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NUMERICAL APPROXIMATIONS OF $L_2([-1,1])$ - INTEGRABLE SOLUTIONS TO URYSON'S EQUATION

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Abstract. This paper considers a numerical method for approximating solutions of Uryson equation

$$y(x) = g(x) + \int_{-1}^{1} K(x, s, y(s))ds, \ x \in [-1, 1],$$

in the functional space $L_2([-1,1])$, where K and g are given functions and y is the solution to be determined. The fundamental feature of the method is the numerical solution of a fixed-point problem concerning an operator defined in the functional space, which has the same solution as Uryson's equation. Here, we constructed the approximate solutions by employing Lagrange's interpolation. Error analysis is done for the approximate solutions and an example is included to verify the theoretical results.

1. Preliminaries

Here, we explain the special notation and some of the concepts used in the paper. Let $I = [-1,1]$ and let $L_2(I)$ denote the space of real square-integrable functions defined on I.

For any $y \in L_2(I)$ we have the L_2 - norm

$$||y||_2 = \left(\int_{-1}^{1} |y(x)|^2 dx \right)^{1/2}.$$

$L_2(I)$ is a Hilbert space.

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Let P_n be the Legendre polynomial of n - th degree defined on I. The set of n zero-points of P_n is denoted by $\Delta_n = \{x_1, \ldots, x_n\}$, where each element is called a colocation point.

The Lagrange interpolation on Δ_n for $y \in R(I)$ is given by

$$L_n(y)(x) := \sum_{i=1}^{n} y(x_i) l_i(x),$$

where

$$l_i(x) = \frac{P_n(x)}{(x - x_i) P_n'(x_i)}.$$

For the Lagrange interpolation on Δ_n, the following result (the Erdos-Turan theorem) is known [2,5]: For any Riemann-integrable function $y \in R(I), L_n(y)$ converges to y in L_2 - norm, that is, $||L_n(y) - y||_2 \to 0$, as n tends to infinity. It is also known [3] that the error of the interpolation $L_n(y)$ for a sufficiently smooth function y is given by

$$R_n(y)(x) = \frac{2^n \cdot n!}{(2n)!} P_n(x) y^{(n)}(\zeta), \ |\zeta| < 1, \ x \in [-1, 1],$$

where

$$R_n(y) = y - L_n y \text{ and } y^{(n)} \text{ is the } n \text{ -th derivative of } y.$$

The real numbers

$$W_i = \int_{-1}^{1} l_i(x) dx = \frac{2(1 - x_i^2)}{n^2 P_{n-1}^2(x_i)} > 0, \ i \in \overline{1, n},$$

are weight coeficients of Legendre - Gauss's quadrature with abscissas Δ_n, where the relation $W_1 + W_2 + \cdots + W_n = 2$ is satisfied [3]. For the Legendre-Gauss quadrature with $\{W_i\}_i$ and abscissas Δ_n, the following convergence assertion known [1]:

$$\sum_{i=1}^{n} W_i y(x_i) \overset{n \to \infty}{\longrightarrow} \int_{-1}^{1} y(x) dx, (\forall) y \in R([-1, 1]).$$

In addition, the quadrature for a sufficiently smooth function y has the error

$$E_n(y) = \frac{2^{2n+1} (n!)^4}{(2n + 1)(2n!)^3} = y^{(2n)}(\eta), |\eta| < 1.$$

Finally, the folowing relation holds for both $\{l_i\}_i$ and $\{W_i\}_i$:

$$\int_{-1}^{1} l_i(x) l_j(x) dx = \sum_{q=1}^{n} W_i \delta_{iq} \delta_{jq}, \ 1 \leq i, j \leq n,$$

where δ_{iq} and $\delta_j q$ are Kronecker's delta functions.

2. Existence of the Solution for Uryson Integral Equations and Composition of Approximations Solutions

Let us consider the integral equation:

$$y(x) = g(x) + \int_a^b K(x, s, y(s))ds, \ x \in [-1, 1]. \tag{1}$$

We make the following assumptions:

(i) $g \in C([-1,1])$ and $|g(x)| \leq A, \ (\forall) x \in [-1, 1]$;

(ii)$|K(x, s, o)| \leq B, \ (\forall) x, s \in [-1, 1]$;

(iii) Let $h > A + 2B$ and we defined

$$\Omega = \{y \in L_2([-1,1])/|y(x)| \leq h \text{ a.e.w. on } [-1,1]\}$$

(iv) $K(x, s, r) \in C([-1,1] \times [-1,1] \times [-h, h])$ such that

$$|K(x, s, r_1) - K(x, s, r_2)| \leq C|r_1 - r_2|, (\forall) x, s \in [-1, 1], (\forall) r_1, r_2 \in [-h, h].$$

Theorem 1 *If* $C \leq \dfrac{h - A - 2B}{2h} \left(< \dfrac{1}{2} \right)$, *then equation (1) has a unique solution in* Ω.

The proof is a straightforward application of the Banach's fixed point theorem.

If we introduce the Legendre-Gauss quadrature formulas we obtain the equation atached:

$$y_n(x) = g(x) + \sum_{j=1}^{n} W_j K(x, s_j, y_n(s_j)), x \in [-1, 1]. \tag{2}$$

We denote $\alpha_i = y_n(s_i)$ and obtain

$$\alpha_i = g(s_i) + \sum W_j K(s_i, s_j, \alpha_j), i \in \overline{1, n}, \tag{3}$$

which represent a nonlinear algebraic sistem.

Theorem 2 *If* $|K(s_i, s, r_1) - K(s_i, s, r_2)| \leq L_i |r_1 - r_2|$,

$$(\forall) \ i \in \overline{1,n}, \ (\forall) \ s \in [-1, 1], \ (\forall) \ r_1, r_2 \in [-h, h] \ \textit{and} \ 2 \left(\sum_{i=1}^{n} L_i^2 \right)^{1/2} < 1,$$

then the nonlinear algebraic sistem (3) has a unique solution in $I_2(n)$.

Proof. Let

$$F : l_2(u) \to l_2(u),\ F(\alpha_1, \ldots, \alpha_n) = (F_1(\alpha_1, \ldots, \alpha_n), \ldots, F_n(\alpha_1, \ldots, \alpha_n)),$$

where

$$F_i(\alpha_1, \ldots, \alpha_n) = f(s_i) + \sum_{j=1}^{n}(W_j K(s_i, s_j, \alpha_j).$$

We have:

$$\begin{aligned}
\|F(\alpha^1) - F(\alpha^2)\|_2 &= \left(\sum_{i=1}^{n} \left|F_i(\alpha^1) - F_i(\alpha^2)\right|^2\right)^{1/2} = \\
&= \left(\sum_{i=1}^{n} \left|\sum_{j=1}^{n} W_j(K(s_1, s_j, \alpha_j^1) - K(s_i, s_j, \alpha_j^2))\right|^2\right)^{1/2} \leq \\
&\leq \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} W_j L_i \left|\alpha_j^1 - \alpha_j^2\right|\right)^2\right)^{1/2} \leq \\
&\leq 2\left(\sum_{i=1}^{n} L_i^2\right)^{1/2} \|\alpha^1 - \alpha^2\|_2.
\end{aligned}$$

Because $2(\sum_{i=1}^{n} L_i^2)^{1/2} < 1$, then F is a contraction operator and the nonlinear algebraic sistem (3) has a unique solution in $I_2(n)$.

Now, we can write that the unique solution of equation (2) is

$$y_n(x) = g(x) + \sum_{j=1}^{n} W_j K(x, s_j, \alpha_j), (\forall)\ x \in [-1, 1].$$

□

Theorem 3 *In conditions of existence the solutions of equations (1) and (2) we have:*

$$\lim_{n\to\infty} \|y - y_n\|_2 = 0.$$

Proof. We have

$$\|y - y_n\|_2 \leq \left(\int_{-1}^{1} \left|\int_{-1}^{1} K(x, s, y(s)) - K(x, s, y_n(s))ds\right|^2 dx\right)^{1/2} + I_n,$$

where

$$I_n = \left(\int_{-1}^{1} \left| \int_{-1}^{1} K(x,s,y_n(s))ds - \sum_{j=1}^{n} W_j K(x,s_j,y_n(s_j)) \right|^2 dx \right)^{1/2}.$$

We find

$$||y - y_n||_2 \leq \sqrt{2}C||y - y_n||_2 + I_n.$$

Let

$$\epsilon > 0.\ (\forall)\ x \in [-1,1],\ (\exists)\ N(x) \in N \text{ so that,}$$

$$\left| \int_{-1}^{1} K(x,s,y_n(s))ds - \sum_{j=1}^{n} W_j K(x,s_j,y_n(s_j)) \right| < \epsilon,\ (\forall) n \geq N(x).$$

Let

$$N = \sup_{x\in[-1,1]} N(x).$$

Then

$$\left| \int_{-1}^{1} K(x,s,y_n(s))ds - \sum_{j=1}^{n} W_j K(x,s_j,y_n(s_j)) \right| < \epsilon, (\forall) x \in [-1,1],\ (\forall) n \geq N.$$

Finally, we have

$$||y - y_n||_2 \leq \frac{\sqrt{2}\cdot\epsilon}{1-\sqrt{2}C},$$

and $||y - y_n||_2 \to 0$ when $n \to \infty$. □

3. **Example**

Let be the integral equation

$$(1) \quad y(x) = \int_{-1}^{1} \sin\left(\frac{x+1}{2}\cdot\pi\right) y^2(s)ds$$

We must solve the nonlinear algebraic sistem:

$$\begin{cases} \alpha_1 = \sin\left(\frac{x_i+1}{2}\pi\right) \sum_{j=1}^{n} W_j \alpha_j^2 \\ \dots \\ \alpha_n = \sin\left(\frac{x_n+1}{2}\pi\right) \sum_{j=1}^{n} W_j \alpha_j^2 \end{cases}$$

We have

$$\begin{cases} \alpha_j = \alpha_1 \dfrac{\sin\left(\dfrac{x^i+1}{2}\pi\right)}{\sin\left(\dfrac{x^1+1}{2}\pi\right)} \\ \alpha_1 = \sin\dfrac{x_1+1}{2}\sum_{j=1}^{n} W_j \alpha_1^2 \dfrac{\sin^2\left(\dfrac{x_j+1}{2}\pi\right)}{\sin^2\left(\dfrac{x_1+1}{2}\pi\right)} \end{cases}$$

and we find $\alpha_1 = 0$; so $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

or $\alpha_1 = \dfrac{\sin\left(\dfrac{x_1+1}{2}\pi\right)}{\sum_{j=1}^{n} W_j \sin^2\left(\dfrac{x_j+1}{2}\pi\right)}$ and $\alpha_i = \dfrac{\sin\left(\dfrac{x^i+1}{2}\pi\right)}{\sum_{j=1}^{n} W_j \sin^2\left(\dfrac{x_j+1}{2}\pi\right)}, i \in \overline{2,n}.$

Equation (2) has two solutions:

$$y_{n1} = 0 \quad (so\ y_1(x) = 0)$$

and

$$y_{n2}(x) = \sum_{i=1}^{n} W_i \left(\frac{\sin\left(\dfrac{x_i+1}{2}\pi\right)}{\sum_{k=1}^{n} W_k \sin^2\left(\dfrac{x_k+1}{2}\pi\right)} \right)^2 \sum_{j=1}^{n} l_j(x)\cdot\sin\left(\frac{x_j+1}{2}\pi\right)$$

or

$$y_{n2}(x) = \sum_{j=1}^{n} l_j(x) \frac{\sin\left(\dfrac{x_j+1}{2}\pi\right)}{\sum_{k=1}^{n} W_k \sin^2\left(\dfrac{x_k+1}{2}\pi\right)}$$

$$y_{n2}(x) = \sum_{j=1}^{n} \alpha_j l_j(x).$$

We have

$$\lim_{n\to\infty} y_{n2}(x) = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j l_j(x) = \lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} W_k \sin^2\left(\dfrac{x_k+1}{2}\pi\right)}\cdot$$

$$\cdot \lim_{n\to\infty} \sum_{j=1}^{n} l_j(x) \sin\left(\frac{x_j+1}{2}\pi\right) =$$

$$= \frac{1}{\displaystyle\int_{-1}^{1} \sin^2\left(\frac{x+1}{2}\pi\right) dx} \cdot \lim_{n\to\infty} L_n\left(\sin\left(\frac{x+1}{2}\pi\right)\right)(x) = 1 \sin\left(\frac{x+1}{2}\pi\right).$$

So,

$$y_2(x) = \sin\left(\frac{x+1}{2}\pi\right).$$

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HOMOGENEOUS NUMERICAL INTEGRATION FORMULAS

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0. Let D be a given region in n-dimensional Euclidian space and f an integrable function on D.

One considers the following numerical integration formula

$$\int_D \cdots \int w(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_1 \dots dx_n = \sum_{i=1}^{m} A_i f(x_{i,1}, \dots, x_{i,n}) + R_m(f),$$

where w is a given weight function (often $w(x_1, \dots, x_n) = 1$), $(x_{i,1}, \dots, x_{i,n}) \in D$, $i = 1, \dots, m$, are the nodes of the formula, A_i, $i = 1, \dots, m$, are its coefficients and $R_m(f)$ is the remainder term.

The problem to construct such a numerical integration formula consists in the determination of the coefficients A_i and of the nodes $(x_{i,1}, \dots, x_{i,n})$, $i = 1, \dots, m$, in some given conditions, and to evaluate the corresponding remainder term.

For particular cases a multi-dimensional numerical integration formula can be obtained using one-dimensional quadrature rules (operators) and product respectively the boolean-sum operations, i.e. so called product and boolean-sum formulas.

The purpose of this note is to derive from the boolean-sum formula, so called, homogeneous formulas in 2 and 3-dimensional rectangular cases.

1. Let D_n be a rectangular domain in $\mathbf{R}^n$, $D_n = \mathop{\times}\limits_{k=1}^{n} [a_k, b_k]$ and $\prod_n$ a rectangular partition of D_n: $\prod_n = \mathop{\times}\limits_{k=1}^{n} \Delta x_k$, where $\Delta_k = \{x_{k,1}, \dots, x_{k,m_k}\}$ with $a_k \leq x_{k,1} < \cdots < x_{k,m_k} \leq b_k$.

One considers the partial quadrature formulas:

$$I^k f = Q_1^k f + R_1^k f$$

for all $k = 1, \ldots, n$, where

$$I^k f = \int_{a_k}^{b_k} f(x_1, \ldots, x_k, \ldots, x_n) dx_k$$

$$Q_1^k f = \sum_{i_k=1}^{m_k} A_{i_k}^k f(x_1, \ldots, x_{k-1}, x_{k,i_k}, x_{k+1}, \ldots, x_n)$$

and $R_1^k f$ is the corresponding remainder term; I^k, Q_1^k and R_1^k act the function f with regard to the variable x_k. Also, one denotes by $ord(Q_1^k)$ the approximation order of the operator Q_1^k, $k = 1, \ldots, n$.

Using the boolean sum and the product of all these quadrature operators $Q_1^1, \ldots, Q_1^n$, there are obtained so called boolean-sum

$$If = Q_s f + R_s f \tag{1}$$

and product

$$If = Q_p f + R_p f \tag{2}$$

integration formula, where

$$If = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

R_s and R_p are the corresponding remainder operators.

Taking into account that

$$\alpha_1 \oplus \cdots \oplus \alpha_n = \sum_{i=1}^{n} \tilde{\alpha}_i \tag{3}$$

with

$$\tilde{\alpha}_1 = \alpha_1, \quad \tilde{\alpha}_i = \alpha_i - \left(\sum_{k=1}^{i-1} \tilde{\alpha}_k \right) \alpha_i, \quad i = 2, \ldots, n,$$

the formulas (1) respectively (2) are based on the following decompositions of the integral operator:

$$I = Q_s + R_p$$

and

$$I = Q_p + R_s$$

where

$$Q_s = Q_1^1 I^{2,\dots,n} + \cdots + Q_1^n I^{1,\dots,n-1} - Q_1^1 Q_1^2 I^{3,\dots,n} - \cdots + (-1)^{n-1} Q_1^1 \dots Q_n^n, \tag{4}$$

$$Q_p = Q_1^1 \dots Q_1^n,$$

and

$$R_s = R_1^1 I^{2,\dots,n} + \cdots + R_1^n I^{1,2,\dots,n-1} - R_1^1 R_1^2 I^{3,\dots,n} - \cdots + (-1)^{n-1} R_1^1 \dots R_1^n \tag{5}$$

with

$$I^{\nu_1,\dots,\nu_p} f = \int_{a_{\nu_1}}^{b_{\nu_1}} \cdots \int_{a_{\nu_p}}^{b_{\nu_p}} f(x_1,\dots,x_n) dx_{\nu_1} \dots dx_{\nu_p}.$$

Remark. The representation (4) and (5) of Q_s respectively R_s derives from the definition (3) of the boolean-sum operation.

It follows that $Q_s f$ of formula (1) contains $(n-1),\dots,2,1$-multiple integrals, i.e. (1) is not a numerical integration formula, while (2) is such a formula. Indeed,

$$Q_p f = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} A_{i_1}^1 \dots A_{i_n}^n f(x_{1,i_1},\dots,x_{n,i_n}).$$

But, the formula (1) has the remarkable property regarding its higheast approximation order.

Our goal is to derive from the boolean-sum formula, using next approximation levels, a numerical formula with the same approximation order.

As an examplification, is taken the 2-dimensional case.

Let

$$If := \int_a^b \int_c^d f(x,y) dx dy = Q_s f + R_p f$$

be the boolean-sum formula given by the quadrature operators Q_1^x and Q_1^y with the corresponding remainder operators R_1^x respectively R_1^y. So, we start with the following decomposition of the operator I:

$$I = Q_1^x I^y + Q_1^y I^x - Q_1^x Q_1^y + R_1^x R_1^y \tag{6}$$

with

$$I^x f = \int_a^b f(x,y) dx, \quad I^y f = \int_c^d f(x,y) dy.$$

In a second level of approximation, the integrals $I^x f$ and $I^y f$ can be approximated using new quadrature operators, say Q_2^x and Q_2^y, i.e.

$$I^x = Q_2^x + R_2^x, \quad I^y = Q_2^y + R_2^y.$$

From (6), one obtains

$$If = (Q_1^x Q_2^y + Q_2^x Q_1^y - Q_1^x Q_1^y)f + (Q_1^x R_2^y + Q_1^y R_2^x + R_1^x R_1^y)f \tag{7}$$

which is a numerical approximation formula, with the approximation operator

$$Q = Q_1^x Q_2^y + Q_2^x Q_1^y - Q_1^x Q_1^y$$

and the remainder operator

$$R_Q = Q_1^x R_2^y + Q_1^y R_2^x + R_1^x R_1^y.$$

As can be seen, the remainder is a sum of the terms.

Is obviously that the remainder operator R_Q of a numerical integration formula $I = Qf + R_Q f$ derived from a boolean sum formula, using many approximation levels, is the sum of many terms. The approximation order of the approximation operator Q must be taken with respect to each term of R_Q. In the above example the order corresponding to the first term $(Q_1^x R_2^y)$ is $ord(Q_2^y) + 1$, $ord(Q_2^x) + 1$ for the second term $(Q_1^y R_2^x)$ and $ord(Q_1^x) + ord(Q_1^y)$ for the thirth term $(R_1^x R_1^y)$.

Definition. Let Q be a numerical integration operator and R_Q the corresponding remainder operator. If the approximation order of the operators corresponding to each term of the remainder R_Q is the same then Q is called a homogeneous approximation operator and

$$If = Qf + R_Q f$$

a homogeneous approximation formula.

Remark. (7) is a homogeneous approximation formula if $ord(Q_2^y) + 1 = ord(Q_2^x) + 1 = ord(Q_1^x) + ord(Q_1^y)$.

So, (7) is a homogeneous approximation formula if the quadrature operators used in the second approximation level are such that

$$ord(Q_2^y) = ord(Q_2^x) = ord(Q_1^x) + ord(Q_1^y) - 1,$$

that is always possible. *It means that, for 2-dimensional case, from the boolean-sum formula can be derived, using a second approximation level, a homogeneous cubature formula with the same approximation order.*

A similar result can be given for 3-dimensional case.

Theorem. *Let $D \in \mathbf{R}^3$ be a rectangular domain, Q_1^x, Q_1^y and Q_1^z quadrature operators with the remainder operators R_1^x, R_1^y, respectively R_1^z and*

$$If = Q_S f + R_P f$$

the corresponding boolean-sum formula. Then, using two next approximation levels, from (7) can be derived a homogeneous numerical approximation formula.

Proof. Let

$$I = (Q_1^x I^{yz} + Q_1^y I^{zx} + Q_1^z I^{xy} - Q_1^x Q_1^y I^z - Q_1^x Q_1^z I^y - Q_1^y Q_1^z I^x + Q_1^x Q_1^y Q_1^z) + R_1^x R_1^y R_1^z \tag{8}$$

be the boolean-sum decomposition of the integral operator I.

In the second level of approximation, for each 2-dimensional integral operator is used a boolean-sum decomposition using a new quadrature operator, as follows:

$$I^{yz} = (Q_1^y I^z + Q_2^z I^y - Q_1^y Q_2^z) + R_1^y R_2^z$$

$$I^{zx} = (Q_1^z I^x + Q_2^x I^z - Q_1^z Q_2^x) + E_1^z R_2^x$$

$$I^{xy} = (Q_1^x I^z + Q_2^y I^x - Q_1^x Q_2^y) + R_1^x R_2^y.$$

The identity (8) becomes:

$$I = (Q_1^x Q_2^z I^y + Q_1^y Q_2^x I^z + Q_1^z Q_2^y I^x - Q_1^x Q_1^y Q_2^z - Q_1^y Q_1^z Q_2^x - Q_1^x Q_1^z Q_2^y + Q_1^x Q_1^y Q_1^z) +$$

$$+ (Q_1^x R_1^y R_2^z + Q_1^y R_1^z R_2^x + Q_1^z R_1^x R_2^y + R_1^x R_1^y R_1^z).$$

Now, using the new quadrature rules

$$I^y = Q_3^y + R_3^y$$

$$I^z = Q_3^z + R_3^z$$

$$I^x = Q_3^x + R_3^x$$

one obtains the final decomposition

$$I = Q + R_Q$$

with

$$Q = Q_1^x Q_2^z Q_3^y + Q_1^y Q_2^x Q_3^z + Q_1^z Q_2^y Q_3^x - Q_1^x Q_1^y Q_2^z - Q_1^y Q_1^z Q_2^x - Q_1^x Q_1^z Q_2^y + Q_1^x Q_1^y Q_1^z$$

and

$$R_Q = Q_1^x Q_2^z R_3^y + Q_1^y Q_2^x R_3^z + Q_1^z Q_2^y R_3^x + Q_1^x R_1^y R_2^z + Q_1^y R_1^z R_2^x + Q_1^z R_1^x R_2^y + R_1^x R_1^y R_1^z).$$

In order to obtain a homogeneous decomposition we must have:

$$\begin{cases} ord(Q_3^x) + 2 = ord(Q_3^y) + 2 = ord(Q_3^z) + 2 = ord(Q_1^x) + ord(Q_1^y) + ord(Q_1^z) \\ ord(Q_1^x) + ord(Q_2^z) + 1 = ord(Q_1^z) + ord(Q_2^x) + 1 = \\ \qquad = ord(Q_1^x) + ord(Q_2^{y+1}) = ord(Q_1^x) + ord(Q_1^y) + ord(Q_1^z) \end{cases}$$

This system has the solution:

$$ord(Q_3^x) = ord(Q_3^y) = ord(Q_3^z) = ord(Q_1^x) + ord(Q_1^y) + ord(Q_1^z) - 2$$

$$ord(Q_2^x) = ord(Q_1^x) + ord(Q_1^y) - 1$$

$$ord(Q_2^y) = ord(Q_1^y) + ord(Q_1^z) - 1$$

$$ord(Q_2^z) = ord(Q_1^z) + ord(Q_1^x) - 1$$

and the proof follows.

For example, a homoegeneous numerical integration formula is obtained for the following quadrature operators: trapezoidal quadrature rules Q_T^x, Q_Y^y, Q_Y^z for the first level $(ord(Q_T) = 3)$, Simpson's quadrature rules Q_S^x, Q_S^y, Q_S^z for the second level $(ord(Q_S) = 5)$ and of Gauss quadrature rules with three nodes Q_G^x, Q_G^y, Q_G^z $(ord(Q_G) = 7)$ for the thirth level.

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ON UNIFORM EXPONENTIAL DICHOTOMY OF C_0-QUASISEMIGROUPS IN BANACH SPACES

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Abstract. Necessary and sufficient conditions for uniform exponential dichotomy of C_0-quasisemigroups in Banach spaces are given. The results obtained may be regarded as generalizations of well-known results of Datko, Pazy, Preda and Megan about asymptotical behaviours of C_0-quasisemigroups.

1. Introduction

Let $S = (S(t))_{t\geq 0}$ be a C_0-quasisemigroup on a Banach space X. In stability theory of C_0-semigroups in Banach spaces, a notable result is the following:

Theorem 1.1. *A C_0-semigroup S is uniformly exponentially stable if and only if there are $M > 0$ and $p \in [1, \infty)$ such that:*

$$\int_0^\infty \|S(t)x\|^p \, dt \leq M \, \|x\|^p$$

for all $x \in X$.

This result was originally proved by Datko [2] for $p = 2$ and Pazy [4] showed that the exponent $p = 2$ may be replaced by $p \in [0, \infty)$.

An extension of Theorem 1.1 is:

Theorem 1.2 *Let S be an uniform dichotomic C_0-semigroup. Then S is uniformly exponentially dichotomic if and only if there are $M > 0$ and a projection operator P such that:*

(i)

$$\int_t^\infty \|T(\tau)Px_0\| \, d\tau \leq M \, \|T(t)Px_0\| \, ;$$

(ii)
$$\int_0^t \|T(\tau)(x_0 - Px_0)\| \, d\tau \le M \|T(t)(x_0 - Px_0)\| ;$$
for all $(t, x_0) \in \mathbf{R}_+ \times X$.

The proof of this result is given in [5].

It is easy to see that from this theorem it results:

Corollary 1.2. *Let S be an uniform dichotomic C_0-semigroup. Then S is uniformly exponentially dichotomic if and only if there are $a, b > 0$ such that:*

(i)
$$\int_t^\infty e^{a(\tau - t)} \|T(\tau)Px_0\| \, d\tau \le b \|T(t)Px_0\| ;$$

(ii)
$$\int_0^t e^{a(t-\tau)} \|T(\tau)(x_0 - Px_0)\| \, d\tau \le b \|T(t)(x_0 - Px_0)\| ;$$
for all $(t, x_0) \in \mathbf{R}_+ \times X$.

The aim of this paper is to extend these results for the case of C_0-quasisemigroups in Banach spaces. Some discrete variants of these results are obtained.

2. Definitions and notations

Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators from X onto itself.

Definition 2.1. An operator-valued map $S : \mathbf{R}_+^2 \to \mathcal{B}(X)$ is called a C_0-*quasisemigroup* on X if it has the following properties:

(q_1) $S(0, t_0) = I$ (the identity operator on X) for every $t_0 \ge 0$;

(q_2) $S(t, s + t_0)S(s, t_0) = S(t + s, t_0)$ for all $(t, s, t_0) \in \mathbf{R}_+^3$;

(q_3) $\lim_{t \to 0} \|S(t, t_0)x_0 - x_0\| = 0$ for all $(t_0, x_0) \in \mathbf{R}_+ \times X$;

(q_4) there exist $M \ge 1$ and $\omega > 0$ such that: $\|S(t, t_0)\| \le Me^{\omega t}$ for all $(t, t_0) \in \mathbf{R}_+^2$.

Remark 2.1. If $S : \mathbf{R}_+^2 \to \mathcal{B}(X)$ is a C_0-semigroup on X then :
$$\widetilde{S} : \mathbf{R}_+^2 \to \mathcal{B}(X), \quad \widetilde{S}(t, t_0) \stackrel{d}{=} S(t)$$
is a C_0-quasisemigroup on X.

Definition 2.2. The C_0-quasisemigroup $S : \mathbf{R}_+^2 \to \mathcal{B}(X)$ is *called uniformly dichotomic* (and we write *u.d.*) if there are a positive constant $N > 0$ and a strongly continuous projection-valued function $P : \mathbf{R}_+ \to \mathcal{B}(X)$ (i.e. $P^2(t) = P(t)$ for every $t \geq 0$), such that for all $(t, s, t_0) \in \mathbf{R}_+^3$, $x_0 \in \operatorname{Im} P(t_0)$ and $y_0 \in Ker P(t_0)$ we have:

d_0) $S(t, t_0)P(t_0) = P(t + t_0)S(t, t_0)$;

d_1) $\|S(t + s, t_0)x_0\| \leq N \|S(s, t_0)x_0\|$;

d_2) $\|S(t, t_0)y_0\| \leq N \|S(t + s, t_0)y_0\|$.

If $P : \mathbf{R}_+ \to \widetilde{X}$ is a projection-valued function then we denote:

$$S_1(t, t_0) = S(t, t_0)P(t_0) \text{ and } S_2(t, t_0) = S(t, t_0)(I - P(t_0)).$$

Remark 2.2. If the C_0-quasisemigroup S is *u.d.* then:

$$S_1(t + s, t_0) = S_1(t, s + t_0)S_1(s, t_0) \text{ and } S_2(t + s, t_0) = S_2(t, s + t_0)S_2(s, t_0)$$

for all $(t, s, t_0) \in \mathbf{R}_+^3$.

Indeed, if $(t, s, t_0) \in \mathbf{R}_+^3$ from the condition (d_0) we obtain:

$$S_1(t + s, t_0) = S(t + s, t_0)P(t_0) = S(t, s + t_0)S(s, t_0)P(t_0) =$$

$$S(t, s + t_0)P(s + t_0)S(s, t_0)P(t_0) = S_1(t, s + t_0)S_1(s, t_0)$$

and $\quad S_2(t + s, t_0) = S(t + s, t_0) - S_1(t + s, t_0) =$

$$S(t + s, t_0) - S(t, s + t_0)S_1(s, t_0) = S_1(t, s + t_0)S_2(s, t_0) =$$

$$S_1(t, s + t_0)S_2(s, t_0) + S_2(t, s + t_0)S_2(s, t_0) = S_2(t, s + t_0)S_2(s, t_0)$$

Definition 2.3. The C_0-quasisemigroup S is said to be *uniformly exponentially dichotomic* (and we write *u.e.d.*) if there are two positive constants $N, \nu > 0$ and a strongly continuous projection-valued function $P : \mathbf{R}_+ \to \widetilde{X}$ such that:

(d_0) $P(t + t_0)S(t, t_0) = S(t, t_0)P(t_0)$;

(ed_1) $e^{\nu t} \|S_1(t + s, t_0)x_0\| \leq N \|S_1(s, t_0)x_0\|$;

(ed_2) $e^{\nu s} \|S_2(t, t_0)x_0\| \leq N \|S_2(t + s, t_0)x_0\|$ for all $(t, s, t_0, x_0) \in \mathbf{R}_+^3 \times X$.

Remark 2.3. It is obvious that if S is *u.e.d.* then it is *u.d.* ;

Remark 2.4. The C_0-quasisemigroup S is *u.e.d.* if and only if it is *u.d.* and the conditions (d_0), (ed_1) and (ed_2) hold for all $(t, t_0, x_0) \in \mathbf{R}_+^2 \times X$ and all $t \geq 1$.

Indeed, if $(t, t_0) \in \mathbf{R}_+^2$ and $t \in [0, 1]$ then:

$$e^{\nu t} \|S_1(t+s, t_0)x_0\| \leq e^{\nu} \|S_1(t+s, t_0)x_0\| \leq$$

$$\leq Ne^{\nu} \|S_1(s, t_0)x_0\| \leq N_1 \|S_1(s, t_0)x_0\|$$

and

$$e^{\nu s} \|S_2(t, t_0)x_0\| \leq Ne^{\nu s} \|S_2(1, t_0)x_0\| \leq N^2 \|S_2(1+s, t_0)x_0\| \leq$$

$$\leq N^2 \|S_2(1-t, t+s+t_0)\| \|S_2(t+s, t_0)x_0\| \leq$$

$$\leq N^2 Me^{\omega} \|S_2(t+s, t_0)x_0\| \leq N_1 \|S_2(t+s, t_0)x_0\|$$

for all $(t, t_0) \in \mathbf{R}_+^2$ and all $x_0 \in X$, where M and ω are given by Definition 2.1. and

$$N_1 = N(e^{\nu} + MNe^{\omega}).$$

Remark 2.5. The C_0-quasisemigroup S is *u.e.d.* if and only if there are a strongly continuous projection-valued function $P : \mathbf{R}_+ \to \mathcal{B}(X)$ and two positive constants $N, \nu > 0$ such that:

(d_0) $P(t+t_0)S(t, t_0) = S(t, t_0)P(t_0)$;

(ed_1') $e^{\nu t} \|S_1(t, t_0)x_0\| \leq N \|x_0\|$;

(ed_2') $e^{\nu s} \|x_0\| \leq N \|S_2(t, t_0)x_0\|$ for all $(t, t_0, x_0) \in \mathbf{R}_+^2 \times X$.

Indeed, from (d_0), (ed_1') and (ed_2') it follows that:

$$e^{\nu t} \|S_1(t+s, t_0)x_0\| = e^{\nu t} \|S_1(t, s+t_0)S_1(s, t_0)x_0\| \leq N \|S_1(s, t_0)x_0\|$$

and

$$e^{\nu s} \|S_2(t, t_0)x_0\| \leq N \|S_2(s, t+t_0)S_2(t, t_0)x_0\| = N \|S_2(t+s, t_0)x_0\|$$

for all $(t, s, t_0, x_0) \in \mathbf{R}_+^3 \times X$.

3. The main result

In this section we shall give characterizations of the uniform exponential dichotomy property.

Theorem 3.1. *Let S be an uniformly dichotomic C_0-quasisemigroup. The following statements are equivalent:*

(i) *S is uniformly exponentially dichotomic;*

(ii) *there are a nondecreasing function $h : \mathbf{R}_+ \to \mathbf{R}_+^*$ with $\lim_{t\to\infty} h(t) = \infty$ and $N > 0$ such that*

(hd_1) $h(t)\,\|S_1(t+s,t_0)x_0\| \le N\,\|S_1(s,t_0)x_0\|$ *and*

(hd_2) $h(s)\,\|S_2(t,t_0)x_0\| \le N\,\|S_2(t+s,t_0)x_0\|$ *for all* $(t,s,t_0,x_0) \in \mathbf{R}_+^3 \times X$.

(iii) *there are $N > 0$ and a nondecreasing function $h : \mathbf{R}_+ \to \mathbf{R}_+^*$ with $\sup_{t\in\mathbf{R}_+} h(t) = \infty$ such that (hd_1) and (hd_2) hold.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (i)$. Let $u > 0$ such that $h(u) > N$. For every $t \in \mathbf{R}_+$ there are $n \in \mathbf{N}$ and $r \in [0,u)$ such that $t = nu + r$.

By (hd_1) it follows that for all $(t,t_0,x_0) \in \mathbf{R}_+^2 \times X$ we have:

$$h(u)^n\,\|S_1(t,t_0)x_0\| \le \omega(u)h(u)^n\,\|S_1(nu,t_0)x_0\| \le$$

$$\le N\omega(u)h(u)^{n-1}\,\|S_1((n-1)u,t_0)x_0\| \le \dots \le N^n\omega(u)\,\|x_0\|,$$

and hence

$$\|S_1(t,t_0)x_0\| \le \omega(u)e^{n\ln\frac{N}{h(u)}}\,\|x_0\| = \omega(u)e^{\frac{r-t}{u}\ln\frac{h(u)}{N}}\,\|x_0\| =$$

$$= \omega(u)e^{1-\nu t}\,\|x_0\| = be^{-\nu t}\,\|x_0\|,$$

for all $(t,t_0,x_0) \in \mathbf{R}_+^2 \times X$ where $\nu = \dfrac{1}{u}\ln\dfrac{h(u)}{N} > 0$ and $b = e\omega(u) > 0$.

On the other hand, by (hd_2) it follows that:

$$N\,\|S_2(t,t_0)x_0\| = N\,\|S(r,t_0+nu)S_2(nu,t_0)x_0\| \ge$$

$$\ge h(u)\,\|S_2(nu,t_0)x_0\| \ge m\,\|S_2(nu,t_0)x_0\| \ge \frac{mh(u)}{N}\,\|S_2((n-1)u,t_0)x_0\| \ge$$

$$\frac{mh^2(u)}{N^2}\,\|S_2((n-2)u,t_0)x_0\| \geq ... \geq \frac{mh^n(u)}{N^n}\,\|x_0\| = m\exp[n\ln\frac{h(u)}{N}]\,\|x_0\| =$$

$$= m\exp[\frac{t-r}{u}\ln\frac{h(u)}{N}]\,\|x_0\| = me^{\nu t}\exp[-\frac{r}{u}\ln\frac{h(u)}{N}]\,\|x_0\| \geq \frac{mN}{h(u)}e^{\nu t}\,\|x_0\|$$

for all $(t,t_0,x_0) \in \mathbf{R}_+^2 \times X$ where

$$m = \inf_{s\in[0,u]} h(s) > 0$$

and

$$\nu = \frac{1}{u}\ln\frac{h(u)}{N} > 0$$

From Remark 2.5. it results that S is uniformly exponential dichotomic. □

A discrete version of Theorem 3.1. is the following:

Corollary 3.1. *Let S be an uniformly dichotomic C_0-quasisemigroup. The following statements are equivalent:*

(i) *S is uniformly exponentially dichotomic;*

(ii) *there is an increasing sequence (a_n) with $a_0 > 0$, $\lim\limits_{n\to\infty} a_n = \infty$ such that:*

(ii)′ $a_n\,\|S_1(n,t_0)x_0\| \leq \|x_0\|$ *and*

(ii)″ $a_n\,\|x_0\| \leq \|S_2(n,t_0)x_0\|$ *for all* $(n,t_0,x_0) \in \mathbf{N}\times\mathbf{R}_+\times X$;

(iii) *there are two increasing sequences (a_n) and (b_n) with the properties:*

(j) $a_0 > 0$ *and* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \infty$;

(jj) $b_0 = 0$ *and* $\overline{\lim}_{n\to\infty}(b_{n+1}-b_n) < \infty$;

(jjj) $a_n\,\|S_1(b_n,t_0)x_0\| \leq \|x_0\|$

and

$$a_n\,\|x_0\| \leq \|S_2(b_n,t_0)x_0\|$$

for all $(n,t_0,x_0) \in \mathbf{N}\times\mathbf{R}_+\times X$;

Proof. The implication $(i) \Rightarrow (ii)$ it follows from Remark 2.5. for $a_n = e^{\nu n}$ and $(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$ The property (j) implies that for every $t \in \mathbf{R}_+$ there is $n \in \mathbf{N}^*$ such that $t \in [b_{n-1}, b_n)$. From (jj) it follows that there exists $b > 0$ such that $b_{n+1} - b_n \leq b$ for all $n \in \mathbf{N}$.

Let $h : \mathbf{R}_+ \to \mathbf{R}_+^*$ be the function defined by $h(t) = a_n$ for $t \in [b_{n-1}, b_n)$. The function is nondecreasing with $\lim\limits_{n\to\infty} h(t) = \infty$ and

$$h(t)\|S_1(t+s,t_0)x_0\| \le a_n \|S(t,s+t_0)S_1(s,t_0)x_0\| \le$$

$$\le a_n \|S(t-b_n, b_n+s+t_0)\| \|S(b_n, s+t_0)S(s,t_0)\, x_0\| \le$$

$$\le a_n\omega(b)\|S_1(s,t_0)x_0\|.$$

and

$$h(s)\|S_2(t,t_0)x_0\| = a_n\|S_2(t,t_0)x_0\| \le \|S_2(b_n, t+t_0)S_2(t,t_0)x_0\| =$$

$$= \|S_2(b_n+t,t_0)x_0\| \le \|S(b_n-s, t+s+t_0)\| \|S_2(t+s,t_0)x_0\| \le$$

$$\le \omega(b)\|S(t+s,t_0)x_0\|$$

for all $(t,s,t_0,x_0) \in \mathbf{R}_+^3 \times X$.

By Theorem 3.1. it follows that S is *u.e.d.* □

Theorem 3.2. *Let S be an uniformly dichotomic C_0-quasisemigroup. Then S is uniformly exponentially dichotomic if and only if there are $a, b > 0$ such that:*

(i) $\int\limits_t^\infty e^{a\tau}\|S_1(\tau,t_0)x_0\|\,d\tau \le be^{at}\|S_1(t,t_0)x_0\|$ *and*

(ii) $\int\limits_0^t \frac{\|S_2(s,t_0)x_0\|}{e^{as}}ds \le b\frac{\|S_2(t,t_0)x_0\|}{e^{at}}$

for all $(t,t_0,x_0) \in \mathbf{R}_+^2 \times X$.

Proof. Necessity. It is a simple verification for $a = \frac{\nu}{2}$ and $b = \frac{2N}{\nu}$ where N and ν are given by Definition 2.3.

Sufficiency. If we denote by $h(t) = e^{at}$ then from (i) we obtain that for all $(t,s,t_0,x_0) \in \mathbf{R}_+^3 \times X$ with $t \ge 1$ we have:

$$\frac{h(t)\|S_1(t+s,t_0)x_0\|}{\omega(1)h(1)} = \int_{t-1}^t \frac{h(t)\|S_1(t+s,t_0)x_0\|}{\omega(1)h(1)}du \le$$

$$\le \int_{t-1}^t \frac{h(t-u)h(u)\|S(t-u,s+u+t_0)x_0\|\|S_1(s+u,t_0)x_0\|}{\omega(1)h(1)}du \le$$

$$\int_0^\infty h(u)\,||S_1(s+u,t_0)x_0||\,du = \int_s^\infty h(\tau-s)\,||S_1(\tau,t_0)x_0||\,d\tau \le b\,||S_1(s,t_0)x_0||\,.$$

Similarly, from (ii) we have

$$\int_0^1 \frac{h(s)\,||S_2(t,t_0)x_0||}{\omega(u)h(1-u)}du = \int_{t-1}^t \frac{h(s)\,||S_2(t,t_0)x_0||}{\omega(u)h(1-u)}du \le$$

$$\le \int_{t-1}^t \frac{h(v-t+1)h(t+s-v-1)\,||S(t-v,v+t_0)||\,||S_2(v,t_0)x_0||}{\omega(t-v)h(v-t+1)}dv \le$$

$$\le \int_{t-1}^t h(t+s+-v-1)\,||S_2(v,t_0)x_0||\,dv \le b\,||S_2(t+s,t_0)x_0||$$

and hence

$$h(s)\,||S_2(t,t_0)x_0|| \le \frac{b}{m}\,||S_2(t+s,t_0)x_0||$$

for all $(t,s,t_0,x_0) \in \mathbf{R}_+^3 \times X$ with $t \ge 1$ where

$$m = \int_0^1 \frac{du}{\omega(u)h(1-u)}$$

From Remark 2.4. it follows that S is *u.e.d.* □

Corollary 3.2. *Let S be an uniformly dichotomic C_0-quasisemigroup. Then S is uniformly exponentially dichotomic if and only if there are $a, b > 0$ such that:*

(j) $\sum\limits_{k=n}^{\infty} e^{ak}\,||S_1(k,t_0)x_0|| \le be^{an}\,||S_1(n,t_0)x_0||$ *and*

(jj) $\sum\limits_{k=0}^{n} \frac{||S_2(k,t_0)x_0||}{e^{ak}} \le b\frac{||S_2(n,t_0)x_0||}{e^{an}}$ *for all* $(n,t_0,x_0) \in \mathbf{N} \times \mathbf{R}_+ \times X$;

Proof. *Necessity.* It is a simple verification for $a = \frac{\nu}{2}$ where $\nu > 0$ is given by Definition 2.3.

Sufficiency. Let $t \in \mathbf{R}_+$ and $n \in \mathbf{N}$ such that $n \le t < n+1$. Them from (j) and Definition 2.2. it results

$$\int_s^\infty e^{a\tau}\,||S_1(\tau,t_0)x_0||\,d\tau \le \int_t^{n+1} e^{a\tau}\,||S_1(\tau,t_0)x_0||\,d\tau +$$

$$+N\sum_{k=n+1}^{\infty}\int_k^{k+1} e^{a\tau}\,||S_1(k,t_0)x_0||\,d\tau \le e^{a(n+1)}N\,||S_1(t,t_0)x_0|| +$$

$$+Ne^a \sum_{k=n+1}^{\infty} e^{ak} \|S_1(k,t_0)x_0\| \leq Ne^a e^{at} \|S_1(t,t_0)x_0\| +$$

$$+bNe^a e^{a(n+1)ak} \|S_1(n+1,t_0)x_0\| \leq N_1 e^{at} \|S_1(t,t_0)x_0\|$$

for all $(t, t_0, x_0) \in \mathbf{R}_+^2 \times X$, where

$$N_1 = Ne^a(1 + Nbe^a).$$

Similarly, from (j) and Definition 2.2. we have:

$$\int_0^t \frac{\|S_2(s,t_0)x_0\|}{e^{as}} ds \leq \int_0^{n+1} \frac{\|S_2(s,t_0)x_0\|}{e^{as}} ds \leq$$

$$\leq \sum_{k=0}^{n} \int_k^{k+1} \frac{\|S_2(s,t_0)x_0\|}{e^{ak}} ds \leq Ne^a \sum_{k=0}^{n} \frac{\|S_2(k+1,t_0)x_0\|}{e^{a(k+1)}} \leq$$

$$\leq Ne^a \sum_{k=0}^{n+1} \frac{\|S_2(k,t_0)x_0\|}{e^{ak}} \leq Ne^a b \frac{\|S_2(n+1,t_0)x_0\|}{e^{a(n+1)}} \leq$$

$$\leq bNe^a \omega(1) \frac{\|S_2(t,t_0)x_0\|}{e^{at}} \leq N_2 \frac{\|S_2(t,t_0)x_0\|}{e^{at}}$$

for all $(t, t_0, x_0) \in \mathbf{R}_+^2 \times X$, where

$$N_2 = N_1 + bNe^a \omega(1) > N_1.$$

From Theorem 3.2. it results that S is *u.e.d.* □

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ON THE BIFURCATION AND VARIATIONAL APPROXIMATION OF THE POSITIVE SOLUTION OF A NONLINEAR REACTION-DIFFUSION PROBLEM

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Abstract. We consider a nonlinear, second-order, two-point boundary value problem that models some reaction-diffusion precesses. When the reaction term has a particular form, $f(u) = u^3$, the problem has a unique positive solution that satisfies an "isoperimetric" condition. We study the bifurcation of this solution with respect to the lenght of the interval, simply by estimating its closed form using a generalized mean value theorem for integrals. It turns out that solution bifurcates from infinity. In order to obtain *directly* accurate numerical approximations to this positive solution, we characterize it by a variational problem involving a conditional extremum. We built up a functional to be minimized introducing the "isoperimetric" condition as a restriction by Lagrange's multiplier method. We then derive a necessary condition of extremum to be satisfied by the weak approximation and carry out some numerical results. They are in good agreement with those previously obtained by indirect methods.

1. Introduction

The two-point boundary value problem

$$\begin{cases} u_{xx} + u^3 = 0, & 0 < x < L \\ u(0) = u(L) = 0 \end{cases} \tag{1}$$

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has a unique positive solution, denoted $\overline{u}$, for $x \in (0, L)$, that satisfies the "isoperimetric" condition

$$\int_0^L \overline{u}(x)dx = \frac{\pi}{\sqrt{2}} \tag{2}$$

These results are outlined in our previous papers [2] and [3]. The first goal of the present paper is an alternative analysis of bifurcation issue for problem (1). The particular form of reaction term in (1) make impossible that analysis in the terms of Stuart-Watson method or two-time technique (Matkowsky) as they are presented in [6]. Consequently, we try to find a functional relationship between $u_{\max}$ (the maximum value of $\overline{u}(x)$ on $(0, L)$) and L. This turn out to be of the form $Lu_{\max} = const.$, which shows that indeed the bifurcation appears at infinity from the null solution. In fact the problem (1) is autonomous, and when one attempts to solve it in a closed form, encounters an integrand of the form $1/\sqrt{1-t^4}$. We estimate subsequent integrals using a generalized mean value theorem suggested in [5].

The second aim of our paper is a direct approximation of the positive solution $\overline{u}(x)$ of (1) which satisfies (2). In this respect we built up a functional defined on $H_0^1(0, L)$ which has a positive lower bound. This functional is then augmented introducing the restriction (2) by means of Lagrange's multiplier method. Eventually, we determine the corresponding Euler's equations for this new functional and use them to obtain the finite elements approximation to $\overline{u}(x)$. Thus, it is underlined the importance of the "isoperimetric" condition (2) in the numerical analysis of reaction-diffusion problem (1).

Our theoretical results are illustrated by a numerical bifurcation diagram in the $L - u_{\max}$ plane and by some numerically computed curves for $\overline{u}(x)$. The later are in good agreement with those previously obtained by indirect methods in [3].

2. Bifurcation from infinity of the positive solution

The nondimensional form of problem (1) reads as follows:

$$\begin{cases} \theta'' + \lambda^2\theta^3 = 0, & 0 < t < 1, \\ \theta(0) = \theta(1) = 0, \end{cases} \tag{3}$$

where $u_m := \max_{x\in[0,L]} u(x), \lambda := Lu_m$ and $t := x/L$.

As the differential equation in (3) is autonomous, usual manipulations and boundary condition in 0 imply:

$$t = \frac{\sqrt{2}}{\lambda}\int_0^\theta \frac{ds}{\sqrt{1-s^4}}, \qquad t \in (0, \frac{1}{2}), \theta \in (0,1). \tag{4}$$

Here we used tacitly the symmetry of the positive solution about the middle of the interval ([2],[3]). To approximate the integral in (4) we use the extension of the mean value theorem for integral suggested in [5]. Thus, there exists θ_t, such that

$$t = \frac{\sqrt{2}}{\lambda}\frac{\theta}{\sqrt{1-\theta_t^4}}, \tag{5}$$

where $0 < \theta_t < \theta, \lim_{\theta\to 0}\frac{\theta_t}{\theta} = \frac{1}{(r+1)^{1/r}}$ for $-1 < r < 0$ for which $\lim_{s\to 0}\frac{1}{s^r\sqrt{1-s^4}} = 0$. Denote $R := \frac{1}{(r+1)^{1/r}}$ and observe that $\theta_t = O(\theta), \theta \to 0$ and $0 < R < 1$. If, motivated by the above asymptotics, we substitute $R\theta$ for θ_t in (5), we obtain the following approximation of the positive (and negative) solution of (3):

$$\theta(t) = \frac{\pm t}{\sqrt{\frac{1}{\lambda^2} + \sqrt{\frac{1}{\lambda^4} + t^4R^4}}}. \tag{6}$$

in a neighborhood of $t = 0$.

Notice that the representation (6) retains all particularities of the exact solution of (3): the symmetry, the smoothness and the asymptotics properties near the boundary points. This entitles us to assimilate the behavior of $\theta'(0)$ obtained from (6) with that corresponding to exact solution. In fact we get

$$Lu_m = \theta'(0)\sqrt{2}. \tag{7}$$

But from [2], as an intermediate result, $\theta'(0)$ does not depend on L. Thus (7) means our bifurcation relationship, see also Figure 1.

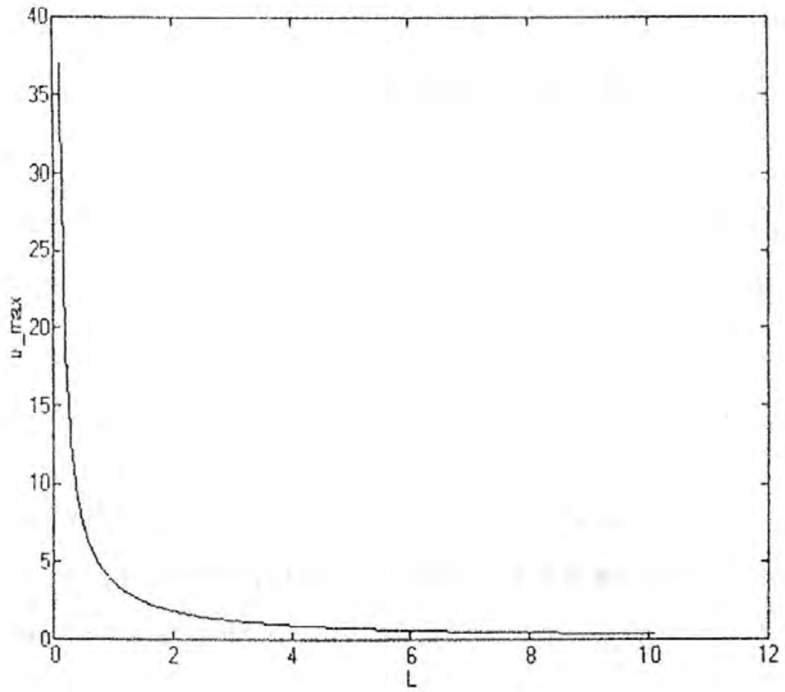

FIGURE 1. Bifurcation diagram in $L - u_m$ plane relevant to the problem (3)

We end this section with the remark that in the lack of a coherent strategy to study bifurcation from infinity we considered an *ad-hoc* method to fill this gape.

3. The variational characterisation of the positive solution

As is apparent from [2], some solvers contributed to a considerable insight into the nature of the solutions of (1). They used exclusively direct manipulation of the equation. With respect to the positive solution, a more penetrating discussion requires to study of an appropriate variational problem whose solution must satisfy (1) and (2).

The obvious choice, namely, the variational problem

$$(V1) \qquad find \quad u \in H_0^1(0,L), \quad \int_0^L (u'v' - u^3 v)\, dx = 0, \forall v \in H_0^1(0,L),$$

of which (1) represents the Euler's equation, proves to be utterly unsuitable for our purposes. The main reason is that, the family of extremals of this problem which pass through a point $(0,0)$ for definiteness, do not form a field, and consequently, the classical sufficient criteria for the existence of extrema, due to Jacobi, become unapplicable (see [1], Ch. 8). We have to notice at this point that we have failed in our attempt to show that the functional

$$J_1(v) := \int_0^L ((v')^2 - v^4)dx, \quad J_1 : H_0^1(0,L) \to \mathbf{R},$$

has a positive minimum.

Instead, using an idea from [4], we introduce the following generalized Rayleigh quotient (functional)

$$J : H_0^1(0,L) \to \mathbf{R}, \quad J(v) := \frac{\left(\int_0^L (v')^2 dx\right)^2}{\int_0^L v^4 dx}, \tag{8}$$

and prove the following result.

LEMMA 1. *The functional $J(v)$ defined in (8) has a positive lower bound on $H_0^1(0,L)$.*

Proof. Recall that for any $y \in H_0^1(0,L)$ the Poincaré's inequality affirms that

$$\pi^2 \int_0^L y^2 dx \le L^2 \int_0^L (y')^2 dx.$$

For $y(x) = v^2(x)$, this implies another useful inequality

$$\pi^2 \int_0^L v^4 dx \le (2L)^2 \int_0^L (vv')^2 dx \tag{9}$$

Cauchy-Schwarz inequality, the left hand side boundary condition, and the inequality (9), enable one to write successively:

$$v^2(x) = \left(\int_0^x v'(t)dt\right)^2 \le x \int_0^x (v')^2 dt < L \int_0^L (v')^2 dt,$$

$$\left(\frac{\pi}{2L}\right)^2 \int_0^L v^4 dx \le L \int_0^L (v')^2 \left(\int_0^L (v')^2 dt\right) dx = L \left(\int_0^L (v')^2 dx\right)^2.$$

This means that

$$J(v) \geq \left(\frac{\pi}{2}\right)^2 \frac{1}{L^3}, \quad \forall v \in H_0^1(0,L). \quad \square$$

Our main result is concentrated in the following theorem.

THEOREM 1. *Given condition (2), a function $u(x)$ that extremizes the functional $J(\cdot)$, defined by (8), satisfies - for an appropriate choice of multiplier μ - Euler's equation corresponding to the functional*

$$J^{**}: H_0^1(0,L) \to \mathbf{R}, \quad J^{**}(v) := J(v) + \mu \int_0^L v dx.$$

Thus, the function $u(x)$ and the multiplier μ can be determined from the system of equations

$$\begin{cases} \int_0^L (u'v' - u^3 v) dx = \frac{\mu}{4} \int_0^L v dx, & \forall v \in H_0^1(0,L), \\ \int_0^L u dx = \frac{\pi}{\sqrt{2}}. \end{cases} \tag{10}$$

Proof. First, we observe that a function u that minimizes $J(\cdot)$ satisfies the necessary condition of extremum

$$\int_0^L \left((u')^2 - u^4 \right) dx = 0. \tag{11}$$

In order to handle the "isoperimetric" condition (2) we have to introduce a new dependent variable $z(x)$ by

$$z(x) := \int_0^x u(s) ds,$$

such that $z(0) = 0, z(L) = \frac{\pi}{\sqrt{2}}$ and $z'(x) = u(x)$. With this, consider the functional

$$J^*: H_0^1(0,L) \times H^1(0,L) \to \mathbf{R}, \quad J^*(v) := J(v) + \int_0^L \mu(x)(v(x) - z'(x)) dx,$$

for a sufficiently regular function $\mu(x)$.

The necessary conditions of extremum for J^* are

$$\left.\frac{dJ^*(u+\varepsilon v,z)}{d\varepsilon}\right|_{\varepsilon=0}=0 \quad and \quad \left.\frac{dJ^*(u,z+\eta y)}{d\eta}\right|_{\eta=0}=0.$$

The first one, in combination with (11), implies

$$\int_0^L (u'v'-u^3v)dx=\frac{1}{4}\int_0^L \mu(x)v(x)dx, \quad \forall v\in H_0^1(0,L), \tag{12}$$

and the second one leads to

$$\int_0^L \mu(x)y'(x)dx=0, \quad \forall y\in H^1(0,L).$$

For sufficiently smooth $y \in H_0^1(0,L)$, such that the fundamental lemma of variational calculus apply, the last integral equality ensures that $\mu'(x)=0$. Consequently, the Lagrange's multiplier μ reduces to a real parameter.

Thus, (12) and (2) imply (10). More than that,

$$\left.\frac{dJ^*(u+\varepsilon v,z)}{d\varepsilon}\right|_{\varepsilon=0}=0$$

means the first equation in the system (10). This completes the proof.□

We have to underline that all problems of extremum we have encountered are meant on the whole Sobolev space $H_0^1(0,L)$ equipped with the usual norm.

4. Numerical results and concluding remarks

The positive solution $u(x)$ of the system (10) is a weak approximation to the positive solution $\overline{u}(x)$ of (1) and (2). To find numerically this approximation we discretize the equations in (10) using classical f.e.m.

The positive solution is approximated by

$$u_h(x)=\sum_{k=1}^{N} c_k\varphi_k(x)$$

where the piecewise linear function $\varphi_k(x)$ satisfies $\varphi_k(x_j)=0$ for $k\neq j$ and $\varphi_k(x_k)=1$, $x_k=kh$ for $k=0,1,...,N+1$ and $h=L/(N+1)$.

For each N, u_h must be a solution of the discrete analogous of (10),

$$\begin{cases} \int_0^L \left(u_h'\varphi_k' - u_h^3\varphi_k\right) dx = \frac{\mu}{4}\int_0^L \varphi_k dx \quad for \quad k = 1, ..., N. \\ \int_0^L u_h dx = \frac{\pi}{\sqrt{2}} \end{cases} \tag{13}$$

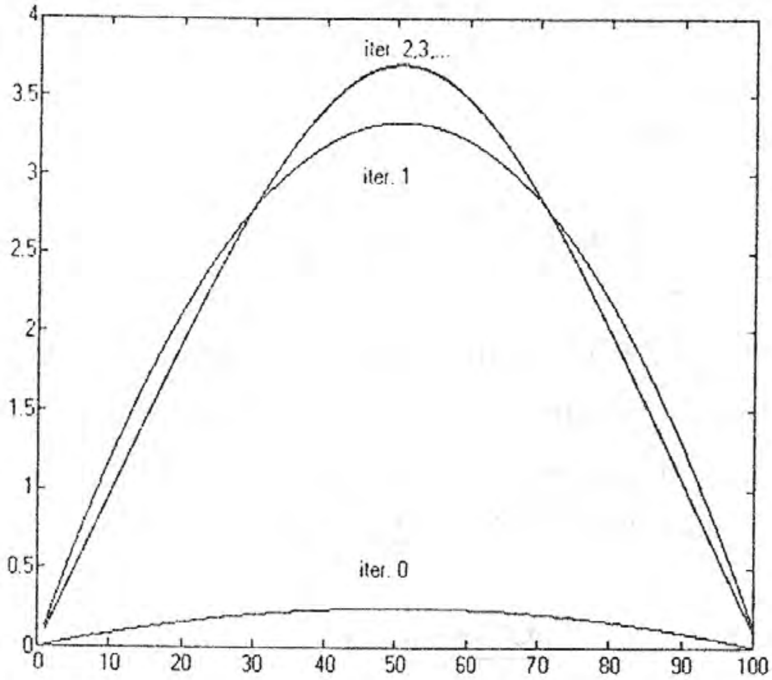

FIGURE 2. Iterations of the Newton method for positive solution

which becomes a nonlinear algebraic system, $F(c) = 0$, for the unknowns $c = (c_1, ..., c_N, \mu)$. Here $F = (F_1, ..., F_N, F_{N+1})$ and

$$F_n(c) = \frac{c_{n-1}}{h} - \frac{2c_n}{h} + \frac{c_{n+1}}{h} + \frac{c_{n-1}^3 h}{20} + \frac{c_{n-1}^2 c_n h}{10} + \frac{3c_{n-1}c_n^2 h}{20} - \tag{14}$$

$$+\frac{2c_n^3 h}{5} + \frac{3c_n^2 c_{n+1} h}{20} + \frac{c_n c_{n+1}^2 h}{10} + \frac{c_{n+1}^3 h}{20} + \frac{\mu}{4}h$$

for $n = 1, ..., N$, where we put $c_0 = c_{N+1} = 0$, in order to impose the boundary conditions and

$$F_{N+1}(c) = h\sum_{n=1}^{N} c_n - \frac{\pi}{\sqrt{2}} \tag{15}$$

We solve this nonlinear system by Newton's method. Starting with an initial guess of the form $c_k^0 = x_k(L - x_k)$, for $k = 1, ..., N$, and $\mu = 10$, Newton's method implies the following sequence of iterations by solving the sequence of linear systems

$$F'(c^p)(c^{p+1} - c^p) = -F(c^p), forp = 0, 1, ...$$

This means that, for every p, we have to solve a linear algebraic system, until for a given ε, $\left\|c^{p+1} - c^p\right\| < \varepsilon$. In spite of that, the method is not so expensive because of the sparsity of the Jacobian $F'(c^p) = \left(\frac{\partial F_i(c^p)}{\partial c_k}\right)_{i,k=1...N+1}$.

We display some iterations $u_h^p(x)$ of the solution $u_h(x)$ in Fig. 2, for $N = 99$, $\varepsilon = 10^{-5}$ and $L = 1$.

We remark that whenever the f.e.m. is applied directly to the problem (1), i.e. the integral condition (2) is ignored, the attracting basin of the positive solution becomes very small. Consequently, Newton's method converges to the positive solution only for initial approximation $u_h^0(x)$ sufficiently closed to this solution. That is why we have incorporated the integral condition (2) in the above algorithm. The numerical experiments confirm the fact that the attracting basin of the positive solution becomes larger. This underlines the importance of the integral condition (2) and furnishes an effective algorithm for the numerical computation of the positive solution to problem (1).

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NONUNIFORM ASYMPTOTIC BEHAVIOURS OF EVOLUTION OPERATORS IN BANACH SPACES

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Abstract. The purpose of this paper is to present a survey of characterizations of exponential stability and exponential dichotomy of evolution operators with nonuniform exponential growth. There are generalized some results obtained by N.van Minh, F.Räbiger, R.Schnaubelt, Y.Latushkin and S.Montgomery-Smith.

1. Introduction

In all what follows we shall consider X a Banach space (real or complex).

Let $D \subset X$ be a dense subspace of X and let $A(t) : D \to X$ be a closed linear operator. We consider the equations:

$$(A) \quad \dot{y} = A(t)y, \qquad t \geq 0$$

and

$$(A, u) \quad \dot{x} = A(t)x + u(t), \quad t \geq 0.$$

We shall denote by

$$C := \{f : \mathbf{R}_+ \to X, f \text{ continuous and bounded}\}.$$

We shall recall some hystorical dates. In 1930 O.Perron proved ([1]) in the case $\dim X < \infty$ that (A) is exponentially stable if and only if for all $u \in C$ the solution x of (A, u) belongs to C.

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Later, in 1954, for $\dim X < \infty$ too, A.D.Maizel proved ([2]) that (A) is exponentially dichotomic if and only if for all $x \in C$ there is $x \in C$ solution of equation (A, u).

In 1966 J.L.Massera and J.J.Schäffer ([5]) have succeeded to extend this last result for the case when X is infinite dimensional, by imposing the additional conditions that $A(t)$ is a bounded operator for every $t \geq 0$ and the map $t \mapsto A(t)$ is locally integrable Bochner on $\mathbf{R}_+$. This result is also presented in the monograph of Y.L.Dalecki and M.G.Krein appeared in 1974.

In what follows we shall present two important results obtained by N.van Minh, F.Räbiger, R.Schnaubelt and we shall give some generalizations for the case of evolution operators with nonuniform exponential growth.

That is why we start with

Definition 1.1. Let $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ be a familly of bounded operators on X. Φ is called *an evolution operator* if the following properties hold:

e_1) $\Phi(t,t) = I$, for all $t \geq 0$;

e_2) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$, for all $t \geq s \geq t_0 \geq 0$;

e_3) for every $t \geq s \geq 0$ and every $x \in X$, the maps $t \mapsto \Phi(t,s)x$ and $s \mapsto \Phi(t,s)x$ are continuous on $[s,\infty)$ and $[0,t]$ respectively .

If Φ is an evolution operator on X then we denote $U(t) = \Phi(t,0)$, for all $t \geq 0$.

Remark 1.1. If $U(t)$ is invertible for all $t \geq 0$ then $\Phi(t,t_0)$ is invertible for all $t \geq t_0 \geq 0$ and

$$\Phi(t,t_0) = U(t)V(t_0), \qquad \text{where } V(t) = U(t)^{-1}.$$

Definition 1.2. The evolution operator Φ is said to be

i) *with exponential growth* if there exist $M, \omega : \mathbf{R}_+ \to (0,\infty)$ such that

$$||\Phi(t,t_0)|| \leq M(t_0)e^{\omega(t_0)(t-t_0)}, \qquad \text{for all } t \geq t_0 \geq 0;$$

ii) *with uniform exponential growth* if there are $M, \omega > 0$ such that

$$||\Phi(t,t_0)|| \leq Me^{\omega(t-t_0)}, \quad \text{for all } t \geq t_0 \geq 0.$$

In what follows we shall denote by

$$C_0 := \{u : \mathbf{R}_+ \to X : u \text{ continuous with } \lim_{t\to\infty} u(t) = 0\}$$

$$C_{00} := \{u \in C_0 : u(0) = 0\}.$$

which are Banach spaces with respect to the norm

$$|||u||| = \sup_{t\geq 0} ||u(t)|||, \text{ for all } u \in \{C_0, C_{00}\}.$$

For $u \in C$ and Φ an evolution operator we consider the equation

$$(E) \quad \varphi_u(t) = \int_0^t \Phi(t,s)u(s)ds, \qquad \text{for all } t \geq 0.$$

Definition 1.3. Let $U, Y \in \{C_{00}, C\}$. The evolution operator Φ is said to be *(U, Y) - stable* (and we denote by $(U, Y) - e.s.$) if for every $u \in U$ we have $\varphi_u \in Y$.

Lemma 1.1. *If the evolution operator Φ is (C_{00}, C) - stable, then there exists $c > 0$ such that*

$$||\varphi_u(t)|| \leq c|||u|||, \quad \textit{for all } (u,t) \in C_{00} \times \mathbf{R}_+.$$

Proof: see [20]. □

For an evolution operator Φ we also consider the equation

$$(E_\Phi) \quad \varphi(t) = \Phi(t,s)\varphi(s) + \int_s^t \Phi(t,\tau)u(\tau)d\tau, \text{ for all } t \geq s \geq 0.$$

Remark 1.2. If $u_1, u_2, \varphi \in C_0$ and $(u_1, \varphi), (u_2, \varphi)$ satisfy the equation (E_Φ) then ([19]) $u_1 = u_2$. Hence it has sense to define the set

$$D(A_\Phi) = \{\varphi \in C_0 : \exists u \in C_0 \text{ such that } (u, \Phi) \text{ verifies } (E_\Phi)\}$$

and the closed linear operator

$$A_\Phi : D(A_\Phi) \subset C_0 \to C_0, \quad A_\Phi\varphi = u.$$

Definition 1.4. Let $U, Y \in \{C_{00}, C_0\}$. The pair (U, Y) is said to be *admissible* for the evolution operator Φ if for every $u \in U$ there exists $\varphi \in Y$ such that (u, φ) satisfies (E_Φ).

Remark 1.3. The pair (C_0, C_0) is admissible for the evolution operator Φ if and only if A_Φ is surjective.

2. Exponential stability evolution operators

Let Φ be an evolution operator on the Banach space X.

Definition 2.1. Φ is said to be

(i) *stable* - and we denote by *s.* - if there exists $N : \mathbf{R}_+ \to (0, \infty)$ such that

$$||\Phi(t, s)|| \leq N(s), \quad \text{for all } t \geq s \geq 0;$$

(ii) *uniformly stable* - and we denote by *u.s.* - if there exists $N > 0$ such that

$$||\Phi(t, s)|| \leq N, \quad \text{for all } t \geq s \geq 0;$$

(iii) *exponentially stable* - and we denote by *e.s.* - if there exist $\nu > 0$ and $N : \mathbf{R}_+ \to (0, \infty)$ such that

$$||\Phi(t, s)|| \leq N(s) e^{-\nu(t-s)}, \quad \text{for all } t \geq s \geq 0.$$

(iv) *uniformly exponentially stable* - and we denote by *u.e.s.* -if there exist $N, \nu > 0$ such that

$$||\Phi(t, s)|| \leq N e^{-\nu(t-s)}, \quad \text{for all } t \geq s \geq 0.$$

Remark 2.1. It is obvious that

$$\begin{array}{ccc} u.e.s. & \Longrightarrow & u.s. \\ \Downarrow & & \Downarrow \\ e.s. & \Longrightarrow & s. \end{array}$$

Example 2.1. If $X = \mathbf{R}$ and $\varphi(t) = t^2 + 1$ then

$$\Phi(t, t_0) = \frac{\varphi(t_0)}{\varphi(t)}, \quad \text{for all } t \geq t_0 \geq 0$$

is an evolution operator which is u.s. and it is not e.s.

An important result is:

Theorem 2.1. *Let Φ be an evolution operator with uniform exponential growth on the Banach space X. The following assertions are equivalent:*

(i) Φ is u.e.s.;

(ii) Φ is (C_{00}, C_{00}) -stable;

(iii) Φ is (C_{00}, C) -stable.

Proof: The proof of the equivalence of (i) and (ii) can be found in [14]. The equivalence between (i) and (iii) has been shown by N.van Minh, F. Räbiger and R.Schnaubelt in [18]. □

The equivalence given by Theorem 2.1. cannot be extended to the nonuniform case.

Example 2.3. If $X = \mathbf{R}$ and $\varphi(t) = e^{t(\frac{3}{2}-sint)}$ then (see [20])

$$\Phi(t,t_0) = \frac{\varphi(t_0)}{\varphi(t)}, \quad \forall t \geq t_0 \geq 0$$

is an evolution operator which is e.s. but it is not (C_{00}, C) -stable.

We shall generalize the result from Theorem 2.1. in certain conditions for the nonuniform case as it follows. First we need the following

Lemma 2.1. *Let $t_0 \in \mathbf{R}_+$ and $t_1 \in \bar{\mathbf{R}}_+$, with $t_0 < t_1$. If $f : [t_0, t_1) \to (0, \infty)$ is a continuous function with the property that there are $M, \omega \in (0, \infty)$ and $c \in (0, 1)$ such that*

$$f(t) \leq Me^{\omega(t-t_0)} \quad \text{and} \int_{t_0}^{t} \frac{f(t)}{f(s)} ds \leq \frac{1}{c}$$

for all $t \in [t_0, t_1)$, then

$$f(t) \leq \frac{M}{c} e^{\omega+c} e^{-c(t-t_0)}, \quad \text{for all } t \in [t_0, t_1).$$

Proof: Let $F : [t_0, t_1) \to (0, \infty)$ be the function defined by:

$$F(t) = \int_{t_0}^{t} \frac{1}{f(s)} ds.$$

We have two possible situations:

1. if $t_1 > t_0 + 1$ and $t \in [t_0 + 1, t_1)$ then

$$F'(t) = \frac{1}{f(t)} \geq cF(t)$$

and hence

$$f(t) \leq \frac{1}{cF(t)} \leq \frac{1}{cF(t_0+1)}e^{-c(t-t_0-1)} \leq \frac{M}{c}e^{\omega+c}e^{-c(t-t_0)}.$$

2. for $t \in [t_0, t_0+1]$ we have:

$$f(t) \leq Me^{\omega} \leq \frac{M}{c}e^{\omega+c}e^{-c(t-t_0)}.$$

□

Now we can give our first main result:

Theorem 2.2. *Let Φ be an evolution operator with exponential growth on the Banach space X. If Φ is (C_{00}, C) - stable then Φ is exponentially stable.*

Proof: From Lemma 1.1. we have that there exists $c \in (0,1)$ such that

$$||\int_0^t \Phi(t,s)u(s)ds|| \leq \frac{1}{c}|||u||| \quad (1)$$

for all $u \in C_0$ and $t \in \mathbf{R}_+$.

Let $t_0 \geq 0$ and $x \in X \setminus \{0\}$. If $t_1 = sup\{t \geq t_0 : \Phi(t,t_0)x \neq 0\}$ let

$$f : [t_0, t_1) \to (0, \infty), \quad f(t) = ||\Phi(t,t_0)x||.$$

For $n \in \mathbf{N}$ sufficiently large we choose a real continuous function $\delta_n : \mathbf{R}_+ \to [0,1]$ such that δ_n has compact support contained in (t_0, t_1) and $\delta_n(t) = 1$ for all $t \in [t_0 + \frac{1}{n}, min\{n, t_1 - \frac{1}{n}\}]$. If we denote by

$$u_n(s) = \begin{cases} \delta_n(t)\frac{\Phi(t,t_0)x}{f(t)} & , \quad \text{for all } t \in [t_0, t_1) \\ 0 & , \quad \text{for all } t \notin [t_0, t_1) \end{cases}$$

we have that $u_n \in C_{00}$ and $|||u_n||| = 1$. For $t \in [t_0, t_1)$:

$$\int_0^t \Phi(t,s)u_n(s)ds = \int_{t_0}^t \Phi(t,s)u_n(s)ds = \int_{t_0}^t \frac{\delta_n(s)}{f(s)}ds\, \Phi(t,t_0)x.$$

By relation (1) we have that

$$\int_{t_0}^t \delta_n(s)\frac{f(t)}{f(s)}ds \leq \frac{1}{c},$$

for all $t \in [t_0, t_1)$. For $n \to \infty$ we obtain

$$\int_{t_0}^{t} \frac{f(t)}{f(s)} ds \leq \frac{1}{c}.$$

Because Φ has exponential growth there exist $M, \omega : \mathbf{R}_+ \to (0, \infty)$ such that

$$||\Phi(t, t_0)x|| \leq M(t_0) e^{\omega(t_0)(t-t_0)} ||x||, \text{ for all } t \geq t_0.$$

From Lemma 2.1. we have that

$$||\Phi(t, t_0)x|| \leq \frac{M(t_0)}{c} e^{\omega(t_0)+c} e^{-c(t-t_0)} ||x||$$

for all $t \geq t_0$. It follows that

$$||\Phi(t, t_0)x|| \leq N(t_0) e^{-c(t-t_0)} ||x||,$$

for all $t \geq t_0 \geq 0$ and $x \in X$, where

$$N : \mathbf{R}_+ \to (0, \infty), \quad N(t_0) = \frac{M(t_0)}{c} e^{\omega(t_0)+c}$$

so Φ is e.s. □

3. Exponential dichotomy of evolution operators

Let Φ be an evolution operator on the Banach space X. In all what follows we shall suppose that

$$X_1 = \{x \in X : \lim_{t \to \infty} U(t)x = 0\}$$

is complementable, i.e. there exists X_2 such that $X = X_1 \oplus X_2$. Let P_1, P_2 be the projections coresponding to this decomposition. We define

$$P_1(t) = I - U(t)P_2 \qquad P_2(t) = U(t)P_2, \quad t \geq 0$$

and

$$\Phi_1(t, t_0) = \Phi(t, t_0)P_1(t_0), \qquad t \geq t_0 \geq 0$$

$$\Phi_2(t, t_0) = \Phi(t, t_0)P_2(t_0), \qquad t \geq t_0 \geq 0$$

Definition 3.1. The evolution operator Φ is said to be *exponentially dichotomic* - and we denote by *e.d.* if there exist $\nu > 0$ and $N : \mathbf{R}_+ \to (0, \infty)$ such that:

(d_1) for all $t \geq 0, P_1(t), P_2(t)$ are projections;

(d_2) $||\Phi_1(t,t_0)x|| \leq N(t_0)e^{-\nu(t-t_0)}||P_1(t_0)x||$, for all $t \geq t_0 \geq 0$ and $x \in X$;

$(d_3)$$N(t)||\Phi_2(t,t_0)x|| \geq e^{\nu(t-t_0)}||P_2(t_0)x||$, for all $t \geq t_0 \geq 0$, and all $x \in X$.

Definition 3.2. The evolution operator Φ is said *uniformly exponentially dichotomic* - and we denote by *u.e.d.* - if Φ is e.d. and the map N from Definition 3.1. is constant.

An important result of characterization of uniform exponential dichotomy is given by N.van Minh, F.Räbiger and R.Schnaubelt in [18].

In order to present this result we remind that if Φ is an evolution operator with uniform exponential growth and with the property that the map

$$(t,s) \mapsto \Phi(t,s)x$$

is continuous for all $t \geq s \geq 0$ and all $x \in X$, and F is one of the spaces C_0 and C_{00}, then one can define

$$(E^t f)(s) = \begin{cases} \Phi(s,s-t)f(s-t) & , \quad 0 \leq t < s \\ \Phi(s,0)f(0) & , \quad 0 \leq s \leq t \end{cases} \qquad \text{for all } t,s \geq 0, f \in F.$$

$\{E^t\}_{t\geq 0}$ is a C_0 -semigroup on F called *the evolution semigroup* on F associated to the evolution operator Φ. We shall denote by G its infinitezimal generator on C_0 and by G_0 its infinitezimal generator on C_{00}. It is obviously that $G_0 = G_{|C_{00}}$.

Now we can give

Theorem 3.1. *Let Φ be an evolution operator on the Banach space X, with uniform exponential growth and with the property that the map $(t,s) \mapsto \Phi(t,s)x$ is continuous for all $x \in X$ and all $t \geq s \geq 0$. Then the following assertions are equivalent:*

(i) Φ is u.e.d.;

(ii) the range of G coincides with C_{00};

(iii) the pair (C_0, C_0) is admissible for Φ.

Proof: can be found in [18]. □

In what follows we shall extend this result for the case of evolution operators with (nonuniform) exponential growth.

Theorem 3.2. *Let Φ be an evolution operator with exponential growth on X. If the pair (C_0, C_0) is admissible for Φ, then Φ is e.d.*

Proof: can be found in [19]. □

As a corollary we obtain a decomposition of the Banach space X as it follows:

Corrolary 3.1. *Let Φ be an evolution operator with exponential growth on the Banach space X. If the pair (C_0, C_0) is admissible for Φ then*

$$X = X_1(t_0) \oplus X_2(t_0), \qquad \text{for all } t_0 \geq 0,$$

where

$$X_1(t_0) = \{x \in X : \lim_{t\to\infty} \Phi(t,t_0)x = 0\}, \text{ for all } t_0 \geq 0$$

and

$$X_2(t_0) = U(t_0)X, \qquad \text{for all } t_0 \geq 0.$$

Proof: see [19]. □

Corollary 3.2. *Let Φ be an evolution operator with exponential growth on the Banach space X. If Φ is e.d. then $\Phi_2(t,t_0) : X_2(t_0) \to X_2(t)$ is invertible, for all $t \geq t_0 \geq 0$.*

Proof: We have that

$$X_2(t) = \Phi_2(t,t_0).X(t_0)$$

and hence $\Phi_2(t,t_0)$ is surjective. Using (d_3) we obtain that $\Phi_2(t,t_0)$ is injective, and then the conclusion. □

Remark 3.1. Comparatively to the uniform case if Φ has (nonuniform) exponential growth and Φ is e.d. one cannot generally obtain that the pair (C_0, C_0) is admissible for Φ. This fact is ilustrated by

Example 3.1. Let $X = \mathbf{R}^2$ with respect to the norm

$$||(x_1, x_2)|| = |x_1| + |x_2|.$$

We define the maps:

$$\delta : \mathbf{R}_+ \to (0,\infty), \qquad \delta(t) = e^{-2t+tsint}$$

$$\gamma : \mathbf{R}_+ \to (0,\infty), \qquad \gamma(t) = e^{t-4tsint}.$$

Thus

$$\Phi(t,t_0)(x_1,x_2)=(\frac{\delta(t)}{\delta(t_0)}x_1,\frac{\gamma(t)}{\gamma(t_0)}x_2),\quad \text{for all } t\geq t_0\geq 0,(x_1,x_2)\in \mathbf{R}^2$$

is easily checked to be an evolution operator on X. Let $P(t)=P$, for all $t\geq 0$, where

$$P:\mathbf{R}^2\to\mathbf{R}^2,\quad P(x_1,x_2)=(x_1,0).$$

Hence

$$X_1(t)=\mathbf{R}\times\{0\},\ X_2(t)=\{0\}\times\mathbf{R},\ \text{for all } t\geq 0.$$

Then we have that:

$$||\Phi_1(t,t_0)x||=\frac{\delta(t)}{\delta(t_0)}|x_1|=e^{-2t+tsint}e^{2t_0-t_0sint_0}||P_1x||$$

$$\leq e^{2t_0}e^{-(t-t_0)}||P_1x||,\quad \text{for all } t\geq t_0\geq 0, x\in X.$$

Moreover:

$$||\Phi_2(t,t_0)x||=\frac{\gamma(t)}{\gamma(t_0)}|x_2|=e^{t-4tsint}e^{4t_0sint_0-t_0}||P_2x||\geq$$

$$\geq e^{-3t}e^{-5t_0}||P_2x||,\quad \text{for all } t\geq t_0\geq 0, x\in X.$$

Thus we have

$$e^{8t}||\Phi_2(t,t_0)x||\geq e^{5(t-t_0)}||P_2x||\geq e^{t-t_0}||x||,\ \text{for all } t\geq t_0\geq 0, x\in X.$$

It follows that Φ is e.d. with $\nu=1$ and $N(t)=e^{8t}$.

Now, if we consider

$$u(t)=(e^{-\frac{t}{2}},0),\qquad t\geq 0$$

then there is no $\varphi\in C_0$ such that (u,φ) satisfies the equation (E_Φ).

Suppose the contrary, i.e. there exists $\varphi=(\varphi_1,\varphi_2)\in C_0$ such that (u,φ) satisfies (E_Φ). Then

$$\varphi_1(t)=\delta(t)\varphi_1(0)+\delta(t)\int_0^t e^{2s-ssins}e^{-\frac{s}{2}}ds,\ \text{for all } t\geq 0.$$

For $t_n=2n\pi+\frac{\pi}{2}:\delta(t_n)=e^{-t_n}$ and hence

$$\int_0^{t_n} e^{2s-ssins-\frac{s}{2}}ds\geq\int_{(2n-1)\pi+\frac{\pi}{6}}^{(2n-1)\pi+\frac{\pi}{2}} e^{2s-ssins-\frac{s}{2}}ds\geq$$

$$\geq \int_{(2n-1)\pi+\frac{\pi}{6}}^{(2n-1)\pi+\frac{\pi}{2}} e^{2s} ds \geq \frac{\pi}{3} e^{(4n-2)\pi}.$$

Thus we would obtain:

$$\varphi_1(t_n) \geq e^{-(2n\pi+\frac{\pi}{2})}(\varphi_1(0) + \frac{\pi}{3}e^{(4n-2)\pi}) \to \infty,$$

for $n \to \infty$, which contradicts the assumption that $\varphi \in C_0$.

So, Φ is e.d. but the pair (C_0, C_0) is not admissible for Φ.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLV**, Number 3, September 2000

ABOUT AN INTEGRAL OPERATOR PRESERVING THE UNIVALENCE

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Abstract. In this paper we prove a new univalence criterion for the analicity and univalence in the unit disc $U = \{z : |z| < 1\}$ of an integral operator.

1. Introduction

Let A be the class of the functions f which are analytic in the unit disc U and $f(0) = f'(0) - 1 = 0$.We denote S the class of the functions $f \in A$ which are univalent in U.

In the theory of univalent functions an interesting problem is to find those integral operators which preserve the univalence of the class S.

Many authors studied the problem of integral operators which preserve the class S. In this sense, important results are due to Y.J. Kim, E.P. Merkes [1], M. Nunokawa[3] and J. Pfaltzgraff[5].

2. Preliminaries

We will need the following theorem in this paper.

Theorem A[4]. Let α be a complex number and $f \in A$. If

$$\left| \frac{1 - |z|^{2\alpha}}{\alpha} \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in U \tag{1}$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha - 1} f'(u) du \right]^{\frac{1}{\alpha}} \tag{2}$$

is in the class S.

3. Main result

Theorem. Let $g \in S$, $\alpha = a + bi$ be a complex number. If

$$a \in \left[\frac{3}{4}, \frac{5}{4}\right], \quad |b| \leq \frac{1}{4}\sqrt{1-16(a-1)^2} \tag{3}$$

then the function

$$G_\alpha(z) = \left\{\alpha \int_0^z [g(u)]^{\alpha-1} du\right\}^{\frac{1}{\alpha}} \tag{4}$$

is in the class S.

Proof. From (4) we have

$$G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u}\right)^{\alpha-1} du\right]^{\frac{1}{\alpha}} \tag{5}$$

Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha-1} du. \tag{6}$$

The function f is regular in U.

From (6) we get

$f'(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1}$, $f''(z) = (\alpha-1)\left(\frac{g(z)}{z}\right)^{\alpha-1} \frac{zg'(z)-g(z)}{z^2}$

and

$$\left|\frac{1-|z|^{2\alpha}}{\alpha}\right| \left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{1-|z|^{2\alpha}}{\alpha}\right| \left|(\alpha-1)\left(\frac{zg'(z)}{g(z)} - 1\right)\right|. \tag{7}$$

For $z = 0$ the inequality (1) is verified. If $z \in U, z \neq 0$, then we have

$\left|\frac{1-|z|^{2\alpha}}{\alpha}\right| = \left|\frac{1-e^{2\alpha ln|z|}}{\alpha}\right| = \left|2ln|z| \int_0^1 e^{2\alpha t ln|z|} dt\right| \leq$

$\leq -2ln|z| \int_0^1 \left|e^{2\alpha t ln|z|}\right| dt = -2ln|z| \int_0^1 e^{2t ln|z| Re\alpha} dt =$

$= \frac{1-|z|^{2Re\alpha}}{Re\alpha}$,

for $Re\alpha > 0$ and hence, for all $z \neq 0$, we get

$\left|\frac{1-|z|^{2\alpha}}{\alpha}\right| \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2Re\alpha}}{Re\alpha} |\alpha - 1| \left|\frac{zg'(z)}{g(z)} - 1\right| \leq$

$$\leq \frac{1-|z|^{2Re\alpha}}{Re\alpha} |\alpha-1| \left(\frac{1+|z|}{1-|z|} + 1\right) = \frac{2|\alpha-1|}{Re\alpha} \frac{1-|z|^{2Re\alpha}}{1-|z|}. \tag{8}$$

We have $Re\alpha = a$ and let us note $|z| = x$, $x \in (0,1)$. Let us consider the function $\phi(x) = \frac{1-x^{2a}}{1-x}$, $x \in (0,1)$. It easy to prove that

$$\phi(x) \leq \begin{cases} 1 & \text{if} \quad a \in (0, \frac{1}{2}) \\ 2a & \text{if} \quad a \in [\frac{1}{2}, \infty) \end{cases} \tag{9}$$

Using (8),(9) and (3) we obtain

$$\left| \frac{1-|z|^{2\alpha}}{\alpha} \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in U. \tag{10}$$

From (6) we have $f'(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1}$ and using (10) by Theorem A it results that the function G_α is in the class S.

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HYDRODYNAMICAL REMARKS ON THE APPROACH OF CLOSE BINARY SYSTEMS

TITUS PETRILA AND RODICA ROMAN

Abstract. As a part of the Roche geometry of a close binary system, the mass transfer from the inner Lagrangian point L_1 is analysed. The hypersonic flow of the matter is taken into consideration. The corresponding stream, which is leaving the Lagrangian conical point L_1, is attracted by the secondary component star and the whole configuration rotates around the mass-center. In a first approximation the stationary character of the corresponding configuration and the Clapeyron's law are accepted. The equations of the flow in the down-stream of the secondary component are established. Then, in order to undertake a numerical approach, some mathematical aspects of the corresponding equations and the validity of the hypothesis for simplification are analysed. Some remarks on the shock wave are underlined too, especially in the turbulent region in close vicinity of the secondary component (behind of the corresponding shock wave). Finally, we mention that this problem even if it was formulated years ago,is still an open problem in the stellar hydrodynamic, and its solution would be very useful for the study of stellar stability in the close binary systems.

1. Introduction

From astronomical point of view, a binary system consists of two bodies, which attract each other and, under their mutual gravitation, they are moving, on the corresponding orbits, around their common mass center. If the separation between the two stars is comparable with their sizes, then we have to deal with a *close binary system*.

Even if the study of the binary systems has started long time ago, the most important results have been obtained in the second half of our century, especially in the frame of *the restricted three - body problem* . Here, from gravitational point of view, the two component stars are considered to be mass - points, that is it is used the well known Roche model (proposed by Roche in 1873).

In addition it is taken into consideration an infinitesimal body (a test particle) which is attracted by each of the two finite bodies, but it does not attract them.

Within the restricted three body problem the motion of the infinitesimal body is resumed, the problem being restricted because the infinitesimal body has not influence on the motions of the two finite bodies. However, even if this is a problem of celestial mechanics, it is of a very great importance for the actual study of the binary systems.

In order to establish the differential equations for the motion of the infinitesimal body, we are using an rotating barycentric coordinate system (0xyz) where the origin O is situated in the common mass - center, and the two component stars are always situated on the x-axis, while (x0y)-plane coincides with the orbital plane. In such conditions the Roche equipotential surfaces - Roche equipotentials - are defined by the formula:

$$G\frac{m_1}{r_1} + G\frac{m_2}{r_2} + \frac{1}{2}\omega^2(x^2 + y^2) = C(= constant) \tag{1}$$

where G is the gravitational constant, $m_1 > m_2$ are respectively the masses of the primary and secondary component, ω is the orbital angular velocity and $r_{1,2}$ are the distances between the infinitesimal body and the two component stars, defined by :

$$r_1^2 = (x + \mu)^2 + y^2 + z^2 \tag{2}$$

$$r_2^2 = (x + \mu - 1)^2 + y^2 + z^2 \tag{3}$$

where

$$\mu = \frac{m_2}{m_1 + m_2} \qquad\qquad 1 - \mu = \frac{m_1}{m_1 + m_2} \tag{4}$$

If C is large enough, the surfaces defined by Eq.(1) consist of two separated closed ovals around each of the two mass points. By diminishing the value of C the ovals become increasing by elongation in the direction of the center of gravity of the system, until, for a certain critical value of C (characteristic of each mass-ratio), the both ovals will join in the Lagrangian point L_1. This last surface is known as *Roche limit* and it is of great importance in the study of stellar evolution for close binary systems.

For much smaller values of C the connecting point L_1 opens up and the corresponding equipotential surfaces would enveloppe both bodies.

According to the currently accepted theory of stellar evolution, the more massive star consumes its hydrogen fuel more quickly and should begin to expand earlier than the less massive one. Nevertheless, the observation shows that it is the lighter component that fills up the Roche limit (Algol Paradox).

When the star's surface is identical with the Roche limit, then, with the exception of the L_1 conical point, the force acting on the stellar matter is oriented inwardly evrywhere on the star's surface. It is sufficient for the star to grow slightly larger than the Roche limit in order that outflow of matter to occur. Therefore, it is shown theoretically the existence of the mass transfer between the two component stars of a binary system.

About the existence of the matter streams there is information for a long time ago, as it is underlined by Joy (1942) and Struve(1958). The corresponding physical existence was already put in evidence by some spectroscopical and photometrical observations, from the presence of the emission lines and from the analyse of the distorsions and assymmetries of the corresponding light curve.

Let us consider now that P denotes one mass-element from the matter stream and let consider that the following forces actting on it: the gravitational force, the centrifugal force, determined by the rotation of the two finite bodies around the common mass-centre, and the Coriolis force. This last one will be the cause of the trajectory curvature of the particle which has escaped from L_1. The centrifugal and Coriolis forces operate in planes which are parallel to the orbital plane.

There are many attempts in order to use the restricted three-body problem for the dynamics study of the matter streams in close binary systems (Kuiper, 1941; Kopal, 1956 and so on) but such methods have the following shortcomings:

- the mean free - path of a constituent particle in stream is by few orders of magnitude less than the size of the corresponding system.
- the group of trajectories of the material particles (discreetly considered) cannot be identified with the stream lines of the stationary flow if two trajectories are crossing.

From the above mentioned considerations it appears the necessity to do a hydrodynamical approach for the streams motion. Nevertheless, for a very complicated study it will be necessary to have in view some supplementary assumptions, in order to symplify the problem. That is why, it is necessary to assume that the corresponding stellar orbits, around the common mass-center, are circular with the angular velocity $\vec{\omega}$. In the above mentioned system of rectangular coordinates the angular velocity $\vec{\omega}$ is oriented upward in the direction of the 0z - axis and its modulus is constant.

Our assumption about a stationary flow is also based on the experimental data provided by the spectroscopic and photometric observations.

2. The constitutive law

Let us consider that the pressure is given by the Clapeyron's law:

$$p = \mathcal{R}\varrho T \tag{5}$$

where $\mathcal{R}$ is the universal constant of the perfect gases. The perfect gas in the stellar jet is assumed as being a monoatomic one (i.e. we may consider $c_v = const$ and $c_p = const$). Now, if the fluid flow is isentropic, then Clapeyron's law becomes an adiabatic constitutive law:

$$p = K\varrho^{\gamma} \tag{6}$$

where

$$K = \frac{p_0}{\varrho_0^{\gamma}} \quad with \quad \gamma = \frac{c_v}{c_p} > 1 \tag{7}$$

If we accept the Fourier's law for the amount of the received heat:

$$\vec{q} = -\mathcal{X}\, gradT \tag{8}$$

where $\mathcal{X}$ is the coefficient of conductivity and $\vec{q}$ is the heat flux through a cross section of unit area. Thus from the first principle of the thermodynamics:

$$\varrho\dot{T} = \dot{p} + div(\mathcal{X}\, gradT) \tag{9}$$

it follows that

$$\varrho\dot{T} = K\gamma\varrho^{\gamma-1}\dot{\varrho} + grad\mathcal{X}\, gradT + \mathcal{X}\triangle T \tag{10}$$

Of course, in order to integrate Eq.(10) it will be necessary that fluid flow has been determined together with the density ϱ.

3. The equation of continuity

The mathematic model for the flow of the hypersonic stellar jet before the shock wave, in stationary state, is based up on the assumption that the flow is potential and plane (in orbital plane, perpendicular on $\vec{\omega}$), so that:

$$rot\vec{v} = 0 \qquad \Longleftrightarrow \qquad \vec{v} = grad\varphi \tag{11}$$

The equation of continuity (conservation of mass) becomes then:

$$div(\varrho\vec{v}) = 0 \quad , \tag{12}$$

thus

$$div\vec{v} = -\frac{gradp}{p\gamma}\vec{v} \tag{13}$$

If we introduce the function $h = h(\varrho)$ defined by the equality:

$$\frac{gradp}{\varrho} = gradh \tag{14}$$

then Eq.(13) leads to:

$$div\vec{v} = -\frac{\varrho}{p\gamma}(\vec{v}\cdot\nabla)h \tag{15}$$

and since $\vec{v} = grad\varphi$, (that is $div\vec{v} = \Delta\varphi$), from (15)we have:

$$\Delta\varphi = -\frac{\varrho}{p\gamma} grad\varphi \cdot gradh \tag{16}$$

Eq.(16) is of an eliptic type and it allows us to determine the value of φ (i.e. the determination of $\vec{v}$ if we have already known the function $\varrho = \varrho(\vec{v})$).

4. The equation of motion

For a viscous and compressible fluid, considered in a force field of density $\vec{f}$, where the viscous coefficients λ and μ are constant and conected by Stokes hypothesis $(3\lambda + 2\mu = 0)$, then the equation of the motion is:

$$\varrho\left[\frac{\partial\vec{v}}{\partial t} + grad\left(\frac{v^2}{2}\right) + rot\vec{v} \times \vec{v}\right] = \varrho\vec{f} - gradp + \mu\Delta\vec{v} + (\mu + \lambda)grad(div\vec{v}) \tag{17}$$

Under the above mentioned conditions, Eq.(17) becomes:

$$grad\left(\frac{v^2}{2}\right) = \vec{f} - gradh - \frac{2\lambda}{\varrho} grad(div\vec{v}) \tag{18}$$

where:

$$\vec{f} = \vec{f}_{stellar\ attraction} + \vec{f}_{Coriolis} + \vec{f}_{centrifugal} \tag{19}$$

The forces of stellar attraction are evidently conservative and so are the centrifugal force:

$$\vec{f}_{centrifugal} = grad\left(\omega^2\frac{r^2}{2}\right), \quad if\ \vec{\omega} = constant \tag{20}$$

Now, in order that $\vec{f}$ can also be considered a conservative force, it is necessary that there is a function ψ which satisfie the ecuation:

$$\vec{f}_{Coriolis} = -2\vec{\omega} \times \vec{v}_r = grad\psi, \tag{21}$$

that is $\vec{f}_{Coriolis} \cdot d\vec{r}$ to be an exact total differential. But for this purpose it would be necessary that at least approximately to have:

$$div\vec{v}_r \cong 0 \tag{22}$$

and since

$$div(\vec{\omega} \times \vec{r}) = 0 \tag{23}$$

i.e.

$$div\vec{v} \cong 0 \tag{24}$$

In what follows, we will assume that $\frac{\varrho}{p}$ or $-\frac{\dot{\varrho}}{\varrho}$ are small enough in order to may write $div\vec{v}_r = 0$ everywhere in the hypersonic jet of stellar matter and consequently there will be a scalar function U such that the total force density:

$$\vec{f} = gradU \tag{25}$$

Taking into consideration that λ is very small (while μ is very large), we can accept that there is also a scalar function $A \neq constant$, so that

$$gradA = -\frac{2\lambda}{\varrho} grad(div\vec{v}) \tag{26}$$

Precisely, from the equation of continuity, written in the form:

$$div\vec{v} = -\frac{\varrho}{p\gamma}\frac{dh}{dt} \tag{27}$$

due to $dh = \frac{dp}{\varrho}$, it follows that:

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\varrho} \tag{28}$$

and hence:

$$gradA = 2\lambda\left(\frac{1}{\varrho^2} grad\,\dot{\varrho} + \frac{\dot{\varrho}}{\varrho} grad\frac{1}{\varrho}\right) \tag{29}$$

Under the above mentioned hypotheses ($div\vec{v}_r \cong 0$) accepted for sake of simplicity, Eq.(29) becomes:

$$A = -2\lambda\frac{d}{dt}\left(\frac{1}{\varrho}\right) + \lambda\frac{d}{dt}\left(\frac{1}{\varrho^2}\right) \tag{30}$$

Consequently from Eq.(18) we obtain the following prime integral of Bernoulli type:

$$\frac{v^2}{2} + \frac{\gamma K}{\gamma - 1}\varrho^{\gamma-1} + 2\lambda\frac{d}{dt}\left(\frac{1}{\varrho}\right) - \lambda\frac{d}{dt}\left(\frac{1}{\varrho^2}\right) - U = constant \tag{31}$$

The value of the constant is the same in the whole jet and it may be determined from the values of the motion parameters, known in a given point (e.g. in L_1). In Eq.(31) we will use the denotation:

$$\frac{\gamma K}{\gamma-1}\varrho^{\gamma-1}-\frac{2\lambda}{\varrho^2}\left(\frac{d\varrho}{dt}-\frac{1}{\varrho}\frac{d\varrho}{dt}\right)=X(v^2,\vec{r}) \tag{32}$$

that is

$$X(v^2,\vec{r})=-\frac{v^2}{2}+U+const. \tag{33}$$

with, obviously $\frac{\partial X}{\partial v^2}<0$. With $\varrho=\varrho(\vec{v},\vec{r})$ and $\frac{d\varrho}{dt}=\vec{v}grad\varrho$ (steadiness), Eq.(32) becomes:

$$div\vec{v}=\frac{\varrho^2X(v^2,\vec{r})-\frac{K\gamma}{\gamma-1}\varrho^{\gamma+1}}{2\lambda(\varrho-1)} \tag{34}$$

and hence:

$$\varrho=\varrho((grad\varphi)^2,\Delta\varphi,x,y). \tag{35}$$

This dependence should be established effectively from Eq.(34), where the values of $\vec{v}$ and $\vec{r}$ may be obtained by the intersection of the curves:

$$y=\varrho^2X(v^2,\vec{r})-2\lambda(\varrho-1)div\vec{v} \tag{36}$$

and

$$y=\frac{K\gamma}{\gamma-1}\varrho^{\gamma+1} \tag{37}$$

From the condition $div\vec{v}\cong 0$ and Eq.(34) we also have:

$$\varrho(v^2,\vec{r})\cong\sqrt[\gamma-1]{\frac{(\gamma-1)X(v^2,\vec{r})}{k\gamma}} \tag{38}$$

As it is easy to see $\frac{\partial\varrho}{\partial v^2}<0$, that ϱ becomes smaller and smaller when the velocity of the hypersonic jet becomes higher and higher.

The equation of continuity (16) together with the relation $div\vec{v}=-\frac{\dot{\varrho}}{\varrho}$ lead to:

$$\Delta\varphi=-\frac{grad\varphi\cdot grad\,\varrho}{\varrho} \tag{39}$$

where we should also consider the Eqs.(33) and (38), so that:

$$\triangle\varphi = -grad\varphi \frac{\vec{f} - grad\frac{v^2}{2}}{(\gamma-1)X(v^2,\vec{r})} \tag{40}$$

5. Boundary conditions

Let $J = 0$ be the equation of the separation jet surface between the fluid and the interstellar medium (the outside of the jet). Since the stress is continuous across a contact surface we may write (under the Stokes'hypothesis):

$$[[(-p + \lambda div\vec{v})\mathbf{I} - 3\lambda\mathbf{D}]] = 0 \tag{41}$$

where $\mathbf{D}$ is the velocity deformation tensor. If $p = p_0$ in the stellar atmosphere where also $\vec{v} \equiv 0$, Eq.(41) becomes:

$$-p\delta_{ij} + \lambda\left(\triangle\varphi\delta_{ij} - 3\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right)\Bigg|_{J=0} = -p_0\delta_{ij} \qquad \forall i,j \in \{1,2\} \tag{42}$$

where $x_1 = x$ and $x_2 = y$ (in the system of coordinates x0y).

At the same time the surface, defined by $J = 0$, being a material surface, we have along it:

$$grad\varphi \cdot grad J = 0 \tag{43}$$

that is the velocity vector $\vec{v}$ is tangent in each point of the separation jet line.

Of course, the integration of the system (40), with the above mentioned boundary conditions is a free boundary value problem. The boundary will be determined together with the corresponding solution. This subject will be approached in the future.

If we determine the hydrodynamical parameters (temperature included) then, from classical formulas for the jump (discontinuity), along a shock wave, it is also possible to determine the values of the corresponding parameters behind the shock wave. Within the conditions of the hypersonic jet, the shock wave will be in the nearest proximity of the primary stellar component, practically being "sticked" together. The very high temperatures which are developped after this shock wave determine the arise of ionisation and particles dissociation and consequently the *laminar* model of the fluids and implicit the known equations of the viscous fluids cannot be used. The mentioned zone of turbulence represents in fact a *complementarity* of the primary star S_1, zone which can be remarked by direct astronomical observations, in accord with the high increasing of the temperature.

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ALGEBRAIC PROPERTIES OF THE OPERATOR A^∞

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Abstract. In this note we study the operator A^∞ from an algebraic point of view.

1. Introduction

Let (X,d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations:

$A^0 := 1_X,\ A^1 := A,\ A^2 := A \circ A, \ldots,\ A^n := A \circ A^{n-1}, \ldots;$

$F_A := \{x \in X | f(x) = x\}$ - the fixed point set of f;

$I(A) := \{Y \subset X | f(Y) \subset Y, Y \neq \emptyset\}$.

Definition 1.1 (see [6] and [7]). An operator $A : X \to X$ is called weakly Picard operator if the sequence $\{A^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.

If an operator $A : X \to X$ is weakly Picard operator, then we can consider the following operator (see [2], [3], [8])

$$A^\infty : X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

The aim of this paper is to study the operator A^∞ from an algebraic point of view.

2. The operator A^∞ on a metric space

Let (X,d) be an metric space and $A : X \to X$ an weakly Picard operator. We have

Theorem 2.1. *The operator A^∞ is a set retraction of X onto F_A.*

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Proof. It is clear that $A^\infty(X) = F_A$. Let $x \in F_A$. Then, $A^n(x) = x$, for all $n \in \mathbf{N}$. This imply that $A^\infty(x) = x$ (see [4]). □

Theorem 2.2. *If the operator A is continuous and $(A^n)_{n\in\mathbf{N}}$ converges uniformly to A^∞ then A^∞ is a topological retraction of X onto F_A.*

Proof. The conditions in this theorem implies that the operator A^∞ is continuous. Now the theorem follows from the Theorem 2.1. □

3. The operator A^∞ on an ordered metric space

Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an weakly Picard operator. We have

Theorem 3.1. *If the operator A is monoton increasing, then the operator A^∞ is monoton increasing.*

Proof. Let $x, y \in X$. Then we have

$$x \leq y \Rightarrow A(x) \leq A(y) \Rightarrow \cdots \Rightarrow A^n(x) \leq A^n(y), n \in \mathbf{N}$$

This imply that $A^\infty(x) \leq A^\infty(y)$. □

From the Theorem 2.1. and the Theorem 3.1. we have

Theorem 3.2. *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an increasing weakly Picard operator. Then the operator A^∞ is an ordered set retraction of X onto F_A.*

In what follow we give an application of the Theorem 3.1. Let $X = C([a, b], \mathbf{R}^n)$, $d(x, y) := \max_{a\leq t\leq b} ||x(t) - y(t)||_{\mathbf{R}^n}$ and $x \leq y$ iff $x(t) \leq y(t)$, for all $t \in [a, b]$.
Let $A : X \to X$ defined by

$$A(x)(t) := x(a) + \int_a^t K(t, s, x(s))ds$$

where $H \in C([a, b] \times [a, b] \times \mathbf{R}^n, \mathbf{R}^n)$ in such that,

$$\exists L_k > 0 : ||K(t, s, u) - K(t, s, v)||_{\mathbf{R}^n} \leq L_k||u - v||_{\mathbf{R}^n},$$

for all $t,\ s \in [a, b],\ u,\ v \in \mathbf{R}^n$.

Let

$$X_\alpha := \{x \in C([a, b], \mathbf{R}^n) | x(a) = \alpha\}, \alpha \in \mathbf{R}^n.$$

Then $X_\alpha \in I(A)$, for all $\alpha \in \mathbf{R}^n$ and

$$C([a,b],\mathbf{R}^n) = \bigcup_{\alpha\in\mathbf{R}^n} X_\alpha$$

Now it is clear that the operator A is weakly Picard operator (see [5],[6],[7],[8]). If

$$u \le v \Rightarrow K(t,s,u) \le K(t,s,v),\ u,v \in \mathbf{R}^n$$

for all $t, s \in [a,b]$, then the operator A is monoton increasing. From the Theorem 3.1. we have

Theorem 3.3. *Let $K \in C([a,b]\times[a,b]\times\mathbf{R}^n,\mathbf{R}^n)$ be such that*

(i) $\exists L_k > 0 : \|K(t,s,u) - K(t,s,v)\|_{\mathbf{R}^n} \le L_k\|u-v\|_{\mathbf{R}^n}$, *for all* $t,\ s \in [a,b],\ u,v \in \mathbf{R}^n$;

(ii) $u \le v \Rightarrow K(t,s,u) \le K(t,s,v),\ u,v \in \mathbf{R}^n$ *for all* $t,s \in [a,b]$.

If y, z are solutions of the following integral equation with deviating argument

$$x(t) = x(a) + \int_a^t K(t,s,x(s))ds, t \in [a,b],$$

then

$$y(a) \le z(a) \text{ implies } y \le z.$$

4. The operator A^∞ on a Banach space

We have

Theorem 4.1. *Let X be a real Banach space and $A : X \to X$ a weakly Picard operator. If A is linear then A^∞ is a linear space retract of X onto F_A.*

Proof. First we prove that the operator A^∞ is a linear operator. Let $x, y \in X$ and $\lambda,\ \mu \in \mathbf{R}$. We have

$$A^n(\lambda x + \mu y) = \lambda A^n(x) + \mu A^n(y)$$

This imply that A^∞ is a linear operator. Now the theorem follows from the Theorem 2.1. □

Theorem 4.2. *Let X be an ordered Banach space and $A : X \to X$ a weakly Picard operator. We suppose that* *(i) A is monoton increasing*

(ii) A is linear.

Then the operator A^∞ is a linear ordered space retract of X onto F_A.

Proof. The proof. follows from the Theorem 4.1. and the Theorem 3.2. □

We end these considerations with some examples of linear weakly Picard operators.

Example 4.1. Let $X = C[a,b]$, $d(x,y) = \max_{a\leq t\leq b} |x(t) - y(t)|$ and

$$A(x)(t) := x(a) + \frac{1}{2(b-a)}(t-a)x(t)$$

We remark that

$$||A^2(x) - A(x)|| \leq \frac{1}{2}||A(x) - x||$$

for all $x \in C[a,b]$.

This imply that (see [7]) the operator A is a linear weakly Picard operator.

Example 4.2. Let $X = C[a,b]$, $H \in C([a,b]\times[a,b])$ and

$$A(x)(t) := x(a) + \int_a^t H(t,s)x(s)ds$$

From the Theorem 3.3. we have that the operator A is a linear weakly Picard operator.

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ON SOME CLASSES OF FUNCTIONS WITH NEGATIVE COEFFICIENTS

G. S. SĂLĂGEAN

Abstract. Let $\mathcal{H}(\mathrm{U})$ denote the set of holomorphic functions in the unit complex disc U and let

$$\mathcal{N}=\left\{f\in\mathcal{H}(\mathrm{U}): f(z)=z-\sum_{n=2}^{\infty}a_nz^n,\ \ a_n\geq 0,\ \ n\in\{2,3,...\}\right\}.$$

In this paper are studied some relations between the classes

$$T(\alpha,\beta)=\{f\in\mathcal{N}:|J(f,\alpha;z)|<\beta, z\in\mathrm{U}\}$$

where $\alpha\in[0,1)$, $\beta\in(0,1]$ and

$$J(f,\alpha;z)=\frac{\dfrac{zf'(z)}{f(z)}-1}{\dfrac{zf'(z)}{f(z)}+1-2\alpha}.$$

1. Introduction

Let $\mathrm{U}(c;r)$ denote the disc in the complex plane with the center at c and the radius r and let U denote the unit disc $\mathrm{U}(0;1)$. Let $\mathcal{H}(\mathrm{U})$ denote the set of holomorphic functions in U and let $\mathcal{N}$ be the class of functions with negative coefficients,

$$\mathcal{N}=\left\{f\in\mathcal{H}(\mathrm{U}): f(z)=z-\sum_{n=2}^{\infty}a_nz^n,\ \ a_n\geq 0,\ \ n\in\{2,3,...\}\right\}.$$

V. P. Gupta and R. K. Jain [1] defined and studied the classes $T(\alpha,\beta)$, of the functions with negative coefficients which are starlike of order α and type

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β, $\alpha \in [0.1)$, $\beta \in (0,1]$, where

$$T(\alpha,\beta) = \{f \in \mathcal{N} : |J(f,\alpha;z)| < \beta, \ \ z \in \mathrm{U}\}$$

and

$$J(f,\alpha;z) = \frac{\dfrac{zf'(z)}{f(z)} - 1}{\dfrac{zf'(z)}{f(z)} + 1 - 2\alpha}. \tag{1.1}$$

It is easy to prove, for $\alpha \in [0,1)$ and $\beta \in (0,1]$, the equivalence

$$|J(f,\alpha;z)| < \beta, \ \ z \in \mathrm{U} \ \ \Leftrightarrow \ \ \frac{zf'(z)}{f(z)} \in \mathrm{U}(c;r), \ \ z \in \mathrm{U} \tag{1.2}$$

where

$$c = c(\alpha,\beta) = \frac{1+\beta^2-2\alpha\beta^2}{1-\beta^2} \tag{1.3}$$

and, for $\beta = 1$ and $\alpha \in [0,1)$ the equivalence

$$|J(f,\alpha;z)| < 1, \ \ z \in \mathrm{U} \ \ \Leftrightarrow \ \ \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha, \ \ \ z \in \mathrm{U}. \tag{1.4}$$

From (1.4) we deduce that $T(\alpha,1)$ is the class of functions with negative coefficients which are starlike of order α; this class was studied by H. Silverman [3].

More generally, if $f \in T(\alpha,\beta)$, $(\alpha \in [0,1), \ \beta \in (0,1])$, then $\mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} > \sigma$, $(z \in \mathrm{U})$, where

$$\sigma(\alpha,\beta) = \frac{1+2\alpha\beta-\beta}{1-\beta}, \tag{1.5}$$

hence the inclusion

$$T(\alpha,\beta) \subseteq T(\sigma,1) \tag{1.6}$$

holds for $\sigma = \sigma(\alpha,\beta)$ given by (1.5).

We need the next characterization of the classes $T(\alpha,\beta)$ proved in [2].

Theorem 1.1. *Let f be a function from $\mathcal{N}$, let $\alpha \in [0,1)$ and $\beta \in (0,1]$; then $f \in T(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} a_n \le 1.$$

2. Main results

By using the definition of the class $T(\alpha,\beta)$ and the equivalence (1.2) we obtain the next inclusions

$$T(\alpha_2,\beta)\subset T(\alpha_1,\beta),\quad \text{when}\quad 0\le\alpha_1<\alpha_2<1$$

and

$$T(\alpha,\beta_1)\subset T(\alpha,\beta_2),\quad \text{when}\quad 0<\beta_1<\beta_2\le 1.$$

In this paper, for $\sigma\in[0,1)$ fixed, we obtain more information concerning the classes $T(\alpha,\beta)$ and $T(\sigma,1)$.

Theorem 2.1. *Let σ be a fixed number from the real interval $[0,1)$, let*

$$\beta\in\left[\frac{1-\sigma}{1+\sigma},1\right]\quad \textit{and let}\ \ \alpha=\alpha(\beta,\sigma)=\frac{(\beta+1)\sigma+\beta-1}{2\beta};$$

then $T(\alpha,\beta)=T(\sigma,1)$.

Proof. Let $f\in T(\sigma,1)$,

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n$$

From Theorem 1.1 we have

$$\sum_{n=2}^{\infty}\frac{n-1+(n+1-2\sigma)}{2(1-\sigma)}a_n\le 1. \tag{2.1}$$

From $\alpha=\alpha(\beta,\sigma)$ we have

$$\sigma=\frac{2\alpha\beta+1-\alpha}{1+\beta}$$

and then

$$\frac{n-1+n+1-2\sigma}{2(1-\sigma)}=\frac{n-\sigma}{n+\sigma}=\frac{n-1+(n+1-2\alpha)\beta}{2\beta(1-\alpha)}$$

and from this we obtain that (2.1) is equivalent to

$$\sum_{n=2}^{\infty}\frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}a_n\le 1 \tag{2.2}$$

which, by Theorem 1.1, implies that $f\in T(\alpha,\beta)$, hence $T(\sigma,1)\subseteq T(\alpha,\beta)$. From this last inclusion and from (1.6) we have $T(\alpha,\beta)=T(\sigma,1)$.

Theorem 2.2. *Let* σ *be a fixed number from* $[0,1)$, *let* $\alpha \in [0,\sigma]$ *and let* $\beta = \beta(\alpha,\sigma)$ *where*

$$\beta(\alpha,\sigma) = \frac{1-\sigma}{1+\sigma-2\alpha};$$

then $T(\alpha,\beta) = T(\sigma,1)$.

Proof. From $\beta = \beta(\alpha,\sigma)$ we have

$$\sigma = \frac{1-\beta+2\alpha\beta}{1+\beta} \quad \text{and} \quad \frac{n-\sigma}{n+\sigma} = \frac{n-1+\beta(n+1-2\alpha)}{2\beta(1-\alpha)}$$

implies that $f \in T(\sigma,1)$ if and only if $f \in T(\alpha,\beta)$.

REMARKS. 1. If $\alpha,\ \sigma \in [0,1)$, $\beta \in (0,1]$ and

$$1-\beta-\sigma+2\alpha\beta-\beta\sigma = 0, \tag{2.3}$$

then $T(\alpha,\beta) = T(\sigma,1)$. But if $f \in T(\alpha,\beta)$, then

$$\frac{zf'(z)}{f(z} \in \mathrm{U}(c;r), \ \text{ for all } z \in \mathrm{U},$$

where $c = c(\alpha,\beta)$ and $r = r(\alpha.\beta)$ are given by (1.3) and if $f \in T(\sigma,1)$, then

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \sigma, \ \text{ for all } z \in \mathrm{U}.$$

Let s and d denote the extremity of the real diameter of the disc $\mathrm{U}(c;r)$; then

$$s = s(\alpha,\beta) = \frac{1+2\alpha\beta-be}{1+\beta} = \sigma \ \text{ and } \ d = d(\alpha,\beta) = \frac{(1-2\alpha)\beta+1}{1-\beta} = \frac{2-(1-\beta)\sigma}{1-\beta}.$$

We note that $\mathrm{U}(c;r) \subset \{w \in \mathbb{C};\ \operatorname{Re} w > \sigma\}$ and $\mathrm{U}(c;r)$ is tangent to the straightline $\{w \in \mathbb{C};\ \operatorname{Re} w = \sigma\}$.

2. For $\sigma \in [0,1)$ fixed, $\alpha \in [0,1)$, $\beta \in (0,1]$ and which satisfy (2.3), the smallest disc $\mathrm{U}(c;r)$ ($c,\ r$ given by (1.3)) is the disc which corresponds to the smallest d. We have

$$[d(\alpha,\beta)]'_\beta = \frac{2(1-\alpha)}{(1-\beta)^2} > 0,$$

hence $d(\alpha,\beta)$ is an increasing function of β. But for fixed σ, we have $\beta \geq \dfrac{1-\sigma}{1+\sigma} = \beta_0$ and then

$$d(\alpha,\beta) \geq \frac{2-(1+\beta_0)\sigma}{1-\beta_0} = \frac{1}{\sigma}.$$

In this case $\alpha = 0$, $c(0;\beta_0) = \dfrac{\sigma^2+1}{2\sigma}$ and $r(0,\beta_0) = \dfrac{1-\sigma^2}{2\sigma}$.

3. For $f \in \mathcal{N}$ and $\sigma \in (0,1)$ we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \sigma,\ \ z \in \mathrm{U} \Leftrightarrow \frac{zf'(z)}{f(z)} \in \mathrm{U}\left(\frac{1+\sigma^2}{2\sigma};\frac{1-\sigma^2}{2\sigma}\right),\ \ z \in \mathrm{U}. \tag{2.4}$$

In the limit case $\sigma \to 0$ the disc $\mathrm{U}\left(\frac{1+\sigma^2}{2\sigma};\frac{1-\sigma^2}{2\sigma}\right)$ become the halfplane $\{w :\ \operatorname{Re} w > 0\}$. But in this case it is known that (see H. Silverman [4])

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\ \ z \in \mathrm{U} \Leftrightarrow \frac{zf'(z)}{f(z)} \in \mathrm{U}(1;1),\ \ z \in \mathrm{U}.$$

We deduce from this that the equivalence (2.4) can be improved.

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(h,ν)-STRUCTURES ON THE DUAL OF A VECTOR BUNDLE

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The differential geometry of the manifold E^*, the dual of the total space E of the vector bundles $\xi = (E, \pi, M)$, represents a generalisation of the geometry of the total spaces T^*M of the contangent bundle (T^*M, π^*, M) or the Hamiltonian space; it is applied in the Hamilton theory of physical fields.

The concept of non-linear connection on E^*, d-tensor fields and d-connections on E^*, the properties of a d-connection curve and torsion, metric types on E^*, and others are studied in the papers [2], [4], [5].

In the present paper, the author analyses the almost complex (h,ν)-structure and almost hermitian (h,ν)-structure on the total space of the vector bundle (E^*, π^*, M^*).

0. Preliminaries

Let $\xi = (E, \pi, M)$ be a real vector bundle, with n-dimensional manifold M as base, a n-dimensional vector space F as type fibre and π is the natural projection. Let us denote with $\xi^* = (E^*, \pi^*, M^*)$ the dual of ξ, where the type fibre is the dual of F. If $U \subset M$ is a domain of local map on M and (x^i, y^a), $I \leq i \leq n$, $I \leq a \leq m$ are the coordinates of the point $e = \pi^{-1}(U)$, $\pi(e) = x$, then these will be transformed by the following laws:

$$\begin{cases} \overline{x}^i = \overline{x}^i(x^1, x^2, \ldots, x^n) \\ \overline{y}_a = M_a^b(\overline{x}) y_b \end{cases} \tag{0.1}$$

where $rank\left(\frac{\partial\overline{x}^i}{\partial x^i}\right) = n$, $rank(M_a^b(x) = m)$ and (x^i, p_a), the coordinates of point $u \in \pi^{-1}(U) \subset E^*$ transforms according to the following laws:

$$\begin{cases} \overline{x}^i = \overline{x}^i(x^1, x^2, \dots, x^n) \\ \overline{p}_a = \tilde{M}_a^n(\overline{x})p_b \end{cases} \tag{0.2}$$

where the matrix $(\tilde{M}_a^b(\overline{x})$ is the inverse of the matrix $(M_a^b(x))$.

Let $\pi^{*T} : TE \to TM$ be the tangent application and VE^* the kernel of this. This application $u \to VE^*$ on E is a differentiable distribution called the vertical distribution of ξ^* and $(\dot{\partial}^a)$, $\left(\dot{\partial}^a = \frac{\partial}{\partial p_a}\right)$, represents a frame of this distribution.

A non-linear connection N^* on E^* represents a differential distribution on E^*, supplementary to the vertical distribution, i.e. $TE_n^* = VE_n^* \oplus N_u^*$, $u \in E^*$, locally defined by $\delta_i = \partial_1 + N_{ai}(x,p)\dot{\partial}^a$, where the functions $N_{ai} : \pi^{-1}(U) \to R$ are called the coefficients of non-linear connection N^*.

If $(\delta, \dot{\partial}^a)$ is an adapted frame to the N^* abd VE^* distributions then $(dx^i, \delta p_a)$, $\delta p_a = dp_a - N_{ai}dx^i$ represents a coframe of this.

For any vector field X on E^*, we denote by X^H and X^V the projection of the horizontal and respectively vertical distribution. We have $X = X^H + X^V$ and for a 1-form $\omega \in \chi^*(E^*)$ we have $\omega = \omega^H + \omega^V$.

Given a non-linear connection N^* on E^* then the linear connection D on E^* is called "distinguished", shortly d-connection, if D preserve by parallelism the horizontal distribution N^* and the vertical distribution VE^* on E^*.

A d-connection D on E^*, uniquely decomposed in the form $D_X = D_X^h + D_X^\nu$, where $D_X^h = D_{X^H}$, $D_X^\nu = D_{X^V}$ are called $h-$ and $\nu-$ covariant derivaties respectively of the d-connection D.

In the local coordinates a d-connection D is characterised by the coefficients: H_{jk}^i, $-\tilde{H}_{bk}^a$, C_j^{ic}, $-\tilde{C}_b^{ac}$, given by

$$\begin{cases} D_{\delta_k}\delta_j = H_{jk}^i(x,p)\delta_i, \quad D_{\delta_k}\dot{\partial}_i = \tilde{H}_{bk}^a(x,p)\dot{\partial}^b \\ D_{\dot{\partial}^c}\delta_j = C_j^{ic}(x,p)\delta_i, \quad D_{\dot{\partial}^c}\dot{\partial}^a = -\tilde{C}_b^{ac}(x,p)\dot{\partial}^b \end{cases} \tag{0.3}$$

1. Almost complex (h,ν)-structures on E^*

Let N^* be a non-linear connection on E^*, $(\delta_i, \dot{\partial}^a)$, $1 \leq i \leq n$, a frame in the module of vector fields $\chi(E^*)$ and $(dx^i, \delta p_a)$, $1 \leq a \leq m$ a frame in the module of covector fields $\chi(E^*)$.

Let's consider the tensor field $F \in \tau_1^1(E^*)$ given by

$$F(x,p) = F_j^i(x,p)\delta_i \otimes dx^a + F_b^a \dot{\partial}^b \otimes \delta_{p_a} \tag{1.1}$$

Definition 1.1. A tensor fields of $(1,1)$ type in the form (1.1) with the property $F^2 = -I$ is called almost complex (h,ν)-structure on the total space of the bundle ξ^*.

Proposition 1.1. *The equation $F^2 = -I$, in the local coordinates is equivalent to*

$$F_k^i F_j^k = -\delta_j^i \quad and \quad F_c^a F_b^c = -\delta_b^a \tag{1.2}$$

From these conditions, it results that $rank(F_j^i) = n = 2n'$ and $rank(F_b^a) = m = 2m'$. The tensor field F given by (1.1) is a d-tensor field on E^*.

Let $D = (H_{jk}^i, -\tilde{H}_{bk}^a, C_j^{ic}, -\tilde{C}_b^{ac})$ be a d-linear connection on E^*.

Definition 1.2. A d-linear connection D is a d-connection compatible with the almost complex (h,ν)-structure F if

$$D_X F = 0 \tag{1.3}$$

If $\overset{\circ}{D}$ is a fixed d-linear connection on E^*, then we have:

Proposition 1.2. *A linear connection $D_X = \overset{\circ}{D}_X + P_X$ is a d-connection on E^* if and only if the tensor field P_X of $(1,1)$ type has the form:*

$$P_X = P_j^i \delta_i \otimes dx_i + P_b^a \dot{\partial}_b \otimes \delta_{p_a} \tag{1.4}$$

The d-linear connection $D_X = \overset{\circ}{D}_X + P_X$ is compatible with the almost complex structuire F if:

$$F \circ P_X - P_X \circ F = \overset{\circ}{D}_X F \tag{1.5}$$

Taking into account that $X = X^H + X^V$, $\forall\, X \in \chi(E^*)$, the equation (1.5) is equivalent to the following equations:

$$\begin{cases} F(\overset{1}{P}_X\,(Y^H)) - \overset{1}{P}_X\,(F(Y^H)) = (\overset{\circ}{D}_X\, F)Y^H \\ F(\overset{2}{P}_X\,(Y^V)) - \overset{2}{P}_X\,(F(Y^V)) = \overset{\circ}{D}_X\, F)Y^V,\ \forall\, X, Y \in \chi(E^*) \end{cases} \tag{1.6}$$

Denote by:

$$\underset{1}{\Omega}{}^{ih}_{rj} = \frac{1}{2}[\delta^i_r\delta^h_j - F^i_r F^h_j],\quad \underset{1}{\Omega}{}^{*ih}_{rj} = \frac{1}{2}[\delta^i_r\delta^h_j + F^i_r F^h_j] \quad \text{and}$$
$$\underset{2}{\Omega}{}^{ac}_{db} = \frac{1}{2}[\delta^a_d\delta^c_b - F^a_d F^c_b],\quad \underset{2}{\Omega}{}^{*ac}_{db} = \frac{1}{2}[\delta^a_d\delta^c_b + F^a_d F^c_b], \quad \text{the Obata operators,} \tag{1.7}$$

the equations (1.6) can be written like

$$\begin{cases} \underset{1}{\Omega}{}^*\ \overset{1}{P}_X = -\frac{1}{2}F\circ \overset{\circ}{D}_X\, F \\ \underset{2}{\Omega}{}^*\ \overset{2}{P}_X = -\frac{1}{2}F\circ \overset{\circ}{D}_X\, F \end{cases} \tag{1.8}$$

As $\underset{1}{\Omega}(F\circ \overset{\circ}{D}_X\, F) = 0$ and $\underset{2}{\Omega}(F\circ \overset{\circ}{D}_X\, F) = 0$, we have

$$\begin{cases} \overset{1}{P}_X = \frac{1}{2}F\circ \overset{\circ}{D}_X\, F + \underset{1}{\Omega}\ \overset{2}{B}\ X \\ \overset{2}{P}_X = \frac{1}{2}F\circ \overset{\circ}{D}_X\, F + \underset{2}{\Omega}\ \overset{2}{B}\ X \end{cases} \tag{1.9}$$

where $\overset{1}{B}_X$ and $\overset{2}{B}_X$ are arbitrary d-tensor fields of the fields type $\overset{1}{P}_X$ and $\overset{2}{P}_X$ respectively.

If $(H^i_{jk}, -\tilde{H}^a_{bk}, C^{ic}_j, -\tilde{C}^{ac}_b)$ is a fixed d-linear connection on E^*, then we have:

Theorem 1.1. *There are compatible d-connections with the almost complex (h,ν)-structure F, one of them is given by:*

Theorem 1.2. *The set of all compatible d-connections with almost complex (h,ν)-structure F on E^* is given by*

$$\overline{D}_X = D_X + \underset{1}{\Omega}\ \overset{1}{B}_X + \underset{2}{\Omega}\ \overset{2}{B}_X, \tag{1.10}$$

where D_X is a compatible d-connection with almost complex structure F, $\overset{1}{B}_X$ and $\overset{2}{B}_X$ are arbitraty d-tensorial fields of $(1,1)$ type, the first is horizontal and the second is vertical.

The local coordinates, the d-connection (1.11) is characterized by the following relations:

$$\begin{cases} \overline{H}^i_{jk} = H^i_{jk} + \Omega^{ij}_{rj} B^r_{kh} \\ \overline{\tilde{H}}^a_{bk} = \tilde{H}^a_{bk} + \Omega^{ac}_{db} B^d_{kc} \\ \overline{C}^{ic}_j = C^{ic}_j + \Omega^{ih}_{rj} B^{rc}_h \\ \overline{\tilde{C}}^{ac}_b = C^{ac}_b + \Omega^{af}_{db} B^{dc}_f \end{cases} \tag{1.11}$$

2. Almost Hermitian (h,ν)-structure on E^*

Let N^* be a nonlinear connection on E^*, $(\delta_i, \dot{\partial}^a)$ and $(dx^i, \delta p_a)$, $1 \leq i \leq n$, $1 \leq a \leq m$, are the frame and coframe respectively, adapted to the horizontal distribution N^* and vertical distribution VE^*.

Let's consider the d-tensor fields $F \in \tau^1_1(E^*)$ and $G \in \tau^0_2(E^*)$ given by

$$F = F^o_j(x,p)\delta_i \otimes dx^j + F^a_b(x,p)\dot{\partial}^b \otimes \delta p_a \tag{2.1}$$

$$G = g_{ij}(x,p)dx^i \otimes dx^j + g^{ab}(x,p)\delta p_a \otimes \delta p_b \tag{2.2}$$

Definition 2.1. An almost hermitian (h,ν)-structure on E^* is defined by the (F,G) pair formed by an almost complex structure F given by (2.1) and a metric structure G given by (2.2) that satisfies the property:

$$G(FX,FY) = G(X,Y),\ \forall\ X,Y \in \chi(E^*) \tag{2.3}$$

In local coordinates the (2.3) condition is equivalent with

$$F^k_i F^s_j g_{ks} = g_{ij} \quad \text{and} \quad F^a_c F^b_d g^{cd} = g^{ab}. \tag{2.4}$$

We attach to the almost hermitian (h,ν)-structure (F,G) on E^* the 2-form ω given by

$$\omega(X,Y) = G(FX,Y),\ \forall\ X,Y \in \chi(E^*) \tag{2.5}$$

In local coordinates the (2.5) condition is written as:

$$\omega_{ij} = g_{is}F^s_j, \quad \omega_{ib} = 0, \quad \omega_{aj} = 0, \quad \omega^{ab} = F^a_c g^{cb}. \tag{2.6}$$

Denote by (g^{ij}) and (g_{ab}) the inverse matrixes of the (g_{ij}) and (g^{ab}) respectively, we have:

$$\begin{cases} g_{ij} = -\omega_{ik}F^k_j, & F^i_j = g^{ik}\omega_{kj} = -\omega^{ik}g_{kj} \\ g^{ij} = F^i_k\omega^{kj}, & \omega^{ij} = g^{ik}F^j_k \end{cases} \tag{2.7}$$

$$\begin{cases} g^{ab} = F^a_c\omega^{cb}, & \omega^{ab} = g^{ak}F^b_c \\ g_{ab} = -\omega_{ac}F^c_b, & F^a_b = g^{ac}\omega_{cb}, \quad F^a_b = -\omega^{ac}g_{cb} \end{cases} \tag{2.8}$$

Definition 2.2. A d-connection on E^* is called almost hermitian d-connection compatible with the almost hermitian (h,ν)-structure (F,G) if

$$D_X F = 0 \quad \text{and} \quad D_X G = 0, \; \forall\, X \in \chi(E^*) \tag{2.9}$$

If there is an almost hermitian d-connection then this is compatible with the induced almost simplectic structures.

Proposition 2.1. A d-connection D on E^* is compatible with the almost hermitian (h,ν)-structures (F,G) if and only if

$$\begin{cases} D^h_X F^H = 0, & D^\nu_X F^V = 0 \\ D^h_X G^H = 0, & D^\nu_X G^V = 0 \end{cases} \quad \forall\, X \in \chi(E^*), \tag{2.10}$$

where $F^H = F^i_j\delta_i \otimes dx^j$ is the h-almost complex part, $F^V = F^a_b\dot{\partial}^b \otimes \delta p_a$ is ν-almost complex part of F, $G^H = g_{ij}dx_i \otimes dx^j$ is h-metric part, $G^H = g^{ab}\delta p_a \otimes \delta p_b$ is ν-metric part of G.

The proof is immediate.

By using the usual method of comapatible connections with pairs of structures we obtained through a direct calculation:

Theorem 2.1. *If D is a fixed linear d-connection on E, characterized by local coordinates* $\left(\overset{\circ}{H}{}^i_{jk}, -\tilde{H}^a_{bk}, \overset{\circ}{C}{}^{jc}_j, -\tilde{C}^{ac}_b\right)$ *and denote $h-$ and $\nu-$ covariant derivatives according to this connection, the the d-connection* $\left(\overset{\circ}{H}{}^i_{jk}, -\tilde{H}^a_{bk}, \overset{\circ}{C}{}^{jc}_j, -\tilde{C}^{ac}_b\right)$ *given by is compatible with the almost hermitian (h,ν)-structure (F,G) on E^*.*

Taking into account the Obata operators of metrics g^{ij} and g_{ab}, Obata operators of almost complex structure F, using d-connection (2.10), we can give the set of all compatible d-connection with the almost hermitian (h,ν)-structure on E^*.

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SOME GENERALIZATIONS OF EDELESTEIN'S THEOREM FOR OPERATORS ON PRODUCT SPACES

MARCEL-ADRIAN ŞERBAN

Abstract. We'll extend the Edelstein's theorem, which generalize the Banach contraction principle, to the operators of the form $f : X^n \to X$.

1. Introduction

There are many generalizations of Banach's contraction principle. I. A. Rus, in [5], present many conditions of contractive type for an operator $f : X \to X$ which ensure that f has a unique fixed point. These conditions can be split in two categories:

1° that conditions given in complete metric spaces which ensure the convergence of the successive approximating sequence to the unique fixed point f, such as:

(i) (Banach (1922))
$d(f(x), f(y)) \leq ad(x, y)$, for all $x, y \in X$ and $a \in [0; 1[$.

(ii) (Ćirić-Reich-Rus 1971)
$d(f(x), f(y)) \leq ad(x, y) + b\left[d(x, f(x)) + d(y, f(y))\right] +$
$+c\left[d(x, f(y)) + d(y, f(x))\right]$
for all $x, y \in X$ and $a, b, c \in \Re_+$, $a + 2b + 2c < 1$.

(iii) (Ćirić (1974))
$d(f(x), f(y)) \leq$
$\leq a \cdot \max\left\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\right\}$
for all $x, y \in X$ and $a \in [0; 1[$.

For other examples see I. A. Rus [5]

2° that conditions given in compact metric spaces which ensure the convergence of the subsequence of successive approximating sequence to the fixed point f, such as:

(i) (Edelstein (1962))

$d(f(x), f(y)) < d(x, y)$, for all $x, y \in X$, $x \neq y$.

(ii) (Reich (1972))

$d(f(x), f(y)) < \frac{1}{2}[d(x, f(x)) + d(y, f(y))]$, for all $x, y \in X$, $x \neq y$.

(iii) (Rhodes (1977))

$d(f^p(x), f^q(y)) <$

$< \max\{d(x, y), d(x, f^p(x)), d(y, f^q(y)), d(x, f^q(y)), d(y, f^p(x))\}$

for all $x, y \in X$, $x \neq y$ and p, q integers.

The first category of conditions has been extended to the operators of type $f : X^n \to X$, a fixed point for such operators means that $x \in X$ such that

$$x = f(x, ..., x).$$

Such results was given by Dhage [1], Rus [4], Tascović [7], Şerban [6].

The main purpose of this paper is to extend the condition from the second category to the operators of type $f : X^n \to X$.

2. Main results

In this paragraph we consider (X, d) a metric space and $f : X^n \to X$ an operator. Related to this operator we can construct the following sequence:

$$\begin{aligned} y_1 &= f(x_1, ..., x_n), \\ y_2 &= f(y_1, ..., y_1), \\ &\cdots\cdots \\ y_m &= f(y_{m-1}, ..., y_{m-1}), \end{aligned} \tag{2.1}$$

for any $x_1, ..., x_n \in X$. We'll show if this sequence has a convergent subsequence then f has a fixed point. First we need the following result.

Theorem 2.1. *(K. Iseki, [3])*

Let $f : X \to X$ be a continuous mapping of a metric space (X, d) such that:

(i) *$d(f^2(x), f(x)) < d(x, f(x))$, for every $x \in X$ with $x \neq f(x)$;*

(ii) *there exists* $x \in X$ *such that the sequence* $(f^n(x))_{n \in N}$ *contains a convergent subsequence* $(f^{n_i}(x))_{i \in N}$.

Then f has a fixed point.

Theorem 2.2. *Let* (X, d) *be a metric space and* $f : X^n \to X$ *an operator such that:*

(i) *f is continuous;*

(ii) $d(f^2(\overline{x}), f(\overline{x})) < \max\limits_{i=\overline{1,n}} \{d(x_i, f(\overline{x}))\}$,

for all $\overline{x} = (x_1, ..., x_n) \in X^n$ *such that* $\exists\ i_0 \in \{1, ..., n\}$ *with* $d(x_{i_0}, f(\overline{x})) \neq 0$, *where*

$$f^2(\overline{x}) \overset{not}{=} f(f(\overline{x}), ..., f(\overline{x}));$$

(iii) $\exists\ \overline{x} \in X^n$ *such that the sequence* $(y_m)_{m \in N}$ *contains a convergent subsequence* $(y_{m_i})_{i \in N}$, $y_{m_i} \to x^* \in X$, *when* $i \to \infty$.

Then $x^* \in F_f$.

Proof. We consider the operator

$$\widetilde{f} : X \to X, \qquad (2.2)$$
$$\widetilde{f}(x) = f(x, ..., x)$$

Condition (ii) implies:

$$d(\widetilde{f}^2(x), \widetilde{f}(x)) < d(x, \widetilde{f}(x)), \text{ for every } x \in X \text{ with } x \neq \widetilde{f}(x)$$

and from condition (iii) we find $y_1 = f(x_1, ..., x_n)$ such that the sequence $\left(\widetilde{f}^m(y_1)\right)_{m \in N}$ contains a convergent subsequence $\left(\widetilde{f}^{m_i}(y_1)\right)_{i \in N}$, $\widetilde{f}^{m_i}(y_1) \to x^*$. Thus all the conditions of Theorem 2.1 are satisfied. □

Theorem 2.3. *Let* (X, d) *be a metric space and* $f : X^n \to X$ *an operator such that:*

(i) *f is continuous;*

(ii) *there exits* $\varphi : \Re_+^n \to \Re_+$ *monotonically increasing such that:*
$\varphi(r, 0, ..., 0) + ... + \varphi(0, ..., 0, r) \leq \varphi(r, ..., r) \leq r$, *for all* $r \in \Re_+$

(iii) $d(f(x_1, ..., x_n), f(x_2, ..., x_{n+1})) < \varphi\left(d(x_1, x_2), ..., d(x_n, x_{n+1})\right)$
for all $x_1, ..., x_{n+1} \in X$ *such that* $\exists\ i_0 \in \{1, ..., n\}$ *with* $d(x_{i_0}, x_{i_0+1}) \neq 0$;

(iv) $\exists\ \overline{x} \in X^n$ *such that the sequence* $(y_m)_{m \in N}$ *contains a convergent subsequence* $(y_{m_i})_{i \in N}$, $y_{m_i} \to x^* \in X$, *when* $i \to \infty$.

Then $F_f = \{x^*\}$.

Proof. We consider again the operator $\widetilde{f}$ defined by relation (2.2). Using condition (ii) and (iii) we deduce

$$\begin{aligned}
&d(\widetilde{f}^2(x), \widetilde{f}(x)) = d\left(f\left(f(x,...,x),...,f(x,...,x)\right), f(x,...,x)\right) \leq \\
&\leq d\left(f\left(f(x,...,x),...,f(x,...,x)\right), f\left(f(x,...,x),...,f(x,...,x),x\right)\right) + \\
&+d\left(f\left(f(x,...,x),...,f(x,...,x),x\right), f\left(f(x,...,x),...,f(x,...,x),x,x\right)\right) + ...+ \\
&+d\left(f\left(f(x,...,x),x,...,x\right)\right), f\left(x,...,x\right)) < \\
&< \varphi\left(0,...,0,d\left(x,f\left(x,...,x\right)\right)\right) + ... + \varphi\left(d\left(x,f\left(x,...,x\right)\right),0,...,0\right) \leq \\
&\leq \varphi\left(d\left(x,f\left(x,...,x\right)\right),...,d\left(x,f\left(x,...,x\right)\right)\right) \leq d\left(x,f\left(x,...,x\right)\right) = d(x,\widetilde{f}(x)),
\end{aligned}$$

which show that condition (i) from Theorem 2.1 is satisfied. Condition (ii) from Theorem 2.1 is easy to obtain from condition (iv), as in the proof of Theorem 2.2.

Suppose that there exist $x^*, y^* \in F_f$, with $x^* \neq y^*$. We have:

$$\begin{aligned}
&d(x^*, y^*) = d\left(f(x^*,...,x^*), f(y^*,...,y^*)\right) \leq \\
&\leq d\left(f(x^*,...,x^*), f(x^*,...,x^*,y^*)\right) + ... + d\left(f(x^*,y^*,...,y^*), f(y^*,...,y^*)\right) < \\
&< \varphi\left(0,...,0,d(x^*,y^*)\right) + ... + \varphi\left(d(x^*,y^*),0,...,0\right) \leq \\
&\leq \varphi\left(d(x^*,y^*),...,d(x^*,y^*)\right) \leq d(x^*,y^*)
\end{aligned}$$

which is a contraction, so this show the uniqueness of fixed point. □

3. Application

In this paragraph we 'll give some examples of conditions for an operator $f : X^n \to X$obtained from Theorem 2.2 and Theorem 2.3.

Theorem 3.1. *Let* (X, d) *be a metric space and* $f : X^n \to X$ *an operator such that:*

(i) f *is continuous;*

(ii) *there exist* $a_i, b_i, c_i, d_i, q_i \in \Re_+$ *such that:*

$$\begin{aligned}
&d(f(\overline{x}), f(\overline{y})) < \sum_{i=1}^{n}[a_i d(x_i, f(\overline{x})) + b_i d(y_i, f(\overline{y})) + \\
&+c_i d(x_i, f(\overline{y})) + d_i d(y_i, f(\overline{x})) + q_i d(x_i, y_i)],
\end{aligned}$$

for all $\overline{x}, \overline{y} \in X^n$ *such that at least one term of sum is nonzero;*

(iii) $\sum_{i=1}^{n}(a_i + b_i + 2d_i + q_i) = 1$

(iv) $\exists\ \overline{x} \in X^n$ *such that the sequence* $(y_m)_{m\in N}$ *given by (2.1) contains a convergent subsequence* $(y_{m_i})_{i\in N}$, $y_{m_i} \to x^* \in X$, *when* $i \to \infty$.

Then $x^* \in F_f$.

Moreover, if

(v) $\sum_{i=1}^{n}(c_i + d_i + q_i) \leq 1$

then $F_f = \{x^*\}$.

Proof. From condition (ii) we obtain:

$$d\left(f^2(\overline{x}), f(\overline{x})\right) < \frac{\sum_{i=1}^{n}(b_i + d_i + q_i)}{1 - \sum_{i=1}^{n}(a_i + d_i)} \cdot \max_{i=\overline{1,n}}\{d(x_i, f(\overline{x}))\}.$$

Conditions (iii) and (iv) show us that all the conditions of Theorem 2.2 are satisfied and thus we obtain the conclusion of the theorem. Uniqueness is easy to obtain from condition (v). □

Theorem 3.2. *Let* (X, d) *be a metric space and* $f : X^n \to X$ *an operator such that:*

(i) f *is continuous;*

(ii) $d(f(\overline{x}), f(\overline{y})) <$
$< \frac{1}{2}\max_{i=\overline{1,n}}\{d(x_i, f(\overline{x})), d(y_i, f(\overline{y})), d(x_i, f(\overline{y})), d(y_i, f(\overline{x})), d(x_i, y_i)\}$,
for all $\overline{x}, \overline{y} \in X^n$ *such that at least one term is nonzero;*

(iii) $\exists\ \overline{x} \in X^n$ *such that the sequence* $(y_m)_{m\in N}$ *given by (2.1) contains a convergent subsequence* $(y_{m_i})_{i\in N}$, $y_{m_i} \to x^* \in X$, *when* $i \to \infty$.

Then $F_f = \{x^*\}$.

Proof. From condition (ii) and using triangle inequality we obtain:

$$d\left(f^2(\overline{x}), f(\overline{x})\right) < \max_{i=\overline{1,n}}\{d(x_i, f(\overline{x}))\}.$$

Uniqueness of fixed point is also implied by condition (ii). Using Theorem 2.2 we deduce the conclusion. □

Theorem 3.3. *Let* (X, d) *be a metric space and* $f : X^n \to X$ *an operator such that:*

(i) *f is continuous;*

(ii) $d(f(\overline{x}), f(\overline{y})) <$
$\max\limits_{i=\overline{1,n}} \left\{d(x_i, f(\overline{x})), d(y_i, f(\overline{y})), d(x_i, f(\overline{y})), \frac{1}{2} \cdot d(y_i, f(\overline{x})), d(x_i, y_i)\right\}$,
for all $\overline{x}, \overline{y} \in X^n$ such that at least one term is nonzero;

(iii) *$\exists\ \overline{x} \in X^n$ such that the sequence $(y_m)_{m\in N}$ given by (2.1) contains a convergent subsequence $(y_{m_i})_{i\in N}$, $y_{m_i} \to x^* \in X$, when $i \to \infty$.*

Then $F_f = \{x^*\}$.

Proof. We make the estimation:

(3.1) $d(f^2(\overline{x}), f(\overline{x})) <$
$< \max\limits_{i=\overline{1,n}}\{d(f^2(\overline{x}), f(\overline{x})), d(x_i, f(\overline{x})), 0, \frac{1}{2}d(x_i, f^2(\overline{x})), d(x_i, f(\overline{x}))\}$.

We denote by

(3.2) $M(\overline{x}, \overline{y}, f) =$
$= \max\limits_{i=\overline{1,n}}\{d(x_i, f(\overline{x})), d(y_i, f(\overline{y})), d(x_i, f(\overline{y}))\frac{1}{2}d(y_i, f(\overline{x})), d(x_i, y_i)\}$

Using this notation we have:

$M((f(\overline{x}), ..., f(\overline{x})), \overline{x}, f) =$
$= \max\limits_{i=\overline{1,n}}\{d(f^2(\overline{x}), f(\overline{x})), d(x_i, f(\overline{x})), \frac{1}{2}d(x_i, f^2(\overline{x})), d(x_i, f(\overline{x}))\}$

We'll show that

(3.3) $$M\left((f(\overline{x}), ..., f(\overline{x})), \overline{x}, f\right) = \max_{i=\overline{1,n}} \left\{d\left(x_i, f(\overline{x})\right)\right\}$$

We have that $M\left((f(\overline{x}), ..., f(\overline{x})), \overline{x}, f\right) > 0$ because of condition (ii). Suppose that:

$$M\left((f(\overline{x}), ..., f(\overline{x})), \overline{x}, f\right) = d\left(f^2(\overline{x}), f(\overline{x})\right)$$

then $M\left((f(\overline{x}), ..., f(\overline{x})), \overline{x}, f\right) = d\left(f^2(\overline{x}), f(\overline{x})\right) < d\left(f^2(\overline{x}), f(\overline{x})\right)$ which is a contradiction.

If we suppose that $M\left((f(\overline{x}), ..., f(\overline{x})), \overline{x}, f\right) = \dfrac{1}{2}\max\limits_{i=\overline{1,n}} \left\{d\left(x_i, f^2(\overline{x})\right)\right\}$ then there exits a $k_0 \in \{1, ..., n\}$ such that

$$\tfrac{1}{2}d(x_{k_0}, f^2(\overline{x})) > d(x_i, \overline{f(x)})\ , \text{ for all } i \in \{1, ..., n\}.$$

The inequality:

$$d(x_{k_0}, f^2(\overline{x})) \leq d(x_{k_0}, f(\overline{x})) + d\left(f^2(\overline{x}), f(\overline{x})\right) < \tfrac{1}{2} \cdot \left[d(x_{k_0}, f^2(\overline{x})) + d(x_{k_0}, f^2(\overline{x}))\right]$$

lead us to a contradiction, thus the equality (3.3) is true. The conclusion of theorem follows from Theorem 2.2. □

Theorem 3.4. *Let (X,d) be a metric space and $f : X^n \to X$ an operator such that:*

(i) *f is continuous;*

(ii) *there exist $\alpha_i, \beta_i, \gamma_i \in \Re_+$ such that:*
$d(f(\overline{x}), f(\overline{y})) < \max\limits_{i=\overline{1,n}}\{\alpha_i d(x_i, y_i) + \beta_i \cdot [d(x_i, f(\overline{x})) + d(y_i, f(\overline{y}))] +$
$+\gamma_i \cdot [d(x_i, f(\overline{y})) + d(y_i, f(\overline{x}))]\}$,
for all $\overline{x}, \overline{y} \in X^n$ such that at least one term of sum is nonzero;

(iii) $\max\limits_{i=\overline{1,n}} \{\alpha_i + \beta_i + \gamma_i\} + \max\limits_{i=\overline{1,n}} \{\beta_i + \gamma_i\} = 1$

(iv) *$\exists\ \overline{x} \in X^n$ such that the sequence $(y_m)_{m\in N}$ given by (2.1) contains a convergent subsequence $(y_{m_i})_{i\in N}$, $y_{m_i} \to x^* \in X$, when $i \to \infty$.*

Then $F_f = \{x^*\}$.

Proof. From condition (ii) we obtain:

$$d\left(f^2(\overline{x}), f(\overline{x})\right) < \frac{\max\limits_{i=\overline{1,n}} \{\alpha_i + \beta_i + \gamma_i\}}{1 - \max\limits_{i=\overline{1,n}} \{\beta_i + \gamma_i\}} \cdot \max_{i=\overline{1,n}} \{d(x_i, f(\overline{x}))\}.$$

Conditions (iii) and (iv) show us that all the conditions of Theorem 2.2 are satisfied and thus we obtain the conclusion of the theorem. Uniqueness is easy to obtain from condition (ii) and (iii). □

Further we give some application of Theorem 2.3.

Theorem 3.5. *Let (X,d) be a metric space and $f : X^n \to X$ an operator such that:*

(i) *f is continuous;*

(ii) *there exist $a_1, ..., a_n \in \Re_+$ such that:*
$d(f(x_1, ..., x_n), f(x_2, ..., x_{n+1})) < \sum\limits_{i=1}^{n} [a_i d(x_i, x_{i+1})]$,
for all $x_1, ..., x_{n+1} \in X$ such that $\exists\ i_0 \in \{1, ..., n\}$ with $d(x_{i_0}, x_{i_0+1}) \neq 0$;

(iii) $\sum\limits_{i=1}^{n} a_i = 1$

(iv) *$\exists\ \overline{x} \in X^n$ such that the sequence $(y_m)_{m\in N}$ given by (2.1) contains a convergent subsequence $(y_{m_i})_{i\in N}$, $y_{m_i} \to x^* \in X$, when $i \to \infty$.*

Then $F_f = \{x^*\}$.

Proof. In this case we consider:

$$\varphi : \Re_+^n \to \Re_+$$
$$\varphi(r_1, ..., r_n) = \sum_{i=1}^{n} a_i r_i$$

$\varphi(r, 0, ..., 0) + ... + \varphi(0, ..., 0, r) = r \sum_{i=1}^{n} a_i = r = \varphi(r, ..., r) \leq r$, for all $r \in \Re_+$, thus condition (ii) of Theorem 2.3 is satisfied, using Theorem 2.3 we get the conclusion. □

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DEFLECTION OF LIGHT RAY IN MANEFF GRAVITATIONAL FIELD

VASILE URECHE

Abstract. The deflection of light ray in Maneff gravitational field is studied. The analysis is made until to the second order in small terms. The term of the first order coincides with the result obtained by Soldner for Newtonian gravitational field, that is the half of the value given by the Theory of General Relativity and by observations. This means that the Maneff gravitational field can not be considered an alternative to the Theory of General Relativity.

1. Introduction

Using some physical considerations, in 1924 G. Maneff proposeded a post-Newtonian nonrelativistic law of gravitation (Maneff 1924, 1925, 1930a, 1930b), assuming that the gravitational interaction between the masses m_1 and m_2 is given by the "force function"

$$U = \frac{Gm_1m_2}{r}\left[1 + \frac{3G(m_1+m_2)}{2c^2r}\right], \tag{1}$$

where r is the distance between m_1 and m_2, G is the Newtonian gravitational constant, and c is the speed of light.

The law of Maneff was forgotten a half of century. It was rediscovered by Hagihara (1970-1972, 1974-1976). Different problems of Celestial Mechanics were recently studied with this law (Diacu 1993, 1996; Diacu et al., 1995; Mioc, Stoica 1995a, 1995b, 1995c; Delgado et al., 1996; Stoica, Mioc, 1996).

In the three previous papers a problem of Astrophysics was considered (Free-fall collapse of a homogeneous sphere, Ureche, 1995) and two general-relativistic tests

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(Perihelion advance, Ureche, 1999 and Gravitational redshift, Ureche, 1998) were studied for the gravitational field of Maneff.

In this paper we analyse the deflection of the light ray in Maneff gravitational field.

2. Deflection of the light ray

We shall consider a massive body of mass M and a photon of mass m_f, in the field of M. The effects of this field on m_f can be descibed by the following patential Φ, which is attached to U from (1) ($m_1 = M$, $m_2 = m_f$):

$$\Phi(r) = -\frac{GM}{r} - \frac{3G^2M^2}{2c^2r^2}. \tag{2}$$

Then, the orbit of the photon in this field is given by the eqution (Ureche, 1999):

$$r = \frac{C^2(1-\alpha)}{GM}\frac{1}{1+e\cos(\sqrt{1-\alpha}\,\theta-\omega)}, \tag{3}$$

where $\alpha < 1$ is given by:

$$\alpha = 3\frac{G^2M^2}{C^2c^2}. \tag{4}$$

The notations (from the cited paper) are usual.

The orbit of the equation (3) is a precessional conic (noncollisional orbits). The constant of areas C will be determined from the initial conditions as follows. Let t_0 be the moment when the photon passes at the perihelion and

$$\begin{aligned} r(t_0) &= r_0 \geq R, \quad \theta(t_0) = \theta_0 = \frac{\omega}{\sqrt{1-\alpha}} \approx \omega\left(1+\frac{1}{2}\alpha\right) \\ v(t_0) &= v_0 = c, \quad (\vec{r}_0, \vec{v}_0) = \frac{\pi}{2}, \end{aligned} \tag{5}$$

where R is the radius of the body.

Then

$$C = |\vec{r}_0 \times \vec{v}_0| = r_0 c, \tag{6}$$

and from the equation of orbit (3) we obtain the value of the eccentricity

$$e = \frac{2r_0}{R_s}\left(1 - \frac{R_S}{2r_0} - \frac{3}{4}\frac{R_S^2}{r_0^2}\right), \tag{7}$$

where R_s is the Schwarzschild gravitational radius, $R_S = 2GM/c^2$.

The type of orbit (of conic) is determined by the values of e One observes that $e = 1$ for

$$r_0 = \frac{3}{2}R_S = R_M, \qquad R_M = \frac{3GM}{c^2}, \tag{8}$$

where R_M is the Maneff gravitational radius (Ureche, 1998).

The eccentricity becomes $e = 0$ for

$$r_0 = \frac{1+\sqrt{13}}{4}R_S, \tag{9}$$

and the orbit becomes circular.

Even for the neutron stars the radius R is higher than the limit (8). From the observational data it results that R is higher than $2 - 3R_S$. From stability considerations, using a realistic equation of state, it results $R > R_S$ (Shapiro, Teukolski, 1983). In the usual astrophysical conditions $r_0 \geq R >> R_S$, so the eccentricity $e >> 1$. This means that the orbit of the photon is a stetched hyperbola. Only near the neutron star or in a black hole the orbit of the photon is closed (or, more exactly bounded).

The deflection of light ray is given by the measure of the angle between the asymptotes of hyperbola (3). Let this be φ. We shall determine the deflection angle directly from the equation (3), avoiding the cartezian representation of the orbit.

For this we shall note ψ the argument of the function cos from the equation of the orbit (3), that is:

$$\psi = \sqrt{1-\alpha}\,\theta - \omega. \tag{10}$$

One observes that the function r from (3) is positively defined. This means that the denominator is not defined for every $\psi \in [0, 2\pi]$, but only for

$$\psi \in (\psi_{min}, \psi_{max}) \tag{11}$$

where

$$\psi_{max} = \arccos\left(-\frac{1}{e}\right) > 0, \quad \psi_{min} = -\psi_{max} \tag{12}$$

The radius vector for $\psi = \psi_{max}$ is parallel with one of the asymptotes of hyperbola, while the radius vector for$\psi = \psi_{min}$ is parallel with the other aymptote.

The first named asymptote forms with the perpendicular to the polar axis the angle

$$\frac{\varphi}{2} = \psi_{max} - \frac{\pi}{2}, \tag{13}$$

and taking into account the symmetry of the hyperbola as well as the fact that φ is small, we have:

$$\varphi = \frac{2}{e} \tag{14}$$

So, up to the second order in small terms we obtain:

$$\varphi = \frac{R_S}{r_0} + \frac{1}{2}\frac{R_S^2}{r_0^2} \tag{15}$$

From (3) and (10) we observe that we have another term of the second order, namely taking into account the effect of the perihelion advance (Ureche, 1999), this is:

$$\varphi_1 = \frac{1}{2}\alpha\left(\psi_{max} - \psi_{min}\right) = \frac{3\pi}{8}\frac{R_S^2}{r_0^2} \tag{16}$$

The total deflection is:

$$\varphi_{tot} = \frac{R_S}{r_0} + \frac{4+3\pi}{8}\frac{R_S^2}{r_0^2}. \tag{17}$$

The maximal deflection is obtained for $r = R$. If only the first order term is retained we obtain:

$$\varphi_{max} \approx \frac{R_S}{R} = \frac{2GM}{c^2 R} \tag{18}$$

The result obtained for Maneff gravitational field coincides with that obtained by Soldner at the beginning of the 19th century for the Newtonian graviational field.

3. Comparison with the observations

With the formula (18) for the Sun we obtain for φ_{max} about 0",87. The observations made in time of solar eclipses by Van Biesbroeck (Ionescu-Pallas, 1980) give the value of 1",75 $\pm$ 0",10. The last value shows an excelent agreement with the results of the Theory of General Relativity, which gives the coefficient 4 instead of the coefficient 2 from (18), that is twice (Tolman, 1969).

In conclusion, the Maneff gravitational field does not verify the test of the deflection of the light ray and cannot be considered an alternative to the Theory of General Relativity.

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BOOK REVIEWS

S i g u r d u r H e l g a s o n, *The Radon Transform,* Second Edition, Progress in Mathematics, Birkhäuser Verlag, Basel-Boston-Berlin 1999, xii+188 pp., ISBN 0-8176-4109-2

Apparently a purely mathematical topic, the Radon transform has found spectacular applications in X-ray technology, nuclear magnetic resonance scanning, and tomography. Beside these technical applications, there are important mathematical applications to partial differential equations (PDE) and group representations.

The study of the Radon transform has its origins in a paper published by J. Radon in 1917 in which he proved that a differentiable function on $\mathbf{R}^2(\mathbf{R}^3)$ can be recovered from its integrals on lines in $\mathbf{R}^2$ (respectively on planes in $\mathbf{R}^3$). A similar result for a symmetric function on the two-sphere $\mathbf{S}^2$ and its great-circle integrals has been obtained by P. Funk in 1916. The term Radon transform was coined by F. John in 1955 in his book *Plane waves and spherical means*, J. Wiley, New York 1955.

For a function f on $\mathbf{R}^n$, integrable on each hyperplane, its Radon transform is defined by the formula $\hat{f}(\xi) = \int_\xi f(x)dm(x)$, for $\xi \in \mathbf{P}^n$, where $\mathbf{P}^n$ denotes the space of hyperplanes topologized in a natural way. The dual Radon transform is defined by $\check{\varphi}(\xi) = \int_{x\in\xi} \varphi(\xi)d\mu(\xi)$, where $\xi \in \mathbf{P}^n$ and φ is a continuous function on $\mathbf{P}^n$. This is the framework in which thwe Radon transform is studied in the first chapter of the book. Special attention is paid to the inversion formula and to applications to PDEs and X-ray transforms.

Starting from the remark that $\mathbf{S}^2$ and the great circles in $\mathbf{R}^2$ as well as $\mathbf{R}^2$ and the lines in $\mathbf{R}^2$ are homogeneous spaces of the orthogonal group $O(3)$ and of the group $\mathbf{M(2)}$ of rigid motions of $\mathbf{R}^2$, respectively, the author of the present monograph started in 1965 the study of the Radon transform and its dual, on general homogeneous spaces. This approach is exposed in the second and the third chapters of the book, with a special emphasis on group-theoretic point of view. Although in this part there are some overlapings with other books of the author (as e.g. *Geometric analysis on symmetric spaces,* American Mathematical Society 1994), the approach given here is more elementary. The third chapter of the book is concerned with orbital integrals and wave operators on isotropic Lorentz spaces.

For the convenience of the reader, a new chapter, Chapter V, a "rapid course" on Fourier transforms and Schwartz theory of distributions, has been added to this new edition.

The first edition of this book, published by Birkhäuser in 1980 and which is out of print for some time, was the first systematic exposition of the subject, accessible to beginners. A Russian translation appeared in 1983. This second edition is a significantly expanded and updated version of the first one, reflecting the progress made in the field since 1980. With respect to the first edition many concrete examples with explicit inversion formulas and range theorems have been added. Some of these examples were worked out by J. Hilgert and H. Schlichtkrull, their contributions being mentioned in specific places in the text. The above mentioned chapter on distributions has been added, too.

Undoubtely that, as the first edition of the book, this second edition will find a large audience, from mathematicians to engineers, physicist and physicians, interested in Radon transform and its manifold applications.

S. Cobzaş

Waltraud Kahle, Elart von Collani, Jürgen Franz, Uwe Jensen (Eds.), *Advances in Stochastic Models for Reliability, Quality and Safety,* (Statistics for Industry and Technology, Series Editor: N. Balakrishnan), Birkhäuser, Boston- Basel-Berlin, 1998, xxvii + 382pp, ISBN: 0–8176–4049–5, 3–7643–4049–5.

This volume contains twenty four original and survey papers presented at the Workshop on *Stochastic Models of Reliability, Quality, and Safety* held in Schierke near Magdeburg, Germany, in 1997.

The editors have considered to be published in this volume a series of papers devoted to the theory of stochastic models, and to its applications to engineering activities. There is a balance between theoretical studies of stochastic processes, the analysis of specific techniques of reliability, quality and control, and the applications to concret problems relevant to engineers.

The contributed papers are divided into four parts.

Part I, *Lifetime Analysis*, covers topics including characterization of self–decomposable random variables via survival distributions, acceptance regions in lifetime estimation, estimation models using censored life time data, estimation methods for repair models, limit theorems in risk theory, least squares and minimum distance estimation methods, and applications of considered methods.

Part II, *Reliability Analysis*, is concerned with some of the methods and models in reliability analysis such as maximum likelihood estimation for the model parameters and the distribution parameters, stochastic models in handling of returns, complex systems including the case of dependent censoring, estimating the parameters of damage processes, deriving exact boundary crossing probabilities of Poisson counting processes with general boundaries, obtaining optimal sequential estimation procedures for the parameters of Markov–additive processes, exact evaluation of the absorption probabilities of Brownian motion in a triangular domain.

Papers in Part III, *Network Analysis*, develop algorithms for determination of network reliability for large networks, computation of the flow probability in flow networks, approximating the probability density function of stochastic mechanical systems under white noise excitation, computing all–terminal reliability of recurrent structures.

Part IV, *Process Control*, is devoted to change–point detection method in quality control, integration of statistical process control and engineering process control, economic approach to optimal control policy of a process model, control charts for time series, process capability indicies.

Petru P. Blaga

H e i n z L ü n e b u r g – *Die euklidische Ebene und ihre Verwandten*, Birkhäuser Verlag, Basel, Boston, Berlin, 1999, 207 + VIII pp., Paperbound, ISBN 3-7643-5685-5.

The book under review is based on the notes prepared by the author for a course about the foundations of the plane geometry, held by him at the Fernuniversität, from Hagen, in 1980. The fact that this material was already presented to a large audience is useful, because the author had the opportunity to improve the methods in which he introduces the notions and demonstrate the classical theorems and, also, he could remove the bugs that may have escaped during the writting.

The first chapter is devoted to the affine and projective planes. The discussion is focused on incidence and central collineation. Related to this last subject, at the end of the chapter we find the theorem of Desargues about the existence of nontrivial central collineation. The main results related to the Desargues planes are discussed in the following chapter. Using the algebraic language, the author describes and obtains the properties and the structure of these planes. In the same way there are characterised the Pappus planes in chapter three.

The connection between conical sections and polarity (polar mappings) is outlined in chapter 4. The author's main goal is to reformulate the classical results of Pascal, Steiner or Segre, concerning conical sections in projective planes. The conical sections are treated as a whole, making no difference between the members of this family, because the circles, ellipses, parabolas and hyperbolas have common properties from the projective point of view.

In the last part of the book we return to the affine planes. In chapter 5 there is investigated the way in which a segment can be divided in a given ratio. The author also pay attention to the properties of the orthogonality relations. The last topic discussed in this chapter is the bisection of an angle. The reader find out in which Pappus planes is possible to construct the bissecting line. In the following chapter there are studied the conical sections in the affine planes. In these planes the members of this family have different properties. Characterising them from this point of view, the author outlines the common properties and the differences between the ellipses, hyperbolas and parabolas, in different kinds of planes. The last chapter of the book is devoted to the geometry of real affine planes. Each chapter is completed by a number of well chosen exercises and problems.

The book is written in a very pedagogical manner and covers an important part of the geometry that is usually taught to undergraduates students in mathematics. It is appropriate to be used as basic textbook for a one semester course in affine and projective geometry.

The book includes an index and a (commented) list of references.

Paul A. Blaga

M i c h a e l M o n a s t y r s k y, *Riemann, Topology and Physics*, Second Edition, Birkhäuser Verlag, Basel-Boston-Berlin 1999, xiii+215 pp., ISBN 0-8176-3789-3

The book is formed by two relatively independent parts: I. Bernhard Riemann, and II. Topological Themes in Contemporary Physiscs, which were published separately in Russian (the first one as an article in the journal Priroda (Nature) in 1976). B. Riemann was one of the few great minds whose work combined the ability to put and solve difficult problems with a deep philosophical penetration into the basic laws of the universe. The last decades have confirmed the modernity and fecundity of his topological ideas, many of them unknown and unpublished during his lifetime, which after lying dormant for almost a century, found now new and fruitful applications in modern physics, as the present book aims to show. This is the reason why Birkhäuser Editors decided, even in the first edition, to publish together these two booklets by M. Monastyrsky.

The first part of the book contains a short and terse biography of B. Riemann and, with respect to the first edition, only minor changes have been done. Some historical and mathematical innaccuracies were corrected and a few facts confirming the modernity and fecundity of Riemann's ideas have been added.

Major changes have been made in the second part, reflecting the progress made since the publication of the first edition (in fact, since the publication of the Russian edition, as the first edition follows closely the original text). In the preface to the second edition, the author expresses his pride that some of his ideas concerning the evolution of modern physics have been confirmed by the last developments in the field – a combination of physics with topology and algebraic geometry. The remarkable results on 4 smooth mannifolds obtained by S. Donaldson, as well as the very recent ones obtained by E. Witten and N. Seiberg, are closely related with the theory of gauge fields. Also the remarkable progress in knot theory is connected with intertwining ideas and methods from field theory, statistical physics, and topology.

The author succeeds to present in a masterly way, but at the same time keeping the popular level of the book, recent discoveries in field theory and condensed matter such as quantum Hall effect, quasicrystals, 't Hooft monopoles, twisted hedgehogs, membranes with notrivial topology.

The monograph is selfcontained and can be understood by people not specialized in modern physics, but interested in the last achievements in this area, which will change our perception of the real world and our philosophical ideas.

S. Cobzaş

J. J. M o r a l e s - R u i z – *Differential Galois Theoryand Non-Integrability of Hamiltonian Systems,* Birkhäuser, Boston - Basel - Berlin (Progress in Mathematics, vol.179, Ferran Sunyer i Balaguer award winning monograph), 1999, 167 + XIV pp., Hardback, ISBN 0-8176-6078-X, ISBN 3-7643-6078-X

As it is well known from algebra , if it is given an algebraic equation with coefficients in a field K, there is a prescribed way to construct an extension L of K (dependent on the equation) and to construct the so-called *Galois group,* which is the group of all automorphisms of L which leave fixed aall the elements of K. Now, the central result of the Galois theory is that the equation can be solved in radicals over K iff its galois group is solvable.

If we consider, instead of an arbitrary field a differential field K (i.e. a field endowed with an additive endomorphism verifying a Leibniz rule), than we can consider, also, linear differential equations over K and construct, in an analogous manner, a (differential) Galois group of the differential equation. The differential Galois theory relates the complete integrability of a linear differential equation to the solvability of its Galois group.

The monograph under review exposes some new results of the author and its collaborators regarding the applications of differential Galois theory to prove the non-integrability of some Hamiltonian systems.

The idea of the approach followed in the book is to replace the Hamiltonian system by a variational equation. It is shown that, in case the Hamiltonian system is completely integrable, then the identity component of differential Galois group of the variational equation should be abelian. Thus, proving the non-integrability is the same with proving the non-commutativity of the above mentioned group.

The book contains an introduction an introduction and seven other chapters. The chapters two and three are very important, because they expose the foundational material on differential Galois theory and Hamiltonian systems. In general, only results are mentioned, without proofs, unless the results are new and, maybe, belong to to the author. Anyway, there is given enough literarture information.

The chapter four is the real core of the book. Here are proven the main general results regarding the non-integrability of different classes of Hamiltonian differential equations.

The rest of the book is devoted to application to aplications to concrete ("non-academic", to use the author's terminology) systems: homogeneous potentials, Bianchi cosmological models, the three-body problem,, Lamé equations, Hénon-Heiles systems, Toda systems aare treated in chapters five and six. Chapter seven discusses a connection between the approach followed in this book and Lerman's real dynamical criterion of non-integrability. Finaly, chapter eight provides some complementary applications as well as some conjectures.

The book is a highly specialized monograph, addressing a narrow domain of research. It can be of a great value for experts in the field and graduate students looking for a way in mathematics. Definitely, there is a lot of activity in the field and the book fills a gap, covering (in an expert manner) a material that can be hardly found in the monograph literature.

Paul A. Blaga

D. L a u g w i t z – *Bernhard Riemann (1826 – 1866) – Turning Points in the Conception of Mathematics,* Birkhäuser, Boston - Basel - Berlin, 1999, 356 + XVIII pp., Hardback, ISBN 0-8176-4040-1, ISBN 0-7643-4040-1

The name and Bernhard Riemann as well as his contribution to mathematics are too well known to anyone with a slightest interest in mathematics, from college student to researcher to require any presentation. However, suprising as it may seem, his life is not that well known, since there are but few books devoted to it and they are not always at hand.

The goal of this book is twofold: first, to give an exposition of the biography of Riemann and, secondly, to examine the impact of the work of Riemann on the later development of mathematics.

The long introduction of the book is ment to achieve the first goal. I should mention, though, that the only part really important for the author is the scientific biography, that is to say, he is trying to explain *how* Riemann became such a brilliant scientist and how his life influenced his formation.

The following three chapters investigate the contributions of Riemann to three different branches of mathematics: complex analysis, real analysis an geometry with its implications to physics and philosophy, while the last chapter is the one to address the problem of the impact of Riemann's work.

The book is very well documented and all the affirmations are accompanied by solid argumentations. I should mention, in this respect, the great number of quotations from Riemann's works, both in English and German. A special attention is paid to the sources of Riemann's results. Riemann doesn't use to acknowldege *all* the sources so it is, in many cases, difficult if not impossible, to decide whether he was aquianted or not with some of the works of his predecessors. Thus, for instance, it is not known for sure if Riemann had read the book of Lobatchevski on Non Euclidean geometry geometry. (Although the book could be found in the Götingen library, apparently none read it in the period when Riemann was working on his thesis).

Another point that should be mentioned is that, while discussing the impact of Riemann's scientific activity, the author makes very interesting interesting observation on the history of different fields of mathematics both before and after Riemann, such that, in the process, he provides an entire panorama of the developments of mathematics during a period of more than one century.

The book is addressed to the entire mathematical community, from students to researcher and it shhould be read by anyone interested in the history of mathematics. It is full of very interesting information, exposed in a very attractive way. I should mention here the great number of photographs spreaded through de book, some of them very rare images of some of the most brilliant mathematicians of the nineneenth century.

The book is in excellent graphical conditions, includes and index and a bibliography.

V. I. V o r o t n i k o v — *Partial Stability and Control*, Birkhäuser, 1998, 430 pp., Hardbound, ISBN 0-8176-3917-9

At the end of the last century and the beginning of our century Lyapunov invented a very effective method for the study of the stability of the solution of ordinary differential equations. At the same time, he considered the case, a lot more difficult, of the stability with respect to only a part of the variables, which is now known as partial stability. This kind of stability proved to be very interesting in many practical applications. However, the Lyapunov functions method for partial stability is difficult to be applied, because is not at all an easy task to construct Lyapunov functions.

In this monograph, the author suggests a new approach to the problem of partial stability and stabilization (or control). The key idea is to transform the original differential equations and construct an auxiliary system. This new system is investigated for stability with respect to all the variables and after that one should be able to obtain information on the partial stability of the original system.

The contents of the book are divided into three parts. The first part (the first four chapters, except for a couple of subsections) is devoted to the development of the new methods and to its application to both theoretical and practical nature. An interesting (and important!) theoretical result is the reduction of the solutions of some nonlinear problem of partial stabilization to the solution of the problem of stabilization for linear systems, specially constructed. Among the practical applications, I was especially pleased by the elegant solution of some problems concerning the stabilization of an artificial satellite in circular and geostationary orbits.

The second part is dealing with linear and nonlinear problems with respect to a part of the variables (the remaining parts of the first four chapters) and with nonlinear game-theoretic problems of control with respect to a part of the variables in the presence of uncontrolable inferences (chapter 5).

Finally, the third part (the chapters 6 and 7), extend the method described in the book from ordinary differential equations to more general equations, such as functional-differential equations and stochastic equations.

The subject of this monograph is very actual and very important for both theoretical and practical points of view. The monograph literature in the field is still relatively poor, that is the reason why I think this is, really, a valuable addition. The book is very well written, well documented, and the author is an expert in the field (actually, many of the results are of his own). I will be of a great help for both experts and advanced graduate students.

The book is very accurate printed and is accompanied by a large and well-organized list of references, as well as by an index.

Paul A. Blaga

K. J a r o s z (Ed.) *Function Spaces,* Proceedings of the Third Conference on Function Spaces, May 19-23, 1998, Southern Illinois University at Edwardsville, Contemporary Mathematics vol. 232, American Mathematical Society 1999

These are the Proceedings of the Third Conference on Function Spaces organized by Professor Krzysztof Jarosz at Southern Illinois University at Edwardsville (SIUE), from May 19 to May 23, 1998. The Proceedings of the previous two conferences, organized also by Professor Jarosz at the same University in Spring 1990 and in Spring 1994, were published by Marcel Dekker as volumes 136 and 172 of the series Lecture Notes in Pure and Applied Mathematics in 1992 and 1995, respectively.

The Third Conference was attended by over 100 participants (the list of participants included in the volume counts 114 mathematicians) from over than 25 countries. The aim of the Conference was to bring together mathematicians interested in various aspects of function spaces or in related areas, as spaces and algebras of analytic functions (of one or several variables) and operators acting on them, L^p-spaces, spaces of continuous functions, spaces of Banach-valued functions, Banach and C^*-algebras, geometry of Banach spaces. Some lectures were of expository nature, presenting in an accessible way to non-experts in the field known (to the experts) results and open problems, establishing links between various areas of investigation and opening possibilities for future joint work. Some of these lectures are included in the present volume. The papers presenting new results are also written in a manner that make them understandable by a broader audience.

The volume contains 36 papers dealing with topics as: norm attaining operators on $L_1(\mu)$ (M. D. Acosta), separating maps on spaces of continuous functions (J. Arauzo, K. Jarosz), extending linear isometries (S. J. Dilworth), fixed point property in $L_1[0,1]$ (P. N. Dowling), smoothness properties of sequence spaces (R. Gonzalo, J. A. Jaramillo) isometries in Orlicz spaces (B. Randriantonina), Banach spaces isometric to their squares (N. J. Kalton), isometries of noncommutative L^p-spaces (K. Watanabe), Weil-spectrum (P. Aiena), normal functions of several complex variables (J. T. Anderson, J. A. Cima), composition and Toeplitz operators on Hardy spaces (P. Avramidou, F. Jafari, K. Stroethoff), a survey on compact-like operators on H^∞ (M. D. Contreras, S. Diaz-Madrigal), a survey on closed ideals in familiar function algebras (P. Gorkin, R. Mostini), conditions for a linear functional to be multiplicative (K. Jarosz), Witt groups on C^*-algebras (C. Badea), the quaternionic Riemann theorem (S. Bernstein), convolution by means of bilinear maps (O. Blasco), Sobolev spaces of holomorphic functions (S. Krantz, M. M. Peloso), a.o.

For the Third Conference on Function Spaces a WEB page containing the abstracts, the schedule and the pictures of the participants, was created by the organizers. It is still available at the address: +http://www.siue.edu/MATH/ conference.htm .+

Undoubtely that the present volume, containig contibutions written by eminent specialists all around the world, will attract a large audience, including people intersted in functional analytic methods in analysis or in their applications to other fields as partial differential equations, optimization, variatonal analysis and optimal control.

The volume is printed in excellent typographical conditions by the American Mathematical Society.

S. Cobzaş

I.B.S. P a s s i (ed.), *Algebra: Some Recent Advances*, Trends in Mathematics, Birkhäuser, 1999, 249 pp., ISBN 3-7643-6058-5 The volume under review is a collection of articles written by well-known algebraists. The aim of the book is to provide a glimpse over an active area of research in algebra through accessible surveys of topics of current research, recent trends, problems and their status.

This collection contains sixteen papers, most of them presenting recent results on group rings. Some of the articles end with extended bibliography which covers a wide variety of papers written in the area of their subject or related topics. The papers of this collection deal with: abelian difference sets (K.T.Arasu and Surinder K. Sehgal), unit groups of group rings (Ashwani K.Bhandari and I.B.S.Passi), projective modules over polynomial rings (S.M.Bhatwadekar), automorphisms of relatively free groups (C.K.Gupta), Jordan decompositions (A.W.Hales and I.B.S.Passi), the normalizer problem (Wolfgang Kimmerle), Galois cohomology of classical groups (R.Parimala), central units in integral group rings (M.M.Parmenter), alternative loop rings and related topics (César Polcino Milies), L-values at zero and the Galois structure of global units (Jürgen Ritter), groups in which the normality or permutability is a transitive relation (Derek J.S.Robinson), the structure of some group rings (Klaus W.Roggenkamp), symmetric elements and identities in group algebras (Sudarshan K.Sehgal), serial modules and rings (Surjeet Singh), subgroups determined by ideals of an integral group ring (L.R.Vermani), a complex irreducible representation of the quaternion group and a non-free projective module over the polynomial ring in two variables over the real quaternions (R.Sridharan).

The articles included in this volume will be of great interest for the scientists involved in the field.

C. Pelea

N. B e l l o m o, L. P r e z i o s i, A. R o m a n o – *Mechanics and Dynamical Systems with Mathematica*, Birkhäuser, 2000, Hardbound, XIII+417 pp., ISBN 0-8176-4007-X, ISBN 3-7643-4007-X

There are a lot of controversies today on how should one teach mathematical methods to students in engineering or natural sciences. Of course, everybody agrees that the "mathematical" way ("definition-theorem-proof") is the worst in this respect. This book proposes an integrated aproach to the teaching of one of the most widely used part of applied mathematics, namely dynamical systems. By "integrated" we mean that there are treated all the three aspects of the problem:

- the building of the model;
- the analytical and qualitative methods of studying the differential equations describing the model;
- the computational tools for performing both the analytical and qualitative studies, provided by the powerful computer algebra system Mathematica.

The book is divided in three parts. The first one is made up of the first three chapters and is devoted to the description of some methods for the study of differential equations. To be fair, we should say that, to a great extent, this part is concerned rather with the construction of models for different physical phenomena, although there are, also, discussed the methods of solving the differential equations, as well as a variety of qualitative methods (especially related to stability and perturbations).

The second part (the chapters 4–6) is an introduction to the most important notions and results from the mechanics of systems with a finite number of degrees of freedom (Newtonian point dynamics, rigid body dynamics, analytical (Lagrangean) mechanics). Finally, the third part (chapters 7–9) is dealing with more advanced topics, in particular infinite dimensional systems, modeled by partial differential equations. Among the subjects treated, we should mention: deterministic and stochastic models, chaotic dynamics, stability and bifurcation, discrete models of continuous systems.

There are, also, three appendices on numerical methods for ordinary differential equations, on kinematics, applied forces, momentum and mechanical energy, as well as on scientific programming.

As we have mentioned from the very beginning, the book is not addressesd to mathematicians so, in general, rigorous proofs are omitted in most cases, however, solid motivations are always given. What is more important, the book includes a great number of examples, taken from practice, many of them being worked out completely. There are, also, an im,portant number of problems, and, actually, here is were the tools of Mathematica enter. We should say that the authors prepared a number of Mathematica programs and notebooks, described in the last appendix and available for downloading from a web page. The reader is asked to use these programs in his investigations of different dynamical systems. We ought to say, though, that the authors' explanations are far from sufficient and the reader is supposed to have a prior experience in working with Mathematica, or, otherwise, he should make use of a specialised book on this topic.

As we said before, the book is ment to be used as course material for different courses in applied mathematics (especially for dynamical systems, of course) and it is very appropriate for this purpose. The prerequisites are rather modest, probably just basic courses in calculus and linear algebra would be just enough. We believe that it is, equally, useful for practicians and evem the mathematicians could find a lot of very interesting examples and motivations.

The book is very accurate printed and includes a great number of line diagrams. There are included, equally, a bibliography and an index.

S. Cobzaş