Anul XLV 2000

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
1

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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLV, Number 1, March 2000

SOME REMARKS ON A SURJECTIVITY RESULT OF KASAHARA

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Abstract. In this paper we will present a new formulation and proof of the surjctivity theorem gives by Kasahara in [5]. Also, some consequences of this theorem are given.

1. Introduction

In [6] McCord gave the following surjectivity theorem for linear mappings:

Theorem 1. *Let X be a Banach space and Y be a normed linear space. Suppose $f : X \to Y$ is a bounded linear mapping for which there are positive real numbers α, β, $\alpha < 1$ such that the following holds. For each y in Y of norm 1, there exists an x in X of norm less or equal to β such that $\|y - f(x)\| \leq \alpha$. Then for each y in Y, there exists an x in X such that $y = f(x)$ and $\|x\| \leq \beta(1-\alpha)^{-1}\|y\|$.*

This theorem can be used for the proof of Tietze extension theorem. Also, this theorem was extending by Kasahara in tow directions. One is a surjectivity theorem for linear mappings in the case of topological vector spaces (using theorem of Pták, (see [7])). Second, is for nonlinear mappings, case for which Kasahara obtained in [5] an interesting theorem of surjectivity.

In this note we give an other formulation and proof for the Kasahara's theorem from [5] and we gives some sufficient conditions such that the assumptions of Kasahara surjectivity theorem will be satisfied.

2. The theorem of Kasahara

From the beginning of this section we mention the surjectivity theorem from [5]

1991 *Mathematics Subject Classification.* 47H10, 47H17, 47H99.

Key words and phrases. surjectivity, Gateaux differentiability .

Theorem 2. *Let (X,d) be a complete metric space, (Y,ϱ) be a metric space and $f : X \to Y$ be a continuous mapping. Suppose that for all y in Y there is $\beta > 0$ and $\alpha \in (0,1)$ such that the following property will be satisfied:*
for each x in X there exists x' in X such that

$$(i) \quad d(x,x') \leq \beta \ \varrho(f(x),y)$$

$$(ii) \quad \varrho(f(x'),y) \leq \alpha \ \varrho(f(x),y)$$

Then for all y in Y there exists x in X such that $f(x)=y$ and

$$d(z,x) \leq \frac{\beta}{1-\alpha} \ \varrho(f(z),y), \ \forall z \in X$$

The above theorem can be reformulate as follows.

Theorem 3. *Let (X,d) be a complete metric space, (Y,ϱ) be a metric space and $f : X \to Y$ be a continuous mapping. Suppose that for all y in Y there are functions $\overline{\alpha} : Y \to (0,1)$, $\overline{\beta} : Y \to \mathbf{R}_+$ and a mapping $h(.;y) : X \to X$ such that the following conditions will be satisfied*

$$(i) \quad d(x, h(.;y)(x)) \leq \overline{\beta}(y) \ \varrho(f(x),y) \ \forall \ x \in X \tag{1}$$

$$(ii) \quad \varrho(f(h(.;y)(x)),y) \leq \overline{\alpha}(y) \ \varrho(f(x),y) \ \forall \ x \in X \tag{2}$$

Then f is a surjectivity mapping. Moreover we have:

$$h^n(.;y)(x) \longrightarrow x_y^* \text{ and } f(x_y^*) = y$$

$$d(x,x_y^*) \leq \frac{\overline{\beta}(y)}{1-\overline{\alpha}(y)} \ \varrho(f(x),y) \ \forall \ x \in X$$

Proof. Let y be an arbitrary element of Y and x an arbitrary element of X. From (1) and (2) we have:

$$\varrho(f(h^n(.;y)(x)) \ ,y) \leq \overline{\alpha}^n(y) \ \varrho(f(x),y) \text{ and} \tag{3}$$

$$d(h^{n+1}(.;y)(x), h^n(.;y)(x)) \leq \overline{\beta}(y)\overline{\alpha}^n(y) \ \varrho(f(x),y) \tag{4}$$

for every n in $\mathbf{N}$. But using (4) we have

$$\begin{aligned} d(h^{n+m}(.;y)(x), h^n(.;y)(x)) &\leq \\ \leq d(h^n(.;y)(x), h^{n+1}(.;y)(x)) &+ \ldots + d(h^{n+m-1}(.;y)(x), h^{n+m}(.;y)(x)) \leq \\ &\leq \frac{\overline{\alpha}^n(y)}{1-\overline{\alpha}(y)}\overline{\beta}(y)\ \varrho(f(x), y) \end{aligned}$$

for every m in $\mathbf{N}$, so $(h^n(x;y))$ is a Cauchy sequence in X and $(h^n(.;y)(x))$ converges to x_y^*. Therefore from (3) the sequence $(f(h^n(.;y)(x)))$ converges to y. But f is a continuous mapping, so $f(x_y^*) = y$.

Moreover we have by (4)

$$\begin{aligned} d(x, x_y^*) &\leq d(x, h(.;y)(x)) + d(h(.;y)(x), h^2(.;y)(x)) + \ldots + d(h^n(.;y)(x), x_y^*) \\ &\leq \overline{\beta}(y)\frac{1-\overline{\alpha}^n(y)}{1-\overline{\alpha}(y)}\ \varrho(f(x), y) + \varrho(h^n(.;y)(x), x_y^*) \end{aligned}$$

for every $n \in \mathbf{N}$, which shows that

$$d(x, x_y^*) \leq \frac{\overline{\beta}(y)}{1-\overline{\alpha}(y)}\ \varrho(f(x), y),\ \forall\, x \in X$$

Remark 1. The **Theorem 3** remains true if we replace the assumption of continuity with that of f has a closed graph.

3. Remarks

In the following we give condition such that the assumptions of the **Theorem 2** will be satisfied in the case of Banach spaces.

Lemma 1. *Let X, Y be Banach spaces and $f : X \to Y$ be Gateaux differentiable mapping on X. Let $\delta : [0, \infty) \to [0, \infty)$ be a continuous function, bounded away from zero and suppose for each x in X that*

$$df_x(B(0,1)) \supseteq B(0, \delta(\|x\|)) \qquad (5)$$

Then for each y in Y there is $\bar{x}$ in X and $\alpha \in (0,1/2]$ such that

$$\|f(\bar{x}) - y\| \leq \alpha\|f(x) - y\| \text{ and} \tag{6}$$

$$\|\bar{x} - x\| \leq \frac{1}{\delta(\|x\|)}\|f(x) - y\| \tag{7}$$

Proof. First observe that if $f(x) = y$, we may take $\bar{x} = x$ and let ε be arbitrary; thus without loss of generality we suppose that $f(x) \neq y$. Set $w = \frac{y - f(x)}{\|y - f(x)\|}\delta(\|x\|)$; so $w \in B(0, \delta(\|x\|))$. By (5) there is $v \in B(0,1)$ such that $df_x(v) = w$. By Gateaux differentiability of f at x we have:

$$\forall\ \bar{v} \in X \ \lim_{t \to 0} \frac{f(x + t\bar{v}) - f(x)}{t} = df_x(\bar{v})$$

which is equivalent with: for all $\bar{v}$ in X and for all $\varepsilon > 0$ there is $\eta = \eta(\varepsilon) > 0$ such that if $|t| < \eta$ we have:

$$\|f(x + t\bar{v}) - f(x) - t\ df_x(\bar{v})\| \leq |t|\ \varepsilon \tag{8}$$

Set $\varepsilon = \frac{\|y - f(x)\|}{2}$ there is $\eta = \eta(\varepsilon) > 0$ such that for $|t| < \eta$ we have:

$$\|f(x + t\bar{v}) - f(x) - t\ df_x(\bar{v})\| \leq |t|\ \frac{\|y - f(x)\|}{2}\ \text{for all } \bar{v} \in X$$

So for $\eta = \eta(\varepsilon) > 0$ there is t_0 such that $|t_0| < \eta$ and $0 < t_0 < 1$ we have:

$$\|f(x + t_0\bar{v}) - f(x) - t_0\ df_x(\bar{v})\| \leq t_0\ \frac{\|y - f(x)\|}{2}\ \text{for all } \bar{v} \in X$$

If we choose $\bar{v} = \frac{1}{\delta(\|x\|)t_0}\|y - f(x)\|v$ we have $df_x(\bar{v}) = \frac{y - f(x)}{t_0}$

$$\left\| f(x + t_0\bar{v}) - f(x) - t_0\frac{y - f(x)}{t_0} \right\| \leq t_0\ \frac{\|y - f(x)\|}{2}$$

So for $\bar{v} = \frac{1}{\delta(\|x\|)t_0}\|y - f(x)\|v,\ \varepsilon = \frac{1}{2}\|y - f(x)\|$ and $|t_0| < \eta,\ 0 < t_0 < 1$ we obtain

$$\|f(x + t_0\bar{v}) - y\| \leq t_0\ \frac{\|y - f(x)\|}{2}$$

Taking $\bar{x} = x + t_0\bar{v}$, and $\alpha = \frac{t_0}{2}$ in $\left(0, \frac{1}{2}\right]$

$$\|f(\bar{x}) - y\| \leq \alpha\|y - f(x)\|$$

when $\alpha \in (0,1)$, so (6) holds. Also since $||v|| \leq 1$ and $\bar{x} = x + t_0\bar{v}$ we have $||\bar{x} - x|| = t_0||\bar{v}|| \leq \frac{1}{\delta(||x||)}||y - f(x)||$, so (7) holds.
Using previous lemma we can obtain a similar theorem to those obtained by W. Ray and A Walker in [8]. For proving theirs theorem, they used a version of Ekeland's theorem [3] and the maximal principle of H. Brezis and F. E. Browder [1].

Theorem 4. *Let X, Y be Banach spaces, and $f : X \to Y$ be continuous and Gateaux differentiable mapping on X. Let $\delta : [0,\infty) \to [0,\infty)$ be a continuous function bounded away from zero and suppose for each x in X that*

$$df_x(B(0,1)) \supseteq B(0,\delta(||x||)) \tag{9}$$

Then f is a surjectivity mapping.

Proof. Let y be an element of Y. From assumptions of the theorem and **Lemma 1** we have that for an element x in X there is $\bar{x}$ in X, α_x in $\left(0, \frac{1}{2}\right]$ and a number $\delta(||x||)$ such that

$$\begin{aligned} ||f(\bar{x}) - y|| &\leq \alpha_x||f(x) - y|| \text{ and} \\ ||\bar{x} - x|| &\leq \frac{1}{\delta(||x||)}||f(x) - y|| \end{aligned}$$

Set $\alpha = sup\{\ \alpha_x \mid x \in X\} \leq \frac{1}{2} < 1$. Because δ is bounded away from zero, there is $\beta > 0$ such that $\beta \leq \delta(||x||)$, for all x in X. So for y in Y there are $\beta > 0$ and α in $(0,1)$ such that the following property will be satisfied:
for each x in X there is $\bar{x}$ in X such that

$$\begin{aligned} ||f(\bar{x}) - y|| &\leq \alpha\ ||f(x) - y|| \text{ and} \\ ||\bar{x} - x|| &\leq \frac{1}{\beta}||f(x) - y|| \end{aligned}$$

After that from **Theorem 2** we obtain that f is a surjectivity mapping.

Consequence 1. *Let X, Y be Banach spaces and $f : X \to Y$ be continuous and Gateaux differentiable mapping on X. Let $\delta : [0,\infty) \to [0,\infty)$ be a continuous function, bounded away from zero and suppose for each x in X that*

$$df_x(B(0,1)) \supseteq B(0,\delta(||x||)$$

Then there are $h : X \to X$, $\overline{\alpha} : Y \to [0,1)$ and $\overline{\beta} : Y \to \mathbf{R}_+$ such that these functions verify conditions (1) and(2) from **Theorem 3**.

Proof. Let y be an arbitrary element of Y and x an arbitrary element of X. If $y = f(x)$ then the problem of surjectivity is clear. Other way, for $f(x) \neq y$ using the notations from the proof of **Lemma 1** and **Theorem 4** we can construct functions h, $\overline{\alpha}$ and $\overline{\beta}$ as follows:

$$\begin{aligned} h(.;y) &\quad : X \to X,\ x \longmapsto x + \bar{v}t_0 \\ \overline{\alpha} &\quad : Y \to [0,1),\ \overline{\alpha}(y) = \alpha, \text{ gives by } \textbf{Theorem 4} \\ \overline{\beta} &\quad : Y \to [0,1),\ \overline{\beta}(y) = \frac{1}{\beta}, \text{ gives by } \textbf{Theorem 4} \end{aligned}$$

where t_0 is taken from the proof of **Lemma 1**. With this functions if we follow the proof of the **Theorem 4** we find:

$$\begin{aligned} ||f(h(.;y)(x)) - y|| &\leq \alpha||f(x) - y|| \\ ||h(.;y)(x) - x|| &\leq \frac{1}{\beta}||f(x) - y|| \end{aligned}$$

which are conditions (1) and (2) from **Theorem 3**, in the case of Banach space.

Acknowledgments

The author express her thanks to Professor I. A. Rus, for his suggestions and constructive comments throughout the preparation of this paper.

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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume **XLV**, Number 1, March 2000

GENERALIZATION OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. We introduce the subclass $T_j(n, m, \lambda, \alpha)$ of analytic functions with negative coefficients by operator D^n. Coefficient estimates, some important properties of the class $T_j(n, m, \lambda, \alpha)$ and distortion theorems are determined. Further, extremal properties of the class $T_j(n, m, \lambda, \alpha)$, radii of close-to-convexity, starlikeness and convexity and integral operators are obtained.

1. Introduction

Let A_j denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in N = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For a function $f(z)$ in A_j, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = z f'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [7]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to A_j is in the

1991 *Mathematics Subject Classification.* 30C45.

Key words and phrases. analytic functions, negative coefficients, distortion theorems.

class $A_j(n,m,\lambda,\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{\dfrac{D^{n+m}f(z)}{D^n f(z)}}{\lambda\dfrac{D^{n+m}f(z)}{D^n f(z)}+(1-\lambda)}\right\}>\alpha \quad (n,m\in N_0=N\cup\{0\}) \tag{1.5}$$

for some α $(0\le\alpha<1)$, λ $(0\le\lambda<1)$ and for all $z\in U$.

Let T_j denote the subclass of A_j consisting of functions of the form

$$f(z)=z-\sum_{k=j+1}^{\infty}a_k z^k \quad (a_k\ge 0,\ j\in N). \tag{1.6}$$

Further, we define the class $T_j(n,m,\lambda,\alpha)$ by

$$T_j(n,m,\lambda,\alpha)=A_j(n,m,\lambda,\alpha)\cap T_j. \tag{1.7}$$

We note that:

(i) $T_1(0,1,0,\alpha)=T^*(\alpha)$ and $T_1(1,1,0,\alpha)=C(\alpha)$ (Silverman [9]).

(ii) $T_1(0,1,\lambda,\alpha)=T(\lambda,\alpha)$ and $T_1(1,1,\lambda,\alpha)=C(\lambda,\alpha)$ (Altintas and Owa [1]).

(iii) $T_1(n,1,0,\alpha)=T(n,\alpha)$ (Hur and Oh [6]).

(iv) $T_j(0,1,0,\alpha)=T_\alpha(j)$ and $T_j(1,1,0,\alpha)=C_\alpha(j)$ (Chatterjee [3] and Srivastava, Owa and Chatterjee [10]).

(v) $T_1(n,1,\lambda,\alpha)=T_n(\lambda,\alpha)$ (Aouf and Cho [2] and Cho and Aouf [4]).

(vi) $T_j(n,m,0,\alpha)=T_j(n,m,\alpha)$ (Sekine [8] and Hossen, Salagean and Aouf [5]).

2. Coefficient estimates

Lemma 1. *Let the function $f(z)$ be defined by (1.6) with $j=1$. Then $f(z)\in T_j(n,m,\lambda,\alpha)$ if and only if*

$$\sum_{k=2}^{\infty}k^n[k^m(1-\alpha\lambda)-\alpha(1-\lambda)]a_k\le(1-\alpha) \tag{2.1}$$

for $n\in N_0$, $m\in N_0$, $0\le\alpha<1$, and $0\le\lambda<1$. The result is sharp.

Proof. Assume the inequality (2.1) holds and let $|z| = 1$. Then we have

$$\left| \frac{\dfrac{D^{n+m} f(z)}{D^n f(z)}}{\lambda \dfrac{D^{n+m} f(z)}{D^n f(z)} + (1-\lambda)} - 1 \right| \leq \frac{(1-\lambda) \sum_{k=2}^{\infty} k^n (k^m - 1) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} k^n [1 + \lambda(k^m - 1)] a_k |z|^{k-1}} \leq$$

$$\leq \frac{(1-\lambda) \sum_{k=2}^{\infty} k^n (k^m - 1) a_k}{1 - \sum_{k=2}^{\infty} k^n [1 + \lambda(k^m - 1)] a_k} \leq (1-\alpha) \tag{2.2}$$

which implies (1.5). Thus it follows from this fact that $f(z) \in T_1(n, m, \lambda, \alpha)$.

Conversely, assume that the function $f(z)$ is in the class $T_1(n, m, \lambda, \alpha)$. Then

$$\operatorname{Re} \left\{ \frac{\dfrac{D^{n+m} f(z)}{D^n f(z)}}{\lambda \dfrac{D^{n+m} f(z)}{D^n f(z)} + (1-\lambda)} \right\} = \operatorname{Re} \left\{ \frac{1 - \sum_{k=2}^{\infty} k^{n+m} a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k^n [1 + \lambda(k^m - 1)] a_k z^{k-1}} \right\} > \alpha \tag{2.3}$$

for $z \in U$. Choose values of z on the real axis so that $\dfrac{\dfrac{D^{n+m} f(z)}{D^n f(z)}}{\lambda \dfrac{D^{n+m} f(z)}{D^n f(z)} + (1-\lambda)}$ is real.

Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we obtain

$$1 - \sum_{k=2}^{\infty} k^{n+m} a_k \geq \alpha \left\{ 1 - \sum_{k=2}^{\infty} k^n [1 + \lambda(k^m - 1)] a_k \right\} \tag{2.4}$$

which gives (2.1). Finally the result is sharp with the extremal function given by

$$f(z) = z - \frac{(1-\alpha)}{k^n [k^m (1 - \alpha\lambda) - \alpha(1-\lambda)]} z^k \quad (k \geq 2). \tag{2.5}$$

□

With the aid of Lemma 1, we prove

Theorem 1. *Let the function $f(z)$ be defined by (1.6). Then $f(z) \in T_j(n, m, \lambda, \alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n[k^m(1-\lambda\alpha) - \alpha(1-\lambda)]a_k \leq (1-\alpha) \tag{2.6}$$

for $n \in N_0$, $m \in N_0$, $0 \leq \alpha < 1$, and $0 \leq \lambda < 1$. The result is sharp for the function

$$f(z) = z - \frac{(1-\alpha)}{k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]} z^k \quad (k \geq j+1). \tag{2.7}$$

Proof. Putting $a_k = 0$ $(k = 2, 3, \ldots, j)$ in Lemma 1, we can prove the assertion of Theorem 1. □

Corollary 1. *Let the function $f(z)$ define by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then*

$$a_k \leq \frac{(1-\alpha)}{k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]} z^k \quad (k \geq j+1). \tag{2.8}$$

The equality in (2.8) is attained for the function $f(z)$ given by (2.7).

3. Some properties of the class $T_j(n, m, \alpha, \lambda)$

Theorem 2. *Let $0 \leq \alpha < 1$, $0 \leq \lambda_1 \leq \lambda_2 < 1$, $n \in N_0$ and $m \in N_0$. Then*

$$T_j(n, m, \lambda_1, \alpha) \subset T_j(n, m, \lambda_2, \alpha).$$

Proof. It follows from Theorem 1, that

$$\sum_{k=j+1}^{\infty} k^n[k^m(1-\alpha\lambda_2) - \alpha(1-\lambda_2)]a_k \leq \sum_{k=j+1}^{\infty} k^n[k^m(1-\alpha\lambda_1) - \alpha(1-\lambda_1)]a_k \leq (1-\alpha)$$

for $f(z) \in T_j(n, m, \lambda_1, \alpha)$. Hence $f(z) \in T_j(n, m, \lambda_2, \alpha)$. □

Theorem 3. *Let $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $n \in N_0$ and $m \in N_0$. Then*

(i) $T_j(n+1, m, \lambda, \alpha) \subset T_j(n, m, \lambda, \alpha)$,

(ii) $T_j(n, m+1, \lambda, \alpha) \subset T_j(n, m, \lambda, \alpha)$,

and

(iii) $T_j(n+1, m+1, \lambda, \alpha) \subset T_j(n, m, \lambda, \alpha)$.

The proof follows immediately from Theorem 1.

4. Distortion theorems

Theorem 4. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then we have*

$$|D^i f(z)| \geq |z| - \frac{(1-\alpha)}{(j+1)^{n-i}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}|z|^{j+1} \tag{4.1}$$

and

$$|D^i f(z)| \leq |z| + \frac{(1-\alpha)}{(j+1)^{n-i}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}|z|^{j+1} \tag{4.2}$$

for $z \in U$, where $0 \leq i \leq n$. The equalities in (4.1) and (4.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{(1-\alpha)}{(j+1)^n[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}z^{j+1}. \tag{4.3}$$

Proof. Note that $f(z) \in T_j(n,m,\lambda,\alpha)$ if and only if $D^i f(z) \in T_j(n-i,m,\lambda,\alpha)$, and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \tag{4.4}$$

Using Theorem 1, we note that

$$(j+1)^{n-i}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)] \sum_{k=j+1}^{\infty} k^i a_k \leq (1-\alpha), \tag{4.5}$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i a_k \leq \frac{(1-\alpha)}{(j+1)^{n-i}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}. \tag{4.6}$$

It follows from (4.4) and (4.6) that

$$|D^i f(z)| \geq |z| - \frac{(1-\alpha)}{(j+1)^{n-i}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}|z|^{j+1} \tag{4.7}$$

and

$$|D^i f(z)| \leq |z| + \frac{(1-\alpha)}{(j+1)^{n-i}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}|z|^{j+1}. \tag{4.8}$$

Finally, we note that the equalities in (4.1) and (4.2) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{(1-\alpha)}{(j+1)^{n-i}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]} z^{j+1}. \tag{4.9}$$

This completes the proof of Theorem 4. □

Corollary 2. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then we have*

$$|f(z)| \geq |z| - \frac{(1-\alpha)}{(j+1)^n[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}|z|^{j+1} \tag{4.10}$$

and

$$|f(z)| \leq |z| + \frac{(1-\alpha)}{(j+1)^n[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}|z|^{j+1} \tag{4.11}$$

for $z \in U$. The equalities in (4.10) and (4.11) are attained for the function $f(z)$ given by (4.3).

Proof. Taking $i=0$ in Theorem 4, we can easily show (4.10) and (4.11). □

Corollary 3. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then we have*

$$|f'(z)| \geq 1 - \frac{(1-\alpha)}{(j+1)^{n-1}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}|z|^j \tag{4.12}$$

and

$$|f'(z)| \leq 1 + \frac{(1-\alpha)}{(j+1)^{n-1}[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}|z|^j \tag{4.13}$$

for $z \in U$. The equalities in (4.12) and (4.13) are attained for the function $f(z)$ given by (4.3).

Proof. Note that $D^1 f(z) = zf'(z)$. Hence, making $i=1$ in Theorem 4, we have the corollary. □

5. Extremal properties of the class $T_j(n,m,\lambda,\alpha)$

Theorem 5. *The class $T_j(n,m,\lambda,\alpha)$ is convex.*

Proof. Let the functions

$$f_\nu(z) = z + \sum_{j=k+1}^{\infty} a_{\nu,k} z^k \quad (a_{\nu,k} \geq 0,\ \nu = 1,2) \tag{5.1}$$

be in the class $T_j(n,m,\lambda,\alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = tf_1(z) + (1-t)f_2(z) \quad (0 \leq t \leq 1) \tag{5.2}$$

is in the class $T_j(n,m,\lambda,\alpha)$. Since

$$h(z) = z - \sum_{k=j+1}^{\infty} [ta_{1,k} + (1-t)a_{2,k}]z^k, \tag{5.3}$$

with the aid of Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]\{ta_{1,k} + (1-t)a_{2,k}\} \leq 1-\alpha \tag{5.4}$$

which implies that $h(z) \in T_j(n,m,\lambda,\alpha)$. Hence $T_j(n,m,\lambda,\alpha)$ is convex. □

As a consequence of Theorem 5 we can obtain the extreme points of the class $T_j(n,m,\lambda,\alpha)$.

Theorem 6. *Let $f_j(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]} z^k \quad (n, m \in N_0,\ k \geq j+1) \tag{5.5}$$

for $0 \leq \alpha < 1$, and $0 \leq \lambda < 1$. Then $f(z)$ is in the class $T_j(n,m,\lambda,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z), \tag{5.6}$$

where $\mu_k \geq 0$ for $k \geq j$ and $\sum_{k=j}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z) = \mu_j f_j(z) + \sum_{k=j+1}^{\infty} \mu_k f_k(z) =$$

$$= \left(1 - \sum_{k=j+1}^{\infty} \mu_k\right) z + \sum_{k=j+1}^{\infty} \mu_k \left\{ z - \frac{1-\alpha}{k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]} \right\} =$$

$$= z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]} z^k. \tag{5.7}$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \frac{k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)](1-\alpha)\mu_k}{(1-\alpha)k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]} = \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \leq 1. \tag{5.8}$$

So be Theorem 1, $f(z) \in T_j(n, m, \lambda, \alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_j(n, m, \lambda, \alpha)$. Then

$$a_k \leq \frac{(1-\alpha)}{k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]} \quad (k \geq j+1). \tag{5.9}$$

Setting

$$\mu_k = \frac{k^n[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]}{(1-\alpha)} a_k \quad (k \geq j+1), \tag{5.10}$$

and

$$\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k$$

we notice that $f(z)$ can be expressed in the form (5.6). This completes the proof of Theorem 6. □

Corollary 4. *The extreme points of the class $T_j(n, m, \lambda, \alpha)$ are the functions $f_k(z)$ $(k \geq j)$ given by Theorem 6.*

6. Radii of close-to-convexity, starlikeness and convexity

Theorem 7. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$; then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1(n, m, \lambda, \alpha, \rho)$, where*

$$r_1(n, m, \lambda, \alpha, \rho) = \inf_k \left\{ \frac{(1-\rho)[k^m(1-\alpha\lambda) - \alpha(1-\lambda)]k^{n+1}}{(1-\alpha)} \right\}^{\frac{1}{(k-1)}} \quad (k \geq j+1). \tag{6.1}$$

The result is sharp, with the extremal function $f(z)$ given by (2.7).

Proof. It is sufficient to show that $|f'(z)-1| = 1-\rho$ $(0 \le \rho < 1)$ for $|z| < r_1(n,m,\lambda,\alpha,\rho)$. We have

$$|f'(z)-1| = \left| -\sum_{k=j+1}^{\infty} k a_k z^{k-1} \right| \le \sum_{k=j+1}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z)-1| \le 1-\rho$ if

$$\sum_{k=j+1}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \le 1. \tag{6.2}$$

Hence, by Theorem 1, (6.2) will be true if

$$\frac{k|z|^{k-1}}{(1-\rho)} \le \frac{k^n[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]}{(1-\alpha)}$$

or if

$$|z| \le \left[\frac{(1-\rho)[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]k^{n-1}}{(1-\alpha)}\right]^{\frac{1}{(k-1)}}, \quad (k \ge j+1). \tag{6.3}$$

The theorem follows easily from (6.3). □

Theorem 8. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$; then $f(z)$ is starlike of order ρ $(0 \le \rho < 1)$ in $|z| < r_2(n,m,\lambda,\alpha,\rho)$, where*

$$r_2(n,m,\lambda,\alpha,\rho) = \inf_k \left\{\frac{(1-\rho)[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]k^n}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{(k-1)}}, \quad (k \ge j+1). \tag{6.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.7).

Proof. We must show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \le (1-\partial)$ for $|z| < r_2(n,m,\lambda,\alpha,\rho)$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum_{k=j+1}^{\infty} (k-1) a_k |z|^{k-1}}{1-\sum_{k=j+1}^{\infty} a_k |z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)} - 1\right| \le (1-\rho)$ if

$$\sum_{k=j+1}^{\infty} \frac{(k-\rho)a_k|z|^{k-1}}{(1-\rho)} \le 1. \tag{6.5}$$

Hence, by Theorem 1, (6.5) will be true if

$$\frac{(k-\rho)a_k|z|^{k-1}}{(1-\rho)} \leq \frac{k^n[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]}{(1-\alpha)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]k^n}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{(k-1)}}, \quad (k \geq j+1). \tag{6.6}$$

The theorem follows easily from (6.6). □

Corollary 5. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3(n,m,\lambda,\alpha,\rho)$, where*

$$r_3(n,m,\lambda,\alpha,\rho) = \inf_k \left\{\frac{(1-\rho)[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]k^{n-1}}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{(k-1)}}, \quad (k \geq j+1). \tag{6.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.7).

7. Integral operators

Theorem 9. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{7.1}$$

also belongs to the class $T_j(n,m,\lambda,\alpha)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \tag{7.2}$$

where

$$b_k = \left(\frac{c+1}{c+k}\right)a_k, \tag{7.3}$$

therefore

$$\sum_{k=j+1}^{\infty} k^n[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]b_k = \sum_{k=j+1}^{\infty} k^n[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]\left(\frac{c+1}{c+k}\right)a_k \leq$$

$$\leq \sum_{k=j+1}^{\infty} k^n[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]a_k \leq (1-\alpha), \tag{7.4}$$

since $f(z) \in T_j(n,m,\lambda,\alpha)$. Hence, by Theorem 1, $F(z) \in T_j(n,m,\lambda,\alpha)$. □

Theorem 10. *Let c be a real number such that $c > -1$. If $F(z) \in T_j(n,m,\lambda,\alpha)$, then the function $f(z)$ defined by (7.1) is univalent in $|z| < R^*$, where*

$$R^* = \inf_k \left[\frac{(c+1)k^{n-1}[k^n(1-\lambda\alpha)-\alpha(1-\lambda)]}{(c+k)(1-\alpha)}\right]^{\frac{1}{(k-1)}} \quad (k \geq j+1). \tag{7.5}$$

The result is sharp.

Proof. Let $F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k$ $(a_k \geq 0)$. It follows from (7.1) that

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{(c+1)} = z - \sum_{k=j+1}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k \quad (c > -1). \tag{7.6}$$

In order to obtain the required result, it suffices to show that $|f'(z)-1| < 1$ in $|z| < R^*$.

Now

$$|f'(z)-1| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1}.$$

Thus $|f'(z)-1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1} < 1. \tag{7.7}$$

Hence, by using (2.6), (7.7) will be satisfied if

$$\frac{k(c+k)|z|^{k-1}}{(c+1)} \leq \frac{k^n[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]}{(1-\alpha)} \quad (k \geq j+1)$$

or if

$$|z| \leq \left[\frac{(c+1)k^{n-1}[k^m(1-\lambda\alpha)-\alpha(1-\lambda)]}{(c+k)(1-\alpha)}\right]^{\frac{1}{(k-1)}} \quad (k \geq j+1). \tag{7.8}$$

Therefore $f(z)$ is univalent in $|z| < R^*$. Sharpness follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n[k^m(1-\lambda\alpha)-\alpha(1-\lambda)](c+1)} z^k \quad (k \geq j+1). \tag{7.9}$$

□

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PAIRS OF METRICAL STRUCTURES IN J_0^2M

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The differential geometry of higher order Lagrange spaces was studied recently by R. Miron and Gh. Atanasiu [6-9], founded on the notions of k-osculator bundle (Osc^kM, π, M), regular Lagrangean $L : Osc^kM \to \mathbf{R}$ and the geometrical model (Osc^kM, G, F) where G is the Sasaki lift of the fundamental tensor field g_{ij} of the space $L^{(k)n}$ and F is the natural $F(3,1)$ structure on Osc^kM. For $k = 2$ we obtain the bundle space of accelerations J_0^2M or the space of the 2-jets.

In this paper we determine the N-linear connections compatible with a pair of metrical structures given in $Osc^2M \equiv J_0^2M$ satisfying the suplementary condition (2.1) (cf. Atanasiu [1] and Atanasiu-Klepp [2], the natural case). We remark the equivalence of this d-structure with other structures and we give finaly the general form of a canonical N-linear connection (2.5).

1. Preliminaries

A transformation of coordinates $(x^i, y^{(1)i}, y^{(2)i}) \to (\tilde{x}^1, \tilde{y}^{(1)i}, \tilde{y}^{(2)i})$ on Osc^2M is given by

$$\tilde{x}^i = \tilde{x}_i(x^1, \dots, x^n), \quad rank \left\| \frac{\partial \tilde{x}^i}{\partial x^j} \right\| = n$$

$$\tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} \tag{1.1}$$

$$2\tilde{y}^{(2)i} = \frac{\partial \tilde{y}^{(1)j}}{\partial x^j} + 2\frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j}$$

The point $u \in Osc^2M$ of coordinates $(x^i, y^{(1)i}, y^{(2)i})$ will be noted, also, with $u = (x^i, y^{(1)i}, y^{(2)i})$.

The bundle of the 1-jets Osc^1M can be identified with the tangent bundle TM.

A nonlinear connection N on $E = Osc^2M$ is characterized by the functions $\underset{(\alpha)}{N^i}_j(x, y^{(1)}, y^{(2)})$ $(\alpha = 1, 2)$ called the coefficients of N which to a transformation of cooordinates on E has effect the rules:

$$
\begin{aligned}
\underset{(1)}{\widetilde{N}^i}_m \frac{\partial \widetilde{x}^m}{\partial x^j} &= \underset{(1)}{\widetilde{N}^m}_j \frac{\partial \widetilde{x}^i}{\partial x^m} - \frac{\partial \widetilde{y}^{(1)i}}{\partial x^j} \\
\underset{(2)}{\widetilde{N}^i}_m \frac{\partial \widetilde{x}^m}{\partial x^j} &= \underset{(2)}{\widetilde{N}^m}_j \frac{\partial \widetilde{x}^i}{\partial x^m} - \underset{(1)}{\widetilde{N}^m}_j \frac{\partial \widetilde{y}^{(1)i}}{\partial x^j} - \frac{\partial \widetilde{y}^{(2)i}}{\partial x^j}
\end{aligned}
\tag{1.2}
$$

We obtain the direct decomposition:

$$
T_uE = N_0(u) \oplus N_1(u) \oplus V_2(u), \quad \forall\, u \in E \quad (N_0 = N) \tag{1.3}
$$

with the local basis adapted to this

$$
\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}} \right\}, \quad (i = 1, \dots, n) \tag{1.4}
$$

given by

$$
\begin{aligned}
\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - \underset{(1)}{N^j}_i \frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N^j}_i \frac{\partial}{\partial y^{(2)j}} \\
\frac{\delta}{\delta y^{(1)i}} &= \frac{\partial}{\partial y^{(1)j}} - \underset{(1)}{N^j}_i \frac{\partial}{\partial y^{(2)j}}
\end{aligned}
\tag{1.5}
$$

The fields of geometrical objects which are important on E are introduced with respect to the direct decomposition (1.3).

The transformation (1.1) implies:

$$
\frac{\delta}{\delta x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\delta}{\delta \widetilde{x}^j}, \quad \frac{\delta}{\delta y^{(1)i}} = \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\delta}{\delta \widetilde{y}^{(1)j}}, \quad \frac{\delta}{\delta y^{(2)i}} = \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\delta}{\delta \widetilde{y}^{(2)j}}
$$

If we consider the projectors h, v_1, v_2 determined by (1.3) and denote $V_\alpha X = X^{V_\alpha}$ $(\alpha = 1, 2)$ we can uniquely write

$$
X = X^H + X^{V_1} + X^{V_2}, \quad \forall\, X \in \chi(E). \tag{1.6}
$$

Thus we have

$$X^H = X^{(0)i}\frac{\delta}{\delta x^i}, \quad X^{V_1} = X^{(1)i}\frac{\delta}{\delta y^{(1)i}}, \quad X^{V_2} = X^{(2)i}\frac{\partial}{\partial y^{(1)i}} \qquad (1.6)'$$

The coordinates $X^{(\alpha)i}$, $(\alpha = 0,1,2)$ change under (1.1) as follows:

$$\widetilde{X}^{(\alpha)i} = \frac{\partial \widetilde{x}^i}{\partial x^j} X^{(\alpha)j}, \quad (\alpha = 0,1,2). \qquad (1.6)''$$

Each of them is called a distinguished vector field, shortly a d-vector field.

Let us consider the dual basis of (1.4):

$$\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}, \quad (i = 1, \ldots, n) \qquad (1.4)'$$

Then for a field of 1-form ω on E we can put:

$$\omega = \omega^H + \omega^{V_1} + \omega^{V_2}, \qquad (1.7)$$

where

$$\omega^H = \omega_i^{(0)} dx^i, \quad \omega^{V_1} = \omega_i^{(1)} \delta y^{(1)i}, \quad \omega_{V_2} = \omega_i^{(2)} \delta y^{(2)i} \qquad (1.7)'$$

and with respect to (1.1) we have:

$$\omega_i^{(\alpha)} = \frac{\partial \widetilde{x}^j}{\partial x^i} \widetilde{\omega}_j^{(\alpha)}, \quad (\alpha = 0,1,2) \qquad (1.7)''$$

Now, we can define a distinguished tensor field on E of type (r,s) (shortly a d-tensor field) as an element $T \in T_s^r(E)$ with the property:

$$T(\underset{1}{X}, \ldots, \underset{s}{X}, \overset{1}{\omega}, \ldots, \overset{r}{\omega}) = T(\underset{1}{X}^H, \ldots, \underset{s}{X}^{V_2}, \overset{1}{\omega}^H, \ldots, \overset{r}{\omega}^{V_2}) \qquad (1.8)$$

$$\forall\ \underset{1}{X}, \ldots, \underset{s}{X} \in \chi(E), \quad \forall\ \overset{1}{\omega}, \ldots, \overset{r}{\omega} \in \chi^*(E).$$

Then in adapted basis (1.4), (1.4)' we obtain:

$$T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x, y^{(1)}, y^{(2)}) \frac{\delta}{\delta x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{(2)i_r}} \otimes dx^{j_1} \otimes \cdots \otimes \delta y^{(2)j_s} \qquad (1.8)'$$

and with respect to (1.1), we get:

$$\widetilde{T}^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial \widetilde{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \widetilde{x}^{i_r}}{\partial x^{m_r}} \cdot \frac{\partial x^{q_1}}{\partial \widetilde{x}^{j_1}} \cdots \frac{\partial x^{q_s}}{\partial \widetilde{x}^{j_s}} T^{m_1 \ldots m_r}_{q_1 \ldots q_s} \qquad (1.8)''$$

Consequently we can give a d-tensor field T, of type (r,s) by its local components $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x, y^{(1)}, y^{(2)})$.

Let us consider the $F(E)$-linear map $J : \chi(E) \to \chi(E)$ given on the natural basis on $\chi(E)$ by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, \quad J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, \quad J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0, \quad (i = 1, \ldots, n) \tag{1.9}$$

We define a N-linear connection on E as a linear connection D on E which preserves by parallelism the horizontal distribution N and which is compatible with the structure J (i.e. $D_X J = 0$, $\forall X \in \chi(E)$).

In the adapted basis (1.4) it is sufficient to give

$$D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}}, \quad D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = \underset{(\beta)}{C}{}^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}} \tag{1.10}$$

$$(\alpha = 0, 1, 2, \quad \beta = 1, 2, \quad y^{(0)i} = x^i)$$

in order to obtain all the coefficients $D\Gamma(N) = (L^i_{jm}, \underset{(1)}{C}{}^i_{jm}, \underset{(2)}{C}{}^i_{jm})$ of a N-linear connection D.

With respect to (1.1) we have for the coefficients $\underset{(\alpha)}{C}{}^i_{jm}(x, y^{(1)}, y^{(2)})$ the transformation (1.8)' of the d-tensor field of type (1.2) and for the coefficients $L^i_{jm}(x, y^{(1)}, y^{(2)})$ the transformation law of an object of connection:

$$\widetilde{L}^i_{pq} \frac{\partial \widetilde{x}^p}{\partial x^r} \frac{\partial \widetilde{x}^q}{\partial x_s} = L^m_{rs} \frac{\partial \widetilde{x}^i}{\partial x^m} - \frac{\partial^2 \widetilde{x}^i}{\partial x^r \partial x^s}$$

The h-covariant derivative noted with $|$ and the v_α-covariant derivative noted with $\overset{(\alpha)}{|}$ $(\alpha = 1, 2)$ in the algebra of the d-tensor fields act, for example, for a d-tensor field $K^i_j(x, y^{(1)}, y^{(2)})$ of the type (1.1) as:

$$\begin{aligned} K^i_{j|m} &= \frac{\delta K^i_j}{\delta x^m} + L^i_{rm} K^r_j - L^s_{jm} K^i_s \\ K^i{}_{j \overset{(\alpha)}{|} m} &= \frac{\delta K^i_j}{\delta y^{(\alpha)m}} + \underset{(\alpha)}{C}{}^i_{rm} K^r_j - \underset{(\alpha)}{C}{}^s_{jm} K^i_s, \quad (\alpha = 1, 2) \end{aligned} \tag{1.11}$$

If $D\Gamma(N) = (L_{ij}^m, \underset{(1)}{C_{ij}}^m, \underset{(2)}{C_{ij}}^m)$ are the local components of a N-linear connection D on E, then the identities of Ricci holds, written for a d-vector field $X^m(x, y^{(1)}, y^{(2)})$:

$$X^m_{|p|q} - X^m_{|q|p} = X^r R^m_{rpq} - T^r_{pq} X^m_{|r} - \overset{(1)}{\underset{(0)}{R^r_{pq}}} X^m \overset{(1)}{\Big|}_r - \overset{(2)}{\underset{(0)}{R^r_{pq}}} X^m \overset{(2)}{\Big|}_r$$

$$X^m{}_{|p}{}^{(\beta)}\Big|_q - X^m{}^{(\beta)}\Big|_{q|p} = X^r \underset{(\beta)}{P^m_{rpq}} - \underset{(\beta)}{C^m_{pq}} X^m_{|r} - \overset{(1)}{\underset{(\beta)}{P^r_{pq}}} X^m \overset{(1)}{\Big|}_r - \overset{(2)}{\underset{(\beta)}{P^r_{pq}}} X^m \overset{(2)}{\Big|}_r, \quad (\beta = 1, 2)$$

$$X^m \overset{(1)}{\Big|}_p \overset{(2)}{\Big|}_q - X^m \overset{(2)}{\Big|}_q \overset{(1)}{\Big|}_p = X^r \underset{(1)(2)}{P^m_{rpq}} - (\underset{(2)}{C^r_{pq}} X^m \overset{(1)}{\Big|}_r - \underset{(1)}{C^r_{qp}} X^m \overset{(2)}{\Big|}_r) - \overset{(2)}{\underset{(1)(2)}{P^r_{pq}}} X^m \overset{(2)}{\Big|}_r \quad (1.12)$$

$$X^m \overset{(1)}{\Big|}_p \overset{(1)}{\Big|}_q - X^m \overset{(1)}{\Big|}_q \overset{(1)}{\Big|}_p = X^r \underset{(1)}{S^m_{rpq}} - \underset{(1)}{S^r_{pq}} X^m \overset{(1)}{\Big|}_r - \overset{(1)}{\underset{(1)}{R^r_{pq}}} X^m \overset{(2)}{\Big|}_r$$

$$X^m \overset{(2)}{\Big|}_p \overset{(2)}{\Big|}_q - X^m \overset{(2)}{\Big|}_q \overset{(2)}{\Big|}_p = X^r \underset{(2)}{S^m_{rpq}} - \underset{(2)}{S^r_{pq}} X^m \overset{(2)}{\Big|}_r$$

where the tensor fields of torsion T^r_{pq}, $\overset{(1)}{\underset{(0)}{R^r_{pq}}}$, $\overset{(2)}{\underset{(0)}{R^r_{pq}}}$, $\underset{(1)}{C^r_{pq}}$, $\underset{(2)}{C^r_{pq}}$, $\overset{(1)}{\underset{(1)}{P^r_{pq}}}$, $\overset{(1)}{\underset{(2)}{P^r_{pq}}}$, $\overset{(2)}{\underset{(2)}{P^r_{pq}}}$, $\overset{(2)}{\underset{(1)}{P^r_{pq}}}$, $\overset{(1)}{\underset{(1)(2)}{P^r_{pq}}}$, $\underset{(1)}{S^r_{pq}}$, $\underset{(2)}{S^r_{pq}}$, $\overset{(1)}{\underset{(1)}{R^r_{pq}}}$, and the tensor fields of curvature R^m_{rpq}, $\underset{(1)}{P^m_{rpq}}$, $\underset{(2)}{P^m_{rpq}}$, $\underset{(1)(2)}{P^m_{rpq}}$, $\underset{(1)}{S^m_{rpq}}$, $\underset{(2)}{S^m_{rpq}}$ appear.

2. Pairs of metrical d-structures

Let $E = Osc^2 M$ be having a pair of metrical d-structures $g_{ij}(x, y^{(1)}, y^{(2)})$ and $G_{ij}(x, y^{(1)}, y^{(2)})$ with the corresponding Obata's operators $\underset{1}{\Omega^{pq}_{ij}}$, $\underset{2}{\Omega^{pq}_{ij}}$ and $\underset{1}{\Lambda^{pq}_{ij}}$, $\underset{2}{\Lambda^{pq}_{ij}}$ satisfying the permutability condition:

$$\underset{1}{\Omega}\,\underset{1}{\Lambda} = \underset{1}{\Lambda}\,\underset{1}{\Omega} \tag{2.1}$$

We have:

$$\underset{1}{\Omega^{pq}_{ij}} = \frac{1}{2}(\delta^p_i \delta^q_j - g_{ij} g^{pq}), \quad \underset{2}{\Omega^{pq}_{ij}} = \frac{1}{2}(\delta^p_i \delta^q_j + g_{ij} g^{pq})$$

$$\underset{1}{\Lambda^{pq}_{ij}} = \frac{1}{2}(\delta^p_i \delta^q_j - G_{ij} G^{pq}), \quad \underset{2}{\Lambda^{pq}_{ij}} = \frac{1}{2}(\delta^p_i \delta^q_j + G_{ij} G^{pq})$$

and

$$(\underset{\alpha}{\Omega}\,\underset{\beta}{\Lambda})^{pq}_{ij} = \underset{\alpha}{\Omega}^{sq}_{ir}\,\underset{\beta}{\Lambda}^{pr}_{sj} \quad (\alpha, \beta = 1, 2)$$

We can prove that the condition (2.1) is equivalent with the existence of a function non-vanishing $\lambda(x, y^{(1)}, y^{(2)}) : E \to \mathbb{R}$ with the property:

$$g_{ir}g_{js}G^{rs} = \lambda G_{ij} \tag{2.1$'$}$$

We have

Theorem 2.1. *If M is a paracompact manifold, and on E exists an almost product d-structure $P^i_j(x, y^{(1)}, y^{(2)})$, then on E exists an above definite structure, called a (g, G, λ)-structure.*

Theorem 2.2. *If M is a paracompact manifold and on E exists an almost Hermitian d-structure of the type -1:*

$$(f^i_j(x, y^{(1)}, y^{(2)}), g_{ij}(x, y^{(1)}, y^{(2)}), \quad g_{rs} f^r_i f^s_j = -g_{ij},$$

then on E exists a (g, G, λ)-structure.

A N-linear connection $D\Gamma(N) = (L^m_{ij}, \underset{(\alpha)}{C^m_{ij}}, (\alpha = 1, 2)$ on E is called a (g, G, λ)-linear connection on E if $g(x, y^{(1)}, y^{(2)})$ and $G(x, y^{(1)}, y^{(2)})$ are $h-$ and v_α-covariant constants with respect to $D\Gamma(N)$, i.e.:

$$g_{ij|m} = 0, \quad G_{ij|m} = 0, \quad g_{ij\overset{(\alpha)}{|}m} = 0, \quad G_{ij\overset{(\alpha)}{|}m} = 0, \quad (\alpha = 1, 2) \tag{2.2}$$

We have

Theorem 2.3. *There exists (g, G, λ)-linear connections on E if and only if the function $\lambda(x, y^{(1)}, y^{(2)})$ is constant on E.*

In the case $\lambda = a^2$ (=cst.) the d-tensor field $P^i_j = aG_{jr}g^{ir}$ determine an almost product d-structure on E, and (P^o_j, g_{ij}) is a metrical almost product d-structure on E.

Theorem 2.4. *If $D\overset{\circ}{\Gamma}(N) = \overset{\circ}{L}{}^m_{ij}, \underset{(\alpha)}{\overset{\circ}{C}{}^m_{ij}}$ $(\alpha = 1, 2)$ is a fixed N-linear connection on E, $\overset{\circ}{|}$ and $\overset{\circ}{\overset{(\alpha)}{|}}$ $(\alpha = 1, 2)$ denote the $h-$ and v_α-covariant derivatives with respect to*

$D\overset{\circ}{\Gamma}(N)$, *then the following N-linear connection:*

$$L_{ij}^m = \overset{\circ}{L}{}_{ij}^m + \frac{1}{4}\Big\{ g^{rm} g_{ri\overset{\circ}{|}j} + G^{rm} G_{ri\overset{\circ}{|}j} - P_i^r P^m_{r\overset{\circ}{|}j} \Big\}$$

$$\underset{(\alpha)}{C_{ij}^m} = \underset{(\alpha)}{\overset{\circ}{C}{}_{ij}^m} + \frac{1}{4}\Big\{ g^{rm} g_{ri\overset{(\alpha)}{\overset{\circ}{|}}j} + G^{rm} G_{ri\overset{(\alpha)}{\overset{\circ}{|}}j} - P_i^r P^m_{r\overset{(\alpha)}{\overset{\circ}{|}}j} \Big\} \quad (\alpha = 1,2) \qquad (2.3)$$

is a (g, G, a^2)*-linear connection on* E.

In the case $\lambda = -a^2$ (=cst.) the d-tensor field $F_j^i = aG_{jr}g^{ir}$ determine an almost complex d-structure on E. It follows $n = 2n'$, $n' \in \mathbb{N}^*$ and (F_j^i, g_{ij}) is an almost Hermitian d-structure on the type -1 on E.

Theorem 2.5. *If* $D\overset{\circ}{\Gamma}(N) = \overset{\circ}{L}{}_{ij}^m, \underset{(\alpha)}{\overset{\circ}{C}{}_{ij}^m}$ $(\alpha = 1,2)$ *is a fixed N-linear connection on* E, $\overset{\circ}{|}$ *and* $\overset{(\alpha)}{\overset{\circ}{|}}$ $(\alpha = 1,2)$ *denote the h– and* v_α*-covariant derivatives with respect to* $D\overset{\circ}{\Gamma}(N)$, *then the following N-linear connection:*

$$L_{ij}^m = \overset{\circ}{L}{}_{ij}^m + \frac{1}{4}\Big\{ g^{rm} g_{ri\overset{\circ}{|}j} + G^{rm} G_{ri\overset{\circ}{|}j} + F_i^r P^m_{r\overset{\circ}{|}j} \Big\}$$

$$\underset{(\alpha)}{C_{ij}^m} = \underset{(\alpha)}{\overset{\circ}{C}{}_{ij}^m} + \frac{1}{4}\Big\{ g^{rm} g_{ri\overset{(\alpha)}{\overset{\circ}{|}}j} + G^{rm} G_{ri\overset{(\alpha)}{\overset{\circ}{|}}j} + F_i^r P^m_{r\overset{(\alpha)}{\overset{\circ}{|}}j} \Big\} \quad (\alpha = 1,2) \qquad (2.4)$$

is a (g, G, a^2)*-linear connection on* E.

Finally we determine the **canonical** (g, G, a^2)-linear connection and the **canonical** $(g, G, -a^2)$-linear connection on E and give an unitary formula for the set of all N-linear connection compatible with such d-structures, in the form:

$$L_{ij}^m = \overset{c}{L}{}_{ij}^m + (\Omega_1 \Lambda_1)_{ir}^{sm} Y_{sj}^r$$

$$\underset{(\alpha)}{C_{ij}^m} + (\Omega_1 \Lambda_1)_{ir}^{sm} \underset{(\alpha)}{Z_{sj}^r} \quad (\alpha = 1,2) \qquad (2.5)$$

where Y_{ij}^m and $\underset{(\alpha)}{Z_{sj}^r}$ are arbitrary d-tensor fields and $D\overset{c}{\Gamma}(N) = \overset{c}{L}{}_{ij}^r, \underset{(\alpha)}{\overset{c}{C}{}_{ij}^m}$ $(\alpha = 1,2)$ is the canonical metrical N-linear connection of g_{ij} of G_{ij} gived by the generalized Christoffel symbol (cf. with Miron-Atanasiu [3-6]).

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SOME DISCRETE FIXED POINT THEOREMS

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Abstract. In this paper we present theorems about fixed points of mappings of partially ordered sets (posets) which generalize some known results.

1. Introduction

Let P a partially ordered set (poset) (i.e. a set with reflexive, antisymmetric and transitive relation $\leq$), 0 and 1 being its least and greatest elements (if they exists). Let X a subset of a poset P. An element $y \in P$ is an *upper (lower) bound* of X iff $x \leq y (y \leq x)$ for all $x \in X$. The terms the least upper bound and the greatest lower bound will be abbreviated to *sup* and *inf*, respectively. A non empty subset X of an ordered set P is called a *chain* if is a totally ordered, that is, for every pair $x, y \in X$ we have

$$x \leq y \text{ or } y \leq x.$$

Let P be a poset. For every $a \in P$

$$B_a := \{x \in P \mid a \leq x\}$$

is a *right section at a*. For $f : P \to P$ and $x \in P$,

$$O_f(x) := \{x,\ f(x),\ f^2(x), ...,\ f^n(x), ...\}$$

is called the *orbit of x*. For any mapping $f : P \to P$ an element $x \in P$ is called fixed point of f if $x = f(x)$ and we write F_f for the fixed point set.

2. Main results

First we state a Lemma which needed in our proofs.

Lemma 2.1 *Let $(P\ ,\leq)$ be a chain complete poset, with least element 0 and $P^* = P \setminus \{0\}$. Let the mapping $\varphi : P^* \longrightarrow P^*$ such that,*

$$x \text{ and } \varphi(x) \text{ is comparable for all } x \in P$$

$$\sup_{x \in B_q} \varphi(x) \text{ exists for all } q \in P$$

$$\sup_{x \in B_q} \varphi(x) \ < \ q \tag{1}$$

If the sequence $\{x_n\} \subset P^*$ satisfies the condition

$$x_{n+1} \leq \varphi(x_n) \qquad \forall n \in N^*$$

then

$$\inf_n x_n \ = \ 0.$$

Proof. First we prove that

$$\varphi(p) < p \qquad \forall p \in P^*.$$

If exist $p_0 \in P^*$ such that

$$\varphi(p_0) \geq p_0 \tag{2}$$

we have $\varphi(p_0) \in B_{p_0}$ and condition (1) implies

$$\varphi^2(p_0) \ = \ \varphi(\varphi(p_0)) < p_0 \tag{3}$$

in other words $p_0 \in B_{\varphi^2(p_0)}$ which implied by (1) that

$$\varphi(p_0) < \varphi^2(p_0) \tag{4}$$

But from (2), (3), (4):

$$p_0 \leq \varphi(p_0) < \varphi^2(p_0) < p_0$$

contradiction. It follows that :

$$x_{n+1} \leq \varphi(x_n) \leq x_n \qquad \forall n \in N^*,$$

(x_n) is non increasing sequence in P^* and chain . Thus they have inf.
If exists $y_0 \in P^*$ such that $y_0 \leq x_n$ $\forall n \in N^*$ we have by (1)

$$y_0 \leq x_{n+1} \leq \varphi(x_n) \leq \sup_{x \in B_{y_0}} \varphi(x) < y_0$$

Which is contradiction. Consequently $\inf_n x_n = 0$. □

Corollary 2.2 (Taskovic [9]) *Let the mapping* $\varphi : R_+ \longmapsto R_+ := (0, \infty)$ *have the properties*

$$\varphi(t) < t \quad \textit{and} \quad \lim \sup_{z \to t+0} \varphi(z) < t \qquad \forall t \in R_+$$

If the sequence $\{x_n\}$ *of non negative real numbers satisfies the condition*

$$x_{n+1} \leq \varphi(x_n) \qquad \forall n \in N^*$$

then

$$\lim_{n \to \infty} x_n = 0 .$$

Proof. Let $P = R_+$, $B_q = \{p \in R_+ \mid q \leq p\} = [q, \infty)$. Because $\varphi(t) < t$ and $\limsup_{z \to t+0} \varphi(z) < t$ we have $\varphi(t) < t$ and $\limsup_{z \in (t, \infty)} \varphi(z) < t$ or $\sup_{z \in B_t} \varphi(x) < t$. If we apply Lemma 2.1 we have $\inf_n x_n = 0$.
But the sequence (x_n) is non decreasing, bounded sequence in R_+

$$\lim_{n \to \infty} x_n = \inf_n x_n = 0.$$

Theorem 2.3 *Let* $(Q, \leq)$ $(P, \leq)$ *partially ordered sets such that every chain has inf and* P *has a least element 0. Let* T , A *and* φ *a mappings such that* $T : Q \longrightarrow Q$, $A : Q \times Q \longrightarrow P$, $\varphi : P \longrightarrow P$ *a mapping such that* φ *satisfies*

$$x \text{ and } \varphi(x) \text{ is comparable for all } x \in P$$

$$\sup_{x \in B_q} \varphi(x) \text{ exists for all } q \in P$$

$$\sup_{x \in B_q} \varphi(x) < q$$

and T *and* A *satisfies the following conditions:*

a)

$$A(Tx,Ty) \leq \varphi(A(x,y)) \quad \forall x,y \in Q$$

b)

$$A(x,y) = 0 \ \textit{implies} \ x = y$$

c)

$$\textit{if} \ \inf_n A\left(T^n x, T^{n+1}x\right) = 0 \quad \{T^n x\} \ \textit{has a chain with} \ \inf$$

d)

$$(x_n) \subset O(x) \ , \ \inf_n x_n = p \ \textit{implied} \ A(p,Tp) \leq \inf_n A(x_n, Tx_n)$$

Then T has a unique fixed point $y_0 \in Q$.

Proof. Let x be an arbitrary point in P. Then we, have

$$A\left(T^n x, T^{n+1}x\right) \leq \varphi\left(A\left(T^{n-1}x, T^n x\right)\right) \quad n \in N^*.$$

Applying the Lemma 2.1 to the sequence $\left\{A\left(T^n x, T^{n+1}x\right)\right\}$ ($x_n := A(T^n x, T^{n+1}x)$, φ satisfies (1) and $x_{n+1} \ \leq \varphi(x_n)$) we obtain

$$inf_n A\left(T^n x, T^{n+1}x\right) = 0.$$

This implies that $\{T^n x\}$ has a chain $\{T^{n_i}x\}$ with inf y_0. Since A satisfies d) we have:

$$A(y_0, Ty_0) \leq \inf_n A\left(T^{n_i}x, T^{n_{i-1}}\right) = \inf_n A\left(T^n x, T^{n+1}x\right) = 0$$

which implies $A(y_0, Ty_0) = 0$ and by b) $y_0 = Ty_0$, and we have shown that for each $x \in P^*$ the inf of $\{T^n x\}$ a fixed point of T. We complete the proof by showing that T can have at most one fixed point. If $y \neq y_0$ is a fixed point, then:

$$A(y,y_0) = A(Ty,Ty_0) \leq \varphi(A(y,y_0)) < A(y,y_0)$$

a contradiction. Consequently $y = y_0$. The proof is complete. □

Theorem 2.4 *Let Q a non empty set with a condition:*

if a sequence $(x_n)_{n\geq 1} \subset Q$ has a property that every sequence the sets $(Q_n)_{n\geq 1}$, which satisfies

q)

$$Q_n \subset Q \ \textit{and} \ x_{n+k} \in Q_n \quad \forall k \geq 0, \ n \ \textit{sufficiently large} \ (n \geq N_0)$$

$$\exists x \in Q \text{ such that } \bigcap_{n \geq N_0} Q_n = \{x\},$$

Let $(P, \leq)$ partially ordered set with least element 0, such that every chain has inf. Let T, A and φ an operators such that $T : Q \longrightarrow Q$, $A : Q \times Q \longrightarrow P$, $\varphi : P^* \longrightarrow P^*$; φ satisfies the conditions of Lemma 2.1 and T and A satisfies the conditions a),b) of Theorem 2.3. Then T has a unique fixed point $y \in Q$, $F_T = \{y\}$.

Proof. Let x be an arbitrary point in P. Then we, have

$$A\left(T^n x, T^{n+1} x\right) \leq \varphi\left(A\left(T^{n-1} x, T^n x\right)\right) \quad n \in N^*. \tag{5}$$

and by lemma 1

$$inf_n A\left(T^n x, T^{n+1} x\right) = 0$$

Let

$$x_n = A\left(T^n x, T^{n+1} x\right)\ ,\ y_n = T^n x,$$

$$Q_n = \{y \in Q \mid A(y, Ty) \leq x_n\}$$

We have $x_{n+k} \in Q_n \quad \forall k \geq 0$ and by q) exist $y \in Q$ such that

$$\bigcap_{n \geq N_0} Q_n = \{y\}$$

which means that $A(y, Ty) \leq x_n$, $\forall n \geq N_0$ and because $\inf_n x_n = 0$, $A(y, Ty) \leq 0$ which implied $A(y, Ty) = 0$ but by b) we have than y a fixed point of T.

The proof of uniqueness is same to Theorem 2.3. □

Corollary 2.5 (M. Taskovic [7]) *Let X be a topological space and let T , A and φ a mappings such that $T : X \longrightarrow X$, $A : X \times X \longrightarrow R$, $\varphi : R_+ \longrightarrow R_+$ such that φ satisfies the condition of Corollary 2.2 and $x \longrightarrow A(x, Tx)$ is T-orbitally lower semicontinous and $A(x, y) = 0$ implies $x = y$. Then T has a unique fixed point $y \in X$ and $T^n x \to y$ for each $x \in X$.*

Theorem 2.4 implies the results of Rus [5] [6], Kannan [4], Dugundji, Granas [3] and other metrical fixed point results which appear in [2], [1] for example.

Corollary 2.6 (Kannan [4], Rus [6]) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ an operator such that*

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Yy)] \quad \forall x, y \in X$$

where $\alpha \in (0, \frac{1}{2})$ a fixed constant. Then T has unique fixed point.

Proof. Let

$$A(x, y) = d(x, y),\ x_n = d(T^n x, T^{n+1} x)$$
$$\varphi(t) = \frac{\alpha}{1-\alpha} t$$

and

$$Q_n = \{x \in X \mid d(x, Tx) \leq x_n\}.$$

It is easy to see that A and φ satisfies the condition of Theorem 2.4. □

Corollary 2.7 (Rus [5]) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ an operator such that*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall x, y \in X$$

where $\varphi : R_+ \longrightarrow R_+$ is an increasing, right upper semiocontinous function for which

$$\varphi(t) < t, \quad \forall t > 0.$$

Then T has an unique fixed point.

Proof. Let

$$A(x, y) = d(x, y),\ x_n = d(T^n x, T^{n+1} x)$$

$$Q_n = \{x \in X \mid d(x, Tx) \leq x_n\}$$

and we apply Theorem 2.4. □

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLV, Number 1, March 2000

SOME REMARKS ON COINCIDENCE THEORY

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Abstract. Coincidence theorems involving two mappings which satisfy some generalized condition of pseudo-contractivity are obtained. The fixed point technique is used.

1. Introduction

We obtain three coincidence theorems involving two mappings which satisfy some generalized conditions of pseudo-contractivity.

The fixed point technique is used. The main idea is to transform the coincidence type equation into a fixed point type equation. It was also used in coincidence theory by Goebel ([4]) and Rus ([11]). We used it in [1] in order to obtain data dependence results on coincidence problems. Also, in [1] we give some applications to differential equations and complementarity problems of the coincidence theorem of Goebel (Theorem 4.1 below). In this theorem condition 2.4 below was used, i.e. condition for a mapping to be contraction with respect to another mapping. In Definition 2.1 we generalize this condition in a Banach space setting. It worse to mention the connection between the notions of Definition 2.1 and the theory of accretive mappings (for more details, see [2]).

We also compare our results with the above mentioned coincidence theorem of Goebel and another coincidence theorem of Kirk and Anderson ([6]).

1991 *Mathematics Subject Classification.* 54H25, 47H10, 47H15.

Key words and phrases. Coincidence theorems, Fixed point technique, Pseudo-contractivity.

2. Preliminaries

Notations. In what follows, X will be a nonempty set, Z will be a Banach space, D a subset of Z and $J : Z \rightsquigarrow Z^*$ the normalized duality mapping of Z

We consider a class Ψ (see also [2]) of functions $\psi : \mathbb{R}_+ \to \mathbb{R}$ such that,

(i) $(I - \psi)(0) = 0$

(ii) $(I - \psi)(r) > 0$ for $r > 0$

(iii) $\liminf_{r\to\infty}(I - \psi)(r) > 0$

(iv) $\liminf_{r\to r_0}(I - \psi)(r) = 0$ implies $r_0 = 0$.

The following notions generalize the well-known pseudo-contractivity, strong pseudo-contractivity, ψ-pseudo-contractivity ([5, 7, 8, 9, 10, 13]) and then, also, contractivity ([4, 1]) and continuity. These known notions corespond, in the context of the next definitions, with the case when X is a subset of Z and $g(x) = x$ for all $x \in X$. Let us notice also that these notions are related to the generalized accretivity defined in [2]. For example, f is pseudo-contraction with respect to g if and only if $(g - f)$ is accretive with respect to g (see [2]).

Definition 2.1. Let f and g be two mappings from X into Z.

a) We say that f is pseudo-contraction with respect to g if for all $x, y \in X$ there exists $j(g(x) - g(y)) \in J(g(x) - g(y))$ such that

$$\langle f(x) - f(y), j(g(x) - g(y))\rangle \leq ||g(x) - g(y)||^2. \tag{2.1}$$

If instead of (2.1) we have

$$\langle f(x) - f(y), j(g(x) - g(y))\rangle \leq t||g(x) - g(y)||^2. \tag{2.2}$$

for some $0 < t < 1$, then we say that f is a strong pseudo-contraction with respect to g.

If instead of (2.1) we have

$$\langle f(x) - f(y), j(g(x) - g(y))\rangle \leq \psi(||g(x) - g(y)||)||g(x) - g(y)||. \tag{2.3}$$

where $\psi \in \Psi$, then we say that f is a ψ-pseudo-contraction with respect to g.
We say that f is locally pseudo-contraction with respect to g on D, where $D \subset g(X)$ is an open set, if for all $x_0 \in X$ there exists a neighborhood $N_{g(x_0)} \subset D$ of $g(x_0)$ such that relation (2.1) holds for all x, y with $g(x), g(y) \in N_{g(x_0)}$.

Remark 1. Let X be a topological space and $g : X \to Z$ be a continuous and open mapping. Then $f : X \to Z$ is locally pseudo-contraction with respect to g on $g(X)$ if and only if for all $x_0 \in X$ there exists N_{x_0} a neighborhood of x_0 such that relation (2.1) holds for all $x, y \in N_{x_0}$.
This can be easily seen if we notice two facts.
1. The openess of the mapping g assures that $g(N_{x_0})$ is a neighborhood of $g(x_0)$ when N_{x_0} is a neighborhood of x_0.
2. The continuity of g assures that $g^{-1}(N_{g(x_0)})$ is a neighborhood of x_0 if $N_{g(x_0)}$ is a neighborhood of $g(x_0)$.

Definition 2.2. ([4, 1]) In the case that X is a nonempty set and (Z, d) is a metric space, we say that $f : X \to Z$ is contraction with respect to $g : X \to Z$ if the following relation holds for some $0 < t < 1$,

$$d(f(x), f(y)) \leq t \cdot d(g(x), g(y)). \tag{2.4}$$

Definition 2.3. ([2]) Let f and g be two mappings from a set X to a Banach space Z. We say that f is continuous with respect to g if $f \circ g^{-1} : g(X) \rightsquigarrow Z$ has a continuous selection.

We have mentioned that our main results are obtained by the fixed point technique. This is the reason for presenting next three fixed point theorems. Theorem 2.1 is another form of the surjectivity Theorem 3 from [9] for the φ-accretive mapping $(I - T)$. The same relation is between Theorem 2.2 and Theorem 8 from [9] (you could see, also, [3]). Theorem 2.3 can be found in this form in [5] and [10].

Theorem 2.1. *If the mapping $T : Z \to Z$ is continuous and ψ-pseudo-contraction then it has a fixed point.*

Theorem 2.2. *Let D be a bounded open subset of Z, $z_0 \in D$. If $T : \overline{D} \to Z$ is a continuous, ψ-pseudo-contraction mapping (with $\lim_{r\to\infty}(r - \psi(r)) = \infty$) such that for all $z \in \partial D$ and $t > 0$ holds*

$$0 \neq t(z - z_0) + z - T(z)$$

then it has a fixed point.

Theorem 2.3. *Let Z be a uniformly convex Banach space, D be a bounded open subset of Z and $z_0 \in D$. If $T : \overline{D} \to Z$ is a continuous and locally pseudo-contraction mapping on D and*

$$||z_0 - T(z_0)|| < ||z - T(z)|| \text{ for all } z \in \partial D$$

then it has a fixed point.

3. Coincidence theorems

The following theorems are our main results.

Theorem 3.1. *If the mappings $f, g : X \to Z$ are such that*
(i) f is continuous with respect to g,
(ii) f is ψ-pseudo-contraction with respect to g,
(iii) g is surjective,
then f and g have a coincidence point.

Proof. From (iii), $g(X) = Z$ and from (i), there exists $T : Z \to Z$ a continuous selection of $f \circ g^{-1}$. We shall prove that T is ψ- pseudo-contraction. In order to do this, let $z_1, z_2 \in Z$. Then $T(z_1) \in f \circ g^{-1}(z_1)$, $T(z_2) \in f \circ g^{-1}(z_2)$, which means that there exist $x \in g^{-1}(z_1)$ and $y \in g^{-1}(z_2)$ such that $T(z_1) = f(x)$, $z_1 = g(x)$, $T(z_2) = f(y)$ and $z_2 = g(y)$. Then relation

$$\langle f(x) - f(y), j(g(x) - g(y))\rangle \leq \psi(||g(x) - g(y)||)||g(x) - g(y)||$$

implies that

$$\langle T(z_1) - T(z_2), j(z_1 - z_2)\rangle \leq \psi(||z_1 - z_2||)||z_1 - z_2||.$$

We apply Theorem 2.1 and we deduce that there exists $z^* \in Z$ such that $z^* = T(z^*)$. Then there exists $x^* \in g^{-1}(z^*)$ such that $z^* = g(x^*) = T(z^*) = f(x^*)$, i.e. x^* is a coincidence point of the mappings f and g. □

Theorem 3.2. *Let D be a bounded open subset of the Banach space Z, $z_0 \in D$. If $f : X \to Z$ and $g : X \to \overline{D}$ are such that*

(i) f is continuous with respect to g,

(ii) f is ψ-pseudo-contraction with respect to g,

(iii) g is surjective,

(iv) for all $x \in X$ with $g(x) \in \partial D$ and $t > 0$ the following relation holds

$$0 \neq t(g(x) - z_0) + g(x) - f(x),$$

then f and g have a coincidence point.

Let us omit the proof, which is similar with those of Theorems 3.1 and 3.3

Theorem 3.3. *Let D be a bounded open subset of the uniformly convex Banach space Z and $x_0 \in X$. If $f : X \to Z$ and $g : X \to \overline{D}$ are such that*

(i) f is continuous with respect to g,

(ii) f is locally pseudo-contraction with respect to g on D,

(iii) g is surjective and $g(x_0) \in D$,

(iv) for all $x \in X$ with $g(x) \in \partial D$ the following relation holds

$$||g(x_0) - f(x_0)|| < ||g(x) - f(x)||,$$

then f and g have a coincidence point.

Proof. Like in the proof of Theorem 3.1, we consider the continuous selection of $f \circ g^{-1}$, $T : \overline{D} \to Z$. From (ii) we deduce that T is locally pseudo-contraction. Let us denote $z_0 = g(x_0) \in D$ and consider $z \in \partial D$. Then there exists $x \in g^{-1}(z)$ such that $z = g(x)$ and $T(z) = f(x)$. Thus, assumption (iv) implies that

$$||z_0 - T(z_0)|| < ||z - T(z)||.$$

The mapping T satisfies the hypothesis of Theorem 2.3. It follows that T has a fixed point, which assures (like in the proof of Theorem 3.1), the existence of a coincidence point of the mappings f and g. □

4. A comparison between Theorem 3.1 and a coincidence theorem of Goebel

Theorem 4.1. *(Goebel [4]) Let X be a nonempty set and Z be a complete metric space. Let $f, g : X \to Z$ be two mappings such that:*
(i) f is contraction with respect to g,
(ii) g is surjective.
Then f and g have a coincidence point.

In the case that Z is a Banach space we can prove the above theorem as a consequence of Theorem 3.1.

This can be easily seen after the following remarks.

Remark 2. If f is contraction with respect to g then f is continuous with respect to g.
Indeed, using relation (3.1) we can notice that $g(x) = g(y)$ implies $f(x) = f(y)$. Then, the mapping $f \circ g^{-1} : Z \to Z$ is single-valued and, also, a contraction. Thus it is continuous, i.e. f is continuous with respect to g.

Remark 3. If f is contraction with respect to g, then f is strong pseudo-contraction with respect to g.
Indeed,

$$\langle f(x) - f(y), j(g(x) - g(y)) \rangle \leq ||f(x) - f(y)|| \cdot ||g(x) - g(y)|| \leq t \cdot ||g(x) - g(y)||^2.$$

5. A comparison between Theorem 3.3 and a coincidence theorem of Kirk and Anderson

Let Y be a Banach space, Z be a uniformly convex Banach space, C be an open subset of Y.

Theorem 5.1. *(Kirk and Anderson, [6]) Let $x_0 \in C$. We suppose that:*
(i) $f : \overline{C} \to Z$ is continuous;
(ii) $g : \overline{C} \to Z$ is continuous, local homeomorphism with $g(C)$ bounded;
(iii) for all $x_1 \in C$ there exists N_{x_1} a neighborhood of x such that for all $x, y \in N_{x_1}$ the following relation holds

$$||f(x) - f(y)|| \leq ||g(x) - g(y)||;$$

(iv) either f or g is proper;
(v) $||g(x_0) - f(x_0)|| < ||g(x) - f(x)||$ for all $x \in \partial C$.
Then there exists $x \in C$ such that $f(x) = g(x)$.

In the case that assumption
(iv) either f or g is proper
is replaced by
(iv)' g is proper
we can prove the above theorem as a consequence of Theorem 3.3.

This can be easily seen after the following remarks.

Remark 4. Let $f, g : \overline{C} \to Z$. If for some $x, y \in \overline{C}$

$$||f(x) - f(y)|| \leq ||g(x) - g(y)||$$

then

$$\langle f(x) - f(y), j(g(x) - g(y)) \rangle \leq ||g(x) - g(y)||^2.$$

Remark 5. If $g : \overline{C} \to Z$ is a local homeomorphism, then it is an open mapping. Indeed, let N be an open subset of $\overline{C}$. We intend to prove that $g(N)$ is open.

Let $y_1 \in g(N)$ and $x_1 \in N$ such that $g(x_1) = y_1$. There exists $N_{x_1} \subset N$ a neighborhood of x_1 such that $g : N_{x_1} \to g(N_{x_1})$ is a homemorphism. Then $g(N_{x_1}) \subset g(N)$ is a neighborhood of y_1. After all these considerations, we conclude that $g(N)$ is open.

Thus, assumptions (ii) and (iii) of Theorem 5.1 implies that f is locally pseudo-contraction with respect to g on the open set $g(C)$, i.e. the hypothesis (ii) of Theorem 3.3 holds.

Remark 6. If $g : \overline{C} \to Z$ is an open mapping and $D = g(C)$ then $g^{-1}(\partial D) \subset \partial C$. Indeed, let us consider $x \in g^{-1}(\partial D)$. Then there exists $y \in \partial D$ such that $y = g(x) \in \partial D$. If x would belong to C then $g(x)$ would belong to D, which leads to a contradiction. Then $x \in \partial C$.

Thus, assumption (v) of Theorem 5.1 implies assumption (iv) of Theorem 3.3.

Remark 7. The closedness of $g(X)$ is an essential hypothesis in Theorem 3.3, while, as is showed in [6], assumption (iv) (i.e. either f or g is proper) of Theorem 5.1 can not be dropped.
Anyway, if $g : \overline{C} \to Z$ is continuous and proper then $g(\overline{C}) = \overline{D}$ is a closed set.

Remark 8. (Theorem 4.G, page 174, [12]) If $g : \overline{C} \to g(\overline{C})$ is a local homeomerphism and is proper then it is a global homeomorphism.
If, in addition, $f : \overline{C} \to Z$ is continuous then f is continuous with respect to g.
Let us mention that this follows by the continuity of the single-valued mapping, $f \circ g^{-1} : g(\overline{C}) \to Z$.

Thus, assumptions (i) and (ii) of Theorem 5.1 and (iv)' implies (i) of Theorem 3.3.

Remark 9. We do not know yet if, in the hypothesis of Theorem 5.1, and in the special case that

f is proper

the claim

f is continuous with respect to g

is true or not.

Now it is clear that, if we replace (iv) by (iv)' in Theorem 5.1, our result is more general and has many advantages.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLV, Number 1, March 2000

NEW METHODS FOR COMPUTING MAXIMUM LYAPUNOV EXPONENT FOR CHAOTIC SYSTEMS

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1. Introduction

1.1. Lyapunov Exponents. One of the main methods for identifying the presence of chaos is based on Lyapunov exponents. They measure the sensitivity of the motion to small changes in the initial conditions. If two trajectories start from close points, then the distance between them grows exponentially according to the law

$$d(t) = d(t_0)2^{\lambda t} \tag{1}$$

where λ is called maximum Lyapunov exponent *corresponding to the interval* $[t_0, t]$. The base in Eq.1 is either 2 or e. If $\lambda > 0$ the motion is chaotic, else it is regular. The quantity λ in Eq.1 depends on t, and the Lyapunov exponent is defined by the following limit:

$$\lambda = \lim_{t\to\infty} \frac{1}{t - t_0} \log \frac{d(t)}{d(t_0)}. \tag{2}$$

If a dynamical system is defined by an operator $x_{n+1} = f(x_n)$, $f : \mathbf{R} \to \mathbf{R}$, $f \in C^1(\mathbf{R})$, then $\forall\ x_0 \in \mathbf{R}$ the Lyapunov exponent in x_0 is defined by

$$\lambda(x_0) = \lim_{n\to\infty} \sup \frac{1}{n} \log |(f^n)'(x_0)| = \lim_{n\to\infty} \sup \frac{1}{n} \sum_{j=0}^{n} |f'(x_j)| \tag{3}$$

where $x_j = f^j(x_0)$, and f^j is the j^{th} iterate of f.

If the dynamical system is defined by an ODEs system of the form

$$y'(t) = A(t)y(t) \tag{4}$$

with matrix A bounded, then the Lyapunov exponents are defined as follows.

Let $y_i = Y(t)p_i$ be n linearly independent solutions, where $Y(t)$ is the fundamental matrix and $\{p_i\}$ is an orthonormal basis in $\mathbf{R}^n$, then Lyapunov exponents are defined by

$$\lambda_i = \limsup_{t\to 0} \frac{1}{t} \log \|Y(t)p_i\| \tag{5}$$

Finally, let consider the Cauchy problem

$$\begin{cases} \dot{x} = f(x,c) \\ x(t_0) = x_0 \end{cases} \tag{6}$$

here f is differentiable, and c is a vector of parameters. Let $x^*(t, t_0, x_0)$ be the solution of problem (6), and δx_0 a peturbation of the initial condition. The perturbed solution will be denoted by $x^* + y$ where y is the solution of the linearized system

$$\begin{cases} \dot{y} = Ay \\ y(t_0) = y_0 \end{cases} \tag{7}$$

where $A = \nabla f(x^*(t_k))$, and y_0 is arbitrary chosen. The distance $d(t)$ will be evaluated by

$$d(t) = \|y(t)\| \tag{8}$$

The given and the linearized system are integrated simultaneously.

1.2. **Computation scheme.** In the actual computation, one picks a large time $TT = t - t_0$, and the limit in Eq.(2) is replaced by

$$\lambda = \frac{1}{TT} \log \frac{d(t_0 + TT)}{d(t_0)} \tag{9}$$

For computational reasons - e.g. the overflow caused by the growth of $d(t)$ - the interval $[t_0, TT]$ is divided in sub-intervals $[t_{K-1}, t_K]$, $K = \overline{1, NT}$, and numbers $\lambda(t)$ are computed on these sub-intervals. Let denote: $\lambda_K =$ the Lyapunov exponent corresponding to the sub-interval $[t_{K_1}, t_K]$;

$d_K = d(t_K)$; $\tau_K =$ sub-interval length, i.e. $\tau_K = t_K - t_{K_1}$, and $\sum_{K=1}^{NT} \tau_K = TT$.

From Eq.(1) with vase e, one obtains successively

$$d_1 = d_0 e^{\lambda_1 \tau_1}$$

$$d_2 = d_1 e^{\lambda_2 \tau_2} = d_0 e^{\lambda_1 \tau_1 + \lambda_2 \tau_2}, \dots,$$

and generally,

$$d_{NT} = d_0 \exp\left(\sum_K \lambda_K \tau_K\right) \tag{10}$$

For convenience, the exponential is now denoted by $\exp(\cdot)$.

From Eq.(2), with $t - t_0 = TT = \sum_K \tau_K$, one obtains

$$d_{NT} = d_0 \exp\left(\lambda \sum_K \tau_K\right) \tag{11}$$

Equating the RHS of Eqs.(10) and (11), yields

$$\lambda = \frac{1}{TT} \sum_K \lambda_K \tau_K. \tag{12}$$

Eq.(12) is the ***sub-interval computation formula*** for λ.

Sub-interval lengths τ_K need not be equal, but usually τ_K are picked even, namely a multiple of the integration time step (assumed constant).

1.3. **The Logarithm method.** Using the above computational scheme, Eq.(9) yields

$$\lambda_K = \frac{1}{\tau_K} \log \frac{d_K}{d_{K_1}}. \tag{13}$$

Using now Eq.(13), the following formula for λ is obtained:

$$\lambda = \frac{1}{TT} \sum_{K=1}^{NT} \log \frac{d_K}{d_{K-1}} \tag{14}$$

If TT is suffficiently large, this value is a good approximation of the limit in Eq.(2).

The computation based on Eq.(14) will be referred as the *Logarithm* method.

Actually, the distance $d_K = d(\tau_K)$ is computed by

$$d_K = ||y(t_K)||,$$

where $y(t)$ is the solution of problem (7).

Note. *Normalization:* In order to avoid overflow (or underflow), after every time τ_K the initial distance is normalized, $d(t_{K-1}) = 1,\ K = 2, NT$. If we normalize also $d(t_0)$, i.e. $d(t_0) = 1$, then Eq.(14) becomes

$$\lambda = \frac{1}{TT} \sum_{K=1}^{NT} \log d_K. \tag{15}$$

2. A variational method for evaluating the distance between two trajectories

The evaluation of the distance is based on the minimization of a functional. Let $F : D \to \mathbf{R}$, $D = I \times \mathbf{R}^2 \subset \mathbf{R}^3$, and $I = [a, b] \subset \mathbf{R}$, $F \in C^3(D)$, and two functions $\varphi, \psi : I \to \mathbf{R}$, $\varphi, \psi \in C^1(I)$. Let γ_1, γ_2 be the curves defined by $y = \varphi(x)$ and $y = \psi(x)$, respectively. Let also, γ be an arc of equation $y = y(x)$, $y : I \to \mathbf{R}$, tying the points $A \in \gamma_1$ and $B \in \gamma_2$ - see Figure 1. Consider the functional $J : C^3[I] \to \mathbf{R}$ defined by

$$J[y] = \int_{\Gamma} F(x, y, y')dx \tag{16}$$

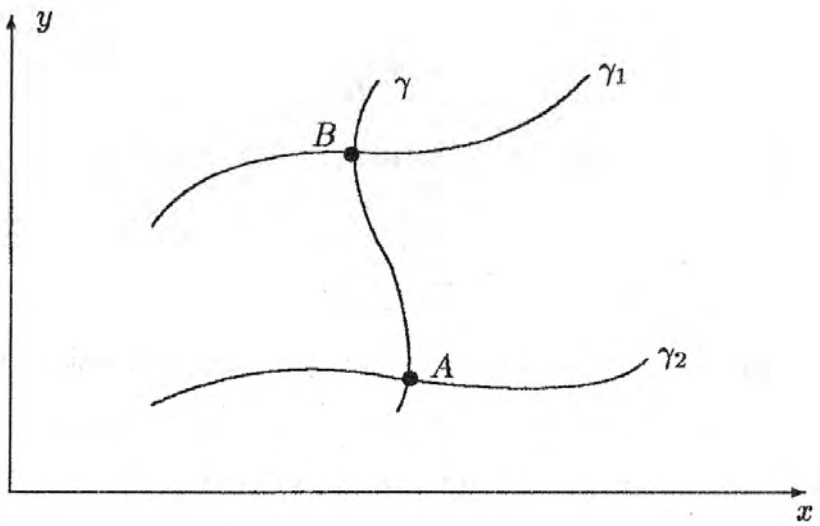

Figure 1

We define the distance between curves γ_1 and γ_2 by

$$d(\gamma_1,\gamma_2)=\min_{\substack{A\in\gamma_1\cap\gamma\\ B\in\gamma_2\cap\gamma}} d(A,B) \tag{17}$$

The problem is to find the extremum of functional (16), when $F(x,y,y')$ is the length of arc AB on γ. Let γ be an extremal for the functional (16), from the set of curves of class C^1 tying two points on γ_1 and γ_2, Then the transversality conditions [6] are:

$$\left[F+(\varphi'-\gamma')F_{y'}\right]\Big|_{x=x_0}=0,\quad \left[F+(\psi'-\gamma')F_{y'}\right]\Big|_{x=x_1}=0 \tag{18}$$

For our problem, the functional J is

$$J[y]=\int_\gamma \sqrt{1+y'^2}dx \tag{19}$$

From Euler equation $F_y(x,y,y')-\dfrac{d}{dx}F_{y'}(x,y,y')=0$, it follows

$$y=f(x,C_1,C_2)$$

Then, from transversality conditions (18) and from

$$f(x_0,C_1,C_2)=\varphi(x_0),\quad f(x_1,C_1,C_2)=\psi(x_1) \tag{20}$$

one can find x_0,x_1,C_1,C_2.

If x_0 is given on γ_1, we look for the solution in the set of C^1 curves that pass through x_0 and intersect γ_2. In this case, the first equation in (18) is fulfilled, and only the second one is used. Then, the ration $d(t_k)/d(t_{k-1})$ is evaluated with the distances found as above.

3. An alternative method to the logarithm formula

3.1. The alternative method. Consider again Eq.(1), with base e, i.e.

$$d(t)=d(t_0)e^{\lambda t} \tag{21}$$

Let suppose that $t - t_0$ is small enough to consider λ = constant on the interval $[t_0, t]$. Then, differentiating Eq.(21) with respect to t yields

$$\dot{d}(t) = d(t_0)\lambda e^{\lambda(t-t_0)} \tag{22}$$

and, dividing side by side Eqs.(22) and (21), the following alternative formula for numbers $\lambda = \lambda(t)$ is obtained: .

$$\lambda(t) = \frac{\dot{d}(t)}{d(t)} \tag{23}$$

If numbers $\lambda(t)$ were constant along the trajectory, it should be sufficient to apply Eq.(23) for a particular value of t, $t \geq t_0$. The computation scheme proceeds as follows: the interval $[t_0, t_0 + TT]$ is divided into sub-intervals $[t_{k-1}, t_k]$ of length equal to the integration step h, in order to meet the assumption that λ is constant on a sub-interval. The number of these sub-intervals will be denoted by $No\_Intervals$. From Eqs.(13), (14), with $\tau_k = h$ and $TT = No\_Intervals \cdot h$, the following alternative formula for MLE is obtained

$$\lambda = \frac{1}{No\_Intervals} \sum_{k=1}^{No\_Intervals} \lambda_k, \tag{24}$$

where

$$\lambda_k = \frac{\dot{d}(t_k)}{d(t_k)} \tag{25}$$

Explicitly, Eq.(24) reads

$$\lambda = \frac{1}{No\_Intervals} \sum_{k=1}^{No\_Intervals} \frac{\dot{d}(t_k)}{d(t_k)} \tag{26}$$

The computation based on Eq.(26) will be referred as the Alternative method. It avoids the use of the logarithm as in Eqs.(15), (16). Minor indices k were used in this Section in order to emphasize that now, numbers λ_k are associated with every integration moment t_k, $k \geq 1$, and not with the end of larger intervals of length τ_k as in Section 1.2. The summation in Eqs.(24) and (26) has the carried out after each integration step.

Actually, λ_k in Eq.(24) is computed as follows. If $y(t)$ is the solution of the linearized system (7), then we have

$$d^2(t) = ||y(t)||_2^2$$

where $||\cdot||_2$ denotes the Euclidean norm. Explicitly,

$$d^2(t) = \sum_{l=1}^{n} y_k^2(t)$$

Differentiating with respect to t and dividing both sides by $d^2(t)$, yields

$$\frac{\dot{d}(t)}{d(t)} = \frac{1}{d^2(t)} \sum_{l=1}^{n} y_l \dot{y}_l$$

or,

$$\lambda(t) = \frac{\langle y(t), \dot{y}(t) \rangle}{\langle y(t), y(t) \rangle} \tag{27}$$

where $\langle a, b \rangle$ denotes the dot product of vectors a and b.

Omitting, for brevity, the argument t in the RHS of Eq.(27), and recalling that $\dot{y}$ is calculated by the RHS of the first Eq.(7), the following computation formula for λ is obtained:

$$\lambda(t) = \frac{\langle y, Ay \rangle}{\langle y, y \rangle}$$

This formula is used in Eq.(24), to compute $\lambda_k = \lambda(t_k)$. It is obvious that scaling Eq.(7) by $y \leftarrow y/C$ does not change the value of λ.

3.2. **Computer program.** A computer program, written in Fortran 90, has been developed. It computes the maximum Lyapunov exponent for an ODEs system, by both Logarithm and Alternative methods. The program description is given in [3]. The source code is provided at the following URL: ftp://bavaria.utcluj.ro/pub/chisalita/Lyapunov.

3.3. Numerical results. Several examples have been analyzed. Some of them are reported hereinafter.

Duffing equation [9,12,8,7]:

$$\ddot{x} + k\dot{x} + x^3 = B\cos(t)$$

Initial conditions: $x_0 = y_0 = [3.0\ 0.0]^T$

Parameter values: $k = 0.1$; $B = 9.9 - 13.3$ (chaos), and $B = 13.5$ (no chaos).

Integration time and step: D1 [7]: $TT = 10053s$; $h = 0.05$. D2 [8]: $TT = 2513s$; $h = 0.01s$.

Lorenz system [11,9,4,10]:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = -xz + rx - y \\ \dot{z} = xy - bz \end{cases}$$

Initial conditions: $x_0 = y_0 = [1.0\ 0.0\ 0.0]^T$

Parameter values and integration time:

L1 [4,10]: $\sigma = 10$; $r = 28$; $n = 8/3$; $TT = 5000s$.

L2 [11,10]: $\sigma = 16$; $r = 45.92$; $b = 4$; $TT = 10000s$.

The numerical results and integration data, are given in the following tables.

Table 1 - MLE, Duffing equation, D1 data

$h = 0.05s$; 201,060 intervals; 12,566 skipped; $No\_Tau = 628$

B	Log Method	Alt Method	Ref.6
9.9	0.1424162E-1	0.1430434E-1	0.015
10	0.1008771	0.1006897	0.094
11	0.1121318	0.1116026	.112
12	0.1393880	0.1392568	0.139
13	0.1627607	0.1621470	.165
13.3	0.1734799	0.1734925	0.168
13.5	-0.4990226	-0.4985679	–
Execution Time* (s)	0.61/0.44	0.61/0.49	–

* Integration time/MLE computation time

Table 2 - MLE, Duffing equation, D2 data

$h = 0.01s$; 251,300 intervals; 31,416 skipped; $No\_Tau = 10$

B	Log Method	Alt Method	Ref.5
9.9	0.8423675E-2	0.8431191E-2	0.012
10	0.1064195	0.1064096	0.094
11	0.1187785	0.1187568	0.114
12	0.1557320	0.1557199	0.143
13	0.1717352	0.1717286	0.167
13.3	0.1764100	0.1763974	0.174
13.5	-0.4979396E-1	-0.4980415E-1	–
Execution Time* (s)	0.77/0.55	0.77/0.60	–

* Integration time/MLE computation time

Table 3 - MLE, Lorenz system

$h = 0.01s$; $Skipped\_Intervals = 100$; $No\_Tau = 20,000$

Data Set (σ, r, b)	Log Method	Alt Method	Ref.[·]
L1 (10; 28; 8/3) 500,000 intervals	0.9046615	0.9044032	0.906 [10]* .9057 [4]**
L2 (16; 45.92; 4) 1,000,000 intervals	1.501490	1.500246	1.497 [11] 1.50 [10]*
Execution Time*** (s): L1	0.99/1.43	0.99/1.70	–
L2	1.92/2.69	1.92/3.18	–

* Integration data: $h = 0.001, 10^6$ intervals

** Value corresponding to MLE=2.16 (base 2)

*** Integration time/MLE computation time

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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLV, Number 1, March 2000

AN EXTENSION OF NADLER'S THEOREM TO A LOCALLY CONVEX SPACE

TZANKO DONCHEV AND VASIL ANGELOV

Abstract. The present note deals with fixed points of multivalued mappings in locally convex (topological vector) spaces. We extend partially the result of Nadler (see [8]) proved for metric spaces. The single valued Banach contraction mapping principle has been extended to locally convex spaces in [7], to uniform spaces in [2,3] and for multifunctions in Banach spaces in [4,9]. In case of uniform spaces the result of Nadler is extended in [1]. We extend Nadler's result in case of locally convex spaces under weaker assumptions than these of [1]. However, our approach is not applicable to nonlinear spaces. Furthermore we require the upper semicontinuity of the multifunction.

Through the paper we denote by E a Hausdorff sequentially complete locally convex spaces (LCS) with topology defined by a saturated family of seminorms $\mathcal{A} = \{|x-y|_\alpha : \alpha \in \mathcal{A}\}$, where $\mathcal{A}$ is an index set. Denote by $CC(E)$ the set of all nonempty convex compact subsets of E. For $X, Y \in CC(E)$ define $D^{(}_\alpha X, Y) = \max_{x\in X} \min_{y \in Y}\{|x - y|_\alpha\}$. Furthermore $CC(E)$ is endowed with a family of Hausdorff pseudometrics:

$$H_{\mathcal{A}} = \{D_\alpha(X,Y) : \alpha \in \mathcal{A}\} \quad (X, Y \in CC(E)),$$

where $D_\alpha(X,Y) = \max\{D^+_\alpha(X,Y), D^+_\alpha(Y,X)\}$ (compare [4,5]). We point out that the family $H_{\mathcal{A}}$ depends essentially on the family $\mathcal{A}$ (see [1] for instance). For a compact $X \subset E$ denote by $\overline{co}X$ the closed convex hull of X.

We will use the following assertion.

Proposition 1. [4,5] *If E is sequentially complete LCS with respect to $\mathcal{A}$ then $CC(E)$ is a sequentially complete uniform space with the uniformity generated by $H_{\mathcal{A}}$.*

This work is partially supported by National Foundation for Scientific Research at the Bulgarian Ministry of Education and Science under contract MM-701/97, MM-807/98

Consider a family $\Phi = \{\Phi_\alpha(t) : \alpha \in \mathcal{A}\}$ of functions $\Phi_\alpha(t) : R^+ \to R^+$ satisfying the property:

F1. $\Phi_\alpha(\cdot)$ is monotone nondecreasing, continuous from the right and $0 < \Phi_\alpha(t) < t$. Moreover $\Phi_\alpha(t_1 + t_2) \leq \Phi_\alpha(t_1) + \Phi_\alpha(t_2)$ for $\alpha \in \mathcal{A}$. Let $j : \mathcal{A} \to \mathcal{A}$ be a mapping from the index set $\mathcal{A}$ into itself such that $j^0(\alpha) = \alpha$ and $j^{n+1}(\alpha) = j(j^n(\alpha))$ for $\alpha \in \mathcal{A}$, where n is a nonnegative integer number.

Definition 1. The multimap $F : E \to CC(E)$ is said to be Φ-contractive when

$$D_\alpha(F(x), F(y)) \leq \Phi_a l(|x-y|_{j(\alpha)}), \quad \alpha \in \mathcal{A} \text{ for every } x, y \in E.$$

Denote by $G(X) = \overline{co}_{x \in X} F(x)$. From Proposition 1 one can easily derive:

Proposition 2. *If $F : E \to CC(E)$ is ϕ-contractive and maps compact sets into relative compacts then the multimap $G(\cdot)$ from $CC(E)$ into itself is also Φ-contraction, i.e.*

$$D_\alpha(F(X), F(Y)) \leq \Phi_\alpha(D_{j(\alpha)}(X, Y)).$$

Proof. Notice first that $D_\alpha(X, Y)$ may be defined for abitrary compact (not necessarily convex) subsets of E. Furthermore $D_\alpha(\overline{co}X, \overline{co}Y) \leq D_\alpha(X, Y)$ as shown in [10]. Moreover for $x \in X$ there exists $y \in Y$ such that $|x-y|_\alpha \leq D^+_\alpha(X, Y)$, i.e. for every $f_x \in F(X)$ there exist $x \in X$, $y \in Y$ and $f_y \in F(Y)$ with

$$|f_x - f_y|_{j(\alpha)} \leq D^+_{j(\alpha)}(F(x), F(y)) \leq D^+_{j(\alpha)}(F(X), F(Y))$$

such that $f_\in F(x)$ and $y \in F(y)$. Consequently $D_\alpha(G(X) m G(Y)) \leq \Phi_\alpha(D_{j(\alpha)}(X, Y))$, i.e. the proof is complete. □

The multimap $F : E \to CC(E)$ is said to be upper semicontinuous (USC) at x_0 when for every open set $B \supset F(x_0)$ there exists a neighbourhood $Y \ni x_0$ such that $F(x) \subset B$ for every $x \in U$. Denote

$$Graph(F)_B = \{(x, F(x)) : x \in B\}.$$

When the last set is closed we say that F admits a closed graph on B.

We prove two fixed point theorems using eseentially the results of [2].

Theorem 1. *Assume F satisfies all the assumptions of Proposition 2. Assume moreover that for each $\alpha \in \mathcal{A}$ there exists a function $\overline{\Phi}_\alpha$ satisfying property* **F1** *such that $\sup\{\Phi_{j^n(\alpha)}(t) : n = 0, 1, 2, \dots\} \leq \overline{\Phi}_\alpha(t)$ and at least one of the following hypotheses hold:*

A1 *$F : M \to M$ for the closed bounded set M and $D^0_{j^n(\alpha)} \leq \rho^0_\alpha$, $(n = 0, 1, 2, \dots)$ where $\rho^0_\alpha = \sup\{|x - y|_\alpha : x, y \in M\} < \infty$ and $D^0_{j^n(\alpha)} = \sup\{D_\alpha(x - y) : x, y \in M\} < \infty$.*

A2 *$\overline{\Phi}_\alpha(\cdot)$ is nondecreasing and there exists $X \in CC(E)$ with*

$$D^+_{j^n(\alpha)}(M, F(M)) \leq \rho(\alpha) < \infty \quad (n = 0, 1, 2, \dots).$$

Then there exists $N \in CC(E)$ such that $G(N) = N$, i.e. $F(N) \subset N$.

Proof. From Proposition 1 we know that $CC(E)$ is a sequentially complete uniform space with respect to the topology generated by $H_{\mathcal{A}}$.

As we have seen $G(X) = \overline{co}F(X)$ is a convex compact valued multifunction in view of Proposition 2. Furthermore G is Φ-contractive on $CC(E)$. Therefore in case of **A1** all the conditions of theorem 1 of [2] hold, while in case of **A2** all the conditions of theorem 4 of [2] hold. Therefore in both cases there exists $Y \in CC(E)$ such that $G(Y) = Y$, i.e. $F(Y) \subset Y$. □

By means of theorem 1 we can prove the main result of the paper.

Theorem 2. *Assume all the conditions of Theorem 1 hold and F has a closed graph on every compact B, then there exists $x \in E$ such that $x \in F(x)$.*

Proof. From the conditions of the theorem follows that $F(X)$ is compact for every compact X. Therefore G maps $CC(E)$ into itself. Thus there exists a convex compact X such that $F(X) \subset X$ thatnks to Theorem 1. Recall that F is convex and compact valued. Furthermoee F is USC on X since admits closed graph on X. Therefore there exists a fixed point $y \in F(y)$ due to Gliksberg's theorem (see [6] for instance). □

Remark 1. In [1] is required that to $f_x \in F(x)$ there exists $f_y \in F(y)$ with $\rho_\alpha(f_x, f_y) \leq \rho_{j(\alpha)}(x, y)$ for every $\alpha \in \mathcal{A}$. Here we assume that for $\alpha \in \mathcal{A}$ to $f_x \in F(x)$ there exists

$f_y \in F(y)$ with $\rho_\alpha(f_x, f_y) \leq \rho_{j(\alpha)}(x, y)$. Therefore theorem 2 generalises theorem 1 of [1] in case of locally convex E.

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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLV, Number 1, March 2000

SOME REMARKS ON THE MIDPOINT RULE IN NUMERICAL INTEGRATION

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Abstract. Using some classical results from the theory of inequalities (Grüss' inequality, Hermite-Hadamard's inequality and others) we point out some quasi-midpoint quadrature formulae, for which the errors of approximation are smaller than the error given for the classical approach.

1. Introduction.

The following inequality is well known in the literature as the midpoint inequality:

$$\left|\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)\right| \le \frac{\|f''\|_\infty}{24}(b-a)^3 \tag{1}$$

where the mapping $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is supposed to be twice differentiable on the interval (a,b) and having the second derivative bounded on (a,b), that is, $\|f''\|_\infty := \sup_{x\in(a,b)} |f''(x)| < \infty$. Now if we assume that $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(x)\,dx$ by the *midpoint quadrature* formula $M_T(f,I_h)$ having an error given by $R_T(f,I_h)$, where

$$M_T(f,I_h) = \sum_{i=0}^{n-1} f\left(\frac{x_{i+1}+x_i}{2}\right)h_i$$

and the remainder $R_T(f,I_h)$ satisfies the estimation

$$|R_T(f,I_h)| \le \frac{\|f''\|_\infty}{24}\sum_{i=0}^{n-1} h_i^3$$

1991 *Mathematics Subject Classification.* 26D15, 41A55.

where $h_i = x_{i+1} - x_i$ for $i = 0, 1, 2, ..., n-1$.

In this paper, by the use of some classical results from the theory of inequalities; Hölder's inequality, Grüss' inequality and the Hermite-Hadamard inequality; we provide some quasi-midpoint quadrature formulae for which the remainder terms are smaller than the classical one given above. For other results in connection with the midpoint inequality see chapter XV of the recent book by Mitrinović et al. [2].

2. Some Integral Inequalities.

The following lemma will be useful in what follows.

Lemma 1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on (a,b). Suppose that $f'' : (a,b) \to \mathbb{R}$ is integrable on (a,b). Then we have the identity:*

$$\int_a^b f(x)\,dx = (b-a) f\left(\frac{a+b}{2}\right) + \int_a^b \phi(x) f''(x)\,dx \tag{2}$$

where $\phi(x)$ is the kernel given by

$$\phi(x) = \begin{cases} \frac{(x-a)^2}{2} & \text{if } x \in \left[a, \frac{a+b}{2}\right] \\ \frac{(b-x)^2}{2} & \text{if } x \in \left(\frac{a+b}{2}, b\right] \end{cases} . \tag{3}$$

Proof. We have successively

$$\int_a^b \phi(x) f''(x)\,dx = \int_a^{\frac{a+b}{2}} \frac{(x-a)^2}{2} f''(x)\,dx + \int_{\frac{a+b}{2}}^b \frac{(b-x)^2}{2} f''(x)\,dx,$$

integrating by parts twice we eventually obtain

$$\begin{aligned} \int_a^b \phi(x) f''(x)\,dx &= \int_a^{\frac{a+b}{2}} f(x)\,dx - \frac{b-a}{2} f\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b f(x)\,dx \\ &\quad + \frac{a-b}{2} f\left(\frac{a+b}{2}\right) \\ &= \int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) \end{aligned}$$

and the identity (2) is proved.■

The following theorem containing an integral inequality, which is known in the literature as the *midpoint inequality*, holds.

Theorem 2. *Let f be as above. Then we have the inequality*

$$\left|\int_a^b f(x)\,dx-(b-a)f\left(\frac{a+b}{2}\right)\right|\le\begin{cases}\frac{(b-a)^3}{24}\|f''\|_\infty, & \text{if } f''\in L_\infty(a,b),\\ \frac{(b-a)^{2+\frac{1}{p}}}{8(2p+1)^{\frac{1}{p}}}\|f''\|_q, & \text{if } f''\in L_q(a,b)\\ & \text{where } \frac{1}{p}+\frac{1}{q}=1, p>1,\\ \frac{(b-a)^2}{8}\|f''\|_1 & \text{if } f''\in L_1(a,b).\end{cases} \tag{4}$$

Proof. Using the representation (2) we have that

$$\left|\int_a^b f(x)\,dx-(b-a)f\left(\frac{a+b}{2}\right)\right|\le\int_a^b|\phi(x)|\,|f''(x)|\,dx.$$

Now, if $f''\in L_\infty(a,b)$, then

$$\begin{aligned}\int_a^b|\phi(x)|\,|f''(x)|\,dx &\le \|f''\|_\infty\int_a^b|\phi(x)|\,dx\\ &=\|f''\|_\infty\left[\int_a^{\frac{a+b}{2}}\frac{(x-a)^2}{2}dx+\int_{\frac{a+b}{2}}^b\frac{(b-x)^2}{2}dx\right]\\ &=\frac{(b-a)^3}{24}\|f''\|_\infty.\end{aligned}$$

If $f''\in L_q(a,b)$ where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ then we have, by Hölder's inequality, that

$$\int_a^b|\phi(x)|\,|f''(x)|\,dx\le\|f''\|_q\left(\int_a^b|\phi(x)|^p\,dx\right)^{\frac{1}{p}}.$$

But

$$\int_a^b |\phi(x)|^p\,dx = \int_a^{\frac{a+b}{2}} \left(\frac{(x-a)^2}{2}\right)^p dx + \int_{\frac{a+b}{2}}^b \left(\frac{(b-x)^2}{2}\right)^p dx$$
$$= \frac{(b-a)^{2p+1}}{8^p(2p+1)}$$

and therefore the second inequality in (4) holds. Finally, if $f'' \in L_1(a,b)$, then

$$\int_a^b |\phi(x)|\,|f''(x)|\,dx \leq \max_{x\in(a,b)} \phi(x)\,\|f''\|_1$$
$$= \frac{(b-a)^2}{8}\|f''\|_1$$

and therefore the last inequality in (4) is also proved.

An example will now be presented to illustrate that the different norms in (4) provide better bounds on the error depending on the behaviour of the integrand. We may take, without loss of generality, in the right hand of (4), $a = 0$ and $b - a = 2\beta$ so that

$$T_1 = \frac{\beta^3}{3} \sup_{t\in(0,2\beta)} |f''(t)|$$

$$T_2 = \frac{\beta^2}{2}\left(\frac{2\beta}{2p+1}\right)^{\frac{1}{p}} \left(\int_0^{2\beta} |f''(t)|^q\,dt\right)^{\frac{1}{q}}, \frac{1}{p}+\frac{1}{q} = 1, p > 1 \text{ and}$$

$$T_3 = \frac{\beta^2}{2}\int_0^{2\beta} |f''(t)|\,dt.$$

Consider the example $f''(t) = e^t$, the Figure 1, shows, on the (p,β) plane, the contours, from the horizontal axis, of the ratios $\frac{T_1}{T_2} = 1, \frac{T_1}{T_3} = 1$ and $\frac{T_2}{T_3} = 1$. The regions A, B, C and D are respectively represented by the inequalities:

$$A : T_1 < T_2 < T_3,\ B : T_1 < T_3 < T_2,$$
$$C : T_2 < T_3 < T_1,\ D : T_3 < T_2 < T_1.$$

Hence, we have demonstrated that each of the bounds T_1, T_2 or T_3 are best in a particular region of (p, β) .

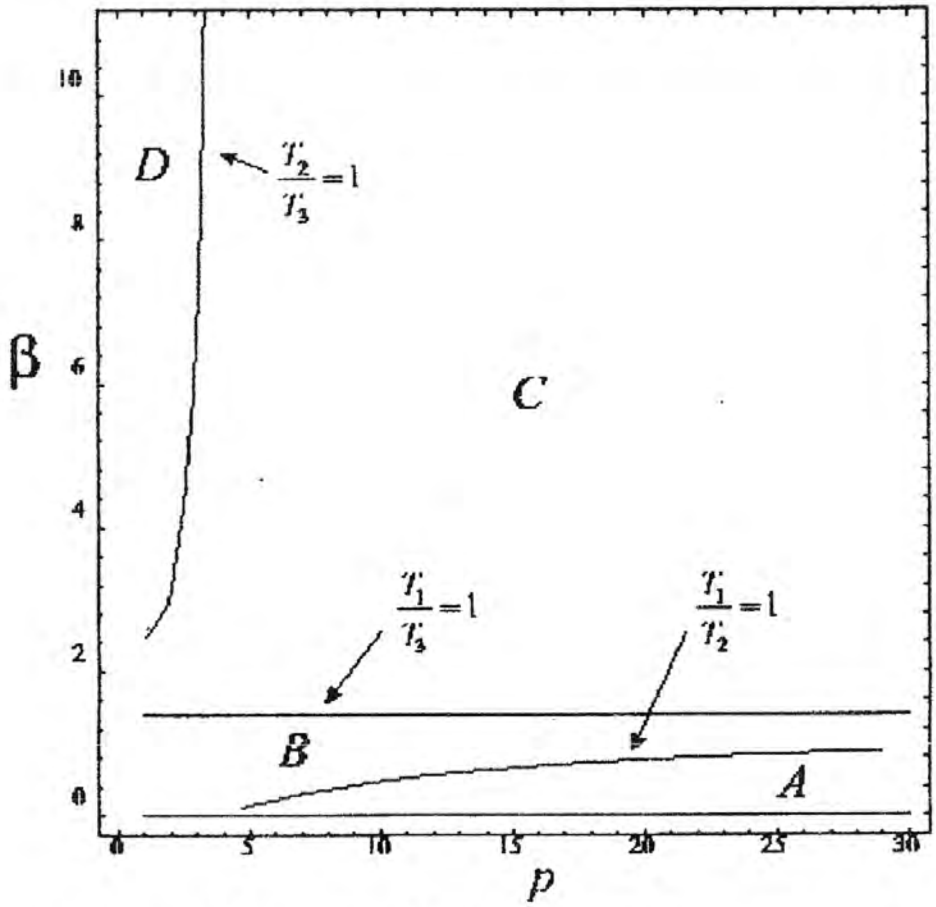

FIGURE 1. The diagram shows regions A, B, C, D of the (p, β) plane, separated by the contours of $\frac{T_1}{T_2} = 1, \frac{T_1}{T_3} = 1$ and $\frac{T_2}{T_3} = 1$.

The following theorem, regarding an integral inequality also holds.

Theorem 3. *Let* $f : [a, b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a, b). *If* $f'' : (a, b) \to \mathbb{R}$ *satisfies the condition*

$$\gamma \leq f''(x) \leq \Gamma \text{ for all } x \in (a, b), \tag{5}$$

then the following inequality is satisfied:

$$\left| \int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}\left(f'(b) - f'(a)\right) \right| \leq \frac{(b-a)^3 (\Gamma - \gamma)}{32}. \tag{6}$$

Proof. Applying Grüss' integral inequality [[1], p.296], we may state that

$$\left|\frac{1}{b-a}\int_a^b \phi(x) f''(x)\,dx - \frac{1}{b-a}\int_a^b \phi(x)\,dx \frac{1}{b-a}\int_a^b f''(x)\,dx\right| \leq \frac{(b-a)^2(\Gamma-\gamma)}{32} \tag{7}$$

as $0 \leq \phi(x) \leq \frac{(b-a)^2}{8}$ for all $x \in [a,b]$. It may be easily seen that $\frac{1}{b-a}\int_a^b \phi(x)\,dx = \frac{(b-a)^3}{24}$ and $\frac{1}{b-a}\int_a^b f''(x)\,dx = \frac{f'(b)-f'(a)}{b-a}$ and hence from (7) we may write

$$\left|\int_a^b \phi(x) f''(x)\,dx - \frac{(b-a)^2}{24}\left(f'(b)-f'(a)\right)\right| \leq \frac{(b-a)^3(\Gamma-\gamma)}{32},$$

which, by identity (2), is clearly equivalent to inequality (6).■

Now, using the celebrated Hermite-Hadamard integral inequality for convex functions, $g:[a,b]\to\mathbb{R}$, which may be written as

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)\,dx \leq \frac{g(a)+g(b)}{2} \tag{8}$$

we obtain the following theorem.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be as in the above theorem; then we have the following double inequality:*

$$\frac{\gamma(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(b-a)^2}{24} \tag{9}$$

and the estimation

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) - \frac{(\gamma+\Gamma)(b-a)^2}{48}\right| \leq \frac{(\Gamma-\gamma)(b-a)^3}{48}. \tag{10}$$

Proof. Let us choose in (8) $g(x) = f(x) - \frac{\gamma x^2}{2}$, then $g(x)$ is a convex function in x, since $g''(x) \geq 0$, and hence

$$f\left(\frac{a+b}{2}\right) - \frac{\gamma(a+b)^2}{8} \leq \frac{1}{b-a}\left(\int_a^b f(x)\,dx - \frac{\gamma(b^3-a^3)}{6}\right)$$

which is equivalent to

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) &\geq \frac{\gamma}{2}\left(\frac{b^3-a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2\right) \\ &= \frac{\gamma(b-a)^2}{24},\end{aligned}$$

and the first part of (9) is therefore obtained. For the second part, let $g(x) = \frac{x^2\Gamma}{2} - f(x)$, and similar manipulations, as previous lead to the second part of (9). The inequality (10) is now obvious by (9), the details have been omitted.■

3. Composite Rules.

We now consider applications of the integral inequalities developed in the previous section, to obtain some midpoint composite rules.

Theorem 5. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on (a,b). If $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$, then we have*

$$\int_a^b f(x)\,dx = A_M(f, I_h) + R_M(f, I_h) \tag{11}$$

where

$$A_M(f, I_h) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

is the midpoint quadrature rule and the remainder $R_M(f, I_h)$ satisfies the inequality

$$|R_M(f,I_h)| \leq \begin{cases} \frac{1}{24}\|f''\|_\infty \sum_{i=0}^{n-1} h_i^3 \\ \frac{1}{8(2p+1)^{\frac{1}{p}}}\|f''\|_q \left(\sum_{i=0}^{n-1} h_i^{2p+1}\right)^{\frac{1}{p}}, \quad \text{where } \frac{1}{p}+\frac{1}{q}=1,\ p>1 \text{ and} \\ \frac{1}{8}\|f''\|_1 \nu^2(I_h) \end{cases} \tag{12}$$

where $h_i := x_{i+1} - x_i$, $i = 0,1,2,...,n-1$ and $\nu(I_h) = \max_{i=0,...,n-1} h_i$.

Proof. Applying the first inequality in (4) on the interval $[x_i, x_{i+1}]$ we obtain

$$\left| \int_{x_i}^{x_{i+1}} f(x)\,dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \right| \le \frac{1}{24} \|f''\|_\infty h_i^3$$

for all $i = 0, 1, 2, ..., n-1$. Summing over i from 0 to $n-1$ we obtain the first part of inequality (12). The second inequality in (4) gives us

$$\left| \int_{x_i}^{x_{i+1}} f(x)\,dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \right| \le \frac{h_i^{2+\frac{1}{p}}}{4(2p+1)^{\frac{1}{p}}} \left(\int_{x_i}^{x_{i+1}} \left|f''(t)\right|^q dt \right)^{\frac{1}{q}}$$

for all $i = 0, 1, 2, ..., n-1$. Summing over all i and using Hölder's discrete inequality, we obtain

$$\left| \int_a^b f(x)\,dx - A_M(f, I_h) \right| \le \frac{1}{8(2p+1)^{\frac{1}{p}}} \sum_{i=0}^{n-1} h_i^{\frac{2p+1}{p}} \left(\int_{x_i}^{x_{i+1}} \left|f''(t)\right|^q dt \right)^{\frac{1}{q}}$$

$$\le \frac{1}{8(2p+1)^{\frac{1}{p}}} \left(\sum_{i=0}^{n-1} \left(h_i^{\frac{2p+1}{p}} \right)^p \right)^{\frac{1}{p}} \left(\sum_{i=0}^{n-1} \left(\left(\int_{x_i}^{x_{i+1}} \left|f''(t)\right|^q dt \right)^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}}$$

$$= \frac{1}{8(2p+1)^{\frac{1}{p}}} \left(\sum_{i=0}^{n-1} h_i^{2p+1} \right)^{\frac{1}{p}} \left(\int_a^b \left|f''(t)\right|^q dt \right)^{\frac{1}{q}}$$

and the second inequality in (12) is proved. In the last part, we have by (4) that

$$\begin{aligned} |R_M(f, I_h)| &\le \frac{1}{8} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} \left|f''(t)\right| dt \right) h_i^2 \\ &\le \frac{1}{8} \max_{i=0,\dots,n-1} h_i^2 \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left|f''(t)\right| dt \\ &= \frac{1}{8} \|f''\|_1 \nu^2(I_h) \end{aligned}$$

and the theorem is proved.■

Remark 1. *It is of some interest to note that in every book on numerical integration, encountered by the authors, only the first estimate in (12) is used. Sometimes, where*

$\left\|f''\right\|_q$ $(q > 1)$ *or* $\left\|f''\right\|_1$ *are easier to compute, it would perhaps be more appropriate to use the second or third estimates.*

We shall now investigate the case where we have an equidistant partitioning of $[a,b]$ given by: $I_h : x_i = a + \left(\frac{b-a}{n}\right) i, i = 0,1,2,...,n-1$. The following result is a consequence of theorem 3.1.

Corollary 6. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b). Then we have*

$$\int_a^b f(x)\,dx = A_{M,n}(f) + R_{M,n}(f)$$

where

$$A_{M,n}(f) := \frac{b-a}{n}\sum_{i=0}^{n-1} f\left(a + \frac{2i+1}{2n}(b-a)\right)$$

and the remainder $R_{M,n}(f)$ satisfies the estimate:

$$|R_{M,n}(f)| \le \begin{cases} \frac{(b-a)^3}{12n^2}\|f''\|_\infty \,, \\ \frac{(b-a)^{2+\frac{1}{p}}}{8(2p+1)^{\frac{1}{p}}n^2}\|f''\|_q \,, \\ \frac{(b-a)^2}{8n^2}\|f''\|_1 \,. \end{cases}$$

The following theorem gives a quasi-midpoint formula which is sometimes more appropriate.

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b). If $f'' : (a,b) \to \mathbb{R}$ satisfies the condition (5) and I_h is an arbitrary partition of $[a,b]$ as above, then we have*

$$\int_a^b f(x)\,dx = A_M\left(f, f', I_h\right) + \widetilde{R}_M\left(f, f', I_h\right)$$

where

$$A_M\left(f, f', I_h\right) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i + \frac{1}{24}\sum_{i=0}^{n-1}\left(f'(x_{i+1}) - f'(x_i)\right) h_i^2$$

is a perturbed midpoint rule and the remainder term, $\widetilde{R}_M\left(f, f', I_h\right)$, *satisfies the estimation*

$$\left|\widetilde{R}_M\left(f, f', I_h\right)\right| \leq \frac{\Gamma - \gamma}{32} \sum_{i=0}^{n-1} h_i^3 \tag{13}$$

where h_i *is as defined above.*

Proof. Writing the inequality (6) on the interval $[x_i, x_{i+1}]$, $i = 0, 1, 2, ..., n-1$ we obtain

$$\left| \int_{x_i}^{x_{i+1}} f(x)\, dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i - \frac{1}{24}\left(f'(x_{i+1}) - f'(x_i)\right) h_i^2 \right| \leq \frac{\Gamma - \gamma}{32} h_i^3,$$

and summing over i from 0 to $n-1$ we easily deduce the desired estimation (13).■

Remark 2. *If we consider a mapping* $f : [a, b] \to \mathbb{R}$ *so that (5) is satisfied and* $\frac{\Gamma-\gamma}{32} \leq \frac{\|f''\|_\infty}{24} = \frac{1}{24}\max\{|\gamma|, |\Gamma|\}$, *that is,*

$$\Gamma - \gamma \leq \frac{4}{3}\max\{|\gamma|, |\Gamma|\} \tag{14}$$

then the estimation provided by (13) is better than the first estimation in (12). Also notice that if $\gamma \geq 0$, *then the condition (14) holds.*

The following corollary is also valid.

Corollary 8. *Let* f *be as defined in the previous theorem, then we have*

$$\int_a^b f(x)\, dx = A_{M,n}\left(f, f'\right) + \widetilde{R}_{M,n}\left(f, f'\right) \tag{15}$$

where

$$\begin{aligned} A_{M,n}\left(f, f'\right) &= \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{2i+1}{2n}(b-a)\right) \\ &\quad + \frac{(b-a)^2}{24n^2}\left(f'(b) - f'(a)\right) \end{aligned}$$

and the remainder, $\widetilde{R}_{M,n}\left(f, f'\right)$, *satisfies the estimation*

$$\left|\widetilde{R}_{M,n}\left(f, f'\right)\right| \leq \frac{(M-m)(b-a)^3}{32n^2},$$

for all $n \geq 1$, *where* $m := \inf_{x \in (a,b)} f'(x) > -\infty$ *and* $M := \sup_{x \in (a,b)} f'(x) < \infty$.

Now, if we apply Theorem 2.3, we can state the following quadrature formula which is a quasi-midpoint formula.

Theorem 9. *Let* f *be as in Theorem 3.2. If* I_h *is a partition of the interval* $[a, b]$ *then we have*

$$\int_a^b f(x)\,dx = A_M(f, \gamma, \Gamma) + R_M(f, \gamma, \Gamma)$$

where

$$A_M(f, \gamma, \Gamma) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i + \frac{\Gamma + \gamma}{48} \sum_{i=0}^{n-1} h_i^2$$

and

$$|R_M(f, \gamma, \Gamma)| \leq \frac{\Gamma - \gamma}{48} \sum_{i=0}^{n-1} h_i^3.$$

Proof. Applying the inequality (10) in $[x_i, x_{i+1}]$ we obtain

$$\left| \int_{x_i}^{x_{i+1}} f(x)\,dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i - \frac{\Gamma + \gamma}{48} h_i^2 \right| \leq \frac{\Gamma - \gamma}{48} h_i^3,$$

and summing over i from 0 to $n - 1$ we have the desired estimation.■

Corollary 10. *Let* f *be as above. Then we have*

$$\int_a^b f(x)\,dx = A_{M,n}(f, \gamma, \Gamma) + R_{M,n}(f, \gamma, \Gamma)$$

where

$$A_{M,n}(f, \gamma, \Gamma) = \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{2i+1}{2n}(b-a)\right) + \frac{(\Gamma + \gamma)(b-a)^2}{48n}$$

and the remainder satisfies the estimation

$$|R_{M,n}(f, \gamma, \Gamma)| \leq \frac{(\Gamma - \gamma)(b-a)^3}{48n^2}.$$

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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLV, Number 1, March 2000

MAPS IN SPACES OF NON-POSITIVE CONSTANT CURVATURE

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Abstract. In this paper it has been shown that the lines curvature have been preserved under Stereographic projection, Inversion map and Lambert projection which are defined in Riemannian manifold of constant sectional curvature -1.

1. Introduction

Let S^n be an n-dimensional sphere in Euclidean $(n+1)$-space R^{n+1}. Özdemir [1] showed that lines of curvature were preserved under Stereographic projection, Inversion map and Lambert projection when these maps are defined from sphere space S^n. Here we will show that similar results hold when the ambient space changes to a simple connected Riemannian manifold of constant sectional curvature -1, that is, hyperbolic space.

Definition 1.1. Let $n \geq 2$ and $0 \leq \nu \leq n$. Then the pseudohyperbolic space of radius $r \geq 0$ in R^{n+1} is the hyperquadric

$$H_\nu^n(r) = \{x \in R^{n+1} : \ g(x,x) = -r^2\}$$

with dimension n and index ν. For $\nu = 0$, $H_0^n(r) = H^n$ is called a hyperbolic space in Euclidean space R^{n+1} [2,5].

Let M be a surface of R^n. Given a curve α on M. If tangent vector at every point of α, is a principal direction then the curve α is called a line of curvature of M [3].

2. Preserved line of curvature under some maps

Definition 2.1. Let M and M_r be two hypersurface in R^{n+1} and let $f_* : \Xi(M) \to \Xi(M_r)$ be an adjoint transformation of $f : M \to M_r$. It is said that the curvature line

α is preserved by f if $S(T) = \lambda T$ and $S_r(f_*(T)) = \mu(f_*(T))$ where α is the line of curvature and T is the tangent vector field of α. Here S and S_r are shape operators of hypersurface M and M_r respectively [2].

Definition 2.2. Given a hyperbolic space H^n and a hyperplane H_n in R^{n+1}. Let $B = (0, \ldots, 0, r) \in H^n$. A map

$$\sigma : H^n \setminus \{B\} \to H_n$$

defined by $\sigma(P) = \overline{P}$ for every $P \in H^n \setminus \{B\}$ and $\overline{P} \in H_n$ is called stereographic projection (see figure 1). The analytic statement of stereographic projection:

$$\overrightarrow{O\overline{P}} = \overrightarrow{OP} + \overrightarrow{P\overline{P}},$$

$$\overrightarrow{O\overline{P}} = \overrightarrow{OP} + \lambda\overrightarrow{PB},$$

$$(\overline{x}_1, \ldots, \overline{x}_n, 0) = (x_1, \ldots, x_{n+1} + \lambda[(0, \ldots, r) + (x_1, \ldots, x_{n+1})],$$

$$\overline{x}_i = (1-\lambda)x_i, \quad 1 \leq x_i \leq n$$

$$0 = x_{n+1} + \lambda(r - x_{n+1})$$

by a simple computation, we get

$$\overline{x}_i = \frac{rx_i}{r - x_{n+1}}$$

in other word

$$\sigma(x_1, \ldots, x_{n+1}) = \left(\frac{rx_1}{r - x_{n+1}}, \frac{rx_2}{r - x_{n+1}}, \ldots, \frac{rx_n}{r - x_{n+1}}, 0\right). \tag{1}$$

Using (1) we obtain the derivative operator σ_* as

$$\sigma_*(p) = \frac{1}{r - x_{n+1}} \begin{bmatrix} 1 & 0 & \ldots & 0 & \frac{x_1}{r - x_{n+1}} \\ 1 & 0 & \ldots & 0 & \frac{x_2}{r - x_{n+1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & \ldots & 0 & \frac{x_n}{r - x_{n+1}} \end{bmatrix}$$

Theorem 2.1. *The stereographic projection preserves the line of curvature.*

Proof. Let us consider stereographic projection $\sigma : H^n \setminus \{B\} \to H_n$. Let $S_h = -\frac{1}{r}$ be a shape operator for hyperbolic space and let S_d be a shape operator for hyperplane (see [4]). We obtain

$$S_h(T) = \lambda T, \quad \forall\, T \in \Xi(H^n \setminus \{B\})$$

when the curve $\alpha : I \to H^{n+1} \setminus \{B\}$ is a line of curvature on hyperbolic space and T is a tangent vector field of α. Since

$$S_d(\sigma_*(T)) = 0.\sigma_*(T),$$

$\sigma_*(T)$, which is the image for the tangent vector field under σ_*, is a principal direction. Hence every curve is a line of curvature on hyperplane. □

Definition 2.3. (Inversion map). Let $\varphi : H^n \to H^n$ be a symmetric map with respect to hyperplane H_n. Then ψ by

$$\psi = \sigma\varphi\sigma^{-1} : M_p^n \to M_p^n,$$

is called inversion map. Where M_p^n is a hyperplane as shown in figure 2.

The analytic statement of ψ:

$$\sigma\varphi\sigma^{-1}(\overline{x}_1, \ldots, \overline{x}_n) = \sigma\varphi\left(\frac{2r^2\overline{x}_1}{r^2 - \sum\limits_{i=1}^{n}\overline{x}_i^2}, \ldots, \frac{2r^2\overline{x}_n}{r^2 - \sum\limits_{i=1}^{n}\overline{x}_i^2}, \frac{r\left[r^2 + \sum\limits_{i=1}^{n}\overline{x}_i^2\right]}{\sum\limits_{i=1}^{n}\overline{x}_i^2 - r^2}\right) =$$

$$= \left(\frac{-r^2\overline{x}_1^2}{\sum\limits_{i=1}^{n}\overline{x}_i^2}, \ldots, \frac{-r^2\overline{x}_n^2}{\sum\limits_{i=1}^{n}\overline{x}_i^2}\right).$$

Thus

$$\psi(\overline{x}) = \frac{-r^2\overline{x}}{\sum\limits_{i=1}^{n}\overline{x}_i^2}.$$

Theorem 2.2. *The inversion map preserves the line of curvatures.*

Proof. $\psi : M_p^n \to M_p^n$ is given as

$$\psi(x) = \frac{-r^2\overline{x}}{||x||^2},$$

and the normal vector field $N = (a_1, \ldots, a_n)$ of hyperplane M^n is a constant vector field. Hence for every $X \in \Xi(M_p^n)$,

$$S(X) = D_X N = X[N]$$

$$S(X) = 0_n.$$

This shape operator is the same at every point of M_p^n. When $\alpha : I \to M_p^n$ is a curvature line and T is tangent vector field of α, it follows that

$$S(T) = 0.T$$

Hence, since

$$S(\psi_*(T)) = 0.\psi_*(T),$$

for $\psi_*(T)$, we obtain that every line is a principal direction and every curve is a curvature line. □

Definition 2.4. (Lambert projection). Let $H^2 = \{(x, y, z) \in R^3 : x^2 + y^2 - z^2 = 1\}$ and $C^2 = \{(x, y, z) \in R^3 : x^2 + y^2 = 1,\ x = t \text{ arbitrary}\}$. Denote by $B = (0, 0, 1)$, $R = (0, 0, -1)$ and let $F : H^2 \setminus \{B\} \cup \{R\} \to C^2$ be defined as follows: For each $p \in H^2 \setminus \{B\} \cup \{R\}$ let the perpendicular from p to the z axis meet Oz at q. Consider the half-line l starting at q and containing p. Then $f(p) = l \cap C^2$ as shown in figure 3 [6].

By similarity triangles $OX'Q' \sim OXQ$

$$\frac{\overline{OX}}{\overline{OX'}} = \frac{\overline{OQ}}{\overline{OQ'}}, \quad \overline{OQ} = 1,$$

so

$$x' = \frac{x}{(x^1 - 1)^{1/2}} \quad \text{and} \quad y' = \frac{y}{(z^2 - 1)^{1/2}}.$$

Hence,

$$f(x, y, z) = \left(\frac{x}{(z^2 - 1)^{1/2}}, \frac{y}{(z^2 - 1)^{1/2}}, z \right).$$

Theorem 2.3. *The Lambert projection preserves the line of curvature.*

Proof. Consider

$$S_h(T) = \lambda T, \quad \lambda = constant$$

for a curvature line, $X : I \to H^2$, and a tangent vector field T. Then it is enough to find lines such that

$$S_c(f_*(T)) = \mu f_*(T),$$

where S_c is a shape operator of cylinder. We will find the lines such that

$$e_3 = f_*(X),$$

for $X \in \Xi(H^2)$. Since $f_*(X) = (0,0,1)$ for $X = (x_1, x_2, x_3)$, it follows that

$$X = \left(\frac{zx}{z^2-1}, \frac{zy}{z^2-1}, 1\right),$$

curvature lines whose image are e_3. Since

$$S_h(X) = \lambda X, \quad S_h = -I$$

$$X = -\lambda X$$

we find that $\lambda = -1$. For the shape operator $S_c = 0_2$. of a cylinder, we write $S_c(f_*(X)) = 0.f_*(X)$. □

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CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

TÜNDE JAKAB

Abstract. In this work the author obtain radii of starlikeness and convexity of order δ for three class of functions: $S^*(\alpha,\beta)$; $C^*(\alpha,\beta)$ and $P^*(\alpha,\beta)$; and a representation formula for the class $P^*(\alpha,\beta)$.

Let S denote the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the unit disc $U = \{z \in \mathbf{C} : |\mathbf{z}| < \mathbf{1}\}$

A function $f \in S$ is said to be in the class $S(\alpha,\beta)$, the class of starlike functions of order α, $(0 \le \alpha < 1)$ and type β, $(0 < \beta \le 1)$ if and only if :

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + (1-2\alpha)} \right| < \beta \qquad (|z| < 1). \tag{2}$$

Further, $f \in S$ is in $K(\alpha,\beta)$, the class of convex functions of order α and type β, $(0 \le \alpha < 1, 0 < \beta \le 1)$ if and only if $zf'(z) \in S(\alpha,\beta)$.

We denote by $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$ the classes obtained by taking intersections, respectively, of the classes $S(\alpha,\beta)$ and $K(\alpha,\beta)$ with the class T, where T is a subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \ge 0). \tag{3}$$

Thus

$$S^*(\alpha,\beta) = S(\alpha,\beta) \cap T$$

and

$$C^*(\alpha,\beta) = K(\alpha,\beta) \cap T.$$

Lemma 1 ([1]). *Let a function $f(z)$ be defined by (3). Then $f(z)$ is in the class $S^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\{(n-1)+\beta(n+1-2\alpha)\}a_n \le 2\beta(1-\alpha).$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)}z^n \qquad (n \ge 2).$$

Remark 1 ([1]). We have $T = S^*(0,1)$.

Lemma 2 ([1]). *Let a function $f(z)$ be defined by (3). Then $f(z)$ is in the class $C^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n\{(n-1)+\beta(n+1-2\alpha)\}a_n \le 2\beta(1-\alpha).$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta(1-\alpha)}{n\{(n-1)+\beta(n+1-2\alpha)\}}z^n, \qquad (n \ge 2).$$

Theorem 1. *If a function $f(z)$ defined by (3) is in the class $S^*(\alpha,\beta)$, then f is starlike of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{n\ge 2}\left\{\frac{(1-\delta)\{(n-1)+\beta(n+1-2\alpha)\}}{2\beta(n-\delta)(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{4}$$

The result is sharp.

Proof. Note that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}} \le 1-\delta$$

if and only if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1} \le 1.$$

By virtue of Lemma 1, we need to find values of $|z|$ for which

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}$$

for $n \ge 2$, which will be true when $|z| < r_1$.

We can see that the result is sharp for the function

$$f(z) = z - \frac{2\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} z^n, \qquad (n \ge 2). \tag{5}$$

□

Corollary 1. *If a function $f(z)$ defined by (3) is in the class $S^*(\alpha, \beta)$, then f is starlike in the disc $|z| < r_2$, where*

$$r_2 = \inf_{n \ge 2} \left\{ \frac{(n-1)+\beta(n+1-2\alpha)}{2\beta n(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{6}$$

The result is sharp for the function $f(z)$ given by (5).

The proof is immediately obtained by taking $\delta = 0$ in Theorem 1.

Theorem 2. *If a function $f(z)$ defined by (3) is in the class $S^*(\alpha, \beta)$, then f is convex of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_3$, where*

$$r_3 = \inf_{n \ge 2} \left\{ \frac{(1-\delta)\{(n-1)+\beta(n+1-2\alpha)\}}{2\beta n(n-\delta)(1-\alpha)} \right\}^{\frac{1}{n-1}}.$$

The result is sharp.

Proof. It suffices to show, that

$$\left| \frac{zf''(z)}{f'(z)} \right| \le 1-\delta \qquad \text{for} \qquad |z| < r_3,$$

which is equivalent to

$$\left| \frac{zf''(z)}{f'(z)} \right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}} \le 1-\delta \qquad \text{for} \qquad |z| < r_3.$$

The result follows if

$$\sum_{n=2}^{\infty} \frac{n(n-\delta)}{1-\delta} a_n |z|^{n-1} \le 1.$$

By virtue of Lemma 1, we need to find values of $|z|$ for which

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1} \leq \frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}$$

for $n \geq 2$, which will be true if $|z| < r_3$. The result is sharp for the function given by (5). □

Corollary 2. *If $f \in S^*(\alpha, \beta)$, then f is convex in the disc $|z| < r_4$, where*

$$r_4 = \inf_{n\geq 2} \left\{ \frac{(n-1)+\beta(n+1-2\alpha)}{2\beta n^2(1-\alpha)} \right\}^{\frac{1}{n-1}}.$$

The result is sharp for the extremal function given by (5).

The proof is immediately obtained by taking $\delta = 0$ in Thorem 2.

In the same way we can prove the following two theorems and two corollaries, using Lemma 2; the extremal function beeing

$$f(z) = z - \frac{2\beta(1-\alpha)}{n\{(n-1)+\beta(n+1-2\alpha)\}} z^n, \qquad (n \geq 2).$$

Theorem 3. *If $f(z)$ defined by (3) is in the class $C^*(\alpha, \beta)$, then f is starlike of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_5$, where*

$$r_5 = \inf_{n\geq 2} \left\{ \frac{n(1-\delta)\{(n-1)+\beta(n+1-2\alpha)\}}{2\beta(n-\delta)(1-\alpha)} \right\}^{\frac{1}{n-1}}.$$

Corollary 3. *If $f \in C^*(\alpha, \beta)$, then f is starlike in the disc $|z| < r_6$, where*

$$r_6 = \inf_{n\geq 2} \left\{ \frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} \right\}^{\frac{1}{n-1}}.$$

Theorem 4. *If $f(z)$ defined by (3) is in $C^*(\alpha, \beta)$, then f is convex of order δ, $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, r_1 given by (4).*

Corollary 4. *If $f \in C^*(\alpha, \beta)$, then f is convex in the disc $|z| < r_2$, r_2 given by (6).*

The Theorem 4 and Corollary 4 result from Theorem 1 and Corollary 1, too.

The class $P^*(\alpha, \beta)$

Definition 1 ([1]). A function $f \in T$ is in the class $P^*(\alpha, \beta)$ if and only if

$$\left| \frac{f'(z)}{f'(z) + (1 - 2\alpha)} \right| < \beta, \qquad (|z| < 1),$$

for α $(0 \leq \alpha < 1)$ and β $(0 < \beta \leq 1)$.

V.P.Gupta and P.K.Jain showed in [2] the following lemma.

Lemma 3 ([2]). *A function $f(z)$ defined by (3) is in the class $P^*(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(1 + \beta) a_n \leq 2\beta(1 - \alpha).$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta(1 - \alpha)}{n(1 + \beta)} z^n, \qquad (n \geq 2).$$

Lemma 4 (A representation formula). *A function $f(z)$ defined by (3) is in the class $P^*(\alpha, \beta)$ if and only if*

$$f(z) = z + 2(1 - \alpha) \int_0^z \frac{V(t)}{1 - V(t)} dt, \tag{7}$$

where $V(z)$ is an analytic function and satisfies $|V(z)| < \beta$ for $|z| < 1$.

Proof. From $f \in P^*(\alpha, \beta)$ we have

$$\left| \frac{f'(z)}{f'(z) + (1 - 2\alpha)} \right| < \beta \qquad \text{for} \qquad |z| < 1.$$

Let

$$V(z) = \frac{f'(z) - 1}{f'(z) + (1 - 2\alpha)},$$

which is equivalent to

$$f'(z) = 1 + 2(1 - \alpha) \frac{V(z)}{1 - V(z)},$$

and we have the representation formula, with $|V(z)| < \beta$ for $|z| < 1$.

The "only if" part is easily obtained by differentiating (7). □

Theorem 5. *If $f \in P^*(\alpha, \beta)$, then f is sarlike of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_7$, where*

$$r_7 = \inf_{n \geq 2} \left\{ \frac{n(1-\delta)(1+\beta)}{2\beta(n-\delta)(1-\alpha)} \right\}^{\frac{1}{n-1}}.$$

The result is sharp.

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta \qquad \text{for} \qquad |z| < r_7.$$

First, we note that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}} \leq 1 - \delta$$

is equivalent to

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n |z|^{n-1} \leq 1.$$

Using Lemma 3, we have

$$\sum_{n=2}^{\infty} \frac{n(1+\beta)}{2\beta(1-\alpha)} a_n \leq 1.$$

Hence, f is srarlike of order δ, if

$$\frac{n-\delta}{1-\delta} |z|^{n-1} \leq \frac{n(1+\beta)}{2\beta(1-\alpha)}$$

for $n \geq 2$; that is

$$|z|^{n-1} \leq \frac{n(1-\delta)(1+\beta)}{2\beta(n-\delta)(1-\alpha)} \qquad \text{for} \qquad n \geq 2.$$

Further, we can see that the result is sharp for the function

$$f(z) = z - \frac{2\beta(1-\alpha)}{n(1+\beta)} z^n \qquad (n \geq 2). \tag{8}$$

□

Corollary 5. *If $f \in P^*(\alpha, \beta)$, then f is starlike in the unit disc.*

Proof. From Theorem 5 we have that f is starlike in the disc $|z| < r_7'$, where

$$r_7' = \inf_{n\geq 2}\left\{\frac{1+\beta}{2\beta(1-\alpha)}\right\}^{\frac{1}{n-1}}.$$

Because

$$\frac{1+\beta}{2\beta(1-\alpha)} > 1,$$

we have

$$r_7' = \left\{\frac{1+\beta}{2\beta(1-\alpha)}\right\}^{\inf\limits_{n\geq 2}\frac{1}{n-1}} = 1.$$

□

Theorem 6. *If $f \in P^*(\alpha,\beta)$, then f is convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_8$, where*

$$r_8 = \inf_{n\geq 2}\left\{\frac{(1-\delta)(1+\beta)}{2\beta(n-\delta)(1-\alpha)}\right\}^{\frac{1}{n-1}}.$$

The result is sharp.

Proof. It suffices to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\delta \qquad \text{for} \qquad |z| < r_8.$$

First, we note that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} na_n|z|^{n-1}}.$$

The result follows if

$$\sum_{n=2}^{\infty}\frac{n(n-\delta)}{1-\delta}a_n|z|^{n-1} \leq 1.$$

By Lemma 3 we have

$$\sum_{n=2}^{\infty}\frac{n(1+\beta)}{2\beta(1-\alpha)}a_n \leq 1.$$

Hence f is convex of order δ if

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1} \leq \frac{n(1+\beta)}{2\beta(1-\alpha)}$$

for $n \geq 2$, which gives that $|z| < r_8$.

Equality holds for the function $f(z)$ given by (8). □

Corollary 6 ([2,Th.5]). *If $f \in P^*(\alpha, \beta)$, then f is convex in the disc $|z| < r = r(\alpha, \beta)$, where*

$$r(\alpha, \beta) = \inf_{n \geq 2} \left\{ \frac{1+\beta}{2\beta n(1-\alpha)} \right\}^{\frac{1}{n-1}} .$$

The result is sharp with the extremal function given by (8).

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STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLV, Number 1, March 2000

THE STABILITY OF THE BETA FUNCTIONAL EQUATION

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1. Introduction

In 1940, S. M. Ulam raised the following problem : *Under what conditions does there exist an additive mapping near an approximately additive mapping?* In 1941, this problem was solved by D. H. Hyers [2]. Thereafter we usually say that the equation $E_1(h) = E_2(h)$ has the Hyers-Ulam stability if for an approximate solution f of this equation, i.e. for a function f with $|E_1(f) - E_2(f)| \leq \delta$, there exists a function g such that $E_1(g) = E_2(g)$ and $|f(x) - g(x)| \leq \epsilon$. This stability problem has been further generalized by Th. M. Rassias [4] in 1978, then it is said the Hyers-Ulam-Rassias stability. In [1], R. Ger proved the stability of equation $E_1(h) = E_2(h)$ for the following type: if for an approximate solution f of this equation, i.e. for a function f such that

$$|\frac{E_1(f)}{E_2(f)} - 1| \leq \psi$$

holds with a given function ψ, there exists a function g such that $E_1(g) = E_2(g)$ and $\alpha \leq \frac{f}{g} \leq \beta$ for some fixed functions α and β.

The aim of the present note is to give two stability theorems, that is, the Hyers-Ulam stability and a stability in the sense of R. Ger for the beta functional equation. Throughout this paper, let $\delta > 0$ be fixed and let n_0 be a given non-negative integer.

Now we consider the beta functional equation. The beta function is given by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

1991 *Mathematics Subject Classification.* 39B22,39B72.

Key words and phrases. Stability of functional equation, Hyers-Ulam stability, beta function.

Then $B(x,y)$ is continuous, and $B(x,y) > 0$ and also

$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$$

for all $x, y > 0$, where $\Gamma(x)$ is the gamma function which is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \quad (x > 0).$$

The equation $f(x+1) = xf(x)$ (for all $x > 0$) is called the gamma functional equation.

It is well known that gamma function satisfies the gamma functional equation, and S-M. Jung [3, 4] obtained stability theorems of the gamma functional equation. Now we define the *beta functional equation* as follows :

$$B(x+1, y+1) = \frac{xy}{(x+y)(x+y+1)}B(x,y). \tag{1}$$

The direct proof for the stability of equation (1) is confronted by the difficult issue which is the construction of Cauchy sequence. So we shall be consider the inverse form of (1):

$$F(x+1, y+1)^{-1} = \frac{(x+y)(x+y+1)}{xy}F(x,y)^{-1}, \tag{1'}$$

which is equivalent to the equation (1).

2. The stability of the Beta functional equation

We investigate the Hyers-Ulam stability of beta functional equation (1′) in the following theorem.

Theorem 1. *If a mapping $F : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ satisfies the inequality*

$$|F(x+1, y+1)^{-1} - \frac{(x+y)(x+y+1)}{xy}F(x,y)^{-1}| \le \delta \tag{2}$$

for some $\delta > 0$ and for all $x, y > n_0$, then there exists a unique mapping $T : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ which satisfies the equation (1′) and the inequality

$$|F(x,y)^{-1} - T(x,y)^{-1}| \le \frac{xy}{(x+y)(x+y+1)}2\delta \tag{3}$$

for all $x, y > n_0$.

Proof. For any $x, y > 0$ and for every positive integer n we define

$$P_n(x,y) = F(x+n, y+n)^{-1} \prod_{i=0}^{n-1} \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)}.$$

By (2) we have

$$\begin{aligned} |P_{n+1}(x,y) - P_n(x,y)| &= |F(x+n+1, y+n+1)^{-1} \\ &\quad - \frac{(x+y+2n)(x+y+2n+1)}{(x+n)(y+n)} F(x+n, y+n)^{-1}| \\ &\quad \cdot \prod_{i=0}^{n} \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)} \\ &\leq \delta \prod_{i=0}^{n} \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)}. \end{aligned}$$

Now we use induction on n to prove

$$|P_n(x,y) - F(x,y)^{-1}| \leq \delta \sum_{j=0}^{n-1} \prod_{i=0}^{j} \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)} \tag{4}$$

for $x, y > n_0$ and for all positive integers n.

The inequality (4) for the case of $n = 1$ is an immediate consequence of (1′). Assume that (4) holds true for some n. It then follows that

$$\begin{aligned} |P_{n+1}(x,y) - F(x,y)^{-1}| &\leq |P_{n+1}(x,y) - P_n(x,y)| + |P_n(x,y) - F(x,y)^{-1}| \\ &\leq \delta \sum_{j=0}^{n} \prod_{i=0}^{j} \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)} \end{aligned}$$

which completes the proof of (4). Let m, n be positive integers with $n \geq m$. Suppose $x, y > n_0$ is given. We have

$$\begin{aligned} |P_n(x,y) - P_m(x,y)| &\leq \sum_{j=m}^{n-1} |P_{j+1}(x,y) - P_j(x,y)| \\ &\leq \delta \sum_{j=m}^{n-1} \prod_{i=0}^{j} \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)} \\ &\leq \delta \sum_{j=m}^{n-1} \frac{1}{2^{j+1}} \longrightarrow 0 \end{aligned}$$

as $m \longrightarrow 0$. Thus $\{P_n(x,y)\}$ is a Cauchy sequence for $x, y > n_0$ and hence we can define a mapping $T_0 : (n_0, \infty) \times (n_0, \infty) \longrightarrow (0, \infty)$ by

$$T_0(x,y)^{-1} = \lim_{n\to\infty} P_n(x,y)$$

for all $x, y > n_0$. It is easy to see

$$\begin{aligned} T_0(x+1,y+1)^{-1} &= \lim_{n\to\infty} P_n(x+1,y+1) \\ &= \lim_{n\to\infty} \frac{(x+y)(x+y+1)}{xy} P_{n+1}(x,y) \\ &= \frac{(x+y)(x+y+1)}{xy} T_0(x,y)^{-1} \end{aligned}$$

for all $x, y > n_0$. On account of (4), for all $x, y > n_0$, we have

$$\begin{aligned} |T_0(x,y)^{-1} - F(x,y)^{-1}| &= \lim_{n\to\infty} |P_n(x,y) - F(x,y)^{-1}| \\ &\le \frac{xy}{(x+y)(x+y+1)} \delta \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &\le \frac{xy}{(x+y)(x+y+1)} 2\delta. \end{aligned}$$

Now, let $G : (n_0, \infty) \times (n_0, \infty) \longrightarrow (0, \infty)$ be another mapping which satisfies (1′) as well as (3) for all $x, y > n_0$. It then follows that

$$\begin{aligned} |T_0(x,y)^{-1} - G(x,y)^{-1}| &= \frac{xy}{(x+y)(x+y+1)} \\ &\quad |T_0(x+1,y+1)^{-1} - G(x+1,y+1)^{-1}| \\ &= \prod_{i=0}^{n} \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)} \\ &\quad |T_0(x+n+1,y+n+1)^{-1} - G(x+n+1,y+n+1)^{-1}| \\ &\le 4\delta \prod_{i=0}^{n+1} \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)} \longrightarrow 0 \end{aligned}$$

for all $x, y > n_0$ and for all positive integers n. This implies the uniqueness of T_0. We can inductively define new mappings $T_i : (n_0 - i, n_0 - i + 1] \times (n_0 - i, n_0 - i + 1] \longrightarrow R$ $(i = 1, 2, \cdots, n_0)$ by

$$T_i(x,y) = \frac{xy}{(x+y)(x+y+1)} T_{i-1}(x+1,y+1).$$

Further, define a mapping $T : (0,\infty)\times(0,\infty) \longrightarrow R$ by $T(x,y) = T_i(x,y)$ for $n_0 - i < x, y \leq n_0 - i + 1$ $(i = 1, 2, \cdots, n_0)$ and $T(x,y) = T_0(x,y)$ for all $x, y > n_0$. Then T satisfies the equation (1′), and also satisfies (3) for $x, y > n_0$. Hence the proof of the theorem is complete. □

In the following theorem we investigate a stability in the sense of R. Ger (more generally, modified Hyers-Ulam-Rassias stability) of beta functional equation (1′).

Theorem 2. *Let $F : (0,\infty)\times(0,\infty) \longrightarrow (0,\infty)$ be a mapping that satisfies the inequality*

$$\left|\frac{xy}{(x+y)(x+y+1)}\frac{F(x,y)}{F(x+1,y+1)} - 1\right| \leq \varphi(x,y) \tag{5}$$

for all $x, y > n_0$, where $\varphi : (0,\infty)\times(0,\infty) \longrightarrow (0,1)$ is a mapping such that

$$\alpha(x,y) := \sum_{i=0}^{\infty} \log(1 - \varphi(x+i, y+i))$$

and

$$\beta(x,y) := \sum_{i=0}^{\infty} \log(1 + \varphi(x+i, y+i))$$

are bounded for $x, y > n_0$. Then there exists a unique solution $T : (0,\infty)\times(0,\infty) \longrightarrow (0,\infty)$ of the beta functional equation (1′) with

$$e^{\alpha(x,y)} \leq \frac{F(x,y)}{T(x,y)} \leq e^{\beta(x,y)} \tag{6}$$

Proof. Let $P_n(x,y)$ be defined as in the proof of Theorem 1. For any $x, y > 0$ and for all positive integers $m, n \geq n_0$ with $n > m$ it holds

$$\begin{aligned}\frac{P_n(x,y)}{P_m(x,y)} &= \frac{F(x+m,y+m)}{F(x+n,y+n)} \cdot \frac{(x+m)(y+m)}{(x+y+2m)(x+y+2m+1)} \\ &\quad \cdot \frac{(x+m+1)(y+m+1)}{(x+y+2(m+1))(x+y+2(m+1)+1)} \\ &\quad \cdots \frac{(x+n-1)(y+n-1)}{(x+y+2(n-1))(x+y+2(n-1)+1)} \\ &= \frac{(x+m)(y+m)}{(x+y+2m)(x+y+2m+1)} \cdot \frac{F(x+m,y+m)}{F(x+m+1,y+m+1)} \\ &\quad \cdot \frac{(x+m+1)(y+m+1)}{(x+y+2(m+1))(x+y+2(m+1)+1)} \cdot \frac{F(x+m+1,y+m+1)}{F(x+m+2,y+m+2)} \\ &\quad \cdots \frac{(x+n-1)(y+n-1)}{(x+y+2(n-1))(x+y+2(n-1)+1)} \cdot \frac{F(x+n-1,y+n-1)}{F(x+n,y+n)}\end{aligned}$$

Note that for all $i \geq n_0$

$$\begin{aligned}0 &< 1 - \varphi(x+i,y+i) \\ &\leq \frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)} \cdot \frac{F(x+i,y+i)}{F(x+i+1,y+i+1)} \\ &\leq 1 + \varphi(x+i,y+i).\end{aligned}$$

Thus

$$\prod_{i=m}^{n-1} (1 - \varphi(x+i,y+i)) \leq \frac{P_n(x,y)}{P_m(x,y)} \leq \prod_{i=m}^{n-1} (1 + \varphi(x+i,y+i))$$

or

$$\begin{aligned}\sum_{i=m}^{n-1} \log(1 - \varphi(x+i,y+i)) &\leq \log P_n(x,y) - \log P_m(x,y) \\ &\leq \sum_{i=m}^{n-1} \log(1 + \varphi(x+i,y+i)).\end{aligned}$$

By assumption, $\{\log P_n(x,y)\}$ is a Cauchy sequence for all $x, y > 0$. Now we can define

$$L(x,y) := \lim_{n\to\infty} \log P_n(x,y)$$

and

$$T(x,y)^{-1} = e^{L(x,y)} = \lim_{n\to\infty} P_n(x,y)$$

for all $x, y > 0$. It is easy to see that

$$\begin{aligned} T(x+1, y+1)^{-1} &= \lim_{n\to\infty} P_n(x+1, y+1) \\ &= \lim_{n\to\infty} \frac{(x+y)(x+y+1)}{xy} P_{n+1}(x, y) \\ &= \frac{(x+y)(x+y+1)}{xy} T(x,y)^{-1} \end{aligned}$$

for any $x, y > 0$. Since

$$\begin{aligned} \frac{P_n(x,y)}{F(x,y)^{-1}} = & \frac{F(x,y)}{F(x+1,y+1)} \frac{xy}{(x+y)(x+y+1)} \\ & \cdot \frac{F(x+1,y+1)}{F(x+2,y+2)} \frac{(x+1)(y+1)}{(x+y+2)(x+y+3)} \\ & \cdots \frac{F(x+n-1,y+n-1)}{F(x+n,y+n)} \frac{(x+n-1)(y+n-1)}{(x+y+2(n-1))(x+y+2(n-1)+1)}, \end{aligned}$$

we get

$$\prod_{i=m}^{n-1} (1-\varphi(x+i,y+i)) \le \frac{P_n(x,y)}{F(x,y)^{-1}} \le \prod_{i=m}^{n-1} (1+\varphi(x+i,y+i))$$

This implies that

$$e^{\alpha(x,y)} \le \frac{T(x,y)^{-1}}{F(x,y)^{-1}} \le e^{\beta(x,y)}.$$

Now it remains only to prove the uniqueness of T. Assume that $G : (0,\infty)\times(0,\infty) \to (0,\infty)$ is an another solution of the equation $(1')$ which satisfies (6). By equation $(1')$,

$$\frac{T(x,y)}{G(x,y)} = \frac{T(x+n,y+n)}{G(x+n,y+n)} = \frac{F(x+n,y+n)^{-1}}{T(x+n,y+n)^{-1}} \cdot \frac{G(x+n,y+n)^{-1}}{F(x+n,y+n)^{-1}}$$

for any $x, y > 0$. Hence we have

$$\frac{e^{\alpha(x+n,y+n)}}{e^{\beta(x+n,y+n)}} \le \frac{T(x,y)}{G(x,y)} \le \frac{e^{\beta(x+n,y+n)}}{e^{\alpha(x+n,y+n)}}$$

for all sufficiently large n. Since $\sum_{i=0}^{\infty} \log(1-\varphi(x+i,y+i))$ converges,

$$\begin{aligned} \alpha(x+n,y+n) &= \sum_{i=0}^{\infty} \log(1-\varphi(x+n+i,y+n+i)) \\ &= \sum_{i=n}^{\infty} \log(1-\varphi(x+i,y+i)) \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow 0$, and similarly $\beta(x+n, y+n) \longrightarrow 0$ as $n \longrightarrow 0$. Hence, it is obvious that $F(x,y) = G(x,y)$ □

Remark 1. If $\varphi(x,y) = \frac{\delta}{x^p}$ for $p > 1$ and $\delta > 0$ as in the case of stability of gamma functional equation, then $\sum_{i=0}^{\infty} \log(1 \pm \varphi(x+i, y+i))$ converges.

But if $\varphi(x,y) = (\frac{xy}{(x+y)(x+y+1)})^p$ for $p > 1$, then $\sum_{i=0}^{\infty} \log(1 + \varphi(x+i, y+i))$ does not converge because

$$\lim_{i \to \infty} \log\left(1 + (\frac{(x+i)(y+i)}{(x+y+2i)(x+y+2i+1)})\right) \neq 0.$$

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ON INVERSE LOEWNER CHAINS

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Abstract. In this note, we obtain conditions in which an inverse Loewner chain in the disc $U_r = \{z \in \mathbb{C} : |z| < r \leq 1\}$ can be extended to a family of analytic and univalent functions in the unit disc $U = U_1$.

1. Introduction

A family of analytic and univalent functions $L(\cdot, t) : U \to \mathbb{C}$, $t \in [0, \infty)$ is a Loewner chain if $L(0, s) = L(0, t)$ and $L(U, s) \subset L(U, t)$, whenever $0 \leq s < t < \infty$.

A basic result in the theory of Loewner chains is due to Ch. Pommerenke.

Theorem 1. [3] *Let $L(z, t) = a_1(t)z + \ldots$, be a function from $U_r \times [0, \infty)$ into $\mathbb{C}$ such that:*

(i) For each $t \geq 0$, $L(\cdot, t)$ is analytic in U_r

(ii) $L(z, t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in U_r$

(iii) $a_1(t) \neq 0$, for all $t \geq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t \geq 0}$ forms a normal family in U_r

(iv) There exists a family of functions $p(z, t)$ such that for all $t \geq 0$

$$\frac{\partial L(z,t)}{\partial t} = zp(z,t)\frac{\partial L(z,t)}{\partial z}, \quad z \in U_r \tag{1}$$

where for each $t \geq 0$, $p(\cdot, t)$ is analytic in U, for each $z \in U$, $p(z, t)$ is a measurable function of $t \in [0, \infty)$ and $\operatorname{Re} p(z, t) > 0$, $(z, t) \in U \times [0, \infty)$.

Then, for all $t \geq 0$ the function $L(z, t)$ has an analytic univalent extension in U.

T. Betker [2] introduced the notion of inverse Loewner chains. A family of analytic and univalent functions $\omega(\cdot, t) : U \to \mathbb{C}$, $t \geq 0$ is an inverse Loewner chain if $\omega(0, s) = \omega(0, t)$ and $\omega(U, s) \supset \omega(U, t)$, whenever $0 \leq s < t < \infty$.

If $\omega(z,t)$, $(z,t) \in U \times [0,\infty)$ is an inverse Loewner chain, then the family $\omega(z,-t)$ is a Loewner chain over $(-\infty, 0]$. It follows that the differential equation for an inverse Loewner chain has the form

$$\frac{\partial \omega(z,t)}{\partial t} = -zq(z,t)\frac{\partial \omega(z,t)}{\partial z}, \quad z \in U, \quad \text{a.e.} \quad t \geq 0.$$

where $\text{Re } q(z,t) > 0, \ z \in U, \ t \geq 0$.

2. Main results

By using Theorem 1 and the method of T. Betker [2], we obtain the following result concerning inverse Loewner chains.

Theorem 2. *Let $\omega(z,t) = b_1(t)z + \ldots$, be a function from $U_r \times [0,\infty)$ into $\mathbb{C}$ such that:*

(i) For each $t \geq 0$, $\omega(\cdot,t)$ is analytic in U_r and $\omega(z,0) = z, \ z \in U_r$

(ii) $\omega(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in U_r$

(iii) $\left\{\dfrac{\omega(z,t)}{b_1(t)}\right\}_{t\geq 0}$ forms a normal family in U_r

(iv) There exists a family of functions $q(z,t)$ such that for a.e. $t \geq 0$

$$\frac{\partial \omega(z,t)}{\partial t} = -zq(z,t)\frac{\partial \omega(z,t)}{\partial z}, \quad z \in U_r \tag{2}$$

where for each $t \geq 0$, $q(\cdot,t)$ is analytic in U, for each $z \in U$, $q(z,t)$ is a measurable function of $t \in [0,\infty)$ and $\text{Re } q(z,t) > 0, \ z \in U, \ t \geq 0$. Also, let $q(0,t)$ be locally integrable in $[0,\infty)$ and $\int_0^\infty \text{Re } q(0,t)dt = \infty$.

Then, for all $t \in [0,\infty)$ the function $\omega(z,t)$ has an analytic univalent extension in U.

Remark. If $\omega(z,t) = b_1(t)z + \ldots$ satisfies the differential equations (2), then

$$|b_1(t)| = e^{-\int_0^t \text{Re } q(0,\tau)d\tau}, \quad t \geq 0.$$

Hence $b_1(t) \neq 0$, for all $t \geq 0$ and $\lim_{t\to\infty} |b_1(t)| = 0$.

Proof. Let $0 < t_1 < t_2 < \ldots, \ \lim_{n\to\infty} t_n = \infty$.

We consider the function $L_n : U_r \times [0,\infty) \to \mathbb{C}$ defined by

$$L_n(z,t) = \begin{cases} \omega(z, t_b - t), & t \in [0, t_n] \\ e^{t-t_n} z, & t \in [t_n, \infty) \end{cases}$$

The function $L_n(z,t)$ satisfies the absolute continuity requirements of Theorem 1. Also, $L_n(z,t)$ satisfies the differential equation (1), where for all $z \in U$, $t \in [0,\infty)$

$$p_n(z,t) = \begin{cases} q(z, t_n - t), & t \in [0, t_n] \\ 1, & t \in (t_n, \infty). \end{cases}$$

We have

$$a_1(t)_n = \frac{\partial L_n(0,t)}{\partial z} = \begin{cases} b_1(t_n - t), & t \in [0, t_n] \\ e^{t-t_n}, & t \in [t_n, \infty). \end{cases}$$

Hence $a_1(t)_n \neq 0$ for all $t \geq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and $\left\{ \frac{L_n(z,t)}{a_1(t)_n} \right\}_{t\geq 0}$ forms a normal family in U_r.

The conditions of Theorem 1 being satisfied, we obtain that for all $t \geq 0$, the function $L_n(z,t)$ has an analytic univalent extension in U. The uniqueness of this extension results from the condition $L_n(z, t_n) = z$. □

We denote by $\mathcal{L}_n(z,t)$ the extension of $L_n(z,t)$ in U. Then for all $t \geq 0$, the function $\Omega(z,t) = \mathcal{L}(z, t_n - t)$, $z \in U$, $t \in [0, t_n]$ is the analytic and univalent extension of $\omega(z,t)$ in U.

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