F_{n,m,\delta}(\eta, g)$ of univalent analytic functions for which we give some properties.

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1. Introduction

Let $\Omega \subset \mathbb{C}$, $H(\Omega)$ the class of holomorphic functions on Ω and denote by $H_d(\Omega)$ the class of holomorphic and univalent functions on Ω. In this paper, we denote by $H(\Delta)$ the class of holomorphic functions in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with $B_\Delta = \{z \in \mathbb{C} : |z| = 1\}$ the boundary of the unit disk. For $\beta \in \mathbb{C}$ and $d \in \mathbb{N}$, we denote

$$H[\beta, d] = \left\{ f \in H(\Delta) : \ f(z) = \beta + \sum_{j=d+1}^{\infty} a_j z^j, \quad z \in \Delta \right\},$$

$$\mathbb{A}_d = \left\{ f \in H(\Delta) : \ f(z) = z + \sum_{j=d+1}^{\infty} a_j z^j, \quad z \in \Delta \right\} \quad \text{with} \quad \mathbb{A}_1 = \mathbb{A},$$

and

$$\mathcal{S} = \{ f \in \mathbb{A} : \ f \text{ is a univalent function in } \Delta \} .$$

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We denote by

$$\mathcal{C}=\left\{f\in\mathbb{A}:\ \Re\left(1+\frac{zf^{''}(z)}{f^{'}(z)}\right)>0,\ z\in\Delta\right\},$$

the set of convex functions in Δ.

Definition 1.1. [4, 11] Let f_1 and f_2 are analytic function in Δ, then f_1 is subordinate to f_2, written $f_1\prec f_2$ if there exists a Schwarz function w, which is analytic in Δ with $w(0)=0$ and $|w(z)|<1$ for all $z\in\Delta$, such that $f_1(z)=f_2(w(z))$. Furthermore, if the function f_2 is univalent in Δ, then we have the following equivalence:

$$f_1(z)\prec f_2(z)\Leftrightarrow f_1(0)=f_2(0)\text{ and }f_1(\Delta)\subset f_2(\Delta).$$

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and propositions:

Definition 1.2. [10] Fuzzy subset of $\mathcal{Y}$ is a pair $(\mathcal{B},\mathcal{F}_{\mathcal{B}})$, with $\mathcal{F}_{\mathcal{B}}:\mathcal{Y}\to[0,1]$ and

$$\mathcal{B}=\{x\in\mathcal{Y}:0<\mathcal{F}_{\mathcal{B}}(x)\le 1\}. \tag{1.1}$$

The support of the fuzzy set $(\mathcal{B},\mathcal{F}_{\mathcal{B}})$ is the set $\mathcal{B}$ and the membership function of $(\mathcal{B},\mathcal{F}_{\mathcal{B}})$ is $\mathcal{F}_{\mathcal{B}}$.

Proposition 1.3. [12] *(i) If $(\mathcal{B},\mathcal{F}_{\mathcal{B}})=(\mathcal{U},\mathcal{F}_{\mathcal{U}})$, then we have $\mathcal{B}=\mathcal{U}$, where*

$$\mathcal{B}=\sup(\mathcal{B},\mathcal{F}_{\mathcal{B}})\ \text{ and }\mathcal{U}=\sup(\mathcal{U},\mathcal{F}_{\mathcal{U}});$$

(ii) If $(\mathcal{B},\mathcal{F}_{\mathcal{B}})\subseteq(\mathcal{U},\mathcal{F}_{\mathcal{U}})$, then we have $\mathcal{B}\subseteq\mathcal{U}$, where

$$\mathcal{B}=\sup(\mathcal{B},\mathcal{F}_{\mathcal{B}})\ \text{ and }\mathcal{U}=\sup(\mathcal{U},\mathcal{F}_{\mathcal{U}}).$$

Let $f,g\in H(\Omega)$, we denote by

$$f(\Omega)=\left\{f(z):\ 0<\mathcal{F}_{f(\Omega)}f(z)\le 1,\ z\in\Omega\right\}=\sup\left(f(\Omega),\mathcal{F}_{f(\Omega)}\right), \tag{1.2}$$

and

$$g(\Omega)=\left\{g(z):\ 0<\mathcal{F}_{g(\Omega)}g(z)\le 1,\ z\in\Omega\right\}=\sup\left(g(\Omega),\mathcal{F}_{g(\Omega)}\right). \tag{1.3}$$

Definition 1.4. [12] Let $z_0\in\Omega$ be a fixed point and let the functions $f,g\in H(\Omega)$. The function f is said to be fuzzy subordinate to g and write $f\prec_{\mathcal{F}}g$ or $f(z)\prec_{\mathcal{F}}g(z)$, if are satisfied the following conditions:

(i) $f(z_0)=g(z_0)$

(ii) $\mathcal{F}_{f(\Omega)}f(z)\le\mathcal{F}_{g(\Omega)}g(z),\quad z\in\Omega.$

Proposition 1.5. [12] *Assume that $z_0\in\Omega$ is a fixed point and the functions $f,\mathrm{g}\in H(\Omega)$. If $f(z)\prec_{\mathcal{F}}g(z)$, $z\in\Omega$, then*

(i) $f(z_0)=g(z_0)$

(ii) $f(\Omega)\subseteq g(\Omega),\quad\mathcal{F}_{f(\Omega)}f(z)\le\mathcal{F}_{\mathrm{g}(\Omega)}g(z),\ z\in\Omega,$

where $f(\Omega)$ and $\mathrm{g}(\Omega)$ are defined by (1.2) and (1.3), respectively.

Definition 1.6. [13] Assume that $\Phi:\mathbb{C}^3\times\Delta\to\mathbb{C}$ and $h\in\mathcal{S}$, with $\Phi(\alpha,0,0;0)=h(0)=\alpha$. If p is analytic in Δ, with $p(0)=\alpha$ and satisfies the second order fuzzy differential subordination

$$\mathcal{F}_{\Phi(\mathbb{C}^3\times\Delta)}\Phi\left(p(z),zp^{'}(z),z^2p^{''}(z);z\right)\le\mathcal{F}_{h(\Delta)}h(z),$$

$$\text{i.e.} \quad \Phi\left(p(z), zp'(z), z^2p''(z); z\right) \prec_{\mathcal{F}} h(z), \quad z \in \Delta, \tag{1.4}$$

then p is said to be a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions for the fuzzy differential subordination if

$$\mathcal{F}_{p(\Delta)}p(z) \leq \mathcal{F}_{q(\Delta)}q(z), \quad \text{i.e.} \quad p(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta$$

for all p satisfying (1.4).
A fuzzy dominant $\widetilde{q}$ that satisfies

$$\mathcal{F}_{\widetilde{q}(\Delta)}\widetilde{q}(z) \leq \mathcal{F}_{q(\Delta)}q(z), \quad \text{i.e.} \quad \widetilde{q}(z) \prec_{\mathcal{F}} q(z), \quad z \in \Delta$$

for all fuzzy dominants q of (1.4) is called the fuzzy best dominant of (1.4).

Making use the binomial series

$$(1-\delta)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \ \delta^i \qquad (n \in \mathbb{N} = \{1, 2, ...\}),$$

for $f \in \mathbb{A}$, we introduced the linear differential operator as follows:

$$\mathcal{D}^0_{n,\delta,g}f(z) = (f * g)(z),$$

$$\begin{aligned}
\mathcal{D}^1_{n,\delta,g}f(z) &= \mathcal{D}_{n,\delta,g}f(z) = (1-\delta)^n (f*g)(z) + [1-(1-\delta)^n]\, z\, (f*g)'(z) \\
&= z + \sum_{j=2}^{\infty} [1 + (j-1)\, c^n(\delta)]\, a_j b_j z^j \\
&\ \ \vdots \\
\mathcal{D}^m_{n,\delta,g}f(z) &= \mathcal{D}_{n,\delta,g}\left(\mathcal{D}^{m-1}_{n,\delta,g}f(z)\right) \\
&= (1-\delta)^n \mathcal{D}^{m-1}_{n,\delta,g}f(z) + [1-(1-\delta)^n]\, z \left(\mathcal{D}^{m-1}_{n,\delta,g}f(z)\right)'
\end{aligned}$$

$$= z + \sum_{j=2}^{\infty} [1 + (j-1)\, c^n(\delta)]^m\, a_j b_j z^j \tag{1.5}$$

$$(\delta > 0,\ n \in \mathbb{N},\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where

$$c^n(\delta) = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \ \delta^i \qquad (n \in \mathbb{N}).$$

From (1.5), we obtain that

$$c^n(\delta)\ z\ \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' = \mathcal{D}^{m+1}_{n,\delta,g}f(z) - [1 - c^n(\delta)]\, \mathcal{D}^m_{n,\delta,g}f(z).$$

By specializing the parameters n, δ and b_j, we note that
(i) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$), then $\mathcal{D}^m_{n,\delta,\frac{z}{1-z}} = \mathcal{D}^m_{n,\delta}$ defined by Yousef et al. [17].
(ii) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$) and $n = 1$, then $\mathcal{D}^m_{1,\delta,\frac{z}{1-z}} = \mathcal{D}^m_{\delta}$ defined by Al-Oboudi [3].

(iii) Putting $b_j = 1$ (or $g(z) = \frac{z}{1-z}$) and $n = \delta = 1$, then $\mathcal{D}^m_{1,1,\frac{z}{1-z}} = \mathcal{D}^m$ defined by Sălăgean.[15].

(iv) Putting $b_j = \left(\frac{\ell+1}{\ell+j}\right)^\alpha$ $(\alpha > 0,\ \ell > -1)$ and $n = 1$, then $\mathcal{D}^m_{1,\delta,g} = \mathcal{I}^{m,\alpha}_{\ell,\delta} f(z)$ defined by El-Deeb and Lupaş [6].

(v) Putting $b_j = \left(\frac{\alpha+1}{\alpha+j}\right)^n \frac{m^{j-1}}{(j-1)!} e^{-m}$ $(m, \alpha \geq 0,\ n \in \mathbb{N}_0)$ and $m = 0$, then $\mathcal{D}^0_{n,\delta,g} = \mathcal{H}^n_{\alpha,m} f(z)$ defined by El-Deeb and Oros [9].

(vi) Putting $b_j = \frac{(-1)^{k-1}\Gamma(\upsilon+1)}{4^{k-1}(k-1)!\Gamma(k+\upsilon)} \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}}$, $(\upsilon > 0,\ \lambda > -1,\ 0 < q < 1)$ studied by El-Deeb and Bulboacă [7] and El-Deeb [5], we obtain the operator $\mathcal{N}^{m,\lambda,q}_{\upsilon,n,\delta}$, defined as follows:

$$\mathcal{N}^{m,\lambda,q}_{\upsilon,n,\delta} f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\, c^n(\delta)\right]^m \frac{(-1)^{j-1}\Gamma(\upsilon+1)}{4^{j-1}(j-1)!\Gamma(j+\upsilon)} a_j z^j$$
$$(\lambda > -1;\ 0 < q < 1;\ \delta, \upsilon > 0;\ n \in \mathbb{N};\ m \in \mathbb{N}_0)\,;$$

(vi) Putting $b_j = \left(\frac{\ell+1}{\ell+j}\right)^\alpha \cdot \frac{[k,q]!}{[\lambda+1,q]_{k-1}}$, $(\alpha > 0,\ n \geq 0,\ \lambda > -1,\ 0 < q < 1)$ studied by El-Deeb and Bulboacă [8] and Srivastava and El-Deeb [16], we obtain the operator $\mathcal{M}^{m,\lambda,q}_{\ell,n,\delta,\alpha}$, defined as follows:

$$\mathcal{M}^{m,\lambda,q}_{\ell,n,\delta,\alpha} f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\, c^n(\delta)\right]^m \left(\frac{n+1}{n+k}\right)^\alpha \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_j z^j$$
$$(\alpha > 0;\ \lambda > -1;\ \ell \geq 0;\ 0 < q < 1;\ \delta > 0;\ n \in \mathbb{N};\ m \in \mathbb{N}_0)\,.$$

2. Preliminary

To prove our results, we need the following lemmas.

Lemma 2.1. [11] *Let* $\psi \in \mathbb{A}$ *and*

$$\mathcal{G}(z) = \frac{1}{z} \int_0^z \psi(t)dt,\ z \in \Delta.$$

If $\Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > \frac{-1}{2},\ z \in \Delta$, *then* $\mathcal{G} \in \mathcal{K}$.

Lemma 2.2. [14, Theorem 2.6] *Let* ψ *be a convex function with* $\psi(0) = \beta$ *and* $\nu \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ *with* $\Re(\nu) \geq 0$. *If* $p \in H[\beta, d]$ *with* $p(0) = \beta$, $\Phi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$,

$$\Phi\left(p(z), zp'(z); z\right) = p(z) + \frac{1}{\nu} zp'(z)$$

is analytic function in Δ *and*

$$\mathcal{F}_{\Phi(\mathbb{C}^2\times\Delta)}\left(p(z) + \frac{1}{\nu} zp'(z)\right) \leq \mathcal{F}_{h(\Delta)} h(z) \quad \to p(z) + \frac{1}{\nu} zp'(z) \prec_{\mathcal{F}} h(z),\quad z \in \Delta,$$

then

$$\mathcal{F}_{p(\Delta)} p(z) \leq \mathcal{F}_{q(\Delta)} q(z) \leq \mathcal{F}_{h(\Delta)} h(z) \ \to\ p(z) \prec_{\mathcal{F}} q(z),\ z \in \Delta,$$

where

$$q(z) = \frac{\nu}{dz^{\frac{\nu}{d}}} \int_0^z \psi(t) t^{\frac{\nu}{d}-1} dt, \ z \in \Delta.$$

The function q is convex and it is the fuzzy best dominant.

Lemma 2.3. [14, Theorem 2.7] *Let g be a convex function in Δ and*

$$\psi(z) = g(z) + d\gamma z g'(z),$$

where $z \in \Delta, \ d \in \mathbb{N}$ and $\gamma > 0$. If

$$p(z) = g(0) + p_d \ z^d + p_{d+1} \ z^{d+1} + ...$$

belongs to $H(\Delta)$, and

$$\mathcal{F}_{p(\Delta)} \left(p(z) + \gamma z p'(z) \right) \leq \mathcal{F}_{\psi(\Delta)} \psi(z) \quad \rightarrow \quad p(z) + \gamma z p'(z) \prec_{\mathcal{F}} \psi(z), \ z \in \Delta,$$

then

$$\mathcal{F}_{p(\Delta)} \left(p(z) \right) \leq \mathcal{F}_{g(\Delta)} g(z) \quad \rightarrow \quad p(z) \prec_{\mathcal{F}} g(z), \quad z \in \Delta.$$

This result is sharp.

For the general theory of fuzzy differential subordination and its applications, we refer the reader to [1, 2].

In the next section, we obtain several fuzzy differential subordinations associated with the diferential operator $\mathcal{D}^m_{n,\delta,g} f(z)$ by using the method of fuzzy differential subordination.

3. Main results

Assume that $\eta \in [0, 1), \ \delta > 0, \ n \in \mathbb{N}, \ m \in \mathbb{N}_0, \ \lambda > 0$ and $z \in \Delta$ are mentioned through this paper.

By using the integral operator $\mathcal{D}^m_{n,\delta,g}$, we define a class of analytic functions and we derive several fuzzy differential subordinations for this class.

Definition 3.1. Let the function $f \in \mathbb{A}$ belongs to the class $\mathcal{H}^F_{n,m,\delta}(\eta, g)$ for all $\eta \in [0, 1), \ n \in \mathbb{N}_0, \ m > 0$ and $\alpha \geq 0$ if it satisfies the inequality:

$$F_{\left(\mathcal{D}^m_{n,\delta,g} f\right)'(\Delta)} \left(\mathcal{D}^m_{n,\delta,g} f(z) \right)' > \eta, \qquad (z \in \Delta).$$

Theorem 3.2. *Let k belongs to $\mathcal{C}$ in Δ and suppose that $h(z) = k(z) + \frac{1}{\lambda+2} z k'(z)$. If $f \in \mathcal{H}^F_{n,m,\delta}(\eta, g)$ and*

$$G(z) = I^\lambda f(z) = \frac{\lambda+2}{z^{\lambda+1}} \int_0^z t^\lambda f(t) dt, \tag{3.1}$$

then

$$F_{\left(\mathcal{D}^m_{n,\delta,g} f\right)'(\Delta)} \left(\mathcal{D}^m_{n,\delta,g} f(z) \right)' \leq F_{h(\Delta)} h(z) \rightarrow \left(\mathcal{D}^m_{n,\delta,g} f(z) \right)' \prec_{\mathcal{F}} h(z), \tag{3.2}$$

implies

$$F_{\left(\mathcal{D}^m_{n,\delta,g}G\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \le F_{k(\Delta)}k(z) \rightarrow \left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \prec_{\mathcal{F}} k(z),$$

and this result is sharp.

Proof. Since

$$z^{\lambda+1}G(z) = (\lambda+2)\int_0^z t^{\lambda}f(t)dt,$$

by differentiating, it obtain

$$(\lambda+1)\,G(z) + zG'(z) = (\lambda+2)\,f(z),$$

and

$$(\lambda+1)\,\mathcal{D}^m_{n,\delta,g}G(z) + z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' = (\lambda+2)\,\mathcal{D}^m_{n,\delta,g}f(z), \tag{3.3}$$

and also, by differentiating (3.3) we obtain

$$\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' + \frac{1}{(\lambda+2)}z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)'' = \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \tag{3.4}$$

By using (3.4), the fuzzy differential subordination (3.2) is

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' + \frac{1}{(\lambda+2)}z\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)''\right)$$
$$\le F_{h(\Delta)}\left(k(z) + \frac{1}{(\lambda+2)}zk'(z)\right). \tag{3.5}$$

We denote

$$q(z) = \left(\mathcal{D}^m_{n,\delta,g}G(z)\right)', \quad \text{so} \quad q \in \mathcal{H}\left[1,n\right]. \tag{3.6}$$

Putting (3.6) in (3.5), we have

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(q(z) + \frac{1}{(\lambda+2)}zq'(z)\right) \le F_{h(\Delta)}\left(k(z) + \frac{1}{(\lambda+2)}zk'(z)\right), \tag{3.7}$$

and applying Lemma (2.3), we have

$$F_{q(\Delta)}q(z) \le F_{k(\Delta)}k(z), \quad \text{i.e} \quad F_{\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \le F_{k(\Delta)}k(z),$$

therefore $\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)' \prec_{\mathcal{F}} k(z)$, and k is the fuzzy best dominant. □

Theorem 3.3. *Assume that* $h(z) = \frac{1+(2\eta-1)z}{1+z}$, $\eta \in [0,1)$, $\lambda > 0$ *and* $\mathcal{I}^{\lambda}$ *is given by (*3.1*), then*

$$\mathcal{I}^{\lambda}\left[\mathcal{H}^F_{n,m,\delta}(\eta,g)\right] \subset \mathcal{H}^F_{n,m,\delta}(\eta^*,g), \tag{3.8}$$

where

$$\eta^* = 2\eta - 1 + (\lambda+2)(2-2\eta)\int_0^1 \frac{t^{\lambda+2}}{t+1}dt. \tag{3.9}$$

Proof. A function h belongs to $\mathcal{C}$ and using the same technique in the proof of Theorem 3.2, we obtain from the hypothesis of Theorem 3.3 that

$$F_{q(\Delta)}\left(q(z)+\frac{1}{(\lambda+2)}zq^{'}(z)\right)\leq F_{h(\Delta)}h(z),$$

where $q(z)$ is defined in (3.6). By using Lemma 2.2, we obtain

$$F_{q(\Delta)}q(z)\leq F_{k(\Delta)}k(z)\leq F_{h(\Delta)}h(z),$$

which implies

$$F_{\left(\mathcal{D}^m_{n,\delta,g}G\right)^{'}(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)^{'}\leq F_{k(\Delta)}k(z)\leq F_{h(\Delta)}h(z),$$

where

$$\begin{aligned} k(z) &= \frac{\lambda+2}{z^{\lambda+2}}\int\limits_0^z t^{\lambda+1}\frac{1+(2\eta-1)\,t}{1+t}dt \\ &= (2\eta-1)+\frac{(\lambda+2)\,(2-2\eta)}{z^{\lambda+2}}\int\limits_0^z\frac{t^{\lambda+1}}{1+t}dt. \end{aligned}$$

k belongs to $\mathcal{C}$ and $k\,(\Delta)$ is symmetric with respect to the real axis, so we conclude

$$F_{\left(\mathcal{D}^m_{n,\delta,g}G\right)^{'}(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}G(z)\right)^{'}\geq\min_{|z|=1}F_{k(\Delta)}k(z)=F_{k(\Delta)}k(1), \tag{3.10}$$

and

$$\eta^*=k(1)=2\eta-1+(\lambda+2)\,(2-2\eta)\int\limits_0^1\frac{t^{\lambda+2}}{t+1}dt.$$ □

Theorem 3.4. *Let k belongs to $\mathcal{C}$ in Δ, $k(0)=1$, and $h(z)=k(z)+zk^{'}(z)$. If $f\in\mathbb{A}$ and satisfies the fuzzy differential subordination*

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)^{'}(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)^{'}\leq F_{h(\Delta)}h(z)\rightarrow\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)^{'}\prec_{\mathcal{F}}h(z), \tag{3.11}$$

then

$$F_{\mathcal{D}^m_{n,\delta,g}f(\Delta)}\frac{\mathcal{D}^m_{n,\delta,g}f\,(z)}{z}\leq F_{k(\Delta)}k(z)\rightarrow\frac{\mathcal{D}^m_{n,\delta,g}f\,(z)}{z}\prec_{\mathcal{F}}k(z). \tag{3.12}$$

The result is sharp.

Proof. For

$$\begin{aligned} q(z) &= \frac{\mathcal{D}^m_{n,\delta,g}f\,(z)}{z}=\frac{z+\sum\limits_{j=2}^{\infty}\left[1+(j-1)\,c^n(\delta)\right]^m a_jb_jz^j}{z} \\ &= 1+\sum_{j=2}^{\infty}\left[1+(j-1)\,c^n(\delta)\right]^m a_jb_jz^{j-1}, \end{aligned}$$

we obtain that

$$q(z)+zq^{'}(z)=\left(\mathcal{D}^m_{n,\delta,g}f\,(z)\right)^{'},$$

so

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \leq F_{h(\Delta)}h(z)$$

implies

$$F_{q(\Delta)}\left(q(z)+zq'(z)\right) \leq F_{h(\Delta)}h(z) = F_{k(\Delta)}\left(k(z)+zk'(z)\right).$$

Applying Lemma 2.3, we have

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z) \to F_{\mathcal{D}^m_{n,\delta,g}f(\Delta)}\frac{\mathcal{D}^m_{n,\delta,g}f(z)}{z} \leq F_{k(\Delta)}k(z),$$

and we get

$$\frac{\mathcal{D}^m_{n,\delta,g}f(z)}{z} \prec_{\mathcal{F}} k(z).$$

The result is sharp. □

Theorem 3.5. *Consider $h \in \mathcal{H}(\Delta)$ with $h(0)=1$, which satisfies*

$$\Re\left(1+\frac{zh''(z)}{h'(z)}\right) > \frac{-1}{2}.$$

If $f \in \mathbb{A}$ and the fuzzy differential subordination

$$F_{\left(\mathcal{D}^m_{n,\delta,g}f\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \leq F_{h(\Delta)}h(z) \to \left(\mathcal{D}^m_{n,\delta,g}f(z)\right)' \prec_{\mathcal{F}} h(z), \tag{3.13}$$

holds, then

$$F_{\mathcal{D}^m_{n,\delta,g}f(\Delta)}\frac{\mathcal{D}^m_{n,\delta,g}f(z)}{z} \leq F_{k(\Delta)}k(z) \qquad i.e \quad \frac{\mathcal{D}^m_{n,\delta,g}f(z)}{z} \prec_{\mathcal{F}} k(z), \tag{3.14}$$

where

$$k(z)=\frac{1}{z}\int_0^z h(t)dt,$$

the function k is convex and it is the fuzzy best dominant.

Proof. Let

$$q(z)=\frac{\mathcal{D}^m_{n,\delta,g}f(z)}{z} = 1+\sum_{j=2}^{\infty}\left[1+(j-1)c^n(\delta)\right]^m a_j b_j z^{j-1}, \quad q \in \mathcal{H}[1,1],$$

where $\Re\left(1+\frac{zh''(z)}{h'(z)}\right) > \frac{-1}{2}$. From Lemma 2.1, we have

$$k(z)=\frac{1}{z}\int_0^z h(t)dt$$

belongs to the class $\mathcal{C}$, which satisfies the fuzzy differential subordination (3.13). Since

$$k(z)+zk'(z)=h(z),$$

it is the fuzzy best dominant.

We have

$$q(z) + zq^{'}(z) = \left(\mathcal{D}^m_{n,\delta,g} f\left(z\right)\right)^{'},$$

then (3.13) becomes

$$F_{q(\Delta)}\left(q(z) + zq^{'}(z)\right) \leq F_{h(\Delta)}h(z).$$

Applying Lemma 2.3, we have

$$F_{q(\Delta)}q(z) \leq F_{k(\Delta)}k(z), \quad \text{i.e.} \quad F_{\mathcal{D}^m_{n,\delta,g} f(\Delta)} \frac{\mathcal{D}^m_{n,\delta,g} f\left(z\right)}{z} \leq F_{k(\Delta)}k(z),$$

then

$$\frac{\mathcal{D}^m_{n,\delta,g} f\left(z\right)}{z} \prec_{\mathcal{F}} k(z). \qquad \square$$

Putting $h(z) = \frac{1+(2\beta-1)z}{1+z}$ in Theorem 3.5, we obtain the following corollary:

Corollary 3.6. *Let* $h = \frac{1+(2\beta-1)z}{1+z}$ *a convex function in* Δ, *with* $h(0) = 1$, $0 \leq \beta < 1$. *If* $f \in \mathbb{A}$ *and verifies the fuzzy differential subordination*

$$F_{\left(\mathcal{D}^m_{n,\delta,g} f\right)'(\Delta)}\left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'} \leq F_{h(\Delta)}h(z), \quad i.e \quad \left(\mathcal{D}^m_{n,\delta,g} f(z)\right)^{'} \prec_{\mathcal{F}} h(z),$$

then

$$k(z) = 2\beta - 1 + \frac{2\left(1-\beta\right)}{z}\ln\left(1+z\right),$$

the function k *is convex and it is the fuzzy best dominant.*

Concluding, all the above results give us information about fuzzy differential subordinations for the operator $\mathcal{D}^m_{n,\delta,g}$, we give some properties for the class $\mathcal{H}^F_{\alpha,m}\left(n,\eta\right)$ of univalent analytic functions.

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Sheza M. El-Deeb
Department of Mathematics, Faculty of Science,
Damietta University, New Damietta 34517, Egypt,
and
Department of Mathematics,
College of Science and Arts in Al-Badaya,
Qassim University, Buraidah, Saudi Arabia
e-mail: shezaeldeeb@yahoo.com; s.eldeeb@qu.edu.sa

Alina Alb Lupaş
Department of Mathematics and Computer Science,
University of Oradea,
Str. Universităţii nr. 1, 410087 Oradea, Romania
e-mail: dalb@uoradea.ro

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Radius of starlikeness through subordination

Asha Sebastian and Vaithiyanathan Ravichandran

Abstract. A normalized function f on the open unit disc is starlike (or convex) univalent if the associated function $zf'(z)/f(z)$ (or $1+zf''(z)/f'(z)$) is a function with positive real part. The radius of starlikeness or convexity is usually obtained by using the estimates for functions with positive real part. Using subordination, we examine the radius of various starlikeness, in particular, radii of Janowski starlikeness and starlikeness of order β, for the function f when the function f is either convex or $(zf'(z) + \alpha z^2 f''(z))/f(z)$ lies in the right-half plane. Radii of starlikeness associated with lemniscate of Bernoulli and exponential functions are also considered.

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1. Introduction

Let $\mathcal{A}$ be the class of all functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be its subclass consisting of univalent functions. The Bieberbach conjecture (and now de Branges theorem [4]) states that the coefficients of $f \in \mathcal{S}$ satisfy the inequality $|a_n| \leq n$ for $n \geq 2$ and it led to the study of several geometrically defined classes such as the class of starlike functions, denoted by $\mathcal{S}^*$ and the class of convex functions, denoted by $\mathcal{K}$. These classes and other subclasses can be unified by subordination and convolution. The concept of subordination was introduced by Lindelöf [9]. A function f analytic in $\mathbb{D}$ is subordinate to an analytic function g in $\mathbb{D}$, written $f \prec g$, if there exists a Schwarz

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function $w : \mathbb{D} \to \mathbb{D}$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. When g is univalent in $\mathbb{D}$, the subordination $f \prec g$ holds if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. The convolution or Hadamard product of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

in $\mathcal{A}$ is defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n.$$

Motivated by earlier works on unifying various subclasses of starlike and convex functions, Shanmugam [26] introduced and studied convolutions properties (using results of [25]) of the class

$$\mathcal{S}_g^*(\varphi) := \{f \in \mathcal{A} : z(f * g)'(z)/(f * g)(z) \prec \varphi(z)\}$$

where φ is a convex function and g is a fixed function in the class $\mathcal{A}$. When $g(z)$ is $z/(1-z)$ and $z/(1-z)^2$, the subclass $\mathcal{S}_g^*(\varphi)$ becomes the classes

$$\mathcal{S}^*(\varphi) := \{f \in \mathcal{A} : zf'(z)/f(z) \prec \varphi(z)\}$$

and

$$\mathcal{K}(\varphi) := \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \prec \varphi(z)\}$$

respectively. Ma and Minda [11] studied the distortion, growth theorems for these classes where φ is a starlike function. We are interested in few special choices of φ. When $\varphi(z) = (1+(1-2\alpha)z)(1-z)^{-1}$, $0 \le \alpha < 1$, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are the classes of starlike and convex functions of order α introduced by Robertson [24]. The classes $\mathcal{S}^*(0) = \mathcal{S}$ and $\mathcal{K}(0) = \mathcal{K}$ are respectively the well-known classes of starlike and convex functions. For example, when $-1 \le B < A \le 1$, the class

$$\mathcal{S}^*[A, B] := \mathcal{S}^*\big((1 + Az)/(1 + Bz)\big)$$

is the class of Janowski starlike functions and the class

$$\mathcal{K}[A, B] := \mathcal{K}\big((1 + Az)/(1 + Bz)\big)$$

is the class of Janowski convex functions considered by several authors [5, 21, 22]. We are also interested in the class $\mathcal{S}_L^* = \mathcal{S}^*(\sqrt{1+z})$ studied by Sokół and Stankiewicz [28] and $\mathcal{S}_e^* = \mathcal{S}^*(e^z)$ studied by Mendiratta *et al.* [12]. These classes were studied in [2, 1, 3, 18, 6, 14].

Let $\alpha > 1$, $0 \le \beta < 1$ and $\beta \ge 1/2 - 1/(2\alpha)$. Let $\varphi_p : \mathbb{D} \to \mathbb{C}$ be defined by

$$\varphi_p(z) := (1-\alpha)\frac{1+(1-2\beta)z}{(1-z)} + \alpha\left(\frac{1+(1-2\beta)z}{(1-z)}\right)^2 + \alpha\frac{2(1-\beta)z}{(1-z)^2}. \tag{1.1}$$

The image of the unit disk $\mathbb{D}$ under the function $\varphi_p(z) = u + \mathrm{i}\,v$ is the exterior of parabola given by

$$v^2 = -\frac{(1-\alpha(1-2\beta))^2(2-2\beta)}{\alpha(3-2\beta)}\left(u - (\alpha\beta\,(\beta - 1/2) + \beta - \alpha/2)\right)$$

with its vertex at $(\alpha\beta(\beta-1/2)+\beta-\alpha/2, 0)$. Note that it includes the right half plane. If $\beta = 1/2 - 1/(2\alpha)$, the region $\varphi_p(\mathbb{D})$ becomes the entire complex plane with a slit along the negative real axis from $-\left((2\alpha^2-\alpha+1)/4\alpha\right)$ to $-\infty$. Also the condition $\beta \geq 1/2 - 1/(2\alpha)$ restricts the range of β to $(0, 1/2)$. We are mainly concerned with the class $\mathcal{S}^*_{\alpha,\beta}$ of all functions $f \in \mathcal{A}$, with $f(z)/z \neq 0$, satisfying

$$\frac{zf'(z)+\alpha z^2 f''(z)}{f(z)} \prec \varphi_p(z) \tag{1.2}$$

where the function φ_p is defined in (1.1). Singh and Gupta [27, Corollary 4.1] have shown that $\mathcal{S}^*_{\alpha,\beta} \subseteq \mathcal{S}^*_{\beta}$. This extends the results of Li and Owa [8], Padmanabhan [17] and Ravichandran *et al.* [23]. These functions were also studied in [7, 10, 15, 16, 19, 20].

For two families $\mathcal{G}$ and $\mathcal{F}$ of $\mathcal{A}$, the $\mathcal{G}$-radius of $\mathcal{F}$, denoted by $R_{\mathcal{G}}(\mathcal{F})$ is the largest number R such that $r^{-1}f(rz) \in \mathcal{G}$ for $0 < r \leq R$, and for all $f \in \mathcal{F}$. Whenever $\mathcal{G}$ is characterised by a geometric property P, the number R is also referred to as the radius of property P for the class $\mathcal{F}$. If the class $\mathcal{F}$ is clear from the context, then we just write $R_{\mathcal{G}}(\mathcal{F})$ as $R_{\mathcal{G}}$. Using the theory of differential subordination developed by Miller and Mocanu [13], we determine radius constants for functions in the classes $\mathcal{S}^*_{\alpha,\beta}$ and $\mathcal{K}$ to belong to various subclass of starlike functions, in particular, to the class of Janowski starlike functions and the starlike functions of order β as well as to the classes of starlike functions associated with lemniscate of Bernoulli and the exponential functions. The results are shown to be sharp by explicitly showing the extremal function. The class $\mathcal{S}^*_{\alpha,\beta}$ for suitable α, β is a subclass of starlike functions of order β and the class of convex functions $\mathcal{K}$ is a subclass of functions starlike of order 1/2. These observations lead us to discuss radius constants of functions in the class $\mathcal{S}^*(\beta)$ in Lemma 1.2. It is then applied to find radius constants for functions in the classes $\mathcal{S}^*_{\alpha,\beta}$ and $\mathcal{K}$.

Various radii constants for the class $\mathcal{S}^*_{\alpha,\beta}$ are given in the following:

Theorem 1.1. *The following sharp radius results hold for the class* $\mathcal{S}^*_{\alpha,\beta}$*:*

(i) *For* $-1 \leq B < A \leq 1$, *the* $\mathcal{S}^*[A,B]$ *radius*

$$R_{\mathcal{S}^*[A,B]} = \min\{1,\ (A-B)/\left(|A+B-2\beta B| + 2(1-\beta)\right)\}.$$

(ii) *For* $0 \leq \gamma < 1,\ \gamma > \beta$, *the* $\mathcal{S}^*(\gamma)$ *radius* $R_{\mathcal{S}^*(\gamma)} = (1-\gamma)/(1+\gamma-2\beta)$.

(iii) *The* $\mathcal{S}_L$ *radius* $R_{\mathcal{S}_L} = (\sqrt{2}-1)/\left(\sqrt{2}+1-2\beta\right)$.

(iv) *The* $\mathcal{S}^*_e$ *radius* $R_{\mathcal{S}^*_e} = (e-1)/(e+1-2\beta)$.

The idea of the proof is to use inclusion results for the class $\mathcal{S}^*_{\alpha,\beta}$ with the class of starlike functions of order β. Singh and Gupta [27, Corollary 4.1] have shown that $\mathcal{S}^*_{\alpha,\beta} \subseteq \mathcal{S}^*(\beta)$. In order to use this inclusion, we first find the various radii for the class of starlike functions of order β in the following:

Lemma 1.2. *The following sharp radius results hold for the class* $\mathcal{S}^*(\beta)$*:*

(i) *For* $-1 \leq B < A \leq 1$, *the* $\mathcal{S}^*[A,B]$ *radius*

$$R_{\mathcal{S}^*[A,B]} = \min\{1,\ (A-B)/\left(|A+B-2\beta B| + 2(1-\beta)\right)\}.$$

(ii) *For* $0 \leq \gamma < 1,\ \gamma > \beta$, *the* $\mathcal{S}^*(\gamma)$ *radius* $R_{\mathcal{S}^*(\gamma)} = (1-\gamma)/(1+\gamma-2\beta)$.

(iii) *The* $\mathcal{S}_L$ *radius* $R_{\mathcal{S}_L} = (\sqrt{2}-1)/\left(\sqrt{2}+1-2\beta\right)$.
(iv) *The* $\mathcal{S}_e^*$ *radius* $R_{\mathcal{S}_e^*} = (e-1)/(e+1-2\beta)$.

Theorem 1.1 follows from this lemma except for the sharpness. To find the extremal function $\tilde{f}$ for the class $\mathcal{S}^*_{\alpha,\beta}$, write $\tilde{f}$ as

$$\tilde{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and determine the coefficients a_n from

$$\frac{z\tilde{f}'(z)}{\tilde{f}(z)}\left(1+\alpha\frac{z\tilde{f}''(z)}{f_1'(z)}\right) = \varphi_p(z) \tag{1.3}$$

where φ_p is given by (1.1). Writing

$$C = 2(2\alpha-\beta) - 4\alpha\beta,\ D = 2(\alpha+\beta) + 2\alpha\beta(2\beta-3) - 1,$$

the equation (1.3) readily gives

$$\begin{aligned} a_2 &= \frac{C+2}{1+2\alpha} = 2(1-\beta) \\ a_n &= \frac{\left(C+2(n-1)+2\alpha(n-1)(n-2)\right)}{(1+n\alpha)(n-1)} a_{n-1} \\ &\quad + \frac{\left(D-(n-2)-\alpha(n-2)(n-3)\right)}{(1+n\alpha)(n-1)} a_{n-2}. \end{aligned}$$

Calculating the coefficients a_n from the above recurrence relation, we see that the extremal function $\tilde{f}$ is the generalised Koebe's function given by

$$\tilde{f}(z) = \frac{z}{(1-z)^{2-2\beta}}. \tag{1.4}$$

Interestingly, it is the extremal of the class $\mathcal{S}^*(\beta)$ and hence the sharpness of our theorem follows trivially.

It is also well-known that a convex function is starlike of order $1/2$ and so the class $\mathcal{K}$ of convex function is contained in the class $\mathcal{S}^*(1/2)$ of starlike functions of order $1/2$. This inclusion and Lemma 1.2 together readily yields the following radii results for the class of convex functions:

Corollary 1.3. *The following sharp radius results hold for the class* $\mathcal{K}$*:*

(i) *For* $-1 \le B < A \le 1$, *the* $\mathcal{S}^*[A,B]$ *radius*

$$R_{\mathcal{S}^*[A,B]} = \min\{1,\ (A-B)/(1+|A|)\}.$$

(ii) *For* $0 \le \gamma < 1,\ \gamma > 1/2$, *the* $\mathcal{S}^*(\gamma)$ *radius* $R_{\mathcal{S}^*(\gamma)} = (1-\gamma)/\gamma$.
(iii) *The* $\mathcal{S}_L$ *radius* $R_{\mathcal{S}_L} = 1 - 1/\sqrt{2} \approx 0.2929$.
(iv) *The* $\mathcal{S}_e^*$ *radius* $R_{\mathcal{S}_e^*} = 1 - 1/e \approx 0.6321$.

The method of convolution can also be applied to find radius problems of various classes. Corollary 1.3 (ii) requires the largest number ρ such that the function $l_\rho : \mathbb{D} \to \mathbb{C}$ is a starlike of order $\gamma \ge 1/2$, where $f_\rho(z) = f(z) * l_\rho(z)$. Here $l(z) = z/(1-z)$ is the convolution identity and the functions $f_\rho,\ l_\rho : \mathbb{D} \to \mathbb{C}$ are defined respectively by $f_\rho(z) = f(\rho z)/\rho$ and $l_\rho(z) = z/(1-\rho z)$. This is equivalent to find the number ρ such that $\mathrm{Re}(\rho z/(1-\rho z)) > \gamma - 1$. It follows by simple computation that $\rho = (1-\gamma)/\gamma$, since the real part of the function $(\rho z/(1-\rho z))$ attains minimum at $z = -1$.

2. Proof of Lemma 1.2

Let the function $f \in \mathcal{S}^*(\beta)$. Then, it follows that

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{(1-z)}.$$

Define the function $f_\rho : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ by $f_\rho(z) := f(\rho z)/\rho$. For this function, we immediately get

$$\frac{zf'_\rho(z)}{f_\rho(z)} \prec \frac{1+(1-2\beta)\rho z}{(1-\rho z)}. \tag{2.1}$$

(i) Let $-1 \le A < B \le 1$ and the functions $p,\ q : \mathbb{D} \longrightarrow \mathbb{C}$ be defined by,

$$p(z) = \frac{1+(1-2\beta)z}{(1-z)} \quad \text{and} \quad q(z) = \frac{1+Az}{1+Bz}. \tag{2.2}$$

From (2.2), it follows that $p^{-1}(w) = (w-1)/(w+1-2\beta)$ and hence

$$p^{-1} \circ q(z) = \frac{q(z)-1}{q(z)+1-2\beta} = \frac{(A-B)z}{(A+B-2\beta B)z + 2(1-\beta)}. \tag{2.3}$$

The values taken by $p^{-1} \circ q(z)$ in (2.3) leads us in finding ρ through two different cases.

Case 1. If $(A-B)/(|A+B-2\beta B| + 2(1-\beta)) \ge 1$, then, by (2.3), we have

$$|p^{-1} \circ q(z)| \ge \frac{A-B}{|A+B-2\beta B| + 2(1-\beta)} \ge 1 \quad (z \in \partial\mathbb{D}).$$

This shows that $z \prec p^{-1}(q(z))$ and hence $p(z) \prec q(z)$. This shows that $\rho = 1$.

Case 2. If $(A-B)/(|A+B-2\beta B| + 2(1-\beta)) \le 1$, then it follows from (2.2) that

$$\begin{aligned} R_{\mathcal{S}^*[A,B]} &= \min_{|z|=1} \left|p^{-1} \circ q(z)\right| \\ &= \min_{|z|=1} \left| \frac{(A-B)z}{(A+B-2\beta B)z + 2 - 2\beta} \right| \\ &= \frac{A-B}{|A+B-2\beta B| + 2(1-\beta)}. \end{aligned} \tag{2.4}$$

Thus, for $0 < \rho \le R_{\mathcal{S}^*[A,B]}$, we have $p(\rho z) \prec q(z)$. By (2.1), it follows that $zf'_\rho(z)/f_\rho(z) \prec p(\rho z) \prec q(z)$ or $f_\rho \in \mathcal{S}^*[A,B]$. Thus, the $\mathcal{S}^*[A,B]$ radius of the class $S^*(\beta)$ is at least $R_{\mathcal{S}^*[A,B]}$.

To show the sharpness, consider the function $\tilde{f} : \mathbb{D} \to \mathbb{C}$ defined by

$$\tilde{f}(z) = \frac{z}{(1-z)^{2-2\beta}}. \tag{2.5}$$

At the point $z = R_{\mathcal{S}^*[A,B]}$, the function $\tilde{f}$ satisfies

$$\left|\frac{(z\tilde{f}'(z)/\tilde{f}(z) - 1}{A - B(z\tilde{f}'(z)/\tilde{f}(z)}\right| = 1$$

and hence the result is sharp.

(ii) Let $0 \leq \gamma < 1$. We consider two cases depending on the values γ, namely, $\gamma \leq \beta$ and $\gamma \geq \beta$. Since $\mathcal{S}^*(\gamma) = \mathcal{S}^*[1-2\gamma, -1]$, substituting $A = 1-2\gamma$ and $B = -1$ in (2.4), we obtain $\rho = 1$ when $\gamma \leq \beta$ and the required $\rho = R_{\mathcal{S}^*(\gamma)}$ when $\gamma \geq \beta$.

(iii) For $0 < a < \sqrt{2}$, by [2, Lemma 2.2], we have

$$\{w \in \mathbb{C} : |w - a| < \sqrt{2} - a\} \subseteq \{w \in \mathbb{C} : \left|w^2 - 1\right| < 1\}. \tag{2.6}$$

Let the functions $p, q : \mathbb{D} \to \mathbb{C}$ be defined by

$$p(z) := \frac{1 + 2(1-\beta)z}{1 - z} \quad \text{and} \quad q(z) := \sqrt{1+z}. \tag{2.7}$$

It is evident from (2.7) that

$$p^{-1}(q(z)) = \frac{\sqrt{1+z} - 1}{\sqrt{1+z} + 1 - 2\beta} = \left(1 + \frac{2(1-\beta)}{\sqrt{1+z} - 1}\right)^{-1}. \tag{2.8}$$

By (2.6), we have $\left|\sqrt{1+z} - 1\right| \geq \sqrt{2} - 1$ and so

$$1 + \frac{2(1-\beta)}{\left|\sqrt{1+z} - 1\right|} \leq 1 + \frac{2(1-\beta)}{\sqrt{2} - 1}. \tag{2.9}$$

Substituting (2.9) in (2.8), it follows that

$$\rho = \min_{|z|=1} \left|p^{-1} \circ q(z)\right| = \min_{|z|=1} \left|\left(1 + \frac{2(1-\beta)}{\sqrt{1+z} - 1}\right)^{-1}\right|. \tag{2.10}$$

This is equivalent to

$$\rho = \left(\max_{|z|=1} \left|1 + \frac{2(1-\beta)}{\sqrt{1+z} - 1}\right|\right)^{-1} = \left(1 + \frac{2(1-\beta)}{\sqrt{2} - 1}\right)^{-1}. \tag{2.11}$$

Therefore, we have

$$\frac{1 + 2(1-\beta)\rho z}{1 - \rho z} \prec \sqrt{1+z}.$$

By (2.1), this proves that the function $f_\rho \in \mathcal{S}_L$.
At the point $z = R_{\mathcal{S}_L}$, the function $\tilde{f}$ defined in (2.5) satisfies

$$\left|\left(\frac{z\tilde{f}'(z)}{\tilde{f}(z)}\right)^2 - 1\right| = \left|\left(1 + \frac{2(1-\beta)z}{1 - z}\right)^2 - 1\right| = 1.$$

(iv) Let the functions p and q be defined as

$$p(z) := \frac{1 + (1-2\beta)z}{1 - z} \quad \text{and} \quad q(z) := e^z, \quad z \in \mathbb{D}. \tag{2.12}$$

It is apparent from (2.12) that

$$p^{-1}(q(z)) = \frac{e^z - 1}{e^z + 1 - 2\beta} = \left(1 + \frac{2(1-\beta)}{e^z - 1}\right)^{-1}. \tag{2.13}$$

Let $\lambda = 2(1-\beta),\ 0 \le \beta \le 1$. On the boundary of the unit disc $\mathbb{D}$, we have

$$\begin{aligned}\left|1 + \frac{\lambda}{e^z - 1}\right|^2 &= \left|1 + \frac{\lambda}{e^{\cos\theta}\cos(\sin\theta) - 1 + \mathrm{i}\, e^{\cos\theta}\sin(\sin\theta)}\right|^2 \\ &= \frac{e^{2\cos\theta} + 2(\lambda-1)e^{\cos\theta}\cos(\sin\theta) + (\lambda-1)^2}{e^{2\cos\theta} - 2e^{\cos\theta}\cos(\sin\theta) + 1}.\end{aligned} \tag{2.14}$$

Substituting $\cos\theta = x$ in (2.14), we get

$$\begin{aligned}\left|1 + \frac{\lambda}{e^z - 1}\right|^2 &= \frac{e^{2x} + 2(\lambda-1)e^x\cos(\sqrt{1-x^2}) + (\lambda-1)^2}{e^{2x} - 2e^x\cos(\sqrt{1-x^2}) + 1} \\ &= \frac{g(x,\lambda)}{g(x,0)}\end{aligned} \tag{2.15}$$

where

$$g(x,\lambda) := e^{2x} - 2e^x\cos(\sqrt{1-x^2}) + 1. \tag{2.16}$$

Let $-1 \le x \le 1,\ 0 \le \lambda \le 2$ and the function S be defined by

$$S(x) := g(x,\lambda)g(1,0) - g(x,0)g(1,\lambda).$$

Using (2.16) in $S(x)$, it can be seen that

$$\begin{aligned}S(x) = {}& 2x(e^2 + \lambda - 1)e^x\cos(\sqrt{1-x^2}) - (2e + \lambda - 2)e^{2x} \\ & - e(2(\lambda-1) - e(\lambda-2)).\end{aligned} \tag{2.17}$$

Define the function s by

$$s(x) := 2x(e^2 + \lambda - 1)e^x\cos(\sqrt{1-x^2}) - (2e + \lambda - 2)e^{2x}.$$

The function $s'(x)$ is an increasing function. Therefore it has at most one zero, say η. Also $s''(x) > 0$, this shows that η is a local minima. Thus, the maximum of s occurs at $x = \pm 1$. At $x = -1$,

$$s(-1) = -2(e - e^{-2}) - \lambda e^{-1}(e^{-1} + 2) \le 0.$$

These observations together with (2.17) lead us to the fact that $S(x) \le 0$, or equivalently, the function h defined by $h(x) := g(x,\lambda)/g(x,0)$ satisfies $h(x) \le h(1)$. Therefore, the maximum of $h(x)$ occurs at $x = 1$, and, by (2.15),

$$\left|1 + \frac{2(1-\beta)}{e^z - 1}\right| \le \left|1 + \frac{2(1-\beta)}{e - 1}\right|. \tag{2.18}$$

From the definition of ρ, it follows from (2.13) that

$$\rho = \min_{|z|=1}\left|p^{-1} \circ q(z)\right| = \min_{|z|=1}\left|\left(1 + \frac{2(1-\beta)}{e^z - 1}\right)^{-1}\right|.$$

From (2.18), it is clear that

$$\rho = \left(\max_{|z|=1} \left| 1 + \frac{2(1-\beta)}{e^z - 1} \right| \right)^{-1} = \left(1 + \frac{2(1-\beta)}{e-1} \right)^{-1}. \tag{2.19}$$

This proves that

$$\frac{1+(1-2\beta)\rho z}{1-\rho z} \prec e^z$$

and so the function $f_\rho \in \mathcal{S}_e^*$.
At the point $z = R_{\mathcal{S}_e^*}$, the function $\tilde{f}$ defined in (2.5) satisfies

$$\left| \log \frac{z\tilde{f}'(z)}{\tilde{f}(z)} \right| = \left| \log \left(1 + \frac{2(1-\beta)z}{1-z} \right) \right| = 1.$$

This completes the proof of the lemma.

Let $-1 \le B < A \le 1$ and $-1 \le D < C \le 1$. In 1997, Gangadharan and Ravichandran [5] discussed the $\mathcal{S}^*[A,B]$ radius of the class $\mathcal{S}^*[C,D]$ and shown that

$$R_{\mathcal{S}^*[A,B]}(\mathcal{S}^*[C,D]) = \min \left\{ 1, (A-B)/(C-D+|AD-BC|) \right\}.$$

Lemma 1.2*(i)* is indeed a particular case when $C = 1-2\delta$ and $D = -1$. The radius determined in Corollary [5, pp.305] is exactly the same as Lemma 1.2*(ii)*. Theorem [2, pp.6562] determined the $\mathcal{S}_L$ radius of $\mathcal{S}^*[A,B]$ when $B \le 0$. When $A = 1-2\delta, B = -1$, their result gives

$$R_{\mathcal{S}_L} = \min \left\{ 1, (\sqrt{2}-1)/(1-\delta+\sqrt{(1-\delta)^2+(\sqrt{2}-1)(\sqrt{2}+1-2\delta)}\ ) \right\}$$

and it is same as the radius in Lemma 1.2*(iii)*. Mendiratta *et al.* [12] discussed subordination theorems and radii constants for the functions in the class $\mathcal{S}^*(e^z)$. They determined the $\mathcal{S}_e^*$ radius of $f \in \mathcal{S}^*[A,B]$. By substituting $A = 1-2\delta, B = -1$ in Theorem [12, pp.381], the radius obtained is our Lemma 1.2*(iv)*.

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Asha Sebastian
Department of Mathematics,
National Institute of Technology,
Tiruchirappalli-620015, India
e-mail: ashasebastian13@gmail.com

Vaithiyanathan Ravichandran
Department of Mathematics,
National Institute of Technology,
Tiruchirappalli-620015, India
e-mail: vravi68@gmail.com; ravic@nitt.edu

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Local existence and blow up of solutions to a logarithmic nonlinear wave equation with time-varying delay

Abdelbaki Choucha and Djamel Ouchenane

Abstract. In this work, we are concerned with a problem of a logarithmic nonlinear wave equation with time-varying delay term. We established the local existence result and we proved a blow up result for the solution with negative initial energy under suitable conditions. This improves earlier results in the literature [11] for time-varying delay.

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Keywords: Wave equation, blow up, logarithmic source, varying delay term.

1. Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) = u|u|^{p-2} ln|u|^k \\ u\left(x,t\right) = 0, x \in \partial\Omega, \\ u_t(x, t-\tau(0)) = f_0(x, t-\tau(0)), (x,t) \in \Omega \times (0, \tau(0)) \\ u\left(x,0\right) = u_0\left(x\right), u_t\left(x,0\right) = u_1\left(x\right), x \in \Omega, \end{cases} \tag{1.1}$$

where

$$(x,t) \in \Omega \times (0, +\infty),$$

and $\tau(t) > 0$ represents the time varying delay and $p \geq 2, k, \mu_1$ are positive constants, μ_2 is a real number.

This type of problems is encountered in many branches of physics such as Nuclear Physics, Optics and Geophysics. It is well known, from the Quantum Field Theory, that such kind of nonlinearity appears naturally in inflation cosmology and in super

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symmetric field theories (see [1], [2], [7], [8], [14]).
In [10], the authors considered the following problem

$$\begin{cases} u_{tt} - \Delta u + u - u \log |u|^2 + u_t + u|u|^2 = 0, x \in \Omega, t \in [0,T] \\ u(x,t) = 0, x \in \partial\Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \end{cases} \tag{1.2}$$

The authors studied the global existence of weak solution. Another related mathematical work involving the logarithmic terms by Cazenave and Haraux [6], where they established the existence and uniqueness of a solution for the following problem in the $(\mathbb{R}^3)$

$$\begin{cases} u_{tt} - \Delta u + u_t - u \log |u|^2 = 0, \\ u(x,t) = 0, x \in \partial\Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{1.3}$$

We can also mention some other works on the logarithmic Schrodinger equation as in [5], [4], [9].
In the case of constant delay, that is for $\tau(t) = \tau$, the system (1.1) has been studied by Kafini and Messaoudi [11], they considered with the following delay wave equation with logarithmic nonlinear source term

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(x, t-\tau) = u|u|^{p-2} \ln |u|^k \quad , \quad x \in \Omega, \quad t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega \\ u_t(x, t-\tau) = f_0(x, t-\tau), \quad t \in (0,\tau) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{1.4}$$

under the assumption $|\mu_2| \leq \mu_1$, they established the local existence by the semigroup theory and proved a finite time blow up result.
The case of time-varying delay in the wave equation has been studied recently by Nicaice et al [13], they proved the exponential stability under the condition

$$\mu_2 < \sqrt{1-d}\mu_1$$

where d is a constant satisfies

$$\tau'(t) \leq d < 1, \forall t > 0 \tag{1.5}$$

For the wave equation ant with a time-varying delay, in [13] the authors which considers the system

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(x,t) = 0 \\ \dfrac{du}{dv}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)), \end{cases}$$

where the time-varying delay $\tau(t) > 0$ satisfies

$$0 \leq \tau(t) \leq \overline{\tau}, \forall t > 0 \tag{1.6}$$

$$\tau'(t) \leq 1, \forall t > 0 \tag{1.7}$$

and

$$\tau(t) \in W^{2,\infty}([0,T]), \forall T > 0 \tag{1.8}$$

They proved the exponential stability, under suitable conditions.
This paper is organized as follows: in the section 2, under the assumption

$$|\mu_2| \leq \sqrt{1-d}\mu_1, \tag{1.9}$$

we establish a local existence and in section 3, we prove a blow-up result under assumption on the delay by the energy method and Lyapunov function.

2. Local existence

In order to prove the existence of a unique solution of problem (1.1)-(2.6), we introduce the new variable

$$z(x,\rho,t) = u_t(x, t-\tau(t)\rho), \tag{2.1}$$

then we obtain

$$\begin{cases} \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0 \\ z(x,0,t) = u_t(x,t) \end{cases} \tag{2.2}$$

consequently, the problem is equivalent to

$$\begin{cases} u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = u|u|^{p-2} ln|u|^k. \\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0 \end{cases} \tag{2.3}$$

where

$$(x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).$$

with the initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \quad in \ \ \partial\Omega \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \\ z(x,\rho,0) = f_0(x, -\rho\tau(0)), \end{cases} \tag{2.4}$$

for all $(x,\rho,t) \in \Omega \times (0,1) \times (0,\infty)$, where the function $\tau(t)$ satisfies (1.5), (1.8) and the condition

$$0 < \tau_0 < \tau(t) < \overline{\tau}, \forall t > 0. \tag{2.5}$$

Let $v = u_t$ and denote by

$$U = (u,v,z)^T, \quad and \quad J(U) = (0, u|u|^{p-2}\ln|u|^k, 0)^T$$

Therefore, (1.1) can be rewritten as

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = J(U(t)), \quad t > 0 \\ U(0) = U_0 \end{cases} \tag{2.6}$$

where $U_0 = (u_0, u_1, f_0(., -\rho\tau(0))^T$ and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} -v \\ -\Delta u + \mu_1 v + \mu_2 z(x,1,t) \\ \frac{(1-\tau'(t))}{\tau(t)} z_\rho. \end{pmatrix} \tag{2.7}$$

We define the energy space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega, (0,1))$$

$\mathcal{H}$ is a Hilbert space with respect to the inner product

$$< U, \overline{U} >_{\mathcal{H}} \quad = \quad \int_\Omega \nabla u \nabla \overline{u} dx + \int_\Omega v \overline{v} dx + \int_\Omega \int_0^1 z \overline{z} d\rho dx \tag{2.8}$$

for all $U = (u, v, z)^T, \overline{U}(\overline{u}, \overline{v}, \overline{z})^T$.
The domain of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = \begin{pmatrix} (u, v, z)^T \in \mathcal{H} \quad / \quad u \in H^2(\Omega), v \in H_0^1(\Omega), z(x, 1, t) \in L^2(\Omega) \\ z, z_\rho \in L^2(\Omega, (0, 1))), z(x, 0, t) = v. \end{pmatrix} \tag{2.9}$$

Before establishing the local existence result, we need the following lemma

Lemma 2.1. *For any $\varepsilon > 0$, there exist $A > 0$, such that the real function*

$$j(s) = |s|^{p-2} \ln |s|, \qquad p > 2$$

satisfies

$$|j(s)| \leq A + |s|^{p-2+\varepsilon}$$

Proof. Since $\lim_{|s| \to +\infty} \left(\frac{\ln |s|}{|s|^\varepsilon} \right) = 0$, then there exists $B > 0$, such that

$$\frac{\ln |s|}{|s|^\varepsilon} < 1, \quad \forall |s| > B$$

So

$$|j(s)| \leq |s|^{p-2+\varepsilon}$$

since $p > 2$, then $|j(s)| \leq A$, for some $A > 0$ and for all $|\varepsilon| < B$
thus

$$|j(s)| \leq A + |s|^{p-2+\varepsilon}$$

then, we have following local existence result. □

Theorem 2.2. *Assume that (1.5)-(1.9) and*

$$\begin{cases} 2 < p < \dfrac{2(n-1)}{n-2}, & if \quad n \geq 3 \\ p > 2, & if \quad n = 1, 2 \end{cases} \tag{2.10}$$

then for all $U_0 \in \mathcal{H}$, problem (2.6) has a unique weak solution $U \in C([0, T], \mathcal{H})$.

Proof. We will show that $\mathcal{A}$ is a monotone maximal operator on $\mathcal{H}$ and J is a locally Lipschitz function on $\mathcal{H}$.
First, for all $U \in \mathcal{D}(\mathcal{A})$, we define the time-dependent inner-product on $\mathcal{H}$, (which is equivalent to the classical inner product).

$$\begin{aligned} < U, \overline{U} >_t \quad = \quad & \int_\Omega \nabla u \nabla \overline{u} dx + \int_\Omega v \overline{v} dx \\ & + \xi \tau(t) \int_\Omega \int_0^1 z(x, \rho) \overline{z}(x, \rho) d\rho dx, \end{aligned} \tag{2.11}$$

where ξ satisfies

$$\frac{|\mu_2|}{\sqrt{1-d}} \leq \xi \leq \left(2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}\right). \tag{2.12}$$

Thanks to hypothesis (1.9).
Let us set

$$\kappa(t) = \frac{(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)}.$$

In this step, we prove the monotony of the operator $\overline{\mathcal{A}}(t) = \mathcal{A}(t) + \tau(t)I$.
For a fixed t and $U = (u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$, we have

$$\begin{aligned} < \mathcal{A}(t)U, U >_t &= \mu_1 \int_\Omega v^2 dx + \mu_2 \int_\Omega vz(x,1)dx \\ &+ \xi \int_\Omega \int_0^1 (1-\tau'(t)\rho) z(x,\rho) z_\rho(x,\rho) d\rho dx. \end{aligned} \tag{2.13}$$

Observe that

$$\begin{aligned} \int_0^1 \int_0^1 (1-\tau'(t)\rho) z(x,\rho) z_\rho(x,\rho) d\rho dx &= \frac{1}{2}\int_0^1 \int_0^1 (1-\tau'(t)\rho)\frac{d}{d\rho} z^2 d\rho dx \\ &= \frac{\tau'(t)}{2} \int_0^1 \int_0^1 z^2(x,\rho) d\rho dx \\ &+ \frac{1}{2}\int_0^1 z^2(x,1)(1-\tau'(t))dx \\ &- \frac{1}{2}\int_0^1 z^2(x,0)dx, \end{aligned} \tag{2.14}$$

whereupon

$$\begin{aligned} < \mathcal{A}(t)U, U >_t &= \mu_1 \int_\Omega v^2 dx + \mu_2 \int_\Omega vz(x,1)dx \\ &+ \frac{\xi\tau'(t)}{2} \int_0^1 \int_0^1 z^2(x,\rho) d\rho dx \\ &+ \frac{\xi}{2}\int_0^1 z^2(x,1)(1-\tau'(t))dx - \frac{\xi}{2}\int_0^1 v^2 dx. \end{aligned} \tag{2.15}$$

By using Cauchy-Schwartz inequality and (1.5), we get

$$\begin{aligned} < \mathcal{A}(t)U, U >_t &= \left(\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\xi}{2}\right) \int_0^1 v^2 dx \\ &+ \left(\xi\frac{(1-d)}{2} - \frac{|\mu_2|\sqrt{1-d}}{2}\right) \int_0^1 z^2(x,1)dx \\ &- \kappa(t) < U, U >_t . \end{aligned}$$

Condition (2.12) allows to write

$$\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\xi}{2} \geq 0 \quad , \quad \xi\frac{(1-d)}{2} - \frac{|\mu_2|\sqrt{1-d}}{2} \geq 0. \tag{2.16}$$

Consequently, the operator $\overline{\mathcal{A}}(t)$ is monotone. To show that $\mathcal{A}$ is maximal, we prove that each

$$F = (f_1, f_2, f_3)^T \in \mathcal{H}$$

there exists $U(u, v, z)^T \in \mathcal{D}(\mathcal{A})$, such that $(I + \mathcal{A})U = F$

$$\begin{cases} u - v = f_1 \\ v - \Delta u + \mu_1 v + \mu_2 z(x, 1, t) = f_2 \\ z + \frac{(1-\tau'(t))}{\tau(t)} z_\rho = f_3. \end{cases} \tag{2.17}$$

Noting that $v = u - f_1$, we have deduce from $(2.17)_3$

$$z(x, 0) = v(x), x \in \Omega. \tag{2.18}$$

Following the same approach as in [11], we obtain

$$\begin{cases} z(x, \rho) = v(x)e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)} \displaystyle\int_0^\rho f_3(x, y)e^{y\tau(t)}dy, \quad if \quad \tau'(t) = 0 \\ z(x, \rho) = v(x)e^{\eta_\rho(t)} + e^{\eta_\rho(t)} \displaystyle\int_0^\rho \frac{\tau(t)}{1 - \tau'(t)y} f_3(x, y)e^{-\eta_y(t)}dy, \quad if \quad \tau'(t) \neq 0, \end{cases}$$

where $\eta_\rho(t) = \frac{\tau(t)}{\tau'(t)} ln(1 - \tau'(t)\rho)$. Whereupon, from $(2.17)_1$, we obtain

$$\begin{cases} z(x, \rho) = u(x)e^{-\rho\tau(t)} - f_1 e^{-\rho\tau(t)} + \tau(t)e^{-\rho\tau(t)} \displaystyle\int_0^\rho f_3(x, y)e^{y\tau(t)}dy, \\ z(x, \rho) = u(x)e^{\eta_\rho(t)} - f_1 e^{\eta_\rho(t)} + e^{\eta_\rho(t)} \displaystyle\int_0^\rho \frac{\tau(t)}{1 - \tau'(t)y} f_3(x, y)e^{-\eta_y(t)}dy, \end{cases} \tag{2.19}$$

and in particular

$$\begin{cases} z(x, 1) = u(x)e^{-\tau(t)} + z_0(x), \quad if \quad \tau'(t) = 0 \\ z(x, 1) = u(x)e^{\eta_1(t)} + z_0(x), \quad if \quad \tau'(t) \neq 0, \end{cases} \tag{2.20}$$

where

$$\begin{cases} z_0(x) = -f_1 e^{-\tau(t)} + \tau(t)e^{-\tau(t)} \displaystyle\int_0^1 f_3(x, y)e^{y\tau(t)}dy, \quad if \quad \tau'(t) = 0 \\ z_0(x) = -f_1 e^{\eta_1(t)} + e^{\eta_1(t)} \displaystyle\int_0^1 \frac{\tau(t)}{1 - \tau'(t)y} f_3(x, y)e^{-\eta_y(t)}dy, \quad if \quad \tau'(t) \neq 0, \end{cases}$$

with

$$z_0 \in L^2(\Omega).$$

Substituting (2.20) in $(2.17)_2$, we get

$$\Gamma u - \Delta u = G,$$

where

$$\begin{cases} \Gamma = 1 + \mu_1 + \mu_2 e^{-\tau(t)}, \quad if \quad \tau'(t) = 0 \\ G = f_2 + (1 + \mu_1)f_1 - \mu_2 z_0 \in L^2(\Omega), \end{cases} \tag{2.21}$$

and

$$\begin{cases} \Gamma = 1 + \mu_1 + \mu_2 e^{\eta_1(t)}, & if \quad \tau'(t) \neq 0 \\ G = f_2 + (1+\mu_1) f_1 - \mu_2 z_0 \in L^2(\Omega). \end{cases} \tag{2.22}$$

Now, we define, over $H_0^1(\Omega)$, the bilinear and linear forms

$$B(u,\phi) = \Gamma \int_\Omega u\phi + \int_\Omega \nabla u.\nabla\phi, \qquad L(\phi) = G\phi$$

It is easy to verify that B is continuous and coercive and L is continuous on $H_0^1(\Omega)$. Then, Lax-Milgram theorem implies that the equation

$$B(u,\phi) = L(\phi), \quad \forall \phi \in H_0^1(\Omega), \tag{2.23}$$

has a unique solution $u \in H_0^1(\Omega)$. Hence, $v = u - f_1 \in H_0^1(\Omega)$.
Consequently, from (2.19), we have $z, z_\rho \in L^2(\Omega \times (0,1))$. Thus, $U \in \mathcal{H}$.
Using (2.23), we get

$$\Gamma \int_\Omega u\phi + \int_\Omega \nabla u.\nabla\phi = \int_\Omega G\phi, \quad \forall \phi \in H_0^1(\Omega).$$

The elliptic regularity theory implies that $u \in H_0^1(\Omega)$ and, in addition, Green's formula and $(2.17)_2$ give

$$\int_\Omega [(1+\mu_1)v - \Delta u + \mu_2 z(x,1,t) - f_2]\phi = 0, \quad \forall \phi \in H_0^1(\Omega).$$

Hence

$$(1+\mu_1)v - \Delta u + \mu_2 z(x,1,t) = f_2 \in L^2(\Omega).$$

Therefore,

$$U = (u,v,z)^T \in \mathcal{D}(\mathcal{A}).$$

Therefore, the operator $I + \mathcal{A}$ is surjective for any fixed $t > 0$. Since $\tau(t) > 0$ and

$$I + \overline{\mathcal{A}}(t) = (1+\kappa(t))I + \mathcal{A}(t),$$

we deduce that the operator $I + \overline{\mathcal{A}}(t)$ is also surjective for any $t > 0$ and then $\overline{\mathcal{A}}(t)$ is maximal.
Consequently, from the above analysis, we deduce that the problem

$$\begin{cases} \overline{U}_t + \overline{\mathcal{A}}(t)\overline{U} = 0 \\ \overline{U}(0) = U_0, \end{cases} \tag{2.24}$$

has a unique solution $\overline{U} \in C([0,\infty), \mathcal{H})$.
Now, let

$$U(t) = e^{\beta(t)}\overline{U}(t),$$

with $\beta(t)=\int_0^t \tau(s)ds$, then we have using (2.24)

$$\begin{aligned} U_t(t) &= \tau(t)e^{\beta(t)}\overline{U}(t)+e^{\beta(t)}\overline{U}_t(t) \\ &= \tau(t)e^{\beta(t)}\overline{U}(t)-e^{\beta(t)}\overline{\mathcal{A}}(t)\overline{U} \\ &= e^{\beta(t)}(\tau(t)\overline{U}(t)-\overline{\mathcal{A}}(t)\overline{U}) \\ &= e^{\beta(t)}\mathcal{A}(t)\overline{U} \\ &= \mathcal{A}(t)e^{\beta(t)}\overline{U} \\ &= \mathcal{A}(t)U(t). \end{aligned}$$

Consequently, $U(t)$ is the unique solution of problem.
Finally, we show that $J:\mathcal{H}\to\mathcal{H}$ is locally Lipschitz. So, if we set

$$F(s)=|s|^{p-2}s\ln|s|^k,$$

then

$$F'(s)=k[1+(p-1)\ln|s|]|s|^{p-2}.$$

Hence

$$\begin{aligned} \|J(U)-J(\overline{U})\|^2_{\mathcal{H}} &= \|(0,u|u|^{p-2}\ln|u|^k-\overline{u}|\overline{u}|^{p-2}\ln|\overline{u}|^k,0,0)\|^2_{\mathcal{H}} \\ &= \|u|u|^{p-2}\ln|u|^k-\overline{u}|\overline{u}|^{p-2}\ln|\overline{u}|^k\|^2_L \\ &= \|F(U)-F(\overline{U})\|^2_L. \end{aligned} \tag{2.25}$$

As a consequence of the mean value theorem, we have, for $0\le\theta\le1$,

$$\begin{aligned} |F(U)-F(\overline{U})| &= |F'(\theta u+(1-\theta)\overline{u})(u-\overline{u})| \\ &\le k[1+(p-1)\ln|\theta u+(1-\theta)\overline{u}|]|\theta u+(1-\theta)\overline{u}|^{p-2}|u-\overline{u}| \\ &\le k|\theta u+(1-\theta)\overline{u}|^{p-2}|u-\overline{u}| \\ &\quad +k(p-1)|j(\theta u+(1-\theta)\overline{u})||u-\overline{u}|. \end{aligned} \tag{2.26}$$

By recalling Lemma 2.1, we arrive at

$$\begin{aligned} |F(U)-F(\overline{U})| &= k|\theta u+(1-\theta)\overline{u}|^{p-2}|u-\overline{u}|+k(p-1)A|u-\overline{u}| \\ &\quad +k(p-1)|\theta u+(1-\theta)\overline{u}|^{p-2+\varepsilon}|u-\overline{u}| \\ &\le k(|u|+|\overline{u}|)^{p-2}|u-\overline{u}|+k(p-1)A|u-\overline{u}| \\ &\quad +k(p-1)(|u|+|\overline{u}|)^{p-2+\varepsilon}|u-\overline{u}|. \end{aligned} \tag{2.27}$$

As $u,\overline{u}\in H^1_0(\Omega)$, we then use Holder's inequality and the Sobolev embedding

$$H^1_0(\Omega)\hookrightarrow L^r(\Omega),\quad \forall 1\le r\le \frac{2n}{n-2},$$

to get

$$\begin{aligned}\int_{\Omega}[(|u|+|\overline{u}|)^{p-2}|u-\overline{u}|]^2dx &= \int_{\Omega}[(|u|+|\overline{u}|)^{2(p-2)}|u-\overline{u}|^2]dx\\ &\le C\left(\int_{\Omega}(|u|+|\overline{u}|)^{2(p-2)}dx\right)^{\frac{(p-2)}{(p-1)}}\\ &\quad\times\left(\int_{\Omega}(|u-\overline{u}|)^{2(p-2)}dx\right)^{1/(p-1)}\\ &\le C[\|u\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}+\|\overline{u}\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}]^{\frac{(p-2)}{(p-1)}}\\ &\quad\times\|u-\overline{u}\|^2_{L^{2(p-1)}(\Omega)}\\ &\le C[\|u\|^{2(p-1)}_{H^1_0(\Omega)}+\|\overline{u}\|^{2(p-1)}_{H^1_0(\Omega)}]^{\frac{(p-2)}{(p-1)}}\\ &\quad\times\|u-\overline{u}\|^2_{H^1_0(\Omega)}.\end{aligned}\tag{2.28}$$

Similarly, we estimate

$$\begin{aligned}\int_{\Omega}[(|u|+|\overline{u}|)^{p-2+\varepsilon}|u-\overline{u}|]^2dx &= \int_{\Omega}[(|u|+|\overline{u}|)^{2(p-2+\varepsilon)}|u-\overline{u}|^2]dx\\ &\le C\left(\int_{\Omega}(|u|+|\overline{u}|)^{\frac{2(p-2+\varepsilon)(p-1)}{(p-2)}}dx\right)^{\frac{(p-2)}{(p-1)}}\\ &\quad\times\left(\int_{\Omega}(|u-\overline{u}|)^{2(p-2)}dx\right)^{1/(p-1)}\\ &\le C\left(\int_{\Omega}(|u|+|\overline{u}|)^{2(p-1)+\frac{2\varepsilon(p-1)}{(p-2)}}dx\right)^{\frac{(p-2)}{(p-1)}}\\ &\quad\times\|u-\overline{u}\|^2_{L^{2(p-1)}(\Omega)}.\end{aligned}\tag{2.29}$$

Since, $p<(n-1)/(n-2)$, we can choose $\varepsilon>0$ so small that

$$p^*=2(p-2)+\frac{2\varepsilon(p-1)}{(p-2)}\le\frac{2n}{n-2}.$$

Hence, we have

$$\begin{aligned}\int_{\Omega}[(|u|+|\overline{u}|)^{p-2+\varepsilon}|u-\overline{u}|]^2dx &= C[\|u\|^{p^*}_{L^{p^*}(\Omega)}+\|\overline{u}\|^{p^*}_{L^{p^*}(\Omega)}]^{\frac{(p-2)}{(p-1)}}\\ &\quad\quad\|u-\overline{u}\|^2_{L^{2(p-1)}(\Omega)}\\ &\le C[\|u\|^{p^*}_{H^1_0(\Omega)}+\|\overline{u}\|^{p^*}_{H^1_0(\Omega)}]^{\frac{(p-2)}{(p-1)}}\\ &\quad\quad\|u-\overline{u}\|^2_{H^1_0(\Omega)}.\end{aligned}\tag{2.30}$$

Therefore, by combining (2.25)-(2.30), we obtain

$$\begin{aligned}\|J(U)-J(\overline{U})\|_{\mathcal{H}}^2 &= [k^2(p-1)^2A^2]\|u-\overline{u}\|_{H_0^1(\Omega)}^2\\ &+C[(\|u\|_{H_0^1(\Omega)}^{2(p-1)}+\|\overline{u}\|_{H_0^1(\Omega)}^{2(p-1)})^{(p-2)/(p-1)}\\ &+(\|u\|_{H_0^1(\Omega)}^{p^*}+\|\overline{u}\|_{H_0^1(\Omega)}^{p^*})^{(p-2)/(p-1)}]\|u-\overline{u}\|_{H_0^1(\Omega)}^2\\ &\leq C(\|u\|_{H_0^1(\Omega)},\|\overline{u}\|_{H_0^1(\Omega)})\|u-\overline{u}\|_{H_0^1(\Omega)}^2. \qquad (2.31)\end{aligned}$$

Therefore, J is locally Lipschitz. Thanks to ([12], [15]), the proof is completed. □

3. Blow up

We introduce the energy functional

Lemma 3.1. *Assume that* (1.9)*holds and the hypotheses* (1.5)*,* (1.8) *and* (2.2) *are satisfied, let* $u(t)$ *be a solution of* (1.1)*, then* $E(t)$ *is non-increasing, that is*

$$\begin{aligned}E(t) &= \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2+\frac{k}{p^2}\|u\|_p^p\\ &+\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\\ &-\frac{1}{p}\int_\Omega |u|^p\ln|u|^k dx. \qquad (3.1)\end{aligned}$$

satisfies

$$E(t)\leq -c_1(\|u_t\|_2^2+\int_\Omega z^2(x,1,t)dx)\leq 0. \qquad (3.2)$$

Proof. By multiplying the equation $(2.3)_1$ by u_t and integrating over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|_2^2+\mu_1\|u_t\|_2^2+\mu_2\int_\Omega u_t z(x,1,t)dx\\ &=\int_\Omega u_t u|u|^{p-2}\ln|u|^k dx. \qquad (3.3)\end{aligned}$$

Now, we multiply $(2.3)_2$ by ξz and integrate the resulting equation over $\Omega\times(0,1)$ with respect to ρ and x, respectively, to obtain

$$\begin{aligned}\frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1\tau(t)z^2(x,\rho,t)d\rho dx &= -\xi\int_\Omega\int_0^1(1-\tau'(t)\rho)zz_\rho d\rho dx\\ &+\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\\ &= -\frac{\xi}{2}\int_\Omega\int_0^1\frac{d}{d\rho}(1-\tau'(t)\rho)z^2(x,\rho,t)d\rho dx\\ &= \frac{\xi}{2}\int_\Omega[z^2(x,0,t)-z^2(x,1,t)]dx\\ &+\frac{\xi\tau'(t)}{2}\int_\Omega z^2(x,1,t)dx. \qquad (3.4)\end{aligned}$$

By (3.3) and (3.4), we get (3.1) and

$$\begin{aligned}\frac{d}{dt}E(t) &= -\left(\mu_1 - \frac{\xi}{2}\right)\|u_t\|_2^2 - \left(\frac{\xi\tau'(t)}{2} - \frac{\xi}{2}\right)\int_\Omega z(x,1,t)dx \\ &\quad -\mu_2\int_\Omega u_t z(x,1,t)dx. \qquad (3.5)\end{aligned}$$

Thanks to Young's inequality, the last term in (3.5) can be estimated as follows

$$\mu_2\int_\Omega u_t z(x,1,t)dx \le \frac{|\mu_2|}{2\sqrt{1-d}}\int_\Omega u_t^2 dx + \frac{|\mu_2|\sqrt{1-d}}{2}\int_\Omega z^2(x,1,t)dx,$$

inserting (3.6) into (3.5), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &\le -\left(\mu_1 - \frac{\xi}{2} - \frac{|\mu_2|}{2\sqrt{1-d}}\right)\int_\Omega u_t^2 dx \\ &\quad -\left(\frac{\xi}{2}(\tau'(t)-1) - \frac{|\mu_2|\sqrt{1-d}}{2}\right)\int_\Omega z(x,1,t)dx. \qquad (3.6)\end{aligned}$$

Then, by using (2.16) and (1.5) our conclusion holds. □

Lemma 3.2. *There exists a positive constant $c > 0$, depending on Ω only such that*

$$\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{s/p} \le c\left(\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right), \qquad (3.7)$$

for any $u \in L^{p+1}(\Omega)$ and $2 \le s \le p$, provided that

$$\int_\Omega |u|^p \ln|u|^k dx \ge 0.$$

Proof. If $\displaystyle\int_\Omega |u|^p \ln|u|^k dx > 1$, then

$$\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{s/p} \le c[\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2]. \qquad (3.8)$$

If $\int_\Omega |u|^p \ln|u|^k dx \le 1$, then we set

$$\Omega_1 = \{x \in \Omega, |u| > 1\}$$

and, for any $\beta \le 2$, we have

$$\begin{aligned}\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{s/p} &\le \left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\beta/p} \le \left(\int_{\Omega_1} |u|^p \ln|u|^k dx\right)^{\beta/p} \\ &\le \left(\int_\Omega |u|^{p+1} dx\right)^{\beta/p} \le \left(\int_{\Omega_1} |u|^{p+1} dx\right)^{\beta/p} = \|u\|_{p+1}^{\beta(p+1)/p}..\end{aligned}$$

We choose $\beta = 2p/(p+1) < 2$ to get

$$\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{s/p} \le \|u\|_{p+1}^2 \le c\|\nabla u\|_2^2. \qquad (3.9)$$

Combining (3.8) and (3.9), we obtain (3.6). □

Lemma 3.3. *There exists a positive constant $C>0$ depending on Ω only such that*

$$\|u\|_p^p \le c\left(\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right), \tag{3.10}$$

for any $u\in L^p(\Omega)$, provided that $\int_\Omega |u|^p \ln|u|^k dx \ge 0$.

Proof. We set

$$\begin{aligned}\Omega_+ &= \{x\in\Omega, |u|>e\}\\ \Omega_- &= \{x\in\Omega, |u|\le e\},\end{aligned}$$

thus

$$\begin{aligned}\|u\|_p^p &= \int_{\Omega_+}|u|^p dx + \int_{\Omega_-}|u|^p dx\\ &\le \int_{\Omega_+}|u|^p\ln|u|^k dx + \int_{\Omega_-} e^p\left|\frac{u}{e}\right|^p dx\\ &\le \int_{\Omega_+}|u|^p\ln|u|^k dx + e^p\int_{\Omega_-}\left|\frac{u}{e}\right|^p dx\\ &\le \int_{\Omega}|u|^p\ln|u|^k dx + e^p\int_{\Omega}\left|\frac{u}{e}\right|^p dx\\ &\le c\left(\int_\Omega |u|^p\ln|u|^k dx + \|\nabla u\|_2^2\right).\end{aligned}$$

Using the fact that $\|u\|_2^2 \le c\|u\|_p^2 \le c(\|u\|_p^p)^{2/p}$, we have □

Corollary 3.4. *There exists a positive constant $C>0$ depending on Ω only such that*

$$\|u\|_2^2 \le c\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{2/p} + \|\nabla u\|_2^{4/p}, \tag{3.11}$$

provided that $\int_\Omega |u|^p\ln|u|^k dx \ge 0$.

Lemma 3.5. *There exists a positive constant $C>0$ depending on Ω only such that*

$$\|u\|_p^s \le c[\|u\|_p^p + \|\nabla u\|_2^2, \tag{3.12}$$

for any $u\in L^{p+1}(\Omega)$ and $2\le s\le p$.

Proof. If $\|u\|_p \ge 1$ then

$$\|u\|_p^s \le \|u\|_p^p$$

If $\|u\|_p\le 1$ then, $\|u\|_p^s \le \|u\|_p^2$. Using Sobolev embedding theorems, we have

$$\|u\|_p^s \le \|u\|_p^2 \le c\|\nabla u\|_2^2.$$ □

Now we are ready to state and prove our main result. For this purpose, we define

$$\begin{aligned}H(t) = -E(t) &= \frac{1}{p}\int_\Omega |u|^p\ln|u|^k dx - \frac{1}{2}\|u_t\|_2^2 - \frac{k}{p^2}\|u\|_p^p\\ &\quad -\frac{1}{2}\|\nabla u_t\|_2^2 - \frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx.\end{aligned} \tag{3.13}$$

Theorem 3.6. *Assume* (1.5)*-*(1.9) *and* (2.10) *hold. Assume further that* $E(0) < 0$*, then the solution of problem* (1.1) *blow up in finite time.*

Proof. From (3.1), we have

$$E(t) \le E(0) \le 0. \tag{3.14}$$

Hence

$$\begin{aligned} H'(t) = -E'(t) &\ge c_1 \left(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)dx \right) \\ &\ge c_1 \int_\Omega z^2(x,1,t)dx \ge 0. \end{aligned} \tag{3.15}$$

and

$$0 \le H(0) \le H(t) \le \frac{1}{p} \int_\Omega |u|^p \ln |u|^k dx. \tag{3.16}$$

We set

$$\mathcal{K}(t) = H^{1-\alpha} + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon \mu_1}{2} \int_\Omega u^2 dx. \tag{3.17}$$

where $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{3.18}$$

By multiplying $(1.1)_1$ by u and taking a derivative of (3.17), we obtain

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha)H^{-\alpha}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 \\ &+\varepsilon \int_\Omega |u|^p \ln |u|^k dx - \varepsilon\mu_2 \int_\Omega u z(x,1,t)dx. \end{aligned} \tag{3.19}$$

Using

$$\varepsilon\mu_2 \int_\Omega u z(x,1,t)dx \le \varepsilon|\mu_2|\{\delta_1\|u\|_2^2 + \frac{1}{4\delta_1}\int_\Omega z^2(x,1,t)dx\}. \tag{3.20}$$

we obtain, from (3.19),

$$\begin{aligned} \mathcal{K}'(t) &\ge (1-\alpha)H^{-\alpha}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 + \varepsilon \int_\Omega |u|^p \ln |u|^k dx \\ &-\varepsilon|\mu_2|\{\delta_1\|u\|_2^2 + \frac{1}{4\delta_1}\int_\Omega z^2(x,1,t)dx\}. \end{aligned} \tag{3.21}$$

Therefore, using (3.15) and by setting δ_1 so that, $\dfrac{|\mu_2|}{4\delta_1 c_1} = \kappa H^{-\alpha}(t)$, substituting in (3.21), we get

$$\begin{aligned} \mathcal{K}'(t) &\ge [(1-\alpha) - \varepsilon\kappa]H^{-\alpha}H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 \\ &+\varepsilon \int_\Omega |u|^p \ln |u|^k dx - \varepsilon\frac{H^\alpha(t)}{4c_1\kappa}|\mu_2|^2\|u\|_2^2. \end{aligned} \tag{3.22}$$

For $0 < a < 1$, from (3.13)

$$\begin{aligned}
\varepsilon \int_\Omega |u|^p \ln |u|^k dx &= \varepsilon p(1-a)H(t) + \frac{\varepsilon p(1-a)}{2}\|u_t\|_2^2 + \frac{\varepsilon(1-a)k}{p}\|u\|_p^p \\
&+ \frac{\varepsilon p(1-a)}{2}\|\nabla u_t\|_2^2 + \varepsilon a \int_\Omega |u|^p \ln |u|^k dx \\
&+ \frac{\varepsilon p(1-a)}{2}\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \qquad (3.23)
\end{aligned}$$

substituting in (3.22), we get

$$\begin{aligned}
\mathcal{K}'(t) &\geq [(1-\alpha) - \varepsilon\kappa]H^{-\alpha}H'(t) + \varepsilon[\frac{p(1-a)}{2} + 1]\|u_t\|_2^2 \\
&+ \varepsilon\left[\left(\frac{p(1-a)}{2} - 1\right)\right]\|\nabla u\|_2^2 \\
&+ a\varepsilon \int_\Omega |u|^p \ln |u|^k dx - \varepsilon\frac{H^\alpha(t)}{4c_1\kappa}|\mu_2|^2\|u\|_2^2 \\
&+ \varepsilon p(1-a)H(t) + \frac{\varepsilon(1-a)k}{p}\|u\|_p^p \\
&+ \frac{\varepsilon p(1-a)}{2}\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \qquad (3.24)
\end{aligned}$$

Using (3.11), (3.16) and Young's inequality, we find

$$\begin{aligned}
H^\alpha(t)\|u\|_2^2 &\leq \int_\Omega |u|^p \ln |u|^k dx)^\alpha \|u\|_2^2 \\
&\leq c\Big\{\int_\Omega |u|^p \ln |u|^k dx)^{\alpha+2/p} + \left(\int_\Omega |u|^p \ln |u|^k dx\right)^\alpha \|\nabla u\|_{4/p}^2\Big\} \\
&\leq c\Big\{\int_\Omega |u|^p \ln |u|^k dx)^{\frac{(p\alpha+2)}{p}} + \|\nabla u\|_2^2 \\
&\quad + \left(\int_\Omega |u|^p \ln |u|^k dx\right)^{\frac{p\alpha}{(p-2)})}\Big\}
\end{aligned} \qquad (3.25)$$

Exploiting (3.18), we have

$$2 < p\alpha + 2 \leq p, \quad and \quad 2 < \frac{\alpha p^2}{p-2} \leq p.$$

Thus, lemma 3.2 yields

$$H^\alpha(t)\|u\|_2^2 \leq c\left(\int_\Omega |u|^p \ln |u|^k dx + \|\nabla u\|_2^2\right). \qquad (3.26)$$

Combining (3.24) and (3.26), we obtain

$$\begin{aligned}\mathcal{K}'(t) \geq\ & [(1-\alpha)-\varepsilon\kappa]H^{-\alpha}H'(t)+\varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|_2^2 \\ &+\varepsilon\left\{\left(\frac{p(1-a)}{2}-1\right)-\frac{c|\mu_2|^2}{4c_1\kappa}\right\}\|\nabla u\|_2^2 \\ &+\varepsilon\left[a-\frac{c|\mu_2|^2}{4c_1\kappa}\right]\int_\Omega |u|^p\ln|u|^k dx \\ &+\varepsilon p(1-a)H(t)+\frac{\varepsilon(1-a)k}{p}\|u\|_p^p \\ &+\frac{\varepsilon p(1-a)}{2}\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \end{aligned} \tag{3.27}$$

At this point, we choose $a>0$ so small that

$$\alpha_1=\frac{p(1-a)}{2}-1>0$$

then we choose κ so large that

$$\alpha_2=\left(\frac{p(1-a)}{2}-1\right)-\frac{c|\mu_2|^2}{4c_1\kappa}>0$$

and

$$\alpha_3=a-\frac{c|\mu_2|^2}{4c_1\kappa}>0$$

Once κ and a are fixed, we pick ε so small so that

$$\alpha_4=(1-\alpha)-\varepsilon\kappa>0$$

Thus, for some $\beta>0$, estimate (3.27) becomes

$$\begin{aligned}\mathcal{K}'(t) \geq\ & \beta\{H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\int_\Omega |u|^p\ln|u|^k dx+\|u\|_p^p \\ &+\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\}. \end{aligned} \tag{3.28}$$

and

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0,\quad t>0. \tag{3.29}$$

Next, using Holder's and Young's inequalities, we have

$$\|u\|_2=\left(\int_\Omega u^2dx\right)^{\frac{1}{2}}\leq\left[\left(\int_\Omega(|u|^2)^{p/2}dx\right)^{\frac{2}{p}}\left(\int_\Omega 1dx\right)^{1-\frac{2}{p}}\right]^{\frac{1}{2}}\leq c\|u\|_p. \tag{3.30}$$

and

$$\left|\int_\Omega uu_tdx\right|\leq\|u\|_2.\|u_t\|_2\leq c\|u\|_p.\|u_t\|_2$$

which implies

$$\begin{aligned} \left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} &\geq c\|u\|_p^{\frac{1}{1-\alpha}}.\|u_t\|_2^{\frac{1}{1-\alpha}} \\ &\leq c\left[\|u\|_p^{\frac{\mu}{1-\alpha}} + \|u_t\|_2^{\frac{\theta}{1-\alpha}}\right]. \end{aligned} \tag{3.31}$$

where $\dfrac{1}{\mu} + \dfrac{1}{\theta} = 1$.
we take $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq p$$

Therefore, for $s = 2/(1-2\alpha)$, we obtain

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} \leq c[\|u\|_p^s + \|u_t\|_2^2].$$

hence, lemma 3.3 gives

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} \leq c\left[\|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p\right]. \tag{3.32}$$

Therefore,

$$\begin{aligned} \mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha} + \varepsilon\int_\Omega uu_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx\right)^{\frac{1}{1-\alpha}} \\ &\leq c\left[H(t) + |\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} + \|u\|_2^{\frac{2}{1-\alpha}}\right] \\ &\quad c[H(t) + \|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p]. \end{aligned} \tag{3.33}$$

According to (3.28) and (3.33), we get

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{3.34}$$

where $\lambda > 0$, depending only on β and c.
A simple integration of (3.34), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda\frac{\alpha}{(1-\alpha)}t}$$

Therefore, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}$$

This completes the proof. □

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Abdelbaki Choucha
Department of Mathematics, Faculty of Exact Sciences,
University of El Oued, B.P. 789, El Oued 39000, Algeria
e-mail: abdelbaki.choucha@gmail.com

Djamel Ouchenane
Laboratory of Pure and Applied Mathematics,
Amar Teledji Laghouat University, Algeria
e-mail: ouchenanedjamel@gmail.com or d.ouchenane@lagh-univ.dz

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Nonlinear two conformable fractional differential equation with integral boundary condition

Somia Djiab and Brahim Nouiri

Abstract. This paper deals with a boundary value problem for a nonlinear differential equation with two conformable fractional derivatives and integral boundary conditions. The results of existence, uniqueness and stability of positive solutions are proved by using the Banach contraction principle, Guo-Krasnoselskii's fixed point theorem and Hyers-Ulam type stability. Two concrete examples are given to illustrate the main results.

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1. Introduction

The subject of fractional as a definition has attracted increasing interest researchers since L'Hospital's letter in 1695. Later on, many definitions are made (the most popular ones are the Riemann-Liouville fractional derivative and Caputo's fractional derivative) and increasingly used in a variety of fields witch prove that the subject of fractional derivative is as important as calculus; see ([11, 17, 15, 6]). Moreover, Khalil et al. in ([10]) introduced new fractional derivative, namely "the conformable fractional derivative", since then, the basic concepts of conformable fractional calculus has been greatly development due to the nature of definition witch is satisfy all the requirements of the standard derivative.

Integral boundary conditions of fractional differential equations is recently approached by various researchers by applying different fixed point theorems, also, there are a few papers concerning conformable fractional differential equations with integral boundary conditions, see ([8, 13, 14, 19]), for example; the authors in ([19]) discussed

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the existence of positive solutions for

$$D_\alpha x(t) = f(t, x(t)),\ t \in [0,1],\ \alpha \in (1,2],$$
$$x(0) = 0,\ x(1) = \lambda \int_0^1 x(t)\,dt,$$

where $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$. By using the fixed point theorem in a cone.

Another aspect has increasingly attracted the attentions of researchers known as stability analysis. Different kinds of stability have been studied for fractional differential equations including exponential, Mittag-Leffler, Lyapunov stability, the Ulam-Hyers-Rassias stability, etc; for instance, M. Houas et all. in ([9]) studied the existence, uniqueness and stability of solutions to the following fractional boundary value problem with two Caputo fractional derivatives involving nonlocal boundary conditions:

$$D^\alpha \left(D^\beta + \lambda\right) x(t) = f(t, x(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s, x(s))\,ds,\ t \in [0,T],$$
$$x(0) = x_0 + g(x),\ x(T) = \theta \int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} x(s)\,ds,\ \eta \in (0,T),$$

where $D^\alpha,\ D^\beta$ denote the Caputo fractional derivatives, with

$$0 < \alpha, \beta \leq 1,\ 1 < \alpha + \beta \leq 2,\ f : [0,T] \times \mathbb{R} \to \mathbb{R}$$

and $g : C([0,T], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, and $\sigma, p > 0, \lambda, x_0, \theta$ are real constants, $g(x)$ may be regarded as

$$g(x) = \sum_{j=0}^{m} k_j x(t_j),$$

where $k_j, j = 1, \ldots, m$ are given constants and $0 < t_0 < \ldots < t_m \leq 1$. The existence, uniqueness and Ulam's stability for conformable fractional differential equations was studied as well; see ([4, 18, 12]).

On the other hand, Avery et all. in ([3]) investigated the existence of positive solution of the following conformable fractional boundary value problem with Sturm-Liouville boundary conditions

$$-D_\beta D_\alpha u(t) = f(t, u(t)),\ t \in (0,1),$$
$$\gamma u(0) - \delta D_\alpha u(0) = 0 = \eta u(1) + \zeta D_\alpha u(1),$$

where $0 < \alpha, \beta \leq 1,\ \gamma, \delta, \eta, \zeta \geq 0$ and $d = \eta\delta + \gamma\zeta + \gamma\eta/\alpha > 0$. By employing a functional compression expansion fixed point theorem.

In this paper, we concern by study the existence, uniqueness and Ulam stability of positive solutions to the following fractional boundary value problem with two conformable fractional derivatives involving integral boundary condition (for short CFBVP)

$$D_\beta D_\alpha x(t) + \lambda f(t, x(t)) = 0,\ t \in [0,1], \tag{1.1}$$
$$D_\alpha x(0) = 0,\ x(1) = \gamma \int_0^1 x(t)\,dt, \tag{1.2}$$

where $0 < \alpha, \beta \leq 1$, $\lambda > 0$, $\gamma \geq 0$, the derivatives are conformable fractional derivatives and the function $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

This paper is organized as follows. In Section 2, we give some basic concepts and properties results that will be used to prove our main results. In Section 3, we obtain the existence and uniqueness of the positive solutions for CFBVP (1.1)-(1.2), by the use of Gou-Krasnosel'skii fixed point theorem and Banach contraction mapping principle. Furthermore, we study different types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and generalized Ulam-Hyers-Rassias stability for CFBVP considered.

2. Preliminaries

In this section, we recall some useful definitions, lemmas and theorems. It is always assumed that $0 < \alpha, \beta \leq 1$ throughout this paper.

Definition 2.1. ([10]). The conformable fractional derivative of a function $x : [0, \infty) \to \mathbb{R}$ of order α is defined by

$$D_\alpha x(t) = \lim_{\epsilon \to 0} \frac{x\left(t + \epsilon t^{1-\alpha}\right) - x(t)}{\epsilon}, \ \textit{for all } t > 0.$$

If $D_\alpha x(t)$ exists on $(0, b), b > 0$, then $D_\alpha x(0) = \lim_{t \to 0} D_\alpha x(t)$.

Definition 2.2. ([10, 1]). The fractional integral of a function $x : [0, \infty) \to \mathbb{R}$ of order α and of order $\alpha\beta$ are defined respectively by

$$\begin{aligned} I_\alpha x(t) &= \int_0^t s^{\alpha-1} x(s)\, ds, \\ I_\alpha I_\beta x(t) &= \frac{1}{\beta} \int_0^t s^{\alpha-1} \left(t^\beta - s^\beta\right) x(s)\, ds. \end{aligned}$$

Lemma 2.3. ([10, 1]).

(i). *If x is a continuous function on $[0, \infty)$, then $D_\alpha (I_\alpha x(t)) = x(t)$.*
(ii). *If $D_\alpha x(t)$ is continuous function on $[0, \infty)$, then $I_\alpha (D_\alpha x(t)) = x(t) - x(0)$.*

Theorem 2.4. ([10, 1]).

(i). *If x is differentiable on $(0, \infty)$, then $D_\alpha x(t) = t^{1-\alpha} x'(t)$.*
(ii). *If x is twice differentiable on $(0, \infty)$, then*

$$D_\beta D_\alpha x(t) = t^{1-\beta} \left[t^{1-\alpha} x'(t)\right]' = (1 - \alpha)\, t^{1-\beta-\alpha} x'(t) + t^{2-\beta-\alpha} x''(t).$$

Remark 2.5. Note that $D_\beta D_\alpha \neq D_\alpha D_\beta$.

Further, we present the following fixed point theorems which will be used in studying of our main results.

Theorem 2.6. (Guo-Krasnoselskii fixed point theorem [7]). *Let E be a Banach space, $P \subset E$ be a cone and Ω_1, Ω_2 are two bounded open subsets of E with $\overline{\Omega}_1 \subset \Omega_2$. Assume that $\mathcal{T} : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ is a completely continuous operator such that either*

$$\begin{aligned} \|\mathcal{T}x\| &\geq \|x\|,\ x \in P \cap \partial\Omega_1 \text{ and } \|\mathcal{T}x\| \leq \|x\|,\ x \in P \cap \partial\Omega_2 \text{ or,} \\ \|\mathcal{T}x\| &\leq \|x\|,\ x \in P \cap \partial\Omega_1 \text{ and } \|\mathcal{T}x\| \geq \|x\|,\ x \in P \cap \partial\Omega_2. \end{aligned}$$

Then $\mathcal{T}$ has at least one fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

Theorem 2.7. *(The Banach contraction principle theorem [5]). Let E be a Banach space, $P \subseteq E$ a nonempty closed subset. If $\mathcal{T} : P \to P$ is a contraction mapping, then $\mathcal{T}$ has a unique fixed point in P.*

To facilitate the use of Theorem 2.6, we provide the following definitions and theorem:

Definition 2.8. ([16]). Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$ and if $x, -x \in P$ then $x = 0$.

Definition 2.9. ([16]). An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Theorem 2.10. *(Ascoli-Arzelà [2]). Let E be a compact space. If $\mathcal{T}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{T}$ is relatively compact.*

Next, we present an integral presentation of the solution for the linearized equation related to the equation (1.1)

$$D_\beta D_\alpha x(t) + \lambda g(t) = 0, \tag{2.1}$$

with the boundary conditions (1.2).

Lemma 2.11. *Let $g \in C[0,1]$, then the CFBVP (2.1)-(1.2) has a unique solution x given by*

$$x(t) = \lambda \int_0^1 G(t,s)\, g(s)\, ds,$$

where

$$G(t,s) = \frac{1}{\beta} \begin{cases} \left[\frac{\beta+1-\gamma}{(\beta+1)(1-\gamma)}\left(1-s^\beta\right) - \left(t^\beta - s^\beta\right)\right] s^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ \frac{\beta+1-\gamma}{(\beta+1)(1-\gamma)}\left(1-s^\beta\right) s^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Proof. By the continuity of g and Lemma 2.3, it follows from (2.1) that

$$x(t) = x(0) + I_\alpha D_\alpha x(0) - \lambda I_\alpha I_\beta g(t),\ t \in [0,1].$$

This, together the boundary conditions, implies

$$x(t) = \gamma \int_0^1 x(t)\, dt + \lambda I_\alpha I_\beta g(1) - \lambda I_\alpha I_\beta g(t),\ t \in [0,1]. \tag{2.3}$$

Now, we integrate (2.3) from 0 to 1 in both sides and by using the Fubini theorem, we get

$$\begin{aligned}\int_0^1 x(t)dt &= \gamma \int_0^1 x(t)\, dt + \frac{\lambda}{\beta} \int_0^1 s^{\alpha-1} \left(1 - s^{\beta}\right) g(s)\, ds \\ &\quad - \frac{\lambda}{\beta(\beta+1)} \int_0^1 s^{\alpha-1} \left(1 - s^{\beta}\right) g(s)\, ds,\end{aligned}$$

which implies

$$\int_0^1 x(t)dt = \frac{\lambda}{(\beta+1)(1-\gamma)} \int_0^1 s^{\alpha-1} \left(1 - s^{\beta}\right) g(s)\, ds. \tag{2.4}$$

Substituting (2.4) into (2.3), which yields

$$\begin{aligned}x(t) &= \frac{\lambda\gamma}{(\beta+1)(1-\gamma)} \int_0^1 s^{\alpha-1} \left(1 - s^{\beta}\right) g(s)\, ds \\ &\quad + \frac{\lambda}{\beta} \int_0^1 s^{\alpha-1} \left(1 - s^{\beta}\right) g(s)\, ds - \frac{\lambda}{\beta} \int_0^t s^{\alpha-1} \left(t^{\beta} - s^{\beta}\right) g(s)\, ds.\end{aligned}$$

□

The Green function G in (2.2) has several important properties given as follows:

Lemma 2.12. *For any* (t, s) *in* $[0,1] \times [0,1]$ *and* $\gamma \in [0,1)$*:*

- **(G1).** $0 \leq G(t,s)$ *and continuous,*
- **(G2).** $G(1,s) \leq G(t,s) \leq G(0,s)$,
- **(G3).** $G(0,s) = G(s,s) = \frac{\beta+1-\gamma}{\gamma\beta} G(1,s)$.

Proof. Obviously that G is positive, continuous and $\frac{\partial G(t,s)}{\partial t} \leq 0$, for $0 \leq t, s \leq 1$, then $G(t,s)$ is decreasing with respect to $t \in [0,1]$, and therefore

$$G(1,s) \leq G(t,s) \leq G(0,s), \ \ for\ 0 \leq t, s \leq 1.$$

A simple calculation shows that

$$\begin{aligned}G(0,s) &= \frac{\beta+1-\gamma}{\beta(\beta+1)(1-\gamma)} \left(1 - s^{\beta}\right) s^{\alpha-1} = G(s,s), \\ G(1,s) &= \frac{\gamma}{(\beta+1)(1-\gamma)} \left(1 - s^{\beta}\right) s^{\alpha-1} = \frac{\gamma\beta}{\beta+1-\gamma} G(0,s).\end{aligned}$$

□

3. Main results

For investigating the existence, uniqueness and stability of positive solutions for the CFBVP (1.1)-(1.2), we define the Banach space $E = C[0,1]$ with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ and the bounded subset Ω_r of E, with $\Omega_r = \{x \in E,\ \|x\| \leq r,\ r > 0\}$. As well, define the cone P in E by

$$P = \left\{x \in E,\ x(t) \geq \frac{\gamma\beta}{\beta+1-\gamma} \|x\|,\ t \in [0,1],\ \gamma \in [0,1)\right\}$$

Furthermore, define

$$\Lambda_1 = \int_0^1 G(0,s)\,ds,\ \ \Lambda_2 = \frac{\gamma\beta}{\beta+1-\gamma}\int_0^1 G(0,s)\,ds.$$

Also, define the operators $\mathcal{T} : E \to E$ as

$$\mathcal{T}x(t) = \lambda\int_0^1 G(t,s)\,f(s,x(s))\,ds,$$

under the properties of G in Lemma 2.12 and our assumptions on f, the operator is well-defined, continuous, positive and has the following properties.

Lemma 3.1. **(i).** *$\mathcal{T}(P) \subset P$.*
(ii). *The operator $\mathcal{T} : P \to P$ is completely continuous.*

Proof. (i) From Lemma 2.12 and the definition of the cone P, we have

$$\begin{aligned}
\mathcal{T}x(t) &= \lambda\int_0^1 G(t,s)\,f(s,x(s))\,ds \\
&\geq \frac{\lambda\gamma\beta}{\beta+1-\gamma}\int_0^1 G(0,s)\,f(s,x(s))\,ds \\
&\geq \frac{\lambda\gamma\beta}{\beta+1-\gamma}\max_{t\in[0,1]}\int_0^1 G(t,s)\,f(s,x(s))ds \\
&\geq \frac{\gamma\beta}{\beta+1-\gamma}\,\|\mathcal{T}x\|,\ \ for\ all\ t\in[0,1].
\end{aligned}$$

Hence $\mathcal{T}x \in P$.

(ii) Let $x \in \Omega_r$, then there exists a positive constant L_0 such that

$$\sup_{\|x\|\leq r}\max_{t\in[0,1]} f(t,x) \leq L_0,$$

then, it holds that

$$\|\mathcal{T}x(t)\| = \max_{t\in[0,1]}\lambda\int_0^1 G(t,s)\,f(s,x(s))\,ds \leq \lambda L_0\int_0^1 G(0,s)\,ds,$$

which implies that $\mathcal{T}(\Omega_r)$ is bounded. Hence, for all $t_1, t_2 \in [0,1]$, $t_1 < t_2$ and by Lemma 2.12, we have

$$\begin{aligned}
\|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)\| &\leq \max_{t\in[0,1]}\int_{t_1}^{t_2} G(t,s)\,f(s,x(s))\,ds \\
&\leq L_0\int_{t_1}^{t_2} G(0,s)\,ds \\
&= \frac{L_0\lambda(\beta+1-\gamma)}{\beta(\beta+1)(1-\gamma)}\int_{t_1}^{t_2}\left(1-s^{\beta}\right)s^{\alpha-1}ds \\
&\leq \frac{L_0\lambda(\beta+1-\gamma)}{\alpha\beta(\beta+1)(1-\gamma)}\left(t_2^{\alpha}-t_1^{\alpha}\right),
\end{aligned}$$

$\|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)\| \to 0$ as $t_1 \to t_2$ which implies that the set $\mathcal{T}(\Omega_r)$ is equicontinuous. By the Arzelà-Ascoli theorem $\mathcal{T} : \Omega_r \to \Omega_r$ is compact. We thus complete the proof. □

Lemma 3.2. *The CFBVP* (1.1)-(1.2) *has a positive solution $x \in E$ if and only if it is a fixed point of $\mathcal{T}$ in P.*

Proof. Let x be a fixed point of $\mathcal{T}$ in P, then

$$\begin{aligned} x(t) &= \lambda \int_0^1 G(t,s) f(s, x(s))\, ds, \; t \in [0,1], \\ &= \gamma \int_0^1 x(t)\, dt + \lambda I_\alpha I_\beta f(t, x(t)), \end{aligned} \tag{3.1}$$

and thus, by the continuity of f and Lemma 2.3, we obtain

$$D_\beta D_\alpha x(t) = \lambda f(t, x(t)).$$

Furthermore, the equality (3.1) directly implies

$$x(1) = \gamma \int_0^1 x(t)\, dt \text{ and } D_\alpha x(0) = 0.$$

Therefore, x is a positive solution of the CFBVP (1.1)-(1.2).
Moreover, the Lemmas 2.11 and 3.1 imply that x is a fixed point of $\mathcal{T}$ in P. □

3.1. The existence of positive solutions of the CFBVP

Before presenting our results, we present some important notations as follows:

$$f^0 = \lim_{x\to 0} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \; f^\infty = \lim_{x\to \infty} \max_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$f_0 = \lim_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x}, \; f_\infty = \lim_{x\to \infty} \min_{t\in[0,1]} \frac{f(t,x)}{x}.$$

Theorem 3.3. *Assume there exists $r_2 > r_1 > 0$, such that*

$$f(t,x) \le \frac{r_2}{\lambda\Lambda_1}, \; x \in [0, r_2], \; t \in [0,1],$$
$$f(t,x) \ge \frac{r_1}{\lambda\Lambda_2}, \; x \in [0, r_1], \; t \in [0,1],$$

then the CFBVP (1.1)-(1.2) *has at least one positive solution.*

Proof. By Lemma 2.12, for $x \in P \cap \partial\Omega_{r_1}$, we have

$$\|\mathcal{T}x\| \ge \mathcal{T}x(t) \ge \frac{\gamma\beta}{\beta+1-\gamma} \int_0^1 G(0,s) \frac{r_1}{\Lambda_2} ds = r_1.$$

For $x \in P \cap \partial\Omega_{r_2}$, we get

$$\|\mathcal{T}x\| = \int_0^1 G(0,s) f(s, x(s))\, ds \le \int_0^1 G(0,s) \frac{r_2}{\Lambda_1} ds = r_2.$$

Applying Theorem 2.6 yields that $\mathcal{T}$ has at least one fixed point $x \in P \cap \left(\overline{\Omega}_{r_2} \backslash \Omega_{r_1}\right)$ with $r_1 \le \|x\| \le r_2$. It follows from Lemma 3.2 that the CFBVP (1.1)-(1.2) has at least one positive solution x. The proof is complete. □

Theorem 3.4. *Let* $f_{\infty}\frac{\gamma\beta}{\beta+1-\gamma}\geq 1$ *and* $f^{0}\leq\frac{\gamma\beta}{\beta+1-\gamma}$ *are satisfied, then for each* $\lambda\in\left(\frac{1}{\Lambda_1},\ \frac{1}{\Lambda_2}\right)$ *the CFBVP* (1.1)-(1.2) *has at least one positive solution.*

Proof. From the definition of f^0, there exists $r_1>0$, such that

$$f\left(t,x\right)\leq f^{0}x,\ \ for\ all\ t\in\left[0,1\right],\ 0<x\leq r_1.$$

For $x\in P\cap\partial\Omega_{r_1}$, we have

$$\begin{aligned}\left\|\mathcal{T}x\right\|&=\lambda\int_0^1 G\left(0,s\right)f\left(s,x\left(s\right)\right)ds\\&\leq\lambda\int_0^1 G\left(0,s\right)f^{0}x\left(s\right)ds\\&\leq\lambda f^{0}\left\|x\right\|\Lambda_1\\&\leq\left\|x\right\|.\end{aligned}$$

Consequently

$$\left\|\mathcal{T}x\right\|\leq\left\|x\right\|,\ x\in P\cap\partial\Omega_{r_1}. \tag{3.2}$$

By the definition of f_{∞}, there exists $r_3>0$, such that

$$f\left(t,x\right)\geq f_{\infty}x,\ \ for\ all\ t\in\left[0,1\right],\ x\geq r_3.$$

If $x\in P\cap\partial\Omega_{r_2}$ with $r_2=\max\left\{2r_1,r_3\right\}$, then by the definition of cone P, we have

$$\begin{aligned}\left\|\mathcal{T}x\right\|&=\lambda\int_0^1 G\left(0,s\right)f\left(s,x\left(s\right)\right)ds\\&\geq\lambda f_{\infty}\int_0^1 G\left(0,s\right)x\left(s\right)ds\\&\geq\lambda\frac{\gamma\beta}{\beta+1-\gamma}f_{\infty}\left\|x\right\|\int_0^1 G\left(0,s\right)ds\\&\geq\left\|x\right\|.\end{aligned}$$

Hence

$$\left\|\mathcal{T}x\right\|\geq\left\|x\right\|,\ x\in P\cap\partial\Omega_{r_2}. \tag{3.3}$$

From (3.2)-(3.3) and Theorem 2.6 we assurance that the operator $\mathcal{T}$ has at least one fixed point $x\in P\cap\left(\overline{\Omega}_{r_2}\backslash\Omega_{r_1}\right)$ with $r_1\leq\left\|x\right\|\leq r_2$. It follows from Lemma 3.2 that the CFBVP (1.1)-(1.2) has at least one positive solution x. □

Theorem 3.5. *If* $\frac{\gamma\beta}{\beta+1-\gamma}f_0\geq 1$ *and* $f^{\infty}\leq\frac{\gamma\beta}{2(\beta+1-\gamma)}$ *are satisfied, then for each* $\lambda\in\left(\frac{1}{\Lambda_1},\frac{1}{\Lambda_2}\right)$ *the CFBVP* (1.1)-(1.2) *has at least one positive solution.*

Proof. From the definition of f_0, there exists $r_1>0$, such that

$$f\left(t,x\right)\geq f_{0}x,\ \ for\ all\ t\in\left[0,1\right],\ 0<x\leq r_1.$$

Further, for $x \in P$ with $\|x\| = r_1$, then as previously

$$\begin{aligned} \|\mathcal{T}x\| &\geq \lambda \int_0^1 G(0,s) f_0 x(s)\, ds \\ &\geq \lambda \frac{\gamma\beta}{\beta+1-\gamma} f_0 \|x\| \int_0^1 G(0,s)\, ds \\ &\geq \|x\|. \end{aligned}$$

Hence

$$\|\mathcal{T}x\| \geq \|x\|, \ x \in P \cap \partial\Omega_{r_1}.$$

By the definition of f^∞, there exists $L > 0$, such that

$$f(t,x) \leq f^\infty x, \ \textit{for all } t \in [0,1], \ x \geq r_4,$$

it follows that there exists $\delta > 0$, such that

$$\delta = \max_{t\in[0,1]} f(t, r_4), \ \textit{for all } t \in [0,1], \ 0 < x \leq r_4.$$

Then

$$f(t,x) \leq f^\infty x + \delta, \ \textit{for all } t \in [0,1], \ x \geq 0.$$

If $x \in P \cap \partial\Omega_{r_2}$, with $r_2 = \max\left\{2r_1, \ \frac{2\gamma\beta\delta}{\beta+1-\gamma}\right\}$, we get

$$\begin{aligned} \|\mathcal{T}x\| &= \lambda \int_0^1 G(0,s) f(s, x(s))\, ds \\ &\leq \lambda \int_0^1 G(0,s) (f^\infty x(s) + \delta)\, ds \\ &\leq \lambda (f^\infty \|x\| + \delta) \Lambda_1 \\ &\leq \|x\|. \end{aligned}$$

Thus

$$\|\mathcal{T}x\| \leq \|x\|, \ x \in P \cap \partial\Omega_{r_2}.$$

Applying Theorem 2.6 yields that $\mathcal{T}$ has at least one fixed point $x \in P \cap \left(\overline{\Omega}_{r_2} \backslash \Omega_{r_1}\right)$ and Lemma 3.2 ensure that the CFBVP (1.1)-(1.2) has at least one positive solution x. □

Example 3.6. Consider the CFBVP (1.1)-(1.2) with $\beta = 1, \ \alpha = \frac{1}{2}, \ \gamma = \frac{3}{4}$ and

$$f(t,x) = \begin{cases} (t+1)x^2, & (t,x) \in [0,1] \times (0,2], \\ 2(t+1)x, & (t,x) \in [0,1] \times (2,\infty), \end{cases}$$

$$F(t,x) = (2t+1)\left(\sin x + e^{-x}\right),$$

the functions f, F are continuous for any $t \in [0,1]$ and any $x > 0$, we have

$$\begin{aligned} f^0 &= \lim_{x\to 0} \max_{t\in[0,1]} \frac{f(t,x)}{x} = 0, \ f_\infty = \lim_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x} = 2, \\ F_0 &= \lim_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \infty, \ F^\infty = \lim_{x\to\infty} \max_{t\in[0,1]} \frac{F(t,x)}{x} = 0. \end{aligned}$$

By simple calculations we obtain $\frac{\gamma\beta}{\beta+1-\gamma}=\frac{3}{5}$. On the other hand, we get

$$\begin{aligned}\Lambda_1 &= \int_0^1 G(0,s)\,ds=\frac{\beta+1-\gamma}{\beta(\beta+1)(1-\gamma)}\int_0^1\left(1-s^{\beta}\right)s^{\alpha}ds=\frac{2}{3},\\ \Lambda_2 &= \frac{\gamma\beta}{\beta+1-\gamma}\int_0^1 G(0,s)\,ds=\frac{2}{5}.\end{aligned}$$

For $\lambda\in\left(\frac{3}{2},\frac{5}{2}\right)$, for specified function f the Theorem 3.4 (or for function F the Theorem 3.5) gives that the CFBVP (1.1)-(1.2) has at least one positive solution x defined on $[0,1]$.

3.2. The uniqueness and Ulam-Hyers stability of positive solution of the CFBVP

In this subsection, we present four types of Ulam stability definition, namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias:

Definition 3.7. The CFBVP (1.1)-(1.2) is Ulam-Hyers stable if there exists $c_f\in\mathbb{R}_+$ such that for each $\varepsilon>0$ and for every solution $y\in C^2([0,1],[0,\infty))$ of the inequality

$$|D_\beta D_\alpha y(t)+\lambda f(t,y(t))|\leq\varepsilon,\ t\in[0,1], \tag{3.4}$$

there exists a unique solution $x\in C^2([0,1],[0,\infty))$ of the CFBVP (1.1)-(1.2) with

$$\|y-x\|\leq c_f\varepsilon,\ t\in[0,1].$$

Definition 3.8. The CFBVP (1.1)-(1.2) is generalized Ulam-Hyers stable if there exists $\theta_f\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\theta_f(0)=0$ such that for each $\varepsilon>0$ and for every solution $y\in C^2([0,1],[0,\infty))$ of the inequality (3.4), there exists a unique solution $x\in C^2([0,1],[0,\infty))$ of the CFBVP (1.1)-(1.2) with

$$\|y-x\|\leq\theta_f(\varepsilon),\ t\in[0,1].$$

Definition 3.9. The CFBVP (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to $\varphi\in C([0,1],\mathbb{R}_+)$ if there exists $c_f\in\mathbb{R}_+$ such that for each $\varepsilon>0$ and for every solution $y\in C^2([0,1],[0,\infty))$ of the inequality

$$|D_\beta D_\alpha y(t)+\lambda f(t,y(t))|\leq\varepsilon\varphi(t),\ t\in[0,1], \tag{3.5}$$

there exists a unique solution $x\in C^2([0,1],[0,\infty))$ of the equations (1.1)-(1.2) with

$$\|y-x\|\leq c_f\varepsilon\varphi(t),\ t\in[0,1].$$

Definition 3.10. The CFBVP (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi\in C([0,1],\mathbb{R}_+)$, if there exists $c_{f,\varphi}\in\mathbb{R}_+$, such that for every solution $y\in C^2([0,1],[0,\infty))$ of the inequality

$$|D_\beta D_\alpha y(t)+\lambda f(t,y(t))|\leq\varphi(t),\ t\in[0,1], \tag{3.6}$$

there exists a unique solution $x\in C^2([0,1],[0,\infty))$ of the equations (1.1)-(1.2) with

$$\|y-x\|\leq c_{f,\varphi}\varphi(t),\ t\in[0,1].$$

Remark 3.11. Clearly,

(i). Definition 3.7 $\Rightarrow$ Definition 3.8.

(ii). Definition 3.9 $\Rightarrow$ Definition 3.10.

Theorem 3.12. *Assume there exists* $L > 0$ *such that*

$$\left|f(t,x) - f(t,y)\right| \leq L\left|x - y\right|, \text{ for almost every } t \in [0,1], \text{ and all } x, y \in E.$$

Then, if

$$\Delta = \lambda L \Lambda_1 < 1, \tag{3.7}$$

the CFBVP (1.1)-(1.2) *has exactly one positive solution defined on* $[0,1]$.

Proof. Using Lemma 2.3, we have

$$\begin{aligned} \left\|\mathcal{T}x(t) - \mathcal{T}y(t)\right\| &\leq \lambda \int_0^1 G(0,s)\left|\left(f(s,x(s)) - f(s,y(s))\right)\right| ds \\ &\leq \lambda L \left\|x - y\right\| \int_0^1 G(0,s)\, ds \\ &= \Delta \left\|x - y\right\|. \end{aligned}$$

Then, Theorem 2.7 and Lemma 3.2 ensure that there is a unique and positive x in E with $x = \mathcal{T}x$. ☐

Theorem 3.13. *Let* (3.7) *holds, then the CFBVP* (1.1)-(1.2) *is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*

Proof. Let $y \in C^2([0,1],[0,\infty))$ be any solution of the inequality (3.4), Thank to Lemma 2.11, we obtain

$$y(t) = \lambda \int_0^1 G(t,s) f(s,y(s))\, ds,$$

which yields

$$\begin{aligned} \left|y(t) - \lambda \int_0^1 G(t,s) f(s,y(s))\, ds\right| &\leq \frac{\varepsilon}{\beta} \int_0^t \left(t - s^\beta\right) s^{\alpha-1} ds \\ &\leq \frac{\varepsilon}{\beta} \int_0^1 \left(1 - s^\beta\right) s^{\alpha-1} ds \\ &\leq \varepsilon \Lambda_1. \end{aligned}$$

Let $x \in C^2([0,1],[0,\infty))$ be the unique solution of the CFBVP (1.1)-(1.2), we have for any $t \in [0,1]$

$$\begin{aligned} \left|y(t) - x(t)\right| &= \left|y(t) - \lambda \int_0^1 G(t,s) f(s,x(s))\, ds\right| \\ &= \left|y(t) - \lambda \int_0^1 G(t,s) f(s,y(s))\, ds + \lambda \int_0^1 G(t,s) f(s,y(s))\, ds \right. \\ &\quad \left. -\lambda \int_0^1 G(t,s) f(s,x(s))\, ds\right| \end{aligned}$$

$$\begin{aligned} &\leq \left| y(t) - \lambda \int_0^1 G(t,s) f(t, y(s))\, ds \right| \\ &\quad + \lambda \left| \int_0^1 G(t,s) \left(f(s, y(s)) - f(s, x(s)) \right) ds \right| \\ &\leq \varepsilon \Lambda_1 + \lambda L \int_0^1 G(0,s) \left| (y(s) - x(s)) \right| ds, \end{aligned}$$

which implies

$$\|y - x\| \leq \varepsilon \Lambda_1 + \lambda L \Lambda_1 \|y - x\|,$$

on simplification it gives

$$\|y - x\| \leq \varepsilon c_f, \ where\ c_f = \frac{\Lambda_1}{1 - \lambda L \Lambda_1},$$

which completes the proof. By putting $\theta_f(\varepsilon) = \varepsilon c_f, \theta_f(0) = 0$, then the CFBVP (1.1)-(1.2) is generalized Ulam-Hyers stable. □

Theorem 3.14. *Let* (3.7) *holds. Assume that, there exists an increasing function* $\varphi \in C([0,1], \mathbb{R}_+) \in E$ *and there exists* $\sigma_\varphi \in \mathbb{R}_+$ *such that for any* $t \in [0,1]$

$$I_\alpha I_\beta \varphi(t) \leq \sigma_\varphi \varphi(t),$$

is satisfied, then the solutions of the CFBVP (1.1)-(1.2) *are Ulam-Hyers-Rassias stable. Further the solutions of the considered CFBVP* (1.1)-(1.2) *are generalized Ulam-Hyers-Rassias stable.*

Proof. Similar to the proof of Theorem 3.13, let $y \in C^2([0,1], [0,\infty))$ be any solution of the inequality (3.5), Thank to Lemma 2.11, we obtain

$$\begin{aligned} \left| y(t) - \lambda \int_0^1 G(t,s) f(s, y(s))\, ds \right| &\leq \varepsilon I_\alpha I_\beta \varphi(t) \\ &\leq \varepsilon \sigma_\varphi \varphi(t). \end{aligned}$$

Let $x \in C^2([0,1], [0,\infty))$ be the unique solution of the CFBVP (1.1)-(1.2), we have for any $t \in [0,1]$

$$\begin{aligned} |y(t) - x(t)| &\leq \left| y(t) - \lambda \int_0^1 G(t,s) f(t, y(s))\, ds \right| \\ &\quad + \lambda \left| \int_0^1 G(t,s) \left(f(s, y(s)) - f(s, x(s)) \right) ds \right| \\ &\leq \varepsilon \sigma_\varphi \varphi(t) + \lambda L \int_0^1 G(0,s) \left| (y(s) - x(s)) \right| ds, \end{aligned}$$

which implies that

$$\|y - x\| \leq c_f \varepsilon \sigma_\varphi \varphi(t), \ where\ c_f = \frac{1}{1 - \lambda L \Lambda_1},$$

which completes the proof of the theorem. Moreover, if we set $\varphi(\varepsilon) = \varepsilon \varphi(t)$, then $\varphi(0) = 0$. Analogously one can easily prove that the solutions of CFBVP (1.1)-(1.2) are generalized Ulam-Hyers-Rassias stable. □

Example 3.15. Consider the CFBVP (1.1)-(1.2) with $\beta = \frac{1}{2}, \alpha = 1, \gamma = \frac{3}{4}$ and

$$f(t,x) = \frac{1}{t+2}\sin x,$$

the function f is continuous for any $t \in [0,1]$ and any $x > 0$, by simple calculations we obtain

$$|f(t,x) - f(t,y)| \le \frac{1}{2}|x-y| \ \ and\ \Lambda_1 = \frac{2}{5}.$$

For $\lambda \in (0,5)$, Theorem 3.12 give that the CFBVP (1.1)-(1.2) has exactly one positive solution x defined on $[0,1]$. Now, let

$$\left| D_{\frac{1}{2}} y'(t) + \frac{3}{5(t+2)}\sin x \right| \le \varepsilon,\ \ t \in [0,1],$$

then, by Theorem 3.13 the CFBVP (1.1)-(1.2) is Ulma-Hyers stable with $c_f = \frac{5}{11}$. On the other hand, Consider the inequality

$$\left| D_{\frac{1}{2}} y'(t) + \frac{3}{5(t+2)}\sin x \right| \le \varepsilon t,\ \ t \in [0,1],$$

by Theorem 3.14 the CFBVP (1.1)-(1.2) is Ulam-Hyers-Rassias stable with

$$c_f = \frac{1}{1 - \lambda L \Lambda_1} = \frac{25}{22},\ \ \sigma_t = \frac{1}{(\alpha+1)\beta} = 1.$$

4. Conclusion

By using the Banach contraction principle, Guo-Krasnoselskii's fixed point theorem and Hyers-Ulam type stability, we discuss problem (1.1)-(1.2), a two conformable fractional differential equation with integral boundary conditions. We present our results of the existence, uniqueness of positive solution and Hyers-Ulam type stability. Two concrete examples are given to better demonstrate our main results.

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Somia Djiab
Laboratory of Pure and Applied Mathematics,
Mohamed Boudiaf University,
Box 166, Ichbilia, 28000, M'sila, Algeria
e-mail: `somia.djiab@univ-msila.dz`

Brahim Nouiri
Laboratory of Pure and Applied Mathematics,
Mohamed Boudiaf University,
Box 166, Ichbilia, 28000, M'sila, Algeria
e-mail: `brahim.nouiri@univ-msila.dz`

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Deficient quartic spline of Marsden type with minimal deviation by the data polygon

Diana Curilă (Popescu)

Abstract. In this work we construct the deficient quartic spline with the knots following the Marsden's scheme and prove the existence and uniqueness of the deficient quartic spline with minimal deviation by the data polygon. The interpolation error estimate of the obtained quartic spline is given in terms of the modulus of continuity. A numerical example is included in order to illustrate the geometrical behaviour of the quartic spline with minimal quadratic oscillation in average in comparison with the two times continuous differentiable deficient quartic spline.

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1. Introduction

Motivated by the nice properties of complete cubic splines, Howell and Vorma extend in [7] the complete splines to quartic degree in such a manner that the tridiagonal shape of the matrix for computing the local derivatives is preserved. The obtained deficient complete quartic spline of Marsden type (see [10]) has in each interval $[x_{i-1}, x_i]$, $i = \overline{1, n}$, the expression:

$$S_i(x) = \frac{(x_i - x)^2 \cdot \left((x_i - x)^2 + 4 \cdot (x_i - x) \cdot (x - x_{i-1}) - 5 \cdot (x - x_{i-1})^2\right)}{h_i^4} \cdot y_{i-1}$$
$$+ \frac{16 \cdot (x - x_{i-1})^2 \cdot (x_i - x)^2}{h_i^4} \cdot y_{i/2}$$
$$+ \frac{(x - x_{i-1})^2 \cdot \left[(x - x_{i-1})^2 + 4 \cdot (x_i - x) \cdot (x - x_{i-1}) - 5 \cdot (x_i - x)^2\right]}{h_i^4} \cdot y_i$$

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$$+\frac{(x_i-x)^2\cdot(x-x_{i-1})\cdot(x_{i-1}+x_i-2\cdot x)}{h_i^3}\cdot m_{i-1}$$
$$+\frac{(x_i-x)\cdot(x-x_{i-1})^2\cdot(x_{i-1}+x_i-2\cdot x)}{h_i^3}\cdot m_i$$
$$=A_i(x)\cdot y_{i-1}+B_i(x)\cdot y_{i-1/2}+C_i(x)\cdot y_i+D_i(x)\cdot m_{i-1}+E_i(x)\cdot m_i, \qquad (1.1)$$

where

$$a=x_0<x_1<...<x_{n-1}<x_n=b$$

is a mesh of $[a,b]$, $h_i=x_i-x_{i-1}$, $i=\overline{1,n}$, and under traditional notations

$$m_i=S'(x_i),\ y_i=S(x_i),\ i=\overline{0,n},$$
$$y_{i-1/2}=S\left(x_{i-1/2}\right),\ i=\overline{1,n},$$

with

$$x_{i-1/2}=\frac{x_{i-1}+x_i}{2}.$$

Since $S\in C^2[a,b]$, the local derivatives $m_i, i=\overline{0,n}$, are obtained by the continuity condition $S''\in C[a,b]$ arriving to the tridiagonal dominant linear system

$$-\frac{1}{h_i}\cdot m_{i-1}+\left(\frac{4}{h_i}+\frac{4}{h_{i+1}}\right)\cdot m_i-\frac{1}{h_{i+1}}\cdot m_{i+1}=\frac{5}{h_i^2}\cdot y_{i-1}-\frac{5}{h_{i+1}^2}\cdot y_{i+1}$$
$$+\left(\frac{11}{h_i^2}-\frac{11}{h_{i+1}^2}\right)\cdot y_i+\frac{16}{h_{i+1}^2}\cdot y_{i+1/2}-\frac{16}{h_i^2}\cdot y_{i/2}\ ,\quad i=\overline{1,n-1} \qquad (1.2)$$

With two endpoint conditions of complete type $m_0{=}f'(a)$, $m_n{=}f'(b)$, the local derivatives are uniquely determined, obtaining the existence and uniqueness of the complete C^2-smooth quartic spline (see Theorem 1 in [7]).

The interpolation error estimates in the case of interpolated functions $f\in C^5[a,b]$ were obtained in [7] (for estimating $\|S-f\|_\infty$) and in[13] (for estimating $\|S'-f'\|_\infty$), with sharp error bounds.

In this brief work we intend to find the local derivatives $m_i,\ i=\overline{0,n}$, in order to minimize the deviation of the quartic spline by the data polygon and preserving a less smooth condition $S\in C^1[a,b]$. The deviation of a parametric spline by the data polygon is described in [5] by using the Hausdorff distance. Another measure of the spline deviation by the data polygon is the quadratic oscillation in average (QOA) and was introduced in [2] obtaining the cubic spline of Hermite type with minimal QOA.

Since the request of interpolating the mid-points could introduce some oscillation of the quartic spline, in this paper we try to obtain the deficient quartic spline $S\in C^1[a,b]$, as in (1.1), with minimal QOA.

Considering the polygonal line $L:[a,b]\longrightarrow\mathbb{R}$ with the pieces

$$L_{|[x_{i-1},x_i]}=L_i, i=\overline{1,n},$$

$$L_i(x)=\frac{x_i-x}{h_i}\cdot y_{i-1}+\frac{x-x_i}{h_i}\cdot y_i,\ x\in[x_{i-1},x_i], i=\overline{1,n},$$

and according to [2], the quadratic oscillation in average is the functional

$$\rho_2(S) = \left(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[S(x) - L_i(x) \right]^2 dx \right)^{\frac{1}{2}},$$

which contains the local derivatives $m_i,\ i = \overline{0,n}$, as unknown parameters.

Concerning optimal properties for cubic splines, recently, in [6], the derivative oscillation was introduced by considering the functional

$$I_1(m_0, m_1, ..., m_n) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[S'(x) - L_i'(x) \right]^2 dx,$$

and the cubic spline with minimal derivative oscillation was obtained. In [6], $I_0(m_0, m_1, ..., m_n) = (\rho_2(S))^2$ and $I_1(m_0, m_1, ..., m_n)$ where considered together as the functionals

$$I_k(m_0, m_1, ..., m_n) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[S^{(k)}(x) - L_i^{(k)}(x) \right]^2 dx,$$

for $k = 0, 1, 2$, with $I_2(m_0, m_1, ..., m_n)$ being the well-known curvature of the cubic spline (see [11]). The minimal curvature of convex preserving cubic splines was considered in [4]. Cubic splines with minimal norms

$$J_k(S) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[S^{(k)}(x) \right]^2 dx,\ k = 0, 1, 2, 3,$$

where determined in [8]. Optimal properties for quartic splines were obtained in [9] and [12], concerning the minimization of the norms $J_k(S)$, $k = 0, 1, 2, 3$, (see [9]), and considering the derivative interpolating quartic splines (see [12]). The derivative interpolating splines of even degree and their optimal properties were investigated for the first time in [3].

In the next sections we prove the existence and uniqueness of the deficient quartic spline with minimal QOA and provide the corresponding interpolation error estimate in terms of the modulus of continuity, considering a numerical experiment as test example for the theoretical result.

2. Quartic spline with minimal quadratic oscillation in average

In order to obtain the quartic spline with minimal QOA we consider the residual type functional $I_0(m_0, m_1, ..., m_n)$ denoted here by $R(m_0, m_1, ..., m_n)$, as follows

$$\begin{aligned} R(m_0, m_1, ..., m_n) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \Big[& A_i(x) \cdot y_{i-1} + B_i(x) \cdot y_{i-1/2} \\ & + C_i(x) \cdot y_i + D_i(x) \cdot m_{i-1} \\ & + E_i(x) \cdot m_i - \frac{x_i - x}{h_i} \cdot y_{i-1} - \frac{x - x_{i-1}}{h_i} \cdot y_i \Big]^2 dx. \end{aligned} \tag{2.1}$$

Theorem 2.1. *There exists unique deficient quartic spline (*1.1*) with minimal quadratic oscillation in average. If* $(m_0, m_1, ..., m_n)$ *are the local derivatives of this spline* S *and if* S *interpolates a continuous function* $f \in C[a,b]$*, then the corresponding interpolation error estimate is obtained:*

$$|S(x) - f(x)| \le \left(\frac{9317}{8192} + \frac{14\sqrt{3}\beta^3}{9}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h), \ \forall x \in [a,b] \quad (2.2)$$

where

$$h = \max\left\{h_i : i = \overline{1,n}\right\}, \ \underline{h} = \min\left\{h_i : i = \overline{1,n}\right\}, \ \beta = \frac{h}{\underline{h}},$$

and

$$\omega(f,h) = \sup\{|f(x) - f(y)| : |x - y| \le h\}$$

is the modulus of continuity

Proof. The system of normal equations

$$\frac{\partial R}{\partial m_i} = 0, \ i = \overline{0,n}$$

is

$$\begin{cases} \frac{h_1^3}{630} \cdot m_0 + \frac{h_1^3}{1260} \cdot m_1 = \\ = -\frac{h_1^2}{63} \cdot y_0 - \frac{2 \cdot h_1^2}{315} \cdot y_{1-1/2} + \frac{h_1^2}{180} \cdot y_1 + \frac{h_1^2}{60} \cdot y_0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{h_i^3}{1260} \cdot m_{i-1} + \left(\frac{h_i^3}{630} + \frac{h_{i+1}^3}{630}\right) \cdot m_i + \frac{h_{i+1}^3}{1260} \cdot m_{i+1} = \\ = -\frac{h_i^2}{180} \cdot y_{i-1} + \frac{2 \cdot h_i^2}{315} \cdot y_{i-1/2} + \frac{h_i^2}{63} \cdot y_i - \frac{h_i^2}{60} \cdot y_{i-1} - \frac{h_{i+1}^2}{63} \cdot y_i - \\ -\frac{2 \cdot h_{i+1}^2}{315} \cdot y_{i+1/2} + \frac{h_{i+1}^2}{180} \cdot y_{i+1} + \frac{h_{i+1}^2}{60} \cdot y_i \ , \ i = \overline{1, n-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{h_n^3}{1260} \cdot m_{n-1} + \frac{h_n^3}{630} \cdot m_n = \\ = -\frac{h_n^2}{180} \cdot y_{n-1} + \frac{2 \cdot h_n^2}{315} \cdot y_{n-1/2} + \frac{h_n^2}{63} \cdot y_n - \frac{h_n^2}{60} \cdot y_n \end{cases} \quad (2.3)$$

which can be written in tridiagonal dominant form

$$\begin{cases} m_0 + \frac{1}{2} \cdot m_1 = d_0 \\ \dots\dots\dots\dots\dots\dots \\ \frac{h_i^3}{2 \cdot \left(h_i^3 + h_{i+1}^3\right)} \cdot m_{i-1} + m_i + \frac{h_{i+1}^3}{2 \cdot \left(h_i^3 + h_{i+1}^3\right)} \cdot m_{i+1} = d_i \ , \ i = \overline{1, n-1} \\ \dots\dots\dots\dots\dots\dots \\ \frac{1}{2} \cdot m_{n-1} + m_n = d_n \end{cases} \quad (2.4)$$

where

$$d_0 = \frac{1}{2\cdot h_1}\cdot\left(y_0 - y_{1/2}\right) + \frac{7}{2\cdot h_1}\cdot\left(y_1 - y_{1/2}\right)$$

....

$$d_i = \frac{h_{i+1}^2}{2\cdot\left(h_i^3+h_{i+1}^3\right)}\cdot\left(y_i - y_{i+1/2}\right) + \frac{7\cdot h_{i+1}^2}{2\cdot\left(h_i^3+h_{i+1}^3\right)}\cdot\left(y_{i+1} - y_{i+1/2}\right) +$$
$$+\frac{14\cdot h_i^2}{h_i^3+h_{i+1}^3}\cdot\left(y_{i-1/2} - y_{i-1}\right) + \frac{10\cdot h_i^2}{h_i^3+h_{i+1}^3}\cdot\left(y_i - y_{i-1/2}\right)\ ,\ i = \overline{1, n-1} \tag{2.5}$$

....

$$d_n = \frac{7}{2\cdot h_n}\cdot\left(y_{n-1/2} - y_{n-1}\right) + \frac{1}{2\cdot h_n}\cdot\left(y_{n-1/2} - y_n\right)$$

Since the matrix A of this system is diagonally dominant we have unique solution $(m_0, m_1, ..., m_n)$ and $\left\|A^{-1}\right\| \leq 2$. The Hessian matrix $\left(\frac{\partial^2 R}{\partial m_i \partial m_j}\right)_{i,j=\overline{0,n}}$ has all the diagonal minors positive and therefore $(m_0, m_1, ..., m_n)$ is the unique minimum point of R. So, the local derivatives m_i , $i = \overline{0,n}$ which minimize the functional R are uniquely determined as the solution of the linear system (2.4), and the quartic spline S with minimal QOA is uniquely determined. When S interpolates $f \in C\,[a,b]$, since

$$|d_0| \leq \frac{\left|y_0 - y_{1/2}\right| + 7\cdot\left|y_1 - y_{1/2}\right|}{2\cdot h_1} \leq \frac{4}{h_1}\cdot\omega\left(f, \frac{h}{2}\right)$$

....

$$|d_i| \leq \frac{h_{i+1}^2\cdot\left(\left|y_i - y_{i+1/2}\right| + 7\cdot\left|y_{i+1} - y_{i+1/2}\right|\right)}{2\cdot\left(h_i^3+h_{i+1}^3\right)} + \frac{h_i^2\cdot\left(14\cdot\left|y_{i-1/2} - y_{i-1}\right| + 10\cdot\left|y_i - y_{i-1/2}\right|\right)}{h_i^3+h_{i+1}^3} \leq$$
$$\leq \frac{28\cdot h^2}{h_i^3+h_{i+1}^3}\cdot\omega\left(f, \frac{h}{2}\right)\ ,\ i = \overline{1, n-1} \tag{2.6}$$

....

$$|d_n| \leq \frac{7\cdot\left|y_{n-1/2} - y_{n-1}\right| + \left|y_{n-1/2} - y_n\right|}{2\cdot h_n} \leq \frac{4}{h_n}\cdot\omega\left(f, \frac{h}{2}\right)$$

we get

$$\|d\|_\infty = \max\left\{|d_i| : i = \overline{0,n}\right\} \leq \frac{14\cdot h^2}{\underline{h}^3}\cdot\omega\left(f, \frac{h}{2}\right).$$

The linear system (2.4) has the vectorial form

$$A\cdot m = d$$

and thus

$$m = A^{-1}\cdot d,$$

with

$$m = (m_0, m_1, ..., m_n)^T,\ d = (d_0, d_1, ..., d_n)^T.$$

Then

$$\|m\| = \max\left\{|m_i| : i = \overline{0,n}\right\} \leq \left\|A^{-1}\right\|\cdot\|d\| \leq \frac{28\cdot h^2}{\underline{h}^3}\cdot\omega\left(f, \frac{h}{2}\right).$$

Since

$$A_i(x) \geq 0,\ B_i(x) \geq 0,\ C_i(x) \leq 0,\ D_i(x) \geq 0,\ E_i(x) \geq 0,\ \forall x \in \left[x_{i-1}, x_{i-1/2}\right]$$

and

$$A_i(x) \leq 0,\ B_i(x) \geq 0,\ C_i(x) \geq 0,\ D_i(x) \leq 0,\ E_i(x) \leq 0,\ \forall x \in \left[x_{i-1/2}, x_i\right],$$

we estimate $|S(x) - f(x)|$ separately on $\left[x_{i-1}, x_{i-1/2}\right]$ and $\left[x_{i-1/2}, x_i\right]$.
On $\left[x_{i-1}, x_{i-1/2}\right]$ we have

$$\begin{aligned} |S(x) - f(x)| &\leq |A_i(x) + B_i(x)| \cdot \max\left\{|y_{i-1} - f(x)|, \left|y_{i-1/2} - f(x)\right|\right\} \\ &+ |C_i(x)| \cdot |y_i - f(x)| + |D_i(x) + E_i(x)| \cdot \max\{|m_{i-1}|, |m_i|\} \end{aligned} \tag{2.7}$$

because $A_i(x) + B_i(x) + C_i(x) = 1, \forall x \in [x_{i-1}, x_i]$, and on $\left[x_{i-1/2}, x_i\right]$ we get

$$\begin{aligned} |S(x) - f(x)| &\leq |A_i(x)| \cdot |y_{i-1} - f(x)| \\ &+ |B_i(x) + C_i(x)| \cdot \max\left\{|y_i - f(x)|, \left|y_{i-1/2} - f(x)\right|\right\} \\ &+ |D_i(x) + E_i(x)| \cdot \max\{|m_{i-1}|, |m_i|\} \end{aligned} \tag{2.8}$$

with

$$|D_i(x) + E_i(x)| = t \cdot (1-t) \cdot |1 - 2 \cdot t| \cdot h,$$

where

$$t = \frac{x - x_{i-1}}{h} \in [0,1],\ i = \overline{1,n}.$$

Elementary calculus lead as to

$$\max_{t \in \left[0, \frac{1}{2}\right]} |A_i(x) + B_i(x)| = \max_{t \in \left[\frac{1}{2}, 1\right]} |B_i(x) + C_i(x)| = \frac{9317}{8192},$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} |C_i(x)| = \max_{t \in \left[\frac{1}{2}, 1\right]} |A_i(x)| = \frac{1125}{8192},$$

and

$$\max_{t \in [0,1]} |D_i(x) + E_i(x)| = \frac{\sqrt{3}}{18} \cdot h_i,\ i = \overline{1,n}.$$

Consequently,

$$|S(x) - f(x)| \leq \frac{9317}{8192} \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h) + \frac{\sqrt{3}}{18} \cdot h,$$

$$\|m\|_\infty \leq \left(\frac{9317}{8192} + \frac{14\sqrt{3} \cdot h^3}{9 \cdot \underline{h}^3}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h),$$

$\forall\ x \in [x_{i-1}, x_i],\ i = \overline{1,n}$, obtaining (2.2). □

If f is L-Lipchitz function, then the error estimate becomes

$$|S(x) - f(x)| \leq \left(\frac{11567}{16384} + \frac{7\sqrt{3}}{9} \cdot \beta^3\right) \cdot Lh,\ \forall x \in [a,b].$$

For uniform partitions, $\beta = 1$ and the error estimate will be

$$|S(x) - f(x)| \le \left(\frac{11567}{16384} + \frac{7\sqrt{3}}{9} \right) \cdot Lh \simeq 2.0532 \cdot Lh, \ \forall x \in [a,b].$$

Remark 2.2. The diagonally dominant linear system (2.4) can be solved easily by using the iterative algorithm provided by the Gaussian elimination technique applied to tridiagonal systems (see [1], pages 14-15).

3. Numerical experiment

In order to illustrate the theoretical result we consider a numerical example where the given data are presented in the following table, with $n = 5$:

TABLE 1. The input data

i :	0	1	2	3	4	5
x_i :	0	2	4	6	8	10
y_i :	16	20	28	21	24	28
$y_{i-1/2}$:		12	23	32	18	30

In the context of Theorem 2.1 we will make a comparison of the geometrical performances for the following three quartic splines: the C^2-smooth deficient quartic spline proposed in [7], the C^1-smooth deficient quartic spline with minimal QOA obtained before, and the C^1-smooth deficient quartic spline that minimize the functional $I_2(m_0, m_1, ..., m_n)$. For the C^2-smooth deficient quartic spline $\overline{S}$ introduced in [7] the computed local derivatives m_i , $i = \overline{0,5}$, are:

$$m_0 = -8.7018, \ m_1 = 7.1929, \ m_2 = 8.2452,$$

$$m_3 = -10.731, \ m_4 = 7.9057, \ m_5 = -4.5236.$$

The C^1-smooth deficient quartic spline S with minimal QOA has the local derivatives

$$m_0 = 13.61, \ m_1 = 2.7799, \ m_2 = 15.27,$$

$$m_3 = -12.361, \ m_4 = 2.6746, \ m_5 = 9.6627.$$

In Figure 1 are represented the C^2-smooth quartic spline with dots line, the quartic spline having minimal QOA with solid line, and the polygonal line joining the data points with dashed line. The graphs and the figure were obtained by using the Matlab application.

Computing for comparison the quadratic oscillation in average (QOA) of the above presented two quartic splines S and $\overline{S}$, and the QOA of the C^1-smooth deficient quartic spline $\widetilde{S}$ that has the local derivatives m_i , $i = \overline{0,5}$, obtained by minimizing the curvature $I_2(m_0, m_1, ..., m_n)$ we get the following results:

	S	$\overline{S}$	$\widetilde{S}$
ρ_2 :	11.173	11.359	11.284
$\mathcal{L}$:	64.703	68.676	68.237

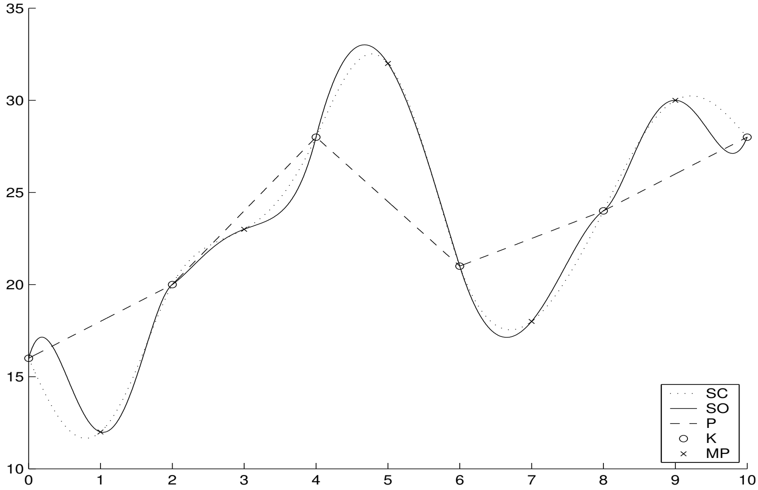

FIGURE 1. The graph of the C^2-smooth quartic spline SC represented by dotted line(....), the graph of the quartic spline SO with minimal QOA represented by solid line (—), the data polygon P represented by dashed line (- -), the knots represented with o and the midpoints represented with x

Here, we have included the length of graph ($\mathcal{L}$) for the three quartic splines, too. The computed local derivatives of $\widetilde{S}$ are

$$m_0 = -7.8476,\ m_1 = 6.9145,\ m_2 = 7.488,$$

$$m_3 = -10.225,\ m_4 = 7.8167,\ m_5 = -4.1417.$$

The geometric properties of the C^1-smooth quartic spline with minimal QOA are illustrated by considering in addition the length of graph,

$$\mathcal{L}(S) = \int_a^b \left[1 + \left(S'(x)\right)^2\right]^{\frac{1}{2}} dx \tag{3.1}$$

the results for $\mathcal{L}(S)$, $\mathcal{L}\left(\overline{S}\right)$, and $\mathcal{L}\left(\widetilde{S}\right)$ being presented above. We see that better results were obtained for the C^1-smooth deficient quartic spline with minimal QOA because smaller QOA and smaller length of graph can be observed for this quartic spline. So, the theoretical result stated in Theorem 2.1 is confirmed.

4. Conclusions

The present work shows us how could be avoided possible wild oscillations induced by the interpolation at midpoints in the case of deficient quartic splines that follows the Marsden scheme of interpolation nodes. In this context we have obtained the unique C^1-smooth deficient quartic spline with minimal quadratic oscillation in average. The tridiagonal dominant linear system of normal equations which provides its local derivatives has the index of diagonal dominance $\frac{1}{2}$, and the corresponding matrix has the condition number $cond\,(A) \leq 3$, that ensures the stability of the procedure for solving this system. The numerical experiment confirm the obtained theoretical result and point out another nice geometric property of the deficient quartic spline with minimal QOA: a smaller length of the graph.

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Diana Curilă (Popescu)
University of Oradea,
Department of Mathematics and Informatics,
1, Universităţii Street,
410087 Oradea, Romania
e-mail: curila_diana@yahoo.com

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Global existence and blow-up of a Petrovsky equation with general nonlinear dissipative and source terms

Mosbah Kaddour and Farid Messelmi

Abstract. This work studies the initial boundary value problem for the Petrovsky equation with nonlinear damping

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u - \Delta u' + |u|^{p-2} u + \alpha g(u') = \beta f(u) \text{ in } \Omega \times [0, +\infty[,$$

where Ω is open and bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$, α, and $\beta > 0$. For the nonlinear continuous term $f(u)$ and for g continuous, increasing, satisfying $g(0) = 0$, under suitable conditions, the global existence of the solution is proved by using the Faedo-Galerkin argument combined with the stable set method in $H_0^2(\Omega)$. Furthermore, we show that this solution blows up in a finite time when the initial energy is negative.

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1. Introduction

This paper devoted to the global existence, uniqueness, and the blow-up of solutions for the nonlinear general Petrovsky equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta^2 u(t) - \Delta u'(t) + |u|^{p-2} u(t) + \alpha g(u'(t)) = \beta f(u(t)), \text{ in } \Omega \times \mathbb{R}^+, \\ u = \partial_\eta u = 0, \text{ on } \Gamma \times [0, +\infty[, \\ u(x,0) = u_0(x), \ u'(x,0) = u_1(x) \text{ in } \Omega. \end{cases} \tag{1.1}$$

Recently, in the absence of the strong damping term $-\Delta u'(t)$ and in the case where

$$\beta f(u(t)) = -q(x) u(x,t) + |u|^{p-2} u(t)$$

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for g continuous, increasing, satisfying $g(0) = 0$, and $q : \Omega \to \mathbb{R}^+$, a bounded function, the problem (1.1) becomes the following

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u(t) + q(x)\, u(x,t) + g(u'(t)) = 0, \text{ in } \Omega \times \mathbb{R}^+.$$

This equation together with initial and boundary conditions of Dirichlet type was considered by Guesmia in [5], he proved a global existence and a regularity result of the solution, the author under suitable growth conditions on g showed that the solution decays exponentially if g behaves like a linear function, whereas the decay is of a polynomial order otherwise. Without the strong damping term $-\Delta u'(t)$ with

$$\alpha g(u'(t)) = |u'(t)|^{\sigma-2} u'(t)$$

and

$$\beta f(u(t)) = (b+1)\, |u(t)|^{p-2} u(t), \ b > 0,$$

the problem (1.1) reduced to the following problem

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u(t) + |u'(t)|^{\sigma-2} u'(t) = b\, |u(t)|^{p-2} u(t), \text{ in } \Omega \times \mathbb{R}^+,$$

this problem has been considered by Messaoudi in [9], where he investigated the global existence and blow-up of solution. More precisely, he showed that solutions with any initial data continue to exist globally in time if $\sigma \geq p$ and blow-up in finite time if $\sigma < p$ and the initial energy is negative. He used a new method introduced by Georgiev and Todorova [4] based on the fixed point theorem for the proof. In [12], Wu and Tsai showed that the solution of the problem considered in [9] is global under some conditions. Also, Chen and Zhou [11] studied the blow-up of the solution of the same problem as in [9]. In the presence of the strong damping, in the case where

$$\beta f(u(t)) = (b+1)\, |u(t)|^{p-2} u(t),$$
$$g(u'(t)) = |u'(t)|^{\sigma-1} u'(t), \ b > 0,$$

general Petrovsky problem as in (1.1) becomes

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u(t) - \Delta u'(t) + |u'(t)|^{\sigma-1} u'(t) = b\, |u(t)|^{p-1} u(t), \tag{1.2}$$

this problem was considered by Li et al. [6], in [10] and in [2], the authors obtained global existence, uniform decay of solutions without any interaction between p and σ, the blow-up of the solution result was established when $\sigma < p$. Very recently, Pişkin and Polat [10] studied the decay of the solution of the problem (1.2). In this paper, our aim is to extend the results of [9], [12] and others' established in a bounded domain to a general problem as in (1.1). The nonlinear term f in (1.1) likes

$$f(u(x,t)) = a(x)\, |u(t)|^{r-2} u(t) - b(x)\, |u(t)|^{q-2} u(t)$$

with $r > q \geq 1$ and $a(x), b(x) > 0$, and g in (1.1) likes

$$g(u'(x,t)) = \alpha(x)\, |u'(t)|^{\sigma-2} u'(t)$$

with $\sigma \geq 2$ for $\alpha : \Omega \to \mathbb{R}^+$ a function, satisfying $\alpha_1 \geq \alpha(x) \geq \alpha_0 > 0$. For these purposes, we must establish the global existence of solution for (1.1), we use the variational approach of Faedo–Galerkin approximation combined with the monotonous, compactness, and the stable set method as in [9], [11] and in [10] with some modification in some passages to derive the blow-up result in the infinite time of the solution.

2. Hypotheses

Let us state the precise hypotheses on p, g, and f. Let p be a positive number with

$$2 < p \leq \frac{2n-6}{n-4} \, (n \geq 5) \, (2 \leq p < \infty \text{ if } n = 1, 2, 3, 4) \,, \tag{H1}$$

g is an odd increasing C^1 function and

$$\begin{cases} xg(x) \geq d_0 |x|^{\sigma}, & \forall x \in \mathbb{R},\ p > \sigma \geq 2, \\ |g(x)| \leq d_1 |x| + d_2 |x|^{\sigma-1}, & \forall x \in \mathbb{R},\ p > \sigma \geq 2,\ d_i \geq 0. \end{cases} \tag{H2}$$

Let $f(x,s) \in C^1(\Omega \times \mathbb{R})$, satisfies:

$$sf(x,s) + k_1(x)|s| \geq pF(x,s),\ p > 2, \tag{H3}$$

and the growth conditions

$$\begin{cases} |f(x,s)| \leq l_1 \left(|s|^{\theta} + k_2(x) \right), \\ |f_s(x,s)| \leq l_1 \left(|s|^{\theta-1} + k_3(x) \right) \text{ in } \Omega \times \mathbb{R}, \end{cases} \tag{H4}$$

where $F(x,s) = \int_0^s f(x,\zeta)\, d\zeta$, with some $l_0,\ l_1 > 0$ and the non-negative functions $k_1(x), k_2(x), k_3(x) \in L^{\infty}(\Omega)$, a.e. $x \in \Omega$, and $1 < \theta \leq \frac{\sigma}{2} < \frac{p}{2}$.

3. Local existence

In this section, we establish a local existence result for (1.1) under the assumptions on f, g, and p.

Theorem 3.1. *Let $(u_0, u_1) \in W \cap L^p(\Omega) \times H_0^2(\Omega) \cap L^{2\sigma-2}(\Omega)$. Assume that (H1)-(H4) hold. Then problem (1.1) has a unique weak solution $u(t)$ satisfying:*

$$u \in L^{\infty}(0,T; W \cap L^p(\Omega)), \tag{3.1}$$

$$u' \in L^{\infty}(0,T; H_0^2(\Omega)), \tag{3.2}$$

$$g(u'(t)).u'(t) \in L^1\left(0,T; L^1(\Omega)\right), \tag{3.3}$$

$$u'' \in L^{\infty}(0,T; L^2(\Omega)), \tag{3.4}$$

where

$$H_0^2(\Omega) = \left\{ \varphi \in H^2(\Omega) : \varphi = \partial_\eta \varphi = 0 \text{ on } \partial\Omega \right\},$$

and

$$W = \left\{ \varphi \in H^4(\Omega) \cap H_0^2(\Omega) : \Delta\varphi = \partial_\eta \Delta\varphi = 0 \text{ on } \partial\Omega \right\}.$$

Note that throughout this paper, C denotes a generic positive constant depending on Ω and as all given constants, which may be different from line to line, and is capable of being examined and modified.

Proof. We adopt the Galerkin method to construct a global solution. Let $T > 0$ be a fixed, and denote by V_m the space generated by $\{\varphi_1,\ \varphi_2, ...,\ \varphi_m\}$, where the set $\{\varphi_m;\ m \in \mathbb{N}\}$ is a basis of $L^2(\Omega)$, $H_0^2(\Omega)$, and $H^4(\Omega) \cap H_0^2(\Omega)$. We construct approximate solutions u_m $(m = 1, 2, 3, ...)$ in the form

$$u_m(t) = \sum_{j=1}^{m} K_{jm}(t) w_j,$$

where K_{jm} are determined by the following ordinary differential equations:

$$\begin{aligned}(u''_m, w_j) + (\Delta u_m, \Delta w_j) + (\nabla u'_m, \nabla w_j)& \\ + \left(|u_m|^{p-2} u_m, w_j\right) + \alpha (g(u'_m), w_j) &= \beta (f(u_m), w_j),\end{aligned} \tag{3.5}$$

$$\begin{aligned}u_m(0) &= u_{0m} = \sum_{i=1}^{m} (u_0, w_j) w_j \overset{\text{as } m \longrightarrow \infty}{\longrightarrow} u_0 \\ &\text{in } H^4(\Omega) \cap H_0^2(\Omega) \cap L^p(\Omega),\end{aligned} \tag{3.6}$$

$$\begin{aligned}u'_m(0) &= u_{1m} = \sum_{i=1}^{m} (u_1, w_j) w_j \overset{\text{as } m \longrightarrow \infty}{\longrightarrow} u_1 \\ &\text{in } H_0^2(\Omega) \cap L^{2\sigma-2}(\Omega),\end{aligned} \tag{3.7}$$

with u_0, u_1 are given functions on Ω, by virtue of the theory of ordinary differential equations, the system (3.5)-(3.7) has a unique local solution on some interval $[0, t_m)$. We claim that for any $T > 0$, such a solution can be extended to the whole interval $[0, T]$, as a consequence of the a priori estimates that shall be proven in the next step. We denote by C, C_k or c_k the constants which are independent of m, the initial data u_0 and u_1.

Multiplying the equation (3.5) by $K'_{jm}(t)$ and performing the summation over $j = 1, ..., m$, the integration par parts gives

$$E'_m(t) + |\nabla u'_m(t)|^2 + \alpha (g(u'_m(t)), u'_m(t)) = 0,\ \forall t \geq 0, \tag{3.8}$$

where

$$E_m(t) = \frac{1}{2} |u'_m(t)|^2 + \frac{1}{2} |\Delta u_m(t)|^2 + \frac{1}{p} \|u_m(t)\|_p^p - \beta \int_\Omega F(x, u_m(t))\, dx, \tag{3.9}$$

by (H3), and Young inequality, we have

$$\begin{aligned}- \int_\Omega F(x, u_m)\, dx &\geq -\frac{1}{p} \int_\Omega k_1(x) |u_m|\, dx - \frac{1}{p} \int_\Omega u_m f(x, u_m)\, dx \\ &\geq -\varepsilon C_*^2 |\Delta u_m(t)|^2 - C_\varepsilon |k_1(x)|^2 - \frac{1}{p} \int_\Omega u_m f(x, u_m)\, dx,\end{aligned} \tag{3.10}$$

by using hypotheses (H4), Young's inequality yields

$$\begin{aligned}
&\frac{1}{p}\int_{\Omega} u_m f(x,u_m)\,dx \le \frac{1}{p}\left|f(x,u_m)\right|\left|u_m\right| \\
&\le \frac{l_1^2}{p}\varepsilon\int_{\Omega}\left(\left|u_m\right|^{2\theta}+\left|k_2(x)\right|^2\right)dx+\frac{c(\varepsilon,p)}{p^2}\int_{\Omega}\left|u_m\right|^2 dx \\
&= \frac{l_1^2}{p}\varepsilon\left\|u_m\right\|_{2\theta}^{2\theta}+\frac{l_1^2}{p}\varepsilon\left|k_2(x)\right|^2+\frac{c(\varepsilon,p)}{p^2}\left\|u_m\right\|_p^2 \qquad (3.11)\\
&\le \frac{l_1^2}{p}\varepsilon\left(\frac{p-2\theta}{p}+\frac{2\theta}{p}\left\|u_m\right\|_p^p\right)+\frac{l_1^2}{p}\varepsilon\left|k_2(x)\right|^2 \\
&\quad +C'(\varepsilon,p)+\frac{1}{p^2}\left\|u_m\right\|_p^p,
\end{aligned}$$

substituting (3.11) in (3.10), and chosen $\varepsilon \le C_0 = \min\left(\frac{1}{2C_*^2};\frac{p}{2\theta l_1^2+1}\right)$, (3.9) becomes

$$E_m(t) \ge \frac{1}{2}\left|u_m'(t)\right|^2 + C_1\left|\Delta u_m(t)\right|^2 + C_2\left\|u_m\right\|_p^p - C_3\left(1+K_1+K_2\right), \qquad (3.12)$$

or

$$\left|u_m'(t)\right|^2+\left|\Delta u_m(t)\right|^2+\left\|u_m\right\|_p^p \le C_4\left(E_m(t)+K_1+K_2+1\right), \qquad (3.13)$$

where

$$\begin{aligned}
&0 < C_1 \le \left(1-C_0C_*^2\right),\ 0 < C_2 \le \left(\frac{1}{p}-\frac{2\theta l_1^2+1}{p^2}C_0\right),\\
&C_3 = \max\left(C_\varepsilon;\frac{l_1^2}{p}\varepsilon;C'(\varepsilon,p)+\frac{l_1^2}{p}\varepsilon\frac{p-2\theta}{p}\right),\\
&C_4 = \max\left(\frac{1}{\min\left(\frac{1}{2},C_1,C_2\right)},C_3\right).
\end{aligned}$$

Thus, it follows from (3.8), and (3.12) that, for any $m = 1, 2, ...,$ and $t \ge 0$,

$$\begin{aligned}
&\left|u_m'(t)\right|^2+\left|\Delta u_m(t)\right|^2+\left\|u_m(t)\right\|_p^p+\int_0^t\left|\nabla u_m'(s)\right|^2 ds \qquad (3.14)\\
&+\alpha\int_0^t\left(g\left(u_m'(s)\right),u_m'(s)\right)ds \le C_4\left(E_m(0)+K_1+K_2+1\right).
\end{aligned}$$

By assumption (H2)-(H4), according to the Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_{\Omega}F(x,u_{0m})\,dx\right| \le \frac{1}{p}\int_{\Omega}k_1(x)\left|u_{0m}\right|dx+\frac{1}{p}\int_{\Omega}u_{0m}f(x,u_{0m})\,dx \qquad (3.15)\\
&\le C\left(\left|u_m(0)\right|^2+\left|k_1(x)\right|^2+\left\|u_m(0)\right\|_p^p+\left|k_2(x)\right|^2+\left|u_m(0)\right|^2\right).
\end{aligned}$$

Then using (3.6), (3.7), (3.8), and (3.9) we obtain that

$$\begin{aligned} E_m(t) \le E_m(0) &= \frac{1}{2}|u_{1m}|^2 + \frac{1}{p}\|u_{0m}\|_p^p \\ &+ \frac{1}{2}|\Delta u_{0m}|^2 - \beta\int_\Omega F(x,u_{0m})\,dx \\ \le C_4\Big(|u_{1m}|^2 + \|u_{0m}\|_p^p &+ |\Delta u_{0m}|^2 + |u_{0m}|^2 + K_1 + K_2\Big) \le C, \end{aligned} \tag{3.16}$$

for some $C > 0$, where $K_1 = \|k_1\|_\infty^2$, $K_2 = \|k_2\|_\infty^2$.
Hence, for any $t \ge 0$, and $m = 1, 2, ...$, from (3.14), and (3.16) we get

$$\begin{aligned} |u_m'(t)|^2 + |\Delta u_m(t)|^2 + \int_0^t |\nabla u_m'(s)|^2\,ds + \|u_m(t)\|_p^p \\ + \alpha\int_0^t\int_\Omega g(u_m'(s))\,u_m'(s)\,dxds \\ \le C. \end{aligned} \tag{3.17}$$

By the growth conditions, the estimate (3.17), and as $2\theta \le p$, we have

$$|f(u_m)|^2 \le Cl_1\left(|u_m|^{2\theta} + |k_2(x)|^2\right) \le C\left(||u_m||_p^{2\theta} + \|k_2\|_\infty^2\right) \le C.$$

With this estimate we can extend the approximate solution $u_m(t)$ to the interval $[0,T]$ and the following a priori estimates

$$\begin{cases} u_m \text{ is bounded in } L^\infty(0,T;L^p(\Omega)), \\ u_m' \text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \nabla u_m' \text{ is bounded in } L^2\left(0,T;L^2(\Omega)\right), \\ g(u_m').u_m' \text{ is bounded in } L^1(\Omega\times(0,T)), \\ \Delta u_m(t) \text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \\ f(u_m) \text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \end{cases} \tag{3.18}$$

hold. □

Lemma 3.2. *There exists a constant $K > 0$ such that*

$$||g(u_m'(t))||_{L^{\frac{\sigma}{\sigma-1}}(\Omega\times[0,T])} \le K,$$

for all $m \in \mathbb{N}$.

Proof. From (H2), Holder's, and Young's inequalities gives

$$\begin{aligned} \int_0^T\int_\Omega |g(u_m')|^{\frac{\sigma}{\sigma-1}}\,dxdt &= \int_0^T\int_\Omega |g(u_m')|\,|g(u_m')|^{\frac{1}{\sigma-1}}\,dxdt \\ \le \int_0^T\int_\Omega |g(u_m'(t))| &\left(d_1|u_m'(t)| + d_2|u_m'(t)|^{\sigma-1}\right)^{\frac{1}{\sigma-1}}\,dxdt \\ \le C\int_0^T\int_\Omega |g(u_m'(t))| &\left(|u_m'(t)|^{\frac{1}{\sigma-1}} + |u_m'(t)|\right)dxdt \\ = C\int_0^T\int_\Omega &|g(u_m'(t))|\,|u_m'(t)|^{\frac{1}{\sigma-1}}\,dxdt \end{aligned}$$

$$+C\int_0^T\int_\Omega |g(u_m'(t))|\,|u_m'(t)|\,dxdt$$
$$\leq \frac{\sigma-1}{\sigma}\int_0^T\int_\Omega |g(u_m')|^{\frac{\sigma}{\sigma-1}}\,dxdt + C(\sigma)\int_0^T\int_\Omega |u_m'(t)|^{\frac{\sigma}{\sigma-1}}\,dxdt$$
$$+C\int_0^T\int_\Omega |g(u_m'(t))|\,|u_m'(t)|\,dxdt,$$

therefore

$$\frac{1}{\sigma}\int_0^T\int_\Omega |g(u_m'(t))|^{\frac{\sigma}{\sigma-1}}\,dxdt \leq C(\sigma)\int_0^T\int_\Omega |u_m'(t)|^{\frac{\sigma}{\sigma-1}}\,dxdt$$
$$+C\int_0^T\int_\Omega |g(u_m'(t))|\,|u_m'(t)|\,dxdt$$
$$\leq C\int_0^T ||u_m'(t)||_2^{\frac{\sigma}{\sigma-1}}\,dt + C\int_0^T\int_\Omega |g(u_m'(t))|\,|u_m'(t)|\,dxdt,$$

hence, by (3.18), we deduce

$$\int_0^T\int_\Omega |g(u_m'(t))|^{\frac{\sigma}{\sigma-1}}\,dxdt \leq K. \qquad \square$$

Lemma 3.3. *There exists a constant $M > 0$ such that*

$$|u_m''(t)| + |\Delta u_m'(t)| + \int_0^T |\nabla u_m''(t)|\,dt \leq M,$$

for all $m \in \mathbb{N}$.

Proof. From (3.5) we obtain

$$|u_m''(0)| \leq |u_{0m}|^{p-1} + \left|\Delta^2 u_{0m}\right| + |\Delta u_{1m}| + \alpha\,|g(u_{1m})| + \beta\,|f(u_{0m})|,$$

by (H4) we have

$$|f(u_{0m})|^2 \leq l_1\left(|u_{0m}|^{2\theta} + |k_2(x)|^2\right) \leq C\left(||\Delta u_{0m}||_2^{2\theta} + \|k_2\|_\infty^2\right),$$

Since $g(u_{1m})$ is bounded in $L^2(\Omega)$ by (H2), from (3.6) and (3.7) we obtain

$$|u_m''(0)| \leq C.$$

Differentiating (3.5) with respect to t, we get

$$(u_m''', w_j) + \left(\Delta^2 u_m', w_j\right) - (\Delta u_m'', w_j) + (p-1)\left(|u_m|^{p-2}\,u_m', w_j\right)$$
$$+\alpha\,(g'(u_m')\,u_m'', w_j) = \beta\,(f'(u_m)\,u_m', w_j). \qquad (3.19)$$

Multiplying it by $K_{jm}''(t)$ and summing over j from 1 to m , according to the Hölder's inequality, to find

$$\frac{1}{2}\frac{d}{dt}\left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2\right) + |\nabla u_m''(t)|^2 + \alpha\,(g'(u_m')\,u_m'', u_m'') \qquad (3.20)$$
$$\leq (p-1)\int_\Omega |u_m|^{p-2}\,|u_m'|\,|u_m''|\,dx + \beta\int_\Omega |f'(u_m)|\,|u_m'|\,|u_m''|\,dx.$$

By choosing λ satisfies the inequalities

$$\begin{cases} \lambda+1 \leq \min\left(\frac{p}{2(\theta-1)}, \frac{n}{n-4}\right) \text{ if } n \geq 5, \\ \lambda+1 \leq \frac{p}{2(\theta-1)} \text{ if } n=1,2,3,4, \end{cases}$$

then by using (H4), estimates (3.18) and generalized Hölder's inequality, we deduce that

$$\begin{aligned} &\int_\Omega |f'(u_m)| \, |u_m'| \, |u_m''| \, dx \\ &\leq \left\| l_1 \left(|u_m|^{\theta-1} + k_3(x) \right) \right\|_{2(\lambda+1)}^{\lambda} \|u_m'\|_{2(\lambda+1)} \|u_m''\|_2 \\ &\leq C \left(\left\| |u_m|^{\theta-1} \right\|_{2(\lambda+1)}^{\lambda} + \|k_3(x)\|_{2(\lambda+1)}^{\lambda} \right) \|u_m'\|_{2(\lambda+1)} \|u_m''\|_2 \\ &\leq C \left(\|u_m\|_p^{\lambda(\theta-1)} + \|k_3(x)\|_p^{\lambda} \right) \|\Delta u_m'\|_2 \|u_m''\|_2 \\ &\leq C_5 \left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2 \right), \end{aligned} \tag{3.21}$$

where C_1 and C_2 are positive constants independent of m and $t \in [0,T]$.
By same manner, using condition (H1), Young's inequality, Sobolev embedding, and estimate (3.18) we reach to

$$\begin{aligned} \int_\Omega |u_m|^{p-2} |u_m'| \, |u_m''| \, dx &\leq \left\| |u_m|^{p-2} \right\|_n \|u_m'\|_{\frac{2n}{n-2}} \|u_m''\|_2 \\ &\leq C \|\Delta u_m'\|_2 \|u_m''\|_2 \leq C_5 \left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2 \right). \end{aligned} \tag{3.22}$$

Combining (3.20), (3.21) and (3.22) we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2 \right) + |\nabla u_m''(t)|^2 + \alpha \left(g'(u_m') u_m'', u_m'' \right) \\ \leq C_6 \left(|u_m''(t)|^2 + |\Delta u_m'(t)|^2 \right). \end{aligned}$$

Integrating the last inequality over $(0,t)$ and applying Gronwall's lemma, we obtain

$$|u_m''(t)| + |\Delta u_m'(t)| + \int_0^t |\nabla u_m''(t)|^2 \, ds \leq C \text{ for all } t \geq 0.$$

Therefore

$$\begin{aligned} u_m'' &\text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \Delta u_m' &\text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \nabla u_m'' &\text{ is bounded in } L^2\left(0,T;L^2(\Omega)\right), \end{aligned} \tag{3.23}$$

it follows from (3.23), (u_m') is bounded in $L^\infty\left(0,T;H_0^2(\Omega)\right)$.
Furthermore, by applying the Lions-Aubin compactness Lemma in [7], we claim that

$$u_m' \text{ is compact in } L^2\left(0,T;L^2(\Omega)\right), \tag{3.24}$$

From (3.18) and (3.23), there exists a subsequence of (u_m), still denote by (u_m), such that

$$\left\{\begin{array}{c} u_m \longrightarrow u \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \\ u_m \longrightarrow u \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right), \\ u_m' \longrightarrow u' \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \\ u_m' \longrightarrow u' \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right), \\ u_m'' \longrightarrow u'' \text{ weak star in } L^\infty\left(0,T;L^2(\Omega)\right), \\ g\left(u_m'\right) \longrightarrow \chi \text{ weak star in } L^{\frac{\sigma}{\sigma-1}}\left(\Omega\times(0,T)\right), \\ f\left(u_m\right) \longrightarrow \zeta \text{ weak star in } L^\infty\left(0,T;L^2(\Omega)\right). \end{array}\right. \tag{3.25}$$

Using the compactness of $H_0^2(\Omega)$ to $L^2(\Omega)$, it is easy to see that

$$\int_0^T\int_\Omega |u_m|^{p-2}u_m v dxdt \to \int_0^T\int_\Omega |u|^{p-2}uvdxdt, \text{ for all } v\in L^\sigma\left(0,T;H_0^2(\Omega)\right),$$

as $m\to\infty$.

By (H2), and estimates (3.25) we have

$$g\left(u_m'\right) \longrightarrow g\left(u'\right) \text{ a.e.in } \Omega\times(0,T).$$

Therefore, from [7, Chapter1,Lemma1.3], we infer that

$$g\left(u_m'\right) \longrightarrow g\left(u'\right) \text{ weak star in } L^{\frac{\sigma}{\sigma-1}}\left(0,T;L^{\frac{\sigma}{\sigma-1}}\right),$$

as $m\to\infty$, and this implies that

$$\int_0^T\int_\Omega g\left(u_m'\right)vdxdt \to \int_0^T\int_\Omega g\left(u'\right)vdxdt \text{ for all } v\in L^\sigma\left(0,T;H_0^2(\Omega)\right).$$

By the same manner using the growth conditions in (H4) and estimate (3.25), we see that

$$\int_0^T\int_\Omega \left|f\left(u_m\right)\right|^{\frac{\theta+1}{\theta}}dxdt$$

is bounded and

$$f\left(u_m\right) \longrightarrow f(u) \text{ a.e.in } \Omega\times(0,T),$$

then

$$f\left(u_m\right) \longrightarrow f(u) \text{ weak star in } L^{\frac{\theta+1}{\theta}}\left(0,T;L^{\frac{\theta+1}{\theta}}\right),$$

as $m\to\infty$, and this implies that

$$\int_0^T\int_\Omega f\left(u_m\right)vdxdt \to \int_0^T\int_\Omega f(u)vdxdt \text{ for all } v\in L^\theta\left(0,T;H_0^2(\Omega)\right).$$

It follows at once from all estimates that for each fixed $v\in L^\theta\left(0,T;H_0^2(\Omega)\right)\cap L^\sigma\left(0,T;H_0^2(\Omega)\right)$,

$$\int_0^T\int_\Omega \left(u_m''+\Delta^2u_m-\Delta u_m'+|u_m|^\rho u_m+\alpha g\left(u_m'\right)-\beta f\left(u_m\right)\right)vdxdt$$
$$\to \int_0^T\int_\Omega \left(u''+\Delta^2u-\Delta u'+|u|^{p-2}u+\alpha g\left(u'\right)-\beta f(u)\right)vdxdt,$$

as $m\to\infty$.

Consequently

$$\int_0^T \int_\Omega \left(u'' + \Delta^2 u - \Delta u' + |u|^{p-2} u + \alpha g(u') - \beta f(u)\right) v dx dt = 0,$$
$$\forall v \in L^\theta\left(0,T; H_0^2(\Omega)\right) \cap L^\sigma\left(0,T; H_0^2(\Omega)\right).$$

This means that the problem admit a weak solution u satisfying (1.1), and (3.1)-(3.4). □

Theorem 3.4. *Under the hypotheses of the Theorem* 3.1, *we have the solution* u *given by Theorem* 3.1, *is unique.*

Proof. Let u and v are two solutions, in the sense of the Theorem 3.1. Then $w = u - v$ satisfies

$$w'' + \left(\Delta^2 u - \Delta^2 v\right) - \Delta w' + \alpha\left(g(u') - g(v')\right)$$
$$+(|u|^{p-2} u - |v|^{p-2} v) = \beta\left(f(u) - f(v)\right), \tag{3.26}$$

$$w(0) = w'(0) = 0 \text{ in } \Omega, \tag{3.27}$$

$$w = \partial_\eta w = 0 \text{ on } \Sigma, \tag{3.28}$$

$$w \in L^p(0,T; W \cap L^p(\Omega)), \tag{3.29}$$

$$w' \in L^2(0,T; H_0^2(\Omega)). \tag{3.30}$$

Let's multiply the two members of (3.26) by w' and integrate on Ω. According to the Green's formula and conditions (3.28), integrating by part the result on $[0,t]$, using conditions (3.27) to find that

$$\frac{1}{2}\left(|w'(t)|^2 + |\Delta w|^2\right) \leq \int_0^t \int_\Omega \left||u|^{p-2} u - |v|^{p-2} v\right| |w'| dx ds \tag{3.31}$$
$$+\beta \int_0^t \int_\Omega |f(u) - f(v)| |w'| dx ds.$$

According to the Hölder's, Young's inequalities, condition (H1), the estimates (3.25) the first term on the right-hand side of (3.31) can be estimated as follows:

$$\int_0^t \int_\Omega \left||u|^{p-2} u - |v|^{p-2} v\right| |w'| dx ds$$
$$\leq (p-1) \int_0^t \left(\left\||u|^{p-2}\right\|_{L^n(\Omega)} + \left\||v|^{p-2}\right\|_{L^n(\Omega)}\right) \|w\|_{L^{\frac{2n}{n-2}}(\Omega)} \|w'\|_{L^2(\Omega)} ds$$
$$\leq C \int_0^t \left(\|u\|_{L^{n(p-2)}(\Omega)}^{p-2} + \|v\|_{L^{n(p-2)}(\Omega)}^{p-2}\right) \|\Delta w\|_{L^2(\Omega)} \|w'\|_{L^2(\Omega)} ds \tag{3.32}$$
$$\leq C \int_0^t \left(\|\Delta u\|_{L^2(\Omega)}^{p-2} + \|\Delta v\|_{L^2(\Omega)}^{p-2}\right) \|\Delta w\|_{L^2(\Omega)} \|w'\|_{L^2(\Omega)} ds$$
$$\leq C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2\right) ds.$$

Now let $U_\varepsilon = \varepsilon u + (1-\varepsilon) v$, $0 \le \varepsilon \le 1$, by the growth conditions, for the second term of the right side to (3.31), we have

$$\begin{aligned}
\left|\int_0^t \int_\Omega |f(u) - f(v)| \, |w'| \, dxdt\right| &= \left|\int_0^t \int_\Omega \int_0^1 \frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon w' dxds\right| \\
&\le \int_0^t \int_\Omega \left|\int_0^1 \frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon\right| |w'| \, dxds \\
&\le \int_0^t \int_\Omega \int_0^1 \left|\frac{d}{d\varepsilon} f(U_\varepsilon) \, d\varepsilon\right| |w'| \, dxds \\
&\le l_1 \int_0^t \int_\Omega \int_0^1 \left(|U_\varepsilon|^{\theta-1} + |k_3(x)|\right) |u - v| \, |w'| \, d\varepsilon dxds \\
&\le C \int_0^t \int_\Omega \left(|u|^{\theta-1} + |v|^{\theta-1} + |k_3(x)|\right) |w(s)| \, |w'(s)| \, dxds = I.
\end{aligned}$$

Using the generalized Hölder's, Young's inequalities, and the estimates (3.25), and choosing λ such that

$$\begin{cases} \lambda + 1 \le \frac{n}{(\theta-1)(n-4)} \text{ if } n \ge 5, \\ 2 \le \lambda + 1 < \infty \text{ if } n = 1, 2, 3, 4, \end{cases}$$

we infer

$$\begin{aligned}
I &\le C \int_0^t \left\| |u|^{\theta-1} + |v|^{\theta-1} + |k_3(x)| \right\|_{2(\lambda+1)}^{\lambda} ||w||_{2(\lambda+1)} \|w'\|_2 \\
&\le C \int_0^t \left(\left\| |u|^{\theta-1} \right\|_{2(\lambda+1)}^{\lambda} + \left\| |v|^{\theta-1} \right\|_{2(\lambda+1)}^{\lambda} + \|k_3(x)\|_{2(\lambda+1)}^{\lambda} \right) ||w||_{2(\lambda+1)} \|w'\|_2 \, ds \\
&\le C \int_0^t \left(\|\Delta u\|_2^{\lambda(\theta-1)} + \|\Delta v\|_2^{\lambda(\theta-1)} + \|k_3(x)\|_\infty^{\lambda} \right) ||\Delta w||_2 \|w'\|_2 \, ds \\
&\le C \int_0^t ||\Delta w||_2 \|w'\|_2 \, ds \le C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2 \right) ds. \qquad (3.33)
\end{aligned}$$

Combining (3.31), (3.32) and (3.33) to obtain

$$|w'(t)|^2 + |\Delta w(t)|^2 \le C \int_0^t \left(|w'(s)|^2 + |\Delta w(s)|^2 \right) ds.$$

The integral inequality and Gronwall's lemma show that $w = 0$. □

4. Global existence

In this section, we discuss the global existence of the solution for problem (1.1). In order to state and prove our main results, we first introduce the following functions

$$I(t) = I(u(t)) = |\Delta u(t)|^2 - \beta \int_\Omega f(u(t)) \, u(x,t) \, dx - \beta \int_\Omega k_1(x) \, |u(x,t)| \, dx, \quad (4.1)$$

$$J(t) = J(u(t)) = \frac{1}{2} |\Delta u|^2 - \beta \int_\Omega F(x,u) \, dx, \quad (4.2)$$

$$E(t) = E(u(t), u'(t)) = J\left(u\left(t\right)\right) + \frac{1}{2}\left|u_t(t)\right|_2^2 + \frac{1}{p}\left\|u(t)\right\|_p^p . \tag{4.3}$$

And the stable set as

$$W = \left\{u : u \in H_0^2\left(\Omega\right), I\left(t\right) > 0\right\} \cup \left\{0\right\}. \tag{4.4}$$

The next lemma shows that our energy functional (4.3) is a nonincreasing function along with the solution of (1.1).

Lemma 4.1. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'\left(t\right) = -\left|\nabla u'\left(t\right)\right|^2 - \alpha\int_\Omega u'\left(t\right) g\left(u'\left(t\right)\right) dx \leq 0. \tag{4.5}$$

Proof. By multiplying equation (1.1) by u' and integrate over Ω, using integrate by parts and summing up the product results,

$$E\left(t\right) - E\left(0\right) = -\int_0^t \left|\nabla u'\left(s\right)\right|^2 ds - \alpha\int_0^t\int_\Omega u'\left(s\right) g\left(u'\left(s\right)\right) dxds \text{ for } t \geq 0. \qquad \square$$

Lemma 4.2. *Suppose that (H1)-(H4) hold, let $u_0 \in W$ and $u_1 \in H_0^2(\Omega)$ such that*

$$\gamma = \beta C_*^{\theta+1}\left(\frac{2p}{p-2}E\left(0\right)\right)^{\frac{\theta-1}{2}}\left(l_1 + l_1\left|\left|k_2\left(x\right)\right|\right|_\infty + \left|\left|k_1\left(x\right)\right|\right|_\infty\right) < 1. \tag{4.6}$$

Then $u \in W$ for each $t \geq 0$, where C_* is the Sobolev–Poincaré embedding such that for all $2 < p \leq \frac{2n}{n-4}$ $(n \geq 5)$, $(2 \leq p < \infty$ if $n = 1, 2, 3, 4)$ we have

$$\left\|u\left(t\right)\right\|_p \leq C_*\left\|\Delta u\left(t\right)\right\|_2, \ \forall u \in H_0^2(\Omega).$$

Proof. Since $I\left(0\right) > 0$, by the continuity, there exists $0 < T_m < T$ such

$$I\left(t\right) \geq 0, \ \forall t \in \left[0, T_m\right],$$

this gives from (4.2), and (H3),

$$E\left(t\right) \geq J\left(t\right) = \frac{1}{p}I\left(t\right) + \frac{p-2}{2p}\left|\Delta u\right|^2$$
$$+\frac{\beta}{p}\left(\int_\Omega f\left(u\right) u dx + \int_\Omega k_1\left(x\right)\left|u\right| dx - p\int_\Omega F\left(x, u\right) dx\right) \geq \frac{p-2}{2p}\left|\Delta u\right|^2. \tag{4.7}$$

By using (4.7), (4.3), and (4.5),

$$\left|\Delta u\right|^2 \leq \frac{2p}{p-2}J\left(t\right) \leq \frac{2p}{p-2}E\left(t\right) \leq \frac{2p}{p-2}E\left(0\right). \tag{4.8}$$

By recalling (H1), (H2), (4.8), (4.6), Cauchy-Schwartz inequality, and Sobolev embedding we have

$$\begin{aligned}
\beta \int_\Omega f(u)\, u dx + \beta \int_\Omega k_1(x)\, |u|\, dx &\le \beta \int_\Omega |f(u)|\, |u|\, dx + \beta \int_\Omega |k_1(x)|\, |u|\, dx \\
&\le \beta l_1 \int_\Omega |u|^{\theta+1} dx + \beta l_1 \int_\Omega |k_2(x)|\, |u|\, dx + \beta \int_\Omega |k_1(x)|\, |u|\, dx \\
&\le \beta l_1 \|u(t)\|_{\theta+1}^{\theta+1} + \beta \left(l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty \right) \|u(t)\|_{\theta+1}^{\theta+1} \\
&\le \beta l_1 C_*^{\theta+1} |\Delta u(t)|^{\theta+1} + \beta C_*^{\theta+1} \left(l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty \right) |\Delta u(t)|^{\theta+1} \qquad (4.9) \\
&= \beta l_1 C_*^{\theta+1} |\Delta u(t)|^{\theta-1} |\Delta u(t)|^2 \\
&\quad + \beta C_*^{\theta+1} \left(l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty \right) |\Delta u(t)|^{\theta-1} |\Delta u(t)|^2 \\
&\le \beta C_*^{\theta+1} \left(\frac{2p}{p-2} E(0) \right)^{\frac{\theta-1}{2}} \left(l_1 + l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty \right) |\Delta u|^2 \\
&< |\Delta u|^2 \text{ on } [0, T_m].
\end{aligned}$$

Therefore, by using (4.1), we conclude that $I(t) > 0$ for all $t \in [0, T_m]$. By repeating this procedure, and using the fact that

$$\lim_{t \to T_m} \beta C_*^{\theta+1} \left(\frac{2p}{p-2} E(t) \right)^{\frac{\theta-1}{2}} \left(l_1 + l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty \right) \le D < 1,$$

T_m is extended to T. □

Lemma 4.3. *Let the assumptions (4.6) holds. Then there exists $\eta = 1 - \gamma$ such that*

$$\beta \int_\Omega f(u)\, u dx + \beta \int_\Omega k_1(x)\, |u|\, dx \le (1 - \eta) |\Delta u|^2, \qquad (4.10)$$

and therefore

$$|\Delta u|^2 \le \frac{1}{\eta} I(t). \qquad (4.11)$$

Proof. From (4.9) we have

$$\beta \int_\Omega f(u)\, u dx + \beta \int_\Omega k_1(x)\, |u|\, dx \le \gamma |\Delta u|^2.$$

We get (4.10) by taking $\eta = 1 - \gamma > 0$, and by using (4.10), from (4.1) we get the result (4.11). □

Theorem 4.4. *Suppose that (H1)-(H4) hold. Let $u_0 \in W$ satisfying (4.6). Then the solution of problem (1.1) is global.*

Proof. It sufficient to show that $\|u_t\|_2^2 + |\Delta u|^2$ is bounded independently to t. To see this we use (4.1), (4.3), and (H3) to obtain

$$
\begin{aligned}
E(0) \geq E(t) &= \frac{1}{2}|\Delta u|^2 - \beta \int_\Omega F(x,u)\,dx + \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{p}\|u(t)\|_p^p \\
&\geq \frac{1}{2}|\Delta u|^2 - \frac{\beta}{p}\int_\Omega f(u)\,u dx - \frac{\beta}{p}\int_\Omega k_1(x)\,|u|\,dx \\
&+ \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{p}\|u(t)\|_p^p = \frac{1}{2}|\Delta u|^2 + \frac{1}{p}\left(I(t) - |\Delta u|^2\right) \\
&+ \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{p}\|u(t)\|_p^p \\
&= \frac{p-2}{2p}|\Delta u|^2 + \frac{1}{p}I(t) + \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{p}\|u(t)\|_p^p \\
&\geq \frac{1}{2}\|u'(t)\|_2^2 + \frac{p-2}{2p}|\Delta u(t)|^2,
\end{aligned}
$$

since $I(t) \geq 0$, and $p > 2$. Therefore

$$\|u'(t)\|_2^2 + |\Delta u|^2 \leq \max\left(2, \frac{2p}{p-2}\right) E(0).$$

These estimates imply that the solution $u(t)$ exist globally in $[0, +\infty[$. □

5. Blow-up of solution

In this section, after some estimates, we show that the solution of problem (1.1) blows up in finite time under the assumption $E(0) < 0$, where

$$E(t) = E(u(t), u'(t)) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|\Delta u(t)|^2 + \frac{1}{p}\|u(t)\|_p^p - \beta \int_\Omega F(x, u(t))\,dx. \quad (5.1)$$

Remark 5.1. We set

$$H(t) = -E(t), \quad (5.2)$$

we multiply Eq.(1.1) by $-u'$ and integrate over Ω, using (H2) to get

$$H'(t) = |\nabla u'(t)|^2 + \alpha \int_\Omega u'(t)\,g(u'(t))\,dx \geq \alpha d_0 \|u'(t)\|_\sigma^\sigma \text{ a.e. } t \in [0, T], \quad (5.3)$$

$H(t)$ is absolutely continuous, hence

$$0 < H(0) \leq H(t) \leq \beta \int_\Omega F(x, u)\,dx, \quad (5.4)$$

when

$$E(0) < 0.$$

We need the following lemma, easy to prove by using the definition of the energy corresponding to the solution

Lemma 5.2. *Let $2 < p \leq \frac{2n}{n-4}$ if $n \geq 5$ and $2 < p < \infty$ if $n \leq 4$. Then there exists a positive constant $C > 1$, depending only on Ω, such that*

$$\|u(t)\|_p^s \leq C\left(\|u(t)\|_p^p + |\Delta u(t)|^2\right), \text{ with } 2 \leq s \leq p, \tag{5.5}$$

for any $u \in H_0^2(\Omega)$. If u is the solution constructed in Theorem 3.1, then

$$\|u(t)\|_p^s \leq C\left(H(t) + \|u(t)\|_p^p + |u'(t)|^2 + \beta \int_\Omega F(x, u(t))\, dx\right), \tag{5.6}$$

with $2 \leq s \leq p$ on $[0,T)$.

Theorem 5.3. *Let the conditions of the Theorem 3.1 be satisfied. Assume further that*

$$E(0) < 0. \tag{5.7}$$

Then the solution (3.1) blows up in a finite time T.

Proof. We pose

$$\begin{cases} L(t) = |u(t)|^2 = \int_\Omega |u(x,t)|^2\, dx, \\ L'(t) = 2(u(t), u'(t)), \\ L''(t) = 2|u'(t)|^2 + 2(u(t), u''(t)), \end{cases}$$

we define the function

$$G(t) = H^{1-a}(t) + \varepsilon L'(t) - 3\varepsilon p e^{T-t}\beta \int_\Omega F(x, u(t))\, dx$$
$$+\gamma_1 \varepsilon t \|k_1(x)\|_\infty + \gamma_2 \varepsilon t \|k_2(x)\|_\infty^\sigma, \ t \geq 0, \tag{5.8}$$

where γ_1, γ_2, $\varepsilon > 0$ are positives constants to be specified later, and

$$0 < a \leq \min\left(\frac{p-2}{2p}, \frac{p-\sigma}{(\theta+1)(\sigma-1)}\right) < 1, \tag{5.9}$$

derivative the Eq. (5.8), using Eq. (1.1), and hypotheses (H3) we obtain

$$\frac{d}{dt}G(t) = (1-a)H^{-a}(t)H'(t) + \varepsilon L''(t) + \gamma_1 \varepsilon \|k_1(x)\|_\infty$$
$$+\gamma_2 \varepsilon \|k_2(x)\|_\infty^\sigma + \frac{d}{dt}\left(-3p\varepsilon e^{T-t}\beta \int_\Omega F(x, u(t))\, dx\right)$$
$$= (1-a)H^{-a}(t)H'(t) + 2\varepsilon|u'(t)|^2 + 2\varepsilon(u(t), u''(t))$$
$$+\gamma_1 \varepsilon \|k_1(x)\|_\infty + \gamma_2 \varepsilon \|k_2(x)\|_\infty^\sigma$$
$$+3p\varepsilon e^{T-t}\beta \int_\Omega F(x, u(t))\, dx - 3p\varepsilon e^{T-t}\beta \int_\Omega f(u(t))\, u'(t)\, dx \tag{5.10}$$
$$= (1-a)H^{-a}(t)H'(t) + 2\varepsilon|u'(t)|^2 + 2\beta\varepsilon \int_\Omega u(t) f(u(t))\, dx - 2\varepsilon|\Delta u(t)|^2$$
$$-2\varepsilon \int_\Omega u(t)\Delta u'(t)\, dx - 2\varepsilon\|u(t)\|_p^p + \gamma_1 \varepsilon \|k_1(x)\|_\infty + \gamma_2 \varepsilon \|k_2(x)\|_\infty^\sigma$$
$$+3p\varepsilon e^{T-t}\beta \int_\Omega F(x, u(t))\, dx - 3p\varepsilon e^{T-t}\beta \int_\Omega f(u(t))\, u'(t)\, dx - 2\alpha\varepsilon \int_\Omega u(t)\, g(u'(t))\, dx.$$

We then exploit Holder's, Young's inequalities, and the hypotheses on g, to estimate the last term in (5.10) as

$$
\begin{aligned}
2\alpha\varepsilon\left|\int_{\Omega} u(t)\, g(u'(t))\, dx\right| &\leq 2\alpha\varepsilon d_1 \int_{\Omega} |u'(t)|\,|u(t)|\, dx + 2\alpha\varepsilon d_2 \int_{\Omega} |u'(t)|^{\sigma-1} |u(t)|\, dx \\
&\leq 2\alpha\varepsilon d_1 \frac{\delta^{\sigma}}{\sigma} ||u(t)||_{\sigma}^{\sigma} + 2\alpha\varepsilon d_1 \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} ||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} \qquad (5.11) \\
&+2\alpha\varepsilon d_2 \frac{\delta^{\sigma}}{\sigma} ||u(t)||_{\sigma}^{\sigma} + 2\alpha\varepsilon d_2 \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} ||u'(t)||_{\sigma}^{\sigma} \\
&= 2(d_1+d_2) \frac{\delta^{\sigma}}{\sigma} \alpha\varepsilon ||u(t)||_{\sigma}^{\sigma} \\
&+2\alpha\varepsilon \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} \left(d_1 ||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + d_2 ||u'(t)||_{\sigma}^{\sigma}\right),\ \delta > 0,
\end{aligned}
$$

because $\frac{\sigma}{\sigma-1} \leq \sigma$, then by (5.3) we have

$$
\begin{aligned}
d_1 ||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + d_2 ||u'(t)||_{\sigma}^{\sigma} &\leq C(\Omega)^{\frac{\sigma-2}{\sigma}} d_1 ||u'(t)||_{\sigma}^{\frac{\sigma}{\sigma-1}} + \frac{d_2}{\alpha d_0} H'(t) \\
&\leq C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} ||u'(t)||_{\sigma}^{\sigma} + \frac{d_2}{\alpha d_0} H'(t) \\
&\leq \frac{1}{\alpha d_0}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right) H'(t). \qquad (5.12)
\end{aligned}
$$

By the boundary conditions we derive the following estimates

$$
\int_{\Omega} u(t)\, \Delta u'(t)\, dx = \int_{\Omega} \Delta u(t)\, u'(t)\, dx \leq \frac{1}{4} |\Delta u(t)|^2 + |u'(t)|^2. \qquad (5.13)
$$

Using hypotheses (H4), Holder's, Young's inequalities, conditions (5.9), and (5.3) we have

$$
\begin{aligned}
\int_{\Omega} |f(u(t))|\,|u'(t)|\, dx &\leq l_1 \int_{\Omega} \left(|u|^{\theta} |u'(t)| + |k_2(x)|\,|u'(t)|\right) dx \\
&\leq l_1 ||u(t)||_{2\theta}^{\theta} ||u'(t)||_2 + l_1 \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} ||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + l_1 \frac{\delta^{\sigma}}{\sigma} \|k_2(x)\|_{\infty}^{\sigma} \\
&\leq \frac{l_1}{\sigma} C(\delta,\sigma)\, \delta^{\sigma} ||u(t)||_{2\theta}^{2\theta} + \frac{1}{\sigma} l_1 \delta^{\frac{\sigma}{1-\sigma}} ||u'(t)||_2^2 \\
&+l_1 \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} ||u'(t)||_{\frac{\sigma}{\sigma-1}}^{\frac{\sigma}{\sigma-1}} + l_1 \frac{\delta^{\sigma}}{\sigma} \|k_2(x)\|_{\infty}^{\sigma} \\
&\leq \frac{l_1}{\sigma} C^* C(\delta,\sigma)\, C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} ||u||_{\sigma}^{\sigma} \\
&+\frac{1}{\sigma} l_1 C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \delta^{\frac{\sigma}{1-\sigma}} ||u'(t)||_{\sigma}^{\sigma} \\
&+l_1 \frac{\sigma-1}{\sigma} \delta^{\frac{\sigma}{1-\sigma}} C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} ||u'(t)||_{\sigma}^{\sigma} + l_1 \frac{\delta^{\sigma}}{\sigma} \|k_2(x)\|_{\infty}^{\sigma} \\
&\leq \frac{l_1}{\alpha d_0} C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \delta^{\frac{\sigma}{1-\sigma}} H'(t) \\
&+\frac{l_1}{\sigma} C(\delta,\sigma)\, \delta^{\sigma} C^* C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} ||u||_{\sigma}^{\sigma} + l_1 \frac{\delta^{\sigma}}{\sigma} \|k_2(x)\|_{\infty}^{\sigma}.
\end{aligned}
$$

By the hypotheses (H3), and the estimate (5.4) we have

$$\begin{aligned} 2\beta \int_\Omega u(t) f(u(t))\,dx &\geq 2\beta p \int_\Omega F(x)\,dx - 2\beta \int_\Omega k_1(x)|u(x)|\,dx \\ &\geq 2pH(t) - 2\beta \int_\Omega k_1(x)|u(x)|\,dx, \end{aligned} \tag{5.14}$$

and by Holder's, Young's inequalities,

$$\int_\Omega k_1(x)|u(x)|\,dx \leq C(\sigma,\alpha)\|k_1(x)\|_\infty + 2\alpha\frac{\delta^\sigma}{\sigma}||u(t)||_\sigma^\sigma. \tag{5.15}$$

By substituting in (5.10), and using (5.11)-(5.15), yields,

$$\begin{aligned} &\frac{d}{dt}G(t) \\ &\geq \left(\begin{array}{c} (1-a)H^{-a}(t) \\ -\frac{1}{\alpha d_0}\left(3p\varepsilon e^{T-t}\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} + 2\alpha\varepsilon\frac{\sigma-1}{\sigma}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right)\right)\delta^{\frac{\sigma}{1-\sigma}} \end{array} \right) H'(t) \\ &+2p\varepsilon H(t) - 2\varepsilon\|u(t)\|_p^p - \frac{5}{2}\varepsilon|\Delta u(t)|^2 + (\gamma_1 - 2\beta C(\sigma,\alpha))\varepsilon\|k_1(x)\|_\infty \\ &+\left(\gamma_2 - 3p\varepsilon e^{T-t}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\varepsilon\|k_2(x)\|_\infty^\sigma + 3p\beta\varepsilon\int_\Omega F(x,u(s))\,dx \\ &-\varepsilon\left(3\theta p e^{T-t}\beta l_1 C(\delta,\sigma)C^* C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} + 2\beta\alpha(d_1+d_2)\right)\frac{\delta^\sigma}{\sigma}||u(t)||_\sigma^\sigma, \\ &\forall\delta,\varepsilon > 0. \end{aligned} \tag{5.16}$$

At this point, for a large positive constant λ to be chosen later, picking δ such that $\delta^{\frac{\sigma}{1-\sigma}} = \lambda H^{-a}(t) > 0$ in (5.16) we arrive for all $t > 0$ at

$$\begin{aligned} &\frac{d}{dt}G(t) \\ &\geq \left(\begin{array}{c} (1-a) \\ -\frac{\lambda}{\alpha d_0}\left(3p\varepsilon e^{T}\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} + 2\alpha\varepsilon\frac{\sigma-1}{\sigma}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right)\right) \end{array} \right) H^{-a}(t)H'(t) \\ &+3\beta p\varepsilon\int_\Omega F(x,u)\,dx - 2\varepsilon\|u(t)\|_p^p - \frac{5}{2}\varepsilon|\Delta u(t)|^2 + 2p\varepsilon H(t) \\ &+(\gamma_1 - 2\beta C(\sigma,\alpha))\varepsilon\|k_1(x)\|_\infty \\ &+\left(\gamma_2 - 3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\varepsilon\|k_2(x)\|_\infty^\sigma \\ &-\varepsilon\left(3\theta p e^{T}\beta l_1 C(\delta,\sigma)C^* C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}} + 2\beta\alpha(d_1+d_2)\right)\frac{\lambda^{1-\sigma}}{\sigma}H^{a(\sigma-1)}(t)||u(t)||_\sigma^\sigma, \\ &\forall\delta,\ \varepsilon > 0. \end{aligned} \tag{5.17}$$

By exploiting (5.4), we have

$$H^{a(\sigma-1)}(t)\,||u(t)||_{\sigma}^{\sigma} \leq \beta^{a(\sigma-1)} \left(\int_{\Omega} F(x,u)\,dx \right)^{a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma}, \tag{5.18}$$

from (H3) we have

$$\begin{aligned} \int_{\Omega} F(x,u)\,dx &\leq \frac{l_1}{p} \left(\int_{\Omega} |u(t)|^{\theta+1}\,dx + (|k_2(x)| + |k_1(x)|)\,|u| \right) \\ &\leq \frac{l_1}{p} ||u(t)||_{\theta+1}^{\theta+1} + C\frac{l_1}{p} (\|k_1(x)\|_\infty + \|k_2(x)\|_\infty)\,||u(t)||_{\theta+1}^{\theta+1}, \\ &\leq C\frac{l_1}{p} ||u(t)||_{\theta+1}^{\theta+1} \end{aligned} \tag{5.19}$$

by condition (5.9), and the estimates (5.6) we confirm that

$$\begin{aligned} &\beta^{a(\sigma-1)} \left| \int_{\Omega} F(x,u)\,dx \right|^{a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma} \\ &\leq C\frac{l_1}{p}\beta^{a(\sigma-1)} \left(||u(t)||_{\theta+1}^{\theta+1} \right)^{a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma} \\ &= C\frac{l_1}{p}\beta^{a(\sigma-1)} ||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)} ||u(t)||_{\sigma}^{\sigma} \\ &\leq C\frac{l_1}{p}\beta^{a(\sigma-1)} ||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)} ||u(t)||_{\theta+1}^{\sigma} \\ &= C\frac{l_1}{p}\beta^{a(\sigma-1)} ||u(t)||_{\theta+1}^{(\theta+1)a(\sigma-1)+\sigma} \\ &\leq \frac{l_1}{p}\beta^{a(\sigma-1)} C \left(H(t) + \|u(t)\|_p^p + |u'(t)|^2 + \beta \int_{\Omega} F(x,u)\,dx \right) \\ &\leq C\frac{l_1}{p}\beta^{a(\sigma-1)} \begin{pmatrix} H(t) + \|u(t)\|_p^p + |u'(t)|^2 + \beta \int_{\Omega} F(x,u)\,dx \\ + \|k_1(x)\|_\infty + \|k_2(x)\|_\infty^{\sigma} \end{pmatrix} \end{aligned} \tag{5.20}$$

substituting (5.20) in (5.17) we obtain

$$\begin{aligned} \frac{d}{dt}G(t) \geq &\left((1-a) - \frac{\lambda}{\alpha d_0} \begin{pmatrix} 3p\varepsilon e^T \beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \\ +2\alpha\varepsilon^{\frac{\sigma-1}{\sigma}} \left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2 \right) \end{pmatrix} \right) H^{-a}(t)\,H'(t) \\ &+3p\beta\varepsilon \int_{\Omega} F(x,u)\,dx - \frac{5}{2}\varepsilon\,|\Delta u(t)|^2 - 2\varepsilon\,\|u(t)\|_p^p \\ &+\varepsilon\,(\gamma_1 - 2\beta C(\sigma,\alpha))\,\|k_1(x)\|_\infty \\ &+\varepsilon \left(\gamma_2 - 3p\varepsilon e^T \beta l_1 \frac{\delta^\sigma}{\sigma} \right) \|k_2(x)\|_\infty^{\sigma} \end{aligned} \tag{5.21}$$

$$+\varepsilon\left(\begin{array}{c} 2pH(t)-\left(3\theta pe^{T}\beta l_1 C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}+2\beta\alpha(d_1+d_2)\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)} \\ \times C\left(\begin{array}{c} H(t)+\|u(t)\|_p^p+|u'(t)|^2+\beta\int_\Omega F(x,u)\,dx \\ +\|k_1(x)\|_\infty+\|k_2(x)\|_\infty^\sigma \end{array}\right)\end{array}\right)$$

or

$$\begin{aligned}\frac{d}{dt}G(t)\geq{}&\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\left(\begin{array}{c}3p\varepsilon e^{T}\beta C^{*}C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha\varepsilon\frac{\sigma-1}{\sigma}\left(C^{*}d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\end{array}\right)H^{-a}(t)H'(t)\\ &+3p\beta\varepsilon\int_\Omega F(x,u)\,dx-\frac{5}{2}\varepsilon|\Delta u(t)|^2-2\varepsilon\|u(t)\|_p^p\\ &+\varepsilon(\gamma_1-2\beta C(\sigma,\alpha))\|k_1(x)\|_\infty \qquad (5.22)\\ &+\varepsilon\left(\gamma_2-3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\|k_2(x)\|_\infty^\sigma\\ &+\varepsilon\left(\begin{array}{c}(5p-1)H(t)\\ -\left(\begin{array}{c}3\theta pe^{T-t}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\\ \times C\left(\begin{array}{c}H(t)+\|u(t)\|_p^p+|u'(t)|^2+\beta\int_\Omega F(x,u)\,dx\\ +\|k_1(x)\|_\infty+\|k_2(x)\|_\infty^\sigma\end{array}\right)\end{array}\right)-\varepsilon(3p-1)H(t).\end{aligned}$$

By using the definition (5.2), the estimate (5.22) gives

$$\begin{aligned}\frac{d}{dt}G(t)\geq{}&\left(\begin{array}{c}(1-a)\\ -\frac{\lambda}{\alpha d_0}\left(\begin{array}{c}3p\varepsilon e^{T}\beta C^{*}C(\Omega)^{\frac{\sigma-2}{2\sigma}}\\ +2\alpha\varepsilon\frac{\sigma-1}{\sigma}\left(C^{*}d_1C(\Omega)^{\frac{\sigma-2}{\sigma}}+d_2\right)\end{array}\right)\end{array}\right)\\ &\times H^{-a}(t)H'(t)\\ &+\varepsilon\left[\begin{array}{c}\left(\frac{3p-1}{2}\right)\\ -\left(C\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\\ &\times|u'(t)|^2\\ &+\left(\frac{3p-1}{2}-\frac{5}{2}\right)\varepsilon|\Delta u(t)|^2\\ &+\varepsilon\left[\begin{array}{c}(\gamma_1-2\beta C(\sigma,\alpha))\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\|k_1(x)\|_\infty\\ &+\varepsilon\left[\begin{array}{c}\left(\gamma_2-3p\varepsilon e^{T}\beta l_1\frac{\delta^\sigma}{\sigma}\right)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_1C(\delta,\sigma)C^{*}C(\Omega)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha(d_1+d_2)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_1}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\|k_2(x)\|_\infty^\sigma\end{aligned}$$

$$+\varepsilon\left[\begin{array}{c}\left(\frac{3p-1}{p}-2\right)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_{1}C\left(\delta,\sigma\right)C^{*}C\left(\Omega\right)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha\left(d_{1}+d_{2}\right)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_{1}}{p}\beta^{a(\sigma-1)}\right)\\ \times\left\Vert u\left(t\right)\right\Vert_{p}^{p}\end{array}\right]$$

$$+\varepsilon\left[\begin{array}{c}3p-\left(3p-1\right)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_{1}C\left(\delta,\sigma\right)C^{*}C\left(\Omega\right)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha\left(d_{1}+d_{2}\right)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_{1}}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]\beta\int_{\Omega}F\left(x,u\right)dx$$

$$+\varepsilon\left[\begin{array}{c}\left(5p-1\right)\\ -C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_{1}C\left(\delta,\sigma\right)C^{*}C\left(\Omega\right)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha\left(d_{1}+d_{2}\right)\end{array}\right)\frac{\lambda^{1-\sigma}}{\sigma}\frac{l_{1}}{p}\beta^{a(\sigma-1)}\right)\end{array}\right]H\left(t\right).$$

pose

$$C_{1}=C\left(\left(\begin{array}{c}3\theta pe^{T}\beta l_{1}C\left(\delta,\sigma\right)C^{*}C\left(\Omega\right)^{\frac{\sigma-2\theta}{2\theta\sigma}}\\ +2\beta\alpha\left(d_{1}+d_{2}\right)\end{array}\right)\frac{1}{\sigma}\frac{l_{1}}{p}\beta^{a(\sigma-1)}\right),$$

we arrive at

$$\begin{array}{c}\frac{d}{dt}G\left(t\right)\geq\left(\begin{array}{c}\left(1-a\right)\\ -\frac{\lambda}{\alpha d_{0}}\varepsilon\left(\begin{array}{c}3pe^{T}\beta C^{*}C\left(\Omega\right)^{\frac{\sigma-2}{2\sigma}}\\ +2^{\frac{\sigma-1}{\sigma}}\left(C^{*}d_{1}C\left(\Omega\right)^{\frac{\sigma-2}{\sigma}}+d_{2}\right)\end{array}\right)\end{array}\right)H^{-a}\left(t\right)H'\left(t\right)\\ +\varepsilon\left[\frac{3p-1}{2}-C_{1}\lambda^{1-\sigma}\right]\left|u'\left(t\right)\right|^{2}+\left(\frac{3p-1}{2}-\frac{5}{2}\right)\varepsilon\left|\Delta u\left(t\right)\right|^{2}\\ +\varepsilon\left(\left(\gamma_{1}-2\beta C\left(\sigma,\alpha\right)\right)-C_{1}\lambda^{1-\sigma}\right)\left\Vert k_{1}\left(x\right)\right\Vert_{\infty}\\ +\varepsilon\left(\left(\gamma_{2}-3p\varepsilon e^{T}\beta l_{1}\frac{\delta^{\sigma}}{\sigma}\right)-C_{1}\lambda^{1-\sigma}\right)\left\Vert k_{2}\left(x\right)\right\Vert_{\infty}^{\sigma}\\ +\varepsilon\left[\frac{p-1}{p}-C_{1}\lambda^{1-\sigma}\right]\left\Vert u\left(t\right)\right\Vert_{p}^{p}+\varepsilon\left[1-C_{1}\lambda^{1-\sigma}\right]\beta\int_{\Omega}F\left(x,u\right)dx\\ +\varepsilon\left(\left(5p-1\right)-C_{1}\lambda^{1-\sigma}\right)H\left(t\right).\end{array}\tag{5.23}$$

chosen $\gamma_1 = 1 + 2\beta C(\sigma,\alpha), \gamma_2 = 1 + 3p\varepsilon e^T\beta l_1\frac{\delta^\sigma}{\sigma}$ and λ satisfying the following inequality

$$\lambda\geq\lambda_{0}=\min\left(\sqrt[\sigma-1]{\frac{2C_{1}}{3p-1}},\ \sqrt[\sigma-1]{\frac{pC_{1}}{p-1}},\ \sqrt[\sigma-1]{C_{1}},\ \sqrt[\sigma-1]{\frac{C_{1}}{5p-1}}\right)$$

so that the coefficients of $H(t)$, $|u'(t)|^2$, $|\Delta u(t)|^2$, $\|u(t)\|_p^p$, $\|k_1(x)\|_\infty$, $\|k_2(x)\|_\infty$ and $\int_\Omega F(x,u)\,dx$ in (5.23) are strictly positive, hence we get

$$\frac{d}{dt}G(t) \geq \left(\begin{array}{c} (1-a) \\ -\frac{\lambda}{\alpha d_0}\varepsilon \left(\begin{array}{c} 3pe^T\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \\ +2\alpha\frac{\sigma-1}{\sigma}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right) \end{array} \right) \end{array} \right) H^{-a}(t)\,H'(t)$$
$$+\omega\varepsilon \left(\begin{array}{c} H(t) + |u'(t)|^2 + \|u(t)\|_p^p + \int_\Omega F(x,u)\,dx \\ + \|k_1(x)\|_\infty + \|k_2(x)\|_\infty^\sigma \end{array} \right), \qquad (5.24)$$

where ω is the minimum of these coefficients. We pick ε small enough, so that

$$0 < \varepsilon \leq \varepsilon_0 = \min \left(\begin{array}{c} \dfrac{1-a}{\frac{\lambda}{\alpha d_0}\left(\begin{array}{c} 3pe^T\beta C^* C(\Omega)^{\frac{\sigma-2}{2\sigma}} \\ +2\alpha\frac{\sigma-1}{\sigma}\left(C^* d_1 C(\Omega)^{\frac{\sigma-2}{\sigma}} + d_2\right) \end{array}\right)}; \\ \dfrac{H^{1-a}(0)}{-L'(0)+3pe^T\beta\int_\Omega F(x,u_0)\,dx} \end{array} \right)$$

therefore (5.24) take the form

$$\frac{d}{dt}G(t) \geq \omega\varepsilon \left(\begin{array}{c} H(t) + |u'(t)|^2 + \|u(t)\|_p^p \\ + \int_\Omega F(x,u)\,dx + \|k_1(x)\|_\infty + \|k_2(x)\|_\infty^\sigma \end{array} \right), \qquad (5.25)$$

hence

$$G(t) \geq G(0) > 0 \text{ for all } t \geq 0.$$

The second term in (5.8), by applying Young's inequality we can estimate as follows

$$\frac{1}{2}L'(t) = (u(t), u'(t)) \leq c\,|u'(t)|\,\|u(t)\|_p \leq c\left(|u'(t)|^{2(1-a)} + \|u(t)\|_p^{\frac{2(1-a)}{1-2a}}\right),$$

so

$$|(u(t), u'(t))|^{\frac{1}{1-a}} \leq C\left(|u'(t)|^2 + \|u(t)\|_p^{\frac{2}{1-2a}}\right)$$

using Lemma (5.2) and the condition (5.9) we obtain

$$|(u(t), u'(t))|^{\frac{1}{1-a}}$$
$$\leq C\left(H(t) + |u'(t)|^2 + \|u(t)\|_p^p + \int_\Omega F(x,u)\,dx\right), \ \forall t \geq 0. \qquad (5.26)$$

Consequently we have

$$G(t)^{\frac{1}{1-a}} = \left(H^{1-a}(t) + 2\varepsilon\int_\Omega u(x,t)\,u'(t)\,dx + \gamma_1\varepsilon t\,\|k_1(x)\|_\infty + \gamma_2\varepsilon t\,\|k_2(x)\|_\infty^\sigma\right)^{\frac{1}{1-a}}$$
$$\leq C\left(H(t) + \left|2\varepsilon\int_\Omega u(x,t)\,u'(t)\,dx\right|^{\frac{1}{1-a}} + |\gamma_1\varepsilon t\,\|k_1(x)\|_\infty|^{\frac{1}{1-a}} + |\gamma_2\varepsilon t\,\|k_2(x)\|_\infty^\sigma|^{\frac{1}{1-a}}\right)$$
$$\leq C\left(H(t) + |u'(t)|^2 + \|u(t)\|_p^p + \int_\Omega F(x,u)\,dx + \|k_1(x)\|_\infty + \|k_2(x)\|_\infty^\sigma\right). \qquad (5.27)$$

We then combine (5.25), (5.26), and (5.27), to arrive at

$$\frac{d}{dt}G(t) \geq \rho G(t)^{\frac{1}{1-a}}, \qquad (5.28)$$

where ρ is a constant depending on C, ω, and ε only, and not depend of u.
Integrate (5.28) over $(0,t)$ to get

$$G(t)^{\frac{a}{1-a}} \geq \frac{1}{G^{\frac{a-1}{a}}(0) - t\frac{a}{(1-a)}\rho}.$$

Therefore $G\ (t)$ blows up in a finite time T^* where

$$T^* \leq \frac{1-a}{a\rho G^{\frac{a}{1-a}}(0)}.$$

□

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Mosbah Kaddour
Faculty of Mathematics and Computer Science,
Mohamed Boudiaf University-M'Sila 28000, Algeria
e-mail: `mosbah_kaddour@yahoo.fr`

Farid Messelmi
Department of Mathematics and LDMM Laboratory Ziane Achour,
University of Djelfa 17000, Algeria
e-mail: `foudimath@yahoo.fr`