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SUMAR – CONTENTS – SOMMAIRE

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THE 60^{TH} ANNIVERSARY OF PROF. VASILE POP

V. URECHE

Prof. Vasile Pop was born on 17^{th} of January 1940 in the village of Bârsana, in the northern region of Romania. He attended primary school in Bârsana and graduated the high school in Sighetu Marmaţiei in 1957. In 1958 he became student at "Babeş-Bolyai" University in Cluj-Napoca, at the Department of Mathematics and Mechanics.

After graduating the University, he is appointed teaching assistent at the Pedagogical Institute in Baia-Mare, where he holds seminaries in Mathematical Analysis and Analytical Geometry. In 1967 he becomes researcher at the Astronomical Observatory of the University of Cluj. In 1971 and 1974 he obtains two research scholarships at astronomical observatories in Czechoslovakia.

His Ph.D. thesis "The study of secondary effects for RR-Lyrae variables" elaborated under the advisement of Prof. Gh. Chiş was sustained in 1978. After he obtained the Ph.D. title, he became Lecturer at the Central Institute for Didactic Personnel Training, and in 1979 he returns to the University of Cluj, as Lecturer at the Department of Mathematics.

In 1990 Prof. Pop became Associate Professor and in 1995 he obtains his actual position of Full Professor. Along the years, Prof. Pop has held several courses and has led various seminaries in Mathematics and Astronomy at the Mathematics Department but also at the Departments of Chemistry, Geology and Physics. His lectures are characterized by clarity and rigourness, and also by enthusiasm and tonic optimism extended to his students.

In 1996 Prof. Pop was elected Assistant Dean of the Department of Mathematics and Computer Science, position that he held until 2000 when he became Head of Mechanics and Astronomy.

As it concerns the academic activity of Prof. Vasile Pop, he published six books on the subjects he teaches, while his research concretized through the publication of over 60 papers. The scientific interest of Prof. Pop has focused on the following subjects:

- The photometric study RR Lyrae variables, for which he made over 10000 observations.
- The use of advanced mathematical concepts in the primary analysis of observational data, with an emphasis on spline functions.
- Mathematical modeling of the internal structure of neutron stars.

Among the most important achievements of Prof. Pop we mention:

- He was part of the team that developed and experimented with the photoelectric photometer at the Cluj-Napoca Observatory.
- The photometric study of the pulsating variables RT Comae Berenices, BE Eridani, XZ Cygni and of several eclipsing binaries
- The spectophotometric study of the pulsating star Beta Cephei
- The photometric study of asteroids 433 Eros and 1 Ceres
- The models of the equation of state neutron stars obtained by spline function fitting
- The development of models for the internal structure of neutron stars.

All the results of his research were communicated at several congresses and symposia, national and international conferences. A token of the national and international recognition of his scientific status is the fact that Prof. Pop was elected as a member of the board of the National Romanian Astronomical Committee, while he is also a member of the International Astronomical Union and also of the European Astronomical Society.

On this festive occasion, when he reaches the age of 60, we wish our colleague, "Many Happy Returns of the Day", health and success in all fields of activity to the thrive of Romanian astronomy.

The scientifical work

a) Books and university lectures.

[1] Iancu, C., Pop, S.M., **Pop, V.**, *Probabilităţi şi statistică. Teorie şi aplicaţii*, Ed. SERVOSAT, 1996, ISBN 973-97561-4-X (269 pagini).

[2] Pal, A., **Pop, V.**, Ureche, V., *Astronomie. Culegere de probleme (cu soluţii)*, Ed. Presa Universitară Clujeană, 1998, ISBN 973-9354-28-9 (270 pagini).

[3] **Pop, V.**, *Complemente de matematici pentru geologi*, 1983, Universitatea "Babeş-Bolyai" Cluj-Napoca (Litografiat, 267 pagini).

[4] Oproiu, T. Pal, A., **Pop, V.**, Ureche, V., *Astronomie. Culegere de exerciţii, probleme şi programe de calcul*, 1985, Universitatea "Babeş-Bolyai" Cluj-Napoca (Litografiat, 348 pagini).

[5] Iancu, C., **Pop, V.**, Ţoca, A., *Matematici superioare* 1987, Universitatea "Babeş-Bolyai" Cluj-Napoca (Litografiat, 273 pagini).

[6] Oproiu, T. Pal, A., **Pop, V.**, Ureche, V., *Astronomie. Culegere de exerciţii, probleme şi programe de calcul*, 1989, Universitatea "Babeş-Bolyai" Cluj-Napoca (Reeditat).

[7] **Pop, V.**, Pop, D., *Trigonometrie plană şi sferică*, 1997, Universitatea "Babeş-Bolyai" Cluj-Napoca (Litografiat, 105 pagini).

b) Scientific works.

[8] Janko, B., **Pop, V.**, *About the solution of operational equations defined in the metric space by means of convergent method of second and third order*, 1965, Lucrările Colocviului de teoria funcţiilor convexe cu aplicaţii la calculul numeric, Cluj-Napoca, 27-28.

[9] Janko, B., **Pop, V.**, *Asupra metodei hiperbolelor tangente aplicate la rezolvarea ecuaţiilor operaţionale neliniare*, 1966, Studii şi Cercetări matematice, t.18, nr.8, 1147-1153.

[10] Janko, B., **Pop, V.**, *Asupra rezolvării ecuaţiilor operaţionale definite în spaţii metrice cu ajutorul metodelor convergente de ordinul K*, 1967, Studii şi Cercetări matematice, t.19, nr.8, 1155-1158.

[11] Chiş, G., **Pop, V.**, *Studiul fotometric al câmpului telescopului de tip Newton de la Observatorul Astronomic din Cluj*, 1970, Studii şi Cerc. Astron., t.15, nr.2, 147-155.

[12] **Pop, V.**, *Curba de lumincă şi elementele fotometrice ale variabilei RT Comae Berenices*, 1971, Studii şi Cerc. Astron., t.16, nr.1, 33-37.

[13] Todoran, I., Ureche, V., **Pop, V.**, Chiş, D., *Fotometru fotoelectric al Observatorului Astronomic din Cluj. Determinarea coeficienţilor de extincţie şi a constantelor de trecere la sistemul standard UBV al lui Johnson*, 1972, Studii şi Cerc. Astron., t.17, nr.1, 33-43.

[14] **Pop, V.**, Todoran, I., *Studiul fotometric al stelei pulsante BE Eridani*, 1973, Studii şi Cerc. Astron., t.18, nr.1, 67-71.

[15] **Pop, V.**, *Efectul Blazhko*, 1974, Anuarul Observatorului din Bucureşti, Ed. Acad. R.S.R., 239-271.

[16] **Pop, V.**, Chiş, D., *The Reduction of the photoelectric observations to the UBV system and the use of the atmospheric extinction data*, 1976, Contributions of the Astronomical Observatory, t.1, 27-32.

[17] **Pop, V.**, Chiş, D., *Photoelectric observations of the asteroids 433 Eros and 1 Ceres*, 1976, Contributions of the Astronomical Observatory, t.1, 33-40.

[18] **Pop, V.**, *Modalităţi de integrare a învăţământului de astronomie*, 1977, Forum, nr.9, 29-31.

[19] **Pop, V.**, *The secondary effects of XZ Cygni*, 1978, Studia Univ. Babeş-Bolyai, Mathem., nr.1, 21-29.

[20] **Pop, V.**, *Studiul efectelor secundare la stele variabile de tip RR Lyrae*, 1978, Teză de doctorat, Univ. Babeş-Bolyai Cluj-Napoca.

[21] **Pop, V.**, Pop, P., *Utilizarea funcţiilor spline de ajustare la studiul curbelor empirice*, 1978, Pregătirea tehnico-productivă, Editori A. Chircev, V. Lăscuş, T. Fodor, Cluj-Napoca, 171-192.

[22] **Pop, V.**, *Photoelectric observations of XZ Cygni*, 1978, Contributions of the Astronomical Observatory, 52-72.

[23] **Pop, V.**, *The triple system VW Cephei*, 1979, Studia Univ. Babeş-Bolyai, Mathem., nr.1, 23-27.

[24] **Pop, V.**, *Stadiul actual al cunoaşterii stelelor pulsante de tip RR Lyrae*, 1980, Anuarul Astronomic, Ed. Acad. R.S.R., 241-287.

[25] Chiş, D., Cristescu, C., Oprescu, G., **Pop, V.**, Şuran, M.D., Todoran, I., *The binary system BS Draconis*, 1980, Preprint, Central Institute of Physics, Centre for Astronomy and Space Sciences, A-7-1980, 1-25.

[26] **Pop, V.**, *Neutron stars and the testing of some gravitation theories*, 1981, National Symposium of Structural Method and Models in Physics and Related Fields, 121-123.

[27] **Pop, V.**, *Efectele secundare la variabile pulsante de tip RR Lyrae*, 1981, Anuarul Astronomic, Ed. Acad. R.S.R., 237-271.

[28] Chiş, D., **Pop, V.**, *The Wolf-Rayet system CQ Cephei*, 1983, Babeş-Bolyai University, Fac. of Math., Research Seminaries, Preprint 4, 152-156.

[29] **Pop, V.**, *Problema variaţiei perioadei de pulsaţie la stele de tip RR Lyrae*, 1983, Direcţii moderne în Astronomie şi Astrofizica, Bucureşti, 55-56.

[30] **Pop, V.**, *The physical parameters of RR Lyrae stars*, 1985, Babeş-Bolyai University, Fac. of Mathematics, Research Seminaries, Preprint 2, 64-78.

[31] **Pop, V.**, Oproiu, T., Iancu, C., Pop, P., *A numerical model for relativistic stars*, 1985, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, Preprint 2, 124-136.

[32] Oproiu, T., Pal, A., **Pop, V.**, *A BASIC Program for calculation of cometary Ephemerides*, 1985, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, Preprint 10, p.3-18.

[33] **Pop, V.**, Oproiu, T., *Numerical integration of a differential equation system occuring in the relativistic modelling*, 1986, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, Preprint 3, (Proceedings of the Conference Differential Equations), 283-286.

[34] **Pop, V.**, Chiş, D., *Remarks on the period variation of XZ Cygni*, 1986, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, Preprint 6, 81-90.

[35] **Pop, V.**, Oproiu, T., Radu, E., *Determination of the apparent positions of stars by means of microcomputers*, 1987, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, Preprint 10, 183-196.

[36] **Pop, V.**, *Tehnica funcţiilor spline in astronomie*, 1988, Lucrările Conferinţei de Matematică aplicată şi Mecanică, 20-23 octombrie, Institutul Politehnic Cluj-Napoca, 107-115.

[37] **Pop, V.**, *Determination of isophoto wavelength of instrumental photometer system*, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, Preprint 5, 269-273.

[38] **Pop, V.**, *The radial velocities of Beta Cephei*, 1989, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, Preprint 5, 149-160.

[39] **Pop, V.**, *Perspective ale cercetării astronomice deschise de telescopul spaţial*, 1990, Babeş-Bolyai University, Fac. of Mathematics and Informatics, Lucrările Seminarului "Didactica Matematicii", vol.6, 253-260.

[40] **Pop, V.**, Iancu, C., Oproiu, T., *Use of cubic splines to astronomical problems*, 1990, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, nos.2& 4, 93-102.

[41] **Pop, V.**, *Professor Ioan Armeanca (1990-1954) - Founder of Romanian Atrophysical Research*, Babeş-Bolyai University, Fac. of Mathematics, Research Seminars, 12, nos.2& 4, 155-158.

[42] **Pop, V.**, *The spectrophotometric observations of Beta Cephei*, 1991, Romanian Astronomical Journal, vol.1, nr.1-2, 57-61.

[43] **Pop, V.**, Iancu, C., Oproiu, T., *Spline technique in empirical data fitting*, 1992, Romanian Astronomical Journal, vol.2, nr.2, 141-147.

[44] **Pop, V.**, *Professor Gheorghe Chiş (1913-1981) An initiator of variable stars research in Romania and of articial satellite observations in Cluj*, 1993, Romanian Astronomical Journal, vol.3, nr.1, 83-84.

[45] **Pop, V.**, Iancu, C., Oproi, T., *Un model politropic pentru stele neutronice*, 1995, Analele ştiinţifice ale Universităţii "Al.I. Cuza" din Iaşi, seria Matematica, t.XLI, 137-143.

[46] **Pop, V.**, Chiş, Gh.D., *Variabila pulsantă XZ Cygni*, 1995, Analele ştiinţifice ale Universităţii "Al.I. Cuza" din Iaşi, seria Matematica, t.XLI, 159-161.

[47] **Pop, V.**, *Suporturi astronomice ale conceptului biblic asupra creaţiei*, 1996, Babeş-Bolyai University, Fac. of Mathematics and Informatics, Lucrările Seminarului "Didactica Matematicii", vol.13, 85-89.

[48] **Pop, V.**, Iancu, C., Oproi, T., *Neutron star models with spline-fitted equation of state*, 1995, Rom. Astr. Journal, vol.5, nr.1, 1-10.

[49] **Pop, V.**, *Profesorul dr. Vasile Ureche la 60 de ani*, 1995, Studia Univ. "Babeş-Bolyai", Mathematica, vol.XL, nr.4, 107-120.

[50] **Pop, V.**, Oproiu, T., *Immersion surface for a numerical model of neutron star*, 1998, Rom. Astr. Journal, vol.8, nr.1, 1-10.

[51] **Pop, V.**, Oproiu, T., *Metoda aproximaţiilor succesive în matematica elementară*, 1998, Didactica Matematicii, vol.13, 279-298.

[52] **Pop, V.**, Iancu, C., Oproi, T., *Determination of some parameters of the total solar eclipse of 11 August 1999 by inverse interpolation splines*, 1998, Buletinul Ştiinţific al Universităţii de Nord Baia Mare, seria B, Fasc. Matematică-Informatică, vol.XIV (6 pagini).

[53] **Pop, V.**, Iancu, C., Oproiu, T., *Tehnica funcţiilor spline de interpolare inversă în determinarea parametrilor de vizibilitate ai eclipsei totale de Soare din 11 August 1999 pentru Timişoara*, 1998, Universitatea de Vest din Timişoara, Lucrarile dedicate semicentenarului Facultăţii de Matematică, (10 pagini).

[54] **Pop, V.**, Oproiu, T., *Profesorul Gheorghe Chiş (1931-1981), iniţiatorul cercetării stelelor variabile în România şi al observării sateliţilor artificiali ai Pământului la Cluj*, 1992, Gazet matematică (Perfecţionare metodică şi metodologică în matematica şi informatică), nr.1-2, 51-53.

[55] Todoran, I., **Pop, V.**, *The period variation of VW Cephei*, 1972, Acta Astronomica, vol.22, nr.3, 267-271.

[56] **Pop, V.**, Pop, P., *The study of light curves with aid of smoothing spline functions*, 1975, în monografia: Variable stars and stellar evolution, editor Plaut, Dordrecht Holland, 567-569.

[57] **Pop, V.**, *On the period variation and Blazhko effect of XZ Cygni*, 1975, Inf. Bull. Variable Stars, nr.990, 1-4.

[58] Chiş, D., **Pop, V.**, Todoran, I., *The new photometric elements of BS Draconis*, 1976, Inf. Bull. Variable Stars, nr.1079, 1-2.

[59] **Pop, V.**, Chiş, D., *Photoelectric light curves of Asteroid 433 Eros*, 1976, Icarus, vol.28, 37-38.

[60] **Pop, V.**, Torodan, I., *The eclipsing binary system V566 Ophiuchi*, 1977, Astronomische Nachrichten, Bd.298, H.2, 117-120.

[61] Chiş, D., Cristescu, C., Oprescu, G., Şuran, M.D., Todoran, I., **Pop, V.**, *The binary system BS Draconis*, 1980, Inf. Bull. Variable Stars, nr.1794.

[62] Cristescu, C., Oprescu, G., Şuran, M.D., Chiş, D., Todoran, I., **Pop, V.**, *The eclipsing variable BS Draconis*, 1981, Acta Astronomica, vol.31, nr.4, 505-513.

[63] **Pop, V.**, Chiş, Gh.D., *About the period variations of XZ Cygni*, 1990, Abstract Book, 12th European Reg. Astron. Meeting IAU, Davos, Switzerland, III-38-41.

[64] **Pop, V.**, Iancu, C., Oproiu, T., *Neutron star models with spline-fitted equation of state*, 1996, Proceedings of the 2nd Astronomical Conference of the Hellenic Astronomical Society, Thessaloniki, edited by M.E. Contadakis, J.D. Hadjidemetriou, L.N. Mavridis, J.H. Seiradakis, Hellenic Astronomical Society, Thessaloniki, 179-182.

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DUFOUR AND SORET EFFECTS ON FREE CONVECTION BOUNDARY-LAYER OVER A VERTICAL SURFACE EMBEDDED IN A POROUS MEDIUM

M. ANGHEL, H.S. TAKHAR, AND I. POP

Dedicated to Professor Vasile Pop at his 60^{th} anniversary

Abstract. The diffusion-thermo (Dufour) and thermal-diffusion (Soret) effects on free convection boundary-layer over a vertical flat plate embedded in a porous medium is studied theoretically. The buoyancy effect is due to the variation of temperature and concentration across the boundary-layer. Results are found in excellent agreement with those from the open literature.

1. Introduction

The subject of convective flow in porous media has attracted considerable attention in the last several decades and is now considered to be an important field of study in the general areas of fluid dynamics and heat transfer. This topic has important applications, such as heat transfer associated with heat recovery from geothermal systems and particularly in the field of large storage systems of agricultural products, heat transfer associated with storage of nuclear waste, exothermic reaction in packed-bed reactors, heat removal from nuclear fuel debris, flows in soils, petroleum extraction, control of pollutant spread in groundwater, solar power collectors and porous material regenerative heat exchangers, to name just a few applications. The growing volume of work devoted to this area is amply documented by the most recent books by Nield and Bejan [1], and Ingham and Pop [2].

Combined heat and mass transfer driven by buoyancy due to temperature and concentration variations is also of great practical importance since there are many possible engineering application, such as the migration of moisture through the air contained in fibrous insulation and grain storage installations, and the dispersion of chemical contaminants through water-saturated soil. A comprehensive review on the phenomena has been recently provided by Trevisan and Bejan [3].

When heat and mass transfer over simultaneously in a moving fluid, the relations between the fluxes and the driving potentials are of more intricate nature. It is known that an energy flux can be generated not only by temperature gradients but also by composition gradients. The energy flux caused by a composition gradient is called the Dufour or diffusion-thermo effect. On the other hand, mass (or concentration) fluxes can also be generated by temperature gradients and this is the Soret or thermal-diffusion effect (see Kafoussias and Williams [4]). In general, Dufour and Soret effects are of a smaller order of magnitude than the effects described by Fourier's or Fick's laws and are often neglected in heat and mass (or concentration) processes. There are, however, exceptions. The Soret effect, for instance, has been utilised for isotope separation and in mixture between gases with very light molecular weight (H_2, He) and of medium molecular weight (H_2, air). Eckert and Drake [5] have found that the Dufour effect is of such magnitude that it cannot be neglected. Therefore, we shall investigate in this paper the Dufour and Soret effects on the free convection boundary-layer over a vertical surface embedded in a porous medium using the Darcy-Boussinesq model. The partial differential equations, governing this problem have been transformed by a similarity transformation into a system of ordinary differential equations, which is solved numerically using a double shooting method proposed by Takhar [6]. Results are shown to be in excellent agreement with those known from the open literature.

2. Basic equations

Consider the steady free convection over a vertical surface of a uniform temperature T_w and a uniform mass (concentration) flux C_w, which is embedded in a fluid-saturated porous medium of ambient temperature T_∞ and concentration C_∞, where $T_w > T_\infty$ and $C_w > C_\infty$, respectively. Under the boundary-layer and Darcy-Boussinesq approximations, the basic boundary-layer equations are, see Nield and

Bejan [1] and Kafoussias and Williams [4],

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u = \frac{gK}{v}[\beta(T - T_\infty) + \beta^*(C - C_\infty)] \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \frac{D_m k_T}{C_s C_p}\frac{\partial^2 C}{\partial y^2} \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_m \frac{\partial^2 C}{\partial y^2} + \frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} \tag{4}$$

which are to be solved along with the following boundary conditions

$$u = 0, \quad T = T_w, \quad C = C_w \quad \text{on} \quad x = 0$$

$$v \to 0, \quad T = T_\infty, \quad C = C_\infty, \quad \text{as} \quad x \to \infty \tag{5}$$

Here x and y are co-ordinates measured normal and along the plate, respectively, as in Bejan and Khair [7], u and v are the velocity components along x and y axes, T is the fluid temperature, C is the mass (or concentration) flux, and the other quantities are defined in the Nomenclature.

To solve Eqs. (1)-(5), we assume the following similarity variables, as defined by Cheng and Minkowycz [8] or Bejan and Khair [7],

$$\psi = \alpha_m Ra_y^{1/2} f(\eta), \quad \theta = (T - T_\infty)/\Delta T$$

$$\phi = (C - C_\infty)/\Delta C, \quad \eta = \frac{x}{y} Ra_y^{1/2} \tag{6}$$

where $\Delta T = T_w - T_\infty$, $\Delta C = C_w - C_\infty$ and $Ra_y = gK\beta\Delta T y/v\alpha_m$ is the local Rayleigh number. Substituting (6) into Eqs. (2)-(4), we get the following ordinary differential equations

$$f'' = -\theta' - N\phi' \tag{7}$$

$$\theta'' - \frac{1}{2} f\theta' + D_f \phi'' = 0 \tag{8}$$

$$\frac{1}{Le}\phi'' - \frac{1}{2}f\phi' + Sr\theta'' = 0 \tag{9}$$

and the boundary conditions (5) become

$$\begin{aligned} &f(0) = 0, \quad \theta(0) = 1, \quad \phi(0) = 1 \\ &f' \to 0, \quad \theta \to 0, \quad \phi \to 0 \quad \text{as} \quad \eta \to \infty \end{aligned} \tag{10}$$

where primes denote differentiation with respect to η. Further, N is the buoyancy ratio parameter, Le is the Lewis number, Df and Sr are the modified Dufour and Soret numbers for porous media, and these quantities are defined as

$$N = \frac{\beta^* \Delta C}{\beta \Delta T}, \quad Le = \frac{\alpha_m}{D_m}, \quad Df = \frac{D_m k_T \Delta C}{C_S C_p \alpha_m \Delta T}, \quad Sr = \frac{D_m k_T \Delta T}{\alpha_m T_w \Delta C} \tag{11}$$

The parameters of interest in this problem are the local Nusselt number and the local Sherwood number, which are given by

$$Nu_y / Ra_y^{1/2} = -\theta'(0), \quad Sh_y / Ra_y^{1/2} = -\phi'(0) \tag{12}$$

We notice that Eqs. (7)-(9) reduce to those of Bejan and Khair [7] when the Dufour and Soret effects are neglected, i.e. for $Df = Sr = 0$.

3. Results and discussion

The system of ordinary differential equations (7)-(10) was solved numerically using a double shooting method as proposed by Takhar [6] for several values of the pertinent parameters N, Le, Df and Sr. To verify the proper treatment of the problem, the present solution for $Df = Sr = 0$ has been compared with that of Bejan and Khair [7], see Table 1. It can be seen from this table that the present results are in excellent agreement with those reported by Bejan and Khair [7]. It should also be mentioned that for $N = Df = Sr = 0$, we found $Nu_y / Ra_y = -0.4439$, which agrees very well with the earlier value of -0.444 reported by Cheng and Minkowycz [8].

TABLE 1

Comparison of the local Nusselt and Sherwood numbers

N	Le	Df	Sr	Bejan and Khair [7]	Present	Bejan and Khair [7]	Present
1	1	0	0	0.628	0.6276	0.628	0.6276
1	2	0	0	0.593	0.5926	0.930	0.9295
1	4	0	0	0.559	0.5586	1.358	1.3575
1	6	0	0	0.541	0.5408	1.685	1.6847
1	8	0	0	0.529	0.5295	1.960	1.9599
1	10	0	0	0.521	0.5215	2.202	2.2021
1	100	0	0	0.470	0.4702	7.139	7.1391

Table 2 shows the values of the local Nusselt and Sherwood numbers for some values of the parameters N, Df and Sr when $Le = 1$. It is seen that in the Cases I and II both the local Nusselt and Sherwood numbers first decrease to minimum values then they increase. However, in the Case III, the local Nusselt number increases, while the local Sherwood number decreases monotonically as Df decreases and Sr increases.

Typical velocity, temperature and concentration profiles are shown in Figs. 1-3 for some values of the parameters N, Le, Df and Sr. By comparing these figures with Figs. 4 and 6 from Bejan and Khair [7], we can conclude that the flow field is appreciably influenced by thermal-diffusion (Soret) as well as the diffusion-thermo (Dufour) effects.

TABLE 2

Values of the local Nusselt and Sherwood numbers for some values of the parameters N, Df and Sr when $Le = 1$

N	Le	Df	Sr	$Nu_y Ra_y^{-1/2}$	$Sh_y Ra_y^{-1/2}$
Case I					
1	1	0.030	2.0	0.7201	0.1447
1	1	0.037	1.6	0.7080	0.0116
1	1	0.050	1.2	0.6966	0.1656
1	1	0.075	0.8	0.6883	0.3155
1	1	0.150	0.4	0.6943	0.4619
1	1	0.600	0.1	0.8206	0.5602
Case II					
-5	1	0.15	0.4	0.9956	0.7037
-4	1	0.15	0.4	1.2361	1.0934
-3	1	0.15	0.4	0.7338	0.5023
-2	1	0.15	0.4	0.5535	0.3394
Case III					
0.2	1	0.150	0.4	0.5392	0.3394
0.5	1	0.075	0.8	0.5895	0.2620
0.8	1	0.030	2.0	0.6760	0.1433

Nomenclature

C	concentration
C_p	specific heat at constant pressure
C_S	concentration susceptibility
Df	Dufour number
D_m	mass diffusivity
f	non-dimensional stream function
g	acceleration due to gravity
k_T	thermal diffusion ratio

K permeability of porous medium

Le Lewis number

N buoyancy ratio parameter

Nu_y local Nusselt number

Ra_y local Rayleigh number

Sh_y local Sherwood number

Sr Soret number

T fluid temperature

T_m mean fluid temperature

u, v velocity components in x and y directions

x, y Cartesian co-ordinates normal to the plate and along it, respectively

Greek symbols

α_m effective thermal diffusivity of the fluid saturated porous medium

β coefficient of thermal expansion

β^* coefficient of expansion with concentration

η similarity variable

θ, ϕ non-dimensional temperature and concentration profiles

ν kinematic viscosity

ψ stream function

Superscript

$'$ differentiation with respect to η

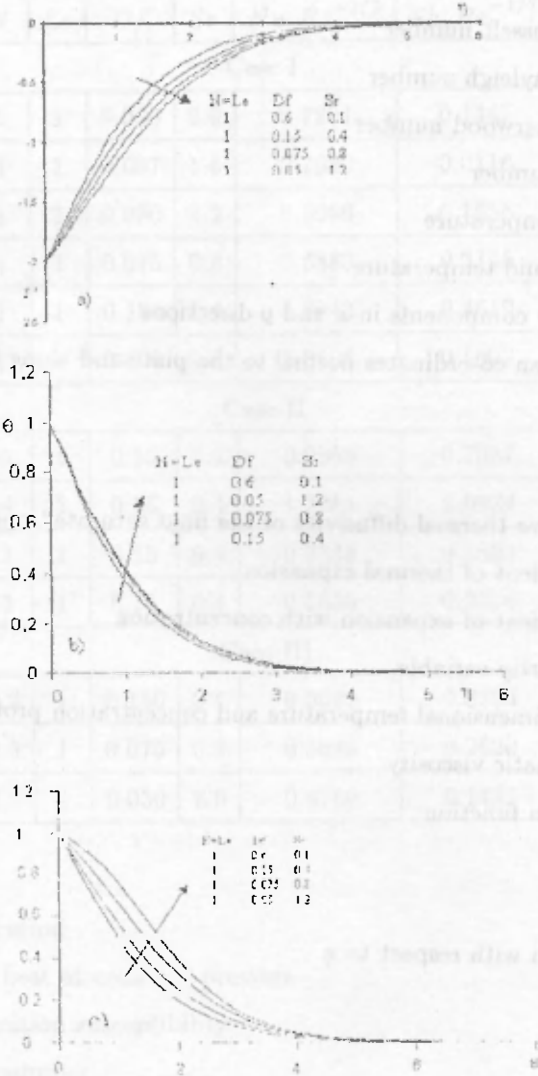

FIG.1

Dimensionless velocity (a), temperature (b) and concentration (c) profiles

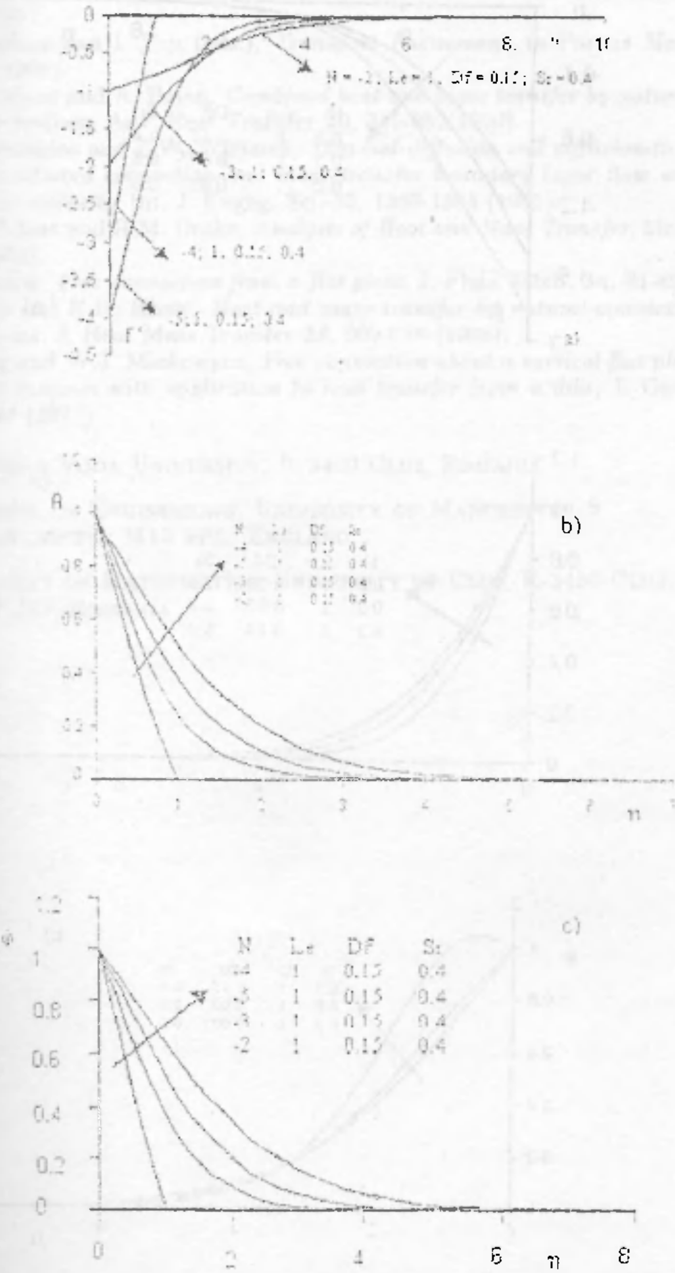

FIG. 2

Dimensionless velocity (a), temperature (b) and concentration (c) profiles

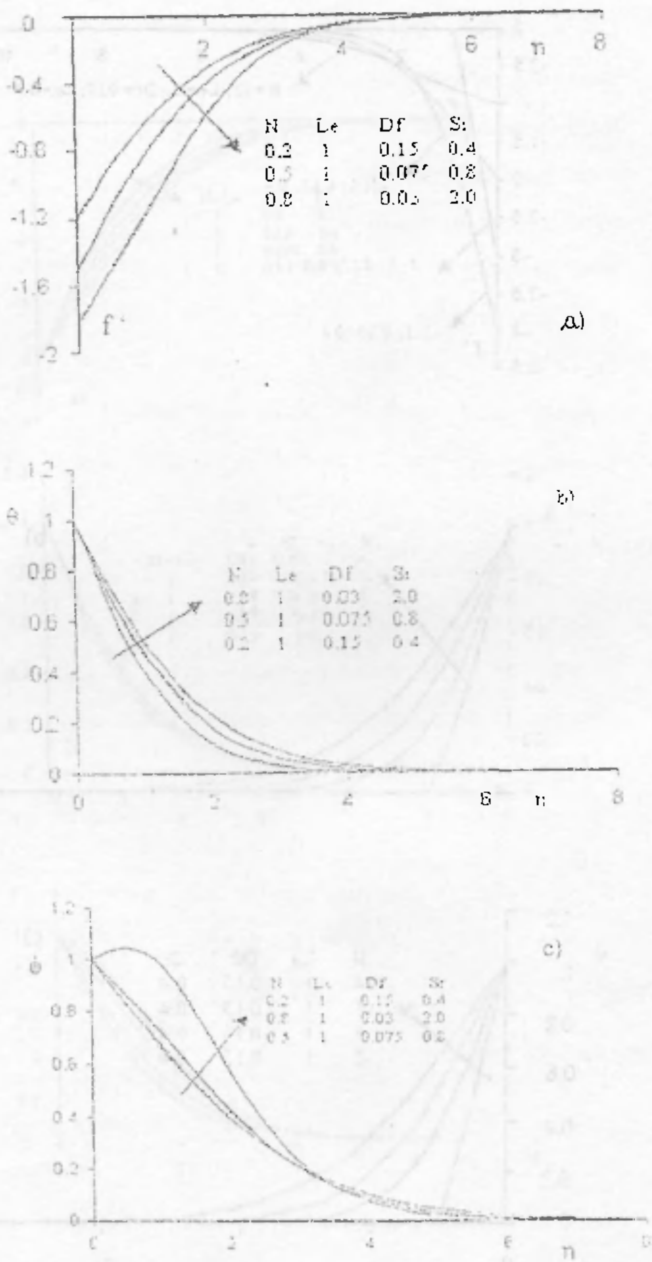

FIG. 3

Dimensionless velocity (a), temperature (b) and concentration (c) profiles

References

[1] D.A. Nield and A. Bejan, *Convection in Porous Media* (2nd edition), Springer, New York (1998).

[2] D.B. Ingham and I. Pop (eds.), *Transport Phenomena in Porous Media*, Pergamon, Oxford (1998).

[3] O.V. Trevisan and A. Bejan, *Combined heat and mass transfer by natural convection in a porous medium*, Adv. Heat Transfer **20**, 315-352(1990).

[4] N.G. Kafoussias and E.W. Williams, *Thermal-diffusion and diffusion-thermal effects on mixed free-forced convection and mass transfer boundary layer flow with temperature dependent viscosity*, Int. J. Engng. Sci. 33, 1369-1384 (1995).

[5] E.R.G. Eckert and R.M. Drake, *Analysis of Heat and Mass Transfer*, McGraw Hill, New York (1972).

[6] H.S. Takhar, *Free convection from a flat plate*, J. Fluid Mech. **34**, 81-89 (1968).

[7] A. Bejan and K.R. Khair, *Heat and mass transfer by natural convection in a porous medium*, Int. J. Heat Mass Transfer **28**, 909-918 (1985).

[8] P. Cheng and W.J. Minkowycz, *Free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike*, J. Geophys. Res. **82**, 2040-2044 (1977).

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A MODEL OF A FLOW IN A RING CANAL

SIMINA BODEA

Dedicated to Professor Vasile Pop at his 60^{th} anniversary

Abstract. We model the nonstationary movement of the incompressible flow in a ring canal with viscosity, gravity and surface tension.

This paper is concerned with the modelling of the nonstationary motion of a viscous, incompressible fluid contained in an uncovered ring canal. The upper surface changes with the motion of the fluid, so we deal with a free boundary problem. The unknown functions are not only the velocity field u and the pressure p, but also the domain Ω. The effect of the surface tension on the upper free boundary is included. The external forces are gravity and wind force which acts on the free boundary and in fact generates the motion of the flow.

We write the equations using the euclidian coordinates (x_1, x_2, x_3) or the cylindrical coordinates (r, θ, z); the components of the velocity field are then denoted by (u_1, u_2, u_3) or (u_r, u_θ, u_z) respectively. In describing the equations of motion we will assume that all variables are nondimensionalized in the usual way.

Let $C = S^1 \times I \times I_l$, $I = [0,1]$, $I_l = [0,l], l > 1$ be the ring canal (see Fig.1) and Ω the domain occupied by the fluid with fixed boundary denoted by Σ and free boundary denoted by Γ. Let $Q = S^1 \times I^2$ be the fluid domain at equilibrium, with the upper boundary S and the rest boundary (the bottom and the walls) denoted by B,

$$S = \{(r, \theta, z) \in \mathbb{R}^3 : r_1 < r < r_2, \theta \in [0, 2\pi], z = 1\}.$$

To describe the free surface of the fluid, we assume small perturbations of the equilibrium surface S and parametrize the free boundary of the liquid with a function $\eta : S \longrightarrow \mathbb{R}$. So, the height of the free surface is a function of horizontal coordinates

$z = \eta(t,\xi)$, $\xi \in S$. The domain occupied by the fluid is

$$\Omega(t) = \{(r,\theta,z) \in \mathbb{R}^3 : r_1 < r < r_2, \theta \in [0,2\pi], 0 < z < 1 + \eta(t,\xi), \xi \in S\}.$$

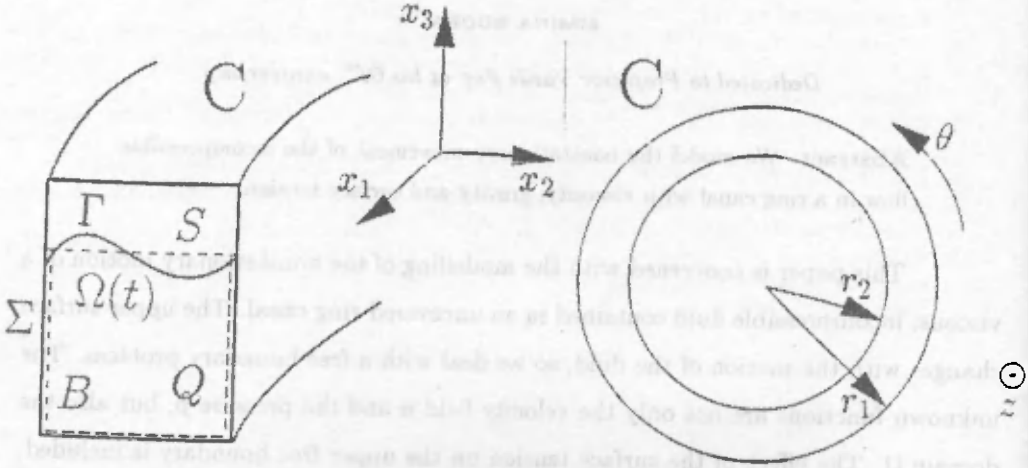

Figure 1

The velocity field is a function

$$u(t,\cdot) : \Omega(t) \longrightarrow \mathbb{R}^3.$$

Let n denote the exterior normal vector of $\partial\Omega$ (or ∂Q) and let τ_i, $i = 1, 2$, denote the tangential directions. We introduce the strain tensor

$$(S_u)_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

and the notations

$$S_u : S_v := \sum_{i,j} (S_u)_{ij}(S_v)_{ij}$$

$$S_u^n := n \cdot S_u \cdot n$$

$$S_u^{\tau_i} := \tau_i \cdot S_u \cdot n$$

The components of the stress tensor are:

$$\sigma_{ij} = p\delta_{ij} - 2\nu(S_u)_{ij}.$$

The motion of the fluid in the interior is governed by the Navier-Stokes equations for an incompressible fluid with viscosity ν (the reciprocal of the Reynolds number):

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla\bar{p} + g\nabla z = 0 \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

where g is the acceleration of gravity. It is natural to consider the pressure $\bar{p} : \Omega(t) \longrightarrow \mathbb{R}$ in the form

$$p = P_0 - gz + p \tag{$\diamond$}$$

where P_0 is the atmospheric pressure above the liquid and p is the hydrostatic pressure. The density does not appear because of the nondimensionalization. After substitution, the gravity term in (1) is eliminated.

On the free surface we have the kinematic boundary condition which states that the fluid particles do not cross the free surface (which is equivalent with the geometric condition that η always parametrizes the free surface):

$$\partial_t\eta = u_z - (\partial_r\eta)u_r - (\partial_\theta\eta)u_\theta \quad \text{on } \Gamma. \tag{3}$$

If we neglected the surface tension, the remaining boundary condition on Γ would be the equality of the stress on the two sides of the surface. The effect of surface tension is to introduce a discontinuity in the normal stress proportional to the mean curvature $H(\eta)$ of the free surface Γ. Our boundary condition on Γ is therefore

$$p - 2\nu S_u^n = g\eta - \beta H(\eta) + f_3$$

$$\tag{4}$$

$$S_u^{\tau_i} = f_i \qquad i = 1, 2$$

where β is the nondimensionalized coefficient of the surface tension and $f = (f_1, f_2, f_3)$ is the wind force. The gravity term $g\eta$ appears because the form ($\diamond$) for the pressure is considered.

From a physical point of view, the usual boundary condition $u = 0$ on Σ can not be considered here because of the contact between the free surface and the fixed boundary (we can not assume that is not moving at all on the walls, so we can not "stick" the free surface on the fixed boundary); but it is natural to consider that the velocity vanishes in the normal dierction, so

$$u \cdot n\Big|_{\Sigma} = 0. \tag{5}$$

Because of the arbitrary (non-smooth) contact between the free surface and the fixed boundary, this condition is not enough to solve the problem, so we have to pose some conditions in the tangential directions too (more precisely we need to describe how the liquid touches the walls of the canal). For the mathematical well-posedness of the problem, there are two possible conditions:

$a)\; S^{\tau_i}_{u}\Big|_{\Sigma} = 0 \qquad i = 1, 2$

$b)$ the solution is periodic in the r direction of the canal, (6)

$$(u, p, \eta)(r_1, \theta, z, t) = (u, p, \eta)(r_2, \theta, z, t) \qquad \forall \theta, z, t.$$

Physically, the condition $a)$ assume no friction on the fixed boundary; the condition $b)$ note that the liquid behaves similarly at the confluence with the two walls, so this condition overlooked the centrifugal force. In order to solve the problem, we shall choose the condition $a)$.

We need also to prescribe the contact angle between the free surface and the fixed boundary. We shall choose it to be $\frac{\pi}{2}$. So, the free surface is moving on the walls, but the value of the contact angle should remain constant. This condition can be writen as:

$$\partial_r \eta\Big|_{r=r_1} = \partial_r \eta\Big|_{r_2} = 0. \tag{7}$$

For similar problems with contact angle 0 or π see [2], [3] and the references presented there.

From a mathematical perspective, we also run into difficulties because the fixed boundary is not smooth. This problem can be avoided by rounding off the corners at the basis of the canal or (at least from the mathematical point of view) by considering the fluid domain going down infinitely. In the second case we loose the advantage of the compact domain (due to which the spectrum of the linearized operator is discrete) and we go too far from the "physical presence" - the canal, so we shall consider that the "contact" between te bottom and the walls of the canal is smooth.

The initial codition is

$$(u,\eta)\Big|_{t=0} = (u_0,\eta_0). \tag{8}$$

The idea of modelling this problem issues from a practical necesity: at the Institute for Environmental Physics of the University of Heidelberg, a canal with 3.5m diameter is being constructed (see Fig.2). For the formulation and solving of similar problems in (semi-)infinite domain, see [1], [2] and [3]. The paper [6] solved a similar problem in a compact domain too, but the whole boundary is free (and close to a sphere).

We collect now all the equations of the movement of a flow in a ring canal:

$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0 \quad \text{in } \Omega$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega$$

$$\partial_t \eta = u_z - (\partial_r \eta)u_r - (\partial_\theta \eta)u_\theta \quad \text{on } \Gamma$$

$$p - 2\nu S_u^n = g\eta - \beta H(\eta) + f_3 \quad \text{on } \Gamma$$

$$S_u^{\tau_i} = f_i \quad (i=1,2) \quad \text{on } \Gamma$$

$$u\cdot n\Big|_\Sigma = 0.$$

$$S_u^{\tau_i}\Big|_\Sigma = 0$$

$$\partial_r\eta\Big|_{r=r_1} = \partial_r\eta\Big|_{r_2} = 0$$

$$(u,\eta)\Big|_{t=0} = (u_0,\eta_0).$$

Figure 2

References

[1] T. Beale, *The Initial Value problem for the Navier-Stokes Equations with a Free Surface*, Comm. Pure Appl. Math. 34, 359-392, 1980.

[2] I.V. Denisova, *A priori estimates of the solution of a linear time-dependent problem connected with the motion of a drop in a fliud medium*, Proc. Steklov Inst. Math. 3, 1-24, 1991.

[3] I.V. Denisova and V.A. Solonnikov *Solvability in Hölder spaces of a model initial-boundary value problem generated by a problem on the motion of two fluids*, J. Math. Sci. 70, 1717-1746, 1994.

[4] M. Golubitsky, I. Stewart and D. Schaeffer, *Singularities and Groups in Bifurcation Theory*, Applied Mathematical Sciences 69, Springer-Verlag, 1981.

[5] M. Renardy and D.D. Josef, *Hopf bifurcation in two-component flow*, SIAM J. Math. Anal. 7, 895-910, 1986.

[6] B. Schweizer, *Oscilatory behavior of liquid drops*, Ph.D. Thesis, Heidelberg, 1995.

[7] M.E. Taylor, *Partial Differential Equations I*, Springer-Verlag, 1996.

[8] R. Temam, *Navier-Stokes Equations*, North-Holland Publishing Company, 1977.

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NEW ESTIMATION OF THE REMAINDER IN THE TRAPEZOIDAL FORMULA WITH APPLICATIONS

S.S. DRAGOMIR AND T.C. PEACHEY

Dedicated to Professor Vasile Pop at his 60^{th} *anniversary*

Abstract. A new inequality for the trapezoidal formula in terms of p-norms is presented with applications to numerical integration and special means.

1. Introduction

Integral inequalities have been used extensively in most subjects involving mathematical analysis. They are particularly useful for approximation theory and numerical analysis in which estimates of approximation errors are involved. In this paper, by the use of an integral identity, we point out some new integral inequalities for the trapezoidal rule and apply these to special means: p-logarithmic means, logarithmic means, identric means etc., and in numerical integration.

Classically, the error bounds for the trapezoidal quadrature rule depend on the maximum norms of the second derivative of the integrand. The new upper bounds for the quadrature rules obtained in this paper have the merit that they depend on only the first derivative of the integrand and thus they are particularly useful for integrals with integrands having bounded first derivatives, but unbounded second derivatives in some norms.

2. The Results

We shall start with the following lemma which contains an interesting integral identity.

Lemma 2.1. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $f' \in L_1[a, b]$. Then we have the identity*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{(b-a)^2}\int_a^b\int_a^b (y-x)f'(y)\,dx dy. \tag{2.1}$$

Proof. Our proof uses the well-known relations

$$\int_a^{t_n} dt_{n-1}\int_a^{t_{n-1}} dt_{n-2}\cdots\int_a^{t_1} f(t_0)\,dt_0 = \int_a^{t_n}\frac{(t_n-u)^{n-1}}{(n-1)!}f(u)\,du, \tag{2.2}$$

and

$$\int_{t_n}^{b} dt_{n-1}\int_{t_{n-1}}^{b} dt_{n-2}\cdots\int_{t_1}^{b} f(t_0)\,dt_0 = \int_{t_n}^{b}\frac{(u-t_n)^{n-1}}{(n-1)!}f(u)\,du \tag{2.3}$$

valid for $f \in L_1[a, b]$ and any positive integer n. We consider

$$\begin{aligned} I &= \int_a^b\int_a^b (y-x)f'(y)\,dx dy \\ &= \int_a^b\int_x^b (y-x)f'(y)\,dy dx - \int_a^b\int_a^x (x-y)f'(y)\,dy dx = T_1 - T_2. \end{aligned}$$

Applying (2.3) to the inner integral in T_1 gives

$$\begin{aligned} T_1 &= \int_a^b dx\int_x^b du\int_u^b f'(t)\,dt = \int_a^b dx\int_x^b [f(b)-f(u)]\,du \\ &= \int_a^b (v-a)[f(b)-f(v)]\,dv = \int_a^b (a-v)f(v)\,dv + \frac{1}{2}(b-a)^2 f(b). \end{aligned}$$

Similarly, applying (2.2) to T_2,

$$T_2 = \int_a^b (b-v)f(v)\,dv - \frac{1}{2}(b-a)^2 f(a).$$

Combining these yields

$$I = T_1 - T_2 = \int_a^b (a-b)f(v)\,dv + \frac{1}{2}(b-a)^2[f(a)+f(b)]$$

and the identity (2.1) follows.

The lemma may be used to prove

Theorem 2.2. *With the above assumptions, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \begin{cases} \frac{b-a}{3}\|f'\|_\infty, \\ \left[\frac{2(b-a)}{(p+1)(p+2)}\right]^{\frac{1}{p}}\|f'\|_q,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f'\|_1. \end{cases} \tag{2.4}$$

Proof. From (2.1)

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{(b-a)^2}\int_a^b\int_a^b |y-x||f'(y)|\,dy\,dx = I.$$

We treat the three cases in turn.

(i) Since

$$\int_a^b\int_a^b |y-x||f'(y)|\,dy\,dx \le \|f'\|_\infty \int_a^b\int_a^b |y-x|\,dy\,dx = \|f'\|_\infty \frac{(b-a)^3}{3}$$

so that $I \le \frac{1}{3}(b-a)\|f'\|_\infty$.

(ii) By Hölder's integral inequality

$$\int_a^b\int_a^b |y-x|\,|f'(y)|\,dx\,dy \le \left(\int_a^b\int_a^b |y-x|^p\,dx\,dy\right)^{\frac{1}{p}}\left(\int_a^b\int_a^b |f'(y)|^q\,dx\,dy\right)^{\frac{1}{q}}$$

$$= K^{\frac{1}{p}}(b-a)^{\frac{1}{q}}\|f'\|_q$$

where

$$K=\int_a^b\int_a^b |y-x|^p\,dx\,dy = 2\int_a^b\int_x^b (y-x)^p\,dy\,dx = \frac{2(b-a)^{p+2}}{(p+1)(p+2)}$$

using the symmetry of the integrand. Thus

$$\int_a^b\int_a^b |y-x|\,|f'(y)|\,dx\,dy \le \left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}+\frac{1}{q}}\|f'\|_q$$

so that with $\frac{1}{p}+\frac{1}{q}=1$ we obtain

$$I \le \left[\frac{2(b-a)}{(p+1)(p+2)}\right]^{\frac{1}{p}}\|f'\|_q$$

as required.

(iii) Finally, we have that

$$\int_a^b \int_a^b |y-x|\,|f'(y)|\,dx\,dy \le \left[\max_{(x,y)\in[a,b]^2} |y-x|\right] \int_a^b \int_a^b |f'(y)|\,dx\,dy = (b-a)^2 \|f'\|_1$$

showing that $I \le \|f'\|_1$. The three cases in (2.4) have now been proved.

Remark 2.3.1. If $p = q = 2$ we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \sqrt{\frac{b-a}{6}}\,\|f'\|_2. \tag{2.5}$$

Remark 2.3.2. In the paper [2], S. S. Dragomir and S. Wang have obtained the following similar result as a particular case of an Ostrowski type inequality.

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{1}{4}(b-a)(\Gamma-\gamma) \le \frac{1}{2}(b-a)\|f'\|_\infty \tag{2.6}$$

where $\gamma := \inf_{t\in(a,b)} f(t) > -\infty$ and $\Gamma := \sup_{t\in(a,b)} f(t) < \infty$.

Remark 2.3.3. In [1] S. S. Dragomir and S. Wang have obtained the following result

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{(b-a)^{\frac{1}{p}}\|f'\|_q}{(p+1)^{\frac{1}{p}}} \tag{2.7}$$

as a particular case of Ostrowski's inequality for q-norms. Since

$$\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} \le \left[\frac{1}{p+1}\right]^{\frac{1}{p}} \text{ for } p > 1,$$

then our estimate in (2.4) is *better* than that embodied in (2.7).

Remark 2.3.4. In [3], S. S. Dragomir and S. Wang obtained the inequality

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \|f'\|_1 \tag{2.8}$$

as a particular case of an Ostrowski type inequality for the L_1 norm.

Remark 2.3.5. In 1938, by means of geometrical considerations, K. S. K. Iyengar [4, p.471] has proved the following inequality

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{(b-a)\|f'\|_\infty}{4} - \frac{(f(b)-f(a))^2}{4(b-a)\|f'\|_\infty} \le \frac{(b-a)\|f'\|_\infty}{4} \tag{2.9}$$

which is a better inequality than our first inequality in (2.2).

In conclusion, Theorem 2.2 gives the following new result

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left[\frac{2(b-a)}{(p+1)(p+2)}\right]^{\frac{1}{p}} \|f'\|_q \tag{2.10}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, and the particular case

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left[\frac{b-a}{6}\right]^{\frac{1}{2}} \|f'\|_2. \tag{2.11}$$

All our further applications for special means and in numerical integration for the trapezoidal formula will be based on these new results.

3. Applications To Special Means

Let us recall first some special means that we will use in the sequel:

(a) The *arithmetic mean:* $A=A(a,b):=(a+b)/2, \quad a,b\ge 0,$

(b) the *geometric mean:* $G=G(a,b):=\sqrt{ab}, \quad a,b\ge 0,$

(c) the *harmonic mean:* $H=H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a,b>0,$

(d) the *logarithmic mean:*

$$L=L(a,b):=\begin{cases} a & \text{if } a=b \\ \dfrac{b-a}{\ln b-\ln a} & \text{if } a\ne b \end{cases} \qquad a,b>0,$$

(e) the *identric mean:*

$$I=I(a,b):=\begin{cases} a & \text{if } a=b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\ne b \end{cases} \qquad a,b>0,$$

(f) the *p-logarithmic mean:*

$$L_p=L_p(a,b):=\begin{cases} a & \text{if } a=b \\ \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\ne b \end{cases} \qquad a,b>0, \text{ and } p\in \mathbb{R}\backslash\{-1,0\}.$$

These means are often used in numerical approximation and in other areas.

The following simple relationships are known:

$$H\le G\le L\le I\le A$$

and L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 := I$ and $L_{-1} := L$.

1. Let us assume that $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^s$, $s \in \mathbb{R}\backslash\{-1,0\}$ and $p,q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then obviously

$$\frac{f(a)+f(b)}{2} = \frac{a^s+b^s}{2} = A(a^s,b^s),$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b x^s\,dx = \frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)} = L_s^s(a,b).$$

Since $f'(x) = sx^{s-1}$,

$$\|f'\|_q = \left[|s|^q \int_a^b x^{q(s-1)}dx\right]^{1/q} = |s|(b-a)^{1/q}\left[\frac{1}{b-a}\int_a^b x^{q(s-1)}dx\right]^{1/q} =$$

$$= |s|(b-a)^{1/q} L_{q(s-1)}^{s-1}$$

so the inequality (2.10) becomes

$$|A(a^s,b^s) - L_s^s(a,b)| \le \left[\frac{2(b-a)}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{q(s-1)}^{s-1}(a,b).$$

That is, we have

$$|A(a^s,b^s) - L_s^s(a,b)| \le |s|(b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{q(s-1)}^{s-1}(a,b), \tag{3.1}$$

for $0 \le a \le b < \infty$. In particular, for $p = q = 2$,

$$|A(a^s,b^s) - L_s^s(a,b)| \le \frac{|s|(b-a)}{\sqrt{6}} L_{2(s-1)}^{s-1}(a,b). \tag{3.2}$$

2. Let us assume that $f : (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ and $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Then

$$\frac{f(a)+f(b)}{2} = \frac{\frac{1}{a}+\frac{1}{b}}{2} = H^{-1}(a,b),$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b \frac{dx}{x} = \frac{\ln b - \ln a}{b-a} = L^{-1}(a,b),$$

$$f'(x) = -\frac{1}{x^2}, \qquad \|f'\|_q = \left[\int_a^b \frac{dx}{x^{2q}}\right]^{1/q} = (b-a)^{\frac{1}{q}} L_{-2q}^{-2}.$$

Then (2.10) becomes

$$\left|H^{-1}(a,b)-L^{-1}(a,b)\right| \leq \left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{-2q}^{-2}(a,b)$$

This yields the inequality

$$0 \leq L-H \leq \frac{(b-a)LH}{L_{-2q}^{2}}\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{q(s-1)}^{s-1} \tag{3.3}$$

for $0 < a < b < \infty$.

In particular, for $p=q=2$ we have

$$0 \leq L-H \leq \frac{(b-a)LH}{\sqrt{6}\,L_{-4}^{2}}. \tag{3.4}$$

3. Let us assume that $f:(0,\infty)\to\mathbb{R}$, $f(x)=\ln x$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\frac{f(a)+f(b)}{2}=\frac{\ln a+\ln b}{2}=\ln G$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{b-a}\int_a^b \ln x\,dx=\frac{1}{b-a}\ln\left(\frac{b^b}{a^a}\right)-1=\ln I,$$

$$f'(x)=\frac{1}{x},\qquad \|f'\|_q=\left[\int_a^b\frac{dx}{x^q}\right]^{1/q}=(b-a)^{\frac{1}{q}}L_{-q}^{-1}.$$

Then inequality (2.10) gives

$$|\ln G-\ln I| \leq (b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{-q}^{-1}$$

Thus

$$1 \leq \frac{I}{G} \leq \exp\left[(b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{-q}^{-1}\right] \tag{3.5}$$

for $0 < a \leq b < \infty$.

In particular, for $p=q=2$ we have

$$1 \leq \frac{I}{G} \leq \exp\left[\frac{(b-a)}{\sqrt{6}\,L_{-2}}\right]. \tag{3.6}$$

4. Applications In Numerical Integration

We discuss here the application of the inequality (2.10) in Numerical Integration to obtain some new estimates of the remainder term in the classical trapezoidal rule.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) and assume that f' is q-integrable on $[a,b]$, that is that $f' \in L_q[a,b]$, $q > 1$. If $I_h : a = x_0 < x_1 < \ldots < x_n = b$ is a partition of $[a,b]$, then we have*

$$\int_a^b f(x)\,dx = T(f, I_h) + R(f, I_h) \tag{4.1}$$

where $T(f, I_h)$ is the trapezoidal quadrature rule, i.e.,

$$T(f, I_h) = \sum_{i=0}^{n-1} \left[\frac{f(x_i) + f(x_{i+1})}{2}\right] h_i \tag{4.2}$$

where $h_i = x_{i+1} - x_i$ for all $i = 0, 1, 2, \ldots, n-1$ and the remainder $R(f, I_h)$ satisfies the inequality

$$|R(f, I_h)| \le \left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} \|f'\|_q \left[\sum_{i=1}^{n-1} h_i^{p+1}\right]^{\frac{1}{p}}. \tag{4.3}$$

Proof. Applying the inequality (2.10) on the interval $[x_i, x_{i+1}]$ where $i = 0, 1, \ldots, n-1$ we have that

$$\left|\frac{f(x_i) + f(x_{i+1})}{2} h_i - \int_{x_i}^{x_{i+1}} f(x)\,dx\right| \le \left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} h_i^{1+\frac{1}{p}} \left(\int_{x_i}^{x_{i+1}} |f'(x)|^q dx\right)$$

for all $i = 0, 1, 2, \ldots, n-1$. Summing these inequalities and using Holder's discrete inequality we have that

$$\begin{aligned}
|R(f, I_h)| &\leq \sum_{i=0}^{n-1} \left| \frac{f(x_i) + f(x_{i+1})}{2} h_i - \int_{x_i}^{x_{i+1}} f(x)\,dx \right| \\
&\leq \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \sum_{i=0}^{n-1} h_i^{\frac{p+1}{p}} \left(\int_{x_i}^{x_{i+1}} |f'(x)|^q dx \right)^{\frac{1}{q}} \\
&\leq \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \left(\sum_{i=0}^{n-1} \left(h_i^{\frac{p+1}{p}} \right)^p \right)^{\frac{1}{p}} \left(\sum_{i=0}^{n-1} \left(\left(\int_{x_i}^{x_{i+1}} |f'(x)|^q dx \right)^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \\
&= \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} h_i^{p+1} \right)^{\frac{1}{p}} \|f'\|_q.
\end{aligned}$$

The theorem is thus proved.

Corollary 4.2. *With the above assumptions, if $f' \in L_2[a,b]$ we have*

$$|R(f, I_h)| \leq \frac{\|f'\|_2}{\sqrt{6}} \left(\sum_{i=1}^{n-1} h_i^3 \right)^{\frac{1}{2}}. \tag{4.4}$$

Suppose now that I_h denotes the equidistant partitioning of $[a,b]$ given by

$$I_h: \quad x_i = a + \frac{b-a}{n} i, \quad i = 0, 1, \ldots, n.$$

For this partition we have the following corollary.

Corollary 4.3. *Under the assumptions of Theorem 4.1,*

$$\int_a^b f(x)\,dx = T_n(f) + R_n(f) \tag{4.5}$$

where $T_n(f)$ is the trapezoidal quadrature rule for the partition I_h, that is

$$T_n(f) = \frac{b-a}{2n} \sum_{i=0}^{n-1} \left[f\left(a + i\frac{b-a}{n} \right) + f\left(a + (i+1)\frac{b-a}{n} \right) \right] \tag{4.6}$$

and the remainder term $R_n(f)$ satisfies the estimate

$$|R_n(f)| \leq \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \frac{(b-a)^{1+\frac{1}{p}} \|f'\|_q}{n} \quad \textit{for } n \geq 1. \tag{4.7}$$

In particular, for $p = 2$, we have

$$|R_n(f)| \le \frac{(b-a)^{\frac{3}{2}}}{\sqrt{6}} \frac{\|f'\|_2}{n}. \tag{4.8}$$

Given any $\epsilon > 0$, we are able using (4.7), to establish the minimum number of nodes such that the error in the numerical integration based on the equidistant trapezoidal rule is smaller than ϵ. This is contained in the following corollary.

Corollary 4.4. *Given any constant $\epsilon > 0$, if $n \ge n_\epsilon$, where*

$$n_\epsilon = \left[\left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \frac{(b-a)^{1+\frac{1}{p}} \|f'\|_q}{\epsilon} \right] + 1$$

then $|R_n(f)| \le \epsilon$.

Example 4.4. We give an example where the bound on R_n provided by (4.8) is better than those previously known. The equivalent bound imposed by (2.7) with $p = 2$ is

$$|R_n(f)| \le \frac{(b-a)^{\frac{3}{2}}}{\sqrt{3}} \frac{\|f'\|_2}{n}, \tag{4.9}$$

that imposed by (2.8) is

$$|R_n| \le \frac{(b-a)\|f'\|_1}{n}, \tag{4.10}$$

while that implied by (2.9) is

$$|R_n| < \frac{(b-a)^2}{4} \frac{\|f'\|_\infty}{n}. \tag{4.11}$$

As the example, we take $a = 0$, $b = 1$ and $f(x) = x^{2/3} e^{-2x/3}$ so that $f'(x) = \frac{2}{3} x^{-1/3}(1-x)e^{-2x/3}$. In this case $\|f'\|_\infty$ is infinite so (4.11) yields nothing useful. Since $f'(x)$ is positive on $(0,1)$, we have $\int_0^1 |f'(x)|\, dx = f(1) - f(0) = e^{-2/3}$. Thus (4.10) is

$$|R_n| \le \frac{e^{-2/3}}{n} \approx \frac{0.513}{n}.$$

Also

$$\|f'\|_2^2 = \int_0^1 \frac{4}{9} e^{-2x/3} \left(\frac{1-x}{x^{1/3}} \right)^2 dx \le \frac{4}{9} \int_0^1 (1-x)^2 x^{-2/3} dx = \frac{4}{9} B(3, \frac{1}{3}) = \frac{6}{7}.$$

Inserting this into (4.9) gives

$$|R_n| \leq \sqrt{\frac{2}{7}}\,\frac{1}{n} \approx \frac{0.535}{n}$$

while (4.8) becomes

$$|R_n| \leq \sqrt{\frac{1}{7}}\,\frac{1}{n} \approx \frac{0.378}{n}.$$

Thus in this example the new bound is superior.

References

[1] S. S. Dragomir aud S. Wang, "A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules", submitted.

[2] S. S. Dragomir and S. Wang, "An inequality of Ostrowski-Grüss' type and its application to the estimation of error bounds for some special means and for some numerical quadrature rules", Computers Math. Applic., Vol 33(11), 1997, pp. 15-20.

[3] S. S. Dragomir and S. Wang, "A new inequality of Ostrowski's type in L_1 norms and applications to some special means and to some numerical quadrature rules", Tamkang J. of Math., to appear.

[4] D. S. Mitronovic, J. E. Pecăric and A. M. Fink, "Inequalities Involving Functions and Their Integrals and Derivatives", Kluwer, Dordrecht, 1991.

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ABOUT THE *P*-MODEL OF THE STOCHASTIC VECTORIAL PROGRAMMING PROBLEMS WITH SIMPLE RECOURSE

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Dedicated to Professor Vasile Pop at his 60th anniversary

Abstract. This work deals with the *P*-model of the stochastic vectorial programming problems with simple recourse in two stages with probabilistic constraints. The problem is formulated in this case and the way of solving it is stated by converting the probabilistic constraints into their deterministic equivalent and searching the deterministic equivalent of the objective functions. The problem is reduced from the multi-objective case to a single objective function by using a synthesis function obtained by the minimization criterion of distances' sum between possible maximum of each objective function and its value in a certain point.

1. Introduction

The *P*-model was considered for the first time by A. Charnes and W. W. Cooper [3] and by Charnes and Kirby [4] for stochastic programming problem with recourse. Stancu-Minasian [11] generalizes the *P*-model presented by Charnes and Cooper from an objective function to r objective functions, introducing the notion of multiple minimum risk solution as a generalization of the minimum risk solution independently introduced by B. Bereanu [1] and Charnes and Cooper [3].

In the present work we want to generalize the model for an objective function, proposed by Charnes and Kirby [4] to the r objective functions. A.Prekopa [10] underlines that the solutions of *P*-model with probabilistic constraints have not been studied and due to the author's knowledge for the vectorial case the problem hasn't been formulated yet.

We formulate the P-model of stochastic programming with multiple objective functions (of minimum risk)

$$\max f_k(x,\xi_2) = P[z_k(x) - q_k^T y \geq u_k], \quad k \in I = \{1,2,\ldots,r\} \tag{1}$$

subject to:

$$P(Ax \leq \xi_1) \geq p_1$$

$$P(Tx + Wy = \xi_2) \geq p_2$$

$$Bx \leq b$$

$$x \geq 0, \quad y \geq 0$$

where $z_k(x) = c_{k1}x_1 + c_{k2}x_2 + \cdots + c_{kn}x_n$, $k \in I$ are linear functions, u_k, $k \in I$ are given values $\xi_i = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{im_i})^T$, $i = 1,2$ is a random vector defined on the probability space (Ω, K, P), A is an $m_1 \times n$ matrix, T is an $m_2 \times n$ matrix, W is the $m_2 \times m_2$ identity matrix, x is an n-vector, y is an m_2-vector, B is an $m_3 \times n$ matrix, p_i, $i = 1,2$ are vectors of m_1 respectively m_2 dimension, having the components some prescribed probabilities, b is an m_3 vector. The matrices A and T might have random elements.

Solving problem (1) means solving of the two so called stages; in order to make notifications easier we begin with the second stage which is:

$$\min q_k^T y, \quad k \in I \tag{2}$$

subject to:

$$P(Tx + Wy = \xi_2) \geq p_2$$

$$x \geq 0, \quad y \geq 0,$$

for the given x and ξ_1.

The first stage of the problem (1) is

$$\text{Vmax } f_k(x,\xi_2) = P[z_k(x) - q_k^T y \geq u_k], \quad k \in I \tag{3}$$

subject to:

$$P(Ax \leq \xi_1) \geq p_1$$

$$x \geq 0, \; x \in K_2$$

where K_2 is the set of those $x \in \mathbb{R}^n$ vectors for which the constraints of the second stage are satisfied, that means the set

$$K_2 = \{x \in \mathbb{R}^n \mid \text{for each } \xi_2, p_2 \text{ exists } y \geq 0 \text{ so that } P(Tx + Wy = \xi_2) \geq p_2\}.$$

We note with K_1 the set of those $x \in \mathbb{R}^n$ vectors for which the constraints of the first stage are satisfied, that means $K_1 = \{x \in \mathbb{R}^n \mid P(Ax < \xi_1) > p_1,\ x \geq 0,\ x \in K_2\}$. We note with K_3 the set of those $x \in \mathbb{R}^n$ vectors that satisfies the constraints $Bx \leq b$. The interpretation of the model is: x represents a decision that has to meet the requirements of the constraints $P(Ax \leq \xi_1) \geq p_1$, $P(Tx = \xi_2) \geq p_2$, $Bx < b$. Because of the fact that the first two constraints contain random elements, it is possible, that due to some causes, the constraints shouldn't be met and we will consider that this happens for the second constraint $P(Tx \leq \xi_2) \geq p_2$; in this case a recourse-decision y is taken and by it this constraint will be fulfilled, so it becomes $P(Tx + Wy = \xi_2) \geq p_2$, where for the problem with simple recourse we get $W = I_{m_2}$. The recourse decision y influences each objective function, this being penalised with $q_k^T y$, $k \in I$ value, where $q_k = (q_{k1}, q_{k2}, \ldots, q_{km_2})$.

Definition 1. A feasible solution for (1) is a vector x such that for any ξ_1 it satisfies the first stage constraints and such that, for any ξ_2, it is always possible to find a feasible solution to the second stage.

The problem that interests is how we can obtain the multiple minimum risk solutions of the problem (1) taking into account that both the objective function and some of the constraints of the problem are probabilistic.

Solving the problem (1) which we write as

$$\text{Vmax } f_k(x, \xi_2) = P[z_k(x) - q_k^T y \geq u_k], \quad k \in I \tag{4}$$

subject to:

$$x \in K_1,$$

means finding the multiple minimum risk solution.

Definition 2. A point $x^0 \in K$ is multiple minimum risk solution if it is an efficient solution for (1), that is if there is no $x \in K$ so that $f_i(x, \xi_2) \geq f_i(x^0, \xi_2)$ for $i \in I$ and for at least $j \in I$, $j \neq i$ we should have $f_j(x, \xi_2) > f_j(x^0, \xi_2)$.

2. Solving the *P*-model stochastic programming problem with multiple objective functions

We introduce the next assumptions for the problem (1):

(i) the random variables ξ_1, ξ_2 have a normal distribution and are independent;

(ii) the feasible set K is a compact and nonempty set in $\mathbb{R}^n$.

In order to find the multiple minimum risk solution we perform the next steps:

a) we determine the deterministic equivalent of the second stage's constraints, meaning the constraints of problem (2);

b) we determine the deterministic equivalent of the constraints of problem (3);

c) we determine the deterministic equivalent of the constraints of problem (1);

d) we determine the optimal solution for a synthesis function F^* of the r objective functions.

a) Let the constraints of the second stage of the problem (1) be

$$P(Tx + Wy = \xi_2) \geq p_2, \quad x \geq 0, \quad y \geq 0,$$

where the first of these inequalities can be written as:

$$P\left[\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{m_21} & g_{m_22} & \cdots & g_{m_2n} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} + \right.$$

$$+\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}\begin{pmatrix} y_{21} \\ y_{22} \\ \dots \\ y_{2n} \end{pmatrix} = \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \dots \\ \xi_{2m_2} \end{pmatrix}\Bigg] \geq \begin{pmatrix} p_{21} \\ p_{22} \\ \dots \\ p_{2m_2} \end{pmatrix}$$

Be $K_2 = G_1 \cap G_2 \cap \dots \cap G_{m_2}$ where

$$G_i = \{x \in \mathbb{R}^n |\ P[(g_{i1}x_1 + g_{i2}x_2 + \dots + g_{in}x_n + y_{2i}) = \xi_{2i}] \geq p_{2i},\ y_{2i} \geq 0\},$$

$i = 1, 2, \dots, m_2$.

We determine the deterministic equivalent of G_i, $i = 1, 2, \dots, m_2$ set:

$$G_i = \{x \in \mathbb{R}^n |\ P(g_{i1}x_1 + g_{i2}x_2 + \dots + g_{in}x_n + y_{2i} = \xi_{2i}) \geq p_{2i},\ y_{2i} \geq 0\} =$$

$$= \{x \in \mathbb{R}^n |\ P(y_{2i} = \xi_{2i} - g_{i1}x_1 - g_{i2}x_2 - \dots - g_{in}x_n) \geq p_{2i},\ y_{2i} \geq 0\} =$$

$$= \{x \in \mathbb{R}^n |\ P(\xi_{2i} - g_{i1}x_1 - g_{i2}x_2 - \dots - g_{in}x_n \geq 0) \geq p_{2i}\} =$$

$$= \{x \in \mathbb{R}^n |\ P(\xi_{2i} < g_{i1}x_1 + g_{i2}x_2 + \dots + g_{in}x_n) \leq p_{2i}\} =$$

$$= \{x \in \mathbb{R}^n |\ F(g_{i1}x_1 + g_{i2}x_2 + \dots + g_{in}x_n) \leq p_{2i}\} =$$

$$= \{x \in \mathbb{R}^n |\ g_{i1}x_1 + g_{i2}x_2 + \dots + g_{in}x_n \leq F^{-1}(p_{2i})\}, \quad i = 1, 2, \dots, m_2,$$

where we note with F the probability distribution function of the random variable ξ_{2i}, $i = 1, 2, \dots, m_2$.

So we obtain the $K_2 = G_1 \cap G_2 \cap \dots \cap G_{m_2}$ set.

b) The determination of the deterministic equivalent of probabilistic constraints of problem (3) is solved in a similar way. Hence, let the constraints be:

$$P(Ax \leq \xi_1) \geq p_1$$

which may be written as:

$$P\left[\begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ \dots & \dots & \dots & \dots \\ h_{m_1 1} & h_{m_1 2} & \dots & h_{m_1 n} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \leq \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \dots \\ \xi_{1m_1} \end{pmatrix}\right] \geq \begin{pmatrix} p_{11} \\ p_{12} \\ \dots \\ p_{1m_1} \end{pmatrix}$$

We consider the sets:

$$H_j = \{x \in \mathbb{R}^n \mid P(h_{j1}x_1 + h_{j2}x_2 + \cdots + h_{jn}x_n \leq \xi_{1j}) \geq p_{1j}\} =$$

$$= \{x \in \mathbb{R}^n \mid P(\xi_{1j} < h_{j1}x_1 + h_{j2}x_2 + \cdots + h_{jn}x_n) < p_{1j}\} =$$

$$= \{x \in \mathbb{R}^n \mid F(h_{j1}x_1 + h_{j2}x_2) + \cdots + h_{jn}x_n) \leq p_{1j}\} =$$

$$= \{x \in \mathbb{R}^n \mid h_{j1}x_1 + h_{j2}x_2 + \cdots + h_{jn}x_n \leq F^{-1}(p_{1j})\}, \quad j = 1, 2, \ldots, m_1.$$

We obtain the $K_1 = H_1 \cap H_2 \cap \cdots \cap H_{m_1}$.

Also we determine the $K_3 = \{x \in \mathbb{R}^n \mid Bx \leq b\}$ set.

c) The determination of the deterministic equivalent of the problem (1) means to obtain a deterministic vectorial programming problem with r objective functions. For this we will search to determine, first of all the deterministic equivalent of each objective function.

Let be problem (1) written as:

$$\text{Vmax } f_k(x, \xi_2) = P[z_k(x) - q_k^T y > u_k], \quad k \in I$$

subject to:

$$x \in K.$$

For any $k \in I$, the objective function $f_k(x, \xi_2)$ is:

$$f_k(x, \xi_2) = P[z_k(x) - q_k^T y > u_k] =$$

$$= P[z_k(x) - q_{k1}(\xi_{21} - g_{11}x_1 - g_{12}x_2 - \cdots - g_{1n}x_n) -$$
$$- q_{k2}(\xi_{22} - g_{21}x_1 - g_{22}x_2 - \cdots - g_{2n}x_n) - \cdots -$$
$$- q_{km_2}(\xi_{2m_2} - g_{m_21}x_1 - g_{m_22}x_2 - \cdots - g_{m_2n}x_n) > u_k].$$

We note

$$\eta_k = q_{k1}\xi_{21} + q_{k2}\xi_{22} + \cdots + q_{km_2}\xi_{2m_2} = \eta_{k1} + \eta_{k2} + \cdots + \eta_{km_2},$$

which is also a random variable, with a normal distribution, which will have the following parameters:

$$\overline{\eta}_k = \overline{\eta}_{k1} + \overline{\eta}_{k2} + \cdots + \overline{\eta}_{km_2} = q_{k1}\overline{\xi}_{21} + q_{k2}\overline{\xi}_{22} + \cdots + q_{km_2}\overline{\xi}_{2m_2}$$

and

$$\sigma_{\eta_k} = \sqrt{D(\eta_k)} = \sqrt{D(\eta_{k1}) + D(\eta_{k2}) + \cdots + D(\eta_{km_2})} =$$

$$= \sqrt{q_{k1}^2 D(\xi_{21}) + q_{k2}^2 D(\xi_{22}) + \cdots + q_{km_2}^2 D(\xi_{2m_2})} =$$

$$= \sqrt{q_{k1}^2 \sigma_{\xi_{21}}^2 + q_{k2}^2 \sigma_{\xi_{22}}^2 + \cdots + q_{km_2}^2 \sigma_{\xi_{2m_2}}^2}$$

We note

$$z_k'(x) = q_{k1}g_{11}x_1 + q_{k1}g_{12}x_2 + \cdots + q_{k1}g_{1n}x_n + \cdots +$$

$$+q_{k2}g_{21}x_1 + q_{k2}g_{22}x_2 + \cdots + q_{k2}g_{1n}x_n + \cdots +$$

$$+q_{km_2}g_{m_21}x_1 + q_{km_2}g_{m_22}x_2 + \cdots + q_{km_2}g_{m_2n}x_n.$$

It follows:

$$P[z_k(x) + z_k'(x) - u_k \geq \eta_k] = P\left(\frac{\eta_k - \overline{\eta}_k}{\sigma_{\eta_k}} < \frac{z_k(x) + z_k'(x) - u_k - \overline{\eta}_k}{\sigma_{\eta_k}}\right) =$$

$$= \left[\Phi\left(\frac{z_k(x) + z_k'(x) - u_k - \overline{\eta}_k}{\sigma_{\eta_k}}\right) + \frac{1}{2}\right]$$

where Φ is Laplace's function.

Because Φ is a strict increasing function, it results that its maximum is obtained for $\max\limits_{x \in K}\left(\dfrac{z_k(x) + z_k'(x) - u_k - \eta_k}{\sigma_{\eta_k}}\right)$ or the equivalent for $\max(z_k(x) + z_k'(x))$ taking into account that the other terms are constants and positives.

Hence we proved the next theorem:

Theorem 1. *The minimum risk solution of the problem*

$$\text{Vmax } f_k(x, \xi_2) = P[z_k(x) - q_k^T y \geq u_k], \quad k \in I$$

subject to:

$$x \in K$$

is given by the linear programming problem solution

$$\max_{x \in K}(z_k(x) + z_k'(x)), \quad k \in I.$$

Hence, for each $k \in I$ we will find a minimum risk solution $x_k = (x_{k1}, x_{k2}, \dots, x_{kn})$, and what remains to be found is the efficient solution for those r objective functions, meaning the one which provides "the best compromise".

d) A lot of choice criteria of the synthesis function are known, but we stopped at the model that minimize the distances' sum between maximum values and the values in an any x point of the objective function:

$$F^*(x) = \min_{x\in K} \sum_{k=1}^{r} \left[\max_{x\in K} \Phi\left(\frac{z_k(x) + z_k'(x) - u_k - \overline{\eta}_k}{\sigma_{\eta_k}}\right) - \right. \tag{5}$$
$$\left. - \Phi\left(\frac{z_k(x) + z_k'(x) - u_k - \overline{\eta}_k}{\sigma_{\eta_k}}\right)\right].$$

The x^0 point in which this minimum is obtained will be the efficient point for problem(1). In the given form in relation (5) the problem can't be solved, that's why we search for an equivalent form that can be calculable. We note:

$$t_k(x) = \frac{z_k(x) + z_k'(x) - u_k - \overline{\eta}_k}{\sigma_{\eta_k}}.$$

For $x \in K$, the $t_k(x)$ function values set will be an interval $[a_k, b_k]$; taking into account that Laplace's function is continuous and strictly increasing, instead of the criterion from relation (5), that mean, instead of the differences between maximum of the objective function $\Phi(b_k)$ and its value in a $t_k(x)$ point, we will use $b_k - t_k(x)$, but if we want the two relations to be equivalent, the function should be linear. Hence we will linearize Laplace's function on given intervals, to provide an approximation precision of probability 0,005.

Be $\Delta = (\Delta_0, \Delta_1, \dots, \Delta_{12})$ an equidistant partition of the $[-3, 3]$ interval,

$$\Delta = [-3; -2,5; -2; -1,5; -1; -0,5; 0; 0,5; 1; 1,5; 2; 2,5; 3].$$

Having this division we obtain the intervals: $I_0 = (-\infty, -3)$; $I_1 = [-3, -2,5)$; ...; $I_{12} = [2,5, 3)$; $I_{13} = [3, +\infty)$.

It is known that for a random variable X with normal distribution, with a mean m and variance σ^2 it follows that:

$$P(\alpha \le X < \beta) = F(\beta) - F(\alpha) = \Phi\left(\frac{\beta - m}{\sigma}\right) - \Phi\left(\frac{\alpha - m}{\sigma}\right)$$

and using the Laplace's function table this probability can be determined. We consider now a division $\Delta' = (\alpha = \Delta'_0, \Delta'_1, \ldots, \Delta'_n = \beta)$. It follow that $P(\alpha \le X < \beta) = p_1 + p_2 + \cdots + p_n$ where we note $p_i = P(\Delta_{i-1} \le X < \Delta_i)$, $i = 1, 2, \ldots, n$.

On this base for the chosen division Δ we determine the coefficients:

$$K_i = \frac{\Phi\left(\frac{\Delta_i - m}{\sigma}\right) - \Phi\left(\frac{\Delta_{i-1} - m}{\sigma}\right)}{\Delta_i - \Delta_{i-1}}, \quad i = 1, 2, \ldots, 12, \tag{6}$$

$$K_0 = \frac{\Phi\left(\frac{\Delta_0 - m}{\sigma}\right) - \Phi\left(\frac{-\infty - m}{\sigma}\right)}{\Delta_0 - (-\infty)} = 0,$$

$$K_{13} = \frac{\Phi\left(\frac{\infty - m}{\sigma}\right) - \Phi\left(\frac{\Delta_{12} - m}{\sigma}\right)}{\infty - \Delta_{12}} = 0.$$

We agree to say that I_s is a significant interval for Laplace's function if $K_s > 0,01$, $s = 0, 1, 2, \ldots, 13$.

In table 1 the values $\Phi(I_s)$ and K_s are written for every real value which $t_k(x)$ can take, taking into account the chosen Δ division.

Table 1

$t_k(x)$	$-\infty$	-3	$-2,5$	-2	-1.5	-1	$-0,5$
Δ		Δ_0	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
$\Phi(I_s)$	$0,0014$	$0,005$	$0,015$	$0,045$	$0,09$	$0,15$	$0,19$
K_s	0	$0,01$	$0,03$	$0,09$	$0,18$	$0,30$	$0,38$

$t_k(x)$	0	$0,5$	1	$1,5$	2	$2,5$	3	$+\infty$
Δ	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}	
$\Phi(I_s)$	$0,19$	$0,15$	$0,09$	$0,045$	$0,015$	$0,005$	$0,0014$	
K_s	$0,38$	$0,30$	$0,18$	$0,09$	$0,03$	$0,01$	0	

Let be $[a_k, b_k]$, $k = 1, 2, \ldots, r$ the codomain of the function $t_k(x)$, $x \in K$, $a_k \in I_{s_0^k}$ and $b_k \in I_{s_1^k}$ and we note

$$a_k = v_0^k, \quad \Delta_{s_0^k} = v_1^k, \quad \Delta_{s_0^k+1} = v_2^k, \ldots, \Delta_{s_0^k+l_k-1} = v_{l_k}^k,$$

$$b_k = v^k_{l_k+1}, \quad l_k = s^k_1 - s^k_0, \quad s^k_0, x^k_1 \in \{0, 1, 2, \dots 13\}, \quad s^k_0 < s^k_1.$$

Let be x^υ the efficient solution of problem (5); this belongs to a certain I_s, interval $s = 0, 1, 2, \dots, 13$, which has a certain length and whose a $\Psi(I_s)$ value correspond, therefore we consider :

- E^k_0 is the event that the solution to belong to an interval of length $v^k_1 - a_k$,
- E_1 is the event that the solution should be on an interval of length $v^k_2 - v^k_1$
- E_2 is the event that the solution should be on an interval of length $v^k_3 - v^k_2$

$\vdots$

- $E^k_{l_k}$ is the event that the solution should be on an interval of length $b_k - v^k_{l_k}$.

The probabilities of those events are:

$$P(E^k_0) = \frac{v^k_1 - a_k}{b_k - a_k}, \quad P(E^k_{l_k}) = \frac{b_k - v^k_{l_k}}{b_k - a_k}. \tag{7}$$

$$P(E^k_i) = \frac{v^k_{i+1} - v^k_i}{b_k - a_k}, \quad i = 1, 2, \dots l_k - 1.$$

We consider :

- F^k_0 is the event that the interval $[t_k(x), v^k_1)$ should be included in $I^k_{s_0}$
- F^k_1 is the event that the interval $[t_k(x), v^k_2)$ should be included in $I_{s^k_0+1}$
- F^k_2 is the event that the interval $[t_k(x), v^k_3)$ should be included in $I_{s^k_0+2}$

$\vdots$

- $F^k_{l_k-1}$ is the event that the interval $[t_k(x), v^k_{l_k})$ should be included in $I_{s^k_0+l_k-1}$
- $F^k_{l_k}$ is the event that the interval $[t_k(x), b_k)$ should be included in $I_{s^k_1}$.

Taking into account the relation (6), probabilities of these events are:

$$(v^k_1 - t_k(x))K_{s^k_0}, (v^k_2 - t_k(x))K_{s^k_0+1}, \dots, (v^k_{l_k+1} - t_k(x))K_{s^k_0+l_k} \tag{8}$$

We consider the incompatible events:

A^k_0 is the event that x^υ should belong the interval $[t_k(x), v^k_1)$,

A^k_1 is the event that x^0 should belong the interval $[t_k, v^k_2)$,

$\vdots$

$A^k_{l_k}$ is the event that x^υ should belong to the interval $[t_k(x), b_k)$.

Hence: $A_0^k = E_0^k \cap F_0^k$, $A_1^k = E_1^k \cap F_1^k, \ldots, A_{l_k}^k = E_{l_k}^k \cap F_{l_k}^k$, $k = 1, 2, \ldots, r$, $l_k = s_1^k - s_0^k$.

For this event the probabilities will be:

$$P(A_{m-1}^k) = P(E_{m-1}^k)P(F_{m-1}^k) = (v_m^k - t_k(x))K_{s_0^k+m-1}\frac{v_m^k - v_{m-1}^k}{b_k - a_k}, \tag{9}$$

$m = 1, 2, \ldots, l_k$.

We introduce the compatible events:

B_0 is the event that x^0 should belong to the interval $[v_1^k, b_k)$,

B_1 is the event that x^0 should belong to the interval $[v_2^k, b_k)$,

⋮

$B_{l_k-1}^k$ is the event that x^0 should belong to the interval $[v_{l_k}^k, b_k)$.

For this events, it implies that:

$$P(B_0^k) = \Phi(b_k) - \Phi(v_1^k) = \Phi(I_{s_0^k+1}) + \Phi(I_{s_0^k+2}) + \cdots + \Phi(I_{s_0^k+l_k}) \tag{10}$$

$$P(B_1^k) = \Phi(b_k) - \Phi(v_2^k) = \Phi(I_{s_0^k+2}) + \cdots + \Phi(I_{s_0^k+l_k})$$

⋮

$$P(B_{l_k-1}^k) = \Phi(b_k) - \Phi(v_{l_k}^k) = \Phi(I_{s_0^k+l_k}),$$

where $\Phi(I_{s_0^k+l_k}) = \Phi(b_k) - \Phi(v_{l_k}^k)$.

We note $V_0^k = (A_0^k \cup B_0^k)$, $V_1^k = (A_1^k \cup B_1^k), \ldots, V_{l_k-1}^k = (A_{l_k-1}^k \cup B_{l_k-1}^k)$, $V_{l_k}^k = (A_{l_k}^k)$ using Poincare's formula, after reordering of the terms, results:

$$\Phi(b_k) - \Phi(t_k(x)) = (V_0^k \cup V_1^k \cup \cdots \cup V_{l_k}^k) = \tag{11}$$

$$= P(A_0^k \cup B_0^k \cup A_1^k \cup B_1^k \cup \cdots \cup A_{l_k-1}^k \cup B_{l_k-1}^k \cup A_{l_k}^k) =$$

$$= P(A_0^k \cup A_1^k \cup \cdots \cup A_{l_k}^k \cup B_0^k \cup B_1^k \cup \cdots \cup B_{l_k-1}^k) =$$

$$= P(A^k \cup B^k) = P(A^k) + P(B^k) - P(A^k)P(B^k)$$

where we noted: $A^k = A_0^k \cup A_1^k \cup \cdots \cup A_{l_k}^k$, $B^k = B_0^k \cup B_1^k \cup \cdots \cup B_{l_k-1}^k$, A^k and B^k being compatible events.

Knowing that $A_0^k, A_1^k, \ldots, A_{l_k}^k$ are incompatible events, and $B_0^k, B_1^k, \ldots, B_{l_k-1}^k$ are compatible, the relation (11) becomes:

$$\Phi(b_k) - \Phi(t_k(x)) = P(A_0^k) + P(A_1^k) + \cdots + P(A_{l_k}^k) + \tag{12}$$

$$+P(B_0^k) + P(B_1^k) + \cdots + P(B_{l_k-1}^k) - \sum_{\substack{i,j=0\\ i\neq j}}^{l_k-1} P(B_i^k \cap B_j^k) -$$

$$- \sum_{\substack{i,j,p=0\\ i\neq j\neq p}}^{l_k-1} P(B_i^k \cap B_j^k \cap B_p^k) - \cdots - (-1)^{l_k-1} P\left(\bigcap_{q=0}^{l_k-1} B_q^k\right) - P(A^k)P(B^k) =$$

$$= \sum_{m=1}^{l_k} (v_m^k - t_k(x)) K_{s_0^k+m-1} \frac{v_m^k - v_{m-1}^k}{b_k - a_k} + \sum_{u=0}^{l_k-1} \sum_{q=u}^{l_k-1} \Phi(I_{s_0^k+q+1}) -$$

$$- \sum_{\substack{i,j=0\\ i\neq j}}^{l_k-1} P(B_i^k \cap B_j^k) - \sum_{\substack{i,j,p=0\\ i\neq j\neq p}}^{l_k-1} P(B_i^k \cap B_j^k \cap B_p^k) - \cdots -$$

$$-(-1)^{l_k-1} P\left(\bigcap_{q=0}^{l_k-1} B_q^k\right) - P(A^k)P(B^k)$$

Replacing in this formula the terms with their corresponding values, given by (10),(11) we calculate the values of the $\Phi(b_k) - \Phi(t_k(x))$, $k = 1, 2, \ldots, r$.

Considering the relation (5) and taking into account the things presented previously, we proved:

Theorem 2. *Problem (1) is equivalent with the following linear programming problem:*

$$F^* = \min_{x\in K} \sum_{k=1}^{r} \left[\sum_{m=1}^{l_k} (v_m^k - t_k(x)) K_{s_0^k+m-1} \frac{v_m^k - v_{m-1}^k}{b_k - a_k} + \right. \tag{13}$$

$$+ \sum_{u=0}^{l_k-1} \sum_{q=u}^{l_k-1} \Phi(I_{s_0^k+q+1}) - \sum_{\substack{i,j=0\\ i\neq j}}^{l_k-1} P(B_i^k \cap B_j^k) - \sum_{\substack{i,j,p=0\\ i\neq j\neq p}}^{l_k-1} P(B_i^k \cap B_j^k \cap B_p^k) - \cdots -$$

$$\left. -(-1)^{l_k-1} P\left(\bigcap_{q=0}^{l_k-1} B_q^k\right) - P(A^k)P(B^k) \right]$$

In this form the synthesis function is calculable and we obtain after replacing the values, a linear programming problem whose solution is efficient solution for (1), and for the x^* point we can calculate the maximum of each objective function.

3. Application

In a factory two types of products P_1 and P_2 are manufactured and in their composition the raw materials M_1, M_2, M_3 and M_4 are included in the amounts indicated in table 1. The maximum amounts that can be assured for the raw materials M_1, M_2 and M_3 are random variables normally distributed: $\xi(12,2)$ for M_1, $\xi_{21}(7,2)$ for M_2 and $\xi_{22}(18,3)$ for M_3 are quantities which depend on the possibilities of delivery of the provider. For a good tide of production it is required that the probabilities that the necessary amount of raw materials M_1, M_2 and M_3 to be lower or equal than the delivered quantities, to be at least 0,2 , 0,3 and 0,6. The necessary quantity of raw material M_4 should be a maximum of 18 units.

We have to establish the quantity of each product P_1 and P_2 so that the probability of currency benefit to be at least 75 units which to be maximum, and the probability that total benefit should be least 65 units which also to be maximum, the values of these benefits for a unit of product are to be found in the table 2 and knowing that, the penalties are follows:

- the currency benefit decreases with one currency unit for each undelivered in time unit from the raw material M_2 which composes P_1 product, and two currency unit for raw material M_2 and P_2 product;

- the total benefit decreases with two units for every undelivered time unit from raw material M_1 which composes P_1 product, and one unit for raw material M_1 and P_2 product.

Table 2

		Raw materials				Currency	Total
		M_1	M_2	M_3	M_4	benefit	benefit
Products	P_1	3	1	3	1	9	7
	P_2	1	1	2	3	9	9

- Solving -

The problem has the next form:

$$\text{Vmax } f_1(x,\xi_2) = P(9x_1 + 9x_2 - q_1^T y_2 \geq 75)$$

$$\text{Vmax } P_2(x,\xi_2) = P(7x_1 + 9x_2 - q_2^T y_2 \geq 65)$$

subject to:

$$\begin{cases} P(3x_1 + x_2 \leq \xi_1) \geq 0,2 \\ P\left[\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix} = \right. \\ \left. = \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix}\right] \geq \begin{pmatrix} 0,3 \\ 0,6 \end{pmatrix} \\ x_1 + 3x_2 \leq 18 \\ x \geq 0,\ y \geq 0 \end{cases} \tag{14}$$

We determine the $K = K_1 \cap K_2 \cap K_3$ set.

$$K_2 = \{x \in \mathbb{R}^n \mid P(x_1 + x_2 + y_{21} = \xi_{21}) \geq 0,3,$$

$$P(3x_1 + 2x_2 + y_{22} = \xi_{22}) \geq 0,6,\ y_{21} \geq 0,\ y_{22} \geq 0\} =$$

$$= \{x \in \mathbb{R}^n \mid P(\xi_{21} < x_1 + x_2) \leq 0,7,\ P(\xi_{22} < 3x_1 + 2x_2) \leq 0,4\} =$$

$$= \{x \in \mathbb{R}^n \mid F(x_1 + x_2) \leq 0,7,\ F(3x_1 + 2x_2) \leq 0,4\} =$$

$$= \{x \in \mathbb{R}^n \mid x_1 + x_2 \leq F^{-1}(0,7),\ 3x_1 + 2x_2 \leq F^{-1}(0,4)\} =$$

$$= \{x \in \mathbb{R}^n \mid x_1 + x_2 \leq 0,5 + 7,\ 3x_1 + 2x_2 \leq -0,3 + 18\} =$$

$$= \{x \in \mathbb{R}^n \mid x_1 + x_2 \leq 7,5,\ 3x_1 + 2x_2 \leq 17,7\}.$$

$$K_1 = \{x \in \mathbb{R}^n \mid P(3x_1 + x_2 \leq \xi_1) \geq 0,2,\ x \in K_2,\ x \geq 0\} =$$

$$= \{x \in \mathbb{R}^n \mid P(\xi_1 < 3x_1 + x_2) \leq 0,8,\ x \in K_2,\ x \geq 0\} =$$

$$= \{x \in \mathbb{R}^n \mid F(3x_1 + x_2) \leq 0,8,\ x \in K_2,\ x \geq 0\} =$$

$$= \{x \in \mathbb{R}^n \mid 3x_1 + x_2 \leq F^{-1}(0,8),\ x \in K_2,\ x \geq 0\} =$$

$$= \{x \in \mathbb{R}^n \mid 3x_1 + x_2 \leq 12,8,\ x \in K_2,\ x \geq 0\}$$

$$K_3 = \{x \in \mathbb{R}^n \mid x_1 + 3x_2 \leq 18\}$$

K is obtained solving the system:

$$\begin{cases} x_1 + x_2 \leq 7,5 \\ 3x_1 + 2x_2 \leq 17,7 \\ 3x_1 + x_2 \leq 12,8 \\ x_1 + 3x_2 \leq 18 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

We determine the deterministic equivalent of the two objective functions and let the first of them be:

$$\text{Vmax } f_1(x, \xi_2) = P(9x_1 + 9x_2 - q_1^T y_2 \geq 75); \tag{15}$$

$$q_1^T y_2 = (q_{11}\ q_{12}) \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix} = (1\ 2) \begin{pmatrix} \xi_{21} - x_1 - x_2 \\ \xi_{22} - 3x_1 - 2x_2 \end{pmatrix} =$$

$$-\xi_{21} - x_1 - x_2 + 2\xi_{22} - 6x_1 - 4x_2 = -7x_1 - 5x_2 + \xi_{21} + 2\xi_{22}.$$

We note $\eta_1 = \eta_{11} + \eta_{12}$ where $\eta_{11} = \xi_{21}$, $\eta_{12} = 2\xi_{22}$.

We calculate the parameters of this random variable:

$$\overline{\eta}_1 = M(\eta_{11} + \eta_{12}) = M(\xi_{21}) + M(2\xi_{22}) = \overline{\xi}_{21} + 2\overline{\xi}_{22} = 7 + 2 \cdot 18 = 43$$

$$\sigma_{\eta_1} = \sqrt{D(\eta_1)} = \sqrt{D(\eta_{11}) + D(\eta_{12})} = \sqrt{q_{11}^2 D(\xi_{21}) + q_{12}^2 D(\xi_{22})} =$$

$$= \sqrt{q_{11}^2 \sigma_{\xi_{21}}^2 + q_{12}^2 \sigma_{\xi_{22}}^2} = \sqrt{1 \cdot 4 + 4 \cdot 9} = \sqrt{40} = 6,325$$

We replace in (15) the determined values and we obtain:

$$\underset{x \in K}{\text{Vmax}} f_1(x, \xi_2) = P[9x_1 + 9x_2 - (-7x_1 - 5x_2 + \xi_{21} + 2\xi_{22}) \geq 75] =$$

$$= P(16x_1 + 14x_2 - \xi_{21} - 2\xi_{22} \geq 75) = P(16x_1 + 14x_2 - \eta_1 \geq 75) =$$

$$= P(16x_1 + 14x_2 - 75 \geq \eta_1) = P\left(\frac{\eta_1 - \overline{\eta}_1}{\sigma_{\eta_1}} \leq \frac{16x_1 + 14x_2 - 75 - \overline{\eta}_1}{\sigma_{\eta_1}}\right) =$$

$$= \left[\Phi\left(\frac{16x_1 + 14x_2 - 118}{6,326}\right) + 0,5\right]$$

Maximum of this function is obtained, according to theorem 1 for problem's solution.

$$\max_{x\in K}(16x_1 + 14x_2) \tag{16}$$

Solving (16) is obtained $\max_{x\in K}(16x_1 + 14x_2) = 110,3$ for $x_1 = 2,65$ and $x_2 = 4,85$ and it results:

$$\underset{x\in K}{\mathrm{Vmax}} f_1(x,\xi_2) = \Phi\left(\frac{110,3-118}{6,325}\right) + 0,5 =$$
$$= \Phi(-1,217) + 0,5 = -0,388 + 0,5 = 0,112.$$

Calculating the minimal and maximal value of the term $t_1(x) = \dfrac{16x_1 + 14x_2 - 118}{6,326}$ we obtain $a_1 = -18,6$, $b_1 = -1,217$. a_1 value was obtained in $x_1 = x_2 = 0$.

We determine the deterministic equivalent of the second objective function:

$$\underset{x\in K}{\mathrm{Vmax}} f_2(x,\xi_2) = P(7x_1 + 9x_2 - q_2^T y_2 \geq 65), \tag{17}$$

where

$$q_2^T y_2 = (q_{21}\ q_{22})\begin{pmatrix} y_{21} \\ y_{22}\end{pmatrix} = (2\ 1)\begin{pmatrix} \xi_{21} - x_1 - x_2 \\ \xi_{22} - 3x_1 - 2x_2\end{pmatrix} =$$
$$= 2\xi_{21} - 2x_1 - 2x_2 + \xi_{22} - 3x_1 - 2x_2 = -5x_1 - 4x_2 + 2\xi_{21} + \xi_{22}$$

Replacing in (17) we obtain:

$$\underset{x\in K}{\mathrm{Vmax}} f_2(x,\xi_2) = P(12x_1 + 13x_2 - 2\xi_{21} - \xi_{22} \geq 65) = \tag{18}$$
$$= P(12x_1 + 13x_2 - \eta_2 \geq 65) = P(\eta_2 \leq 12x_1 + 13x_2 - 65)$$

Using the same relations as for the first objective function we obtain: $\overline{\eta}_2 = 32$, $\sigma_{\eta_2} = 5$, and (18) becomes

$$\underset{x\in K}{\mathrm{Vmax}} f_2(x,\xi_2) = P\left(\frac{\eta_2 - \overline{\eta}_2}{\sigma_{\eta_2}} < \frac{12x_1 + 13x_2 - 65 - \overline{\eta}_2}{\sigma_{\eta_2}}\right) =$$
$$= \left[\Phi\left(\frac{12x_1 + 13x_2 - 97}{5}\right) + 0,5\right]$$

whose maximum is obtained for the problem's solution:

$$\max_{x\in K}(12x_1 + 13x_2) \tag{19}$$

Solving (19) we obtain $\max_{x\in K}(12x_1+13x_2) = 95,25$ for $x_1 = 2,25$ and $x_2 = 5,25$ and it results that:

$$\operatorname{Vmax}_{x\in K} f_2(x,\xi_2) = \Phi\left(\frac{95,25-97}{5}\right) + 0,5 = \Phi(-0,35) + 0,5 =$$

$$= -0,137 + 0,5 = 0,363$$

having $a_2 = -19,4$, $b_2 = -0,35$.

From (7)-(12) relations and theorem 2, we calculate the efficient solution of (14):

For f_1 function in according with (11) results

$$\Phi(b_1) - \Phi(t_1(x)) = P(A^1) + P(B^1) - P(A^1)P(B^1)$$

where:

$$P(A^1) = \left(-3 - \frac{16x_1+14x_2-118}{6,325}\right)\frac{15,6}{17,383}\cdot 0+$$

$$+\left(-2,5 - \frac{16x_1+14x_2-118}{6,325}\right)\frac{0,5}{17,383}\cdot 0,01+$$

$$+\left(-2 - \frac{16x_1+14x_2-118}{6,325}\right)\frac{0,5}{17,383}\cdot 0,03+$$

$$+\left(-1,5 - \frac{16x_1+14x_2-118}{6,325}\right)\frac{0,5}{17,383}\cdot 0,09+$$

$$+\left(-1,217 - \frac{16x_1+14x_2-118}{6,325}\right)\frac{0,283}{17,383}\cdot 0,18$$

$$P(B^1) = (0,06+0,19+0,11) + (0,19+0,11) + 0,11-$$

$$-(0,19+0,11) - 0,11 - 0,11 - 0,11 = 0,14$$

and we obtain:

$$\Phi(b_1) - \Phi(t_1(x)) = P(A^1) + P(B^1) - P(A^1)P(B^1) =$$

$$= -\frac{16x_1+14x_2-118}{6,325}\cdot 0,048 + 0,2807$$

For f_2 it follows that:

$$P(A^2) = \left(-3 - \frac{12x_1+13x_2-97}{5}\right)\frac{16,4}{19,05}\cdot 0+$$

$$+\left(-2,5 - \frac{12x_1+13x_2-97}{5}\right)\frac{0,5}{19,05}\cdot 0,01+$$

$$+\left(-2-\frac{12x_1+13x_2-97}{5}\right)\frac{0,5}{19,05}\cdot 0,03+$$

$$+\left(-1,5-\frac{12x_1+13x_2-97}{5}\right)\frac{0,5}{19,05}\cdot 0,09+$$

$$+\left(-1-\frac{12x_1+13x_2-97}{5}\right)\frac{0,5}{19,05}\cdot 0,18+$$

$$+\left(-0,5-\frac{12x_1+13x_2-97}{5}\right)\frac{0,5}{19,05}\cdot 0,3+$$

$$+\left(-0,35-\frac{12x_1+13x_2-97}{5}\right)\frac{0,15}{19,05}\cdot 0,38=$$

$$=-\frac{12x_1+13x_2-97}{5}\cdot 0,0191-0,0141$$

$$P(B^2)=0,1428$$

$$\Phi(b_2)-\Phi(t_2(x))=-\frac{12x_1+13x_2-97}{5}\cdot 0,016+0,131$$

The efficient solution is given by the linear programming problem's solution:

$$F^*=\min_{x\in K}[\Phi(b_1)-\Phi(t_1(x))+\Phi(b_2)-\Phi(t_2(x))]=$$

$$=\min_{x\in K}\left[-\frac{16x_1+14x_2-118}{6,325}\cdot 0,048+0,2807-\right.$$

$$\left.-\frac{12x_1+13x_2-97}{5}\cdot 0,016+0,131\right]=$$

$$=\min_{x\in K}(-1,9812x_1-1,7415x_2+14,9191)$$

Solving the problem, we obtain the solution $x=(2,65;4,85)$.

We determine for the two objective functions the maximum values which are obtained in the point $x=(2,65;4,85)$:

$$f_1(x)=\Phi(-1,217)+0,5=-0,388+0,5=0,112$$

$$f_2(x)=\Phi\left(\frac{12\cdot 2,65+13\cdot 4,85-97}{5}\right)+0,5=\Phi(0,43)+0,5=0,667$$

References

[1] Bereanu B., *Soluţii de risc minim în programarea liniară*, Analele Univ. Bucureşti, matematică-mecanică, 13(1), (1964), 121-140.
[2] Ciucu D., Craiu V., Ştefanescu M.V., Ştefanescu A., *Statistică matematică şi cercetări operaţionale*, Editura didactică şi pedagogică, 1982.
[3] Charnes A., Cooper W.W., *Deterministic equivalents for optimizing and satisfying under chance constraints*, Operations Res., 11, (1963), 18-39.
[4] Charnes A., Kirby M., *Some special P-model in chance-constrained programming*, Management Science 14(1966), 183-195.
[5] Everitt R., Ziemba W.T., *Two-period stochastic programs with simple recourse*, Oper Res. 27(1979), 485-502.
[6] Frauendorfer K., *Stochastic Two-Stage Programming*, vol.392(1992) of Lecture Notes in Econ. Math. Syst. Springer Verlag, Berlin.
[7] Geoffrion A.M., *Proper efficiency and the theory of vector maximization*, J.M.A.A. 22, 3(1968), 618-630.
[8] Kall P., Wallace S.W., *Stochastic programming*, John Wiley & Sons, 1994.
[9] Kataoka S., *A stochastic programming model*, Econometrica 31(1963), 181-196.
[10] Prekopa A., *Stochastic programming*, Akademiai Kiado, Budapest, 1995.
[11] Stancu-Minasian I.M., *Programarea stochastică cu mai multe funcţii obiectiv*, Editura Academiei R.S.R., Bucureşti 1980.
[12] Stancu-Minasian I.M., Ţigan S., *The minimum-risk approach to special problems of mathematical programming. The distribution function of the optimal value*, Rev. Anal. Numer. Theor. Approx., 13, 2(1984), 175-187.
[13] Wets R.J.B., *Programming under uncertainty the equivalent convex program*, Siam J. Appl. Math. 14, No.1(1966), 89-105.
[14] Wets R.J.B., *Stochastic programming with fixed recourse: the equivalent deterministic program*, SIAM Review 16, No.3(1974), 309-339.

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ON DIFFERENTIABLE FUNCTIONS PRESERVING RATIONALITY AND IRRATIONALITY

ALEXANDRU HORVÁTH

Dedicated to Professor Vasile Pop at his 60^{th} anniversary

Abstract. In this paper we construct nonlinear differentiable functions f, with the property $f(\mathbf{Q}) \subset \mathbf{Q}$ and $f(\mathbf{R} \setminus \mathbf{Q}) \subset \mathbf{R} \setminus \mathbf{Q}$. The construction is based on some convergence properties of the sequences of convex functions and on a criterion of irrationality for numbers expressed as series of rational numbers. The main theorem of the paper states that the set of functions with this property is dense in $(C(\mathbf{R}), \|\cdot\|)$.

Assume that a and b are rational numbers and $a \neq 0$. Then obviously, the value of the linear function $f(x) = ax + b$ is rational if and only if x is rational. Let us say a function $f : \mathbf{R} \to \mathbf{R}$ has the property of preserving rationality if

$$f(\mathbf{Q}) \subset \mathbf{Q} \quad \text{and} \quad f(\mathbf{R} \setminus \mathbf{Q}) \subset \mathbf{R} \setminus \mathbf{Q}. \tag{1}$$

It is natural to investigate the existence of a differentiable functions with this property, which is not linear. The question in fact is: how rich the set of functions with this property is?

Just the linearity is not a consequence of the differentiability and the property (1), as is shown by the function $f : \mathbf{R} \to \mathbf{R}$ given by:

$$f(x) = \begin{cases} 1/x & \text{if } x \geq 1 \\ 2 - x & \text{if } x < 1 \end{cases}. \tag{2}$$

However, the linear and the above function as well are both monotone. In addition, the derived function of each of them is nowhere vanishing. This kind of constructions seams to lead us always to monotone functions.

The goal of this paper is to give the answer, how rich the class of differentiable functions with the property (1) is. We prove the following

Theorem 1. *The set of differentiable functions f, with the property that $f(x)$ is rational if and only if x is rational, is dense in the space of continuous functions, with respect to the uniform convergence topology.*

In the body of the paper all functions are defined on $\mathbf{R}$ and take their values in $\mathbf{R}$.

After some preliminary facts and results, we recall a criterion of irrationality (Lemma 3), which is followed by the main result and its proof.

We start with a well known property of the convex functions:

Lemma 1. *Let f be a convex function. Then the left derivative $f'(x-0)$ is left continuous and the right derivative $f'(x+0)$ is right continuous.*

For the proof see, for example, [2].

Lemma 2. *Let $\{f_n\}_n$ be a sequence of left (right) continuous functions. If $\{f_n\}_n$ converges uniformly the sequence $\{f_n\}_n$ is left (right) equicontinuous.*

The proof is similar to that of Theorem 7.24 [4], which discusses the case of continuous functions.

Proposition 1. *Let $\{f_n\}_n$ be a sequence of convex functions which is pointwise convergent to f. Assume that $\{f_n(x-0)\}_n$ and $\{f_n(x+0)\}_n$ converges uniformly to $l_-(x)$ and $l_+(x)$ respectively. Then $l_-(x) \leq f'(x-0)$ and $f'(x+0) \leq l_+(x)$.*

In particular, if $\lim_{n\to\infty}(f_n'(x+0) - f_n'(x-0)) = 0$, pointwise for any x, then f is a differentiable function, and its derivative f' is the common limit of the sequences of the left and right derivatives of the functions f_n.

Proof. Set an arbitrary $\epsilon > 0$. Then there exists an integer n_ϵ such that for any x and any $n \geq n_\epsilon$ we have

$$l_(x) - \epsilon \leq f_n'(x-0).$$

Fix a point x. Applied for the sequence $\{f_n'(x-0)\}_n$, Lemma 1 and 2 provide an $\eta > 0$ with the following property: for any $0 < h < \eta$ and any n one has

$$f_n'(x-0) \le f_n'(x-h-0) + \epsilon.$$

On the other hand

$$f_n'(x-h-0) + \epsilon \le f_n'(x-h+0) + \epsilon \le \frac{f_n(x) - f_n(x-h)}{h} + \epsilon.$$

If we let $n \to \infty$, then the above inequalities give:

$$l_-(x) - \epsilon \le \frac{f(x) - f(x-h)}{h} + \epsilon.$$

Finally, we have

$$l_-(x) - \epsilon \le f'(x-0) + \epsilon \quad \text{as} \quad h \to 0.$$

Since ϵ was arbitrary, the first inequality follows. By an analogous way we obtain the second one. The last assertion is now obvious.

Notice, that the above proposition provides differentiable functions as the limit of a sequence of non-differentiable functions. This is one of the key points of the paper. The second one is the use of the following irrationality criterion. For its proof see [5].

Lemma 3 (Criterion of irrationality). *Let $\{p_n\}_n$ and $\{q_n\}_n$ be sequences of integer numbers and $u_n = \sum_{k=1}^{n} \frac{p_k}{q_k}$. Assume that the following conditions are satisfied:*

(1) $q_n \ge 1$ *for any n, and $p_n \ne 0$ for infinitely many n,*

(2) *there exists a sequence of real numbers $\{a_n\}_n$, such that $|u_m - u_n| \le a_n$ for any n and $m > n$, and*

(3) $\lim_{n\to\infty} a_n[q_1, ..., q_n] = 0$, *where $[q_1, ..., q_n]$ denotes the least common multiple of the numbers $q_1, ..., q_n$.*

Then the series $\sum_{k=1}^{\infty} \frac{p_k}{q_k}$ converges, and its value is an irrational number.

Now we start to construct a sequence of convex, piecewise linear functions. Choose two non-zero numbers $m_1 < m_2$ and a point in the plain with coordinates (x_0, y_0). Define the function f_0 as follows:

$$f_0(x) = \begin{cases} y_0 + m_1(x - x_0), & \text{for} \quad x \leq x_0 \\ y_0 + m_2(x - x_0), & \text{for} \quad x > x_0. \end{cases}$$

This function is obviously convex and piecewise linear. Let $\{\epsilon_n\}_n$ be a sequence of positive real numbers, such that $\epsilon_{n+1} < \frac{1}{2}\epsilon_n$ for any integer $n \geq 1$. Consider the points

$$M_1^1(x_0 - \epsilon_1, y_0 - \epsilon_1 m_1) \text{ and } M_2^1(x_0 + \epsilon_1, y_0 + \epsilon_1 m_2)$$

on the graph of the function f_0. Define the function f_1, in such a way that its restriction to $(-\infty, x_0 - \epsilon_1] \cup [x_0 + \epsilon_1, \infty)$ is equal to the restriction of f_0 to this subset. On the other hand, the graph of its restriction to $[x_0 - \epsilon_1, x_0 + \epsilon_1]$ is the segment $M_1^1 M_2^1$. The graph of the function f_1 has two angular points (i.e. f_1 is not differentiable at $x_0 - \epsilon_1$ and $x_0 + \epsilon_1$).

Repeat the above construction for both of these angular points, using now ϵ_2 instead of ϵ_1. The new function f_2 has four angular points on its graph, namely

$$M_1^2(x_0 - \epsilon_1 - \epsilon_2, y_0 - \epsilon_1 m_1 - \epsilon_2 m_1),$$

$$M_2^2(x_0 - \epsilon_1 + \epsilon_2, y_0 - \epsilon_1 m_1 + \epsilon_2 \frac{m_1 + m_2}{2}),$$

$$M_3^2(x_0 + \epsilon_1 - \epsilon_2, y_0 + \epsilon_1 m_2 - \epsilon_2 \frac{m_1 + m_2}{2}),$$

$$M_4^2(x_0 + \epsilon_1 + \epsilon_2, y_0 + \epsilon_1 m_2 + \epsilon_2 m_2),$$

and it still remains convex and piecewise linear. Iterating the above step, we obtain for every n, a convex piecewise linear function f_n, which has on its graph 2^n angular points $M_i^n, i = 1, ..., 2^n$. The coordinates of these points have the following form:

$$x = x_0 + s_1\epsilon_1 + s_2\epsilon_2 + \ldots + s_n\epsilon_n, \tag{3}$$

$$y = y_0 + s_1\epsilon_1\mu_1 + s_2\epsilon_2\mu_2 + \ldots + s_n\epsilon_n\mu_n, \tag{4}$$

where $s_k = \pm 1$ and μ_k is a certain combination of m_1 and m_2 for any $k = 1, ..., n$. Notice that they depend on n and i as well.

Now we are able to formulate the main result of the paper:

Proposition 2. *The sequence $\{f_n\}_n$ constructed above converges uniformly to a function f. The limit f is a nonlinear differentiable function. Moreover, $f(x)$ is rational if and only if x is rational, provided that the parameters m_1, m_2, x_0, y_0 and ϵ_n are properly chosen in the construction.*

Proof. By the above construction $\{f_n\}_n$ is an increasing sequence of continuous, convex functions. Define a continuous function g as follows: its restriction to $X = (-\infty, x_0 - 2\epsilon_1] \cup [x_0 + 2\epsilon_1, \infty)$ is equal to the restriction of f_0 to this set, and the graph of its restriction to $[x_0 - 2\epsilon_1, x_0 + 2\epsilon_1]$ is a segment. Then $f_n \leq g$ for any n. In particular the sequence $\{f_n\}_n$ is upper bounded, hence pointwise convergent, too. Actually, it converges uniformly because $f_n = f_0$ on X for any n and the functions are convex (or directly because of the formula (5)).

Therefore the limit function f is convex and continuous. Since $f_0 \leq f \leq g$, it is also nonlinear.

Now we will prove that f is differentiable. It is a matter of simple computation to prove by induction the following formulas, to make (1) and (2) more precise:

$$s_k = (-1)^{[(i-1)/2^{n-k}]}$$

$$\mu_k = \frac{[2^{k-1} - \frac{1}{2}[(i-1)/2^{n-k}]]m_1 + (2^{k-1} - [2^{k-1} - \frac{1}{2}[(i-1)/2^{n-k}]])m_2}{2^{k-1}}.$$

Here $[x]$ denotes the integer part of x. Moreover, it is not difficult to see that the slopes of the line segments of the function f_n are

$$m_{i,n} = m_1 + \frac{i}{2^n}(m_2 - m_1), \quad \text{where} \quad i = 0, 1, ..., 2^n \tag{5}$$

in order, so that $m_{0,n} = m_1$ and $m_{2^n,n} = m_2$.

Consider now the functions $f'_n(x-0)$ and $f'_n(x+0)$. For arbitrary x and n one has

$$|f'_{n+1}(x-0) - f'_n(x-0)| \leq \frac{m_2 - m_1}{2^{n+1}},$$

hence for any $p > 0$

$$|f'_{n+p}(x-0) - f'_n(x-0)| \leq \frac{m_2 - m_1}{2^n},$$

which means the uniform convergence of $\{f'_n(x-0)\}_n$. Obviously this is valid for $\{f'_n(x+0)\}_n$, too. Notice that

$$|f'_n(x+0) - f'_n(x-0)| \leq \frac{m_2 - m_1}{2^n},$$

hence we conclude the differentiability of f, by Proposition 1.

Finally, we have to investigate the behavior of the function f with respect to the rational points on its graph. Assume that $\epsilon_n = 2^{-n^2}$, and the numbers x_0, y_0, m_1, m_2 are rational, such that for any i and n the slope $m_{i,n}$ is non-zero, i.e.

$$(2^n - i)m_1 + im_2 \neq 0, n \in \mathbb{N} \text{ and } i = 0, 1, ..., 2^n \tag{6}$$

Assume that x is not an accumulation point of the set of x-values expressed as in (1). Then the sequence $\{f_n(x)\}_n$ is constant for n sufficiently large, in particular $f(x)$ is rational if and only if x is rational. Now consider the accumulation points of the above set. These points are exactly those, which have the form

$$x = x_0 + \sum_{n=1}^{\infty} s_n \epsilon_n, \quad \text{where} \quad \epsilon_n = \pm 1.$$

But Lemma 3, with the choice $a_n = 2^{[illegible]}$, assures us that all these numbers are irrational. Similarly, $y = f(x)$ has the form

$$y = y_0 + \sum_{n=1}^{\infty} s_n \epsilon_n \mu_n, \quad \text{where} \quad \epsilon_n = \pm 1.$$

Now with the choice $a_n = (|m_1| + |m_2|)2^{-(n^2+2n)}$. Lemma 3 gives the irrationality of y. Hence Proposition 2 is proved.

It remains the proof of the main Theorem, stated in the introductory part of the paper.

Proof (of the main Theorem 1). First, notice that all the above results are valid for concave, instead of convex, functions too (i.e. for $m_1 > m_2$).

Consider an arbitrary continuous function g and fix an arbitrary $\epsilon > 0$. Then g can be approximated by a piecewise linear continuous function f, such that

(a) $|g(x) - f(x)| < \epsilon/2$, for any x,

(b) the angular points of f are all isolated points and have rational coordinates,

(c) the slopes are different, non-zero rational numbers,

(d) the consecutive slopes satisfy the condition (3).

Now in a sufficiently small neighborhood of the angular points repeat our main construction. The (easy) details are left to the reader.

Finally, here it is a conjecture for further investigations. Notice at the first, that our construction allow the derivative of the function vanish. Indeed, if one chooses $x_0 = y_0 = 0$, $m_1 = -2$ and $m_2 = 3$ then the function f will be differentiable with derivative -2 on the left-hand side and derivative 3 on the right-hand side. Because of the Darboux property of derivatives, f' must be 0 somewhere in between.

Conjecture The derivative of any differentiable function with property (1) can only vanish on an irrational point.

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References

[1] Ralph P. Boas, Jr., *A Primer of Real Functions*, Math. Assoc. of America, Rahway, New Jersey, (1960), vol. 13.
[2] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Uniw. Slaski - P.W.N., Warszawa-Krakow-Katowice, (1985).
[3] I. Niven, *Irrational Numbers*, Carus Math. Monographs, Rahway, New Jersey, 11(1956).
[4] W. Rudin, *Principles of Mathematical Analysis*, Muszaki Konyvkiado, Budapest, (1978), Hungarian translation.
[5] J. Sándor, *An Irrational series*, Astra Matematica 5(1990), 10-14.

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THE CONTINUITY OF THE OPTIMAL VALUE OF A LINEAR-FRACTIONAL PROGRAM

DOINA IONAC

Dedicated to Professor Vasile Pop at his 60^{th} anniversary

Abstract. In the paper we give some sufficient conditions for the continuity and upper and lower semi-continuity of the optimal value of a linear-fractional programming problem with respect to the problem coefficients. For this we use some regularity and strictly consistency conditions of the linear-fractional problem as well as some similar continuity results obtained by Cojocaru and Dragomirescu for the general linear programs.

1. Here, we apply some known results for the continuity of the optimal value of the linear program [2], [4], [8] in order to obtain sufficient conditions for the continuity and semi-continuity of the optimal values of a linear-fractional programming problem, verifying some regularity or pseudo-regularity conditions. Another aspects of the optimal value function continuity for linear programs can be found in [4], [5], [8]. In the case of nonlinear optimization problems some topological sufficient conditions for the optimal value continuity and upper and lower semi-continuity are given in [6], [7].

2. Let consider the linear-fractional programming problem:

PF(z). Find

$$\max\left\{ f(x) = \frac{cx + c_0}{dx + d_0} | Ax \leq b,\, x \geq 0 \right\},$$

where $A = (a_{ij})$ is a $m \times n$ dimensional matrix ($m < n$) of rank m with real elements, c and d are real vectors of dimension n, b is a given real vector of dimension m and c_0 and d_0 are given real numbers.

By $z = (A, b, c', d')$ we denote a point in the $(m \times n + m + 2(n+1))$ dimensional Euclidean space of the problem PF(z) coefficients,

$$Z = \{z \in (A, b, c', d') | A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c' \in \mathbb{R}^{n+1}, d' \in \mathbb{R}^{n+1}\},$$

where $c' = (c^T, c_0)^T$, $d' = (d^T, d_0)^T$.

Next, we will designate by f the objective function of the linear fractional problem (PF) and by X the feasible set of the problem PF(z).

Next we suppose on the problem PF(z) that: $dx + d_0 > 0, \forall x \in X$.

Let $F : Z \to \mathbb{R} \cup \{\pm\infty\}$ be the optimal value function of PF(z):

$$F(A, b, c', d') = \begin{cases} \sup\{f(x) \,|\, x \in X\}, & \text{for } X \neq \emptyset \\ -\infty, & \text{for } X = \emptyset \end{cases}$$

and let $Z' = \{z \in Z | -\infty < F(z) < +\infty\}$.

Definition 1. *(i) If the feasible set $X \neq \emptyset$ the problem PF(z) is called consistent.*

(ii) If $X^* = \{x \in \mathbb{R}^n | Ax < b, x > 0\} \neq \emptyset$ the problem PF(z) is called strictly consistent.

Problem PF(z) can be converted by the variable change $y = tx$ into:

PL(z). Find

$$Max\{(cy + c_0 t) | Ay - bt \leq 0, dy + d_0 t = 1, y \geq 0, t \geq 0\}.$$

The objective of linear program PL(z) will be denoted by $g(y, t) = cy + c_0 t$, the feasible set by XL and optimal value function by $S(A, b, c', d')$. We can associate to problem PF(z) a dual problem DL(z), which is the dual problem of the linear program PL(z):

DL(z). Find

$$Min\{w | A^t u + dw \geq c, -bu + d_0 w \geq c_0, u \geq 0, w \in \mathbb{R}\}.$$

Definition 2. *Problem PF(z) is a regular problem if the following conditions hold:*

i). $X \neq \emptyset$,

ii). the objective function f is not constant on X,

iii). there exists $M > 0$ such that $0 < dx + d_0 < M$, for any $x \in X$.

Definition 3. *Problem $PF(z)$ is a pseudo-regular problem if the following conditions hold:*

i). $X \neq \emptyset$,

ii). f is not constant function on X,

iii). $dx + d_0 > 0$, for any $x \in X$.

3. The following theorems will establish a relationship between the problems PF(z) and PL(z) concerning strictly consistency of these problems.

Theorem 1. *Let problem $PF(z)$ be pseudo-regular. Then $PF(z)$ is strictly consistent if and only if $PL(z)$ is strictly consistent.*

Proof. The proof essentially is based on Charnes-Cooper variable change [1], [5]. □

Theorem 2. *If problem $PF(z)$ is regular then the following properties hold:*

(i) If the dual problem $DL(z)$ is strictly consistent then the optimal value function F is upper semicontinuous at z with respect to Z'.

(ii) If the primal problem $PF(z)$ is strictly consistent then the optimal value function F is lower semi-continuous at z with respect to Z'.

(iii) If both problems $PF(z)$ and $DL(z)$ are strictly consistent then the optimal value function F is continuous at z with respect to Z'.

Proof. (i) Since problem DL(z) is strictly consistent, it results, by Theorem 3.3 [2], that the optimal value function of the problem PL(z) is upper semi-continuous at z with respect to Z'. The problem PF(z) is regular, so that by Theorem 2.2.3 [3], it follows that the problems PL(z) and PF(z) have the same optimal value function, that is $F(z') = S(z')$, $\forall z' \in Z'$. Therefore, F is upper semi-continuous at z with respect to Z'.

(ii) Because, problem PF(z) is strictly consistent, it follows, in conformity with Theorem 1, that PL(z) is strictly consistent. Then by Theorem 3.3 [2], it results that the optimal value function S of the problem PL(z) is lower semi-continuous at z with respect to Z'. The problem PF(z) is regular, so that by Theorem 2.2.3 [3], it follows that the problems PL(z) and PF(z) have the same optimal value function. Therefore, F is upper semi-continuous at z with respect to Z'.

(iii) If the problem PF(z) is strictly consistent, in virtue of Theorem 1, it follows that problem PL(z) is strictly consistent too. Since problems PL(z) and DL(z) are strictly consistent, according to Theorem 3.3 [2], it results that the optimal value function S of problem PL(z) is continuous on z with respect to Z'. The problem PF(z) is regular, so that by Theorem 2.2.3 [3], it follows that the problems PL(z) and PF(z) have the same optimal value function. Therefore, F is continuous at z with respect to Z'. □

References

[1] Charnes A., Cooper W. W., Programming with linear fractional functionals, Naval Res. Logist. Quart., 9, 1-2 (1962), 181-186.

[2] Cojocaru I., Dragomirescu M., The continuity of the optimal value of a general linear program, Revue Roumaine de mathematiques pures et appliquees, XXVII, No. 6 (1982), 663-676.

[3] Ionac D., Metode de rezolvare a problemelor de programare matematică, teză de doctorat, Univ. Babeş-Bolyai Cluj-Napoca, 1998.

[4] Martin D. H., On the continuity of the maximum in parametric linear programming, J. Optimization Theory and Appl., 17, 3-4, (1975), 205-210.

[5] Stancu-Minasian I. M., Metode de rezolvare a problemelor de programare fracţionară, Editura Academiei Române, Bucureşti, 1992.

[6] Tigan S., Sur quelques proprietes de stabilite concernant les problemes d'optimisation avec contraintes, Analyse Numer. Theor. Approx., 6, no. 2 (1977), 203-225.

[7] Tigan S., Asupra extinderii unor proprietăţi de stabilitate pentru problemele de optimizare cu restricţii, în vol. "Lucrările Seminarului itinerant de ecuaţii funcţionale, aproximare şi convexitate", Univ. Babeş-Bolyai Cluj-Napoca, 1979, 197-203.

[8] Wiliams A. C., Marginal values in linear programming, SIAM J. Appl. Math. 11, 1 (1963), 82-94.

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FROBENIUS FUNCTORS AND FUNCTOR CATEGORIES

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Abstract. If F is a functor having the functor G both as a left and right adjoint, then one can associate to F a transfer map. The pair (F, G) is called a Frobenius pair. In this paper we discuss of Frobenius functors between functor categories and their transfer maps.

1. Introduction

The functors $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ are said to form a Frobenius pair if G is both a left and a right adjoint of F.

If (F, G) is a Frobenius pair, then one may define (see [3], [2] natural the tranformations

$$\begin{aligned} \mathrm{Tr_F} &: \mathrm{Hom}_{\mathcal{B}}(\mathrm{F}(-), \mathrm{F}(-)) \to \mathrm{Hom}_{\mathcal{A}}(-,-), \\ \mathrm{Tr_G} &: \mathrm{Hom}_{\mathcal{A}}(\mathrm{G}(-), \mathrm{G}(-)) \to \mathrm{Hom}_{\mathcal{B}}(-,-) \end{aligned}$$

which satisfy the usual properties of the trace map Tr_H^G from group representation theory. These transformations we also investigated in [5],

This paper is a sequel of [5], and it is inspired by the work of J.A. Green [4] on functor categories over group algebras. We show that a pair of Frobenius functors between A–mod and B–mod, where A and B are finite dimensional k-algebras, induce Frobenius functors $\mathcal{F}$ and $\mathcal{G}$ between the the categories A-Mmod and B-Mmod of contravariant functors from A–mod, respectively B–mod to k–Mod. Therefore Higman's theorem characterizing relatively $\mathcal{F}$-projective objects holds in this case too.

All our categories and functors will be additive. Rings are associative with unity, and modules are unital and left, unless otherwise specified. Benson's book [1] contains the needed facts on functor categories, and in [6] many of Green's results were generalized to group graded algebras.

2. The transfer map

2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, $F : \mathcal{A} \to \mathcal{B}$ an (additive) functor and $G : \mathcal{B} \to \mathcal{A}$ a right adjoint of F. Denote by

$$\alpha_{-,-} : \mathrm{Hom}_{\mathcal{B}}(\mathrm{F}(-), -) \to \mathrm{Hom}_{\mathcal{A}}(-, \mathrm{G}(-))$$

the adjunction isomorphism, and let

$$\sigma : \mathrm{id}_{\mathcal{A}} \to G \circ F, \quad \sigma_A = \alpha_{A,F(A)}(\mathrm{id}_{F(A)}) : A \to (G \circ F)(A)$$

be the unit, and

$$\rho : F \circ G \to \mathrm{id}_{\mathcal{B}}, \quad \rho_B = \alpha^{-1}_{G(B),B}(\mathrm{id}_{G(B)}) : (F \circ G)(B) \to B$$

the counit of this adjunction, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

2.2. If G is also a left adjoint of F, then F and G are called *Frobenius functors* and (F, G) is a *Frobenius pair*. In this case we also have the adjunction isomorphism

$$\gamma_{-,-} : \mathrm{Hom}_{\mathcal{A}}(\mathrm{G}(-), -) \to \mathrm{Hom}_{\mathcal{B}}(-, \mathrm{F}(-))$$

with unit

$$\xi : 1_{\mathcal{B}} \to F \circ G, \quad \xi_B = \gamma_{B,G(B)}(\mathrm{id}_{G(B)}) : B \to (F \circ G)(B),$$

and counit

$$\tau : G \circ F \to \mathrm{id}_{\mathcal{A}}, \quad \tau_A = \gamma^{-1}_{F(A),A}(\mathrm{id}_{F(A)}) : (G \circ F)(A) \to A$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

2.3. The functors F and G induce natural the transformations

$$\mathrm{Res}_{\mathrm{F}} = \mathrm{Res}_{\mathrm{F}}(-,-) \ : \ \mathrm{Hom}_{\mathcal{A}}(-,-) \to \mathrm{Hom}_{\mathcal{B}}(\mathrm{F}(-), \mathrm{F}(-)), \quad \mathrm{f} \mapsto \mathrm{F}(\mathrm{f})$$

$$\mathrm{Res}_{\mathrm{G}} = \mathrm{Res}_{\mathrm{G}}(-,-) \ : \ \mathrm{Hom}_{\mathcal{B}}(-,-) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{G}(-), \mathrm{G}(-)), \quad \mathrm{g} \mapsto \mathrm{G}(\mathrm{g})$$

for any morphism f in $\mathcal{A}$, and for any morphism g in $\mathcal{B}$.

Since F and G are Frobenius functors, we may define natural transformations in the opposite direction

2.4. The map

$$\mathrm{Tr_F} = \mathrm{Tr_F}(A, A') : \mathrm{Hom}_{\mathcal{B}}(F(A), F(A')) \to \mathrm{Hom}_{\mathcal{A}}(A, A'),$$

$$f \mapsto \mathrm{Tr_F}(f) = \tau_{A'} \circ G(f) \circ \sigma_A$$

is the *transfer* (or trace) map associated to F. Similarly,

$$\mathrm{Tr_G} = \mathrm{Tr_G}(B, B') : \mathrm{Hom}_{\mathcal{A}}(G(B), G(B')) \to \mathrm{Hom}_{\mathcal{B}}(B, B'),$$

$$g \mapsto \mathrm{Tr_G}(g) = \rho_{B'} \circ F(g) \circ \xi_B$$

is the transfer map associated to G. For every object A of $\mathcal{A}$ denote also

$$e_A = e(F)_A = \tau_A \circ \sigma_A \in \mathrm{End}_{\mathcal{A}}(A).$$

We recall from [5, Lemma 1.4 and Proposition 1.7] some properties of the trace map.

2.5. Proposition a) *For all* $f \in \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$,

$$\mathrm{Tr_F}(f) = \tau_{A'} \circ \alpha_{A,F(A)} = \gamma^{-1}_{F(A),A'} \circ \sigma_A.$$

b) *For all* $f: A_2 \to A_3$ *in* $\mathcal{A}$, $u: F(A_1) \to F(A_2)$ *in* $\mathcal{B}$, *and* $v: F(A_3) \to F(A_4)$ *in* $\mathcal{B}$,

$$\mathrm{Tr_F}(v \circ F(f) \circ u) = \mathrm{Tr_F}(v) \circ f \circ \mathrm{Tr_F}(u).$$

In particular, $\mathrm{ImTr_F}$ *is an "ideal" of* $\mathrm{Hom}_{\mathcal{A}}(A_2, A_3)$.

c) *We have that* $e_A = (\mathrm{Tr_F} \circ \mathrm{Res_F})(\mathrm{id_A})$. *Moreover,* e_A *is a central element of* $\mathrm{End}_{\mathcal{A}}(A)$, *and for any morphism* $f: A_1 \to A_2$ *in* $\mathcal{A}$ *we have*

$$(\mathrm{Tr_F} \circ \mathrm{Res_F})(f) = e_{A_2} \circ f = f \circ e_{A_1}.$$

d) (Transitivity) *Let* (F,G) *and* (F',G') *be Frobenius pairs, where* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{F'} \mathcal{C}$ *and* $\mathcal{C} \xrightarrow{G'} \mathcal{B} \xrightarrow{G} \mathcal{A}$. *Then* $(F' \circ F, G \circ G')$ *is a Frobenius pair, and* $\mathrm{Tr}_{F' \circ F} = \mathrm{Tr}_F \circ \mathrm{Tr}_{F'}$.

3. Functor categories and transfer

3.1. Let k be a field and A a finite dimensional k-algebra. We denote by A-Mmod the (abelian) category A-Mmod $=$Mod($A-$mod) of k-linear contravariant functors $T\colon A-\mathrm{mod} \to k-\mathrm{Mod}$ and natural transformations $\alpha\colon T \to T'$.

For any (finitely generated) module $M \in A-$mod, $\mathrm{Hom_A}(-,\mathrm{M}) \in$ A-Mmod is the functor *represented* by M. Yoneda's lemma states that for any $T \in A$-Mmod, the correspondence

$$\mathrm{Hom}(\mathrm{Hom_R}(-,\mathrm{M}),\mathrm{T}) \to \mathrm{T}(\mathrm{M}), \quad \phi \mapsto \phi_{\mathrm{M}}(\mathrm{id_M})$$

is a k-isomorphism, and this implies that every morphism $\phi : \mathrm{Hom_A}(-,\mathrm{M}) \to \mathrm{Hom_A}(-,\mathrm{M}')$ has the form $\phi = \mathrm{Hom}(-,\mathrm{f})$ for a unique morphism $f : M \to M'$ in $A-$mod. Define the k-linear covariant functor

$$\mathcal{Y}^A : A-\mathrm{mod} \to A\text{-Mmod}, \quad M \mapsto \mathrm{Hom_A}(-,\mathrm{M}), \quad \mathrm{f} \mapsto \mathrm{Hom_A}(-,\mathrm{f}).$$

Then for any $M, M' \in A-$mod, the map

$$\mathcal{Y}^A_{M,M'} : \mathrm{Hom_A}(\mathrm{M},\mathrm{M}') \to \mathrm{Hom}(\mathrm{Hom_A}(-,\mathrm{M}),\mathrm{Hom_A}(-,\mathrm{M}'))$$

is a k-isomorphism, so $\mathcal{Y}^A$ is a full embedding of $A-$mod in A-Mmod.

3.2. Let A and B be finite dimensional k-algebras and $F : A-\mathrm{mod} \to B-\mathrm{mod}$ a k-linear covariant functor. Denote $\mathcal{A} = A$-Mmod and $\mathcal{B} = B$-Mmod.

One can associate to F a functor

$$\mathbf{F} : \mathcal{B} \to \mathcal{A}$$

as follows. If U is an object and $\beta\colon U \to U'$ a morphism in $\mathcal{B}$, define

$$\mathbf{F}(U)(X) = U(F(X)), \quad \mathbf{F}(U)(f) = U(F(f)),$$

for any A-module X and morphism $f: X \to X'$ in A-mod, and define

$$\mathbf{F}(\beta)(X) = \beta_{F(X)} : U(F(X)) \to U'(F(X)),$$

for all $X \in A$-mod. It is not difficult to show that $\mathbf{F}$ is a k-linear, covariant and exact functor. Actually, we have a k-linear contravariant functor $F \mapsto \mathbf{F}$, $\phi \mapsto \phi$ from $\mathbf{Fun}(A\text{-mod}, B\text{-mod})$ to $\mathbf{Fun}(\mathcal{B}, \mathcal{A})$, where, if $\phi : F_1 \to F_2$ is a natural transformation between the functors $F_1, F_2 : A\text{-mod} \to B\text{-mod}$, then the natural transformation $\phi: \mathbf{F}_2 \to \mathbf{F}_1$ is defined by

$$\phi_U(X) = U(\phi_X), \text{ for all } U \in \mathcal{B}, X \in A\text{-mod}.$$

Moreover, if $F: A\text{-mod} \to B\text{-mod}$, $F': B\text{-mod} \to C\text{-mod}$ and $F'' = F' \circ F$, then $\mathbf{F}'' = \mathbf{F} \circ \mathbf{F}'$.

3.3. Let (F, G) be a Frobenius pair, where $F: A\text{-mod} \to B\text{-mod}$ and $G: B\text{-mod} \to A\text{-mod}$, and consider the functors

$$\mathcal{G} = \mathbf{F} : \mathcal{B} \to \mathcal{A} \quad \text{and} \quad \mathcal{F} = \mathbf{G} : \mathcal{A} \to \mathcal{B}.$$

It is easy to see that these definitions are compatible with the Yoneda embeddings $\mathcal{Y}^A$ and $\mathcal{Y}^B$, that is, we have the isomorphisms (coming from 2.1 and 2.2)

$$\gamma_{-,X} : \mathcal{F}\mathrm{Hom_A}(-, \mathrm{X}) \simeq \mathrm{Hom_B}(-, \mathrm{F(X)}) \quad \text{in} \quad \mathcal{B},$$

$$\alpha_{-,Y} : \mathcal{G}\mathrm{Hom_B}(-, \mathrm{Y}) \simeq \mathrm{Hom_A}(-, \mathrm{G(Y)}) \quad \text{in} \quad \mathcal{A}.$$

3.4. Lemma. *$(\mathcal{F}, \mathcal{G})$ is a Frobenius pair.*

Proof. The natural transformations σ, ρ, ξ and τ defined in 2.1 and 2.2 induce the natural transformations $\sigma : \mathcal{G} \circ \mathcal{F} \to \mathrm{id}_\mathcal{A}$, $\rho : \mathrm{id}_\mathcal{B} \to \mathcal{F} \circ \mathcal{G}$, $\xi : \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_\mathcal{B}$, and $\tau: \mathrm{id}_\mathcal{A} \to \mathcal{G} \circ \mathcal{F}$. Since F is a left adjoint of G, the compositions

$$F \xrightarrow{F\circ\sigma} F \circ G \circ F \xrightarrow{\rho\circ F} F \quad \text{and} \quad G \xrightarrow{\sigma\circ G} G \circ F \circ G \xrightarrow{G\circ\rho} G$$

are the identity transformations of F and G respectively.

It follows from 3.2 and 3.3 that the compositions

$$\mathcal{G} \xrightarrow{\mathcal{G}\circ\rho} \mathcal{G} \circ \mathcal{F} \circ \mathcal{G} \xrightarrow{\sigma\circ\mathcal{G}} \mathcal{G} \quad \text{and} \quad \mathcal{F} \xrightarrow{\mathcal{F}\circ\sigma} \mathcal{F} \circ \mathcal{G} \circ \mathcal{F} \xrightarrow{\rho\circ\mathcal{F}} \mathcal{F}$$

are the identity transformations of $\mathcal{G}$ and $\mathcal{F}$ respectively, hence $\mathcal{G}$ is a left adjoint of $\mathcal{F}$, the unit of this adjunction is ρ and the counit is σ. Similarly, $\mathcal{F}$ is a left adjoint of $\mathcal{G}$, with unit τ and counit ξ.

3.5. Since $(\mathcal{F}, \mathcal{G})$ is a Frobenius pair, for any objects $T \in \mathcal{A}$ and $U \in \mathcal{B}$ there are natural isomorphisms

$$\begin{aligned} \alpha_{T,U} &: \operatorname{Hom}_{\mathcal{B}}(\mathcal{F}(T), U) \to \operatorname{Hom}_{\mathcal{A}}(T, \mathcal{G}(U)), \\ \gamma_{U,T} &: \operatorname{Hom}_{\mathcal{A}}(\mathcal{G}(U), T) \to \operatorname{Hom}_{\mathcal{B}}(U, \mathcal{F}(T)). \end{aligned}$$

We want to establish a connection between $\alpha_{T,U}$ and $\gamma_{U,T}$, and $\alpha_{X,Y}$, $\gamma_{Y,X}$. From a basic result on adjoint functors we know that $\alpha_{T,U} = \mathcal{G}(f) \circ \tau_T$ as composition of natural transformations. By 3.2, we have for any $X \in A$–mod:

$$\tau_T(X) = T(\tau_X) = T(\gamma^{-1}_{F(X),X}(\mathrm{id}_{F(X)})),$$

$$\mathcal{G}(f)(X) = \mathbf{F}(f)(X) = f_F(X) : \mathcal{F}(T)(F(X)) \to U(F(X)),$$

where $\mathcal{F}(T)(F(X)) = \mathbf{G}(T)(F(X)) = T((G \circ F)(X))$. From these equalities we get:

$$\begin{aligned} \alpha_{T,U}(X) &= (\mathcal{G}(f) \circ \tau_T)(X) = \mathcal{G}(f)(X) \circ \tau_T(X) \\ &= f_F(X) \circ T(\gamma^{-1}_{F(X),X}(\mathrm{id}_{F(X)})) \end{aligned}$$

Similary, for any $Y \in B$–mod, we have

$$\gamma_{U,T}(Y) = g_G(Y) \circ U(\alpha_{Y,G(Y)}(\mathrm{id}_{G(Y)})).$$

3.6. Since $(\mathcal{F}, \mathcal{G})$ is a Frobenius pair, for any objects $T, T' \in A$-Mmod we may define the transfer map

$$\operatorname{Tr}_{\mathcal{F}} = \operatorname{Tr}_{\mathcal{F}}(T, T') : \operatorname{Hom}_{\mathcal{B}}(\mathcal{F}(T), \mathcal{F}(T')) \to \operatorname{Hom}_{\mathcal{A}}(T, T'),$$

$$f \mapsto \operatorname{Tr}_{\mathcal{F}}(f) = \sigma_{T'} \circ G(f) \circ \tau_T,$$

where τ and σ are defined in the proof of the preceding lemma.

For every object T of $\mathcal{A}$ denote

$$e_T = e(\mathcal{F})_T = \sigma_T \circ \tau_T \in \operatorname{End}_{\mathcal{A}}(T).$$

As in Proposition 2.5 c), we have that $\mathbf{e}_T = (\mathrm{Tr}_{\mathcal{F}} \circ \mathrm{Res}_{\mathcal{F}})(\mathrm{id_T})$ and for any morphism $\phi : T_1 \to T_2$ in $\mathcal{A}$

$$(\mathrm{Tr}_{\mathcal{F}} \circ \mathrm{Res}_{\mathcal{F}})(\phi) = \mathbf{e}_{\mathrm{T}_2} \circ \phi = \phi \circ \mathbf{e}_{\mathrm{T}_1}.$$

Moreover, we have that $\mathbf{e}_T(X) = T(e_X)$ for any $X \in A$–mod. Indeed,

$$\begin{aligned} \mathbf{e}_T(X) &= (\boldsymbol{\sigma}_T \circ \boldsymbol{\tau}_T)(X) = \boldsymbol{\sigma}_T(X) \circ \boldsymbol{\tau}_T(X) \\ &= T(\sigma_X) \circ T(\tau_X) = T(\tau_X \circ \sigma_X) = T(e_X). \end{aligned}$$

Clearly, the statements of [5, Proposition 1.7] and Higman's criterion [5, Theorem 2.2] hold in this new context.

3.7. Next, we establish the connection between the transfer maps $\mathrm{Tr}_{\mathcal{F}}$ and Tr_{F}.

Let $M, M' \in A$–mod and let

$$\phi : \mathcal{F}\mathrm{Hom_A}(-, \mathrm{M}) \to \mathcal{F}\mathrm{Hom_A}(-, \mathrm{M'}).$$

Then there is a unique morphism

$$\mathrm{Hom_B}(-, \mathrm{F(M)}) \to \mathrm{Hom_B}(-, \mathrm{F(M')})$$

such that the following diagram is commutative; Yoneda's lemma implies that there is a unique $h \in \mathrm{Hom_B}(\mathrm{F(M)}, \mathrm{F(M')})$ such that this morphism has the form $\mathrm{Hom_B}(-, \mathrm{h})$.

$$\begin{array}{ccc} \mathcal{F}\mathrm{Hom}_A(-, M) & \xrightarrow[\gamma_{-,M}]{} & \mathrm{Hom}_B(-, F(M)) \\ \Big\downarrow \phi & & \Big\downarrow \mathrm{Hom}_B(-, h) \\ \mathcal{F}\mathrm{Hom}_A(-, M) & \xrightarrow{\gamma_{-,M'}} & \mathrm{Hom}_B(-, F(M')) \end{array}$$

3.8. Proposition. *Let ϕ and h as above. Then*

$$\mathrm{Tr}_{\mathcal{F}}(\phi) = \mathrm{Hom_A}(-, \mathrm{Tr_F(h)}) : \mathrm{Hom_A}(-, \mathrm{M}) \to \mathrm{Hom_A}(-, \mathrm{M'}).$$

Proof. Denote $T = \mathrm{Hom_A}(-, \mathrm{M})$ and $T' = \mathrm{Hom_A}(-, \mathrm{M'})$, and let X be an A-module. By definition, we have

$$\mathrm{Tr}_{\mathcal{F}}(\phi) = \boldsymbol{\sigma}_{\mathrm{T'}} \circ \mathcal{G}(\phi) \circ \boldsymbol{\tau}_{\mathrm{T}},$$

where recall that we have the equalities

$$\tau_T(X) = T(\tau_X) = \mathrm{Hom_A}(\tau_X, \mathrm{id_M}),$$

$$\sigma_{T'}(X) = T(\sigma_X) = \mathrm{Hom_A}(\sigma_X, \mathrm{id_{M'}}).$$

Moreover,

$$\begin{aligned} \mathcal{G}(\phi)(X) &= \mathbf{F}(\phi)(X) = \phi_{F(X)} \\ &= \gamma_{F(X),M'} \circ \mathrm{Hom_B}(\mathrm{id_{F(X)}}, \mathrm{h}) \circ \gamma_{\mathrm{F(X),M}}. \end{aligned}$$

By Proposition 2.5 a) it follows that

$$(\mathrm{Tr}_{\mathcal{F}}\phi)(\mathrm{X}) = \mathrm{Tr_F}(\mathrm{X}, \mathrm{M}') \circ \mathrm{Hom_B}(\mathrm{id_{F(X)}}, \mathrm{h}) \circ \mathrm{Res_F}(\mathrm{X}, \mathrm{M}),$$

and for any $f \in \mathrm{Hom_A(X, M)}$,

$$\begin{aligned} (\mathrm{Tr}_{\mathcal{F}}\phi)(\mathrm{X})(\mathrm{f}) &= \mathrm{Tr_F}(\mathrm{Res_F}(\mathrm{f}) \circ \mathrm{h}) = \mathrm{f} \circ \mathrm{Tr_F}(\mathrm{h}) \\ &= \mathrm{Hom_A}(\mathrm{id_X}, \mathrm{Tr_F}(\mathrm{h})). \end{aligned}$$

3.9. Let $T \in A$-mmod. By a well-known theorem of Auslander and Reiten, T is finitely presented if and only if there exist $M, X \in A-$mod and a morphism

$$a\colon \mathrm{Hom_A}(-, \mathrm{M}) \to \mathrm{DHom_A}(\mathrm{X}, -)$$

in A-Mmod such that $T \simeq \mathrm{Im} a$. (Here D denotes the k-dual.)

Denote $t_a = a_M(\mathrm{id}_M) \in D\mathrm{Hom_A(X, M)}$. By Yoneda's lemma we have that t_a also determines a, since

$$a_Y(f) = D\mathrm{Hom_A}(\mathrm{id_X}, \mathrm{f})(\mathrm{t_a}),$$

for all $Y \in A-$mod and $f \in \mathrm{Hom_A(Y, M)}$.

In the last part of this paper we give similar specifications for $\mathcal{F}(T)$ and $\mathcal{G}(U)$.

3.10. We have that $\mathcal{F}(T) \simeq \mathrm{Im}\mathcal{F}(\mathrm{a}) \simeq \mathrm{Im} b$, where b is the unique morphism which makes the following diagram commutative.

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{B}}(-, F(M)) & \xleftarrow{\gamma_{-,M}} & \mathcal{F}\mathrm{Hom}_{\mathcal{A}}(-, M) \\
\Big\downarrow b & & \Big\downarrow \mathcal{F}(a) \\
D\mathrm{Hom}_{\mathcal{B}}(F(X), -) & \xleftarrow[D\alpha_{X,-}]{} & \mathcal{F}D\mathrm{Hom}_{\mathcal{A}}(X, -)
\end{array}
$$

where $D\alpha_{X,-} : \mathcal{F}D\mathrm{Hom}_{\mathrm{A}}(\mathrm{X}, -) \to \mathrm{DHom}(\mathrm{F}(\mathrm{X}), -)$ is the $\mathcal{B}$ morphism defined for every $Y \in B-$mod by:

$$D\alpha_{X,Y}(\lambda)(s) = \lambda(\alpha_{X,Y}(s)),$$

for all $\lambda \in D\mathrm{Hom}_{\mathrm{A}}(\mathrm{X}, \mathrm{G}(\mathrm{Y})), \mathrm{s} \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{F}(\mathrm{X}), \mathrm{Y})$.

3.11. Proposition. *With the above notations, $t_b - t_a \circ \mathrm{Tr}_{\mathrm{F}}$, that is, t_b is the composite map*

$$\mathrm{Hom}_{\mathcal{B}}(\mathrm{F}(\mathrm{X}), \mathrm{F}(\mathrm{M})) \xrightarrow{\mathrm{Tr}_{\mathrm{F}}} \mathrm{Hom}_{\mathcal{A}}(\mathrm{X}, \mathrm{W}) \xrightarrow{t_a} \mathrm{k}.$$

Proof. From the above diagram we have:

$$t_b = b_{F(M)}(\mathrm{id}_{F(M)}) = D\alpha_{X,F(M)} \circ \mathcal{F}(a)_{F(M)} \circ \gamma^{-1}_{F(M),M}(\mathrm{id}_{F(M)}).$$

Then, using 3.9, we have for $s \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{F}(\mathrm{X}), \mathrm{F}(\mathrm{M}))$:

$$
\begin{aligned}
t_b(s) &= (D\alpha_{X,F(M)} \circ (\mathcal{F}(a)_{F(M)} \circ \gamma^{-1}_{F(M),M}(\mathrm{id}_{F(M)}))(s) \\
&= D\alpha_{X,F(M)}(a_{(G\circ F)(M)}(\tau_M))(s) \\
&= a_{(G\circ F)(M)}(\tau_M)(\alpha_{X,F(M)}(s)) \\
&= t_a(\tau_M \circ \alpha_{X,F(M)})(s)) \\
&= (t_a \circ \mathrm{Tr}_{\mathrm{F}})(\mathrm{s}).
\end{aligned}
$$

3.12. If U is an object of $\mathcal{B}$, we have a similar formula to calculate $\mathcal{G}(U)$.

There are B-modules Y, N and a morphism

$$z : \mathrm{Hom}_{\mathrm{B}}(-, \mathrm{N}) \to \mathrm{DHom}_{\mathrm{B}}(\mathrm{Y}, -)$$

such that $U \simeq \mathrm{Im} z$, and let $t_z = z_N(\mathrm{id_N}) \in \mathrm{DHom_B(Y,N)}$. It follows that

$$\mathcal{G}(U) \simeq \mathrm{Im}\mathcal{G}(z) \simeq \mathrm{Im} x,$$

where x is the unique morphism such that the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Hom}_A(-, G(N)) & \xleftarrow{\alpha_{-,N}} & \mathcal{G}\mathrm{Hom}_B(-, N) \\ \Big\downarrow x & & \Big\downarrow \mathcal{G}(z) \\ D\mathrm{Hom}_A(G(Y), -) & \xleftarrow[D\gamma_{Y,-}]{} & \mathcal{G}D\mathrm{Hom}_B(Y, -) \end{array}$$

With the same proof as above (just exchanging F and G), we obtain

3.13. Proposition. *With the above notations we have* $t_x = t_z \circ \mathrm{Tr_G}$.

3.14. We shall use these results to discuss semisimplicity of functors. Let M be an A-module, $a\colon \mathrm{Hom_A(-,M)} \to \mathrm{D(Hom_A(M,-)}$ be a morphism in $\mathcal{A} = A$-mmod, and let $t_a = a_M(\mathrm{id}_M)$. Then by [4, Lemma 7.1], we have that $T = \mathrm{Im} a$ is a semisimple functor if and only if $t_a(J(\mathrm{End_A(M)}) = 0$.

If in addition M is indecomposable and $S_M = \mathrm{Hom_A(-,M)/Rad_A(-,M)}$ denotes the simple functor associated to M, then $\mathrm{Im} a \simeq \mathrm{S_M}$ is and only if

$$t_a \neq 0, \qquad t_a(J(\mathrm{End_A(M)}) = 0.$$

If M is indecomposable, then it follows by 3.3 that $\mathrm{Head}\mathcal{F}(\mathrm{S_M})$ is a direct summand of $\mathrm{HeadHom_B(-,F(M))}$. Here we denoted $\mathrm{HeadT = T/RadT}$ and by J the Jacobson radical.

3.15. Theorem. *If M is an indecomposable A-module, then $\mathcal{F}(S_M)$ is semisimple if and only if* $\mathrm{Tr_F(J(End_B(F(M)))) \subseteq J(End_A(M))}$.

Proof. Let M be an indecomposable A-module. Consider a morphism t_a : $\mathrm{End_A(M)} \to \mathrm{k}$, such that

$$t_a \neq 0, \quad t_a(J(\mathrm{End_A(M)})) = 0.$$

(that is $\mathrm{Ker}a = \mathrm{J}(\mathrm{End}_\mathrm{A}(\mathrm{M}))$. Then by 3.9, t_a induces a morphism $a: \mathrm{Hom}_\mathrm{A}(-, \mathrm{M}) \to \mathrm{D}(\mathrm{Hom}_\mathrm{A}(\mathrm{M}, -)$ such that $\mathrm{Im}a \simeq \mathrm{S}_\mathrm{M}$. From 3.10 and Proposition 3.11 we know that there exists a morphism $b: \mathrm{Hom}_\mathrm{B}(-, \mathrm{F}(\mathrm{M})) \to \mathrm{D}(\mathrm{Hom}_\mathrm{B}(\mathrm{F}(\mathrm{M}), -)$ such that $\mathcal{F}(S_M) \simeq \mathrm{Im}b$ and $t_b = t_a \circ \mathrm{Tr}_\mathrm{F}$. If $\mathcal{F}(S_M)$ is semisimple, then by 3.14 we obtain that $t_a(\mathrm{Tr}_\mathrm{F}(\mathrm{J}(\mathrm{End}_\mathrm{B}(\mathrm{F}(\mathrm{M}))))) = 0$. It follows from the definition of t_a that

$$\mathrm{Tr}_\mathrm{F}(\mathrm{J}(\mathrm{End}_\mathrm{B}(\mathrm{F}(\mathrm{M})))) \subseteq \mathrm{Ker}a = \mathrm{J}(\mathrm{End}_\mathrm{A}(\mathrm{M})).$$

Conversely, if $\mathrm{Tr}_\mathrm{F}(\mathrm{J}(\mathrm{End}_\mathrm{B}(\mathrm{F}(\mathrm{M})))) \subseteq \mathrm{J}(\mathrm{End}_\mathrm{A}(\mathrm{M}))$, we deduce that $t_b(\mathrm{End}_\mathrm{B}(\mathrm{F}(\mathrm{M}))) = 0$, which means that $\mathrm{Im}b \simeq \mathcal{F}(\mathrm{S}_\mathrm{M})$ is semisimple.

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References

[1] D.J. Benson, *Representations and cohomology I*, Cambridge Univ. Press, 1991.
[2] M. Broue, *On representations of symmetric algebras: an introduction, Notes by M. Stricker*, ETH Zurich, 1991.
[3] L.G. Chouinard, *Transfer maps*, Comm. Algebra 8(1980), 1519-1537.
[4] J.A. Green, *Functors on categories of finite group representations*, J. Pure Appl. Algebra **37**(1985), 265–298.
[5] A. Madar and A. Marcus, *Frobenius functors and transfer*, to appear in Mathematica (Cluj).
[6] A. Marcus, *Representation Theory of Group Graded Algebras*, Nova Science Publishers, Commack, NY, 1999.

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SOME APPLICATIONS OF THE FIBER CONTRACTION THEOREM

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Dedicated to Professor Vasile Pop at his 60^{th} anniversary

Abstract. The main results of this paper are an existence, uniqueness and continuous data dependence theorem and a theorem for the differentiability with respect to parameters for the solutions of an integral equation with deviating arguments. The Picard operators' technique and the Fiber contraction theorem are used.

1. Introduction

The purpose of this paper is to study the data dependence and the differentiability with respect to parameters for the solution of the following integral equation:

$$u(t) = \begin{cases} \varphi(0) + \int_0^t K(t,s,u(s),u(\lambda s),u(s-\omega))ds, \ t \in [0,b], \ \lambda \in [0,1], \ \omega \in [0,c] \\ \varphi(t), \ t \in [-c,0], \end{cases}$$

where $K \in C([0,b] \times [0,b] \times \mathbb{R}^3)$ and $\varphi \in C[-c,0]$.

We apply the Contraction principle and the Fiber contraction theorem given by Hirsch and Pugh ([5]) and generalized by Rus ([12] and [13]). For $\lambda = 0$ or $\lambda = 1$ we obtain an integral equation with delay. This kind of equations has studied in many papers as [1]-[4]. For $\omega = 0$ we obtain an integral equation with linear modification of the argument. This kind of equation has been studied in many papers too and we quote here [6]-[9]. Both kinds of equations appear in many practical problems from biology, chemistry, astronomy, technical sciences.

The following Cauchy problem ([7]):

$$u'(t) = K(t, u(t), u(\lambda t), u(t-\omega)),\ t \in [0, b],\ 0 < \lambda < 1,\ \omega > 0$$

$$u(t) = \varphi(t),\ t \in [-\omega, 0],$$

provide us an integral equation of such kind as that studied in this paper.

2. Fiber contraction theorem

In order to obtain our results, we need the Contraction principle and the Fiber contraction theorem. In what follows we will present these theorems.

Let X be a nonempty set and $T : X \to X$ an operator. We denote by F_T the fixed point set of T, that is

$$F_T = \{x \in X |\ T(x) = x\}.$$

Definition 2.1. (Rus [11]) Let (X, d) be a metric space. An operator $T : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:

(a) $F_T = \{x^*\}$;

(b) the sequence $(T^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.

We have

Theorem 2.1. (Contraction principle) *Let (X, d) be a complete metric space and $T : X \to X$ a contraction, with the constant c. Then T is a Picard operator and*

$$d(x^*, T^n(x_0)) \leq \frac{c^n}{1-c} d(x_0, T(x_0))$$

for all $x_0 \in X$.

Theorem 2.2. (Hirsch-Pugh [5], Rus [12]) (Fiber contraction theorem) *Let (X, d) be a metric space, (Y, ρ) be a complete metric space and $T : X \times Y \to X \times Y$ be a continuous operator.*

We suppose that:

(i) $T(x, y) = (T_1(x), T_2(x, y))$;

(ii) $T_1 : X \to X$ is a Picard operator;

(iii) there exists $c \in]0, 1[$ such that

$$\rho(T_2(x, y), T_2(x, z)) \leq c\rho(y, z),$$

for all $x \in X$ and all $y, z \in Y$.

Then the operator T is a Picard operator.

Remark 2.1. Theorem 2.1 was generalized by Rus in [12] as Fiber Picard operators theorem and in [13] as a Fiber generalized contraction theorem. Theorem 2.2 and its generalizations can be used for proving solution of operatorial equations to be differentiable.

3. Main results

Let $\lambda \in [0,1]$ and $\omega \in [0,c]$ be. We consider the following integral equation:

$$u(t) = \begin{cases} \varphi(0) + \int_0^t K(t,s,u(s),u(\lambda s),u(s-\omega))ds, & t \in [0,b] \\ \varphi(t), \ t \in [-c,0], \end{cases} \tag{3.1}$$

where $K \in C([0,b] \times [0,b] \times \mathbb{R}^3)$ and $\varphi \in C[-c,0]$ are given functions.

We have

Theorem 3.1. (existence and uniqueness in space) *We suppose that the following Lipschitz condition is satisfied:*

(i) there exist $L_i > 0$, $i = \overline{1,3}$, such that

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \leq \sum_{i=1}^{3} L_i |u_i - v_i|,$$

for all $t, s \in [0,b]$ and all $u_i, v_i \in \mathbb{R}$, $i = \overline{1,3}$.

Then

(a) for all $\lambda \in [0,1]$ and all $\omega \in [0,c]$, the equation (3.1) has in $C[-c,b]$ a unique solution $z^* = z^*(\cdot,\lambda,\omega)$;

(b) for all $z_0 \in C([-c,b])$, the sequence $(z_n)_{n \in \mathbb{N}}$ defined by

$$z_{n+1}(t) := \begin{cases} \varphi(0) + \int_0^t K(t,s,z_n(s),z_n(\lambda s),z_n(s-\omega))ds, & t \in [0,b] \\ \varphi(t), \ t \in [-c,0] \end{cases}$$

converges uniformly to z^* on $[-c,b]$;

(c) the function $z^* : [-c, b] \times [0,1] \times [0,c] \to \mathbb{R}$, $(t, \lambda, \omega) \to z^*(t, \lambda, \omega)$ *is continuous.*

Proof. Let $\| \cdot \|_B$ be a Bielecki norm on $C[-c,b]$ defined by

$$\|z\|_B = \max_{t \in [-c,b]} |z(t)| e^{-\tau t},$$

where $\tau > 0$.

We consider the operator $A : C[-c,b] \to C[-c,b]$, defined by

$$(A(z))(t) := \begin{cases} \varphi(0) + \int_0^t K(t, s, z(s), z(\lambda s), z(s-\omega)) ds, & t \in [0,b] \\ \varphi(t), \ t \in [-c, 0]. \end{cases}$$

We have $|(A(z))(t) - (A(w))(t)| = 0$, for all $t \in [-c, 0]$.

By using the condition (i), we obtain

$$|(A(z))(t) - (A(w))(t)| \le L_1 \int_0^t |z(s) - w(s)| e^{-\tau s} e^{\tau s} ds +$$

$$+ L_2 \int_0^t |z(\lambda s) - w(\lambda s)| e^{-\tau \lambda s} e^{\tau \lambda s} ds + L_3 \int_0^t |z(s-\omega) - w(s-\omega)| e^{-\tau(s-\omega)} e^{\tau(s-\omega)} ds,$$

for all $t \in [0, b]$.

Therefore,

$$|(A(z))(t) - (A(w))(t)| \le$$

$$\le \left[\frac{L_1}{\tau}(e^{\tau t} - 1) + \frac{L_2 e^{\tau t}}{\lambda \tau}(e^{\tau(\lambda t - t)} - e^{-\tau t}) + \frac{L_3 e^{\tau t}}{\tau}(e^{-\tau \omega} - e^{\tau(-\omega - t)}) \right] \|z - w\|_B <$$

$$\le \frac{e^{\tau t}}{\tau} \|z - w\|_B \left[L_1 + \frac{L_2}{\lambda} \max_{t \in [0,b]} (e^{\tau(\lambda t - t)} - e^{-\tau t}) + L_3 \max_{t \in [0,b]} (1 - e^{\tau(-\omega - t)}) \right] \le$$

$$\le \frac{e^{\tau t}}{\tau} \|z - w\|_B \left(L_1 + \frac{L_2}{\lambda} + L_3 \right),$$

for all $t \in [0, b]$.

So we obtain

$$|(A(z))(t) - (A(w))(t)| e^{-\tau t} \le \frac{1}{\tau} \left(L_1 + \frac{L_2}{\lambda} + L_3 \right) \|z - w\|_B,$$

for all $t \in [0, b]$.

It follows that

$$\|A(z) - A(w)\|_B \le L_A \|z - w\|_B,$$

for all $z, w \in C[-c, b]$, where

$$L_A = \frac{1}{\tau}\left(L_1 + \frac{L_2}{\lambda} + L_3\right).$$

We can choose τ large enough so that $L_A < 1$.

By applying Theorem 2.1 to the operator A, we obtain (a), (b) and (c).

In what follows we consider the following integral equation:

$$u(t,\lambda,\omega) = \begin{cases} \varphi(0) + \int_0^t K(t,s,u(s,\lambda,\omega),u(\lambda s,\lambda,\omega),u(s-\omega,\lambda,\omega))ds, \\ \qquad t \in [0,b],\ \lambda \in [0,1],\ \omega \in [0,c] \\ \varphi(t),\ t \in [-c,0], \end{cases} \tag{3.2}$$

where $K \in C([0,b] \times [0,b] \times \mathbb{R}^3)$ and $\varphi \in C[-c,0]$ are given functions.

So we have:

Theorem 3.2. *We suppose that:*

(i) $K \in C^1([0,b] \times [0,b] \times \mathbb{R}^3)$;

(ii) $\varphi \in C^1[-c,0]$ *is a given function such that*

$$\varphi'(0) = K(0,0,\varphi(0),\varphi(0),\varphi(-\omega)),$$

for all $\omega \in [0,c]$;

(iii) there exist $M_i > 0$ *such that*

$$\left|\frac{\partial K}{\partial u_i}(t,s,u_1,u_2,u_3)\right| \le M_i, \quad i = \overline{1,3},$$

for all $t, s \in [0,b]$ *and all* $u_i \in \mathbb{R}$, $i = \overline{1,3}$.

Then

(a) the equation (3.2) has in $C([-\omega,b] \times [0,1] \times [0,c])$ *a unique solution* u^*;

(b) for all $u_0 \in C([-\omega,b] \times [0,1] \times [0,c])$, *the sequence* $(u_n)_{n\in\mathbb{N}}$, *defined by*

$$u_{n+1}(t) := \begin{cases} \varphi(0) + \int_0^t K(t,s,u_n(s,\lambda,\omega),u_n(\lambda s,\lambda,\omega),u_n(s-\omega,\lambda,\omega))ds, \\ \qquad t \in [0,b],\ \lambda \in [0,1],\ \omega \in [0,c] \\ \varphi(t),\ t \in [-\omega,0] \end{cases}$$

converges uniformly to u^* *on* $[-\omega, b] \times [0,1] \times [0,c]$ *and*

$$\|u_n - u^*\|_B \leq \frac{L^n}{1-L}\|u_1 - u_0\|_B,$$

where

$$L = \left(M_1 + \frac{M_2}{\lambda} + M_3\right)\left(M_1 + \frac{M_2}{\lambda} + M_3 + 1\right)^{-1}$$

and $\|\cdot\|_B$ *is a Bielecki norm on* $C([0,b] \times [0,1] \times [0,\omega])$ *defined by*

$$\|u\|_B = \max_{t\in[-c,b],\lambda\in[0,1],\omega\in[0,c]} |u(t,\lambda,\omega)|e^{-\tau t},$$

where $\tau > 0$*;*

(c) $u^* \in C([-c,b] \times [0,1] \times [0,c])$*;*

(d) $u^*(\cdot,\cdot,\omega) \in C^1([-c,b] \times [0,1])$*, for all* $\omega \in [0,c]$.

Proof. Let $X = C([-c,b] \times [0,1] \times [0,c])$ be and $\|\cdot\|_B$ the Bielecki norm on X defined before.

We consider the operator $T_1 : X \to X$, defined by

$$(T_1(u))(t,\lambda,\omega) := \begin{cases} \varphi(0) + \int_0^t K(t,s,u(s,\lambda,\omega),u(\lambda s,\lambda,\omega),u(s-\omega,\lambda,\omega))ds, \\ \qquad t \in [0,b],\ \lambda \in [0,1],\ \omega \in [0,c] \\ \varphi(t),\ t \in [-c,0]. \end{cases}$$

By the same proof as of the Theorem 3.1 we obtain (a), (b) and (c).

Let u^* be the solution of the equation (3.2). So we have

$$u^*(t,\lambda,\omega) = \begin{cases} \varphi(0) + \int_0^t K(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))ds, \\ \qquad t \in [0,b],\ \lambda \in [0,1],\ \omega \in [0,c] \\ \varphi(t),\ t \in [-c,0]. \end{cases} \tag{3.3}$$

We remark that

$$\frac{\partial u^*}{\partial t}(t,\lambda,\omega) = \begin{cases} K(t,t,u^*(t,\lambda,\omega),u^*(\lambda t,\lambda,\omega),u^*(t-\omega,\lambda,\omega)) + \\ + \int_0^t \frac{\partial K}{\partial t}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))ds, \\ \qquad t \in [0,b],\ \lambda \in [0,1],\ \omega \in [0,c] \\ \varphi'(t),\ t \in [-c,0]. \end{cases} \tag{3.4}$$

Because of conditions (i) and (ii) we have that $\frac{\partial u^*}{\partial t}$ is a continuous function

Let we prove that there exists $\frac{\partial u^*}{\partial \lambda}$ and $\frac{\partial u^*}{\partial \lambda} \in C([-c,b]\times[0,1]\times[0,c])$.

If we suppose that there exists $\frac{\partial u^*}{\partial \lambda}$, then from (3.3) we obtain

$$\frac{\partial u^*}{\partial \lambda}(t,\lambda,\omega) = 0, \quad \text{for} \quad t\in[-c,0],$$

and

$$\frac{\partial u^*}{\partial \lambda}(t,\lambda,\omega) = \int_0^t \frac{\partial K}{\partial u_1}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))\frac{\partial u^*}{\partial \lambda}(s,\lambda,\omega)ds+$$
$$+\int_0^t \frac{\partial K}{\partial u_2}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))\frac{\partial u^*}{\partial \lambda}(\lambda s,\lambda,\omega)ds+$$
$$+\int_0^t \frac{\partial K}{\partial u_2}(t,s,u^*(s,\lambda,\omega,u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))s\frac{\partial u^*}{\partial z}(z,\lambda,\omega)\Big|_{z:=\lambda s}ds+$$
$$+\int_0^t \frac{\partial K}{\partial u_3}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))\frac{\partial u^*}{\partial \lambda}(s-\omega,\lambda,\omega)ds,$$

for $t\in[0,b]$, where $K=K(t,s,u_1,u_2,u_3)$.

Because of the following relationship

$$\frac{\partial u^*}{\partial z}(z,\lambda,\omega)\Big|_{z:=\lambda s} = K(\lambda s,\lambda s,u^*(\lambda s,\lambda,\omega),u^*(\lambda^2 s,\lambda,\omega)u^*(\lambda s-\omega,\lambda,\omega))+$$
$$+\int_0^{\lambda s}\frac{\partial K}{\partial t}(\lambda s,s_1,u^*(s_1,\lambda,\omega),u^*(\lambda s_1,\lambda,\omega),u^*(s_1-\omega,\lambda,\omega))ds_1,$$

we have that

$$\frac{\partial u^*}{\partial \lambda}(t,\lambda,\omega) = \int_0^t \frac{\partial K}{\partial u_1}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))\frac{\partial u^*}{\partial \lambda}(s,\lambda,\omega)ds+$$
$$+\int_0^t \frac{\partial K}{\partial u_2}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))\frac{\partial u^*}{\partial \lambda}(\lambda s,\lambda,\omega)ds+$$
$$+\int_0^t s\frac{\partial K}{\partial u_2}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))\cdot$$
$$\cdot[K(\lambda s,\lambda s,u^*(\lambda s,\lambda,\omega),u^*(\lambda^2 s,\lambda,\omega),u^*(\lambda s-\omega,\lambda,\omega))]ds+$$
$$+\int_0^t s\frac{\partial K}{\partial u_2}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))\cdot$$
$$\cdot\left[\int_0^{\lambda s}\frac{\partial K}{\partial t}(\lambda s,s_1,u^*(s_1,\lambda,\omega),u^*(\lambda s_1,\lambda,\omega),u^*(s_1-\omega,\lambda,\omega))ds_1\right]ds+$$
$$+\int_0^t \frac{\partial K}{\partial u_3}(t,s,u^*(s,\lambda,\omega),u^*(\lambda s,\lambda,\omega),u^*(s-\omega,\lambda,\omega))\frac{\partial u^*}{\partial \lambda}(s-\omega,\lambda,\omega)ds,$$

for $t\in[0,b]$.

This relation suggest us to consider the operator $T_2 . X \times X \to X$, defined by

$$(T_2(u,y))(t,\lambda,\omega) := \begin{cases} 0, \text{ for } t \in [-c,0]; \\ \int_0^t \frac{\partial K}{\partial u_1}(t,s,u(s,\lambda,\omega),u(\lambda s,\lambda,\omega),u(s-\omega,\lambda,\omega))y(s,\lambda,\omega)ds+ \\ +\int_0^t \frac{\partial K}{\partial u_2}(t,s,u(s,\lambda,\omega),u(\lambda s,\lambda,\omega),u(s-\omega,\lambda,\omega))y(\lambda s,\lambda,\omega)ds+ \\ +\int_0^t s\frac{\partial K}{\partial u_2}(t,s,u(s,\lambda,\omega),u(\lambda s,\lambda,\omega),u(s-\omega,\lambda,\omega))\cdot \\ \cdot[K(\lambda s,\lambda s,u(\lambda s,\lambda,\omega),u(\lambda^2 s,\lambda,\omega),u(\lambda s-\omega,\lambda,\omega))]ds+ \\ +\int_0^t s\frac{\partial K}{\partial u_2}(t,s,u(s,\lambda,\omega),u(\lambda s,\lambda,\omega),u(s-\omega,\lambda,\omega))\cdot \\ \cdot\left[\int_0^{\lambda s}\frac{\partial K}{\partial t}(\lambda s,s_1,u(s_1,\lambda,\omega),u(\lambda s_1,\lambda,\omega),u(s_1-\omega,\lambda,\omega))ds_1\right]ds+ \\ +\int_0^t \frac{\partial K}{\partial u_3}(t,s,u(s,\lambda,\omega),u(\lambda s,\lambda,\omega),u(s-\omega,\lambda,\omega))y(s-\omega,\lambda,\omega)ds, \\ \qquad \text{for } t\in[0,b],\ \lambda\in[0,1],\ \omega\in[0,c]. \end{cases}$$

By using (iii), we obtain

$$\|T_2(u,y)-T_2(u,z)\|_B \le \left(M_1+\frac{M_2}{\lambda}+M_3\right)\tau^{-1}\|y-z\|_B,$$

for all $u,y,z\in X$.

Choosing $\tau = M_1+\frac{M_2}{\lambda}+M_3+1$ we have that T_2 is a contraction.

If we take the operator $T : X\times X \to X\times X$, $T=(T_1,T_2)$, then we are in the conditions of the Theorem 2.2.

It follows from this theorem that T is a Picard operator. So, the sequences $(u_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, where

$$u_{n+1}(t,\lambda,\omega) := \begin{cases} \varphi(0)+\int_0^t K(t,s,u_n(s,\lambda,\omega),u_n(\lambda s,\lambda,\omega),u_n(s-\omega,\lambda,\omega))ds, \\ \qquad t\in[0,b],\ \lambda\in[0,1],\ \omega\in[0,c] \\ \varphi(t),\ t\in[-c,0] \end{cases}$$

respectively

$$y_{n+1}(t,\lambda,\omega) := \int_0^t \frac{\partial K}{\partial u_1}(t,s,u_n(s,\lambda,\omega),u_n(\lambda s,\lambda,\omega),u_n(s-\omega,\lambda,\omega))y_n(s,\lambda,\omega)ds+$$

$$+\int_0^t \frac{\partial K}{\partial u_2}(t,s,u_n(s,\lambda,\omega),u_n(\lambda s,\lambda,\omega),u_n(s-\omega,\lambda,\omega))y_n(\lambda s,\lambda,\omega)ds+$$

$$+\int_0^t s\frac{\partial K}{\partial u_2}(t,s,u_n(s,\lambda,\omega),u_n(\lambda s,\lambda,\omega),u_n(s-\omega,\lambda,\omega))\cdot$$

$$[K(\lambda s,\lambda s,u_n(\lambda s,\lambda,\omega),u_n(\lambda^2 s,\lambda,\omega),u_n(\lambda s-\omega,\lambda,\omega))]ds+$$

$$+\int_0^t s\frac{\partial K}{\partial u_2}(t,s,u_n(s,\lambda,\omega),u_n(\lambda s,\lambda,\omega),u_n(s-\omega,\lambda,\omega))\cdot$$

$$\cdot\left[\int_0^{\lambda s}\frac{\partial K}{\partial t}(\lambda s,s_1,u_n(s_1,\lambda,\omega),u_n(\lambda s_1,\lambda,\omega),u_n(s_1-\omega,\lambda,\omega))ds_1\right]ds+$$

$$+\int_0^t \frac{\partial K}{\partial u_3}(t,s,u_n(s,\lambda,\omega),u_n(\lambda s,\lambda,\omega),u_n(s-\omega,\lambda,\omega))y_n(s-\omega,\lambda,\omega)ds,$$

for $t\in[0,b]$, $\lambda\in[0,1]$, $\omega\in[0,c]$ and

$$y_{n+1}(t,\lambda,\omega):=0,\ \text{for } t\in[-c,0],$$

converge uniformly on $[-c,b]\times[0,1]\times[0,c]$ to u^* respectively to y^*, for all $u_0, y_0\in X$ and $(u^*,y^*)\in F_T$.

If we take $u_0=0$, $y_0=0$, then $y_1=\frac{\partial u_1}{\partial\lambda}$.

By induction we can prove that $y_n=\frac{\partial u_n}{\partial\lambda}$.

Thus $(u_n)_{n\in\mathbb{N}}$ converges uniformly to u^* and $\left(\frac{\partial u_n}{\partial\lambda}\right)_{n\in\mathbb{N}}$ converges uniformly to y^*. By using a Weierstrass argument, we conclude that $\frac{\partial u^*}{\partial\lambda}$ exists and $\frac{\partial u^*}{\partial\lambda}=y^*$.

These imply that $\frac{\partial u^*}{\partial\lambda}$ is a continuous function.

Remark 3.1. In the paper [14], A. Tămăşan has studied the differentiability with respect to λ for nonlinear pantograph equation.

References

[1] S.R. Bernfeld, V. Lakshmikantham, *An introduction to nonlinear boundary value problems*, Acad. Press, New York, 1974.

[2] L.F. Elsgoltz, S.B. Norkin, *Introduction to the theory of differential equations with deviating arguments*, MIR, Moscow, 1971 (in Russian).

[3] J.K. Hale, *Theory of functional differential equations*, Springer Verlag, 1977.

[4] J.K. Hale, L.A.C. Ladeira, *Differentiability with respect to delays*, J. Diff. Eq. 92(1991), 14-26.

[5] M.W. Hirsch, C.C. Pugh, *Stable manifolds and hyperbolic sets*, Proc. Symp. in Pure Math., 14(1970), 133-163.

[6] T. Kato, J.B. Mc Leod, *The functional-differential equation $y'(x)=ay(\lambda x)+by(x)$*, Bull. Amer. Math. Soc., 77, 6(1971), 891-937.

[7] V. Mureşan, *Differential equations with affine modification of the argument*, Transilvania Press, Cluj-Napoca, 1997 (in Romanian).
[8] J.R. Ockendon, A.B. Tayler, *The dynamics of a curent collection system for an electric locomotive*, Proc. Roy. Soc. London, A 322, 1971, 447-468.
[9] J.R. Ockendon, *Differential equations and industry*, The Math. Scientist, vol.5, nr.1(1980), 1-12.
[10] I.A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Caroline, 34, 3(1993), 769-773.
[11] I.A. Rus, *Picard operators and applications*, Babeş-Bolyai University Cluj-Napoca, Preprint 3(1996).
[12] I.A. Rus, *An abstract point of view for some integral equations from applied mathematics*, Proceedings of the International Conference on Analysis and Numerical Computation of Solutions of Nonlinear Systems, Univ. Timişoara (1997), 256-270.
[13] I.A. Rus, *A fiber generalized contraction theorem and applications* (to appear).
[14] A. Tămăşan, *Differentiability with respect to lag for nonlinear pantograph equations*, PUMA Vol.9(1998), No.1-2, 215-220.

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ON THE EXISTENCE AND UNIQUENESS OF A SOLUTION FOR AN ELLIPTIC PROBLEM

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Dedicated to Professor Vasile Pop at his 60^{th} anniversary

Abstract. Existence, uniqueness and a maximum principle for the solution of a nonlinear elliptic problem is obtained. The underlying ideea combines the fix point theorem and a maximum priciple regularization.

1. Introduction

In this work we show existence and uniqueness of a solution for a quasilinear elliptic problem. Under the assumptions below, a maximum principle is also obtained. Let Ω be a bounded domain in $\mathbb{R}^d (d \geq 1)$ with a Lipschitz continuous boundary. We deal here with the following nonlinear elliptic problem

Problem P.

$$\begin{aligned} \beta(u) - \Delta u + V \cdot \nabla u = f, \qquad &\text{in } \Omega, \\ u = 0, \qquad &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

$\beta : \mathbb{R} \to \mathbb{R}$ being a strictly increasing function. Throughout this paper we assume, without loss of generality, that $\beta(0) = 0$. However, there are no growth conditions imposed on β in the vicinity of 0, where its derivative may become infinite. Such problems can be investigated through the theory of monotone operators (see, e.g. [5]). Other possible approaches are the construction of sub- an supersolutions ([3]). We give here a different proof, based on the fix point theorem. First, the original problem is approximated by a sequence of problems, for which the differential operator is bounded on a closed convex subset. For the approximating problems, the Banach contraction principle guarantees the existence and uniqueness of a solution. Finally, the sequence of solutions are shown to converge to the unique solution of Problem P.

About the problem in (1.1), we make the following assumptions.

(A1) β is continuous and differentiable a.e. (almost everywhere), $\beta(0) = 0$, $\beta'(u) \geq l_\beta > 0$ and β' may be infinite in 0.

(A2) V is $L^\infty(\Omega)$ and divergence free, or it satisfies the condition

$$\|V\|_\infty < 4l_\beta.$$

(A3) $f \in L^\infty(\Omega)$ is positive a.e.. The nonlinearity function β is strongly monotone. Its smoothness may be weakened, but the above assumption makes the presentation easier. However, we do not impose any growth condition in 0, therefore the corresponding operator may be unbounded. The assumption in (A1) implies that the derivative of β is finite almost everywhere, excluding 0. Then there are two positive constants such that

$$0 < l_\beta \leq \beta'(u) \leq C(\varepsilon) < \infty, \tag{1.2}$$

for any real $u \geq \varepsilon$,

The speed of the first order term is essentially bounded. It is either divergence free, or controlled by the inequality in (A2). We are interested here in positive solutions, so the right hand side of the problem in (1.1) is positive (a. e. in Ω). Moreover, we assume f essentialy bounded, so a constant $M > 0$ can be found s.t. $f(x) \leq M$ a.e. in Ω.

The Laplace operator in (1.3) can be replaced by $\nabla A \nabla u$, A being a symmetric, positive definite matrix. There would be no changes in what follows, excepting the essential bounds for V, if this is not divergence free. In this case the inequality in (A2) becomes $\|V\|_\infty < 4l_\beta \lambda_{min}$, where λ_{min} stands for the minimal eigenvalue of A. Analogous, other boundary conditions may also be considered here, but only if they provide a positive solution. This restriction is fulfilled in many of the cases of practical interest.

A usual definition of the solution for (1.1) is

Definition 1.1. u is called a solution of the problem in (1.1) iff $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ the equation holds true

$$(\beta(u), \varphi) + (\nabla u, \nabla \varphi) + (V \cdot \nabla u, \varphi) = (f, \varphi). \tag{1.3}$$

In this paper we show that the problem in (1.3) has a unique solution, which is positive and essentially bounded. This is stated in the following

Theorem. Under the assumptions (A1), (A2) and (A3), the problem in (1.1) has a unique weak solution satisfying

$$0 \leq u(x) \leq M/l_\beta$$

almost everywhere in Ω.

Here and below we use the standard notation. In particular, we let $(\cdot, \cdot)$ stand for the inner product on $L^2(\Omega)$ and $\|\cdot\|$ for the norm in $L^2(\Omega)$. Correspondingly, $\|\cdot\|_1$ denotes the norm in $H^1(\Omega)$. If necessary, $\|\cdot\|_{k,p,\Omega}$ denotes the norm in $H^{k,p}(\Omega)$.

The paper is organized as follows. In the next section we prove a maximum principle for the perturbed nonlinear problems approximating the one in (1.3). Section 3 is devoted to the proof of existence of a solution for the above mentioned problems. Finally the main result is obtained.

2. A maximum principle

In this section we consider a perturbation of the nonlinear problems in (1.1). Concretely, let ε be an artificial positive small number and consider the perturbed problem

Problem P_ε.

$$\begin{aligned} \beta(u+\varepsilon) - \Delta u + V \cdot \nabla u &= f + \beta(\varepsilon), \qquad \text{in } \Omega, \\ u &= 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

In fact, Problem P_ε is similar to Problem P, but the boundary data are perturbed by ε. This idea has been applied in the analysis of some class of degenerate equations (see, e.g., [2]). The resulting Dirichlet problem is reduced to P_ε, with homogeneous boundary conditions. As for the original problem in (1.1), a weak form reads

Definition 2.1. u is called a solution of the problem in (2.1) iff $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ the equation holds true

$$(\beta(u+\varepsilon), \varphi) + (\nabla u, \nabla\varphi) + (V \cdot \nabla u, \varphi) = (f + \beta(\varepsilon), \varphi). \tag{2.2}$$

A solution for Problem P_ε is sought in the following set

$$W = \left\{ \varphi \in H_0^1(\Omega) : 0 \leq \varphi \leq \frac{M}{l_\beta}, \quad a.e. \right\}. \tag{2.3}$$

In this case u is positive a.e. in Ω and the derivative of β is bounded, as given in (1.2). It is clear that, W is closed and convex. In fact, by considering the above set, a maximum principle has been stated. This is shown in the following

Theorem 2.1. *Assume (A1), (A2) and (A3). Then, if a weak solution of Problem P_ε exists, it belongs to W.*

Proof. We start with the proof of the lower bound because this is essential in our approach. This is done by reductio ad absurdum, following the ideas used in [1] for the proof of the weak maximum principle. Assume that

$$\inf_\Omega u < 0,$$

(here the infimum should be understood as the essential one). Let φ be a positive (a.e.) function in $H_0^1(\Omega)$. Since $f \geq 0$, for any $\delta < 0$, the monotonicity of β leads to the inequality

$$0 \leq (\beta(\varepsilon) - \beta(\varepsilon+\delta), \varphi) + (f, \varphi).$$

Thus, if u solves Problem P_ε, the following holds true

$$(\beta(u+\varepsilon) - \beta(\varepsilon+\delta), \varphi) + (\nabla u, \nabla\varphi) + (V \cdot \nabla u, \varphi) \geq 0$$

for all $\varphi \geq 0$ in $H_0^1(\Omega)$. Moreover, $\varphi_\delta = [\delta - u]_+$ is an $H_0^1(\Omega)$ function, which is also positive a.e. in Ω (here $[x]_+ = x$ if $x \geq 0$, otherwise $[x]_+ = 0$). Taking φ_δ in the above inequality yields

$$\int_{u<\delta} (\beta(u+\varepsilon) - \beta(\varepsilon+\delta))(\delta - u) + \int_{u<\delta} \nabla u \cdot \nabla(\delta - u) + \int_{u<\delta} (\delta - u) V \cdot \nabla u \geq 0.$$

Denoting by Ω_δ the support of $[\delta - u]_+$ and applying the Cauchy inequality, the above relation becomes

$$\int_{\Omega_\delta} (\beta(u+\varepsilon) - \beta(\varepsilon+\delta))(u - \delta) + \|\nabla u\|^2_{0,2,\Omega_\delta} \leq$$
$$\leq \|V\|_\infty \|\nabla u\|_{0,2,\Omega_\delta} \|u - \delta\|_{0,2,\Omega_\delta \cap supp\{\nabla u\}},$$

where $supp\{\nabla u\}$ is the support of ∇u. Now, the monotonicity of β yields

$$\|\nabla \varphi_\delta\|^2_{0,2,\Omega_\delta} \leq \|V\|_\infty \|\varphi_\delta\|_{0,2,\Omega_\delta \cap supp\{\nabla u\}} \|\nabla \varphi_\delta\|_{0,2,\Omega_\delta}.$$

Hence, the following holds true (for any $\delta < 0$)

$$\|\nabla \varphi_\delta\|_{0,2,\Omega_\delta} \leq \|V\|_\infty \|\varphi_\delta\|_{0,2,\Omega_\delta \cap supp\{\nabla u\}}.$$

Next, the Sobolev embedding theorem is applied. If $d > 2$ (where d stands for the dimension of the domain Ω), because φ_δ lies in $H_0^1(\Omega)$, we get

$$\|\varphi_\delta\|_{0,\frac{2d}{d-2},\Omega} \leq C\|\nabla \varphi_\delta\|_{0,2,\Omega} \leq \|V\|_\infty \|\varphi_\delta\|_{0,2,\Omega_\delta \cap supp\{\nabla u\}}.$$

Now, the Holder inequality for the last term yields

$$\|\varphi_\delta\|_{0,\frac{2d}{d-2},\Omega} \leq \|V\|_\infty \left(meas\{\Omega_\delta \cap supp\{\nabla u\}\}\right)^{\frac{1}{d}} \|\varphi_\delta\|_{0,\frac{2d}{d-2},\Omega},$$

where the inclusion $\Omega_\delta \subset \Omega$ was used. Therefore

$$meas\{\Omega_\delta \cap supp\{\nabla u\}\} \geq C, \tag{2.4}$$

where $C > 0$ does not depend on δ. Analogous, if $d = 2$, the same property of the above set can be obtained. This shows that the essential infimum of u is finite. Moreover, since the constant in (2.4) does not depend on δ, the inequality must hold as δ tends to $\inf_{\Omega} u$. That is, the function u attains its infimum in Ω on a set of positive measure, where at the same time its gradient vanishes (since the function is constant almost everywhere there). This contradicts the inequality in (2.4) and therefore the assumption on the infimum of u is false. A similar argument shows that u has a finite essential supremum. The equality in (2.1) can be rewritten as

$$(\beta(u+\varepsilon)-\beta(\delta+\varepsilon),\varphi)+(\nabla u,\nabla\varphi)+(V\cdot\nabla u,\varphi)=(f+\beta(\varepsilon)-\beta(\delta+\varepsilon),\varphi),$$

for any real number δ. Assume now that the essential supremum of u lies above $\frac{M}{l_\beta}$, taking an arbitrary $\delta > M/l_\beta$ and $\varphi_\delta = [u-\delta]_+ \in H_0^1(\Omega)$, everything follows as before. In this case we have

$$\beta(\delta+\varepsilon)-\beta(\varepsilon)\geq l_\beta\delta\geq M,$$

and therefore

$$(f+\beta(\varepsilon)-\beta(\delta+\varepsilon),\varphi_\delta)\leq 0.$$

The rest of the proof relies on similar arguments as for the lower bound. □

Remark 2.1. Everywhere in the proof ε can be taken 0 since the upper bound for β' in (1.2) is never used in the proof. Therefore the above maximum principle remains valid also for the problem in (1.1), in its weak form.

3. Existence and uniqueness

In this part existence and uniqueness of a solution for Problem P_ε is obtained. This is done by a fix point argument. To do so let K be a constant which will be given below and define, for $\psi,\varphi\in H_0^1(\Omega)$ and $\chi\in W$,

$$\begin{aligned} a_K(\psi,\varphi) &= K(\psi,\varphi)+(\nabla\psi,\nabla\varphi)+(V\cdot\nabla\psi,\varphi),\\ f_{K,\varepsilon}(\chi;\varphi) &= K(\chi,\varphi)-(\beta(\chi+\varepsilon)-\beta(\varepsilon),\varphi)+(f,\varphi), \end{aligned}$$

which are linear. Clearly, a_K is also bounded. The same holds for $f_{K,\varepsilon}$ since χ is positive a.e. in Ω and therefore we can get

$$\begin{aligned} |f_{K,\varepsilon}(\chi;\varphi)| &\leq \left(K\|\chi\|+\|(\beta(\chi+\varepsilon)\|+\beta(\varepsilon)\ meas\{\Omega\}^{\frac{1}{2}}+\|f\|\right)\|\varphi\|\\ &\leq \left[(K+C(\varepsilon))\|\chi\|+\beta(\varepsilon)\ meas\{\Omega\}^{\frac{1}{2}}+\|f\|\right]\|\varphi\|, \end{aligned}$$

where $C(\varepsilon)$ is the constant in (1.2). Recalling the notation in (2.3), an iteration can be induced through $T: W\to W$, the operator giving the solution of the following problem **Problem Aux:** Let $\psi \in W$. Find $T\psi \in W$ such that

$$a_K(T\psi, \varphi) = f_{K,\varepsilon}(\psi; \varphi) \tag{3.1}$$

for all $\varphi \in H_0^1(\Omega)$.

Now the iteration can be defined as

$$\psi^{i+1} = T\psi^i \tag{3.2}$$

for $i \geq 0$ and $\psi^0 \in W$ arbitarily choosen. This will help us in the proof of existence and uniqueness for the solution of Problem P_ε. These properties cannot be obtained as a direct consequence of a nonlinear Lax-Milgram lemma since the differential operator appearing here is bounded only on a subset of $H_0^1(\Omega)$. If K satisfies

$$K \geq C(\varepsilon), \quad \text{and} \quad K \geq l_\beta \tag{3.3}$$

(the last inequality being implied by the first one if ε is small), the coercivity of a_K follows as a consequence of the Cauchy inequality

$$\begin{aligned} a_K(\psi, \psi) &= K\|\psi\|^2 + \tau\|\nabla\psi\|^2 - \|V\|_\infty \|\nabla\psi\| \|\psi\| \\ &\geq \left(K - \frac{\eta}{2}\right)\|\psi\|^2 + \left(1 - \frac{\|V\|_\infty^2}{2\eta}\right)\|\nabla\psi\|^2, \end{aligned} \tag{3.4}$$

where $\eta = \dfrac{4l_\beta + \|V\|_\infty^2}{4}$. Clearly, this choice of η satisfies the inequality

$$\|V\|_\infty^2 < 2\eta < 4l_\beta, \tag{3.5}$$

and therefore (A2) and (3.3) ensures the coercivity of a_K.

Remark 3.1. If V is divergence free, applying the Gauß divergence theorem, a simple calculation shows that, for any H_0 function φ we get

$$(V \cdot \nabla\varphi, \varphi) = (\nabla \cdot (V\varphi), \varphi) = 0 - (V\varphi, \nabla\varphi) = -(\varphi, V \cdot \nabla\varphi),$$

and therefore $(V \cdot \nabla\varphi, \varphi) = 0$. In this case the coercivity of a_K holds true without imposing any bounds for $\|V\|_\infty$, namely

$$a_K(\psi, \psi) \geq K\|\psi\|^2 + \|\nabla\psi\|^2.$$

If $\psi \in W$, the Lax-Milgram lemma can be applied to get a unique solution $T\psi \in H_0^1(\Omega)$ of Problem Aux, which is linear. In this way, we have defined an operator T from W to $H_0^1(\Omega)$. In fact, a similar proof as the one given for Theorem 2.1 leads to the following result

Lemma 3.1. *Assume (A1), (A2), (A3) and* $\psi^0 \in W$. *Then* $TW \subset W$.

Having defined the iterative scheme, a convergence result for the corresponding sequence is necessary. Below we will show that, under the restrictions in (3.3), T is a contraction mapping on the convex and closed set W with an appropriate norm, so u_ε can be taken as

$$u_\varepsilon = \lim_{i\to\infty} \psi^i .$$

In this case, assuming the above limit makes sense, it is easy to see that we have obtained a solution in W of Problem P_ε.

The existence of $\lim_{i\to\infty} \psi^i$ in W can be immediately seen by applying the fixed point theorem to T. This statement is supported by the following lemma

Lemma 3.2. *Assume (A1), (A2), (A3) and* $\psi^0 \in W$. *If* K *satisfies the inequalities in (3.3), then there is a norm on* $H_0^1(\Omega)$ *equivalent to the usual one, such that* T *is contractive on the closed set* W.

Proof. We consider here only the case V is not divergence free. Lemma 3.1 shows that W is preserved by T. Note that

$$\|\varphi\|_K^2 \equiv \left(K - \frac{\eta}{2}\right)\|\varphi\|^2 + \left(1 - \frac{\|V\|_\infty^2}{2\eta}\right)\|\nabla\varphi\|^2$$

is a norm on $H_0^1(\Omega)$, where η was choosen in (3.4). Moreover, $\|\varphi\|_K \le \sqrt{a_K(\varphi,\varphi)}$. With this new norm, T is a contraction mapping on the closed subset W, as follows from (3.1)

$$\begin{aligned} |a_K(T\psi_1 - T\psi_2, \varphi)| = \ & |K(\psi_1 - \psi_2, \varphi) - (\beta(\psi_1+\varepsilon) - \beta(\psi_2+\varepsilon), \varphi)| \\ \le \ & |((K - \beta'(\theta))(\psi_1 - \psi_2), \varphi)| \\ \le \ & (K - l_\beta)\|\psi_1 - \psi_2\|\|\varphi\|, \end{aligned}$$

for any $\psi_1, \psi_2 \in W$ and θ between $\psi_1 + \varepsilon$ and $\psi_2 + \varepsilon$. In the above inequalities the mean value theorem, the inequalities in (3.3), (1 2), and the positiveness of ψ_1, ψ_2 have been used. Hence, since

$$\|\varphi\|_K \geq \sqrt{K - \frac{\|V\|_\infty^2}{2\eta}}\,\|\varphi\|,$$

recalling the inequalities in (3.5), by taking $\varphi = T\psi_1 - T\psi_2$ the proof is completed. □

Remark 3.2. If V is divergence free, the norm $\|\cdot\|_K$ can be defined as

$$\|\varphi\|_K^2 \equiv K\|\varphi\|^2 + \|\nabla\varphi\|^2.$$

Then, in the above proof we get

$$\|T\psi_1 - T\psi_2\|_K^2 \leq (K - l_\beta)\|\psi_1 - \psi_2\|\|T\psi_1 - T\psi_2\| \leq \frac{K - l_\ell}{K}\|\psi_1 - \psi_2\|_K\|T\psi_1 - T\psi_2\|_K.$$

Remark 3.3. The above iteration relies on the operator T, which has been used in [3], pp. 96. But we show that T is a contraction mapping, at least for the present problems, in the setting defined in the proof of Lemma 3.2. This approach was considerd in [4]. The monotonicity of T can lead to an alternative proof similar to the one in [3].

The above results show that Problem P_ε has a unique solution u_ε in W. But in the frame set in Lemma 3.2, the uniqueness holds in the whole $H_0^1(\Omega)$.

Lemma 3.3. *Assume (A1), (A2), (A3) and $\psi^0 \in W$. If K satisfies the inequalities in (3.3), Problem P_ε has at most one solution $u_\varepsilon \in H_0^1(\Omega)$.*

Proof. Assume first V is not of divergence 0. Let $\psi, \theta \in H_0^1(\Omega)$, then $\psi - \theta \in H_0^1(\Omega)$. Assuming that both solve Problem P_ε with the same ψ^0, testing with $\varphi = \psi - \theta$ and subtracting the corresponding equalities for both solutions yields

$$\begin{aligned} 0 = {} & (\beta(\psi+\varepsilon) - \beta(\theta+\varepsilon), \psi - \theta) + \|\nabla(\psi-\theta)\|^2 + (V\cdot\nabla(\psi-\theta), \psi-\theta) \\ \geq {} & l_\beta\|\psi-\theta\|^2 + \|\nabla(\psi-\theta)\|^2 - \|V\|_\infty\|\nabla(\psi-\theta)\|\|\psi-\theta\| \\ \geq {} & \left(1 - \frac{\|V\|_\infty^2}{4l_\beta}\right)\|\nabla(\psi-\theta)\|^2, \end{aligned}$$

where (1.2) has been used. Assumption (A2) on V guarantees that the multiplication constant above is strictly positive, therefore $\|\nabla(w-\theta)\| = 0$, so w and θ coincide a.e. in Ω.

The case $\nabla \cdot V = 0$ can be handled as before. In this situation the first order term $(V \cdot \nabla(w-\theta), \psi - \theta)$ vanishes as shown in Remark 3.1. □

Remark 3.4. A similar approach shows that the original Problem P admits at most one solution in $H_0^1(\Omega)$.

Up to now we have considered Problem P_ε, which, as seen before, admits a unique solution u_ε satisfying a maximum principle. Moreover, $\|u_\varepsilon\|_1$ is uniformly bounded as ε goes to 0, as resulting from

$$\begin{aligned}(\beta(u_\varepsilon+\varepsilon), u_\varepsilon) + \|\nabla u_\varepsilon\|^2 = & -(V \cdot \nabla u_\varepsilon, u_\varepsilon) + (f + \beta(\varepsilon), u_\varepsilon) \\ \leq & \|V\|_\infty \|\nabla u_\varepsilon\| \|u_\varepsilon\| + (M + \beta(\varepsilon))\ meas\{\Omega\}^{\frac{1}{2}} \|u_\varepsilon\| \\ \leq & \frac{1}{2}\|\nabla u_\varepsilon\|^2 + C,\end{aligned}$$

where the constant C depends on M, l_β, Ω, but can be choosen the same for all ε less than a given one. The uniform boundedness results immediately since the first term of the left-hand side is positive. Therefore the sequence $\{u_\varepsilon, \varepsilon > 0\}$ is compact in H_0^1 and admits a weakly convergent subsequence, denoted - for simplicity - $\{u_\varepsilon\}$. If we denote the weak limit by u, the subsequence converges strongly in L^2 and a.e. in Ω to u. Thus, by the continuity of β, we get

$$\begin{aligned}(\beta(u_\varepsilon+\varepsilon), \varphi) &\longrightarrow (\beta(u), \varphi)\\ (\nabla u_\varepsilon, \nabla\varphi) &\longrightarrow (\nabla u, \nabla\varphi)\\ (\nabla u_\varepsilon, V\varphi) &\longrightarrow (\nabla u, V\varphi)\\ (f+\beta(\varepsilon), \varphi) &\longrightarrow (f, \varphi),\end{aligned} \tag{3.6}$$

for all $\varphi \in H_0^1(\Omega)$. Therefore the limit u is a solution of Problem P in the sense of Definition 1.1. This, together with the maximum principle and the uniqueness result mentioned in Remarks 2.1 and 3.4 demonstrates the main result stated in the introduction

Theorem 3.4. *Under the assumptions (A1), (A2) and (A3), Problem P has a unique weak solution satisfying*

$$0 \leq u(x) \leq M/l_\beta$$

almost everywhere in Ω.

References

[1] D. Gilbarg & N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren der Mathematischen Wissenschaften, 224, *Springer Verlag, Berlin*, 2^{nd} ed., 1983.

[2] O. A. Oleinik & A. S. Kalashnikov & Y.-L. Zhou, The Cauchy Problem and Boundary Value Problems for Equations of the Type of Unsteady Filtration, *Izv. Akad. Nauk SSSR Ser. Mat.*, 22(1958), 667 - 704.

[3] J. Smoller, Shock Waves and Reaction-Diffusion Equations, *Springer Verlag, New York*, 1983.

[4] W. A. Yong & I. S. Pop, A Numerical Approach to Porous Medium Equations, *Preprint 96-50 (SFB 359)* (1996), IWR, University of Heidelberg.

[5] E. Zeidler, Applied Functional Analysis, Vols. I, II, Applied Mathematical Sciences 108, 109, *Springer Verlag, New York*, 1995.

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FIXED POINTS, ZEROS AND SURJECTIVITY

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Dedicated to Professor Vasile Pop at his 60^{th} anniversary

Abstract. Let $(X,+)$ be an abelian group and $f : X \to X$ an operator. In this paper we study the connections between fixed points, zeros and surjectivity property of operator f.

1. Introduction

There are many papers which use the theory of fixed point to prove the surjectivity of an oparator (see Deimling [9], Dugundji-Granas [10], Rus [14, 16, 17, 18], Zeidler [20], Aldea [1, 2],...). Also there are papers which study when a surjective operators has fixed point (see Bae-Cho-Yeam [3], Browder [4], Ciric [5], Cramer-Ray [6], Danes [7], Danes-Kolomy [8], Kasahara [12], Morales [13], Rus [15], Wang-Li-Gao-Iseri [19],...). Let $(X,+)$ be an abelian group and $\mathcal{F} \subset \mathbb{M} := \{f : X \to X \mid \text{an operator}\}$. By definition:

(i) X has the fixed point property with respect to $\mathcal{F}$, if $f \in \mathcal{F}$ implies $F_f \neq \emptyset$.

(ii) X has the zero point property with respect to $\mathcal{F}$, if $f \in \mathcal{F}$ implies $Z_f \neq \emptyset$.

(iii) X has the surjectivity property with respect to $\mathcal{F}$, if $f \in \mathcal{F}$ implies f is surjective.

The aim of this paper is to study the following problem:

Problem 1. (see Rus, [18]) Determine the condition on X and $\mathcal{F}$ such that the following statements are equivalent:

(a) X has the fixed point property with respect to $\mathcal{F}$.

(b) X has the zero point property with respect to $\mathcal{F}$.

(c) X has the surjectivity property with respect to $\mathcal{F}$.

For study of this problem we use the following definition.

Definition 1.1. *(Rus, [18]) Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Function φ is a comparison function if satisfies:*

(i) φ is monotone increasing;

(ii) $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 for all $t \geq 0$.

Definition 1.2. *Let X be a Banach space. An operator $f : X \to X$ is a dilating operator if there is $k > 1$ such that*

$$||f(x) - f(y)|| \geq k \cdot ||x - y|| \tag{1}$$

for all $x,\ y \in X$.

A generalization of the last definition is the following.

Definition 1.3. *(Rus, [18]) Let X be a Banach space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function. An operator $f : X \to X$ is a φ-dilating operator if*

$$||f(x) - f(y)|| \geq \varphi(||x - y||) \tag{2}$$

for all $x,\ y \in X$.

Let $f : X \to X$ be an operator. In present paper we use the followimg notation:

$$F_f = \{x \in X \mid f(x) = x\}$$

$$Z_f = \{x \in X \mid f(x) = 0\}$$

2. Main result

In this section we give an answer to Problem 1. Other answer from the same problem was given by Gilespie and Williams in [11]. They have studied the same problem for X a Banach spaces and $\mathcal{F} = \{f : X \to X \mid f$ continuous and dilating operator$\}$. From our answer we can obtain the Gillespie- Williams answer's.

Theorem 2.1. *Let $(X, +)$ be an abelian group. Let $\mathcal{F} \subset \mathbb{M}$ a familly of operators $f : X \to X$. We suppose that:*

(i) $f \in \mathcal{F}$ implies f is injective operator;

(ii) $f \in \mathcal{F}$ *and* $f(X) = X$ *implies* $F_{f^{-1}} \neq \emptyset$;

(iii) For all $f \in \mathcal{F}$ *there is* $n_0(f) \in \mathbb{N}$ *such that* $f^{n_0} + 1_X \in \mathcal{F}$;

(iv) For all $f \in \mathcal{F}$ *and* $y_0 \in X$ *imply* $f + y_0 \in \mathcal{F}$.

Then the following statement are equivalent:

(a) X *has the fixed point property with respect to* $\mathcal{F}$.

(b) X *has the zero point property with respect to* $\mathcal{F}$.

(c) X *has the surjectivity property with respect to* $\mathcal{F}$.

Proof. **(a)**$\Longrightarrow$ **(b)** Let $f \in \mathcal{F}$. Let $g : X \to X$ be an operator define by $x \mapsto f^{n_0}(x) + x$. From (iii), $g \in \mathcal{F}$. Hence $F_g \neq \emptyset$. This implies $Z_{f_0^n} \neq \emptyset$. So $Z_f \neq \emptyset$.

(b)$\Longrightarrow$ **(c)** Let $y_0 \in X$ and $f \in \mathcal{F}$. We take $g : X \to X$, $x \mapsto f(x) - y_0$. It is clear that $g \in \mathcal{F}$. From (b) we have that $Z_g \neq \emptyset$. But this implies that there exists $x_0 \in X$ such that $f(x_0) = y_0$.

(c) $\Longrightarrow$ **(b)** Let $f \in \mathcal{F}$. From (ii) and (c) we have that $F_{f^{-1}} \neq \emptyset$. So $F_f \neq \emptyset$.

Now the problem is to find conditions for operator f (a generic element of $\mathcal{F}$) such that the hypothesis of Theorem 2.1 are satisfied.

If we take the set X a Banach space and $\mathcal{F} = \{f : X \to X \mid f \text{ is } \varphi\text{-dilating operator}\}$ we obtain:

Theorem 2.2. *Let* X *be a Banach spaces,* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a function, and* $\mathcal{F} = \{f : X \to X \mid f \text{ is } \varphi - \text{dilating operator}\}$. *Suppose that* φ *satisfies :*

(i) $\varphi(0) = 0$;

(ii) φ *is bijective;*

(iii) φ^{-1} *comparison function;*

(iv) there is $n \in \mathbb{N}$ *such that* $\varphi^n(t) - t \geq \varphi(t)$ *for all* $t > 0$.

Then the conclusion of Theorem 2.1 holds.

Proof. Using the hypothesis we will prove that assumptions (i)-(iv) from Theorm 2.1 are satisfied.

$(i)_{Th.2.1}$ From (i) and (ii) we have that for all x, y from X with $x \neq y$ we have

$$0 < \varphi(||x - y||) \leq ||f(x) - f(y)|| \Longrightarrow f(x) \neq f(y)$$

so f is injective.

$(ii)_{Th.2.1}$ Because f is bijective we have

$$\varphi(\|f^{-1}(x)-f^{-1}(y)\|)\leq \|x-y\|$$

for all $x,\ y\in X$.

But φ^{-1} is a comparison function, so

$$\|f^{-1}(x)-f^{-1}(y)\|\leq \varphi^{-1}(\|x-y\|)$$

for all $x,\ y\in X$.

We apply fixed point theorem for φ-contraction ([18]) and we have $F_{f^{-1}}\neq\emptyset$.

$(iii)_{Th.2.1}$ From (2) for all $n\in\mathbb{N}$ we have

$$\|f^n(x)-f^n(y)\|\geq \varphi^n(\|x-y\|) \tag{3}$$

for all $x,\ y$ from X.

Also

$$\|f^n(x)+x-(f^n(y)+y)\|\geq \|f^n(y)+x-(f^n(x)+x)\|-$$

$$-\|f^n(y)+y-(f^n(y)+x)\|=\|f^n(x)-f^n(y)\|-\|x-y\| \tag{4}$$

for all $n\in\mathbb{N}$ and $x,\ y\in X$.

From (3) and (4) we have

$$\|f^n(x)+x-(f^n(y)+y)\|\geq \varphi^n(\|x-y\|)-\|x-y\| \tag{5}$$

for all $n\in\mathbb{N}$ and $x,\ y\in X$.

From (iv) we have that there is $n_0\in\mathbb{N}$ such that

$$\|f^{n_0}(x)+x-(f^{n_0}(y)+y)\|\geq \varphi^{n_0}(\|x-y\|)-\|x-y\|\geq\varphi(\|x-y\|) \tag{6}$$

So there is $n_0\in\mathbb{N}$ such that $f^{n_0}+1_X\in\mathcal{F}$.

$(iv)_{Th.2.1}$ That property is verified using the definition of $\mathcal{F}$.

So the hypothesis of Theorem 2.1 are fulfilled which implies that the conclusion of Theorem 2.2 holds.

Also from Theorem 2.1 we obtain following theorem.

Theorem 2.3. *(Gillespie-Wiliams, [11]) Let X be a Banach space and $\mathcal{F} = \{f : X \to X \mid f$ continuous dilating oparator$\}$. Then the conclusion of Theorem 2.1 holds.*

Proof. We will prove that for X Banach spaces and operators class $\mathcal{F} = \{f : X \to X \mid f$ continuous dilating operator$\}$ statements (i)-(iv) from Teorema 2.1 are satisfied. Let $f \in \mathcal{F}$

(i) Let x, y be in X, $x \neq y$ we have that

$$0 < k \cdot \|x - y\| \leq \|f(x) - f(y)\| \implies f(x) \neq f(y),$$

so f is injective.

(ii) Because f is dilating and surjective we have that f is bijective and

$$\|f^{-1}(x) - f^{-1}(y)\| \leq \frac{1}{k}\|x - y\|$$

for all x, y from X.

So f^{-1} is contraction. Then $F_{f^{-1}} \neq \emptyset$ (Banach contraction principle).

(iii) If f is dilatating wiht constant k, then f^n is dilatating operator with constant k^n, for all $n \in \mathbb{N}$. We choose $n_0 \geq 1$ such that $k^{n_0} > 2$. Then

$$\|f^{n_0}(x) + x - f^{n_0}(y) - y\| \geq (k^{n_0} - 1) \cdot \|x - y\|$$

so there is $n_0 \in \mathbb{N}$ such that $f^{n_0} + 1_X$ is dilating operator.

(iv) Using the definition of dilating operator statement (iv) is obvious.

Using the facts proved and Theorem 2.1 we have (a) $\iff$ (b) $\iff$ (c).

Gillespie and Wiliams showed in [11] that in the finite dimensional Banach spaces operators $f : X \to X$ which are dilating and continuous are surjective. From Theorem 2.1 the surjective of an operator is equivalent with fixed point property and with zero point property.

Theorem 2.4. *Let X be a finite dimensional Banach space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

(i) $\varphi(0) = 0$;

(ii) φ is bijective;

(iii) φ^{-1} comparison function.

We consider the set $\mathcal{F} = \{f : X \to X \mid f$ is φ – dilating and continuous operator$\}$. Then X has the fixed point property with respect to $\mathcal{F}$.

Proof. It is known that any finite dimensional spaces is izomorph with $\mathbb{R}^n$ and the fixed point property is kept in the case of izomorph spaces. It is enough to prove the theorem for $X = \mathbb{R}^n$.

Because f is φ-dilating operator and φ satisfies (i) and (ii) we have that f is injective. Also f is continuous , so $f(\mathbb{R}^n)$ is open set.

(The last statement is true because in finite dimensional spaces an operator which is local injective and continuous is open.)

In order to prove that $f(\mathbb{R}^n)$ is a closed set, we consider the sequence y_n from $f(\mathbb{R}^n)$ convergent to y^*. We will prove that $y^* \in f(\mathbb{R}^n)$. Because

$$||f(x) - f(y)|| \geq \varphi(||x - y||),$$

for all $x,\ y \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to f(\mathbb{R}^n)$ is bijective, we have that:

$$||u - v|| \geq \varphi(||f^{-1}(u) - f^{-1}(v)||) \tag{7}$$

for all $u,\ v \in f(\mathbb{R}^n)$.

Function φ is bijective and φ^{-1} is comparison function so,

$$\begin{aligned} ||f^{-1}(u) - f^{-1}(v)|| &\leq \varphi^{-1}(||u - v||) \\ &< ||u - v|| \end{aligned} \tag{8}$$

for all $u,\ v$ from $f(\mathbb{R}^n) \iff f^{-1}$ is uniform continuous.

Because $y_n \in f(\mathbb{R}^n)$ we have that there is $x_n \in \mathbb{R}^n$ such that $y_n = f(x_n) \longrightarrow y^*$.

Operator f^{-1} is uniform continuous so

$$f^{-1}(f(x_n)) \to z^* \iff x_n \to z^* \in \mathbb{R}^n$$

Using continuity of f we obtein

$$f(x_n) \to f(z^*) \text{ but}$$

$$f(x_n) \quad \to \quad y^* \tag{9}$$

which means $f(z^*) = y^* \iff y^* \in f(\mathbb{R}^n)$. So, $f(\mathbb{R}^n)$ closed set. Because we proved that $f(\mathbb{R}^n)$ is open set we have that $f(\mathbb{R}^n) = \mathbb{R}^n \iff f$ is surjective.

Operator $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, φ^{-1} is comparison function and (8) take place, then from fixed point theorem for φ-contraction (see Rus, [18]) we have that $F_{f^{-1}} \neq \emptyset \implies F_f \neq \emptyset$ for all $f \in \mathcal{F} \iff X$ has the fixed point property with respect to $\mathcal{F}$.

Theorem 2.5. *(Gillespie-Williams, [11]) If X is a finit dimensional Banach space and f is continuous dilating operator. Then $F_f \neq \emptyset$.*

Proof. Because f is dilating operator there is $k > 1$ such that f is φ-dilating operator with $\varphi(t) = k \cdot t$. Function φ verifies assumptions (i), (ii) and (iii) from Teorema 2.4 and $f \in \mathcal{F}$. Using Teorema 2.1 we have that $F_f \neq \emptyset$.

For infinit dimensional case there is the following open problem.

Open problem. (Gillespie-Williams) Let X be an infinit dimensional space and f a continuous dilating operator. Is f a surjective operator?

References

[1] F. Aldea, *Surjectivily via fixed point structures,* Seminar on Fixed Point Theory "Babe's-Bolyai" Univ., 1997, 9-14.

[2] F. Aldea, *Surjectivity theorems for normcontraction operators,* to appear in Mathematica.

[3] J. S. Bae, E. W. Cho, S. H. Yeam, *A generalization of the Caristi-Kirk fixed point theorem and its applications to mappings theorems,* J. Korean Math. Soc., 31 (1994), 29-48.

[4] F.Browder, *Fixed Point Theory and Nonlinear Problems,* Bull. Amer. Math. Soc., 9(1983), 1-39.

[5] L. Ciric, *A new class of nonexpansive type mappings and fixed points,* Czeck. Math. J., 49(1999), 891-899.

[6] W.J. Cramer, W.O. Ray, *Solvability of nonlinear operator equations,* Pacific J. Math., 95(1981), 37-50.

[7] J. Danes, *Surjectivity and fixed point theorems,* C. M. U. C., 13(1972), 181-184.

[8] J. Danes, J. Kolomy, *Fixed points, surjectivity and invariance of domain theorems for weakly continuous mappings,* Boll. U.M.I., 5 (1976), 369-394.
[9] K. Deimling, *Nonlinear functional analysis,* Springer-Verlag, 1985.
[10] J. Dugundji, A. Granas, *Fixed point theory,* Warszawa, Polish Scientific Publishers, 1982.
[11] A. Gillespie, B. Williams, *Fixed point theorems for expanding maps,* Univ. Texas at Arlington, Preprint of Math. Technical Report 154(1981).
[12] S. Kasahara, *Surjectivity and fixed point of nonlinear mappings,* Math. Sem. Notes, 2 (1974), 119-126.
[13] C. Morales, *Zeros for strongly accretive set-valued mappings,* C.M.U.C., 27(1986), 445-469.
[14] I. A. Rus, *Surjectivity and iterated mappings,* Math. Sem. Note, 10(1981), 179-181.
[15] I. A. Rus, *Fixed point and surjectivity for (α, φ) contractions,* Itinerant Seminar on Functional Equations Approximation and Convexity, Cluj-Napoca, 1983, 143-146.
[16] I. A. Rus, *Generalized contraction,* Seminar on fixed point theory, 1983, 1-131.
[17] I. A. Rus, *Normcontraction mappings outside a bounded set,* Itinerant Seminar on Functional Equations Approximation and Convexity, Cluj-Napoca, 1986.
[18] I.A. Rus, *Contracții generalizate,* to appear.
[19] S. Wang, B. Li, Z. Gao, K. Iseri *Some fixed point theorems on expansion mappings,* Math. Japonica 29(1984), 631-636.
[20] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed points theorems,* Springer, 1990.

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BOOK REVIEWS

T. Banakh, T. Radul, M. Zarichnyi, *Absorbing Sets in Infinite-Dimensional Manifolds*, Mathematical Studies, Monograph Series, Volume 1, VNTL Publishers, Ukraine, 1996.

The book is devoted to the theory of absorbing sets and its applications, most of them consisting in beautiful and short characterizations of many remarkable spaces. The term *space*, stands for *separable, metrizable topological space*.

The first chapter of the book contains an exposition of the basic theory of absorbing sets.

A space X is said to be $\mathcal{C}$-*absorbing* (with respect to a given class $\mathcal{C}$ of spaces) if:

a) X is strongly $\mathcal{C}$-*universal ANR* (absolute neighborhood retract) satisfying *SDAP* (strong discrete approximation property)

b) $X \in \sigma\mathcal{C}$ (i.e. X is a countable union of spaces from $\mathcal{C}$)

c) X is a Z_σ space (i.e. X is a countable union of Z-spaces in $\mathcal{C}$.

[recall that a set $A \subseteq X$ is a Z-space if for every open cover $\mathcal{U}$ of X, there exists a continuous map $f : X \to X$ such that $f(X) \cap A = \emptyset$ and f, id_X are $\mathcal{U}$-close, i.e. for each $x \in X$ with $f(x) \neq x$, there exists $U \in \mathcal{U}$ such that $x \in U, f(x) \in U$].

The second chapter, "Construction of absorbing sets", contains examples of absorbing sets with respect to several classes of sets, sometimes defined by dimensional conditions.

The third chapter contains some even more technical results concerning strong universality for pairs and for spaces.

The last two chapters include applications to infinite products, topological groups, convex sets, spaces of probability measures.

The results included in the 232 pages of the book integrate the work of the authors with work of many other mathematicians such as K. Borsuk, C. Bessaga, A. Pelczińsky, H. Torunczyk, M. Bestvina, J. Mogilski, T. Dobrowolski, O. Keller.

The book contains many exercises and notes and comments at the end of each chapter.

V. Anisiu

Steven G. Krantz, *Handbook of Complex Variables*, Birkhauser, Boston-Basel-Berlin 1999, 290 pp., ISBN 0-8176-4011-8.

The present book consists an excellent introduction and reference for undergraduate students, graduate students, engineers and all researchers interested in the study of basic concepts of complex analysis of one variable. The book is divided into sixteen chapters. The first two contain a detailed introduction and overview in the theory of complex numbers, holomorphic functions and Cauchy integrals. Chapter 3 is devoted to some applications of the Cauchy theory, such as: the Cauchy estimates, Liouville's theorem, the fundamental theorem of algebra, the zeros of holomorphic functions, uniqueness of analytic continuation, etc. Isolated singularities and Laurent series are studied in Chapter 4. In this chapter it is clearly presented the calculus of residues and its applications to the calculation of definite integrals and sums. Chapter 5 is concerned with some problems and questions concerning the argument principle, that have a geometric and qualitative nature rather than an analytical one. These questions center around the issue of the local geometric behavior of a holomorphic function. The purpose of chapter 6 is to study the notion of conformal mapping, while the concepts of harmonic functions are presented in chapter 7. Chapters 8, 9 are concerned with the theory of infinite series and products and their applications to obtain the Weierstrass and Mittag-Leffler factorization theorems, Jensen's formula, etc. Chapter 10 and 11 deal with some problems about analytic continuation and rational approximation theory. In chapter 12 contains a brief discussion about the notion of schlicht (one-to-one) functions and a historical discussion concerning *Bieberbach conjecture*. Chapter 13 treats the most important special functions: the Gamma function of Euler, the Beta function of Legendre, and the Zeta function of Riemann.

On the other hand, the author has made a special effort to include a detailed material on areas of engineering and physics in which complex variable theory is applied. This discussion is included in both chapters 14 and 15. The book concludes with a brief discussion of the software that is useful for understanding several concepts of one complex variable theory.

Written by an eminent specialist in the field, in a clear and rigorous manner, the book contains a lot of results which are very often illustrated by particular examples followed by diagrams and figures (as, for instance, the appendix to chapter 14). A special emphasis is put on the applications of complex analysis to engineering and physics. The author has made also special efforts to give through references to all topics presented.

It can be used as a standard reference in the field. I recommend it warmly to all desiring to learn, to teach or to use complex variable theory.

G. Kohr

D. Repovs and P. V. Semenov, *Continuous Selections of Multivalued Mappings* Mathematics and Its Applications Vol. 455, viii + 356 pp, Kluwer Academic Publishers, Dordrecht Boston London 1998, ISBN: 0-7923-5277-7.

Multivalued analysis is a field which has been intensively developing in the last years, mainly in connection with its deep applications to optimization theory, convex analysis and mathematical economics. Although there are several good books dealing with multivalued analysis (as, e.g., J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser 1990, and G. Beer, *Topologies on Closed and Convex Sets*, Kluwer A.P. 1993), this is the first monograph devoted to a systematic treatment of various aspects of the theory of continuous selections, a subject which has its roots in the pioneering work done by E. Michael in the mid 1950's. The book focuses on continuous selections for lower semi-continuous multivalued mappings on paracompact spaces, but measurable selections are considered too, via the unifying approach proposed by G. Mägerl in 1978. In fact, the paracompactness and lower semicontinuity are the natural framework in which continuous selections can be treated: the existence of a continuous selection for a multivalued mapping $F: X \to Y$, X, Y topological spaces, forces F to be lower semicontinuous, and if any convex-valued lower semicontinuous mapping F with closed convex values from a topological space X to a Banach space Y admits a continuous selection then the space X must be paracompact.

The book is divided into three parts; A. *Theory*, B. *Results*, and C. *Applications*. The first part of the book develops the basic of the theory, being aimed to students in mathematics in their senior year or at post-graduate level. It starts by an introductory section surveying the results from topology and functional analysis, needed for the understanding of the book. The proofs in this part are given in details and are presented in a structured form - part one *Construction* and part two *Verification*. The first part contains the steps of the proof and the properties which have to be verified, whereas in the second part these properties are effectively checked. Although the book has no exercises, these steps of the proofs can be used as exercises for self-training by the beginner. The verification part can be skipped by the experienced reader.

The second part is devoted to specialists in the area and its goal is to give a comprehensive survey on the existing results on continuous selections. In this part, one proves selection theorems for more general multivalued maps than lower semicontinuous, as almost lower semicontinuous, quasi lower semicontinuous, etc. The condition on the map to have convex values is also relaxed by considering paraconvex sets, topological convex structure or working with an axiomatic approach to convexity. A section is devoted to measurable selections. In this part most of the proof are omitted or only sketched, in order to cover maximum of results in a reasonable number of pages.

The third part of the book, a very interesting one, contains applications of the developed machinery. Among these we mention: Bartle-Graves type theorems and liftings, the homeomorphism of separable Banach spaces, fixed-point theorems, metric projections and differential inclusions.

The bibliography of the book counts 422 items.

Written by two eminent specialist in the area and including their original results, the book is a valuable contribution to multivalued analysis and its applications, appealing to a large audience, including graduate students and research mathematicians interested in general topology, functional analysis, convex sets and convex analysis, optimization theory, and mathematical economics.

S. Cobzaş

Myroslav Sheremeta, *Analytic Functions of Bounded Index*, Mathematical Studies - Monograph Series, vol.6, VNTL Publishers, Lvov 1999, 141 pp., ISBN 966-7148-77-7.

In this monograph the author investigates properties of analytic functions of bounded index in arbitrary domains. In the case of entire functions of bounded index there are many results which are not typical for more general classes of analytic functions. Some results of such kind are included in the present monograph. It consists of seven chapters. The first chapter deals with the notion of a function with bounded index, and includes some properties of such functions. It is also included a theorem of Hayman in the case of entire functions of bounded index (Theorem 1.5), as well as some applications of this result. Chapter 2 is devoted to some problems concerning value distribution of functions of bounded l-index. The aim of chapter 3 is to obtain some growth of entire and analytic functions of bounded l-index in a disc $\{z \in \mathbf{C} : |z| < R\}$. Moreover, in the same chapter are pointed several path of growth investigations of these functions. Chapter 4 is concerned with the notion of a function of bounded $l-M-$index while Chapter 5 studies several properties of analytic solutions of some linear differential equations, related to boundedness of l-index. In chapter 6 there are investigated conditions on a positive continuous function l on $[0, \infty)$ ensuring the existence of an entire function of prescribed growth and bounded l-index. Chapter 7 concludes with the case of entire functions of bounded 1-index.

The book is clearly written, introducing the reader to the area of analytic functions of bounded index. The final part of the book contains very useful historical comments about this subject. It can be recommended warmly to researchers interested in the concept of analyticity of bounded index.

G. Kohr

Nik Weaver, *Lipschitz Algebras,* World Scientific, Singapore 1999, 223 pp., ISBN; 981-02-3873-8.

Although important and having a rich and beautiful theory, the spaces of Lipschitz functions have been less studied than other function spaces of functional analysis. A possible explanation, suggested by the author in the Preface, could be the fact that the algebraic properties of the spaces of Lipschitz functions on a metric space X are intimately related with the metric structure of the underlying space X, while the properties of the spaces $C(X)$ and $L^\infty(X)$ are the functional-analytic realizations of topological or measure structures. The fact that the metric structures are finer and deeper than topological ones, makes the subject harder to handle.

The main objects of the present book are the space $\mathrm{Lip}_0(X)$ of scalar-valued Lipschitz functions vanishing at a distinguished point $e \in X$, normed by $L(f) = \sup\{|f(p) - f(q)|/\rho(p,q) : p, q \in X,\ p \neq q\}$, and the space $\mathrm{Lip}(X)$ of all scalarly-valued bounded Lipschitz functions on the metric space X, equipped with the norm $\|f\|_L = \max\{L(f), \|f\|_\infty\}$. Both of them are Banach spaces and if X is of finite diameter then $\mathrm{Lip}_0(X)$ is a Banach algebra. $\mathrm{Lip}(X)$ is also a Banach algebra satisfying the (unessentially) weaker law $\|fg\|_L \leq 2\|f\|_L\|g\|_L$. The spaces of Lipschitz functions have some universality properties, making them important tools of functional analysis. For instance, if A is a commutative unital Banach algebra then the Gefand transform takes A nonexpansively into $C(\Delta(A))$, where $\Delta(A)$ is the carrier space (the field of Banach algebra homomorphisms from A to the base field $\mathbf{F}$). Endowed with the relative $*$-topology $\Delta(A)$ is a compact Hausdorff space. Considering $\Delta(A)$ with the inherited metric, the Gelfand transform takes A nonexpansively into $\mathrm{Lip}(\Delta(A))$. Also for a metric space X with distiguished element e there is a Banach space E and an isometrical imbedding of X into E such that every nonexpansive map f from X into another Banach space F satisfying $f(e) = 0$, extends uniquely to a nonexpansive linear maap from E to F. Furthermore the dual of the space E is isometrically isomorphic to $\mathrm{Lip}_0(X)$. There are two realizations of $\mathrm{Lip}_0(X)$ as a dual Banach space, one by de Leeuw and other by Arens-Eells, which are carefully presented in the second chapter of the book, Ch. 2, *The predual.* The first chapter of the book, Ch. 1 *Lipschitz spaces,* contains the basic results on metric spaces and Lipschitz functions.

The objects of the third chapter, Ch. 3 *Little Lipschitz spaces,* are functions in $\mathrm{Lip}_0(X)$, X a compact metric space with distinguished point, satisfying the condition: for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(p) - f(q)| \le \epsilon\rho(p,q)$ whenever $\rho(p,q) < \delta$. Such functions are called little Lipschitz functions and the corresponding subspaces of $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}(X)$ are denoted by $\mathrm{lip}_0(X$ and $\mathrm{lip}(X)$, respectively. Under some hypotheses we have $\mathrm{Lip}_0(X) = \mathrm{lip}_0(X)^{**}$, showing that $\mathrm{Lip}_0(X)$ can be even a double dual Banach space.

Banach algebra properties of the spaces $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}(X)$, meaning ideals, carrier spaces, spectral synthesis, derivations, are studied in Ch. 4 *Lipschitz algebras,* while Ch. 5 *Lipschitz lattices* is concerned with lattice properties of these spaces. Ch. 6, *Measurable Lipschitz algebras,* is concerned with measurable metric spaces (X, μ) and the spaces $\mathrm{Lip}(X, \mu)$ of bounded measurable Lipschitz functions.

In the final chapter of the book, Ch. 6 *Derivations,* the author studies derivations on Lipschitz algebras, giving the right extension to a very general setting of the well known fact that a differentiable function is Lipschitz if and only if its differential is bounded. The last section of this chapter contains a brief account on noncommutaive Lipschitz algebras, a field which is still in constructions, deserving further investigation.

The book is clearly written and contains a wealth of material on spaces of Lipschitz functions, available for the first time in book form. It is fairly self-contained, accessible to students acquainted with the basics of measure theory and functional analysis. The open problems, posed in various places in the book, open new research opportunities for the diligent reader.

S. Cobzaş