Anul XLVI **2001**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
2

Redacţia: 3400 Cluj-Napoca, str. M. Kogalniceanu nr. 1 • Telefon: 405300

SUMAR – CONTENTS – SOMMAIRE

PROFESSOR PETRU T. MOCANU ON HIS 70^{th} BIRTHDAY

Professor Petru T. Mocanu was born on June 1, 2001 in Brăila, Romania. He attended primary and secondary school in Brăila, then went on to take his undergraduate (1950-1953) and postgraduate degrees (1953-1957) at the Faculty of Mathematics, University of Cluj (now the Babeş-Bolyai University of Cluj-Napoca). In 1959 he defended his doctoral dissertation which was written under the guidance of the great Romanian mathematician G. Călugareanu. Professor Mocanu's doctoral thesis was entitled *Variational methods in the theory of univalent functions.*

Professor Mocanu has worked at Babeş-Bolyai University as Assistant Professor (1953-1957), Lecturer (1957-1962), Associate Professor (1962-1970) and Full Professor (1970 to the present). He was Visiting Professor at Conakry, Guinea (1966-1967) and at Bowling Green State University, Ohio, USA (fall semester, 1992), and invited to give lectures to international audiences at many different universities since 1966. These have included various universities in the United States and Germany, including University of Michigan, Iowa University, and the University of Hagen. Professor Mocanu has also taught at the University of Rouen in France, the Universities of Lodz and Lublin in Poland, the University of Jyvaskyla in Finland and other universities in Hungary and Moldavia.

Professor Mocanu has held a number of distinguished positions at Babeş-Bolyai University. He has served as Dean of the Faculty of Mathematics (1968-1976 and 1984-1987), Head of the Sub-Department of Function Theory (1976-1984 and 1990-2000), Head of Department of Mathematics and Vice-Rector of the University (1990-1992). He is also the Chief Editor of *Mathematica (Cluj)*, a member of the Editorial Board of *Studia Universitatis Babeş-Bolyai* and *Bulletin de Mathématiques S.S.M.R.*, the Chairman of the Seminar of Geometric Function Theory, Department of Mathematics at Babeş-Bolyai University and the Head of the Romanian School of Univalent Functions.

Since 1972 Professor Mocanu has been an active supervisor of doctoral degrees; under his guidance 25 students have completed PhD degrees and another 10 are currently preparing their dissertations. He has a wide range of teaching interests and many students have benefited from his expertise.

Among the subjects he has offered is a basic course on Complex Analysis and he has developed many other specialized courses (Univalent Functions, Differential Subordinations, Geometric Function Theory, Measure Theory, Hardy Spaces etc.) He is also the author of two handbooks on Complex Analysis that have become standard texts for Romanian students of mathematics.

Professor Mocanu is also:

- Corresponding Member of Romanian Academy,
- President of the Romanian Mathematical Society,
- Member of the American Mathematical Society,
- Doctor Honoris Causa, University "Lucian Blaga", Sibiu (Romania) - 1998,
- Doctor Honoris Causa, University of Oradea (Romania) - 2000.

In terms of published research Professor Mocanu's output has been prodigious. He is the author more than 155 papers in the field of Geometric Function Theory (Univalent Functions) and has written two important monographs: *Geometric Theory of Univalent Functions* (in Romanian), Ed. Casa Cărţii de Ştiinţă, Cluj-Napoca, 1999, 410 pages (with T. Bulboacă and Gr. Şt. Sălăgean) and *Differential Subordinations: Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000, 459 pages (with S. S. Miller). Approximately 200 mathematicians worldwide have cited his research in more than 500 papers. The work of Professor Mocanu and Professor Miller on the method of differential subordinations (admissible function method) has an international reputation and has proved very influential within research activity in the field of Geometric Function Theory.

Professor Mocanu obtained important results in the following domains (see "Scientific Papers"):

- extremal problems in the theory of univalent functions [1-5, 8, 9, 12, 13, 15, 17, 18, 20-22, 24, 25, 27, 37, 42, 47, 72, 91, 125, 144]

- new classes of univalent functions (well known is the class of alpha-convex functions) [7, 10, 14-16,19, 23, 26-32, 34-36, 38, 40, 41, 43, 46, 65, 70, 95, 99, 102, 103, 109, 114, 124, 126, 130, 131, 142]

- integral operators on classes of univalent functions [44, 45, 48, 55-58, 62-64, 66, 67, 69-71, 73, 76, 78, 80-83, 88-90, 92-94, 96-98, 100, 102, 103, 106, 110, 111, 115, 123, 129, 132, 136, 146, 151]

- differential subordinations [49, 50, 52, 53, 61, 64, 68, 74, 77, 79, 85-87, 101, 104, 105, 107, 111-113, 119, 122, 137, 139, 141]

- conditions of diffeomorphism in the complex plane [51-54, 59, 60, 75, 95,128, 143, 154]

- sufficient conditions for injectivity, starlikeness or convexity [116-118, 120, 121, 128, 133-135, 138, 140, 145, 147-150, 152, 153, 155].

Scientific Papers

[1] O generalizare a teoremei contracţiei n clasa S de funcţii univalente, Stud. Cerc. Mat., Cluj , 8 (1957), 303-312.

[2] Asupra unei generalizări a teoremei contracţiei n clasa funcţiilor univalente, Stud. Cerc. Mat., Cluj 9(1958), 149-159.

[3] Despre o teoremă de acoperire în clasa funcţiilor univalente, Gaz. Mat. Fiz., Ser. A (N.S) 10(63)(1958), 473-477.

[4] O problemă variaţională relativă la funcţiile univalente, Studia Univ. Babeş-Bolyai, III, 3(1958), 119-127.

[5] O problemă extremală în clasa funcţiilor univalente, Stud. Cerc. Mat., Cluj 11(1960), 99-106.

[6] O teoremă asupra funcţiilor univalente, Studia Univ. Babeş-Bolyai, I, 1 (1960), 91-95.

[7] Asupra razei de stelaritate a funcţiilor univalente, Stud. Cerc. Mat., Cluj 11 (1960), 337-341.

[8] Asupra unui domeniu extremal în clasa funcţiilor univalente, Studia Univ. Babeş-Bolyai, I, 1(1961), 221-224.

[9] Domenii extremale în clasa funcţiilor univalente, Stud. Cerc. Mat., Cluj, 2, 12(1961), 303-314.

[10] Sur le rayon d'étoilement et le rayon de convexité de fonctions holomorphes, Mathematica (Cluj), 4(27)(1962), 57-63.

[11] Despre raza de stelaritate şi raza de convexitate a funcţiilor olomorfe, Stud. Cerc. Mat. Cluj 13(1962), 93-100.

[12] Asupra unei probleme extremale relativă la funcţiile univalente, Stud. Cerc. Mat., Cluj 14(1963), 85-91.

[13] On the equation $f(z) = af(a)$ in the class of univalent functions, Mathematica (Cluj), 6(29)(1964), 63-79.

[14] Asupra razei de convexitate a funcţiilor olomorfe, Studia Univ. Babeş-Bolyai, Math. Phys., 9, 2(1964), 31-33.

[15] Funcţii univalente pe sectoare, Stud. Cerc. Mat., Cluj, 17(1965), 625-931.

[16] Convexity and starlikeness of conformal mappings, Mathematica(Cluj), 8(31)(1966), 91-102.

[17] Generalized radii of starlikeness and convexity of analytic functions, Studia Univ. Babeş-Bolyai, Math. Phys., 11, 2(1966), 43-50.

[18] About the radius of starlikeness of the exponential function, Studia Univ. Babeş-Bolyai, Math. Phys. 14, 1(1969), 35-40.
[19] Une propriété de convexité généralisée dans la théorie de la représentation conforme, Mathematica (Cluj), 11(34)(1969), 43-50.
[20] Sur la géometrie de la représentation conforme, Mathematica(Cluj), 12(35)(1970), 299-308.
[21] An extremal problem for univalent functions associated with the Darboux formula, Ann.Univ. M.Curie-Sklodowska, A, 18(1968/1969/1970), 131-135.
[22] Sur deux notions de convexité generalisée dans la représentation conforme, Studia Univ. Babeş-Bolyai , Math. Mech., 16, 2(1971), 13-19.
[23] On generalized convexity in conformal mappings, Rev. Roumaine Math. Pures Appl., 16(1971), 1541-1544. (with M.O.Reade).
[24] On the holomorphic product of Haar measures, Mathematica (Cluj), 13(36)(1971), 229-233.
[25] Equations fonctionnelles aux implications, Studia Univ. Babeş-Bolyai, Math. 17, 1(1972), 33-36.
[26] All a-convex functions are starlike, Rev. Roumaine Math. Pures Appl. 17, 9(1972), 1395-1397 (with S.S.Miller and M.O. Reade).
[27] A generalized property of convexity in conformal mappings, Rev. Roumaine Math. Pures Appl. 17, 9(1972), 1391-1394.
[28] Sur une propriété d'étoilement dans la théorie de la representation conforme , Studia Univ. Babeş-Bolyai, Math., 17, 2(1972), 55-58.
[29] On Bazilevic functions, Proc. Amer. Math. Soc., 39, 1(1973), 173-174. (with M.O. Reade and E. Zlotkiewicz).
[30] All a-convex functions are univalent and starlike, Proc. Amer. Math. Soc., 37, 2(1973), 553-554. (with S.S. Miller and M.O. Reade).
[31] Numerical computation of the a-convex Koebe functions, Studia Univ. Babeş-Bolyai, Math. Mech. 19, 1(1974), 37-46. (with Gr. Moldovan and M.O. Reade).
[32] Bazilevic functions and generalized convexity, Rev. Roumaine Math. Pures Appl., 19, 2(1974), 213-224. (with S.S. Miller and M.O. Reade).
[33] On the functional $f(z_1)/f'(z_2)$ for typically-real functions, Rev. Anal. Numer. Théorie Approx. 3, 2(1974), 209-214. (with M.O. Reade and E. Zlotkiewicz).
[34] On a subclass of Bazilevic functions, Proc. Amer. Math. Soc., 45, 1(1974), 88-92. (with P. Eenigenburg , S. Miller and M. Reade).
[35] The radius of alpha-covexity for the class of starlike univalent functions, alpha-real, Rev. Roumaine Math. Pures Appl., 20, 5(1975), 561-565. (with M.O. Reade).
[36] Alpha-convex functons and derivatives in the Nevanlinna class, Studia Univ. Babeş-Bolyai, Ser. Math, 20(1975), 35-40. (with S.S. Miller).
[37] An extremal problem for the transfinite diameter of a continuum, Mathematica (Cluj), 17(40), 2(1975), 191-196. (with D. Ripeanu).
[38] The radius of alpha-convexity for the class of starlike univalent functions, alpha-real, Proc. Amer. Math. Soc., 51, 2(1975), 395-400. (with M.O. Reade).
[39] The Hardy class for functions in the class $MV[a, k]$, J. of Math. Analysis and Appl., 51, 1(1975), 35-42. (with S.S. Miller and M.O. Reade).
[40] Janowski alpha-convex functions, Ann. Univ. M. Curie-Sklodowska, 29, A(1975), 93-98. (with S.S. Miller and M.O. Reade).
[41] On generalized convexity in conformal mappings II, Rev. Roumaine Math. Pures Appl., 21, 2(1976), 219-225. (with S.S. Miller and M.O. Reade).
[42] The Hardy class of functions of bounded argument rotations, J. Austral. Math. Soc. A, 21, 1(1976) 72-78. (with S.S. Miller).
[43] On the radius of alpha-convexity, Studia Univ. Babeş-Bolyai, Ser. Math., 22, 1(1977), 35-39. (with S.S. Miller and M.O. Reade).
[44] The order of starlikeness of a Libera integral operator, Mathematica (Cluj), 19(42), 1(1977), 67-73. (with M.O. Reade and D. Ripeanu).
[45] A particular starlike integral operator, Studia Univ. Babeş-Bolyai, Math., 22, 2(1977), 44-47. (with S.S. Miller and M.O. Reade).

[46] The order of starlikeness of alpha-convex functions, Mathematica (Cluj), 20(43), 1(1978), 25-30. (with S.S. Miller and M.O. Reade).
[47] Second order of differential inequalities in the complex plane, J. of Math. Analysis and Appl., 65, 2(1978), 289-305. (with S.S. Miller).
[48] Starlike integral operators, Pacific J. of Math., 79, 1(1978), 157-168. (with S.S. Miller and M.O. Reade).
[49] Proprietăţi de subordonare ale unor operatori integrali, Sem. Itin. Ec. Funct. Aprox. Conv., Cluj-Napoca (1980), 83-90.
[50] Subordonări diferenţiale şi teoreme de medie în planul complex, Sem. Itin. Ec. Funct. Aprox. Conv., Timişoara (1980), 181-185.
[51] Starlikeness and convexity for non-analytic functions in the unit disc, Mathematica (Cluj), 22(45), 1(1980), 77-83.
[52] On classes of functions subordinate to the Koebe function, Rev. Roumaine Math. Pures Appl., 26, 1(1981), 95-99. (with S.S. Miller).
[53] On a differential inequality for analytic functions in the unit disc, Studia Univ. Babeş-Bolyai, Math. 26, 2(1981), 62-64.
[54] Sufficient conditions of univalency for complex functions in the class C^1 , Rev. Anal. Numer. Theorie Approximation, 10, 1(1981), 75-79.
[55] On the order of starlikeness of convex function of order alpha, Rev. Anal. Numer. Theorie Approximation, 10, 2(1981), 195-199. (with D. Ripeanu and I. Şerb).
[56] The order of starlikeness of certain integral operators, Mathematica (Cluj), 23(46), 2(1981), 225-230. (with D. Ripeanu and I. Şerb).
[57] Operatori integrali care conservă convexitatea şi aproape convexitatea, Sem. Itin. Ec. Funct. Aprox. Conv., Cluj-Napoca (1981), 257-266.
[58] On the order of starlikeness of the Libera transform of starlike functions of order alpha, Sem. of Functional Analysis and Numerical Analysis, Babeş-Bolyai Univ. Cluj-Napoca, preprint no. 4(1981) , 85-92. (with D. Ripeanu and I. Şerb).
[59] Spirallike non-analytic functions, Proc. Amer. Soc., 82, 1(1981), 61-65. (with H. Al-Amiri).
[60] Certain sufficient conditions for univalency of the class C^1, J. of Math. Analysis and Appl., 80, 2(1981), 387-392. (with H. Al-Amiri).
[61] Differential subordinaton and univalent functions, Michigan Math. J., 28(1981), 157-171. (with S.S. Miller).
[62] The order of starlikenessof the Libera transform of the class of starlike functions of order , Mathematica (Cluj), 24(47), 1-2(1982), 73-78. (with D. Ripeanu and I. Şerb).
[63] Convexitatea unor funcţii olomorfe, Sem. Itin. Ec. Funct. Aprox. Conv., Cluj-Napoca (1982), 207-210.
[64] Sur l'ordre de stelarité d'une classe de fonctions analytiques, Seminar of Functional Analysis and Numerical Methods, Babeş-Bolyai Univ. Cluj-Napoca, Preprint no. 1(1983), 89-106. (with D. Ripeanu and I. Şerb).
[65] On some particular classes of starlike integral operators, Seminar of Geometric Function Theory, Babeş-Bolyai Univ. Cluj-Napoca, Preprint no. 4(1982/1983), 159-165. (with S.S. Miller and M.O. Reade).
[66] General second order inequalities in the complex plane, Idem, 96-114. (with S.S. Miller).
[67] Some integral operators and starlike functions, Idem, 115-128.
[68] On a Briot-Bouquet differential subordination, General Inequalities, 3(1983), 339-348. (with P. Eenigenburg, S.S. Miller and M.O. Reade).
[69] Convexity and close-to-convexity preserving integral operators, Mathematica (Cluj), 25(48), 2(1983), 177-182.
[70] On starlike functions with respect to symmetric points, Bull. Math. Soc. Math., RSR, 28(76), 1(1984), 46-50.
[71] On some classes of regular functions, Studia Univ. Babeş-Bolyai, Ser. Math., 29(1984), 61-65. (with Gr. Sălăgean).

[72] Sur un problem extremal, Seminar of Functional Analysis and Numerical Methods, Babeş-Bolyai Univ., Cluj-Napoca, Preprint no. 1(1984), 105-122. (with M. Iovanov and D. Ripeanu).
[73] Convexity of some particular functions, Studia Univ. Babeş-Bolyai, Ser. Math., 29(1984), 70-73.
[74] On a Briot-Bouquet differential subordination, Rev. Roumaine Math. Pures Appl., 29, 7(1984), 567-573. (with P. Eenigenburg, S.S. Miller and M.O. Reade).
[75] On some starlike nonanalytic functions, Itin. Sem. On Funct. Eq., Approx. and Convexity, Cluj-Napoca (1984), 107-112.
[76] Subordination-preserving integral operators, Transactions of the Amer. Math. Soc., 283, 2(1984), 605-615. (with S.S. Miller and M.O. Reade).
[77] Univalent solutions of Briot-Bouquet differential equations, J. of Diff. Equations, 56, 3(1985), 297-309. (with S.S. Miller).
[78] On starlike functions of order alpha, Itin. Sem. On Funct. Eq., Approx. and Convexity, Cluj-Napoca, 6(1985), 135-138.
[79] On some classes of first order differential subordinations, Michigan Math. J., 32(1985), 185-195. (with S.S. Miller).
[80] Starlikeness conditions for Alexander integral, Itin. Sem. On Funct. Eq., Approx. and Convexity, Cluj-Napoca, 7(1986), 173-178.
[81] Some integral operators and starlike functions, Rev. Roumaine Math. Pures Appl., 21, 3(1986), 231-235.
[82] On a class of spirallike integral operators, Idem, 225-230. (with S.S. Miller).
[83] On starlikeness of Libera transform, Mathematica (Cluj), 28(51), 2(1986), 153-155.
[84] On a theorem of M. Robertson, Seminar on Geometric Function Theory, Babeş-Bolyai Univ., Cluj-Napoca, 5(1986), 77-82.
[85] Mean-value theorems in the complex plane, Idem, 63-76. (with S.S. Miller).
[86] The effect of certain integral operator on functions of bounded turning and starlike functions, Idem, 83-90. (with M. Iovanov).
[87] Convexity of the order of starlikeness of the Libera transform of starlike functions of order alpha, Idem, 99-104. (with D. Ripeanu and I. Serb).
[88] Best bound of the agument of certain functions with positive real part, Idem, 91-98. (with D. Ripeanu and M. Popovici).
[89] Subordination by convex functions, Idem, 105-108. (with V. Selinger).
[90] On strongly-starlike and strongly-convex functions, Studia Univ. Babeş-Bolyai, Ser. Math., 31, 4(1986), 16-21.
[91] Differential subordination and inequalities in the complex plane, J. of Diff. Equations, 67, 2(1987), 199-211. (with S.S. Miller).
[92] Marx-Strohhaecker differential subordinations systems, Proc. Amer. Math. Soc., 99, 3(1987), 527-534. (with S.S. Miller).
[93] On a close-to-convexity preserving integral operator, Studia Univ. Babeş-Balyai, Ser. Math., 32, 2(1987), 53-56.
[94] On starlike images by Alexander integral, Itin. Sem. on Funct. Eq., Approx. and Convexity, Cluj-Napoca, 6(1987), 245-250.
[95] Alpha-convex nonanalytic functions, Mathematica (Cluj), 29(52), 1(1987), 49-55.
[96] Best bound for the argument of certain analytic functions with positive real part (II), Seminar on Functional Analysis and Numerical Methods, Babeş-Bolyai Univ., Cluj-Napoca, 1(1987), 75-91. (with M. Popovici and D. Ripeanu).
[97] Some starlikeness conditions for analytic functions, Rev. Roumaine Math. Pures Appl., 33(1988), 1-2, 117-124.
[98] Integral operators and starlike functions, Itin. Sem. On Funct. Eq., Approx. and Convexity, Cluj-Napoca, 6(1988), 233-236.
[99] Conformal mappings and refraction law, Babeş-Bolyai Univ., Fac. of Math., Research Seminars, 2(1988), 113-116.
[100] On an inequality concerning the order of starlikeness of the Libera transform of starlike functions of order alpha, Seminar of Mathematical Analysis, Babeş-Bolyai Univ., Fac. of Math., Research Seminars, 7(1988), 29-32. (with D. Ripeanu and I. Şerb).

[101] Second order averaging operators for analytic functions, Rev. Roumaine Math. Pures Appl., 33(1988), 10, 875-881.
[102] Alpha-convex integral operator and strongly-starlike functions, Studia Univ. Babeş-Bolyai, Ser. Math, 34, 2(1989), 16-24.
[103] Alpha-convex integral operator and starlike functions of order beta, Itin. Sem. On Funct. Eq., Approx. and Convexity, Cluj-Napoca, (1989), 231-238.
[104] The theory and applications of second-order differential subordinations, Studia Univ. Babeş-Bolyai, Ser. Math., 34, 4(1989), 3-33. (with S.S. Miller).
[105] On a simple sufficient condition for starlikeness, Mathematica (Cluj), 31(54), 2(1989), 97-101. (with V. Anisiu).
[106] Integral operators on certain classes of analytic functions, Univalent Functions, Fractional Calculus and their Applications, 1989, 153-166. (with S.S. Miller).
[107] On an integral inequality for certain analytic functions, Mathematica Panonica 1, 1(1990), 111-116.
[108] Certain classes of starlike functions with respect to symmetric points, Mathematica, 32(55), 2(1990), 153-157.
[109] Univalence of Gaussian and confluent hypergeometric functions, Proc. Amer. Math. Soc. 110, 2(1990), 333-342 (with S.S. Miller).
[110] Integral operators and meromorphic starlike functions, Mathematica 32 (55), 2(1990), 147-152 (with G. Sălăgean).
[111] On a class of first-order differential subordinations, Babeş-Bolyai Univ., Fac. of Math., Res. Seminar on Mathematical Analysis, Preprint 7(1991), 37-46.
[112] On a Marx-Strohhacker differential subordination, Studia Univ. Babeş-Bolyai, Math. 36, 4(1991), 77-84.
[113] On certain differential and integral inequalities for analytic functions, Studia Univ.Babeş-Bolyai, 36, 1(1991), 17-22 (with X. Xanthopoulos).
[114] On certain analytic functions with positive real part, Studia Univ. Babeş-Bolyai, Math. 36, 2(1991),45-50 (with X. Xanthopoulos).
[115] Classes of univalent integral operators, J. Math. Anal. Appl., 157, 1(1991), 147-165 (with S.S. Miller).
[116] Two simple sufficient conditions for starlikeness, Mathematica, 34(57), 2(1992), 175-181.
[117] Two simple sufficient conditions for convexity, Studia Univ. Babeş-Bolyai, Math. 37, 4(1992) (with G. Kohr and M. Kohr).
[118] Bounds on the argument of a certain meromorphic derivative implying starlikeness, Studia Univ. Babeş-Bolyai, Math.37, 4(1992), 35-42 (with H. Al-Amiri).
[119] A special differential subordination and its application to univalency conditions, Current Topics in Analytic Function Theory, World Scientific, Singapore, London (1992), 171-185(with S.S. Miller).
[120] On a starlikeness condition, Mathematica, 35(58), 2(1993), 175-178.
[121] Some simple criteria for starlikeness and convexity, Libertas Mathematica, 13(1993), 27-40.
[122] Averaging operators and a generalized Robinson differential inequality, J. Math. Anal. Appl., 13, 2(1993), 459-469 (with S.S. Miller).
[123] Starlikeness of certain integral operators, Mathematica, 36(59) 2(1994), 179-184.
[124] On certain subclasses of starlike functions Studia Univ. Babeş-Bolyai , Math., 34, 4(1994), 3-9.
[125] Starlikeness of certain analytic functions, Mathematica, 36(59), 1(1994), 73-79 (with H. Al-Amiri, K.C. Chan, J. Gresser and S. Seubert).
[126] On certain subclasses of meromorphic close-to-convex functions, Bull. Math. Soc. Sci. Math. Roumanie, 38(86), 3-15 (with H. Al-Amiri).
[127] New extensions of a theorem of R.Singh and S. Singh, Mathematica, 37(60)(1995), 171-182.
[128] A sufficient condition for injectivity in the complex plane, PU.M.A 6 (1995), 2, 231-238.

[129] Integral operators on certain subclasses of univalent functions, Revue Roumaine Math. Pures Appl., 60, 3-4(1995), 339-348 (with H. Al-Amiri, K.C. Chan, J. Gresser and S. Seubert).
[130] Starlike and close-to-convex functions with respect to symmetric-conjugate points, Glasnik Mat., 30(50)(1995), 209-219. (with H. Al-Amiri and D. Coman).
[131] Some properties of starlike functions with respect to symmetric-conjugate points, Internat. J. Math. & Math. Sci. 18, 3 (1995), 469-474.(with H. Al-Amiri, D. Coman).
[132] Integral operators preserving meromorphic starlike functions, Mathematica, 37(60) (1995), 3-10. (with H.Al. Amiri).
[133] Some simple criteria of starlikeness and convexity for meromorphic functions, Mathematica, 37(60)(1995), 11-21. (with H. Al-Amiri).
[134] A sharp criterion for starlikeness, Indian J. Pures Appl. Math. 27, 11(1996), 1111-1117 (with V. Anisiu and I. Şerb).
[135] A sharp sufficient condition for a subclass of strlike functions, Libertas Mathematica, 16 (1996), 55-60. (with V. Anisiu and I. Şerb).
[136] A class of nonlinear averaging integral operators, J. Math. Anal. Appl., 197(1996), 318-323. (with S.S. Miller).
[137] Differential inequalities and boundedness preserving integral operators,Studia Univ. Babeş-Bolyai, Math. 41, 4(1996), 81-88. (with S.S. Miller).
[138] Starlikeness conditions for meromorphic functions, Academia Română, Memoriile Secţiilor Ştiinţifice, Seria IV, tomul XIX (1996), 7-12.
[139] Briot-Bouquet differential equations and differential subordinations, Complex Variables, 33(1997), 217-237. (with S.S. Miller).
[140] A sharp simple criterion for a subclass of starlike functions, Complex Variables, 32(1997), 161-168. (with I. Şerb).
[141] Some new applications of differential subordinations, The Annual Conference of the Romanian Society of Mathematical Sciences , Bucharest, May 1997, Romania, 23-27.
[142] Real star-convex functions, Studia Univ. Babeş-Bolyai, Math. 42, 3(1997), 65-80 (with I. Şerb and Gh. Toader).
[143] Conditions for diffeomorphism in the complex plane, Analysis and Topology, eds. C. Andreian - Cazacu, O. Lehto and Th. M. Rassias, World Scientific Publ. Co., 525-540.
[144] On a majorization-subordination problem for univalent functions, (with St. Ruscheweyh), J. Analysis 6(1998), 43-48.
[145] Sufficient conditions for starlikeness and strongly-starlikeness, The Annual Conference of the Romanian Society of Mathematical Sciences 1998, Cluj-Napoca, 119-123.
[146] On starlike images by Alexander integral II, PU.M.A., 9 (1998), 1-2, 139-147.
[147] Sufficient conditions for starlikeness, Studia Univ. Babeş-Bolyai, Math., 43, 1(1998), 57-62 (with Gh. Oros).
[148] Sufficient conditions for starlikeness II, Studia Univ. Babeş-Bolyai, Math., 43, 2 (1998), 49-53 (with Gh. Oros).
[149] Some starlikeness and convexity conditions for meromorphic functions, Conference on Analysis, Functional Equations, Approximation and Convexity, Cluj-Napoca, Oct. 15-16 (1999), 181-192.
[150] A simple convexity condition for meromorphic functions, Bull. Math. Soc. Sc. Math. Roumanie 42(90), 2(1999), 115-119.
[151] On an integral operator and starlike functions of order alpha, Bull. Math. Soc. Sc. Math. Roumanie 42(90), 3(1999), 253-256 (with Gh. Oros).
[152] On a starlikeness condition for meromorphic functions, Mathematica (Cluj), 41(64), 2(1999), 221-225 (with S.S. Miller and Gh. Oros).
[153] On a certain condition for starlikeness, Rev. Roumaine Math. Pures Appl., 44, 4(1999), 617-622 (with Gh. Oros).
[154] Certain conditions for diffeomorphism in the complex plane - the duality theorem, Scripta Scientiarum Mathematicarum, 1, 2(1999), 292-304.
[155] Some extensions of Aksentiev's univalence condition for meromorphic functions, Complex Variables, 41(2001), 29-38.

ON A CLASS OF CERTAIN ANALYTIC FUNCTIONS OF COMPLEX ORDER

M. K. AOUF, H. E. DARWISH AND A. A. ATTIYA

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. We introduce a class, namely, $H_n(b, M)$ of certain analytic functions. For this class we determine sufficient condition in terms of coefficients, coefficient estimate, maximization theorem concerning the coefficients, and radius problem.

1. **Introduction**

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. We use Ω to denote the class of functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. For a function $f(z)$ in A, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1, 2, \ldots, \}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [11]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to A is in the

1991 *Mathematics Subject Classification.* 30C45.

Key words and phrases. analytic, Salagean operator, complex order.

class $F_n(b, M)$ if and only if

$$\left| \frac{b - 1 + \dfrac{D^{n+1} f(z)}{D^n f(z)}}{b} - M \right| < M, \quad z \in U, \tag{1.5}$$

where $M > \frac{1}{2}$ and $b \neq 0$, complex.

It follows by Kulshrestha [6] that $g(z) \in H_0\ (1, M) = F(1, M)$ if and only if for $z \in U$

$$\frac{zg'(z)}{g(z)} = \frac{1 + w(z)}{1 - mw(z)}, \tag{1.6}$$

where $m = 1 - \frac{1}{M}$ $\left(M > \frac{1}{2}\right)$ and $w(z) \in \Omega$.

One can easily show that $f(z) \in H_n(b, M)$ if and only if there is a function $g(z) \in H_0(1, M) = F(1, M)$ such that

$$D^n f(z) = z \left[\frac{g(z)}{z}\right]^b. \tag{1.7}$$

Thus from (1.6) and (1.7) it follows that $f(z) \in H_n(b, M)$ if and only if for $z \in U$

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + [b(1 + m) - m]w(z)}{1 - mw(z)}, \tag{1.8}$$

where $m = 1 - \frac{1}{M}$ $\left(M > \frac{1}{2}\right)$ and $w(z) \in \Omega$.

By giving specific values to n, b and M, we obtain the following important subclasses studied by various authors in earlier works:

(1) $H_0(b, M) = F(b, M)$ (Nasr and Aouf [7]) and $H_1(b, M) = G(b, M)$ (Nasr and Aouf [8]).

(2) $H_0(\cos\lambda e^{-i\lambda}, M) = F_{\lambda,M}$ and $H_1(\cos\lambda e^{-i\lambda}, M) = G_{\lambda,M}$ $\left(|\lambda| < \frac{\pi}{2}\right)$ (Kulshrestha [4]).

(3) $H_0((1-\alpha)\cos\lambda e^{-i\lambda}, \infty) = S^\lambda(\alpha)$ $\left(|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1\right)$ (Libera [6]) and $H_1((1-\alpha)\cos\lambda e^{-i\lambda}, \infty) = C^\lambda(\alpha)$ $\left(|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1\right)$ (Chichra [3] and Sizuk [14]).

(4) $H_0(b, M) = S(1 - b)$ (Nasr and Aouf [9]) and $H_1(b, M) = C(b)$ (Wiatrowski [15] and Nasr and Aouf [10]).

(5) $H_0((1 - \alpha)\cos\lambda e^{-i\lambda}, M) = F_M(\lambda, \alpha)$ and $H_1((1 - \alpha)\cos\lambda e^{-i\lambda}, M) = G_M(\lambda, \alpha)$ $\left(|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1\right)$ (Aouf [1,2]).

(6) $H_0(1,1) = F(1,1)$ Singh [12] and $H_0(1,M) = F(1,M)$ (Singh and Singh [13]).

From the definitions of the classes $F(b,M)$ and $H_n(b,M)$, we observe that

$$f(z) \in H_n(b,M) \text{ if and only if } D^n f(z) \in F(b,M). \tag{1.9}$$

The purpose of the present paper is to determine sufficient condition in terms of coefficients for function belong to $H_n(b,M)$, coefficient estimate, and maximization of $|a_3 - \mu a_2^2|$ on the class $H_n(b,M)$ for complex value of μ. Further we obtain the radius of disc in which $\operatorname{Re}\left\{\dfrac{D^{n+1}f(z)}{D^n f(z)}\right\} > 0$, wherever $f(z)$ belongs to $H_n(b,M)$.

2. A sufficient condition for a function to be in $H_n(b,M)$

Theorem 1. *Let the function $f(z)$ defined by (1.1) and let*

$$\sum_{k=2}^{\infty}\{(k-1) + |b(1+m) + m(k-1)|\}k^n|a_k| \le |b(1+m)|, \tag{2.1}$$

holds, then $f(z)$ belongs to $H_n(b,M)$, where $m = 1 - \dfrac{1}{M}$ $\left(M > \dfrac{1}{2}\right)$.

Proof. Suppose that the inequality (2.1) holds. Then we have for $z \in U$

$$|D^{n+1}f(z) - D^n f(z)| - |b(1+m)D^n f(z) + m(D^{n+1}f(z) - D^n f(z))|$$

$$= \left|\sum_{k=2}^{\infty} k^n(k-1)a_k z^k\right| - \left|b(1+m)\left\{z + \sum_{k=2}^{\infty} k^n a_k z^k\right\} + m\sum_{k=2}^{\infty} k^n(k-1)a_k z^k\right| \le$$

$$\le \sum_{k=2}^{\infty} k^n(k-1)|a_k|r^k - \left\{|b(1+m)|r - \sum_{k=2}^{\infty}|b(1+m) + m(k-1)|k^n|a_k|r^k\right\} =$$

$$= \sum_{k=2}^{\infty}\{(k-1) + |b(1+m) + m(k-1)|\}k^n|a_k|r^k - |b(1+m)|r.$$

Letting $r \to {}^{-}1$, then we have

$$|D^{n+1}f(z) - D^n f(z)| - |b(1+m)D^n f(z) + m(D^{n+1}f(z) - D^n f(z))| =$$

$$= \sum_{k=2}^{\infty}\{(k-1) + |b(1+m) + m(k-1)|\}k^n|a_k| - |b(1+m)| \le 0, \text{ by (2.1)}.$$

Hence it follows that

$$\left|\frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{b(1+m) + m\left\{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right\}}\right| < 1, \quad z \in U.$$

Letting

$$w(z) = \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{b(1+m) + m\left\{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right\}},$$

then $w(0) = 0$, $w(z)$ is analytic in $|z| < 1$ and $|w(z)| < 1$. Hence we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1 + [b(1+m) - m]w(z)}{1 - mw(z)}$$

which shows that $f(z)$ belongs to $H_n(b, M)$.

3. Coefficient estimate

Theorem 2. *Let the function $f(z)$ defined by (1.1) be in the class $H_n(b, M)$, $z \in U$.*

(a) If $2m(k-1)\text{Re}\,\{b\} > (k-1)^2(1-m) - |b|^2(1+m)$, let

$$N = \left[\frac{2m(k-1)\text{Re}\,\{b\}}{(k-1)^2(1-m) - |b|^2(1+m)}\right], \quad k = 2, 3, \ldots, j-1.$$

Then

$$|a_j| \le \frac{1}{j^n(j-1)!}\prod_{k=2}^{j} |b(1+m) + (k-2)m|, \tag{3.1}$$

for $j = 2, 3, \ldots, N+2$; and

$$|a_j| \le \frac{1}{j^n(j-1)(N+1)!}\prod_{k=2}^{N+3} |b(1+m) + (k-2)m|, \quad j > N+2. \tag{3.2}$$

(b) If $2m(k-1)\text{Re}\,\{b\} \le (k-1)^2(1-m) - |b|^2(1+m)$, then

$$|a_j| \le \frac{(1+m)|b|}{j^n(j-1)}, \text{ for } j \ge 2, \tag{3.3}$$

where $m = 1 - \dfrac{1}{M}$ $\left(M > \dfrac{1}{2}\right)$ and $b \neq 0$, complex.

The inequalities (3.1) and (3.3) are sharp.

Proof. Since $f(z) \in H_n(b, M)$, so from (1.8) we have that

$$\sum_{k=2}^{\infty} k^n(k-1)a_k z^k = \left\{b(1+m)z + \sum_{k=2}^{\infty} k^n[b(1+m) + m(k-1)]a_k z^k\right\} w(z) \tag{3.4}$$

which is equivalent to

$$\sum_{k=2}^{j} k^n(k-1)a_k z^k + \sum_{k=2}^{\infty} d_k z^k =$$

$$= \left\{ b(1+m)z + \sum_{k=2}^{j-1} k^n [b(1+m) + m(k-1)] \right\} a_k z^k w(z),$$

where d_j's are some complex numbers.

Then since $|w(z)| < 1$, we have

$$\left| \sum_{k=2}^{j} k^n (k-1) a_k z^k + \sum_{j=k+1}^{\infty} d_k z^k \right| \leq$$

$$\leq \left| b(1+m)z + \sum_{k=2}^{j-1} k^n [b(1+m) + m(k-1)] a_k z^k \right|. \tag{3.5}$$

Squaring both sides of (3.5) and integrating round $|z| = r < 1$, we get, after taking the limit when $r \to 1$

$$j^{2n}(j-1)^2 |a_j|^2 \leq (1+m)^2 |b|^2 +$$

$$+ \sum_{k=2}^{j-1} k^{2n} \{ |b(1+m) + m(k-1)|^2 - (k-1)^2 \} |a_k|^2. \tag{3.6}$$

Now there may be following two cases:

(a) Let $2m(k-1)\text{Re}\,\{b\} > (k-1)^2(1-m) - (1+m)|b|^2$. Suppose that $j \leq N+2$; then for $j = 2$, (3.7) gives

$$|a_2| \leq \frac{(1+m)|b|}{2^n}$$

which gives (3.1) for $j = 2$. We establish (3.1), by mathematical induction. Suppose (3.1) is valid for $k = 2, 3, \ldots, j-1$. Then it follows from (3.6)

$$j^{2n}(j-1)^2 |a_j|^2 \leq (1+m)^2 |b|^2 +$$

$$+ \sum_{k=2}^{j-1} k^{2n} \{ |b(1+m) + m(k-1)|^2 - (k-1)^2 \} \frac{1}{k^{2n}((k-1)!)^2} \prod_{p=2}^{k} |b(1+m) + (p-2)m|^2 =$$

$$= \frac{1}{((j-1)!)^2} \prod_{k=2}^{j} |b(1+m) + (k-2)m|^2.$$

Thus, we get

$$|a_j| \leq \frac{1}{j^n (j-1)!} \prod_{k=2}^{j} |b(1+m) + (k-2)m|,$$

which completes the proof of (3.1).

Next, we suppose $j > N+2$. Then (3.6) gives

$$j^{2n}(j-1)^2 |a_j|^2 \leq (1+m)^2 |b|^2 +$$

$$+\sum_{k=2}^{N+1} k^{2n}\{|b(1+m)+m(k-1)|^2-(k-1)^2\}|a_k|^2+$$

$$+\sum_{k=N+3}^{j-1} k^{2n}\{|b(1+m)+m(k-1)|^2-(k-1)^2\}|a_k|^2 \leq$$

$$\leq (1+m)^2|b|^2+\sum_{k=2}^{N+2}\{|b(1+m)+m(k-1)|^2-(k-1)^2\}|a_k|^2.$$

On substituting upper estimates for $a_2, a_3, \ldots, a_{N+2}$ obtained above, and simplifying, we obtain (3.2).

(b) Let $2m(k-1)\text{Re }\{b\} \leq (k-1)^2(1-m)-(1+m)|b|^2$, then it follows from (2.7)

$$j^{2n}(j-1)^2|a_j|^2 \leq (1+m)^2|b|^2, \quad (j \geq 2)$$

which proves (3.3).

The bounds in (3.1) are sharp for the function $f(z)$ given by

$$D^n f(z) = \begin{cases} \dfrac{z}{(1-mz)^{\frac{b(1+m)}{m}}}, & m \neq 0, \\ z\exp(bz), & m = 0. \end{cases} \tag{3.7}$$

The bounds in (3.3) are sharp for the function $f_k(z)$ given by

$$D^n f_k(z) = \begin{cases} \dfrac{z}{(1-mz^{k-1})\dfrac{b(1+m)}{m(k-1)}}, & m \neq 0, \\ \\ z\exp\left(\dfrac{b}{k-1}z^{k-1}\right), & m = 0. \end{cases} \tag{3.8}$$

4. Maximization of $|a_3 - \mu a_2^2|$

We shall need in our discussion the following lemma:

Lemma 1. [5] *Let $w(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega$, if μ is any complex number, then*

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\} \tag{4.1}$$

for any complex μ. Equality in (4.1) may be attained with the functions $w(z) = z^2$ and $w(z) = z$ for $|\mu| < 1$ and $|\mu| \geq 1$, respectively.

Theorem 3. *If a function $f(z)$ defined by (1.1) is in the class $H_n(b, M)$ and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m)|}{2 \cdot 3^n} \max\{1, |d|\}, \tag{4.2}$$

where

$$d = \frac{2 \cdot 3^n \mu b(1+m)}{2^{2n}} - [b(1+m) + m]. \tag{4.3}$$

The result is sharp.

Proof. Since $f(z) \in H_n(b, m)$, we have

$$w(z) = \frac{D^{n+1} f(z) - D^n f(z)}{[b(1+m) - m] D^n f(z) + m D^{n+1} f(z)} =$$

$$= \frac{\sum_{k=2}^{\infty} k^n (k-1) a_k z^{k-1}}{b(1+m) + \sum_{k=2}^{\infty} k^n [b(1+m) + m(k-1)] a_k z^{k-1}} =$$

$$= \frac{\sum_{k=2}^{\infty} k^n (k-1) a_k z^{k-1}}{b(1+m)} \left[1 + \frac{\sum_{k=2}^{\infty} k^n [b(1+m) + m(k-1)] a_k z^{k-1}}{b(1+m)} \right]^{-1}. \tag{4.4}$$

Now compare the coefficients of z and z^2 on both sides of (4.4). We thus obtain

$$a_2 = \frac{b(1+m)}{2^n} c_1, \tag{4.5}$$

and

$$a_3 = \frac{b(1+m)}{2 \cdot 3^n} \{c_2 + [b(1+m) + m] c_1^2\}. \tag{4.6}$$

Hence

$$a_3 - \mu a_2^2 = \frac{b(1+m)}{2 \cdot 3^n} [c_2 - d c_1^2], \tag{4.7}$$

where

$$d = \frac{2 \cdot 3^n \mu b(1+m)}{2^{2n}} [b(1+m) + m].$$

Taking modulus both sides in (4.7), we have

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m)|}{2 \cdot 3^n} |c_2 - d c_1^2|. \tag{4.8}$$

Using Lemma 1 in (4.8), we have

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m)|}{2 \cdot 3^n} \max\{1, |d|\}.$$

Finally, the assertion (4.2) of Theorem 3 is sharp in view of the fact that the assertion (4.1) of Lemma 1 is sharp.

5. Radius Theorem

The following theorem may be obtained with the help of (1.9) and Theorem 3 of Nasr and Aouf [7].

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $H_n(b, M)$. Then*

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > 0 \text{ for } |z| < r_n,$$

where

$$r_n = 2\left\{|b|(1+m) + \left[|b|^2(1+m)^2 - 4\left\{\operatorname{Re}\,(b)\left(\frac{1+m}{m}\right) - 1\right\} - 1\right]^{\frac{1}{2}}\right\}^{-1} \tag{5.1}$$

such that

$$|b|^2(1+m)^2 \geq 4\left\{\operatorname{Re}\,(b)\left(\frac{1+m}{m}\right) - 1\right\}.$$

The result is sharp for the function $f_0(t)$, where

$$D^n f_0(z) = z(1-mz)^{-b\left(\frac{1+m}{m}\right)} \tag{5.2}$$

and

$$t = \frac{r\left[r - m\left(\frac{\bar{b}}{b}\right)^{\frac{1}{2}}\right]}{m\left[1 - mr\left(\frac{\bar{b}}{b}\right)^{\frac{1}{2}}\right]}.$$

Remarks on Theorem 4.

(i) Putting $n = 0$, we get the sharp radius of starlikeness of the class $F(b, M)$ studied by Nasr and Aouf [7].

(ii) Putting $n = 1$, we get the sharp radius of convexity of the class $G(b, M)$ which is investigated by Nasr and Aouf [8].

References

[1] M.K. Aouf, *Bounded p-valent Robertson functions of order α*, Indian P. Pure Appl. Math. 16(1985), 775-790.

[2] M.K. Aouf, *Bounded spiral-like functions with fixed second coefficient*, Internat. J. Math. Sci. 12(1989), no.1, 113-118.

[3] P.N. Chichra, *Regular functions $f(z)$ for which $zf'(z)$ is α-spirallike*, Proc. Amer. Math. Soc. 49(1975), 151-160.

[4] P.K. Kulshrestha, *Bounded Robertson functions*, Rend. Mat. (7)9(1976), 137-150.

[5] F.R. Keogh and E.P. Merkes, *A coefficient inequalitu for certain classes of analytic functions*, Proc. Amer. Math. Soc. 20(1969).
[6] R.J. Libera, *Univalent α-spiral functions*, Canad. J. Math. 19(1967), 449-456.
[7] M.A. Nasr and M.K. Aouf, *Bounded starlike functions of complex order*, Proc. Indian Acad. Sci. (Math. Sci.) 92(1983), 97-102.
[8] M.A. Nasr and M.K. Aouf, *Bounded convex functions of complex order*, Bull. Fac. Sci. Mansoura Univ. 10(1983), 513-527.
[9] M.A. Nasr and M.K. Aouf, *Starlike functions of complex order*, J. Natur. Sci. Math. 25(1985), 1-12.
[10] M.A. Nasr and M.K. Aouf, *On convex functions of complex order*, Bull. Fac. Sci. Mansoura Univ. 10(1983), 513-527.
[11] G.S. Salagean, *Subclasses of univalent functions*, Lecture Notes in Math. 1013, pp.362-372, Springer-Verlag, Berlin, Heidelberg and New York.
[12] R. Singh, *On a class of starlike functions*, J. Math. Soc. 32(1968), 208-213.
[13] R. Singh and V. Singh, *On a class of bounded starlike functions*, Indian J. Pure Appl. Math. 5(1974), 733-754.
[14] P.I. Sizuk, *Regular functions $f(z)$ for which $zf'(z)$ is α-spirallike*, Proc. Amer. Math. Soc. 49(1975), 151-160.
[15] P. Wiatrowski, *The coefficients of a certain family of holomorphic functions*, Zeszyty Nauk. Univ. Lodzk. Nauki. Math. Przyrod 39(1971), 75-85.

Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt
E-mail address: `sinfac@mum.mans.eun.eg`

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 2, June 2001

STARLIKE, CONVEX AND ALPHA-CONVEX FUNCTIONS OF HYPERBOLIC COMPLEX AND OF DUAL COMPLEX VARIABLE

SORIN G. GAL

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

1. Introduction

The study of functions of hyperbolic complex and of dual complex variable was done in [11-12] and continued in the very recent papers [1-6].

In this paper we begin the study of a geometric theory for such of functions, in the general setting of nonanalytic functions.

It is known that for the functions of usual complex variable, the geometric theory is based on the identification of the field of usual complex numbers with the two-dimensional Euclidean plane. But according to the Cayley-Klein scheme, there are nine plane geometries, corresponding to all possible combinations which can be formed for the three kinds of measures of angles and the three kinds of measure of distances (see [13, p. 195-219], [14, p. 214-288]):

1) Elliptic geometry, Euclidean geometry, Hyperbolic geometry, based on the same elliptic (usual) measure of angles but having three different kinds of measures for distances, i.e. elliptic measure, parabolic measures and hyperbolic measure, respectively.

The analytic model for these geometries are the usual complex numbers.

2) Co-Euclidean geometry, Galilean geometry, Co-Minkowskian geometry, based on the same parabolic measure of angles but having the three different kinds of measures for distances as in the case 1, respectively.

The analytic model for these geometries are the dual complex numbers.

3) Cohyperbolic geometry, Minkowskian geometry, doubly hyperbolic geometry, based on the same hyperbolic measure of angles but again having the three different kinds of measures of distances, as above, respectively.

The analytic model for these last three geometries are the hyperbolic complex numbers.

A geometric theory for (analytic) functions of usual complex variable, based on the hyperbolic geometry was done in [7].

In the next sections we will consider a few geometrical aspects for (nonanalytic) functions of hyperbolic complex and of dual complex variables, based on the Minkowskian geometry and on the Galilean geometry, respectively.

Besides the fact that in this way we introduce several plane transformations with new geometrical properties, our method permits an unitary treatment for the geometric theories of functions of usual complex, of hyperbolic complex and of dual complex variables.

Section 2 contains some preliminaries facts.

In the Sections 3 and 4 we introduce and study the classes of starlike, convex and alpha-convex functions of hyperbolic complex and of dual complex variable, respectively.

The methods were suggested by the classical ones in [8-10].

2. **Preliminaries**

First let us recall some known facts about the complex-type numbers (see e.g. [6], [13-14]). It is known that excepting an isomorphism, three kinds of complex numbers are important:

(i) $\mathbb{C}_q$, $q \notin \mathbb{R}$, $q^2 = -1$, called the system of usual complex numbers,

(ii) $\mathbb{C}_q$, $q \notin \mathbb{R}$, $q^2 = 0$, called the system of complex numbers,

(iii) $\mathbb{C}_q$, $q \notin \mathbb{R}$, $q^2 = +1$, called the system of hyperbolic complex numbers, where $\mathbb{C}_q = \{z = x + qy;\ x, y \in \mathbb{R}\}$.

For simplicity, let us denote $q = i$ if $q^2 = -1$, $q = d$ if $q^2 = 0$ and $q = h$ is $q^2 = +1$.

If $q = i$, then $\mathbb{C}_q$ is a field, if $q = d$ then $\mathbb{C}_q$ is a ring with the set of divisors of zero given by $\mathbb{Z}_q = \{z = x + qy;\ x = 0,\ y \in \mathbb{R}\}$ and if $q = h$ then $\mathbb{C}_q$ is a ring with the zero divisors $\mathbb{Z}_q = \{z = x + qy; x, y \in \mathbb{R},\ |x| = |y|\}$. Obviously $\mathbb{Z}_q = \{z \in \mathbb{C}_q;\ \rho_q(z) = 0\}$, where $\rho_q(z)$ is defined below, for all $q \in \{i, d, h\}$.

For $z = x + qy \in \mathbb{C}_q$, let us denote $\overline{z} = x - qy$, (so $z\overline{z} = x^2 - q^2y^2 \in \mathbb{R}$), $\rho_q(z) = \sqrt{|z\overline{z}|}$, for $r > 0$ let us denote $U_r^{(q)} = \{z \in \mathbb{C}_q;\ \rho_q(z) < r\}$, $C_r^{(q)} = \{z \in \mathbb{C}_q;\ \rho_q(z) = r\}$, for all $q \in \{i, d, h\}$.

In the Euclidean geometry, $C_r^{(i)}$ is a circle of radius r and of center $(0, 0)$, $C_r^{(d)}$ represents the straigth lines $x = -r$ and $x = +r$, and $C_r^{(h)}$ represents the hyperbolas $x^2 - y^2 = -r^2$, $x^2 - y^2 = r^2$.

The polar coordinates and the exponentials are defined as follows. Let $z = x + qy \in \mathbb{C}_q$. For $q = i$ they are well-known.

For $q = d$ we have $|z|_d = x$, $\varphi = \arg_d z = \dfrac{y}{x}$, $z \notin \mathbb{Z}_d$, and $z = |z|_d(1 + d\varphi) = |z|_d e_d^{d\varphi}$, where $e_d^z = e^x e_d^{dy} = e^x(1 + dy)$.

For $q = h$ we have $e_h^{hy} = \cosh(y) + h\sinh(y)$, $e_h^z = e^x e_h^{hy} = e^x \cosh y + he^x \sinh y$, $|z|_h = (\mathrm{sgn}x)\sqrt{x^2 - y^2}$, $\varphi = \arg_h z = \mathrm{arcth}\dfrac{y}{x}$, $z = |z|_h e_h^{h\varphi}$, for $x^2 - y^2 > 0$, and $|z|_h = (\mathrm{sgn}y)\sqrt{y^2 - x^2}$, $\varphi = \mathrm{arcth}\dfrac{x}{y}$, $z = q|z|_h e_h^{h\varphi}$, for $y^2 - x^2 > 0$. In the first case z is called of first kind (1-kind) and in the other case it is called of second kind (2-kind).

Note that $\mathbb{Z}_q = \{z \in \mathbb{C}_q;\ |z|_q = 0\}$, for all $q \in \{i, d, h\}$.

Let $q \in \{i, d, h\}$ and $\gamma : I \to \mathbb{C}_q$, $\gamma(t) = x(t) + qy(t)$, $t \in I$ (bounded or unbounded interval) be a differentiable path in $\mathbb{C}_q$, such that $\gamma'(t) \notin \mathbb{Z}_q$, $t \in I$. Then $\arg_q[\gamma'(t)]$ represents the "Q"-angle with the positive sense of Ox-axis, of the "Q"-tangent at the path γ in the point $\gamma(t)$, where by convention, everywhere in the paper "Q" means the words Euclidean, Galilean and Minkowskian, for $q = i, d$ and h, respectively.

Let us denote $D_h(f)(z) = z\dfrac{\partial f}{\partial z} - \overline{z}\dfrac{\partial f}{\partial \overline{z}}$, $\mathcal{D}_h(f)(z) = z\dfrac{\partial f}{\partial z} + \overline{z}\dfrac{\partial f}{\partial \overline{z}}$, $f = U + hV$, $z = x + hy$, where (see [7])

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right] + \frac{h}{2}\left[\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right],$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right] + \frac{h}{2}\left[\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right],$$

i.e.

$$D_h(f)(z) = x\frac{\partial V}{\partial y} + y\frac{\partial V}{\partial x} + h\left[x\frac{\partial U}{\partial y} + y\frac{\partial U}{\partial x}\right],$$

$$\mathcal{D}_h(f)(z) = x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} + h\left[x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right].$$

It is easy to verify the following formulas

$$D_h(\overline{f}) = -\overline{D_h(f)}, \quad \mathcal{D}_h(\overline{f}) = \overline{\mathcal{D}_h(f)}, \quad D_h(\text{Re } f) = h\text{Im } [D_h(f)],$$

$$\mathcal{D}_h[\text{Re } f] = \text{Re } \mathcal{D}_h(f), \quad D_h(\text{Im } h) = h\text{Re } D_h(f), \quad \mathcal{D}_h[\text{Im } f] = \text{Im } \mathcal{D}_h(f),$$

$$\frac{\partial f}{\partial \varphi} = hD_h(f), \quad \frac{\partial f}{\partial |z|_h} = \frac{1}{|z|_h}\mathcal{D}_h(f), \quad D_h(|f|_h) = h|f|_h\text{Im } \frac{D_h(f)}{f},$$

$$\mathcal{D}_h(|f|_h) = |f|_h\text{Re } \frac{\mathcal{D}_h(f)}{f}, \quad D_h(\arg_h f) = h\text{Re } \frac{D_h(f)}{f}, \quad \mathcal{D}_h(\arg_h f) = \text{Im } \frac{\mathcal{D}_h(f)}{f},$$

which immediately imply

$$\frac{\partial |f|_h}{\partial \varphi} = |f|_h\text{Im } \frac{D_h(f)}{f}, \quad \frac{\partial |f|_h}{\partial |z|_h} = \frac{|f|_h}{|z|_h}\text{Re } \frac{\mathcal{D}_h(f)}{f}, \tag{1}$$

$$\frac{\partial \arg_h f}{\partial \varphi} = \text{Re } \frac{D_h(f)}{f}, \quad \frac{\partial \arg_h f}{\partial |z|_h} = \frac{1}{|z|_h}\text{Im } \frac{\mathcal{D}_h(f)}{f}, \tag{2}$$

where in all the above formulas $\varphi = \arg_h z$, $|z|_h \neq 0$, $|f(z)|_h \neq 0$.

Also, if $h \in C^1(\mathbb{R})$, then $D_h(h(z\overline{z})) = 0$ and $\mathcal{D}_h[h(\arg_h z)] = 0$.

Note that these formulas are valid for all the cases when z and $f(z)$ are of first or of second kind. On the other hand, in comparison with the case $q = i$ in [8], among the above formulas only three differ (by sign) from those in [8], namely those which give formulas for $D_h(\text{Im } f)$, $D_h(\arg_h f)$ and $\frac{\partial |f|_h}{\partial \varphi}$.

3. **Starlike functions**

Let $f : U_1^{(q)} \to \mathbb{C}_q$ be of C^1-class on $U_1^{(q)}$, $f = U + qV$, where q is any between i, d and h.

Definition 3.1. We say that f is Symmetrically Uniformly (shortly (SU)) - "Q" starlike function on $U_1^{(q)}$, if f is univalent on $U_1^{(q)} \setminus \mathbb{Z}_q$, $f(z) \in \mathbb{Z}_q$ iff $z \in \mathbb{Z}_q$ and moreover, for any fixed $\rho \in (-1, 1) \setminus \{0\}$, we have

$$\frac{\partial}{\partial \arg_q z}(\arg_q f(z)) > 0, \text{ for all } |z|_q = \rho. \tag{3}$$

The univalency of f is required only on $U_1^{(q)} \setminus \mathbb{Z}_q$ (and not on the whole $U_1^{(q)}$), because the geometric condition in (3) holds only on $U_1^{(q)} \setminus \mathbb{Z}_q$.

Remarks. 1) If $q = i$, then we obtain the classical conditions in [8]: f is (SU)-Euclidean starlike, if f is univalent on the whole $U_1^{(i)}$, $f(0) = 0$ and

$$\text{Re}\ \frac{D_i f}{f} > 0,\ \text{for all}\ z \in U_1^{(i)} \setminus \{0\}, \tag{4}$$

where $D_i f = z\dfrac{\partial f}{\partial z} - \overline{z}\dfrac{\partial f}{\partial \overline{z}}$ and $\dfrac{\partial f}{\partial z}, \dfrac{\partial f}{\partial \overline{z}}$, $z = x + iy$, are given in [8].

¿From [8] it follows that (4) implies the starlikeness of all the sets $f(U_r^{(i)})$, $0 < r < 1$, which suggested us the denomination of "Symmetrically Uniformly" for f.

In fact it is well-known that (see e.g. [10, Theorem 3.1]) if f is analytic and $f'(0) = 0$, then f is (SU)-starlike if and only if f is starlike (in the classical sense).

Since simple calculations show that $D_i(f) = x\dfrac{\partial V}{\partial y} - y\dfrac{\partial V}{\partial x} + i\left(y\dfrac{\partial U}{\partial x} - x\dfrac{\partial U}{\partial y}\right)$ and

$$\text{Re}\ \frac{D_i(f)}{f} = \frac{1}{U^2 + V^2}\left\{x\left(U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y}\right) + y\left(V\frac{\partial U}{\partial x} - U\frac{\partial V}{\partial x}\right)\right\},$$

it follows that f generates the injective vectorial transform defined on $U_1^{(i)}$ (in fact on the Euclidean image of $U_1(i)$), $F(x, y) = (U(x, y), V(x, y))$, with $U(0, 0) = V(0, 0) = 0$ and satisfying

$$x\left[U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y}\right] + y\left[V\frac{\partial U}{\partial x} - U\frac{\partial V}{\partial x}\right] > 0,\ \forall\ x^2 + y^2 \le 1,\ x \ne 0,\ y \ne 0 \tag{5}$$

(since obviously (4) is equivalent with (5)).

2) Let $q = d$. First, in this case the condition "$f(z) \in \mathbb{Z}_d$ iff $z \in \mathbb{Z}_d$", means that "$U(x, y) = 0$ iff $x = 0$". For $z \in U_1^{(d)} \setminus \mathbb{Z}_d$ we have $z = |z|_d(1 + d\varphi)$, $\varphi = \arg_d z \in \mathbb{R}$, $x = |z|_d = r \ne 0$, $y = r\varphi$ (r fixed in $(-1, 1) \setminus \{0\}$), and (3) becomes

$$\frac{\partial}{\partial \varphi}(\arg_d f) = \frac{\partial}{\partial \varphi}\left(\frac{V}{U}\right) = \frac{UV'_\varphi - VU'_\varphi}{U^2} > 0,$$

where $V'_\varphi = \dfrac{\partial V}{\partial x} \cdot \dfrac{\partial x}{\partial \varphi} + \dfrac{\partial V}{\partial y} \cdot \dfrac{\partial y}{\partial \varphi} = x\dfrac{\partial V}{\partial y}$, $U'_\varphi = x\dfrac{\partial U}{\partial y}$.

As an immediate conclusion it follows that a (SU)-Galilean starlike function f generates the injective vectorial transform on $U_1^{(d)} \setminus \mathbb{Z}_d$, $F(x, y) = (U(x, y), V(x, y))$, with $U(x, y) = 0$ iff $x = 0$ and satisfying

$$x\left(U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y}\right) > 0,\ \forall\ x \in (-1, 1) \setminus \{0\},\ y \in \mathbb{R}. \tag{6}$$

Note that (6) is equivalent with the inequality

$$x\frac{\partial}{\partial y}\left(\frac{V}{U}\right) > 0,\ \forall\ x \in (-1, 1) \setminus \{0\},\ y \in \mathbb{R}.$$

3) Let $q = h$. The condition "$f(z) \in \mathbb{Z}_h$ iff $z \in \mathbb{Z}_h$" is equivalent with "$|U(x,y)| = |V(x,y)|$ iff $|x| = |y|$". Let $z \notin \mathbb{Z}_h$, then $\arg_h f(z) = \text{arcth}\dfrac{V}{U}$, for $U^2 - V^2 > 0$ and $\arg_h f(z) = \dfrac{U}{V}$, for $V^2 - U^2 > 0$. Denoting $\arg_h z = \varphi \in \mathbb{R}$, $|z|_h = r \in (-1,1) \setminus \{0\}$ and (3) becomes

$$\left[\text{arcth}\left(\frac{V}{U}\right)\right]'_\varphi = \frac{\left(\dfrac{V}{U}\right)'_\varphi}{1 - \left(\dfrac{V}{U}\right)^2} = \frac{UV'_\varphi - VU'_\varphi}{U^2 - V^2} > 0, \text{ if } U^2 - V^2 > 0$$

and

$$\left[\text{arcth}\left(\frac{U}{V}\right)\right]'_\varphi = \frac{\left(\dfrac{U}{V}\right)'_\varphi}{1 - \left(\dfrac{U}{V}\right)^2} = \frac{UV'_\varphi - VU'_\varphi}{U^2 - V^2} > 0, \text{ if } V^2 - U^2 > 0.$$

Now, taking into account that for fixed r and independent of the fact that z is of the first or second kind, we have $\dfrac{\partial x}{\partial \varphi} = y$ and $\dfrac{\partial y}{\partial \varphi} = x$, by simple calculations it follows that a (SU)-Minkowskian starlike function f, generates the injective vectorial transform on $U_1^{(h)} \setminus \mathbb{Z}_h$, $F(x,y) = (U(x,y), V(x,y))$ with $|U(x,y)| = |V(x,y)|$ iff $|x| = |y|$, satisfying the differential inequality

$$\frac{1}{U^2 - V^2}\left\{x\left[U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y}\right] - y\left[V\frac{\partial U}{\partial x} - U\frac{\partial V}{\partial x}\right]\right\} > 0, \ \forall\, |x^2 - y^2| < 1, \ |x| \neq |y|. \tag{7}$$

On the other hand, taking into account the relations satisfied by $D_h(f)(z)$ in Section 2, we easily obtain that (7) (and therefore (3)) is equivalent with

$$\text{Re}\,\frac{D_h(f)(z)}{f(z)} > 0, \text{ for all } z \in U_1^{(h)} \setminus \mathbb{Z}_h. \tag{8}$$

4) It is immediate that by the conditions in Definition 3.1, f has in addition the following property of univalency: if $z_1 \neq z_2$, $z_1 \in \mathbb{Z}_q$, $z_2 \in U_1^{(q)} \setminus \mathbb{Z}_q$, then $f(z_1) \neq f(z_2)$.

5) The differential inequalities (5), (6), (7), suggest us that each kind of starlikeness in Definition 3.1 is completely independent in respect with the other two, as can be seen in the following simple examples.

Note that in all these examples, U and V are of C^1-class on the whole $\mathbb{R}^2$.

Example 1. Let $U(x,y) = x$, $V(x,y) = x^{100}e^y$. The function $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$, is (SU)-Galilean starlike in $U_1^{(d)}$, since it is univalent on $U_1^{(d)} \setminus \mathbb{Z}_d$, $U(x,y) = 0$ iff $x = 0$, and (6) is satisfied. But even if $f(z) = U(x,y) + iV(x,y)$, $z = x + iy$, satisfies $f(0) = 0$, however f cannot be (SU)-Euclidean starlike, because (5) is not satisfied in any $U_r^{(i)}$, $r \in (0,1]$, and f is not univalent on the whole $U_1^{(i)}$.

Also, $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, cannot be (SU)-Minkowskian starlike in $U_1^{(h)}$, firstly because it is not satisfied the condition $|U(x,y)| = |V(x,y)|$ iff $|x| = |y|$, secondly (7) is not satisfied, and thirdly f is not univalent on $U_1^{(h)} \setminus \mathbb{Z}_h$.

Example 2. Let $U(x,y) = x + \frac{1}{2}(x^2 - y^2)$, $V(x,y) = y - xy$. By [8], $f(z) = U(x,y) + iV(x,y) = z + \frac{1}{2}\overline{z}^2$, $z = x + iy$, is (SU)-Euclidean starlike in $U_1^{(i)}$. But $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$, cannot be (SU)-Galilean starlike in $U_1^{(d)}$ (for example, (6) does not hold) and $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, cannot be (SU)-Minkowskian starlike in $U_1^{(d)}$ (for example, (7) does not hold).

Example 3. Let $U(x,y) = xe^{x^2}$, $V(x,y) = ye^{y^2}$. The vectorial function $F(x,y) = (U(x,y), V(x,y))$ is injective on the whole $\mathbb{R}^2$. Let $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$. Then f is (SU)-Galilean starlike on $U_1^{(d)}$, because $U(x,y) = 0$ iff $x = 0$, and (6) becomes

$$x^2 e^{x^2}(1 + 2y^2)e^{y^2} > 0, \text{ for all } x \neq 0, \ y \in \mathbb{R}.$$

Let us denote $g(t) = te^{t^2}$. Since $g'(t) = e^{t^2}(1+2t^2) > 0$, g is strictly increasing on $\mathbb{R}$, and as consequence we obtain $|U(x,y)| = |V(x,y)|$ iff $|x|e^{|x|^2} = |y|e^{|y|^2}$ iff $g(|x|) = g(|y|)$ iff $|x| = |y|$.

The function $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, also is (SU)-Minkowskian starlike on $U_1^{(h)}$, because (7) becomes

$$\frac{e^{x^2} e^{y^2}(x^2 - y^2)}{H(x^2) - H(y^2)} > 0, \text{ for all } x^2 - y^2 \neq 0,$$

taking into account that $H(t) = te^{2t}$ is strictly increasing on $\mathbb{R}_+$.

Now, let us denote $f(z) = U(x,y) + iV(x,y)$, $z = x + iy$. We have $f(0) = 0$ and (5) becomes

$$e^{x^2} e^{y^2}[x^2 + y^2 + 4x^2y^2] > 0, \text{ for all } x \neq 0, \ y \neq 0,$$

which means that f is (SU)-Euclidean starlike too (on $U_1^{(i)}$).

Remark. Let $q = d$ or h. We will say that a region $G \subset \mathbb{C}_q$ is (SU)-"Q" starlike if there exists $f : U_1^{(q)} \to \mathbb{C}_q$ as in Definition 3.1, such that $G = f[U_1^{(q)}]$. Then it would be of interest to give internal geometric characterizations (in Euclidean language) of the (SU)-"Q" starlike regions.

In the following we will obtain some sufficient conditions for (SU)-"Q" starlikeness. Thus, because $U_1^{(d)}$ is an usual convex domain, combining [6, Corollary 3.2] with Definition 3.1 and relation (6), we obtain

Theorem 3.1. *Let $f : U_1^{(d)} \to \mathbb{C}_d$, $f(z) = U(x, y) + dV(x, y)$, $z = x + dy$, be of C^1-class. If f satisfies the conditions*

(i) $U(x, y) = 0$ iff $x = 0$,

(ii) $xU\dfrac{\partial V}{\partial y} > 0$ on $U_1^{(d)} \setminus \mathbb{Z}_d$,

(iii) $\dfrac{\partial V}{\partial y} \neq 0$, $\dfrac{\partial U}{\partial x} > 0$, $\dfrac{\partial U}{\partial y} = 0$ on $U_1^{(d)}$ (conditions of univalency),

then f is (SU)-Galilean starlike on $U_1^{(d)}$.

An example of f satisfying Theorem 3.1 is for $U(x, y) = x$, $V(x, y) = (x + 1)^{100}e^y$.

Note that this f is univalent on the whole $U_1^{(d)}$.

Another example is $f(z) = \dfrac{z}{(1 + z)^2}$, $z = x + dy$, which can be written in the form $f = U + dV$, with $U(x, y) = \dfrac{x}{(1 + x)^2}$, $V(x, y) = \dfrac{y(1 - x)}{(1 + x)^3}$.

Now, as in the case $q = i$ in [8], it is of interest to see how the geometric conditions together with the local univalency (imposed by using the Jacobian) could imply the (global) univalency, in the cases $q = d$ and $q = h$ too.

The ideas of proof of Theorem 1 in [8] can be summarized by two properties which must to be checked:

$$f \text{ is univalent on } C_r^{(q)}, \text{ for any fixed } r \in (0, 1), \tag{9}$$

$$f(C_{r_1}^{(q)}) \cap f(C_{r_2}^{(q)}) = \emptyset, \text{ for any } r_1, r_2 \in (0, 1),\ r_1 \neq r_2. \tag{10}$$

But the case $q = i$ is essentially different from the cases $q = d$ and $q = h$, because while for $q = i$, $f(C_r^{(i)})$, $r \in (0, 1)$, are Jordan curves, in the cases $q = d$ and $q = h$ (because of the zero divisors) they are not anymore, which will require additional conditions on f, as can be seen in the following results.

Theorem 3.2. *Let $f : U_1^{(d)} \to \mathbb{C}_d$, $f(z) = U(x, y) + dV(x, y)$, $z = x + dy$, be of C^1-class. If f satisfies the conditions:*

(i) $|f(x)|_d = 0$ iff $|z|_d = 0$,

(ii) $J(f)(z) > 0$, *for all* $z \in U_1^{(d)} \setminus \mathbb{Z}_d$, *(here* $J(f)(z)$ *denotes the Jacobian of* f*),*

(iii) $x \dfrac{\partial}{\partial y}\left(\dfrac{V}{U}\right) > 0$, *for all* $x \in (-1,1) \setminus \{0\}$, $y \in \mathbb{R}$,

(iv) Denoting $L_-(x) = \lim\limits_{y \to -\infty} \arg_d f(z)$, $L_+(x) = \lim\limits_{y \to +\infty} \arg_d f(z)$, $\arg_d f(z) = \dfrac{V(x,y)}{U(x,y)}$, $z = x + dy \in U_1^{(d)} \setminus \mathbb{Z}_d$ *(by (iii),* $L_-(x)$, $L_+(x)$ *exist finite or infinite),*

$$I(x) = (L_-(x), L_+(x)) \text{ if } x > 0, \quad I(x) = (L_+(x), L_-(x)) \text{ if } x < 0,$$

and supposing

$$I(\alpha) \cap I(\beta) = \emptyset, \text{ for all } \alpha \in (0,1),\ \beta \in (-1,0), \bigcap_{x \in (0,1)} I(x) \neq \emptyset, \bigcap_{x \in (-1,0)} I(x) \neq \emptyset, \tag{11}$$

then f *is* (SU)*-Galilean starlike on* $U_1^{(d)}$.

Proof. We have to prove that f is univalent on $U_1^{(d)} \setminus \mathbb{Z}_d$. In this sense we will show that for $q = d$, (9) and (10) hold.

For any $r \in (0,1)$ we have $C_r^{(d)} = C_r^{(d^+)} \cup C_r^{(d^-)}$, $C_r^{(d^+)} \cap C_r^{(d^-)} = \emptyset$, where

$$C_r^{(d^+)} = \{z = x + dy;\ x = r\}, \quad C_r^{(d^-)} = \{z = x + dy;\ x = -r\}.$$

Note that $C_r^{(d^+)} \cap \mathbb{Z}_d = \emptyset$, $C_r^{(d^-)} \cap \mathbb{Z}_d = \emptyset$ and that by (i) it follows that $f(C_r^{(d^+)}) \cap \mathbb{Z}_d = \emptyset$, $f(C_r^{(d^-)}) \cap \mathbb{Z}_d = \emptyset$.

In order to prove (9), let $z_1, z_2 \in C_r^{(d)}$, $z_1 \neq z_2$. $r \in (0,1)$ be fixed. If $|z_1|_d = -|z_2|_d$, then by (11) it follows $\arg_d f(z_1) \neq \arg_d f(z_2)$, i.e. $f(z_1) \neq f(z_2)$. So let $|z_1|_d = |z_2|_d$. We have two possibilities:

a) $|z_1|_d = |z_2|_d = r$;

b) $|z_1|_d = |z_2|_d = -r$.

In both cases $\varphi_1 = \arg_d z_1 \neq \arg_d z_2 = \varphi_2$ and by (iii) we get

$$\frac{\partial}{\partial \varphi}[\arg_d f(z)] > 0,\ \varphi = \arg_d z, \text{ i.e. } \arg_d f(z_1) \neq \arg_d f(z_2),$$

which proves (9).

Now, let $r_1, r_2 \in (0,1)$, $r_1 \neq r_2$. We will prove that

$$f(C_{r_1}^{(d^-)}) \cap f(C_{r_2}^{(d^+)}) = \emptyset, \quad f(C_{r_1}^{(d^+)}) \cap f(C_{r_2}^{(d^-)}) = \emptyset \tag{12}$$

and

$$f(C_{r_1}^{(d^+)}) \cap f(C_{r_2}^{(d^+)}) = \emptyset, \quad f(C_{r_1}^{(d^-)}) \cap f(C_{r_2}^{(d^-)}) = \emptyset, \tag{13}$$

which obviously will imply (10).

Indeed, (12) is immediate by (11). Let $\theta \in \bigcap_{x\in(0,1)} I(x)$ be fixed.

For any $\rho \in (0,1)$, by (9) it follows that the system

$$\arg_d f(z) = \theta, \quad |z|_d = \rho \tag{14}$$

yields a unique point $z = \rho e_d^{d\varphi}$, $\varphi = \varphi(\rho)$. Differentiating in respect with ρ, we obtain

$$\left[\frac{\partial}{\partial x}\left(\frac{V}{U}\right)\right](\rho, \rho\varphi(\rho)) + [\rho\varphi(\rho)]' \left[\frac{\partial}{\partial y}\left(\frac{V}{U}\right)\right](\rho, \rho\varphi(\rho)) = 0. \tag{15}$$

On the other hand, for the values of z in (14), denoting $R(\rho) = |f(z)|_d = U(\rho, \rho\varphi(\rho))$, we obtain

$$R'(\rho) = \frac{\partial U}{\partial x}(\rho, \rho\varphi(\rho)) + (\rho\varphi(\rho))' \frac{\partial U}{\partial y}(\rho, \rho\varphi(\rho)). \tag{16}$$

Eliminating $(\rho\varphi(\rho))'$ between (15) and (16) and taking into account (i), (ii) and (iii), we get

$$R'(\rho) = \frac{J(f)(\rho, \rho\varphi(\rho))}{\left[U \frac{\partial}{\partial y}\left(\frac{V}{U}\right)\right](\rho, \rho\varphi(\rho))} \neq 0, \text{ for all } \rho \in (0,1),$$

i.e. $R'(\rho)$ keeps the same sign on (0,1), which immediately implies that $f(C_{r_1}^{(d^+)}) \cap f(C_{r_2}^{(d^+)}) = \emptyset$.

Now, let $\theta \in \bigcap_{x\in(-1,0)} I(x)$. For any $\rho \in (-1,0)$, reasoning as above, we obtain that $f(C_{r_1}^{(d^-)}) \cap f(C_{r_2}^{(d^-)}) = \emptyset$, which proves (13) and therefore the theorem.

Remarks. 1) From the proof we can see how the geometric condition (iii), together with the condition of local univalency in (ii) imply the global univalency on $U_1^{(d)} \setminus \mathbb{Z}_d$. In comparison with Theorem 1 in [8], because of the zero divisors $\mathbb{Z}_d$ in this case appears the additional condition (11).

2) The function in the previous Example 1 satisfies Theorem 3.2. Another example is $f = U + dV$, with $U(x,y) = x^2$ and $V(x,y) = xe^y$.

Analysing the proof of Theorem 3.2, we see that the condition (11) can be replaced by others. Thus we easily obtain

Corollary 3.1. *Let $f : U_1^{(d)} \to \mathbb{C}_d$, $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$, be of C^1-class. If f satisfies the conditions (i), (ii), (iii) in the statement of Theorem*

3.2 and

$$\bigcap_{x\in(0,1)} I(x) \neq \emptyset, \qquad \bigcap_{x\in(-1,0)} I(x) \neq \emptyset, \quad |f(z_1)|_d \neq |f(z_2)|_d,$$

for all $z_1 = x_1 + dy_1$, $z_2 = x_2 + dy_2 \in U_1^{(d)} \setminus \mathbb{Z}_d$, *with* $x_1 x_2 < 0$, *then* f *is* (SU)*-Galilean starlike on* $U_1^{(d)} \setminus \mathbb{Z}_d$.

Remark. The function f in Example 3 and $f(z) = \dfrac{z}{(1+z)^2}$ satisfy Corollary 3.1.

For functions of hyperbolic complex variable we can prove

Theorem 3.3. *Let* $f : U_1^{(h)} \to \mathbb{C}_h$, $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, *be of* C^1*-class. If* f *satisfies the conditions:*

(i) $|f(z)|_h = 0$ *iff* $|z|_h = 0$,

(ii) $J(f)(z) > 0$, *for all* $z \in U_1^{(h)} \setminus \mathbb{Z}_h$,

(iii) $\operatorname{Re} \dfrac{D_h f(z)}{f(z)} > 0$, *for all* $z \in U_1^{(h)} \setminus \mathbb{Z}_h$,

(iv) $(x^2 - y^2)[U^2(x,y) - V^2(x,y)] > 0$, *on* $U_1^{(h)} \setminus \mathbb{Z}_h$,

(v) if $x_1 x_2 < 0$ *then* $U(x_1,y_1)U(x_2,y_2) < 0$ *and if* $y_1 y_2 < 0$ *then*

$$V(x_1,y_1)V(x_2,y_2) < 0, \ \text{on } U_1^{(h)} \setminus \mathbb{Z}_h,$$

(vi) Denoting

$$A_1^s(r) = \operatorname{arcth}\left[\lim_{\varphi\to-\infty} \frac{V(sr\cosh\varphi, sr\sinh\varphi)}{U(sr\cosh\varphi, sr\sinh\varphi)}\right],$$

$$B_1^s(r) = \operatorname{arcth}\left[\lim_{\varphi\to+\infty} \frac{V(sr\cosh\varphi, sr\sinh\varphi)}{U(sr\cosh\varphi, sr\sinh\varphi)}\right],$$

$$A_2^s(r) = \operatorname{arcth}\left[\lim_{\varphi\to-\infty} \frac{U(sr\sinh\varphi, sr\cosh\varphi)}{V(sr\sinh\varphi, sr\cosh\varphi)}\right],$$

$$B_2^s(r) = \operatorname{arcth}\left[\lim_{\varphi\to+\infty} \frac{U(sr\sinh\varphi, sr\cosh\varphi)}{V(sr\sinh\varphi, sr\cosh\varphi)}\right],$$

$s \in \{-1,+1\}$, $r \in (0,1)$, *(by (iii), (iv) these numbers exist, finite or infinite and* $A_p^s(r) < B_p^s(r)$, $p \in \{1,2\}$, $s \in \{-1,+1\}$, $r \in (0,1)$*) and supposing that*

$$\bigcap_{r\in(0,1)} (A_p^s(r), B_p^s(r)) \neq \emptyset, \quad p \in \{1,2\}, \ s \in \{-1,+1\},$$

then f *is* (SU)*-Minkowskian starlike on* $U_1^{(h)}$.

Proof. We have to prove that f is univalent on $U_1^{(h)} \setminus \mathbb{Z}_h$, in this sense showing that (9) and (10) hold for $q = h$.

First, it is obvious that for any $r \in (0,1)$ we have

$$C_r^{(h)} = C_r^{(h_1^+)} \cup C_r^{(h_1^-)} \cup C_r^{(h_2^+)} \cup C_r^{(h_2^-)}, \ \text{where for } p = 1,2$$

$$C_r^{(h_p^+)} = \{z \in \mathbb{C}_h;\ z \text{ if of } p\text{-kind and } |z|_h = r\},$$

$$C_r^{(h_p^-)} = \{z \in \mathbb{C}_h;\ z \text{ if of } p\text{-kind and } |z|_h = -r\},$$

the four sets being disjoint two by twos.

The univalency of f on each between the above four sets, easily follows from (iii) (since it is equivalent with (3)).

On the other hand, by (iv) we get

$$f(C_r^{(h_1^+)}) \cap f(C_r^{(h_2^+)}) = \emptyset, \quad f(C_r^{(h_1^-)}) \cap f(C_r^{(h_2^-)}) = \emptyset,$$

$$f(C_r^{(h_1^-)}) \cap f(C_r^{(h_2^+)}) = \emptyset, \quad f(C_r^{(h_1^+)}) \cap f(C_r^{(h_2^-)}) = \emptyset,$$

and by (v) we get

$$f(C_r^{(h_1^+)}) \cap f(C_r^{(h_1^-)}) = \emptyset, \quad f(C_r^{(h_2^+)}) \cap f(C_r^{(h_2^-)}) = \emptyset,$$

which immediately proves (9).

Now, let $r_1, r_2 \in (0,1)$, $r_1 \neq r_2$. In order to prove (10), we have to check sixteen relations of the form

$$f(C_{r_1}^{(d_p^s)}) \cap f(C_{r_2}^{(d_l^t)}) = \emptyset, \tag{17}$$

with $p, l \in \{1,2\}$, $s, t \in \{+,-\}$.

For $p \neq l$, (17) follows by (iv). For $s \neq t$, (17) follows by (v). Therefore it remains to prove the following four relations

$$\begin{aligned} f(C_{r_1}^{(h_1^+)}) \cap f(C_{r_2}^{(h_1^+)}) = \emptyset, \quad f(C_{r_1}^{(h_1^-)}) \cap f(C_{r_2}^{(h_1^-)}) = \emptyset, \\ f(C_{r_1}^{(h_2^+)}) \cap f(C_{r_2}^{(h_2^+)}) = \emptyset, \quad f(C_{r_1}^{(h_2^-)}) \cap f(C_{r_2}^{(h_2^-)}) = \emptyset. \end{aligned} \tag{18}$$

In order to obtain the first relation, let $\theta \in (A_1^{+1}, B_1^{+1})$ be fixed.

For any $\rho \in (0,1)$, by (7) we get that the system

$$\arg_h f(z) = \theta, \quad z = x + hy, \quad |z|_h = \rho, \tag{19}$$

yields a unique point $z = \rho e_h^{h\varphi}$, $\varphi = \varphi(\rho)$. For this value of z let us denote $R(\rho) = |f(z)|_h$. We will show that $R(\rho)$, $\rho \in (0,1)$, is strictly monotonous on (0,1), i.e.

$$\frac{d|f|_h}{d|z|_h} = \frac{dR}{d\rho} \text{ keeps the same sign on } (0,1), \tag{20}$$

which will imply the desired conclusion.

In this sense we follow the ideas of proof in [8, Theorem 1].

Differentiating (19) in respect with ρ and using (2), we obtain

$$\frac{1}{\rho}\text{Im }\frac{\mathcal{D}_h(f)}{f}+\varphi'(\rho)\text{Re }\frac{D_h(f)}{f}=0. \tag{21}$$

Then by (1) we get

$$\frac{dR}{d\rho}=R\left(\frac{1}{\rho}\text{Re }\frac{\mathcal{D}_h(f)}{f}+\varphi'(\rho)\text{Im }\frac{D_h(f)}{f}\right). \tag{22}$$

Eliminating $\varphi'(\rho)$ between (21) and (22) (since Re $\frac{D_h(f)}{f}\neq 0$), we obtain

$$\frac{dR}{d\rho}\text{Re }\frac{D_h(f)}{f}=\frac{R}{\rho}\left[\text{Re }\frac{D_h(f)}{f}\text{Re }\frac{\mathcal{D}_h(f)}{f}-\text{Im }\frac{D_h(f)}{f}\text{Im }\frac{\mathcal{D}_h(f)}{f}\right]=$$

$$=\frac{R}{\rho}\text{Re }\left[\frac{D_h(f)}{f}\left(\overline{\frac{\mathcal{D}_h(f)}{f}}\right)\right]=\frac{R}{\rho}\cdot\frac{1}{U^2-V^2}\text{Re }[D_h(f)\cdot\overline{\mathcal{D}_h(f)}].$$

Since by direct calculation Re $[D_h(f)\cdot\overline{\mathcal{D}_h(f)}]=(x^2-y^2)J(f)$, we get the formula

$$\frac{dR}{d\rho}\text{Re }\frac{D_h(f)}{f}=\frac{R}{\rho}\cdot\frac{x^2-y^2}{U^2-V^2}J(f),$$

which can be written in the form

$$\frac{d|f(z)|_h}{d|z|_h}\text{Re }\frac{D_h(f)}{f}=\frac{|f(z)|_h}{|z|_h}\cdot\frac{x^2-y^2}{U^2-V^2}J(f). \tag{23}$$

As conclusion, the sign of $\frac{d|f(x)|_h}{d|z|_h}$ is the same with the sign of $\frac{|f(z)|_h}{|z|_h}$. But because $U_1^{(h_1^+)}=\{z=x+hy\in U_1^{(h)};\ x^2-y^2>0,\ x>0\}$ is obviously a connected set (in $\mathbb{R}^2$), by the hypothesis it follows that the continuous function $F:U_1^{(h_1^+)}\to\mathbb{R}$, $F(z)=\frac{|f(z)|_h}{\rho}=\frac{|f(z)|_h}{|z|_h}$ keeps the same sign on $U_1^{(h_1^+)}$, which proves the first relation in (18).

Taking now $\theta\in(A_1^{-1},B_1^{-1})$ and again considering (18) but for $\rho\in(-1,0)$, by similar reasonings we obtain (23), which will imply that $f(C_{r_1}^{(h_1^-)})\cap f(C_{r_2}^{(h_1^-)})=\emptyset$, $r_1\neq r_2$.

Analogously we can prove the last two relations in (18), which completes the proof.

Remarks. 1) The previous Example 3 satisfies Theorem 3.3, while $f(z)=z^2\overline{z}$ do not satisfies it, but still is starlike.

2) By the relations $\cosh\varphi = \sqrt{1+\sinh^2\varphi}$ and denoting $\sinh\varphi = t$, it is easy to see that the conditions in Theorem 3.3,(vi), can be written as

$$A_1^s(r) = \text{arcth}\left[\lim_{t\to-\infty}\frac{V(sr\sqrt{1+t^2}, srt)}{U(sr\sqrt{1+t^2}, srt)}\right],$$

$$B_1^s(r) = \text{arcth}\left[\lim_{t\to+\infty}\frac{V(sr\sqrt{1+t^2}, srt)}{U(sr\sqrt{1+t^2}, srt)}\right],$$

and similarly for $A_2^s(r)$, $B_2^s(r)$, $s\in\{-1,+1\}$, $r\in(0,1)$.

3) Condition (iv) in Theorem 3.3 assures that the kind of $z\in U_1^{(h)}$ is not changed by f. On the other hand, it is obvious that (iv), (v), (vi) can be replaced by other conditions.

4. **Convex and alpha-convex functions**

Let q be any between i, d, h, $f: U_1^{(q)} \to \mathbb{C}_q$, $f(z) = U(x,y) + qV(x,y)$, $z = x + qy$, f of C^2-class on $U_1^{(q)}$. For any fixed $r\in(0,1)$, let us consider the differentiable path in $\mathbb{C}_q$, $\gamma_q(\varphi) = f(C_r^{(q)})$, where $\varphi = \arg_q z$ is variable and $|z|_q$ is constant ($|z|_q = r$ if $q = i$, $|z|_q = \pm r$ if $q = d, h$).

Then

$$\gamma_q'(\varphi) = \frac{\partial U}{\partial x}\cdot\frac{\partial x}{\partial\varphi} + \frac{\partial U}{\partial y}\cdot\frac{\partial y}{\partial\varphi} + q\left[\frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial\varphi} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial\varphi}\right], \tag{24}$$

and $\arg_q[\gamma_q'(\varphi)]$ represents the "Q"-angle (with the positive sense of Ox-axis) of the "Q"-tangent at the path $f(C_r^{(q)})$ in $\gamma_q(\varphi)$.

Definition 4.1. We say that f is (SU)-"Q" convex on $U_q^{(q)}$ if f is univalent on $U_1^{(q)}\setminus\mathbb{Z}_q$, $\gamma_q'(\varphi)\in\mathbb{Z}_q$ iff $z\in\mathbb{Z}_q$ and moreover, for any fixed r with $A_r = \{z\in\mathbb{C}_q;\ |z|_q = r\}\cap(U_1^{(q)}\setminus\mathbb{Z}_q)\neq\emptyset$, we have

$$\frac{\partial}{\partial\varphi}(\arg_q\gamma_q'(\varphi)) > 0, \text{ for all } z\in A_r. \tag{25}$$

Remarks. 1) Let $q = i$. Then by (24) and by $x = r\cos\varphi$, $y = r\sin\varphi$, $\varphi\in(0,2\pi]$, we get that (25) is equivalent with the inequality $\frac{\partial}{\partial\varphi}[\arg_i D_i(f)] > 0$, and we obtain the equivalent inequality in [8]

$$\text{Re}\,\frac{D_i^2(f)(z)}{D_i(f)(z)} > 0,\quad z\in U_1^{(i)}\setminus\{0\}.$$

2) Let $q = d$. In this case $z = x(1 + d\varphi)$, where $x = \pm r$, $y = x\varphi$, $\varphi \in \mathbb{R}$,

$$\gamma_d'(\varphi) = x\frac{\partial U}{\partial y} + q\left[x\frac{\partial V}{\partial y}\right], \quad \arg_d(\gamma_d'(\varphi)) = \frac{\partial V}{\partial y}/\frac{\partial U}{\partial y},$$

for $x \neq 0$, and simple calculations show that a (SU)-Galilean convex function f, generates the injective vectorial transform on $U_1^{(d)} \setminus \mathbb{Z}_d$, $F(x,y) = (U(x,y), V(x,y))$, with $\frac{\partial U}{\partial y}(x,y) = 0$ iff $x = 0$ and satisfying

$$x\left(\frac{\partial U}{\partial y}\cdot\frac{\partial^2 V}{\partial y^2} - \frac{\partial V}{\partial y}\cdot\frac{\partial^2 U}{\partial y^2}\right) > 0,\ \forall\, x \in (-1,1)\setminus\{0\},\ y \in \mathbb{R}. \tag{26}$$

Obviously that (26) is equivalent with

$$x\frac{\partial}{\partial y}[(\partial V/\partial y)/(\partial U/\partial y)] > 0, \quad x \in (-1,1)\setminus\{0\},\ y \in \mathbb{R}.$$

A simple example of (SU)-Galilean convex function is $f = U + dV$, with $U(x,y) = xe^y$, $V(x,y) = -y$. Note that f is univalent on the whole $U_1^{(d)}$.

3) Let $q = h$. In this case, we obtain: $z = |z|_h(\cosh\varphi + h\sinh\varphi)$ if z is of first kind, $z = |z|_h(\sinh\varphi + h\cosh\varphi)$ if z is of second kind, $\varphi \in \mathbb{R}$, $|z|_h = \pm r$ (constant), and in both cases $\frac{\partial x}{\partial \varphi} = y$, $\frac{\partial y}{\partial \varphi} = x$.

Then by (24) we obtain

$$\gamma_h'(\varphi) = x\frac{\partial U}{\partial y} + y\frac{\partial U}{x} + h\left(x\frac{\partial V}{\partial y} + y\frac{\partial V}{\partial x}\right) = qD_h(f)(z),$$

which immediately implies $\arg_h[\gamma_h'(\varphi)] = \arg_h[D_h(f)(z)]$.

Reasoning exactly as in the case of starlikeness, we can say that f is (SU)-Minkowskian convex on $U_1^{(h)}$, if f is univalent on $U_1^{(h)} \setminus \mathbb{Z}_h$, $D_h(f)(z) \in \mathbb{Z}_h$ iff $z \in \mathbb{Z}_h$ and

$$\operatorname{Re}\frac{D_h^2(f)(z)}{D_h(f)(z)} > 0,\ \text{for all } z \in U_1^{(h)} \setminus \mathbb{Z}_h. \tag{27}$$

A simple example of (SU)-Minkowskian convex function is $f(z) = z^2\overline{z}$, $z = x + hy$.

4) Let $q = d$ or h. We will say that a region $G \subset \mathbb{C}_q$ is (SU)-"Q" convex, if there exists $f : U_1^{(q)} \to \mathbb{C}_q$, (SU)-"Q" convex function on $U_1^{(q)}$ such that $G = f(U_1^{(q)})$. An interesting question would be to find internal geometric characterization (in Euclidean language) of the (SU)-"Q" convex regions.

By using the ideas in [9], at end we can introduce the concept of alpha-convex functions.

The Remarks 2 after the Definitions 3.1 and 4.1, suggest

Definition 4.2. Let $f : U_1^{(d)} \to \mathbb{C}_d$, $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$, be of C^2-class on $U_1^{(d)}$ and α a real number. The function f is called (SU)-Galilean α-convex if f is univalent on $U_1^{(d)} \setminus \mathbb{Z}_d$, $U(x,y) = 0$ iff $x = 0$, $\frac{\partial U}{\partial y}(x,y) = 0$ iff $x = 0$, and for all $x \in (-1,1) \setminus \{0\}$, $y \in \mathbb{R}$, we have

$$(1-\alpha)\frac{\partial[D(U,V)]}{\partial y} + \alpha \frac{\partial\left[D\left(\frac{\partial U}{\partial y}, \frac{\partial V}{\partial y}\right)\right]}{\partial y} > 0,$$

where $D(U,V) = x\left(\frac{V}{U}\right)$.

Note that $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$, with $U(x,y) = xe^y$, $V(x,y) = e^{2y}$ is (SU)-Galilean α-convex, for any $\alpha > -1$.

By the relations (8) and (27) we can introduce

Definition 4.3. Let $f : U_1^{(h)} \to \mathbb{C}_h$, $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, be of C^2-class on $U_1^{(h)}$ and α a real number. The function f is called (SU)-Minkowskian α-convex if f is univalent on $U_1^{(h)} \setminus \mathbb{Z}_h$, $|U(x,y)| = |V(x,y)|$ iff $|x| = |y|$, $\left|x\frac{\partial V}{\partial y} + y\frac{\partial V}{\partial x}\right| = \left|x\frac{\partial U}{\partial y} + y\frac{\partial U}{\partial x}\right|$ iff $|x| = |y|$ and on $U_1^{(h)} \setminus \mathbb{Z}_h$ we have

$$\text{Re}\left[(1-\alpha)\frac{D_h(f)(z)}{f(z)} + \alpha\frac{D_h^2(f)(z)}{D_h(f)(z)}\right] > 0.$$

Note that $f(z) = z^2\overline{z}$, $z = x + hy$, is (SU)-Minkowskian α-convex, for any $\alpha \in \mathbb{R}$.

Remark. A deeper study of the function classes introduced in this paper together with a corresponding theory for spirallike functions will be done in another paper.

Also, the method in this paper can be extended to functions of hypercomplex variables, as for example of quaternionic variable, or even in abstract Clifford algebras, and will be done elsewhere.

References

[1] P. Fjelstad and S.G. Gal, *n-dimensional dual complex numbers*, Adv. Appl. Clifford Algebras, 8, 2(1998) (in press).

[2] P. Fjelstad and S.G. Gal, *n-dimensional hyperbolic complex numbers*, Adv. Appl. Clifford Algebras, 8, 1(1998), 47-68.

[3] P. Fjelstad and S.G. Gal, *Note on the monogeneity of functions of complex-type variable*, submitted.

[4] S.G. Gal, *On the sufficient conditions of monogeneity for functions of complex-type variable*, Rev. Un. Mat. Argentina, 40(1996), 33-42.

[5] S.G. Gal, *Approximation and interpolation of functions of hyperbolic complex variable*, Rev. Un. Mat. Argentina, 40(3-4)(1997), 25-35.
[6] S.G. Gal, *Univalent functions of hyperbolic complex and of dual complex variable*, Mathematica (Cluj), submitted.
[7] W. Ma and D. Minda, *Hyperbolically convex functions*, Ann. Polon. Math., 60(1994), 81-100.
[8] P.T. Mocanu, *Starlikeness and convexity for non-analytic functions in the unit disc*, Mathematica (Cluj), 22(45), 1(1980), 77-83.
[9] P.T. Mocanu, *Alpha-convex nonanalytic functions*, Mathematica (Cluj), 29(52), 1(1987), 49-55.
[10] P.T. Mocanu, *Geometrical aspects in the theory of functions of complex variable* (in Romanian), in Complex Analysis, Classical and Modern Aspects (C. Andreian-Cazacu Ed.), Ed. St. Encicl., Bucharest, 1988, 17-34.
[11] J.C. Vignaux, *The theory of functions of one and of several dual complex variables* (in Spanish), Univ. Nac. La Plata, Publ. Fac. Ci. Fisicomat. Contrib., 107(1936), 221-282; 109(1937), 389-406.
[12] J.C. Vignaux and A. Durañona Vedia, *On the theory of functions of a hyperbolic complex variable* (in Spanish), Univ. Nac. La Plata, Publ. Fac. Ci. Fisicomat. Contrib., 104(1935), 139-183.
[13] I.M. Yaglom, *Complex Numbers in Geometry*, Academic Press, New York, 1968.
[14] I.M. Yaglom, *A Simple non-Euclidean Geometry and its Physical Basis*, Springer-Verlag, New York, 1979.

Department of Mathematics, University of Oradea, Str. Armatei Române 5, 3700 Oradea, Romania e-mail: galso@hs.uoradea.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 2, June 2001

GENERALIZATION OF CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

H. M. HOSSEN

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. The object of the present paper is to obtain coefficient estimates, some properties, distortion theorem and closure theorems for the classes $R_n^*(\alpha)$ of analytic and univalent functions with negative coefficients, defined by using the n-th order Ruscheweyh derivative. We also obtain several interesting results for the modified Hadamard product of functions belonging to the classes $R_n^*(\alpha)$. Further, we obtain radii of close-to-convexity, starlikeness and convexity and integral operators for the classes $R_n^*(\alpha)$.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. We denote by S the subclass of univalent functions $f(z)$ in A. The Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$ will be denoted by $f * g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

then

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

Let

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \tag{1.4}$$

1991 *Mathematics Subject Classification.* 30C45.

Key words and phrases. analytic, univalent, Ruscheweyh derivative, modified Hadamard product.

for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $z \in U$, where $\mathbb{N} = \{1, 2, 3, \dots\}$. This symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [3]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. By using the Hadamard product, Ruscheweyh [5] observed that if

$$D^{\beta} f(z) = \frac{z}{(1-z)^{\beta+1}} * f(z) \quad (\beta \geq -1) \tag{1.5}$$

then (1.4) is equivalent to (1.5) when $\beta = n \in \mathbb{N}_0$.

It is easy to see that

$$D^n f(z) = k + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \tag{1.6}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{1.7}$$

Note that

$$z(D^n f(z))' = (n+1) D^{n+1} f(z) - n D^n f(z) \quad \text{(cf. [5])}. \tag{1.8}$$

Let $R_n(\alpha)$ denote the classes of functions $f(z) \in A$ which satisfy the condition

$$\operatorname{Re} \left\{ \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \alpha, \quad (z \in U) \tag{1.9}$$

for some α $(0 \leq \alpha < 1)$ and $n \in \mathbb{N}_0$. The class $R_n(\alpha)$ was studied by Ahuja [1,2].

From (1.8) and (1.9) it follows that a function $f(z)$ in A belongs to $R_n(\alpha)$ is and only if

$$\operatorname{Re} \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \frac{n+\alpha}{n+1} \quad (z \in U). \tag{1.10}$$

Let T denote the subclass of S consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.11}$$

In the present paper we introduce the following classes $R_n^*(\alpha)$ by using the n-th order Ruscheweyh derivative of $f(z)$, defined as follows:

Definition. We say that $f(z)$ is in the class $R_n^*(\alpha)$ $(0 \leq \alpha < 1,\ n \in \mathbb{N}_0)$, if $f(z)$ defined by (1.11) satisfies the condition (1.10).

We note that $R_n^*(0) = R_n^*$ was studied by Owa [4] and $R_0^*(\alpha) = T^*(\alpha)$ (the class of starlike functions of order α) and $R_1^*(\alpha) = C(\alpha)$ (the class of convex functions of order α), were studied by Silverman [7]. Hence $R_n^*(\alpha)$ is a subclass of $T^*(\alpha) \subset S$. Further, we can show that $R_{n+1}^*(\alpha) \subset R_n^*(\alpha)$ for every $n \in \mathbb{N}_0$.

2. **Coefficient Estimates**

Theorem 1. *Let the function $f(z)$ be defined by (1.11). Then $f(z)$ is in the class $R_n^*(\alpha)$ if and only if*

$$\sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)a_k \le 1-\alpha. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z| = 1$. Then we get

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| = \left|\frac{-\sum_{k=2}^{\infty}(\delta(n+1,k)-\delta(n,k))a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\delta(n,k)a_k z^{k-1}}\right| \le$$

$$\le \frac{\sum_{k=2}^{\infty}\left(\frac{k-1}{n+1}\right)\delta(n,k)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}\delta(n,k)a_k|z|^{k-1}} \le \frac{\sum_{k=2}^{\infty}\left(\frac{k-1}{n+1}\right)\delta(n,k)a_k}{1-\sum_{k=2}^{\infty}\delta(n,k)a_k} \le \frac{1-\alpha}{n+1}.$$

This shows that the values of $\frac{D^{n+1}f(z)}{D^n f(z)}$ lies in a circle centered at $w = 1$ whose radius is $\frac{1-\alpha}{n+1}$. Hence $f(z)$ satisfies the condition (1.10) hence further, $f(z) \in R_n^*(\alpha)$.

For the converse, assume that the function $f(z)$ defined by (1.11) belongs to the class $R_n^*(\alpha)$. Then we have

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = \operatorname{Re}\left\{\frac{1-\sum_{k=2}^{\infty}\delta(n+1,k)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\delta(n,k)a_k z^{k-1}}\right\} > \frac{n+\alpha}{n+1} \tag{2.2}$$

for $0 \le \alpha < 1$ and $z \in U$. Choose values of z on the real axis so that $\frac{D^{n+1}f(z)}{D^n f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we get

$$(n+1)\left(1-\sum_{k=2}^{\infty}\delta(n+1,k)a_k\right) \ge (n+\alpha)\left(1-\sum_{k=2}^{\infty}\delta(n,k)a_k\right) \tag{2.3}$$

which gives (2.1). Finally the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(k-\alpha)\delta(n,k)} z^k \quad (k \geq 2) \tag{2.4}$$

is an extremal function for the theorem.

Corollary 1. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$. Then*

$$a_k \leq \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \quad (k \geq 2). \tag{2.5}$$

The equality in (2.5) is attained for the function $f(z)$ given by (2.4).

3. Some properties of the class $R_n^*(\alpha)$

Theorem 2. *Let $0 \leq \alpha_1 \leq \alpha_2 < 1$ and $n \in \mathbb{N}_0$. Then we have*

$$R_n^*(\alpha_1) \supseteq R_n^*(\alpha_2). \tag{3.1}$$

Proof. Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha_2)$ and $\alpha_1 = \alpha_2 - \varepsilon$. Then, by Theorem 1, we have

$$\sum_{k=2}^{\infty} (k-\alpha_2)\delta(n,k)a_k \leq 1-\alpha_2$$

and

$$\sum_{k=2}^{\infty} \delta(n,k)a_k \leq \frac{1-\alpha_2}{2-\alpha_2} < 1. \tag{3.2}$$

Consequently

$$\sum_{k=2}^{\infty} (k-\alpha_1)\delta(n,k)a_k = \sum_{k=2}^{\infty} (k-\alpha_2)\delta(n,k)a_k + \varepsilon \sum_{k=2}^{\infty} \delta(n,k)a_k \leq 1-\alpha_1. \tag{3.3}$$

This completes the proof of Theorem 2 with the aid of Theorem 1.

Theorem 3. *$R_{n+1}^*(\alpha) \subseteq R_n^*(\alpha)$ for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$.*

Proof. Let the function $f(z)$ defined by (1.11) be in the class $R_{n+1}^*(\alpha)$; then

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n+1,k)a_k \leq 1-\alpha \tag{3.4}$$

and since

$$\delta(n,k) \leq \delta(n+1,k) \text{ for } k \geq 2, \tag{3.5}$$

we have

$$\sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)a_k \le \sum_{k=2}^{\infty}(k-\alpha)\delta(n+1,k)a_k \le 1-\alpha. \tag{3.6}$$

The result follows from Theorem 1.

4. **Distortion theorem**

Theorem 4. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$. Then we have for $|z| = r < 1$*

$$r - \frac{1-\alpha}{(2-\alpha)(n+1)}r^2 \le |f(z)| \le r + \frac{1-\alpha}{(2-\alpha)(n+1)}r^2 \tag{4.1}$$

and

$$1 - \frac{2(1-\alpha)}{(2-\alpha)(n+1)}r \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{(2-\alpha)(n+1)}r. \tag{4.2}$$

The result is sharp.

Proof. In view of Theorem 1, we have

$$\sum_{k=2}^{\infty} a_k \le \frac{1-\alpha}{(2-\alpha)(n+1)}. \tag{4.3}$$

Consequently, we have

$$|f(z)| \ge r - r^2\sum_{k=2}^{\infty} a_k \ge r - \frac{1-\alpha}{(2-\alpha)(n+1)}r^2 \tag{4.4}$$

and

$$|f(z)| \le r + r^2\sum_{k=2}^{\infty} a_k \le r + \frac{1-\alpha}{(2-\alpha)(n+1)}r^2 \tag{4.5}$$

which prove the assertion (4.1).

From (4.3) and Theorem 1, it follows also that

$$\sum_{k=2}^{\infty} ka_k \le \frac{1-\alpha}{n+1} + \alpha\sum_{k=2}^{\infty} a_k \le \frac{2(1-\alpha)}{(2-\alpha)(n+1)}. \tag{4.6}$$

Consequently, we have

$$|f'(z)| \ge 1 - r\sum_{k=2}^{\infty} ka_k \ge 1 - \frac{2(1-\alpha)}{(2-\alpha)(n+1)}r \tag{4.7}$$

and

$$|f'(z)| \le 1 + r\sum_{k=2}^{\infty} ka_k \le 1 + \frac{2(1-\alpha)}{(2-\alpha)(n+1)}r, \tag{4.8}$$

which prove the assertion (4.2). This completes the proof of Theorem 4.

The bounds in (4.1) and (4.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(2-\alpha)(n+1)} z^2 \quad (z = \pm r). \tag{4.9}$$

Corollary 2. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$. Then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{(2-\alpha)(n+1) - (1-\alpha)}{(2-\alpha)(n+1)} \tag{4.10}$$

The result is sharp with extremal function $f(z)$ given by (4.9).

5. **Closure theorems**

Let the functions $f_i(z)$ be defined, for $i = 1, 2, \ldots, m$, by

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (a_{k,i} \geq 0,\ k \geq 2) \tag{5.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the classes $R_n^*(\alpha)$.

Theorem 5. *Let the functions $f_i(z)$ defined by (5.1) be in the class $R_n^*(\alpha)$ for every $i = 1, 2, \ldots, m$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \quad (c_i \geq 0) \tag{5.2}$$

is also in the class $R_n^*(\alpha)$, where

$$\sum_{i=1}^{m} c_i = 1. \tag{5.3}$$

Proof. According to the definition of $h(z)$, we can write

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{m} c_i a_{k,i} \right) z^k. \tag{5.4}$$

Further, since $f_i(z)$ are in $R_n^*(\alpha)$ for every $i = 1, 2, \ldots, m$, we get

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) a_{k,i} \leq 1 - \alpha \tag{5.5}$$

for every $i = 1, 2, \ldots, m$. Hence we can see that

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) \left(\sum_{i=1}^{m} c_i a_{k,i} \right) = \sum_{i=1}^{m} c_i \left(\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) a_{k,i} \right) \leq$$

$$= \left(\sum_{i=1}^{m} c_i\right)(1-\alpha) \leq 1-\alpha \tag{5.6}$$

with the aid of (5.5). This proves that the function $h(z)$ is in the class $R_n^*(\alpha)$ by means of Theorem 1. Thus we have the theorem.

Theorem 6. *The class $R_n^*(\alpha)$ is closed under convex linear combinations.*

Proof. Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (5.1) be in the class $R_n^*(\alpha)$. Then it is sufficient to prove that the function

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \leq \lambda \leq 1) \tag{5.7}$$

is in the class $R_n^*(\alpha)$. Since, for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{k=2}^{\infty} \{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} z^k, \tag{5.8}$$

we readily have

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k)\{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} \leq 1-\alpha, \tag{5.9}$$

by means of Theorem 1, which implies that $h(z) \in R_n^*(\alpha)$.

Theorem 7. *Let*

$$f_1(z) = z \tag{5.10}$$

and

$$f_k(z) = z - \frac{1-\alpha}{(k-\alpha)\delta(n,k)} z^k \quad (k \geq 2) \tag{5.11}$$

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$. Then $f(z)$ is in the class $R_n^*(\alpha)$ if and only if can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{5.12}$$

where $\lambda_k \geq 0$ and

$$\sum_{k=1}^{\infty} \lambda_k = 1. \tag{5.13}$$

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \lambda_k z^k. \tag{5.14}$$

Then we have

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \cdot \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \lambda_k = \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1. \tag{5.15}$$

So by Theorem 1, $f(z) \in R_n^*(\alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.11) belongs to the class $R_n^*(\alpha)$. Again, with the aid of Theorem 1, we get

$$a_k \leq \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \quad (k \geq 2). \tag{5.16}$$

Setting

$$\lambda_k = \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_k \quad (k \geq 2), \tag{5.17}$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k. \tag{5.18}$$

Hence, we can see that $f(z)$ can be expressed in the form (5.12). This completes the proof of Theorem 7.

Corollary 3. *The extreme points of the class $R_n^*(\alpha)$ are the functions $f_1(z)$ and $f_k(z)$ $(k \geq 2)$ given by Theorem 7.*

6. Modified Hadamard product

Let the functions $f_i(z)$ $(i = 1, 2)$ be defined (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \tag{6.1}$$

Theorem 8. *Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (5.1) be in the class $R_n^*(\alpha)$. Then $f_1 * f_2(z) \in R_n^*(\beta(n, \alpha))$, where*

$$\beta(n,\alpha) = \frac{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)^2}{(n+1) - \left(\frac{1-\alpha}{2-\alpha}\right)^2}. \tag{6.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [4], we need to find the largest $\beta = \beta(n, \alpha)$ such that

$$\sum_{k=2}^{\infty} \frac{(k-\beta)\delta(n,k)}{1-\beta} a_{k,1} a_{k,2} \leq 1. \tag{6.3}$$

Since

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_{k,1} \leq 1 \tag{6.4}$$

and

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_{k,2} \leq 1, \tag{6.5}$$

by the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \sqrt{a_{k,1} a_{k,2}} \leq 1. \tag{6.6}$$

Thus it is sufficient to show that

$$\frac{(k-\beta)\delta(n,k)}{1-\beta} a_{k,1} a_{k,2} \leq \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \sqrt{a_{k,1} a_{k,2}} \quad (k \geq 2), \tag{6.7}$$

that is, that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{(k-\alpha)(1-\beta)}{(k-\beta)(1-\alpha)} \quad (k \geq 2). \tag{6.8}$$

Note that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \quad (k \geq 2). \tag{6.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{(k-\alpha)\delta(n,k)} \leq \frac{(k-\alpha)(1-\beta)}{(k-\beta)(1-\alpha)} \quad (k \geq 2), \tag{6.10}$$

or, equivalently, that

$$\beta \leq \frac{\delta(n,k) - k\left(\dfrac{1-\alpha}{k-\alpha}\right)^2}{\delta(n,k) - \left(\dfrac{1-\alpha}{k-\alpha}\right)^2} \quad (k \geq 2). \tag{6.11}$$

Since

$$A(k) = \frac{\delta(n,k) - k\left(\dfrac{1-\alpha}{k-\alpha}\right)^2}{\delta(n,k) - \left(\dfrac{1-\alpha}{k-\alpha}\right)^2} \tag{6.12}$$

is an increasing function of k $(k \geq 2)$, letting $k = 2$ in (6.12), we obtain

$$\beta \leq A(2) = \frac{(n+1) - 2\left(\dfrac{1-\alpha}{2-\alpha}\right)^2}{(n+1) - \left(\dfrac{1-\alpha}{2-\alpha}\right)^2}, \tag{6.13}$$

which completes the proof of the theorem. Finally, by taking the functions $f_i(z)$ given by

$$f_i(z) = z - \frac{1-\alpha}{(2-\alpha)(n+1)} z^2 \quad (i = 1, 2), \tag{6.14}$$

we can see that the result is sharp.

Corollary 4. *For $f_1(z)$ and $f_2(z)$ as in Theorem 8, we have*

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1} a_{k,2}}\, z^k \tag{6.15}$$

belongs to the class $R_n^*(\alpha)$.

The result follows from the inequality (6.6). It is sharp for the same functions $f_i(z)$ $(i = 1, 2)$ as in Theorem 8.

Theorem 9. *Let $f_1(z) \in R_n^*(\alpha)$ and $f_2(z) \in R_n^*(\beta)$, then $f_1 * f_2(z) \in R_n^*(\gamma(n, \alpha, \beta))$, where*

$$\gamma(n, \alpha, \beta) = \frac{(n+1) - 2\left(\dfrac{1-\alpha}{2-\alpha}\right)\left(\dfrac{1-\beta}{2-\beta}\right)}{(n+1) - \left(\dfrac{1-\alpha}{2-\alpha}\right)\left(\dfrac{1-\beta}{2-\beta}\right)}. \tag{6.16}$$

The result is sharp for the functions

$$f_1(z) = z - \frac{1-\alpha}{(2-\alpha)(n+1)} z^2 \tag{6.17}$$

and

$$f_2(z) = z - \frac{1-\beta}{(2-\beta)(n+1)} z^2. \tag{6.18}$$

Proof. Proceeding as in the proof of Theorem 8, we get

$$\gamma \leq B(k) = \frac{\delta(n, k) - k\left(\dfrac{1-\alpha}{k-\alpha}\right)\left(\dfrac{1-\beta}{k-\beta}\right)}{\delta(n, k) - \left(\dfrac{1-\alpha}{k-\alpha}\right)\left(\dfrac{1-\beta}{k-\beta}\right)}. \tag{6.19}$$

Since the function $B(k)$ is an increasing function of k ($k \geq 2$), setting $k = 2$ in (6.19), we obtain

$$\gamma \leq B(2) = \frac{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)}{(n+1) - \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)}. \tag{6.20}$$

This completes the proof of Theorem 9.

Corollary 5. *Let the functions $f_i(z)$ $(i = 1, 2, 3)$ defined by (5.1) be in the class $R_n^*(\alpha)$, then $f_1 * f_2 * f_3(z) \in R_n^*(\zeta(n, \alpha))$, where*

$$\zeta(n, \alpha) = \frac{(n+1)^2 - 2\left(\frac{1-\alpha}{2-\alpha}\right)^3}{(n+1)^2 - \left(\frac{1-\alpha}{2-\alpha}\right)^3}. \tag{6.21}$$

The result is best possible for the functions

$$f_i(z) = z - \frac{1-\alpha}{(2-\alpha)(n+1)} z^2 \quad (i = 1, 2, 3). \tag{6.22}$$

Proof. From Theorem 8, we have $f_1 * f_2(z) \in R_n^*(\beta)$, where β is given by (6.2). We use now Theorem 9, we get $f_1 * f_2 * f_3(z) \in R_n^*(\zeta(n, \alpha))$, where

$$\zeta(n, \alpha) = \frac{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)}{(n+1) - \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)} = \frac{(n+1)^2 - 2\left(\frac{1-\alpha}{2-\alpha}\right)^3}{(n+1)^2 - \left(\frac{1-\alpha}{2-\alpha}\right)^3}.$$

This completes the proof of Corollary 5.

Theorem 10. *Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (5.1) be in the class $R_n^*(\alpha)$. Then the function*

$$h(z) = z - \sum_{k=2}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{6.23}$$

belongs to the class $R_n^*(\varphi(n, \alpha))$, where

$$\varphi(n, \alpha) = \frac{(n+1) - \left(\frac{2(1-\alpha)}{2-\alpha}\right)^2}{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)^2}. \tag{6.24}$$

The result is sharp for the functions $f_i(z)$ $(i = 1, 2)$ defined by (6.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{(k-\alpha)\delta(n,k)}{1-\alpha}\right]^2 a_{k,1}^2 \le \left[\sum_{k=2}^{\infty}\frac{(k-\alpha)\delta(n,k)}{1-\alpha}a_{k,1}\right]^2 \le 1 \tag{6.25}$$

and

$$\sum_{k=2}^{\infty}\left[\frac{(k-\alpha)\delta(n,k)}{1-\alpha}\right]^2 a_{k,2}^2 \le \left[\sum_{k=2}^{\infty}\frac{(k-\alpha)\delta(n,k)}{1-\alpha}a_{k,2}\right]^2 \le 1. \tag{6.26}$$

It follows from (6.25) and (6.26) that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{(k-\alpha)\delta(n,k)}{1-\alpha}\right]^2 (a_{k,1}^2 + a_{k,2}^2) \le 1. \tag{6.27}$$

Therefore, we need to find the largest $\varphi = \varphi(n,\alpha)$ such that

$$\frac{(k-\varphi)\delta(n,k)}{1-\varphi} \le \frac{1}{2}\left[\frac{(k-\alpha)\delta(n,k)}{1-\alpha}\right]^2 \quad (k \ge 2), \tag{6.28}$$

that is

$$\varphi \le \frac{\delta(n,k) - 2k\left(\dfrac{1-\alpha}{k-\alpha}\right)^2}{\delta(n,k) - 2\left(\dfrac{1-\alpha}{k-\alpha}\right)^2} \quad (k \ge 2). \tag{6.29}$$

Since

$$D(k) = \frac{\delta(n,k) - 2k\left(\dfrac{1-\alpha}{k-\alpha}\right)^2}{\delta(n,k) - 2\left(\dfrac{1-\alpha}{k-\alpha}\right)^2} \tag{6.30}$$

is an increasing function of k $(k \ge 2)$, we readily have

$$\varphi \le D(2) = \frac{(n+1) - \left(\dfrac{2(1-\alpha)}{2-\alpha}\right)^2}{(n+1) - 2\left(\dfrac{1-\alpha}{2-\alpha}\right)^2}, \tag{6.31}$$

and Theorem 10 follows at once.

Theorem 11. *Let $f_1(z) \in R^*_{n_1}(\alpha)$, and $f_2(z) \in R^*_{n_2}(\alpha)$. Then the modified Hadamard product $f_1 * f_2(z) \in R^*_{n_1}(\alpha) \cap R^*_{n_2}(\alpha)$.*

Proof. Since $f_2(z) \in R^*_{n_2}(\alpha)$, we have from (4.3) that

$$a_{k,2} \le \frac{1-\alpha}{(2-\alpha)(n_2+1)}. \tag{6.32}$$

From Theorem 1, since $f_1(z) \in R^*_{n_1}(\alpha)$, we have

$$\sum_{k=2}^{\infty}\frac{(k-\alpha)\delta(n_1,k)}{1-\alpha}a_{k,1} \le 1. \tag{6.33}$$

Now, from (6.32) and (6.33), we have

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n_1,k)}{1-\alpha} a_{k,1}a_{k,2} \leq \frac{1-\alpha}{(2-\alpha)(n_2+1)} \sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n_1,k)}{1-\alpha} a_{k,1} \leq$$

$$\leq \frac{1-\alpha}{(2-\alpha)(n_2+1)} \leq 1.$$

Hence $f_1 * f_2(z) \in R^*_{n_1}(\alpha)$. Interchanging n_1 and n_2 by each other in the above, we get $f_1 * f_2(z) \in R^*_{n_2}(\alpha)$. Hence the theorem.

7. Radii of close-to-convexity, starlikeness and convexity

Theorem 12. *Let the function $f(z)$ defined by (1.11) be in the class $R^*_n(\alpha)$, then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1(n,\alpha,\rho)$, where*

$$r_1(n,\alpha,\rho) = \inf_k \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.1}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

Proof. We must show that $|f'(z)-1| \leq 1-\rho$ for $|z| < r_1(n,\alpha,\rho)$. We have

$$|f'(z)-1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z)-1| \leq 1-\rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \tag{7.2}$$

Hence, by Theorem 1, (7.2) will be true if

$$\frac{k|z|^{k-1}}{1-\rho} \leq \frac{(k-\alpha)\delta(n,k)}{1-\alpha}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.3}$$

The theorem follows easily from (7.3).

Theorem 13. *Let the function $f(z)$ defined by (1.11) be in the class $R^*_n(\alpha)$, then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2(n,\alpha,\rho)$, where*

$$r_2(n,\alpha,\rho) = \inf_k \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$ for $|z| < r_2(n, \alpha, \rho)$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$ if

$$\sum_{k=2}^{\infty} \frac{(k-\rho)a_k|z|^{k-1}}{1-\rho} \leq 1. \tag{7.5}$$

Hence, by Theorem 1, (7.5) will be true if

$$\frac{(k-\rho)|z|^{k-1}}{1-\rho} \leq \frac{(k-\alpha)\delta(n,k)}{1-\alpha}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.6}$$

The theorem follows easily from (7.6).

Corollary 6. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$, then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3(n, \alpha, \rho)$, where*

$$r_3(n, \alpha, \rho) = \inf_k \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{k(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

8. Integral operators

Theorem 14. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$ and let the function $F(z)$ be defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt. \tag{8.1}$$

Then

(i) for every c, $c > -1$, $F(z) \in R_n^*(\alpha)$

and

(ii) for every c, $-1 < c \leq n$, $F(z) \in R_{n+1}^*(\alpha)$.

Proof. (i) From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{8.2}$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k. \tag{8.3}$$

Therefore,

$$\sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)b_k = \sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)\left(\frac{c+1}{c+k}\right) a_k \leq$$

$$\leq \sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)a_k \leq 1-\alpha,$$

since $f(z) \in R_n^*(\alpha)$. Hence, by Theorem 1, $F(z) \in R_n^*(\alpha)$.

(ii) In view of Theorem 1 it is sufficient to show that

$$\sum_{k=2}^{\infty}(k-\alpha)\delta(n+1,k)\left(\frac{c+1}{c+k}\right) a_k \leq 1-\alpha. \tag{8.4}$$

Since

$$\delta(n,k) - \delta(n+1,k)\left(\frac{c+1}{c+k}\right) \geq 0 \text{ if } -1 < c \leq n \ (k = 2, 3, \dots)$$

the result follows from Theorem 1.

Putting $c = 0$ in Theorem 14, we get

Corollary 7. *Let the function $f(z)$ defined by (1.6) be in the class $R_n^*(\alpha)$ and let the function $F(z)$ be defined by*

$$F(z) = \int_0^z \frac{f(t)}{t} dt. \tag{8.5}$$

Then $F(z) \in R_{n+1}^*(\alpha)$.

Theorem 15. *Let the function $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$ $(a_k \geq 0)$ be in the class $R_n^*(\alpha)$, and let c be a real number such that $c > -1$. Then the function $f(z)$ defined by (8.1) is univalent in $|z| < r^*$, where*

$$r^* = \inf_k \left[\frac{(c+1)(k-\alpha)\delta(n,k)}{k(c+k)(1-\alpha)}\right]^{\frac{1}{k-1}}, \quad (k \geq 2). \tag{8.6}$$

The result is sharp.

Proof. From (8.1), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} \quad (c > -1) = z - \sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right) a_k z^k. \tag{8.7}$$

In order to obtain the required result it suffices to show that

$$|f'(z) - 1| < 1 \text{ in } |z| < r^*.$$

Now

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$, if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \tag{8.8}$$

But Theorem 1 confirms that

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_k \leq 1. \tag{8.9}$$

Hence (8.8) will be satisfied if

$$\frac{k(c+k)|z|^{k-1}}{c+1} < \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \quad (k \geq 2)$$

or if

$$|z| < \left[\frac{(c+1)(k-\alpha)\delta(n,k)}{k(c+k)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{8.10}$$

Therefore $f(z)$ is univalent in $|z| < r^*$. Sharpness follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{(k-\alpha)\delta(n,k)(c+1)} z^k \quad (k \geq 2). \tag{8.11}$$

Acknowledgements. The author wishes to thank Prof. Dr. M.K. Aouf for his kind encouragement and help in the preparation of this paper.

References

[1] O.P. Ahuja, *On the radius problems of certain analytic functions*, Bull. Korean Math. Soc. 22(1985), no.1, 31-36.
[2] O.P. Ahuja, *Integral operator of certain univalent functions*, Internat. J. Math. Sci. 8(1985), no.4, 653-662.
[3] H.S. Al-Amiri, *On Ruscheweyh derivatives*, Ann. Polon. Math. 38(1980), 87-94.
[4] S. Owa, *On new classes of univalent functions with negative coefficients*, Bull. Korean Math. Soc. 22(1985), no.1, 43-52.
[5] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. 49(1975), 109-115.
[6] A. Schild and H. Silverman, *Convolution of univalent functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklodowska, Sect.A, 29(1975), 99-107.
[7] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51(1975), 109-116.

Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt
E-mail address: sinfac@mum.mans.eun.eg

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLVI, Number 2, June 2001

ON CERTAIN CLASSES OF FUNCTIONS DEFINED BY CONVOLUTIONS

H. M. HOSSEN AND M. K. AOUF

Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. The object of the present paper is to obtain several interesting results involving coefficient estimates for analytic normalized functions belonging to certain classes defined in terms of the convolution with the extremal function for the class of starlike functions of order α, $0 \leq \alpha < 1$.

1. Introduction

Let A_1 denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. And let S denote the subclass of A_1 consisting of analytic and univalent functions $f(z)$ in the unit disc U. A function $f(z)$ of S is said to be starlike of order α if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$.

Now, the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{1.3}$$

is the well-known extremal function for $S^*(\alpha)$. Setting

$$C(\alpha, n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!} \quad (n = 2, 3, \dots), \tag{1.4}$$

1991 *Mathematics Subject Classification.* 30C45.

Key words and phrases. analytic, starlike, convolution.

$S_\alpha(z)$ can be written in the form

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n. \tag{1.5}$$

Note that $C(\alpha, n)$ is a decreasing function of α, $0 \le \alpha < 1$, and satisfies

$$\lim_{n\to\infty} C(\alpha, n) = \begin{cases} \infty, & \alpha < 1/2 \\ 1, & \alpha = 1/2 \\ 0, & \alpha > 1/2. \end{cases} \tag{1.6}$$

Let $(f * g)(z)$ denote the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.7}$$

then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.8}$$

Recently, many classes defined by convolution of $f(z)$ and $S_\alpha(z)$ have been studied by Ahuja and Silverman [1], Owa and Ahuja [11, 12], Sheil-Small, Silverman, and Silvia [15], Silverman and Silvia [16], and Ruscheweyh and Singh [14].

We denote by $P_\alpha(\beta, \gamma, A, B)$ the class of functions $f(z)$ in A_1 that satisfy the condition

$$(f * S_\alpha)'(z) \prec \frac{1 + [B + (A - B)(1 - \beta)]\gamma z}{1 + B\gamma z} \quad (z \in U) \tag{1.9}$$

for some α $(0 \le \alpha < 1)$, β $(0 \le \beta < 1$, γ $(0 < \gamma \le 1)$, and $-1 \le A < B \le 1$, $0 < B \le 1$, where $\prec$ means subordination. For $f \in P_\alpha(\beta, \gamma, A, B)$, the values of $(f * S_\alpha)'(z)$ lie in a disc centered at $\dfrac{1 - [B + (A - B)(1 - \beta)]B\gamma^2}{1 - B^2\gamma^2}$ whose radius is $\dfrac{(B - A)\gamma(1 - \beta)}{1 - B^2\gamma^2}$.

We observe that, by specializing the parameters α, β, γ, A and B, we obtain the following subclasses studied by various authors:

(1) $P_{1/2}(0, 1, -1, 1) = \{f \in A_1 : \text{ Re } f'(z) > 0,\ z \in U\}$ (Mac-Gregor [8]).

(2) $P_{1/2}(0, \gamma, -1, 1) = \left\{ f \in A_1 : f'(z) \prec \dfrac{1 - \gamma z}{1 + \gamma z},\ z \in U \right\}$ (Padmanabhan [13] and Caplinger and Causey [4]).

(3) $P_{1/2}(\beta, \gamma, -1, 1) = \left\{ f \in A_1 : f'(z) \prec \dfrac{1 + (2\beta - 1)\gamma z}{1 + \gamma z},\ z \in U \right\}$ (Juneja and Mogra [7]).

(4) $P_{1/2}(0,1,A,B)=\left\{f\in A_1:\ f'(z)\prec \dfrac{1+Az}{1+Bz},\ z\in U\right\}$ (Mehrok [9]).

(5) $P_{1/2}(\beta,\gamma,A,B)=\left\{f\in A_1:\ f'(z)\prec \frac{1+[B+(A-B)(1-\beta)]\gamma z}{1+B\gamma z},\ z\in U\right\}$ (Aouf and Owa [3]).

(6) $P_\alpha(\beta,\gamma,-1,1)\ =\ \left\{f\in A_1:\ (f*S_\alpha)'(z)\prec \dfrac{1+(2\beta-1)\gamma z}{1+\gamma z},\ z\in U\right\}$ (Owa and Ahuja [12]).

(7) $P_\alpha(0,1,-1,1)=\{f\in A_1:\ \text{Re}\ (f*S_\alpha)'(z)>0,\ z\in U\}$ (Ahuja and Owa [2]).

It is well-known that the functions in $P_{1/2}(0,1,-1,0)$ and $P_{1/2}(0,1,A,B)$ are univalent in U.

Further, we say that a function $f(z)$ in A_1 belongs to the class $Q_\alpha(\beta,\gamma,A,B)$ if and only if $zf'(z)\in P_\alpha(\beta,\gamma,A,B)$ for all $z\in U$. Finally, denote by $R_\alpha(\beta,\gamma,A,B)$ the class of functions $f(z)$ in A_1 that satisfy the condition

$$\frac{1}{z}(f*S_\alpha)(z)\prec \frac{1+[B+(A-B)(1-\beta)]\gamma z}{1+B\gamma z} \tag{1.10}$$

for some α,β,γ,A, and B as defined above. Note that

$$R_{1/2}(0,1,A,B)=\left\{f\in A_1:\ \frac{f(z)}{z}\prec \frac{1+Az}{1+Bz},\ z\in U\right\}.$$

In Section 2, we first prove that

$$Q_\alpha(\beta,\gamma,A,B)\subset P_\alpha(\beta,\gamma,A,B)\subset R_\alpha(\beta,\gamma,A,B),$$

and we then determine coefficient inequalities for the functions belonging to these classes. Finally, the coefficient inequalities for some subclasses of $P_\alpha(\beta,\gamma,A,B)$ and $Q_\alpha(\beta,\gamma,A,B)$ are obtained.

2. **Coefficient Inequalities**

First we examine the relationship between $P_\alpha(\beta,\gamma,A,B)$ and $Q_\alpha(\beta,\gamma,A,B)$. We need the following very useful result due to Jack [6], and Suffridge [17].

Lemma 1. *Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z_0)|=\max_{|z|=r}|w(z)|$, then we have*

$$z_0w'(z_0)=kw(z_0),$$

where k is a real number and $k\geq 1$.

Theorem 1. $Q_\alpha(\beta, \gamma, A, B) \subset P_\alpha(\beta, \gamma, A, B)$.

Proof. Let $f \in Q_\alpha(\beta, \gamma, A, B)$. Then $zf'(z) \in P_\alpha(\beta, \gamma, A, B)$ and therefore

$$(zf' * S_\alpha)'(z) \prec g(z), \tag{2.1}$$

where $g(z) = \dfrac{1 + [B + (A - B)(1 - \beta)]\gamma z}{1 + B\gamma z}$ is convex univalent in U. In view of the principle of subordination and the Schwarz's Lemma [10], it follows that (2.1) is equivalent to

$$\left| \frac{(zf' * S_\alpha)'(z) - 1}{B\gamma(zf' * S_\alpha)'(z) - [B + (A - B)(1 - \beta)]\gamma} \right| < 1. \tag{2.2}$$

Define $w(z)$ by

$$\frac{(zf' * S_\alpha)(z)}{z} = \frac{1 + [B + (A - B)(1 - \beta)]\gamma w(z)}{1 + B\gamma w(z)}. \tag{2.3}$$

We observe that

$$\frac{(zf' * S_\alpha)(z)}{z} = (f * S_\alpha)'(z).$$

Thus (2.3) can be written as

$$w(z) = \frac{(f * S_\alpha)'(z) - 1}{[B + (A - B)(1 - \beta)]\gamma - B\gamma(f * S_\alpha)'(z)}. \tag{2.4}$$

Note that $w(z)$ is analytic in U and $w(0) = 0$. We need to show that $|w(z)| < 1$ for all $z \in U$. On the contrary, suppose $|w(z)| \not< 1$. Then by Lemma 1, there exists a point $z_0 \in U$ such that $|w(z_0)| = 1$, $z_0 w'(z_0) = kw(z_0)$ for some $k \geq 1$. Therefore, (2.3) yields

$$(z_0 f' * S_\alpha)'(z_0) - 1 = \frac{(A - B)(1 - \beta)\gamma w(z_0)(1 + T(z_0))}{1 + B\gamma w(z_0)},$$

and

$$B\gamma(z_0 f' * S_\alpha)'(z_0) - [B + (A - B)(1 - \beta)]\gamma =$$
$$= \frac{(B - A)(1 - \beta)\gamma[1 - B\gamma w(z_0)T(z_0)]}{1 + B\gamma w(z_0)}$$

where $T(z_0) = \dfrac{k}{1 + B\gamma w(z_0)}$, and hence (2.2) implies that

$$\left| \frac{1 + T(z_0)}{1 - B\gamma w(z_0)T(z_0)} \right| < 1.$$

This last inequality gives

$$(1 - B^2\gamma^2)|T(z_0)|^2 < -2\text{Re}\ [(1 + B\gamma w(z_0))T(z_0)]. \tag{2.5}$$

Since the right side of (2.5) is equal to $-2k$ and $k \geq 1$, we conclude that (2.5) is not possible. This contradiction thereby shows that $|w(z)| < 1$ for all $z \in U$, and hence (2.4) immediately proves that $f \in P_\alpha(\beta, \gamma, A, B)$. The proof of the theorem is completed.

Theorem 2. $P_\alpha(\beta, \gamma, A, B) \subset R_\alpha(\beta, \gamma, A, B)$.

Proof. Let $f \in P_\alpha(\beta, \gamma, A, B)$. Then it follows that

$$\frac{1}{z}(zf' * S_\alpha)(z) \prec g(z),$$

where $g(z) = \dfrac{1 + [B + (A - B)(1 - \beta)]\gamma z}{1 + B\gamma z}$ is convex univalent in U and hence $h(z) = zf'(z) \in R_\alpha(\beta, \gamma, A, B)$. Therefore, in view of (1.10), we have

$$\frac{1}{z}\left(\int_0^z \frac{h(t)}{t} dt * S_\alpha\right)(z) = \int_0^1 \frac{(h * S_\alpha)(tz)}{tz} dt \prec g(z).$$

This implies that

$$f(z) = \int_0^z \frac{h(t)}{t} dt \in R_\alpha(\beta, \gamma, A, B),$$

which completes the proof of the theorem.

Corollary 1. *If $f(z) \in P_\alpha(\beta, \gamma, A, B)$, then we have*

$$\left|\arg \frac{1}{z}(f * S_\alpha)(z)\right| \leq \sin^{-1}\left(\frac{(B - A)\gamma(1 - \beta)|z|}{1 - B\gamma^2[B + (A - B)(1 - \beta)]|z|^2}\right).$$

The bound is sharp.

We next obtain a sufficient condition in terms of the modulus of the coefficients for a function to be in $P_\alpha(\beta, \gamma, A, B)$.

Theorem 3. *Let the function $f(z)$ defined by (1.1) satisfies the condition*

$$\sum_{n=2}^{\infty} n(1 + B\gamma)C(\alpha, n)|a_n| \leq (B - A)\gamma(1 - \beta) \tag{2.6}$$

for $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $0 < \gamma \leq 1$, and $-1 \leq A < B \leq 1$, $0 < B \leq 1$. Then $f(z)$ is in the class $P_\alpha(\beta, \gamma, A, B)$.

Proof. We use a method of Clunie and Keogh [5]. Assuming the inequality (2.6), we have

$$|(f * S_\alpha)'(z) - 1| - \gamma|B(f * S_\alpha)'(z) - [B + (A - B)(1 - \beta)]| =$$

$$= \left|\sum_{n=2}^{\infty} nC(\alpha, n)a_n z^{n-1}\right| - \gamma\left|(B - A)(1 - \beta) + \sum_{n=2}^{\infty} BnC(\alpha, n)a_n z^{n-1}\right| \leq$$

$$\leq \sum_{n=2}^{\infty} nC(\alpha,n)|a_n||z|^{n-1} - \gamma\left\{(B-A)(1-\beta) - \sum_{n=2}^{\infty} BnC(\alpha,n)|a_n||z|^{n-1}\right\} \leq$$

$$\leq \sum_{n=2}^{\infty} n(1+B\gamma)C(\alpha,n)|a_n| - (B-A)\gamma(1-\beta) \leq 0$$

for all $z \in U$. Consequently, by the maximum modulus theorem, it follows that $f(z) \in P_\alpha(\beta,\gamma,A,B)$. The equality in (2.6) is attained for the functions of the form

$$f_n(z) = z + \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} z^n \quad (n \geq 2).$$

Example. The function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ given by

$$(f * S_\alpha)(z) = z + \sum_{n=2}^{\infty} C(\alpha,n) a_n z^n =$$

$$= -\frac{[B+(A-B)(1-\beta)]}{B} z + \frac{(B-A)(1-\beta)}{B^2\gamma} \ln(1-B\gamma z) \tag{2.7}$$

belongs to $P_\alpha(\beta,\gamma,A,B)$ but

$$\sum_{n=2}^{\infty} \frac{n(1+B\gamma)C(\alpha,n)}{(B-A)\gamma(1-\beta)}|a_n| = \sum_{n=2}^{\infty} \frac{n(1+B\gamma)C(\alpha,n)}{(B-A)\gamma(1-\beta)} \frac{(B-A)(1-\beta)}{nC(\alpha,n)} B^{n-2}\gamma^{n-1} =$$

$$= \sum_{n=2}^{\infty} (1+B\gamma)(B\gamma)^{n-2} > 1$$

for each α,β,γ,A,B $(0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1,\ -1 \leq A < B \leq 1,\ 0 < B \leq 1)$. This example shows that the converse of Theorem 3 may not be true.

Motivated by Theorem 3 and the above Example, we now consider a class $H_\alpha(\beta,\gamma,A,B)$ of precisely those functions in A_1 which are characterized by the condition (2.6): that is, $f(z) \in H_\alpha(\beta,\gamma,A,B)$ if and only if $f(z)$ satisfies (2.6) for some α,β,γ,A,B $(0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1,\ -1 \leq A < B \leq 1,\ 0 < B \leq 1)$. Clearly, $H_\alpha(\beta,\gamma,A,B) \subset P_\alpha(\beta,\gamma,A,B)$. This containment is proper because $f(z)$ given by (2.7) belongs to $P_\alpha(\beta,\gamma,A,B) - H_\alpha(\beta,\gamma,A,B)$. We next prove a theorem about convolutions of functions in $H_\alpha(\beta,\gamma,A,B)$. But first we need the following

Lemma 2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H_\alpha(\beta,\gamma,A,B)$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A_1$ *with* $|b_n| \leq 1$ *for every* n, *then* $(f * g)(z) \in H_\alpha(\beta,\gamma,A,B)$.

Proof. The result follows from (2.6) upon noting that

$$\sum_{n=2}^{\infty} \frac{n(1+B\gamma)C(\alpha,n)}{(B-A)\gamma(1-\beta)}|a_n||b_n| \leq \sum_{n=2}^{\infty} \frac{n(1+B\gamma)C(\alpha,n)}{(B-A)\gamma(1-\beta)}|a_n| \leq 1.$$

Remark. The condition $|b_n| \leq 1$ is best possible because if $|b_n| > 1$ for some n, then

$$\left(z + \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} z^n\right) * g(z) \notin H_\alpha(\beta,\gamma,A,B).$$

Theorem 4. *If $f, g \in H_\alpha(\beta,\gamma,A,B)$ with*

$$\alpha \leq \frac{1+B\gamma\beta}{1+B\gamma} \tag{2.8}$$

then $f * g(z) \in H_\alpha(\beta,\gamma,A,B)$.

Proof. According to Lemma 2, it suffices to show that the modulus of the n-th coefficient, $|b_n|$, is bounded above by 1. Note that

$$C(\alpha,n) = \frac{\prod_{k=2}^{\infty}(k-2\alpha)}{(n-1)!} \geq \frac{2(1-\alpha)}{(n-1)!}\prod_{k=3}^{n}(k-2) > \frac{(B-A)(1-\alpha)}{Bn}.$$

Thus from (2.6) we have

$$|b_n| \leq \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} <$$

$$< \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)}\frac{Bn}{(B-A)(1-\alpha)} = \frac{B\gamma(1-\beta)}{(1-\alpha)(1+B\gamma)}. \tag{2.9}$$

This last expression is bounded above by 1 if (2.8) holds and the proof is completed.

Remark. The condition (2.8) cannot be eliminated. The function

$$f_n(z) = z + \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} z^n = z + a_n z^n \quad (n \geq 2)$$

is in $H_\alpha(\beta,\gamma,A,B)$ but $f_n * f_n(z) \notin H_\alpha(\beta,\gamma,A,B)$ for α close enough to 1 to assure that $a_n > 1$.

With the aid of Theorem 3, we have

Theorem 5. *Let the function $f(z)$ defined by (1.1) satisfies the condition*

$$\sum_{n=2}^{\infty} n^2(1+B\gamma)C(\alpha,n)|a_n| \leq (B-A)\gamma(1-\beta) \tag{2.10}$$

for $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1$ and $-1 \leq A < B \leq 1,\ 0 < B \leq 1$. Then $f(z)$ is in the class $Q_\alpha(\beta,\gamma,A,B)$.

Proof. We note that $f(z) \in Q_\alpha(\beta,\gamma,A,B)$ if and only if $zf'(z) \in P_\alpha(\beta,\gamma,A,B)$. Since $zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$, we may replace a_n by na_n in Theorem 3. Further, the equality in (2.10) holds for the functions of the form

$$f_n(z) = z + \frac{(B-A)\gamma(1-\beta)}{n^2(1+B\gamma)C(\alpha,n)} z^n \quad (n \geq 2). \tag{2.11}$$

Following the method of Theorem 3, we obtain a sufficient condition in terms of the modulus of the coefficients for a function to be in $R_\alpha(\beta,\gamma,A,B)$.

Theorem 6. *Let the function $f(z)$ defined by (1.1) satisfies the condition*

$$\sum_{n=2}^{\infty} (1+B\gamma)C(\alpha,n)|a_n| \leq (B-A)\gamma(1-\beta) \tag{2.12}$$

for $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $0 < \gamma \leq 1$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$. Then $f(z)$ is in the class $R_\alpha(\beta,\gamma,A,B)$. The equality in (2.12) is attained for the functions of the form

$$f_n(z) = z + \frac{(B-A)\gamma(1-\beta)}{(1+B\gamma)C(\alpha,n)} z^n \quad (n \geq 2). \tag{2.13}$$

Remark. The proof of Theorem 6 is omitted. Furthermore, analogous to subclass $H_\alpha(\beta,\gamma,A,B)$ of $P_\alpha(\beta,\gamma,A,B)$ and Theorem 4, it is a simple exercise to introduce and study corresponding subclasses of $Q_\alpha(\beta,\gamma,A,B)$ and $R_\alpha(\beta,\gamma,A,B)$.

The next theorem gives the coefficient bounds for the functions in the class $P_\alpha(\beta,\gamma,A,B)$.

Theorem 7. *Let the function $f(z)$ defined by (1.1) be in the class $P_\alpha(\beta,\gamma,A,B)$. Then we have*

$$|a_n| \leq \frac{(B-A)\gamma(1-\beta)}{nC(\alpha,n)} \quad (n \geq 2). \tag{2.14}$$

These bounds are sharp.

Proof. Let $f(z) \in P_\alpha(\beta,\gamma,A,B)$. Then it follows from the definition of subordination

$$(f * S_\alpha)'(z) = \frac{1+[B+(A-B)(1-\beta)]\gamma w(z)}{1+B\gamma w(z)}, \tag{2.15}$$

where $w(z) = \sum_{k=1}^{\infty} t_k z^k$ is analytic and satisfies the condition $|w(z)| < 1$ for all z in U. On simplification, (2.15) gives

$$\gamma \left[(B-A)(1-\beta) + \sum_{n=2}^{\infty} BnC(\alpha,n)a_n z^{n-1} \right] \left[\sum_{n=1}^{\infty} t_n z^n \right] =$$

$$= -\sum_{n=2}^{\infty} nC(\alpha,n)a_n z^{n-1}. \tag{2.16}$$

Equating corresponding coefficients on both sides of (2.16) we find that the coefficient a_n on the right side depends only on the coefficients $a_2, a_3, \ldots, a_{n-1}$ on the left side. Therefore, since $|w(z)| < 1$, (2.16) gives

$$\gamma \left| (B-A)(1-\beta) + \sum_{k=2}^{n-1} BkC(\alpha,k)a_k z^{k-1} \right| \geq \left| \sum_{k=2}^{n} kC(\alpha,k)a_k z^{k-1} - \sum_{k=n+1}^{\infty} b_k z^{k-1} \right|$$

for all $n \geq 2$. Writting $z = re^{i\theta}$, $r < 1$, squaring both sides of the preceeding inequality and then integrating, we obtain

$$\gamma^2 \left[(B-A)^2(1-\beta)^2 + \sum_{k=2}^{n-1} B^2 k^2 (C(\alpha,k))^2 |a_k|^2 r^{2(k-1)} \right] \geq$$

$$\geq \sum_{k=2}^{n} k^2 (C(\alpha,k))^2 |a_k|^2 r^{2(k-1)} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2(k-1)}.$$

Taking the limit as $r \to 1^-$, we have

$$\gamma^2 \left[(B-A)^2(1-\beta)^2 + \sum_{k=2}^{n-1} B^2 k^2 (C(\alpha,k))^2 |a_k|^2 \right] \geq$$

$$\geq n^2 (C(\alpha,n))^2 |a_n|^2 + \sum_{k=2}^{n-1} k^2 (C(\alpha,k))^2 |a_k|^2. \tag{2.17}$$

Since $0 < \gamma \leq 1$ and $0 < B \leq 1$, (2.17) immediately yields

$$(B-A)^2 \gamma^2 (1-\beta)^2 \geq n^2 (C(\alpha,n))^2 |a_n|^2$$

which proves (2.14). The bounds in (2.14) are sharp for the functions $f(z)$ defined by

$$(f * S_\alpha)(z) = \int_0^z \frac{1 - [B + (A-B)(1-\beta)]\gamma t^{n-1}}{1 - B\gamma t^{n-1}} dt \tag{2.18}$$

for $n \geq 2$ and for all $z \in U$.

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $Q_\alpha(\beta, \gamma, A, B)$. Then we have*

$$|a_n| \leq \frac{(B-A)\gamma(1-\beta)}{n^2 C(\alpha, n)} \quad (n \geq 2). \tag{2.19}$$

These bounds are sharp.

Proof. We need only replace a_n by na_n in Theorem 7.

Remark. We can show that the inclusion $Q_\alpha(\beta, \gamma, A, B) \subset P_\alpha(\beta, \gamma, A, B)$ and $H_\alpha(\beta, \gamma, A, B) \subset P_\alpha(\beta, \gamma, A, B)$ are both proper. In particular, for $f(z)$ given by (2.18) it follows that

$$f(z) = z + \frac{(B-A)\gamma(1-\beta)}{nC(\alpha, n)} z^n + \cdots = z + a_n z^n + \ldots$$

is in $P_\alpha(\beta, \gamma, A, B)$ but $f \notin Q_\alpha(\beta, \gamma, A, B)$ and $f \notin H_\alpha(\beta, \gamma, A, B)$ because a_n exceeds the coefficients bounds of the above Corollary 2 and (2.6).

By using the arguments similar to Theorem 7, we obtain the following

Theorem 8. *Let the function $f(z)$ defined by (1.1) be in the class $R_\alpha(\beta, \gamma, A, B)$. Then we have*

$$|a_n| \leq \frac{(B-A)\gamma(1-\beta)}{C(\alpha, n)} \quad (n \geq 2). \tag{2.20}$$

These bounds are sharp for the function $f(z)$ given by

$$(f * S_\alpha)(z) = \left(\frac{1 - [B + (A-B)(1-\beta)]\gamma z^{n-1}}{1 - B\gamma z^{n-1}} \right) z. \tag{2.21}$$

3. Subclasses of $P_\alpha(\beta, \gamma, A, B)$ and $Q_\alpha(\beta, \gamma, A, B)$

In view of Theorem 3, we introduce the following classes. Let $P_\alpha(\beta, \gamma, A, B; k)$ be the subclasses of $P_\alpha(\beta, \gamma, A, B)$ consisting of functions of the form

$$f(z) = z + \sum_{i=1}^{k} B_i p_i z^i + \sum_{n=k+1}^{\infty} a_n z^n, \tag{3.1}$$

where $0 \leq p_i < 1$, $0 \leq \sum_{i=2}^{k} p_i < 1$, and

$$B_i = \frac{(B-A)\gamma(1-\beta)}{i(1+B\gamma)C(\alpha, i)} \quad (i = 2, 3, \ldots, k). \tag{3.2}$$

Further, let $Q_\alpha(\beta,\gamma,A,B;k)$ be the subclass of $Q_\alpha(\beta,\gamma,A,B)$ consisting of functions of the form

$$f(z) = z + \sum_{i=2}^{k} E_i p_i z^i + \sum_{n=k+1}^{\infty} a_n z^n, \tag{3.3}$$

where $0 \le p_i < 1;\ 0 \le \sum_{i=2}^{k} p_i < 1$, and

$$E_i = \frac{(B-A)\gamma(1-\beta)}{i^2(1+B\gamma)C(\alpha,i)} \quad (i = 2,3,\dots,k). \tag{3.4}$$

Theorem 9. *Let the function $f(z)$ defined by (3.1) satisfies the condition*

$$\sum_{n=k+1}^{\infty} n(1+B\gamma)C(\alpha,n)|a_n| \le (B-A)\gamma(1-\beta)\left(1 - \sum_{i=2}^{k} p_i\right) \tag{3.5}$$

for $0 \le \alpha < 1,\ 0 \le \beta < 1,\ 0 < \gamma \le 1$, and $-1 \le A < B \le 1,\ 0 < B \le 1$. Then $f(z)$ is in the class $P_\alpha(\beta,\gamma,A,B;k)$.

Proof. By virtue of Theorem 3, we note that

$$f(z) = z + \sum_{i=2}^{k} B_i p_i z^i + \sum_{n=k+1}^{\infty} a_n z^n$$

belongs to the class $P_\alpha(\beta,\gamma,A,B;k)$ if

$$\sum_{i=2}^{k} i(1+B\gamma)C(\alpha,i)B_i p_i + \sum_{n=k+1}^{\infty} n(1+B\gamma)C(\alpha,n)|a_n| \le (B-A)\gamma(1-\beta), \tag{3.6}$$

or if

$$\sum_{i=2}^{k}(B-A)\gamma(1-\beta)p_i + \sum_{n=k+1}^{\infty} n(1+B\gamma)C(\alpha,n)|a_n| \le (B-A)\gamma(1-\beta). \tag{3.7}$$

This is equivalent to the condition (3.5). Further, by taking the function given by

$$f(z) = z + \sum_{i=2}^{k} B_i p_i z^i + \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} z^n \quad (n \ge k+1), \tag{3.8}$$

we can show that the result (3.5) is sharp.

Putting $p_i = 0$ $(i = 2,3,\dots,k)$ in Theorem 9, we have

Corollary 3. *Let the function $f(z)$ defined by (3.1) with $p_i = 0$ $(i = 2,3,\dots,k)$. If f satisfies*

$$\sum_{n=k+1}^{\infty} n(1+B\gamma)C(\alpha,n)|a_n| \le (B-A)\gamma(1-\beta) \tag{3.9}$$

for $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1,$ *and* $-1 \leq A < B \leq 1,\ 0 < B \leq 1,$ *then* $f(z) \in P_\alpha(\beta, \gamma, A, B; k)$.

Similarly, we obtain

Theorem 10. *Let the function* $f(z)$ *defined by (3.2) satisfies the condition*

$$\sum_{n=k+1}^{\infty} n^2(1+B\gamma)C(\alpha,n)|a_n| \leq (B-A)\gamma(1-\beta)\left(1-\sum_{i=2}^{k} p_i\right) \tag{3.10}$$

for $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1,$ *and* $-1 \leq A < B \leq 1,\ 0 < B \leq 1.$ *Then* $f(z)$ *is in the class* $Q_\alpha(\beta, \gamma, A, B; k)$.

Corollary 4. *Let the function* $f(z)$ *be defined by (3.2) with* $p_i = 0$ $(i = 2, 3, \ldots, k)$. *If* $f(z)$ *satisfies*

$$\sum_{n=k+1}^{\infty} n^2(1+B\gamma)C(\alpha,n)|a_n| \leq (B-A)\gamma(1-\beta) \tag{3.11}$$

for $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1,$ *and* $-1 \leq A < B \leq 1,\ 0 < B \leq 1,$ *then* $f(z) \in Q_\alpha(\beta, \gamma, A, B; k)$.

Remark. Putting $A = -1$ and $B = 1$ in the above theorems we get the results obtained by Ahuja and Owa [2].

References

[1] O.P. Ahuja and H. Silverman, *Convolutions of prestarlike functions*, Internat. J. Math. Sci. 6(1983), 59-78.

[2] O.P. Ahuja and S. Owa, *Classes of functions defined by convolutions*, Math. Japon, 33(1988), no.4, 485-499.

[3] M.K. Aouf and S. Owa, *On a class of univalent functions*, J. Math. Res. Exposition, 2(1991), 193-201.

[4] T.R. Caplinger and W.M. Causey, *A class of univalent functions*, Proc. Amer. Math. Soc. 39(1973), 357-361.

[5] J. Clunie and F.R. Keogh, *On starlike and convex schlicht functions*, J. London Math. Soc. 35(1960), 229-233.

[6] I.S. Jack, *Functions convex and starlike of order* α, J. London Math. Soc. 2(1971), 3, 469-474.

[7] O.P. Juneja and M.L. Mogra, *A class of univalent functions*, Bull. Sci. Math. 2 serie 103(1979), 435-447.

[8] T.H. MacGregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. 104(1962), 532-537.

[9] B.S. Mehrok, *A class of univalent functions*, Tamkang J. Math. 13(1982), no.2, 141-155.

[10] Z. Nehari, *Conformal Mapping*, McGraw Hill Book Co. Inc., 1952.

[11] S. Owa and O.P. Ahuja, *An application of the fractional calculus*, Math. Japon 30(1985), 947-955.

[12] S. Owa and O.P. Ahuja, *A class of functions defined by using Hadamard product*, Hokkaido Math. J. 15(1986), 217-232.

[13] K.S. Padmanabhan, *On a certain class of functions whose derivatives have a positive real part in the unit disc*, Ann. Polon. Math. 23(1970-71), 73-81.

[14] S. Ruscheweyh and V. Singh, *Convolution theorems for a class of bounded convex functions*, Rocky Mountain J. Math. 16(1986), 173-146.
[15] T. Sheil-Small, H. Silverman and E.M. Silvia, *Convolution multipliers and starlike functions*, J. Anal. Math. 41(1982), 181-192.
[16] H. Silverman and E.M. Silvia, *Prestarlike functions with negative coefficients*, Internat. J. Math. Sci. 2(1979), 427-439.
[17] T.J. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J. 37(1979), 775-777.

Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt
E-mail address: sinfac@mum.mans.eun.eg

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 2, June 2001

CONVOLUTIONS OF PRESTARLIKE FUNCTIONS

TÜNDE JAKAB

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

1.Introduction

We denote the class of starlike functions of order α by $S^*(\alpha)$, and the class of convex functions of order α by $K(\alpha)$. The function

$$s_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n$$

is the well-known extremal function for $S^*(\alpha)$, where

$$C(\alpha, n) = \frac{\prod_{k=2}^{n} (k - 2\alpha)}{(n-1)!} \qquad (n \geq 2).$$

Let $(f * g)(z)$ denote the Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ and $g(z)$ are given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let T denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0), \tag{1}$$

which are analytic in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$.

If $f(z)$ is given by (1) and

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geq 0),$$

then the Hadamard product of f and g is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $R[\alpha,\beta]$ be a subclass of T, consisting functions which satisfies

$(f * s_\alpha)(z) \in S^*(\beta)$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. Futher let $C[\alpha,\beta]$ be a subclass of T of functions satisfying $zf'(z) \in R[\alpha,\beta]$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. $R[\alpha,\beta]$ is called the class of functions α-prestarlike of order β with negative coefficients.

LEMMA 1.[7] *Let the function $f(z)$ be defined by* (1). *Then $f(z)$ is in the class $R[\alpha,\beta]$ if and only if*

$$\sum_{n=2}^{\infty}(n-\beta)C(\alpha,n)a_n \leq 1-\beta.$$

LEMMA 2.[3] *Let the function $f(z)$ be defined by* (1). *Then $f(z)$ is in the class $C[\alpha,\beta]$ if and only if*

$$\sum_{n=2}^{\infty}n(n-\beta)C(\alpha,n)a_n \leq 1-\beta.$$

Since $f(z)$ defined by (1) is univalent in the unit disc if $\sum\limits_{n=2}^{\infty} na_n \leq 1$; we can see that $f \in R[\alpha,\beta]$ is univalent if $0 \leq \alpha \leq \frac{1}{2}$; and a function $f(z) \in C[\alpha,\beta]$ is univalent in the unit disc if $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$.

LEMMA 3.[2, Th.8] *Let $f(z)$ a function defined by* (1) *be in the class $C[\alpha,\beta]$. Then f belongs to the class $R[\alpha,\gamma]$, where*

$$\gamma = \frac{2}{3-\beta}.$$

2.Convolutions

THEOREM 1. *If a function $f(z)$ defined by* (1) *belongs to the class $R[\alpha,\beta]$ with $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$, then $(\underbrace{f * f * ... * f}_{m})(z)$, $m \in \mathbf{N}= \{1,2,...\}$ belongs to the class $R[\alpha,\beta]$, too.*

Proof. Using Lemma 1 we have

$$\sum_{n=2}^{\infty}(n-\beta)C(\alpha,n)a_n^m \leq \left[\frac{1-\beta}{2(1-\alpha)(2-\beta)}\right]^{m-1}(1-\beta) \leq 1-\beta$$

with $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$.

THEOREM 2. *If a function $f(z)$ defined by* (1) *belongs to the class $C[\alpha,\beta]$ cu $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{7-3\beta}{4(2-\beta)}$, then $(\underbrace{f * f * ... * f}_{m})(z) \in C[\alpha,\beta]$,*

$(m \in \mathbf{N})$.

Proof. Using Lemma 2 we have

$$\sum_{n=2}^{\infty} n(n-\beta)C(\alpha,n)a_n^m \leq \left[\frac{1-\beta}{4(1-\alpha)(2-\beta)}\right]^{m-1}(1-\beta) \leq 1-\beta$$

with $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{7-3\beta}{4(2-\beta)}$.

THEOREM 3. *Let a function $f(z)$ defined by* (1) *be in the class $R[\alpha,\beta]$ with $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$; and let the function $g(z)$ defined by*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geq 0)$$

be in the class $C[\alpha,\beta]$ with $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$. Then we have

$$(\underbrace{f * f * ... * f}_{p} * \underbrace{g * g * ... * g}_{q})(z) \in C[\alpha,\beta], \qquad p,q \in \mathbf{N}.$$

Proof. Applying Lemma 1 and Lemma 2 we have

$$\sum_{n=2}^{\infty} n(n-\beta)C(\alpha,n)a_n^p b_n^q \leq$$

$$\leq \left[\frac{1-\beta}{2(1-\alpha)(2-\beta)}\right]^{p}\left[\frac{1-\beta}{4(1-\alpha)(2-\beta)}\right]^{q-1}(1-\beta) \leq 1-\beta$$

if $0 \leq \beta < 1$, $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$ and $0 \leq \alpha \leq \frac{7-3\beta}{4(2-\beta)}$.

But we have $\frac{3-\beta}{2(2-\beta)} < \frac{7-3\beta}{4(2-\beta)}$, and results $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$.

We need the following notation

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \qquad (a_{n,i} \geq 0, i = 1,2) \tag{2}$$

and the following results from [1]:

THEOREM 4.[1] *Let the function $f_1(z)$ defined by* (2) *be in the class $R[\alpha,\beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$ and let the function $f_2(z)$ defined by* (2) *be in the class $R[\alpha,\tau]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \tau < 1$. Then $(f_1 * f_2)(z) \in R[\alpha,\psi]$, where*

$$\psi = 1 - \frac{(1-\beta)(1-\tau)}{2(1-\alpha)(2-\beta)(2-\tau)-(1-\beta)(1-\tau)}.$$

The result is sharp for the functions

$$f_1(z) = z - \frac{1-\beta}{2(1-\alpha)(2-\beta)}z^2 \qquad \text{and} \qquad f_2(z) = z - \frac{1-\tau}{2(1-\alpha)(2-\tau)}z^2.$$

Theorem 5.[1] *Let the function* $f_1(z)$ *defined by* (2) *be in the class* $C[\alpha,\beta]$ *with* $0 \le \alpha \le \frac{1}{2}$ *and* $0 \le \beta < 1$ *and let the function* $f_2(z)$ *defined by* (2) *be in the class* $C[\alpha,\tau]$ *with* $0 \le \alpha \le \frac{1}{2}$ *and* $0 \le \tau < 1$. *Then* $(f_1 * f_2)(z) \in C[\alpha,\psi]$, *where*

$$\psi = 1 - \frac{(1-\beta)(1-\tau)}{4(1-\alpha)(2-\beta)(2-\tau)-(1-\beta)(1-\tau)}.$$

The result is sharp for the functions

$$f_1(z) = z - \frac{1-\beta}{4(1-\alpha)(2-\beta)}z^2 \qquad \text{and} \qquad f_2(z) = z - \frac{1-\tau}{4(1-\alpha)(2-\tau)}z^2.$$

The following two theorems are generalizations of the Theorem 4 and Theorem 5.

Theorem 6. *Let the functions* $f_i(z)$ $(i = 1,2,...,m)$ *defined by* (2) *be in the classes* $R[\alpha,\beta_i]$ *with* $0 \le \alpha \le \frac{1}{2}$ *and* $0 \le \beta_i < 1$ *for all* $i = 1,2,...,m$. *Then*$(f_1 * f_2 * ... * f_m)(z)$ *belongs to the class* $R[\alpha,\psi]$, *where*

$$\psi = 1 - \frac{\prod_{i=1}^{m}(1-\beta_i)}{2^{m-1}(1-\alpha)^{m-1}\prod_{i=1}^{m}(2-\beta_i)-\prod_{i=1}^{m}(1-\beta_i)}.$$

The result is sharp for the extremal functions defined by

$$f_i(z) = z - \frac{1-\beta_i}{2(1-\alpha)(2-\beta_i)}z^2 \qquad (i = 1,2,...m).$$

Proof. We apply the method of the mathematical induction.

For $m = 2$ and $\beta_1 = \beta$, $\beta_2 = \tau$, our theorem is reduced to Theorem 4, which is true. Suppose that

$$f_i(z) \in R[\alpha,\beta_i] \qquad (i = 1,2,...,k; k \in \mathbf{N}, k \ge 2) \Rightarrow$$

$$\Rightarrow (f_1 * f_2 * ... * f_k)(z) \in R[\alpha,\psi'],$$

where

$$\psi' = 1 - \frac{\prod_{i=1}^{k}(1-\beta_i)}{2^{k-1}(1-\alpha)^{k-1}\prod_{i=1}^{k}(2-\beta_i)-\prod_{i=1}^{k}(1-\beta_i)}.$$

If $f_{k+1} \in R[\alpha,\beta_{k+1}]$, then from Theorem 4, we have

$$((f_1 * f_2 * ... * f_k) * f_{k+1})(z) \in R[\alpha,\psi],$$

where

$$\psi = 1 - \frac{(1-\psi')(1-\beta_{k+1})}{2(1-\alpha)(2-\psi')(2-\beta_{k+1}) - (1-\psi')(1-\beta_{k+1})},$$

which is equivalent to

$$\psi = 1 - \frac{\prod_{i=1}^{k+1}(1-\beta_i)}{2^k(1-\alpha)^k \prod_{i=1}^{k+1}(2-\beta_i) - \prod_{i=1}^{k+1}(1-\beta_i)}.$$

This means that if the theorem is true for $m = k$, then it is true for $m = k+1$, so that it is true for all $m \geq 2$.

THEOREM 7. *Let the functions* $f_i(z)$ $(i = 1, 2, ..., m)$ *defined by* (2) *be in the classes* $C[\alpha, \beta_i]$ *with* $0 \leq \alpha \leq \frac{1}{2}$ *and* $0 \leq \beta_i < 1$ *for all* $i = 1, 2, ..., m$. *Then*$(f_1 * f_2 * ... * f_m)(z)$ *belongs to the class* $C[\alpha, \psi]$, *where*

$$\psi = 1 - \frac{\prod_{i=1}^{m}(1-\beta_i)}{4^{m-1}(1-\alpha)^{m-1} \prod_{i=1}^{m}(2-\beta_i) - \prod_{i=1}^{m}(1-\beta_i)}.$$

The result is sharp for the functions

$$f_i(z) = z - \frac{1-\beta_i}{4(1-\alpha)(2-\beta_i)} z^2 \qquad (i = 1, 2, ...m).$$

Proof. For $m = 2$ and $\beta_1 = \beta$, $\beta_2 = \tau$, our theorem is reduced to Theorem 5, which is true.

Suppose that

$$f_i(z) \in C[\alpha, \beta_i] \qquad (i = 1, 2, ..., k; k \in \mathbf{N}, k \geq 2) \Rightarrow$$

$$\Rightarrow (f_1 * f_2 * ... * f_k)(z) \in C[\alpha, \psi'],$$

where

$$\psi' = 1 - \frac{\prod_{i=1}^{k}(1-\beta_i)}{4^{k-1}(1-\alpha)^{k-1} \prod_{i=1}^{k}(2-\beta_i) - \prod_{i=1}^{k}(1-\beta_i)}.$$

If $f_{k+1} \in C[\alpha, \beta_{k+1}]$, then from Theorem 5, we have

$$((f_1 * f_2 * ... * f_k) * f_{k+1})(z) \in C[\alpha, \psi],$$

where

$$\psi = 1 - \frac{(1-\psi')(1-\beta_{k+1})}{4(1-\alpha)(2-\psi')(2-\beta_{k+1}) - (1-\psi')(1-\beta_{k+1})},$$

which is equivalent to

$$\psi = 1 - \frac{\prod_{i=1}^{k+1}(1-\beta_i)}{4^k(1-\alpha)^k \prod_{i=1}^{k+1}(2-\beta_i) - \prod_{i=1}^{k+1}(1-\beta_i)},$$

which means that the theorem is true for all $m \geq 2$.

THEOREM 8. *If* $f(z) \in C[\alpha, \beta_i]$ $(i = 1, 2, ..., m)$ *with*$0 \leq \alpha \leq \frac{1}{2}$ *and* $0 \leq \beta_i < 1$ *for all* $i = 1, 2, ..., m$, *then*$(f_1 * f_2 * ... * f_m)(z) \in R[\alpha, \tau]$, *where*

$$\tau = 1 - \frac{\prod_{i=1}^{m}(1-\beta_i)}{2 \cdot 4^{m-1}(1-\alpha)^{m-1} \prod_{i=1}^{m}(2-\beta_i) - \prod_{i=1}^{m}(1-\beta_i)}.$$

The result is sharp.

From Theorem 6 (or Theorem 7) and Lemma 3 we obtain the result.

THEOREM 9. *Let the functions* $f_i(z)$ $(i = 1, 2)$ *defined by* (2) *be in the class* $C[\alpha, \beta]$ *with* $0 \leq \alpha \leq \frac{1}{2}$ *and* $0 \leq \beta < 1$. *Then the function* $h(z)$ *defined by*

$$h(z) = z - \sum_{n=2}^{\infty} \left[a_{n,1}^2 + a_{n,2}^2\right] z^n$$

belongs to the class $R[\alpha, \gamma]$, *where*

$$\gamma = 1 - \frac{(1-\beta)^2}{4(1-\alpha)(2-\beta)^2 - (1-\beta)^2}.$$

The result is sharp.

Using Theorem 9 (or Theorem 10) from [1] and Lemma 3 we obtain immediately the result.

References

[1] M. K. Aouf and G. S. Sălăgean, *Certain subclasses of prestarlike functions with negative coefficients*, Studia Univ. Babeş-Bolyai, Math., 39 (1994), no. 1, 19-30.

[2] T. Jakab, *Subclasses of prestarlike functions with negative coefficients* (to appear).

[3] S. Owa and B. A. Uralegaddi, *A class of functions α-prestarlike of order β*, Bull. Korean Math. Soc. 21 (1984), No.2, 77-85.

[4] S. M. Sarangi and B. A. Uralegaddi, *Certain generalization of prestarlike functions with negative coefficients*, Ganita, 34 (1983), 99-105.

[5] T. Sheil-Small, H. Silverman and E. Silvia, *Convolution multipliers and starlike functions*, J. Analyse Math. 41 (1982), 181-192.

[6] H. Silverman and E. Silvia, *Prestarlike functions with negative coefficients*, Internat. J. Math. & Math. Sci. (3) 2 (1979), 427-439.

[7] H. Silverman and E. Silvia, *Subclasses of prestarlike functions*, Math. Japon. 29 (1984), No.6, 929-935.

Babeş-Bolyai University, 3400 Cluj-Napoca, Kogălniceanu Nr.1, Romania

NORM ESTIMATES, COEFFICIENT ESTIMATES AND SOME PROPERTIES OF SPIRAL-LIKE FUNCTIONS

YÛSUKE OKUYAMA

Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. This is a survey of the author's talk at the VIIIth Romanian-Finnish Seminar in Iassy, Romania, in 23-27 August 1999. We shall state the sharp estimates of the norms of pre-Schwarzian and Schwarzian derivatives of spiral-like functions and about the optimal growth estimates of coefficients of them. We shall also remark that some spiral-like function $f(z) = z + a_2 z^2 + \cdots$ is normalized and univalent on the unit disk $\mathbb{D}$ but satisfies $a_2 f(z) + 1 = 0$ for some $z \in \mathbb{D}$.

1. Introduction

We consider an analytic function f on the unit disk $\mathbb{D}$ normalized so that $f(0) = f'(0) - 1 = 0$. For a constant $\beta \in (-\pi/2, \pi/2)$, such a function f is called *β-spiral-like* if f is univalent on $\mathbb{D}$ and for any $z \in \mathbb{D}$, the β-logarithmic spiral $\{f(z)\exp(-e^{i\beta}t); t \geq 0\}$ is contained in $f(\mathbb{D})$. It is equivalent to the analytic condition that $\Re(e^{-i\beta} z f'(z)/f(z)) > 0$ in $\mathbb{D}$. We denote by $SP(\beta)$ the set of β-spiral-like functions. We call $f_\beta(z) := z(1-z)^{-2e^{i\beta}\cos\beta} \in SP(\beta)$ the β-spiral Koebe function. Note that $SP(0)$ is the set of starlike functions and that $f_0(z) = z(1-z)^{-2}$ is the Koebe function. The β-spiral Koebe function conformally maps the unit disk onto the complement of the β-logarithmic spiral $\{f_\beta(-e^{-2i\beta})\exp(-e^{i\beta}t); t \leq 0\}$ in $\mathbb{C}$. For the known results about these classes of the functions, see, for example, [1].

1991 *Mathematics Subject Classification.* 30C45.

Key words and phrases. spiral-like function, pre-Schwarzian derivative, Schwarzian derivative, growth estimate, strongly normalized univalent function.

Partially supported by JSPS Research Fellowships for Young Scientists.

Paper presented at the International Conference on Complex Analysis and The 8th Romanian-Finnish Seminar, Iaşi, 23-27 August 1999.

2. Norm estimates

For a locally univalent holomorphic function f, we define

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = (T_f)' - \frac{1}{2}(T_f)^2,$$

which are said to be the *pre-Schwarzian derivative* (or nonlinearity) and the *Schwarzian derivative* of f, respectively. For a locally univalent function f in $\mathbb{D}$, we define the norms of T_f and S_f by

$$\|T_f\|_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)| \quad \text{and} \quad \|S_f\|_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f(z)|,$$

respectively.

As well as $\|S_f\|_2$, the norm $\|T_f\|_1$ has a significant meaning in the theory of Teichmüller spaces. For example, see [8], [2] and [13].

We shall give the best possible estimate of the norms of pre-Schwarzian derivatives for the class $SP(\beta)$.

Main Theorem 1 ([9]). *For any $f \in SP(\beta)$, where $\beta \in (-\pi/2, \pi/2)$, we have the following.*

I) *In the case $|\beta| \le \pi/3$, we have*

$$\|T_f\|_1 \le \|T_{f_\beta}\|_1 = 2|2+e^{2i\beta}|. \tag{1}$$

II) *In the case $|\beta| > \pi/3$, we have $\|T_f\|_1 \le \|T_{f_\beta}\|_1$, where*

$$\|T_{f_\beta}\|_1 = \max_{0\le m\le \frac{4}{3}\sin|\beta|} 2m\cos\beta\left(1+\sqrt{\frac{m^2+4-4m\sin|\beta|}{m^2+1-2m\sin|\beta|}}\right) \quad \text{and} \tag{2}$$

$$2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1 < 2\left(1+\frac{4}{3}\sin 2|\beta|\right). \tag{3}$$

In particular, $\|T_{f_\beta}\|_1 \to 2$ as $|\beta| \to \pi/2$.

In both cases, the equality $\|T_f\|_1 = \|T_{f_\beta}\|_1$ holds if and only if f is a rotation of the β-spiral Koebe function, i.e., $f(z) = (1/\varepsilon)f_\beta(\varepsilon z)$ for some $|\varepsilon| = 1$.

The proof of Main Theorem 1 is in [9]. From the proof, if $|\beta| \le \pi/3$, the function $(1-|z|^2)|T_{f_\beta}(z)|$ does not attain its supremum in $\mathbb{D}$. However if $|\beta| > \pi/3$, it does since

$$\max_{\partial\mathbb{D}\ni z_0} \limsup_{\mathbb{D}\ni z\to z_0}(1-|z|^2)|T_{f_\beta}(z)| = 2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1.$$

This phenomenon of *phase transition* seems to be quite interesting.

Remark. Clearly, the β-spiral Koebe function f_β converges to $id_{\mathbb{D}}$ (which is bounded) locally uniformly on $\mathbb{D}$ as $|\beta| \to \pi/2$ but does not converge to it with respect to the norm $\|\cdot\|_1$ since $\lim_{|\beta|\to\pi/2} \|T_{f_\beta}\|_1 = 2$. On the other hand, it is known that a normalized analytic function f is bounded if $\|T_f\|_1 < 2$. In fact, the value 2 is the least one of the norms of unbounded normalized analytic functions.

We would also like to mention the related works about norm estimates of pre-Schwarzian derivatives in other classes by Shinji Yamashita [11] and Toshiyuki Sugawa [10].

Theorem 2.1. *Let* $0 \le \alpha < 1$ *and* f *be a normalized analytic function.*

If f *is starlike of order* α, *i.e.,* $\Re(zf'(z)/f(z)) > \alpha$, *then* $\|T_f\|_1 \le 6 - 4\alpha$.

If f *is convex of order* α, *i.e.,* $\Re(1+zf''(z)/f'(z)) > \alpha$, *then* $\|T_f\|_1 \le 4(1-\alpha)$.

If f *is strongly starlike of order* α, *i.e.,* $\arg(zf'(z)/f(z)) < \pi\alpha/2$, *then* $\|T_f\|_1 \le M(\alpha)+2\alpha$, *where* $M(\alpha)$ *is a specified constant depending only on* α *satisfying* $2\alpha < M(\alpha) < 2\alpha(1+\alpha)$.

All of the bounds are sharp.

On the other hand, we also obtain the estimate of the norms of Schwarzian derivatives of β-spiral-like functions.

Main Theorem 2 ([9]). *Assume* $|\beta| < \pi/2$. *For any* $f \in SP(\beta)$, $\|S_f\|_2 \le \|S_{f_\beta}\|_2 = 6$.

In the rest of this article, we shall state two remarks about spiral-like functions.

3. **Order estimates of the coefficients**

Knowing the norm $\|T_f\|_1$ enables us to estimate the growth of coefficients of f. For example, the following holds.

Theorem 3.1 (cf.[7]). *Let* $(3/2) < \lambda \le 3$. *For a normalized analytic function* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *such that* $\|T_f\|_1 \le 2\lambda$, *it holds that* $a_n = O(n^{\lambda-2})$ *as* $n \to +\infty$. *This order estimate is best possible.*

However the sharp estimate of coefficients of $f \in SP(\beta)$ has been already obtained by Zamorski [12] in 1960. We would like to remark that we can derive the sharp growth estimate of coefficients of $f \in SP(\beta)$ from this.

Theorem 3.2 (Zamorski). *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is in $SP(\beta)$ and $|\beta| < \pi/2$, then*

$$|a_n| \leq \prod_{k=1}^{n-1} \left|1 + \frac{e^{2i\beta}}{k}\right| \tag{4}$$

for $n \geq 2$. The equality in (4) holds for some $n \geq 2$ if and only if f is a rotation of the β-spiral Koebe function f_β.

Remark. This is also shown in terms of generalized spiral-like functions by C. Burniak, J. Stankiewicz and Z. Stankiewicz [4](1980).

Corollary 3.1. *Let $|\beta| < \pi/2$ and $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be a β-spiral-like function. Then it holds that*

$$a_n = O(n^{\cos 2\beta}) \quad (n \to +\infty). \tag{5}$$

This order estimate is sharp.

Remark. In the case $|\beta| < \pi/4$, this is shown by Basgöze and Keogh in [3](1970).

4. **Strongly normalized univalent functions are not always holomorphic.**

The following is known.

Theorem 4.1. *For a holomorphic function ϕ on a simply connected domain A, there exists a locally univalent meromorphic function f on A such that*

$$S_f = \phi.$$

The solution is unique up to postcomposition of an arbitrary Möbius transformation.

We assume $A = \mathbb{D}$. Nehari showed that if $\|\phi\|_2 = \sup_{z\in\mathbb{D}} |\phi(z)|(1-|z|^2)^2 \leq 2$, then f is univalent and meromorphic on $\mathbb{D}$. It is well-known that if f is *strongly normalized*, i.e., $f(0) = f'(0) - 1 = f''(0) = 0$, then f is holomorphic on $\mathbb{D}$. Since for a normalized analytic function $f(z) = z + a_2z^2 + \cdots$, $g := f/(a_2f + 1)$ is strongly normalized and $\|S_f\|_2 = \|S_g\|_2$, we have the following.

Proposition 4.1 ([6], [5] Corollary 2). *If a normalized analytic function* $f(z) = z + a_2 z^2 + \cdots$ *satisfies* $\|S_f\|_2 \leq 2$*, then* f *is univalent and* $a_2 f + 1 \neq 0$ *on* $\mathbb{D}$.

In [5] Chuaqui and Osgood remark that a strongly normalized univalent function f is not always holomorphic if $\|S_f\|_2 > 2$. Spiral-like functions are examples for this fact.

Theorem 4.2. *If* $|\beta|$ *is sufficiently close to* $\pi/2$*, the* β*-spiral-Koebe function* $f_\beta(z) = z + a_2 z^2 + \cdots$ *satisfies* $a_2 f_\beta(z) + 1 = 0$ *for some* $z \in \mathbb{D}$.

References

[1] Ahuja, O. P. and Silverman, H. A survey on spiral-like and related function classes, *Math. Chronicle*, **20** (1991), 39–66.

[2] Astala, K. and Gehring, F. W. Injectivity, the *BMO* norm and the universal Teichmüller space, *J. Analyse Math.*, **46** (1986), 16–57.

[3] Basgöze, T. and Keogh, F. B. The Hardy class of a spiral-like function and its derivative, *Proc. Amer. Math. Soc*, **26** (1970), 266–269.

[4] Burniak, C., Stankiewicz, J. and Stankiewicz, Z. The estimations of coefficients for some subclasses of spirallike functions, *Demonstratio Math.*, **15** (1982), 427–441.

[5] Chuaqui, M. and Osgood, B. Sharp distortion theorems associated with the Schwarzian derivative, *J. London Math. Soc.*, **48** (1993), 289–298.

[6] Gehring, F. W. and Pommerenke, C. On the Nehari univalence criterion and quasicircles, *Comment. Math. Helv.*, **59** (1984), 226–242.

[7] Kim, Y. C. and Sugawa, T. Growth and coefficient estimates for uniformly locally univalent functions on the unit disk, *Sūrikaisekikenkyūsho Kōkyūroku*, **1062** (1998), 34–46.

[8] Lehto, O. *Univalent Functions and Teichmüller Spaces*, Springer-Verlag (1987).

[9] Okuyama, Y. The Norm Estimates of Pre-Schwarzian Derivatives of Spiral-like Functions, *Complex Variables Theory Appl.*, **42** (2000), 225–239.

[10] Sugawa, T. On the norm of the pre-Schwarzian derivatives of strongly starlike functions, *Ann. Univ. Mariae Curie-Skłodowska, Sectio A*, **52** (1998), 149–157.

[11] Yamashita, S. Norm estimates for function starlike or convex of order alpha, *Hokkaido Math*, **28** (1999), 217–230.

[12] Zamorski, J. About the extremal spiral schlicht functions, *Ann. Polon. Math*, **9** (1961), 265–273.

[13] Zhuravlev, I. V. A model of the universal Teichmüller space, *Siberian Math. J.*, **27** (1986), 691–697.

Department of Mathematics, Graduate School of Science Kyoto University, Kyoto 606-8502, Japan
E-mail address: okuyama@kusm.kyoto-u.ac.jp

ON THE UNIVALENCE OF CONVEX FUNCTIONS OF COMPLEX ORDER

HORIANA OVESEA, IRINEL RADOMIR

Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. In this note we study the univalence of the functions f who belong to the class of convex functions of complex order introduced by Nasr and Aouf [2]. The results obtained improve the results from paper [3].

1. Introduction

Let A be the class of functions f analytic in the unit disk $U = \{z \in C : |z| < 1\}$ and such that $f(0) = 0,\ f'(0) = 1$.
Let S denote the class of functions $f \in A,\ f$ univalent in U.
Nasr and Aouf defined the class of functions $f \in A,\ f'(z) \neq 0$ in U, for which $Re[1 + zf''(z)/(\alpha f'(z))] > 0$, where $\alpha \in C$. For a fixed complex number α, $\alpha \neq 0$, let us denote this class by $N(\alpha)$,

$$N(\alpha) = \left\{ f \in A : \ Re\left(1 + \frac{1}{\alpha}\frac{zf''(z)}{f'(z)}\right) > 0, \quad f'(z) \neq 0, \quad (\forall) z \in U \right\} \tag{1}$$

Theorem 1.1 ([3]). *Let α be a complex number, $\alpha \neq 0$ and let $f \in N(\alpha)$. If $\alpha \in D$, where*

$$D = D_1 \cup D_2 \cup [-1/2,\ -1/4] \cup [1/4,\ 3/2] \quad \text{and} \tag{2}$$

$$D_1 = \{\alpha \in C : \quad |\alpha| \leq 1/4\}$$

$$D_2 = \{\alpha \in C : \ |\alpha - 1/2| \leq 1/2 \quad \text{and} \quad \pi/3 \leq |\arg \alpha| \leq \pi/2\},$$

then the function f is univalent in U.

2. Preliminaries

Theorem 2.1 ([4]). *Let $f \in A$. Let α, β, c be complex numbers, $Re\beta > 0$, $Re(2\alpha+\beta) > 0$, $Re\frac{\alpha}{\beta} > -1/2$, $|c(\alpha+\beta)+\alpha| + |\alpha| \le |\alpha+\beta|$. If there exists an analytic function g, $g \in A$, such that*

$$\left|(1+c)\frac{f'(z)}{g'(z)} - 1\right| < 1, \qquad (\forall) z \in U,$$

$$\left|\left[(1+c)\frac{f'(z)}{g'(z)} - 1\right]|z|^{2(\alpha+\beta)} + \frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zg''(z)}{g'(z)} - \alpha\right)\right| \le 1$$

for all $z \in U \setminus \{0\}$, then the function

$$F(z) = \left(\beta \int_0^z u^{\beta-1} f'(u) du\right)^{1/\beta}$$

is analytic and univalent in U.

The results obtained are proved by using Theorem 2.1 in the particular case $f \equiv g$ and $\alpha = 1-\beta$. For this choise, from Theorem 2.1 we get the following

Corollary 2.1. *Let $f \in A$ and let β, c be complex numbers. If $|\beta - 1| < 1$, $|c| < 1$, $|c+1-\beta| + |\beta-1| \le 1$ and*

$$\left|c|z|^2 + (1-|z|^2)\left(\frac{zf''(z)}{f'(z)} + \beta - 1\right)\right| \le 1, \qquad (\forall) z \in U, \tag{3}$$

then the function

$$F(z) = \left(\beta \int_0^z u^{\beta-1} f'(u) du\right)^{1/\beta} \tag{4}$$

is analytic and univalent in U.

Theorem 2.2 ([1]). *If g is a starlike function in U and $-1/2 \le \alpha \le 3/2$, then the function*

$$G(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha} du$$

is a close-to-convex function in U.

Lemma 2.1. *If g is a starlike function in U and a is a fixed point from the unit disk U, then the function*

$$h(z) = \frac{a \cdot z}{(a+z)(1+\overline{a}z)g(a)} \cdot g\left(\frac{a+z}{1+\overline{a}z}\right) \tag{5}$$

is a starlike function in U.

3. Main results

Theorem 3.1. *Let α, β be complex numbers, $\alpha \neq 0$, $|\beta - 1| < 1$ and let $f \in N(\alpha)$. If*

$$|\alpha| < \frac{1 - |\beta - 1|}{2}, \tag{6}$$

then it exists an univalent function F in U, such that

$$f(z) = \int_0^z \left(\frac{F(u)}{u}\right)^{\beta - 1} F'(u)du \ , \quad z \in U. \tag{7}$$

Proof. Let us consider the function

$$g(z) = z \cdot (f'(z))^{1/\alpha} \ , \ \ \alpha \neq 0.$$

We have

$$\frac{zg'(z)}{g(z)} = 1 + \frac{1}{\alpha}\frac{zf''(z)}{f'(z)} \tag{8}$$

Because $f \in N(\alpha)$ it follows that $Re[zg'(z)/g(z)] > 0$ in U and hence g is a starlike function in U. Let h be the function defined by (5), $h(z) = z + a_2 z^2 + \ldots$. We obtain

$$a_2 = \frac{h''(0)}{2} = (1 - |a|^2)\frac{g'(a)}{g(a)} - \frac{1 + |a|^2}{a}$$

and then

$$\frac{zg'(z)}{g(z)} = \frac{1 + a_2 z + |z|^2}{1 - |z|^2} \tag{9}$$

The relations (8) and (9) lead to

$$\frac{zf''(z)}{f'(z)} = \alpha\left(\frac{zg'(z)}{g(z)} - 1\right) = \alpha\frac{a_2 z + 2|z|^2}{1 - |z|^2} \tag{10}$$

Taking into account (10) it results

$$c|z|^2 + (1 - |z|^2)\left(\frac{zf''(z)}{f'(z)} + \beta - 1\right) = \tag{11}$$

$$= (c + 2\alpha + 1 - \beta)|z|^2 + \alpha a_2 z + \beta - 1 \ .$$

If $c = \beta - 1 - 2\alpha$, from (6) it follows that $|c| < 1$ and also

$$|c + 1 - \beta| + |\beta - 1| = |2\alpha| + |\beta - 1| < 1 \ .$$

Since h is a starlike function, then $|a_2| \leq 2$ and in view of (6) , the relation (11) becomes

$$\left| c|z|^2 + (1 - |z|^2)\left(\frac{zf''(z)}{f'(z)} + \beta - 1\right)\right| =$$

$$= |\alpha a_2 z + \beta - 1| \leq 2|\alpha| + |\beta - 1| < 1 \,.$$

From Corollary 2.1 we conclude that the function

$$F(z) = \left(\beta \int_0^z u^{\beta-1} f'(u) du\right)^{1/\beta}$$

is analytic and univalent in U.

We have $F^{\beta-1}(z)F'(z) = z^{\beta-1}f'(z)$ and therefore

$$f'(z) = \left(\frac{F(z)}{z}\right)^{\beta-1} F'(z).$$

It follows that the function f is given by (7), where F is analytic and univalent in U.

If in Theorem 3.1 we take $\beta = 1$, then we have $f(z) = F(z)$ and we get the following result

Corollary 3.1. *Let α be a complex number, $\alpha \neq 0$ and let $f \in N(\alpha)$. If $|\alpha| < 1/2$, then the function f is univalent in U.*

Theorem 3.2. *Let α be a complex number , $\alpha \neq 0$ and let $f \in N(\alpha)$. If $\alpha \in D$, where*

$$D = D_1 \cup [1/2,\ 3/2] \cup \{-1/2\} \,, \tag{12}$$

$$D_1 = \{\alpha \in C : \ |\alpha| < 1/2\} \,,$$

then the function f is univalent in U.

If α is a real number, $\alpha \notin D$, then the function

$$f(z) = \int_0^z (1-u)^{-2\alpha} du \tag{13}$$

belongs to the class $N(\alpha)$ but it is not univalent in U.

Proof. If $\alpha \in D_1$, from Corollary 3.1 it follows that f is an univalent function. Let α be a real number, $\alpha \in [-1/2,\ 3/2] \setminus \{0\}$. In the same manner as in Theorem 3.1 we consider the function $g(z) = z(f'(z))^{1/\alpha}$. The function g being a starlike function, from Theorem 2.2 it follows that the function

$$G(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha} du = \int_0^z f'(u) du = f(z)$$

is a close-to-convex function. For the function f defined by (13) a short computation gives

$$1 + \frac{1}{\alpha} \frac{z f''(z)}{f'(z)} = \frac{1+z}{1-z}$$

For $z \in U$ we have $Re(1+z)/(1-z) > 0$ and hence $f \in N(\alpha)$.

For $\beta \in R$, $\beta \neq 0$, we know that the function $h(z) = (1-z)^{\beta}$ is univalent in U if and only if $\beta \in [-2, 2]$. From (13) we get

$$f(z) = \frac{1}{2\alpha - 1} \left[(1-z)^{-2\alpha+1} - 1 \right], \qquad \alpha \neq 1/2$$

and then the function f is not univalent if $\alpha < -1/2$ or $\alpha > 3/2$.

References

[1] A.W.Goodman, *Univalent functions* , Mariner Publishing Company Inc., 1984.
[2] M.A.Nasr, M.K.Aouf, *Caracterisation for convex and starlike functions of complex order in U* , Bull. Fac. Sci. Assiut Univ., A11(1982),nr.2, 117-121.
[3] H.Ovesea, N.N.Pascu, I.Radomir, *On the univalence of Nasr's function*, Sem.geom. function theory, Preprint 2(1991), Braşov, 73-76.
[4] H. Ovesea, *An extension of Lewandowski's univalence criteria*, Demonstratio Mathematica, XXXIX, 4(1996),699-706.

Department of Mathematics, "Transilvania" University, 2200 Braşov, Romania

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 2, June 2001

ON UNIVALENT FUNCTIONS IN A HALF-PLANE

NICOLAE N. PASCU

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. A basic result in the theory of univalent functions is well-known inequality $\left|-2\left|z^2\right| + \left(1-|z|^2\right) zf''(z)/f'(z)\right| \leq 4|z|$ where f is an univalent function in the unit disc. In this note we obtain a similar result for univalent functions in the upper half-plane.

1. Introduction.

Let U be the unit disc $\{z : z \in C, |z| < 1\}$ and let A be the class of analytic and univalent functions in U. We denote by S the class of the functions f, $f \in A$, normalized by conditions $f(0) = f'(0) - 1 = 0$.

As a corollary of the inequality of the second coefficient, for the functions in the class S, it results the following well-known theorem:

Theorem A. If the function f belongs to the class A, then for all $z \in U$ we have

$$\left|-2|z|^2 + \left(1-|z|^2\right) zf''(z)/f'(z)\right| \leq 4|z|.$$

The Theorem A is the starting point for solving some problems (distortion theorem, rotation theorem) in the class S.

We denote by D the upper half-plane $\{z : Im(z) > 0\}$ and by S_D the class of analytic and univalent functions in the domain D. In this note we obtain a result, similar to the Theorem A, for functions in the class S_D.

Paper presented at the International Conference on Complex Analysis and The 8th Romanian-Finnish Seminar, Iaşi, 23-27 August 1999.

2. Main results.

Let $g : U \to D$ be the function defined from

$$g(z) = i\frac{1-z}{1+z} \tag{1}$$

The function g belongs to the class A and $g(U) = D$.

We denote by D_r the disc $g(U_r)$, where $r \in (0,1]$, $U_r = \{z : |z| < r\}$ and $U_1 = U$. We observe that, for all $0 < r < s \leq 1$ we have $D_r \subset D_s \subset D_1 = D$ and hence for all $\xi \in D$, there exists $r_0 \in (0,1)$ such that $\xi \in D_r$, for all $r \in (r_0, 1)$.

Let ξ_r and R_r be the numbers defined from

$$\xi_r = i\frac{1+r^2}{1-r^2}, \quad R_r = \frac{2r}{1-r^2} \tag{2}$$

For $\xi = g(z)$, where $|z| = r$, we have

$$|\xi - \xi_r|^2 = \frac{4\left|z+r^2\right|^2}{|1+z|^2\left(1-r^2\right)^2} \tag{3}$$

Because for all z, $|z| = r < 1$, we have

$$\left|z+r^2\right| = |r+rz| \tag{4}$$

it result that

$$|\xi - \xi_r| = R_r \tag{5}$$

and hence D_r is the disc with the center at the point ξ_r and the radius R_r.

Lemma. For all fixed point $\xi \in D$ there exists $r_0 \in (0,1)$ and $u_r \in U$ such that for all $r \in (r_0, 1)$

$$\xi = \xi_r + R_r u_r \tag{6}$$

and

$$\lim_{r\to 1} u_r = -i, \quad \lim_{r\to 1}[R_r(1-|u_r|)] = \operatorname{Im}(\xi). \tag{7}$$

Proof. If $\xi \in D$, then $\left|g^{-1}(\xi)\right| < 1$ and hence for all r_0, $\left|g^{-1}(\xi)\right| < r_0 < 1$ we have $\xi \in D_r$, for all r, $r_0 < r < 1$.

For $x_r = \operatorname{Re}(u_r)$, $y_r = \operatorname{Im}(u_r)$, $X = \operatorname{Re}(\xi)$, $Y = \operatorname{Im}(\xi)$ we have

$$X = x_r\frac{2r}{1-r^2}, \quad Y = \frac{1+r^2}{1-r^2} + y_r\frac{2r}{1-r^2} \tag{8}$$

for all $r, r_0 < r < 1$ and hence

$$\lim_{r\to 1} x_r = \lim_{r\to 1}\frac{\left(1-r^2\right)X}{2r} = 0, \quad \lim_{r\to 1} y_r = \lim_{r\to 1}\frac{\left(1-r^2\right)Y - 1 - r^2}{2r} = -1 \tag{9}$$

From (8) we have

$$\left(1-|u_r|^2\right)R_r=\left[1-\frac{\left(1-r^2\right)^2X^2+\left(\left(1-r^2\right)Y-\left(1+r^2\right)\right)^2}{4r^2}\right]\cdot\frac{2r}{1-r^2} \tag{10}$$

It result that

$$\lim_{r\to 1}\left(1-|u_r|^2\right)R_r=\lim_{r\to 1}\frac{2\left(1+r^2\right)\mathrm{Im}\left(\xi\right)-\left(1-r^2\right)|\xi|^2-1+r^2}{2r}=2\mathrm{Im}\left(\xi\right) \tag{11}$$

and hence

$$\lim_{r\to 1}[(1-|u_r|)\,R_r]=\mathrm{Im}\left(\xi\right) \tag{12}$$

Theorem. If the function f is analytic and univalent in the domain D, for all $\xi\in D$ we have

$$\left|i-\mathrm{Im}\left(\xi\right)\frac{f''\left(\xi\right)}{f'\left(\xi\right)}\right|\leq 2 \tag{13}$$

Proof. Let ξ be a fixed point in the domain D. From Lemma it result that there exists $r_0\in(0,1)$ such that $\xi\in D_r$ for all $r\in(r_0,1)$. We consider the function $g_r:U\to C$ defined from

$$g_r\left(u\right)=f\left(\xi_r+R_ru\right) \tag{14}$$

where $r\in(r_0,1)$.

For all fixed $r, r\in(r_0,1)$ the function g_r is analytic and univalent in U and from Theorem A it result that

$$\left|-2\left|u\right|^2+\left(1-|u|^2\right)R_r\frac{uf''\left(\xi_r+R_ru\right)}{f'\left(\xi_r+R_ru\right)}\right|\leq 4\left|u\right| \tag{15}$$

From Lemma it result that for fixed point $\xi\in D$ there exists $u_r\in U$ such that $\xi=\xi_r+R_ru_r$ and hence, from (15) we obtain

$$\lim_{r\to 1}\left|-2\left|u_r\right|^2+\left(1-|u_r|^2\right)R_r\frac{u_rf''\left(\xi\right)}{f'\left(\xi\right)}\right|\leq 4\lim_{r\to 1}\left|u_r\right| \tag{16}$$

Because $\lim\limits_{r\to 1}u_r=-i$ and $\lim\limits_{r\to 1}[(1-|u_r|)\,R_r]=\mathrm{Im}\left(\xi\right)$, form (16) we obtain the inequality (13).

Remark. The function f defined from

$$f\left(\xi\right)=\xi^2 \tag{17}$$

is analytic and univalent in the domain D and

$$\left|i-\mathrm{Im}\left(\xi\right)\frac{f''\left(\xi\right)}{f'\left(\xi\right)}\right|=\left|i-\mathrm{Im}\left(\xi\right)\frac{1}{\xi}\right| \tag{18}$$

If we observe that $\left| i - \operatorname{Im}(\xi)\,\dfrac{1}{\xi} \right| = 2$ for $\xi = i$, it result that the inequality (13) is best possible.

Univ. "Transilvania" Braşov, Faculty of Sciences, Romania

ON CONVEX FUNCTIONS IN AN ELLIPTICAL DOMAIN

NICOLAE N. PASCU, DORINA RĂDUCANU, MIHAI N. PASCU, RADU N. PASCU

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. In this note we define the notions of convexity for analytic functions in the ellipse $E = \left\{ z = x + iy \in \mathbb{C} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 < 0 \right\}$, $a > b > 0$. We obtain sufficient conditions for an analytic function to be a convex function in the ellipse E.

1. **Introduction and preliminaries**

Let g be a complex function defined in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For $z = x + iy \in U$ we consider $u(x, y) = \text{Re}g(z)$ and $v(x, y) = \text{Im}g(z)$. The function g belongs to the class $C^1(U)$, respectively $C^2(U)$ if the functions u and v of the real variables x and y have continuous first order, respectively second order, partial derivatives in U [1].

For $g \in C^1(U)$ the following operators are defined

$$Dg(z) = z\frac{\partial g}{\partial z} - \overline{z}\frac{\partial g}{\partial \overline{z}} \quad \text{and} \quad Jg = \left|\frac{\partial g}{\partial z}\right|^2 - \left|\frac{\partial g}{\partial \overline{z}}\right|^2$$

where

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right) \quad \text{and} \quad \frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right).$$

P.T. Mocanu [1] obtained sufficient conditions for a non-analytic function in the unit disc, to be univalent and convex.

Definition 1. [1] A function g of the class $C^1(U)$ is a convex function in U if it is univalent and $g(U)$ is a convex domain.

A sufficient condition for convexity is given in the following theorem.

Theorem 1. [1] *If the function* $g \in C^1(U)$ *satisfies the conditions*

(i) $g(0) = 0$, $Dg \in C^1(U)$ *and* $g(z)Dg(z) \neq 0$, *for all* $z \in U \setminus \{0\}$,

Paper presented at the International Conference on Complex Analysis and The 8th Romanian-Finnish Seminar, Iaşi, 23-27 August 1999.

(ii) $Jg(z) > 0$, *for all* $z \in U$

(iii) $\mathrm{Re}\dfrac{D^2 g(z)}{Dg(z)} > 0$, *for all* $z \in U \setminus \{0\}$

then g *is a convex function in* U.

2. Main results

Let f be an analytic function in the ellipse E.

Definition 2. The function f is a convex function in E if it is an univalent function in E and $f(E)$ is a convex domain.

In the next two theorems, sufficient conditions for an analytic function in E to be convex in E, are given.

Theorem 2. *If the analytic function* $f : E \to \mathbb{C}$ *satisfies the conditions*

(i) $f(0) = 0$ *and* $f'(z) \neq 0$, *for all* $z \in E$,

(ii) the inequality

$$(a^2 + b^2)\mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] - (a^2 - b^2)\mathrm{Re}\left[\frac{\overline{z}f''(z)}{f'(z)} + 1\right] > 0 \qquad (1)$$

holds for all $z \in E$, *then* f *is a convex function in* E.

Proof. Let $h : U \to E$ be the function defined by

$$h(z) = \frac{a+b}{2}z + \frac{a-b}{2}\overline{z}. \qquad (2)$$

Then h belongs to the class $C^1(U)$, is an univalent function in U and $h(U) = E$.

We consider the functions $g : U \to \mathbb{C}$, $g = f \circ h$. In order to prove that f is a convex function in E it is sufficient to show that the function g satisfies the conditions from theorem 1. We have

$$Dg(z) = f'(u)\left(\frac{a+b}{2}z - \frac{a-b}{2}\overline{z}\right) \qquad (3)$$

where $u = h(z) \in E$. Since $f'(u) \neq 0$, for all $u \in E$, then $g(z)Dg(z) \neq 0$, for all $z \in U \setminus \{0\}$. The Jacobian of g is

$$Jg(z) = ab|f'(u)|^2 > 0, \quad \text{for all} \quad z \in U.$$

We also have

$$\frac{D^2 g(z)}{Dg(z)} = \frac{f''(u)}{f'(u)}\left(\frac{a+b}{2}z - \frac{a-b}{2}\overline{z}\right) + \frac{(a+b)z + (a-b)\overline{z}}{(a+b)z - (a-b)\overline{z}}. \qquad (4)$$

From $u = \dfrac{a+b}{2}z + \dfrac{a-b}{2}\overline{z}$ and $\overline{u} = \dfrac{a-b}{2}z + \dfrac{a+b}{2}\overline{z}$ we obtain

$$z = \frac{1}{2ab}[(a+b)u - (a-b)\overline{u}] \tag{5}$$

and hence $\text{Re}\dfrac{D^2g(z)}{Dg(z)} > 0$, for all $z \in U$, holds only if

$$(a^2+b^2)\text{Re}\left[\frac{uf''(u)}{f'(u)} + 1\right] - (a^2-b^2)\text{Re}\left[\frac{\overline{u}f''(u)}{f'(u)} + 1\right] > 0, \quad \text{for all} \quad u \in E.$$

Remark. For $a = b$ ($E = U$), the conditions from above are the same with the well-known conditions for convexity for analytic functions in the unit disc.

Theorem 3. *If the analytic function $f : E \to \mathbb{C}$ satisfies the conditions*

(i) $f(0) = 0$ and $f'(z) \neq 0$, for all $z \in E$,

(ii) the inequalities

$$\text{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > \frac{1}{2} \tag{6}$$

and

$$\left|\arg\left[\frac{zf''(z)}{f'(z)} + 1\right]\right| \leq \arccos\frac{3(a^2-b^2)}{a^2+b^2} \tag{7}$$

are true, for all $z \in E$, then f is a convex function in E.

Proof. In order to prove that the function f is convex in E it is sufficient to show that the inequality (1) is true. From (6) we have

$$\left|\frac{zf''(z)}{f'(z)} + 1\right| \geq \left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\overline{z}f''(z)}{f'(z)}\right| \geq \text{Re}\frac{\overline{z}f''(z)}{f'(z)} \tag{8}$$

and

$$\left|\frac{zf''(z)}{f'(z)} + 1\right| > \frac{1}{2}, \tag{9}$$

for all $z \in E$.

From (17) we also have

$$\frac{\text{Re}\left[\dfrac{zf''(z)}{f'(z)} + 1\right]}{\left|\dfrac{zf''(z)}{f'(z)} + 1\right|} > \frac{3(a^2-b^2)}{a^2+b^2}, \tag{10}$$

for all $z \in E$.

Using the inequalities (8), (9) and (10) we obtain

$$(a^2+b^2)\text{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] - (a^2 - b^2\text{Re}\left[\frac{\overline{z}f''(z)}{f'(z)} + 1\right] >$$

$$> (a^2+b^2)\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]-(a^2-b^2)\left[\left|\frac{zf''(z)}{f'(z)}\right|+1\right]>$$

$$> (a^2+b^2)\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]-(a^2-b^2)\left[\left|\frac{zf''(z)}{f'(z)}+1\right|+1\right]>$$

$$> 3(a^2-b^2)\left|\frac{zf''(z)}{f'(z)}+1\right|-(a^2-b^2)\left[\left|\frac{zf''(z)}{f'(z)}+1\right|+1\right]>0,$$

for all $z\in E$.

References

[1] P.T. Mocanu, *Starlikeness and convexity for non-analytic functions in the unit disc*, Mathematica 22(45), no.1(1980), 77-83.

[2] W.C. Royster, *Convexity and starlikeness of analytic functions*, Duke Math., 19(1952), 447-457.

"Transilvania" University of Braşov, Department of Mathematics,
2200 Braşov, Romania

NON-ANALYTIC FUNCTIONS IN AN ELLIPSE

MIHAI N. PASCU AND VERONICA ŞERBU

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

The purpose of this paper is to generalize some results about functions of class C^1 on the unit disc obtained by P.T.Mocanu in [1], considering functions of class C^1 on an elliptic domain. We also obtained a sufficient condition for univalency, by introducing the notion of starlikeness with respect to the origin for functions of class C^1 on the elliptic domain.

Let E denote the elliptic domain
$E := \left\{ z = x + iy \in C : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 < 0 \right\}$.

Consider a complex function defined on E of the form $f(z) = u(x,y) + iv(x,y)$. For $r \in (0,1)$ and $\theta \in [0, 2\pi]$, the elliptic coordinates of a point $z = x + iy$ from E are
$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta. \end{cases}$

Definition 1. The function $f : E \to C$ is said to be of class $C^1(E)$ if the real functions $u = Ref$ and $v = Imf$, of the real variables $x = Rez$, $y = Imz$, are continuous and have continuous first order partial derivatives in E.

For $f \in C^1(E)$, we denote

$$Df(z) = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}} \tag{1}$$

$$\mathcal{D}f(z) = \frac{z(a^2+b^2) - \overline{z}(a^2-b^2)}{2ab}\frac{\partial f}{\partial z} + \frac{\overline{z}(a^2+b^2) - z(a^2-b^2)}{2ab}\frac{\partial f}{\partial \overline{z}} \tag{2}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2ab}\left(a^2\frac{\partial}{\partial x} - ib^2\frac{\partial}{\partial y} \right)$$

Paper presented at the International Conference on Complex Analysis and The 8th Romanian-Finnish Seminar, Iaşi, 23-27 August 1999.

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2ab}\left(a^2\frac{\partial}{\partial x} + ib^2\frac{\partial}{\partial y}\right).$$

The linear differential operators defined by (1) and (2) verify the rules of the differential calculus, for example:

$$D(f+g) = Df + Dg,$$
$$D(fg) = fDg + gDf,$$
$$D(\tfrac{f}{g}) = \tfrac{gDf - fDg}{g^2},$$
$$D(f \circ g) = \tfrac{\partial f}{\partial g}Dg + \tfrac{\partial f}{\partial \overline{g}}D\overline{g};$$

For $a = 1$ and $b = 1$, from (1) and (2), we obtain the differential operators defined in [Mo].

The two operators have the following properties:

$$D\overline{f} = -\overline{Df} \qquad \mathcal{D}\overline{f} = \overline{\mathcal{D}f}$$
$$D\,\mathrm{Re}\,f = i\,\mathrm{Im}\,Df \qquad \mathcal{D}\,\mathrm{Re}\,f = \mathrm{Re}\,\mathcal{D}f$$
$$D\,\mathrm{Im}\,f = -i\,\mathrm{Re}\,Df \qquad \mathcal{D}\,\mathrm{Im}\,f = \mathrm{Im}\,\mathcal{D}f$$
$$D\,|f| = i\,|f|\,\mathrm{Im}\,\tfrac{Df}{f} \qquad \mathcal{D}\,|f| = |f|\,\mathrm{Re}\,\tfrac{\mathcal{D}f}{f}$$
$$D\arg f = -i\,\mathrm{Re}\,\tfrac{Df}{f} \qquad \mathcal{D}\arg f = \mathrm{Im}\,\tfrac{\mathcal{D}f}{f}$$

We also have:

$$\frac{\partial f}{\partial \theta} = iDf \qquad \text{and} \qquad \frac{\partial f}{\partial r} = \frac{1}{r}\mathcal{D}f$$

where $z = r(a\cos\theta + ib\sin\theta)$.

From here we deduce that

$$\frac{\partial\,|f|}{\partial \theta} = -\,|f|\,\mathrm{Im}\,\frac{Df}{f} \qquad \text{and} \qquad \frac{\partial\,|f|}{\partial r} = \frac{|f|}{r}\,\mathrm{Re}\,\frac{\mathcal{D}f}{f} \tag{3}$$

$$\frac{\partial}{\partial \theta}\arg f = \mathrm{Re}\,\frac{Df}{f} \qquad \text{and} \qquad \frac{\partial}{\partial r}\arg f = \frac{1}{r}\,\mathrm{Re}\,\frac{\mathcal{D}f}{f} \tag{4}$$

The Jacobian of the function $f \in C^1(E)$ is given by

$$Jf = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2 .$$

It is known that a function verifying $Jf > 0$, $z \in E$, is locally univalent and preserves the orientation.

Definition 2. The continuous function $f : E \to C$, $f(0) = 0$, is called starlike in E with respect to the origin if it is univalent in E and $f(E)$ is a starlike set.

Theorem 3. *A function$f \in C^1(E)$ that satisfies the conditions:*

(i) $f(0) = 0$ *and* $f(z) \neq 0$ *for all* $z \in E\backslash\{0\}$,

(ii)$Jf(z) > 0$ *for all* $z \in E$,

(iii)$\mathrm{Re}\,\frac{Df(z)}{f(z)} > 0$ *for all* $z \in E\backslash\{0\}$,

is starlike in E.

Proof. We denote $E_r := \left\{z = x + iy \in C : \frac{x^2}{(ar)^2} + \frac{y^2}{(br)^2} - 1 < 0\right\}$ and $C_{\mathrm{r}} = f(\partial E_{\mathrm{r}})$ for $r \in (0,1)$. From (4) and (iii) we deduce that

$$\frac{\partial}{\partial \theta} \arg f(r(a\cos\theta + ib\sin\theta)) > 0, \text{ for all } \theta \in [0, 2\pi] \text{ and all } r \in (0,1).$$

Therefore C_r is a starlike curve (not necesarry simple) with respect to the origin, for all $r \in (0,1)$.

In order to prove the univalency of f it is enough to show that C_r are Jordan curves and they are each two disjoint. From the condition (i) follows that the curves $C_r, r \in (0,1)$, are homotopic in $C\backslash\{0\}$, therefore the index of C_r with respect to the origin is the same, for each $r \in (0,1)$, i.e. $n(C_r, 0) = const$. Because of the condition (ii) there exists a neighbourhood of the origin such that f is univalent and preserves orientation in this neighbourhood. Thus we have an $r_0 \in (0,1)$ such that for every $r < r_0$, $n(C_r, 0) = 1$, meaning that the variation of the argument along C_r is 2π. We conclude that C_r is a Jordan curve, for each $r \in (0,1)$.

In order to prove that every two different curves C_r and $C_{r'}$ are disjoint, we will show that for any ray starting from the origin, the modulus of the unique point of intersection of this ray with the curve C_r is a strictly increasing function of r, as r increases in the interval $(0,1)$.

Let us fix $\varphi \in (0, 2\pi)$. The system

$$\begin{cases} \arg f(z) = \varphi \\ z = r(a\cos\theta + ib\sin\theta) \end{cases}, r \in (0,1)$$

has a unique solution $\theta = \theta(r)$, that gives us the unique point $z = r(a\cos\theta + ib\sin\theta)$. For this value of z we consider

$$R(r) = |f(z)| \tag{5}$$

We will show that $R(r)$ is strictly increasing in $(0,1)$.

From (5), by differentiating with respect to r, we get

$$\frac{dR}{dr} = R\left(\frac{1}{r}\operatorname{Re}\frac{\mathcal{D}f}{f} - \frac{d\theta}{dr}\operatorname{Im}\frac{Df}{f}\right). \tag{6}$$

From the relation $\arg f(z) = \varphi$, we obtain

$$\frac{1}{r}\operatorname{Im}\frac{\mathcal{D}f}{f} + \frac{d\theta}{dr}\operatorname{Re}\frac{Df}{f} = 0. \tag{7}$$

By eliminating $\frac{d\theta}{dr}$ between the equations (6) and (7) we get

$$\frac{dR}{dr}\operatorname{Re}\frac{Df}{f} = \frac{R}{r}\left(\operatorname{Re}\frac{Df}{f}\operatorname{Re}\frac{\mathcal{D}f}{f} + \operatorname{Im}\frac{Df}{f}\operatorname{Im}\frac{\mathcal{D}f}{f}\right)$$

or

$$\frac{dR}{dr}\operatorname{Re}\frac{Df}{f} = \frac{1}{r}\operatorname{Re}\left(Df\overline{\mathcal{D}f}\right)$$

A simple calculus shows that $\operatorname{Re}\left(Df\overline{\mathcal{D}f}\right) = abr^2Jf$, therefore

$$\frac{dR}{dr}\operatorname{Re}\frac{Df}{f} == abrJf,$$

Because $\frac{dR}{dr} > 0$, R is a strictly increasing function in $(0,1)$. We proved the univalency of f.

We have that the domain $f(U_r)$ is starlike for each $r \in (0,1)$ and $f(U_r) \subset f(U_{r\prime})$ for $0 < r < r\prime < 1$. It follows that $f(U)$ is also a starlike domain. Our theorem is proved.

References

[1] **Petru T. Mocanu**, *Starlikeness and convexity for non-analytic functions in the unit disc*, Mathematica 22(45), 1, 1980, 77-83.

"Transilvania University", Braşov

"Babeş-Bolyai University", Cluj-Napoca

NEW UNIVALENCE CRITERION FOR CERTAIN INTEGRAL OPERATOR

VIRGIL PESCAR

Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. In this work we prove a new univalence criterion for the analicity and univalence in the unit disc $U = \{z \in C : |z| < 1\}$ of an integral operator.

1. INTRODUCTION

Let A be the class of the functions f which are analytic in the unit disc and $f(0) = f'(0) - 1 = 0$. We denote by S the class of the functions $f \in A$ which are univalent in U.

In the theory of univalent functions an interesting problem is to find those integral operators which preserve the univalence of the class S.

Many authors studied the problem of integral operators which preserve the class S. In this sense, important results are due to Y. J. Kim, E.P. Merkes [1], M. Nunokawa [3] and J. Pfaltzgraff [5].

2. PRELIMINARIES

We will need the following theorem in this paper.

THEOREM A[4]. Let α be a complex number, $Re\alpha > 0$ and $f \in A$.

If

$$\left(1 - |z|^{2Re\alpha}\right) \left|\frac{zf''(z)}{f'(z)}\right| \leq Re\alpha \tag{1}$$

for all $z \in U$, then the function

Paper presented at the International Conference on Complex Analysis and The 8th Romanian-Finnish Seminar, Iaşi, 23-27 August 1999.

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du\right]^{\frac{1}{\alpha}} \tag{2}$$

is in the class S.

3. MAIN RESULT

THEOREM. Let $g \in S$ and $\alpha = a + bi$ be a complex number and $a \in (0, 4]$. If

$$a^4 + a^2 b^2 - 4 \geq 0, a \in \left(0, \frac{1}{2}\right) \; and \quad a^2 + b^2 - 16 \geq 0, \quad a \in \left[\frac{1}{2}, 4\right] \tag{3}$$

then the function

$$H_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\alpha}} \tag{4}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du. \tag{5}$$

The function f is regular in U.From (5) we have
$f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}}, f''(z) = (\frac{1}{\alpha}\left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}-1} \frac{zg'(z)-g(z)}{z^2}$
and

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}}\left(\frac{z(g'(z))}{g(z)} + 1\right). \tag{6}$$

for all $z \in U$.

From (6) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}}\left(\frac{1+|z|}{1-|z|} + 1\right). \tag{7}$$

and hence we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2}{a\sqrt{a^2+b^2}} \frac{1-|z|^{2a}}{1-|z|} \tag{8}$$

for all $z \in U$.

Let us note $|z| = x$, $x \in (0,1)$ and $\phi(x) = \frac{1-x^{2a}}{1-x}$, $a > 0$. It easy to prove that

$$\phi(x) \leq \begin{cases} 1 & \text{if} \quad a \in \left(0, \frac{1}{2}\right) \\ 2a & \text{if} \quad a \in [\frac{1}{2}, \infty) \end{cases} \tag{9}$$

Using $a \in (0,4]$ and the relations (8),(9),(3) we obtain

$$\left(\frac{1-|z|^{2a}}{a}\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{10}$$

for all $z \in U$.

From (5) we have $f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}}$ and using (10) by Theorem A it results that the function H_α is in the class S.

References

[1] Y.J. Kim, E.P.Merkes, On an integral of powers of a spirallike function, Kyungpook Math. J. Vol. 12, No. 2, December 1972, 249-253.

[2] Z. Nehari, "Conformal mapping", Mc Graw-Hill Book Comp., New York, 1952 (Dover. Publ. Inc., 1975)

[3] M. Nunokowa, On the univalence of a certain integral, Transactions of the American Mathematical Society, Vol. 146, December 1969, 439-446.

[4] N.N. Pascu, On a univalence criterion (II), Itinerant seminar on functional equations approximation and convexity, Cluj-Napoca, Preprint Nr. 6, 1985, 153-154.

[5] J. Pfaltzgraff, Univalence of the integral $\int_0^z [f'(t)]^c \, dt$, Bull. London Math. Soc. 7(1975), No. 3, 254-256.

[6] C. Pommerenke, "Univalent functions", Vandenhoeck Ruprecht in Gottingen, 1975.

"Transilvania" University of Braşov, Faculty of Sciences, Department of Mathematics, 2200 Braşov, Romania

DATA DEPENDENCE OF THE FIXED POINTS SET OF MULTIVALUED WEAKLY PICARD OPERATORS

IOAN A. RUS, ADRIAN PETRUŞEL, ALINA SÎNTĂMĂRIAN

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. The purpose of this paper is to present data dependence results for some multivalued weakly Picard operatorors such as: Reich-type operators, graphic-contractions.

1. **Introduction**

The purpose of this paper is to study the following problem (see Lim [9], Rus [21], Rus-Mureşan [23], etc).

Problem. Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$ two multivalued operators. If the fixed points sets F_{T_1} and F_{T_2} are nonempty and there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$, estimate $H(F_{T_1}, F_{T_2})$, where H is the Hausdorff-Pompeiu generalized functional on $P(X)$.

Throughout the paper we follow the terminologies and the notations from Rus [20]. For the convenience of the reader, we recall some of them.

Let (X, d) be a metric space. We denote:

$$P(X) := \{A | A \text{ is a nonempty subset of } X\}, \quad P_{cl}(X) := \{A \in P(X) | A \text{ - closed}\},$$

$$P_b(X) := \{A \in P(X) |\ A - \text{bounded}\}, \quad P_{cp}(X) := \{A \in P(X) | A \text{ - compact}\},$$

$$P_{b,cl}(X) := P_b(X) \cap P_{cl}(X).$$

If $A, B \in P(X)$, then we define the functional:

$$D(A, B) := \inf\{d(a, b) | a \in A,\ b \in B\},$$

1991 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. Multivalued Reich-type contraction, Weakly Picard operator, Fixed point.

and the following generalized functionals:

$$\rho(A,B) := \sup\{D(a,B)|a \in A\}, \quad H(A,B) := \max\{\rho(A,B), \rho(B,A)\}.$$

In this note we need the following well known properties of the functionals D and H (see Nadler [13], Reich [15], Rus [19], [20],...).

Lemma 1.1 *Let $A, B \in P(X)$ and $q \in \mathbb{R}$, $q > 1$, be given.*

Then for every $a \in A$, there exists $b \in B$ such that $d(a,b) \leq qH(A,B)$.

Lemma 1.2. *Let $A, B \in P(X)$. We suppose that there exists $\eta \in \mathbb{R}$, $\eta > 0$, such that*

(i) for each $a \in A$ there is $b \in B$ such that $d(a,b) \leq \eta$;

(ii) for each $b \in B$ there is $a \in A$ such that $d(a,b) \leq \eta$.

Then $H(A,B) \leq \eta$.

Lemma 1.3. *Let $A \in P(X)$ and $x \in X$. Then $D(x,A) = 0$ iff $x \in \overline{A}$.*

If $T : X \to P(X)$ is a multivalued operator, then we denote by F_T the fixed points set of T, i. e.

$$F_T := \{x \in X | x \in T(x)\}.$$

2. Multivalued weakly Picard operators

Let us start the section by recalling an important notion.

Definition 2.1. Let (X,d) be a metric space and $T : X \to P_{cl}(X)$ a multivalued operator. By definition, T is a *weakly Picard operator (*briefly *w.P.o.)* iff for all $x \in X$ and all $y \in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that*:*

*(*i*)* $x_0 = x$, $x_1 = y$,

*(*ii*)* $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$,

*(*iii*)* the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Remark 2.2. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the condition *(*ii*)* and *(*iii*)*, in the Definition 2.1 is, by definition, a sequence of successive approximations of T starting from x_0.

Example 2.3. [see Rus [22]] If $t : X \to X$ is a singlevalued w.P.o., then the multivalued operator $T : X \to P_{cl}(X)$, $T(x) := \{t(x)\}$, for each $x \in X$, is a multivalued w.P.o.

Example 2.4. Let $t_i : X \to X$, $i \in \{1, 2, \dots, n\}$, be singlevalued contractions. Then the multivalued operator $T : X \to P_{cl}(X)$, $T(x) = \{t_1(x), \dots, t_n(x)\}$, for each $x \in X$, is a multivalued w.P.o.

Example 2.5. [see Covitz-Nadler [4] and Reich [15]] Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued contraction. Then T is a multivalued w.P.o.

Other examples will be given in the next paragraphs.

3. Data dependence of the fixed points set of Reich-type operators

The first main result of this paper is the following:

Theorem 3.1. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$, be two multivalued operators. We suppose that:*

(i) *there exist $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}_+$, $\alpha_i + \beta_i + \gamma_i < 1$, such that*

$$H(T_i(x), T_i(y)) \le \alpha_i d(x, y) + \beta_i D(x, T_i(x)) + \gamma_i D(y, T_i(y)),$$

for all $x, y \in X$ and $i \in \{1, 2\}$;

(ii) *there exists $\eta > 0$ such that*

$$H(T_1(x), T_2(x)) \le \eta, \text{ for all } x \in X.$$

Then

(a) $F_{T_i} \in P_{cl}(X)$, $i \in \{1, 2\}$,

(b) *the operators T_1, T_2 are w.P.o. and*

$$H(F_{T_1}, F_{T_2}) \le \eta(1 - \min\{\gamma_1, \gamma_2\})(1 - \max\{\alpha_1 + \beta_1 + \gamma_1, \alpha_2 + \beta_2 + \gamma_2\})^{-1}.$$

Proof. (a) From a theorem of Reich (Theorem 5 in [15]), we have that $F_{T_i} \in P(X)$, $i \in \{1, 2\}$. Let us prove that the fixed points set of a multivalued operator T, satisfying a condition of type (i) (with $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < 1$) is closed. For this purpose let $x_n \in F_T$, $n \in \mathbb{N}$, such that $x_n \to x^*$, as $n \to +\infty$. We have:

$$D(x^*, T(x^*)) \le d(x^*, x_n) + D(x_n, T(x^*)) \le d(x^*, x_n) + H(T(x_n), T(x^*)) \le$$

$$\le d(x^*, x_n) + \alpha d(x_n, x^*) + \beta D(x_n, T(x_n)) + \gamma D(x^*, T(x^*)).$$

From this relation we have that

$$D(x^*, T(x^*)) \leq (1+\alpha)(1-\gamma)^{-1} d(x^*, x_n) \to 0, \text{ as } n \to \infty.$$

Hence, by Lemma 1.3, $x^* \in T(x^*)$.

(b) Let $q \in]1, \min\{(\alpha_1+\beta_1+\gamma_1)^{-1}, (\alpha_2+\beta_2+\gamma_2)^{-1}\}[$. Let $x_0 \in F_{T_1}$ and $x_1 \in T_2(x_0)$ such that

$$d(x_0, x_1) \leq qH(T_1(x_0), T_2(x_0)) \leq q\eta.$$

Using again Lemma 1.1, there exists $x_2 \in T_2(x_1)$ such that

$$d(x_1, x_2) \leq q(\alpha_2+\beta_2)(1-q\gamma_2)^{-1} d(x_0, x_1).$$

By induction, we prove that there exists a sequence of successive approximations of T_2, starting from $x_0 \in F_{T_1}$, such that

$$d(x_n, x_{n+1}) \leq L_2(q) d(x_{n-1}, x_n), \ n \in \mathbb{N}^*,$$

where $L_2(q) = q(\alpha_2+\beta_2)(1-q\gamma_2)^{-1} < 1$.

This relation implies that $x_n \to x^*$, as $n \to \infty$. By standard argument we prove that $x^* \in F_{T_2}$ and

$$d(x_n, x^*) \leq [1-L_2(q)]^{-1}[L_2(q)]^n q\eta, \ n \in \mathbb{N}.$$

For $n = 0$, we obtain

$$d(x_0, x^*) \leq [1-L_2(q)]^{-1} q\eta. \tag{1}$$

By a similar way, we have that for all $y_0 \in F_{T_2}$ and $y_1 \in T_1(y_0)$, there exists a sequence of successive approximations of T_1 such that

$$y_n \to y^* \in F_{T_1}, \text{ as } n \to \infty$$

and

$$d(y_n, y^*) \leq [1-L_1(q)]^{-1}][L_1(q)]^n q\eta, \ n \in \mathbb{N},$$

where $L_1(q) := q(\alpha_1+\beta_1)(1-q\gamma_1)^{-1} < 1$.

For $n = 0$, we have

$$d(y_0, y^*) \leq [1-L_1(q)]^{-1} q\eta. \tag{2}$$

By Lemma 1.2, using (1) and (2) we have

$$H(F_{T_1}, F_{T_2}) \leq [1 - \max\{L_1(q), L_2(q)\}]^{-1} q\eta.$$

Letting $q \searrow 1$, we get the conclusion. □

Remark 3.2. For $\beta_i = \gamma_i = 0$ we have a result given by Lim *[9]*. See also Rus *[21]*.

4. Data dependence of the fixed points set of multivalued graphic-contraction-type operators

A multivalued graphic-contraction-type operator is a multivalued operator $T : X \to P_{cl}(X)$ satisfying a contraction-type condition for all $x \in X$ and $y \in T(x)$. We have:

Theorem 4.1. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ such that:*

(i) *there exist $\alpha_i, \beta_i \in \mathbb{R}_+$, $\alpha_i + \beta_i < 1$ such that*

$$H(T_i(x), T_i(y)) \leq \alpha_i d(x, y) + \beta_i D(y, T_i(y)),$$

for every $x \in X$, every $y \in T_i(x)$ and for $i \in \{1, 2\}$;

(ii) *there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.*

If:

(iii) *T_1, T_2 are closed multivalued operators*

or

(iv) *there exist two continuous functions $\psi_1, \psi_2 : \mathbb{R}^5_+ \to \mathbb{R}_+$ such that:*

(iv$_1$) *$H(T_i(x), T_i(y)) \leq \psi_i(d(x, y), D(x, T_i(x)), D(y, T_i(y)), D(x, T_i(y)), D(y, T_i(x)))$, for all $x, y \in X$ and for $i \in \{1, 2\}$;*

(iv$_2$) *$\psi_i(0, 0, s, s, 0) < s$, if $s > 0$, $i \in \{1, 2\}$;*

(iv$_3$) *If $u_1 \leq u_2$ and $v_1 \leq v_2$ then $\psi_i(u, u_1, v, w, v_1) \leq \psi_i(u, u_2, v, w, v_2)$, for all $u_i, v_i, u, v, w \in \mathbb{R}_+$ and $i \in \{1, 2\}$,*

then

(a) *$F_{T_i} \in P_{cl}(X)$, for $i \in \{1, 2\}$;*

(b) *T_i are w.P.o., for $i \in \{1, 2\}$;*

(c) *$H(F_{T_1}, F_{T_2}) \leq \eta(1 - \min\{\beta_1, \beta_2\})(1 - \max\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\})^{-1}$.*

Proof. Let us have (i), (ii) and (iii). From Lemma 2 in Rus [19] and (iii) we have T_i are w.P.o. and $F_{T_i} \in P(X)$, for $i \in \{1,2\}$. Let us prove that $F_{T_i} \in P_{cl}(X)$, $i \in \{1,2\}$. For this purpose, let $(x_n)_{n\in\mathbb{N}} \subset F_{T_i}$ be a convergent sequence to an element $x^* \in X$. It is sufficient to prove that $x^* \in F_{T_i}$. We have: $x_n \in T_i(x_n)$, $n \in N$. From (iii) it follows that $x^* \in T_i(x^*)$, for $i \in \{1,2\}$.

Let us have (i), (ii) and (iv). Using Theorem 1 in [19] we obtain $F_{T_i} \in P(X)$, for $i \in \{1,2\}$. Let us prove again that F_{T_i} is closed in X for each $i \in \{1,2\}$. As before, let $(x_n)_{n\in\mathbb{N}} \subset F_{T_i}$ be a convergent sequence to a point $x^* \in X$. Then:

$$D(x^*, T_i(x^*)) \le d(x^*, x_n) + D(x_n, T_i(x^*)) \le d(x^*, x_n) + H(T_i(x_n), T_i(x^*)) \le$$

$$\le d(x_n, x^*) + \psi_i(d(x_n, x^*), D(x_n, T_i(x_n)), D(x^*, T_i(x^*)), D(x_n, T_i(x^*)), D(x^*, T_i(x_n))) \le$$

$$\le d(x^*, x_n) + \psi_i(d(x_n, x^*), 0, D(x^*, T_i(x^*)), D(x_n, T_i(x^*)), d(x^*, x_n)).$$

Letting $n \to \infty$, we have:

$$D(x^*, T_i(x^*)) \le \psi_i(0, 0, D(x^*, T_i(x^*)), D(x^*, T_i(x^*)), 0).$$

From (iv_2) it follows that $D(x^*, T_i(x^*)) = 0$ and hence $x^* \in F_{T_i}$, for $i \in \{1,2\}$.

So, we get the conclusions (a) and (b). For (c) let $x_0 \in F_{T_1}$.

For every $q > 1$, there exists $x_1 \in T_2(x_0)$ such that $d(x_0, x_1) \le qH(T_1(x_0), T_2(x)) \le q\eta$. For $x_1 \in T_2(x_0)$ and $1 < q < \min\left\{\frac{1}{\alpha_1+\beta_1}, \frac{1}{\alpha_2+\beta_2}\right\}$ there is $x_2 \in T_2(x_1)$ such that $d(x_1, x_2) \le qH(T_2(x_0), T_2(x_1)) \le q[\alpha_2 d(x_0, x_1) + \beta_2 D(x_1, T_2(x_1))] \le q[\alpha_2 d(x_0, x_1) + \beta_2 d(x_1, x_2)]$ and hence

$$d(x_1, x_2) \le \frac{q\alpha_2}{1 - q\beta_2} d(x_0, x_1).$$

By induction, one prove that there exists a sequence of successive approximations for T_2, starting from $x_0 \in F_{T_1}$ such that $d(x_n, x_{n+1}) \le p_2(q) d(x_{n-1}, x_n)$, where $p_2(q) = \frac{q\alpha_2}{1 - q\beta_2} < 1$. This implies that:

1) $x_n \to x^*$, as $n \to \infty$,

2) $x^* \in F_{T_2}$,

3) $d(x_n, x^*) \le \frac{[p_2(q)]^n}{1 - p_2(q)} d(x_0, x_1) \le \frac{[p_2(q)]^n}{1 - p_2(q)} q\eta$, $n \in \mathbb{N}$.

Interchanging the roles, one can prove that for each $y_0 \in F_{T_2}$, there exists a sequence of successive approximations for T_1, starting from y_0 such that

1') $y_n \to y^*$, as $n \to \infty$,

2') $y^* \in F_{T_1}$,

3') $d(y_n, y^*) \leq \frac{[p_1(q)]^n}{1-p_1(q)} d(y_0, y_1) \leq \frac{[p_1(q)]^n}{1-p_1(q)} q\eta$, $n \in \mathbb{N}$, (where $p_1(q) = \frac{q\alpha_1}{1-q\beta_1} < 1$).

For $n = 0$ we get $d(x_0, x^*) \leq \frac{q\eta}{1-p_2(q)}$ and $d(y_0, y^*) \leq \frac{q\eta}{1-p_1(q)}$. As consequence $H(F_{T_1}, F_{T_2}) \leq q\eta[1 - \max\{p_1(q), p_2(q)\}]^{-1}$.

Letting $q \searrow 1$, the conclusion follows. □

5. **Applications**

We shall prove now a data dependence result for the following equation:

$$\phi(u) + \psi(u) = v, \quad u \in U. \tag{3}$$

Let us denote by $S_{\psi,v}$ the solutions set for (3). We have:

Theorem 5.1. *Let* $(U, \|\cdot\|_U)$ *and* $(V, \|\cdot\|_V)$ *be real Banach spaces and let* $\phi : U \to V$ *be a continuous linear operator from* U *onto* V*. Put* $\alpha = \sup\{\inf\{\|u\|_U | u \in \phi^{-1}(v)\}, v \in V, \|v\|_V \leq 1\}$.

Then, for every $v_1, v_2 \in V$ *and every lipschitzian operators* $\psi_1, \psi_2 : U \to V$ (*with the same Lipschitz constant* $L > 0$) *satisfying the following assertions*:

i) *there is* $\eta_1 > 0$ *such that* $\|v_1 - v_2\|_V \leq \eta_1$;

ii) *there exists* $\eta_2 > 0$ *such that* $\|\psi_1(u) - \psi_2(u)\|_V \leq \eta_2$, *for each* $u \in U$;

iii) $\alpha L < 1$

are true the conclusions:

a) $S_{\psi_i, v_i} \in P_{cl}(U)$, *for* $i \in \{1, 2\}$;

b) $H(S_{\psi_1, v_1}, S_{\psi_2, v_2}) \leq \frac{\alpha(\eta_1 + \eta_2)}{1 - \alpha L}$.

Proof. From a result given by B. Ricceri (see [17], Theorem 4) it follows that $S_{\psi_i, v_i} \neq \emptyset$ and $S_{\psi_i, v_i} = FixF_i$, where $F_i : U \to P_{cl}(U)$ is a multivalued αL-contraction, given by the formula $F_i(u) = \phi^{-1}(v_i - \psi_i(u))$, for $i \in \{1, 2\}$ (see also [18]). From Theorem 3.1 one have:

$$H(S_{\psi_1, v_1}, S_{\psi_2, v_2}) \leq \frac{1}{1 - \alpha L} \sup_{u \in U} H(F_1(u), F_2(u)).$$

But $H(F_1(u), F_2(u)) = H(\phi^{-1}(v_1 - \psi_1(u)), \phi^{-1}(v_2 - \psi_2(u))) \leq \alpha\|v_1 - \psi_1(u) - v_2 + \psi_2(u)\| \leq \alpha(\eta_1 + \eta_2)$, for each $u \in U$ and hence the conclusion follows. □

Let us consider now the following functional equations of n-th order:

$$\varphi(x) \in G_1(x, \varphi(f_1(x)), \dots, \varphi(f_n(x))), \quad x \in X, \tag{4}$$

$$\varphi(x) \in G_2(x, \varphi(g_1(x)), \dots, \varphi(g_n(x))), \quad x \in X, \tag{5}$$

where φ is an unknown function and the multivalued operators G_1, G_2 and the singlevalued functions f_k, g_k ($k \in \{1, 2, \dots, n\}$) are given. Let us denote by S_i ($i \in \{1, 2\}$) the space of continuous solutions for problems (4) and (5) respectively.

Theorem 5.2. *Let X be a compact metric space and Y be a nonempty, closed, convex subset of a Banach space. Let $G_1, G_2 : X \times Y^n \to P_{cl,cv}(Y)$ be multivalued operators and $f_k, g_k : X \to X$, $k \in \{1, 2, \dots, n\}$ functions. We assume the following conditions on the given operators:*

i) *there exist two functions $\beta_i : \mathbb{R}_+^n \to \mathbb{R}_+$ non-decreasing with respect to each variable with the property $\beta_i(t, t, \dots, t) \leq a_i t$, for each $t > 0$, with $0 \leq a_i < 1$ such that one have:*

$$H(G_i(x, y_1, \dots, y_n), G_i(x, z_1, \dots, z_n)) \leq \beta_i(\|y_1 - z_1\|, \dots, \|y_n - z_n\|),$$

for $x \in X$, $y_k, z_k \in Y$ ($k \in \{1, 2, \dots, n\}$) and for $i \in \{1, 2\}$;

ii) *$f_k, g_k : X \to X$ are continuous, $k \in \{1, 2, \dots, n\}$;*

iii) *G_1, G_2 are lower semicontinuous (l.s.c.);*

iv) *there exist η_k, $\tilde{\eta} > 0$ such that $\|f_k(x) - g_k(x)\| \leq \eta_k$ for $k \in \{1, 2, \dots, n\}$ and $H(G_1(x, y_1, \dots, y_n), G_2(x, y_1, \dots, y_n)) \leq \tilde{\eta}$, for $x \in X$ and $y_1, \dots, y_n \in Y$.*

Then:

a) *$S_i \in P_{cl}(\mathcal{C})$, for $i \in \{1, 2\}$ (where $\mathcal{C} = C(X, Y)$ is the space of continuous functions from X to Y);*

b) *$H(S_1, S_2) \leq (1 - \max\{a_1, a_2\}) [\beta(\eta_1, \dots, \eta_n) + \tilde{\eta}]$.*

Proof. From Theorem 4.1 in Węgrzyk [26] we get that $S_i = F_{T_i}$, where $T_i : \mathcal{C} \to P_{cl,cv}(\mathcal{C})$, $i \in \{1, 2\}$ are multivalued operators given by the formulae:

$$T_1(\varphi) = \{\psi \in \mathcal{C} | \psi(x) \in G_1(x, \varphi(f_1(x)), \dots, \varphi(f_n(x))),\ x \in X\}$$

and

$$T_2(\varphi) = \{\psi \in \mathcal{C} |\ \psi(x) \in G_2(x, \varphi(g_1(x)), \dots, \varphi(g_n(x))),\ x \in X\}.$$

From Lemma 4.1 in the same paper [26], we have that $H(T_i(\varphi_1), T_i(\varphi_2)) \leq \gamma_i(\overline{d}(\varphi_1, \varphi_2))$, for $\varphi_1, \varphi_2 \in \mathcal{C}$, where $\gamma_i(t) = \beta_i(t, \ldots, t)$, for $t \in \mathbb{R}_+$ and $\overline{d}(\varphi_1, \varphi_2) = \sup\{\|\varphi_1(x) - \varphi_2(x)\| \mid x \in X\}$.

By $i)$ it follows that T_i are multivalued a_i-contractions, for $i \in \{1, 2\}$. Then, we obtain:

$$S_i \in P_{cl}(\mathcal{C}), \text{ for } i \in \{1, 2\}$$

and

$$H(S_1, S_2) = H(F_{T_1}, F_{T_2}) \leq [1 - \max\{a_1, a_2\}] \sup_{\varphi \in \mathcal{C}} H(T_1(\varphi), T_2(\varphi)). \tag{6}$$

On the other side, let us estimate $H(T_1(\varphi), T_2(\varphi))$.

For this purpose, let $\varphi_1 \in T_1(\varphi)$. Then $\varphi_1(x) \in G_1(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x)))$, $x \in X$. We have

$$D(\varphi_1(x), G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))) \leq H(G_1(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x))),$$

$$G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))) \leq H(G_1(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x))),$$

$$G_1(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))) + H(G_1(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))),$$

$$G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x)))) \leq \beta(\|\varphi(f_1(x)) - \varphi(g_1(x))\|, \ldots, \|\varphi(f_n(x)) - \varphi(g_n(x))\|) + \tilde{\eta}.$$

From the uniform continuity of φ on the compact space X and from iv) we get that

$$\|\varphi(f_k(x)) - \varphi(g_k(x))\| \leq \eta_k, \text{ for each } x \in X.$$

Hence we conclude that

$$D(\varphi_1(x), G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta},$$

for each $x \in X$.

Then, for a fixed $\varepsilon > 0$ and for every $x \in X$ there exists $z_x \in G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x)))$ such that

$$\|\varphi_1(x) - z_x\| \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta} + \varepsilon.$$

Using the same argument like in the proof of Lemma 4.1 from [26] we infer that for every $\varepsilon > 0$ there exists a continuous function $\varphi_2 \in T_2(\varphi)$ such that

$$\overline{d}(\varphi_1, \varphi_2) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta} + \varepsilon.$$

It follows $D(\varphi_1, T_2(\varphi)) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}$. From the analogous inequality: $D(\varphi_2, T_1(\varphi)) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}$, for every $\varphi_2 \in T_2(\varphi)$ we get that

$$H(T_1(\varphi), T_2(\varphi)) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}.$$

Making use of the estimate (6), we obtain

$$H(S_1, S_2) \leq (1 - \max\{a_1, a_2\}) [\beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}]. \quad \square$$

Remark 5.3. For other applications see *[2], [3], [7], [8], [11], [24].*

References

[1] A. Alesina, S. Massa, D. Roux, *Punti uniti di multifunzioni con condizioni di tipo Boyd-Wong*, Boll. U. M. I., 8 (1973), 29-34.

[2] A. Constantin, *Some existence results for nonlinear integral equations*, in Qualitative problems for diff. eq. and control theory, Editor: C. Corduneanu, World Scientific, 1995, 105-111.

[3] A. Constantin, *Stability of solutions sets for differential equations with multivalued right-hand side*, J. Diff. Eq., 114 (1994), 243-252.

[4] H. Covitz, S. B. Nadler jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., 8 (1970), 5-11.

[5] P. Z. Daffer, H. Haneko, *Remarks on contractive-type mappings*, Scientiae Math., 1 (1998), 125-131.

[6] P. Z. Daffer, H. Haneko, W. Li, *On a conjecture of S. Reich*, Proceed. Amer. Math. Soc., 124 (1996), 3159-3162.

[7] M. Frigon, A. Granas, *Resultats du type de Leray-Schauder pour les contractions multivoques*, Topol. Methods in Nonlinear Analysis, 4 (1994), 197-208.

[8] C. J. Himmelberg, F. S. Van Vleck, *Lipschitzian generalized differential equations*, Rend. Sem. Mat. Univ. Padova, 48 (1973), 159-168.

[9] T. C. Lim, *On fixed point stability for set-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl., 110 (1985), 436-441.

[10] J. T. Markin, *Continuous dependence of fixed points sets*, Proc. A. M. S., 38 (1973), 545-547.

[11] J. T. Markin, *Stability of solutions set for generalized differential equations*, Journal Math. Anal. Appl., 46 (1974), 289-291.

[12] S. Massa, D. Roux, *On the minimal displacement under generalized nonexpansive multivalued mappings*, Rend. di Mat., 12 (1979), 577-585.

[13] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math., 30 (1969), 475-488.

[14] V. A. Plotnikov, L. I. Plotnikova, *Approximated equations: theory and applications*, in Approximation and Optimization, ICAOR, Vol. II, 203-216, Cluj-Napoca, 1996.

[15] S. Reich, *Kannan's fixed point theorem*, Boll. U. M. I., 4 (1971), 1-11.

[16] S. Reich, *Some problems and results in fixed points theory*, Contemporary Math., 21 (1983), 179-187.

[17] B. Ricceri, *Une propriété topologique de l'ensemble des points fixed d'une contraction multivoque à valeurs convexes*, Atti. Acc. Lincei Rend., LXXXI (1987), 283-286.

[18] S. M. Robinson, *Normed convex processes*, Trans. A. M. S., 174 (1972), 127-140.

[19] I. A. Rus, *Fixed point theorems for multivalued mappings in complete metric spaces*, Math. Japonica, 20 (1975), 21-24.

[20] I. A. Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001.

[21] I. A. Rus, *Basic problems of the metric fixed point theory revisited*, (II), Studia Univ. Babeş-Bolyai, 36 (1991), 81-99.

[22] I. A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Carolinae, 34, 4 (1993), 769-773.
[23] I. A. Rus, S. Mureşan, *Data dependence of the fixed points set of weakly Picard operators*, Studia Univ. Babeş-Bolyai, 43 (1998), 79-83.
[24] J. Saint-Raymond, *Multivalued contractions*, Set Valued Analysis, 2 (1994), 559-571.
[25] C. Yu-Qing, *On a fixed point problem of Reich*, Proc. A. M. S., 124 (1996), 3085-3088.
[26] R. Węgrzyk, *Fixed point theorems for multivalued functions and their applications to functional equations*, Dissertationes Math., 21 (1982), 1-28.

Babeş-Bolyai University, 3400 Cluj-Napoca, Kogălniceanu Nr.1, Romania

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 2, June 2001

ON SOME CLASSES OF HOLOMORPHIC FUNCTIONS

DORINA RĂDUCANU

Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. In this note we define two classes of functions, which are called α-starlike and α-harmonic starlike and we obtain some properties concerning these classes.

1. **Introduction and preliminaries**

Let $\mathbb{C}^n$ be the space of n-complex variables $z = (z_1, \dots, z_n)$ with the norm $\|z\| = \max\limits_{1\le k\le n} |z_k|$. The unit polydisc $\{z \in \mathbb{C}^n : \ \|z\| < 1\}$ is denoted by P.

Let $H(P)$ be the family of all holomorphic functions from P into $\mathbb{C}$. The Fréchet derivative of $f \in H(P)$ is

$$Df(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\right), \quad z \in P$$

and $D^2 f(z) = \left(\dfrac{\partial^2 f}{\partial z_k \partial z_j}(z)\right)_{1\le k,j\le n}$ is the Fréchet derivative of the second order of f.

Let A denote the class of all functions $f \in H(P)$ which satisfy the conditions $f(0) = 0$ and $\dfrac{\partial f}{\partial z_k}(0) = 1$, $1 \le k \le n$.

In several papers K. Dobrowolska, J. Dziubinski, R. Sitarski [1], [2] and E. Janiec [4] have studied the subclasses of the class A consisting in starlike and convex functions.

Let $S^*(P)$ be the class of all functions $f \in A$, $f(z) \neq 0$ for all $z \in P \setminus \{0\}$, satisfying the condition

$$Re \frac{zDf(z)'}{f(z)} > 0, \quad \text{for} \quad z \in P \tag{1}$$

Paper presented at the International Conference on Complex Analysis and The 8th Romanian-Finnish Seminar, Iaşi, 23-27 August 1999.

where $Df(z)'$ is the transpose of $Df(z)$. The functions of this class are called starlike on P.

Let $S^c(P)$ be the class of all functions $f \in A$, $zDf'(z) \neq 0$, $z \in P \setminus \{0\}$, for which

$$Re\left(1 + \frac{zD^2f(z)z'}{zDf(z)'}\right) > 0, \quad \text{for} \quad z \in P \tag{2}$$

where z' is the transpose of z. The class $S^c(P)$ is the class of convex functions on P.

We shall use the following theorem to prove our results.

Theorem 1. [3] *Let q be a holomorphic and univalent function on $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ without at most one point $\zeta \in \partial U$, which is a simple pole. Let $p : P \to \mathbb{C}$ be a holomorphic function on P with $p(0) = q(0)$. If $p(P) \not\subset q(U)$, then there exist $\zeta_0 \in \partial U$, $r_0 \in (0,1)$, $z_0 \in r_0\overline{P}$ and $m \geq 1$ such that*

$$p(z_0) = q(\zeta_0) \tag{3}$$

$$z_0 Df(z_0)' = m\zeta_0 q'(\zeta_0) \tag{4}$$

$$Re\left(1 + \frac{z_0D^2f(z_0)z_0'}{z_0Df(z_0)'}\right) \geq m\, Re\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right). \tag{5}$$

2. **Main results**

Let α be a complex number. A function $f \in A$, $f(z) \neq 0$, $z \in P \setminus \{0\}$ is called α-starlike on P if the function

$$G(z) = (1-\alpha)f(z) + \alpha z Df(z)', \quad \text{for} \quad z \in P \tag{6}$$

is a starlike function on P. We denote by $S^*_\alpha(P)$ the class of α-starlike functions on P.

Since $G \in S^*(P)$, from (1) and (6) it follows that a function f is α-starlike on P if

$$Re\left[p(z) + \alpha\frac{zDp(z)'}{1-\alpha+\alpha p(z)}\right] > 0, \quad \text{for all} \quad z \in P, \tag{7}$$

where $p(z) = \dfrac{zDf(z)'}{f(z)}$.

The definitions of the classes $S^*(P)$, $S^c(P)$ and $S^*_\alpha(P)$ imply immediately $S^*_0(P) = S^*(P)$ and $S^*_1(P) = S^c(P)$.

Theorem 2. *If $f \in S^*_\alpha(P)$ and $\alpha \in \mathbb{C}$ with $\left|\alpha - \frac{1}{2}\right| \leq \frac{1}{2}$, then $f \in S^*(P)$.*

Proof. We assume that $Re\dfrac{zDf(z)'}{f(z)} \not> 0$ for some $z \in P$. Let $q : \overline{U} \setminus \{1\} \to \mathbb{C}$ be the function defined by $q(z) = \dfrac{1+z}{1-z}$.

If $p(z) = \dfrac{zDf(z)'}{f(z)}$, $z \in P$ then we have $p(0) = q(0) = 1$ and $p(P) \not\subset q(U)$. From Theorem 1 there exist $\xi_0 \in \partial U$, $r_0 \in (0,1)$ and $z_0 \in r_0\overline{P}$ such that $p(z_0) = q(\zeta_0)$ and $z_0 Dp(z_0)' = m\zeta_0 q'(\zeta_0)$, $m \geq 1$. It follows $Re\, p(z_0) = Re\, q(\zeta_0) = 0$ and $z_0 Dp(z_0)' < 0$. We obtain

$$Re\left[p(z_0) + \alpha \frac{z_0 Dp(z_0)'}{1-\alpha+\alpha p(z_0)}\right] = \frac{z_0 Dp(z_0)'}{|1-\alpha+\alpha p(z_0)|^2} Re(\alpha - |\alpha|^2).$$

Since $\left|\alpha - \dfrac{1}{2}\right| \leq \frac{1}{2}$ it follows $Re\left[p(z_0) + \dfrac{\alpha z_0 Dp(z_0)'}{1-\alpha+\alpha p(z_0)}\right] \leq 0$ which contradicts (7). We get $Re\dfrac{zDf(z)'}{f(z)} > 0$ for all $z \in P$ and then $f \in S^*(P)$.

The notion of α-starlikeness was introduced with the help of the generalized arithmetical mean of the functions $f(z)$ and $zDf(z)'$. We now consider a new class of functions using the generalized harmonic mean of the functions $f(z)$ and $zDf(z)'$.

Let α be a complex number. The function $f \in A$, $f(z) \neq 0$, $zDf(z)' \neq 0$ for $z \in P \setminus \{0\}$ is called α-harmonic starlike if the function $F : P \to \mathbb{C}$ defined by

$$\frac{1}{F(z)} = \frac{1-\alpha}{f(z)} + \frac{\alpha}{zDf(z)'}, \quad \text{for} \quad z \in P \tag{8}$$

is a starlike function on P.

We denote by $SH^*_\alpha(P)$ the class of α-harmonic starlike functions on P. We have $SH^*_0(P) = S^*(P)$ and $SH^*_1(P) = S^c(P)$. Using (1) and (8) it follows that a function f belongs to the class $SH^*_\alpha(P)$ if

$$Re\left[p(z) + \frac{zDp(z)'}{p(z)} - (1-\alpha)\frac{zDp(z)'}{\alpha+(1-\alpha)p(z)}\right] > 0, \quad \text{for all} \quad z \in P, \tag{9}$$

where $p(z) = \dfrac{zDf(z)'}{f(z)}$.

Theorem 3. *If $f \in SH^*_\alpha(P)$ and $\alpha \in \mathbb{C}$ with $\left|\alpha - \dfrac{1}{2}\right| \geq \dfrac{1}{2}$ then $f \in S^*(P)$.*

The proof is similar with the proof of Theorem 2.

Remark. The classes $S^*_\alpha(P)$ and $SH^*_\alpha(P)$ are the extensions of the α-starlike and α-harmonic starlike functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ which were obtained by N.N. Pascu [5] and N.N. Pascu, D. Răducanu [6].

References

[1] K. Dobrowolska, J. Dziubinski, *On starlike and convex functions of many variables*, Demonstratio Math. 11(1978).
[2] J. Dziubinski, R. Sitarski, *On classes of holomorphic functions of many variables starlike and convex on some hypersurfaces*, Demonstratio Math. 13(1980).
[3] S. Gong, S.S. Miller, *Partial differential subordinations and inequalities defined on complex circular domains*, Comm. in Partial Diff. Equations, 11(1986).
[4] E. Janiec, *On starlike and convex maps of a Banach space into the complex plane*, Demonstratio Math. 2(1994).
[5] N.N. Pascu, *Alpha-starlike functions*, Bull. Univ. Braşov, CXIX, 1977.
[6] N.N. Pascu, D. Răducanu, *Generalized means and generalized convexity*, Seminar of Geometric Function Theory, Preprint 3(1993).

Department of Mathematics, Faculty of Sciences, "Transilvania"
University of Braşov, Iuliu Maniu, 50, 2200 Braşov, Romania