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SOME PROPERTIES OF THE INTEGRAL OPERATORS IN UNIVALENT FUNCTION

R. AGHALARY AND S.R. KULKARNI

Abstract. In this paper we have obtained some properties of the integral operators on the lines of Miller and Mocanu [2], Nour [4], after generalizing several lemmas of the above mentioned authors needed in the course of research.

1. **Introduction**

Let $\mathcal{A}$ denote the class of functions analytic in the unit disc $U = \{z : \ |z| < 1\}$ and normalized by $f(0) = f'(0)-1 = 0$. Also let S denote the subclass of $\mathcal{A}$ consisting of (normalized) functions f which are univalent in U. A function $f(z)$ in S is said to be starlike of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U, \ 0 \le \alpha < 1).$$

Let $S^*(\alpha)$ denote the class of all functions which are starlike of order α in U. It is well known that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$.

Let f, g be analytic in the unit disc U. We call the function f is a subordinate to g, written $f \prec g$, if there exists an analytic function ϕ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f(z) = g(\phi(z))$.

Let $\rho(A, B)$ consist of all functions g that are analytic in U with $g(0) = 1$ and satisfy the condition

$$g(z) \prec \frac{1 + Az}{1 + Bz} \quad (-1 \le B < A \le 1).$$

Finally a function $f(z) \in \mathcal{A}$ is said to be in the class $S^*(A, B)$ if and only if

$$\frac{zf'(z)}{f(z)} \in \rho(A, B).$$

In the present paper we will investigate some properties of the integral operators. We shall make use of the results due to Miller and Mocanu [2] and Noor [4]. For the sake of convenience, we recall those results as the following lemmas:

Lemma 1 (Miller and Mocanu [2]). *Let $\alpha \geq 0$, $\beta > 0$ and $\alpha + \delta = \beta + \gamma > 0$ and let the functiona $\varphi(z)$ and $\phi(z)$ be in the class D defined by*

$$D := \{\theta : \ \theta(z) \text{ analytic in } U, \ \theta(z) \neq 0, \ \text{and } \theta(0) = 1\}.$$

Suppose also that

$$\delta + Re\left\{\frac{z\varphi'(z)}{\varphi(z)}\right\} \geq \gamma \quad \text{and} \quad Re\left\{\frac{z\phi'(z)}{\phi(z)}\right\} \leq \beta w(0)$$

where $w(\rho)$ is given, in terms of the Gaussian hypergeometric function ${}_2F_1$, by

$$w(\rho) = \frac{1}{\beta}\left[\frac{(\beta+\gamma)2^{-2\beta(1-\rho)}}{{}_2F_1[2\beta(1-\rho), \beta+\gamma; \beta+\gamma+1; -1]} - \gamma\right]$$
$$(\max\{(\beta-\gamma-1)/2\beta, -\gamma/\beta\} \leq \rho < 1)$$

Then for the integral operator I defined by

$$I(f)(z) = \left(\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z \{f(t)\}^{\alpha}\varphi(t)t^{\delta-1}dt\right)^{1/\beta}$$

we have

$$I(S^*) \subset \begin{cases} S^* & (\phi(z) \not\equiv 1) \\ S^*(w(0)) & (\phi(z) \equiv 1) \end{cases}$$

Lemma 2 (Noor [4]). *Let $\rho_j(z) \in \rho(A, B)$, $(j = 1, 2)$. Then, for $\alpha > 0$ and $\beta > 0$,*

$$\frac{\alpha\rho_1(z) + \beta\rho_2(z)}{\alpha+\beta} \in \rho(A, B).$$

2. Some results related to the function space $\rho(A, B)$

Lemma 3. *Let $\alpha \geq 0$ and $D(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the region. Let $N(z)$ be analytic in E with $N(0) = D(0) = 0$.*

Then

$$(1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz} \ \Rightarrow \ \frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}$$

where $(1 \leq B < A \leq 1)$.

Proof. Let

$$\frac{N(z)}{D(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}.$$

Clearly $w(0) = 0$. We will prove that $|w(z)| < 1$, $\forall\ z \in U$ for, if not, by Jack's lemma [1] there exists $z_0 \in U$, such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. We consider

$$\varphi(z) = (1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)}$$

since

$$\frac{N'(z)}{D'(z)} = \frac{N(z)}{D(z)} + \frac{D(z)}{D'(z)}\left(\frac{(A-B)w'(z)}{(1+Bw(z))^2}\right).$$

So

$$\varphi(z_0) = (1-\alpha)\frac{N(z_0)}{D(z_0)} + \alpha\frac{N'(z_0)}{D'(z_0)} =$$

$$= \frac{N(z_0)}{D(z_0)} + \alpha\left(\frac{D(z_0)}{z_0 D'(z_0)}\right)\left(\frac{(A-B)kw(z_0)}{(1+Bw(z_0))^2}\right).$$

Now

$$\left|\frac{\varphi(z_0)-1}{B\varphi(z_0)-A}\right| = \left|\frac{\frac{(A-B)w(z_0)}{1+Bw(z_0)}\left(1 + \frac{D(z_0)}{z_0 D'(z_0)}\frac{k\alpha}{1+Bw(z_0)}\right)}{\frac{(B-A)}{1+Bw(z_0)}\left(1 - \frac{D(z_0)k\alpha Bw(z_0)}{z_0 D'(z_0)(1+Bw(z_0))}\right)}\right|$$

or

$$\left|\frac{\varphi(z_0)-1}{B\varphi(z_0)-A}\right| = \left|\frac{1 + \frac{D(z_0)k\alpha}{z_0 D'(z_0)(1+Bw(z_0))}}{1 - \frac{D(z_0)k\alpha\beta w(z_0)}{z_0 D'(z_0)(1+\beta w(z_0))}}\right|$$

Therefore

$$\left|\frac{\varphi(z_0)-1}{B\varphi(z_0)-A}\right| > 1 \ \Leftrightarrow\ \left|1 + \frac{k\alpha D(z_0)}{z_0 D'(z_0)(1+Bw(z_0))}\right| >$$

$$> \left|1 - \frac{k\alpha w(z_0)D(z_0)}{z_0 D'(z_0)(1+Bw(z_0))}\right|$$

But

$$\left|1 + \frac{k\alpha D(z_0)}{z_0 D'(z_0)(1+Bw(z_0))}\right|^2 - \left|1 - \frac{k\alpha Bw(z_0)D(z_0)}{z_0 D'(z_0)(1+Bw(z_0))}\right|^2 =$$

$$= (1-B)^2\left|\frac{D(z_0)}{z_0 D'(z_0)}\right|^2\left|\frac{k\alpha}{1+Bw(z_0)}\right|^2 + 2k\alpha Re\left(\frac{D(z_0)}{z_0 D'(z_0)}\right) > 0.$$

Hence

$$\left|\frac{\varphi(z_0)-1}{B\varphi(z_0)-A}\right| > 1$$

and this is contradiction with this fact that $\varphi(z) \prec \dfrac{1+Az}{1+Bz}$ so $|w(z)| < 1$ and the proof is complete.

By putting $\alpha = 0$ we get the result due to Miller and Mocanu [3] as:

Corollary 1. *Let the functions $M(z)$ and $N(z)$ be analytic in U with $M(0) = N(0) = 0$ and let γ be a real number. Suppose also that $N(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the origin. Then*

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} > \gamma, \quad (z \in U) \ \Rightarrow \ Re\left(\frac{M(z)}{N(z)}\right) > \gamma, \ (z \in U).$$

Lemma 4. *Let $\alpha \geq 0$ and $D(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the region. Let $N(z)$ be analytic in E with $N(0) = D(0) = 0$ and $\dfrac{N'(0)}{D'(0)} = k$ then*

$$(1-\alpha)\frac{N(z)}{kD(z)} + \alpha\frac{N'(z)}{kD'(z)} \prec \frac{1+Az}{1+Bz} \ \Rightarrow \ \frac{N(z)}{kD(z)} \prec \frac{1+Az}{1+Bz}$$

(where $-1 \leq B < A \leq 1$).

Proof. Proceeding as in the proof of Lemma 3 we get our result.

By putting $\alpha = 0$ we get the result due to Reddy and Padmanabhan [5] as:

Corollary 2. *Let the functions $N(z)$ and $D(z)$ be analytic in U and let $D(z)$ maps U onto a many-sheeted starlike region. Suppose also that $N(0) = D(0) = 0$, $\dfrac{N'(0)}{D'(0)} = k$ and $\dfrac{N'(z)}{kD'(z)} \in \rho(A,B)$, $(k \geq 1)$ then $\dfrac{N(z)}{kD(z)} \in \rho(A,B)$.*

Lemma 5. *Let $\alpha > 0$ and $f \in \mathcal{A}$. Then*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z) + \lambda\left(\frac{f(z)}{z}\right)^{\alpha} \in \rho(A,B) \ \Rightarrow \ \left(\frac{f(z)}{z}\right)^{\alpha} \in \rho(A,B)$$

(where $-1 \leq B < A \leq 1$ and $0 \leq \lambda \leq 1$).

Proof. Let

$$\left(\frac{f(z)}{z}\right)^{\alpha} = \frac{1+Aw(z)}{1+Bw(z)}.$$

Clearly $w(0) = 0$. We will prove $|w(z)| < 1, \ \forall \ z \in U$. For, if not, by Jack's lemma [1] there exists $z_0 \in E$, such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. Let

$$\psi(z) = (1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z) + \lambda\left(\frac{f(z)}{z}\right)^{\alpha}.$$

But

$$\alpha\left(\frac{zf'(z)-f(z)}{z^2}\right)\left(\frac{f(z)}{z}\right)^{\alpha-1}=\frac{(A-B)w'(z)}{(1+Bw(z))^2}$$

or

$$f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}=\frac{1+Aw(z)}{1+Bw(z)}+\frac{(A-B)zw'(z)}{\alpha(1+Bw(z))^2}$$

Hence

$$\psi(z_0)=\frac{1+Aw(z_0)}{1+Bw(z_0)}+\frac{(1-\lambda)kw(z_0)(A-B)}{\alpha(1+Bw(z_0))^2}$$

If we take $\phi(z)=\dfrac{(1-\lambda)k}{\alpha(1+Bw(z))}$ then we have

$$\left|\frac{\psi(z_0)-1}{B\psi(z_0)-A}\right|=\left|\frac{\dfrac{(A-B)w(z_0)}{1+Bw(z_0)}\left(1+\dfrac{(1-\lambda)k}{\alpha(1+Bw(z_0))}\right)}{\dfrac{B-A}{1+Bw(z_0)}\left(1-\dfrac{(1-\lambda)kw(z_0)B}{\alpha(1+Bw(z_0))}\right)}\right|=$$

$$=\left|\frac{1+\phi(z_0)}{1-\phi(z_0)Bw(z_0)}\right|$$

But the right hand side of above equivality is greater than 1, because

$$|1+\phi(z_0)|^2-|1-Bw(z_0)\phi(z_0)|^2=(1-B^2)|\phi(z_0)|^2+\frac{2(1-\lambda)k}{\alpha}>0$$

and this is contradiction with hypothesis, so $|w(z)|<1$ and the proof is complete. By putting $\lambda=0$ we get the result due to Noor [4] as

Corollary 3. *If* $f(z)\in\mathcal{A}$ *and* $\left(\frac{f(z)}{z}\right)^{\alpha-1}f'(z)\in\rho(A,B)$ *then* $\left(\frac{f(z)}{z}\right)^{\alpha}\in\rho(A,B)$ *(where* $\alpha\in N=\{1,2,3,\dots\}$*).*

3. Some properties of the integral operators

Theorem 1. *Let* $g\in S^*(A,B)$*, then the function* $F(z)$ *defined by*

$$F(z)=\left[\alpha^{-1}\int_0^z g(t)^{1/\alpha}t^{-1}dt\right]^{\alpha}$$

is in the class $S^*(A,B)$*,* $(\alpha>0)$*.*

Proof. We know from Lemma 1 that $F(z)\in S^*$. But with direct calculation we can write

$$\frac{zg'(z)}{g(z)}=(1-\alpha)\frac{zF'(z)}{F(z)}+\alpha\left(1+\frac{zF''(z)}{F'(z)}\right)$$

So, by hypothesis,

$$(1-\alpha)\frac{zF'(z)}{F(z)}+\alpha\left(1+\frac{zF''(z)}{F'(z)}\right)\in\rho(A,B). \tag{3.1}$$

We consider $N(z) = zF'(z)$ and $D(z) = F(z)$, then functions $N(z)$ and $D(z)$ satisfy the conditions of Lemma 3. Now from (3.1) we have

$$(1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)} \in \rho(A,B).$$

So, by lemma 3,

$$\frac{zF'(z)}{F(z)} = \frac{N(z)}{D(z)} \in \rho(A,B)$$

and this completes the proof.

Theorem 2. *Let $\alpha > 0$, $\gamma > 0$, $f(z) \in \mathcal{A}$ and $F(z)$ be defined by*

$$F(z) = \left(\frac{\alpha+\gamma}{z^{\gamma}}\int_0^z f(t)^{\alpha}t^{\gamma-1}dt\right)^{1/\alpha}$$

then

$$\left(\frac{f(z)}{z}\right)^{\alpha} \in \rho(A,B) \;\Rightarrow\; \left(\frac{F(z)}{z}\right)^{\alpha} \in \rho(A,B).$$

Proof. Since

$$\alpha F'(z) = \left(\frac{-\gamma(\alpha+\gamma)}{z^{\gamma+1}}\int_0^z f(t)^{\alpha}t^{\gamma-1}dt + \frac{\alpha+\gamma}{z^{\gamma}}f(z)^{\alpha}z^{\gamma-1}\right)F(z)^{1-\alpha} =$$

$$= \left(-\frac{\gamma}{z}F(z)^{\alpha} + \frac{\alpha+\gamma}{z}f(z)^{\alpha}\right)F(z)^{1-\alpha}$$

or

$$\frac{\alpha}{\alpha+\gamma}\left(\frac{F(z)}{z}\right)^{\alpha-1} + \frac{\gamma}{\alpha+\gamma}\left(\frac{F(z)}{z}\right)^{\alpha} = \left(\frac{f(z)}{z}\right)^{\alpha} \tag{3.2}$$

But, by hypothesis, $\left(\frac{f(z)}{z}\right)^{\alpha} \in \rho(A,B)$. Therefore from (3.2) we have

$$\frac{\alpha}{\alpha+\gamma}\left(\frac{F(z)}{z}\right)^{\alpha-1}F'(z) + \frac{\gamma}{\alpha+\gamma}\left(\frac{F(z)}{z}\right)^{\alpha} \in \rho(A,B) \tag{3.3}$$

Hence from (3.3) and Lemma 5 we get the desired result.

Theorem 3. *Let $\alpha > 1$, $f, g \in \mathcal{A}$ and function $F(z)$ is defined by*

$$F(z) = \left[\alpha^{-1}\int_0^z f(t)^{1/\alpha}g(t)^{(\alpha-1)/\alpha_t-1}dt\right]^{\alpha}. \tag{3.4}$$

Then $\frac{zg'(z)}{g(z)} \in \rho(A,B)$ *and* $\frac{zf'(z)}{f(z)} \in \rho(A,B) \;\Rightarrow\; \frac{1}{\alpha}\frac{zF'(z)}{F(z)} \in \rho(A,B)$.

Proof. It is clear, by Lemma 1, $F \in S^*$. By differentiation from (3.4) we get

$$F'(z) = (f(z)^{1/\alpha}g(z)^{(\alpha-1)/\alpha_z-1})(F(z))^{(\alpha-1)/\alpha}$$

or

$$zF(z)^{(1-\alpha)/\alpha}F'(z) = f(z)^{1/\alpha}g(z)^{(\alpha-1)/\alpha}. \tag{3.5}$$

By differentiation from (3.5) we get

$$\left(1+\frac{zF''(z)}{F'(z)}\right)+\left(\frac{1-\alpha}{\alpha}\right)\frac{zF'(z)}{F(z)} = \frac{1}{\alpha}\frac{zf'(z)}{f(z)}+\frac{\alpha-1}{\alpha}\frac{zg'(z)}{g(z)}.$$

But the right habd side of the above equality belongs to $\rho(A,B)$, by lemma 2. So we have

$$\left(1+\frac{zF''(z)}{F'(z)}\right)+\left(\frac{1-\alpha}{\alpha}\right)\frac{zF'(z)}{F(z)} \in \rho(A,B). \tag{3.6}$$

Let $N(z) = zF'(z)$ and $D(z) = \alpha F(z)$ then functions $N(z)$ and $D(z)$ satisfy the condition of Lemma 3. But

$$\left(1+\frac{zF''(z)}{F'(z)}\right)+\frac{1-\alpha}{\alpha}\frac{zF'(z)}{F(z)} = \alpha\frac{N'(z)}{D'(z)}+(1-\alpha)\frac{N(z)}{D(z)} \tag{3.7}$$

So from relations (3.6), (3.7) and lemma 3 we have $\dfrac{N(z)}{D(z)} = \dfrac{zF'(z)}{\alpha F(z)} \in \rho(A,B)$ and the proof is complete.

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ON 6-DIMENSIONAL HERMITIAN SUBMANIFOLDS OF CAYLEY ALGEBRA

BANARU MIKHAIL

Abstract. Results have been obtained concerning one of the most important characteristics of a Hermitian manifold which is a Ricci curvature.

One of the most beautiful and substantial examples of Hermitian manifolds are the 6-dimensional oriented submanifolds of Cayley octave algebra. In the present work a number of results on the characteristics of such manifolds are shown. Let's remember that Hermitian is named the manifold M^{2n}, that has an almost complex structure J and Riemannian metric $g = \langle \cdot, \cdot \rangle$ when meeting the conditions:

1) $\langle JX, JY \rangle = \langle X, Y \rangle, \ X, Y \in \aleph(M)$;

2) $[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0$.

1.

It is well-known that Ricci tensor ric of a Riemannian manifold is named the tensor whose components are connected with the components of the tensor of Riemannian curvature (Riemann-Christoffel tensor) as follows [4]:

$$ric_{ij} = R^k_{ijk}.$$

This tensor is symmetric; the value of the corresponding quadratic form on vector X, $X \in \aleph(M)$ is called Ricci curvature and is denoted $S(X)$. Thus,

$$S(X) = ric_{ij} X^i X^j, \quad \|X\| = 1.$$

Let's use the values of the spectrum of the Riemann-Christoffel tensor of 6-dimensional Hermitian submanifolds of octave algebra [1].

$$R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0;$$

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$$R_{\hat{a}b\hat{c}d} = -\sum_{\varphi} T^{\varphi}_{\hat{a}\hat{c}} T^{\varphi}_{bd}, \tag{1}$$

where T^{φ}_{ij} are the components of configuration tensor (or, in other words, the tensor of Euler curvature). Here $a, b, c, d = 1, 2, 3$; $\hat{a} = a + 3$; $\varphi = 7, 8$, $i, j = 1, 2, 3, 4, 5, 6$.

Let's calculate the Ricci tensor spectrum for the 6-dimensional Hermitian submanifolds of Cayley algebra. Taking into account (1), we get:

$$ric_{ab} = R^{c}_{abc} + R^{\hat{c}}_{ab\hat{c}} = R_{\hat{c}abc} + R_{cab\hat{c}} = 0;$$

$$ric_{\hat{a}b} = R^{c}_{\hat{a}bc} + R^{\hat{c}}_{\hat{a}b\hat{c}} = R_{\hat{c}\hat{a}bc} + R_{c\hat{a}b\hat{c}} =$$

$$= R_{c\hat{a}b\hat{c}} = R_{\hat{a}c\hat{c}b} = -\sum_{\varphi} T^{\varphi}_{\hat{a}\hat{c}} T^{\varphi}_{cb};$$

$$ric_{a\hat{b}} = R^{c}_{a\hat{b}c} + R^{\hat{c}}_{a\hat{b}\hat{c}} = R_{\hat{c}a\hat{b}c} + R_{ca\hat{b}\hat{c}} =$$

$$= R_{\hat{c}a\hat{b}c} = -\sum_{\varphi} T^{\varphi}_{\hat{c}\hat{b}} T^{\varphi}_{ac};$$

$$ric_{\hat{a}\hat{b}} = R^{c}_{\hat{a}\hat{b}c} + R^{\hat{c}}_{\hat{a}\hat{b}\hat{c}} = R_{\hat{c}\hat{a}\hat{b}c} + R_{c\hat{a}\hat{b}\hat{c}} = 0.$$

In view of the reality of the Ricci tensor,

$$ric_{ab} = \overline{ric_{\hat{a}\hat{b}}}; \quad ric_{\hat{a}b} = \overline{ric_{a\hat{b}}}.$$

Therefore, the Ricci tensor spectrum is calculated as follows:

$$ric_{ab} = 0; \quad ric_{\hat{a}b} = -\sum_{\varphi} T^{\varphi}_{\hat{a}\hat{c}} T^{\varphi}_{bc}. \tag{2}$$

Then, the Ricci curvature of Hermitian 6-dimensional submanifolds of octave algebra is calculated as follows:

$$S(X) = -2\sum_{\varphi} T^{\varphi}_{\hat{a}\hat{c}} T^{\varphi}_{bc} X^{b} X_{a} = -2\sum_{\varphi} (T^{\varphi}_{\hat{a}\hat{c}} X_{a})(T^{\varphi}_{bc} X^{b}) =$$

$$= -2\sum_{\varphi} (T^{\varphi}_{ab} X^{b})(\overline{T^{\varphi}_{ab} X^{b}}) = -2\sum_{\varphi,a,b} |T^{\varphi}_{ab} X^{b}|^{2}.$$

Thus,

$$S(X) = -2\sum_{\varphi,a,b} |T^{\varphi}_{ab} X^{b}|^{2},$$

and so, we conclude that is correct

Theorem 1. *The 6-dimensional Hermitian submanifold of Cayley algebra has a nonpositive Ricci curvature, moreover the above mentioned curvature vanishes in geodesic points and only in them.*

Consequence. *The 6-dimensional Hermitian submanifold of Cayley algebra is Ricci flat manifold then and only then, when it is a domain on the Kählerian plane.*

2.

Let's calculate the scalar curvature of the 6-dimensional Hermitian submanifolds of Cayley algebra. Taking into account (2) we get

$$K = ric_i^i = -2 \sum_{\varphi,a,b} |T^{\varphi}_{ab}|^2 \leq 0.$$

Evidently, the scalar curvature of the 6-dimensional Hermitian submanifolds of octave algebra is also nonpositive and becomes zero exclusively in geodesic points. In this sense, the scalar curvature "repeats" both the Ricci curvature and the bisectional holomorphic curvature [2] of such manifolds.

If the considered manifold is a manifold of constant scalar curvature ($K = const$), then we get, that

$$\sum_{\varphi,a,b} |T^{\varphi}_{ab}|^2 = const$$

and therefore is correct.

Theorem 2. *The 6-dimensional Hermitian submanifold of Cayley algebra is a manifold of the constant scalar curvature in the case and only when the configuration tensor has a constant length.*

Let's note that both the theorems sum up the well-known results obtained by V. Kirichenko [3] on the 6-dimensional Kahlerian submanifolds of octave algebra.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 1, March 2001

A KOROVKIN-TYPE THEOREM FOR THE APPROXIMATION OF n-VARIATE B-CONTINUOUS FUNCTIONS

DAN BĂRBOSU

Abstract. The aim of this note is to extend the results from [1], [4] to the case of n variate B-continuous functions in the sense of Bogel [5]. In the section 1 we present the notions of n-variate B-continuous function and uniform n-variate B-continuous function. Some relationship among these notions are also presented. In the section 2, we discuss a Korovkin-type criterion for the approximation by means of linear positive operators of the B-continuous functions of n-variables. The main result of the paper is the theorem 2.1. In the section 3 we present some applications of the theorem 2.1.

1. Let $\mathbf{R}^{I^n}$ be the space of functions $f : I^n \to \mathbf{R}$, where I=[0,1] and n is a positive integer. The notion of B-continuous function was introduced in [5] using the operator $\Delta_2 : \mathbf{R}^{I^2} \to \mathbf{R}^{I^2}$

$$\Delta_2 [f; M, M'] = \Delta_{s_1,s_2} [f; M, M'] =$$

$$= f\ (s_1, s_2) - f\ (s_1, x_2) - f\ (x_1, s_2) + f\ (x_1, x_2) \tag{1.1}$$

for any $f \in \mathbf{R}^{I^2}$ and any points $M (x_1, x_2), M' (s_1, s_2) \in I^2$.

Let $\Delta_2 : \mathbf{R}^I \to \mathbf{R}^I$ be the univariate operator given by

$$\Delta_2 [f; M, M'] = \Delta_{s_1} [f; x_1] = f\ (s_1) - f\ (x_1) \tag{1.2}$$

for any $f \in \mathbf{R}^I$ and any points $M (x_1), M' (s_1) \in I$.

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If $f \in \mathbf{R}^{I^2}$ and ${}_{s_1}\Delta, {}_{s_2}\Delta$ are the parametric extension of the operator (1,2), then the following equality holds:

$$\Delta_{s_1,s_2}[f;x_1,x_2] = ({}_{s_1}\Delta \circ {}_{s_2}\Delta)[f;x_1,x_2]. \tag{1.3}$$

The last remark allows us to define the operator of n-variate difference by

Definition 1.1: Let $f \in \mathbf{R}^{I^n}$ be a given function and ${}_{s_1}\Delta, \ldots, {}_{s_n}\Delta$ be the parametric extensions of the operator (1,2). The operator $\Delta_n : \mathbf{R}^{I^n} \to \mathbf{R}^{I^2}$ given by

$$\Delta_n[f;M,M'] = \Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n] = ({}_{s_1}\Delta \circ \ldots \circ_{s_n}\Delta)[f;x_1,\ldots,x_n] \tag{1.4}$$

for any functions $f \in \mathbf{R}^{I^n}$and any points $M(x_1,\ldots,x_n), M'(s_1,\ldots s_n) \in I^n$ is called operator of n-variate difference.

Remark 1.1: It is easy to see that the representation

$$\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_2] = f(s_1,\ldots,s_n) - \sum_{i=1}^{n} f(x_1,\ldots,x_{i-1},s_i,x_{i+1},\ldots,x_n) +$$

$$+ \sum_{i,j=1}^{n} f(x_1,\ldots,x_{i-1},s_i,x_{i+1},\ldots,x_{j-1},s_j,x_{j+1},\ldots,x_n) - \ldots + (-1)^n f(x_1,\ldots,x_n) \tag{1.5}$$

is valid.

Definition 1.2: The function $f \in \mathbf{R}^{I^n}$ is called B-continuous in the point $M(x_1,\ldots,x_n) \in I^n$ is the equality

$$\lim_{(s_1,\ldots s_n)\to(x_1,\ldots,x_n)} \Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_2] = 0 \tag{1.6}$$

holds.

If $f \in \mathbf{R}^{I^n}$ is B-continuous at every point of I^n one says that f is B-continuous on I^n and the set of all B-continuous functions on I^n is denoted by $C_b(I^n)$.

Definition 1.3: The function $f \in \mathbf{R}^{I^n}$ is uniform B-continuous on I^n if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ so that for any point $(x_1,\ldots,x_n), (s_1,\ldots s_n) \in I^n$ for which one has

$$|x_1 - s_1| < \delta, \ldots, |x_n - s_n| < \delta \tag{1.7}$$

the inequality

$$\left|\Delta_{s_1,\dots,s_n}\left[f;x_1,\dots,x_2\right]\right| < \varepsilon \tag{1.8}$$

holds.

Definition 1.4: The function $f \in \mathbf{R}^{I^n}$ is B-bounded on I^n if there exists a positive number K so that:

$$\left|\Delta_{s_1,\dots,s_n}\left[f;x_1,\dots,x_2\right]\right| \leq K, (\forall)\left(x_1,\dots,x_n\right), \left(s_1,\dots s_n\right) \in I^n \tag{1.9}$$

The relationships between B-continuous, uniform B-continuous and B-bounded functions are immediately and are contained in the following two lemmas:

Lemma 1.1: If $f \in C_b\left(I^n\right)$ then f is uniform B-continuous on I^n.

Lemma 1.2: If $f \in C_b\left(I^n\right)$ then f is B-bounded on I^n.

2. We will establish a Korovkin type theorem for the approximation on $C_b\left(I^n\right)$. First, we establish an auxiliary result.

Lemma 2.1: Let $f \in C_b\left(I^n\right)$ be arbitrarily chosen. For any positive number $\varepsilon > 0$ there exist n positives numbers $A_i = A_i\left(\varepsilon\right), i = \overline{1,n}$ so that for any $\left(x_1,\dots,x_n\right), \left(s_1,\dots s_n\right) \in I^n$ one has:

$$\left|\Delta_{s_1,\dots,s_n}\left[f;x_1,\dots,x_n\right]\right| \leq \frac{\varepsilon}{n+1} + \sum_{i=1}^{n} A_i\left(\varepsilon\right)\left(x_i - s_i\right)^2. \tag{2.1}$$

Proof. Because $f \in C_b\left(I^n\right)$, from lemma 1.1 it follows that f is uniform B-continuous on I^n i.e. for $\left(x_1,\dots,x_n\right), \left(s_1,\dots s_n\right) \in I^n$ with $\left|x_1 - s_1\right| < \delta\left(\varepsilon\right), \dots, \left|x_n - s_n\right| < \delta\left(\varepsilon\right)$ one has

$$\left|\Delta_{s_1,\dots,s_n}\left[f;x_1,\dots,x_n\right]\right| < \frac{\varepsilon}{n+1}. \tag{2.2}$$

Let $\varepsilon > 0$ be a given positive number and $\left(x_1,\dots,x_n\right), \left(s_1,\dots s_n\right) \in I^n$. The inequalities $\left|x_i - s_i\right| < \delta\left(\varepsilon\right)$ can be valid for all $i \in \{1,2,\dots,n\}$, for $(n-1)$ values of $i \in \{1,2,\dots,n\}$, ..., for one value of $i \in \{1,2,\dots,n\}$ or for none of the values $i \in \{1,2,\dots,n\}$.

If $\left|x_i - s_i\right| < \delta\left(\varepsilon\right)$ for any $i \in \{1,2,\dots,n\}$, from (2.2) one deduces that

$$\left|\Delta_{s_1,\dots,s_n}\left[f;x_1,\dots,x_n\right]\right| < \frac{\varepsilon}{n+1} \tag{2.3}$$

Because $f \in C_b(I^n)$, from lemma 1.2 it follows that there exists a positive number K such that

$$|\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n]| \le K. \tag{2.4}$$

We suppose that there is only one value $j \in \{1,2,\ldots,n\}$ so that $|x_j - s_j| \ge \delta(\varepsilon)$.

If j = 1 then $|x_1 - s_1| \ge \delta(\varepsilon), |x_2 - s_2| < \delta(\varepsilon), \ldots, |x_n - s_n| < \delta(\varepsilon)$.

For $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ with these properties we have

$$|\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n]| \le K \cdot [\delta(\varepsilon)]^{-2}(x_1 - s_1)^2. \tag{2.5}$$

This way, for the points $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ for which there is only one value $j \in \{1,2,\ldots,n\}$ such that $|x_j - s_j| \ge \delta(\varepsilon)$ we have:

$$|\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n]| \le K \cdot [\delta(\varepsilon)]^{-2}\sum_{i=1}^{n}(x_i - s_i)^2. \tag{2.6}$$

In a similar way, for the points $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ for which there are only two values $i,j \in \{1,2,\ldots,n\}$, $i \ne j$ such that $|x_i - s_i| \ge \delta(\varepsilon), |x_j - s_j| \ge \delta(\varepsilon)$, we have:

$$|\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n]| \le K \cdot [\delta(\varepsilon)]^{-2^2}\sum_{i,j=1,i\ne j}^{n}(x_i - s_i)^2(x_j - s_j)^2 \tag{2.7}$$

For all the points $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ for which $|x_i - s_i| \ge \delta(\varepsilon)$, $(\forall)\, j \in \{1,2,\ldots,n\}$ we have:

$$|\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n]| \le K \cdot [\delta(\varepsilon)]^{-2^n}(x_1 - s_1)^2 \ldots (x_n - s_n)^2. \tag{2.8}$$

With these observations, for any $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ the next relation holds:

$$|\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n]| \le \frac{\varepsilon}{n+1} + K \cdot [\delta(\varepsilon)]^{-2}\sum_{i=1}^{n}(x_i - s_i)^2 +$$

$$+K \cdot [\delta(\varepsilon)]^{-2^2}\sum_{i,j=1,i\ne j}^{n}(x_i - s_i)^2(x_j - s_j)^2 + \ldots +$$

$$+K \cdot [\delta(\varepsilon)]^{-2^n}(x_1 - s_1)^2 \ldots (x_n - s_n)^2 \tag{2.9}$$

Because $s_k, x_k \in [0,1]$ $(\forall)\, k \in \{1,2,\ldots,n\}$ we have that $(x_k - s_k)^2 \le 1$.

Using this observation and (2.9), one obtains:

$$|\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n]| \le$$

$$\le \frac{\varepsilon}{n+1} + K\cdot[\delta(\varepsilon)]^{-2}\left\{1+[\delta(\varepsilon)]^{-2}+\ldots+[\delta(\varepsilon)]^{-2^n+2}\right\}(x_1-s_1)^2 +$$

$$+K\cdot[\delta(\varepsilon)]^{-2}\left\{1+[\delta(\varepsilon)]^{-2}+\ldots+[\delta(\varepsilon)]^{-2^n+2^2}\right\}(x_2-s_2)^2+\ldots+$$

$$+K\cdot[\delta(\varepsilon)]^{-2}(x_n-s_n)^2. \tag{2.10}$$

Choosing then

$$A_{1=}K\cdot[\delta(\varepsilon)]^{-2}\left\{1+[\delta(\varepsilon)]^{-2}+\ldots+[\delta(\varepsilon)]^{-2^n+2}\right\}$$
$$A_{2=}K\cdot[\delta(\varepsilon)]^{-2}\left\{1+[\delta(\varepsilon)]^{-2}+\ldots+[\delta(\varepsilon)]^{-2^n+2^2}\right\}$$
$$\ldots$$
$$A_{n=}K\cdot[\delta(\varepsilon)]^{-2}$$

it follows that (2.1) is valid.

Now we can establish the main result of the paper. We consider the following functions on I^n :

$$e_0(s_1,\ldots s_n)=1, e_i(s_1,\ldots s_n)=s_i, i=\overline{1,n}\ ,(s_1,\ldots s_n)\in I^n.$$

Theorem 2.1: *Let* $\{L_{m_1,m_2,\ldots,m_n}\}$ *be a sequence of positive linear operators mapping the functions of* R^{I^n} *into functions of* R^{I^n} *such that for all* $(x_1,\ldots x_n)\in I^n$ *one has*

i) $L_{m_1,m_2,\ldots,m_n}(e_0;x_1,\ldots x_n)=1$;

ii) $L_{m_1,m_2,\ldots,m_n}(e_0;x_1,\ldots x_n)=x_i+\alpha^{(i)}_{m_1,m_2,\ldots,m_n}(x_1,\ldots x_n)$,

$i\in\{1,2,\ldots,n\}$;

iii) $L_{m_1,m_2,\ldots,m_n}\left(\sum\limits_{i=1}^{n} e_i^2;x_1,\ldots x_n\right)=\sum\limits_{i=1}^{n} x_i^2+\delta_{m_1,m_2,\ldots,m_n}(x_1,\ldots x_n)$,

where the sequences $\left\{\alpha^{(i)}_{m_1,m_2,\ldots,m_n}(x_1,\ldots x_n)\right\}, \{\delta_{m_1,m_2,\ldots,m_n}(x_1,\ldots x_n)\}$ tend to zero uniformly on I^n as $m_1,m_2,\ldots m_n$ tend to infinity.

If $f(\cdot,\ldots,\cdot)\in C_b(I^n)$, we introduce the notation:

$(\star)$ $U_{m_1,m_2,\ldots,m_n}(f;x_1,\ldots x_n)=L_{m_1,m_2,\ldots,m_n}(f\ (x_1,\ldots x_n)-\Delta_{\cdot,\ldots,\cdot}[f;x_1,\ldots x_n])$

In the hypothesis i), ii), iii) the sequence $\{U_{m_1,m_2,\dots,m_n}(f)\}$ converges to f uniformly on I^n, for any $f \in C_b(I^n)$.

Proof. It is obvious that $U_{m_1,m_2,\dots,m_n}$ is a well-defined operator on $C_b(I^n)$. Let $f \in C_b(I^n)$ be arbitrarily chosen, $(x_1,\dots x_n) \in I^n$ and $\varepsilon > 0$ given.

Because $L_{m_1,m_2,\dots,m_n}$ is a linear operator reproducing the constant functions (from the condition i)), we have:

$$f\ (x_1,\dots x_n) - U_{m_1,m_2,\dots,m_n}(f; x_1,\dots x_n) =$$

$$= L_{m_1,m_2,\dots,m_n}(\ \Delta_{\cdot,\dots,\cdot}[f; x_1,\dots x_n]) \tag{2.11}$$

From the positivity of $L_{m_1,m_2,\dots,m_n}$ we have

$$|L_{m_1,m_2,\dots,m_n}(\ g\,; x_1,\dots x_n)| =$$

$$\max\{L_{m_1,m_2,\dots,m_n}(\ g;\ x_1,\dots x_n), L_{m_1,m_2,\dots,m_n}(\ -g\,; x_1,\dots x_n)\} \tag{2.12}$$

for any $g \in C_b(I^n)$.

Applying this result to $G(s_1,\dots s_n) = \Delta_{s_1,\dots,s_n}[f; x_1,\dots,x_2]$ and using the monotonicity of $L_{m_1,m_2,\dots,m_n}$ and the lemma 2.1, we find (with $A(\varepsilon) = \max\{A_1(\varepsilon),\dots,A_n(\varepsilon)\}$) the inequality:

$$|f\ (x_1,\dots x_n) - U_{m_1,m_2,\dots,m_n}(f; x_1,\dots x_n)| \le$$

$$\le L_{m_1,m_2,\dots,m_n}\left[\frac{\varepsilon}{n+1} + A(\varepsilon)\sum_{i=1}^{n}(x_i - \cdot)^2; x_1,\dots,x_n\right]. \tag{2.13}$$

After some transformation of (2.13) we obtain:

$$|f\ (x_1,\dots x_n) - U_{m_1,m_2,\dots,m_n}(f; x_1,\dots x_n)| \le$$

$$\le \frac{\varepsilon}{n+1} + A(\varepsilon)\, L_{m_1,m_2,\dots,m_n}\left(\sum_{i=1}^{n} e_i^2; x_1,\dots,x_n\right) -$$

$$-2\cdot A(\varepsilon)\sum_{i=1}^{n} x_i \cdot L_{m_1,m_2,\dots,m_n}(e_i; x_1,\dots,x_n) +$$

$$+A(\varepsilon)\cdot L_{m_1,m_2,\ldots,m_n}(e_0;x_1,\ldots,x_n)\sum_{i=1}^{n}x_i^2. \tag{2.14}$$

Using now the hypothesis of the theorem, we can write:

$$|f\ (x_1,\ldots x_n)-U_{m_1,m_2,\ldots,m_n}(f;x_1,\ldots x_n)|\leq$$

$$\leq\frac{\varepsilon}{n+1}+A(\varepsilon)\cdot\left\{\delta_{m_1,\ldots,m_n}(x_1,\ldots,x_n)-2\cdot\sum_{i=1}^{n}x_i\alpha_{m_1,\ldots,m_n}^{(i)}(x_1,\ldots,x_n)\right\}. \tag{2.15}$$

Taking into account that $\left\{\alpha_{m_1,\ldots,m_n}^{(i)}(x_1,\ldots,x_n)\right\},\{\delta_{m_1,\ldots,m_n}(x_1,\ldots,x_n)\}$ tend to zero uniformly on I^n as $m_1,m_2,\ldots,m_n$ tend to infinity, from (2.15) we obtain the desired result.

Remark 2.1: The positive operator $L_{m_1,m_2,\ldots,m_n}:C_b(I^n)\to C_b(I^n)$ is the product of the parametric extension $L_{m_1}^{x_1},\ldots,L_{m_n}^{x_n}$ of the positive linear univariate operator $L_m:\mathbf{R}^I\to\mathbf{R}^I$.

Remark 2.2: In the case n=2, the theorem 2.1 reduced to the Korovkin-type theorem established in [1]. The idea of the proof of the theorem 2.1 is suggested by the idea from [1].

3. We shall present two applications of the theorem 2.1. For simplicity, we consider the case n=3.

Example 1: We consider the Bernstein-Stancu´s operator $B_{m_1}^{\langle\alpha\rangle},B_{m_2}^{\langle\beta\rangle},B_{m_3}^{\langle\gamma\rangle}:\mathbf{R}^I\to\mathbf{R}^I$, given by

$$B_{m_1}^{\langle\alpha\rangle}(f)(x)=\sum_{i=1}^{m_1}f\left(\frac{i}{m_1}\right)\cdot\omega_{m_1,i}(x,\alpha),x\in I,$$

$$B_{m_2}^{\langle\beta\rangle}(g)(y)=\sum_{j=1}^{m_2}f\left(\frac{j}{m_2}\right)\cdot\omega_{m_2,j}(y,\beta),y\in I,$$

$$B_{m_3}^{\langle\gamma\rangle}(h)(z)=\sum_{k=1}^{m_3}f\left(\frac{k}{m_3}\right)\cdot\omega_{m_3,k}(z,\gamma),z\in I,$$

where $\omega_{m_1,i}(x,\alpha)$,$\omega_{m_2,j}(y,\beta),\omega_{m_3,k}(z,\gamma)$ are the fundamental polynomials of Bernstein-Stancu type, i.e.

$$\omega_{m_1,i}(x,\alpha)=\begin{pmatrix}m_1\\i\end{pmatrix}\frac{x^{[i,-\alpha]}\cdot(1-x)^{[m_1-i,-\alpha]}}{1^{[m_1,-\alpha]}},$$

$$\omega_{m_2,j}(y,\beta)=\begin{pmatrix}m_2\\j\end{pmatrix}\frac{y^{[j,-\beta]}\cdot(1-y)^{[m_2-j,-\beta]}}{1^{[m_2,-\beta]}},$$

$$\omega_{m_3,k}(z,\gamma) = \begin{pmatrix} m_3 \\ k \end{pmatrix} \frac{z^{[k,-\gamma]} \cdot (1-z)^{[m_3-k,-\gamma]}}{1^{[m_3,-\gamma]}}.$$

In the precedent relation, $x^{[i,-\alpha]}$ denotes the factorial power of x with the exponent i and the increment -α, i.e.$x^{[i,-\alpha]} = x(x+\alpha)\dots(x+(i-1)\alpha)$.

In same relation, the parameter α, β and γ satisfy the condition $\alpha = \alpha(m_1) \geq 0$, $\beta = \beta(m_2) \geq 0$, $\gamma = \gamma(m_3) \geq 0$.

Let suppose that $f \in C_b(I^3)$; the operators $L_{m_1}, L_{m_2}, L_{m_3} : C_b(I^3) \to C_b(I^3)$ are the parametric extensions of the operator $B^{\langle\alpha\rangle}_{m_1}, B^{\langle\beta\rangle}_{m_2}, B^{\langle\gamma\rangle}_{m_3}$:

$L_{m_1}(f)(x,y,z) = \sum_{i=1}^{m_1} f\left(\frac{i}{m_1}, y, z\right) \cdot \omega_{m_1,i}(x,\alpha), x \in I;$

$L_{m_2}(f)(x,y,z) = \sum_{i=1}^{m_1} f\left(x, \frac{j}{m_2}, z\right) \cdot \omega_{m_2,j}(y,\beta), y \in I;$

$L_{m_3}(f)(x,y,z) = \sum_{k=1}^{m_3} f\left(x, y, \frac{k}{m_3}\right) \cdot \omega_{m_3,k}(z,\gamma), z \in I.$

The operator L_{m_1,m_2,m_3} is the product of the operators $L_{m_1}, L_{m_2}, L_{m_3}$ and it is defined by

$$L_{m_1,m_2,m_3}(f)(x,y,z) = \\ = \sum_{i=0}^{m_1} \sum_{i=0}^{m_1} \sum_{k=0}^{m_3} f\left(\frac{i}{m_1}, \frac{j}{m_2}, \frac{k}{m_3}\right) \cdot \omega_{m_1,i}(x,\alpha) \cdot \omega_{m_2,j}(y,\beta) \cdot \omega_{m_3,k}(z,\gamma).$$

By direct computation, one obtains

$L_{m_1,m_2,m_3}(e_0)(x,y,z) = 1, L_{m_1,m_2,m_3}(e_1)(x,y,z) = x,$

$L_{m_1,m_2,m_3}(e_2)(x,y,z) = y, L_{m_1,m_2,m_3}(e_3)(x,y,z) = z,$

$L_{m_1,m_2,m_3}(e_1^2 + e_2^2 + e_3^2)(x,y,z) = x^2 + y^2 + z^2 +$
$+ \frac{x(1-x)}{1+\alpha}\left[\frac{1}{m_1} + \alpha\right] + \frac{y(1-y)}{1+\beta}\left[\frac{1}{m_2} + \beta\right] + \frac{z(1-z)}{1+\gamma}\left[\frac{1}{m_3} + \gamma\right]$

for any $(x,y,z) \in I^3$. It follows that the sequence $\{L_{m_1,m_2,m_3}\}_{m_1,m_2,m_3 \in \mathbf{N}}$ satisfies the hypothesis of the theorem 2.1 with

$$\alpha^{(1)}_{m_1,m_2,m_3} = \alpha^{(2)}_{m_1,m_2,m_3} = \alpha^{(3)}_{m_1,m_2,m_3} = 0$$

and

$$\delta_{m_1,m_2,m_3}(x_1,x_2,x_3) = \frac{x(1-x)}{1+\alpha}\left[\frac{1}{m_1} + \alpha\right] + \\ + \frac{y(1-y)}{1+\beta}\left[\frac{1}{m_2} + \beta\right] + \frac{z(1-z)}{1+\gamma}\left[\frac{1}{m_3} + \gamma\right]$$

If $\alpha(m_1), \beta(m_2), \gamma(m_3)$ tend to zero as m_1, m_2, m_3 tend to infinity, from the theorem 2.1 one obtains that the sequence $\{U_{m_1,m_2,m_3}(f)\}$, defined by

$$U_{m_1,m_2,m_3}(f)(x,y,z) =$$

$$= \sum_{i=0}^{m_1} \sum_{i=0}^{m_1} \sum_{k=0}^{m_3} f \left(\frac{i}{m_1}, \frac{j}{m_2}, \frac{k}{m_3} \right) \cdot \omega_{m_1,i}(x, \alpha) \cdot \omega_{m_2,j}(y, \beta) \cdot \omega_{m_3,k}(z, \gamma) \cdot$$

$$\cdot \left\{ f \left(\frac{i}{m_1}, y, z \right) + f \left(x, \frac{j}{m_2}, z \right) + f \left(x, y, \frac{k}{m_3} \right) - \right.$$

$$\left. -f \left(\frac{i}{m_1}, \frac{j}{m_2}, z \right) - f \left(\frac{i}{m_1}, y, \frac{k}{m_3} \right) - f \left(x, \frac{j}{m_2}, \frac{k}{m_3} \right) + f \left(\frac{i}{m_1}, \frac{j}{m_2}, \frac{k}{m_3} \right) \right\}$$

converges to f, uniformly on I^3, as m_1, m_2, m_3 tend to infinity, for any $f \in C_b\left(I^3\right)$.

This result was obtained first in the paper [1.3] without the theorem 2.1.

Example 2: In this example one consider the operator of Bleimann, Butzer and Hahn $\widetilde{L}, \overline{L}, \overline{\overline{L}} : \mathbf{R}^I \to \mathbf{R}^I$, given by

$$\widetilde{L}(f)(x) = \sum_{i=0}^{m_1} f \left(\frac{i}{m_1 - i + 1} \right) \cdot p_{m_1,i}(x), \qquad p_{m_1,i}(x) = \begin{pmatrix} m_1 \\ i \end{pmatrix} \cdot \frac{x^i}{(1+x)^{m_1}};$$

$$\overline{L}(g)(y) = \sum_{j=0}^{m_2} g \left(\frac{j}{m_2 - j + 1} \right) \cdot \overline{q}_{m_2,j}(y), \qquad \overline{q}_{m_2,j}(y) = \begin{pmatrix} m_2 \\ j \end{pmatrix} \cdot \frac{y^j}{(1+y)^{m_2}};$$

$$\overline{\overline{L}}(h)(z) = \sum_{k=0}^{m_3} h \left(\frac{k}{m_3 - k + 1} \right) \cdot \overline{\overline{r}}_{m_3,k}(z), \qquad \overline{\overline{r}}_{m_3,k}(z) = \begin{pmatrix} m_3 \\ k \end{pmatrix} \cdot \frac{z^{jk}}{(1+z)^{m_3}}.$$

The operator $L_{m_1}, L_{m_2}, L_{m_3}$ are the parametric extensions of the operators from above, i.e.

$$L_{m_1}(f)(x, y, z) = \sum_{i=0}^{m_1} f \left(\frac{i}{m_1 - i + 1}, y, z \right) \cdot p_{m_1,i}(x),$$

$$L_{m_2}(f)(x, y, z) = \sum_{j=0}^{m_2} g \left(x, \frac{j}{m_2 - j + 1}, z \right) \cdot \overline{q}_{m_2,j}(y),$$

$$L_{m_3}(f)(x, y, z) = \sum_{k=0}^{m_3} h \left(x, y, \frac{k}{m_3 - k + 1} \right) \cdot \overline{\overline{r}}_{m_3,k}(z).$$

The product of these extensions is the positive linear operator

$$L_{m_1,m_2,m_3}(f)(x, y, z) = \sum_{i=0}^{m_1} \sum_{i=0}^{m_1} \sum_{k=0}^{m_3} f \left(\frac{i}{m_1}, \frac{j}{m_2}, \frac{k}{m_3} \right) \cdot p_{m_1,i}(x) \cdot \overline{q}_{m_2,j}(y) \cdot \overline{\overline{r}}_{m_3,k}(z).$$

It is easy to see that $\{L_{m_1,m_2,m_3}\}$ satisfies the hypothesis of theorem 2.1. Applying then this theorem, it follows that the sequence $\{U_{m_1,m_2,m_3}(f)\}$, where

$$U_{m_1,m_2,m_3}(f) = \sum_{i=0}^{m_1}\sum_{i=0}^{m_1}\sum_{k=0}^{m_3} f\left(\frac{i}{m_1},\frac{j}{m_2},\frac{k}{m_3}\right)\cdot p_{m_1,i}(x)\cdot \overline{q}_{m_2,j}(y)\cdot \overline{\overline{r}}_{m_3,k}(z)\cdot$$

$$\cdot \qquad \cdot\left\{f\left(\frac{i}{m_1},y,z\right)+f\left(x,\frac{j}{m_2},z\right)+f\left(x,y,\frac{k}{m_3}\right)-\right.$$

$$\left.-f\left(\frac{i}{m_1},\frac{j}{m_2},z\right)-f\left(\frac{i}{m_1},y,\frac{k}{m_3}\right)-f\left(x,\frac{j}{m_2},\frac{k}{m_3}\right)+f\left(\frac{i}{m_1},\frac{j}{m_2},\frac{k}{m_3}\right)\right\}$$

converges to f, uniformly on I^3 as m_1, m_2, m_3 tend to infinity.

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A NOTE ON STATE ESTIMATION FROM DOUBLY STOCHASTIC POINT PROCESS OBSERVATION

DANG PHUOC HUY AND TRAN JUNG THAO

0. Introduction

In this note we study a state estimation of a Markovian semimartingale from a doubly stochastic point process observation.

All stochastic processes below are supposed to be defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ where $(\mathcal{F}_t)$ is a filtration satisfying usual conditions.

Consider a state estimation problem where the signal process is a real-valued continuous semimartingale X that is also a Markov process given by

$$X_t = X_0 + \int_0^t H_s ds + B_t, \quad t \in \mathbb{R}^+, \tag{0.1}$$

where H_t is a continuous process and B_t is a standard Brownian motion, and the observation is a doubly stochastic point process N_t driven by X_t: N_t is a point process of intensity $\lambda_t = \lambda(X_t)$ where λ is a nonnegative boolean function.

Denote by Z_t^u the process $\exp(iuX_t)$. We want to investigate the best state estimation

$$\pi_t(Z_t^u) = E[Z_t^u | \mathcal{F}_t^N] \tag{0.2}$$

where $\mathcal{F}_t^N$ is the natural filtration of the process N_t i.e. $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$. In the sequel the notation $\pi_t(\dots)$ stands for the conditional expectation given $\mathcal{F}_t^N$.

1. A stochastic differential equation for the best state estimation of Z_t^u

Theorem 1. *$\pi_t(Z_t^u)$ satisfies the following equation:*

$$\pi_t(Z_t^u) = E[Z_0^u] + iu\int_0^t \pi_s(Z_s^u H_s)ds - \frac{u^2}{2}\int_0^t \pi_s(Z_s^u) +$$

$$+ \int_0^t \lambda_s^{-1}\pi_s[(Z_s^u - \pi_s(Z^u))(\lambda_s - \pi_s(\lambda_s))](dN_s - \pi_s(\lambda_s)ds) \tag{1.1}$$

Proof. Applying the Ito formula to $z_t^u = \exp(iuX_t)$ we have

$$Z_t^u = Z_0^u + \int_0^t \left(iuH_s - \frac{u^2}{2} \right) ds + iu \int_0^t Z_s^u dB_s.$$

Z_t^u is in fact a semimartingale, and the filtering equation from point process observation [2] applied to Z_t^u:

$$Z_t(Z^u) = E[Z_0^u] + \int_0^t \pi_s \left[Z_s^u \left(iuH_s - \frac{u^2}{2} \right) \right] ds +$$

$$+ \int_0^t \pi_s^{-1}(\lambda)[\pi_s(Z^u \lambda_s) - \pi_s(Z_s^u)\pi_s(\lambda_s)][dN_s - \pi_s(\lambda_s)ds].$$

Now

$$\pi_s\{[Z_s^u - \pi_s(Z_s^u)][\lambda_s - \pi_s(\lambda_s)]\} =$$

$$= \pi_s[Z_s^u \lambda_s - Z_s^u \pi_s(\lambda_s) - \pi_s(Z_s^u)\lambda_s + \pi_s(Z_s^u)\pi_s(\lambda_s)] =$$

$$= \pi_s(Z_s^u \lambda_s) - \pi_s[Z_s^u \pi_s(\lambda_s)] - \pi_s[\pi_s(Z_s^u)\lambda_s] + \pi_s(Z_s^u)\pi_s(\lambda_s). \tag{1.2}$$

It follows from

$$\pi_s[Z_s^u \pi_s(\lambda_s)] = E[Z_s^u E(\lambda_s|\mathcal{F}_s^N)|\mathcal{F}_s^N] =$$

$$= E(\lambda_s|\mathcal{F}_s^N)E(Z_s^u|\mathcal{F}_s^N) = \pi_s(\lambda_s)\pi_s(Z^u),$$

and also from

$$\pi_s[\pi_s(Z_s^u)\lambda_s] = \pi_s(Z_s^u)\pi_s(\lambda_s)$$

that it remains only the first and the second terms in the left hand side of (1.2) and we have:

$$\pi_s(Z_s^u \lambda_s) - \pi_s(Z_s^u)\pi_s(\lambda_s) = \pi_s[(Z_s^u - \pi_s(Z_s^u))(\lambda_s - \pi_s(\lambda_s))]$$

and the equation (1.1) is thus completely proved.

Remark. In the multidimensional case, the signal process is a vector process given by

$$X_t = X_0 + \int_0^t H_s ds + B_t$$

where X, H, B are multidimensional process. By Z_t^u we denote now the process $\exp(i\langle u, X_t \rangle)$, where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $X_t = (X_t^1, \ldots, X_t^n)$ and $\langle , \rangle$ stands for the scalar product in $\mathbb{R}^n$. And the best state estimation for Z_t^u based on an observation process that is a doubly stochastic point of intensity $\lambda_t = \lambda(X_t)$ is

$$\pi_t(Z_t^u) \equiv E[Z_t^u | \mathcal{F}_t^N] = E[\exp i\langle u, X_t\rangle | \mathcal{F}_t^N]. \tag{1.3}$$

The stochastic differential equation for $\pi_t(Z_t^u)$ is the same as (1.1) with $Z_t^u = \exp\langle u, X_t\rangle$.

In the next Section, we will establish a connection between the characteristic function of X_t and the filter of Z_t^u and so we will see that the laws of the signal X_t can be completely determined by $\pi_t(Z_t^u)$.

2. **Characteristic function of X_t**

Put

$$\psi_t(u) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\exp(iu\Delta X_t) - 1 | X_t] \tag{2.1}$$

is the limit in the right hand side exists, where $E[\cdot | X_t]$ is the conditional expectation given X_t.

Denote by $\varphi_t(u)$ the characteristic function of X_t:

$$\varphi_t(u) = E[\exp(iuX_t)] = E[Z_t^u].$$

We note that

$$\varphi_{t+\Delta t}(u) = E[\exp(iuX_{t+\Delta t})] = E[\exp iu(X_t + \Delta X_t)] =$$

$$= E[\exp(iuX_t \exp iu\Delta X_t] =$$

$$= E[\exp(iuX_t E(\exp iu\Delta X_t | X_t))]$$

$$\varphi_{t+\Delta t}(u) - \varphi_t(u) = E\{(\exp(iuX_t) E[\exp iu\Delta X_t - 1 | X_t]\}$$

It follows that

$$\frac{\partial \varphi_t(u)}{\partial t} = \lim_{\Delta t \downarrow 0} E\left\{(\exp iuX_t) \frac{1}{\Delta t} E[\exp iu\Delta X_t - 1 | X_t]\right\}.$$

We have now:

$$\frac{\partial \varphi_t(u)}{\partial t} = E[Z_t^u \psi_t(u)]$$

$$\tag{2.2}$$

$$\varphi_0(u) = E[Z_0^u]$$

Next, we denote by $\mathcal{F}^N_{t-\varepsilon}$ the σ-algebra generated by all N_s, $s \le t-\varepsilon$ for all small $\varepsilon > 0$. In noticing that by definition (2.1) $\psi_t(u)$ is conditioning to the random variable χ_t so it is independent of $\mathcal{F}^N_{t-\varepsilon}$ and we have:

$$E[Z^u_t\psi(u)] = E[E(Z^u_t\psi(u)|\mathcal{F}^N_{t-\varepsilon})] = E[\psi_t(u)E(Z^u_t|\mathcal{F}^N_{t-\varepsilon})].$$

Because of the left continuity of $(\mathcal{F}^N_t)$ we have by letting $\varepsilon \to 0$

$$E[Z^u_t\psi_t(u)] = E[\psi_t(u)E(Z^u_t|\mathcal{F}^N_t)] = E[\psi_t(u)\pi_t(Z^u_t)]$$

then we have the following

Proposition 1. *The law of the signal X_t can be determined in term of filtering by the following equation:*

$$\frac{\partial\varphi_t(u)}{\partial t} = E[\psi_t(u)\pi_t(Z^u_t)]$$

$$\varphi_0(u) = E[Z^u_0] \tag{2.3}$$

We will see in next Section that X_t can be recognized by filtering and the process H_t.

3. An expression of the function $\psi_t(u)$

The equation (1.1) can be rewritten as:

$$dX_t = H_t dt + dB_t \tag{3.1}$$

or

$$\Delta X_t = H_t\Delta t + \Delta B_t \tag{3.2}$$

where $\Delta X_t = X_{t+\Delta t} - X_t$, $\Delta B_t = B_{t+\Delta t} - B_t$, B_t is a Brownian motion and since $EB_tB_s = \min(t,s)$, we have

$$E[\exp iu\Delta X_t - 1|X_t] = \exp[iuH_t(X_t)\Delta t]E[\exp iu\Delta B_t|X_t] - 1$$

It follows from the fact that ΔB_t is normally distributed with mean 0 and covariance Δt

$$E[iu\Delta B_t|X_t] = \exp\left[-\frac{1}{2}u^2\Delta t\right].$$

Hence,

$$\psi_t(u) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\exp iu\Delta X_t - 1 | X_t] =$$
$$= \exp\left(iuH_t - \frac{u^2}{2}\right)$$

or

$$\psi_t(u) = \exp\left(iuH_t - \frac{u^2}{2}\right). \tag{3.3}$$

A substitution of this expression of ψ_t into (2.3) yields

Proposition 2.

$$\frac{\partial \varphi_t(u)}{\partial t} = E\left\{\left[\exp\left(iuH_t - \frac{u^2}{2}\right)\right] \pi_t(Z_t^u)\right\} \tag{3.4}$$

$$\varphi_0(u) = E[Z_0]$$

4. A Bayes formula for the best state estimation of Z_t^u

We know that by a change of reference probability $P \to Q$ such that $P_t \ll Q_t$ for all restriction P_t and Q_t of P and Q respectively to $(\Omega, \mathcal{F}_t)$, we have [1]

$$E_P[U_t | \mathcal{G}_t] = \frac{E_Q[U_t L_t | \mathcal{G}_t]}{E_Q[L_t | \mathcal{G}_t]}$$

where U_t is a real-valued bounded process adapted to $\mathcal{F}_t, \mathcal{G}_t$ is any sub σ-field of $\mathcal{F}_t : \mathcal{G}_t \subset \mathcal{F}_t$ and $L_t = \dfrac{dP_t}{dQ_t}$.

Now, for a doubly stochastic point process Y_t of intensity $\lambda_t = \lambda(X_t)$ we have

$$L_t = \left(\prod_{0 \le s \le t} \lambda(X_s)\Delta N_s\right) \exp\left\{\int_0^t (1 - \lambda(X_s)) ds\right\}.$$

We note that under Q the process N_t is a Poisson process of intensity 1. And we have

$$\pi_t(Z_t^u) = \frac{E_Q[Z_t^u L_t | \mathcal{F}_t^N]}{E_Q[L_t | \mathcal{F}_t^N]} = \frac{E_Q[L_t \exp iuX_t | \mathcal{F}_t^N]}{E_Q[L_t | \mathcal{F}_t^N]}.$$

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ON THE OSTROWSKI'S INTEGRAL INEQUALITY FOR LIPSCHITZIAN MAPPINGS AND APPLICATIONS

S.S. DRAGOMIR

Abstract. A generalization of Ostrowski's inequality for lipschitzian mappings and applications in Numerical Analysis and for Euler's Beta function are given.

1. INTRODUCTION

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [2, p. 469].

THEOREM 1.1. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a,b]$ we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty.$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In this paper we prove that Ostrowski's inequality also holds for lipschitzian mappings and apply it in obtaining a Riemann's type quadrature formula for this class of mappings. Applications for Euler's Beta function are also given.

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2. OSTROWSKI'S INEQUALITY FOR LIPSCHITZIAN MAPPINGS

The following inequality for lipschitzian mappings holds:

THEOREM 2.1. *Let $u : [a, b] \to R$ be an $L-$lipschitzian mapping on $[a, b]$, i.e.,*

$$|u(x) - u(y)| \leq L|x - y| \text{ for all } x, y \in [a, b].$$

Then we have the inequality

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \leq L(b-a)^2 \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right]. \tag{2.1}$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^x (t-a)du(t) = u(x)(x-a) - \int_a^x u(t)dt$$

and

$$\int_x^b (t-b)du(t) = u(x)(b-x) - \int_x^b u(t)dt.$$

If we add the above two equalities, then we get

$$u(x)(b-a) - \int_a^b u(t)dt = \int_a^b p(x,t)du(t) \tag{2.2}$$

where

$$p(x,t) := \begin{cases} t-a \text{ if } t \in [a, x) \\ t-b \text{ if } x \in [x, b] \end{cases},$$

for all $x, t \in [a, b]$.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i \in \{0,...,n-1\}}(x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$. If $p : [a, b] \to R$ is Riemann integrable on $[a, b]$ and $v : [a, b] \to R$ is L-lipschitzian on $[a, b]$, then

$$\left| \int_a^b p(x)dv(x) \right| = \left| \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)})[v(x_{i+1}^{(n)}) - v(x_i^{(n)})] \right|$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| p(\xi_i^{(n)}) \right| (x_{i+1}^{(n)} - x_i^{(n)}) \left| \frac{v(x_{i+1}^{(n)}) - v(x_i^{(n)})}{x_{i+1}^{(n)} - x_i^{(n)}} \right|$$

$$\leq L \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} |p(\xi_i^{(n)})|(x_{i+1}^{(n)} - x_i^{(n)}) = L \int_a^b |p(x)|dx. \tag{2.3}$$

Applying the inequality (2.3) for $p(x,t)$ as above and $v(x) = u(x), x \in [a,b]$, we get

$$\left| \int_a^b p(x,t)du(t) \right| \leq L \int_a^b |p(x,t)|dt$$

$$= L[\int_a^x \left[t - a|dt + \int_x^b |t-b|dt \right] = \frac{L}{2}[(x-a)^2 + (b-x)^2]$$

$$= L(b-a)^2[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}] \tag{2.4}$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).

Now, assume that the inequality (2.1) holds with a constant $C > 0$, i.e.,

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \leq L(b-a)^2 \left[C + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \tag{2.5}$$

for all $x \in [a,b]$.

Consider the mapping $f : [a,b] \to R, f(x) = x$ in (2.5). Then

$$\left| x - \frac{a+b}{2} \right| \leq C + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}$$

for all $x \in [a,b]$; and then for $x = a$, we get

$$\frac{b-a}{2} \leq \left(C + \frac{1}{4} \right)(b-a)$$

which implies that $C \geq \frac{1}{4}$ and the theorem is completely proved. □

The following corollary holds:

COROLLARY 2.2. *Let $u : [a,b] \to R$ be as above. Then we have the inequality:*

$$\left| \int_a^b u(t)dx - u(\frac{a+b}{2})(b-a) \right| \leq \frac{1}{4}L(b-a)^2. \tag{2.6}$$

Remark 2.3. It is well known that if $f:[a,b]\to R$ is a convex mapping on $[a,b]$, then *Hermite-Hadamard's* inequality holds

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{2.7}$$

Now, if we assume that $f:I\subset R\to R$ is convex on I and $a,b\in Int(I), a<b$, then f'_+ is monotonous nondecreasing on $[a,b]$ and by Theorem 2.1 we get

$$0\leq\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\leq\frac{1}{4}f'_+(b)(b-a) \tag{2.8}$$

which gives a counterpart for the first membership of Hadamard's inequality.

3. A QUADRATURE FORMULA OF RIEMANN TYPE

Let $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$ and $\xi_i\in[x_i,x_{i+1}]$ $(i=0,...,n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f,I_n,\xi)=\sum_{i=0}^{n-1}f(\xi_i)h_i$$

where $h_i:=x_{i+1}-x_i$.

We have the following quadrature formula

THEOREM 3.1. *Let $f:[a,b]\to R$ be an $L-$lipschitzian mapping on $[a,b]$ and I_n,ξ_i $(i=0,...,n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx=R_n(f,I_n,\xi)+W_n(f,I_n,\xi) \tag{3.1}$$

where the remainder satisfies the estimation

$$|W_n(f,I_n,\xi)|\leq\frac{1}{4}L\sum_{i=0}^{n-1}h_i^2+L\sum_{i=0}^{n-1}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2$$
$$\leq\frac{1}{2}L\sum_{i=0}^{n-1}h_i^2 \tag{3.2}$$

for all ξ_i $(i=0,...,n-1)$ as above.

The constant $\frac{1}{4}$ is sharp in (3.2).

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i\right| \leq L\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]. \tag{3.3}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$|W_n(f, I_n, \xi)| \leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i\right|$$
$$\leq L\sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right].$$

Now, as

$$\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \leq \frac{1}{4}h_i^2$$

for all $\xi_i \in [x_i, x_{i+1}] (i = 0, ..., n-1)$ the second part of (3.2) is also proved. □

Note that the best estimation we can get from (3.2) is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$ obtaining the following midpoint formula:

COROLLARY 3.2. *Let f, I_n be as above. Then we have the midpoint rule*

$$\int_a^b f(x)dx = M_n(f, I_n) + S_n(f, I_n)$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$|S_n(f, I_n)| \leq \frac{1}{4}L\sum_{i=0}^{n-1} h_i^2.$$

Remark 3.3. If we assume that $f : [a, b] \to R$ is differentiable on (a, b) and whose derivative f' is bounded on (a, b) we can put instead of L the infinity norm $\|f'\|_\infty$ obtaining the estimation due to Dragomir-Wang from the paper [1].

4. APPLICATIONS FOR EULER'S BETA MAPPING

Consider the mapping Beta for real numbers

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \ p,q>0$$

and the mapping $e_{p,q}(t) := t^{p-1}(1-t)^{q-1}, t \in [0,1]$.

We have for $p,q>1$ that

$$e'_{p,q}(t) = e_{p-1,q-1}(t)[p-1-(p+q-2)t].$$

If $t \in \left[0, \frac{p-1}{p+q-2}\right)$ then $e'_{p,q}(t) > 0$ and if $t \in \left(\frac{p-1}{p+q-2}, 1\right]$ then $e'_{p,q}(t) < 0$ which shows that for $t_0 = \frac{p-1}{p+q-2}$ we have a maximum for $e_{p,q}$ and then:

$$\sup_{t\in[0,1]} e_{p,q}(t) = e_{p,q}(t_0) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}; \ p,q>1.$$

Consequently

$$\begin{aligned} |e'_{p,q}(t)| &\le \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \max_{t\in[0,1]} |p-1-(p+q-2)t| \\ &= \max\{p-1,q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}; \ p,q>2 \end{aligned}$$

for all $t \in [0,1]$ and then

$$\| e'_{p,q}\|_\infty \le \max\{p-1,q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \ p,q>2. \tag{4.1}$$

The following inequality for Beta mapping holds

PROPOSITION 4.1. *Let $p,q>2$ and $x \in [0,1]$. Then we have the inequality*

$$\begin{aligned} &|B(p,q) - x^{p-1}(1-x)^{q-1}| \\ &\le \max\{p-1,q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} [\frac{1}{4} + (x-\frac{1}{2})^2]. \end{aligned} \tag{4.2}$$

The proof follows by Theorem 2.1 applied for the mapping $e_{p,q}$ and taking into account that $\| e'_{p,q}\|_\infty$ satisfies the inequality (4.1).

COROLLARY 4.2. *Let $p, q > 2$. Then we have the inequality*

$$\left| B(p,q) - \frac{1}{2^{p+q-2}} \right| \leq \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}.$$

Now, if we apply Theorem 3.1 for the mapping $e_{p,q}$ we get the following approximation of Beta mapping in terms of Riemann sums.

PROPOSITION 4.3. *Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ a sequence of intermediate points for I_n and $p, q > 2$. Then we have the formula*

$$B(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1}(1-\xi_i)^{q-1} h_i + T_n(p,q)$$

where the remainder $T_n(p,q)$ satisfies the estimation

$$|T_n(p,q)| \leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}$$

$$\times \left[\frac{1}{4} \sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]$$

$$\leq \frac{1}{2} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2.$$

Particularly, if we choose above $\xi_i = \frac{x_i + x_{i+1}}{2}$ $(i = 0, ..., n-1)$ then we get the approximation

$$B(p,q) = \frac{1}{2^{p+q-2}} \sum_{i=0}^{n-1} (x_i + x_{i+1})^{p-1}(2 - x_i - x_{i+1})^{q-1} + V_n(p,q)$$

where

$$|V_n(p,q)| \leq \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2.$$

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EXPONENTIAL STABILITY OF EVOLUTION OPERATORS

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Abstract. The aim of this paper is to give some sufficient, respectively necessary and sufficient conditions, for the exponential stability of evolution operators in infinite-dimensional spaces. The obtained results are like those, of Datko-type, for evolutionary processes which are linear operators-valued.

1. Introduction

Let X be a Banach space and let $(X_t)_{t\geq 0}$ be a family of parts of X.

Definition 1. The family of applications $\Phi(t,t_0) : X_{t_0} \to X_t$, $t \geq t_0 \geq 0$, will be called an evolution operator in X, if the following conditions are satisfied:

i) $\Phi(t,t)x = x$, for all $t \geq 0$ and $x \in X_t$.

ii) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$, for all $t \geq s \geq t_0 \geq 0$.

iii) $\Phi(\cdot,s)x : [s,\infty) \to X$ is continuous, for all $s \geq 0$ and $x \in X_s$.

iv) There is a nondecreasing function $p(\cdot) : \mathbb{R}_+ \to (0,\infty)$, such that

$$\|\Phi(t,s)x\| \leq p(t-s)\|x\|, \text{ for all } t \geq s \geq 0 \text{ and } x \in X_s.$$

Remark 1. Condition iv) can be replaced by

v) There are $M, \omega > 0$ such that

$$\|\Phi(t,s)x\| \leq Me^{\omega(t-s)}\|x\|,$$

for all $t \geq s \geq 0$ and $x \in X_s$.

Proof. Let iv) be satisfied and let $t \geq s \geq 0$ and $x \in X_s$. Then there are $n \in \mathbb{N}$ and $r \in [0,1)$ such that $t - s = n + r$. We have

$$\|\Phi(t,s)x\| \leq p(t-s-n)\|\Phi(s+n,s)x\| \leq p(1)^{n+1}\|x\|.$$

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Let $\omega > \max\{0, \ln p(1)\}$. Then

$$\|\Phi(t,s)x\| \leq p(1)e^{\omega n}\|x\| \leq p(1)e^{\omega(t-s)}\|x\|.$$

The converse is obviously. □

In the sequel we will denote by M and ω those constants which satisfy condition v).

Definition 2. The evolution operator $\Phi(\cdot,\cdot)$ will be called exponentially stable, if there are $\nu > 0$ and a function $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_0)x\| \leq N(t_0)e^{-\nu(t-t_0)}\|x\|,$$

for all $t \geq t_0 \geq 0$ and $x \in X_{t_0}$.

Remark 2. Let $\Phi(\cdot,\cdot)$ be an evolution operator. The following assertions are equivalent:

(1) $\Phi(\cdot,\cdot)$ is exponentially stable.

(2) There are $\nu > 0$ and $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_0)x\| \leq N(s)e^{-\nu(t-s)}\|\Phi(s,t_0)x\|,$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X_{t_0}$.

Definition 3. The evolution operator $\Phi(\cdot,\cdot)$ will be called uniformly exponentially stable, if there are $N, \nu > 0$ such that

$$\|\Phi(t,t_0)x\| \leq Ne^{-\nu(t-t_0)}\|x\|,$$

for all $t \geq t_0 \geq 0$ and $x \in X_{t_0}$.

Remark 3. The evolution operator $\Phi(\cdot,\cdot)$ is uniformly exponentially stable if and only if there are $N, \nu > 0$ such that

$$\|\Phi(t,t_0)x\| \leq Ne^{-\nu(t-s)}\|\Phi(s,t_0)x\|,$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X_{t_0}$.

Lemma. *Let $\Phi(\cdot,\cdot)$ be an evolution operator. If there are $r > 0$ and a continuous function $g : [r,\infty) \to (0,\infty)$ such that*

$$\begin{cases} \inf\limits_{t>r} g(t) < 1, \\ \|\Phi(t,t_0)x\| \leq g(t-t_0)\|x\|, \text{ for all } t_0 \geq 0,\ t \geq t_0 + r \text{ and } x \in X_{t_0}, \end{cases}$$

then $\Phi(\cdot,\cdot)$ is uniformly exponentially stable.

Proof. Let $\delta > r$ such that $g(\delta) < 1$.

For $t \geq t_0 \geq 0$ there is $n \in \mathbb{N}$ such that $n\delta \leq t - t_0 < (n+1)\delta$.

Let $x \in X_{t_0}$. Then

$$\|\Phi(t,t_0)x\| \leq Me^{\omega(t-n\delta-t_0)}\|\Phi(t_0+n\delta,t_0)x\| \leq$$

$$\leq Me^{\omega(t-n\delta-t_0)}g(\delta)^n\|x\|.$$

Denoting $\nu = \dfrac{-\ln g(\delta)}{\delta} > 0$, it follows that

$$\|\Phi(t,t_0)x\| \leq Me^{\omega\delta}e^{\nu\delta}e^{-\nu(t-t_0)}\|x\|.$$

Denoting $N = Me^{(\omega+\nu)\delta}$, we obtain

$$\|\Phi(t,t_0)x\| \leq Ne^{-\nu(t-t_0)}\|x\|,$$

for $t \geq t_0 \geq 0$ and $x \in X_{t_0}$. □

Theorem 1. *The evolution operator $\Phi(\cdot,\cdot)$ is uniformly exponentially stable if and only if there is $K \in (0,\infty)$ such that*

$$\int_t^\infty \left(\int_u^{u+1} \|\Phi(s,t)x\|ds\right) du \leq K\|x\|, \text{ for all } t \geq 0 \text{ and } x \in X_t.$$

Proof. Let $\Phi(\cdot,\cdot)$ be an evolution operator which satisfy, for a $K > 0$, the condition of the hypothesis. We have

$$\|\Phi(t,t_0)x\| \leq Me^{\omega(t-s)}\|\Phi(s,t_0)x\|,$$

so

$$e^{\omega s}\|\Phi(t,t_0)x\| \leq Me^{\omega t}\|\Phi(s,t_0)x\|, \text{ for } t \geq s \geq t_0 \geq 0 \text{ and } x \in X_{t_0}.$$

Let $t \geq t_0 + 1$. Integrating successively the last relation we obtain

$$\frac{1}{\omega}(e^\omega - 1)e^{\omega u}\|\Phi(t,t_0)x\| \leq Me^{\omega t}\int_u^{u+1} \|\Phi(s,t_0)x\|ds,$$

for $u \in [t_0, t-1]$, and so

$$\frac{e^\omega - 1}{\omega^2}(e^{\omega t-\omega} - e^{\omega t_0})\|\Phi(t,t_0)x\| \leq$$

$$\leq Me^{\omega t}\int_{t_0}^{t-1} \left(\int_u^{u+1} \|\Phi(s,t_0)x\|ds\right) du \leq MKe^{\omega t}\|x\|.$$

It follows that

$$e^{-\omega}\|\Phi(t,t_0)x\| \leq e^{-\omega(t-t_0)}\|\Phi(t,t_0)x\| + \frac{MK\omega^2}{e^{\omega}-1}\|x\| \leq$$

$$\leq M\left(1+\frac{K\omega^2}{e^{\omega}-1}\right)\|x\|.$$

For $t_0 \leq t < t_0+1$ and $x \in X_{t_0}$ we have

$$\|\Phi(t,t_0)x\| \leq Me^{\omega(t-t_0)}\|x\| \leq Me^{\omega}\|x\|.$$

Denoting $L = Me^{\omega}\left(1+\dfrac{K\omega^2}{e^{\omega}-1}\right)$. we obtain

$$\|\Phi(t,t_0)x\| \leq L\|x\|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}.$$

It follows that, for $t \geq s \geq t_0 \geq 0$, $x \in X_{t_0}$, we have

$$\|\Phi(t,t_0)x\| = \|\Phi(t,s)\Phi(s,t_0)x\| \leq L\|\Phi(s,t_0)x\|.$$

When $t \geq t_0+1$, we obtain

$$\|\Phi(t,t_0)x\| \leq L\int_u^{u+1}\|\Phi(s,t_0)x\|ds, \text{ for all } u \in [t_0, t-1],$$

and so

$$(t-1-t_0)\|\Phi(t,t_0)x\| \leq L\int_{t_0}^{t-1}\left(\int_u^{u+1}\|\Phi(s,t_0)x\|ds\right)du \leq LK\|x\|.$$

It follows by the preceding lemma that $\Phi(\cdot,\cdot)$ is uniformly exponentially stable.

The converse is immediately by direct calculation. □

Theorem 2. *Let $\Phi(\cdot,\cdot)$ be an evolution operator. If there are $\alpha > 0$ and a function $H(\cdot): \mathbb{R}_+ \to (0,\infty)$ such that*

$$\int_t^{\infty}\left(\int_u^{u+1} e^{\alpha(s-t)}\|\Phi(s,t)x\|ds\right)du \leq H(t)\|x\|, \textit{ for all } t \geq 0 \textit{ and } x \in X_t,$$

then there is a function $N(\cdot): \mathbb{R}_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_0)x\| \leq N(t_0)e^{-\alpha(t-t_0)}\|x\|, \textit{ for all } t \geq t_0 \geq 0 \textit{ and } x \in X_{t_0}.$$

Hence $\Phi(\cdot,\cdot)$ will be exponentially stable.

Proof. Let $t_0 \geq 0$, $t \geq t_0+1$ and $x \in X_{t_0}$. We have

$$\|\Phi(t,t_0)x\| \leq Me^{\omega(t-s)}\|\Phi(s,t_0)x\|, \text{ for } s \in [t_0, t].$$

It follows that

$$e^{-\alpha t_0} e^{(\omega+\alpha)s} \|\Phi(t,t_0)x\| \le M e^{\omega t} e^{\alpha(s-t_0)} \|\Phi(s,t_0)x\|,$$

and by integration, for $u \in [t_0, t-1]$, we have

$$e^{-\alpha t_0} \frac{e^{\omega+\alpha}-1}{\omega+\alpha} e^{(\omega+\alpha)u} \|\Phi(t,t_0)x\| \le M e^{\omega t} \int_u^{u+1} e^{\alpha(s-t_0)} \|\Phi(s,t_0)x\| ds,$$

and so

$$(e^{(\omega+\alpha)(t-1)} - e^{(\omega+\alpha)t_0}) \|\Phi(t,t_0)x\| \le$$
$$\le M e^{\omega t} \int_{t_0}^{t-1} \left(\int_u^{u+1} e^{\alpha(s-t_0)} \|\Phi(s,t_0)x\| ds \right) du,$$

from which

$$(e^{\alpha(t-t_0)-(\omega+\alpha)} - e^{-\omega(t-t_0)}) \|\Phi(t,t_0)x\| \le M \frac{(\omega+\alpha)^2}{e^{\omega+\alpha}-1} H(t_0) \|x\|.$$

It follows that

$$e^{\alpha(t-t_0)} \|\Phi(t,t_0)x\| \le e^{\omega+\alpha} \left(M \frac{(\omega+\alpha)^2}{e^{\omega+\alpha}-1} H(t_0) + M \right) \|x\|.$$

Denoting $N(t_0) = M e^{\omega+\alpha} \left(\frac{(\omega+\alpha)^2}{e^{\omega+\alpha-1}} H(t_0) + 1 \right)$, we obtain

$$\|\Phi(t,t_0)x\| \le N(t_0) e^{-\alpha(t-t_0)} \|x\|.$$

For $t_0 \le t < t_0 + 1$ and $x \in X_{t_0}$ we have

$$\|\Phi(t,t_0)x\| \le M e^{\omega} e^{\alpha} e^{-\alpha(t-t_0)} \|x\|.$$

So, it follows that

$$\|\Phi(t,t_0)x\| \le N(t_0) e^{-\alpha(t-t_0)} \|x\|, \text{ for all } t \ge t_0 \ge 0 \text{ and } x \in X_{t_0}. \ \square$$

Using in the proofs of the theorems the p power of the norm, respectively the p power of the inner integral ($p \in [1,\infty)$), we obtain the following results.

Corollary 1. *Let $\Phi(\cdot,\cdot)$ be an evolution operator and $p \in [1,\infty)$ be arbitrarily. The following assertions are equivalent.*

1) $\Phi(\cdot,\cdot)$ is uniformly exponentially stable.

2) There is $K \in (0,\infty)$ such that

$$\int_t^{\infty} \left(\int_u^{u+1} \|\Phi(s,t)x\|^p ds \right) du \le K \|x\|^p, \text{ for all } t \ge 0 \text{ and } x \in X_t.$$

3) There is $K \in (0,\infty)$ such that

$$\int_t^\infty \left(\int_u^{u+1} \|\Phi(s,t)x\| ds\right)^p du \leq K\|x\|^p, \text{ for all } t \geq 0 \text{ and } x \in X_t.$$

Corollary 2. *Let $\Phi(\cdot,\cdot)$ be an evolution operator and $p \in [1,\infty)$ be arbitrarily.*

1) If there are $\alpha > 0$ and a function $H(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that

$$\int_t^\infty \left(\int_u^{u+1} e^{\alpha(s-t)} \|\Phi(s,t)x\|^p ds\right) du \leq H(t)\|x\|^p, \text{ for all } t \geq 0 \text{ and } x \in X_t,$$

then there is $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_0)x\| \leq N(t_0)e^{-\frac{\alpha}{p}(t-t_0)}\|x\|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}.$$

2) If there are a function $H(\cdot) : \mathbb{R}_+ \to (0,\infty)$ and $\alpha > 0$ such that

$$\int_t^\infty \left(\int_u^{u+1} e^{\alpha(s-t)} \|\Phi(s,t)x\| ds\right)^p du \leq H(t)\|x\|^p, \text{ for all } t \geq 0 \text{ and } x \in X_t,$$

then there is $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_0)x\| \leq N(t_0)e^{-\alpha(t-t_0)}\|x\|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}.$$

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ON THE CONVERGENCE OF ITERATIVE PROCESS FOR NON-LOCAL PROBLEM SOLUTION IN SURGERY

I. OZTURK AND M.M. LAFISHEVA

Abstract. In this study a priori estimate for the solution of parabolic differential equation in surgery was found. Then this estimate was used to prove the convergence of iterative process.

1. In domain $Q = \Omega x\,(0,T)\,, \Omega \equiv \{x = (x_1, x_2) : r_0 < x_1 < R,\ 0 < x_2 < \pi\}$, consider the following problem

$$\frac{\partial u}{\partial t} = \frac{1}{x_1}\frac{\partial}{\partial x_1}\left(x_1 k\,(x,t)\,\frac{\partial u}{\partial x_1}\right) + \frac{1}{x_1^2}\frac{\partial}{\partial x_2}\left(k\,(x,t)\,\frac{\partial u}{\partial x_2}\right) + f\,(x,t)\,, \tag{1}$$

$$\begin{cases} k\frac{\partial u}{\partial x_1}|_{x_1=r_0} = \int_{r_0}^{\alpha} u dx_1 - \mu_1\,(t, x_2)\,, & x_1 = r_0, \\ -k\frac{\partial u}{\partial x_1} = \beta u - \mu_2\,(t, x_2)\,, & x_1 = R, \end{cases} \tag{2}$$

$$\begin{cases} k\frac{1}{x_1}\frac{\partial u}{\partial x_2} = \gamma_1 u - \chi_1\,(x_1, t)\,, & x_2 = 0, \\ -k\frac{1}{x_1}\frac{\partial u}{\partial x_2} = \gamma_2 u - \chi_2\,(x_1, t)\,, & x_2 = \pi, \end{cases} \tag{3}$$

$$u\,(x, 0) = u_0\,(x)\,, \tag{4}$$

where α is a certain number of the interval (r_0, R), $k\,(x,t) \geq c_1 > 0, u_0\,(x)\,, f, \beta, \mu_\upsilon, \gamma_\upsilon, \chi_\upsilon,\ (\upsilon = 1, 2)$ are well known functions which satisfies smoothness conditions necessary for the solution to exist [1]. Problem (1)-(4) appeared during mathematical simulation of the processes of the heat transference into tissue [5]. In [1], the existence of the solution of the problem (1)-(4) is proved by using the potential method.

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Key words and phrases. a priori estimate, parabolic differential equation.

Our nearest goal is to get a priori estimate for the solution of problem (1)-(4) whence particularly uniqueness of the solution will be shown. Then the obtained priori estimate will be used to prove iterative process convergence.

Since the problem with non-local condition (2) generates a non self-conjugate problem and the sign of corresponding differential operator is not defined, the general theory developed for stability and difference scheme cannot be applied to the above mentioned problem. Besides, non-local condition does not allow to use of any disjunction scheme for the solution of two-dimensional problem (1)-(4).

All these difficulties can be overcome, if the following sequence of problems is considered instead of problem (1)-(4).

$$\begin{cases} \frac{\partial \overset{s}{u}}{\partial t} = L\overset{s}{u} + f, L\overset{s}{u} \equiv \frac{1}{x_1}\frac{\partial}{\partial x_1}\left(x_1 k \frac{\partial \overset{s}{u}}{\partial x_1}\right) + \frac{1}{x_1^2}\frac{\partial}{\partial x_2}\left(k\frac{\partial \overset{s}{u}}{\partial x_2}\right), \\ \begin{cases} k\frac{\partial \overset{s}{u}}{\partial x_1} = \int_{r_0}^{\alpha} \overset{s-1}{u}\, dx_1 - \mu_1, & x_1 = r_0, \\ -k\frac{\partial \overset{s}{u}}{\partial x_1} = \beta \overset{s}{u} - \mu_2, & x_1 = R, \end{cases} \\ \begin{cases} k\frac{1}{x_1}\frac{\partial \overset{s}{u}}{\partial x_2} = \gamma_1 \overset{s}{u} - \chi_1, & x_2 = 0, \\ -k\frac{1}{x_1}\frac{\partial \overset{s}{u}}{\partial x_2} = \gamma_2 \overset{s}{u} - \chi_2, & x_2 = \pi, \end{cases} \\ \overset{s}{u}(x,0) = u_0(x), & s = 1, 2, ... \end{cases} \tag{5}$$

where s is iterative index.

In each iteration, the problem (5) becomes regular, so for the numerical solution of (5), local one-dimensional schemes can be used [2]. Now let's multiply equation (1) with the scalar product $x_1 u$ and integrate by parts, we obtain

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\left(x_1, u^2\right) + \left(x_1, k u_{x_1}^2\right) + \left(\frac{1}{x_1}, k u_{x_2}^2\right) + \int_0^{\pi} R\beta u^2\left(R, x_2, t\right) dx_2 +$$

$$+ \int_0^{\pi} r_0 u\left(r_0, x_2, t\right)\left(\int_{r_0}^{\alpha} u\left(x_1, x_2, t\right) dx_1\right) dx_2 + \int_{r_0}^{R} \gamma_2 u^2\left(x_1, \pi, t\right) dx_1 +$$

$$+ \int_{r_0}^{R} \gamma_1 u^2\left(x_1, 0, t\right) dx_1 = \left(f, x_1 u\right) + \int_0^{\pi} R\mu_2 u\left(R, x_2, t\right) dx_2 + \int_0^{\pi} r_0 \mu_1 u\left(r_0, x_2, t\right) dx_2 +$$

$$+ \int_{r_0}^{R} \chi_2\left(x_1, t\right) u\left(x_1, \pi, t\right) dx_1 + \int_{r_0}^{R} \chi_1\left(x_1, t\right) u\left(x_1, 0, t\right) dx_1. \tag{6}$$

Let's estimate the right hand-side integrals of (6). Using S.L. Sobolev's embedding theorem [3], we get

$$\int_0^\pi r_0\mu_1 u(r_0,x_2,t)dx_2 \le \frac{r_0}{2}\left(\int_0^\pi u^2(r_0,x_2,t)\,dx_2 + \int_0^\pi \mu_1^2 dx_2\right) \le$$
$$\le \frac{r_0\varepsilon}{2}\int_0^\pi \|u_{x_1}\|^2_{L_2(x_1)}\,dx_2 + \frac{r_0 C_\varepsilon}{2}\int_0^\pi \|u_{x_1}\|_{L_2(x_1)}\,dx_2 + \frac{r_0}{2}\int_0^\pi \mu_1^2 dx_2 \le$$
$$\le \frac{\varepsilon}{2}\|\sqrt{x_1}u_{x_1}\|_0^2 + \frac{C_\varepsilon}{2}\|\sqrt{x_1}u\|_0^2 + \frac{r_0}{2}\int_0^\pi \mu_1^2 dx_2, \tag{7}$$

where $\|u\|_{L_2(x_1)}$ means that the norm is taken in correspondence with variable $x_1, \varepsilon > 0, C_\varepsilon > 0$ are positive constants.

In the same way we find the following

$$\int_0^\pi R\mu_2 u(R,x_2,t)\,dx_2 \le \frac{R}{2r_0}\varepsilon\|\sqrt{x_1}u_{x_1}\|_0^2 + \frac{R}{2r_0}C_\varepsilon\|\sqrt{x_1}u\|_0^2 + \frac{R}{2}\int_0^\pi \mu_2^2 dx_2, \tag{8}$$

$$\int_{r_0}^R \chi_2 u(x_1,\pi,t)\,dx_1 \le \frac{1}{2}\int_{r_0}^R \left(\varepsilon\|u_{x_2}\|^2_{L_2(x_2)} + C_\varepsilon\|u\|^2_{L_2(x_2)}\right)dx_1 + \frac{1}{2}\int_{r_0}^R \chi_2^2 dx_1 \le$$
$$\le \frac{R\varepsilon}{2}\left\|\frac{1}{\sqrt{x_1}}u_{x_2}\right\|_0^2 + \frac{C_\varepsilon}{2r_0}\|\sqrt{x_1}u\|_0^2 + \frac{1}{2}\int_{r_0}^R \chi_2^2 dx_1, \tag{9}$$

$$\int_{r_0}^R \chi_1 u(x_1,0,t)\,dx_1 \le \frac{R\varepsilon}{2}\left\|\frac{1}{\sqrt{x_1}}u_{x_2}\right\|_0^2 + \frac{C_\varepsilon}{2r_0}\|\sqrt{x_1}u\|_0^2 + \frac{1}{2}\int_{r_0}^R \chi_1^2 dx_1. \tag{10}$$

Let's estimate the left hand-side integrals of (6) which corresponds to non-local condition (2):

$$r_0\int_0^\pi u(r_0,x_2,t)\left(\int_{r_0}^\alpha u(x_1,x_2,t)\,dx_1\right)dx_2 \le$$
$$\le \frac{r_0}{2}\left(\int_0^\pi\int_{r_0}^\alpha u^2(x_1,x_2,t)\,dx_1dx_2 + \frac{\alpha-r_0}{2}\int_0^\pi u^2(r_0,x_2,t)\,dx_2\right) \le$$
$$\le \frac{1}{2}\int_0^\pi\int_{r_0}^R x_1u^2(x_1,x_2,t)\,dx_1dx_2 + \frac{r_0(\alpha-r_0)}{2}\int_0^\pi\left(\varepsilon\|u_{x_1}\|^2_{L_2(x_1)} + C_\varepsilon\|u\|^2_{L_2(x_1)}\right)dx_2$$
$$\le \frac{1}{2}(1+(\alpha-r_0)C_\varepsilon)\|\sqrt{x_1}u\|_0^2 + \frac{(\alpha-r_0)}{2}\varepsilon\|\sqrt{x_1}u_{x_1}\|_0^2, \tag{11}$$

$$(f, x_1 u) \leq \frac{1}{2} \left\| \sqrt{x_1} f \right\|_0^2 + \frac{1}{2} \left\| \sqrt{x_1} u \right\|_0^2, \tag{12}$$

where $\|u\|_0^2 = \int_0^{\pi} \int_{r_0}^{R} u^2 dx_1 dx_2$.

Having substituted inequalities (7)-(12) in (6) and chosen sufficiently small ε, we find

$$\frac{\partial}{\partial t} \left\| \sqrt{x_1} u \right\|_0^2 + \nu_1 \left\| \sqrt{x_1} u_{x_1} \right\|_0^2 + \nu_2 \left\| \frac{1}{\sqrt{x_1}} u_{x_2} \right\|_0^2 \leq$$

$$\leq \left\| \sqrt{x_1} f \right\|_0^2 + M(\varepsilon) \left\| \sqrt{x_1} u_{x_1} \right\|_0^2 + R \left| \mu \right|^2 + \left| \chi \right|^2, \tag{13}$$

where $\nu_1 = 2c_1 - \varepsilon \left[1 + (\alpha - r_0) + \frac{R}{r_0} \right] > 0, \nu_2 = 2 (c_1 - R\varepsilon) > 0, M(\varepsilon) = 1 + \left(1 + \frac{2+R}{r_0} \right) C_\varepsilon$,

$$\left| \mu \right|^2 = \int_0^{\pi} (\mu_1^2 + \mu_2^2) dx_2, \left| \chi \right|^2 = \int_{r_0}^{R} (\chi_1^2 + \chi_2^2) dx_1.$$

Integrating the inequality (13) from 0 to t we get

$$\left\| \sqrt{x_1} u \right\|_0^2 + \nu_1 \left\| \sqrt{x_1} u_{x_1} \right\|_{2,Q_t}^2 + \nu_2 \left\| \tfrac{1}{\sqrt{x_1}} u_{x_2} \right\|_{2,Q_t}^2 \leq$$

$$\leq \left\| \sqrt{x_1} f \right\|_{2,Q_t}^2 + M(\varepsilon) \int_0^t \left\| \sqrt{x_1} u \right\|_0^2 d\tau + \left\| \sqrt{x_1} u(x, 0) \right\|_0^2 +$$

$$+ R \int_0^t \left| \mu(\tau) \right|^2 d\tau + \int_0^t \left| \chi(\tau) \right|^2 d\tau, \tag{14}$$

where $\|u\|_{2,Q_t}^2 = \int_0^t \|u\|_0^2 d\tau$.

From inequality (14), we have

$$\left\| \sqrt{x_1} u \right\|_0^2 \leq M(\varepsilon) \int_0^t \left\| \sqrt{x_1} u \right\|_0^2 d\tau + F(t) \tag{15}$$

where $F(t) = \left\| \sqrt{x_1} f \right\|_{2,Q_t}^2 + \left\| \sqrt{x_1} u(x, 0) \right\|_0^2 + R \int_0^t \left| \mu(\tau) \right|^2 d\tau + \int_0^t \left| \chi(\tau) \right|^2 d\tau$.

Using well-known Lemma 1.1 [4], from (15) it is obtained

$$y(t) \le e^{M(\varepsilon)t} tF(t)\,, y(t) = \int_0^t \|\sqrt{x_1}u\|_0^2 d\tau. \tag{16}$$

By making use of (16), we obtain a necessary estimate from (14)

$$\|\sqrt{x_1}u\|_0^2 + \nu_1 \|\sqrt{x_1}u_{x_1}\|_{2,Q_t}^2 + \nu_2 \left\| \frac{1}{\sqrt{x_1}} u_{x_2} \right\|_{2,Q_t}^2 \le$$

$$\le M(t) \left[\|\sqrt{x_1}f\|_{2,Q_t}^2 + \|\sqrt{x_1}u(x,0)\|_0^2 + \int_0^t |\mu(\tau)|^2 d\tau + \int_0^t |\mu(\tau)|^2 d\tau \right]. \tag{17}$$

Since $r_0 < x_1 < R$, $\sqrt{x_1}$ can be ignored in estimate (17). So we have

$$\|u\|_0^2 + \nu_1 \|u_{x_1}\|_{2,Q_t}^2 + \nu_2 \|u_{x_2}\|_{2,Q_t}^2 \le M(t) \left[\|f\|_{2,Q_t}^2 + \|u_0(x)\|_0^2 + \int_0^t \left(|\mu(\tau)|^2 + |\chi(\tau)|^2 \right) d\tau \right]. \tag{18}$$

It is clear from estimate (18) that the problem (1)-(4) has a unique solution.

2. Let's designate $\overset{s}{z} = \overset{s}{u} - u$, then for $\overset{s}{z}$, we have

$$\frac{\partial \overset{s}{z}}{\partial t} = L \overset{s}{z},$$

$$\begin{cases} k \frac{\partial \overset{s}{z}}{\partial x_1} = \int_{r_0}^{\alpha} \overset{s-1}{z} dx_1 & \\ -k \frac{\partial \overset{s}{z}}{\partial x_1} = \beta \overset{s}{z}, & x_1 = R, \end{cases}$$

$$\begin{cases} k \frac{1}{x_1} \frac{\partial \overset{s}{z}}{\partial x_2} = \gamma_1 \overset{s}{z}, & x_2 = 0, \\ -k \frac{1}{x_1} \frac{\partial \overset{s}{z}}{\partial x_2} = \gamma_2 \overset{s}{z}, & x_2 = \pi, \end{cases}$$

$$\overset{s}{z}(x,0) = 0.$$

Using estimate (18) for $\overset{s}{z}$, we find

$$\left\| \overset{s}{z} \right\|_0^2 \le M(t) \int_0^t \int_0^\pi \left(\int_{r_0}^{\alpha} \overset{s-1}{z} dx_1 \right)^2 dx_2 dt \le M(T)(\alpha - r_0) \int_0^T \left\| \overset{s-1}{z} \right\|_0^2 dt,$$

or after integration with respect to t from 0 to T , we have

$$\left\|\overset{s}{z}\right\|_{2,Q_T} \leq \sqrt{M\left(T\right)T\left(\alpha - r_0\right)}\left\|\overset{s-1}{z}\right\|_{2,Q_T}, M\left(T\right) = Te^{M(\varepsilon)T}. \tag{19}$$

Suppose $q = Te^{M(\varepsilon)T}\sqrt{\alpha - r_0} < 1$. From (19), we obtain the estimate

$$\left\|\overset{s}{z}\right\|_{2,Q_T} \leq q^s \left\|\overset{0}{z}\right\|_{2,Q_T}.$$

Thus the iterative process is convergent in the norm $\|\cdot\|_{2,Q_T}$, for sufficiently small T or small $\alpha - r_0$.

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PERTURBATIONS OF CERTAIN NONLINEAR PARTIAL FINITE DIFFERENCE EQUATIONS

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Abstract. In the present paper we establish some new variation of constants formulae for nonlinear perturbed partial finite difference equations in two independent variables. We also present some applications to convey the importance of our results in the qualitative theory of certain partial finite difference equations.

1. **Introduction**

During the past few years the abundance of applications is stimulating a rapid development of the theory of finite difference equations. A variety of new methods and tools are developed by different investigators to study the various types of finite difference equations. In the theory of ordinary finite difference equations the method of variation of parameters is a very useful tool in studying the properties of solutions of perturbed finite difference equations. Motivated and inspired by the results given in [5], see also [1-4, 6-10], in the present paper we establish some representation formulae related to the solutions of a certain nonlinear partial finite difference equation and its perturbed partial finite difference equation in two independent variables. We also use these formulae to study certain properties of the solutions of the corresponding perturbed partial finite difference equation.

2. **Statement of results**

In what follows, we let $N_0 = \{0, 1, 2, \dots\}$, and

$$N(x_0) = \{x_0, x_0 + 1, x_0 + 2, \dots\}, \quad N(y_0) = \{y_0, y_0 + 1, y_0 + 2, \dots\},$$

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for x_0, y_0 in N_0. The empty sums and products are taken to be 0 and 1 respectively. For any functions $z(x,y), w(m,n), z(x,y,w(m,n))$, x,y,m,n in N_0, we define

$$\Delta_1 z(x,y) = z(x+1,y) - z(x,y), \quad \Delta_2 z(x,y) = z(x,y+1) - z(x,y),$$

$$\Delta_2\Delta_1 z(x,y) = \Delta_1 z(x,y+1) - \Delta_1 z(x,y)),$$

$$\Delta_m z(x,y,w(m,n)) = z(x,y,w(m+1,n)) - z(x,y,w(m,n)),$$

$$\Delta_n z(x,y,w(m,n)) = z(x,y,w(m,n+1)) - z(x,y,w(m,n)).$$

We denote the product $N(x_0)\times N(y_0)$ by $N(x_0,y_0)$. For $(x_0,y_0),(x,y)$ in $N(x_0,y_0)$ we define

$$\phi(x,y,x_0,y_0,w(x,y)) = \Delta_w z(x,y,x_0,y_0,w(x,y)),$$

where

$$\Delta_w z(x,y,x_0,y_0,w(x,y))(w(x+1,y)-w(x,y))) =$$

$$= z(x,y,x_), y_0, w(x+1,y)) - z(x,y,x_0,y_0,w(x,y)).$$

We consider the nonlinear partial finite difference equation

$$\Delta_1\Delta_1 u(x,y) = f(x,y,u(x,y)), \quad u(x,y_0) = u(x_0,y) = u_0, \tag{E}$$

and its perturbed nonlinear finite difference equation

$$\Delta_2\Delta_1 v(x,y) = f(x,y,v(x,y)) + g(x,y,v(x,y)), \quad v(x,y_0) = v(x_0,y) = u_0, \tag{P}$$

for (x,y) in $N(x_0,y_0)$, where u,v are real-valued functions defined on $N(x_0,y_0)$, f,g are real-valued functions defined on $N(x_0,y_0)\times R$, R denotes the set of real numbers, and u_0 is a constant. We use $u(x,y,x_0,y_0,u_0)$ and $v(x,y,x_0,y_0,u_0)$ to denote the solutions of (E) and (P) respectively passing through the point $x(x_0,y_0)\in N(x_0,y_0)$.

A useful nonlinear variation of constants formula is established in the following theorem.

Theorem 1. *Suppose that $u(x,y,x_0,y_0,u_0)$ is the unique solution of (E) and $\phi(x+1,y,x_0,y_0,w(x,y))$, $\phi^{-1}(x+1,y,x_0,y_0,w(x,y))$ exist for (x,y) in $N(x_0,y_0)$. Then any solution $v(x,y,x_0,y_0,u_0)$ of (P) satisfies the relation*

$$v(x,y,x_0,y_0,u_0) = u(x,y,x_0,y_0,u_0 + \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1}\phi^{-1}(s+1,t,x_0,y_0,w(s,t))\times$$
$$\times[A(s,t,x_0,y_0,w(s,t)) + g(s,t,v(s,t,x_0,y_0,u_0))]) \tag{2.1}$$

where $w(x,y)$ is a solution of the equation

$$\Delta_2\Delta_1 w(x,y) = \phi^{-1}(x+1,y,x_0,y_0,w(x,y))[A(x,y,x_0,y_0,w(x,y))+$$

$$+g(x,y,v(x,y,x_0,y_0,u_0))], \quad w(x,y_0) = w(x_0,y) = u_0, \tag{2.2}$$

for (x,y) in $N(x_0,y_0)$ and

$$A(x,y,x_0,y_0,w(x,y)) = -\{[\Delta_1 u(x,y+1,x_0,y_0,w(x,y+1))-$$

$$-\Delta_1 u(x,y+1,x_0,y_0,w(x,y))] + [\Delta_w u(x+1,y+1,x_0,y_0,w(x,y+1))-$$

$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))]\Delta_1 w(x,y+1)\}, \tag{2.3}$$

for $(x,y) \in N(x_0,y_0)$.

Another interesting and useful representation formula is given in the following theorem.

Theorem 2. *Suppose that $u(x,y,x_0,y_0,u_0)$ is the unique solution of (E) and $\phi(x+1,y,x_0,y_0,w(x,y))$, $\phi^{-1}(x+1,y,x_0,y_0,w(x,y))$ exist for $(x,y) \in N(x_0,y_0)$. Then any solution $v(x,y,x_0,y_0,u_0)$ of (P) satisfies the relation*

$$v(x,y,x_0,y_0,u_0) = u(x,y,x_0,y_0,u_0) + \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} B(x,y,x_0,y_0,w(s,t))+$$

$$+\sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} \phi(x,y,x_0,y_0,w(x,y))\times$$

$$\times\phi^{-1}(s+1,t,x_0,y_0,w(s,t))[A(s,t,x_0,y_0,w(s,t))+$$

$$+g(s,t,v(s,t,x_0,y_0,u_0))], \tag{2.4}$$

where $A(x,y,x_0,y_0,w(x,y))$ is given by (2.3) and

$$B(x,y,x_0,y_0,w(s,t)) = [\Delta_w u(x,y,x_0,y_0,w(s,t+1))-$$

$$-\Delta_w u(x,y,x_0,y_0,w(s,t))]\Delta_1 w(s,t+1), \tag{2.5}$$

where $w(x,y)$ is a solution of (2.2).

3. **Proof of Theorem 1**

Since $u(x,y,x_0,y_0,u_0)$ is the solution of (E), by the method of variation of parameters we can find the solution of (P) by the relation

$$v(x,y,x_0,y_0,u_0)=u(x,y,x_0,y_0,w(x,y)),\quad w(x_0,y)=w(x,y_0)=u_0, \tag{3.1}$$

where the function $w(x,y)$ is yet to be determined. For this it is necessary that

$$\Delta_1 v(x,y,x_0,y_0,u_0)=u(x+1,y,x_0,y_0,w(x+1,y))-u(x,y,x_0,y_0,w(x,y))=$$

$$=\Delta_1 u(x,y,x_0,y_0,w(x,y))+$$

$$+\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y). \tag{3.2}$$

From (3.2) we have

$$\Delta_2\Delta_1 v(x,y,x_0,y_0,u_0)=\Delta_1 u(x,y+1,x_0,y_0,w(x,y+1))-$$

$$-\Delta_1 u(x,y,x_0,y_0,w(x,y))+\Delta_w u(x+1,y+1,x_0,y_0,w(x,y+1))\Delta_1 w(x,y+1)-$$

$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y)=$$

$$=\Delta_1 u(x,y+1,x_0,y_0,w(x,y))-\Delta_1 u(x,y,x_0,y_0,w(x,y))+$$

$$+\Delta_1 u(x,y+1,x_0,y_0,w(x,y+1))-\Delta_1 u(x,y+1,x_0,y_0,w(x,y))+$$

$$+\Delta_w u(x+1,y+1,x_0,y_0,w(x,y+1))\Delta_1 w(x,y+1)-$$

$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y+1)+$$

$$+\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y+1)-$$

$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y)=$$

$$=\Delta_2\Delta_1 u(x,y,x_0,y_0,w(x,y))+\{[\Delta_1 u(x,y+1,x_0,y_0,w(x,y+1))-$$

$$-\Delta_1 u(x,y+1,x_0,y_0,w(x,y))]+[\Delta_w u(x+1,y+1,x_0,y_0,w(x,y+1))-$$

$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))]\Delta_1 w(x,y+1)\}+$$

$$+\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_2\Delta_1 w(x,y)=$$

$$=\Delta_2\Delta_1 u(x,y,x_0,y_0,w(x,y))-A(x,y,x_0,y_0,w(x,y))+$$

$$+\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_2\Delta_1 w(x,y). \tag{3.3}$$

Now from (E), (P) and (3.3) we have

$$f(x,y,v(x,y,x_0,y_0,u_0))+g(x,y,v(x,y,x_0,y_0,u_0))=$$

$$= f(x, y, u(x, y, x_0, y_0, w(x, y)) - A(x, y, x_0, y_0, w(x, y))+$$

$$+\phi(x + 1, x_0, y_0, w(x, y))\Delta_2\Delta_1 w(x, y),$$

which because of (3.1) and the fact that $\phi^{-1}(x + 1, y, x_0, y_0, w(x, y))$ exists, reduces to

$$\Delta_2\Delta_1 w(x, y) = \phi^{-1}(x + 1, y, x_0, y_0, w(x, y))[A(x, y, x_0, y_0, w(x, y))+$$

$$+g(x, y, v(x, y, x_0, y_0, u_0))], \quad w(x_0, y) = w(x, y_0) = u_0, \tag{3.4}$$

which determined the required function $w(x, y)$. The solutions of (3.4) then determine $w(x, y)$. Further from (3.4) we have

$$w(x, y) = u_0 + \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} \phi^{-1}(s + 1, t, x_0, y_0, w(s, t))[A(s, t, x_0, y_0, w(s, t))+$$

$$+g(s, t, v(s, t, x_0, y_0, w(s, t))]. \tag{3.5}$$

From (3.5) and (3.1), (2.1) is immediate. The proof is complete.

4. **Proof of Theorem 2**

For $x_0 \le m \le x$, $y_0 \le n \le y$, $x_0, m, x \in N(x_0)$, $y_0, n, y \in N(y_0)$, we have

$$\Delta_m u(x, y, x_0, y_0, w(m, n)) = u(x, y, x_0, y_0, w(m + 1, n)) - u(x, y, x_0, y_0, w(m, n)) =$$

$$= \Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n). \tag{4.1}$$

From (4.1) we have

$$\Delta_n\Delta_m u(x, y, x_0, y_0, w(m, n)) = \Delta_w u(x, y, x_0, y_0, w(m, n + 1))\Delta_1 w(m, n + 1)-$$

$$-\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n) =$$

$$= \Delta_w u(x, y, x_0, y_0, w(m, n + 1))\Delta_1 w(m, n + 1)-$$

$$-\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n + 1)+$$

$$+\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n + 1)-$$

$$-\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n) =$$

$$= [\Delta_w u(x, y, x_0, y_0, w(m, n + 1))-$$

$$-\Delta_w u(x, y, x_0, y_0, w(m, n))]\Delta_1 w(m, n + 1)+$$

$$+\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_2\Delta_1 w(m, n) =$$

$$= B(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n + 1)+$$

$$+\phi(x, y, x_0, y_0, w(m, n))\Delta_2\Delta_1 w(m, n). \tag{4.2}$$

Now keeping x, y, m fixed in (4.2), set $n = t$ and sum over t from y_0 to $y-1$, and then keeping x, y, t fixed in the resulting inequality, set $m = s$ and sum over s from x_0 to $x-1$, to obtain

$$u(x, y, x_0, y_0, w(x, y)) = u(x, y, x_0, y_0, u_0) + \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} B(x, y, x_0, y_0, w(s, t))+$$

$$+\sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} \phi(x, y, x_0, y_0, w(s, t))\Delta_2\Delta_1 w(s, t). \tag{4.3}$$

If $w(x, y)$ is any solution of (2.2), then the result (2.4) follows from (4.3), (3.1) and (2.2). The proof is complete.

5. **Some applications**

In this section we use the formulae given in Theorems 1 and 2 to study the boundedness of the solutions of perturbed finite difference equation (P) under some suitable conditions on the functions involved in (P). We say that the solution $u(x, y, x_0, y_0, u_0)$ of (E) is globally uniformly stable if there exists a constant $M > 0$ such that $|u(x, y, x_0, y_0, u_0)| \leq M|u_0|$, for $f(x, y) \in N(x_0, y_0)$ and $|u_0| < \infty$.

We shall need the following special version of the inequality established be Pachpatte in [8,Theorem 1].

Lemma. *Let $u(x, y)$ and $h(x, y)$ be real-valued nonnegative functions defined on N_0^2 and $c \geq 0$ be a constant. If*

$$u(x, y) \leq c + \sum_{s=0}^{x-k}\sum_{t=0}^{y-1} h(s, t)u(s, t),$$

for $x, y \in N_0$, then

$$u(x, y) \leq c\prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} h(s, t)\right],$$

for $x, y \in N_0$.

We first give the following application of the variation of constants formula established in Theorem 1.

Theorem 3. *Let the solution $u(x, y, x_0, y_0, u_0)$ of (E) be globally uniformly stable and the hypothesis of Theorem 1 hold. Further, suppose that*

$$|\phi^{-1}(x+1, y, x_0, y_0, w(x, y))[A(x, y, x_0, y_0, w(x, y))+$$
$$+g(x, y, v(x, y, x_0, y_0, u_0))| \le p(x, y)|w(x, y)|], \tag{5.1}$$

for $(x, y) \in N(x_0, y_0)$ where $p(x, y)$ is a real-valued nonnegative function defined on $N(x_0, y_0)$ and

$$\prod_{s=x_0}^{x-1} \left[1 + \sum_{t=y_0}^{y-1} p(s, t)\right] < \infty, \tag{5.2}$$

for $(x, y) \in N(x_0, y_0)$. Then any solution $v(x, y, x_0, y_0, u_0)$ to (P) is bounded for $(x, y) \in N(x_0, y_0)$.

Proof. By Theorem 1, any solution $v(x, y, x_0, y_0, u_0)$ of (P) satisfies

$$v(x, y, x_0, y_0, u_0) = u(x, y, x_0, y_0, w(x, y)), \quad w(x, y_0) = w(x_0, y) = u_0, \tag{5.3}$$

where $w(x, y)$ is given by (3.5) is a solution of (2.2). Using (3.5) and (5.1) we have

$$|w(x, y)| \le |u_0| + \sum_{s=x_0}^{x-1} \sum_{t=y_0}^{y-1} |\phi^{-1}(s+1, t, x_0, y_0, w(x, t))\times$$
$$\times[A(s, t, x_0, y_0, w(s, t)) + g(s, t, v(s, t, x_0, y_0, y_0))]| \le$$
$$\le |u_0| + \sum_{s=x_0}^{x-1} \sum_{t=y_0}^{y-1} p(s, t)|w(s, t)|. \tag{5.4}$$

Now a suitable application of Lemma to (5.4) yields

$$|w(x, y)| \le |u_0| \prod_{s=x_0}^{x-1} \left[1 + \sum_{t=y_0}^{y-1} p(s, t)\right]. \tag{5.5}$$

The right hand side of (5.5) can be made sufficiently small by using (5.2) and assuming that $|u_0|$ is sufficiently small, i.e.

$$|w(x, y)| \le \varepsilon, \tag{5.6}$$

where $\varepsilon > 0$ is arbitrary, constant. From (5.3) we have

$$|v(x, y, x_0, y_0, u_0)| = |u(x, y, x_0, y_0, w(x, y)| \tag{5.7}$$

From the global uniform stability of the solution $u(x, y, x_0, y_0, u_0)$ of (E) and (5.6) and (5.7) we have

$$|v(x, y, x_0, y_0, u_0)| \le M\varepsilon,$$

which implies the boundedness of the solution of (P). The proof is complete.

We next give the following application of the variation of constants formula established in Theorem 2.

Theorem 4. *Assume that the hypotheses of Theorem 2 hold and the functions involved in (2.4) satisfy*

$$\sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1}|B(x,y,x_0,y_0,w(x,y))| \leq M_1, \tag{5.8}$$

$$|\phi(x,y,x_0,y_0,w(x,y))\phi^{-1}(s+1,t,x_0,y_0,w(s,t))| \leq M_2, \tag{5.9}$$

$$|A(x,y,x_0,y_0,w(x,y))+g(x,y,v(x,y,x_0,y_0,u_0))| \leq$$

$$\leq p(x,y)|v(x,y,x_0,y_0,u_0)|, \tag{5.10}$$

where M_1 and M_2 are nonnegative constants and $p(x,y)$ is a real-valued nonnegative function defined on $N(x_0,y_0)$ and

$$\prod_{s=x_0}^{x-1}\left[1+\sum_{t=y_0}^{y-1}p(s,t)\right] < \infty, \tag{5.11}$$

for $(x,y) \in N(x_0,y_0)$. Then for every bounded solution $u(x,y,x_0,y_0,u_0)$ of (E) for $(x,y) \in N(x_0,y_0)$, the corresponding solution $v(x,y,x_0,y_0,u_0)$ of (P) is bounded for $(x,y) \in N$.

The proof of this theorem follows by using (5.8)-(5.10) in (2.4) and applying Lemma and condition (5.11). Here we omit the details.

We note that the results given in Theorem 1-4 can very easily be extended when the perturbation term g involved in (P) is of the more general type i.e. when the equation (P) is of the form

$$\Delta_2\Delta_1 v(x,y) = f(x,y,v(x,y)) + g(x,y,v(x,y),Tv(x,y)),$$

$$v(x,y_0) = v(x_0,y) = u_0, \tag{P'}$$

where

$$Tv(x,y) = \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} h(x,y,s,t,v(s,t)).$$

The formulations of such results corresponding to the equations (E) and (P') are very close to that of the results given in the above theorems with suitable modifications and hence we do not discuss the details.

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HYDRODYNAMICAL CONSIDERATIONS ON THE GAS STREAM IN THE LAGRANGIAN POINT L_1 OF A CLOSE BINARY SYSTEM

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Abstract. Taking into consideration Euler's equation for an ideal fluid and having in view some basic hypotheses specified for hydrodynamic approach, in the case when the potential of massic forces is of Roche type, an integral of Bernoulli type is established. It is shown that it is impossible for a fluid to surpass a certain maximum velocity v_{max} and that the critical velocity of the sound in a fluid depends on the parameters of the Roche model. The conditions in which the fluid motion is subsonic or supersonic are analyzed. In addition the density, the pressure of the fluid and the sound velocity are expressed as function of the fluid velocity and the potential of the massic forces. Then, the Lagrangian point L_1 is considered as a source with its output q and the fluid motion is analyzed in the corresponding close vicinity. The obtained results could also be used as initial conditions for the integration of the mass transfer equations.

1. **Introduction**

The problem of the mass transfer in stellar binary systems is relatively old. It was approached by Kuiper (1941) in his pioneer work. Then, Kopal (1958), Kruszewsky (1964), Plavec et al. (1964, 1965) and many other authors have reviewed this problem, especially with considerations on the orbital period changes. In the above-mentioned papers, the problem of the mass transfer was approached without any hydrodynamic considerations. With computed orbits of single particles, some of the above mentioned authors have demonstrated, theoretically, the existence of rings around the primary component, or some gas streams in the corresponding systems. Other authors also tried to use hydrodynamical considerations: Prendergast (1960), Biermann (1971), Prendergast and Taam (1974). Nevertheless, it was used an arbitrary way to establish the initial conditions for the integration of the differential equations of the correspondent motion. That is why, by taking into consideration some basic hypotheses we have studied the problem of the mass transfer in a close vicinity of the inner Lagrangian point, L_1 and the corresponding results could be used for a better estimation of the initial conditions.

2. **Basic hypotheses**

For the study of the mass transfer in close binary systems, through the methods appropriate to the hydrodynamics, we are obliged to make some hypotheses in order to draw the theoretical model very near to the physical reality. These basic hypotheses will be reviewed in the present section:

a) The two component stars of a binary systems are revolving in circular orbits, about their common mass center. Such an approximation is suited for a great majority of the close binary systems.

b) The fluid flow is assumed as being stationary. This hypothesis is good enough for the detached and semi-detached binary systems, whose light curves have the same behaviour in each cycle, with some small irregularities. Such an assumption is not suited for those binary systems whose stellar components are in contact and the corresponding irregularities are very frequent and well marked.

c) The gas flow is considered only in the orbital plane and an approach of the two dimensions problem may be accepted. This assumption is based on the fact that the resultant force of the corresponding effective forces lies in the orbital plane and it has an endeavour to press the gas stream towards this plane. Indeed, the effective forces, which are operating on the gas stream, are: the forces of the gravitational attraction , the centrifugal force and the Coriolis force . Here we have in view a rotating barycentric coordinates system (M,x,y,z) where the origin M is situated in the common mass-center and the two component stars S_1 and S_2 are always situated on the x-axis, while (x,M,y) plane coincides with the orbital plane. In such conditions the gravitational forces are given by:

$$\overrightarrow{F}_{atr_1} = -G\frac{m_1 m}{r_1^2}\frac{\overrightarrow{r}_1}{r_1}, \qquad \overrightarrow{F}_{atr_2} = -G\frac{m_2 m}{r_2^2}\frac{\overrightarrow{r}_2}{r_2}, \tag{1}$$

with: $\overrightarrow{r}_1 = (x + R_1)\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$, $\overrightarrow{r}_2 = (x - R_2)\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$,
where R_1 and R_2 are the distances of the two stellar components from the common mass-center.

In addition we have:

$$\overrightarrow{F}_{centrif} = -m\overrightarrow{\omega} \times (\overrightarrow{\omega} \times \overrightarrow{r}) = m\omega^2 \left(x\,\overrightarrow{i} + y\,\overrightarrow{j}\right) \quad (2)$$

$$\overrightarrow{F}_{Coriolis} = -2m\overrightarrow{\omega} \times \overrightarrow{v_r} = 2m\omega \left(y\,\overrightarrow{i} - x\,\overrightarrow{j}\right) \quad (3)$$

with: $\overrightarrow{v}_r = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$.

In Eqs. (1) - (3) $m_{1,2}$ are the masses of the two stellar components (S_1 and S_2), while m represents the mass of a gas particle. As it is shown by Biermann (1971), the use of two dimensions only (in the orbital plane) is equivalent either to a cylindrical model of the system or, to a gas flow of constant thickness. The corresponding thickness may be considered as a function of temperature.

d) The gas flow is assumed as being adiabatic. Such an assumption is suited if:

- the mean free path for photons is small compared to the characteristic dimensions of the considered binary - system;

- the thermal time scale of the gas is long compared to the transfer time scale.

Now, from theoretical considerations on the mass transfer in close binary - systems (eg. Kippenhahn et al., 1967) it follows the fact that the corresponding problem has two phases: the first one is characterized by a fast flow of the gas, to the thermal time scale of the star which is losing mass. The second phase is characterized by a slow gas flow, to the nuclear time scale. In the slow phase of the mass transfer, the corresponding gases may be considered as being transparent and their thermal behaviour is determined by the radiation field of the two stellar components .

In the phase of fast mass transfer, the mean free path for photons is small enough while the thermal time scale is long enough in order to surmise that adiabacy is a good approximation.

e) We are considering a binary system as being semi-detached because the assumptions b) and d) are not suited for contact systems.

f) We are assuming that for pressure the law of perfect gas may be adopted:

$$P = \frac{k\varrho T}{\mu m_a} \tag{4}$$

where ϱ is the fluid density, T is the temperature, $k = 1,38054 * 10^{-23} JK^{-1}$ is Boltzmann's, constant, μ is the relative molecular mass (in units of atomic mass) and m_a is the proton mass (see Ureche, 1987).

g) The gas flow is assumed as being laminar throughout, without turbulence and irrotational . As it is known, (Biermann, 1971), the Reynolds number can be written as the product of the Mach number of the flow and ratio of a characteristic length scale to the gas-dynamical mean free path. The Reynolds number for the gas flow in binary systems is evaluated as being of the order of 10^8 (Kopal,1958). On the other hand it is known from experiments (e.g. Biermann, 1971) that the supersonic turbulence is strong connected with the properties of the boundary layers. Nevertheless, there are no fixed boundaries in a close binary system and, consequently, there is no a simple theory to discuss the properties of possible boundary layers. In such conditions it is very difficult to draw an important conclusion concerning the turbulence. For simplicity the gas flow is assumed as being laminar throughout.(obvious with the exception of the area situated behind the shock wave) .

h) The magnetic fields are neglected, even if they could be important in some binary systems. But, as it is shown (Biermann, 1971) in the phase of fast mass transfer, the magnetic fields are important only if their strength is of the order of 10^3 Gauss or greater . In the phase of slow mass transfer 10 Gauss have already appreciable effects. But for the phase of fast mass transfer no observed example is known to have such a magnetic field. On the other hand, for the phase of slow mass transfer which can be identified with many observed binary systems, the value of 10 Gauss is below observational limits. Anyhow, if we take them into consideration, the problem becomes more complicated, because in the motion equations we have to add a supplemental term of form $\overrightarrow{j} \times \overrightarrow{B}$, where $\overrightarrow{j}$ is the density of the stream while $\overrightarrow{B}$ is the magnetic induction. In addition, in equation of the energy we should have a supplemental term of the form $\overrightarrow{j} \cdot \overrightarrow{E}$, where $\overrightarrow{E}$ represents the strength of the electric field (Ureche, 1987).

i) For the gravitational field of each stellar component, the Roche potential is assumed. Since the stars, which are losing mass, are evolving far from main sequence, they have an increased concentration of density. Therefore, the tides do not change very much the gravitational potential. Hence, the Roche potential may be considered as a good approximation.

j) Finally, we are assuming that there is a synchronization between the axial rotation of the two stellar components and the corresponding orbital motion, that is, for the angular velocity, we can write: $\omega = \frac{2\pi}{P} = const.$

3. **Subsonic, supersonic and hypersonic motions in the jet of stellar matter**

From the Euler's equation, written for an ideal fluid, that is:

$$\varrho\left[\frac{\partial\overrightarrow{v}}{\partial t} + \frac{1}{2}grad\left(v^2\right) + rot\,\overrightarrow{v}\times\overrightarrow{v}\right] = \varrho\overrightarrow{f} - grad\,P,$$

in the hypothesis of a steady state, we can write:

$$\varrho\left[\frac{1}{2}grad\left(v^2\right) + rot\,\overrightarrow{v}\times\overrightarrow{v}\right] = \varrho\overrightarrow{f} - grad\,P. \tag{5}$$

If we are considering now that there is a scalar function U (potential) so that

$$\overrightarrow{f} = grad\,U,$$

then Eq.(5) can be written in the form:

$$\varrho\left[\frac{1}{2}grad\left(v^2\right) + rot\,\overrightarrow{v}\times\overrightarrow{v}\right] = \varrho\,grad\,U - grad\,P. \tag{6}$$

In addition, if we assume that the compressible fluid is a barotropic one, we may consider that there is a scalar function h , so that $grad\,h = \frac{grad\,P}{\varrho}$. Therefore, from Eq. (6) it is evident that on any stream line we have:

$$\frac{v^2}{2} - U + h = C_1 = const. \tag{7}$$

which, in fact, is the Bernoulli's integral.

In the hypothesis of a perfect gas and in an izentropic evolution, we can write: $P = k\,\varrho^\gamma$, where $\gamma > 1$ represents the adiabatic exponent.

Moreover, from the relationship: $h = \int dh = \int \frac{dP}{\varrho}$ we have at once:

$$h = \frac{\gamma}{\gamma - 1}\frac{P}{\varrho}, \tag{8}$$

and the Hugoniot formula leads to:

$$c^2 = \frac{dP}{d\varrho} = \gamma\,\frac{P}{\varrho}. \tag{9}$$

Therefore Eqs. (8) and (9) lead to:

$$h = \frac{c^2}{\gamma - 1} \tag{10}$$

In such conditions, Eq. (7) becomes:

$$\frac{v^2}{2} - U + \frac{c^2}{\gamma - 1} = C_1\,. \tag{11}$$

Here the constant C_1 may be determined if we use Eq. (11) for an arbitrary point situated somewhere on the Roche equipotential surface. On such a surface the velocity is null and the corresponding potential is $U_{Roche} = const$. On the other hand, it is known that on such equipotential surfaces, we also have a constant density, hence $\varrho_{Roche} = const$.

Now, from relationship $P = k\,\varrho^{\gamma}$, written for an arbitrary point of the Roche equipotential surface, we have:

$$P_{Roche} = k\,\varrho^{\gamma}_{Roche} = constant$$

and Eq. (9) leads to:

$$c^2_{Roche} = \gamma\,\frac{P_{Roche}}{\varrho_{Roche}} = constant.$$

For an arbitrary point of the Roche equipotential surface we are able to determine the value of the constant C_1 , that is:

$$-U_{Roche} + \frac{c^2}{\gamma - 1} = C_1.$$

and Eq. (11) may be written in the form:

$$\frac{v^2}{2} - U + \frac{c^2}{\gamma - 1} = -U_{Roche} + \frac{c^2_{Roche}}{\gamma - 1} \tag{12}$$

whence we have at once:

$$c^2 = c^2_{Roche}\left[1 - \frac{(\gamma - 1)(v^2 - 2U + 2U_{Roche})}{2c^2_{Roche}}\right]. \tag{13}$$

The solution of Eq.(13) will be found in the range of real numbers only if it is satisfied the condition:

$$1 \geq \frac{(\gamma - 1)(v^2 - 2U + 2U_{Roche})}{2c^2_{Roche}},$$

or:

$$v^2 \leq \frac{2c^2_{Roche}}{\gamma - 1} + 2\left(U - U_{Roche}\right).$$

Therefore, in its motion, a fluid cannot surpass (exceed) a maximum velocity v_{max} , given by the relationship:

$$v^2_{max} = \frac{2c^2_{Roche}}{\gamma - 1} + 2\left(U - U_{Roche}\right). \tag{14}$$

If in a point situated somewhere on a stream line, the fluid velocity becomes equal to the sound velocity in the same point, that is if we can write $v = c = c_*$, then from Eq. (12) we have:

$$c^2_* = \frac{\gamma - 1}{\gamma + 1}\left[2\frac{c^2_{Roche}}{\gamma - 1} + 2\left(U_* - U_{Roche}\right)\right].$$

where c_* represents the critical sound velocity in fluid, while U_* is the corresponding potential.

If $v > c_*$ we have the case of the supersonic motion.

If $v < c_*$ we have the case of the subsonic motion. Furthermore, from Eq.(9) we can write: $c^2 = k\gamma\varrho^{\gamma-1}$ or $c^2_{Roche} = k\gamma\varrho^{\gamma-1}_{Roche}$ and Eq.(13) becomes:

$$\varrho = \varrho_{Roche}\left[1 - \frac{(\gamma - 1)\left(v^2 - 2U + 2U_{Roche}\right)}{2c^2_{Roche}}\right]^{\frac{1}{\gamma-1}}. \tag{15}$$

Here we have in view the relationship: $P = k\varrho^\gamma$ or: $\varrho = \left(\frac{P}{k}\right)^{\frac{1}{\gamma}}$ and Eq. (8) leads to:

$$P = P_{Roche}\left[1 - \frac{(\gamma - 1)(v^2 - 2U + 2U_{Roche})}{2c^2_{Roche}}\right]^{\frac{\gamma}{\gamma-1}}. \tag{16}$$

In conclusion, the relationships (13), (15) and (16) give us the explicit functions c(v), $\varrho(v)$ and P(v). During the study of the mass transfer, it was put in evidence a very luminous patch - a hot spot - in that place where the jet of matter hit the atmosphere of the star which is receiving mass. The existence of such a patch was also detected by observational methods. Consequently it was created a true theory of such named "hot spot", in order to explain some irregularities (fluctuations) observed in the light curves of the eclipsing binary systems. Prendergast and Taam (1974) try to explain such a hot spot as a consequence of the heating determined by the shock wave which arise in that place and have estimated a temperature of the order of 35000 K. In front of the shock wave the jet motion is hypersonic, that is the jet matter must be accelerated by the gravitational attraction until to velocities characterized by the

Mach number $M \geq 5$. If we assume that the fluid motion is *hypersonic*, *izentropic* and *steady* (the jet matter being assumed as a perfect gas), for each stream line we have dS = 0. Further, from Eq. (7) we have at once:

$$v\,dv - dU + dh = 0$$

but $dh = \frac{dP}{\varrho}$ and consequently we can write:

$$\frac{dP}{P} = \frac{\varrho}{P}dU - \frac{\varrho v}{P}dv.$$

Here we can use the following relationship: $\frac{\varrho}{P} = \frac{\gamma}{c^2}$ and consequently we have:

$$\frac{dP}{P} = \frac{\gamma}{c^2}dU - \frac{\gamma v}{c^2}dv$$

Finally, if we have in view the Mach number $M = \frac{v}{c}$, we can write :

$$\frac{dP}{P} = \frac{\gamma M^2}{v^2}dU - \gamma M^2\,\frac{dv}{v}. \tag{17}$$

As it was before mentioned, we can use the relationship: $c^2 = \frac{\gamma P}{\varrho}$ and, if we accept Clapeyron law: $P = \varrho \mathcal{R} T$ it follows that:

$$c^2 = \gamma \mathcal{R} T \tag{18}$$

or, by differentiation it follows that:

$$2c\,dc = \gamma dT \mathcal{R} \tag{19}$$

Now, from Eqs. (18) and (19) we have at once:

$$\frac{dT}{T} = 2\frac{dc}{c}, \tag{20}$$

and from Eq.(11) on a stream line we can write:

$$v\,dv - dU + \frac{2c\,dc}{\gamma - 1} = 0. \tag{21}$$

In such conditions, Eq. (20) becomes:

$$\frac{dT}{T} = -\frac{(v\,dv - dU)\,(\gamma - 1)}{c^2}$$

or, if we have in view the Mach number, M:

$$\frac{dT}{T} = -(\gamma - 1)\,M^2\left(\frac{dv}{v} - \frac{dU}{v^2}\right). \tag{22}$$

From Eqs. (17) and (22) it is evident that, since M^2 is a great number (the fluid motion being assumed as hypersonic one) , to a small change in the velocity on a stream line could correspond a great change for the pressure and temperature. From the relationship: $M = \frac{v}{c}$ we can write:

$$\frac{dM}{M} = \frac{dv}{v}\left(1 - M\,\frac{dc}{dv}\right) \tag{23}$$

By differentiation, from Eq. (11) it follows:

$$v - \frac{dU}{dv} + \frac{2c}{\gamma - 1}\,\frac{dc}{dv} = 0$$

and Eq.(23) can be written in the form:

$$\frac{dM}{M} = \frac{dv}{v\left[1 - M\frac{\gamma-1}{2c}\left(-v + \frac{dU}{dv}\right)\right]},$$

or

$$\frac{dM}{M} = \left(1 + \frac{\gamma - 1}{2}M^2 - M\,\frac{\gamma - 1}{2c}\,\frac{dU}{dv}\right)\frac{dv}{v}. \tag{24}$$

Moreover, from Eq.(21) we can obtain a relationship for dv , and Eq.(24) becomes:

$$\frac{dM}{M} = \left(1 + \frac{\gamma - 1}{2}M^2 - M\,\frac{\gamma - 1}{2c}\,\frac{dU}{dv}\right)\left[\frac{dU}{v^2} - \frac{2}{M^2(\gamma - 1)}\,\frac{dc}{c}\right]. \tag{25}$$

If we consider the fluid motion as being a hypersonic one, we can use the following approximation:

$$\frac{dU}{v^2} \approx 0.$$

and Eqs. (17), (22) and (25) lead to:

$$\frac{dP}{P} = -\gamma M^2\,\frac{dv}{v} \tag{26}$$

$$\frac{dT}{T} = -(\gamma - 1)M^2\,\frac{dv}{v} \tag{27}$$

$$\frac{dM}{M} = \left(1 + \frac{\gamma - 1}{2}M^2 - M\frac{\gamma - 1}{2c}\,\frac{dU}{dv}\right)\left[-\frac{2}{M^2(\gamma - 1)}\,\frac{dc}{c}\right]. \tag{28}$$

The shock wave, which arises in the close vicinity of the secondary component, practically is stuck to this star. The very high temperatures from behind of the shock determine ionization and dissociation of the particles and, consequently, the laminar model of the fluid cannot be used. Behind of the shock wave arises a zone of turbulence that, in fact, is a zone of the complementarity of the secondary star. Moreover, as a strong increasing of the temperature, this zone can be put in evidence through the direct astronomical observations.

4. The study of the fluid motion in the close vicinity of the point L_1

Further, in the present section we shall consider that in the Lagrangian point L_1 we have a mass source, with the corresponding output q . In such a condition, the continuity equation (see L. Dragos, 1981) will be written in the form:

$$div\,(\varrho \overrightarrow{v}) = q\,\delta\,(\overrightarrow{x})\,, \tag{29}$$

where $\delta\,(\overrightarrow{x})$ is Dirac‘s distribution. Let us consider that the fluid motion take place in the orbital plane, where $r = |\overrightarrow{x}|$. Thus, from the study of the Roche equipotentials it is known that the tangents, in the orbital plane, drawn in L_1, have the corresponding slopes θ_0 and $-\theta_0$, where:

$$tg^2\theta_0 = \frac{2x_{L_1}^{-3} + 2\frac{m_2}{m_1}\left(1 - x_{L_1}\right)^{-3} + \left(1 + \frac{m_2}{m_1}\right)}{x_{L_1}^{-3} + \frac{m_2}{m_1}\left(1 - x_{L_1}\right)^{-3} - \left(1 + \frac{m_2}{m_1}\right)} \tag{30}$$

(The corresponding numerical values are listed by Plavec and Kratochvil (1964), for the mass ratio $\frac{m_2}{m_1}$).

In the hypothesis that in the range $\theta_0 \in [-\theta_0, \theta_0]$ there are not preferential directions, Eq. (29) becomes (see L. Dragos, 1981):

$$\frac{1}{r}\,\frac{d}{dr}(\varrho r v) = \frac{q}{2\pi r}\,\delta(r). \tag{31}$$

The corresponding homogenous equation of Eq.(31) is:

$$\frac{1}{r}\,\frac{d}{dr}(\varrho r v) = 0,$$

which have the solution:

$$\varrho v = \frac{C}{r}. \tag{32}$$

The value of the constant C will be determined in such a way that the equation of the continuity to be total satisfied, and not only in a certain range which do not contain the origin and is specified by the relationship: $C = \frac{q}{2\pi}$. In such a case, the solution (32) becomes:

$$\varrho v = \frac{q}{2\pi r} \tag{33}$$

with $r^2 = x^2 + y^2$. From the relationships: $c^2 = \frac{dP}{d\varrho}$ and $P = k\varrho^\gamma$ we obtain $\varrho = \left(\frac{c^2}{k\gamma}\right)^{\frac{1}{\gamma-1}}$ and from Eq. (33) we have at once:

$$v = \frac{q}{2\pi(k\gamma)^{\frac{1}{1-\gamma}}} \frac{c^{\frac{2}{1-\gamma}}}{r}. \tag{34}$$

Now, from Eq. (12) we have:

$$c^2 = (\gamma - 1)\left(U - U_{Roche} + \frac{c^2_{Roche}}{\gamma - 1} - \frac{v^2}{2}\right),$$

and Eq.(34) can be written in the form:

$$v = \frac{q}{2\pi(k\gamma)^{\frac{1}{1-\gamma}}} \frac{(\gamma-1)^{\frac{1}{1-\gamma}}}{r} \left(U - U_{Roche} + \frac{c^2_{Roche}}{\gamma - 1} - \frac{v^2}{2}\right)^{\frac{1}{1-\gamma}}. \tag{35}$$

Therefore if we consider now a point $A\,(x_A, y_A)$ on a stream line, we know the value U_A and $r_A^2 = x_A^2 + y_A^2$, and from Eq.(35) we obtain the value of v_A .

If $v_A > c_*$, we have a supersonic motion.

If $v_A < c_*$ the motion is subsonic.

Finally, Eq. (35) could also be used in order to obtain the initial value of the velocity, which is useful in order to perform the integration of the equation of fluid motion at a great distance from L_1, but taking a suitable value for r . That is why, the study of the fluid motion could be performed on a natural way, the initial conditions being not imposed in an arbitrary way.

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INEQUALITIES FOR GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS II

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In the first part [23] of this series on Inequalities for Generalized Convexity we have studied the most important results and ideas of the author (and coauthors) related to the Jensen inequality. In this part we shall study Hadamard's (or Jensen-Hadamard's, or Hermite-Hadamard's) integral inequality for convex or generalized convex functions. This inequality was applied for the first time by Hadamard in the study of the Riemann zeta function [4]. Many new applications in geometry, special functions, number theory, theory of means, etc. have been published by the author (for References, see [9-25] and Part I). We plan to publish in Part IV of these series some of these applications (Part III will be devoted to Jessen's inequality). As we have stated in the first part [23], in many cases only the new results will be presented with a proof; the other results will be stated only, with connections and/or applications to known theorems. In the course of this survey many new results, new connections, hints, or applications will be pointed out.

2. **Hadamard's inequality**

Let $f : [a,b] \to \mathbb{R}$ be a convex function (in the classical sense). Then Hadamard's inequality (or "inequalities") states that

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\left[\frac{f(a)+f(b)}{2}\right]. \tag{1}$$

This is in fact Corollary 1.1 of Theorem 1.1 from [23]. In the literature (which is quite extensive) there exist papers where the left-side of (1) is called as "Jensen's inequality", while the right-side is due to Hadamard (or vice-versa). In the last time many papers call (1) as the Hermite-Hadamard inequality, since it seems that Hermite was the first discoverer of these relations ([6]). In that period, Jensen also has an important role in the theory of convexity and inequalities of type (1) ([16]).

A. The first extension of the left side of (1) for generalized convex functions has been discovered in 1982 by the author.

Theorem 2.1. ([9]) *Let $f \in C^{2k}[a,b]$ ($k \geq 1$, integer) be a $2k$-convex function on (a,b). Then*

$$\sum_{j=0}^{k-1} \frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!} f^{(2j)}\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx. \tag{2}$$

This result became widely known after its publication in an international journal [10].

For a particular case, namely $k = 2$ one gets:

Corollary 2.1. *Let $f : [a,b] \to \mathbb{R}$, $f \in C^4[a,b]$ and $f^{(4)}(t) \underset{(>)}{\geq} 0$ on (a,b). Then*

$$\int_a^b f(x)dx \underset{(>)}{\geq} (b-a)\left[f\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right]. \tag{3}$$

Remark 2.1. To show the power of this inequality, let us consider, as an immediate application, $a > 0$, $b = a+1$ and let $f_1(x) = \dfrac{1}{x}$, $f_2(x) = -\ln x$ $(x > 0)$ which fulfill the above conditions. After certain elementary computations one can deduce the double-inequality

$$\frac{2a+2}{2a+1} e^{1/6(2a+1)^2} < \frac{e}{\left(1+\dfrac{1}{a}\right)^a} < \sqrt{1+\frac{1}{a}} \cdot e^{-1/3(2a+1)^2} \tag{4}$$

for all real numbers $a > 0$. Clearly, this implies the weaker relations

$$\frac{2a+2}{2a+1} < \frac{e}{\left(1+\dfrac{1}{a}\right)^a} < \sqrt{1+\frac{1}{a}} \tag{5}$$

which in turn are quite strong to imply, or improve certain known results. For example, the much studied inequality by Pólya and Szegö [8], namely

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^n < \frac{e}{2n+1} \quad (n \geq 1, \text{ integer}) \tag{6}$$

follows immediately, even in improved form from (5). All inequalities of [2] are particular cases, or implications of relations (5). For applications to Stirling's theorem and other inequalities for the number e we quote the recent papers [21], [24], [25]. We note that when f is strictly $2k$-convex, we have strict inequality in (2) (the same in the particular case (3)).

In 1989 H. Alzer [1] has extended the right side of (1):

Theorem 2.2. *Let f be as in Theorem 2.1. Then*

$$\int_a^b f(x)dx \le \frac{1}{2}\sum_{i=1}^{2k-1}\frac{(b-a)^i}{i!}[f^{(i-1)}(a)+(-1)^{i-1}f^{(i-1)}(b)] \tag{7}$$

When f is strictly $2k$-convex, then (7) holds true with strict inequality.

Remark 2.2. By using (7), the following rational approximation of the exponential function can be deduced ([1]):

For all $x>0$ and all integers $n\ge 0$ we have

$$\frac{1+\frac{1}{2}\sum_{i=0}^{2n}\frac{(-x)^{i+1}}{(i+1)!}}{1+\frac{1}{2}\sum_{i=0}^{2n}\frac{x^{i+1}}{(i+1)!}} < e^{-x} < \frac{1+\frac{1}{2}\sum_{i=0}^{2n+1}\frac{(-x)^{i+1}}{(i+1)!}}{1+\frac{1}{2}\sum_{i=0}^{2n+1}\frac{x^{i+1}}{(i+1)!}} \tag{8}$$

Inequalities of this type have applications in irrationality proofs (see [11]).

In 1991 the author obtained common generalizations of Theorem 2.1 and 2.2.

Theorem 2.3. ([17]) *Let f be as in Theorem 2.1. Let $t\in[a,b]$ arbitrary chosen. Then*

$$\int_a^b f(x)dx \ge \sum_{i=1}^{2k}\left[\frac{(t-a)^i-(t-b)^i}{i!}\right]\cdot(-1)^{i-1}f^{(i-1)}(t)+$$

$$+\frac{1}{(2k)!}\left\{\frac{(b-a)^{2k}}{2^{2k-1}}[f^{(2k-1)}(t)-f^{(2k-1)}(a)]+S_{k,a,b}(t)\right\}, \tag{9}$$

respectively

$$\int_a^b f(x)dx \le \sum_{i=1}^{2k}\left[\frac{(t-a)^i-(t-b)^i}{i!}\right]\cdot(-1)^{i-1}f^{(i-1)}(t)+$$

$$+\frac{1}{(2k)!}\{(b-a)^{2k}[f^{(2k-1)}(t)-f^{(2k-1)}(a)]+S_{k,a,b}(t)\}, \tag{10}$$

where

$$S_{k,a,b}(t)=\int_a^b (b-x)^{2k}f^{(2k)}(x)dx-2\int_a^b (b-x)^{2k}f^{(2k)}(x)dx.$$

When f is strictly $2k$-convex, then all inequalities in (9) and (10) are strict.

Remark 2.3. Clearly, this result has a lot of particular cases. For example, by putting $t = a$ and $t = b$ resp. in (10), after addition we get Alzer's inequality (7). By doing the same thing in (9), we get

Theorem 2.4. *With the same conditions,*

$$\frac{1}{2}\sum_{i=1}^{2k}\frac{(b-a)^i}{i!}[f^{(i-1)}(a)+(-1)^{i-1}f^{(i-1)}(b)]+$$

$$+\frac{(b-a)^{2k}}{2^{2k-2}(2k)!}[f^{(2k-1)}(b)-f^{(2k-1)}(a)] \le \int_a^b f(x)dx. \tag{11}$$

By applying (9) and (10) for $t = \frac{a+b}{2}$, and remarking that

$$(x-a)^{2k} \le \left(\frac{b-a}{2}\right)^{2k} \quad \text{for } x \in \left[a, \frac{a+b}{2}\right],$$

while

$$(b-x)^{2k} \le \left(\frac{b-a}{2}\right)^{2k} \quad \text{for } x \in \left[\frac{a+b}{2}, b\right],$$

we get firstly our result (2) as well as the following:

Theorem 2.5. *With the same conditions,*

$$\int_a^b f(x)dx \le \sum_{j=0}^{k-1}\frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!}f^{(2j)}\left(\frac{a+b}{2}\right)+$$

$$+\frac{1}{(2k)!2^{2k}}(b-a)^{2k}[f^{(2k-1)}(b)-f^{(2k-1)}(a)]. \tag{12}$$

In what follows, let us use the following notations:

$$A_k \equiv A_k(a,b,f) = f^{(2k-1)}(b) - f^{(2k-1)}(a);$$

$$B_k = B_k(a,b,f) = f^{(k-1)}(a) + (-1)f^{(k-1)}(b).$$

The following two auxiliary results will be necessary:

Lemma 2.1. (”Green-Lagrange identity”) *For $f, g \in C^n[a,b]$ one has the identity*

$$\int_a^b g^{(n)}(x)f(x)dx = [g^{(n-1)}(x)f(x) - \cdots + (-1)^{n-1}g(x)f^{(n-1)}(x)]\Big|_a^b +$$

$$+(-1)^n\int_a^b g(x)f^{(n)}(x)dx. \tag{13}$$

Lemma 2.2. (Chebisheff's integral inequality) *Let $u, v : [a,b] \to \mathbb{R}$ be two synchrone functions (i.e. functions having the same type of monotonicity). Then*

$$\frac{1}{b-a}\int_a^b u(x)v(x)dx \geq \frac{1}{b-a}\int_a^b u(x)dx \cdot \frac{1}{b-a}\int_a^b v(x)dx \tag{14}$$

When u and v are asynchrone functions (having different type of monotonicity), then the inequality sign in (14) is reversed. It is known that equality holds in (14) only when one of u and v is constant on $[a,b]$, eventually excepting a numerable subset of $[a,b]$ (see [5]).

We now are able to state the following result:

Theorem 2.6. *Let $f^{(2k)}$ ($k \geq 1$, integer) be a continuous, decreasing function on $[a,b]$. Then*

$$\int_a^b f(x)dx \leq \sum_{j=1}^{2k} \frac{1}{2^j \cdot j!}(b-a)^j B_j + \frac{1}{2^{2k}} \cdot \frac{(b-a)^{2k}}{(2k+1)!} A_k. \tag{15}$$

If $f^{(2k)}$ is monotone increasing, then the sign of inequality in (15) reverses.

Proof. Let $g(x) = \left(x - \frac{a+b}{2}\right)^n$ in (13). By remarking that

$$g^{(k)}(x) = n(n-1)\dots(n-k+1)\left(x - \frac{a+b}{2}\right)^{n-k},$$

after certain elementary computations one can deduce the following identity

$$\int_a^b f(x)dx = \sum_{j=1}^{n} \frac{1}{2^j \cdot j!} B_j + \frac{(-1)^n}{n!}\int_a^b \left(x - \frac{a+b}{2}\right)^n f^{(n)}(x)dx. \tag{16}$$

Let now $n := 2k$ in (16) and put $u(x) := f^{(2k)}(x)$, $v(x) := \left(x - \frac{a+b}{2}\right)^{2k}$ in (14). Since u and v are monotone increasing functions, we have

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2k} f^{(2k)}(x)dx \leq \frac{1}{2^{2k}(2k+1)!}(b-a)^{2k} A_k,$$

and the result follows.

Theorem 2.7. *Let $f^{(2k-1)}$ be increasing and continuous on $[a,b]$. Then*

$$\int_a^b f(x)dx \leq \sum_{j=1}^{2k-1} \frac{1}{2^j \cdot j!}(b-a)^j B_j \tag{17}$$

When $f^{(2k-1)}$ is decreasing, (17) holds true with reversed inequality.

Proof. Apply (16) with $n := 2k+1$ and put

$$u(x) := f^{(2k-1)}(x), \quad v(x) := \left(x - \frac{a+b}{2}\right)^{2k-1}.$$

Remarking that

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2k-1} dx = 0$$

we obtain from Lemma 2.2 that

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2k-1} f^{(2k-1)}(x)dx \leq 0$$

and (17) follows.

B. Hadamard's inequality has the following geometrical interpretation: the area below the graph of f on $[a,b]$ lies between the areas of two trapeziums, namely the one formed by the points of coordinates $(a, f(a))$; $(b, f(b))$ with the Ox axis, the second one formed by the tangent to the graph of f at the point $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$ with the Ox axis. By rotating these trapeziums round about the Ox axis, we get three volumes,

$$V = \pi \int_a^b f^2(x)dx,$$

$$V_1 = \frac{\pi(b-a)}{3}[f^2(a) + f(a)f(b) + f^2(b)],$$

$$V_2 = \frac{\pi(b-a)}{3}\left[3f^2\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{4}\left(f'\left(\frac{a+b}{2}\right)\right)^2\right].$$

Since, when f is positive and convexe, we have $V \leq V_1$, and under certain conditions $V_2 \leq V$, one can deduce the following result.

Theorem 2.8. *Let $f : [a,b] \to \mathbb{R}$ be nonnegative and convex. Then*

$$\frac{1}{b-a}\int_a^b f^2(x)dx \leq \frac{1}{3}[f^2(a) + f(a)f(b) + f^2(b)]. \tag{18}$$

If, in addition f is differentiable in $x_0 := \frac{a+b}{2}$, and the following condition is satisfied:

(i) $f\left(\frac{a+b}{2}\right) - \frac{b-a}{2}f'\left(\frac{a+b}{2}\right) > 0$ *and* $f'\left(\frac{a+b}{2}\right) > 0$, *then*

$$\frac{1}{b-a}\int_a^b f^2(x)dx \geq f^2\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{12}\left[f'\left(\frac{a+b}{2}\right)\right]^2. \tag{19}$$

Proof. The above stated geometric arguments for the proof of (18) and (19) can be made rigorous. Indeed, for (18), let $K : [a,b] \to \mathbb{R}$ be a linear function having the properties $f(a) = K(a)$, $f(b) = K(b)$. Therefore,

$$K(t) = \frac{t-a}{b-a} f(b) + \frac{b-t}{b-a} f(a), \quad t \in [a,b].$$

Since f is convex and positive, $f^2(t) \le K^2(t)$. Since it is immediate that

$$\int_a^b K^2(t)dt = \frac{b-a}{3}[f^2(a) + f(a)f(b) + f^2(b)],$$

the result follows. For the proof of (19) let us remark that $f(x) \ge f(x_0)+(x-x_0)f'(x_0)$ for all $x \in [a,b]$, $x_0 \in (a,b)$. Put $x_0 := \dfrac{a+b}{2}$ and write that

$$f^2(x) \ge \left[f(x_0) + \left(x - \frac{a+b}{2} \right) f'(x_0) \right]^2,$$

where $f'(x_0) > 0$. An elementary computation shows that

$$\int_a^b \left[f(x_0) + \left(x - \frac{a+b}{2} \right) f'(x_0) \right]^2 dx = f^2 \left(\frac{a+b}{2} \right) + \frac{(b-a)^2}{12} \left[f' \left(\frac{a+b}{2} \right) \right]^2,$$

and this finishes the proof.

Remark 2.4. Without differentiability one can assume only that

$$f'_+ \left(\frac{a+b}{2} \right) > 0 \quad \text{and} \quad f \left(\frac{a+b}{2} \right) - \frac{b-a}{2} f'_+ \left(\frac{a+b}{2} \right) > 0.$$

When f is nonnegative differentiable, and concave, without any condition one has

$$\frac{1}{b-a} \int_a^b f^2(x)dx \le f^2 \left(\frac{a+b}{2} \right) + \frac{(b-a)^2}{12} \left[f' \left(\frac{a+b}{2} \right) \right]^2 \tag{20}$$

Indeed, by $0 < f(x) \le f(x_0) + f'(x_0)(x-x_0)$, by taking squares and integrating, we obtain (20). Without differentiablity (20) holds with $f'_+ \left(\dfrac{a+b}{2} \right)$ in place of $f' \left(\dfrac{a+b}{2} \right)$.

Since $\dfrac{x^2+xy+y^2}{3} \le \dfrac{x^2+y^2}{2}$, inequality (18) refines the right side of Hadamard's inequality applied to the convex function f^2. Inequality (18) has been applied in the Theory of means ([14]).

Let $p : [a,b] \to \mathbb{R}$ be a strictly positive monotone function, and define

$$E_{p,f}(a,b) = E_{p,f} = \int_a^b p(x)f(x)dx \Big/ \int_a^b p(x)dx.$$

In paper [20] the following results have been proved:

Theorem 2.9. *Let f be a convex function. Then*

$$E_{p,f} \geq f(A) + f'_+(A)C_p, \tag{21}$$

where $A = \dfrac{a+b}{2}$, *and* $C_p = C_p(a,b)$. *If p is increasing, the $C_p \geq 0$; while for decreasing p one has $C_p \leq 0$.*

Remark 2.5. Therefore, when $f'_+(A) \geq 0$ one can deduce

$$E_{p,f} \geq f(A) + f'_+(A)C_p \geq f(A), \text{ for increasing } p.$$

This generalizes and refines the left side of Hadamard's inequality.

Theorem 2.10. *Let f be convex, with $f(b) \geq f(a)$. If p is a decreasing function, then*

$$E_{p,f} \leq f(a) + \frac{f(b)-f(a)}{b-a}\int_a^b (x-a)p(x)dx \leq \frac{f(a)+f(b)}{2} \tag{22}$$

The same is valid if $f(b) \leq f(a)$ and p increasing.

Finally, as a generalization of (18) we quote (see [20]):

Theorem 2.11. *Let f be positive and convex, with $f(b) \geq f(a)$. Let p be a decreasing function. Then*

$$E_{p,f^n} \leq \sum_{k=0}^{n} \binom{n}{k} f^k(a) \left(\frac{f(b)-f(a)}{b-a}\right)^{n-k} \int_a^b (x-a)^{n-k}p(x)dx \leq$$

$$\leq \sum_{k=0}^{n} \binom{n}{k} \frac{f^k(a)(f(b)-f(a))^{n-k}}{n-k+1} \tag{23}$$

(Here $n \geq 1$ is a positive integer and $\binom{n}{k}$ denotes a binomial coefficient.)

Theorem 2.12. *Let f be positive and concave. Then*

$$E_{p,f^n} \leq \sum_{k=0}^{n} \binom{n}{k} f^{n-k}(A)(f'_+(A))^k \int_a^b (x-A)^k p(x)dx. \tag{24}$$

If $f'_+(A) \geq 0$ and p is decreasing, the right side of (24) can be majored by

$$\sum_{k=0}^{n} \binom{n}{k} f^{n-k}(A)(f'_+(A))^k (b-a)^k \frac{1+(-1)^k}{(k+1)\cdot 2^{k+1}}. \tag{25}$$

Remark 2.6. For $p \equiv 1$, $n = 2$, f positive and *concave* one obtains the inequality

$$\frac{1}{b-a}\int_a^b f^2(x)dx \leq f^2(A) + (f'_+(A))^2 \frac{(b-a)^2}{12}. \tag{26}$$

C. In the precedent paragraph in certain cases we have obtained refinements of the Hadamard inequality (or for one part of it).

Let now suppose that the continuous function $f : [a,b] \to \mathbb{R}$ has a strictly increasing derivative on (a,b). By Lagrange's mean-value theorem easily follows $f(x) - f(y) < f'(x)(x-y)$ for all $x, y \in (a,b)$. By integrating with respect to x we get

$$\int_a^b f(x)dx < (b-a)f(y) - y[f(b) - f(a)] + \lambda = g(y),$$

where

$$\lambda = \int_a^b xf'(x)dx = bf(b) - af(a) - \int_a^b f(x)dx$$

and $g : [a,b] \to \mathbb{R}$ is defined as above. Clearly, $g'(y) = (b-a)f'(y) - [f(b) - f(a)]$, so by the Lagrange mean-value theorem, $g'(y_0) = 0$ for some $y_0 \in (a,b)$. Since f' is strictly increasing, obviously $g'(y) > g'(y_0) = 0$ for $y > y_0$ and $g'(y) < g'(y_0) = 0$ for $y < y_0$. Therefore y_0 is a minimum-point of the function g, that is $g(y_0) \le g(y)$ for all $y \in [a,b]$. Thus we have obtained the following result, which in fact appeared in [19]:

Theorem 2.13. *If f satisfies the above conditions, then*

$$\int_a^b f(x)dx < \frac{b-a}{2}\left\{f(y_0) - y_0\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(b)-af(a)}{b-a}\right\} \tag{27}$$

where y_0 is defined by the equality

$$f'(y_0) = \frac{f(b)-f(a)}{b-a}. \tag{28}$$

For this choice of $y = y_0$, inequality (27) is optimal.

Remark 2.7. Clearly, inequality (27) is valid for all $y_0 \in (a,b)$, but for y_0 given by (28) we obtain the strongest result. By selecting $y_0 = \dfrac{a+b}{2}$ in (27) we get the following refinement of the right side of Hadamard's inequality:

$$\int_a^b f(x)dx < \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] < \frac{b-a}{2}[f(a)+f(b)]. \tag{29}$$

This is due to P.S. Bullen [7].

Indeed, the first inequality is a consequence of (7), while the second one is equivalent to $f\left(\dfrac{a+b}{2}\right) < \dfrac{f(a)+f(b)}{2}$.

Remark 2.8. Inequality (27) is valid also for a strictly convex function f, and has been rediscovered in [3]. For applications in the theory of means, see [19], [3].

The following refinements of the Hadamard inequalities have been published by the author in cooperation with J.E. Pečarić and S.S. Dragomir [15]:

Theorem 2.14. *Let $n \geq 1$ be a positive integer and let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b \cdots \int_a^b f\left(\sum_{i=1}^{n+1} \frac{x_i}{n+1}\right) dx_1 \ldots dx_{n+1}}{(b-a)^{n+1}} \leq$$

$$\leq \frac{\int_a^b \cdots \int_a^b f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) dx_1 \ldots dx_n}{(b-a)^n} \leq \cdots \leq \frac{\int_a^b \int_a^b f\left(\frac{x_1+x_2}{2}\right) dx_1 dx_2}{(b-a)^2} \leq$$

$$\leq \frac{\int_a^b f(x)dx}{b-a} \leq \frac{f(a)+f(b)}{2}. \tag{30}$$

Remark 2.9. When $n = 1$ we have the following simple relations:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^2} \iint_{[a,b]^2} f\left(\frac{x+y}{2}\right) dxdy \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{31}$$

In applications (e.g. in the theory of Euler Gamma function), this inequality has a special importance.

D. We will conclude our survey with the study of certain mappings associated to the Hadamard inequalities.

Let $f : [a, b] \to \mathbb{R}$ be a convex function, and define the following mappings: $H, G, L : [0, 1] \to \mathbb{R}$, given by

$$H(t) = \frac{1}{b-a} \int_a^b f\left[tx + (1-t)\frac{a+b}{2}\right] dx, \tag{32}$$

$$G(t) = \frac{1}{2}\left\{f\left[ta + (1-t)\frac{a+b}{2}\right] + f\left[(1-t)\frac{a+b}{2} + tb\right]\right\}, \tag{33}$$

$$L(t) = \frac{1}{2(b-a)} \int_a^b \{f[ta + (1-t)x] + f[(1-t)x + tb]\}dx. \tag{34}$$

The following three theorems contain certain properties of these mappings (see [18]).

Theorem 2.15. *Let H be defined by (32). Then* *(i)* $f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx \le \int_0^1 H(t)dt \le$

$$\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)dx\right] \tag{35}$$

and H is a convex mapping.

(ii) If f is differentiable (and convex), then

$$0 \le \frac{1}{b-a}\int_a^b f(t)dt - H(t) \le (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right] \tag{36}$$

and

$$0 \le \frac{f(a)+f(b)}{2} - H(t) \le \frac{[f'(b)-f'(a)](b-a)}{4}, \quad t \in [0,1]. \tag{37}$$

Remark 2.10. Relation (36) gives a new refinement of the right side of (1).

Theorem 2.16. *Let G be defined by (33). Then*

(i) G is convex and increasing on $[0,1]$;

(ii) $\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$; $\quad \sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2}$;

(iii) $H(t) \le G(t)$ for all $t \in [0,1]$;

(iv) $\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx \le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \le \int_0^1 G(t)dt \le$

$$\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right];$$

(v) If f is differentiable (and convex), then

$$0 \le H(t) - f\left(\frac{a+b}{2}\right) \le G(t) - H(t) \text{ for all } t \in [0,1].$$

Remark 2.11. Since $H(1) = \frac{1}{b-a}\int_a^b f(x)dx$, (iii) gives a generalization, while (v) a refinement of Hadamard's inequalities.

Theorem 2.17. *Let L be defined by (34). Then*

(i) L is a convex mapping on $[0,1]$;

(ii) $G(t) \le L(t) \le \frac{1-t}{b-a}\int_a^b f(x)dx + t\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2}$ *for all* $t \in [0,1]$; *and* $\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}$;

(iii) $H(1-t) \le L(t)$ *and* $\frac{H(t)+H(1-t)}{2} \le L(t)$ *for all* $t \in [0,1]$.

Remark 2.12. Since $L(0) = \frac{1}{b-a}\int_a^b f(x)dx$, relation (ii) offers a generalization and new refinement of Hadamard's inequalities.

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ON A HYBRID FDTD-MoM TECHNIQUE: 2-D CASE

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Abstract. FDTD techniques offer fast simulations and small memory requirements while MoM is more suitable for free-space field simulations but needs more processing power and memory. A hybrid method that combines the advantages of both would be highly useful in free-space scattering simulation.

1. Introduction

The finite-difference time-domain (FDTD) solution of Maxwell's curl equations is analogous to existing finite-difference solutions of scalar wave propagation and fluid-flow problems in that the numerical model is based upon a direct solution of the governing partial differential equations.

The simplicity and the ability to handle complex geometry make the FDTD method flexible to implement. It is successfully applied for a wide variety of electromagnetic wave interaction problems. FDTD is a nontraditional approach to numerical electromagnetic wave modeling of complex structures for engineering applications, where the method of moments has dominated for many years.

A. Some general characteristic of proposed technique. The goal of this paper is to develop a hybrid technique using FDTD method and MoM technique, combining the benefits of both while ensuring the stability of the method. The analysis is done for a combined two-dimensional conducting and dielectric electromagnetic structure.

We must preserve a certain ratio between the spectral component of the considered impulse with the highest significant frequency and the FDTD grid step, respectively the time step.

This condition must be also respected inside the dielectric.

The FDTD method introduces a non-physical dispersion (artificial, numerical reason) for the phase velocity, dispersion which is more important as the highest frequency component of the incident wave is represented using less points.

For a pure FDTD simulation, this dispersion must behave like a deformation of the original impulse, as small as possible to avoid the reflections on the boundary of the analyzed domain.

The reflection on the boundary of the analyzed domain appear because both the values of incident wave and MoM - computed scattered field suppose an ideal behavior of dispersion while the values resulted from FDTD are affected by the numerical dispersion.

Therefore, a grid not fine enough with respect to the shape and duration of the incident wave impulse leads to unnatural reflections on the boundary, even if inside the domain the FDTD behavior is acceptable.

B. FDTD algorithm - two dimensional case. The field is described by Maxwell's curl equations:

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu}\nabla \times E \tag{1}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon}\nabla \times H - \frac{\sigma}{\varepsilon}E \tag{2}$$

where E is the electric field in volts/meter; H is the magnetic field in amperes/meter; ε is the electrical permittivity in farads/meter; σ is the electrical conductivity in siemens/meter; μ is the magnetic permeability in henrys/meter.

The FDTD algorithm for electromagnetic wave interactions for TM case, with E_z, H_x and H_y field component only:

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu}\frac{\partial E_z}{\partial y} \tag{3}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu}\frac{\partial E_z}{\partial x} \tag{4}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z\right) \tag{5}$$

Then, we use the centered finite-difference expression for the space and time derivatives:

$$F^n(i,j) = F(idx, jdy, ndt)$$

$$\frac{\partial F^n(i,j)}{\partial x} = \frac{F^n(i+1/2,j) - F^n(i-1/2,j)}{dx} + O(dx^2)$$

$$\frac{\partial F^n(i,j)}{\partial t} = \frac{F^{n+1/2}(i,j) - F^{n-1/2}(i-1/2,j)}{dt} + O(dt^2)$$

and finally we have the formula:

$$\psi^{n+4}(i,j) = -dt\frac{\sigma(i,j)}{\varepsilon(i,j)}\psi^{n+3}(i,j) + dt\frac{\sigma(i,j)}{\varepsilon(i,j)}\psi^{n+1}(i,j) - \psi^n(i,j)+$$

$$+\frac{1}{\mu_0\varepsilon(i,j)}\left(\frac{dt}{dx}\right)^2(\psi^{n+2}(i+1,j) + \psi^{n+2}(i-1,j))+$$

$$+\frac{1}{\mu_0\varepsilon(i,j)}\left(\frac{dt}{dy}\right)^2(\psi^{n+2}(i,j+1) + \psi^{n+2}(i,j-1))+$$

$$+2\left(1 - \frac{1}{\mu_0\varepsilon(i,j)}\left(\frac{dt}{dx}\right)^2 - \frac{1}{\mu_0\varepsilon(i,j)}\left(\frac{dt}{dy}\right)^2\right)\psi^{n+2}(i,j) \tag{6}$$

where $\psi = E_z$.

To ensure the stability of the time-stepping algorithm, dt must be chosen to satisfy the inequality:

$$c_{\max}dt \leq (1/dx^2 + 1/dy^2)^{-1/2} \tag{7}$$

where $c_{\max}$ is the maximum electromagnetic wave phase velocity within the media being modeled.

2. Formulation of the problem

Let us consider a 2D problem with the layout shown in Fig.1. The analyzed domain contains a dielectric zone with arbitrary shape and parameters placed in the proximity of a perfectly conductive material of linear cross-section. The entire system is illuminated with a plane wave propagating towards the origin at an arbitrary angle. The wave will produce secondary scattering waves on the dielectric and will reflect completely on the perfect conductor surface.

We will use during the following computations two boundaries: ∂S_∞ the boundary placed at infinite and ∂S_c united with ∂S_d the boundary of the analyzed domain. For simplicity of geometry generation, the conductor is parallel with Oy axis, but the method allows any placement of it.

Fig.1. Studied case of scattering problem

3. **Mathematical formulation**

A. General facts. The wave speed in free space: $c = 3 * 10^8 m/s$

The free space permittivity: $\varepsilon_0 = 8.8419 * 10^{-12} F/m$

The free space permeability: $\mu = 4 * \pi * 10^{-7} H/m$

The dielectric conductivity: $\sigma = 10^{-9} S/m$

The dimensions of the analized domain are $L_x = 30m$, $L_y = 20m$.

The incident wave is travelling leftwards in the air. It is one cosine impulse length $iw = 4 * 10^{-9}s$, starting tangential in zero.

The incidence angle of the wave, counterclockwise from Ox axis is $\alpha = \pi/4$.

We must have a fine grid to respect the shape and duration of the incident wave impulse. This condition will prevent the unnatural reflections on the boundary.

The place occupied in space by the impulse (for 45 degrees incidence):

Impulse length: $iw * c = 4 * 10^{-9} * 3 * 10^8 = 1.2s$

The grid step: $dx = dy = dl = 0.1m$

Number of nodes: $iw/(dl * \sqrt{2}) = 1.2/(0.1 * 1.414\ldots) = 8.4$

The phase of the wave, which starts at the far end of domain:

$$Tph = (L_x * \cos(\alpha) + L_y * \sin(\alpha))/c$$

The scattering problem is solved for two shapes of dielectric. More complex structures of dielectric shapes can be solved in a similar mode.

In this paper, the shapes of dielectric analized are cylindrical with rectangular and respectively circular cross-sections. The rectangular dielectric is placed in the centre of analized zone and it has the dimensions $L_x/2m$,respectively $L_y/2m$. The relative permitivity is $\varepsilon_{rel} = 10$. The circular cross-section dielectric is also placed in the centre of the analized zone, with radius $L_y/3m$. The relative permitivity is $\varepsilon_{rel} = 2$.

B. Green's theorem. Derivations of integral equations. For the computation of the field values from the contour we have two Helmholz equations:

$$\nabla^2 \psi_\omega^s + \frac{\omega^2}{c^2}\psi_\omega^s = 0 \tag{8}$$

$$\nabla^2 G_\omega(r, r') + \frac{\omega^2}{c^2} G_\omega(r, r') = -\delta(r - r') \tag{9}$$

where $G_\omega(r, r')$ is the Fourier Transform for Green function, ψ_ω^s is the scattered field and r, r' are the vectors of the position.

$\delta(r - r')$ is Dirac function.

On the conductor, the scattered field is zero: $\psi_\omega^s = 0$.

For the field computation, we apply the Green's theorem in time domain on the area outside $\partial S_d, \partial S_c$ and inside ∂S_∞.

$\psi^s(r,t)$	scattered field
$\psi_0(r,t)$	incident wave
$\psi(r,t)$	total field $\psi(r,t) = E_z(r,t)$
$G_\omega(r,r')$	Fourier Transform for Green function
$G(r,r',t-t')$	2D free-space Green function

$$G(r, r', t - t') = \frac{H(t - t' - u/c)}{\sqrt{\left|(t - t')^2 - \dfrac{u^2}{c^2}\right|}} \tag{10}$$

$$u = |r - r'|$$

$H(x)$ Heaviside unit step function

$$\iint\limits_{s} (G_\omega(r,r')\nabla^2\psi^s_\omega(r') - \nabla^2 G_\omega(r,r')\psi^s_\omega(r))dxdy =$$

$$= \int\limits_{\partial S_\infty} \left(G_\omega(r,r') \frac{\partial \psi^s_\omega(r)}{\partial n} - \frac{\partial G_\omega(r,r')}{\partial n} \psi^s_\omega(r) \right) dl +$$

$$+ \int\limits_{\partial S_d} \left(G_\omega(r,r') \frac{\partial \psi^s_\omega(r)}{\partial n} - \frac{\partial G_\omega(r,r')}{\partial n} \psi^s_\omega(r) \right) dl +$$

$$+ \int\limits_{\partial S_c} \left(G_\omega(r,r') \frac{\partial \psi^s_\omega(r)}{\partial n} - \frac{\partial G_\omega(r,r')}{\partial n} \psi^s_\omega(r) \right) dl \tag{11}$$

The first integral is zero because the field vanishes at infinity.

Considering $\partial S_d + \partial S_c = \partial S$, we have:

$$\iint\limits_{S} (\delta(r-r')\psi^s_\omega(r'))dx'dy' =$$

$$\int\limits_{\partial S} \left(G_\omega(r,r') \frac{\partial \psi^s_\omega(r)}{\partial n} - \frac{\partial G_\omega(r,r')}{\partial n} \psi^s_\omega(r) \right) dl \tag{12}$$

Now we use the Inverse Fourier Transform to transform the Green function in time domain and we obtain:

$$\psi(r,t) = \psi_0(r,t) +$$

$$+ \int\limits_{\partial S_d} dl \left(G(r,r',t-t') * \frac{\partial \psi^s(r,t)}{\partial n} - \frac{\partial G(r,r',t-t')}{\partial n} * \psi^s(r,t) \right) +$$

$$+ \int\limits_{\partial S_c} dl \left(G(r,r',t-t') * \frac{\partial \psi^s(r,t)}{\partial n} \right) \tag{13}$$

The system of integral equations has the number of equations equal with the number of points from discretization made on ∂S_d and ∂S_c. For the points on ∂S_c, the field values $\psi(r,t)$ are zero.

C. Computation of the convolution integrals. For the derivative of G and ψ, we have the expressions:

$$\frac{\partial G(r_c, ndt - t')}{\partial n} = \frac{G_{cont}(i_c, j_c) - G_{int}(i_i, j_i)}{dn};$$
$$\frac{\partial \psi^s(r_c, ndt)}{\partial n} = \frac{\psi^s_{cont}(i_c, j_c) - \psi^s_{int}(i_i, j_i)}{dn}, \tag{14}$$

where G_{cont} is the Green function relative to contour nodes, G_{int} is the Green function relative to interior-contour nodes, ψ_{cont} is the field values on the contour nodes, ψ_{int} is the field values on the interior-contour nodes and the boundary ∂S is equal with $\partial S_d + \partial S_c$.

Considering the symbol $*$ for the convolution operator, the integral becomes:

$$\int\limits_{\partial S} dl \left(G(r, r', t - t') * \frac{\partial \psi^s(r, t)}{\partial n} - \frac{\partial G(r, r', t - t')}{\partial n} * \psi^s(r, t) \right) =$$
$$= \int\limits_{\partial S} dl \frac{1}{dn} (G_{cont}(r, r', t - t') * \psi^s_{int}(r, t) - G_{int}(r, r', t - t') * \psi^s_{cont}(r, t)) \cong$$
$$\cong \sum_{j \in contour} \frac{dl}{dn} (G_{cont}(r, r', t - t') * \psi^s_{int}(r, t)|_j - G_{int}(r, r', t - t') * \psi^s_{cont}(r, t)|_j) \tag{15}$$

The approach from (15) is correct because we know that for an arbitrary function $f(r)$, the contour integral over $f(r)$ means sum over all points on the contour:

$$\int\limits_C f(r) dl \cong \sum_{j \in contour} dl f(r)|_j \tag{16}$$

Moreover, the values for the Green function relative to interior nodes and contour, respectively, do not change at different time steps and they can be computed before the loops begin in the program.

Each node pair may use a finite number of such non-zero Green function values since the Green function for a given $u = |r - r'|$ decays with the inverse of time. Therefore, a precision of 10^{-1} will require approximately ten values of the Green function per node pair.

The definition for the convolution is:

$$G(r, r', t - t') * \psi^s(r, t) = \int_{-\infty}^{t} G(r, r', t - t') \psi^s(r, t') dt' = I(t) \tag{17}$$

We calculate the integral at time $t = (n + 1)dt$, so we will make a notation and then we will write the integral as sum of integrals. The sum can be separated in three terms. The first two integrals are evaluated and the rest implies the calcul of Green function.

$$I(t)|_{t=(n+1)dt,r} = I^{(n+1)}(r)$$

$$I^{(n+1)}(r) = \int_{ndt+\frac{dt}{2}}^{(n+1)dt} G(r,r',t-t')\psi^s(r,t')dt' +$$

$$+ \int_0^{\frac{dt}{2}t} G(r,r',t-t')\psi^s(r,t')dt' + \sum_{k=1}^{n} \psi^s|_{r,k'} \int_{kdt-\frac{dt}{2}}^{kdt+\frac{dt}{2}t} G(r,r',t-t')dt' \qquad (18)$$

The first integral from (18) is zero for all points except for the reference point which we neglect. The second integral from (18) is zero because at the beginning the field value is zero. The last integral is analytically calculated.

For the Green function, we know that:

$$\begin{gathered} G(r,r',t-t') = 0 \text{ for } t-t' < |r-r'|/c; \\ (\text{or } (n+1)dt - t' < |r-r'|/c) \end{gathered} \qquad (19)$$

and

$$\begin{gathered} G(r,r',t-t') = 1/\sqrt{(t-t') - |r-r'|^2/c^2}, \\ \text{for } t-t' > |r-r'|/c; \\ (\text{or } (n+1)dt - t' > |r-r'|/c) \end{gathered} \qquad (20)$$

4. **Numerical results**

A. Estimation of the error. Estimation of the error arising from neglecting of

$$I_1 = \int_{ndt+dt/2}^{(n+1)dt} G(r,r',t-t')\psi(r,t')dt'$$

form sum (18) has the analytical expression:

$$Er(dt) = \pi c dt/8$$

For example, if the time step $dt = 10^{-10}$ we have

$$Er(dt) = \pi \cdot 3 \cdot 10^8 \cdot 10^{-10}/8 \cong 0.0117$$

The above computing coefficient multiplies the field on the interior contour at time $(n+1)dt$ which is not yet known and will be computed using the values from the whole summation, fed into FDTD method. This leaves us no other choice than to ignore the whole term when computing the scattered field on the contour at time $(n+1)dt$.

As we know, the relation for computing the scattered field suppose the convolution between the wave shape in a point and the correspondent Green function over time. For computational reason (the occupied memory, the time needed for the computing) the sum which calculates this integral must be truncated. The sum truncation is another error source that can lead to parasite reflections on the boundary or to a reduction of the response induced by the conductor on the system.

B. Analysis of the Stability. For the test of the stability, we must run the program for different steps in space and time. Because the results must be comparable, we must take the measure to fix the initial conditions of the simulation.

The elements that are fixed (must be fixed) are:

1. the dimension of the analyzed domain
2. the absolute position of the conductor
3. the dimension of the conductor
4. the absolute position of the dielectric
5. the absolute dimension of the dielectric
6. the position of the wave front at starting moment of the simulation.

The absolute position of the conductor doesn't pose any problem being fixed. Also the position of the wave front could be chosen at point (L_x, L_y) at moment $t = 0$.

The simulation implementations in the program groups the sum terms in order to take advantage of the Green function values duplications for pairs of points placed at same relative distance.

Let us consider for example the tested geometry. The domain has 31×21 points (grid values) and the conductor has 21×2 points (grid values) which normally would require to compute the Green function for $2 \times 140 \times 140$ points rising to approximately 39.000 sets of values.

Using the fact that the points on the contour are regularly spaced and the distances between the pairs of points are repeated, the number of Green function to compute and, of course, to store is reduced to approximately only 770.

This significant reduction of the necessary memory (of about 50 times), allows simulations with a step seven times finer ($7 \times 7 \sim= 50$) than without the optimization.

The price paid for these optimizations is solving the problem on a grid with the step on x equal with the step on y, and uniform distributed on the boundary.

The advantages obtained by indexed computations of Green function remain valid, but in a lesser way, for non-equal x and y steps, uniform distributed.

The stability of the method is proven by Fig.2 and Fig.3, containing field plots over space at the same moment of time but being computed with different space steps. The same stands true for Fig.4 and Fig.5 in the circular section case.

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THE ROLE OF APPLIED MATHEMATICS

WOLFGANG L. WENDLAND

Magnificus, Spectabiles, Professors, dear friends, ladies and gentlemen!

This is a very touching moment for my dear wife and me. The honorous doctorate of the Babeş-Bolyai University is a very great honour for me and I want to express my sincere gratitude to my dear colleagues, in particular to Professor Micula - thank you so much - to Professor Rus and to Professor Ţâmbulea. Dear Prorector Szilágyi, thank you so much for your kindness and for your flattering laudatio which makes me rather uncomfortable but also very happy. And Magnificus Simon, thank you much for this great honour! I will try to be a good scholar and ambassador.

Now, let me say a few words on my profession.

Mathematics is one of the oldest cultural achievements of mankind, as early as literature, art and music. "Mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, intelligible only in a social context" [1].

In fact, mathematics has always been developed or discovered (?) in combination with other fields. Here are some historical examples:

Pythagoras (582-505 b. Chr.) discovered irrational numbers from the profane problem of partitioning a given surface, which triggered his studies of geometry and numbers. He introduced the concepts of axioms, theorems, proofs and logical reasoning. As we know, because of troubles with tyrant Polykrates of Samos, he became a political refugee immigrating to Italy.

Archimedes (287-212 b. Chr.) was regarded by his king Hieron of Syracuse as top engineer because of his numerous inventions and discoveries such as fundamental laws of hydrostatics, the spiral pump, sets of pulleys, catapults and war machines,

Given at the University Babeş-Bolyai in Cluj-Napoca, Romania, on April 7th 1999, on the occasion of his nomination as "Doctor Honoris Causa".

the concave mirror. King Hieron was very angry at Archimedes' absurd mathematical activities like approximation methods for computing roots and π, developing the concept of the Riemann integral and of differentiation. We all know about the tragic end of Archimedes and of Greek sophistication due to the brutal Roman conquest of Syracuse.

I. Newton (1643-1727) was a famous physicist who developed differential and integral calculus for deriving Kepler's laws from the fundamental law of gravity, and his theory of light arose in connection with his work on optical instruments. However, he himself rated his mathematical work much higher than his achievements in physics and astronomy.

G. Leibniz (1646-1716) was an allround genius, a famous lawyer, historian and philosopher who enjoyed mathematics almost like a hobby. He developed differential and integral calculus, theory of determinants, power series, successive iteration, division of polynomials and two mechanical calculators; one of them based on dual numbers. Modern analysis owes him the elegant and simple rules of calculus.

C.F. Gauss (1777-1855) started his career with number theory and the proof of the fundamental theorem of algebra as a student - he was 22 years old when getting his PhD in absentia at Helmstedt - but professionally he was hired as an astronomer, continued as the director of surveying engineers, then developed the theory of electromagnetism and with Weber constructed the telegraph. But mathematics fascinated him most: He pioneered complex function theory, differential geometry and geodesy, least squares methods and the solution of linear equations, theory of algebraic equations and potential theory - he even proved the existence of solutions to boundary integral equations (my hobby). He was a close friend of Farkas Bolyai, who worked here in Cluj and published a great two-volume book on mathematics. His son Janos Bolyai from Cluj invented the non-Euclidean geometry independently of Gauss which became a necessary prerequisite for Einstein's theory of relativity. This university is named after F. and J. Bolyai!

S. Kowalewskaya (1850-1891) corrected a conjecture of Weierstrass and gave a constructive proof for the convergence of Cauchy's algorithm providing the most general existence proof for systems of partial differential equations. She then made significant contributions to complex function theory, potential theory, Abelian integrals and algebraic geometry. But she achieved her greatest success in the theory of the gyroscope apart from contributions to optics, celestial mechanics and astrophysics.

V. von Neumann (1903-1957) began with complex function theory and was then working with D. Hilbert on the foundation of mathematics, continuing with topology, functional analysis, measure theory, quantum field theory and stochastics, treating problems in meteorology, hydro- and aerodynamics, spherical shock waves, game theory (for the Navy) and the foundation of computer science. He developed the ENIAC in 1944 and the MANIAC in 1951 which was the basic computational tool for America's hydrogen bomb.

Mathematics as a profession by itself, however, is not older than 150 years. And the distinction between pure and applied mathematics is even younger - to me this distinction seems rather artificial. The greatest mathematicians do both: pure and applied mathematics; each side fertilizes the other.

But what is pure and what is applied mathematics? Mathematical theories in their complex and abstract structures grow at their frontiers through continuous interaction between mathematicians and all kinds of researchers. The advance is fed by information and desire from all areas of real and intellectual life. In order to make predictions about future possible states, mathematical models provide the corresponding tools. But "no real phenomenon is perfectly described by any mathematical model. There's usually a choice among several incomplete models, each more or less suitable" [1]. And H.M. Enzensberger says [2]: "The unforeseen utility of mathematical models is somewhat puzzling. It is no means clear why highly precise mental productions, devised entirely in isolation from empirical reality... should be so capable of explaining and manipulating the real world around us. Many have marveled at "the unreasonable effectiveness of mathematics". ... One explanation that presents itself - though not especially popular among the guardians of tradition - might be that one and the same evolutionary process has produced the universe at large and our brain, so that a weak anthropic principle determines that we observe the same operating rules in physical reality and in our own thought processes."

So, applied mathematics can be seen as that part of mathematics that deals with models of real life problems. This kind of work is done by many people, not only professional mathematicians. Among the Nobel Prize winners since 1970, I found at least 14 in physics, 3 each in chemistry, medicine and economy who used and developed deep mathematics for their outstanding achievements. Examples:

1979: A.M. Cormack, Sir G.N. Hounsfield: Computer assisted tomography.

1979: S.L. Glashow, A. Salan, S. Weinberg: Formulation of the standard model unifying weak and electromagnetic interaction.

1982: K.G. Wilson: Theory of critical phenomena in connection with phase transitions.

1990: J.I. Friedman, H.W. Kendall, R.E. Taylor: Scattering of electrons, protons, bound neutrons and the development of the quark model.

1990: H.M. Markowitz, M.M. Miller, W.F. Sharpe: Theory of financial economics.

1998: J.A. Pole: Computational methods in quantum chemistry.

Mathematics, however, is explicitly excluded from the Nobel prize; probably since Mittag-Leffler, who was at the same time a famous mathematician at the Royal Academy and the University of Stockholm disliked the playboyish A. Nobel. These feelings were probably mutual.

During the first half of this century, the terminus applied mathematics was used for teaching mathematics to engineering students. But in the second half, "the most striking development in engineering... has been the increasing use of mathematics in the analysis of engineering problems. No longer is skill in the use of a slide rule sufficient mathematical equipment for a practising engineer. For instance, control engineers use sophisticated, and very often abstract, mathematical concepts, some electrical engineers have to be acquainted with quantum mechanics, others with transform theory, and civil and mechanical engineers reading research papers on continuum mechanics encounter a bewilderingly wide range of mathematical techniques;" writes I. Sneddon in [5]. Nowadays exist engineering sciences.

Sneddon's Encyclopaedic Dictionary [5] and also the Mathematical Handbook [6] by G.A. and T.M. Korn covers mathematical analysis of very high level, I doubt that I am familiar with more than 20% of it. Thus, our engineering colleagues do applied mathematics to a great extent, too.

In [5] and [6] one can also find references to the work of two famous mathematicians and professors of this university: To L. Fejer (1880-1959) who taught in Cluj from 1905 till 1911 and whom we owe fundamental results in harmonic analysis, complex function theory and interpolation, and to F. Riesz (1880-1956) who worked here 1911-1920 and who has made decisive inventions in the functional analysis of Hilbert and Banach spaces, harmonic analysis and approximation theory.

In all engineering fields, the modern tools of electronic computers led to the new branch of scientific computing and simulation in applied mathematics [3] which now is indispensable in computer tomography, geometric design, reconstruction and visualization, direct and inverse scattering of electromagnetic and acoustic fields, heat transfer and radiation, stress and damage analysis in solid mechanics, all branches of fluid mechanics from aircraft design to sedimentation and groundwater pollution, signal processing, network analysis and planning, chemical processing, etc., etc. - and the combination of several field models to multifield problems resulting in new technologies such as the intelligent wing, nondestructive thermography, noise reduced helicopters or most effective sex-segregated baby diapers.

Nowadays industry uses mathematics often directly. This resulted in the fashionable new field of industrial mathematics [4]. The E & E Chief of Exxon, E.E. David in a report for the NSF of the US [7]: "Apparently, too few people recognize that the high technology that is so celebrated today is essentially mathematical technology... Mathematics is, or should be playing an integral role in America's industry's approach to its challenges, at home and abroad."

Indeed, mathematics seems to be the key technology of our future (see [8]) and also the "language of engineers".

The enormous demand for applied mathematics creates completely new mathematical concepts such as qualitative reliability and corresponding tools for critical evaluation of physical and constutive models; cost and time efficient computational algorithms, new solution methods due to new computer technologies (e.g. parallelization), attractive presentation of computational results, adequate learning and teaching of the new mathematical technologies.

These challenges will also influence mathematics in general since "applied mathematics is not illegitimate or marginal. Advances in mathematics for science and technology often are inseparable from advances in pure mathematics" [1].

As it seems, we are living in the high time of mathematics!

However, professional mathematicians are somehow excluded from corresponding benefits. After a stimulating and optimistic start into a seemingly new mathematical era about 20 years ago public opinion has changed more recently. Of course, mathematicians with their persistent obsession of truth and typical slowness in acceptance of empirical relations or new - often false - algorithms of pushing economists and engineers, are then seen as blocking obstacles of progress. One of the leading engineers in finite element analysis claims that mathematicians are rubber stampers but not inventors - they "prove" 10 years behind nothing but that engineers are doing right. Mathematicians think in counterexamples whereas engineers think in examples; and a mathematician always hesitates to publish anything that is not yet completely and rigorously justified within a mathematical theory. However, there are several examples where mathematical cautiousness would have saved a lot of money and even human lives. I recall the destruction of cooling towers at Ferrybridge in 1966, the desastrous sinking of a new Norwegian oil platform in 1993, or less spectacular, the elk test affair of the new Mercedes compact cars, convergence of finite element approximations towards wrong solutions because of negligence of distributional derivatives in modern analysis. In all these cases, design engineers blindly trusted professional software. So, on the path from applied mathematics to the numerical algorithm used for the software, critical information was lost since responsible people did not know enough mathematics.

As was mentioned before, here is a new challenge for applied mathematics: Every mathematical method that is used for the simulation of real life problems should nowadays also provide reliable information on the validity of the model and also reliable error inclusions since every mathematical model is based on some idealization that covers the true problem only partly and every computational, numerical method is incorrect to a certain extent. Of course, such worries are highly unwelcome in modern rushing life which unfortunately seems to be driven by the golden calf of financial profit.

"It is often not obvious how closely "pure" and "applied" mathematics are intertwined; this may be one of the reasons why the status of mathematical research is hugely undervalued in today's society. In addition, there is surely no other field in which the cultural lag is no enormous... One can state dispassionately that great segments of the population have never progressed beyond the mathematical level of the ancient Greeks. An equivalent backwardness in other fields - e.g. in medicine, or physics - would be perilous. This is very dangerous since never has a civilization been so infused with mathematical methology - right down to its everyday life - and so dependent on it as ours!" [2].

The danger for mathematics as a profession can rather clearly be seen nowadays. In my state which is booming in high technology, the State Commission for the Future Structural Development of the State Universities recommends reduction of the mathematical staff by 25% within the next 9 years, in spite of official state expert's recommendation in 1996 to expand scientific computing and numerical mathematics significantly. We currently discuss whether an open key professorship in this field should remain in the mathematical department or better be moved into some field of engineering applications or to computer science. "Der Spiegel" chose "Nobel prizes for nonsense" as headline of an article on the world congress of mathematics 1998 in Berlin. The number of freshmen and graduate students in mathematics dropped significantly during the last years in spite of their marvelous job chances.

Hence, it is just not enough to do good mathematics, pure and applied. We professional mathematicians must reform the education of school teachers in mathematics - it is simply improper to drill intelligent children with boring formalities. We also must reform our university teaching by showing our students from the beginning the power and beauty of applied mathematics, by treating difficult real life problems instead of agonizing them with ancient calculations. We must learn how to talk to our colleagues in other fields and must direct our research interests to challenging open problems of real life applications. The applied part of mathematics is livelier than ever but to me it is not so clear that professional mathematicians are really gripping the situation.

I hope that there will come a time when the following joke due to Professor Collatz will not be understood anymore: Two men in a balloon lost direction in fog.

Fortunately by the time they saw a man on the ground and asked him shouting: Where are we? It took a while before the man replied: You are in the gondola of a balloon! Disappointed, the balloon captain remarked to his friend: This was bad luck to meet a mathematician in our situation. - How did you know, he was a mathematician? - Firstly, replied the captain, before he made any statement he obviously was thinking hard. Secondly, his statement was clear and perfectly correct but - completely useless!

I am sure that applied mathematics is extremely useful!

Mulţumesc!

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BOOK REVIEWS

Ravi P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications,* Second Edition, Revised and Expanded, M. Dekker, Inc., Basel - New York 2000, xiii+971 pages, ISBN: 0-8247-9007-3.

Although difference equations manifest themselves as mathematical models describing real life situations in probability theory, queueing problems, statistics, stochastics time series, combinatorial analysis, number theory, electrical networks, quanta in radiation, genetics, economics, psychology, sociology, etc., these are sometimes considered as the discrete analogs of differential equations. It is an indisputable fact that difference equations appeared much earlier than differential equations and were very important for their development. It is only recently that difference equations have started receiving the attention they deserve. Perhaps this is largely due to the advent of computers, where differential equations are solved by using their approximate difference equation formulations.

The monograph under review is a virtual encyclopedia of results concerning difference equations. This second edition discusses solutions to linear and nonlinear difference equations, highlights discrete versions of Rolle's theorem, the mean value theorem, Taylor's formula, Hospital's rule, Kneser's theorem. The author investigates the stability and oscillatory properties of solutions of difference equations, explores a unified treatment of boundary value problems, introduces difference inequalities in several independent variables, explains Duffing's, van der Pol's, and Hill's equations (among other classical equations), evaluates Sturm-Liouville problems and related inequalities.

In this new edition, beside a new chapter on the qualitative properties of solutions of neutral difference equations, new material has been added in all existing chapters of the first edition. This includes a variety of interesting examples from real world applications, new theorems, over 200 additional problems and 400 further references.

J. Sándor

Jonathan M. Borwein and Adrian S. Lewis, *Convex Analysis and Nonlinear Optimization, Theory and Examples,* Canadian Mathematical Society (CMS) Books in Mathematics, Vol. 3, Springer-Verlag, New York Berlin Heidelberg, 2000, ISBN:0-387-98940-4.

The book is a concise account of convex analysis, its applications and extensions. It is aimed primarily at first-year graduate students, so that the treatment is restricted to Euclidean space, a framework equivalent, in fact, to the space $\mathbb{R}^n$, but the coordinate free notation, adopted by the authors, is more flexible and elegant. The proof techniques are chosen, whenever possible, in such a way that the extension to infinite dimensions be obvious for readers familiar with functional analysis (Banach space theory). Some of the challenges arising in infinite dimensions are discussed in Chapter 9, *Postscript: Infinite versus finite dimensions,* in which case the results involve deeper geometric properties of Banach spaces. The last section of this chapter contains notes on previous chapters, explaining which results extend to infinite dimension and which not, as well as sources where these extensions can be found.

The authors adopted a succint style, avoiding as much as possible complicated technical details, their goal being "to showcase a few memorable principles rather than to develop the theory to its limits". The book consists of short, self-contained sections, each followed by a rather extensive set of exercises grouped into three categories: examples that illustrate the ideas in the text or easy expansions of sketched proofs (no mark); important pieces of additional theory or more testing examples (marked by one asterisk); and longer, harder examples or peripheral theory (marked by two asterisks). Some bibliographical comments are also included along with these exercises, an approach which allow the authors to cover a large variety of topics. A good idea on the included material is given by the headings of the chapters and the presentation of some topics included in the main text or in exercises.

Ch. 1, *Background* - Euclidean spaces, symmetric matrices, in the main text, and Radstrom cancellation, recession cones, affine sets, inequalities for matrices, in exercises.

Ch. 2, *Inequality constraints* - optimality conditions, theorems of alternative, max-functions, in the main text, and nearest points, coercivity, Carathéodory's theorem, Kirchoff's law, Schur convexity, steepest descent, in exercises.

Ch. 3, *Fenchel duality* - subgradients and convex functions, the value function, the Fenchel conjugate, in the main text, and normal cones, Bregman distances, Log-convexity, Duffin's duality gap, Psenichnii-Rockafellar condition, order-convexity and order subgradients, symmetric Fenchel duality, in exercises.

Ch. 4, *Convex analysis* - continuity of convex functions, Fenchel biconjugation, Lagrangian duality, in the main text, and polars and polar calculus, extreme and exposed points, Pareto minimization, von Neumann minimax theorem, Kakutani's saddle point theorems, Fisher information function, in exercises.

Ch. 5, *Special cases* - polyhedral convex sets and functions, functions of eigenvalues, duality, convex process duality, in the main text, and polyhedral algebra, polyhedral cones, convex spectral functions, DC functions, normal cones, order epigraphs, multifunctions, in exercises.

Ch. 6, *Nonsmooth optimization* - generalized derivatives, regularity and strict differentiability, tangent cones, the limiting subdifferential, in the main text, and Dini derivatives and subdifferentials, mean value theorem, regularity and nonsmooth calculus, subdifferentials of eigenvalues, contingent and Clarke cones, Clarke's subdifferentials, in exercises.

Ch. 7, *Karush-Kuhn-Tucker theory* - metric regularity, the KKT theorem, metric regularity and the limiting subdifferential, second order conditions,in the main text, and Lipschitz extension, closure and Ekeland's principle, Liusternik theorem, Slater condition, Hadamard's inequality, Guignard optimality conditions, higher order conditions, in exercises.

Ch. 8, *Fixed points* - the Brouwer fixed point theorem, selection and the Kakutani-Fan fixed point theorem, variational inequalities, in the main text, and nonexpansive mappings and Browder-Kirk fixed point theorem, Knaster-Kuratowski-Mazurkiewicz principle, hairy ball theorem, hedgehod theorem, Borsuk-Ulam theorem, Michael's selection theorem, Hahn-Katetov-Dowker sandwich theorem, single-valuedness and maximal monotonicity, cuscos and variational inequalities, Fan minimax inequality, Nash equilibrium, Bolzano-Poincaré-Miranda intermediate value theorem, in exercises.

There is a chapter, Chapter 10, containing a list of named results and notation, organized by sections. Beside this, the book contains also an Index.

The bibliography counts 168 items.

Written by two experts in optimization theory and functional analysis, the book is an ideal introductory teaching text for first-year graduate students. By the wealth of highly non-trivial exercises, many of which are guided, it can serve for self-study too.

Ştefan Cobzaş

Function Spaces– The Fifth Conference, Edited by Henryk Hudzik and Leszek Skrzypczak, Lecture Notes in Pure and Applied Mathematics Vol. 213, Marcel Dekker, Inc New York-Basel 2000, xiv + 511 pp. ISBN: 0-8247-0419-3.

These are the proceedings, edited by H. Hudzyk and L. Skrzypczak, of the fifth conference **Function Spaces**, held in Poznan, Poland, one of the satellite conferences associated with of the International Congress of Mathematicians, Berlin 1998. The conference was attended by 121 mathematicians from Poland and abroad. During the conference two special sessions were organized: one dedicated to Wladislaw Orlicz (1903-1990), the founder of the Poznan school of functional analysis, and the other dedicated to Genadiĭ Lozanovsky (1937-1976), a member of the famous St. Petersburg school of lattice theory. The personality of W. Orlicz and his mathematical achievements are evoked in two papers; *Wladyslaw Orlicz: his life and contributions to mathematics* by L. Maligranda and W. Wnuk, and *Recent developments of some ideas and results of W. Orlicz on unconditional convergence* by L. Drewnowski. There are also two papers dedicated to G. Lozanovsky: *G. Ya. Lozanovsky: his life* by Rita

Lozanovskaya, and *G. Ya. Lozanowsky: his contributions to the theory of Banach lattices* by Y. A. Abaramovich and A. I. Veksler.

Beside these expository papers, the volume contains also 40 research papers, covering a large variety of topics from general theory to particular spaces, topological and geometrical properties, order structures and the interpolation of operators. A major theme of the volume is the geometry of Banach spaces, focusing on Orlicz spaces and fixed point theory. Other topics are disjointness preserving operators, integral operators, Hardy inequalities and Hardy operators, Hardy dyadic spaces, Köthe-Bochner function spaces, polynomial and multilinear properties of Banach spaces, and much more.

Among the contributors to the volume we mention: Y. A. Abramovich, A. K. Kitover, Bor-Luh Lin, Z. Cieselski, Shutao Chen, R. Urbanski, L. Maligranda, J. Kakol, G. Lewicki, Pei-Kee Lin, J. G. Llavona, R. Taberski, N. Popa, a.o.

The volume is a valuable addition to the existing literature on function spaces and will be an indispensable tool for researchers in functional analysis and its applications.

Ştefan Cobzaş

George Isac, *Topological Methods in Complementarity Problems,* Nonconvex Optimization and its Applications Vol. 41, Kluwer Academic Publishers, Dordrecht 2000, 704 pp, ISBN: 0-7923-6274-8

As far as we know, after the author's volume "Complementarity Problems" in Springer's Lecture Notes in Mathematics (Nr. 1528, 1992) this monography is the second one dedicated wholly to this subject. It is especially dedicated to the study of nonlinear complementarity problems in infinite dimensional spaces. Since the literature on this subject is very large, here only theoretical problems are considered, and first of all those which are related to topological methods.

The first chapter concerns on the notion of the cone in a topological vector space, on the order relation it introduces. The relation of the order and the topology is essential and in this respect some fundamental types of cones (and ordered topological vector spaces) such as normal, regular, completely regular, well based, polyedral cones are defined and their properties are studied.

In the second chapter the reader get the definition of the complementarity problem, the history of the term and a philosophy about the its importance. We remind the simplest form of the Nonlinear Complementarity Problem (NCP): Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces and let $K \subset E$ be a a closed pointed convex cone, $K^* \subset E^*$ be its dual cone. Suppose that $f : E \to E^*$ is a function. Find $x_0 \in K$ such that $f(x_0) \in K^*$ and $\langle x_0, f(x_0) \rangle = 0$. Further a classification of the complementarity problems is given. Inside the two main class: Topological complementarity problems and Order complementarity problems, and especially in the first one a great variety of types are distinguished, such as various linear and nonlinear problems. The chapter ends with the list of the main problems which can be stated about complementarity problems (existence, unicity, dependence on parameters, sensitivity etc.).

Chapter 3 deals with the mode of appearance of the complementarity in mathematical programming, game theory, variational inequalities, etc. Besides purely mathematical formulations a lot of concrete economical, mechanical and technical problems are considered such as various equilibrium questions in the economy, in traffic flows, problem of maximizing oil production, problems in structural engineering, fluid flows, elasticity etc.

A short chapter is devoted to the equivalence of the complementarity problem with fixed point problems, with variational inequalities, minimization problems etc.

Chapter 5 deals with the solvability of the various types of NCP-s. Beginning with the classical existence and uniqueness theorems of Dorn, Cottle and Karamardian the chapter continues with the more recent results of existence and uniqueness results of NCP-s and their equivalent formulations. Global solvability, feasibility, boundendess of solution set are also considered.

The tietles of the following chapters are suggestive enough to reflect their special content: 6. Topological degree and complementarity. 7. Zero-epi mapping and complementarity. 8. Exceptional family of elements and complementarity. 9.Conditions $(S)_+$ and $S(S)^1_+$: application to complementarity theory. 10. Fixed points, coincidence equations on cones and complementarity. 11. Other topological results on complementarity theory.

Each chapter is followed by references, and a global reference list exists at the end of the volume. A Glossary of notations and an Index completes the monography.

The exposition implies the usage of a very large number of auxiliary results from topology and functional analysis. These results are only stated with bibliographic indications. Only the results strictly related to the subject are proved. The resulting text becomes this way a good reference book in the field, which can be used also as a guide for lectures or for the introduction in the subject. It is intended for mathematicians, engineers, economists and for anybody interested in the subject.

A.B. Németh