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DATA DEPENDENCE OF THE FIXED POINTS SET OF MULTIVALUED WEAKLY PICARD OPERATORS IN UNIFORM SPACES

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Abstract. In this paper we study data dependence of the fixed points set for a special class of multivalued weakly Picard operators in uniform spaces.

0. Introduction

The aim of this paper is to study data dependence of the fixed points set for a class of multivalued weakly Picard operators in uniform spaces.

1. Notations

By (X, A) we mean a sequentially complete T_2-separated uniform space whose uniformity is generated by a saturated family of pseudometrics $A = \{\rho_a(x,y) : a \in A\}$, where A is an index set (cf. [8]). By $CB(X)$ we denote the family of all non-empty bounded closed subsets of X ([5], [3]). $CB(X)$ is regarded as a uniform space endowed with a family of Hausdorff-Pompeiu pseudodistances $H_A = \{H_a(M,L) : a \in A\}$ for $M, L \in CB(X)$, where

$$H_a(M,L) = \max\{E_a(M,L), E_a(L,M)\};$$

$E_a(M,L)$ is the excess of M over L, that is,

$$E_a(M,L) = \sup\{\rho_a(x,L) : x \in M\}, \quad \rho_a(x,L) = \inf\{\rho_a(x,y) : y \in L\} \quad \text{(see [4]).}$$

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If L is a compact set and $m_0 \in M$ is arbitrary chosen, then there exists $l_0 \in L$ such that $\rho_a(m_0, l_0) \leq H_a(M, L)$. The element $l_0 \in L$ depends on $a \in A$. The totality of all compact subsets of X we denote by $COM(X)$, while F_T is the set of all fixed points of the operator $T : X \to P(X)$, $P(X) = \{A : A \text{ is nonempty subset of } X\}$.

2. Preliminaries

We recall some basic statements.

Theorem 1. *If X is sequentially complete with respect to A, so are $CB(X)$ and $COM(X)$ with respect to H_A (cf. [5], [3]).*

Theorem 2. *If $\lim\limits_n x_n = x_0$ in (X, A), $\lim\limits_n M_n = M_0$ in $(CB(X), H_A)$ or in $(COM(X), H_A)$ and $x_n \in M_n$, $n \in N$ then $x_0 \in M_0$ ([4]).*

Introduce a family of comparison function $\{\Phi\} = \{\Phi_a(t) : a \in A\}$ with the properties:

(Φ1) $\Phi_a(t)$ is continuous from the right and $0 \leq \Phi_a(t) < t$ for $t > 0$.

(Φ2) $\Phi_a(t_1 + t_2) \leq \Phi_a(t_1) + \Phi_a(t_2)$ for $t_1, t_2 > 0$ and $a \in A$.

(Φ3) for each $a \in A$ there is a function $\overline{\Phi}_a(t) : R^1_+ \to R^1_+$ (satifying (Φ1), (Φ2)) which is strictly increasing and $\sum\limits_{n=0}^{\infty} n\overline{\Phi}^n_a(t) < \infty$ such that $\{\Phi_{j^n(a)}(t) : n = 0, 1, 2, \ldots\} \leq \overline{\Phi}_a(t)$ and $\dfrac{\overline{\Phi}_a(t)}{t}$ is nondecreasing.

Here $\overline{\Phi}^n_a(t)$ stands for the n-th iterate of $\overline{\Phi}_a$, i.e. $\overline{\Phi}^n_a(t) = \underbrace{\overline{\Phi}_a(\overline{\Phi}_a(\ldots\overline{\Phi}_a(t)\ldots))}_{n-times}$, while $j : A \to A$ is a mapping of the index set A into itself. Its iterates are defined as follows:

$$j^0(a) = a, \quad j^k(a) = j(j^{k-1}(a)), \quad (k = 1, 2, \ldots), \quad a \in A.$$

Remark 1. Throughout the remainder of the paper we assume that $\rho_{j(a)}(x, y) \leq \rho_a(x, y)$. In the theory of functional differential equations the delays generate such a map j which satisfies the above inequality [1].

Remark 2. The convergence of $\sum\limits_{n=0}^{\infty} \overline{\Phi}^n_a(t)$ implies a convergence of $\sum\limits_{n=0}^{\infty} n\overline{\Phi}^n_a(t)$. Indeed

$$\frac{\overline{\Phi}^{n+1}_a(t)}{\overline{\Phi}^n_a(t)} = \frac{\overline{\Phi}_a(\overline{\Phi}^n_a(t))}{\overline{\Phi}^n_a(t)} \leq \frac{\Phi_a(t)}{t} < 1.$$

Therefore for sufficiently large $n \Rightarrow \dfrac{(n+1)\overline{\Phi}_a^{n+1}(t)}{n\overline{\Phi}_a^{n}(t)} < 1$. A mapping $T : X \to COM(X)$ is said to be Φ-contractive if for every $x, y \in X$ the following inequality holds true: $H_a(T(x), T(y)) \leq \Phi_a(\rho_{j(a)}(x, y))$ for each fixed $a \in A$. It is known that if $M, L \in range(T)$ are compact sets then for every $m \in M$ there is $l \in L$ such that $\rho_a(m, l) \leq H_a(M, L)$. We need a slightly more restrictive condition introduced in [2]. Suppose the following condition is fulfilled.

(N) For every $M, L \in range(T) \subset COM(X)$ and $m \in M$, there exists $l \in L$ such that $\rho_a(m, l) \leq H_a(M, L)$ and for every $\varepsilon > 0$ there is $l_1 \in L$ such that $\rho_{j(a)}(m, l_1) \leq H_{j(a)}(M, L)$ and $\rho_{j(a)}(l, l_1) < \varepsilon$ for every $a \in A$.

Theorem 3. [2] *Let $T : X \to COM(X)$ be multivalued Φ-contractive mapping which satisfies condition (N). If there is $x_0 \in X$ and a constant $Q > 0$ such that $\rho_{j^n(a)}(x_0, x_1) \leq Q < \infty$ $(n = 0, 1, 2, \ldots)$ for every $x_1 \in T(x_0)$, then T has at least one fixed point in (X, A).*

We call a multivalued operator $T : X \to COM(X)$ U-weakly Picard one if (see [6] and [7] in the case of a metric space):

(i) T satisfies condition (N)

(ii) there is $x_0 \in X$ and $Q = const. > 0$ such that $\rho_{j^n(a)}(x_0, x_1) \leq Q$ $(n = 0, 1, \ldots)$ for every $x_1 \in T(x_0)$

(iii) the sequence $\{x_n\}_{n=1}^{\infty}$, where $x_{n+1} \in T(x_n)$, is convergent in (X, A) and its limit is a fixed point of T.

Introduce the sets:

$C_{T_k} = \{x_0 \in X :$ there exist a constant $Q_k > 0$ such that $\rho_{j^n(a)}(x_0, x_1) \leq Q_k$ $(n = 0, 1, 2, \ldots)$ for every $x_1 \in T_k(x_0)\}$.

3. The main result

In what follows we formulate the main result:

Theorem 4. *Let (X, A) be a sequentially complete Hausdorff uniform space and let $T_k : (X, A) \to (COM(X), H_A)$ $(k = 1, 1)$ be two-multivalued operators. If:*

4.1. there exist two families of comparison functions $(\Phi)^{(k)}$ $(k=1,2)$ such that

$$H_a(T_k(x),T_k(y)) \leq \Phi_a^{(k)}(\rho_{j(a)}(x,y)) \quad (k=1,2)$$

for all $x,y \in X$ and $a \in A$.

4.2. T_k $(k=1,2)$ satisfy condition (N).

4.3. the sets C_{T_k} $(k=1,2)$ are nonempty and $F_{T_1} \subset C_{T_2}$, $F_{T_2} \subset C_{T_1}$.

4.4. for every fixed $a \in A$ and all $x \in X$ there is a constant $\eta_n > 0$ such that $H_a(T_1(x),T_2(x)) \leq \eta_a$,

then:

1) F_{T_k} $(k=1,2)$ are closed subsets of X

2) T_k $(k=1,2)$ are U-weakly Picard operators and

$$H_a(F_{T_1},F_{T_2}) \leq \frac{\eta_a}{1-\max\left\{\sup\left\{\frac{\overline{\Phi}_a^{(1)}(t)}{t} : t>0\right\}, \sup\left\{\frac{\overline{\Phi}_a^{(2)}(t)}{t} : t>0\right\}\right\}} \tag{1}$$

Proof. From Theorem 3 follows that the set F_{T_k} are non-empty. We show that they are closed. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $\lim\limits_{n\to\infty} x_n = x_n^*$ in (X,A). Recall that $D_a(A,B)$ for every $A,B \in P(X)$ is defined in the following way: $D_a(A,B) = \inf\{\rho_a(x,y) : x \in A \text{ and } y \in B\}$.

Then for every fixed $a \in A$ we obtain

$$D_a(x^*,T_k(x^*)) \leq \rho_a(x^*,x_n) + D_a(x_n,T_k(x^*)) \leq \rho_a(x^*,x_n) + H_a(T_k(x_n),T_k(x^*)) \leq$$

$$\leq \rho_a(x^*,x_n) + \Phi_a(\rho_{j(a)}(x_n,x^*)) < \rho_a(x^*,x_n) + \rho_{j(a)}(x_n,x^*) \tag{2}$$

But the convergence in (X,A) means $\lim\limits_{n\to\infty} \rho_a(x^*,x_n) = 0$ for every $a \in A$. Therefore $\lim\limits_{n\to\infty} \rho_{j(a)}(x^*,x_n) = 0$ also and then $D_a(x^*,T(x^*)) = 0$ for every $a \in A$. This means that $x^* \in T_k(x^*)$, that is, $x^* \in F_{T_k}$. Consequently F_{T_k} is a closed set.

Now we form a sequence of successive approximations for T_2, starting from an arbitrary $x_0 \in F_{T_1}$ in view of 4.3, that is, $x_{n+1} \in T_2(x_n)$ and $\lim\limits_{n\to\infty} x_n = x^* \in F_{T_2}$ in view of Theorem 2. We also have (recalling that $T_k(x_0)$ are compact sets):

$$\rho_a(x_), x^*) \leq H_a(T_1(x_0),T_2(x^*)) \leq H_a(T_1(x_0),T_2(x_0)) + H_a(T_2(x_0),T_1(x^*)) \leq$$

$$\leq \eta_a + \overline{\Phi}_a^{(2)}(\rho_{j(a)}(x_0, x^*)) \leq \eta_a + \overline{\Phi}_a^{(2)}(\rho_a(x_0, x^*)) \leq$$

$$\leq \eta_a + \sup\left\{\frac{\overline{\Phi}_a^{(2)}(t)}{t} : t > 0\right\} \cdot \rho_a(x_0, x^*) \tag{3}$$

It follows that T_k are U-weakly Picard operators and

$$\rho_a(x_0, x^*) \leq \frac{\eta_a}{1 - \max\left\{\sup\left\{\frac{\overline{\Phi}_a^{(1)}(t)}{t} : t > 0\right\}, \sup\left\{\frac{\overline{\Phi}_a^{(2)}(t)}{t} : t > 0\right\}\right\}}$$

which implies (1).

In a similar way we choose $y_0 \in F_{T_2}$ and $y_1 \in T_1(y_0)$ and form a sequence $\{y_n\}_{n=0}^{\infty}$ of successive approximation for T_1 such that $\lim_{n\to\infty} y_n = y^* \in F_{T_1}$. Then (1) can be obtained proceeding as the above. □

In order to prove an analogous theorem for multivalued mappings with bounded and closed images we need a family of comparison function (Φ) with slightly more different properties. Namely, $(\Phi 1)$ and $(\Phi 2)$ are the same ones. We replace $(\Phi 3)$ by the following one:

$(\Phi 3 - 1)$ there exists a family (Ψ) of comparison functions $\phi_a(t) : R_a^1 \to R_a^1$ which possess properties $(\Phi 1)$ and $(\Phi 2)$ and $\Phi_a(t) < \Psi_a(t)$ for $t > 0$ $\left(\Psi_a(t) = \frac{1}{2}(t + \Phi_a(t))\right)$ and $\sup\{\Psi_{j^n(a)}(t) : n = 0, 1, \dots\} \leq \overline{\Psi}_a(t)$ where $\sum_{n=0}^{\infty} \overline{\Psi}_a^n(t) < \infty$ and $\frac{\overline{\Psi}_a(t)}{t}$ is non-decreasing.

Condition (N) can be reformulated in the following way:

(N1) Let $M, L \in range\, T$ be arbirtrarily chosen. For every $m \in M$, $\varepsilon > 0$ and $\xi_a > 0$ there exist $l, l_1 \in L$ such that $\rho_a(m, l) \leq H_a(M, L) + \xi_a$, $\rho_{j(a)}(m, l_1) \leq H_{j(a)}(M, L)$ and $\rho_{j(a)}(l, l_1) < \varepsilon$ for every $a \in A$.

(N2) Let $M, L \in range\, T$ be arbitrarily chosen. For every $m\hat{\imath}M$, $\varepsilon > 0$ and $\eta_a, \xi_a > 0$ there exist $l, l_1 \in L$ such that $\rho_a(m, l) \leq H_a(M, L) + \eta_a$, $\rho_{j(a)}(m, l_1)(M, L) + \xi_a$ and $\rho_{j(a)}(l, l_1) < \varepsilon$ for every $a \in A$.

Obviously (N2) is more general than condition (N1) from [2].

Remark 3. The inequality $\rho_a(m, l) \leq H_a(M, L) + \xi_a$ can be written in the form $\rho_a(m, l) \leq q_a H_a(M, L)$ with $q_a > 1$. Indeed, for a given $\xi_a > 0$, q_a can be determined from the equation $H_a(M,L)+\xi_a = q_a H_a(M,L)$, that is, $q_a = \dfrac{H_a(M,L)+\xi_a}{H_a(M,L)} > 1$, provided $H_a(M,L) > 0$. Conversely, for a given $q_a > 1$ we have $\xi_a = (q_a - 1)H_a(M,L)$.

Theorem 5. [2] *If $T : X \to CB(X)$ is Φ-contractive, satisfies (N1) and there is $x_0 \in X$ and $Q > 0$ such that $\rho_{j^n(a)}(x_0, x_1) \le Q < \infty$ $(n = 0,1,2,\dots)$ for every $x_1 \in T_{x_0}$, then T has a fixed point in X.*

Theorem 6. *Let (X, A) be a sequentially complete Hausdorff uniform space and let $T_k : X \to CB(X)$ $(k = 1,2)$ be multi-valued operators. If:*

6.1. T_k are Φ-contrative, i.e. $H_a(T_k(x), T_k(y)) \le \Phi_a^{(k)}(\rho_{j(a)}(x,y))$ for all $x, y \in X$, $a \in A$, where $(\Phi)^{(k)}$ being two families of comparison functions $(k = 1,2)$.

6.2. T_k $(k = 1,2)$ satisfy condition (N1).

6.3. the sets C_{T_k} $(k = 1,2)$ are nonempty and $F_{T_1} \subset C_{T_2}$, $F_{T_2} \subset C_{T_1}$.

6.4. for every fixed $a \in A$ and all $x \in X$ there is a constant $\eta_a > 0$ such that $H_a(T_1(x), T_2(x)) \le \eta_a$,

then:

1) F_{T_k} $(k = 1,2)$ are closed subsets of X.

2) T_k are U-weakly Picard operators and

$$H_a(F_{T_1}, F_{T_2}) \le \frac{\eta_a}{1 - \max\left\{\sup\left\{\dfrac{\overline{\Psi}_a^{(1)}(t)}{t} : t > 0\right\}, \sup\left\{\dfrac{\overline{\Psi}_a^{(2)}(t)}{t} : t > 0\right\}\right\}}$$

Proof. We give those elements which essentially differs from the proof of Theorem 4.

Let us consider the analog of (3).

$$\rho_a(x_), x^*) \le q_a H_a(T_1(x_0), T_2(x^*) \le q_a H_a(T_1(x_0), T_2(x_0)) + q_a H_a(T_2(x_0), T_2(x^*)) \le$$

$$\le q_a \eta_a + q_a \overline{\Psi}_a^{(2)}(\rho_{j(a)}(x_0, x^*)) \le q_a \eta_a + q_a \sup\left\{\frac{\overline{\Psi}_a^{(2)}(t)}{t} : t > 0\right\} \cdot \rho_a(x_0, x^*)$$

or

$$\rho_a(x_0, x^*) \le \frac{q_a \eta_a}{1 - q_a \sup\left\{\dfrac{\overline{\Psi}_a^{(2)}(t)}{t} : t > 0\right\}}.$$

When $q_a \to 1$ we obtain the required conclusion. □

Remark 4. In the case of metric space, from our results, we have some results in [7].

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ON INEQUALITIES IN NORMED LINEAR SPACES AND APPLICATIONS

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Abstract. In this paper we obtain inequalities involving the norms of finite sequences of vectors in normed linear spaces, which are new even for complex numbers.

1. Introduction

Let $(X, \|\cdot\|)$ be a normed linear space. The following inequality is well known in the literature as the polygonal inequality, which is a generalization of the triangle inequality:

$$\Big\|\sum_{i\in I} x_i\Big\| \leq \sum_{i\in I} \|x_i\|; \tag{P}$$

I is a finite set of indices and x_i $(i \in I)$ are vectors in X. In papers [1–3], authors explored relations between the following refinements and generalizations of (P):

Theorem A. *Let $(X, \|\cdot\|)$ be a normed linear space and I be a finite set of indices, $x_i \in X$, $i \in I$ and $p \in \mathbf{R}$, $p \geq 1$. Then we have the following inequalities:*

$$\begin{aligned}
\Big\|\sum_{i\in I} x_i\Big\|^p &\leq [\mathrm{card}(I)]^{p-k-1} \sum_{i_1,\ldots,i_{k+1}\in I} \left\|\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right\|^p \\
&\leq [\mathrm{card}(I)]^{p-k} \sum_{i_1,\ldots,i_k\in I} \left\|\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right\|^p \\
&\leq \cdots \\
&\leq [\mathrm{card}(I)]^{p-2} \sum_{i_1,i_2\in I} \left\|\frac{x_{i_1}+x_{i_2}}{2}\right\|^p \\
&\leq [\mathrm{card}(I)]^{p-1} \sum_{i\in I} \|x_i\|^p,
\end{aligned} \tag{1}$$

where $k \in N$, $k \geq 1$.

The following result is a refinement of (P) for weighted mean:

Theorem B. *With the above assumptions, if g_j $(j = 1, 2, 3, \ldots, k)$ are nonnegative weights such that $\sum_{j=1}^{k} g_j > 0$, then we have the following inequality:*

$$\begin{aligned}\Big\|\sum_{i\in I} x_i\Big\|^p &\le [\mathrm{card}(I)]^{p-k} \sum_{i_1,\ldots,i_k\in I} \Big\|\frac{x_{i_1}+\cdots+x_{i_k}}{k}\Big\|^p \\ &\le [\mathrm{card}(I)]^{p-k} \sum_{i_1,\ldots,i_k\in I} \Big\|\frac{g_1x_{i_1}+\cdots+g_kx_{i_k}}{g_1+\cdots+g_k}\Big\|^p \\ &\le [\mathrm{card}(I)]^{p-1} \sum_{i\in I} \|x_i\|^p. \qquad (2)\end{aligned}$$

Note that both the above inequalities were obtained from more general results for convex mappings defined on convex subsets in normed linear spaces.

In this paper, using only techniques from normed linear spaces, we obtain new inequalities for the norms of finite sequences.

2. The Results

We start with the following theorem:

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_i \in X$ with $i \in I$ (I is finite) and $x = \sum_{i\in I} x_i$. Then we have the following inequalities:*

$$\Big\|\|x\|x - \sum_{i\in I}\|x \pm x_i\|x_i\Big\| \le \sum_{i\in I}\|x_i\|^2 \le \tfrac{1}{2}\Big(\|x\| + \sum_{i\in I}\|x_i\|\Big)\sum_{i\in I}\|x_i\|. \qquad (3)$$

Proof. For a fixed $i \in I$, we have

$$\begin{aligned}\|x_i\| = \Big\|x - \sum_{j\in I, j\ne i} x_j\Big\| &\le \|x\| + \Big\|\sum_{j\in I, j\ne i} x_j\Big\| \\ &\le \|x\| + \sum_{j\in I, j\ne i}\|x_j\| = \|x\| + \sum_{j\in I}\|x_j\| - \|x_i\|,\end{aligned}$$

which implies that, for all $i \in I$,

$$\|x_i\| \le \tfrac{1}{2}\Big(\|x\| + \sum_{j\in I}\|x_j\|\Big).$$

Multiplying by $\|x_i\| \geq 0$ and summing over $i \in I$, we deduce

$$\sum_{i\in I} \|x_i\|^2 \leq \tfrac{1}{2}\Big(\|x\| + \sum_{j\in I}\|x_j\|\Big)\sum_{i\in I}\|x_i\|,$$

and so the second inequality in (1) is proved. Also, for a fixed $i \in I$ we have

$$\|x_i\| = \|-x_i\| = \|x \pm x_i - x\| \geq \Big|\|x \pm x_i\| - \|x\|\Big|.$$

Multiplying by $\|x_i\| \geq 0$, we have

$$\|x_i\|^2 \geq \Big|\|x \pm x_i\| - \|x\|\Big|\,\|x_i\| = \Big\|(\|x \pm x_i\| - \|x\|)x_i\Big\|$$

for all $i \in I$. Summing over $i \in I$, we have

$$\sum_{i\in I}\|x_i\|^2 \geq \sum_{i\in I}\Big\|\|x \pm x_i\|x_i - \|x\|x_i\Big\| \geq \Big\|\sum_{i\in I}\|x \pm x_i\|x_i - \|x\|x_i\Big\|,$$

and so the theorem is proved. □

Corollary 2.2. *With the above assumptions, if $\sum_{i\in I} x_i = 0$, then*

$$\sum_{i\in I}\|x_i\|^2 \leq \tfrac{1}{2}\Big(\sum_{i\in I}\|x_i\|\Big)^2. \tag{4}$$

The following result generalizes Theorem 2.1. In its proof we need the formula

$$\sum_{i_1,\ldots,i_k\in I}(x_{i_1} + \cdots + x_{i_k}) = k[\mathrm{card}(I)]^{k-1}\sum_{i\in I} x_i, \tag{5}$$

which can be proved by induction on k.

Theorem 2.3. *With the above assumptions, we have the following inequalities:*

$$\begin{aligned}
k\Big\|&\sum_{i_1,\ldots,i_k\in I}\|x \pm (x_{i_1} + \cdots + x_{i_k})\|x_{i_1} - n^{k-1}\|x\|x\Big\| \\
&\leq \sum_{i_1,\ldots,i_k\in I}\|x_{i_1} + \cdots + x_{i_k}\|^2 \\
&\leq \tfrac{1}{2}\Big(\sum_{i_1,\ldots,i_k\in I}\|x_{i_1} + \cdots + x_{i_k}\|^2 + k\sum_{i_1,\ldots,i_k\in I}\|x_{i_1}\|\,\|x_{i_1} + \cdots + x_{i_k}\|\Big) \\
&\leq \tfrac{1}{2}\Big(\|x\| + \sum_{i\in I}\|x_i\|\Big)\sum_{i_1,\ldots,i_k\in I}\|x_{i_1} + \cdots + x_{i_k}\|,
\end{aligned} \tag{6}$$

where $k \geq 1$ *and* $\mathrm{card}(I) = n$.

Proof. For fixed $i_1, \dots, i_k \in I$, we have

$$\begin{aligned}
\|x_{i_1} + \cdots + x_{i_k}\| &= \Big\| x - \sum_{j \notin \{i_1,\dots,i_k\}} x_j \Big\| \\
&\le \|x\| + \Big\| \sum_{j \notin \{i_1,\dots,i_k\}} x_j \Big\| \\
&\le \|x\| + \sum_{j \notin \{i_1,\dots,i_k\}} \|x_j\| \\
&= \|x\| + \sum_{i \in I} \|x_i\| - \|x_{i_1}\| - \cdots - \|x_{i_k}\|,
\end{aligned}$$

from which it follows

$$\|x_{i_1} + \cdots + x_{i_k}\| + \|x_{i_1}\| + \cdots + \|x_{i_k}\| \le \|x\| + \sum_{i \in I} \|x_i\|.$$

Multiplying by $\|x_{i_1} + \cdots + x_{i_k}\| \ge 0$ and summing over $i_1, \dots, i_k$ in I, we have

$$\begin{aligned}
\sum_{i_1,\dots,i_k \in I} \|x_{i_1} + \cdots + x_{i_k}\|^2 &+ k \sum_{i_1,\dots,i_k \in I} \|x_{i_1}\| \, \|x_{i_1} + \cdots + x_{i_k}\| \\
&\le \Big(\|x\| + \sum_{i \in I} \|x_i\| \Big) \sum_{i_1,\dots,i_k \in I} \|x_{i_1} + \cdots + x_{i_k}\|,
\end{aligned}$$

and so the third inequality in (6) follows. By the triangle inequality

$$\|x_{i_1} + \cdots + x_{i_k}\| \le \tfrac{1}{2}(\|x_{i_1} + \cdots + x_{i_k}\| + \|x_{i_1}\| + \cdots + \|x_{i_k}\|).$$

Multiplying by $\|x_{i_1} + \cdots + x_{i_k}\|$ and summing over $i_1, \dots, i_k$ in the index set I, we derive

$$\begin{aligned}
&\sum_{i_1,\dots,i_k \in I} \|x_{i_1} + \cdots + x_{i_k}\|^2 \\
&\qquad \le \tfrac{1}{2} \Big(\sum_{i_1,\dots,i_k \in I} \|x_{i_1} + \cdots + x_{i_k}\|^2 + k \sum_{i_1,\dots,i_k \in I} \|x_{i_1}\| \|x_{i_1} + \cdots + x_{i_k}\| \Big)
\end{aligned}$$

and so the second inequality in (6) is also proved.

Finally, we prove the first inequality in (6). Since

$$\begin{aligned}
\|x_{i_1} + \cdots + x_{i_k}\| &= \| - (x_{i_1} + \cdots + x_{i_k})\| \\
&= \|x \pm (x_{i_1} + \cdots + x_{i_k}) - x\| \\
&\ge \Big| \|x \pm (x_{i_1} + \cdots + x_{i_k})\| - \|x\| \Big|,
\end{aligned}$$

then

$$\|x_{i_1}+\cdots+x_{i_k}\|^2 \ge \Big\|(\|x\pm(x_{i_1}+\cdots+x_{i_k})\|-\|x\|)(x_{i_1}+\cdots+x_{i_k})\Big\|$$
$$= \Big\|\|x\pm(x_{i_1}+\cdots+x_{i_k})\|(x_{i_1}+\cdots+x_{i_k})-\|x\|(x_{i_1}+\cdots+x_{i_k})\Big\|.$$

Summing over $i_1,\ldots,i_k$ in the index set I, we use (5) to obtain

$$\begin{aligned}
&\sum_{i_1,\ldots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|^2\\
&\ge \sum_{i_1,\ldots,i_k\in I}\Big\|\|x\pm(x_{i_1}+\cdots+x_{i_k})\|(x_{i_1}+\cdots+x_{i_k})-\|x\|(x_{i_1}+\cdots+x_{i_k})\Big\|\\
&\ge \Big\|k\sum_{i_1,\ldots,i_k\in I}\|x\pm(x_{i_1}+\cdots+x_{i_k})\|x_{i_1}-kn^{k-1}\|x\|\sum_{i\in I}x_i\Big\|\\
&= k\Big\|\sum_{i_1,\ldots,i_k\in I}\|x\pm(x_{i_1}+\cdots+x_{i_k})\|x_{i_1}-n^{k-1})\|x\|x\Big\|
\end{aligned}$$

and thus the theorem is proved. □

Corollary 2.4. *With the above assumptions, if* $\sum_{i\in I}x_i=0$, *then*

$$\begin{aligned}
&k\Big\|\sum_{i_1,\ldots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|x_{i_1}\Big\|\\
&\le \sum_{i_1,\ldots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|^2\\
&\le \tfrac{1}{2}\Big(\sum_{i_1,\ldots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|^2+k\sum_{i_1,\ldots,i_k\in I}\|x_{i_1}\|\,\|x_{i_1}+\cdots+x_{i_k}\|\Big)\\
&\le \tfrac{1}{2}\sum_{i\in I}\|x_{i_1}\|\sum_{i_1,\ldots,i_k\in I}\|x_{i_1}+\cdots+x_{i_k}\|. \qquad (7)
\end{aligned}$$

3. Applications to Complex Numbers

The inequalities obtained in Section II can be restated to obtain new inequalities for complex numbers:

1. Let $z_i\in C$, $i\in I$ (I is finite) and let $z=\sum_{i\in I}z_i$. Then

$$\Big||z|z-\sum_{i\in I}|z\pm z_i|z_i\Big|\le\sum_{i\in I}|z_i|^2\le\tfrac{1}{2}\Big(|z|+\sum_{i\in I}|z_i|\Big)\sum_{i\in I}|z_i|.$$

In particular, if $z=0$, then

$$\sum_{i\in I}|z_i|^2\le\tfrac{1}{2}\Big(\sum_{i\in I}|z_i|\Big)^2.$$

2. With the above assumptions, we have

$$
\begin{aligned}
&k\Big|\sum_{i_1,\dots,i_k\in I}|z\pm(z_{i_1}+\dots+z_{i_k})|z_{i_1}-n^{k-1}|z|z\Big| \\
&\leq \sum_{i_1,\dots,i_k\in I}(z_{i_1}+\dots+z_{i_k})^2 \\
&\leq \tfrac{1}{2}\Big(\sum_{i_1,\dots,i_k\in I}|z_{i_1}+\dots+x_{i_k}|^2+k\sum_{i_1,\dots,i_k\in I}|z_{i_1}||z_{i_1}+\dots+z_{i_k}|\Big) \\
&\leq \tfrac{1}{2}\Big(|z|+\sum_{i\in I}|z_i|\Big)\sum_{i_1,\dots,i_k\in I}|z_{i_1}+\dots+z_{i_k}|,
\end{aligned}
$$

where $k\in N$, $k\geq 1$ and $\mathrm{card}(I)=n$.

In particular, if $z=0$, we have

$$
\begin{aligned}
k\Big|\sum_{i_1,\dots,i_k\in I}|z_{i_1}+\dots+z_{i_k}|z_{i_1}\Big| &\leq \sum_{i_1,\dots,i_k\in I}|z_{i_1}+\dots+z_{i_k}|^2 \\
&\leq \tfrac{1}{2}\Big(\sum_{i_1,\dots,i_k\in I}|z_{i_1}+\dots+x_{i_k}|^2+k\sum_{i_1,\dots,i_k\in I}|z_{i_1}(z_{i_1}+\dots+z_{i_k})|\Big) \\
&\leq \tfrac{1}{2}\sum_{i\in I}|z_i|\sum_{i_1,\dots,i_k\in I}|z_{i_1}+\dots+z_{i_k}|.
\end{aligned}
$$

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SUBCLASSES OF PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. This work presents some properties of functions α - prestarlike of order β with negative coefficients, the class of these functions note by $R[\alpha,\beta]$. The author obtain a coefficient inequality; theorems which show us that the classes $R[\alpha,\beta]$ and $C[\alpha,\beta]$ are closed under "arithmetic mean"; radii of starlikeness and convexity of order δ for $R[\alpha,\beta]$; and a connection between the classes $R[\alpha,\beta]$ and $C[\alpha,\beta]$.

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$.

A function $f(z)$ of S is said to be starlike of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U) \tag{2}$$

for some α $(0 \le \alpha < 1)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$.

A function $f(z)$ of S is said to be convex of order α if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U) \tag{3}$$

for some α $(0 \le \alpha < 1)$. We denote the class of all convex functions of order α by $K(\alpha)$.

Now, the function

$$s_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{4}$$

is the well-known extremal function for $S^*(\alpha)$.

Seting

$$C(\alpha, n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!} \qquad (n \geq 2), \tag{5}$$

$s_\alpha(z)$ can be written in the form

$$s_\alpha(z) = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n. \tag{6}$$

We can see that $C(\alpha, n)$ is decreasing function in α and satisfies

$$\lim_{n\to\infty} C(\alpha, n) = \begin{cases} \infty & \text{if} \quad \alpha < \frac{1}{2} \\ 1 & \text{if} \quad \alpha = \frac{1}{2} \\ 0 & \text{if} \quad \alpha > \frac{1}{2} \end{cases}. \tag{7}$$

Let $(f * g)(z)$ denote the Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $R(\alpha, \beta)$ a subclass of analytic functions of the form (1), which satisfies $(f * s_\alpha)(z) \in S^*(\beta)$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. Futher let $C(\alpha, \beta)$ a subclass of analytic functions of the form (1), satisfying $zf'(z) \in R(\alpha, \beta)$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$.

$R(\alpha, \beta)$ is called the class of functions α-prestarlike of order β and was introduced by Sheil-Small, Silverman and Silvia [7].

The class $R(\alpha, \alpha) = R(\alpha) = R_\alpha$, called class of functions prestarlike of order α, was introduced by Ruscheweyh [4] and has been studied by several other authors (see [1] and [8]).

In [8] it is shows that $R_\alpha \subset S$ if and only if $\alpha \leq \frac{1}{2}$.

In [10] it is shown that

$$R_\alpha \subset R_\beta \qquad \text{for} \qquad \alpha < \beta < 1.$$

We can observ that for $\alpha = \frac{1}{2}$ and $0 \leq \beta < 1$ we get $R(\frac{1}{2}, \beta) = S^*(\beta)$, the class of starlike functions of order β, and $C(\frac{1}{2}, \beta) = K(\beta)$, the class of convex functions of order β.

Let T denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0), \tag{8}$$

which are analytic in the unit disc U.

We denote $R[\alpha, \beta]$ and $C[\alpha, \beta]$ the classes obtained by taking intersections, respectively, of the classes $R(\alpha, \beta)$ and $C(\alpha, \beta)$ with T, that is

$$R[\alpha, \beta] = R(\alpha, \beta) \cap T$$

and

$$C[\alpha, \beta] = C(\alpha, \beta) \cap T.$$

The class $R[\alpha, \beta]$ was studied by Silverman and Silvia [9], Sarangi and Uralegaddi [6]; and the class $C[\alpha, \beta]$ was studied by Owa and Uralegaddi [3].

Lemma 1 ([9]). *Let the function $f(z)$ be defined by (8). Then $f(z)$ is in the class $R[\alpha, \beta]$ if and only if*

$$\sum_{n=2}^{\infty} (n - \beta) C(\alpha, n) a_n \leq 1 - \beta.$$

The result is sharp for the function

$$f(z) = z - \frac{1 - \beta}{2(1 - \alpha)(2 - \beta)} z^2.$$

Lemma 2 ([3]). *Let the function $f(z)$ be defined by (8). Then $f(z)$ is in the class $C[\alpha, \beta]$ if and only if*

$$\sum_{n=2}^{\infty} n(n - \beta) C(\alpha, n) a_n \leq 1 - \beta.$$

The result is sharp for the function

$$f(z) = z - \frac{1 - \beta}{4(1 - \alpha)(2 - \beta)} z^2.$$

Theorem 1. *Let a function $f(z)$ defined by (8) be in the class $R[\alpha,\beta]$. Then we have*

$$a_n \leq \frac{1-\beta}{(n-\beta)C(\alpha,n)}$$

for $n \geq 2$.

Equality holds for the function

$$f(z) = z - \frac{1-\beta}{(n-\beta)C(\alpha,n)} z^n \qquad (n \geq 2).$$

The proof results immediately from Lemma 1.

Theorem 2 ([3]). *Let a function $f(z)$ defined by (8) be in the class $C[\alpha,\beta]$. Then we have*

$$a_n \leq \frac{1-\beta}{n(n-\beta)C(\alpha,n)}$$

for $n \geq 2$.

Equality holds for the function

$$f(z) = z - \frac{1-\beta}{n(n-\beta)C(\alpha,n)} z^n \qquad (n \geq 2).$$

The following two theorems show us that the classes $R[\alpha,\beta]$ and $C[\alpha,\beta]$ are closed under "arithmetic mean".

Theorem 3. *If $f(z)$ defined by (8) and $g(z)$ defined by*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geq 0) \tag{9}$$

are in the class $R[\alpha,\beta]$, then

$$h(z) = z - \frac{1}{2}\sum_{n=2}^{\infty}(a_n + b_n) z^n \tag{10}$$

is also in $R[\alpha,\beta]$.

Proof. From f and g beeing in $R[\alpha,\beta]$, we have

$$\sum_{n=2}^{\infty}(n-\beta)C(\alpha,n)a_n \leq 1-\beta \tag{11}$$

and

$$\sum_{n=2}^{\infty}(n-\beta)C(\alpha,n)b_n \leq 1-\beta. \tag{12}$$

It is sufficient, for $h(z)$, to be a member of $R[\alpha,\beta]$, to show that

$$\frac{1}{2}\sum_{n=2}^{\infty}(n-\beta)C(\alpha,n)(a_n+b_n) \leq (1-\beta),$$

which will follow immediately from inequalities (11) and (12). □

Theorem 4. *If $f(z)$ defined by (8) and $g(z)$ defined by (9) are in the class $C[\alpha,\beta]$, then $h(z)$ defined by (10) is also in the class $C[\alpha,\beta]$.*

The proof follows immediately by appealing to Lemma 2.

Since $f(z)$ defined by (8) is univalent in the unit disc if $\sum_{n=2}^{\infty} na_n \leq 1$; we can see that $f \in R[\alpha,\beta]$ belongs to the class S if $0 \leq \alpha \leq \frac{1}{2}$.

Theorem 5. *Let the function $f(z)$ defined by (8) be in the class $R[\alpha,\beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$. Then f is starlike of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{n\geq 2}\left\{\frac{(n-\beta)(1-\delta)C(\alpha,n)}{(1-\beta)(n-\delta)}\right\}^{\frac{1}{n-1}}.$$

The result is sharp.

Proof. We employ the same technique as used by Sarangi and Uralegaddi ([5]). We note that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}} \leq 1-\delta$$

if and only if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1} \leq 1.$$

By virtue of Lemma 1, we need only find values of $|z|$ for which

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \leq \frac{(n-\beta)C(\alpha,n)}{1-\beta} \qquad (n \geq 2);$$

which will be true when $|z| < r_1$.

Further, we can see that the result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{(n-\beta)C(\alpha,n)} z^n \qquad (n \geq 2). \tag{13}$$

$\square$

Corollary 1. *Let $f(z)$ defined by (8) be in the class $R[\alpha,\beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$. Then f is univalent and starlike for $|z| < r_2$, where*

$$r_2 = \inf_{n\geq 2} \left\{ \frac{(n-\beta)C(\alpha,n)}{n(1-\beta)} \right\}^{\frac{1}{n-1}}.$$

The result is sharp.

Proof. In Theorem 5 we put $\delta = 0$. $\square$

Theorem 6. *Let a function $f(z)$ defined by (8) be in the class $R[\alpha,\beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$. Then $f(z)$ is convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_3$, where*

$$r_3 = \inf_{n\geq 2} \left\{ \frac{(n-\beta)(1-\delta)C(\alpha,n)}{n(1-\beta)(n-\delta)} \right\}^{\frac{1}{n-1}}.$$

The result is sharp for the function $f(z)$ given by (13).

Proof. Note that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} na_n|z|^{n-1}} \leq 1-\delta$$

if and only if

$$\sum_{n=2}^{\infty} \frac{n(n-\delta)}{1-\delta} a_n |z|^{n-1} \leq 1.$$

We need to find values of $|z|$ for which

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1} \leq \frac{(n-\beta)C(\alpha,n)}{1-\beta} \qquad (n \geq 2);$$

which will be true if $|z| < r_3$. $\square$

Corollary 2. *Let $f(z)$ defined by (8) be in the class $R[\alpha,\beta]$ with $0 \le \alpha \le \frac{1}{2}$ and $0 \le \beta < 1$. Then $f(z)$ is univalent and convex for $|z| < r_4$, where*

$$r_4 = \inf_{n\geq 2} \left\{ \frac{(n-\beta)C(\alpha,n)}{n^2(1-\beta)} \right\}^{\frac{1}{n-1}}.$$

The result is sharp.

Proof. In Theorem 6. we put $\delta = 0$. □

Theorem 7 ([9]). *If $f(z)$ defined by (8) is in the class $R[\alpha,\beta]$ with $0 \le \alpha \le \frac{1}{2}$; $0 \le \beta < 1$; then f is starlike of order*

$$\frac{2-2\alpha(2-\beta)}{3-\beta-2\alpha(2-\beta)}.$$

The result is sharp with the extremal function

$$f(z) = z - \frac{1-\beta}{2(2-\beta)(1-\alpha)} z^2.$$

Theorem 8. *Let $f(z)$ a function defined by (8) be in the class $C[\alpha,\beta]$. Then f belongs to the class $R[\alpha,\gamma]$, where*

$$\gamma = \frac{2}{3-\beta}.$$

The result is sharp for the extremal function

$$f(z) = z - \frac{1-\beta}{4(1-\alpha)(2-\beta)} z^2. \tag{14}$$

Proof. From

$$f \in C[\alpha,\beta] \Leftrightarrow \sum_{n=2}^{\infty} \frac{n(n-\beta)C(\alpha,n)}{1-\beta} a_n \le 1$$

and

$$f \in R[\alpha,\gamma] \Leftrightarrow \sum_{n=2}^{\infty} \frac{n(n-\gamma)C(\alpha,n)}{1-\gamma} a_n \le 1$$

results that we have to find the largest γ such that

$$\frac{(n-\gamma)C(\alpha,n)}{1-\gamma} \le \frac{n(n-\beta)C(\alpha,n)}{1-\beta},$$

which is equivalent to

$$\gamma \leq 1 - \frac{(n-1)(1-\beta)}{n(n-\beta)-(1-\beta)}.$$

Let the function $s(n)$ defined by

$$s(n) = 1 - \frac{(n-1)(1-\beta)}{n(n-\beta)-(1-\beta)}.$$

From $s(n) \leq s(n+1)$ for $n \geq 2$ results that $s(n)$ is an increasing function of n. Therefore we have

$$\gamma = s(2) = \frac{2}{3-\beta}.$$

Equality holds for the function given by (14). □

Corollary 3. *Let a function $f(z)$ be defined by (8) be in the class $C[\alpha,\beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$. Then f belongs to the class $S^*(\mu)$, where*

$$\mu = 1 - \frac{1-\beta}{4(1-\alpha)(2-\beta)-(1-\beta)}. \tag{15}$$

The result is sharp for the function $f(z)$ given by (14).

Proof. Using Theorem 8, we have

$$f \in R[\alpha,\gamma], \qquad \text{where} \qquad \gamma = \frac{2}{3-\beta};$$

and from Theorem 7 results that

$$f \in S^*(\mu), \qquad \text{where} \qquad \mu = \frac{2-2\alpha(2-\gamma)}{3-\gamma-2\alpha(2-\gamma)}.$$

After some easy calculus we obtain for γ the expression given by (15). □

Corollary 4. *Let a function $f(z)$ be defined by (8) convex of order β $(0 \leq \beta < 1)$. Then f is starlike of order $\frac{2}{3-\beta}$.*

Proof. If in Theorem 8 or in Corollary 3 we put $\alpha = \frac{1}{2}$, we obtain our result. □

Corollary 5. *A convex function with negative coefficients is starlike of order $\frac{2}{3}$.*

Proof. In Corollary 4 taking $\beta = 0$ we have our result. □

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AVERAGING OF NEUTRAL DIFFERENTIAL INCLUSIONS WHEN THE MIDDLING OF THE RIGHT-HAND SIDE DOES NOT EXIST

TERESA JANIAK AND ELZBIETA LUCZAK-KUMOREK

Abstract. We consider the problem of application of the averaging method to the asymptotic approximation of solutions of functional-differential equations of neutral type $\dot{x}(t) \in F(t, x_t, \dot{x}_t)$ in the case where the average of the right-hand side does not exist. Our paper generalized results of W.A. Plotnikow, W.M. Sawczenko from ([8]) where the generalized system $\dot{x}(t) \in F(t, x)$ was investigated.

1. Introduction and notations

There has recently been a great deal of interest in the field of Bogolubov's type theorems. Many authors [1,5,6,7] have discussed the middling theorem for differential equations or functional-differential inclusions.

In this paper we shall study the functional-differential inclusions of the form:

$$\begin{cases} \dot{x}(t) \in \varepsilon F(t, x_t, \dot{x}_t) & \text{for a.e} \quad t \geq 0 \\ x(t) = \varphi(t) & \text{for} \qquad t \in [-r, 0] \end{cases} \tag{1}$$

where F is a multifunction with values that are nonempty compact convex subsets of n-dimensional Euclidean space R^n, $\varphi : [-r, 0] \to R^n$ is a given absolutely continuous function and $\varepsilon > 0$ is a small parameter.

For a given function $u : [-r, T] \to R^n$ and fixed $t \in [0, T]$, we denote $u_t(s) = u(t + s)$ for $s \in [-r, 0]$, $r \geq 0$, $T > 0$.

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In the method of averaging [3] we considered (1) together with the middling inclusions:

$$\begin{cases} \dot{y}(t) \in \varepsilon F_0(y_t, \dot{y}_t) & \text{for a.e} \quad t \geq 0 \\ y(t) = \varphi(t) & \text{for} \quad t \in [-r, 0] \end{cases} \tag{2}$$

where existed a limit

$$F_0(x_t, \dot{x}_t) = \lim_{L \to \infty} \frac{1}{L} \int_0^L F(t, x_t, \dot{x}_t) dt \tag{3}$$

uniformly with respect to $(x_t, \dot{x}_t) \in C_0 \times L_0$ where C_0 and L_0 denote the Banach spaces of all continuous and Lebesgue integrable functions, respectively, of $[-r, 0]$ into R^n with the norms $\|x\|_0 = \sup_{-r \leq t \leq 0} |x(t)|$ and $|y|_0 = \int_{-r}^0 |y(t)| dt$ for $x \in C_0$ and $y \in L_0$, respectively, where $|\cdot|$ denotes the Euclidean norm. The integral in the limit (3) is meant in Aumann's-Hukuhara's sense.

The aim of this paper is to consider the problem of application of the averaging method to the asymptotic approximation of solution of (1) in the case where the limit (3) does not exist.

Let $F^-, F^+ : C_0 \times L_0 \to convR^n$ denote a multifunctions such that for every $\eta > 0$ there exists a $L^0 > 0$ that for $L > L^0$ the following inclusions are satisfied

$$\begin{aligned} &F^-(x_t, \dot{x}_t) \subset \frac{1}{L} \int_0^L F(t, x_t, \dot{x}_t) dt + S_\eta(0) \\ &\frac{1}{L} \int_0^L F(t, x_t, \dot{x}_t) dt \subset F^+(x_t, \dot{x}_t) + S_\eta(0) \end{aligned} \tag{4}$$

We shall consider (1) together with the following inclusions:

$$\begin{cases} y(t) = \varphi(t) & \text{for} \quad t \in [-r, 0] \\ \dot{y}(t) \in \varepsilon F^-(y_t, \dot{y}_t) & \text{for a.e.} \quad t \geq 0 \end{cases} \tag{5}$$

and

$$\begin{cases} z(t) = \varphi(t) & \text{for} \quad t \in [-r, 0] \\ \dot{z}(t) \in \varepsilon F^-(z_t, \dot{z}_t) & \text{for a.e.} \quad t \geq 0 \end{cases} \tag{6}$$

Finally, let us denote by $(compR^n, H)$ and $(convR^n, H)$ the metric spaces of all nonempty compact and nonempty compact, convex, respectively, subsets of n-dimensional Euclidean space R^n with the Hausdorff metric H.

2. The main result

Let $F : [0,\infty) \times C_0 \times L_0 \to convR^n$ satisfy the following conditions:

(a) $F(\cdot, u, v) : [0,\infty) \to convR^n$ is measurable for fixed $(u, v) \in C_0 \times L_0$

(b) $F(t, \cdot, \cdot) : C_0 \times L_0 \to convR^n$ satisfies for fixed $t \in [0,\infty)$ the Lipschitz condition of the form:

$$H(F(t,u,v), F(t,\overline{u},\overline{v})) \leq k(\|u-\overline{u}\|_0 + |v-\overline{v}|_0)$$

where $k > 0$, $u, \overline{u} \in C_0$ and $v, \overline{v} \in L_0$

(c) there exists a $M > 0$ such $H(F(t,u,v), \{0\}) \leq M$ for $(t,u,v) \in [0,\infty) \times C_0 \times L_0$ and furthermore let

(d) uniformly with respect to $(u,v) \in C_0 \times L_0$ are satisfy the conditions (4) for a $F^-, F^+ : C_0 \times L_0 \to convR^n$.

Theorem 1. *Let $F : [0,\infty) \times C_0 \times L_0 \to convR^n$ and $F^-, F^+ : C_0 \times L_0 \to convR^n$ satisfy the conditions (a)-(c) and (d) respectively suppose that given are problems (1), (5) and (6) together with the initial conditions $x(t) = y(t) = z(t) = \varphi(t) = const$. Then, for each $\eta > 0$ and $T > 0$ there exists a $\varepsilon^0(\eta, T) > 0$ such that for every $\varepsilon \in (0, \varepsilon^0)$ the following conditions are satisfied:*

(i) for each solution $y(\cdot)$ of (5) there exists a solution $x(\cdot)$ of (1) such that

$$|y(t) - x(t)| \leq \eta \quad for \quad t \in \left[-r, \frac{T}{\varepsilon}\right] \tag{7}$$

(ii) for each solution $x(\cdot)$ of (1) there exists a solution $z(\cdot)$ of (6) such that

$$|x(t) - z(t)| \leq \eta \quad for \quad t \in \left[-r, \frac{T}{\varepsilon}\right]. \tag{8}$$

To prove this theorem we shall use the Filippov's type theorem for functional-differential equations of neutral type ([4]) of the form:

Theorem 2. *Let $\delta : [0,T] \to R$ be a nonnegative Lebesgue integrable function and let $\varphi \in C_0$ be absolutely continuous. Suppose $F : [0,T] \times C_0 \times L_0 \to compR^n$ satisfies (a), (b) and (c) of the form:*

$$H(F(t,u,v), F(t,\overline{u},\overline{v})) \leq k(t)(\|u-\overline{u}\|_0 + |v-\overline{v}_0)$$

where $k : [0,T] \to R^+$ is a Lebesgue integrable function, $u, \overline{u} \in C_0$ and $v, \overline{v} \in L_0$. Furthermore let $z : [-r,T] \to R^n$ be an absolutely continuous mapping such that

(e) $z(t) = \varphi(t)$ for $t \in [-r,0]$

(f) $d(\dot{z}(t), F(t, z_t, \dot{z}_t)) \leq \delta(t)$ for a.e. $t \in [0,T]$.

Then there is a solution $x(\cdot)$ of an initial-value problem

$$\dot{x}(t) \in F(t, x_t, \dot{x}_t) \quad \textit{for a.e.} \quad t \in [0,T], \quad x(t) = \varphi t \quad \textit{for} \quad t \in [-r,0]$$

such that $|x(t) - z(t)| \leq \xi(t)$ for $t \in [0,T]$ and $|\dot{x}(t) - \dot{z}(t)| \leq \delta(t) + 2k(t)\xi(t)$ for a.e. $t \in [0,T]$ where $\xi(t) = \int_0^t \delta(s) e^{2[m(t)-m(s)]} ds$ and $m(s) = \int_0^t k(r)dr$.

Proof of the Theorem 1. In the first step of the proof we show Lipschitzean with respect to (u,v) of mappings $F^- : C_0 \times L_0 \to convR^n$. Observe that:

$$H(F^-(u,v), F^-(\overline{u},\overline{v})) \leq H\left(F^-(u,v), \frac{1}{L}\int_0^L F(t,u,v)dt + S_\eta(0)\right) +$$

$$+H\left(\frac{1}{L}\int_0^L F(t,u,v)dt + S_\eta(0), \frac{1}{L}\int_0^L F(t,\overline{u},\overline{v})dt + S_\eta(0)\right) +$$

$$+H\left(\frac{1}{L}\int_0^L F(t,\overline{u},\overline{v})dt + S_\eta(0), F^-(\overline{u},\overline{v})\right) \leq$$

$$\leq H\left(\frac{1}{L}\int_0^L F(t,u,v)dt, \frac{1}{L}\int_0^L F(t,\overline{u},\overline{v})dt\right) \leq k(\|u-\overline{u}\|_0 + |v-\overline{v}|_0)$$

Adopting the procedure presented above we obtain the Lipschitz condition the mapping F^+. Furthermore the mappings F^-, F^+ are bounded i.e. there exists a $M > 0$ such that $H(F^-(u,v),\{0\}) \leq M$ and $H(F^+(u,v),\{0\}) \leq M$ for $(u,v) \in C_0 \times L_0$.

Let $y(\cdot)$ be a solution of (5). To prove this theorem we shall consider the solution $x(\cdot)$ of the inclusion (1) in such that, for $t \in [-r,0]$, $x(t) = y(t) = \varphi(t)$ hence

$|x(t)-y(t)| = 0 < \eta$ and so that for $t \in \left[0, \dfrac{T}{\varepsilon}\right]$ inequality (7) was satisfied too. To do this let us divide the interval $\left[0, \dfrac{T}{\varepsilon}\right]$ on m subintervals $[t_i, t_{i+1}]$ where $t_i = \dfrac{iT}{\varepsilon m}$, $i = 0, 1, 2, \ldots, m-1$, and write a solution $y(\cdot)$ is the form:

$$\begin{cases} y(t) = const. & \text{for} \quad t \in [-r, 0] \\ y(t) = y(t_i) + \varepsilon \displaystyle\int_{t_i}^{t} v(\tau)d\tau & \text{for} \quad t \in [t_i, t_{i+1}] \end{cases} \tag{9}$$

where $v(t) \in F^-(y_t, \dot{y}_t)$.

Let us consider a function $y^1(\cdot)$ defined by

$$\begin{cases} y^1(t) = const. & \text{for} \quad t \in [-r, 0] \\ y^1(t) = y^1(t_i) + \varepsilon u^1(t_i)(t - t_i) & \text{for} \quad t \in [t_i, t_{i+1}] \end{cases} \tag{10}$$

where $u^1(\cdot)$ are measurable functions such that $u^1(t) \in F^-(y^1_{t_i}, \dot{y}^1_{t_i})$ and

$$\left|\frac{T}{\varepsilon m}u^1(t_i) - \int_{t_i}^{t_{i+1}} v(t)dt\right| = \min\left\{\left|\frac{T}{\varepsilon m}z - \int_{t_i}^{t_{i+1}} v(t)dt\right|, \text{ where } z \in F^-(y^1_{t_i}, \dot{y}^1_{t_i})\right\}.$$

Mapping u^1 exists because set-valued function F^- is measurable and has compact and convex values ([2]). Let us denote by $\delta_i = |y(t_i) - y^1(t_i)|$, $i = 1, 2, \ldots, m-1$ when by virtue of (9) for every $t \in [t_i, t_{i+1}]$ we have:

$$|y(t) - y^1(t_i)| = \left|y(t_i) + \varepsilon \int_{t_i}^{t} v(\tau)d\tau - y^1(t_i)\right| \le \delta_i + \varepsilon M(t - t_i). \tag{11}$$

Furthermore for $t \in [t_i, t_{i+1}]$

$$|v(t) - u^1(t_i)| \le H(F^-(y_t, \dot{y}_t), F^-(y^1_{t_i}, \dot{y}^1_{t_i})) \le k(\|y_t - y^1_{t_i}\|_0 + |\dot{y}_t - \dot{y}^1_{t_i}|_0).$$

Adopting now the procedure presented in the proof of theorem 1 ([3]) we have:

$$\|y_t - y^1_{t_i}\|_0 \le \frac{MT}{m} + \delta_i + 2\varepsilon Mr \quad \text{and} \quad |\dot{y}_t - \dot{y}^1_{t_i}| \le 2\varepsilon Mr.$$

Hence for $t \in [t_i, t_{i+1}]$ we obtain

$$|v(t) - u^1(t_i)| \le k\left(\frac{MT}{m} + \delta_i + 4\varepsilon Mr\right). \tag{12}$$

By virtue of (9), (10) and (12) it follows

$$\delta_i = |y(t_i) - y^1(t_i)| \le |y(t_{i-1}) - y^1(t_{i-1})| + \varepsilon \int_{t_{i-1}}^{t_i} |v(\tau) - u^1(t_{i-1})|d\tau \le$$

$$\leq \delta_{i-1} + \varepsilon k \left(\frac{MT}{m} + \delta_{i-1} + 4\varepsilon Mr \right) (t_i - t_{i-1}) =$$

$$= \delta_{i-1} + \varepsilon k \left(\frac{MT}{m} + \delta_{i-1} + 4\varepsilon Mr \right) \frac{T}{\varepsilon m} = \delta_{i-1} \left(1 + \frac{a}{m}\right) + \frac{b}{m}$$

where $a = kT$ and $b = \dfrac{kMT}{m}(T + 4\varepsilon mr)$. Hence

$$\delta_{i-1} \leq \delta_{i-1} \left(1 + \frac{a}{m}\right) + \frac{b}{m} \leq \left(1 + \frac{a}{m}\right) \left[\delta_{i-2} \left(1 + \frac{a}{m}\right) + \frac{b}{m}\right] + \frac{b}{m} =$$

$$= \left(1 + \frac{a}{m}\right)^2 \delta_{i-2} + \left(1 + \frac{a}{m}\right) \frac{b}{m} \leq \cdots \leq \left(1 + \frac{a}{m}\right)^i \delta_0 + \left(1 + \frac{a}{m}\right)^{i-2} \frac{b}{m} + \cdots + \frac{b}{m} =$$

$$= \frac{b}{m} \left(1 + \left(1 + \frac{a}{m}\right) + \cdots + \left(1 + \frac{a}{m}\right)^{i-1}\right) = \frac{b}{a} \left(\left(1 + \frac{a}{m}\right)^i - 1\right) \leq$$

$$\leq \frac{b}{a}(e^a - 1) = \frac{M}{m}(T + 4\varepsilon mr)(e^{kT} - 1) \quad \text{where} \quad i = 0, 1, \ldots, m-1. \tag{13}$$

For $t \in [t_i, t_{i+1}]$ we have:

$$|y(t) - y(t_i)| = \left| \varepsilon \int_{t_i}^{t} v(\tau) d\tau \right| \leq \varepsilon M(t - t_i) \leq \frac{MT}{m}$$

and

$$|y^1(t) - y^1(t_i)| = |\varepsilon u^1(t_i)(t - t_i)| \leq \frac{MT}{m}.$$

Hence, we obtain:

$$|y(t) - y^1(t)| \leq |y(t) - y(t_i)| + |y(t_i) - y^1(t_i)| + |y^1(t_i) - y^1(t)| \leq$$

$$\leq \frac{2MT}{m} + \frac{M}{m}(T + 4\varepsilon mr)(e^{kT} - 1) \tag{14}$$

Now we shall consider the function

$$\begin{cases} y^2(t) = const & \text{for} \quad t \in [-r, 0] \\ y^2(t) = y^2(t_i) + \varepsilon \int_{t_i}^{t} u^2(\tau) d\tau & \text{for} \quad t \in [t_i, t_{i+1}] \end{cases} \tag{15}$$

where $t_i = \dfrac{iT}{\varepsilon m}$, $i = 0, 1, \ldots, m-1$, $u^2(t) \in F(t, y_{t_i}, \dot{y}_{t_i})$.

Let us notice that by virtue of condition (d) for each $\eta > 0$ there exists a $L_0(\eta)$ such that for every $L > L_0$ we have inclusions:

$$F^-(y^1_{t_i}, \dot{y}^1_{t_i}) \subset \frac{1}{L} \int_0^L F(t, y^1_{t_i}, \dot{y}^1_{t_i}) dt + S_\eta(0).$$

In particular, for $\dfrac{T}{\varepsilon m} > L_0$ and for every $i \in \{0, 1, \ldots, m\}$ we have

$$F^{-}(y^1_{t_i}, \dot{y}^1_{t_i}) \subset \frac{\varepsilon m}{iT} \int_0^{\frac{iT}{\varepsilon m}} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt + S_\eta(0) \tag{16}$$

and

$$F^{-}(y^1_{t_i}, \dot{y}^1_{t_i}) \subset \frac{\varepsilon m}{(i+1)T} \int_0^{\frac{(i+1)T}{\varepsilon m}} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt + S_\eta(0). \tag{17}$$

Let us observe that $t_{i+1} = \dfrac{(1+i)T}{\varepsilon m}$ and $t_i = \dfrac{iT}{\varepsilon m}$. By virtue (16), (17) and the Hausdorff metric condition (see Lemma 3(vi), [1]) we have

$$\begin{aligned} H\left(\int_{t_i}^{t_{i+1}} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt, F^{-}(y^1_{t_i}, \dot{y}^1_{t_i})\right) &\leq H\left(\int_0^{t_i} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt, \int_0^{t_i} F^{-}(y^1_{t_i}, \dot{y}^1_{t_i})dt\right) + \\ &+ H\left(\int_0^{t_{i+1}} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt, \int_0^{t_i} F^{-}(y^1_{t_i}, \dot{y}^1_{t_i})dt\right) = \\ &= \frac{iT}{\varepsilon m} H\left(\frac{\varepsilon m}{iT}\int_0^{t_i} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt, F^{-}(y^1_{t_i}, \dot{y}^1_{t_i})\right) + \\ &+ \frac{(1+i)T}{\varepsilon m} H\left(\frac{\varepsilon m}{(1+i)T}\int_0^{t_{i+1}} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt, F^{-}(y^1_{t_i}, \dot{y}^1_{t_i})\right) \leq \\ &\leq \frac{iT}{\varepsilon m}\eta + \frac{(1+i)T}{\varepsilon m}\eta = \frac{T\eta}{\varepsilon m}(2i+1) \leq \frac{T\eta}{\varepsilon m}(2m+1). \end{aligned}$$

Hence

$$H\left(\int_{t_i}^{t_{i+1}} F(t, y^1_{t_i}, \dot{y}^1_{t_i})dt, F^{-}(y^1_{t_i}, \dot{y}^1_{t_i})\right) \leq \frac{\eta_1 T}{\varepsilon m},$$

where $\eta_1 = (2m+1)\eta$ for $\dfrac{T}{\varepsilon m} > L_0\left(\dfrac{\eta_1}{2m+1}\right)$, $\varepsilon < \varepsilon_0(\eta_1, m) = \dfrac{T}{mL_0\left(\frac{\eta_1}{2m+1}\right)}$.

Hence, it follows that

$$\left|\int_{t_i}^{t_{i+1}} u^2(\tau) - u^1(\tau)dt\right| \leq \frac{\eta_1 T}{\varepsilon m} \tag{18}$$

and

$$\begin{aligned} |y^1(t_{i+1} - y^2(t_{i+1})| &\leq |y^1(t_i) - y^2(t_i)| + \varepsilon\left|\int_{t_i}^{t_{i+1}} u^1(t_i) - u^2(\tau)d\tau\right| \leq \\ &\leq |y^1(t_i) - y^2(t_i)| + \frac{\eta_1 T}{m} \leq \cdots \leq m\frac{\eta_1 T}{m} = \eta_1 T, \end{aligned} \tag{19}$$

where $i = 0, 1, \ldots, m-1$.

Using the inequality (19) and the fact that for $t \in [t_i, t_{i+1}]$, $|y^2(t) - y^2(t_i)| \leq \frac{MT}{m}$ and $|y^1(t) - y^1(t_i)| \leq \frac{MT}{m}$ we have

$$|y^1(t) - y^2(t)| \leq |y^1(t) - y^1(t_i)| + |y^1(t_i) - y^2(t_i)| + |y^2(t_i) - y^2(t)| \leq \frac{2MT}{m} + \eta_1 T. \tag{20}$$

By assumption (b) it follows that

$$H(F(t, y_t^2, \dot{y}_t^2), F(t, y_{t_i}^1, \dot{y}_{t_i}^1)) \leq k(\|y_t^2 - y_{t_i}^1\| + |\dot{y}_y^2 - \dot{y}_{t_i}^1|_0).$$

Using the definitions of the norm $\|\cdot\|_0$ and $|\cdot|_0$ and properties F^- and making use of the inequality (20) we obtain $\|y_t^2 - y_{t_i}^1\|_0 \leq \frac{3MT}{m} + \eta_1 T$ and $|\dot{y}_t^2 - \dot{y}_{t_i}^1|_0 \leq 2\varepsilon Mr$ and

$$H(F(t, y_t^2, \dot{y}_t^2), F(t, y_{t_i}^1, \dot{y}_{t_i}^1)) \leq k\left(\frac{3MT}{m} + \eta_1 T + 2\varepsilon Mr\right).$$

Hence and by virtue of (15) we have

$$d(\dot{y}(t), \varepsilon F(t, y_t^2, \dot{y}_t^2)) \leq d(\dot{y}(t), \varepsilon F(t, y_{t_i}^1, \dot{y}_{t_i}^1)) + H(\varepsilon F(t, y_{t_i}^1, \dot{y}_{t_i}^1), \varepsilon F(t, y_t^2, \dot{y}_t^2)) \leq$$
$$\leq \varepsilon k\left(\frac{3MT}{m} + \eta_1 T + 2\varepsilon Mr\right).$$

Now, on the ground of Filippov's type theorem (see Theorem 2) there exists the solution $x(\cdot)$ of (1) that for $t \in \left[0, \frac{T}{\varepsilon}\right]$

$$|y^2(t) - x(t)| \leq \int_0^t \varepsilon k\left(\frac{3MT}{m} + \eta_1 T + 2\varepsilon Mr\right) e^{2\varepsilon k(t-s)} ds \leq$$
$$\leq \left(\frac{3MT}{\varepsilon m} + \frac{\eta_1 T}{2} + \varepsilon Mr\right)(e^{2kT} - 1). \tag{21}$$

By inequalities (21), (20) and (14) it follows

$$|x(t) - y(t)| \leq |x(t) - y^2(t)| + |y^2(t) - y^1(t)| + |y^1(t) - y(t)| \leq$$
$$\leq \left(\frac{3MT}{2m} + \frac{\eta_1 T}{2} + \varepsilon Mr\right)(e^{2kT} - 1) + \frac{2MT}{m} + \eta_1 T + \frac{2MT}{m} +$$
$$+ \frac{M}{m}(T + 4\varepsilon mr)(e^{2kT} - 1) \leq \frac{5MT}{2m}(1 + e^{2kT}) + \frac{\eta_1 T}{2}(1 + e^{kT}) + 5M\varepsilon r(e^{2kT} - 1). \tag{22}$$

Therefore, choosing $m > \dfrac{15MT(1+e^{2kT})}{2\eta}$, $\eta_1 = \dfrac{2\eta}{3T(1+e^{kT})}$,

$\varepsilon < \dfrac{\eta}{15Mr(e^{2kT}-1)}$ we get the inequality $|x(t)-y(t)| \le \eta$ for $t \in \left[0, \dfrac{T}{\varepsilon}\right]$. In this way the proof of condition (i) is completed for $t \in \left[-r, \dfrac{T}{\varepsilon}\right]$.

Adopting now the procedure present above we get condition (ii). □

Corollary. *In the case there exists a limit* $F_0(u,v) = \lim_{L\to\infty} \int_0^L F(t,u,v)dt$ *we have*

$$\overline{F^-(u,v)} = \overline{F^+(u,v)} = F_0(u,v)$$

where

$$\overline{F^-(u,v)} = \varliminf_{L\to\infty} \frac{1}{L}\int_0^L F(t,u,v)dt$$

and

$$\overline{F^+(u,v)} = \varlimsup_{L\to\infty} \frac{1}{L}\int_0^L F(t,u,v)dt.$$

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A MINIMUM PRINCIPLE FOR THE STOCHASTIC NAVIER-STOKES EQUATION

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Abstract. We consider the stochastic Navier-Stokes equation controlled by bounded controls. We formulate a stochastic minimum principle for this equation and give the equations for the adjoint processes.

1. Introduction

Related to the problem of optimal controls for stochatic evolution equations there are investigations about the existence of optimal and ε-optimal controls (for example in [9], [24], [25], [14], [15]), formulations of minimum or maximum principles (see [18],[3], [2]), studies about the Bellman equation (see [10],[19]).

The optimal control of stochastic evolution equations has applications in risk and game theory, filtering theory, dynamic programming, in physical and chemical studies, in technical, industrial and economical decisions. A minimum or maximum principle gives us necessary conditions for optimality.

Our paper deals with the problem of optimal control for the stochastic Navier-Stokes equation controlled by different external forces, which are bounded controls. We formulate a stochastic minimum principle.

The stochastic Navier-Stokes equation describes the behavior of a viscous velocity field of an incompressible liquid. The equation on the domain of flow $G \subset \mathbb{R}^n$ ($n \geq 2$ a natural number) is given by

$$\frac{\partial U}{\partial t} - \nu\Delta U = -(U, \nabla)U + \Phi - \nabla p + \mathcal{C}(U)\frac{\partial w}{\partial t} \qquad (1)$$

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$$\operatorname{div} U = 0, \quad U(0,x) = U_0(x), \quad U(t,x)\,|_{\partial G} = 0,\ t > 0,\ x \in G,$$

where U is the velocity field, ν is the viscosity, Δ is the Laplacian, ∇ is the gradient, Φ is an external force, p is the pressure, and U_0 is the initial condition. Realistic models for flows should contain a random noise part, because external perturbations and the internal Browninan motion influence the velocity field. For this reason equation (1) contains a random noise part $\mathcal{C}(U)\dfrac{\partial w}{\partial t}$. Here the noise is defined as the distributional derivative of a Wiener process $\Big(w(t)\Big)_{t\in[0,T]}$, whose intensity depends on the state U. There are many papers written about this equation, for example the Galerkin method for this equation can be found in [6], [7], [21], [23]; other approximation methods for this equation are given in [4], [16], [11]; about attractors, invariant measures and ergodicity are the papers [12], [13]. There are not many papers written about the problem of optimal control for the stochastic Navier-Stokes equation. H. Choi, R. Temam, P. Moin, J. Kim in [8] investigated this problem for the stochastic Burgers equation (one dimensional Navier-Stokes equation).

The results of our paper are different and can be applied in hydromechanics for the two dimensional ($n = 2$) stochastic Navier-Stokes equation (1). A minimum or maximum principle for this type of equation was not formulated before. Since the considered equation is nonlinear, we are dealing with a nonconvex optimization problem.

The structure of the paper is as follows: In Section 2 we give the assumptions for the stochastic Navier-Stokes equation and formulate the control problem. In Section 3 we calculate the Gateaux derivative of the cost functional (see Theorem 3.4), in Section 4 we formulate a stochastic minimum principle (see Theorem 4.2) and in Section 5 we investigate the equation which characterizes the adjoint processes (see Theorem 5.1). In the appendix of Section 6 we give some properties which we need in our proofs.

Notations

- $\rightharpoonup$ weak convergence (in the sense of functional analysis)
- $(\Omega, \mathcal{F}, P)$ complete probability space

- EX mathematical expectation of the random variable X
- $(\mathcal{F}_t)_{t\in[0,T]}$ right continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets
- $\langle v^*, v\rangle$ the application of $v^* \in V^*$ on $v \in V$
- $\mathcal{L}(V)$ space of all linear and continuous operators from the Banach space V to itself
- $\mathcal{L}^2_V[0,T]$ space of all $B([0,T])$-measurable functions $u : [0,T] \to V$ with $\int\limits_0^T \|u(t)\|^2_V dt < \infty$
- $\mathcal{L}^2_V(\Omega)$ space of all $\mathcal{F}$-measurable random variables $u : \Omega \to V$ with $E\|u\|^2_V < \infty$
- $\mathcal{L}^2_V(\Omega \times [0,T])$ space of all $\mathcal{F} \times B([0,T])$-measurable processes $u : \Omega \times [0,T] \to V$ that are adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $E\int\limits_0^T \|u(t)\|^2_V dt < \infty$
- $\Delta_Q(t)$ is the notation for $\exp\Big\{-\frac{b}{\nu}\int\limits_0^t \|Q(s)\|^2_V ds\Big\}$, where $\Big(Q(t)\Big)_{t\in[0,T]}$ is a V-valued stochastic process; b, ν are positive constants
- Λ Lebesgue measure

2. Assumptions and Formulation of the Control Problem

First we state the assumptions about the stochastic evolution equation that will be considered.

(i) $(\Omega, \mathcal{F}, P)$ is a complete probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ is a right continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. $(w(t))_{t\in[0,T]}$ is a real valued standard $\mathcal{F}_t$-Wiener process.

(ii) (V, H, V^*) is an evolution triple (see [28], p. 416), where $(V, \|\cdot\|_V)$ and $(H, \|\cdot\|)$ are separable Hilbert spaces, and the embedding operator $V \hookrightarrow H$ is assumed to be compact. We denote by $(\cdot,\cdot)$ the scalar product in H.

(iii) $\mathcal{A} : V \to V^*$ is a linear operator such that $\langle \mathcal{A}v, v\rangle \geq \nu\|v\|^2_V$ for all $v \in V$ and $\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}v, u\rangle$ for all $u, v \in V$, where $\nu > 0$ is a constant and $\langle\cdot,\cdot\rangle$ denotes the dual pairing.

(iv) $\mathcal{B} : V \times V \to V^*$ is a bilinear operator such that $\langle \mathcal{B}(u,v), v\rangle = 0$ for all $u, v \in V$ and for which there exists a positive constant $b > 0$ such that

$$|\langle \mathcal{B}(u,v), z\rangle|^2 \leq b||z||_V^2 ||u|| ||u||_V ||v|| ||v||_V \quad \text{for all } u, v, z \in V.$$

(v) $\mathcal{C} : [0,T] \times H \to H$ is a mapping such that

(a) $||\mathcal{C}(t,u) - \mathcal{C}(t,v)||^2 \leq \lambda ||u - v||^2$ for all $t \in [0,T]$, $u, v \in H$, where λ is a positive constant;

(b) $\mathcal{C}(t,0) = 0$ for all $t \in [0,T]$;

(c) $\mathcal{C}(\cdot, v) \in \mathcal{L}_H^2[0,T]$ for all $v \in H$;

(d) for each $t \in [0,T]$ the mapping $\mathcal{C}(t,\cdot)$ is Fréchet differentiable and $\mathcal{C}'(t,x) \in \mathcal{L}(H)$; the Fréchet derivative of $\mathcal{C}(t,\cdot)$, at the point x, satisfies

$$||\mathcal{C}'(t,x)(y)|| \leq k_{\mathcal{C}'} ||y|| \quad \text{for all } t \in [0,T], x, y \in H,$$

where $k_{\mathcal{C}'}$ is a positive constant independent of t and x.

(vi) $\Phi \in \mathcal{L}_H^2(\Omega \times [0,T])$ satisfies

$$||\Phi(\omega,t)|| \leq \rho \quad \text{for } P \times \Lambda \text{ a.e. } (\omega,t) \in \Omega \times [0,T],$$

where $\rho > 0$ is a given constant.

(vii) x_0 is a H-valued $\mathcal{F}_0$-measurable random variable such that $E||x_0||^4 < \infty$.

(viii) $\mathcal{L} : [0,T] \times H \times H \to \mathbb{R}_+$, $\mathcal{K} : H \to \mathbb{R}_+$ are mappings satisfying the following conditions:

(a) for all $t \in [0,T]$, $x_1, x_2, y_1, y_2 \in H$

$$|\mathcal{L}(t,x_1,y_1) - \mathcal{L}(t,x_2,y_2)| \leq c_{\mathcal{L}}\Big(||x_1 - x_2||^2 + ||y_1 - y_2||^2\Big),$$

$$|\mathcal{K}(x_1) - \mathcal{K}(x_2)| \leq c_{\mathcal{K}} ||x_1 - x_2||^2,$$

where $c_{\mathcal{L}}, c_{\mathcal{K}}$ are positive constants;

(b) for all $x, y \in H$ we assume that $\mathcal{L}(\cdot, x, y) \in \mathcal{L}_H^2[0,T]$;

(c) the mappings $\mathcal{L}(t,\cdot,\cdot), \mathcal{K}(\cdot)$ are Fréchet differentiable for each fixed $t \in [0,T]$;

(d) the mappings $\mathcal{L}_x(t,\cdot,\cdot)$, $\mathcal{L}_y(t,\cdot,\cdot), \mathcal{K}'(\cdot)$ are Lipschitz continuous and

$$||\mathcal{L}_x(t,x,y)|| + ||\mathcal{L}_y(t,x,y)|| \leq k_{\mathcal{L}}(1 + ||x|| + ||y||), \qquad ||\mathcal{K}'(x)|| \leq k_{\mathcal{K}}(1 + ||x||)$$

for all $t \in [0,T], x, y \in H$, where $k_{\mathcal{L}}, k_{\mathcal{K}}$ are positive constants;

(e) for all $x, y \in H$ we have $\mathcal{L}_x(\cdot, x, y), \mathcal{L}_y(\cdot, x, y) \in \mathcal{L}^2_H[0,T]$.

Definition 2.1. We call a process $\Big(U_\Phi(t)\Big)_{t\in[0,T]}$ from the space $\mathcal{L}^2_V(\Omega \times [0,T])$ with $E\|U_\Phi(t)\|^2 < \infty$ for all $t \in [0,T]$ a **solution of the stochastic Navier-Stokes equation** if it satisfies the equation:

$$(U_\Phi(t), v) + \int_0^t \langle \mathcal{A}U_\Phi(s), v\rangle ds = (x_0, v) + \int_0^t \langle \mathcal{B}(U_\Phi(s), U_\Phi(s)), v\rangle ds + \tag{2}$$
$$+ \int_0^t (\Phi(s), v) ds + \int_0^t (\mathcal{C}(s, U_\Phi(s)), v) dw(s)$$

for all $v \in V$, $t \in [0,T]$, a.e. $\omega \in \Omega$.

Remark 2.2. If we set $n = 2$, $G \subset \mathbb{R}^n$ (the domain of flow), $V = \{u \in \overset{\circ}{W}{}^1_2(G) : \operatorname{div} u = 0\}$, $H = \bar{V}^{L^2(G)}$ and

$$\langle \mathcal{A}u, v\rangle = \int_G \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx, \quad \langle \mathcal{B}(u,v), z\rangle = -\int_G \sum_{i,j=1}^n u_i \frac{\partial v_j}{\partial x_i} z_j dx$$

for $u, v, z \in V, t \in [0,T]$, then equation (2) can be transformed into (1).

Theorem 2.2. *(i) The Navier-Stokes equation* (2) *has a solution* $U_\Phi \in \mathcal{L}^2_V(\Omega \times [0,T])$, *which is almost surely unique and has almost surely continuous trajectories in* H.

(ii) There exists a positive constant c such that

$$E \sup_{t\in[0,T]} \|U_\Phi(t)\|^4 + E\Big(\int_0^T \|U_\Phi(s)\|^2_V ds\Big)^2 \le cE\|x_0\|^4.$$

The proof of this result can be found in [6] (Theorem 2.2, Lemma 3.5).

We consider the stochastic Navier-Stokes equation controlled by **bounded controls** Φ. We denote by $\mathcal{U}^b$ the set of all admissible controls Φ satisfying assumption **(vi)** from above.

We take a cost functional of the form

$$\mathcal{J}(\Phi) = E\int_0^T \mathcal{L}[s, U_\Phi(s), \Phi(s)]ds + E\mathcal{K}[U_\Phi(T)], \qquad \Phi \in \mathcal{U}^b \tag{3}$$

and denote by $(\mathcal{P}^b)$ the problem of minimizing $\mathcal{J}$ among the admissible controls of the set $\mathcal{U}^b$.

Results about the existence of optimal controls can be found in [5], Chapter 2.

3. The Gateaux derivative of the cost functional

Using the properties of $\mathcal{B}$ it can be proved that the mapping $x \in V \mapsto \mathcal{B}(x,x) \in V^*$ is Fréchet differentiable and

$$\mathcal{B}'(x)(y) = \mathcal{B}(x,y) + \mathcal{B}(y,x) \quad \text{for all } x, y \in V.$$

Let $\Phi, \Upsilon \in \mathcal{U}^b$ be such that for sufficiently small $\theta > 0$ we have $\Phi + \theta\Upsilon \in \mathcal{U}^b$. We denote by

$$X_\theta := \frac{U_{\Phi+\theta\Upsilon} - U_\Phi - \theta Z_\Upsilon}{\theta}. \tag{4}$$

Throughout this section we assume that the parameters b, ν, γ, ρ, T of our fluid are chosen in such a way that

$$E\Delta_{U_\Phi}^{-4}(T) < K < \infty.$$

We recall here the results mentioned in Remark 2.5.2 in [5].

Let $\Upsilon \in \mathcal{L}_H^2(\Omega \times [0,T])$. We consider the stochastic evolution equation

$$\begin{aligned}(Z_\Upsilon(t), v) \quad &+ \quad \int_0^t \langle \mathcal{A}Z_\Upsilon(s), v\rangle ds = \int_0^t \langle \mathcal{B}'(U_\Phi(s))(Z_\Upsilon(s)), v\rangle ds \\ &+ \quad \int_0^t (\Upsilon(s), v)ds + \int_0^t (\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s)), v)dw(s)\end{aligned} \tag{5}$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$.

Lemma 3.1. *There exists a V-valued, $\mathcal{F} \times \mathcal{B}_{[0,T]}$-measurable process $\left(Z_\Upsilon(t)\right)_{t\in[0,T]}$ adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, satisfying (5) and which has almost surely continuous trajectories in H. The solution is almost surely unique and there exists a constant $c > 0$ (independent of Υ) such that*

$$E\Delta_{U_\Phi}(T)\|Z_\Upsilon(T)\|^2 + E\int_0^T \Delta_{U_\Phi}(t)\|Z_\Upsilon(t)\|_V^2\,dt \le cE\int_0^T \|\Upsilon(t)\|^2\,dt$$

and

$$E\Delta_{U_\Phi}^2(T)\|Z_\Upsilon(T)\|^4 + E\left(\int_0^T \Delta_{U_\Phi}(t)\|Z_\Upsilon(t)\|_V^2\,dt\right)^2 \le cE\int_0^T \|\Upsilon(t)\|^4\,dt.$$

We apply results from [5] (Theorem 1.3.1 and Remark 2.5.2 on $X = Y := U_\Phi, a_0 := 0, \Psi := \Upsilon, \Gamma := 0, \mathcal{G}(s,h) := \mathcal{C}'(s, U_\Phi(s))(h), Z_{\Psi,\Gamma} := Z_\Upsilon$).

Lemma 3.2. *(i) There exists a positive constant c independent of θ such that*

$$E\Delta_{U_\Phi}^2(T)\left\|\frac{U_{\Phi+\theta\Upsilon}(T) - U_\Phi(T)}{\theta}\right\|^4 + E\left(\int_0^T \Delta_{U_\Phi}(s)\left\|\frac{U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)}{\theta}\right\|_V^2 ds\right)^2 \le$$

$$\le cE\int_0^T \|\Upsilon(s)\|^4\,ds$$

and

$$\lim_{\theta\searrow 0} \frac{1}{\theta^2} E\int_0^T \Delta_{U_\Phi}^2(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|_V^2 \|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|^2\,ds = 0.$$

(ii) The following convergences hold

$$\lim_{\theta\searrow 0} E\|X_\theta(T)\|^2 = 0 \quad and \quad \lim_{\theta\searrow 0} E\int_0^T \|X_\theta(s)\|_V^2\,ds = 0,$$

where X_θ is defined in (4).

Proof. For all $t \in [0,T]$ and a.e. $\omega \in \Omega$ let

$$e_1(t) = \Delta_{U_\Phi}(t)\exp\{-(2\lambda + 1)t\}.$$

(i) We use the Ito formula and the properties of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ to get

$$e_1(t)\|U_{\Phi+\theta\Upsilon}(t) - U_\Phi(t)\|^2 + \nu\int_0^t e_1(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|_V^2 ds \tag{6}$$

$$\leq \theta^2 \int_0^t e_1(s)\|\Upsilon(s)\|^2 ds + 2\int_0^t e_1(s)(\mathcal{C}(s, U_{\Phi+\theta\Upsilon}(s)) - \mathcal{C}(s, U_\Phi(s)), U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s))dw(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Using Proposition 6.2 we obtain

$$E\left\{\sup_{t\in[0,T]} \Delta_{U_\Phi}^2(t)\left\|\frac{U_{\Phi+\theta\Upsilon}(t) - U_\Phi(t)}{\theta}\right\|^4\right\} + \nu^2 E\left(\int_0^T \Delta_{U_\Phi}(s)\left\|\frac{U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)}{\theta}\right\|_V^2 ds\right)^2$$

$$\leq cE\int_0^T \|\Upsilon(s)\|^4 ds,$$

where c is a positive constant independent of θ. We write

$$E\int_0^T \Delta_{U_\Phi}^2(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|_V^2 \|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|^2 ds$$

$$\leq \theta^4\left[E\left\{\sup_{t\in[0,T]} \Delta_{U_\Phi}^2(t)\left\|\frac{U_{\Phi+\theta\Upsilon}(t) - U_\Phi(t)}{\theta}\right\|^4\right\} E\left(\int_0^T \Delta_{U_\Phi}(s)\left\|\frac{U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)}{\theta}\right\|_V^2 ds\right)^2\right]^{\frac{1}{2}}$$

Consequently,

$$\lim_{\theta\searrow 0} \frac{1}{\theta^2} E\int_0^T \Delta_{U_\Phi}^2(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|_V^2 \|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|^2 ds = 0.$$

(ii) By the Ito formula and the properties of $\mathcal{A}, \Phi, \Upsilon$ we get

$$
\begin{aligned}
& e_1(T)\|X_\theta(T)\|^2 + 2\nu \int_0^T e_1(s)\|X_\theta(s)\|_V^2 \, ds \\
\leq \; & 2\int_0^T e_1(s)\langle \mathcal{B}(X_\theta(s), U_\Phi(s)) + \frac{1}{\theta}\mathcal{B}(U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s), U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)), X_\theta(s)\rangle ds \\
- \; & (2\lambda+1)\int_0^T e_1(s)\|X_\theta(s)\|^2 ds - \frac{b}{\nu}\int_0^T e_1(s)\|U_\Phi(s)\|^2\|X_\theta(s)\|^2 ds \\
+ \; & 2\int_0^T \frac{e_1(s)}{\theta}\Big(\mathcal{C}(s, U_{\Phi+\theta\Upsilon}(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s), X_\theta(s)\Big) dw(s) \\
+ \; & \int_0^T \frac{e_1(s)}{\theta^2}\Big\|\mathcal{C}(s, U_{\Phi+\theta\Upsilon}(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s))\Big\|^2 ds.
\end{aligned}
$$

Using the properties of $\mathcal{B}, \Phi, \Upsilon, \mathcal{C}$, it follows that

$$
\begin{aligned}
& Ee_1(T)\|X_\theta(T)\|^2 + \frac{3\nu}{4}E\int_0^T e_1(s)\|X_\theta(s)\|_V^2 ds \\
\leq \; & \frac{4b}{\nu\theta^2}E\int_0^T e_1(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|^2\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s))\|_V^2 ds \\
+ \; & \frac{2}{\theta^2}E\int_0^T e_1(s)\|\mathcal{C}(s, U_\Phi(s) + \theta Z_\Upsilon(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s))\|^2 ds.
\end{aligned}
$$

Applying **(i)**, the properties of $\mathcal{C}$ and the Lebesgue Theorem we conclude

$$
\lim_{\theta\searrow 0} E\Delta_{U_\Phi}(T)\|X_\theta(T)\|^2 = 0, \quad \lim_{\theta\searrow 0} E\int_0^T \Delta_{U_\Phi}(s)\|X_\theta(s)\|_V^2 ds = 0. \tag{7}
$$

We see that

$$E\Delta^2_{U_\Phi}(T)\|X_\theta(T)\|^4 + E\Big(\int_0^T \Delta_{U_\Phi}(s)\|X_\theta(s)\|_V^2\,ds\Big)^2 \tag{8}$$

$$\leq \quad 8\Bigg[E\Delta^2_{U_\Phi}(T)\Bigg\|\frac{U_{\Phi+\theta\Upsilon}(T)-U_\Phi(T)}{\theta}\Bigg\|^4 + E\Delta^2_{U_\Phi}(T)\|Z_\Upsilon(T)\|^4$$

$$+ \quad E\Bigg(\int_0^T \Delta_{U_\Phi}(s)\Bigg\|\frac{U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)}{\theta}\Bigg\|_V^2 ds\Bigg)^2 + E\Big(\int_0^T \Delta_{U_\Phi}(s)\|Z_\Upsilon(s)\|_V^2\,ds\Big)^2\Bigg].$$

By using the Schwarz inequality we obtain

$$E\|X_\theta(T)\|^2 \leq \Big(E\Delta_{U_\Phi}(T)\|X_\theta(T)\|^2\Big)^{\frac{1}{2}}\Big(E\Delta^{-4}_{U_\Phi}(T)\Big)^{\frac{1}{4}}\Big(E\Delta^2_{U_\Phi}(T)\|X_\theta(T)\|^4\Big)^{\frac{1}{4}}.$$

Taking into account (7), (8), **(i)**, Theorem 1.3.1 from [5], and the condition $E\Delta^{-4}_{U_\Phi}(T) < K < \infty$ it follows that

$$\lim_{\theta\searrow 0} E\|X_\theta(T)\|^2 = 0.$$

Analogously we can prove that

$$\lim_{\theta\searrow 0} E\int_0^T \|X_\theta(s)\|_V^2\,ds = 0.$$

□

Remark 3.3. For the proof of **(i)** in Lemma 3.2 we do not need the condition $E\Delta^{-4}_{U_\Phi}(T) < \infty$.

Theorem 3.4. *The cost functional $\mathcal{J}$ is Gateaux differentiable with*

$$\frac{\mathrm{d}\mathcal{J}(\Phi+\theta\Upsilon)}{\mathrm{d}\theta}\Bigg|_{\theta=0} \quad = \quad E\int_0^T \Big(\mathcal{L}_x[t, U_\Phi(t), \Phi(t)], Z_\Upsilon(t)\Big)dt \tag{9}$$

$$+ \quad E\int_0^T \Big(\mathcal{L}_y[t, U_\Phi(t), \Phi(t)], \Upsilon(t)\Big)dt + E\Big(\mathcal{K}'[U_\Phi(T)], Z_\Upsilon(T)\Big).$$

Proof. We have

$$\mathcal{K}(x) - \mathcal{K}(\tilde{x}) = \int_0^1 \Big(\mathcal{K}'[\tilde{x} + r(x-\tilde{x}))], x - \tilde{x}\Big)dr$$

and

$$\mathcal{L}(t,x,y) - \mathcal{L}(t,\tilde{x},\tilde{y}) = \int_0^1 \Big(\mathcal{L}_x[t,\tilde{x}+r(x-\tilde{x}),\tilde{y}+r(y-\tilde{y})], x-\tilde{x}\Big)dr \quad (10)$$
$$+ \int_0^1 \Big(\mathcal{L}_y[t,\tilde{x}+r(x-\tilde{x}),\tilde{y}+r(y-\tilde{y})], y-\tilde{y}\Big)dr$$

for all $x,\tilde{x},y,\tilde{y} \in H,\ t \in [0,T]$. Equation (3) implies

$$\mathcal{K}[U_{\Phi+\theta\Upsilon}(T)] - \mathcal{K}[U_\Phi(T)] = \int_0^1 \theta\Big(\mathcal{K}'[U_\Phi(T)+r\theta(X_\theta(T)+Z_\Upsilon(T))], X_\theta(T)+Z_\Upsilon(T)\Big)dr.$$

Using (10) we obtain

$$\mathcal{L}[t,U_{\Phi+\theta\Upsilon}(t),(\Phi+\theta\Upsilon)(t)] - \mathcal{L}[t,U_\Phi(t),\Phi(t)]$$

$$= \int_0^1 \Big\{\theta\Big(\mathcal{L}_x[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)], X_\theta(t)+Z_\Upsilon(t)\Big)$$
$$+ \Big(\mathcal{L}_y[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)], \theta\Upsilon(t)\Big)dr$$
$$= \int_0^1 \theta\Big\{\Big(\mathcal{L}_x[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)], X_\theta(t)+Z_\Upsilon(t)\Big)$$
$$+ \Big(\mathcal{L}_y[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)], \Upsilon(t)\Big)\Big\}dr,$$

for all $t \in [0,T]$. Using the properties of $\mathcal{L},\mathcal{K}$, and Lemma 3.2 it follows that relation (9) holds. □

4. A stochastic minimum principle

We will state a stochastic minimum principle in the case of problem $(\mathcal{P}^b)$. Let $\Phi^* \in \mathcal{U}^b$ be an optimal control with $E\Delta^{-4}_{U_{\Phi^*}}(T) < \infty$, $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in$

$\mathcal{L}^2_H(\Omega \times [0,T])$ and let $Z_{\Psi,\Gamma}$ be the solution of

$$\begin{aligned}(Z_{\Psi,\Gamma}(t),v) &+ \int_0^t \langle \mathcal{A}Z_{\Psi,\Gamma}(s),v\rangle ds = \int_0^t \langle \mathcal{B}'(U_{\Phi^*}(s))(Z_{\Psi,\Gamma}(s)),v\rangle ds \\ &+ \int_0^t \langle \Psi(s),v\rangle ds + \int_0^t (\mathcal{C}'(s,U_{\Phi^*}(s))(Z_{\Psi,\Gamma}(s)),v)dw(s) + \int_0^t (\Gamma(s),v)dw(s)\end{aligned} \tag{11}$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$. This equation is $(P_{\Psi,\Gamma})$ from [5] applied for $X = Y := U_{\Phi^*}, a_0 := 0, \mathcal{G}(s,h) := \mathcal{C}'(s,U_{\Phi^*}(s))(h)$.

The mapping

$$\begin{aligned}(\Psi,\Gamma) &\in \mathcal{L}^2_{V^*}(\Omega\times[0,T]) \times \mathcal{L}^2_H(\Omega\times[0,T]) \mapsto \\ &\mapsto E\int_0^T \Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big) \in \mathbb{R}\end{aligned}$$

is linear and continuous, because

$$(\Psi,\Gamma) \in \mathcal{L}^2_{V^*}(\Omega\times[0,T]) \times \mathcal{L}^2_H(\Omega\times[0,T]) \mapsto Z_{\Psi,\Gamma} \in \mathcal{L}^2_V(\Omega\times[0,T])$$

and

$$(\Psi,\Gamma) \in \mathcal{L}^2_{V^*}(\Omega\times[0,T]) \times \mathcal{L}^2_H(\Omega\times[0,T]) \mapsto Z_{\Psi,\Gamma}(T) \in \mathcal{L}^2_H(\Omega)$$

are linear. By using the properties for $\mathcal{L},\mathcal{K},U_{\Phi^*},\Phi^*$ and Theorem 1.3.1 from [5] we get

$$\begin{aligned}&\Big|E\int_0^T \Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big)\Big| \le \\ \le\ &\Big(E\Delta^{-2}_{U_{\Phi^*}}(T)\Big)^{1/4}\Big\{\Big[E\Big(\int_0^T \|\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)]\|^2dt\Big)^2\Big]^{1/4} + \Big(E\|\mathcal{K}'[U_{\Phi^*}(T)]\|^4\Big)^{1/4}\Big\} \\ \times\ &\Big\{\Big(E\int_0^T \Delta_{U_{\Phi^*}}(t)\|Z_{\Psi,\Gamma}(t)\|^2dt\Big)^{1/2} + \Big(E\Delta_{U_{\Phi^*}}(T)\|Z_{\Psi,\Gamma}(T)\|^2\Big)^{1/2}\Big\} \\ \le\ &\tilde{c}\Big(E\Delta^{-2}_{U_{\Phi^*}}(T)\Big)^{1/4}\Big(E\int_0^T \|\Psi(t)\|^2_{V^*}dt + E\int_0^T \|\Gamma(t)\|^2dt\Big)^{1/2},\end{aligned}$$

where $\tilde{c}$ is a positive constant independent of Ψ and Γ. By the Riesz Theorem it follows that there exist in a unique way processes

$$p \in \mathcal{L}^2_V(\Omega \times [0,T]), \quad q \in \mathcal{L}^2_H(\Omega \times [0,T])$$

such that

$$\begin{aligned} E\int_0^T \langle \Psi(t), p(t)\rangle dt &+ E\int_0^T \Big(\Gamma(t), q(t)\Big) dt \qquad (12)\\ &= E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big) dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{\Psi,\Gamma}(T)\Big)\end{aligned}$$

for all $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$.

Let

$$\mathcal{H}(t, v, \tilde{v}, x, y) := \mathcal{L}(t, x, y) + \langle -\mathcal{A}x + \mathcal{B}(x, x), v\rangle + (\mathcal{C}(t, x), \tilde{v}) + (y, v)$$

for $v, x \in V, v, \tilde{v}, y \in H$.

Lemma 4.1. *For all $\Upsilon \in \mathcal{U}^b$ we have*

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} = E\int_0^T \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big) dt \geq 0. \qquad (13)$$

Proof. Since $\mathcal{U}^b$ is convex it follows that $\Phi^* + \theta(\Upsilon - \Phi^*) \in \mathcal{U}^b$ for all $\theta \in [0,1]$. Equation (12) and Theorem 3.4 implies

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} = E\int_0^T \Big(\Upsilon(t) - \Phi^*(t), p(t)\Big) dt +$$

$$+E\int_0^T \Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big) dt.$$

Since Φ^* is an optimal control, we have

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} \geq 0.$$

Taking into account the definition of $\mathcal{H}$, it follows that (13) holds. □

The statement of **the stochastic minimum principle** is contained in the following theorem.

Theorem 4.2. *If $\Phi^* \in \mathcal{U}^b$ is an optimal control, then for all $h \in H$ with $||h|| \leq \rho$ the inequality*

$$\Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)] + p(t), h - \Phi^*(t)\Big) \geq 0 \tag{14}$$

holds for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0, T]$.

Proof. Let $h \in H$ with $||h|| \leq \rho$. We denote by

$$\xi(\omega, t) := \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big), \quad \mathcal{S} := \{(\omega, t) \in \Omega \times [0, T] | \xi(\omega, t) < 0\},$$

and $\mathcal{S}_t := \{\omega \in \Omega | \xi(\omega, t) < 0\}$ for each $t \in [0, T]$. Obviously, for each $t \in [0, T]$ the set $\mathcal{S}_t$ is $\mathcal{F}_t$-measurable. We take

$$\Upsilon(\omega, t) = \begin{cases} h & , \ \omega \in \mathcal{S}_t \\ \Phi^*(\omega, t) & , \ \omega \notin \mathcal{S}_t. \end{cases}$$

We see that $\Upsilon \in \mathcal{U}^b$ and

$$\Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big) = I_{\mathcal{S}_t}(\omega)\xi(\omega, t) \leq 0. \tag{15}$$

It follows from Lemma 4.1 and (15) that

$$0 \leq E \int_0^T \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big) dt$$

$$= \int_{\mathcal{S}} \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big) d(P \times \Lambda) = \int_{\mathcal{S}} \xi(\omega, t) d(P \times \Lambda) \leq 0.$$

Consequently, $(P \times \Lambda)(\mathcal{S}) = 0$ and therefore

$$\Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big) = \Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)] + p(t), h - \Phi^*(t)\Big) \geq 0$$

for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0, T]$ and all $h \in H$ with $||h|| \leq \rho$. $\square$

5. Equation of the adjoint processes

To complete the statement of the stochastic minimum principle, we need to derive the equation for the processes $\left(p(t)\right)_{t\in[0,T]}$ and $\left(q(t)\right)_{t\in[0,T]}$, called **adjoint equation**. We will use an approximation procedure and we will derive the equation for the approximation processes $\left(p_n(t)\right)_{t\in[0,T]}$, $\left(q_n(t)\right)_{t\in[0,T]}$ $(n \in \mathbb{N})$.

In this section we specialize the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ namely by $\left(\mathcal{F}_{[w(r):r\leq t]}\right)_{t\in[0,T]}$, which is the filtration generated by the Wiener process $\left(w(t)\right)_{t\in[0,T]}$.

Let $n \in \mathbb{N}$, $\psi \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$ and let $\Phi^* \in \mathcal{U}^b$ be an optimal control with $E\Delta^{-4}_{U_{\Phi^*}}(T) < \infty$ (Remark 2.5.2 in [5] contains sufficient conditions for this inequality). We consider $Z_{n,\psi,\gamma}$ to be the solution of

$$\begin{aligned}(Z_{n,\psi,\gamma}(t), v) + \int_0^t (\mathcal{A}_n Z_{n,\psi,\gamma}(s), v)ds &= \int_0^t (\mathcal{B}'_n(U^n_{\Phi^*}(s))(Z_{n,\psi,\gamma}(s)), v)ds \\ + \int_0^t (\psi(s), v)ds + \int_0^t (\mathcal{C}'_n(s, U_{\Phi^*}(s))(Z_{n,\psi,\gamma}(s)), v)dw(s) &+ \qquad (16) \\ + \int_0^t (\gamma(s), v)dw(s) &\end{aligned}$$

for all $v \in H_n$, $t \in [0,T]$ and a.e. $\omega \in \Omega$, where $\mathcal{B}'_n(x)(y) := \sum_{i=1}^n \langle \mathcal{B}'(x)(y), h_i\rangle h_i$ for all $x, y \in V$, $U^n_{\Phi^*} := \Pi_n U_{\Phi^*}$, $\mathcal{C}'_n := \Pi_n \mathcal{C}'$. This equation is $(P_{n,\psi,\gamma})$ from [5] applied to $a_0 := 0, X = Y := U_{\Phi^*}, \mathcal{G}_n(s,h) := \mathcal{C}'_n(s, U_{\Phi^*}(s))(h)$.

The mapping

$$\begin{aligned}(\psi,\gamma) \quad &\in \quad \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]) \times \mathcal{L}^2_{H_n}(\Omega \times [0,T]) \mapsto \\ &\mapsto \quad E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{n,\psi,\gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{n,\psi,\gamma}(T)\Big) \in \mathbb{R}\end{aligned}$$

is linear and continuous.

By the Riesz Theorem it follows that there exist in a unique way processes

$$p_n \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), q_n \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$$

such that

$$E\int_0^T \Big(\psi(t), p_n(t)\Big)dt \quad + \quad E\int_0^T \Big(\gamma(t), q_n(t)\Big)dt \tag{17}$$

$$= \quad E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{n,\psi,\gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{n,\psi,\gamma}(T)\Big)$$

for all $\psi \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), \gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$.

Let $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ and set

$$\Psi_n := \sum_{i=1}^n \langle \Psi, h_i \rangle h_i, \qquad \Gamma_n := \Pi_n \Gamma.$$

We have

$$E\int_0^T \langle \Psi(t), p_n(t) \rangle dt \quad + \quad E\int_0^T \Big(\Gamma(t), q_n(t)\Big)dt \tag{18}$$

$$= \quad E\int_0^T \Big(\Psi_n(t), p_n(t)\Big)dt + E\int_0^T \Big(\Gamma_n(t), q_n(t)\Big)dt$$

$$= \quad E\int_0^T \Big(\Delta^{-1}_{U_{\Phi^*}}(t)\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], \Delta_{U_{\Phi^*}}(t) Z_{n,\Psi_n,\Gamma_n}(t)\Big)dt$$

$$+ \quad E\Big(\Delta^{-1}_{U_{\Phi^*}}(T)\mathcal{K}'[U_{\Phi^*}(T)], \Delta_{U_{\Phi^*}}(T) Z_{n,\Psi_n,\Gamma_n}(T)\Big).$$

From the properties of the solution of the Navier-Stokes equation (see Theorem 2.2) and from the hypothesis on $\mathcal{L}$ and $\mathcal{K}$ we deduce that

$$\Delta^{-1}_{U_{\Phi^*}}(t)\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)] \in \mathcal{L}^2_H(\Omega \times [0,T]), \quad \Delta^{-1}_{U_{\Phi^*}}(T)\mathcal{K}'[U_{\Phi^*}(T)] \in \mathcal{L}^2_H(\Omega).$$

We have $\Psi = \lim_{n\to\infty} \Psi_n$ in the space $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$ and $\Gamma = \lim_{n\to\infty} \Gamma_n$ in the space $\mathcal{L}^2_H(\Omega \times [0,T])$. Now we use Lemma 1.3.2 from [5] and (12) in (18) to obtain that

$$\lim_{n\to\infty} \left\{ E\int_0^T \langle \Psi(t), p_n(t) \rangle dt + E\int_0^T \Big(\Gamma(t), q_n(t)\Big)dt \right\}$$

$$= \quad E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big)dt + +E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{\Psi,\Gamma}(T)\Big)$$

$$= \quad E\int_0^T \langle\Psi(t), p(t)\rangle dt + E\int_0^T \Big(\Gamma(t), q(t)\Big)dt,$$

for all $\Psi \in \mathcal{L}^2_{V^*}(\Omega\times[0,T]), \Gamma \in \mathcal{L}^2_H(\Omega\times[0,T])$. Hence, for $n \to \infty$ we have

$$p_n \rightharpoonup p \quad \text{in} \quad \mathcal{L}^2_V(\Omega\times[0,T]) \quad \text{and} \quad q_n \rightharpoonup q \quad \text{in} \quad \mathcal{L}^2_H(\Omega\times[0,T]). \qquad (19)$$

In (17) we take $\psi := p_n, \gamma := q_n$, use the weak convergence from above and Lemma 1.3.2 from [5]. Then

$$\lim_{n\to\infty}\Big\{E\int_0^T \|p_n(t)\|^2 dt \quad + \quad E\int_0^T \|q_n(t)\|^2 dt\Big\} = E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\mathbf{J}p,q}(t)\Big)dt \qquad (20)$$

$$+ \quad E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{\mathbf{J}p,q}(T)\Big) = E\int_0^T \|p(t)\|^2_V dt + E\int_0^T \|q(t)\|^2 dt.$$

From (19) and (20) it follows that the following strong convergences hold:

$$\lim_{n\to\infty} p_n = p \quad \text{in } \mathcal{L}^2_V(\Omega\times[0,T]) \quad \text{and} \quad \lim_{n\to\infty} q_n = q \quad \text{in } \mathcal{L}^2_H(\Omega\times[0,T]). \qquad (21)$$

Now we derive the equations for $\Big(p_n(t)\Big)_{t\in[0,T]}$ and $\Big(q_n(t)\Big)_{t\in[0,T]}$ $(n \in \mathbb{N})$ and then by passing to the limit obtain the equation for $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$.

We consider the following matrices:

$$\tilde{\mathcal{A}}_n := \Big(\langle\mathcal{A}h_j, h_i\rangle\Big)_{i,j=1,n}, \ I_n := \Big(\delta_{i,j}\Big)_{i,j=1,n},$$

$$\tilde{\mathcal{B}}_n(s) := \Big(\langle\mathcal{B}'(U^n_{\Phi^*}(s))(h_j), h_i\rangle\Big)_{i,j=1,n}, \ \tilde{\mathcal{C}}_n(s) := \Big((\mathcal{C}'(s, U_{\Phi^*}(s))(h_j), h_i)\Big)_{i,j=1,n}.$$

The last two matrices depend on s and ω and are $\mathcal{F}_s$-measurable.

For each natural number n we introduce the $n\times n$ matrix processes

$$\Big(X_n(t)\Big)_{t\in[0,T]} = \Bigg(\Big(X^{i,j}_n(t)\Big)_{i,j=1,n}\Bigg)_{t\in[0,T]},$$

$$\Big(Y_n(t)\Big)_{t\in[0,T]} = \Bigg(\Big(Y^{i,j}_n(t)\Big)_{i,j=1,n}\Bigg)_{t\in[0,T]}$$

as the solutions of the stochastic matrix equations

$$X_n(t) + \int_0^t \tilde{\mathcal{A}}_n X_n(s)ds = I_n + \int_0^t \tilde{\mathcal{B}}_n(s)X_n(s)ds + \int_0^t \tilde{\mathcal{C}}_n(s)X_n(s)dw(s) \tag{22}$$

and

$$\begin{aligned} Y_n(t) - \int_0^t Y_n(s)\tilde{\mathcal{A}}_n(s)ds &= I_n - \int_0^t Y_n(s)\tilde{\mathcal{B}}_n(s)ds + \int_0^t Y_n(s)\tilde{\mathcal{C}}_n(s)\tilde{\mathcal{C}}_n(s)ds + \\ &- \int_0^t Y_n(s)\tilde{\mathcal{C}}_n(s)dw(s) \end{aligned} \tag{23}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. To prove the existence and (almost surely) uniqueness of the solution of (22) and (23), respectively, we consider the above equations as linear evolution equations with the unknown variable X_n, respectively Y_n. Their coeficients may depend on ω and s (see $\tilde{\mathcal{B}}_n, \tilde{\mathcal{C}}_n$). We use the same techniques as in the investigation of the equations $(P_{\Psi,\Gamma})$, $(P_{n,\psi,\gamma})$ in Section 1.3. For each $i,j \in \{1,\dots,n\}$ the process $\left(Y_n^{i,j}(t)\right)_{t\in[0,T]}$ has continuous trajectories in $\mathbb{R}$.

Using the Ito formula we obtain

$$Y_n(t)X_n(t) = I_n \quad \text{for all } t \in [0,T], \text{ a.e. } \omega \in \Omega \tag{24}$$

and hence

$$X_n(t)Y_n(t) = I_n \quad \text{for all } t \in [0,T], \text{ a.e. } \omega \in \Omega. \tag{25}$$

If $M := \left(M_{i,j}\right)_{i,j=1,n}$ is a matrix of real numbers and $h \in H_n$, then we write

$$Mh := \sum_{i,j=1}^{n} M_{i,j}(h,h_j)h_i.$$

We write $\widehat{M}$ for the transposed matrix of M.

Theorem 5.1. *The processes* $\Big(p(t)\Big)_{t\in[0,T]}$ *and* $\Big(q(t)\Big)_{t\in[0,T]}$ *satisfy the adjoint equation*

$$(\mathcal{K}'[U_{\Phi^*}(T)] - p(t), v) - \int_t^T \langle \mathcal{A}v, p(s)\rangle ds$$

$$= \; -\int_t^T \langle \mathcal{B}(U_{\Phi^*}(s), v) + \mathcal{B}(v, U_{\Phi^*}(s)), p(s)\rangle ds - \int_t^T (\mathcal{L}_x[s, U_{\Phi^*}(s), \Phi^*(s)], v) ds$$

$$- \; \int_t^T (\mathcal{C}'(s, U_{\Phi^*}(s))(v), q(s)) ds + \int_t^T (q(s), v) dw(s),$$

for all $t \in [0,T], v \in V$ *and a.e.* $\omega \in \Omega$. *The processes* $\Big(p(t)\Big)_{t\in[0,T]}$ *and* $\Big(q(t)\Big)_{t\in[0,T]}$ *are uniquely characterized by this equation.*

Proof. Let $\psi \in \mathcal{D}_V(\Omega \times [0,T])$, $\gamma \in \mathcal{D}_H(\Omega \times [0,T])$ and we define for each $K \in \mathbb{N}$ the stopping time

$$\mathcal{T}_K^n := \min\{\mathcal{T}_K^{Y_n^{i,j}} \; : \; 1 \le i, j \le n\}$$

and we take

$$\psi_K := I_{[0,\mathcal{T}_K^n]}\psi, \;\; \gamma_K := I_{[0,\mathcal{T}_K^n]}\gamma.$$

We consider the H_n-valued process

$$W_n^K(t) := Y_n(t) Z_{n,\psi_K,\gamma_K}(t)$$

where the process $\Big(Z_{n,\psi_K,\gamma_K}(t)\Big)_{t\in[0,T]}$ is the solution of (16) (with ψ_K and γ_K instead of ψ and γ). Using (25) we obtain for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ that

$$Z_{n,\psi_K,\gamma_K}(t) = X_n(t) W_n^K(t). \tag{26}$$

Using (16), (23), and the Ito formula it follows that the process $\Big(W_n^K(t)\Big)_{t\in[0,T]}$ satisfies

$$(W_n^K(t), h) = \int_0^t (Y_n(s)\psi_K(s), h) ds - \int_0^t (Y_n(s)\tilde{\mathcal{C}}_n(s)\gamma_K(s), h) ds + \tag{27}$$

$$+ \int_0^t (Y_n(s)\gamma_K(s), h) dw(s)$$

for all $t \in [0,T]$, $h \in H_n$ and a.e. $\omega \in \Omega$.

We use (17) and (26) to obtain

$$E\int_0^T \Big(\psi_K(t), p_n(t)\Big)dt + E\int_0^T \Big(\gamma_K(t), q_n(t)\Big)dt \tag{28}$$

$$= \quad E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{n,\psi_K,\gamma_K}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{n,\psi_K,\gamma_K}(T)\Big)$$

$$= \quad E\int_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)], W_n^K(t)\Big)dt + E\Big(\widehat{X}_n(T)\mathcal{K}'_n[U_{\Phi^*}(T)], W_n^K(T)\Big),$$

where $\mathcal{L}_x^n(t,x,y) := \Pi_n\mathcal{L}_x(t,x,y), \mathcal{K}'_n(x) := \Pi_n\mathcal{K}'(x),\ t \in [0,T], x, y \in H$. Let us define the H_n-valued random variable

$$\xi_n = \widehat{X}_n(T)\mathcal{K}'_n[U_{\Phi^*}(T)] + \int_0^T \widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)]dt, \tag{29}$$

and the H_n-valued process

$$\zeta_n(t) = -\int_0^t \widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)]dt + E(\xi_n|\mathcal{F}_t)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. By the representation theorem of Levy (see [20], Theorem 4.15, p. 182 and Problem 4.17, p. 184) we have

$$E(\xi_n|\mathcal{F}_t) = E\xi_n + \int_0^t G_n(s)dw(s) \tag{30}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$, where $G_n \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$. Without loss of generality we can assume that the process $\Big(\zeta_n(t)\Big)_{t\in[0,T]}$ has continuous trajectories in H. We see that $\zeta_n(T) = \widehat{X}_n(T)\mathcal{K}'_n[U_{\Phi^*}(T)]$ for a.e. $\omega \in \Omega$.

By using (29), (5), and (30) we deduce by Ito's calculus that

$$E\Big(\zeta_n(T), W_n^K(T)\Big) = -E\int_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)], W_n^K(t)\Big)dt$$

$$+E\int_0^T \Big\{(\zeta_n(t), Y_n(t)\psi_K(t) - Y_n(t)\tilde{C}_n(t)\gamma_K(t)) + (G_n(t), Y_n(t)\gamma_K(t))\Big\}dt.$$

Here we have omitted to write explicitly an intermediate step, namely to consider stopping times for G_n. After taking the mathematical expectation in the above relation (with $\mathcal{T}_M^{G_n}$ instead of T) we let these stopping times to tend to T and use the almost surely continuity of the trajectories of ζ_n and W_n^K. Then we obtain the above equality.

Hence,

$$E\Big(\widehat{X}_n(T)\mathcal{K}_n'[U_{\Phi^*}(T)], W_n^K(T)\Big) + E\int_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)], W_n^K(t)\Big)dt$$

$$= E\int_0^T \Big(\widehat{Y}_n(t)\zeta_n(t), \psi_K(t)\Big)dt + E\int_0^T \Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}_n}(t)\widehat{Y}_n(t)\zeta_n(t), \gamma_K(t)\Big)dt.$$

The processes ψ, γ were arbitrary fixed, and by (17) and (26) it follows that

$$I_{[0,\mathcal{T}_K^n]}(t)p_n(t) = I_{[0,\mathcal{T}_K^n]}(t)\widehat{Y}_n(t)\zeta_n(t) \quad \text{for } P\times\Lambda \text{ a.e. } (\omega,t)\in\Omega\times[0,T] \tag{31}$$

and

$$I_{[0,\mathcal{T}_K^n]}(t)q_n(t) = I_{[0,\mathcal{T}_K^n]}(t)\Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}_n}(t)\widehat{Y}_n(t)\zeta_n(t)\Big) =$$
$$= I_{[0,\mathcal{T}_K^n]}(t)\Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}_n}(t)p_n(t)\Big)$$

for $P\times\Lambda$ a.e. $(\omega,t)\in\Omega\times[0,T]$. Since $\lim_{K\to\infty}\mathcal{T}_K^n = T$ for a.e. $\omega\in\Omega$ (see Proposition 6.1) and by using (31) we have

$$0 = \lim_{K\to\infty} E\int_0^{\mathcal{T}_K^n} \|p_n(t) - \widehat{Y}_n(t)\zeta_n(t)\|dt = E\int_0^T \|p_n(t) - \widehat{Y}_n(t)\zeta_n(t)\|dt.$$

This implies

$$p_n(t) = \widehat{Y}_n(t)\zeta_n(t) \quad \text{for } P\times\Lambda \text{ a.e. } (\omega,t)\in\Omega\times[0,T].$$

Analogously we obtain

$$q_n(t) = \widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}_n}(t)p_n(t)\Big) \quad \text{for } P\times\Lambda \text{ a.e. } (\omega,t)\in\Omega\times[0,T].$$

We can identify $\Big(p_n(t)\Big)_{t\in[0,T]}$ with a process which has continuous trajectories in H. Then for all $t\in[0,T]$ we have

$$p_n(t) = \widehat{Y}_n(t)\zeta_n(t) \quad \text{for all } t\in[0,T] \quad \text{and} \quad p_n(T) = \mathcal{K}_n'[U_{\Phi^*}(T)] \quad \text{for a.e } \omega\in\Omega.$$

By using the equations for $\left(\widehat{Y}_n(t)\right)_{t\in[0,T]}$ and $\left(\zeta_n(t)\right)_{t\in[0,T]}$ it follows by the Ito calculus that $\left(p_n(t)\right)_{t\in[0,T]}$ satisfies for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ the n-dimensional evolution equation

$$
\begin{aligned}
p_n(T) - p_n(t) - \int_t^T \widehat{\tilde{\mathcal{A}}}_n p_n(s)ds &= -\int_t^T \left\{\widehat{\tilde{\mathcal{B}}}_n(s)p_n(s) + \mathcal{L}_x^n[s, U_{\Phi^*}(s), \Phi^*(s)]\right\}ds \\
&- \int_t^T \widehat{\tilde{\mathcal{C}}}_n(s)q_n(s)ds + \int_t^T q_n(s)dw(s) \qquad (32)
\end{aligned}
$$

with $p_n(T) = \mathcal{K}_n'[U_{\Phi^*}(T)]$. Equation (32) can be written equivalently as

$$
\begin{aligned}
&(p_n(T) - p_n(t), v) - \int_t^T \langle \mathcal{A}v, p_n(s)\rangle ds \\
= &-\int_t^T \langle \mathcal{B}(U_{\Phi^*}^n(s), v) + \mathcal{B}(v, U_{\Phi^*}^n(s)), p_n(s)\rangle ds - \int_t^T (\mathcal{L}_x^n[s, U_{\Phi^*}(s), \Phi^*(s)], v)ds \\
- &\int_t^T \left(\mathcal{C}'(s, U_{\Phi^*}(s))(v), q_n(s)\right)ds + \int_t^T (q_n(s), v)dw(s),
\end{aligned}
$$

for all $t \in [0,T]$, $v \in H_n$ and a.e. $\omega \in \Omega$. In this equation we take the limit for $n \to \infty$, use (21), and obtain

$$
\begin{aligned}
&(\mathcal{K}'[U_{\Phi^*}(T)] - p(t), v) - \int_t^T \langle \mathcal{A}v, p(s)\rangle ds \qquad (33) \\
= &-\int_t^T \langle \mathcal{B}(U_{\Phi^*}(s), v) + \mathcal{B}(v, U_{\Phi^*}(s)), p(s)\rangle ds - \int_t^T (\mathcal{L}_x[s, U_{\Phi^*}(s), \Phi^*(s)], h)ds \\
- &\int_t^T \left(\mathcal{C}'(s, U_{\Phi^*}(s))(v), q(s)\right)ds + \int_t^T (q(s), v)dw(s),
\end{aligned}
$$

for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0,T]$ and all $v \in V$. We can identify $\left(p(t)\right)_{t\in[0,T]}$ with a process which has continuous trajectories in H and satisfies (33) for all $t \in [0,T]$ and a.e. $\omega \in \Omega$.

In order to show that (33) characterize in a unique way the adjoint processes $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$, let us take any processes $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$ which satisfy (33). Let $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T])$, $\Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ and let $Z_{\Psi,\Gamma}$ be the solution of (11). Then we have

$$E\Big(p(T), Z_{\Psi,\Gamma}(T)\Big) = E\int_0^T \Big\{\langle \mathcal{A}Z_{\Psi,\Gamma}(t), p(t)\rangle - \langle \mathcal{B}'(U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), p(t)\rangle$$

$$- \quad \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big) - \Big(\mathcal{C}'(t, U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), q(t)\Big)$$

$$-\langle \mathcal{A}Z_{\Psi,\Gamma}(t), p(t)\rangle + \langle \mathcal{B}'(U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), p(t)\rangle + \langle \Psi(t), p(t)\rangle$$

$$+\Big(\mathcal{C}'(t, U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), q(t)\Big) + (\Gamma(t), q(t))\Big\}dt.$$

Hence for all $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ we get

$$E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{\Psi,\Gamma}(T)\Big) \quad + \quad E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big)dt$$

$$= \quad E\int_0^T \langle \Psi(t), p(t)\rangle dt + E\int_0^T (\Gamma(t), q(t))dt.$$

Therefore, $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$ must be the processes that are uniquely defined in (12). □

6. Appendix

Let $\Big(Q(t)\Big)_{t\in[0,T]}$ be a V-valued process with

$$\int_0^T \|Q(s)\|^2_V ds < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \|Q(t)\|^2 < \infty$$

for a.e. $\omega \in \Omega$. For each $M \in \mathbb{N}$ we introduce the following **stopping times**

$$\widetilde{T}^Q_M = \begin{cases} T\,, & \text{if} \quad \sup\limits_{t\in[0,T]} \|Q(t)\|^2 < M \\ \inf\Big\{t \in [0,T] : \|Q(t)\|^2 \geq M\Big\}, & \text{otherwise,} \end{cases}$$

$$\widehat{\mathcal{T}}_M^Q = \begin{cases} T\,, & \text{if } \displaystyle\int_0^T \|Q(s)\|_V^2\,ds < M \\ \inf\Big\{t \in [0,T] : \displaystyle\int_0^t \|Q(s)\|_V^2\,ds \geq M\Big\}, & \text{otherwise.} \end{cases}$$

We define

$$\mathcal{T}_M^Q := \min\{\widetilde{\mathcal{T}}_M^Q, \widehat{\mathcal{T}}_M^Q\}.$$

We see that for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ we have

$$\|Q(t \wedge \mathcal{T}_M^Q)\|^2 \leq M, \qquad \int_0^{t\wedge\mathcal{T}_M^Q} \|Q(s)\|_V^2\,ds \leq M.$$

Proposition 6.1. *The following convergences hold:*

$$\lim_{M\to\infty} P(\mathcal{T}_M^Q < T) = 0$$

and

$$\lim_{M\to\infty} \mathcal{T}_M^Q = T \quad \textit{for a.e. } \omega \in \Omega.$$

Proof. Using some elementary inequalities we obtain

$$\begin{aligned} &\lim_{M\to\infty} P(\mathcal{T}_M^Q < T) \leq \lim_{M\to\infty} P(\widetilde{\mathcal{T}}_M^Q < T) + \lim_{M\to\infty} P(\widehat{\mathcal{T}}_M^Q < T) \\ \leq\;& \lim_{M\to\infty} P\Big(\sup_{t\in[0,T]} \|Q(t)\|^2 \geq M\Big) + \lim_{M\to\infty} P\Big(\int_0^T \|Q(s)\|_V^2\,ds\Big) \geq M\Big) \\ \leq\;& P\Big(\bigcap_{M=1}^{\infty} \Big\{\sup_{t\in[0,T]} \|Q(t)\|^2 \geq M\Big\}\Big) + P\Big(\bigcap_{M=1}^{\infty} \Big\{\int_0^T \|Q(s)\|_V^2 \geq M\Big\}\Big) = 0. \end{aligned}$$

The sequence $\left(T - \mathcal{T}_M^Q\right)$ is monotone decreasing (for a.e. $\omega \in \Omega$). We have proved above that it converges in probability to zero. Therefore it converges to zero for almost every $\omega \in \Omega$. $\square$

Proposition 6.2. *We assume that the following assumptions are fulfilled:*

(1) $k_1, k_2 > 0$ *are real numbers;*

(2) a_0 *is a* H*-valued* $\mathcal{F}_0$*-measurable random variable with* $E\|a_0\|^4 < \infty$*;*

(3) $F_1 \in \mathcal{L}^1_{\mathbb{R}}(\Omega \times [0,T]), F_2 \in \mathcal{L}^2_H(\Omega \times [0,T])$*;*

(4) $F_3 : [0,T] \times H \to H$ *is a mapping such that for all* $t \in [0,T], x \in H$ *we have* $\|F_3(t,x)\| \leq k_{F_3}\|x\|$ *with* k_{F_3} *a positive constant and* $F_3(\cdot, x) \in \mathcal{L}^2_H[0,T]$ *for all* $x \in H$;

(5) $\Big(Q(t)\Big)_{t\in[0,T]}$ *is a* V*-valued process with*

$$\int_0^T \|Q(s)\|_V^2 ds < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \|Q(t)\|^2 < \infty, \quad \text{for a.e. } \omega \in \Omega,$$

satisfying the inequality

$$\begin{aligned} \|Q(t)\|^2 + k_1 \int_0^t \|Q(s)\|_V^2 ds \quad &\leq \quad \|a_0\|^2 + k_2 \int_0^t \|Q(s)\|^2 ds \\ &+ \quad \int_0^t |F_1(s)| ds + \int_0^t (F_2(s) + F_3(s, Q(s)), Q(s)) dw(s) \end{aligned}$$

for all $t \in [0,T]$ *and a.e.* $\omega \in \Omega$.

Then there exists a positive constant c *(depending on* k_1, k_2, k_{F_3}, T*) such that*

$$\begin{aligned} &E \sup_{t\in[0,T]} \|Q(t)\|^2 + E\int_0^T \|Q(s)\|_V^2 ds \leq c\Big[E\|a_0\|^2 + E\int_0^T |F_1(s)| ds \\ &+E\int_0^T \|F_2(s)\|^2 ds\Big] \end{aligned} \tag{34}$$

and if $E\int_0^T |F_1(s)|^2 ds < \infty$, $E\int_0^T \|F_2(s)\|^4 ds < \infty$ *then*

$$\begin{aligned} &E \sup_{t\in[0,T]} \|Q(t)\|^4 + E\Big(\int_0^T \|Q(s)\|_V^2 ds\Big)^2 \leq c\Big[E\|a_0\|^4 + E\int_0^T |F_1(s)|^2 ds \\ &+E\int_0^T \|F_2(s)\|^4 ds\Big]. \end{aligned} \tag{35}$$

Proof. We consider the stopping times $\mathcal{T}_M := \mathcal{T}_M^Q$, $M \in \mathbb{N}$. Using (5) it follows that for all $t \in [0,T]$

$$\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^2 + k_1 \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds \le 2\|a_0\|^2 + 2k_2 \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds$$
$$+ \quad 2 \int_0^{t\wedge\mathcal{T}_M} |F_1(s)|ds + 2 \sup_{s\in[0,t\wedge\mathcal{T}_M]} \Big| \int_0^s (F_2(r) + F_3(r,Q(r)), Q(r)) dw(r) \Big|.$$

and

$$\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^4 + k_1^2 \Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds\Big)^2 \le 16\|a_0\|^4 + 16k_2^2 \Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds\Big)^2$$
$$+ \quad 16 \Big(\int_0^{t\wedge\mathcal{T}_M} |F_1(s)|ds\Big)^2 + 16 \sup_{s\in[0,t\wedge\mathcal{T}_M]} \Big| \int_0^s (F_2(r) + F_3(r,Q(r)), Q(r)) dw(r) \Big|^2.$$

Now we use the Burkholder inequality (see [20], p. 166) and the Schwarz inequality to obtain

$$E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^2 + k_1 E \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds \le 2E\|a_0\|^2 + 2k_2 E \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds$$
$$+2E \int_0^{t\wedge\mathcal{T}_M} |F_1(s)|ds + \frac{1}{2} E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^2 + c_1 E \int_0^{t\wedge\mathcal{T}_M} \|F_2(s) + F_3(s,Q(s))\|^2 ds$$

and

$$E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^4 + k_1^2 E \Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds\Big)^2 \le 16E\|a_0\|^4 + 16k_2^2 T E \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^4 ds$$
$$+16TE \int_0^{t\wedge\mathcal{T}_M} |F_1(s)|^2 ds + \frac{1}{2} E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^4 + c_2 E \int_0^{t\wedge\mathcal{T}_M} \|F_2(s) + F_3(s,Q(s))\|^4 ds$$

for all $t \in [0,T]$, where c_1, c_2 are positive constants. Consequently, for all $t \in [0,T]$, we have

$$E \sup_{s\in[0,t]} I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|^2 + 2k_1 E\int_0^t I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|_V^2\,ds \le 4E\|a_0\|^2$$

$$+4(k_2 + c_1 k_{F_3})E\int_0^t \sup_{r\in[0,s]} I_{[0,\mathcal{T}_M]}(r)\|Q(r)\|^2 dr + 4E\int_0^T |F_1(s)|ds + 4c_1 E\int_0^T \|F_2(s)\|^2 ds$$

and

$$E \sup_{s\in[0,t]} I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|^4 + 2k_1^2 E\Big(\int_0^t I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|_V^2\,ds\Big)^2 \le 32E\|a_0\|^4$$

$$+(32k_2^2 T + 16c_2 k_{F_3})E\int_0^t \sup_{r\in[0,s]} I_{[0,\mathcal{T}_M]}(r)\|Q(r)\|^4 dr + 32TE\int_0^T |F_1(s)|^2 ds$$

$$+16c_2 E\int_0^T \|F_2(s)\|^4 ds.$$

By Gronwall's Lemma it follows that there exists a positive constant c^* (independent of M) such that

$$E \sup_{s\in[0,T\wedge\mathcal{T}_M]} \|Q(s)\|^2 + 2k_1 E \int_0^{T\wedge\mathcal{T}_M} \|Q(s)\|_V^2\,ds \le c^*\Big[E\|a_0\|^2 + E\int_0^T |F_1(s)|ds + E\int_0^T \|F_2(s)\|^2 ds\Big]$$

and

$$E \sup_{s\in[0,T\wedge\mathcal{T}_M]} \|Q(s)\|^4 + 2k_1^2 E\Big(E \int_0^{T\wedge\mathcal{T}_M} \|Q(s)\|_V^2\,ds\Big)^2 \le$$

$$\le c^*\Big[E\|a_0\|^4 + E\int_0^T |F_1(s)|^2 ds + E\int_0^T \|F_2(s)\|^4 ds\Big].$$

Now we use Proposition 6.1, take the limit $M \to \infty$ in the above inequalities and obtain (34) and (35). $\square$

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SOME APPLICATIONS OF THE $C-$ WEAKLY PICARD OPERATORS TO AN INTEGRAL EQUATION WITH DEVIATING ARGUMENTS

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Abstract. For an integral equation with deviating arguments theorems which prove existence, data dependence and differentiability with respect to a parameter of the solutions set are given.

1. Preliminaries

The aim of this paper is to study the existence, data dependence and differentiability with respect to a parameter of the solutions for an integral equation with deviating argument applying some results for $c-$ weakly Picard operators.

Throughout this paper we follow the notations from [3] and [4]. For the convenience we recall some of them.

Let (X, d) be a metric space and $f : X \to X$ an operator. We denote:

$F_f = \{x \in X \mid f(x) = x\}$;

$P(X) = \{Y \subseteq X \mid Y \text{ is nonempty}\}$;

$P_{cl}(X) = \{Y \subseteq X \mid Y \text{ is closed}\}$

$I(f) = \{A \in P(X) \mid f(A) \subseteq A\}$.

The functional $H : P_{cl}(X) \times P_{cl}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left(\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\right)$$

is the generalized Pompeiu - Hausdorff distance.

Definition 1.1.([3], [5], [4]) The operator $f : (X, d) \to (X, d)$ is called:

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i) *Picard operator* (P.o.) if $F_f = \{x^*\}$ and for all $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^*;

ii) *weakly Picard operator* (w.P.o.) if, for all $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges and his limit is a fixed point of f denoted by $f^\infty(x)$.

We present a theorem given by Rus in [6] inspired by Fiber Contraction Theorem (Hirsch, Pugh in [1]) which offer examples of P.o. and w.P.o.

Theorem 1.1. *Let (X,d) and (Y,ρ) be two metric spaces and the operators*

$$B : X \to X,\ C : X \times Y \to Y,$$

$$A : X \times Y \to X \times Y,\ A(x,y) = (B(x), C(x,y))$$

We suppose that:

i) (Y,ρ) *is a complete metric space,*

ii) *there exists $\alpha \in [0,1)$ such that, for all $x \in X$, $C(x,.)$ is $\alpha-$ contraction.*

We have:

a) *if B is P.o. then A is P.o.*

b) *if B is w.P.o. then A is w.P.o.*

For other examples of P.o. and w.P.o. see [3], [4] and [5].

Definition 1.2. Let $c \in (0,\infty)$. An operator $f : X \to X$ is $c-$ *weakly Picard operator* if it is w.P.o. and

$$d(x, f^\infty(x)) \leq c \cdot d(x, f(x)),\ \text{for any } x \in X.$$

Example 1.1. Let (X,d) be a complete metric space and $f : X \to X$ an $a-$contraction.

Then f is $\frac{1}{1-a}-$ w.P.o.

Example 1.2. Let (X,d) be complet metric space, $a \in (0,1)$ and $f : X \to X$ a continous operator such that

$$d(f(x), f^2(x)) \leq a \cdot d(x, f(x))\ \text{for all } x \in X.$$

Then f is $\frac{1}{1-a}-$ w.P.o.

Example 1.3. Let (X,d) be complet metric space and $f : X \to X$. Let $X = \bigcup\limits_{i \in I} X_i$ a partition of X such that $X_i \in I(f), i \in I$. If $f|_{X_i}: X_i \to X_i$, $i \in I$, are $c-$ P.o., then f is $\frac{1}{1-c}-$ w.P.o.

For $c-$ w.P.o. we have the following data dependence theorem (see [8] and also [7], [9]):

Theorem 1.2. *Let (X,d) be a metric space and $f_i : X \to X, i = 1,2$, two operators. We suppose that:*

i) the operators f_i are c_i- w.P.o., $i = 1,2$;

ii) there exists $\eta > 0$ such that $d(f_1(x), f_2(x)) \leq \eta$, for all $x \in X$.

Then $H_d(F_{f_1}, F_{f_2}) \leq \eta \cdot \max(c_1, c_2)$.

2. Data dependence of the solutions set of an integral equation with deviating arguments.

We consider the following integral equation:

$$x(t) = h(t, x(a), x(t)) + \int_a^t K(t, s, x(g(s)))ds, t \in [a,b]. \tag{1}$$

For the conditions:

I) $h \in C([a,b] \times \mathbb{R} \times \mathbb{R})$ and $h(a, \alpha, \alpha) = \alpha$, for all $\alpha \in \mathbb{R}$;

II) there exists $L_h \in (0,1)$ such that

$$|h(t, \alpha, u) - h(t, \alpha, v)| \leq L_h \cdot |u - v|,$$

for all $t \in [a,b]$ and for all $\alpha, u, v \in \mathbb{R}$;

III) $K \in C([a,b] \times [a,b] \times \mathbb{R})$,

IV) there exists $L_K > 0$ such that:

$$|K(t, s, u) - K(t, s, v)| \leq L_K \cdot |u - v|,$$

for all $t, s \in [a,b]$ and for all $u, v \in \mathbb{R}$;

V) $g \in C[a,b]$ and $a \leq g(t) \leq t \leq b$,

we have:

Theorem 2.1. *In the conditions $I) - V)$ the equation (1) has at least one solution which is a limit of a Picard iteration sequence $(x_n)_{n \in \mathbb{N}}$ in $C[a,b]$ for any $x_0 \in C[a,b]$.*

Proof. We consider the Banach space $X = C[a,b]$ with Bielecki norm

$$\|x\|_\tau = \max_{a \leq t \leq b} |x(t)| \cdot e^{-\tau(t-a)}, \ \tau > 0.$$

Let $\alpha \in \mathbb{R}$ and $X_\alpha = \{x \in C[a,b] \mid x(a = \alpha)\}$.
Let $B_\alpha : X_\alpha \to C[a,b]$,

$$B_\alpha(x)(t) = h(t, x(a), x(t)) + \int_a^t K(t,s,x(g(s)))ds, t \in [a,b].$$

From $I)$ we have that $X_\alpha \in I(B_\alpha)$, for all $\alpha \in \mathbb{R}$.
If we take $\tau > 0$ such that $L_h + \frac{L_K}{\tau} < 1$ then, from $II)$ and $IV)$, B_α is a $L_h + \frac{L_K}{\tau}-$ contraction,for all $\alpha \in \mathbb{R}$.
But $X = \bigcup\limits_{\alpha \in \mathbb{R}} X_\alpha$ is a partition of X and then, from Example 1.3, the operator

$$B : X \to X, \ B(x) = B_\alpha(x), \text{ for all } x \in X_\alpha$$

is a $\frac{1}{1 - L_h - \frac{L_K}{\tau}}-$ w.P.o. □

We will consider now the following equations

$$x(t) = h_i(t, x(a), x(t)) + \int_a^t K_i(t,s,x(g(s)))ds, t \in [a,b]. \ (i = 1,2) \qquad (2)$$

Supposing that h_i, K_i and g, for $i = 1,2$, are in the conditions $I) - V)$ we have the following data dependence theorem:

Theorem 2.2. *Let S_i be the solutions sets of the equations (2), $(\ i = 1,2)$. If there exists $\eta_1, \eta_2 > 0$ such that*

$$|h_1(t,\alpha,u) - h_2(t,\alpha,u)| \leq \eta_1,$$

for all $t \in [a,b]$ *and for all* $\alpha, u \in \mathbb{R}$ *and*

$$|K_1(t,s,u) - K_2(t,s,u)| \leq \eta_2,$$

for all $t, s \in [a,b]$ *and for all* $u \in \mathbb{R}$,
then

$$H_{\|\cdot\|_\tau}(S_1, S_2) \leq \frac{\eta_1 + (b-a)\eta_2}{1 - L_h - \frac{L_K}{\tau}},$$

for all $\tau > 0$ *with* $L_h + \frac{L_K}{\tau} < 1$.

Proof. With X_α from previous theorem we define, for $i = 1, 2$, the operators $B_\alpha^i : X_\alpha \to X_\alpha$, where, for all $x \in X_\alpha$,

$$B_\alpha^i(x)(t) = h_i(t, x(a), x(t)) + \int_a^t K_i(t, s, x(g(s)))ds, t \in [a,b]$$

and

$$B^i : X \to X, B^i(x) = B_\alpha^i(x), \text{ for all } x \in X_\alpha.$$

From Theorem 2.1 B^i are $\frac{1}{1-L_h-\frac{L_K}{\tau}}-$ w.P.o so S_i are nonempty, for $i = 1, 2$. Also, for all $x \in X$, we have

$$\left\|B^1(x) - B^2(x)\right\|_\tau \leq \left|B^1(x)(t) - B^2(x)(t)\right| \leq \eta_1 + (b-a)\eta_2.$$

Now we apply Theorem 1.2. □

3. Differentiability of the solutions set for an integral equation with respect to a parameter.

For all $\lambda \in [c,d]$ we consider the following integral equation

$$x(t,\lambda) = h(t, x(a,\lambda), x(t,\lambda)) + \int_a^t K(t, s, x(g(s), \lambda), \lambda)ds, t \in [a,b]. \tag{3}$$

If we suppose

I') $h \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$ and $h(a, \alpha, \alpha) = \alpha$, for all $\alpha \in \mathbb{R}$;

II') there exists $L_h \in (0,1)$ such that

$$|h(t,\alpha,u) - h(t,\alpha,v)| \le L_h \cdot |u-v|,$$

for all $t \in [a,b]$ and for all $\alpha, u, v \in \mathbb{R}$;

III') $K \in C^1([a,b] \times [a,b] \times \mathbb{R} \times [c,d])$,

IV') there exists $L_K > 0$ such that:

$$|K(t,s,u,\lambda) - K(t,s,v,\lambda)| \le L_K \cdot |u-v|,$$

for all $t, s \in [a,b]$, for all $u, v \in \mathbb{R}$ and for all $\lambda \in [c,d]$;

V') $g \in C[a,b]$ and $a \le g(t) \le t \le b$,

we have:

Theorem 3.1. *In the conditions $I') - V')$ equation (3) have at least one solution which is differentiable with respect to a parameter λ.*

Proof. In the same way to the proof of Theorem 2.1, we consider the Banach space $X = C([a,b] \times [c,d])$ with Bielecki norm

$$\|x\|_\tau = \max_{a \le t \le b, c \le \lambda \le d} |x(t,\lambda)| \cdot e^{-\tau[(t-a)+(\lambda-c)]},\ \tau > 0 \tag{4}$$

and a partition of this space, $X = \bigcup_{\alpha \in \mathbb{R}} X_\alpha$, where, for $\alpha \in \mathbb{R}$, we have

$$X_\alpha = \{x \in C([a,b] \times [c,d]) \mid x(t,\lambda) = a,\ \text{for all } \lambda \in [c,d]\}.$$

For all $\alpha \in \mathbb{R}$, we define the operator

$$B_\alpha : X_\alpha \to C^1([a,b] \times [c,d]),$$

$$B_\alpha(x)(t,\lambda) = h(t, x(a,\lambda), x(t,\lambda)) + \int_a^t K(t,s,x(g(s),\lambda).\lambda)ds.$$

From $I')$ and $III')$ we have that $X_\alpha \in I(B_\alpha)$ and from $II')$ and $IV')$ we have that B_α is a $L_h + \frac{L_K}{\tau}-$ contraction (where $\tau > 0$ is such that $L_h + \frac{L_K}{\tau} < 1$) for all $\alpha \in \mathbb{R}$. From Example 1.3 we obtain that

$$B : X \to X, B(x) = B_\alpha(x),\ \text{for all } x \in X_\alpha,$$

is a $\frac{1}{1-L_h-\frac{L_K}{\tau}}-$ w.P.o so for any $x_0 \in X$ the sequence of Picard iterations $(x_{n+1})_{n\in\mathbb{N}} \subset X_{x_0(a,\lambda)}$, with $x_{n+1} = B(x_n)$ converges uniformly to $B^{\infty}(x_0) \in X_{x_0(a,\lambda)}$ which is a solution for the equation (3).

We consider now the Banach space $Y = C([a,b] \times [c,d])$ with the same Bielecki norm (4) and define

$$C : X \times Y \to Y,$$

$$C(x,y)(t,\lambda) = \frac{\partial h(t,x(a,\lambda),x(t,\lambda))}{\partial x(a,\lambda)} \cdot \frac{\partial x(a,\lambda)}{\partial \lambda} + \frac{\partial h(t,x(a,\lambda),x(t,\lambda))}{\partial x(t,\lambda)} \cdot y(t,\lambda) - \\ + \int_a^t \frac{\partial K(t,s,x(g(s),\lambda),\lambda)}{\partial x} \cdot y(s,\lambda)\,ds + \int_a^t \frac{\partial K(t,s,x(g(s),\lambda),\lambda)}{\partial \lambda} ds.$$

For any $\tau > 0$ such that $L_h + \frac{L_K}{\tau} < 1$ and for all $x \in X$, from II') and IV'), the operator $C(x,\cdot)$ is $L_h + \frac{L_K}{\tau}-$ contraction.

Applying Theorem 1.1, the operator

$$A : X \times Y \to X \times Y,\ A(x,y) = (B(x), C(x,y))$$

is a w.P.o. which implies that the sequence of the Picard iterations $\left(A^{n+1}(x_0,y_0)\right)_{n\in\mathbb{N}}$ converges uniformly to $A^{\infty}(x_0,y_0) \in F_A$ for all $(x_0,y_0) \in X \times Y$. For all $n \in \mathbb{N}$, $A^{n+1}(x_0,y_0)$ has the form (x_{n+1},y_{n+1}) where

$$x_{n+1} = B(x_n) \text{ and } y_{n+1} = C(x_n,y_n).$$

For any $x_0 \in X$ (in fact $x_0 \in X_{\alpha}$, with $\alpha = x_0(t,\lambda)$) if we put $y_0 = \frac{\partial x_0}{\partial \lambda}$ then, by induction, $y_{n+1} = \frac{\partial x_n}{\partial \lambda}$, for all $n \in \mathbb{N}$.

Because $x_n \overset{unif}{\longrightarrow} B^{\infty}(x_0)$ and $y_n \overset{unif}{\longrightarrow} y^*$, using a Wierstrass argument, we have that $B^{\infty}(x_0)(t,\lambda)$ is differentiable with respect to λ , $\frac{\partial B^{\infty}(x_0)(t,\lambda)}{\partial \lambda} = y^*(t,\lambda)$ and

$$A^{\infty}\left(x_0, \frac{\partial x_0}{\partial \lambda}\right) = \left(B^{\infty}(x_0), \frac{\partial B^{\infty}(x_0)(t,\lambda)}{\partial \lambda}\right) \in F_A.$$

□

4. **Examples.**

1. For equation (1) if we consider

$$h(t,x(a),x(t)) = \overline{h}(t,x(a))$$

and the condition $I)$ is replaced by

$I_1)$ $\overline{h} \in C([a,b] \times \mathbb{R})$ and $\overline{h}(a,\alpha) = \alpha$, $\forall \alpha \in \mathbb{R}$,

and suppose that $I_1)$, $II) - V)$ holds then, using Theorems 2.1 and 2.2, we obtain data dependence of the solutions set of the equation

$$x(t) = \overline{h}(t, x(a)) + \int_a^t K(t, s, x(g(s)))\, ds$$

given in [8].

2. For the equation

$$x(t) = f(x(t)) + x(0) - f(x(0)) + \int_0^t k \cdot a(s) \cdot x\left(\frac{s}{k}\right) ds,$$

where $t \in [0,b]$ and $k \geq 1$, if we consider the following:

i) $f \in C(\mathbb{R})$;

ii) there exists $L > 0$ such that:

$$|f(u) - f(v)| \leq L \cdot |u - v|, \text{ for all } u, v \in \mathbb{R};$$

iii) $a \in C[0,b]$,

then we can apply Theorem 2.1 for the existence of the solutions set, where $h(t,\alpha,u) = f(u) + \alpha - f(\alpha)$, $K(t,s,u) = k \cdot a(s) \cdot u$, $g(s) = \frac{s}{k}$, $L_h = L$ and $L_K = \max_{0 \leq t \leq b} |a(t)|$.

Moreover, if we replaced $i)$ and $ii)$ with

i') $f \in C^1(\mathbb{R})$;

iii') $a \in C^1[0,b]$,

respectively, and consider that $ii)$ and

$iv)$ $k \in [c,d] \subset [1,\infty)$ also holds

then, from Theorem 3.1, we obtain differentiability with respect to k.

For data dependence of the equation's solutions we will consider the following equations

$$x(t) = f_i(x(t)) + x(0) - f_i(x(0)) + \int_0^t k \cdot a_i(s) \cdot x\left(\frac{s}{k}\right) ds, \ i = 1, 2, \tag{5}$$

where $t \in [0, b]$ and $k \geq 1$.

Supposing that

- f_i and a_i are in the conditions $i), ii)$ and $iii)$, for $i = 1, 2$
- there exists $\eta_1, \eta_2 > 0$ such that

$$|f_1(u) - f_2(u)| \leq \eta_1, \ \forall u \in \mathbb{R},$$

$$|a_1(t) - a_2(t)| \leq \eta_2 \ \forall t \in [0, b]$$

then, from Theorem 2.2, we have that

$$H_{\|\cdot\|_\tau}\left(S_1', S_2'\right) \leq \frac{2\eta_1 + b \cdot k \cdot \eta_2}{1 - L - \frac{M}{\tau}},$$

where $M = \max_{0 \leq t \leq b} (|a_1(t)|, |a_2(t)|)$, $\tau > 0$ is such that $1 - L - \frac{M}{\tau} < 1$ and S_i' denoted the solutions sets of the equations (5).

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ON THE STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION

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Abstract. In this paper, we will survey a number of valuable results concerning the stability of the quadratic functional equation (3) including the cases when the relevant domains are restricted. In particular, the stability problems of approximately quadratic functions defined on bounded subsets of $\mathbf{R}^n$ will be emphasized.

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms (cf. [50]):

> *Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d\left(h(xy), h(x)h(y)\right) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d\left(h(x), H(x)\right) < \varepsilon$ for all $x \in G_1$?*

For the case where the answer is affirmative, the functional equation for homomorphisms will be called *stable*.

The first result concerning the stability of functional equations was presented by D. H. Hyers. He has excellently answered the question of Ulam for the case where G_1 and G_2 are Banach spaces (see [13]). This result of Hyers states as follows:

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Theorem (D. H. Hyers). *If a function $f : E_1 \to E_2$ defined between Banach spaces satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \delta \tag{1}$$

for some $\delta > 0$ and for all $x, y \in E_1$, then the limit

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \tag{2}$$

exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \le \delta$$

for every $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Taking this famous result into consideration, the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is said to have the *Hyers-Ulam stability on* (E_1, E_2) if for each function $f : E_1 \to E_2$ satisfying the inequality (1) for some $\delta \ge 0$ and for all $x, y \in E_1$ there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on E_1.

The method in (2) which was provided by Hyers and which produces the additive function A will be called a *direct method*. This method is the most important and the most powerful tool for studying the stability of various functional equations.

In 1978, Th. M. Rassias addressed the Hyers's stability theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference $f(x+y) - f(x) - f(y)$ and generalized the result of Hyers by making use of a direct method:

Theorem (Th. M. Rassias). *If a function $f : E_1 \to E_2$ between Banach spaces satisfies the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\,(\|x\|^p + \|y\|^p)$$

for some $\theta \ge 0$, some p with $0 \le p < 1$ and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^p}\, \|x\|^p$$

for all $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Since then, a number of results concerning the stability of various functional equations have been obtained (see the references below). The stability problems of the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{3}$$

have been extensively investigated by many mathematicians (see [6], [7], [8], [14], [15], [21], [27], [28], [29], [30], [40], [47], [48] and [49]). One can verify without difficulty that the quadratic function $f(x) = cx^2$ $(x \in \mathbf{R})$, where c is a real constant, satisfies the equation (3). Hence, the equation (3) is called the *quadratic functional equation*. We promise that by a *quadratic function* we mean a solution of the quadratic functional equation (3).

A function $f : E_1 \to E_2$ between real vector spaces is a quadratic function if and only if there exists a unique symmetric biadditive function $B : E_1^2 \to E_2$ such that $f(x) = B(x,x)$. The biadditive function B is sometimes called the *polar* of f and given by

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)).$$

In this paper, a number of valuable results in connection with the stability of the quadratic functional equation will be introduced. In particular, the stability problems of approximately quadratic functions defined on bounded subsets of $\mathbf{R}^n$ will be extensively presented. In the last section, we will also deal with the stability problems for restricted unbounded domains.

We will use the following notations. $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{Z}$ denote the set of positive integers, of rational numbers, of real numbers and of integers, respectively.

2. Modified Hyers-Ulam stability

The Hyers-Ulam stability of the quadratic functional equation (3) was first proved by F. Skof [47] for functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2

is a Banach space. P. W. Cholewa [5] demonstrated that Skof's theorem is also valid if E_1 is replaced by an abelian group G. The Skof-Cholewa theorem states as follows:

Theorem 1. *Let G be an abelian group and let E be a Banach space. If a function $f : G \to E$ satisfies the inequality*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leq \delta$$

for some $\delta \geq 0$ and for all $x, y \in G$, then there exists a unique quadratic function $Q : G \to E$ such that

$$\|f(x)-Q(x)\| \leq \frac{1}{2}\delta$$

for any x in G.

The proof is a special case of that of Theorem 3 by S. Czerwik [6], and hence, it is omitted.

In this section, we will mainly review the paper [6] of S. Czerwik. Before starting the theorem of Czerwik, we need a lemma provided by the same author.

Lemma 2. *Let E_1 and E_2 be normed spaces. Assume that there exist $\delta, \theta \geq 0$ and $p \in \mathbf{R}$ such that a function $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leq \delta + \theta(\|x\|^p + \|y\|^p) \tag{4}$$

for all $x, y \in E_1 \setminus \{0\}$. Then for $x \in E_1 \setminus \{0\}$ and $n \in \mathbf{N}$

$$\|f(2^n x) - 4^n f(x)\| \leq \frac{1}{3}(4^n - 1)(\delta + c) + 2 \cdot 4^{n-1}\theta\, \|x\|^p (1 + a + \cdots + a^{n-1}) \tag{5}$$

and

$$\left\| f(x) - 4^n f\Big(\frac{x}{2^n}\Big)\right\| \leq \frac{1}{3}(4^n - 1)(\delta + c) + 2^{1-p}\theta\, \|x\|^p (1 + b + \cdots + b^{n-1}), \tag{6}$$

where $a = 2^{p-2}$, $b = 2^{2-p}$ and $c = \|f(0)\|$.

Proof. Putting $x = y \neq 0$ in (4) yields

$$\|f(2x) - 4f(x)\| \leq \|f(0)\| + \delta + 2\theta\, \|x\|^p$$

which proves (5) for $n = 1$. Now, assume that (5) holds for $k \leq n$ and $x \in E_1 \setminus \{0\}$. Then for $n + 1$ we have

$$\begin{aligned}
\|f(2^{n+1}x) - 4^{n+1}f(x)\| &\leq \|f(2 \cdot 2^n x) - 4f(2^n x)\| + 4\|f(2^n x) - 4^n f(x)\| \\
&\leq \delta + c + 2\theta\,\|2^n x\|^p + \frac{4}{3}(\delta + c)(4^n - 1) \\
&\quad + 2 \cdot 4^n \theta\,\|x\|^p(1 + a + \cdots + a^{n-1}) \\
&= \frac{1}{3}(4^{n+1} - 1)(\delta + c) + 2 \cdot 4^n \theta\,\|x\|^p(1 + a + \cdots + a^n),
\end{aligned}$$

which proves the validity of the inequality (5).

Similarly, taking $x = y = t/2$, we can verify the inequality (6) for $n = 1$. Applying the induction principle we get the result for all $n \in \mathbf{N}$, which completes the proof. □

In the following 2 theorems, we prove the stability of the quadratic functional equation (3) for the cases $p < 2$ and $p > 2$ separately.

Theorem 3. *Let E_1 be a normed space and let E_2 be a Banach space. If a function $f : E_1 \to E_2$ satisfies the inequality (4) for some $\delta,\ \theta \geq 0,\ p < 2$ and for all $x, y \in E_1 \setminus \{0\}$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{3}(\delta + c) + \frac{2\theta}{4 - 2^p}\,\|x\|^p \tag{7}$$

for any $x \in E_1 \setminus \{0\}$.

Proof. Let us define

$$Q_n(x) = \frac{1}{4^n} f(2^n x) \tag{8}$$

for $x \in E_1$ and $n \in \mathbf{N}$. Then $\{Q_n(x)\}$ is a Cauchy sequence for every $x \in E_1$. Really, for $x = 0$ it is trivial. Let $x \in E_1 \setminus \{0\}$. For $n > m$ we obtain by (5)

$$\begin{aligned}
\|Q_n(x) - Q_m(x)\| &= \frac{1}{4^n}\,\|f(2^{n-m} \cdot 2^m x) - 4^{n-m} f(2^m x)\| \\
&\leq \frac{1}{4^n}\frac{1}{3}(4^{n-m} - 1)(\delta + c) \\
&\quad + \frac{2\theta}{4^{m+1}}\,\|2^m x\|^p(1 + a + \cdots + a^{n-m-1})
\end{aligned}$$

and hence

$$||Q_n(x) - Q_m(x)|| \leq \frac{1}{3}\frac{1}{4^m}(\delta + c) + \frac{1}{2^{m(2-p)+1}}\frac{\theta}{1-a}\,||x||^p. \tag{9}$$

The assumption $p < 2$ implies that $\{Q_n(x)\}$ is a Cauchy sequence. Since E_2 is complete, we can define

$$Q(x) = \lim_{n\to\infty} Q_n(x)$$

for any $x \in E_1$. We will check that Q is a quadratic function. It is clear for $x = y = 0$, since $Q(0) = 0$. For $y = 0$ and $x \neq 0$ we have

$$Q(x+0) + Q(x-0) - 2Q(x) - 2Q(0) = 0.$$

Let us now consider the case $x, y \in E_1 \setminus \{0\}$. Then,

$$\begin{aligned}
&||Q_n(x+y) + Q_n(x-y) - 2Q_n(x) - 2Q_n(y)|| \\
&= \frac{1}{4^n}\,||f(2^n(x+y)) - f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)|| \\
&\leq \frac{1}{4^n}\,(\delta + \theta(||2^n x||^p + ||2^n y||^p)) \\
&\leq \frac{\delta}{4^n} + \frac{\theta}{2^{n(2-p)}}(||x||^p + ||y||^p).
\end{aligned}$$

By letting $n \to \infty$ we get the equality

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0.$$

Consider the case $x = 0$ and $y \neq 0$. If we put $x = y$ in the last equality, then we get $Q(2x) = 4Q(x)$ for each x in E_1. Moreover, putting $y = -x$ we obtain $Q(-x) = Q(x)$ for $x \in E_1$. Therefore,

$$Q(y) + Q(-y) - 2Q(0) - 2Q(y) = 0,$$

i.e., Q is a quadratic function.

The inequality (7) immediately follows from the inequality (5).

To prove the uniqueness, assume that there exist two quadratic functions $q_i : E_1 \to E_2$ $(i = 1, 2)$ such that

$$||f(x) - q_i(x)|| \leq c_i + b_i||x||^p$$

for any $x \in E_1 \setminus \{0\}$ and $i = 1,2$, where c_i, b_i $(i = 1,2)$ are given non-negative constants. Then,

$$q_i(2^n x) = 4^n q_i(x)$$

for any $x \in E_1$, $n \in \mathbf{N}$ and $i = 1,2$. Now, we have

$$\begin{aligned} \|q_1(x) - q_2(x)\| &\leq \frac{1}{4^n}\left(\|q_1(2^n x) - f(2^n x)\| + \|f(2^n x) - q_2(2^n x)\|\right) \\ &\leq \frac{1}{4^n}(c_1 + c_2) + \frac{1}{2^{n(2-p)}}(b_1 + b_2)\,\|x\|^p. \end{aligned}$$

If we let $n \to \infty$, we get $q_1(x) = q_2(x)$ for all x in E_1. □

Theorem 4. *Let E_1 be a normed space and let E_2 be a Banach space. If a function $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{10}$$

for some $\theta \geq 0$, $p > 2$ and for all $x, y \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{2^p - 4}\,\|x\|^p \tag{11}$$

for all $x \in E_1$.

Proof. Define the sequence

$$Q_n(x) = 4^n f\left(\frac{x}{2^n}\right) \tag{12}$$

for all $x \in E_1$ and all $n \in \mathbf{N}$. Since $f(0) = 0$, applying (6), we get

$$\|Q_n(x) - Q_m(x)\| \leq \frac{1}{2^{p-1}}\frac{1}{2^{m(p-2)}}\frac{\theta}{1-b}\,\|x\|^p \tag{13}$$

for $x \in E_1$ and $n > m$. Hence, the assumption $p > 2$ implies that $\{Q_n(x)\}$ is a Cauchy sequence for every $x \in E_1$. Define

$$Q(x) = \lim_{n\to\infty} Q_n(x)$$

for each $x \in E_1$. Then, in a similar way as in the proof of Theorem 3, we may verify that Q is a quadratic function. Using (6) again, we obtain (11).

It is not difficult to prove that

$$Q(rx) = r^2 Q(x) \tag{14}$$

for all $x \in E_1$ and for all $r \in \mathbf{Q}$. Now, assume that there exist two quadratic functions $q_i : E_1 \to E_2$ $(i = 1, 2)$ such that

$$\|f(x) - q_i(x)\| \le d_i \|x\|^p$$

for all $x \in E_1$, where d_i $(i = 1, 2)$ are non-negative constants. By (14) we get

$$\begin{aligned} \|q_1(x) - q_2(x)\| &= 4^n \left\| q_1\left(\frac{x}{2^n}\right) - q_2\left(\frac{x}{2^n}\right) \right\| \\ &\le 4^n (d_1 + d_2) \left\| \frac{x}{2^n} \right\|^p \\ &= \frac{1}{2^{n(p-2)}} (d_1 + d_2) \|x\|^p, \end{aligned}$$

from which we conclude that $q_1 = q_2$. □

The following corollary is derived from the book [14] written by D. H. Hyers, G. Isac and Th. M. Rassias.

Corollary 5. *If in Theorem 3 and Theorem 4 the function f is continuous on E_1, then the quadratic function Q is also continuous at each point $x \in E_1 \setminus \{0\}$. When $p > 0$, this restriction is unnecessary.*

Proof. Suppose that f is continuous at each point x in E_1 and that $x_0 \in E_1 \setminus \{0\}$. Put $s = \|x_0\|/2$ and define an open ball by

$$B(x_0, s) = \{x \in E_1 \mid \|x - x_0\| < s\}.$$

For $x \in B(x_0, s)$ we have $s < \|x\| < 3s$. By letting $n \to \infty$ in inequalities (9) or (13), we get

$$\|Q(x) - Q_m(x)\| \le \frac{1}{3}\frac{1}{4^m}(\delta + c) + \frac{1}{2^{m(2-p)+1}}\frac{\theta}{1-a}\|x\|^p \quad (\text{for } p < 2)$$

or

$$\|Q(x) - Q_m(x)\| \le \frac{1}{2^{p-1}}\frac{1}{2^{m(p-2)}}\frac{\theta}{1-b}\|x\|^p \quad (\text{for } p > 2).$$

For $x \in B(x_0, s)$ we have $s^p > \|x\|^p > (3s)^p$ when $p < 0$, while the inequalities are reversed when $p > 0$. Consequently, Q_m converges uniformly to Q on $B(x_0, s)$ as $m \to \infty$. Since each function Q_m is continuous on $B(x_0, s)$, it follows that the limit Q is also continuous on $B(x_0, s)$. Thus, the quadratic function Q is continuous at any point $x_0 \neq 0$ in E_1. Obviously, the restriction $x_0 \neq 0$ is not needed when $p > 0$. □

It should be noted that S. Czerwik [6] proved the following corollary under the weaker assumption that $f(tx)$ is Borel measurable in t for each fixed x in E_1.

Corollary 6. *Let E_1 and E_2 be a normed space and a Banach space, respectively. If a function $f : E_1 \to E_2$ satisfies either the inequality (4) for $x, y \neq 0$ or the inequality (10) for $x, y \in E_1$, according to $p < 2$ or $p > 2$, and if moreover $f(tx)$ is continuous in t for each fixed x in E_1, then the unique quadratic function $Q : E_1 \to E_2$ defined by*

$$Q(x) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x) \quad (\text{for } p < 2)$$

or

$$Q(x) = \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big) \quad (\text{for } p > 2)$$

satisfies $Q(tx) = t^2 Q(x)$ for all $t \in \mathbf{R}$ and $x \in E_1$.

Proof. It is obvious that $Q(rx) = r^2 Q(x)$ holds for all $x \in E_1$ and for all $r \in \mathbf{Q}$ (ref. (14) in the proof of Theorem 4). To prove that Q is homogeneous of degree 2 for all real numbers as well, it suffices to prove that $Q(tx)$ is continuous in t for each fixed x in E_1. By hypothesis, $f(tx)$ is continuous in t for each fixed x in E_1. Apply Corollary 5 to the case where $E_1 = \mathbf{R}$ to show that if $x \neq 0$ and $t_0 \neq 0$ then $Q(tx)$ is continuous at $t = t_0$. Thus, $Q(t_0 x) = t_0^2 Q(x)$ for all $x \neq 0$ and $t_0 \neq 0$. But this equality is also valid for $x = 0$ or $t_0 = 0$. Therefore, we conclude that $Q(tx) = t^2 Q(x)$ holds for all $t \in \mathbf{R}$ and $x \in E_1$. □

If $p = 2$ in the inequality (4) or (10), the quadratic functional equation (3) is not stable as we see in the following theorem. This counterexample is a modification of the example contained in [9] (see [14] also).

Theorem 7. *Let us define a function $f : \mathbf{R} \to \mathbf{R}$ by*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{4^n} \varphi(2^n x),$$

where the function $\varphi : \mathbf{R} \to \mathbf{R}$ is given by

$$\varphi(x) = \begin{cases} a & \text{for } |x| \geq 1, \\ ax^2 & \text{for } |x| < 1 \end{cases}$$

with a positive number a. The function f satisfies the inequality

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq 32a(x^2 + y^2) \tag{15}$$

for all $x, y \in \mathbf{R}$. Moreover, there exists no quadratic function $Q : \mathbf{R} \to \mathbf{R}$ such that the image set of $|f(x) - Q(x)|/x^2$ $(x \neq 0)$ is bounded.

Proof. For $x = y = 0$ or for $x, y \in \mathbf{R}$ such that $x^2 + y^2 \geq 1/4$ it is clear that the inequality (15) holds true because f is bounded by $(4/3)a$. Consider the case $0 < x^2 + y^2 < 1/4$. Then there exists a $k \in \mathbf{N}$ such that

$$\frac{1}{4^{k+1}} \leq x^2 + y^2 < \frac{1}{4^k}, \tag{16}$$

whence $4^{k-1}x^2 < 1/4$ and $4^{k-1}y^2 < 1/4$ and consequently

$$2^{k-1}x,\ 2^{k-1}y,\ 2^{k-1}(x+y),\ 2^{k-1}(x-y) \in (-1, 1).$$

Therefore, for each $n = 0, 1, ..., k-1$, we have

$$2^n x,\ 2^n y,\ 2^n(x+y),\ 2^n(x-y) \in (-1, 1)$$

and

$$\varphi(2^n(x+y))) + \varphi(2^n(x-y)) - 2\varphi(2^n x) - 2\varphi(2^n y) = 0$$

for $n = 0, 1, ..., k-1$. Using (16) we obtain

$$\begin{aligned} & |f(x+y) + f(x-y) - 2f(x) - 2f(y)| \\ \leq\ & \sum_{n=0}^{\infty} \frac{1}{4^n} |\varphi(2^n(x+y)) + \varphi(2^n(x-y)) - 2\varphi(2^n x) - 2\varphi(2^n y)| \\ \leq\ & \sum_{n=k}^{\infty} \frac{6a}{4^n} \ =\ 2 \cdot 4^{1-k} a \ \leq\ 32a(x^2 + y^2), \end{aligned}$$

i.e., the inequality (15) holds true.

Assume that there exist a quadratic function $Q : \mathbf{R} \to \mathbf{R}$ and a constant $b > 0$ such that

$$|f(x) - Q(x)| \leq bx^2$$

for all $x \in \mathbf{R}$. Since Q is locally bounded, it is of the form $Q(x) = cx^2$ $(x \in \mathbf{R})$ where c is a constant (see [34]). Therefore, we have

$$|f(x)| \leq (b + |c|)x^2 \tag{17}$$

for all $x \in \mathbf{R}$. Let $k \in \mathbf{N}$ satisfy $ka > b + |c|$. If $x \in (0, 2^{1-k})$, then $2^n x \in (0,1)$ for $n \leq k-1$ and we have

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{4^n}\varphi(2^n x) \geq \sum_{n=0}^{k-1} \frac{a}{4^n}(2^n x)^2 = kax^2 > (b + |c|)x^2,$$

which in comparison with (17) is a contradiction. □

Theorems 3 and 4 can be generalized without difficulty to the situation where the right-hand side of the inequality (4) or (10) is replaced by $\theta H(\|x\|, \|y\|)$, in which $H : [0, \infty)^2 \to [0, \infty)$ is positive homogeneous of degree $p \neq 2$. This statement is an analog for approximately quadratic functions of a theorem of Th. M. Rassias and P. Šemrl [42] for approximately additive functions.

3. Stability on bounded subsets of R

Before initially proving the stability of the quadratic functional equation for functions defined on a bounded subset of $\mathbf{R}$, we treat the stability problem for biadditive functions on a restricted domain.

F. Skof and S. Terracini [49] proved the following stability theorem for symmetric biadditive functions based on results from F. Skof [46].

Theorem 8. *Let E be a Banach space and let $c, \delta > 0$ be given. If a symmetric function $f : [0, c)^2 \to E$ satisfies the inequality*

$$\|f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y)\| \leq \delta$$

for all x_1, x_2, y in $[0,c)$ with $x_1 + x_2 < c$, then there exists a symmetric and biadditive function $B : [0,c)^2 \to E$ such that

$$\|f(x,y) - B(x,y)\| \leq 9\delta$$

for all $x, y \in [0,c)$.

Proof. By hypothesis, for each fixed y in $[0,c)$, the function $f_y(x) = f(x,y)$ satisfies the inequality

$$\|f_y(x_1 + x_2) - f_y(x_1) - f_y(x_2)\| \leq \delta$$

for all $x_1, x_2 \in [0,c)$ with $x_1 + x_2 < c$. We define the function $f_y^* \; [0,\infty) \to E$ for fixed y by

$$f_y^*(x) = f_y(\mu) + nf_y\left(\frac{c}{2}\right)$$

for $x = \mu + (c/2)n$, $n \in \mathbf{N}$ and $0 \leq \mu < c/2$. Thus, we have

$$\|f_y(x) - f_y^*(x)\| \leq \delta \tag{18}$$

for any x in $[0,c)$. This function is extended to $\mathbf{R}$ by putting $f_y^*(x) = -f_y^*(-x)$ when $x < 0$. It follows that f_y^* $(0 \leq y < c)$ satisfies

$$\|f_y^*(x_1 + x_2) - f_y^*(x_1) - f_y^*(x_2)\| \leq 2\delta$$

for all $x_1, x_2 \in \mathbf{R}$ (cf. [46] or [32]), and hence, by a theorem of D. H. Hyers [13], there exists a unique additive function

$$A_y^*(x) = \lim_{n\to\infty} \frac{1}{2^n} f_y^*(2^n x),$$

for all $x \in \mathbf{R}$ and $y \in [0,c)$, such that

$$\|f_y^*(x) - A_y^*(x)\| \leq 2\delta \tag{19}$$

for any $x \in \mathbf{R}$ and $y \in [0,c)$.

Now, let us define $A(x,y) = A_y^*(x)$ for $x, y \in [0,c)$. $A(x,y)$ is additive in the first variable.

Fix $x, y, z \in [0, c)$ with $y + z < c$. Put $2^n x = \mu_n + (c/2)k_n$, where $k_n \in \mathbf{N}$ and $0 \leq \mu_n < c/2$, so that $k_n = (2/c)(2^n x - \mu_n)$. Then,

$$\begin{aligned} A(x, y+z) - A(x, y) - A(x, z) &= \lim_{n\to\infty} \frac{1}{2^n}(f^*_{y+z}(2^n x) - f^*_y(2^n x) - f^*_z(2^n x)) \\ &= \lim_{n\to\infty} \frac{1}{2^n}\left(f_{y+z}(\mu_n) - f_y(\mu_n) - f_z(\mu_n)\right) \\ &\quad + \lim_{n\to\infty} \frac{1}{2^n}k_n \left(f_{y+z}\left(\frac{c}{2}\right) - f_y\left(\frac{c}{2}\right) - f_z\left(\frac{c}{2}\right)\right) \\ &= 2\frac{x}{c}\left(f_{y+z}\left(\frac{c}{2}\right) - f_y\left(\frac{c}{2}\right) - f_z\left(\frac{c}{2}\right)\right). \end{aligned}$$

Hence,

$$||A(x, y+z) - A(x, y) - A(x, z)|| \leq 2\delta$$

for all $x, y, z \in [0, c)$ with $y + z < c$.

Next, we extend A to a function $A' : [0, c) \times \mathbf{R} \to E$. With $x \in [0, c)$ and $y \geq 0$, let $y = \mu + (c/2)n$, where $n \in \mathbf{N}$ and $0 \leq \mu < c/2$, and define A' by

$$A'(x, y) = A(x, \mu) + nA\left(x, \frac{c}{2}\right)$$

and put

$$A'(x, y) = -A'(x, -y)$$

for $y < 0$. Similarly as in the first part of the present proof, we obtain

$$||A'(x, y+z) - A'(x, y) - A'(x, z)|| \leq 4\delta$$

for all $y, z \in \mathbf{R}$ and

$$||A(x, y) - A'(x, y)|| \leq 2\delta \tag{20}$$

for all $x, y \in [0, c)$. Also, for each $y \in \mathbf{R}$, A' is additive in the first variable, i.e.,

$$A'(x_1 + x_2, y) = A'(x_1, y) + A'(x_2, y) \tag{21}$$

for $x_1, x_2 \in [0, c)$ with $x_1 + x_2 < c$, since A has this property.

For each fixed $x \in [0,c)$, it follows from a theorem of Hyers [13] that the function

$$B(x,y) = \lim_{n\to\infty} \frac{1}{2^n} A'(x, 2^n y) \tag{22}$$

is additive in y and satisfies

$$\|A'(x,y) - B(x,y)\| \le 4\delta \tag{23}$$

for all $x \in [0,c)$ and $y \in \mathbf{R}$. By (21) and (22) we have

$$B(x_1 + x_2, y) = B(x_1, y) + B(x_2, y)$$

for $x_1, x_2 \in [0,c)$ with $x_1 + x_2 < c$. By (18), (19), (20) and (23), we obtain

$$\|f(x,y) - B(x,y)\| \le 9\delta \tag{24}$$

for $x, y \in [0,c)$.

Since f is symmetric, it follows from (24) that

$$\|B(x,y) - B(y,x)\| \le 18\delta$$

for any $x, y \in [0,c)$. If $y = 0$, we have $B(x,0) = 0 = B(0,x)$ for $x \in [0,c)$. For a given $y \in (0,c)$ put

$$b_y(x) = B(y,x) - B(x,y)$$

for any $x \in [0,c)$. Now, $b_y(x)$ is bounded and additive, more precisely,

$$b_y(x_1 + x_2) = b_y(x_1) + b_y(x_2)$$

for $x_1, x_2 \in [0,c)$ with $x_1 + x_2 < c$, so it is the restriction to $[0,c)$ of a function of x of the form $b_y(x) = a(y)x$. But $b_y(y) = 0$ for all $y \in (0,c)$, so $a(y) \equiv 0$. Hence, B is symmetric on $[0,c)^2$. $\square$

Using the result of the last theorem, Skof and Terracini [49] proved the following theorem.

Theorem 9. *Let E be a Banach space and let $c, \delta > 0$ be given. If a function $f : [0, c) \to E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta \tag{25}$$

for all $x \geq y \geq 0$ with $x + y < c$, then there exists a quadratic function $Q : \mathbf{R} \to E$ such that

$$\|f(x) - Q(x)\| \leq \frac{79}{2}\delta$$

for any $x \in [0, c)$.

Proof. Putting $x = y\,(= 0)$ in (25) yields

$$\|f(2x) + f(0) - 4f(x)\| \leq \delta \quad \text{and} \quad \|f(0)\| \leq \frac{\delta}{2},$$

and thus

$$\|f(2x) - 4f(x)\| \leq \frac{3}{2}\delta \tag{26}$$

for $x \in [0, c/2)$. Define

$$g(x) = \begin{cases} f(x) & \text{for } x \in [0, c), \\ f(-x) & \text{for } x \in (-c, 0) \end{cases}$$

and put $\mu(x, y) = \|g(x+y) + g(x-y) - 2g(x) - 2g(y)\|$. From (25) it follows that

$$\mu(x, y) \leq \delta \quad (\text{for } x \geq y \geq 0 \text{ with } x + y < c). \tag{27}$$

For $y \geq x \geq 0$ with $y + x < c$ we have

$$\mu(x, y) = \mu(y, x) = \|f(x+y) + f(-x+y) - 2f(x) - 2f(y)\|.$$

Hence, from (25) again we get

$$\mu(x, y) \leq \delta \quad (\text{for } y \geq x \geq 0 \text{ with } y + x < c). \tag{28}$$

If $x < 0$, $y \geq 0$ and $y - x < c$, then $\mu(x, y) = \mu(-x, y)$, since g is even. For this case it holds either $-x \geq y \geq 0$ with $-x + y < c$ or $y \geq -x > 0$ with $y - x < c$. So, from (27) or (28) we obtain

$$\mu(x, y) \leq \delta \quad (\text{for } x < 0,\ y \geq 0 \text{ and } y - x < c). \tag{29}$$

Finally, assume that $y < 0$, $|x+y| < c$ and $|x-y| < c$. It then holds either $x \geq -y > 0$ with $x - y < c$ or $-y \geq x \geq 0$ with $x - y < c$ or $x < 0$, $-y > 0$ and $-y - x < c$. Therefore, it follows from (27), (28) or (29) that

$$\mu(x, y) \leq \delta \quad (\text{for } y < 0,\ |x + y| < c \text{ and } |x - y| < c) \tag{30}$$

since $\mu(x, -y) = \mu(x, y)$. According to (27), (28), (29) and (30), we conclude that

$$||g(x + y) + g(x - y) - 2g(x) - 2g(y)|| \leq \delta \tag{31}$$

for all $(x, y) \in D(c)$, where we set

$$D(c) = \{(x, y) \in \mathbf{R}^2 \mid |x + y| < c,\ |x - y| < c\}.$$

Let us define the auxiliary function $h : D(c) \to E$ by

$$h(x, y) = \frac{1}{4}(g(x + y) - g(x - y)).$$

Clearly, $h(x, y) = h(y, x)$ for all $(x, y) \in D(c)$. When $y \in [0, c/2)$, h satisfies the inequality

$$||h(x_1 + x_2, y) - h(x_1, y) - h(x_2, y)|| \leq \delta$$

for any $x_1, x_2 \in [0, c/2)$ with $x_1 + x_2 < c/2$ (see below), and by the interchange of x and y, we also have, when $x \in [0, c/2)$,

$$||h(x, y + z) - h(x, y) - h(x, z)|| \leq \delta$$

for all $y, z \in [0, c/2)$ with $y + z < c/2$. From the definition of h it follows that

$$\begin{aligned}
&4(h(x_1 + x_2, y) - h(x_1, y) - h(x_2, y)) \\
&= \quad (g(x_1 + x_2 + y) + g(x_1 - x_2 - y) - 2g(x_1) - 2g(x_2 + y)) \\
&\quad - (g(x_1 - y - x_2) + g(x_1 - y + x_2) - 2g(x_1 - y) - 2g(x_2)) \\
&\quad - (g(x_1 + y) + g(x_1 - y) - 2g(x_1) - 2g(y)) \\
&\quad + (g(x_2 + y) + g(x_2 - y) - 2g(x_2) - 2g(y))
\end{aligned}$$

for any $(x_1, x_2 + y)$, $(x_1 - y, x_2)$, (x_1, y), (x_2, y) in $D(c)$. Hence, by (31),

$$\|h(x_1 + x_2, y) - h(x_1, y) - h(x_2, y)\| \leq \delta.$$

Thus, h satisfies the hypothesis of Theorem 8 with $c/2$ in place of c, so there exists a symmetric and biadditive function $B : [0, c/2)^2 \to E$ such that

$$\|h(x, y) - B(x, y)\| \leq 9\delta \tag{32}$$

for $(x, y) \in [0, c/2)^2$. When $x \in [0, c/2)$, we have

$$\begin{aligned}
\|f(x) - h(x, x)\| &= \frac{1}{4}\|4f(x) - f(2x) + f(0)\| \\
&\leq \frac{1}{4}\|4f(x) - f(2x) - f(0)\| + \frac{1}{2}\|f(0)\|,
\end{aligned}$$

and it follows from (25) that

$$\|f(x) - h(x, x)\| \leq \frac{\delta}{2} \tag{33}$$

for $x \in [0, c/2)$.

The function $B(x, x)$ is quadratic for $x \geq y \geq 0$ with $x + y < c/2$. According to a theorem by F. Skof [47], it may be extended to a quadratic function $Q : \mathbf{R} \to E$ such that $Q(x) = B(x, x)$ for all $x \in [0, c/2)$. Thus, from (32) and (33) we get

$$\begin{aligned}
\|f(x) - Q(x)\| &= \|f(x) - B(x, x)\| \\
&\leq \|f(x) - h(x, x)\| + \|h(x, x) - B(x, x)\| \\
&\leq \frac{19}{2}\delta
\end{aligned}$$

for $x \in [0, c/2)$. Now, let $x \in [c/2, c)$. Taking the account of (26) and the last inequality, we have

$$\begin{aligned} \|f(x) - Q(x)\| &\le \left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| + \left\|4f\left(\frac{x}{2}\right) - 4Q\left(\frac{x}{2}\right)\right\| \\ &\le \frac{79}{2}\delta, \end{aligned}$$

and this ends the proof. □

With the help of Theorem 9, F. Skof and S. Terracini [49] proved the following theorem.

Theorem 10. *Let E be a Banach space and let $c, \delta > 0$ be given. If a function $f : (-c, c) \to E$ satisfies the inequality (25) for all $x, y \in \mathbf{R}$ with $|x + y| < c$ and $|x - y| < c$, then there exists a quadratic function $Q : \mathbf{R} \to E$ such that*

$$\|f(x) - Q(x)\| \le \frac{81}{2}\delta$$

for any $x \in (-c, c)$.

Proof. It holds

$$\begin{aligned} 2\|f(y) - f(-y)\| &\le \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \\ &\quad + \| - f(x - y) - f(x + y) + 2f(x) + 2f(-y)\| \end{aligned}$$

and hence, it follows from the hypothesis that

$$\|f(y) - f(-y)\| \le \delta \tag{34}$$

for any $y \in (-c, c)$ because for each $y \in (-c, c)$ there exists an $x \in (-c, c)$ such that $|x+y| < c$ and $|x-y| < c$. Let us denote by f_0 the restriction of f to $[0, c)$. According to Theorem 9, there exists a quadratic function $Q : \mathbf{R} \to E$ such that

$$\|f_0(x) - Q(x)\| \le \frac{79}{2}\delta \tag{35}$$

for $x \in [0, c)$. For $x \in (-c, 0)$, the inequalities (34) and (35) imply

$$\begin{aligned} \|f(x) - Q(x)\| &\leq \|f(x) - f(-x)\| + \|f(-x) - Q(-x)\| \\ &\leq \delta + \|f_0(-x) - Q(-x)\| \\ &\leq \frac{81}{2}\delta, \end{aligned}$$

since Q is even. □

4. Stability on bounded subsets of $\mathbf{R}^n$

Throughout this section, let n be a given positive integer, $t > 0$ a constant, and let E be a Banach space. For convenience, we use the notation $I^n = [-t, t]^n$. Furthermore, we denote by $\{e_1, ..., e_n\}$ the canonical basis for $\mathbf{R}^n$.

Recently, S.-M. Jung and B. Kim [30] generalized the result of F. Skof and S. Terracini (see Theorem 10 above). In this section, the results of Jung and Kim will be presented. First, some useful lemmas concerning the approximately additive functions defined on I^n will be introduced.

Lemma 11. *If a function $\phi : I^n \to E$ satisfies the functional inequality*

$$\|\phi(x + y) - \phi(x) - \phi(y)\| \leq \delta \tag{36}$$

for some $\delta > 0$ and for all $x, y \in I^n$ with $x + y \in I^n$, then we have

$$\|\phi(kx) - k\phi(x)\| \leq (|k| + 1)\delta \tag{37}$$

for any integer k and $x \in I^n$ with $kx \in I^n$.

Proof. If we put $y = -x$ in (36), then we have

$$\|\phi(x) + \phi(-x)\| \leq \delta + \|\phi(0)\| \leq 2\delta \tag{38}$$

for all $x \in I^n$. Let us define a sign function by

$$\sigma = \begin{cases} 1 & \text{for } k \geq 0, \\ -1 & \text{for } k < 0. \end{cases}$$

Then, we see that $k = \sigma|k|$.

The inequality (38) and $|k|-1$ times applications of (36) yield

$$\begin{aligned}
\|\phi(kx)-k\phi(x)\| &\leq \|\phi(\sigma|k|x)-\sigma\phi(|k|x)\|+\|\sigma\phi(|k|x)-k\phi(x)\| \\
&= \|\phi(\sigma|k|x)-\sigma\phi(|k|x)\|+\|\phi(|k|x)-|k|\phi(x)\| \\
&\leq 2\delta+(|k|-1)\delta \\
&= (|k|+1)\delta
\end{aligned}$$

for the case $|k|>1$. For $k=-1,0,1$, we can also see that (37) holds true. □

Using a method of F. Skof [47], we will construct a function $\tilde{\phi}:\mathbf{R}^n\to E$ by extending the given approximately additive function $\phi: I^n\to E$ 'nearly' linearly.

Lemma 12. *Let $\phi: I^n\to E$ be the same function given in Lemma 11. Let us define a function $\tilde{\phi}:\mathbf{R}^n\to E$ by*

$$\tilde{\phi}(x)=\sum_{i=1}^{n}k_i\phi\left(\frac{t}{2}e_i\right)+\phi\left(\sum_{i=1}^{n}\mu_i e_i\right),$$

where $x\in\mathbf{R}^n$ is given by

$$x=\sum_{i=1}^{n}k_i\frac{t}{2}e_i+\sum_{i=1}^{n}\mu_i e_i \tag{39}$$

with $k_i\in\mathbf{Z}$ and $0\leq\mu_i<t/2$ for $i=1,\ldots,n$. Then, it holds

$$\|\phi(x)-\tilde{\phi}(x)\|\leq 4n\delta$$

for each $x\in I^n$.

Proof. Assume that an arbitrary $x \in I^n$ is given by (39) with $k_i \in \{-2,-1,0,1,2\}$. By applying (36) repeatedly and by Lemma 11, we obtain

$$\begin{aligned}
\|\phi(x) - \tilde{\phi}(x)\| &= \left\|\phi(x) - \sum_{i=1}^{n} k_i \phi\left(\frac{t}{2}e_i\right) - \phi\left(\sum_{i=1}^{n} \mu_i e_i\right)\right\| \\
&\le \left\|\phi(x) - \phi\left(\sum_{i=1}^{n} k_i \frac{t}{2} e_i\right) - \phi\left(\sum_{i=1}^{n} \mu_i e_i\right)\right\| \\
&\quad + \left\|\phi\left(\sum_{i=1}^{n} k_i \frac{t}{2} e_i\right) - \sum_{i=1}^{n} k_i \phi\left(\frac{t}{2} e_i\right)\right\| \\
&\le \delta + \left\|\phi\left(\sum_{i=1}^{n} k_i \frac{t}{2} e_i\right) - \sum_{i=1}^{n} \phi\left(k_i \frac{t}{2} e_i\right)\right\| \\
&\quad + \left\|\sum_{i=1}^{n} \phi\left(k_i \frac{t}{2} e_i\right) - \sum_{i=1}^{n} k_i \phi\left(\frac{t}{2} e_i\right)\right\| \\
&\le \delta + (n-1)\delta + \sum_{i=1}^{n} \left\|\phi\left(k_i \frac{t}{2} e_i\right) - k_i \phi\left(\frac{t}{2} e_i\right)\right\| \\
&\le 4n\delta,
\end{aligned}$$

since $|k_i| \le 2$. □

From a given symmetric, approximately biadditive function, we will construct an approximately additive function by the same method in Lemma 12.

Lemma 13. *Let $h : I^n \times I^n \to E$ be a symmetric function satisfying the inequality*

$$\|h(x, u+v) - h(x,u) - h(x,v)\| \le \delta \tag{40}$$

for some $\delta > 0$ and for all $x, u, v \in I^n$ with $u + v \in I^n$. For a fixed $z \in I^n$, let us define a function $\tilde{h}_z : \mathbf{R}^n \to E$ by

$$\tilde{h}_z(x) = \sum_{i=1}^{n} k_i h_z\left(\frac{t}{2}e_i\right) + h_z\left(\sum_{i=1}^{n} \mu_i e_i\right)$$

when $x \in \mathbf{R}^n$ can be expressed in the form

$$x = \sum_{i=1}^{n} k_i \frac{t}{2} e_i + \sum_{i=1}^{n} \mu_i e_i \quad \left(k_i \in \mathbf{Z};\ 0 \le \mu_i < \frac{t}{2}\right),$$

and where we define $h_z(y) = h(y, z)$ for $y \in I^n$. Then, it holds

$$\|\tilde{h}_z(x+y) - \tilde{h}_z(x) - \tilde{h}_z(y)\| \leq (3n+1)\delta$$

for any $x, y \in \mathbf{R}^n$.

Proof. Let $x, y \in \mathbf{R}^n$ be given by

$$x = \sum_{i=1}^{n} k_i \frac{t}{2} e_i + \sum_{i=1}^{n} \mu_i e_i \quad \text{and} \quad y = \sum_{i=1}^{n} l_i \frac{t}{2} e_i + \sum_{i=1}^{n} \nu_i e_i,$$

where $k_i, l_i \in \mathbf{Z}$ and $0 \leq \mu_i, \nu_i < t/2$ for $i = 1, ..., n$. Then, there exist $m_i \in \{0, 1\}$ $(i = 1, ..., n)$ such that

$$x + y = \sum_{i=1}^{n} (k_i + l_i + m_i) \frac{t}{2} e_i + \sum_{i=1}^{n} \left(\mu_i + \nu_i - m_i \frac{t}{2} \right) e_i,$$

where $0 \leq \mu_i + \nu_i - m_i(t/2) < t/2$ for $i = 1, ..., n$.

Thus, it follows from (40) and Lemma 11 that

$$\begin{aligned}
&\|\tilde{h}_z(x+y) - \tilde{h}_z(x) - \tilde{h}_z(y)\| \\
= &\ \left\| \sum_{i=1}^{n} (k_i + l_i + m_i) h_z\left(\frac{t}{2} e_i\right) + h_z\left(\sum_{i=1}^{n} \left(\mu_i + \nu_i - m_i \frac{t}{2} \right) e_i \right) \right. \\
&\ \left. - \sum_{i=1}^{n} k_i h_z\left(\frac{t}{2} e_i\right) - h_z\left(\sum_{i=1}^{n} \mu_i e_i\right) - \sum_{i=1}^{n} l_i h_z\left(\frac{t}{2} e_i\right) - h_z\left(\sum_{i=1}^{n} \nu_i e_i\right) \right\| \\
= &\ \left\| \sum_{i=1}^{n} m_i h_z\left(\frac{t}{2} e_i\right) + h_z\left(\sum_{i=1}^{n} \left(\mu_i + \nu_i - m_i \frac{t}{2} \right) e_i \right) \right. \\
&\ \left. - h_z\left(\sum_{i=1}^{n} \mu_i e_i\right) - h_z\left(\sum_{i=1}^{n} \nu_i e_i\right) \right\| \\
\leq &\ \left\| \sum_{i=1}^{n} m_i h_z\left(\frac{t}{2} e_i\right) + h_z\left(\sum_{i=1}^{n} (\mu_i + \nu_i) e_i \right) + h_z\left(-\sum_{i=1}^{n} m_i \frac{t}{2} e_i \right) \right. \\
&\ \left. - h_z\left(\sum_{i=1}^{n} \mu_i e_i\right) - h_z\left(\sum_{i=1}^{n} \nu_i e_i\right) \right\| + \delta \\
\leq &\ \left\| \sum_{i=1}^{n} m_i h_z\left(\frac{t}{2} e_i\right) + h_z\left(\sum_{i=1}^{n} (-m_i) \frac{t}{2} e_i \right) \right\|
\end{aligned}$$

$$
\begin{aligned}
& + \left\| h_z\left(\sum_{i=1}^{n}(\mu_i+\nu_i)e_i\right) - h_z\left(\sum_{i=1}^{n}\mu_i e_i\right) - h_z\left(\sum_{i=1}^{n}\nu_i e_i\right)\right\| + \delta \\
\le\ & \left\| \sum_{i=1}^{n} m_i h_z\left(\frac{t}{2}e_i\right) + h_z\left(\sum_{i=1}^{n}(-m_i)\frac{t}{2}e_i\right)\right\| + 2\delta \\
\le\ & \left\| \sum_{i=1}^{n} m_i h_z\left(\frac{t}{2}e_i\right) + \sum_{i=1}^{n} h_z\left(-m_i\frac{t}{2}e_i\right)\right\| + (n-1)\delta + 2\delta \\
\le\ & \sum_{i=1}^{n} \left\| m_i h_z\left(\frac{t}{2}e_i\right) + h_z\left(-m_i\frac{t}{2}e_i\right)\right\| + (n+1)\delta \\
\le\ & \sum_{i=1}^{n}(m_i+1)\delta + (n+1)\delta \\
\le\ & (3n+1)\delta,
\end{aligned}
$$

since we can show

$$
\left\| h_z\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} h_z(x_i)\right\| \le (n-1)\delta
$$

by applying (40) $n-1$ times repeatedly. □

In the previous lemma, we have constructed an approximately additive function $\tilde{h}_z : \mathbf{R}^n \to E$ from a given symmetric, approximately biadditive function $h : I^n \times I^n \to E$. We now use the direct method of D. H. Hyers [13] to construct a function $H : I^n \times I^n \to E$, which is approximately additive in the second variable, from the function $\tilde{h}_z : \mathbf{R}^n \to E$.

Lemma 14. *Let $h, \tilde{h}_z$ be defined in the same way as in Lemma 13. Let us define a function $H : I^n \times I^n \to E$ by*

$$
H(x,y) = \lim_{m\to\infty} \frac{1}{2^m}\tilde{h}_y(2^m x)
$$

for all $x, y \in I^n$. It then holds

$$
\|H(x,y+z) - H(x,y) - H(x,z)\| \le 2n\delta
$$

for all $x, y, z \in I^n$ with $y+z \in I^n$.

Proof. Given $x \in I^n$ and $m \in \mathbf{N}$, let

$$2^m x = \sum_{i=1}^{n} k_{mi} \frac{t}{2} e_i + \sum_{i=1}^{n} \mu_{mi} e_i,$$

where k_{mi}'s are integers and $0 \le \mu_{mi} < t/2$ for $i = 1, ..., n$. It then follows from (40) and the definition of $\tilde{h}_z$ that

$$\begin{aligned}
&\|H(x, y+z) - H(x, y) - H(x, z)\| \\
= \; & \left\| \lim_{m \to \infty} \frac{1}{2^m} \left[\tilde{h}_{y+z}(2^m x) - \tilde{h}_y(2^m x) - \tilde{h}_z(2^m x) \right] \right\| \\
\le \; & \lim_{m \to \infty} \frac{1}{2^m} \sum_{i=1}^{n} |k_{mi}| \left\| h_{y+z}\left(\frac{t}{2} e_i\right) - h_y\left(\frac{t}{2} e_i\right) - h_z\left(\frac{t}{2} e_i\right) \right\| \\
& + \lim_{m \to \infty} \frac{1}{2^m} \left\| h_{y+z}\left(\sum_{i=1}^{n} \mu_{mi} e_i\right) - h_y\left(\sum_{i=1}^{n} \mu_{mi} e_i\right) - h_z\left(\sum_{i=1}^{n} \mu_{mi} e_i\right) \right\| \\
\le \; & \lim_{m \to \infty} \frac{1}{2^m} \sum_{i=1}^{n} |k_{mi}| \, \delta \\
\le \; & 2n\delta,
\end{aligned}$$

since we have

$$x = \sum_{i=1}^{n} \frac{1}{2^m} k_{mi} \frac{t}{2} e_i + \sum_{i=1}^{n} \frac{1}{2^m} \mu_{mi} e_i,$$

or the ith coordinate of x is

$$x_i = \frac{1}{2^m} k_{mi} \frac{t}{2} + \frac{1}{2^m} \mu_{mi}.$$

Taking limit as $m \to \infty$ on both sides of the above equality, we get

$$\lim_{m \to \infty} \frac{1}{2^m} |k_{mi}| = \frac{2\,|x_i|}{t} \le 2$$

because of $|x_i| \le t$ for any $x \in I^n$. $\square$

For each symmetric, approximately biadditive function $h : I^n \times I^n \to E$, we can choose a symmetric biadditive function $F : I^n \times I^n \to E$ such that $\|h(x, y) - F(x, y)\|$ is bounded on $I^n \times I^n$.

Theorem 15. *Let $h : I^n \times I^n \to E$ be a symmetric function which satisfies the inequality* (40) *for some $\delta > 0$ and for all $x, u, v \in I^n$ with $u + v \in I^n$. Then there exists a symmetric biadditive function $F : I^n \times I^n \to E$ such that*

$$\|h(x, y) - F(x, y)\| \leq (14n^2 + 9n + 1)\delta$$

for all $x, y \in I^n$.

Proof. Let $H : I^n \times I^n \to E$ be defined in the same way as in Lemma 14. By a well known theorem of D. H. Hyers, H is additive in the first variable, i.e.,

$$H(x + y, z) = H(x, z) + H(y, z)$$

for all $x, y, z \in I^n$ with $x + y \in I^n$ (ref. [13]).

We now define a function $G : I^n \times \mathbf{R}^n \to E$ by

$$G(x, y) = \sum_{i=1}^{n} k_i H\left(x, \frac{t}{2}e_i\right) + H\left(x, \sum_{i=1}^{n} \mu_i e_i\right)$$

for $x \in I^n$ and $y \in \mathbf{R}^n$ with

$$y = \sum_{i=1}^{n} k_i \frac{t}{2} e_i + \sum_{i=1}^{n} \mu_i e_i \tag{41}$$

where k_i's are integers and $0 \leq \mu_i < t/2$ for $i = 1, ..., n$.

Let $x, y \in I^n$ be given in the form (41) with $k_i \in \{-2, -1, 0, 1, 2\}$ for $i = 1, ..., n$. By applying Lemma 14 repeatedly and using Lemma 11, we have

$$\begin{aligned}
\|H(x, y) - G(x, y)\| &\leq \left\| H(x, y) - H\left(x, \sum_{i=1}^{n} k_i \frac{t}{2} e_i\right) - H\left(x. \sum_{i=1}^{n} \mu_i e_i\right) \right\| \\
&\quad + \left\| H\left(x, \sum_{i=1}^{n} k_i \frac{t}{2} e_i\right) - \sum_{i=1}^{n} H\left(x, k_i \frac{t}{2} e_i\right) \right\| \\
&\quad + \left\| \sum_{i=1}^{n} H\left(x, k_i \frac{t}{2} e_i\right) - \sum_{i=1}^{n} k_i H\left(x, \frac{t}{2} e_i\right) \right\| \\
&\leq 2n\delta + (n-1)2n\delta + \sum_{i=1}^{n} (|k_i| + 1)2n\delta \\
&\leq 8n^2\delta.
\end{aligned} \tag{42}$$

Assume that $x \in I^n$, $y \in \mathbf{R}^n$ are expressed in the form (41) and that $z \in \mathbf{R}^n$ has the form

$$z = \sum_{i=1}^{n} l_i \frac{t}{2} e_i + \sum_{i=1}^{n} \nu_i e_i,$$

where l_i's are integers and $0 \leq \nu_i < t/2$ for $i = 1, ..., n$. Then, there are $m_i \in \{0, 1\}$ $(i = 1, ..., n)$ with

$$y + z = \sum_{i=1}^{n} (k_i + l_i + m_i) \frac{t}{2} e_i + \sum_{i=1}^{n} \left(\mu_i + \nu_i - m_i \frac{t}{2} \right) e_i,$$

where $0 \leq \mu_i + \nu_i - m_i(t/2) < t/2$ for $i = 1, ..., n$.

We can obtain

$$\begin{aligned}
&\|G(x, y+z) - G(x,y) - G(x,z)\| \\
&= \left\| \sum_{i=1}^{n} (k_i + l_i + m_i) H\left(x, \frac{t}{2} e_i\right) + H\left(x, \sum_{i=1}^{n} \left(\mu_i + \nu_i - m_i \frac{t}{2}\right) e_i\right) \right. \\
&\quad - \sum_{i=1}^{n} k_i H\left(x, \frac{t}{2} e_i\right) - H\left(x, \sum_{i=1}^{n} \mu_i e_i\right) \\
&\quad \left. - \sum_{i=1}^{n} l_i H\left(x, \frac{t}{2} e_i\right) - H\left(x, \sum_{i=1}^{n} \nu_i e_i\right) \right\| \\
&= \left\| \sum_{i=1}^{n} m_i H\left(x, \frac{t}{2} e_i\right) + H\left(x, \sum_{i=1}^{n} \left(\mu_i + \nu_i - m_i \frac{t}{2}\right) e_i\right) \right. \\
&\quad \left. - H\left(x, \sum_{i=1}^{n} \mu_i e_i\right) - H\left(x, \sum_{i=1}^{n} \nu_i e_i\right) \right\| \\
&\leq \sum_{i=1}^{n} (m_i + 1) 2n\delta + (n+1) 2n\delta \\
&\leq (6n^2 + 2n)\delta
\end{aligned} \tag{43}$$

by a similar way as in the proof of Lemma 13.

Since H is additive in the first variable, G is also additive in that variable, i.e.,

$$G(x_1 + x_2, y) = G(x_1, y) + G(x_2, y) \tag{44}$$

for all $x_1, x_2 \in I^n$ with $x_1 + x_2 \in I^n$ and for all $y \in \mathbf{R}^n$.

From (43) and the theorem of D. H. Hyers (ref. [13]), we conclude that if we define a function $F : I^n \times \mathbf{R}^n \to E$ by

$$F(x,y) = \lim_{m\to\infty} \frac{1}{2^m} G(x, 2^m y) \tag{45}$$

for $x \in I^n$ and $y \in \mathbf{R}^n$, then $F(x,y)$ is additive in y and further F satisfies

$$\|G(x,y) - F(x,y)\| \leq (6n^2 + 2n)\delta \tag{46}$$

for $(x,y) \in I^n \times \mathbf{R}^n$. Furthermore, the additivity of $G(x,y)$ in x implies that of $F(x,y)$ also. Therefore, if we denote the restriction of F on $I^n \times I^n$ by F again, we see that F is biadditive.

From Lemmas 12, 13, 14 and the theorem of Hyers [13] with a minor modification, (42) and from (46), it follows that

$$\begin{aligned} \|h(x,y) - F(x,y)\| &\leq \|h(x,y) - \tilde{h}_y(x)\| + \|\tilde{h}_y(x) - H(x,y)\| \\ &\quad + \|H(x,y) - G(x,y)\| + \|G(x,y) - F(x,y)\| \\ &\leq 4n\delta + (3n+1)\delta + 8n^2\delta + (6n^2 + 2n)\delta \\ &= (14n^2 + 9n + 1)\delta \end{aligned}$$

for any $x, y \in I^n$.

It only remains to prove that F is symmetric. Since h is symmetric, we have

$$\|F(x,y) - F(y,x)\| \leq (28n^2 + 18n + 2)\delta \tag{47}$$

for all $x, y \in I^n$. If we define a function $\psi_y : I^n \to E$ by

$$\psi_y(x) = F(x,y) - F(y,x), \tag{48}$$

then ψ_y is additive, because F is biadditive. Hence, (47) implies that ψ_y is a bounded additive function, which must have the form

$$\psi_y(x) = a(y)x$$

for $x \in I^n$. If we put $x = y \neq 0$ in the above equality, then it follows from (48) that

$$a(y)y = \psi_y(y) = 0.$$

Thus, we have $a(y) = 0$, i.e., $F(x,y) = F(y,x)$ for $x \neq 0$ and $y \neq 0$. For the case $x = 0$ or $y = 0$, it follows from (44) and (45) that $F(x,0) = F(0,y) = 0$. Altogether, we conclude that F is symmetric. $\square$

We will now prove that there exists an even, approximately quadratic function $\hat{f} : I^n \to E$ near each approximately quadratic function $f : I^n \to E$.

Lemma 16. *If a function $f : I^n \to E$ satisfies the functional inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta \tag{49}$$

for some $\delta > 0$ and for all $x, y \in I^n$ with $x+y,\ x-y \in I^n$, then there exists an even function $\hat{f} : I^n \to E$ such that

$$\|\hat{f}(x) - f(x)\| \leq 2\delta \tag{50}$$

for any $x \in I^n$ and

$$\|\hat{f}(x+y) + \hat{f}(x-y) - 2\hat{f}(x) - 2\hat{f}(y)\| \leq 13\delta \tag{51}$$

for all $x, y \in I^n$ with $x+y,\ x-y \in I^n$.

Proof. We may select such a subset I' of I^n that $I' \cup (-I') \cup \{0\} = I^n$ and $I' \cap (-I') = \emptyset$. (For example, let $J_i := \{(x_1, ..., x_i, 0, ..., 0) \in I^n \mid x_i > 0\}$ for $i = 1, ..., n$ and $I' := \bigcup_{i=1}^n J_i$).

Define a function $\hat{f} : I^n \to E$ by

$$\hat{f}(x) = \begin{cases} f(x) & \text{for } x \in I' \cup \{0\}, \\ f(-x) & \text{for } x \in -I'. \end{cases}$$

It is not difficult to see that $\hat{f}$ is even. Put $x = y = 0$ in (49) to obtain

$$\|f(0)\| \leq \frac{1}{2}\delta \tag{52}$$

Putting $x = 0$ in (49), replacing y by x in the resulting inequality and using (52), we get

$$\|f(-x) - f(x)\| \leq 2\delta \tag{53}$$

for any $x \in I^n$. Hence, it follows from (53) that

$$\|\hat{f}(x) - f(x)\| \le \|f(-x) - f(x)\| \le 2\delta \tag{54}$$

for $x \in I^n$. Thus, by (49) and (54), we have

$$\begin{aligned}
&\|\hat{f}(x+y) + \hat{f}(x-y) - 2\hat{f}(x) - 2\hat{f}(y)\| \\
\le\ & \|\hat{f}(x+y) - f(x+y)\| + \|\hat{f}(x-y) - f(x-y)\| \\
&+ \|-2\hat{f}(x) + 2f(x)\| + \|-2\hat{f}(y) + 2f(y)\| \\
&+ \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\
\le\ & 13\delta
\end{aligned}$$

for $x, y \in I^n$ with $x+y,\ x-y \in I^n$. $\square$

F. Skof presented in the paper [47] that for every quadratic function $f : [0, a) \to E$, there exists a unique quadratic extension $q : \mathbf{R} \to E$ of f. In the following lemma, we will generalize the result of F. Skof.

Lemma 17. *Given $a > 0$, suppose $F : [-a, a]^n \times [-a, a]^n \to E$ is a symmetric biadditive function. Then there exists a unique symmetric biadditive function $\tilde{F} : \mathbf{R}^n \times \mathbf{R}^n \to E$ such that $\tilde{F}(x, y) = F(x, y)$ for all $(x, y) \in [-a, a]^n \times [-a, a]^n$.*

Proof. We define a function $\tilde{F} : \mathbf{R}^n \times \mathbf{R}^n \to E$ by

$$\tilde{F}(x, y) = \sum_{i,j=1}^{n} \left[k_i l_j F(ae_i, ae_j) + F(\mu_i e_i, \nu_j e_j) + k_i F(ae_i, \nu_j e_j) + l_j F(\mu_i e_i, ae_j)\right]$$

for all $x, y \in \mathbf{R}^n$ with the expressions

$$x = \sum_{i=1}^{n} k_i a e_i + \sum_{i=1}^{n} \mu_i e_i \quad \text{and} \quad y = \sum_{i=1}^{n} l_i a e_i + \sum_{i=1}^{n} \nu_i e_i$$

where $k_i, l_i \in \mathbf{Z}$ and $0 \le \mu_i, \nu_i < a$.

We will show

(*i*) $\tilde{F}(x, y) = F(x, y)$ for all $(x, y) \in [-a, a]^n \times [-a, a]^n$;

(*ii*) $\tilde{F}$ is symmetric and biadditive in $\mathbf{R}^n \times \mathbf{R}^n$.

First, we show the first statement. If $(x, y) \in [-a, a]^n \times [-a, a]^n$, then $k_i, l_i = -1, 0$ or 1 and in any case we have

$$\begin{aligned} k_i l_j F(ae_i, ae_j) &= F(k_i ae_i, l_j ae_j), \\ k_i F(ae_i, \nu_j e_j) &= F(k_i ae_i, \nu_j e_j), \\ l_j F(\mu_i e_i, ae_j) &= F(\mu_i e_i, l_j ae_j). \end{aligned}$$

Therefore, we get

$$\begin{aligned} \tilde{F}(x, y) &= \sum_{i,j=1}^{n} [k_i l_j F(ae_i, ae_j) + F(\mu_i e_i, \nu_j e_j) + k_i F(ae_i, \nu_j e_j) + l_j F(\mu_i e_i, ae_j)] \\ &= \sum_{i,j=1}^{n} F(k_i ae_i + \mu_i e_i, l_j ae_j + \nu_j e_j) \\ &= F\left(\sum_{i=1}^{n} (k_i ae_i + \mu_i e_i), \sum_{j=1}^{n} (l_j ae_j + \nu_j e_j) \right) \\ &= F(x, y). \end{aligned}$$

Now, we will prove the second statement. It is obvious that $\tilde{F}$ is symmetric because F is so. Hence, it is enough to show the additivity of $\tilde{F}$ in the first argument. In other words, we will show

$$\tilde{F}(x_1 + x_2, y) = \tilde{F}(x_1, y) + \tilde{F}(x_2, y).$$

Let

$$x_1 = \sum_{i=1}^{n} k_i ae_i + \sum_{i=1}^{n} \mu_i e_i, \quad x_2 = \sum_{i=1}^{n} l_i ae_i + \sum_{i=1}^{n} \nu_i e_i, \quad y = \sum_{i=1}^{n} m_i ae_i + \sum_{i=1}^{n} \omega_i e_i.$$

Then, we obtain

$$\begin{aligned} x_1 + x_2 &= \sum_{i=1}^{n} (k_i + l_i) ae_i + \sum_{i=1}^{n} (\mu_i + \nu_i) e_i \\ &= \sum_{i=1}^{n} (k_i + l_i + \zeta_i) ae_i + \sum_{i=1}^{n} (\mu_i + \nu_i - \zeta_i a) e_i, \end{aligned}$$

where $\zeta_i = 0$ if $\mu_i + \nu_i < a$ and $\zeta_i = 1$ if $\mu_i + \nu_i \geq a$, for $i = 1, ..., n$. Now

$$\begin{aligned}
\tilde{F}(x_1 + x_2, y) &= \sum_{i,j=1}^{n} [(k_i + l_i + \zeta_i) m_j F(ae_i, ae_j) + F((\mu_i + \nu_i - \zeta_i a)e_i, \omega_j e_j) \\
&\qquad + (k_i + l_i + \zeta_i) F(ae_i, \omega_j e_j) + m_j F((\mu_i + \nu_i - \zeta_i a)e_i, ae_j)] \\
&= \sum_{i,j=1}^{n} [k_i m_j F(ae_i, ae_j) + l_i m_j F(ae_i, ae_j) + \zeta_i m_j F(ae_i, ae_j) \\
&\qquad + F(\mu_i e_i, \omega_j e_j) + F(\nu_i e_i, \omega_j e_j) - F(\zeta_i ae_i, \omega_j e_j) \\
&\qquad + k_i F(ae_i, \omega_j e_j) + l_i F(ae_i, \omega_j e_j) + \zeta_i F(ae_i, \omega_j e_j) \\
&\qquad + m_j F(\mu_i e_i, ae_j) + m_j F(\nu_i e_i, ae_j) - m_j F(\zeta_i ae_i, ae_j)] \\
&= \tilde{F}(x_1, y) + \tilde{F}(x_2, y),
\end{aligned}$$

noting that the terms involving ζ_i cancel each other.

Finally to show the uniqueness of such extension, suppose F' is another such extension. If we fix $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$, we can find pairs (m, x_1) and (m', y_1) such that $m, m' \in \mathbf{N}$, $x_1, y_1 \in [-a, a]^n$ and $x = mx_1$, $y = m'y_1$. Then

$$\begin{aligned}
F'(x, y) &= F'(mx_1, m'y_1) = mm'F'(x_1, y_1) = mm'F(x_1, y_1) = mm'\tilde{F}(x_1, y_1) \\
&= \tilde{F}(mx_1, m'y_1) = \tilde{F}(x, y).
\end{aligned}$$

This proves the uniqueness of such extension. □

According to the last lemma, the function $f : [-a, a]^n \to E$ defined by $f(x) = F(x, x)$ has the unique quadratic extension $q : \mathbf{R}^n \to E$ given by $q(x) = \tilde{F}(x, x)$.

According to the following main result of this section, for each approximately quadratic function $f : I^n \to E$ we can choose a true quadratic function $q : \mathbf{R}^n \to E$ such that $\|f(x) - q(x)\|$ is bounded on I^n.

Theorem 18. *If a function $f : I^n \to E$ satisfies the inequality (49) for some $\delta > 0$ and for all $x, y \in I^n$ with $x + y$, $x - y \in I^n$, then there exists a quadratic function $q : \mathbf{R}^n \to E$ such that*

$$\|f(x) - q(x)\| < (2912n^2 + 1872n + 334)\delta$$

for $x \in I^n$.

Proof. According to Lemma 16, there exists an even function $\hat{f} : I^n \to E$ satisfying the inequalities (50) and (51) for any $x, y \in I^n$ with $x - y$, $x + y \in I^n$. Putting $x = y = 0$ in (49), together with the definition of $\hat{f}$ in the proof of Lemma 16, yields

$$\|\hat{f}(0)\| = \|f(0)\| \leq \frac{1}{2}\delta. \tag{55}$$

If we replace y by x in (51) and if we consider (55), then we have

$$\|\hat{f}(2x) - 4\hat{f}(x)\| \leq \frac{27}{2}\delta \tag{56}$$

for all $x \in I^n_{1/2} = [-\frac{t}{2}, \frac{t}{2}]^n$.

Now, we define a function $h : I^n_{1/4} \times I^n_{1/4} \to E$ by

$$h(x, y) = \frac{1}{4}[\hat{f}(x + y) - \hat{f}(x - y)], \tag{57}$$

where $I^n_{1/4} = [-\frac{t}{4}, \frac{t}{4}]^n$. Then, the evenness of $\hat{f}$ implies that h is symmetric. For any $u, v, y \in I^n_{1/4}$ with $u + v \in I^n_{1/4}$, the inequality (51) and the following relation

$$\begin{aligned}
& 4[h(u + v, y) - h(u, y) - h(v, y)] \\
= \quad & [\hat{f}(u + v + y) + \hat{f}(u - v - y) - 2\hat{f}(u) - 2\hat{f}(v + y)] \\
& - [\hat{f}(u - y - v) + \hat{f}(u - y + v) - 2\hat{f}(u - y) - 2\hat{f}(v)] \\
& - [\hat{f}(u + y) + \hat{f}(u - y) - 2\hat{f}(u) - 2\hat{f}(y)] \\
& + [\hat{f}(v + y) + \hat{f}(v - y) - 2\hat{f}(v) - 2\hat{f}(y)]
\end{aligned}$$

imply the inequality

$$\|h(u + v, y) - h(u, y) - h(v, y)\| \leq 13\delta.$$

Therefore, according to Theorem 15, there exists a symmetric biadditive function $F : I^n_{1/4} \times I^n_{1/4} \to E$ such that

$$\|h(x, y) - F(x, y)\| \leq 13(14n^2 + 9n + 1)\delta \tag{58}$$

for all $x, y \in I^n_{1/4}$. It follows from (55), (56) and (57) that

$$\begin{aligned} \|h(x,x) - \hat{f}(x)\| &= \frac{1}{4}\|\hat{f}(2x) - \hat{f}(0) - 4\hat{f}(x)\| \\ &\leq \frac{1}{4}(\|\hat{f}(2x) - 4\hat{f}(x)\| + \|\hat{f}(0)\|) \\ &\leq \frac{7}{2}\delta \end{aligned} \tag{59}$$

for any $x \in I^n_{1/4}$.

By Lemma 17, the quadratic function $F(x,x) : I^n_{1/4} \to E$ can be extended to the quadratic function $q : \mathbf{R}^n \to E$ such that

$$q(x) = F(x,x)$$

for each $x \in I^n_{1/4}$. By (58) and (59), we have

$$\begin{aligned} \|\hat{f}(x) - q(x)\| &\leq \|\hat{f}(x) - h(x,x)\| + \|h(x,x) - F(x,x)\| \\ &\leq \left(182n^2 + 117n + \frac{33}{2}\right)\delta \end{aligned} \tag{60}$$

for any $x \in I^n_{1/4}$.

Since if $x \in I^n_{1/2}$, then $(1/2)x \in I^n_{1/4}$, we can use (56) and (60) to obtain

$$\begin{aligned} \|\hat{f}(x) - q(x)\| &\leq \left\|\hat{f}(x) - 4\hat{f}\left(\frac{x}{2}\right)\right\| + \left\|4\hat{f}\left(\frac{x}{2}\right) - 4q\left(\frac{x}{2}\right)\right\| \\ &\leq \left(728n^2 + 468n + \frac{159}{2}\right)\delta \end{aligned} \tag{61}$$

for $x \in I^n_{1/2}$.

We can also apply (56) and (61) to get

$$\begin{aligned} \|\hat{f}(x) - q(x)\| &\leq \left\|\hat{f}(x) - 4\hat{f}\left(\frac{x}{2}\right)\right\| + \left\|4\hat{f}\left(\frac{x}{2}\right) - 4q\left(\frac{x}{2}\right)\right\| \\ &< (2912n^2 + 1872n + 332)\delta \end{aligned} \tag{62}$$

for each $x \in I^n$.

Finally, it follows from (50) and (62) that

$$\begin{aligned} \|f(x) - q(x)\| &\leq \|f(x) - \hat{f}(x)\| + \|\hat{f}(x) - q(x)\| \\ &< (2912n^2 + 1872n + 334)\delta \end{aligned}$$

for all $x \in I^n$. □

5. Stability on restricted (unbounded) domains

On the other hand, S.-M. Jung [27] proved the Hyers-Ulam stability of the quadratic functional equation (3) on a restricted (unbounded) domain and applied this result to a study of an asymptotic behavior of the quadratic functions.

In this section, let E_1 and E_2 be a real normed space and a real Banach space, respectively.

Theorem 19. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a function $f : E_1 \to E_2$ satisfies the inequality* (25) *for all $x, y \in E_1$ with $\|x\| + \|y\| \geq d$. Then there exists a unique quadratic function $Q : E_1 \to E_2$ such that*

$$\|f(x) - Q(x)\| \leq \frac{7}{2}\delta \tag{63}$$

for all $x \in E_1$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in E_1$ then $Q(tx) = t^2 Q(x)$ for all $x \in E_1$ and all $t \in \mathbf{R}$.

Proof. Assume $\|x\| + \|y\| < d$. If $x = y = 0$ then we choose a $z \in E_1$ with $\|z\| = d$. Otherwise, let $z = (1 + d/\|x\|)\, x$ for $\|x\| \geq \|y\|$ or $z = (1 + d/\|y\|)\, y$ for $\|x\| < \|y\|$. Clearly, we see

$$\begin{aligned} &\|x - z\| + \|y + z\| \geq d; \;\; \|x + z\| + \|y + z\| \geq d; \\ &\|y + z\| + \|z\| \geq d; \;\; \|x\| + \|y + 2z\| \geq d; \;\; \|x\| + \|z\| \geq d. \end{aligned} \tag{64}$$

From (25), (64) and from the relation

$$\begin{aligned} &f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &\quad = \;\; f(x+y) + f(x-y-2z) - 2f(x-z) - 2f(y+z) \\ &\qquad + f(x+y+2z) + f(x-y) - 2f(x+z) - 2f(y+z) \\ &\qquad - 2f(y+2z) - 2f(y) + 4f(y+z) + 4f(z) \\ &\qquad - f(x+y+2z) - f(x-y-2z) + 2f(x) + 2f(y+2z) \\ &\qquad + 2f(x+z) + 2f(x-z) - 4f(x) - 4f(z) \end{aligned}$$

we get

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le 7\delta. \tag{65}$$

Obviously, the inequality (65) holds true for all $x, y \in E_1$. According to (65) and Theorem 1, there exists a unique quadratic function $Q : E_1 \to E_2$ which satisfies the inequality (63) for all $x \in E_1$. Our last assertion is trivial in view of Theorem 1. □

Define $B = \{(x,y) \in E_1^2 \mid \|x\| < d,\ \|y\| < d\}$ for some $d > 0$. In view of the fact that $\{(x,y) \in E_1^2 \mid \|x\| + \|y\| \ge 2d\} \subset E_1^2 \setminus B$, the following corollary is a direct consequence of Theorem 19.

Corollary 20. *Assume that a function $f : E_1 \to E_2$ satisfies the inequality* (25) *for some $\delta \ge 0$ and for all $(x,y) \in E_1^2 \setminus B$. Then there exists a unique quadratic function $Q : E_1 \to E_2$ satisfying the inequality* (63) *for all $x \in E_1$.*

The uniqueness of the quadratic function Q in the previous theorem and corollary enables us to prove an asymptotic behavior of the quadratic functions:

Corollary 21. *A function $f : E_1 \to E_2$ is quadratic if and only if*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \to 0 \tag{66}$$

as $\|x\| + \|y\| \to \infty$.

Proof. Due to the asymptotic condition (66), there exists a sequence (δ_n) monotonically decreasing to 0 such that

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \delta_n \tag{67}$$

for all $x, y \in E_1$ with $\|x\| + \|y\| \ge n$. Hence, it follows from (67) and Theorem 19 that there exists a unique quadratic function $Q_n : E_1 \to E_2$ such that

$$\|f(x) - Q_n(x)\| \le \frac{7}{2}\delta_n \tag{68}$$

for all $x \in E_1$. Let $\ell, m \in \mathbb{N}$ satisfy $m \ge \ell$. Obviously, by (68) we obtain

$$\|f(x) - Q_m(x)\| \le \frac{7}{2}\delta_m \le \frac{7}{2}\delta_\ell$$

for all $x \in E_1$, since (δ_n) is a monotonically decreasing sequence. The uniqueness of Q_n implies $Q_m = Q_\ell$. Hence, by letting $n \to \infty$ in (68), we conclude that f is quadratic. The reverse assertion is trivial. $\square$

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PAIRS OF ADJOINT SETS OF RIEMANN-STIELTJES INTEGRABLE FUNCTIONS

TIBERIU TRIF

Abstract. We prove that the adjoint of $CBV[a,b]$, the set of all continuous functions of bounded variation on $[a,b]$, with respect to the Riemann-Stieltjes integral is the set of all bounded functions on $[a,b]$ whose points of discontinuity form an at most countable set.

1. Introduction and notation

Let a and b be real numbers with $a < b$, let $\mathcal{F}([a,b];\mathbf{R})$ be the set of all real-valued functions defined on $[a,b]$, and let A and B be nonempty subsets of $\mathcal{F}([a,b];\mathbf{R})$. The sets A and B are said to be *adjoint with respect to the Riemann-Stieltjes integral*, if the following conditions are satisfied:

(i) the Riemann-Stieltjes integral $(R-S)\int_a^b f dg$ exists whenever $f \in A$ and $g \in B$;

(ii) if $f \in \mathcal{F}([a,b];\mathbf{R})$ is such that the Riemann-Stieltjes integral $(R-S)\int_a^b f dg$ exists for all $g \in B$, then $f \in A$;

(iii) if $g \in \mathcal{F}([a,b];\mathbf{R})$ is such that the Riemann-Stieltjes integral $(R-S)\int_a^b f dg$ exists for all $f \in A$, then $g \in B$.

The fact that A and B are adjoint with respect to the Riemann-Stieltjes integral will be denoted by $A * B\,(R-S)$.

We shall use the following notations for some subsets of $\mathcal{F}([a,b];\mathbf{R})$:

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$C[a,b]$	set of all continuous functions $f \in \mathcal{F}([a,b];\mathbf{R})$;
$C_{\aleph_0}[a,b]$	set of all bounded functions $f \in \mathcal{F}([a,b];\mathbf{R})$, whose points of discontinuity form an at most countable set;
$R[a,b]$	set of all Riemann integrable functions $f \in \mathcal{F}([a,b];\mathbf{R})$;
$BV[a,b]$	set of all functions of bounded variation $f \in \mathcal{F}([a,b];\mathbf{R})$;
$CBV[a,b]$	$C[a,b] \cap BV[a,b]$;
$AC[a,b]$	set of all absolutely continuous functions $f \in \mathcal{F}([a,b];\mathbf{R})$.

It is well known (see, for instance, [3, Theorem II.12.1]) that the following result holds true.

Theorem A. $C[a,b] * BV[a,b]\,(R-S)$.

In a recent paper, H. Chen [1] proved that

Theorem B. $R[a,b] * AC[a,b]\,(R-S)$.

Having in mind that $CBV[a,b]$ lies between $AC[a,b]$ and $BV[a,b]$, a natural question arises: which is the adjoint of $CBV[a,b]$ with respect to the Riemann-Stieltjes integral? The answer to this question will be given in the next section of this paper. We shall also give a shorter and less complicated proof for Chen's result than that given in [1].

Throughout the paper, the set of points of discontinuity of any function $f \in \mathcal{F}([a,b];\mathbf{R})$ will be denoted by disc$\,(f)$. We denote by ν_f the Lebesgue-Stieltjes measure generated by any function $f \in CBV[a,b]$. For $f = \mathbf{1}_{[a,b]}$, ν_f is the usual Lebesgue measure and will be denoted by λ.

2. Main result

In the proofs below, we shall use the following results:

Proposition 1 ([3], Theorem II.10.9). *Let $f, g \in \mathcal{F}([a,b];\mathbf{R})$ be bounded functions on $[a,b]$. If f is Riemann-Stieltjes integrable with respect to g, then f and g have no common discontinuities on $[a,b]$.*

Proposition 2 ([4], Théorème 14, p. 33). *Let $f \in \mathcal{F}([a,b];\mathbf{R})$ be a bounded function, and let $g \in CBV[a,b]$. In order that f be Riemann-Stieltjes integrable with respect to g it is necessary and sufficient that $|\nu_g|(\operatorname{disc}(f)) = 0$.*

Before stating the main result of the paper, we give a short

Proof of Theorem B. (i) Let $f \in R[a,b]$ and $g \in AC[a,b]$ be arbitrarily chosen. Since f is Riemann integrable, it follows (by virtue of the necessity part of Proposition 2) that $\lambda(\operatorname{disc}(f)) = 0$. On the other hand, ν_g is absolutely continuous with respect to λ because g is absolutely continuous. Consequently, $|\nu_g|(\operatorname{disc}(f)) = 0$. Applying now the sufficiency part of Proposition 2, we can conclude that $(R-S)\int_a^b f dg$ exists.

(ii) Suppose that $f \in \mathcal{F}([a,b];\mathbf{R})$ is Riemann-Stieltjes integrable with respect to any function $g \in AC[a,b]$. Taking $g = \mathbf{1}_{[a,b]} \in AC[a,b]$, it follows that $f \in R[a,b]$.

(iii) Suppose that $g \in \mathcal{F}([a,b];\mathbf{R})$ is such that the Riemann-Stieltjes integral $(R-S)\int_a^b f dg$ exists for all $f \in R[a,b]$. In particular, $(R-S)\int_a^b f dg$ exists for all $f \in C[a,b]$. According to Theorem A, it follows that $g \in BV[a,b]$. We prove that $g \in C[a,b]$. Indeed, let x_0 be any point in $[a,b]$, and let $f_0 : [a,b] \to \mathbf{R}$ be the function defined by

$$f_0(x) = \begin{cases} 0 & \text{if } a \leq x \leq x_0 \\ 1 & \text{if } x_0 < x \leq b. \end{cases}$$

Since $f_0 \in R[a,b]$, f_0 must be Riemann-Stieltjes integrable with respect to g. Taking account of Proposition 1, it follows that g must be continuous at x_0. Since x_0 was arbitrarily chosen in $[a,b]$, we can conclude that $g \in C[a,b]$. Consequently, $g \in CBV[a,b]$.

In order to prove that $g \in AC[a,b]$, it suffices to prove that ν_g is absolutely continuous with respect to λ. To do this, let A be any subset of $[a,b]$ with $\lambda(A) = 0$. Since ν_g is regulated, there exists a sequence $(F_n)_{n\geq 1}$ of closed sets such that $F_n \subseteq A$ for all $n \geq 1$ and

$$|\nu_g|(F_n) \to |\nu_g|(A) \qquad \text{for } n \to \infty. \tag{1}$$

Let n be any positive integer. Obviously, $\lambda(F_n) = 0$. On the other hand, according to [2, Example 23, pp. 30-31], there exists a bounded function $f_n \in \mathcal{F}([a,b];\mathbf{R})$ such that $\operatorname{disc}(f_n) = F_n$. By virtue of the sufficiency part of Proposition 2 it follows that $f_n \in R[a,b]$. Therefore f_n must be Riemann-Stieltjes integrable with respect to g. Applying now the necessity part of Proposition 2 we deduce that

$$|\nu_g|(\operatorname{disc}(f_n)) = |\nu_g|(F_n) = 0.$$

Since n was arbitrarily chosen, we can conclude that $|\nu_g|(F_n) = 0$ for each positive integer n. Taking account of (1) we get $|\nu_g|(A) = 0$. Consequently, ν_g is absolutely continuous with respect to λ.

The main result of the paper is the following theorem:

Theorem C. $C_{\aleph_0}[a,b] * CBV[a,b]\,(R-S)$.

Proof. (i) Let $f \in C_{\aleph_0}[a,b]$ and $g \in CBV[a,b]$ be arbitrarily chosen. Since g is continuous, it follows that $|\nu_g|(\{x\}) = 0$ for each $x \in [a,b]$. Taking into account that $\operatorname{disc}(f)$ is at most countable, we deduce that $|\nu_g|(\operatorname{disc}(f)) = 0$. Applying the sufficiency part of Proposition 2 we can conclude that $(R-S)\int_a^b f dg$ exists.

(ii) Suppose that $f \in \mathcal{F}([a,b];\mathbf{R})$ is Riemann-Stieltjes integrable with respect to any function $g \in CBV[a,b]$. Then f is Riemann integrable, hence bounded on $[a,b]$. Applying the necessity part of Proposition 2 it follows that

$$|\nu_g|(\operatorname{disc}(f)) = 0 \qquad \text{for all } g \in CBV[a,b]. \tag{2}$$

Suppose that $\operatorname{disc}(f)$ is uncountable. Since $\operatorname{disc}(f)$ is an F_σ set, there exists a sequence $(F_m)_{m\geq 1}$ of closed sets such that

$$\operatorname{disc}(f) = \bigcup_{m=1}^{\infty} F_m.$$

The set $\operatorname{disc}(f)$ being uncountable, it follows that there exists a positive integer m such that F_m is uncountable. According to the well known Cantor-Bendixon Theorem (see, for instance, [6, p. 47]), F_m is the union of a perfect set A and an at most countable set B. Obviously, A must be uncountable. On the other hand, from (2)

Relation (3) ensures that A contains no intervals. Let $\alpha := \min A$ and $\beta := \max A$. The set $]\alpha, \beta[\setminus A$ being open, there exists a sequence $(]\alpha_n, \beta_n[)_{n\geq 1}$ of disjoint open intervals such that

$$]\alpha, \beta[\setminus A = \bigcup_{n=1}^{\infty}]\alpha_n, \beta_n[.$$

For each positive integer n let I'_n and I''_n be the sets defined by

$$I'_n := \{k \in \mathbf{N} \mid \beta_k < \alpha_n\}$$

and

$$I''_n := \{k \in \mathbf{N} \mid \beta_n < \alpha_k\},$$

respectively. For distinct positive integers m and n, let $I_{m,n}$ be the set defined by

$$I_{m,n} := \begin{cases} \{k \in \mathbf{N} \mid \beta_m < \alpha_k < \beta_k < \alpha_n\} & \text{if } \beta_m < \alpha_n \\ \{k \in \mathbf{N} \mid \beta_n < \alpha_k < \beta_k < \alpha_m\} & \text{if } \beta_n < \alpha_m. \end{cases}$$

Since A is perfect and contains no intervals, it is easily seen that all the sets I'_n, I''_n and $I_{m,n}$ are infinite.

We construct inductively a sequence $(J_n)_{n\geq 1}$ of ordered sets $J_n = \{k_1, \ldots, k_p\}$, $p = 2^n - 1$ $(n \in \mathbf{N})$ of positive integers as follows: put $J_1 := \{1\}$ and suppose that J_n has already been constructed. Let $J_n = \{k_1, \ldots, k_p\}$, $p = 2^n - 1$, where the numbers $k_1, \ldots, k_p$ are ordered such that

$$\alpha_{k_1} < \beta_{k_1} < \alpha_{k_2} < \beta_{k_2} < \cdots < \alpha_{k_p} < \beta_{k_p}.$$

Set $l_0 := \min I'_{k_1}$, $l_p := \min I''_{k_p}$, and $l_i := \min I_{k_i, k_{i+1}}$ $(1 \leq i \leq p-1)$. We define the set J_{n+1} by

$$J_{n+1} := \{l_0, k_1, l_1, k_2, l_2, \ldots, k_{p-1}, l_{p-1}, k_p, l_p\}.$$

Let $n \in \mathbf{N}$, let $J_n = \{k_1, \ldots, k_p\}$, $p = 2^n - 1$, and let $g_n : [a, b] \to \mathbf{R}$ be the function defined as follows:

$g_n(x) = 0$ for each $x \in [a, \alpha]$,

$g_n(x) = 1$ for each $x \in [\beta, b]$,

$g_n(x) = i/2^n$ for each $x \in]\alpha_{k_i}, \beta_{k_i}[$, $i = 1, \ldots, p$,

g_n is linear on each subinterval of the set $]\alpha, \beta[\setminus \cup_{i=1}^{p}]\alpha_{k_i}, \beta_{k_i}[$.

Further, let $g : [a, b] \to \mathbf{R}$ be the Cantor-type function defined by

$$g(x) := \lim_{n\to\infty} g_n(x).$$

Obviously, g is continuous and increasing, hence $g \in CBV[a, b]$. By the construction of the sequence $(J_n)_{n\geq 1}$ it follows that

$$\bigcup_{n=1}^{\infty} \left(\bigcup_{k\in J_n}]\alpha_k, \beta_k[\right) =]\alpha, \beta[\setminus A.$$

Consequently, g is constant in any interval $]\alpha_n, \beta_n[$, hence $|\nu_g|(]\alpha_n, \beta_n[) = 0$ for all $n \in \mathbf{N}$. Therefore, $|\nu_g|(A) = |\nu_g|([\alpha, \beta]) = 1$, in contradiction with (3). This contradiction shows that disc (f) must be at most countable, hence $f \in C_{\aleph_0}[a, b]$.

(iii) Suppose that $g \in \mathcal{F}([a, b]; \mathbf{R})$ is such that the Riemann-Stieltjes integral $(R - S) \int_a^b f dg$ exists for all $f \in C_{\aleph_0}[a, b]$. Reasoning like in the proof of Theorem B (iii), it can be shown that $g \in CBV[a, b]$.

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