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SUMAR – CONTENTS – SOMMAIRE

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ON A CERTAIN FAMILIES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. We introduce the subclass $T_j(n,m,\lambda,\alpha)$ of analytic functions with negative coefficients defined by Salagean operators D^n and D^{n+m}. In this paper we give some properties of functions in the class $T_j(n,m,\lambda,\alpha)$ and obtain numerous sharp results including (for example) coefficient estimates, distortion theorems, closure theorems and modified Hadamard products of several functions belonging to the class $T_j(n,m,\lambda,\alpha)$. We also obtain radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class $T_j(n,m,\lambda,\alpha)$ and consider integral operators associated with functions belonging to the class $T_j(n,m,\lambda,\alpha)$.

1. Introduction

Let $A(j)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in N = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For a function $f(z)$ in $A(j)$, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z) \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [5]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to $A(j)$ is in

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the class $S_j(n, m, \lambda, \alpha)$ if and only if

$$\mathrm{Re}\left\{\frac{(1-\lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1-\lambda)D^n f(z) + \lambda D^{n+m} f(z)}\right\} > \alpha \quad (n, m \in N_0 = N \cup \{0\}) \tag{1.5}$$

for some α ($0 \leq \alpha < 1$) and λ ($0 \leq \lambda \leq 1$), and for all $z \in U$. The operator D^{n+m} was studied by Sekine [7] and Aouf and Salagean [2].

Let $T(j)$ denote the subclass of $A(j)$ consisting of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0;\ j \in N). \tag{1.6}$$

Further, we define the class $T_j(n, m, \lambda, \alpha)$ by

$$T_j(n, m, \lambda, \alpha) = S_j(n, m, \lambda, \alpha) \cap T(j). \tag{1.7}$$

We note that by specializing the parameters j, n, m, λ and α, we obtain the following subclasses studied by various authors:

(i) $T_j(n, 1, \lambda, \alpha) = P(j, \lambda, \alpha, n)$, $T_j(n, m, 0, \alpha) = P(j, \alpha, n)$ and $T_j(n, 1, 1, \alpha) = P(j, \alpha, n+1)$ (Aouf and Srivastava [3]);

(ii) $T_j(0, 1, \lambda, \alpha) = P(j, \lambda, \alpha)$ (Altintas [1]);

(iii) $T_j(0, 0, 0, \alpha) = T_\alpha(j)$ and $T_j(0, 1, 1, \alpha) = T_j(1, 0, 1, \alpha) = C_\alpha(j)$ (Chatterjea [4] and Srivastava et al. [9]);

(v) $T_j(n, m, 1, \alpha) = T_j(n, m, \alpha)$, where $T_j(n, m, \alpha)$ represents the class of functions $f(z) \in T(j)$ satisfying the condition

$$\mathrm{Re}\left\{\frac{z(D^{n+m} f(z))'}{D^{n+m} f(z)}\right\} > \alpha \quad (n, m \in N_0;\ 0 \leq \alpha < 1;\ z \in U); \tag{1.8}$$

(iv) $T_1(0, 0, 0, \alpha) = T^*(\alpha)$ and $T_1(0, 1, 1, \alpha) = T_1(1, 0, 1, \alpha) = C(\alpha)$ (Silverman [8]).

2. Coefficient estimates and other properties of the class $T_j(n, m, \lambda, \alpha)$

Theorem 1. *Let the function $f(z)$ be defined by (1.6). Then $f(z) \in T_j(n, m, \lambda, \alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n (k-\alpha)[1 + (k^m - 1)\lambda] a_k \leq 1 - \alpha. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds true. Then we find that

$$\left|\frac{(1-\lambda)z(D^nf(z))'+\lambda z(D^{n+m}f(z))'}{(1-\lambda)D^nf(z)+\lambda D^{n+m}f(z)}-1\right| \leq$$

$$\leq \frac{\sum_{k=j+1}^{\infty} k^n(k-1)[1+(k^m-1)\lambda]a_k|z|^{k-1}}{1-\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k|z|^{k-1}} \leq$$

$$\leq \frac{\sum_{k=j+1}^{\infty} k^n(k-1)[1+(k^m-1)\lambda]a_k}{1-\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k} \leq 1-\alpha.$$

This show that the values of the function

$$\Phi(z)=\frac{(1-\lambda)z(D^nf(z))'+\lambda z(D^{n+m}f(z))'}{(1-\lambda)D^nf(z)+\lambda D^{n+m}f(z)} \tag{2.2}$$

lie in a circle which is centered at $w=1$ and whose radius is $1-\alpha$. Hence $f(z)$ satisfies the condition (1.5).

Conversely, assume that the function $f(z)$ is in the class $T_j(n,m,\lambda,\alpha)$. Then we have

$$\operatorname{Re}\left\{\frac{(1-\lambda)z(D^nf(z))'+\lambda z(D^{n+m}f(z))'}{(1-\lambda)D^nf(z)+\lambda D^{n+m}f(z)}\right\}=$$

$$=\operatorname{Re}\left\{\frac{1-\sum_{k=j+1}^{\infty} k^{n+1}[1+(k^m-1)\lambda]a_kz^{k-1}}{1-\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_kz^{k-1}}\right\}>\alpha, \tag{2.3}$$

for some α $(0\leq\alpha<1)$, λ $(0\leq\lambda\leq1)$, $n,m\in N_0$ and for all $z\in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z\to 1^-$ through real values, we can see that

$$1-\sum_{k=j+1}^{\infty} k^{n+1}[1+(k^m-1)\lambda]a_k \geq \alpha\left\{1-\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k\right\}. \tag{2.4}$$

Thus we have the inequality (2.1).

Finally, the function $f(z)$ given by

$$f(z)=z-\frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]}z^k \quad (k\geq j+1;\ j\in N) \tag{2.5}$$

is an extremal function for the assertion of Theorem 1.

Corollary 1. *Let the function* $f(z)$ *defined by (1.6) be in the class* $T_j(n,m,\lambda,\alpha)$. *Then*

$$a_k \leq \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \quad (k \geq j+1). \tag{2.6}$$

The equality in (2.6) is attained for the function $f(z)$ *given by (2.5).*

Theorem 2. *Let* $0 \leq \alpha_1 \leq \alpha_2 < 1,\ 0 \leq \lambda \leq 1,\ j \in N$ *and* $n, m \in N_0$. *Then*

$$T_j(n,m,\lambda,\alpha_1) \supseteq T_j(n,m,\lambda,\alpha_2). \tag{2.7}$$

Proof. Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha_2)$ and let $\alpha_1 = \alpha_2 - \delta$. Then, by Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]a_k \leq 1-\alpha_2 \tag{2.8}$$

and

$$\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k \leq \frac{1-\alpha_2}{j+1-\alpha_2} < 1. \tag{2.9}$$

Consequently,

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha_1)[1+(k^m-1)\lambda]a_k = \sum_{k=j+1}^{\infty} k^n(k-\alpha_2)[1+(k^m-1)\lambda]a_k +$$

$$+\delta \sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k \leq 1-\alpha_1. \tag{2.10}$$

This completes the proof of Theorem 2 with the aid of Theorem 1.

Theorem 3. *Let* $0 \leq \alpha < 1,\ 0 \leq \lambda_1 \leq \lambda_2 \leq 1,\ j \in N$ *and* $n, m \in N_0$. *Then*

$$T_j(n,m,\lambda_1,\alpha) \supseteq T_j(n,m,\lambda_2,\alpha). \tag{2.11}$$

Proof. It follows from Theorem 1 that

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda_1]a_k \leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda_2]a_k \leq 1-\alpha$$

for $f(z) \in T_j(n,m,\lambda_2,\alpha)$.

Theorem 4. For $0 \leq \alpha < 1,\ 0 \leq \lambda \leq 1,\ j \in N$ and $n, m \in N_0$,

$$T_j(n+1,m,\lambda,\alpha) \subseteq T_j(n,m,\lambda,\alpha). \tag{2.12}$$

The proof of Theorem 4 follows also from Theorem 1.

3. Growth and distortion theorems

Theorem 5. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then for $|z|=r<1$,*

$$|D^i f(z)| \geq r - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j+1} \tag{3.1}$$

and

$$|D^i f(z)| \leq r + \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j+1} \tag{3.2}$$

for $z \in U$ and $0 \leq i \leq n$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(j+1)^{n}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} z^{j+1} \quad (z=\pm r). \tag{3.3}$$

Proof. Note that $f(z) \in T_j(n,m,\lambda,\alpha)$ if and only if

$$D^i f(z) \in T_j(n-i,m,\lambda,\alpha)$$

and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \tag{3.4}$$

By Theorem 1, we know that

$$(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda] \sum_{k=j+1}^{\infty} k^i a_k \leq$$

$$\leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]a_k \leq 1-\alpha, \tag{3.5}$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i a_k \leq \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]}. \tag{3.6}$$

The assertions (3.1) and (3.2) of Theorem 5 would now follow readily from (3.4) and (3.6).

Finally, we note that the equalities in (3.1) and (3.2) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} z^{j+1}. \tag{3.7}$$

This completes the proof of Theorem 5.

Corollary 2. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then, for $|z| = r < 1$,*

$$|f(z)| \geq r - \frac{1-\alpha}{(j+1)^{n}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j+1} \tag{3.8}$$

and

$$|f(z)| \leq r + \frac{1-\alpha}{(j+1)^{n}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j+1}. \tag{3.9}$$

The equalities in (3.8) and (3.9) are attained for the function $f(z)$ given by (3.3).

Proof. Taking $i = 0$ in Theorem 5, we immediately obtain (3.8) and (3.9).

Corollary 3. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then for $|z| = r < 1$,*

$$|f'(z)| \geq \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j} \tag{3.10}$$

and

$$|f'(z)| \leq 1 + \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j} \quad (z \in U). \tag{3.11}$$

The equalities in (3.10) and (3.11) are attained for the function $f(z)$ given by (3.3).

Proof. Setting $i = 1$ in Theorem 5, and making use of the definition (1.3), we arrive at Corollary 3.

4. **Convex linear combinations**

In this section, we shall prove that the class $T_j(n,m,\lambda,\alpha)$ is closed under convex linear combinations.

Theorem 6. *$T_j(n,m,\lambda,\alpha)$ is a convex set.*

Proof. Let the functions

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0;\ v = 1, 2) \tag{4.1}$$

be in the class $T_j(n,m,\lambda,\alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \le \mu \le 1) \tag{4.2}$$

is also in the class $T_j(n,m,\lambda,\alpha)$. Since, for $0 \le \mu \le 1$,

$$h(z) = z - \sum_{k=j+1}^{\infty} [\mu a_{k,1} + (1-\mu) a_{k,2}] z^k, \tag{4.3}$$

with the aid of Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n (k-\alpha)[1+(k^m-1)\lambda][\mu a_{k,1} + (1-\mu) a_{k,2}] \le 1-\alpha, \tag{4.4}$$

which implies that $f(z) \in T_j(n,m,\lambda,\alpha)$. Hence $T_j(n,m,\lambda,\alpha)$ is a convex set.

Theorem 7. *Let*

$$f_j(z) = z \tag{4.5}$$

and

$$f_k(z) = z - \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} z^k \quad (k \ge j+1;\ n, m \in N_0) \tag{4.6}$$

for $0 \le \alpha < 1$ *and* $0 \le \lambda \le 1$. *Then* $f(z)$ *is in the class* $T_j(n,m,\lambda,\alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z), \tag{4.7}$$

where

$$\mu_k \ge 0 \ (k \ge j) \quad \text{and} \quad \sum_{k=j}^{\infty} \mu_k = 1. \tag{4.8}$$

Proof. Assume that

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z) =$$

$$= z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \mu_k z^k. \tag{4.9}$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \cdot \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \mu_k =$$

$$= \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \le 1. \tag{4.10}$$

So, by Theorem 1, $f(z) \in T_j(n, m, \lambda, \alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_j(n, m, \lambda, \alpha)$. Then

$$a_k \leq \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \quad (k \geq j+1;\ n, m \in N_0). \tag{4.11}$$

Setting

$$\mu_k = \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_k \quad (k \geq j+1;\ n, m \in N_0) \tag{4.12}$$

and

$$\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k, \tag{4.13}$$

we can see that $f(z)$ can be expressed in the form (4.7). This completes the proof of Theorem 7.

5. Radii of close-to-convexity, starlikeness, and convexity

Theorem 8. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1$, where*

$$r_1 = r_1(n, m, \lambda, \alpha, \rho) = \inf_k \left[\frac{(1-\rho)k^{n-1}(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{5.1}$$

The result is sharp, the extremal function $f(z)$ begin given by (2.5).

Proof. We must show that

$$|f'(z) - 1| \leq 1 - \rho \text{ for } |z| < r_1(n, m, \lambda, \alpha, \rho),$$

where $r_1(n, m, \lambda, \alpha, \rho)$ is given by (5.1). Indeed we find from the definition (1.6) that

$$|f'(z) - 1| \leq \sum_{k=j+1}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$|f'(z) - 1| \leq 1 - \rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \tag{5.2}$$

But, by Theorem 1, (5.2) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \leq \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)k^{n-1}(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{5.3}$$

Theorem 8 follows easily from (5.3).

Theorem 9. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2(n,m,\lambda,\alpha,\rho) = \inf_f \left[\frac{(1-\rho)k^n(k-\alpha)[1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{5.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho \text{ for } |z| < r_2(n,m,\lambda,\alpha,\rho),$$

where $r_2(n,m,\lambda,\alpha,\rho)$ is given by (5.4). Indeed we find, again from the definition (1.6), that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=j+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=j+1}^{\infty}a_k|z|^{k-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho$$

if

$$\sum_{k=j+1}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k|z|^{k-1} \leq 1. \tag{5.5}$$

But, by Theorem 1, (5.5) will be if

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1} \leq \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)k^n(k-\alpha)[1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{5.6}$$

Theorem 9 follows easily from (5.6).

Corollary 4. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = r_3(n,m,\lambda,\alpha,\rho) = \inf_k \left[\frac{(1-\rho)k^{n-1}(k-\alpha)[1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{5.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

6. Modified Hadamard products

Let the functions $f_v(z)$ $(v = 1,2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=j+1}^{\infty} a_{k,1}a_{k,2}z^k. \tag{6.1}$$

Theorem 10. *Let each of the functions $f_v(z)$ $(v = 1,2)$ defined by (4.1) be in the class $T_j(n,m,\lambda,\alpha)$. Then*

$$f_1 * f_2(z) \in T_j(n,m,\lambda,\beta(j,n,m,\lambda,\alpha)),$$

where

$$\beta(j,n,m,\lambda,\alpha) = 1 - \frac{j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+\lambda[(j+1)^m-1]]-(1-\alpha)^2}. \tag{6.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to find the largest $\beta = \beta(j,n,m,\lambda,\alpha)$ such that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\beta)[1+(k^m-1)\lambda]}{1-\beta} a_{k,1}a_{k,2} \le 1. \tag{6.3}$$

Since

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,1} \le 1 \tag{6.4}$$

and

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,2} \le 1, \tag{6.5}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \sqrt{a_{k,1}a_{k,2}} \le 1. \tag{6.6}$$

Thus it is sufficient to show that

$$\frac{k^n(k-\beta)[1+(k^m-1)\lambda]}{1-\beta}a_{k,1}a_{k,2} \leq \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\sqrt{a_{k,1}a_{k,2}} \ (k \geq j+1), \tag{6.7}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(k-\alpha)(1-\beta)}{(k-\beta)(1-\alpha)} \quad (k \geq j+1). \tag{6.8}$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \quad (k \geq j+1). \tag{6.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \leq \frac{(k-\alpha)(1-\beta)}{(k-\beta)(1-\alpha)} \quad (k \geq j+1), \tag{6.10}$$

or, equivalently, that

$$\beta \leq 1 - \frac{(k-1)(1-\alpha)^2}{k^n(k-\alpha)^2[1+(k^m-1)\lambda]-(1-\alpha)^2} \quad (k \geq j+1). \tag{6.11}$$

Since

$$A(k) = 1 - \frac{(k-1)(1-\alpha)^2}{k^n(k-\alpha)^2[1+(k^m-1)\lambda]-(1-\alpha)^2} \tag{6.12}$$

is an increasing function of k $(k \geq j+1)$, letting $k = j+1$ in (6.12) we obtain

$$\beta \leq A(j+1) = \frac{j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+[(j+1)^m-1]\lambda]-(1-\alpha)^2}, \tag{6.13}$$

which proves the main assertion of Theorem 10.

Finally, by taking the functions

$$f_v(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m-1]\lambda]}z^{j+1} \quad (v = 1,2), \tag{6.14}$$

we can see that the result is sharp,

Theorem 11. *Let* $f_1(z) \in T_j(n,m,\lambda,\alpha)$ *and* $f_2(z) \in T_j(n,m,\lambda,\gamma)$*. then*

$$f_1 * f_2(z) \in T_j(n,m,\lambda,\xi(j,n,m,\lambda,\alpha,\gamma)),$$

where

$$\xi(j,n,m,\lambda,\alpha,\gamma) = \tag{6.15}$$

$$= 1 - \frac{j(1-\alpha)(1-\gamma)}{(j+1)^n(j+1-\alpha)(j+1-\gamma)[1+[(j+1)^m-1]\lambda]-(1-\alpha)(1-\gamma)}.$$

The result is best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} z^{j+1} \tag{6.16}$$

and

$$f_2(z) = z - \frac{1-\gamma}{(j+1)^n(j+1-\gamma)[1+[(j+1)^m-1]\lambda]} z^{j+1}. \tag{6.17}$$

Proof. Proceeding as in the proof of Theorem 10, we get

$$\xi \leq 1 - \frac{(k-1)(1-\alpha)(1-\gamma)}{k^n(k-\alpha)(k-\gamma)[1+(k^m-1)\lambda]-(1-\alpha)(1-\gamma)} \quad (k \geq j+1). \tag{6.18}$$

Since the right hand side of (6.18) is an increasing function of k, setting $k = j+1$ in (6.18) we obtain (6.15). This completes the proof of Theorem 11.

Corollary 5. *Let the functions $f_v(z)$ defined by*

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0,\ v = 1,2,3) \tag{6.19}$$

be in the class $T_j(n,m,\lambda,\alpha)$. Then

$$f_1 * f_2 * f_3(z) \in T_j(n,m,\lambda,\delta(j,n,m,\lambda,\alpha)),$$

where

$$\delta(j,n,m,\lambda,\alpha) = 1 - \frac{j(1-\alpha)^3}{(j+1)^{2n}(j+1-\alpha)^3[1+[(j+1)^m-1]\lambda]^2-(1-\alpha)^3}. \tag{6.20}$$

The result is best possible for the functions

$$f_v(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} z^{j+1} \quad (v = 1,2,3). \tag{6.21}$$

Proof. From Theorem 10, we have

$$f_1 * f_2(z) \in T_j(n,m,\lambda,\beta(j,n,m,\lambda,\alpha)),$$

where β is given by (6.2). Now, using Theorem 11, we get

$$f_1 * f_2 * f_3(z) \in T_j(n,m,\lambda,\delta(j,n,m,\lambda,\alpha)),$$

where

$$\delta(j,n,m,\lambda,\alpha) =$$

$$= 1 - \frac{j(1-\alpha)(1-\beta)}{(j+1)^n(j+1-\alpha)(j+1-\beta)[1+[(j+1)^m-1]\lambda]-(1-\alpha)(1-\beta)} =$$

$$= 1 - \frac{j(1-\alpha)^3}{(j+1)^{2n}(j+1-\alpha)^3[1+[(j+1)^m-1]\lambda] - (1-\alpha)^3}.$$

This completes the proof of Corollary 5.

Theorem 12. *Let the functions $f_v(z)$ $(v = 1, 2)$ defined by (4.1) be in the class $T_j(n, m, \lambda, \alpha)$, then the function*

$$h(z) = z - \sum_{k=j+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{6.22}$$

belongs to the class $T_j(n, m, \lambda, \eta(j, n, m, \lambda, \alpha))$, where

$$\eta(j, n, m, \lambda, \alpha) = 1 - \frac{2j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+[(j+1)^m-1]\lambda] - 2(1-\alpha)^2}. \tag{6.23}$$

The result is sharp for the functions $f_v(z)$ $(v = 1, 2)$ defined by (6.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=j+1}^{\infty} \left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^2 a_{k,1}^2 \leq \tag{6.24}$$

$$\leq \left[\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,1}\right]^2 \leq 1$$

and

$$\sum_{k=j+1}^{\infty} \left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^2 a_{k,2}^2 \leq \tag{6.25}$$

$$\leq \left[\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,2}\right]^2 \leq 1.$$

It follows from (6.24) and (6.25) that

$$\sum_{k=j+1}^{\infty} \frac{1}{2} \left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \tag{6.26}$$

Therefore, we need to find the largest $\eta = \eta(j, n, m, \lambda, \alpha)$ such that

$$\frac{k^n(k-\eta)[1+(k^m-1)\lambda]}{1-\eta} \leq \frac{1}{2}\left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^2 \quad (k \geq j+1), \tag{6.27}$$

that is,

$$\eta \leq 1 - \frac{2(k-1)(1-\alpha)^2}{(k-\alpha)^2 k^n[1+(k^m-1)\lambda] - 2(1-\alpha)^2} \quad (k \geq j+1). \tag{6.28}$$

Since

$$B(k) = 1 - \frac{2(k-1)(1-\alpha)^2}{k^n(k-\alpha)^2[1+(k^m-1)\lambda] - 2(1-\alpha)^2}. \tag{6.29}$$

is an increasing function of k $(k \geq j+1)$, we readily have

$$\eta \leq B(j+1) = 1 - \frac{2j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+[(j+1)^m-1]\lambda] - 2(1-\alpha)^2}, \quad (6.30)$$

and Theorem 12 follows at once.

7. A family of integral operators

Theorem 13. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (c > -1) \quad (7.1)$$

also belongs to the class $T_j(n,m,\lambda,\alpha)$.

Proof. From the representation (7.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k.$$

Therefore, we have

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]b_k = \sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]\left(\frac{c+1}{c+k}\right) a_k \leq$$

$$\leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]a_k \leq 1-\alpha,$$

since $f(z) \in T_j(n,m,\lambda,\alpha)$. Hence, by Theorem 1, $F(z) \in T_j(n,m,\lambda,\alpha)$.

Theorem 14. *Let the function*

$$F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0,\ j \in N)$$

be in the class $T_j(n,m,\lambda,\alpha)$, and let c be a real number such that $c > -1$. Then the function $f(z)$ given by (7.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_k \left[\frac{(k-\alpha)k^{n-1}[1+(k^m-1)\lambda](c+1)}{(1-\alpha)(c+k)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \quad (7.2)$$

The result is sharp.

Proof. From (7.1), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{k=j+1}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k.$$

In order to obtain the required result, it suffices to show that

$$|f'(z) - 1| < 1 \text{ whenever } |z| < R^*,$$

where R^* is given by (7.2). Now

$$|f'(z) - 1| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \tag{7.3}$$

But Theorem 1 confirms that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_k \leq 1. \tag{7.4}$$

Hence (7.3) will be satisfied if

$$\frac{k(c+k)}{c+1} |z|^{k-1} < \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha},$$

that is, if

$$|z| < \left[\frac{(k-\alpha)k^{n-1}[1+(k^m-1)\lambda](c+1)}{(1-\alpha)(c+k)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{7.5}$$

Therefore, the function $f(z)$ given by (7.1) is univalent in $|z| < R^*$. Sharpness of the result follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n(k-\alpha)[1+(k^m-1)\lambda](c+1)} z^k \quad (k \geq j+1). \tag{7.6}$$

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PROJECTORS WITH RESPECT TO SOME SPECIAL SCHUNCK CLASSES

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Abstract. Let π be a set of primes and π' the complement to π in the set of all primes. The paper deals with establishing the projectors with respect to some special Schunck classes: the class $\underline{S}_\pi$ of all solvable π-groups, the class $\underline{N}$ of all finite nilpotent groups and the class $\underline{G}_{\pi'}$ of all π'-groups. A new proof is given for some of W. Gaschütz's results from [7] which show that, in any finite solvable group, the $\underline{S}_\pi$-projectors are the Hall π-subgroups and the $\underline{N}$-projectors are the Carter subgroups. Finally, we prove that, in any finite π-solvable group, the $\underline{G}_{\pi'}$-covering subgroups are exactly the Hall π'-subgroups. Hence, we deduce that, in any finite π-solvable group, the $\underline{G}_{\pi'}$-projectors coincide with the Hall π'-subgroups.

1. Preliminaries

All groups considered in the paper are finite. We denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.

We remind some useful definitions:

Definition 1.1. a) We call $\underline{X}$ a class of groups if:

(1) $\{1\} \in \underline{X}$;

(2) if $G \in \underline{X}$ and f is an isomorphism of G then $f(G) \in \underline{X}$.

b) A class $\underline{X}$ of groups is a homomorph if $\underline{X}$ is closed under homomorphisms, i.e. if $G \in \underline{X}$ and N is a normal subgroup of G then $G/N \in \underline{X}$.

c) A group G is primitive if there is a maximal subgroup W of G with

$$core_G W = \{1\},$$

where

$$core_G W = \cap\{W^g / g \in G\}.$$

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d) A homomorph $\underline{X}$ is a Schunck class if $\underline{X}$ is primitively closed, i.e. if any group G, all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

Definition 1.2. Let $\underline{X}$ be a class of groups, G a group and $H \leq G$.

a) H is $\underline{X}$-maximal in G if:

(1) $H \in \underline{X}$;

(2) $H \leq K \leq G,\ K \in \underline{X} \ \Rightarrow \ H = K$.

b) H is an $\underline{X}$-projector of G if for any normal subgroup N of G, HN/N is $\underline{X}$-maximal in G/N.

c) H is an $\underline{X}$-covering subgroup of G if:

(1) $H \in \underline{X}$;

(2) $H \leq K \leq G,\ K_0 \trianglelefteq K,\ K/K_0 \in \underline{X} \ \Rightarrow \ K = HK_0$.

Definition 1.3. a) A group G is π-solvable if any chief factor of G is either a solvable π-group or a π'-group. If π is the set of all primes, we obtain the notion of solvable group.

b) A class $\underline{X}$ of groups is π-closed if:

$$G/O_{\pi'}(G) \in \underline{X} \ \Rightarrow \ G \in \underline{X},$$

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G. We shall call π-homomorph, respectively π-Schunck class a π-closed homomorph, respectively a π-closed Schunck class.

Theorem 1.4. ([6]) *Let $\underline{X}$ be a homomorph, G a group and $H \leq K \leq G$. If H is an $\underline{X}$-covering subgroup of G, then H is an $\underline{X}$-covering subgroup of K.*

The connection between the special subgroups introduced above is given in [7] and in [4] and is resumed in the following theorem:

Theorem 1.5. *a) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G. If H is an $\underline{X}$-covering subgroup or an $\underline{X}$-projector of G then H is $\underline{X}$-maximal in G.*

b) Let $\underline{X}$ be a homomorph, G a group and H a subgroup of G. H is an $\underline{X}$-covering subgroup of G is and only if H is an $\underline{X}$-projector in any subgroup K with $H \leq K \leq G$. Particularly, any $\underline{X}$-covering subgroup of G is an $\underline{X}$-projector of G.

c) Let $\underline{X}$ be a Schunck class, G a finite solvable group, S and $\underline{X}$-projector of G and $S \leq H \leq G$. Then S is an $\underline{X}$-projector in H.

d) Let $\underline{X}$ be a Schunck class, G a finite solvable group and S a subgroup of G. The following conditions are equivalent:

(1) S is an $\underline{X}$-projector of G;

(2) S is an $\underline{X}$-covering subgroup of G.

The following properties of projectors are also of special interest for this paper:

Theorem 1.6. ([7]) *Let $\underline{X}$ be a homomorph, G a group and $H \leq G$.*

a) H is an $\underline{X}$-projector of G is and only if:

(1) H is $\underline{X}$-maximal in G;

(2) for any minimal normal subgroup M of G, HM/M is an $\underline{X}$-projector in G/M.

b) If H is an $\underline{X}$-projector of G and N is a normal subgroup of G, then HN/N is an $\underline{X}$-projector of G/N. This property holds for $\underline{X}$-covering subgroups too.

Theorem 1.7. ([7]) *Let $\underline{X}$ be a Schunck class, G a solvable group and B a normal abelian subgroup of G. If:*

(1) SB/B is an $\underline{X}$-projector of G/B;

(2) S is $\underline{X}$-maximal in SB,

then S is an $\underline{X}$-projector of G.

In [2], [3] and [4], we established for finite π-solvable groups the following result:

Theorem 1.8. ([2]) *Let $\underline{X}$ be a π-homomorph.*

a) $\underline{X}$ is a Schunck class if and only if any π-solvable group has $\underline{X}$-covering subgroups.

b) Any two $\underline{X}$-covering subgroups of a π-solvable group G are conjugate in G.

Theorem 1.9. ([3], [4]) *Let $\underline{X}$ be a π-homomorph. Then $\underline{X}$ is a Schunck class if and only if any π-solvable group has $\underline{X}$-projectors.*

Corollary 1.10. *Let $\underline{X}$ be a π-homomorph. The following conditions are equivalent:*

(1) $\underline{X}$ is a Schunck class;

(2) any π-solvable group has $\underline{X}$-covering subgroups;

(3) any π-solvable group has $\underline{X}$-projectors.

Theorem 1.11. ([3]) *If $\underline{X}$ is a π-Schunck class, then any two $\underline{X}$-projectors of a π-solvable group G are conjugate in G.*

Particularly, for π the set of all primes, theorems 1.8-1.11 give well-known results from [7] and [10] referring to finite solvable groups.

2. Projectors with respect to the class $\underline{S}_\pi$

Denote by $\underline{S}_\pi$ the class of all solvable π-groups. We give a new proof for the following result given by W. Gaschütz in [7]: In any finite solvable group, the $\underline{S}_\pi$-projectors are exactly the Hall π-subgroups.

A positive integer n is said to be a π-number if for any prime divisor p of n we have $p \in \pi$.

Definition 2.1. a) A finite group G is a π-group if $|G|$ is a π-number.

b) A subgroup S of a group G is a π-subgroup if S is a π-group.

c) A subgroup S of a group G is an Hall π-subgroup if:

(1) S is a π-subgroup;

(2) $(|S|, |G : S|) = 1$, i.e. $|G : S|$ is a π'-number.

We shall use the following properties of the Hall subgroups ([9]):

Proposition 2.2. If G is a group, S is an Hall π-subgroup of G and H is a subgroup of G such that $S \leq H \leq G$, then S is an Hall π-subgroup of H.

Proposition 2.3. If G is a group, S is an Hall π-subgroup of G and N is a normal subgroup of G, then SN/N is an Hall π-subgroup of G/N.

The Hall subgroups were given in [8]. Ph. Hall studied them for finite solvable groups. In [5], S.A. Čunihin extended this study for finite π-solvable groups.

Theorem 2.4. (Ph. Hall, S.A. Čunihin, [9]) a) Any finite π-solvable group G has Hall π-subgroups and Hall π'-subgroups.

b) If G is a finite π-solvable group, then:

(i) any two Hall π-subgroups of G are conjugate in G;

(ii) any two Hall π'-subgroups of G are conjugate in G.

Particularly, any finite solvable group G has Hall π-subgroups and they are conjugate in G.

In preparation to the main theorem of this section, we prove the following results:

Theorem 2.5. *$\underline{S}_\pi$ is a homomorph.*

Proof. Let $G \in \underline{S}_\pi$ and N be a normal subgroup of G. G being solvable, G/N is also solvable. G being a π-group, from $|G/N|$ divides $|G|$ we obtain that G/N is a $\sqrt{\pi}$-group too. □

Theorem 2.6. *If G is a finite solvable group and S is an Hall π-subgroup of G, then S is $\underline{S}_\pi$-maximal in G.*

Proof. By induction on $|G|$. We verify the two conditions from 1.2.a).

(1) $S \in \underline{S}_\pi$. Indeed, S is a π-group and S is solvable being a subgroup of a solvable group.

(2) Let $S \leq H \leq G$ and $H \in \underline{S}_\pi$. We prove that $S = H$. By 2.2, S is an Hall π-subgroup of H. We consider two cases:

a) $H = G$. Then $G \in \underline{S}_\pi$ and G is its own Hall π-subgroup. But S is also an Hall π-subgroup of G. Applying 2.4.b), S and G are conjugate in G, hence $S = G = H$.

b) $H \neq G$. By the induction, S is $\underline{S}_\pi$-maximal in H. But $H \in \underline{S}_\pi$. Hence $S = H$. □

The main theorem in this section is from [7] and we give here a new proof.

Theorem 2.7. *Let G be a finite solvable group and S a subgroup of G. S is an $\underline{S}_\pi$-projector of G if and only if S is an Hall π-subgroup of G.*

Proof. By induction on $|G|$.

Let S be an $\underline{S}_\pi$-projector of G. We prove like in [7] that S is an Hall π-subgroup of G. Clearly S is a π-subgroup of G. We show that $|G : S|$ is a π'-number. Let M be a minimal normal subgroup of G. G being solvable, we have $|M| = p^k$, where p is a prime. Put $S' = SM$. By 1.6, S'/M is an $\underline{S}_\pi$-projector of G/M. Hence, by the induction, S'/M is an Hall π-subgroup of G/M. Two cases are considered:

a) $p \in \pi$. Then $S' \in \underline{S}_\pi$. But, by 1.5.a), S is $\underline{S}_\pi$-maximal in G. So $S = S'$. Then

$$|G : S| = |G : S'| = |G/M : S'/M| \text{ is a } \pi'\text{-number.}$$

b) $p \notin \pi$. Then $M \cap S = \{1\}$. So $|G : S|$ is a π'-number, because:

$$|G : S| = |G : S'||S' : S| = |G/M : S'/M||SM : S|,$$

where $|G/M : S'M|$ is a π'-number and $|SM : S| = |M : M \cap S| = |M| = p^k$, where $p \notin \pi$, hence $p \in \pi'$.

The converse has an original proof based on 1.6.a) and 2.6. Let S be an Hall π-subgroup of G. We shall prove that S is an $\underline{S}_\pi$-projector of G. We use 1.6.a).

(1) S is $\underline{S}_\pi$-maximal in G, by 2.6.

(2) Let M be a minimal normal subgroup of G. We prove that SM/M is an $\underline{S}_\pi$-projector of G/M. Indeed, by 2.3, SM/M is an Hall π-subgroup of G/M. Hence, by the induction, we obtain that SM/M is an $\underline{S}_\pi$-projector of G/M. □

Corollary 2.8. *$\underline{S}_\pi$ is a Schunck class.*

Proof. From 1.9, 2.7 and 2.4. □

Corollary 2.9. *Let G be a finite solvable group and S a subgroup of G. The following conditions are equivalent:*

(1) S is an Hall π-subgroup of G;

(2) S is an $\underline{S}_\pi$-projector of G;

(3) S is an $\underline{S}_\pi$-covering subgroup of G.

Proof. From 2.7, 2.8 and 1.5.d). □

3. **Projectors with respect to the class $\underline{N}$**

Denote by $\underline{N}$ the class of all finite nilpotent groups. We shall give a new proof for the following W. Gaschütz's result from [7]: In any finite solvable group, the $\underline{N}$-projectors are exactly the Carte subgroups.

Definition 3.1. Let G be a group and S a subgroup of G. S is a Carter subgroup of G if:

(1) S is nilpotent;

(2) $N_G(S) = S$.

The following properties given in [1] are important for our considerations:

Proposition 3.2. If G is a group, S is a Carter subgroup of G and H is a subgroup of G such that $S \leq H \leq G$, then S is a Carter subgroup of H.

Proposition 3.3. If G is a group, S is a Carter subgroup of G and N is a normal subgroup of G, then SN/N is a Carter subgroup of G/N.

Theorem 3.4. (R. Carter [1]) *Let G be a finite solvable group. Then:*

a) G has Carter subgroups;

b) any two Carter subgroups of G are conjugate in G.

Theorem 3.5. *$\underline{N}$ is a Schunck class.*

Proof. Obviously $\underline{N}$ is a homomorph. Further, $\underline{N}$ is primitively closed. Indeed, if G is a group such that all primitive factor groups $G/N \in \underline{N}$, we can prove that $G \in \underline{N}$. For this, we use Wieland's criterium for finite groups to be nilpotent: we show that any maximal subgroup W of G is normal in G. Denote by $N = core_G W$. Then G/N is primitive, hence $G/N \in \underline{N}$. But W/N is maximal in G/N. Applying now Wieland's criterium for the nilpotent group G/N, we obtain that W/N is normal in G/N, hence W is normal in G. □

Theorem 3.6. *If G is a finite solvable group and S is a Carter subgroup of G, then S is $\underline{N}$-maximal in G.*

Proof. By induction on $|G|$.

(1) $S \in \underline{N}$, because S is a Carter subgroup.

(2) Let $S \leq H \leq G$ with $H \in \underline{N}$. We prove that $S = H$. Indeed, we have by 3.2 that S is a Carter subgroup of H. Consider two cases:

a) $H = G$. Then $G \in \underline{N}$ and so G is its own Carter subgroup. Applying 3.4.b), S and G are conjugate in G. It follows that $S = G = H$.

b) $H \neq G$. By the induction, S is $\underline{N}$-maximal in H. But $H \in \underline{N}$. Hence $S = H$. □

The main result is:

Theorem 3.7. *Let G be a finite solvable group and S a subgroup of G. S is an $\underline{N}$-projector of G if and only if S is a Carter subgroup of G.*

Proof. Let S be an $\underline{N}$-projector of G. We prove that S is a Carter subgroup of G like in [7]. Clearly S is nilpotent. In order to show that $N_G(S) = S$, let us suppose that $S \neq N_G(S)$. Then S is maximal in H, where $H \leq N_G(S)$. So S is a normal subgroup in H. Now H solvable and S maximal in H imply $|H : S| = p^k$, with p prime. So H/S is a finite p-group, hence H/S is nilpotent. By 1.5.c), S is an $\underline{N}$-projector in H. From S normal in H, follows by 1.6.b) that SS/S is an $\underline{N}$-projector in H/S, hence SS/S is $\underline{N}$-maximal in $H/S \in \underline{N}$. So $S = H$, in contradiction with the choice of H.

The converse has an original proof, based on 1.7 and 3.6. Let S be a Carter subgroup of G. We prove that S is an $\underline{N}$-projector of G by using 1.7. Let B be a minimal normal subgroup of G. G being solvable, B is abelian. By 3.5, $\underline{N}$ is a Schunck class. By 3.6 we have that S is $\underline{N}$-maximal in G and so S is $\underline{N}$-maximal in SB. Further, because, by 3.3, SB/B is a Carter subgroup of G/B, we can use the induction for G/B and so SB/B is an $\underline{N}$-projector of G/B. We apply now 1.7 and so S is an $\underline{N}$-projector of G. □

Corollary 3.8. *Let G be a finite solvable group and S a subgroup of G. The following conditions are equivalent:*

(1) S is a Carter subgroup of G;

(2) S is an $\underline{N}$-projector of G;

(3) S is an $\underline{N}$-covering subgroup of G.

Proof. From 3.7 and 1.5.d). □

4. Projectors with respect to the class $\underline{G}_{\pi'}$

In this section we establish the $\underline{G}_{\pi'}$-projectors of a finite π-solvable group, proving that they coincide with the Hall π'-subgroups.

Remind that π is an arbitrary set of primes and π' is the complement to π in the set of all primes. Denote by $\underline{W}_\pi$ the class of all finite π-solvable groups and by $\underline{G}_{\pi'}$ the class of all π'-groups. Obviously

$$\underline{G}_{\pi'} \subseteq \underline{W}_\pi.$$

All groups considered in this section are finite π-solvable groups.

Theorem 4.1. *The class $\underline{G}_{\pi'}$ is a π-homomorph.*

Proof. Let $G \in \underline{G}_{\pi'}$ and N a normal subgroup of G. Then $|G/N|/|G|$, hence $G/N \in \underline{G}_{\pi'}$. So $\underline{G}_{\pi'}$ is a homomorph. $\underline{G}_{\pi'}$ is π-closed. Indeed, if $G/O_{\pi'}(G) \in \underline{G}_{\pi'}$, then

$$|G| = |G/O_{\pi'}(G)||O_{\pi'}(G)|$$

is a π'-number and so $G \in \underline{G}_{\pi'}$. □

Theorem 4.2. *Let G be a finite π-solvable group and H a subgroup of G. H is a $\underline{G}_{\pi'}$-covering subgroup of G is and only if H is an Hall π'-subgroup of G.*

Proof. Let H be a $\underline{G}_{\pi'}$-covering subgroup of G. We prove by induction on $|G|$ that H is an Hall π'-subgroup of G.

(1) H is a π'-subgroup of G, because $H \in \underline{G}_{\pi'}$.

(2) $(|H|, |G : H|) = 1$. Indeed, let M be a minimal normal subgroup of G. H being a $\underline{G}_{\pi'}$-covering subgroup of G, HM/M is by 1.6.b) a $\underline{G}_{\pi'}$-covering subgroup of G/M, hence, by the induction, HM/M is an Hall π'-subgroup of G/M. Being a minimal normal subgroup of the π-solvable group G, M is either a solvable π-group or a π'-group. We consider now two cases:

a) $G/M \in \underline{G}_{\pi'}$. Then, HM/M being $\underline{G}_{\pi'}$-maximal in G/M (see 1.5.a)), we have $HM/M = G/M$ and so $HM = G$. Let us suppose now that M is a solvable π-group. Then

$$|G : H| = |HM : H| = |M : H \cap M|/|M|$$

is a π-number. Suppose that M is a π'-group. We know that $G/M = HM/M \in \underline{G}_{\pi'}$. Then $|G| = |G/M||M|$ is a π'-number. So $G \in \underline{G}_{\pi'}$. But H is $\underline{G}_{\pi'}$-maximal in G. Hence $H = G$ is its own Hall π'-subgroup.

b) $G/M \notin \underline{G}_{\pi'}$. Then $HM/M \neq G/M$, hence $HM \neq G$. By 1.4, H is a $\underline{G}_{\pi'}$-covering subgroup in HM and it follows, by the induction, that H is a Hall π'-subgroup of HM. Then $|HM : H|$ is a π-number and $|G : HM| = |G/M : HM/M|$ is also a π-number. It follows that

$$|G : H| = |G : HM||HM : H|$$

is a π-number.

Conversely, let H be an Hall π'-subgroup of G. We shall prove that H is an $\underline{G}_{\pi'}$-covering subgroup of G.

(1) $H \in \underline{G}_{\pi'}$ is clear.

(2) $H \leq L \leq G$, $L_0 \trianglelefteq L$, $L/L_0 \in \underline{G}_{\pi'}$ imply $L = HL_0$. We prove this by induction on $|G|$. Two cases are considered:

(i) $L \neq G$. By 2.2, H is an Hall π'-subgroup of L. Applying the induction, from $H \leq L = L$, $L_0 \trianglelefteq L$, $L/L_0 \in \underline{G}_{\pi'}$ follows $L = HL_0$.

(ii) $L = G$. Again two cases are considered. If $L_0 = 1$, then $G = L \cong L/L_0 \in \underline{G}_{\pi'}$ and G is its own Hall π'-subgroup. By 2.4.b), H and G are conjugate in G. Then

$$L = G = H = HL_0.$$

If $L_0 \neq 1$, we have that there is a minimal normal subgroup M of G such that $M \leq L_0$. From 2.3, HM/M is an Hall π'-subgroup of G/M. We apply the induction for G/M. Then, from

$$HM/M \leq G/M = G/M,\ L_0/M \trianglelefteq G/M,\ (G/M)/(L_0/M) \cong G/L_0 = L/L_0 \in \underline{G}_{\pi'},$$

it follows that

$$G/M = (HM/M)(L_0/M).$$

Hence $L = G = HML_0 = HL_0$. □

Corollary 4.3. *The class $\underline{G}_{\pi'}$ is a π-Schunck class.*

Proof. Follows from 1.10, 4.1, 4.2 and 2.4.a). □

Corollary 4.4. *Let G be a finite π-solvable group and $H \leq G$. If H is an Hall π'-subgroup of G, then H is a $\underline{G}_{\pi'}$-projector in G.*

Proof. H being a Hall π'-subgroup of G, H is by 4.2 a $\underline{G}_{\pi'}$-covering subgroup of G. Then, H is by 1.5.b) a $\underline{G}_{\pi'}$-projector in G. □

The main theorem in this section is the following:

Theorem 4.5. *Let G be a finite π-solvable group and H a subgroup of G. H is a $\underline{G}_{\pi'}$-projector of G if and only if H is an Hall π'-subgroup of G.*

Proof. If H is an Hall π'-subgroup of G, then H is by 4.4 a $\underline{G}_{\pi'}$-projector of G.

Conversely, let H be a $\underline{G}_{\pi'}$-projector of G. We prove by induction on $|G|$ that H is an Hall π'-subgroup of G.

(1) H is a π'-subgroup of G because, H being $\underline{G}_{\pi'}$-maximal in G (see 1.5.a)), we have $H \in \underline{G}_{\pi'}$.

(2) $|G : H|$ is a π-number. Indeed, let M be a minimal normal subgroup of G. HM/M is by 1.6.b) a $\underline{G}_{\pi'}$-projector in G/M, hence by the induction HM/M is an Hall π'-subgroup of G/M. It follows that

$$|G : HM| = |G/M : HM/M|$$

is a π-number. M being a minimal normal subgroup of the π-solvable group G, M is either a solvable π-group or a π'-group.

a) If M is a solvable π-group, then $|G : H| = |G : HM||HM : H|$, where $|HM : H| = |M : H \cap M|/|M|$ is a π-number. It follows that $|G : H|$ is a π-number.

b) If M is a π'-group, using that HM/M is a π'-group we notice that HM is a π'-group. But H being a $\underline{G}_{\pi'}$-projector in G, H is by 1.5.a) $\underline{G}_{\pi'}$-maximal in G.

Hence from $H \leq HM \leq G$ and $HM \in \underline{G}_{\pi'}$ follows that $H = HM$. Then

$$|G : H| = |G : HM| = |G/M : HM/M|$$

is a π-number. □

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A NOTE ON τ-QUASI-INJECTIVE MODULES

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Abstract. Let τ be a hereditary torsion theory. We mention a characterization of τ-quasi-injective modules, as fully invariant submodules of their τ-injective hull, and we give some properties for such modules. Moreover, the paper studies when τ-quasi-injective modules are quasi-injective or not, in the case of the hereditary torsion theory τ_D whose τ_D-torsion class consists of all semiartinian modules and τ_D-torsionfree class consists of all modules with zero socle.

1. Preliminaries

Throughout this paper we will denote by R an associative ring with non-zero identity and by τ a hereditary torsion theory on the category R-mod of left R-modules. All modules considered in the paper will be left unital R-modules.

A module A is said to be semiartinian if every non-zero homomorphic image of A contains a simple submodule [6, Chapter I, Definition 11.4.6]. Let A be a module and let B be a submodule of A. Then A is semiartinian if and only if B and A/B are semiartinian [6, Chapter I, Proposition 11.4.8].

A submodule B of a module A is said to be τ-dense (τ-closed) in A if A/B is τ-torsion (τ-torsionfree). A non-zero module A is called τ-cocritical if A is τ-torsionfree and each of its non-zero submodules is τ-dense in A.

A module A is said to be τ-injective if $Ext^1_R(B, A) = 0$ for every τ-torsion module B. A module A is τ-injective if and only if A is a τ-closed submodule of its injective hull [5, Proposition 8.2]. The class of τ-injective modules is closed under taking direct products, direct summands and extensions [5, Proposition 8.4]. For any module A, we will denote by $E(A)$ and $E_\tau(A)$ the injective hull and the τ-injective hull of A respectively.

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In this paper, a non-zero module which is the τ-injective hull of each of its non-zero submodules will be called minimal τ-injective.

For additional information on torsion theories we refer to [5].

2. Some properties

A module A is said to be τ-quasi-injective if whenever B is a τ-dense submodule of A, any $g \in Hom_R(B, A)$ can be extended to $h \in End_R(A)$ [1, Definition 4.1.19].

Remarks. a) Every quasi-injective module is τ-quasi-injective.

b) Every τ-injective module is τ-quasi-injective.

c) A ring R is a τ-quasi-injective R-module if and only if it is τ-injective.

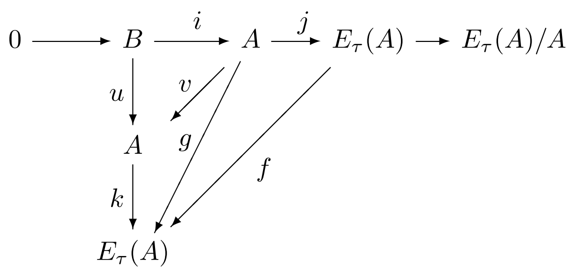
d) If A is a τ-torsion τ-quasi-injective module, then A is quasi-injective.

The following theorem gives a characterization of τ-quasi-injective modules similar to the well known characterization of quasi-injective modules, which are fully invariant submodules of their injective hulls.

Theorem 2.1. *Let A be a module. Then A is τ-quasi-injective if and only if A is a fully invariant submodule of $E_\tau(A)$.*

Proof. We may suppose that $A \neq 0$. Denote $K = End_R(E_\tau(A))$.

Assume first that A is τ-quasi-injective and let $f \in K$. Denote $g = f|_A$ and $B = g^{-1}(A)$. Consider the following commutative diagram



where i, j, k are inclusion monomorphisms and $u : B \to A$ is defined by $u(b) = g(b)$ for every $b \in B$.

We will show that B is a τ-dense submodule of A. The homomorphism g induces a monomorphism $w : A/B \to E_\tau(A)/A$, defined by $w(a + B) = g(a) + A$ for every $a \in A$. Then A/B is τ-torsion because $E_m(A)/A$ is τ-torsion. Hence B is a τ-dense submodule of A.

Since A is τ-quasi-injective, there exists $v \in End_R(A)$ such that $vi = u$. By τ-injectivity of $E_\tau(A)$, there exists $h \in K$ such that $hj = kv$. Thus $h(A) \subseteq A$. Assume $(h - f)(A) \neq 0$. Then $(h - f)(A) \cap A \neq 0$ and there exist $x, y \in A$, $y \neq 0$ such that $y = (h - f)(x)$. It follows that $(h - f)(x) = v(x) - f(x) = y$, hence $f(x) = v(x) - y \in A$. Then $x \in B$ and $y = v(x) - f(x) = 0$, contradiction. Therefore, $(h - f)(A) = 0$, i.e. $f(A) = h(A) \subseteq A$. Hence A is a fully invariant submodule of $E_\tau(A)$.

Suppose now that A is a fully invariant submodule of $E_\tau(A)$. Let B be a τ-dense submodule of A and let $g \in Hom_R(B, A)$. The module $E_\tau(A)/B$ is τ-torsion because $E_\tau(A)/A$ and A/B are τ-torsion. Then g extends to $h \in K$ because $E_\tau(A)$ is τ-injective. Since $h(A) \subseteq A$, g extends to an endomorphism of A. Therefore A is τ-quasi-injective. □

Corollary 2.2. *If every τ-injective module is injective, then every τ-quasi-injective module is quasi-injective.*

Proof. By Theorem 2.1, if A is a τ-quasi-injective module, then A is a fully invariant submodule of $E_\tau(A)$. But $E_\tau(A) = E(A)$. Hence A is a fully invariant submodule of $E(A)$, i.e. A is quasi-injective. □

Remark. By Theorem 2.1 and in a similar way as for quasi-injective modules, it can be easily shown that the class of τ-quasi-injective modules is closed under taking direct summands and any finite direct sum of copies of a τ-quasi-injective module is τ-quasi-injective.

Theorem 2.3. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of modules and let $h : B \to A \oplus D$ be a monomorphism, where D is a module. If $(hf)(A)$ is a τ-dense submodule of $A \oplus D$ and $A \oplus D$ is τ-quasi-injective, then the above sequence splits.

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle h} & & & & \\ & & A \oplus D & \overset{\theta}{\dashleftarrow} & A \oplus D & & & & \end{array}$$

where $\alpha : A \to A \oplus D$ is the canonical injection. Since $(A \oplus D)/(hf)(A)$ is τ-torsion and $A \oplus D$ is τ-quasi-injective, there exists an endomorphism $\theta : A \oplus D \to A \oplus D$ such that $\theta hf = \alpha$. Let $p : A \oplus D \to A$ be the canonical projection and define $\gamma : B \to A$ by $\gamma = p\theta h$. Then $\gamma f = p\theta hf = p\alpha = 1_A$, hence the above sequence splits. □

Corollary 2.4. *Let $f : A \to B$ be a monomorphism of modules. If B is τ-torsion and $A \oplus B$ is τ-quasi-injective, then $A \oplus B$ is τ-injective if and only if B is τ-injective.*

Proof. The "if" part is obvious.

For the "only if" part, in the Theorem 2.3, let $h : B \to A \oplus B$ be the canonical injection. Since B is τ-torsion, A and $B/f(A)$ are τ-torsion. Hence $(A \oplus B)/(hf)(A) \cong (A \oplus B)/f(A)$ is τ-torsion. By Theorem 2.3, $f(A)$ is a direct summand of B, hence A is τ-injective. Therefore $A \oplus B$ is τ-injective. □

3. The Dickson torsion theory

In this section we will establish further results in the case of a particular hereditary torsion theory, namely the Dickson torsion theory.

For let $\mathcal{T}$ be the class of all semiartinian R-modules and let $\mathcal{F}$ be the class of all R-modules with zero socle. Then $\tau_D = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory. The corresponding Gabriel filter F consists of all τ_D-dense left ideals of R (i.e. all left ideals of R with R/I left semiartinian as an R-module).

An R-module D is τ_D-injective if any homomorphism from any left ideal $I \in F$ to D extends to R or equivalently if D is injective with respect to every short exact sequence of modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, where C is τ_D-torsion (i.e. C is semiartinian).

We consider now the following generalization of injectivity for modules. An R-module D is said to be m-injective if for every maximal left ideal M of R the R-module D is injective with respect to the inclusion monomorphism $u : M \to R$ [2, Definition 1].

The notions of τ_D-injectivity and m-injectivity are in fact the same [2, Theorem 6]. By this reason, in the sequel we will use the notation m instead of τ_D. For instance, injective and quasi-injective modules with respect to the Dickson torsion theory will be called m-injective and m-quasi-injective modules respectively.

From the general context of torsion theories it follows that every module A has an m-injective hull, denoted by $E_m(A)$, contained in $E(A)$, unique up to an isomorphism.

We have seen that every quasi-injective module is τ-quasi-injective. For the Dickson torsion theory we will give several cases when quasi-injectivity and m-quasi-injectivity are or are not the same.

Proposition 3.1. *Let R be either left semiartinian or left m-cocritical. Then every m-quasi-injective R-module is quasi-injective.*

Proof. In both cases every, every non-zero left ideal is m-dense in R, hence every m-injective module is injective. Now the result follows by Corollary 2.2. □

Corollary 3.2. *Let R be a commutative noetherian domain with $\dim R \leq 1$. Then every m-quasi-injective R-module is quasi-injective.*

Proof. By hypotheses, every m-injective module is injective [2, Corollary 13]. Now the result follows by Corollary 2.2. □

In the sequel we will see that there exist m-quasi-injective modules which are not m-injective and even quasi-injective modules which are not m-injective.

Theorem 3.3. *Let A be an m-quasi-injective module which is not m-injective and denote $M = E_m(A)$. Consider the Loewy series of M/A*

$$0 = S_0(M/A) \subseteq S_1(M/A) \subseteq \cdots \subseteq S_\alpha(M/A) \subseteq S_{\alpha+1}(M/A) \subseteq \dots$$

where, for each ordinal $\alpha \geq 0$,

$$S_{\alpha+1}(M/A)/S_\alpha(M/A) = Soc((M/A)/S_\alpha(M/A))$$

and if α is a limit ordinal, then

$$S_\alpha(M/A) = \bigcup_{0\leq\beta<\alpha} S_\beta(M/A)\,.$$

For every ordinal $\alpha \geq 0$, let M_α be a submodule of M be such that $S_\alpha(M/A) = M_\alpha/A$.

Then every non-zero proper submodule M_α of M is m-quasi-injective, but not m-injective.

Proof. Let $\alpha \geq 1$ be an ordinal such that M_α is a proper submodule of M and let $f \in End_R(M)$. Since A is m-quasi-injective, $f(A) \subseteq A$ by Theorem 2.1. Then f induces an endomorphism $f^* \in End_R(M/A)$. Since $M_\alpha/A = S_\alpha(M/A)$ is fully invariant [4, 3.11, p.25], $f^*(M_\alpha/A) \subseteq M_\alpha/A$, therefore $f(M_\alpha) \subseteq M_\alpha$, i.e. M_α is m-quasi-injective. On the other hand, M_α is a proper submodule of $E_m(A) = M$, hence M_α is not m-injective. □

Theorem 3.4. *Let S be a simple module which is not m-injective and denote $M = E_m(S)$. Consider the Loewy series of M*

$$0 = S_0(M) \subseteq S_1(M) \subseteq \cdots \subseteq S_\alpha(M) \subseteq S_{\alpha+1}(M) \subseteq \ldots$$

where, for each ordinal $\alpha \geq 0$, $S_{\alpha+1}(M)/S_\alpha(M) = Soc(M/S_\alpha(M))$ and if α is a limit ordinal, then $S_\alpha(M) = \bigcup_{0\leq\beta<\alpha} S_\beta(M)$.

Then every non-zero proper submodule $S_\alpha(M)$ of M is quasi-injective, but not m-injective.

Proof. Let $\alpha \geq 1$ be an ordinal such that $S_\alpha(M)$ is a proper submodule of M. Then $S_\alpha(M)$ is a fully invariant submodule of M [4, 3.11, p.25], therefore m-quasi-injective by Theorem 2.1. Also $S_\alpha(M)$ is semiartinian as a submodule of the semiartinian module M. It follows that $S_\alpha(M)$ is quasi-injective. Since $M = E_m(S)$ is minimal m-injective, $S_\alpha(M)$ is not m-injective. □

We have noted that every quasi-injective module is m-quasi-injective. The converse is not true, as we can see in the following example.

Example 3.5. Let R be a unique factorization domain such that every maximal ideal of R is not principal. Then R is an m-injective R-module which is not injective [2, Theorem 15]. Hence R is m-quasi-injective. Since R is quasi-injective if and only if R is injective, it follows that R is not quasi-injective.

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A DIFFUSION PROBLEM IN A CIRCULAR DOMAIN IN A POROUS LAYER

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1. Introduction

Transport and flow phenomena in porous media or industrial synthetic porous materials, arise in many diverse fields of science and engineering, ranging from agricultural, biomedical, construction, ceramic, chemical, and petroleum engineering to food and soil science, and powder technology. Fifty percent of more of the original oil-in-plan is left in a typical oil reservoir by traditional recovery techniques. Oil recovery processes constitute only a small fraction of an enormous, and still rapidly growing, literature on porous media. In addition to oil recovery processes, the closely related areas of soil science and hydrology are perhaps the best – established topics. The study of groundwater flow and the restoration of aquifers that have been contaminated by various pollutants are important current areas of research in porous media problems. The construction industry, transmission of water by building materials is also an important problem that uses porous media. Phenomena involving porous media are also numerous. Recent books by Ingham and Pop [1], Nield and Bejan [2], Vafai [3] and Pop and Ingham [4] on transport phenomena in porous media clearly demonstrate that flows in porous media are becoming a classical subject, once where earlier developments have been confirmed by a large number of studies.

The present paper studies a diffusion problem in a porous layer of circular form and thickness Δz. We suppose that the pressure p does not vary with height. Thus, the fluid motion is reduced to two – dimensional flow in a circular domain. We assume that the domain's boundary is impermeable, that at the moment $t = 0$ the fluid has an initial pressure, p_i, and that a negative source is placed in the centre of the domain. We will study the evolution of the pressure on time.

2. **Basic Equation**

We consider the two – dimensional flow of a viscous and compressible fluid generated by a negative source of debit q placed in the porous layer. We study the fluid motion in a circular domain where the source is placed in the centre of the domain. The problem is described by the continuity equation, Darcy's law and the state equation as established by Cretu [5] or Ungureanu et al. [6]:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) - M_s(x, y, t) = -\frac{\partial}{\partial t}(m\rho) \tag{1}$$

$$u = -\frac{K}{\mu}\frac{\partial p}{\partial x}, v = -\frac{K}{\mu}\frac{\partial p}{\partial y} \tag{2}$$

$$\rho = \rho_0 e^{\beta(p - p_0)} \tag{3}$$

where x and y are Cartesian coordinates, u and v are velocity components along x and y axes, respectively, K is the permeability of porous medium, ρ is the density, μ is the viscosity, p is the pressure, ρ_0 is te density at the atmospheric pressure p_0 and β is the compressibility coefficient defined as

$$\beta = \frac{1}{\rho}\frac{d\rho}{dp} \tag{4}$$

Because the compressibility coefficient, β , is small equation (3) can be expressed as:

$$\rho \approx \rho_0 \left[1 + \beta (p - p_0)\right] \tag{5}$$

If we assume that the porous medium is homogeneous (K is constant in x and y directions, respectively), μ is independent of the pressure p and that $M_s = q$ (constant), Eqs.(1) – (5) reduces, after some algebra to the following equation:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial p}{\partial y}\right) - q = c\frac{\partial p}{\partial t} \tag{6}$$

and it describes the flow of a viscous fluid trough porous medium. In this equation K_x and K_y denotes the permeability in x and y direction, respectively, q is the debit and c is the hydraulic capacity.

3. **Application**

Because the flow domain is circular the flow is symmetric. Thus, Eq. (5) is written in polar coordinates (r,θ) as follows (see, for example, Kohr [7]):

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\mu}{k}q = \frac{m\beta\mu}{k}\frac{\partial p}{\partial t} \tag{7}$$

where $\partial/\partial\theta = 0$ has been used. Equation (7) is now written in the non – dimensional form by using the new variables

$$r^* = \frac{r}{R}, p^* = \frac{p}{p_i}, t^* = \frac{kt}{R^2 m\beta\mu}, q^* = \frac{R^2\mu q}{kp_i} \tag{8}$$

where R is the radius of the circular domain. Substituting the variables (8) into Eq. (7), it becomes

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + q = \frac{\partial p}{\partial t} \tag{9}$$

where the star has been dropped.

We shall assume now that the boundary of the circular domain is impermeable and that at t = 0 the initial pressure, p_i, is constant and equal with one. Thus, the initial and boundary conditions of Eq. (8) are

$$p(r,0) = 1, \frac{\partial p}{\partial r}(1,t) = 0 \tag{10}$$

Further, we notice that at r = 0, we have

$$\lim_{r\to 0}\frac{\frac{\partial p}{\partial r}}{r} = \frac{\partial^2 p}{\partial r^2} \tag{11}$$

Therefore, Eq. (9) can be written as

$$r = 0: \quad 2\frac{\partial^2 p}{\partial r^2} + q = \frac{\partial p}{\partial t} \tag{12}$$

$$r \neq 0: \quad \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + q = \frac{\partial p}{\partial t} \tag{13}$$

We will use the finite difference operators for the derivatives, which appear in Eqs. (12) and (13) (see Ixaru [8]):

$$\begin{aligned}\frac{\partial p}{\partial r} &= \frac{1}{2h}\left(p_{i+1}-p_{i-1}\right)\\ \frac{\partial^2 p}{\partial r^2} &= \frac{1}{h^2}\left(p_{i-1}-2p_i+p_{i+1}\right)\\ \frac{\partial p}{\partial r} &= \frac{1}{\Delta t}\left(p_{i,n+1}-p_{i,n}\right)\end{aligned} \tag{14}$$

For r = 0 (i = 0) we have

$$\frac{\partial p}{\partial r} = \frac{1}{2h}\left(p_1-p_{-1}\right)=0 \tag{15}$$

and we find that $p_1 = p_{-1}$ and the second order derivative becomes

$$\frac{\partial^2 p}{\partial r^2} = \frac{1}{h^2}\left(p_{-1}-2p_0+p_1\right)=\frac{2}{h^2}\left(p_1-p_0\right) \tag{16}$$

For r = 1 (i = n) we obtain from the condition of boundary impermeability

$$\frac{\partial p}{\partial r} = \frac{1}{2h}\left(p_{n+1}-p_{n-1}\right)=0 \tag{17}$$

so that $p_{n+1} = p_{n-1}$, and the second order derivative becomes

$$\frac{\partial^2 p}{\partial r^2} = \frac{1}{h^2}\left(p_{n-1}-2p_n+p_{n+1}\right)=\frac{2}{h^2}\left(p_{n-1}-p_n\right) \tag{18}$$

The debit function has a nonzero value only in the origin, so that we have:

$$q(r,t)=\begin{cases}\dfrac{R^2\mu q}{kp_i}0 & \text{for} \quad r=0\\ 0 & \text{for} \quad r\neq 0\end{cases} \tag{19}$$

Using (14) – (19) the equations (12) – (13) become:

$$\begin{aligned}j &= 0: p_{0,n+1}=\frac{2\Delta t}{h^2}\left(p_1-p_0\right)+q\Delta t+p_{0,n}\\ j &> 0: p_{j,n+1}=\frac{\Delta t}{h^2}\left(p_{j-1}-2p_j+p_{j+1}\right)+\frac{\Delta t}{jh}\left(p_{j+1}-p_{j-1}\right)+p_{j,n}\\ j &= n: p_{n,n+1}=\frac{2\Delta t}{h^2}\left(p_{n-1}-p_n\right)+p_{n,n}\end{aligned} \tag{20}$$

Equations (20) form an explicit scheme of finite difference for our problem. For the study of convergence we have used a Fourier analysis. We write the pressure p like a Fourier series (see Morton and Mayers [9]):

$$p_{j,n} = \lambda^n e^{ik(jh)} \tag{21}$$

where λ is the amplification parameter and i $=\sqrt{-1}$. After some algebra using (20) and (21) we found:

$$\lambda = 1 - 4\frac{\Delta t}{h^2} \tag{22}$$

Because the condition of converge is that the amplification parameter must be between –1 and 1 we have the condition

$$\frac{\Delta t}{h^2} \leq \frac{1}{2} \tag{23}$$

4. **Results and Discussion**

The Eqs. (20) have been integrated using the time steep $\Delta t = 0.01$ and the spatial steep h = 0.2 and we can see from Eq. (23) that the convergence condition is satisfied. In the Table 1. we have presented the results for q = -1. Obviously the value of the pressure is decreasing in the entire domain, but the effect is more present in the center of the domain. At different moment of time the pressure shape is the same, but the values are lowers, as can see in Figure 1. This behaviour is similar to the one described by Cretu [5] for a rectangular domain:

TABLE 1. The values of the pressure at different moment of time

Δt	h = 0	h = 0.2	h = 0.4	h = 0.6	h = 0.8	h = 1
0	1	1	1	1	1	1
0.5	0.93955	0.95839	0.96902	0.97565	0.97930	0.98042
1	0.91090	0.92975	0.94042	0.94707	0.95074	0.95188
2	0.85372	0.87257	0.88324	0.88989	0.89355	0.89470
3	0.79654	0.81540	0.82606	0.83271	0.83628	0.83752
4	0.73936	0.75822	0.76888	0.77553	0.77920	0.78034
5	0.68218	0.70104	0.71170	0.71835	0.72202	0.72316
6	0.62500	0.64386	0.65452	0.66117	0.66484	0.66598
7	0.56782	0.58668	0.59734	0.60399	0.60766	0.60880
8	0.51064	0.52950	0.54016	0.54681	0.55048	0.55162
9	0.45346	0.47232	0.48298	0.48963	0.49330	0.49444
10	0.39628	0.41514	0.42580	0.43245	0.43612	0.43726

Fig.1. The variation of pressure p at different moment of time

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INVARIANT SETS OF RANDOM VARIABLES IN COMPLETE METRIC SPACES

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1. Introduction

The most known fractals are invariant sets with respect to a system of contraction maps, especially the so called self-similar sets. In a famous work, Hutchinson [6] first studied systematically the invariant sets in a general framework. He proved among others the following: *Let X be a complete metric space and* $f_1, \ldots, f_m : X \to X$ *be contraction maps. Then there exists a unique compact set* $K \subseteq X$ *such that* $K = \bigcup_{i=1}^{m} f_i(K)$.

If the maps f_i are similitudes, this invariant set K is said to be *self-similar.*

Our aim in this work is to generalize the above theorem of Hutchinson for random variables in complete metric spaces using some results from the theory of probabilistic metric spaces.

The theory of probabilistic metric spaces, introduced in 1942 by K. Menger [11], was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [14]. The study of contraction mappings for probabilistic metric spaces was initiated by V. M. Sehgal [16],[17], and H. Sherwood [19].

Falconner [4],Graf [5], and Hutchinson and Rüschendorf [6] used contraction methods to obtain random self-similar fractal sets by essential applying ordinary metrics to a.e. realization in the random setting. The same ideas were used by Arbeiter[1], Olsen [12], and Hutchinson and Rüschendorf [7],[8],[9], to obtain random self similar fractal measures. In these works a finite first moment condition of the distance function is essential. Using probabilistic metric space techniques, we can weak this first moment condition, as will be shown for fractal sets in Section 4.

2. **Preliminaries**

Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{R}_+ := \{x \in \mathbf{R} : x \geq 0\}$. A mapping $F : \mathbf{R} \to [0, 1]$ is called a *distribution function* if it is non-decreasing, left continuous with $\inf F = 0$.(see [2]) By Δ we shall denote the set of all distribution functions F. Let Δ be ordered by the relation "$\leq$": $F \leq G$ if and only if $F(t) \leq G(t)$ for all real t. Also $F < G$ if and only if $F \leq G$ but $F \neq G$. We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

Throughout this paper H will denote the Heviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Let X be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x, y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbf{R}$ by $F_{x,y}(t)$, respectively. The pair $(X, \mathcal{F})$ is a *probabilistic metric space* (briefly *PM space*) if X is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1^0. $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbf{R}$;

2^0. $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3^0. if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s + t) = 1$.

A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if the following conditions are satisfied:

4^0. $T(a, 1) = a$ for every $a \in [0, 1]$;

5^0. $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$

6^0. if $a \geq c$ and $b \geq d$ then $T(a, b) \geq T(c, d)$;

7^0. $T(a, T(b, c)) = T(T(a, b), c)$ for every $a, b, c \in [0, 1]$.

A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a probabilistic metric space, where T is a t-norm, and instead of 3^0 we have the stronger condition

8^0. $F_{x,y}(s + t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbf{R}_+$.

The (t,ϵ)-topology in a Menger space was introduced in 1960 by B. Schweizer and A. Sklar [13]. The base for the neighbourhoods of an element $x \in X$ is given by

$$\{U_x(t,\epsilon) \subseteq X : t > 0, \epsilon \in]0,1[\},$$

where

$$U_x(t,\epsilon) := \{y \in X : F_{x,y}(t) > 1-\epsilon\}.$$

If the t-norm T satisfies the condition

$$sup\{T(a,a) : a \in [0,1[\} = 1,$$

then the (t,ϵ) -topology is metrizable (see [15]).

In 1966, V.M. Sehgal [16] introduced the notion of a contraction mapping in PM spaces. The mapping $f : X \to X$ is said to be a *contraction* if there exists $r \in]0,1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbf{R}_+$.

A sequence $(x_n)_{n\in\mathbf{N}}$ from X is said to be *fundamental* if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1$$

for all $t > 0$. The element $x \in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbf{N}}$, and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent.

Set

$$\mathcal{D}^+ = \{F \in \Delta^+ : \sup_{t\in R} F(t) = 1\}.$$

In the following we always suppose that $(X, \mathcal{F}, T)$ is a Menger space with $\mathcal{F} : X \times X \to \mathcal{D}^+$ and T is continuous.

Let A be a nonempty subset of X. The function $D_A : \mathbf{R} \to [\mathbf{0},\mathbf{1}]$ defined by

$$D_A(t) := \sup_{s<t} \inf_{x,y\in A} F_{x,y}(s)$$

is called the *probabilistic diameter of A*. It is easy to check that $D_A \in \Delta^+$. The set $A \subseteq X$ is *probabilistic bounded* if $D_A \in \mathcal{D}^+$. If B and C are two subsets of X with $B \cap C \neq \emptyset$, then

$$D_{B\cup C}(s+t) \geq T(D_B(s), D_C(t)),\ s,t \in \mathbf{R} \tag{1}$$

(see [3, Theorem 10]). In particular, every finite subset of X is probabilistic bounded.

We also define the *probabilistic radius* $E_A : \mathbf{R} \to [0,1]$ of the set A:

$$E_A(t) := \sup_{s<t} \sup_{y\in A} \inf_{x\in A} F_{x,y}(s).$$

By definition it is easy to verify the following property:

Lemma 2.1.

$$E_A(t) \geq D_A(t),$$

and

$$D_A(2t) \geq T(E_A(t), E_A(t)),\ \textit{for all}\ t > 0.$$

Let A and B nonempty subsets of X. The *probabilistic Hausdorff-Pompeiu distance* between A and B is the function $F_{A,B} : \mathbf{R} \to [\mathbf{0}, \mathbf{1}]$ defined by

$$F_{A,B}(t) := \sup_{s<t} T(\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)).$$

Lemma 2.2. *For the nonempty subsets A and B of X we have*

$$E_A(t_1 + 2t_2) \geq T(D_B(t_1), F_{A,B}(t_2))\ \textit{for all}\ \ t_1, t_2 > 0.$$

Proof. Let $x, y \in A$, $z, u \in B$ and $s_1, s_2 > 0$. By 8^0 we have

$$F_{x,y}(s_1 + 2s_2) \geq T(F_{x,z}(s_1 + s_2), F_{z,y}(s_2)) \geq$$

$$\geq T(T(F_{x,u}(s_2), F_{u,z}(s_1)), F_{y,z}(s_2)) \geq T(T(F_{x,u}(s_2), D_B(s_1)), F_{y,z}(s_2)) =$$

$$= T(D_B(s_1)), T(F_{x,u}(s_2), F_{y,z}(s_2))).$$

Simple calculations show

$$\sup_{y\in A} \inf_{x\in A} F_{x,y}(s_1 + 2s_2) \geq T(D_B(s_1), T(\inf_{x\in A} \sup_{u\in B} F_{x,u}(s_2), \inf_{z\in B} \sup_{y\in A} F_{y,z}(s_2))).$$

If we take the supremum by $s_1 < t_1$ and $s_2 < t_2$ we obtain the required inequality. □

Proposition 2.1. *If $\mathcal{C}$ is a nonempty collection of nonempty closed bounded sets in a Menger space $(X, \mathcal{F}, T)$ with T continuous, then $(\mathcal{C}, F_C, T)$ is also Menger space, where $\mathcal{F}_C$ is defined by $\mathcal{F}_C(A, B) := F_{A,B}$ for all $A, B \in \mathcal{C}$.*

Proof. See [3],[10]. □

Proposition 2.2. *Let $T_m(a, b) := \max\{a + b - 1, 0\}$. If $(X, \mathcal{F}, T_m)$ is a complete Menger space and $\mathcal{C}$ is the collection of all nonempty closed bounded subsets of X in $(t, \epsilon)-$ topology, then $(\mathcal{C}, \mathcal{F}_C, T_m)$ is also a complete Menger space.*

Proof. Let $(A_n)_{n\in\mathbf{N}}$ be a fundamental sequence in $\mathcal{C}$ and let

$$A = \{x \in X : \forall n \in \mathbf{N}, \exists x_n \in A_n, \forall t > 0, \lim_{n\to\infty} F_{x_n,x}(t) = 1\}. \tag{2}$$

Let $\overline{A}$ denote the closure of A. By [3, Theorem 15] we have $F_{A_n,A} = F_{A_n,\overline{A}}$, so it is enough to show that (i) $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $t > 0$, and (ii) $\overline{A} \in \mathcal{C}$.

(i) Let $t > 0$ and $\epsilon > 0$ be given. Then there exists $n_\epsilon(t) \in \mathbf{N}$ such that $n, m > n_\epsilon$ implies $F_{A_n,A_m}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$. Let $n > n_\epsilon(t)$. We claim that $F_{A_n,A}(t) \geq 1 - \epsilon$.

If $x \in A$, then there is a sequence $(x_k)_{k\in\mathbf{N}}$ with $x_k \in A_k$ and $\lim_{k\to\infty} F_{x_k,x}(\frac{t}{4}) = 1$. So, for large enough $k > n_\epsilon(t)$, we have $F_{x_k,x}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$. Since $F_{A_n,A_k}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$, there exist $y \in A_n$ such that $F_{x_k,y}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$. By 8^0 we have $F_{x,y}(\frac{t}{2}) > 1 - \frac{\epsilon}{4}$, hence

$$\sup_{s<t} \inf_{x\in A} \sup_{y\in A_n} F_{x,y}(s) > 1 - \frac{\epsilon}{2}. \tag{3}$$

Now suppose that $y \in A_n$ is arbitrary. Choose integers $k_1 < k_2 < ... < k_i < ...$ so that $k_1 = n$ and

$$F_{A_k,A_{k_i}}(\frac{t}{2^{i+2}}) > 1 - \frac{\epsilon}{2^{i+2}},$$

for all $k > k_i$. We have $\inf_{z\in A_{k_i}} \sup_{x\in A_k} F_{x,z}(\frac{t}{2^{i+1}}) > 1 - \frac{\epsilon}{2^{i+2}}$. Then define a sequence (y_k) with $y_k \in A_k$ as follows. For $k < n$, let $y_k \in A_k$be arbitrarily and $y_n = y$. If y_{k_i} has been chosen and $k_i < k \leq k_{i+1}$, take $y_k \in A_k$ with $F_{y_{k_i},y_k}(\frac{t}{2^{i+2}}) > 1 - \frac{\epsilon}{2^{i+2}}$. Then, for $k_i < k \leq k_{i+1} < ... < k_j < l \leq k_{j+1}$, we have

$$F_{y_l,y_k}(\frac{t}{2^i}) \geq F_{y_k,y_{k_i}}(\frac{t}{2^{i+1}}) + F_{y_{k_i},y_{k_{i+1}}}(\frac{t}{2^{i+2}}) + ... + F_{y_{k_{j-1}},y_{k_j}}(\frac{t}{2^{j+1}}) +$$

$$+ F_{y_{k_j},y_l}(\frac{t}{2^{j+1}}) - (j - i + 1) > 1 - \frac{\epsilon}{2^{i+1}}.$$

Let $0 < r$, $0 < \eta < 1$, and choose i so that $\frac{t}{2^i} < r$ and $\frac{\epsilon}{2^{i+1}} < \eta$. We have

$$F_{y_k,y_l}(r) \geq F_{y_k,y_l}(\frac{t}{2^i}) > 1 - \frac{\epsilon}{2^{i+1}} > 1 - \eta.$$

Hence (y_k) is a fundamental sequence, so it converges. Let x be its limit. Therefore $x \in A$, and we have

$$F_{x,y}(\frac{t}{2}) \geq F_{x,y_k}(\frac{t}{4}) + F_{y_k,y}(\frac{t}{4}) - 1.$$

Select $k > n$ such that $F_{x,y_k}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$. Since $F_{y,y_k}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$, it follows that $F_{x,y}(\frac{t}{2}) > 1 - \frac{\epsilon}{2}$. Therefore we have

$$\sup_{s<t} \inf_{y \in A_n} \sup_{x \in A} F_{x,y}(s) > 1 - \frac{\epsilon}{2}. \tag{4}$$

By (3), the lather implies

$$F_{A_n,A}(t) = \sup_{s<t} T_m(\inf_{x \in A} \sup_{y \in A_n} F_{x,y}(s), \inf_{y \in A_n} \sup_{x \in A} F_{x,y}(s)) > 1 - \epsilon.$$

Thus $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $t > 0$,hence part (i) is proved..

(ii) Taking $\epsilon = 1$ in the last argument, we have proved that A is nonempty.

Next we have to show that A is bounded. Since $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $\epsilon > 0$ and $t_0 > 0$ there exists $n_0 \in N$ such that, for every $n > n_0$, we have $\inf_{x\in A} \sup_{w\in A_n} F_{x,w}(t_0) > 1-\epsilon$ and $\inf_{y\in A_n} \sup_{x\in A} F_{x,y}(t_0) > 1-\epsilon$. The set A_n being probabilistic bounded, for all $\epsilon > 0$ there is $t_\epsilon > t_0$ such that $\inf_{u,v\in A_n} F_{u,v}(t_\epsilon) > 1-\epsilon$.

On the other hand, $x, y \in A$ there exist $u, v \in A_n$ such that

$$F_{x,u}(t_0) > 1 - \epsilon,\ F_{y,v}(t_0) > 1 - \epsilon.$$

We have

$$F_{x,y}(3t_\epsilon) \geq T_m(F_{x,u}(t_\epsilon), F_{u,y}(2t_\epsilon)) \geq T_m(F_{x,u}(t_0), T_m(F_{u,v}(t_\epsilon), F_{v,y}(t_0))) > 1 - 3\epsilon.$$

Therefore $D_A(3t_\epsilon) \geq 1-3\epsilon$, consequently we have $\sup_{t\in\mathbf{R}} D_A(t) = 1$. By [3], it follows that $D_A = D_{\overline{A}}$, hence $\overline{A} \in \mathcal{C}$. □

3. Invariant sets in E-spaces

The notion of E-space was introduced by Sherwood [20] in 1969. Next we recall this definition. Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (M, d) be a metric space. The ordered pair $(\mathcal{E}, F)$ is an *E-space over the metric space* (M, d) (briefly, an E-space) if the elements of $\mathcal{E}$ are random variables from Ω into M and $\mathcal{F}$ is the mapping from $\mathcal{E} \times \mathcal{E}$ into Δ^+ defined via $\mathcal{F}(x,y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega |\ d(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbf{R}$. Usually $(\Omega, \mathcal{K}, P)$ is called the base and (M, d) the target space of the E-space. If $\mathcal{F}$ satisfies the condition

$$\mathcal{F}(x,y) \neq H, \quad for \;\; x \neq y,$$

with H defined in section 2, then $(\mathcal{E}, \mathcal{F})$ is said to be a *canonical E-space.* H. Sherwood [20] proved that every canonical $\mathcal{E}$-space is a Menger space under $T = T_m$, where $T_m(a,b) = \max\{a + b - 1, 0\}$. In the following we suppose that E is a canonical E-space.

The convergence in an $\mathcal{E}$-space is exactly the probability convergence. The E-space $(\mathcal{E}, \mathcal{F})$ is said to be complete if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete.

Proposition 3.1. *If (M,d) is a complete metric space then the E-space* $(\mathcal{E}, F)$ *is also complete.*

Proof. This property is well-known if $M = R$ (see e.g. [21, Theorem VII.4.2.]). In the general case the proof is analogous and we omit it. □

Proposition 3.2. *If A is a nonempty probabilistic bounded subset of* $\mathcal{E}$ *and* $f : \mathcal{E} \to \mathcal{E}$ *is a contraction with ratio r then f(A) is also probabilistic bounded, where*

$$f(A) = \{f(x) \,|\, x \in A\}.$$

Proof. We have

$$D_{f(A)}(t) = \sup_{s<t} \inf_{u,v \in f(A)} F_{u,v}(s) =$$

$$= \sup_{s<t} \inf_{x,y \in A} P(\{\omega \in \Omega |\ d(f(x)(\omega), f(y)(\omega)) < s\}) \geq$$

$$\geq \sup_{s<t} \inf_{x,y \in A} P(\{\omega \in \Omega |\ d(x(\omega), y(\omega)) < \frac{s}{r}\}) \geq$$

$$\geq \sup_{s<t} \inf_{x,y \in A} F_{x,y}(s) = D_A(t).$$

Since $\sup_{t>0} D_A(t) = 1$, it follows that $\sup_{t>0} D_{f(A)}(t) = 1$. □

The main result of this paper is the following:

Theorem 3.1. *Let $(\mathcal{E}, F)$ be a complete E- space, $N \in \mathbf{N}^*$,, and let $f_1, ..., f_N : \mathcal{E} \to \mathcal{E}$ be contractions with ratio $r_1, ...r_N$, respectively. Suppose that there exists an element $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega | d(z(\omega), f_i(z(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \tag{5}$$

for all $i \in \{1, .., N\}$ and for all $t > 0$. Then there exists a unique nonempty closed bounded subset K of $\mathcal{E}$ such that

$$f_1(K) \cup ... \cup f_N(K) = K.$$

Proof. Let $\Phi : 2^{\mathcal{E}} \to 2^{\mathcal{E}}$ be defined by

$$\Phi(A) := f_1(A) \cup f_2(A) \cup ... \cup f_N(A).$$

Let $A_0 = \{z\}$ and $A_n = \Phi(A_{n-1})$ for $n \geq 1$. Let $r = \max\{r_1, ..., r_N\}$, J be the finite alphabet $\{1, .., N\}$, and, for $\sigma = \sigma_1 ... \sigma_n \in J^n$, set $f_\sigma = f_{\sigma_1} \circ f_{\sigma_2} \circ ... \circ f_{\sigma_n}$. We have:

$$A_n = \cup_{\sigma \in J^n} f_\sigma(A_0).$$

First we show that $(A_n)_{n \in N}$ is a fundamental sequence in $(\mathcal{C}, F_C, T_m)$.

Since $A_{n+k} = \Phi^n(A_k)$ and $A_n = \Phi^n(A_0)$, we have

$$\inf_{u \in A_n} \sup_{v \in A_{k+n}} F_{u,v}(s) = \inf_{u \in \cup_{\sigma \in J^n} f_\sigma(A_0)} \sup_{v \in \cup_{\sigma \in J^n} f_\sigma(A_k)} F_{u,v}(s).$$

Observe, there exists $\sigma' \in J^n$ such that

$$\inf_{u \in A_n} \sup_{v \in A_{k+n}} F_{u,v}(s) = \inf_{u \in f_{\sigma'}(A_0)} \sup_{v \in \cup_{\sigma \in J^n} f_\sigma(A_k)} F_{u,v}(s) \geq$$

$$\geq \inf_{u \in f_{\sigma'}(A_0)} \sup_{v \in f_{\sigma'} A_k} F_{u,v}(s) = \inf_{x \in A_0} \sup_{y \in A_k} F_{f_{\sigma'}(x), f_{\sigma'}(y)}(s) \geq$$

$$\geq \sup_{y \in A_k} P(\{\omega \in \Omega | \ r^n d(z(\omega), y(\omega)) < s\}) =$$

$$= \max_{y \in \cup_{\tau \in J^k} f_\tau(A_0)} P(\{\omega \in \Omega | \ r^n d(z(\omega), y(\omega)) < s\}) \geq$$

$$\geq \max_{y \in \cup_{\tau \in J^k} f_\tau(A_0)} P(\{\omega \in \Omega | \ r^n d(z(\omega), y(\omega)) < s \cdot (1 + \sqrt{r} + \cdots + \sqrt{r}^{k-1})(1 - \sqrt{r})\}) \geq$$

$$\geq \max_{\tau \in J^k} P(\{\omega \in \Omega | \ r^n [d(z(\omega), f_{\tau_1}(z(\omega))) + d(f_{\tau_1}(z(\omega)), f_{\tau_1 \tau_2}(z(\omega))) + \cdots$$

$$\cdots + d(f_{\tau_1 \cdots \tau_{k-1}}(z(\omega)), f_{\tau_1 \cdots \tau_k}(z(\omega)))] < s \cdot (1 + \sqrt{r} + \cdots + \sqrt{r}^{k-1})(1 - \sqrt{r})\}) \geq$$

$$\geq \max_{\tau\in J^k}[P(\{\omega\in\Omega|\ d(z(\omega), f_{\tau_1}(z(\omega))) < \frac{s(1-\sqrt{r})}{r^n}\})+$$

$$+P(\{\omega\in\Omega|\ d(f_{\tau_1}(z(\omega)), f_{\tau_1\tau_2}(z(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}}{r^n}\}) + \cdots$$

$$\cdots P(\{\omega\in\Omega|\ d(f_{\tau_1\ldots\tau_{k-1}}(z(\omega)), f_{\tau_1\ldots\tau_k}(z(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^n}\})] - (k-1) \geq$$

$$\max_{\tau\in J^k}[P(\{\omega\in\Omega|\ d(z(\omega), f_{\tau_1}(z(\omega))) < \frac{s(1-\sqrt{r})}{r^n}\})+$$

$$+P(\{\omega\in\Omega|\ rd(z(\omega)), f_{\tau_2}(z(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}}{r^n}\}) + \cdots +$$

$$+P(\{\omega\in\Omega|\ r^{k-1}d(z(\omega), f_{\tau_m}(z(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^n}\})] - (k-1) =$$

$$= 1 - \min_{\tau\in J^m}[P(\{\omega\in\Omega|\ d(z(\omega), f_{\tau_1}(z(\omega))) \geq \frac{s(1-\sqrt{r})}{r^n}\})+$$

$$+P(\{\omega\in\Omega|\ d(z(\omega), f_{\tau_2}(z(\omega))) \geq \frac{s(1-\sqrt{r})\sqrt{r}}{r^{n+1}}\}) + \cdots +$$

$$+P(\{\omega\in\Omega|\ d(z(\omega), f_{\tau_k}(z(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^{n+k-1}}\})] \geq$$

$$\geq 1 - \gamma\cdot r^n\left(\frac{1}{s(1-\sqrt{r})} + \frac{r^{1/2}}{s(1-\sqrt{r})} + \ldots + \frac{r^{(k-1)/2}}{s(1-\sqrt{r})}\right) >$$

$$> 1 - \gamma\frac{r^n}{s(1-\sqrt{r})^2}.$$

Since

$$\lim_{n\to\infty}\left(1 - \gamma\frac{r^n}{s(1-\sqrt{r})^2}\right) = 1,$$

we have, for $t > 0$,

$$\lim_{n\to\infty} F_{A_n, A_{k+n}}(t) = 1,$$

uniformly with respect to k. The space $(\mathcal{E}, F)$ being complete, (A_n) is convergent. Let K be its limit.

Next we show that K is a fixed point of Φ. For $i\in\{1,...,N\}$, $x\in A_{n-1}, y\in K$ and $s > 0$, we have

$$F_{f_i(x), f_i(y)}(s) \geq F_{x,y}(s).$$

There exists $i\in J$ such that

$$\inf_{u\in\Phi(A_{n-1})}\ \sup_{v\in\Phi(K)} F_{u,v}(s) = \inf_{u\in f_i(A_{n-1})}\ \sup_{v\in\Phi(K)} F_{u,v}(s) \geq$$

$$\geq \inf_{x\in A_{n-1}}\ \sup_{y\in K} F_{f_i(x), f_i(y)}(s) \geq \inf_{x\in A_{n-1}}\ \sup_{y\in K} F_{x,y}(s).$$

In a similar way

$$\inf_{v\in\Phi(K)}\sup_{u\in\Phi(A_{n-1})}F_{u,v}(s)\geq\inf_{y\in K}\sup_{x\in A_{n-1}}F_{x,y}(s).$$

Then it follows

$$F_{A_n,\Phi(K)}(\frac{t}{2})\geq F_{A_{n-1},K}(\frac{t}{2})\ \text{ for all } t>0.$$

Using 8^0 one obtains

$$F_{K,\Phi(K)}(t)\geq T_m(F_{K,A_n}(\frac{t}{2}),F_{A_n,\Phi(K)}(\frac{t}{2}))\geq T_m(F_{K,A_n}(\frac{t}{2}),F_{A_{n-1},K}(\frac{t}{2})).$$

Since $\lim_{n\to\infty}A_n=K$, we have

$$F_{K,\Phi(K)}(t)=1\ \ for\ all\ \ t>0,$$

therefore

$$K=\Phi(K).$$

For the uniqueness we suppose that there exist closed and bounded K and K' such that $\Phi(K)=K$ and $\Phi(K')=K'$. For $x\in K$, $y\in K'$, $\sigma\in J^n$, and $s>0$, we have

$$F_{f_\sigma(x),f_\sigma(y)}(s)\geq F_{x,y}(\frac{s}{r^n}).$$

Let $\sigma'\in J^n$ be such that

$$\inf_{v\in\cup_{\sigma\in J^n}f_\sigma(K')}\ \sup_{u\in\cup_{\sigma\in J^n}f_\sigma(K)}F_{v,u}(s)=\inf_{x\in f'_\sigma(K')}\ \sup_{u\in\cup_{\sigma\in J^n}f_\sigma(K)}F_{v,u}(s)\geq$$

$$\geq\inf_{v\in f_{\sigma'}(K')}\ \sup_{u\in f_{\sigma'}(K)}F_{v,u}(s)\geq\inf_{y\in K'}\sup_{x\in K}F_{x,y}(\frac{s}{r^n}).$$

Similarly,

$$\inf_{v\in\cup_{\sigma\in J^n}f_\sigma(K')}\ \sup_{u\in\cup_{\sigma\in J^n}f_\sigma(K)}F_{v,u}(s)\geq\inf_{x\in K}\sup_{y\in K'}F_{x,y}(\frac{s}{r^n}).$$

Since $K=\Phi^n(K)=\cup_{\sigma\in J^n}f_\sigma(K),\ K'=\Phi^n(K')=\cup_{\sigma\in J^n}f_\sigma(K')$, we have

$$F_{K,K'}(t)\geq F_{K,K'}(\frac{t}{r^n})\ \text{ for all } t>0.$$

Using $\lim_{n\to\infty}r^n=0$, we have

$$F_{KK'}(t)=1\ \text{ for all } t>0,$$

therefore $K=K'$. □

Corollary 3.1. *Let $(\mathcal{E}, F)$ be a complete E- space, and let $f : \mathcal{E} \to \mathcal{E}$ be a contraction with ratio r. Suppose there exists $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega |\ d(z(\omega), f(z)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \text{ for all } t > 0.$$

Then there exists a unique $x_0 \in \mathcal{E}$ such that $f(x_0) = x_0$.

Remark: If the mean values $\int_\Omega d(z(\omega), f_i(x(\omega)))dP$ for $i \in \{1, ..., N\}$ are finite, then by the Chebisev inequality, condition (5) is satisfied. However, the condition (5) can also be satisfied for $\int_\Omega d(z(\omega), f(z(\omega)))dP = \infty$. For example, let $\Omega =]0, 1]$ with the Lebesque measure and let $f(x) = \frac{x(\omega)}{3} + \frac{1}{\omega}$. Then for $z(\omega) \equiv 0$, the above expectation is ∞, but, for $\gamma = 1$, the condition (5) holds.

As in [6], the invariant set can be modeled by strings. Let $N \geq 1$, and define

$$\{1, ..., N\}^* = \cup_{k \in \mathbf{N}} \{1, ..., N\}^k.$$

If $\tau \in \{1, ..., N\}^*$, $\tau = \tau_1.\tau_2...\tau_k$, then $|\ \tau| = k$ is the length of τ. Set $f_\tau : \mathcal{E} \to \mathcal{E}$, $f_\tau := f_{\tau_1} \circ f_{\tau_2} \circ ... \circ f_{\tau_k}$. If $A \subset \mathcal{E}$, we set $A_{\tau_1...\tau_k} := f_\tau(A)$.

Let $\{1, ..., N\}^{\mathbf{N}}$ carry the product of the discrete topology on $\{1, ..., N\}$. For $\sigma \in \{1, ..., N\}^* \cup \{1, ..., N\}^{\mathbf{N}}$ with k$\leq |\ \sigma|$ let $\sigma_{|k} = \sigma_1.\sigma_2...\sigma_k$ be the restriction of σ to its first k entries.

Let K be the invariant set from Theorem 3. As in [6], we can show that

a) $K_{\sigma_1...\sigma_k} = \cup_{\sigma_{k+1}=1}^n K_{\sigma_1...\sigma_k \sigma_{k+1}}$

b) $K \supset K_{\sigma_1} \supset ... \supset K_{\sigma_1...\sigma_k} \supset ....$

Proposition 3.3. *Let the hypotheses of Theorem 3 be satisfied. Then, for all $t > 0$, we have*

$$lim_{k \to \infty} D_{f_{\sigma_{|k}}(K)}(t) = 1.$$

Proof. Let A_n be the set defined in the proof of Theorem 3. If f is an r-contraction, then $F_{f(A_n), f(K)}(t) \geq F_{A_n, K}(t)$ for $t > 0$. Let $\sigma \in \{1, ..., N\}^* \cup \{1, ..., N\}^{\mathbf{N}}$. Since $\lim_{n \to \infty} F_{A_n, K}(t) = 1$ for $t > 0$, it follows that

$$\lim_{n \to \infty} F_{f_{\sigma|k}(A_n), f_{\sigma|k}(K)}(t) = 1 \tag{6}$$

uniformly with respect to k.

We have

$$D_{f_{\sigma|k}(A_n)}(t) = \sup_{s<t} \inf_{x,y\in f_{\sigma|k}(A_n)} P(\{\omega \in \Omega|\ d(x(\omega), y(\omega)) < s\}) =$$

$$= \sup_{s<t} \inf_{u,v\in A_n} P(\{\omega \in \Omega|\ d(f_{\sigma_1...\sigma_k}(u)(\omega), f_{\sigma_1...\sigma_k}(v)(\omega)) < s\}) \geq$$

$$\geq \sup_{s<t} \inf_{u,v\in A_n} P(\{\omega \in \Omega|\ r_{\sigma_1}...r_{\sigma_k} d(u(\omega), v(\omega)) < s\}) \geq$$

$$\geq \sup_{s<t} \inf_{u,v\in A_n} P(\{\omega \in \Omega|\ r^k d(u(\omega), v(\omega)) < s\}) \geq$$

$$\geq \sup_{s<t} \inf_{u,v\in A_n} [P(\{\omega \in \Omega|\ d(u(\omega), v(\omega)) < \frac{s}{2r*k}\})+$$

$$+P(\{\omega \in \Omega|\ d(z(\omega), v(\omega)) < \frac{s}{2r^k}\})] - 1 \geq$$

$$\geq 1 - \frac{\gamma}{(1-\sqrt{r})^2} \cdot r^n.$$

Hence

$$\lim_{k\to\infty} D_{f_{\sigma\rceil k}(A_n)}(t) = 1 \text{ for all } t > 0 \text{ and } n \in \mathbf{N}.$$

By Lemma 2.2 we have

$$D_{f_{\sigma|k}(K)}(t) \geq D_{f_{\sigma|k}(A_n)}(t) + F_{f_{\sigma|k}(A_n), f_{\sigma|k}(K)}(t) - 1.$$

Using (6) it follows the assertion. $\square$

Proposition 3.4. *For all* $\sigma \in \{1, ..., N\}^{\mathbf{N}}$ *there exists a unique element* $x_\sigma \in \cap_{n\in N} \overline{K}_{\sigma_1...\sigma_n}$

Proof. For every $n \in \mathbf{N}$ we choose an element $x_n \in K_{\sigma_1...\sigma_n}$. Let $m < n$, then $x_m, x_n \in K_{\sigma_1...\sigma_m}$. Since

$$lim_{k\to\infty} D_{f_{\sigma|k}(K)}(t) = 1 \text{ for } t > 0, \text{ and } \epsilon > 0,$$

there exists $m_0 \in N$ such that, for all $m > m_0$,

$$\inf_{x,y\in K_{\sigma_1...\sigma_m}} P\{\omega \in \Omega | d(x(\omega), y(\omega)) < t\} > 1 - \epsilon.$$

It follows, for $m, n > m_0$, $P(\{\omega \in \Omega | d(x_n(\omega), x_m(\omega)) < t\}) > 1 - \epsilon$, therefore $(x_n)_{n\in N}$ is a Cauchy sequence. Since the space $(\mathcal{E}, \mathcal{F})$ is complete, it follows the convergence of $(x_n)_{n\in N}$. Let x_σ be its limit. Then $x_\sigma \in \cap_{n\in N} \overline{K}_{\sigma_1...\sigma_n}$.

Since $\lim_{n\to\infty} D_{\overline{K}_{\sigma_1...\sigma_n}}(t) = 1$ for all $t > 0$, it follows that x_σ is the unique element of the intersection. $\square$

Proposition 3.5. *The map $\pi : \{1, ..., N\}^{\mathbf{N}} \to K$ given by $\pi(\sigma) = x_\sigma$ is a continuous map onto K.*

Proof. Let $\sigma = \sigma_1...\sigma_n... \in \{1, ..., N\}^{\mathbf{N}}$ and let $\epsilon > 0$. Since $\pi(\sigma) = x_\sigma \in \cap_{n\in N}\overline{K}_{\sigma_1...\sigma_n}$ and $\lim_{n\to\infty} D_{\overline{K}_{\sigma_1...\sigma_n}}(t) = 1$ for all $t > 0$, there exists $n_0 \in N$ such that

$$D_{\overline{K}_{\sigma_1...\sigma_n}}(t) > 1 - \epsilon \text{ for all } n > n_0.$$

For $y \in \overline{K}_{\sigma_1...\sigma_n}$ we have

$$P(\{\omega \in \Omega |\ d(y, \pi(\sigma)) < t\}) > 1 - \epsilon,$$

hence $\overline{K}_{\sigma_1...\sigma_n} \subset U_{\pi(\sigma)}(t, \epsilon)$ for $n > n_0$. Since $\overline{K_{\sigma_1...\sigma_n}}$ contains the image of the open set $\{\beta|\beta_i = \sigma_i, if\ i \leq n\}$, it follows π is continuous.

Let $K' = \pi(\{1, ..., N\}^{\mathbf{N}})$. Observe $K' \subset K$ and K' is a compact set. We show that K' is an invariant set. If $y \in K'$, then there exists $\sigma \in \{1, ..., N\}^{\mathbf{N}}$ such that $y = \pi(\sigma) \in f_{\sigma_1}(K')$. So $K' \subset \cup_{i=1}^{l} f_i(K')$.

If $y \in \cup_{i=1}^{l} f_i(K')$ then there exists $j \in \{1, ..., l\}$ such that $y \in f_j(K')$, hence, for any $\sigma' \in \{1, ..., N\}^{\mathbf{N}}$, $y = f_j(\pi(\sigma')) = \pi(j\sigma') \in K'$.

Since the closed bounded invariant set is unique, it follows $K = K'$. □

Corollary 3.2. *The invariant set in Theorem 3 is compact.*

4. Self similar fractal sets

Recently Hutchinson and Rüschendorf [9] gave a simple proof for the existence and uniqueness of invariant random sets using the L^∞-metric. The underlying probability space for the iteration procedure is generated by selecting independent and identically distributed scaling laws. A *scaling law* $\mathbf{S}$ is an N-tuple $(S_1, ..., S_N)$, $N \geq 2$, of Lipschitz maps $S_i : \mathbf{R}^n \to \mathbf{R}^n$. Let $r_i = LipS_i$. *A random scaling law* $\mathbf{S} = (S_1, S_2, ..., S_N)$ is a random variable whose values are scaling laws. We write $\mathcal{S} = dist\mathbf{S}$ for the probability distribution determined by $\mathbf{S}$ and $\stackrel{d}{=}$ for the equality in distribution.

If K is a random set, then the random set $\mathbf{S}K$ is defined (up to probability distribution) by

$$\mathbf{S}K = \cup_i S_i K^{(i)},$$

where $\mathbf{S}, K^{(1)}, ..., K^{(N)}$ are independent of one another and $K^{(i)} \stackrel{d}{=} K$.

We say K *satisfies the scaling law* $\mathbf{S}$, or is a *self-similar random fractal set,* if

$$\mathbf{S}K \stackrel{d}{=} K, \text{ or equivalently } \mathcal{S}K = K.$$

Let $\mathbf{C}$ be the set of random compact sets K such that

$$ess \sup_{\omega} d_{\mathcal{H}}(K^{\omega}, \delta_B^{\omega}) < \infty, \tag{7}$$

for some, and hence any, fixed compact set $B \subset R^n$, where $d_{\mathcal{H}}$ is the hausdorff metric. By δ_B we mean the random set equal a.s. to B. In [9], Hutchinson and Rüschendorf generate random sets in the following manner:

Beginning with a nonrandom set K_0 one defines a sequence of random sets

$$\mathbf{S}K_0 = \cup_i S_i K_0,$$

$$\mathbf{S}^2 K_0 = \cup_{i,j} S_i \circ S_j^i K_0,$$

$$\mathbf{S}^3 K_0 = \cup_{i,j,k} S_i \circ S_j^i \circ S_k^{ij} K_0,$$

etc.: where $\mathbf{S}^{\mathbf{i}} = (S_1^i, S_2^i, ..., S_N^i)$, for $i \in \{1, ..., N\}$, are independent of each other and of $\mathbf{S}$, the $\mathbf{S}^{ij} = (S_1^{ij}, S_2^{ij}, ..., S_N^{ij})$, for $i, j \in \{1, ..., N\}$ are independent of each other and of $\mathbf{S}$ and $\mathbf{S}^{\mathbf{i}}$, etc.

A *construction tree* (or a construction process) is a map $\omega : \{1, ..., N\}^* \to \Gamma$, where Γ is the set of (nonrandom) scaling laws. The sample space of all construction trees is denoted by $\tilde{\Omega}$. The underlying probability space $(\tilde{\Omega}, \tilde{\mathcal{K}}, \tilde{P})$ for the iteration procedure is generated by selecting identical distributed and independent scaling laws $\omega(\sigma) \stackrel{d}{=} \mathbf{S}$ for each $\sigma \in \{1, ..., N\}^*$ (see [9]). It is well known the following result:

Theorem 4.1. ([4],[5],[9]) *If* $\mathbf{S} = (S_1, S_2, ..., S_N)$ *is a random scaling law with*

$$\lambda := ess \sup_{\omega} r^{\omega} < 1 \tag{8}$$

(where $r^{\omega} = \max_i Lip S_i^{\omega}$*), then for any (nonrandom) compact set* K_0*,*

$$ess \sup_{\omega} d_{\mathcal{H}}(\mathbf{S}K_0, K^*) \leq \frac{\lambda^k}{1-\lambda} ess \sup_{\omega} d_{\mathcal{H}}(K_0, \mathbf{S}K_0) \to 0$$

as $k \to \infty$*, where* K^* *does not depend on* K_0*. In particular,* $\mathbf{S}^k K_0 \to K^*$ *a.s.*

Moreover, up to probability distribution, K^* *is the unique random compact set which satisfies* $\mathbf{S}$.

However, using contraction method in probabilistic metric spaces,instead of (6) we can give weaker conditions for the existence and uniqueness of invariant sets.

Theorem 4.2. *Let $\mathcal{E}$ be the set of nonempty random compact sets $A \subset R^n$, and let* $\mathbf{S}$ *be a random scaling law with $\lambda = ess \sup_\omega \ r^\omega < 1$. Suppose there exists $Z \in \mathcal{E}$ and a positive number γ such that*

$$P(\{\omega \in \Omega | \ d_{\mathcal{H}}(Z(\omega), \mathbf{S}(Z(\omega))) \geq t\}) \leq \frac{\gamma}{t} \ \textit{for all}\, t > 0. \tag{9}$$

Then there exists $K^* \in \mathcal{E}$ such that $\mathbf{S}(K^*) = K^*$.

Moreover, K^* is unique up to probability distribution.

Proof. Define $f : \mathcal{E} \to \mathcal{E}, \ f(A) = \mathbf{S}A$. For $A, B \in \mathcal{E}, \ A^i \overset{d}{=} A, B^i \overset{d}{=} B, i \in \{1, ..., N\}$, one checks that

$$F_{f(A),f(B)}(t) = P(\{\omega \in \Omega | \ d_{\mathcal{H}}(f(A), f(B)) < t\}) =$$

$$= P(\{\omega \in \Omega | \ d_{\mathcal{H}}(\cup_{i=1}^N S_i(\omega)(A^i(\omega)), \cup_{i=1}^N S_i(\omega)(B^i(\omega))) < t\}) \geq$$

$$\geq P(\{\omega \in \Omega | \ \lambda \cdot \max_i \cdot d_{\mathcal{H}}(A^i(\omega)), B^i(\omega)) < t\}) =$$

$$= P(\{\omega \in \Omega | \ \lambda \cdot d_{\mathcal{H}}(A(\omega), B(\omega)) < t\}) = F_{A,B}(\frac{t}{\lambda}) \ \text{ for all } t > 0.$$

It follows that f is a contraction with ratio λ and we can apply the Corollary 3.1 for $r = \lambda$. For the uniqueness, let $\mathcal{C}$ the set of probability distribution of members of $\mathbf{C}$, i.e.

$$\mathcal{C} = \{distA | A \in \mathbf{C}\}.$$

We define on $\mathcal{C}$ the probability metric by

$$F_{\mathcal{A},B}(t) = \sup_{s<t} \sup\{F_{A,B}(s) | A \overset{d}{=} \mathcal{A}, \ B \overset{d}{=} \mathcal{B}\}.$$

It is easy to verify that $\mathcal{S}$ is a contraction map:

$$F_{\mathcal{S}A,\mathcal{S}B}(t) \geq F_{\mathcal{A},B}(\frac{t}{\lambda}) \text{ for all } t > 0.$$

Let $\mathcal{K}^*$ and $\mathcal{K}^{**}$ such that

$$\mathcal{S}K^* = K^* \ and \ \mathcal{S}K^{**} = K^{**}.$$

As in the proof of the Theorem 3, one can show that

$$F_{\mathcal{K}^*,K^{**}}(t) = 1 \text{ for all } t > 0. \qquad \square$$

Remark. If condition (6) is satisfied, then (9) holds.

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M-LINEAR CONNECTION ON THE SECOND ORDER REONOM BUNDLE

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Abstract. The $T^2M \times R$ bundle represents the total space of a time dependent geometry of second order. In this bundle it is studied a special class of derivation rules, named M-linear connections.. There are given their characterization and it is proved their existence. Finally there are studied geometrical properties of one M-linear connection.

1. **Introduction**

The study of the time dependent Lagrange geometry (geometry of the reonom Lagrange spaces) was imposed from considerations of mechanic ,a systematically study of this is finding in the M.Anastasiei and H.Kawaguchi paper [1],[2],[3].

On the other hand, research from the last years imposed into attention the considerations in the superior order geometries where the total space is the prolongation of k order of the TM tangent bundle of a differential manifold or an associated bundle named the osculator bundle of k order ([5],[8],[13]). From calculation reasons we will approach here the case $k = 2$.

The study of the second order reonom bundle $E = T^2M \times R$ was done by us in a previous work([6],[7]).

Let M be a differentiable manifold, $dimM = n$, $x = (x^i)$ the local coordinates in a map (U, φ). We are considering T^2M the 2-jets bundle to the tangent curves in $x \in M$. Locally on T^2M the coordinates are $u = (x^i, y^i, z^i)$ with the following rule of change on the intersection of two local maps:

$$\widetilde{x}^i = \widetilde{x}^i(x^j) \tag{1.1}$$

$$\widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^i$$

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$$\widetilde{z}^i = \frac{1}{2}\frac{\partial \widetilde{y}^i}{\partial x^k} y^k + \frac{\partial \widetilde{x}^i}{\partial x^k} z^k$$

T^2M has a structure of fibre bundle over R^{2n} space ,which is not vectorial one.

The reonom bundle of second order is the bundle of direct product $E = T^2M \times R$, in which variable on R is denoted by t and it is considered in applications as being the time. In respect to the (1.1) changes on E we will haw also and $\widetilde{t} = t$.

Taking as a base the E manifold, we will develop a geometrical techniques of derivation the sections on TE. The tangent space T_uE present approaching difficulties due to the fact that the natural bases $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z^i}, \frac{\partial}{\partial t}\}$ it is changing with the two order derivatives of $\frac{\partial \widetilde{x}^i}{\partial x^j}$.

In order to eliminate this inconvenient we will consider an adapted base of a nonlinear connection on E.

Let $\Pi_2 : E \to M$ the canonical projection and Π_2^* the cotangent map,$\mathcal{V}^2E = Ker\Pi_2^*$ the vertical subbundle of second order. We are considering also the bundle $\Pi_{12} : E \to TM \times R$ and $\mathcal{V}E = Ker\Pi_{12}^*$ the vertical subbundle of first order, that at his turn, is subbundle of the vertical bundle of second order, through his natural structure. Local bases in $\mathcal{V}E$ and $\mathcal{V}^2E$ are respectively $\{\frac{\partial}{\partial x^i}, \frac{\partial}{t}\}$ and $\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial z^i}, \frac{\partial}{\partial t}\}$.

Definition 1. A *nonlinear connection* on E is a splitting of the TE in the sum $TE = \mathcal{V}^2E \oplus \mathcal{N}E$,where $\mathcal{N}E$ will be named the normal subbundle of E.

Locally, a base in $u \to \mathcal{N}_uE$ distribution is given by $\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \mathcal{N}_i^j \frac{\partial}{\partial y^j} - \mathcal{M}_i^j \frac{\partial}{\partial z^j} - \mathcal{K}_I^0 \frac{\partial}{t}\}$ We are imposing further the conditions of global definition of the adapted fields $\{\frac{\delta}{\delta y^i}\}$ and $\{\frac{\delta}{\delta x^i}\}$,

$$\frac{\delta}{\delta x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\delta}{\delta \widetilde{x}^j} \quad and \quad \frac{\delta}{\delta y^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\delta}{\delta \widetilde{y}^j} \tag{1.2}$$

Consequently, we are obtaining the next changing rules of the nonlinear connection coefficients on E.

$$\widetilde{\mathcal{N}}\,{}^r_k \frac{\partial \widetilde{x}^r}{\partial x^k} = \frac{\partial \widetilde{x}^r}{\partial x^k}\mathcal{N}_i^k - \frac{\partial^2 \widetilde{x}^r}{\partial x^i \partial x^k} z^k + \frac{\partial^2 \widetilde{x}^r}{\partial x^i \partial x^k} y^i - \frac{1}{2}\frac{\partial^3 \widetilde{x}^r}{\partial x^i \partial x^j \partial x^k} y^i y^k.. \tag{1.3}$$

$$\widetilde{\mathcal{M}}\,{}^r_k \frac{\partial \widetilde{x}^k}{\partial x^i} = \frac{\partial \widetilde{x}^r}{\partial x^k}\mathcal{M}_i^k - \frac{\partial^2 \widetilde{x}^r}{\partial x^i \partial x^k} y^k \tag{1.4}$$

$$\widetilde{\mathcal{K}}\,{}^{0}_{i}\frac{\partial \widetilde{x}^k}{\partial x^i} = \mathcal{K}^0_i \tag{1.5}$$

and analogue with (1.3) and (1.5) for $\mathcal{H}^j_i$ and $\mathcal{H}^0_i$ .In consequence we will take $\mathcal{H}^j_i = \mathcal{M}^j_i$ and $\mathcal{H}^0_i = \mathcal{K}^0_i$ in the following.

Giving a nonlinear connection on E is obtaining the next adapted local base for T_uE : $\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\partial}{\partial z^i}, \frac{\partial}{\partial t}\right\}$ that is changing as the vectors as it results from(1.2) if there are verified the conditions (1.3) ,(1.4) ,(1.5) .

Considering a nonlinear connection fixed on E , we name *d-tensor of* (r, s) *type* a real function $t^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x, y, z, t)$ that is changing after rule:

$$\widetilde{t}^{h_1\dots h_r}_{k_1\dots k_s}(\widetilde{u}) = \frac{\partial \widetilde{x}^{h_1}}{\partial x^{i_1}}\cdots\frac{\partial \widetilde{x}^{h_r}}{\partial x^{i_r}}\cdot\frac{\partial x^{j_1}}{\partial \widetilde{x}^{k_1}}\cdots\frac{\partial x^{j_s}}{\partial \widetilde{x}^{k_s}}t^{i_1\dots i_r}_{j_1\dots j_s}(u). \tag{1.6}$$

On E we can introduce relatively to the given nonlinear connection , the following geometrical structures.

$$F^i_j = dx^i \otimes \frac{\delta}{\delta y^j} + \delta y^i \otimes \frac{\partial}{\partial z^j} + \delta t \otimes \frac{\partial}{\partial t} \tag{1.7}$$

and his dual

$$F^{*i}_j = \delta y^i \otimes \frac{\delta}{\delta x^j} + \delta z^i \otimes \frac{\delta}{\delta y^j} + \delta t \otimes \frac{\partial}{\partial t}. \tag{1.7$'$}$$

The triplet $(F, \frac{\partial}{\partial t}, \delta t)$ verifies the conditions : $F^3 = \delta t \otimes \frac{\partial}{\partial t}$, $\delta t(\frac{\partial}{\partial t}) = 1$ and $rank\ F = 2n + 1$ and it is named the cotangent structure of second order ([12])

Structure $\varphi = F - F^3$ it is an almost tangent of second order structure on E ([12]) , $rank\ \varphi = 2n.$

The triplet $(F^*, \frac{\partial}{\partial t}, \delta t)$ it is also a cotangent structure of second order named adjoint to F .

Analogue $\varphi^* = F^* - F^3$ it is a tangent structure of second order adjoint to φ .Easily there can be deduced links between these structures ([6]) ..

2. **Linear d-connections on E**

Let $E = T^2M \times R$ be the reonom bundle of second order endowed with a nonlinear connection conveniently chosen $N\Gamma = (\mathcal{M}^i_j, \mathcal{N}^i_j, \mathcal{K}^i_j)$ that determines the $TE = \mathcal{V}E \otimes \mathcal{H}E \otimes \mathcal{N}E$ decomposition, with the corresponding projectors .A field $X \in \mathcal{X}(E)$ will be decomposed in $X = vX + hX + nX.$

Definition 2. It is named *linear d-connection* on E a D linear connection on E that preserves trough parallelism the distributions $\mathcal{V}E, \mathcal{H}E, \mathcal{N}E$.

Theorem 1. *A linear connection D on E is a d-connection if and only if there are verified one of the following conditions :*

a) $(v+h)D_X nY = 0$, $(v+n)D_X hY = 0$, $(h+n)D_X vY = 0$

b) $D_X Y = vD_X vY + hD_X hY + nD_X nY$

c) $Dv = Dh = Dn = 0$

d) $DP_1 = 0, DP_2 = 0$ $DP_3 = 0$ *where* $P_1 = (n+h) - v$,$P_2 = (n+v) - h$, $P_3 = (v+h) - n$ *there are almost product structure on* E.

The proof results from the fact that: $D_X nY \in \mathcal{N}E$, $D_X hY \in \mathcal{H}E$, $D_X vY \in \mathcal{V}E$.

Because D is a R-linear application that can be extended to the whole d-tensors algebra, it results that :

Proposition 2. *It is only one operator of covariant derivation D^n_X named normal derivation thus that :*

$$D^n_X Y = D_{nX} Y \text{ and } D^n_X f = (nx)f : \ \forall X, Y \in \mathcal{X}(E), \ f \in \mathcal{F}(E). \tag{2.1}$$

Locally D^n can be expressed the following way :

$$\begin{gathered} D^n_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = \overset{1}{L}{}^i_{jk} \frac{\delta}{\delta x^i} \\ D^n_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^j} = \overset{2}{L}{}^i_{jk} \frac{\delta}{\delta y^i} \\ D^n_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^j} = \overset{3}{L}{}^i_{jk} \frac{\partial}{\partial z^i} + L^0_{jk} \frac{\partial}{\partial t} \quad ; \quad D^n_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial t} = L^0_{0k} \frac{\partial}{\partial z^i} + L^0_{ok} \frac{\partial}{\partial t} \end{gathered} \tag{2.2}$$

Analogous it is defined the D^h covariant $h-$derivation with the following local expressions.

$$\begin{gathered} D^h_{\frac{\delta}{\delta y^k}} \frac{\delta}{\delta x^j} = \overset{1}{F}{}^i_{jk} \frac{\delta}{\delta x^i} \quad ; \quad D^h_{\frac{\delta}{\delta z^k}} \frac{\partial}{\partial z^j} = \overset{3}{F}{}^i_{jk} \frac{\partial}{\partial z^i} + F^0_{jk} \frac{\partial}{\partial t} \\ D^h_{\frac{\delta}{\delta y^k}} \frac{\delta}{\delta y^j} = \overset{2}{F}{}^i_{jk} \frac{\delta}{\delta y^i} \quad ; \quad D^h_{\frac{\delta}{\delta y^k}} \frac{\partial}{\partial t} = F^i_{0k} \frac{\partial}{\partial z^i} + F^0_{0k} \frac{\partial}{\partial t} \end{gathered} \tag{2.3}$$

and in totally the same way it is introduced D^v covariant $v-$derivation with local expressions

$$D^v_{\frac{\partial}{\partial z^k}} \cdot \frac{\delta}{\delta x^j} = \overset{1}{C}{}^i_{jk} \frac{\delta}{\delta x^i} \quad ; \quad D^v_{\frac{\partial}{\partial z^k}} \cdot \frac{\partial}{\partial t} = C^i_{0k} \frac{\partial}{\partial z^i} + C^0_{0k} \frac{\partial}{\partial t}$$

$$D^v_{\frac{\partial}{\partial z^k}} \cdot \frac{\delta}{\delta y^j} = \overset{2}{C}{}^i_{jk} \frac{\delta}{\delta y^i} \quad ; \quad D^v_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = C^0_{00} \frac{\partial}{\partial t} \tag{2.4}$$

$$D^v_{\frac{\partial}{\partial z^k}} \cdot \frac{\partial}{\partial z^j} = \overset{3}{C}{}^i_{jk} \frac{\partial}{\partial z^i}$$

The curvatures and torsions of a linear $d-$connection are written and are finding their local expressions through the direct calculation.([6])

3. M-linear connection on E

Let D be a linear $d-$ connection on E with local coefficients given by (2.1);(2.2);(2.3).

Definition 3. A $d-$ linear connection D on E it is said that it is a $M-$ *linear connection (Miron -connection)* if:

$$\overset{1}{L}{}^i_{jk} = \overset{2}{L}{}^i_{jk} = \overset{3}{L}{}^i_{jk};\ \overset{1}{F}{}^i_{jk} = \overset{2}{F}{}^i_{jk} = \overset{3}{F}{}^i_{jk};\ \overset{1}{C}{}^i_{jk} = \overset{2}{C}{}^i_{jk} = \overset{3}{C} \tag{3.1}$$

Let F and φ the almost cotangent structures of second order and respectively second order tangent locally given by (1.7) and $\varphi = F - F^3$,and (F^*, φ^*) their adjoint structures:

Definition 4. a) A D- linear connection on E is a *F-linear connection(respectively F^**) if $D = 0$ and $D\frac{\partial}{\partial t} = 0$ (respectively $DF^* = 0, D\frac{\partial}{\partial t} = 0$).

b) A $D-$ linear connection on E is a *(φ, φ^*) -linear connection* on E if $DF = DF^* = 0$ and $D\frac{\partial}{\partial t} = 0$

c) A $D-$ linear connection on E is a *$\varphi-$linear connection* (respectively φ^*- linear connection) if $D\varphi = 0$ (respectively $D\varphi^* = 0$)

d) A $D-$ linear connection on E is a (φ, φ^*) $-$ *linear connection* if $D\varphi = D\varphi^* = 0$

Proposition 3. *A D -linear connection on E is a (F, F^*) -linear connection if and only if is a (φ, φ^*) -linear connection.*

Proof. From $DF=0 \Rightarrow DF^3=0 \Rightarrow D(F-F^3)=0 \Rightarrow D\varphi=0$ and from $DF^*=0 \Rightarrow D(F-F^*)=0 \Rightarrow D\varphi^*=0$. Reciprocal , we have $\varphi^* F^3=0$ and $F^3\varphi^*=0$ (taking into account that $F^3(X_u)\in \mathcal{V}_u E)$ .It results that $DF^3=0$ and together with $D\varphi=0$, $D\varphi^*=0$ we are obtaining that $D(\varphi+F^3)=DF=0$ and $(D\varphi^*+F^3)=DF^*=0$.

Proposition 4. *A $(F.F^*)$-linear connection is a $d-$linear connection on E (F,F^*) .*

Proof:Is a $(F,F^*)-$ linear connection is a $(\varphi,\varphi^*)-$linear connection and using the fact that $v=\varphi^2\varphi^{*2}$,$\varphi^{*2}=n$ and $\varphi^*\varphi-\varphi^{*2}\varphi=h$ it results that $Dn=Dh=Dh=0$ is a $d-$ linear connection on E .

Theorem 5. *A D linear connection on E is a M $-$linear connection if and only if is a (F,F^*)-linear connection .*

Proof:From the proposition 3.2 it results that if D is a (F,F^*)-linear connection than it is also a d-linear connection.

Because

$$(D_{\frac{\delta}{\delta x^k}} F)(\frac{\delta}{\delta x^j})=(D^n_{\frac{\delta}{\delta x^k}} F)(\frac{\delta}{\delta x^j})=(D^n_{\frac{\delta}{\delta x^k}} F)(\frac{\delta}{\delta x^j})-FD^n_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j}=$$

$$=D^n_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta y^j}-\overset{3}{L}{}^i_{jk}\, F(\frac{\delta}{\delta x^i})=(\overset{2}{L}{}^i_{jk}-\overset{3}{L}{}^i_{jk})\frac{\delta}{\delta y^i}.$$

We are obtaining that $(D_{\frac{\delta}{\delta x^k}})(\frac{\delta}{\delta x^j})=0 \Leftrightarrow \overset{2}{L}=\overset{3}{L}$ .In an analogue way, taking these values of the adapted base fields, yields that $DF=DF^*=0$,and hence D is a $M-$ linear connection on E.

We are waking the notifications $F^3=p$ and $q=I-p$.

Theorem 6. *There exists $M-$linear connections on E. One of them is given by :*

$$\overset{B}{D}_X Y=\overset{B}{D}_{qX} qY+\overset{B}{D}_{qX} pY+\overset{B}{D}_{pX} qY+\overset{B}{D}_{pX} pY \tag{3.2}$$

where:

$$\overset{B}{D}_{qX} qY=\varphi^2\left[\left(v+\frac{h}{2}\right)X,\varphi^{*2}y\right]+v\left[\left(n+\frac{h}{2}\right)X,vY\right]+$$

$$\varphi n\left[\left(n+\frac{h}{2}\right)X,\varphi^{*}hX\right]+\ \varphi^{*}v\left[\left(n+\frac{h}{2}\right)X,\varphi hY\right]+\varphi^{*2}\left[\left(n+\frac{h}{2}\right)X,nY\right]$$

$$\overset{B}{D}_{qX}\ pY=p\left[qX\ ,\ pY\right] \tag{3.3}$$

$$\overset{B}{D}_{pX}\ qY=\frac{1}{2}\{\varphi^{2}\left[pX\ ,\varphi^{2}Y\right]+\varphi^{*2}\left[pX\ ,\varphi^{2}Y\right]+(\frac{h}{2}+n)\left[pX\ ,(v+\frac{h}{2})Y\right]\}+$$

$$+\ \frac{1}{4}\{\varphi n\left[pX\ ,\varphi^{*}hY\right]+\varphi^{*}v\left[pX\ ,hY\right].\}$$

$$\overset{B}{D}_{pX}\ pY=\overset{0}{\nabla}_{pX}\ pY-\delta t(X)\delta t(Y)\ \overset{0}{\nabla}_{\frac{\partial}{\partial t}}\ \frac{\partial}{\partial t} \tag{3.4}$$

and $\overset{0}{\nabla}$ is a linear connection on E .

Proof. Trough the direct calculation it is verified that D is a linear connection and that $D\varphi=D\varphi^{*}=0$, so D is a $M-$ linear connection.

Given to X and Y values of the adapted base, from(3.3) results :

Corollary 7. The following functions on E

$$L_{ij}^{k}=\frac{\partial\mathcal{M}_{j}^{l}}{\partial z^{i}}\mathcal{M}_{l}^{k}+\frac{\partial\mathcal{N}_{j}^{k}}{\partial z^{i}}\ ;\qquad F_{ij}^{k}=\frac{\partial\mathcal{M}_{j}^{k}}{\partial z^{i}}\ ;\qquad C_{ij}^{k}=0$$

$$L_{i0}^{k}=L_{0j}^{k}=F_{0j}^{k}=C_{i0}^{0}=C_{0j}^{0}=C_{00}^{0}=0\ . \tag{3.5}$$

defining the coefficients of a M −linear connection on E ,named *Berwald connection* in the reonom bundle of second order

An interesting problem is the determination of the $M-$linear connection compatible with respect to a given metric structure on E . We will approach this in a coming paper .

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A FORMULA FOR THE MEAN CURVATURE OF AN IMPLICIT REGULAR SURFACE

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Abstract. In this paper we will find a formula for the absolute value of the mean curvature of an implicit regular surface $(S)\, f(x,y,z)=a$, expressed in terms of the partial derivatives of the function f.

1. **Introduction**

The most used formulas for the Gaussian curvature or for the mean curvature of a regular surface are those that are expressed locally in terms of the coefficients of the first and second fundamental forms.

However for an implicit regular surface (S) $f(x,y,z)=a$ there exists a formula for the Gaussian curvature expressed in terms of the partial derivatives of the function f, that is,

$$K = -\frac{1}{||\overrightarrow{\nabla} f||^4}\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} & f_x \\ f_{yx} & f_{yy} & f_{yz} & f_y \\ f_{zx} & f_{zy} & f_{zz} & f_z \\ f_x & f_y & f_z & 0 \end{vmatrix} \tag{1}$$

In this paper we are going to prove a similar formula for the absolute value of the mean curvature of an implicit regular surface.

For the mean curvature H of a regular surface S we have the following local formula

$$H = \frac{1}{2}\cdot\frac{eG-2fF+gE}{EG-F^2} \tag{2}$$

where

$$E = \overrightarrow{r}_u\cdot\overrightarrow{r}_u = ||\overrightarrow{r}_u||^2,\ F=\overrightarrow{r}_u\cdot\overrightarrow{r}_v,\ G=\overrightarrow{r}_v\cdot\overrightarrow{r}_v = ||\overrightarrow{r}_v||^2$$

are the coefficients of the first fundamental form and

$$e = \frac{(\overrightarrow{r}_u,\overrightarrow{r}_v,\overrightarrow{r}_{uu})}{\sqrt{EG-F^2}},\ f=\frac{(\overrightarrow{r}_u,\overrightarrow{r}_v,\overrightarrow{r}_{uv})}{\sqrt{EG-F^2}},\ g=\frac{(\overrightarrow{r}_u,\overrightarrow{r}_v,\overrightarrow{r}_{vv})}{\sqrt{EG-F^2}}$$

are the coefficients of the second fundamental form with respect to the local parametrization $r : U \to S$, compatible with the orientation of the surface.

Let $V \subseteq \mathbf{R}^3$ be an open set, $f : V \to \mathbf{R}$ be a differentiable function and $a \in Im\, f$ be a regular value of f. It is well known that $S = f^{-1}(a)$ is an orientable regular surface. For $p \in S$, then one of the partial derivatives $f_x(p), f_y(p), f_z(p)$ is non zero, at least. If $f_z(p) \neq 0$, for instance, then, according to the implicit function theorem, the last variable z can be unically expressed by means of the first two variable x and y. In other words the regular surface $S = f^{-1}(a)$ is locally, around the point p, the graph of a function $z = z(x, y)$, $(x, y) \in U$, where U is a conveniently chosen open set. Therefore the mapping $r : U \to S$, $r(x, y) = (x, y, z(x, y))$ is a local parametrization of S at p, namely $f(x, y, z(x, y)) = a$, $\forall (x, y) \in U$. This is the type of local parametrization that we are going to use for all over this paper.

It is very easy to see that $\overrightarrow{r}_x \times \overrightarrow{r}_y = \frac{1}{f_z} \overrightarrow{\nabla} f$ which means that the local parametrization $r : U \to S$, $r(x, y) = (x, y, z(x, y))$ of S at p is compatible with the orientation $\frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}$ of S iff $f_z(p) > 0$ and of course uncompatible iff $f_z(p) < 0$.

In any case the relation

$$2|H| = |\frac{eG - 2fF + gE}{EG - F^2}| \tag{3}$$

holds.

2. The main formula

In this section we will prove the already anounced formula for the absolute value of the mean curvature of an implicit regular surface.

Theorem 2.1. *Let $V \subseteq \mathbf{R}^3$ be an open set, $f : V \to \mathbf{R}$ be a smooth function and $a \in Im\, f$ be a regular value of the f. For the absolute value of the mean curvature H of the implicit regular surface (S) $f(x, y, z) = a$, at the point $p \in S$, we have the following formula*

$$|H| = \frac{1}{2||\overrightarrow{\nabla} f||} \left| \left[\Delta f - (Hess\, f)\Big(\frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}, \frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}\Big) \right] \right|, \tag{4}$$

where $\overrightarrow{\nabla} f$ is the gradient of f, Δ is the Laplace's operator and $Hess\, f$ is the Hessian of f, all of them being considered at the point p.

PROOF. Assuming that for $p \in f^{-1}(a)$ we have $f_z(p) \neq 0$, it follows that S is locally, around the point p, the graph of a function $z = z(x,y)$, $(x,y) \in U$ and consider the above stated local parametrization $r : U \to S$, $r(x,y) = (x,y,z(x,y))$.

The coefficients of the two fundamental forms are

$$E = 1 + z_x^2, \qquad F = z_x \cdot z_y, \qquad G = 1 + z_y^2$$
$$e = \frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}}, \quad f = \frac{z_{xy}}{\sqrt{1+z_x^2+z_y^2}}, \quad g = \frac{z_{yy}}{\sqrt{1+z_x^2+z_y^2}}$$

$$2|H| = |\frac{eG - 2fF + gE}{EG - F^2}| = |\frac{(1+(f_x)^2)f_{yy} - 2f_x f_y f_{xy} + (1+f_y)^2 f_{xx}}{[1 + z_x^2 + z_y^2]^{3/2}}|. \tag{5}$$

Because $f(x,y,z(x,y)) = a$, $\forall\,(x,y) \in U$, it follows that

$$\begin{cases} f_x + z_x f_z = 0 \\ f_y + z_y f_z = 0 \end{cases} \text{that is} \begin{cases} z_x = -\frac{f_x}{f_z} \\ z_y = -\frac{f_y}{f_z}. \end{cases} \tag{6}$$

¿From relations (6) we get

$$\begin{cases} z_{xx} = -\frac{\partial}{\partial x}\left[\frac{f_x(x,y,z(x,y))}{f_z(x,y,z(x,y))}\right] = -\frac{f_z^2 f_{xx} - 2f_x f_z f_{xz} + f_x^2 f_{zz}}{f_z^3} \\ z_{xy} = -\frac{\partial}{\partial y}\left[\frac{f_x(x,y,z(x,y))}{f_z(x,y,z(x,y))}\right] = -\frac{f_z^2 f_{xy} - f_y f_z f_{xz} - f_x f_z f_{yz} + f_x f_y f_{zz}}{f_z^3} \\ z_{yy} = -\frac{\partial}{\partial x}\left[\frac{f_y(x,y,z(x,y))}{f_z(x,y,z(x,y))}\right] = -\frac{f_z^2 f_{yy} - 2f_y f_z f_{yz} + f_y^2 f_{zz}}{f_z^3}. \end{cases} \tag{7}$$

Replacing the partial derivatives $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ given by the relations (6), (7)in the formula (5) we obtain

$$2|H| =$$

$$|\frac{\frac{(f_y^2+f_z^2)(f_{xx}f_z^2 - 2f_x f_z f_{xz} + f_{zz} f_x^2)}{f_z^5} - 2\frac{f_x f_y (f_x f_y f_{zz} - f_y f_z f_{xz} - f_x f_z f_{yz} + f_{xy} f_z^2)}{f_z^5} + \frac{(f_x^2+f_z^2)(f_{yy} f_z^2 - 2f_y f_z f_{yz} + f_{zz} f_y^2)}{f_z^5}}{\left[\frac{f_x^2+f_y^2+f_z^2}{f_z^2}\right]^{3/2}}|$$

$$= |\frac{|f_z|^3}{f_z^5}\Big[\frac{f_y^2 f_z^2 f_{xx} - 2f_x f_y^2 f_z f_{xz} + f_x^2 f_y^2 f_{zz} + f_z^4 f_{xx} f_x f_z^3 f_{xz} + f_x^2 f_z^2 f_{zz} - 2f_x^2 f_y^2 f_{zz} + 2f_x f_y^2 f_z f_{xz}}{\sqrt{f_x^2 + f_y^2 + f_z^2}^3} +$$

$$+ \frac{2f_x^2 f_y f_z f_{yz} - 2f_x f_y f_z^2 f_{xy} + f_x^2 f_z^2 f_{yy} - 2f_x^2 f_y f_z f_{yz} + f_x^2 f_y^2 f_{zz} + f_z^4 f_{yy} - 2f_y f_z^3 f_{yz} + f_y^2 f_z^2 f_{zz}}{\sqrt{f_x^2 + f_y^2 + f_z^2}^3}\Big]| =$$

$$= |\frac{|f_z|^3}{f_z^5}\frac{f_z^2(f_y^2 f_{xx} + f_z^2 f_{xx} - 2f_x f_z f_{xz} + f_x^2 f_{zz} - 2f_x f_y f_{xy} + f_x^2 f_{yy} + f_z^2 f_{yy} - 2f_y f_z f_{yz} + f_y^2 f_{zz})}{||\overrightarrow{\nabla} f||^3}| =$$

$$= |\frac{f_x^2(f_{yy} + f_{zz}) + f_y^2(f_{xx} + f_{zz}) + f_z^2(f_{xx} + f_{yy}) - (Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) + f_x^2 f_{xx} + f_y^2 f_{yy} + f_z^2 f_{zz}}{||\overrightarrow{\nabla} f||^3}|$$

where

$$(Hesss\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) = (f_x, f_y, f_z)\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} =$$

$$= f_{xx}f_x^2 + f_{yy}f_y^2 + f_{zz}f_z^2 + 2f_{xy}f_xf_y + 2f_{xz}f_xf_z + 2f_{yz}f_yf_z.$$

Therefore for the absolute value of the mean curvature wee have

$$|H| = \frac{1}{2}\Big|\frac{f_x^2(f_{xx}+f_{yy}+f_{zz}) + f_y^2(f_{xx}+f_{yy}+f_{zz} + f_z^2(f_{xx}+f_{yy}+f_{zz}) - (Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f)}{||\overrightarrow{\nabla} f||^3}\Big| =$$

$$= \frac{1}{2}\Big|\frac{(f_x^2+f_y^2+f_z^2)(f_{xx}+f_{yy}+f_{zz}) - (Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f)}{||\overrightarrow{\nabla} f||^3}\Big| =$$

$$= \frac{1}{2}\Big|\frac{||\overrightarrow{\nabla} f||^2\cdot\Delta f - (Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f)}{||\overrightarrow{\nabla} f||^3}\Big| =$$

$$= \frac{1}{2||\overrightarrow{\nabla} f||}\Big|\Big[\Delta f - (Hess\, f)\Big(\frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}, \frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}\Big)\Big]\Big|.\square$$

Corollary 2.2. *If $V \subseteq \mathbf{R}^3$ is an open set, $f : V \to \mathbf{R}$ is a smooth harmonic mapping and $a \in Im\, f$ is a regular value of f, then for the absolute value of the mean curvature of the implicit regular surface $(S)\; f(x,y,z) = a$ we have the following formula:*

$$|H| = \frac{1}{2||\overrightarrow{\nabla} f||^3}\big|(Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f)\big|. \tag{8}$$

3. Example

It is well know that the locus of the orthogonal projections of the center of the ellipsoid $(E)\; \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ on its tangent planes is the so called *pedal surface* of E, that is the regular surface

$$S = \{(x,y,z) \in \mathbf{R}^3 \mid (x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2\}\setminus\{0\}.$$

We will compute the absolute value of the mean curvature of the pedal surface of E in its points.

For this purpose consider $p = (x_0, y_0, z_0) \in S$, the function

$$f : \mathbf{R}^3\setminus\{0\} \to \mathbf{R},\; f(x,y,z) = (x^2+y^2+z^2)^2 - a^2x^2 - b^2y^2 - c^2z^2$$

and observe that $S = f^{-1}(0)$.

The partial derivatives of first and second order of f are

$$f_x = 4x(x^2+y^2+z^2) - 2a^2x$$
$$f_y = 4y(x^2+y^2+z^2) - 2b^2y$$
$$f_z = 4z(x^2+y^2+z^2) - 2c^2z$$

$$\begin{array}{lll} f_{xx} = 4(x^2+y^2+z^2) + 8x^2 - 2a^2 & f_{xy} = f_{yx} = 8xy & f_{xz} = f_{zx} = 8xz \\ f_{yy} = 4(x^2+y^2+z^2) + 8y^2 - 2b^2 & f_{yz} = f_{zy} = 8yz & \\ f_{zz} = 4(x^2+y^2+z^2) + 8z^2 - 2c^2 & & . \end{array}$$

Therefore in the points (x,y,z) of the regular surface S we have $||\overrightarrow{\nabla} f||^2 = 4(a^4x^2 + b^4y^2 + c^4z^2)$, or equivalent $||\overrightarrow{\nabla} f|| = 2(a^4x^2+b^4y^2+c^4z^2)^{1/2}$. Observe that $||\overrightarrow{\nabla} f|| \neq 0$ in all the points of the surface $S = f^{-1}(0)$. Therefore the critical set of f doesn't intersects the level set $S = f^{-1}(0)$, this being of course an argument on the regularity of S.

On the other hand $\Delta f = 20(x^2+y^2+z^2) - 2(a^2+b^2+c^2)$ and

$$(Hessf)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) = f_{xx}f_x^2 + f_{yy}f_y^2 + f_{zz}f_z^2 + 2f_{xy}f_xf_y + 2f_{xz}f_xf_z + 2f_{yz}f_yf_z =$$

$$= (4(x^2+y^2+z^2)+8x^2-2a^2)[16x^2(x^2+y^2+z^2)^2 - 16a^2x^2(x^2+y^2+z^2) + 4a^4x^2] +$$

$$+(4(x^2+y^2+z^2)+8y^2-2b^2)[16y^2(x^2+y^2+z^2)^2 - 16b^2y^2(x^2+y^2+z^2) + 4b^4y^2] +$$

$$+(4(x^2+y^2+z^2)+8z^2-2c^2)[16z^2(x^2+y^2+z^2)^2 - 16c^2z^2(x^2+y^2+z^2) + 4c^4z^2] +$$

$$+16xy[4x(x^2+y^2+z^2) - 2a^2x][4y(x^2+y^2+z^2) - 2a^2y] +$$

$$+16xz[4x(x^2+y^2+z^2) - 2a^2x][4z(x^2+y^2+z^2) - 2c^2z] +$$

$$+16yz[4y(x^2+y^2+z^2) - 2a^2y][4z(x^2+y^2+z^2) - 2c^2z] =$$

$$= 48(x^2+y^2+z^2)(a^4x^2+b^4y^2+c^4z^2).$$

Replacing all of these values considered in p, in the formula (4), we obtain

$$|H_{S_1}(p)| =$$

$$= \frac{1}{4(a^4x_0^2+b^4y_0^2+c^4z_0^2)^{1/2}} |20(x_0^2+y_0^2+z_0^2) - 2(a^2+b^2+c^2) - \frac{48(x_0^2+y_0^2+z_0^2)(a^4x_0^2+b^4y_0^2+c^4z_0^2)}{4(a^4x_0^2+b^4y_0^2+c^4z_0^2)}| =$$

$$= \frac{|4(x_0^2+y_0^2+z_0^2) - (a^2+b^2+c^2)|}{2(a^4x_0^2+b^4y_0^2+c^4z_0^2)^{1/2}} = |2\sqrt{\frac{a^2x_0^2+b^2y_0^2+c^2z_0^2}{a^4x_0^2+b^4y_0^2+c^4z_0^2}} - \frac{1}{2}\frac{a^2+b^2+c^2}{\sqrt{a^4x_0^2+b^4y_0^2+c^4z_0^2}}|.$$

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FIRST ORDER DIFFERENTIAL SUBORDINATIONS AND INEQUALITIES IN A BANACH SPACE

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Abstract. Let E be a complex Banach space and let $B = \{x \in E : \|x\| < 1\}$ be the unit ball in E. Let $p : B \to \mathbb{C}$ be holomorphic in B and let q be holomorphic and univalent in the unit disc U. We prove that if p satisfies some differential subordinations and inequalities, then $p(B) \subset q(U)$. Applications of these results are presented.

1. Introduction

S. Gong and S.S. Miller [1] have dealt with holomorphic functions defined on a complete circular domain in $\mathbb{C}^n$, which satisfy certain partial differential inequalities or subordinations. In this paper we consider similar relationships for holomorphic functions from the unit ball B into $\mathbb{C}$.

The following sets $\{x \in E : \ \|x\| < r \le 1\}$ and $\{x \in E : \ \|x\| \le r \le 1\}$ will be denoted B_r, respectively $\overline{B}_r$.

Let $H(B_r)$, $r \in (0, 1]$ be the class of functions $f : B_r \to \mathbb{C}$ that are holomorphic in B_r, i.e. have the Fréchet derivative $f'(x)$ in each point $x \in B_r$.

2. First order differential subordinations

Lemma 1. *Let $r_0 \in (0, 1)$ and let $f \in H(\overline{B}_{r_0})$ with $f(0) = 0$ and $f(x) \not\equiv 0$. If $x_0 \in \overline{B}_{r_0}$ and*

$$|f(x_0)| = \max\{|f(x)| : \ x \in \overline{B}_{r_0}\} \tag{1}$$

then there exists $m \in \mathbb{C}$ with Re $m \ge 1$ *such that*

$$f'(x_0)(x_0) = mf(x_0). \tag{2}$$

Proof. We have $zx \in B_{r_0}$ for all $z \in U$ and $x \in \overline{B}_{r_0}$. We consider the function $g(z) = \dfrac{f(zx_0)}{f(x_0)}$, for $z \in U$. From (1) we obtain

$$|g(z)| = \left|\frac{f(zx_0)}{f(x_0)}\right| < 1, \quad \text{for all} \quad z \in U.$$

Since $g(0) = 0$, we can apply Schwarz's lemma to obtain $|g(z)| \leq |z|$, $z \in U$ and thus

$$\left|\frac{f(zx_0)}{f(x_0)}\right| \leq |z|, \quad \text{for} \quad z \in U.$$

At the point $z = r$, $r \in (0,1)$ we have

$$\operatorname{Re} \frac{f(rx_0)}{f(x_0)} \leq r. \tag{3}$$

A simple calculation leads to

$$\frac{f'(x_0)(x_0)}{f(x_0)} = \frac{d}{dr}\left[\frac{f(rx_0)}{f(x_0)}\right]\Bigg|_{r=1} = \lim_{r \nearrow 1} \frac{f(rx_0) - f(x_0)}{(r-1)f(x_0)} = \lim_{r \nearrow 1}\left[1 - \frac{f(rx_0)}{f(x_0)}\right]\frac{1}{1-r}.$$

Taking real parts and using (3) we obtain

$$\operatorname{Re} \frac{f'(x_0)(x_0)}{f(x_0)} \geq \lim_{r \nearrow 1}(1-r)\frac{1}{1-r} = 1,$$

which proves the lemma.

We will extend the ideas in Lemma 1, but first we need to consider the following class of functions.

Definition 1. We denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $E(q) = \{\zeta \in \partial U : \lim\limits_{z \to \zeta} q(z) = \infty\}$ and are such that $q'(\zeta) \neq 0$, for $\zeta \in \partial U \setminus E(q)$.

Lemma 2. *Let $q \in Q$ and let $p \in H(B)$ with $p(0) = q(0)$. If $p(B) \not\subset q(U)$ then there exist $r_0 \in (0,1)$, $x_0 \in \overline{B}_{r_0}$ and $\zeta_0 \in \partial U \setminus E(q)$ such that*

(i) $p(x_0) = q(\zeta_0)$

(ii) $p'(x_0)(x_0) = m\zeta_0 q'(\zeta_0)$, *where* $\operatorname{Re} m \geq 1$.

Proof. Since $p(0) = q(0)$ and $p(B) \not\subset q(U)$ there exists $r_0 \in (0,1)$ such that $p(B_{r_0}) \subset q(U)$ and $p(\overline{B}_{r_0}) \cap q(\partial U) \setminus E(q) \neq \emptyset$. Hence there exist $x_0 \in \overline{B}_{r_0}$ and $\zeta_0 \in \partial U \setminus E(q)$ such that $p(x_0) = q(\zeta_0)$. If we let $f(x) = q^{-1}(p(x))$, for $x \in \overline{B}_{r_0}$, then f is holomorphic in $\overline{B}_{r_0}$ and satisfies $|f(x_0)| = |\zeta_0| = 1$, $f(0) = 0$ and $|f(x)| \leq 1$, for $x \in \overline{B}_{r_0}$. Thus f satisfies the conditions of Lemma 1 and we obtain that there eixsts $m \in \mathbb{C}$, with $\operatorname{Re} m \geq 1$ such that $f'(x_0)(x_0) = mf(x_0)$. Since

$p(x) = q(f(x))$, we have $p'(x) = q'(f(x))f'(x)$ and using $\zeta_0 = f(x_0)$, we obtain $p'(x_0)(x_0) = q'(f(x_0))f'(x_0)(x_0) = m\zeta_0 q'(\zeta_0)$.

Definition 2. Let Ω be a set in $\mathbb{C}$, $q \in Q$. We define $\psi[\Omega, q]$ to be the class of functions $\psi : \mathbb{C}^2 \times B \to \mathbb{C}$ that satisfy the condition:

$$\psi(r, s; x) \notin \Omega, \quad \text{whenever} \quad r = q(\zeta), \quad s = m\zeta q'(\zeta),$$

$$x \in B, \quad \zeta \in \partial U \setminus E(q) \quad \text{and} \quad \operatorname{Re} m \geq 1.$$

We are now prepared to present the main result of this section.

Theorem 1. *Let $\psi \in \psi[\Omega, q]$. If $p \in H(B)$ with $p(0) = q(0)$ and if p satisfies*

$$\psi(p(x), p'(x)(x); x) \in \Omega, \quad \textit{for} \quad x \in B \tag{4}$$

then $p(B) \subset q(U)$.

Proof. Assume $p(B) \not\subset q(U)$. By Lemma 2 there exist $x_0 \in B$, $\zeta_0 \in \partial U \setminus E(q)$ and $m \in \mathbb{C}$ with $\operatorname{Re} m \geq 1$ that satisfy (i), (ii) of Lemma 2. Using these conditions with $r = p(x_0)$, $s = p'(x_0)(x_0)$ and $x = x_0$ in Definition 2 we obtain

$$\psi(p(x_0), p'(x_0)(x_0); x_0) \notin \Omega.$$

Since this contradicts (4) we must have $p(B) \subset q(U)$.

We next apply Theorem 1 to two important particular cases corresponding to $q(U)$ being the unit disc and $q(U)$ being the right half-plane.

If we take $q(z) = z$ in Definition 2 and Theorem 1 we obtain the following result.

Corollary 1. *Let Ω be a set in $\mathbb{C}$ and let $\psi : \mathbb{C}^2 \times B \to \mathbb{C}$ be such that*

$$\psi(e^{i\theta}; me^{i\theta}; x) \notin \Omega, \quad \textit{whenever} \quad x \in B, \quad \theta \in \mathbb{R} \quad \textit{and} \quad \operatorname{Re} m \geq 1. \tag{5}$$

If $p \in H(B)$ with $p(0) = 0$ and if p satisfies

$$\psi(p(x), p'(x)(x); x) \in \Omega, \quad \textit{for} \quad x \in B$$

then $|p(x)| < 1$, for $x \in B$.

If we take $q(z) = \dfrac{1+z}{1-z}$ in Definition 2 and Theorem 1 we obtain:

Corollary 2. *Let Ω be a set in $\mathbb{C}$ and let $\psi : \mathbb{C}^2 \times B \to \mathbb{C}$ be such that*

$$\psi(ai, s; x) \notin \Omega, \quad \textit{whenever} \quad x \in B, \quad a \in \mathbb{R}, \quad \textit{and} \quad \operatorname{Re} s \leq -\frac{1+a^2}{2}. \tag{6}$$

If $p \in H(B)$ *with* $p(0) = 1$ *and if* p *satisfies*

$$\psi(p(x), p'(x)(x); x) \in \Omega, \quad \text{for} \quad x \in B$$

then $\text{Re } p(x) > 0$*, for* $x \in B$*.*

3. Examples

In this section we present a seris of examples of differential inequalities by applying the two corollaries of the previous section.

Example 1. Let $\Omega = U$ and let $\psi(r, s; x) = \alpha(|r| + |s|) + \beta\|x\|$, where $\alpha \geq \frac{1}{2}$ and $\beta \geq 0$. If $p \in H(B)$ with $p(0) = 0$, then

$$\alpha(|p(x)| + |p'(x)(x)|) + \beta\|x\| < 1 \;\Rightarrow\; |p(x)| < 1.$$

Proof. To use Corollary 1 we need to shoe that the condition (5) is satisfied. This follows since

$$|\psi(e^{i\theta}, me^{i\theta}; x)| = \Big|\alpha(1 + |m|) + \beta\|x\|\Big| \geq \alpha(1 + |m|) \geq \alpha(1 + \text{Re } m) \geq 2\alpha \geq 1.$$

Remark. When $\alpha = \frac{1}{2}$ and $\beta = 0$ we have

$$|p(x)| + |p'(x)(x)| < 2 \;\Rightarrow\; |p(x)| < 1.$$

The proof of the following example also follows from Corollary 1.

Example 2. Let $\Omega = U$ and let $\psi(r, s; x) = \alpha(x)r + \beta s$, where $\beta \geq 0$ and $\alpha : B \to \mathbb{C}$ such that $\text{Re } \alpha(x) \geq 1 - \beta$. If $p \in H(B)$ with $p(0) = 0$, then

$$|\alpha(x)p(x) + \beta p'(x)(x)| < 1 \;\Rightarrow\; |p(x)| < 1.$$

Example 3. Let $\Omega = \{z \in \mathbb{C} : \text{ Re } z > 0\}$ and let $\psi(r, s; x) = r^2 + s$. If $p \in \mathcal{H}(B)$ with $p(0) = 1$, then

$$\text{Re } [p^2(x) + p'(x)(x)] > 0 \;\Rightarrow\; \text{Re } p(x) > 0.$$

Proof. To use Corollary 2 we need to show that the condition (6) is satisfied. This follows since

$$\text{Re } \psi(ai, s; x) = -a^2 + \text{Re } s \leq \frac{-3a^2 - 1}{2} < 0.$$

The proof of the following example also follows from Corollary 2.

Example 4. Let $\Omega = \{z \in \mathbb{C} : \text{ Re } z > 0\}$ and let $\psi(r, s; x) = \alpha(x)r + \beta s$, where $\beta \geq 0$ and $\alpha : B \to \mathbb{C}$ such that $|\text{Im } \alpha(x)| \leq \beta$. If $p \in H(B)$ with $p(0) = 1$, then

$$\text{Re } [\alpha(x)p(x) + \beta p'(x)(x)] > 0 \;\Rightarrow\; \text{Re } p(x) > 0.$$

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A NOTE ON STANDARD TOPOLOGICAL CONTEXTS WITH PSEUDOMETRIC

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Abstract. Standard topological contexts are a valuable tool in representing several classes of ordered algebraic structures. While investigating Contextual Topology, pseudometric contexts were introduced as a tool in approximating objects by their attributes. Here we describe the interaction between these two classes, i.e., pseudometric contexts and standard topological contexts, pointing out whether the Hartung duality extends in the case of metric lattices or not. Moreover, the meaning of being a contraction or being continuous in the case of multivalued pseudometric morphisms is investigated.

1. **Introduction**

Formal Concept Analysis was introduced first in an attempt of restructuring lattice theory (see [Wi82]). Since then, Formal Concept Analysis developed continuously to a theory of interpreting data by revealing the fundamental patterns of it. These patterns are then synthesized in a structure called **concept lattice**. Ten years later, standard topological contexts were introduced as a valuable tool in representing 0–1 lattices via Formal Concept Analysis ([Ha92]). This representation could be also considered as the first step in investigating the links between Topology and Formal Concept Analysis.

In [Sa00a] pseudometric and metric formal contexts were introduced as a generalization of the well known concepts of a metric on a set. By this generalization, the notion of metric extends on a formal context by the mathematization of a well known fact: Formal contexts are representing data sets. Usually, a data set is a record of several measurements or informations about a set of objects and a set of attributes of interest. These attributes are specific for the topic in study but some of these are more characteristic than others. (Pseudo)metric contexts, and uniform contexts as well, captures at best this phenomenon, i.e., that an attribute is more or less characteristic for an object as another one. For more informations, see for example [Sa00b].

This paper describes the links between standard topological contexts and pseudo metric contexts, investigating whether the well known duality between standard topological contexts and 0–1 lattices remains valid if we consider 0–1 pseudometric lattices. Moreover, we shall describe how some properties of 0–1 pseudometric lattices like being a contraction or being continuous reflects in the category of standard topological contexts with pseudometric.

2. **Basic Definitions and Results**

We briefly sketch the duality between bounded lattices and standard topological contexts developed in [Ha92] and [Ha93]. We recall some definitions and basic facts, for other definitions and results we refer to [GW96].

By (X, τ) we denote a topological space, where X is the underlying set and $\mathcal{T}$ is the family of all closed sets of that space. We start with a triple $\mathbb{K}^{\mathcal{T}} := ((G, \rho), (M, \sigma), I)$ consisting of two topological spaces $(G, \rho), (M, \sigma)$ and a binary relation $I \subseteq G \times M$. For $A \subseteq G$ and $B \subseteq M$, we define two derivations by

$$A' := \{m \in M |\ gIm \text{ for all } g \in A\} \text{ and}$$

$$B' := \{g \in G |\ gIm \text{ for all } m \in B\}.$$

These form a Galois-connection which gives rise to a complete lattice

$$\underline{\mathcal{B}}(\mathbb{K}^{\mathcal{T}}) := \{(A, B) |\ A \subseteq G, B \subseteq M, A' = B, B' = A\}$$

which is known as the **concept lattice** of the **context** $\mathbb{K}^{\mathcal{T}}$. The elements of $\underline{\mathcal{B}}(\mathbb{K}^{\mathcal{T}})$ are called (formal) **concepts**. If (A, B) is a concept of $\mathbb{K}^{\mathcal{T}}$, the sets A and B are called the **extent** and the **intent** of the concept (A, B). For two concepts, the relation subconcept–superconcept is given by

$$(A, B) \leq (C, D) \Leftrightarrow A \subseteq B (\Leftrightarrow B \supseteq B).$$

A **closed concept** is a concept consisting in each component of a closed set with respect to the corresponding topology. The set of all closed concepts is denoted by

$$\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) := \{(A,B) \in \underline{\mathcal{B}}(\mathbb{K}^{\mathcal{T}})|\ A \in \rho \text{ and } B \in \sigma\}.$$

The triple $\mathbb{K}^{\mathcal{T}} := ((G,\rho),(M,\sigma),I)$ is called a **topological context** if the following two conditions are satisfied:

(i) $A \in \rho \Rightarrow A'' \in \rho; B \in \sigma \Rightarrow B'' \in \sigma$.

(ii) $\mathcal{S}_\rho := \{A \subseteq G|\ (A,A') \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})\}$ is a subbasis of ρ and
$\mathcal{S}_\sigma := \{B \subseteq M|\ (B,B') \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})\}$ is a subbasis of σ.

Under these assumptions, $\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with the induced order is a 0–1 lattice in which infima and suprema can be described as follows

$$(A_1,B_1)\wedge(A_2,B_2) = (A_1 \cap A_2,(B_1 \cup B_2)'');$$

$$(A_1,B_1)\vee(A_2,B_2) = ((A_1 \cup A_2)'',B_1 \cap B_2).$$

For each $g \in G$, the concept $\gamma g := (g'',g')$ is called the **object concept** of G and for each $m \in M$, the concept $\mu m := (m',m'')$ is called the **attribute concept** of m. We call a context **clarified** if $g,h \in G$ with $g' = h'$ implies $g = h$ and $m,n \in M$ with $m' = n'$ implies $m = n$. A clarified context is called **reduced** if each object concept is completely join-irreducible and each attribute concept is completely meet-irreducible. For each context $\mathbb{K} := (G,M,I)$, every $g \in G$ and $m \in M$, we define:

$$g \swarrow m \Leftrightarrow g \not{I} m \text{ and } (g' \subset h' \Rightarrow m \in h');$$

$$g \nearrow m \Leftrightarrow g \not{I} m \text{ and } (m' \subset n' \Rightarrow g \in n');$$

$$g \swarrow\!\!\!\!\nearrow m \Leftrightarrow g \swarrow m \text{ and } g \nearrow m.$$

We call two contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ **isomorphic** if there are bijective maps $\alpha : G_1 \to G_2$ and $\beta : M_1 \to M_2$ such that for all $g \in G_1$ and $m \in M_1$, the following condition is fulfilled:

$$gI_1m \Leftrightarrow \alpha(g)I_2\beta(m).$$

For each $H \subseteq G$ and $N \subseteq M$, the context $(H,N,I \cap (H \times N))$ is called a **subcontext** of $\mathbb{K}$. This subcontext is **compatible** if $(A,B) \in \underline{\mathcal{B}}(\mathbb{K})$ implies $(A \cap H, B \cap N) \in \underline{\mathcal{B}}(H,N,I \cap (H \times N))$.

Proposition 2.1. *A subcontext* $(H, N, I \cap (H \times N))$ *of* $\mathbb{K}$ *is compatible if and only if*

$$\Pi_{H,N} : \underline{\mathcal{B}}(\mathbb{K}) \to \underline{\mathcal{B}}(H, N, I \cap (H \times N)) \text{ with } (A, B) \mapsto (A \cap H, B \cap N)$$

is a surjective complete lattice homomorphism.

A subcontext $(H, N, I \cap (H \times N))$ of a purified context $\mathbb{K}$ is called **arrow–closed** if for $h \in H$, the relation $h \swarrow m$ implies $m \in N$ and for $n \in N$, the relation $g \nearrow n$ implies $g \in H$.

A topological context is called a **standard topological context** if in addition the following hold:

(R) $\mathbb{K}^{\mathcal{T}}$ is reduced;

(S) $gIm \Rightarrow \exists (A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with $g \in A$ and $m \in B$;

(Q) $(\complement I, (\rho \times \sigma)_{|\complement I})$ is a quasicompact space where $\complement I := (G \times M) \backslash I$ and $\rho \times \sigma$ denotes the product topology on $G \times M$.

Let now L be a 0–1 lattice. A nonempty lattice filter F of L is called an **I-maximal filter** [Ur78] if there exists a nonempty lattice ideal I of L such that $F \cap I = \emptyset$ and every proper superfilter $\tilde{F} \supset F$ already contains an element of I. We denote the set of all I-maximal proper filters of L by $\mathcal{F}_0(L)$. Dually, the set $\mathcal{I}_0(L)$ of all F-maximal ideals is introduced. The dual space of L, called the standard topological context of L is defined by

$$\mathbb{K}^{\mathcal{T}}(L) := ((\mathcal{F}_0(L), \rho_0), (\mathcal{I}_0(L), \sigma_0), \Delta)$$

where $F \Delta I :\Leftrightarrow F \cap I \neq \emptyset$ and the topologies ρ_0 and σ_0 are given by the subbasis

$$\mathcal{S}_{\rho_0} := \{F_a |\ a \in L\}; F_a := \{F \in \mathcal{F}_0(L) |\ a \in F\},$$

$$\mathcal{S}_{\sigma_0} := \{I_a |\ a \in L\}; I_a := \{I \in \mathcal{I}_0(L) |\ a \in I\}.$$

$\mathbb{K}^{\mathcal{T}}(L)$ is the reduced context of all filters and ideals of L and it is a standard topological context. The 0–1 lattice L is isomorphic to $\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$ via the isomorphism $\iota_A : L \to \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)); \quad \iota_A(a) = (F_a, I_a)$.

Conversely, every standard topological context $\mathbb{K}^{\mathcal{T}}$ is isomorphic to $\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ via the pair of homeomorphisms

$$\psi_{\mathbb{K}^{\mathcal{T}}} : G \to \mathcal{F}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})),\ g \mapsto \{(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) | g \in A\},$$

$$\phi_{\mathbb{K}^{\mathcal{T}}} : M \to \mathcal{I}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})),\ m \mapsto \{(A,B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) | m \in B\}.$$

Let $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ be standard topological contexts. A pair of maps (α, β) with $\alpha : G_1 \to G_2$ and $\beta : M_1 \to M_2$ is called a **context embedding of** $\mathbb{K}_1^{\mathcal{T}}$ **into** $\mathbb{K}_2^{\mathcal{T}}$ if the contexts $\mathbb{K}_1^{\mathcal{T}}$ and $((\alpha(G_1), \rho_{2|\alpha(G_1)}), (\beta(M_1), \sigma_{2|\beta(M_1)}), I_2 \cap (\alpha(G_1) \times \beta(M_1)))$ are isomorphic as topological contexts with respect to (α, β).

If $\mathbb{K}^{\mathcal{T}}$ is a topological context, a subcontext $((H, \rho_{|H}), N, \sigma_{|N}), I \cap H \times N))$ is called **weakly compatible** if

$$(A,B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \Rightarrow (A \cap H, B \cap N) \in \underline{\mathcal{B}}(H, N, I \cap (H \times N)).$$

A context embedding (α, β) between two standard topological contexts $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ is called a **standard embedding of** $\mathbb{K}_1^{\mathcal{T}}$ **into** $\mathbb{K}_2^{\mathcal{T}}$ if the following conditions are satisfied:

(a) $((\alpha(G_1), \rho_{2|\alpha(G_1)}), (\beta(M_1), \sigma_{2|\beta(M_1)}), I_2 \cap (\alpha(G_1) \times \beta(M_1)))$ is a weakly compatible subcontext of $\mathbb{K}_2^{\mathcal{T}}$;

(b) For $(A,B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$, there exists $(C,D) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$ such that

$$(\alpha(A), \beta(B)) = ((C \cap \alpha(G_1)), (D \cap \beta(M_1))).$$

Preimages of I-maximal filters (resp. ideals) are not maximal again, so we have to define appropriate morphisms between standard topological contexts to improve a categorical dual equivalence between the category of bounded lattices and the category of standard topological contexts.

A multivalued function $F : X \to Y$ from a set X to a set Y is a binary relation $F \subseteq X \times Y$ such that $pr_X(F) = X$, where pr_X denotes the projection onto X. For $A \subseteq X$ and $B \subseteq Y$ we define

$$\begin{aligned} FA &:= pr_Y(F \cap (A \times Y)) = \{y \in Y \mid (a,y) \in F \text{ for some } a \in A\}; \\ F^{-1}B &:= pr_X(F \cap (X \times B)) = \{x \in X \mid (x,b) \in F \text{ for some } b \in B\}; \\ F^{[-1]}B &:= \{x \in X \mid Fx \subseteq B\}. \end{aligned}$$

Note that $FA = \bigcup_{a \in A} Fa$ and $F^{-1}B = \bigcup_{b \in B} F^{-1}b$. If $F : X \to Y$ and $G : Y \to Z$ are multivalued functions their relational product

$$G \circ F := \{(x,z) \in X \times Z \mid (x,y) \in F \text{ and } (y,z) \in G \text{ for some } y \in Y\}$$

is a multivalued function from X to Z.

We shall call a **multivalued standard morphism** from $\mathbb{K}_1^{\mathcal{T}}$ to $\mathbb{K}_2^{\mathcal{T}}$ a pair $(R, S) : \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$, where $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ are standard topological contexts, R is a multivalued function from G_1 to G_2 and S is a multivalued function from M_1 to M_2 satisfying the following conditions:

(i) $(R^{[-1]}A, S^{[-1]}B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$ for every $(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$;

(ii) $Rg = Rg'' = \overline{Rg}$ for every $g \in G_1$ and

$Sm = Sm'' = \overline{Sm}$ for every $m \in M_1$.

Remark 1. Condition (ii) can be understood in lattice theoretical terms. Every element $g \in G_1$ correspond to exactly one I-maximal filter of $\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$. The demand on Rg to be a closed extent means that Rg corresponds to a lattice filter of $\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$.

Every multivalued standard morphism induces a 0-1 lattice homomorphism and vice versa. In order to make this assignment functorial we have to modify the relational composition of multivalued standard morphisms, since the relational composition of two multivalued standard morphisms is not necessarily a multivalued standard morphism.

Let $(R_1, S_1) : \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$ and $(R_2, S_2) : \mathbb{K}_2^{\mathcal{T}} \to \mathbb{K}_3^{\mathcal{T}}$ be multivalued standard morphisms between standard topological contexts. We define

$$(R_2, S_2)\Box(R_1, S_1) := (R_2\Box R_1, S_2\Box S_1)$$

where

$$(R_2\Box R_1)g_1 := ((R_2 \circ R_1)g_1)'' \text{ and } (S_2\Box S_1)m_1 := ((S_2 \circ S_1)m_1)''$$

and $\circ$ denotes the relational product, i.e.

$$(R_2 \circ R_1)g_1 := \{g_3 \in G_3 \mid g_3 \in R_2 g_2 \text{ for some } g_2 \in R_1 g_1\} \text{ and, dually,}$$

$$(S_2 \circ S_1)g_1 := \{m_3 \in M_3 \mid m_3 \in S_2 m_2 \text{ for some } m_2 \in S_1 m_1\}.$$

The class of all standard topological contexts together with the multivalued standard morphisms with $\Box$ as composition yields a category which is dually equivalent to the category of 0-1 lattices with 0-1 lattice homomorphisms.

3. Standard Topological Contexts with Pseudometric

If we want to represent several classes of ordered algebraic structures, standard topological contexts are the best tool to do this. On the other hand, if we want to approximate objects by their attributes in a given formal context, we have to modify this approach towards a topological formal concept analysis and to investigate a generalization on formal contexts of the classical notion of a metric (see [Sa00b]).

Definition 3.1. Let G and M be two sets. We call **pseudometric between** G and M a map $d : G \times M \to \mathbb{R}$ satisfying the following rectangle condition:

$$(R) \qquad d(g,m) \leq d(g,n) + d(h,m) + d(h,n),\ \ g,h \in G,\ \ m,n \in M,$$

and, for every $g \in G$ and $\varepsilon > 0$, there is an attribute $m \in M$ with $d(g,m) < \varepsilon$. Dually, for every $m \in M$ and every $\varepsilon > 0$, there is an object $g \in G$ with $d(g,m) < \varepsilon$.

If d is a pseudometric between G and M, then $d^\vee : G \times G \to \mathbb{R}$ defined by $d^\vee(g,h) := \inf_{m \in M}(d(g,m) + d(h,m)), g,h \in G$ is a pseudometric on G. Dually, $d^\wedge : M \times M \to \mathbb{R}$ defined by $d^\wedge(m,n) := \inf_{g \in G}(d(g,m) + d(g,n)), m,n \in M$ is a pseudometric on M.

Definition 3.2. A formal context $\mathbb{K} := (G,M,I)$ is called a **pseudometric context** if there is a pseudometric $d : G \times M \to \mathbb{R}$ between G and M satisfying the following two conditions, called ε-conditions:

$$\forall \varepsilon \geq 0\ \forall g \in G\ \exists m \in M:\ \ gIm \text{ and } d(g,m) < \varepsilon,$$

$$\forall \varepsilon \geq 0\ \forall m \in M\ \exists g \in G:\ \ gIm \text{ and } d(g,m) < \varepsilon.$$

We shall call a pseudometric context **standard** if $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\} = 0$ for every concept (A,B) of $\mathbb{K}$.

Let $\mathbb{K} := (G,M,I;d)$ be a pseudometric context. We consider G and M as topological spaces with the pseudometric topology given by $d^\vee$ and $d^\wedge$, respectively. As we have seen before, a topological context is a triple (G,M,I) where G and M are topological spaces and $I \subseteq G \times M$ is a binary relation between them, satisfying some compatibility conditions with the topologies on G and M. If in addition the topological context satisfies some separation and compactness properties, it is called a standard topological context and it was shown by G. Hartung that the category of standard topological contexts is dual equivalent to that of 0–1 lattices.

A question arises naturally: what are the connections between pseudometric contexts and standard topological ones? As in the topological algebra, there are two possibilities. On the one hand, we can demand that the topologies on G and M are generated by the pseudometrics induced by d. We shall call such a context a **compatible pseudometric context**, i.e., $\mathbb{K}^{\mathcal{T}} := ((G, \mathcal{T}_{d^\vee}), (M, \mathcal{T}_{d^\wedge}, I)$ is a standard topological context where $\mathcal{T}_{d^\vee}$ denotes the pseudometric topology on G, and $\mathcal{T}_{d^\wedge}$ denotes the pseudometric topology on M.

On the other hand, we can simply consider **standard topological contexts with pseudometric**, i.e., no compatibility conditions between the topologies on G and M and the given pseudometric are required.

In the following we shall investigate the categories of standard topological contexts with a compatible pseudometric or not and we shall take a look whether an extension of the Hartung duality to the pseudometric case is possible or not. Beside of this extension, we are mainly interested on how some properties of pseudometric lattice homomorphisms are reflected into the properties of standard topological context morphisms.

Remark 2. Before starting these investigations, remember that $\overline{\mathbb{R}}$ can be understood as the concept lattice of the context $(\mathbb{Q}, \mathbb{Q}, \leq)$. Since $(\mathbb{Q}, d)$ is a metric space, where d is the natural metric on $\mathbb{Q}$, then $(\mathbb{Q}, \mathbb{Q}, \leq)$ is a metric context. The metric on $\overline{\mathbb{R}} \simeq \underline{\mathcal{B}}(\mathbb{Q}, \mathbb{Q}, \leq)$ can be understood as a kind of "reflection" of the contextual metric d on $(\mathbb{Q}, \mathbb{Q}, \leq)$ on the concept lattice. The following Lemma ([Sa00b]) synthesizes this phenomenon in its full generality.

Lemma 3.1. *Let $\mathbb{K} := (G, M, I; \rho)$ be a pseudometric context. The map $d : \underline{\mathcal{B}}(G, M, I) \times \underline{\mathcal{B}}(G, M, I) \to \mathbb{R}$, defined by*

$$d((A, B), (C, D)) := \max\{\rho(A, D), \rho(C, B)\},$$

is a pseudometric on $\underline{\mathcal{B}}(G, M, I)$, the concept lattice of $\mathbb{K}$.

Let $\mathbb{K}_1 := (G_1, M_1, I_1; d_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2; d_2)$ be standard topological contexts with pseudometric. A morphism between them is defined as a pair of multivalued functions $R : G_1 \to G_2$ and $S : M_1 \to M_2$ (i.e., R and S are binary relations on $G_1 \times G_2$ and $M_1 \times M_2$, respectively, satisfying $\mathrm{pr}_{G_1} R = G_1$ and $\mathrm{pr}_{M_1} S = M_1$) with the properties:

(i) $(R^{[-1]}A, S^{[-1]}B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$ for every $(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$;

(ii) $Rg = Rg'' = \overline{Rg}$ for every $g \in G_1$, and

$Sm = Sm'' = \overline{Sm}$ for every $m \in M_1$;

(iii) $d_2(Rg, Sm) \leq d_1(g, m)$ for every $g \in G_1$ and $m \in M_1$.

The pair (R, S) will be called multivalued pseudometric morphism.

Lemma 3.2. *The class of all standard topological contexts with pseudometric together with the multivalued pseudometric morphisms between them yields a category denoted by* $\mathbf{T}opCon_d$*. The class of all compatible pseudometric contexts is a full subcategory* **CCM** *of* $\mathbf{T}opCon_d$*.*

Proof. Let $\mathbb{K} := (G, M, I; d)$ be a pseudometric context, the identity (R_e, S_e) where $R_e : G \to G$ and $S_e : M \to M$ are defined by $R_e g := g''$ and $S_e m := m''$, respectively, is a multivalued standard morphism. Since $d(R_e g, S_e m) = d(g'', m'') \leq d(g, m)$ for every $g \in G$ and $m \in M$, we conclude that (R_e, S_e) is a multivalued pseudometric morphism, i.e., the identity in the category of standard topological contexts is also identity in $\mathbf{T}opCon_d$.

Let now $(R_1, S_1) : (\mathbb{K}_1, d_1) \to (\mathbb{K}_2, d_2)$ and $(R_2, S_2) : (\mathbb{K}_2, d_2) \to (\mathbb{K}_3, d_3)$ be multivalued metric morphisms. We shall prove that their composition $(R_2, S_2)\square(R_1, S_1) := (R_2\square R_1, S_2\square S_1)$ is again a multivalued pseudometric morphism.

By definition of $\square$, $(R_2\square R_1)g_1 := ((R_2 \circ R_1)g_1)''$ for every $g_1 \in G_1$. Dually, we have $(S_2\square S_1)m_1 := ((S_2 \circ S_1)m_1)''$ for every $m_2 \in M_2$. The following holds:

$$\begin{aligned} d_3((R_2\square R_1)g_1, (S_2\square S_1)m_1) &= d_3(((R_2 \circ R_1)g_1)'', ((S_2 \circ S_1)m_1)'') \\ &\leq d_3((R_2 \circ R_1)g_1, (S_2 \circ S_1)m_1) \\ &= d_3(R_2(R_1 g_1), S_2(S_1 m_1)) \\ &= d_3(\{g_3 \in G_3 \mid g_3 \in R_2 g_2 \text{ for some } g_2 \in R_1 g_1\}, \\ &\qquad \{m_3 \in M_3 \mid m_3 \in R_2 m_2 \text{ for some } m_2 \in R_1 m_1\}). \end{aligned}$$

But (R_1, S_1) and (R_2, S_2) are multivalued pseudometric morphisms, and so

$$d_2(R_1 g_1, S_1 m_1) \leq d_1(g_1, m_1) \text{ for every } g_1 \in G_1 \text{ and } m_1 \in M_1$$

$$d_3(R_2 g_2, S_2 m_2) \leq d_2(g_2, m_2) \text{ for every } g_2 \in G_2 \text{ and } m_2 \in M_2.$$

For every $g_2 \in R_1 g_1$ and $m_2 \in S_1 m_1$, we have $d_3(R_2 g_2, S_2 m_2) \leq d_2(g_2, m_2)$ which implies

$$\begin{aligned} d_3(R_2(R_1 g_1), S_2(S_1 m_1)) &= d_3(\bigcup_{g_2 \in R_1 g_1} R_2 g_2, \bigcup_{m_2 \in S_1 m_1} S_2 m_2) \\ &= \inf\{d_3(g_3, m_3) \mid g_3 \in \bigcup_{g_2 \in R_1 g_1} R_2 g_2, \\ &\qquad m_3 \in \bigcup_{m_2 \in S_1 m_1} S_2 m_2\} \\ &\leq \inf\{d_3(g_3, m_3) \mid g_3 \in R_2 g_2, m_3 \in S_2 m_2\}, \\ &\qquad \text{for every } g_2 \in R_1 g_1 \text{ and every } m_2 \in S_1 m_1 \\ &= d_3(R_2 g_2, S_2 m_2) \text{ for every } g_2 \in R_1 g_1, m_2 \in S_1 m_1 \\ &\leq d_2(g_2, m_2) \text{ for every } g_2 \in R_1 g_1, m_2 \in S_1 m_1. \end{aligned}$$

Hence $d_3(R_2(R_1 g_1), S_2(S_1 m_1)) \leq \inf\{d_2(g_2, m_2) \mid g_2 \in R_1 g_1, m_2 \in S_1 m_1\} = d_2(R_1 g_1, S_1 m_1) \leq d_1(g_1, m_1)$. Since associativity is naturally inherited, the above condition completes our proof. □

Lemma 3.3. *If (L, ρ) is a 0-1-lattice and $\rho : L \times L \to \mathbb{R}$ is a pseudometric on L, then $\mathbb{K}^{\mathcal{T}}(L)$, the standard topological context of L, is a standard pseudometric context.*

Proof. As we have seen before, to every 0-1-lattice L, we can define a standard topological context denoted by $\mathbb{K}^{\mathcal{T}}(L) := (\mathcal{F}_0(L), \mathcal{I}_0(L), \Delta)$ where $F \Delta I :\Leftrightarrow F \cap I \neq \emptyset$.

We shall define a pseudometric $d : \mathcal{F}_0(L) \times \mathcal{I}_0(L) \to \mathbb{R}$ on $\mathbb{K}^{\mathcal{T}}(L)$, by $d(F, I) := \inf\{\rho(g, m) \mid g \in F, m \in I\} = \rho(F, I)$. Let $F \in \mathcal{F}_0(L)$. Then $d(F, F') = d(F, \{I \in \mathcal{I}_0(L) \mid F \cap I \neq \emptyset\}) = 0$ since $d(F, I) = 0$ for every $I \in \mathcal{I}_0(L)$ with $F \cap I \neq \emptyset$, i.e., $I \in F'$. Let us prove the rectangle inequality for d. Let $(F, I), (F, J), (K, J)$ and (K, I) in $\mathcal{F}_0(L) \times \mathcal{I}_0(L)$ be arbitrary chosen. We have to prove that

$$d(F, I) \leq d(F, J) + d(K, J) + d(K, I).$$

Then

$$\begin{aligned} d(F,I) &= \inf\{\rho(f,i) \mid f \in F, i \in I\} \\ &\leq \inf\{\rho(f,i)+\rho(k,j)+\rho(k,i) \mid f \in F, i \in I\} \text{ for } j \in J, k \in K \\ &\leq \inf\{\rho(f,i)+\rho(k,j)+\rho(k,i) \mid f \in F, i \in I, j \in J, k \in K\} \\ &\leq \inf\{\rho(f,i) \mid f \in F, i \in I\} + \inf\{\rho(k,j) \mid k \in K, j \in J\} \\ &+ \inf\{\rho(k,i) \mid k \in K, i \in I\} \\ &= d(F,J)+d(K,J)+d(K,I). \end{aligned}$$

If $(A,B) \in \underline{\mathcal{B}}(\mathbb{K}^{\mathcal{T}}(L))$, we conclude that $d(A,B)=0$ by the definition of the incidence relation and of the set distance; hence $(\mathbb{K}^{\mathcal{T}}(L),d)$ is a standard pseudometric context. □

Remark 3. Since $d^{\vee}$ is the pseudometric induced on $\mathcal{F}_0(L)$ by d, we have $d^{\vee}(F_1,F_2)=\inf\{d(F_1,I)+d(F_2,I) \mid I \in \mathcal{I}_0\}$; hence we conclude that generally, the pseudometric $d^{\vee}$ does not induce the topology on $\mathcal{F}_0(L)$ (which has as subbasis of closed sets the family $\{F_a \mid a \in L\}$, where $F_a := \{F \in \mathcal{F}_0(L) \mid a \in F\}$). Indeed, for two filters $F_1, F_2 \in \mathcal{F}_0(L)$ we will often be able to find an ideal $I \in \mathcal{I}_0(L)$ which has a non empty intersection to F_1 and F_2 and therefore $d^{\vee}(F_1,F_2)=0$.

In the following we shall consider only the case where $\mathbb{K}^{\mathcal{T}}$ is a standard topological context with pseudometric. Let $\mathbb{K}^{\mathcal{T}} := (G,M,I)$ be a standard topological context and let $(P_\varepsilon)_{\varepsilon \geq 0}$ be a family of non empty relations, $P_\varepsilon \subseteq G \times M$ with $\varepsilon \geq 0$, which are satisfying the following conditions:

(M') $$P_\varepsilon(x,y) \to P_\delta(x,y),\ \delta \geq \varepsilon$$

$$P_\varepsilon \wedge P_\delta(k,z) \wedge P_\eta(k,y) \to P_{\varepsilon+\delta+\eta}(x,y).$$

(M_∞) $$\forall \delta \geq \varepsilon : P_\delta(x,y) \to P_\varepsilon(x,y), \varepsilon \geq 0.$$

(M_0) $$\forall g\ \forall \varepsilon\ \exists m : P_\varepsilon(x,y).$$

A morphism $(R,S) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon \geq 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon \geq 0}$ has to satisfy the following compatibility condition

(C) $$P_\varepsilon(g_1,m_1) \Rightarrow \exists g_2 \in Rg_1\ \exists m_2 \in Sm_1 : Q_\varepsilon(g_2,m_2), \varepsilon \geq 0.$$

Lemma 3.4. *The class of multicontexts* $(\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0}$*, where* $\mathbb{K}^{\mathcal{T}}$ *is a standard topological context and* $(P_\varepsilon)_{\varepsilon\geq 0}$ *a family of binary relations on* $G \times M$ *satisfying* (M')*,* (M_∞) *and* (M_0)*, together with the multivalued standard morphisms which are satisfying condition* (C) *yields a category denoted by* $\mathbf{TopCon}_\varepsilon$*.*

Proof. Let $\varepsilon \geq 0$ be arbitrary chosen and $\mathbb{K}^{\mathcal{T}}$ be a standard topological context. The identity morphism $(R_\varepsilon, S_\varepsilon) : \mathbb{K}^{\mathcal{T}} \to \mathbb{K}^{\mathcal{T}}$ where $R_e g := g''$ and $S_e m := m''$ is obviously satisfying condition (C). Let us now consider $(R_1, S_1) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0}$ and $(R_2, S_2) : (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_3^{\mathcal{T}}, R_\varepsilon)_{\varepsilon\geq 0}$ two morphisms between objects in $\mathbf{TopCon}_\varepsilon$. We shall prove that their composition in **Topcon**, i.e., $(R_2, S_2)\Box(R_1, S_1) = (R_2\Box R_1, S_2\Box S_1)$ is a morphism between objects of $\mathbf{TopCon}_\varepsilon$, i.e.,

$$P_\varepsilon(g_1, m_1) \Rightarrow \exists g_3 \in (R_2\Box R_1)g_1 \; \exists m_3 \in (S_2\Box S_1)m : R_\varepsilon(g_3, m_3).$$

Since the given morphisms are satisfying condition (C) and, by definition, $(R_2\Box R_1)g_1 := ((R_2 \circ R_1)g_1)''$ and $(S_2\Box S_1)m_1 = ((S_2\Box S_1)m_1)''$, we conclude that $P_\varepsilon(g_1, m_1)$ implies the existence of a $g_2 \in R_1 g_1$ and an $m_2 \in S_1 m_1$ with $Q_\varepsilon(g_2, m_2)$, which implies the existence of elements $g_3 \in R_2(R_1 g_1)$ and $m_3 \in S_2(S_1 m_1)$ with $R_\varepsilon(g_3, m_3)$. Since $g_3 \in R_2(R_1 g_1) \subseteq (R_2 \circ R_1)g_1''$ and $m_3 \in S_2(S_1 m_1) \subseteq (S_2 \circ S_1)m_1''$ our proof is complete. □

Proposition 3.5. *The category* $\mathbf{TopCon_d}$ *of standard topological contexts with pseudometric is equivalent to* $\mathbf{TopCon}_\varepsilon$*.*

Proof. Let $F : \mathbf{TopCon_d} \to \mathbf{TopCon}_\varepsilon$ defined on objects by $F(\mathbb{K}^{\mathcal{T}}, d) = (\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0}$ and on morphisms in an obvious way. The functor F is obviously faithful. Let $(R, S) : F(\mathbb{K}_1^{\mathcal{T}}, d_1) \to F(\mathbb{K}_2^{\mathcal{T}}, d_2)$ be a morphism of $\mathbf{TopCon}_\varepsilon$, that means $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0}$. We only have to prove that $d_2(Rg_1, Sm_1) \leq d_1(g_1, m_1)$ for every $g_1 \in G_1$ and $m_1 \in M_1$.

Let $g_1 \in G_1$ and $m_1 \in M_1$ be arbitrary chosen and define $\varepsilon := d_1(g_1, m_1)$. It follows that $P_\varepsilon(g_1, m_1)$ and by (C), there is a $g_2 \in Rg_1$ and an $m_2 \in Sm_1$ with $Q_\varepsilon(g_2, m_2)$, i.e., $d_2(g_2, m_2) \leq \varepsilon$. Hence $d_2(Rg_1, Sm_1) = \inf\{d_2(g_2, m_2) \mid g_2 \in Rg_1, m_2 \in Sm_1\} \leq \varepsilon$, i.e., F is full.

If $(\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0}$ is an object in $\mathbf{TopCon}_\varepsilon$, we define a pseudometric $d : G \times M \to [0, +\infty]$ by $d(g, m) := \inf\{\delta \geq 0 \mid P_\delta(g, m)\}$. As we have seen in the precedent section, d is well-defined and is a pseudometric between G and M. Hence, we have found an object $(\mathbb{K}^{\mathcal{T}}, d)$ in $\mathbf{TopCon_d}$ with $F(\mathbb{K}^{\mathcal{T}}, d) \cong (\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon \geq 0}$ which concludes the proof. □

We shall now investigate whether the duality between standard topological contexts and 0–1 lattices can be extended to the metric case. Even this will not be generally true, it is of interest to investigate how some properties of morphisms between pseudometric 0–1 lattices are reflected in the category $\mathbf{T}opCon_d$ as equivalent properties of standard multivalued morphisms.

Let $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ be a morphism in $\mathbf{TopCon_d}$. This morphism induces $f_{RS} : (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}), \rho_2) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}), \rho_1)$ a 0–1 lattice homomorphism defined by $f_{RS}(A, B) = (R^{[-1]}A, S^{[-1]}B)$ where $R^{[-1]}A = \{g_1 \in G_1 \mid Rg_1 \subseteq A\}$ and $S^{[-1]}B = \{m_1 \in M_1 \mid Sm_1 \subseteq D\}$. By definition,

$$\begin{aligned}\rho_1(f_{RS}(A, B), f_{RS}(C, D)) &= \rho_1((R^{[-1]A}, S^{[-1]}B), (R^{[-1]}C, S^{[-1]}D) \\ &= \max\{d_1(R^{[-1]}A, S^{[-1]}D), d_1(R^{[-1]}C, S^{[-1]}B)\}.\end{aligned}$$

The morphism (R, S) is in $\mathbf{TopCon_d}$, i.e., it satisfies $d_2(Rg_1, Sm_1) \leq d_1(g_1, m_1)$ for every $g_1 \in G_1$ and $m_1 \in M_1$; hence $d_1(g_1, m_1) \geq d_2(Rg_1, Sm_1) \geq d_2(A, D)$ for every $g_1 \in R^{[-1]}A$ and every $m \in S^{[-1]}D$, and so $d_1(R^{[-1]}A, S^{[-1]}D) \geq d_2(A, D)$. By a similar calculus, we obtain $d_1(R^{[-1]}C, S^{[-1]}B) \geq d_2(C, B)$ which implies the following inequality:

$$\rho_1(f_{RS}(A, B), f_{RS}(C, D)) \geq \rho_2((A, B), (C, D)).$$

As we can see, condition (iii) has as consequence that on the "lattice side" the mappings are not the usually contractions. To avoid this, we will impose for context morphisms the following compatibility condition

(iv) $\quad d_1(R^{-1}g_2, S^{-1}m_2) \leq d_2(g_2, m_2)$ for every $g_2 \in G_2$ and $m_2 \in M_2$.

In fact, let $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ be such a morphism and consider $f_{RS} : (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}), \rho_2) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}), \rho_1)$ the induced 0–1 lattice homomorphism. Then

$$\rho_1(f_{RS}(A, B), f_{RS}(C, D)) = \max\{d_1(R^{[-1]}A, S^{[-1]}D), d_1(R^{[-1]}C, S^{[-1]}B)\}.$$

Since $R^{[-1]}A = \{g_1 \in G_1 \mid Rg_1 \subseteq A\}$ where $A \subseteq G_2$, we have $R^{[-1]}A = \bigcup_{T\subseteq A, T\neq\emptyset} \bigcap_{a\in T} R^{-1}a$ and $S^{[-1]}D = \bigcup_{T'\subseteq D, T'\neq\emptyset} \bigcap_{d\in T'} S^{-1}d$. It follows

$$\begin{aligned} d_1(R^{[-1]}A, S^{[-1]}D) &= d_1(\bigcup_{T\subseteq A, T\neq\emptyset} \bigcap_{a\in T} R^{-1}a, \bigcup_{T'\subseteq D, T'\neq\emptyset} \bigcap_{d\in T'} S^{-1}d) \\ &\leq d_1(\bigcap_{a\in T} R^{-1}a, \bigcap_{d\in T'} S^{-1}d) \end{aligned}$$

for every non empty subsets $T \subseteq A$ and $T' \subseteq D$. Choose $T := \{a\}$ with $a \in A$ and $T' := \{d\}$ with $d \in D$, then

$$d_1(R^{[-1]}A, S^{[-1]}D) \leq d_1(R^{-1}a, S^{-1}d) \leq d_2(a, d)$$

for every $a \in A$ and $d \in D$; hence $d_1(R^{[-1]}A, S^{[-1]}D) \leq d_2(A, D)$.

Analogously, we are able to prove that $d_1(R^{[-1]}C, S^{[-1]}B) \leq d_2(C, B)$; hence f_{RS} is a contraction.

Remark 4. The dual inequality to (iv), i.e.,

$$d_1(R^{-1}g_2, S^{-1}m_2) \geq d_2(g_2, m_2)$$

implies (iii). Indeed, for every $g_1 \in G_1$ and $m_1 \in M_1$, we have

$$d_2(Rg_1, Sm_1) = \inf d_2(g_2, m_2) \leq \inf d_1(R^{-1}g_2, S^{-1}m_2) \leq d_1(g_1, m_1).$$

Lemma 3.6. *The class of standard topological contexts with metric together with all multivalued standard morphisms between them satisfying condition (iv) yields a category denoted by* $\mathbf{TopCon'_d}$.

Proof. The unit morphism $(R_e, S_e) : (\mathbb{K}^{\mathcal{T}}, d) \to (\mathbb{K}^{\mathcal{T}}, d)$ defined by $R_e g := g''$ and $S_e m := m''$ satisfies (C'), since $R_e^{-1}h = \{g \in G \mid h \in g''\}$ and $S_e^{-1}n = \{m \in M \mid n \in m''\}$. In particular, $h \in R^{-1}h$ and $n \in S^{-1}n$, hence $d(R_e^{-1}h, S_e^{-1}n) \leq d(h, n)$ for every $h \in G$ and $n \in M$.

Let $(R_1, S_1) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ and $(R_2, S_2) : (\mathbb{K}_2^{\mathcal{T}}, d_2) \to (\mathbb{K}_3^{\mathcal{T}}, d_3)$ be two multivalued standard morphisms which are satisfying (iv). We shall prove that their composition $(R_2, S_2)\Box(R_1, S_1) = (R_2\Box R_1, S_2\Box S_1) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ is also satisfying (iv).

For $g_3 \in G_3$ and $m_3 \in M_3$ we have

$$\begin{aligned}(R_2 \Box R_1)^{-1} g_3 &= \{g_1 \in G_1 \mid g_3 \in ((R_2 \circ R_1) g_1)''\} \\ &\supseteq \{g_1 \in G_1 \mid g_3 \in R_2(R_1 g_1)\} \\ &= \{g_1 \in G_1 \mid g_1 \in (R_2 \circ R_1)^{-1} g_3\} \\ &= \{g_1 \in G_1 \mid g_1 \in R_1^{-1}(R_2^{-1} g_3))\} \\ &= \{g_1 \in G_1 \mid \exists g_2 \in G_2 : (g_2, g_1) \in R^{-1}, (g_3, g_2) \in R_2^{-1}\} \\ &= \{g_1 \in G_1 \mid \exists g_2 \in G_2 : \ g_1 \in R_1^{-1} g_2, g_2 \in R_2^{-1} g_3\}.\end{aligned}$$

In a similar manner, we are able to prove that

$$(S_2 \Box S_1)^{-1} m_3 \supseteq \{m_1 \in M_1 \mid \exists m_2 \in M_2 : \ m_1 \in S^{-1} m_2, m_2 \in S_2^{-1} m_3\},$$

hence

$$\begin{aligned}&d_1((R_2 \Box R_1)^{-1} g_3, (S_2 \Box S_1)^{-1} m_3) \\ &\qquad \le d_1(\{g_1 \in G_1 \mid \exists g_2 \in G_2 : \ g_1 \in R_1^{-1} g_2, \ g_2 \in R_2^{-1} g_3\}, \\ &\qquad\qquad \{m_1 \in M_1 \mid \exists m_2 \in M_2 : \ m_1 \in S_1^{-1} m_2, \ m_2 \in S_2^{-1} m_2\}) \\ &\qquad = \inf\{d_1(g_1, m_1) \mid \exists g_2 \in G_2 : \ g_1 \in R_1^{-1} g_2, \ g_2 \in R_2^{-1} g_3, \\ &\qquad\qquad \exists m_2 \in M_2 : \ m_1 \in S_1^{-1} m_2, m_2 \in S_2^{-1} m_3\} \\ &\qquad = \inf\{d_1(R_1^{-1} g_2, S_2^{-1} m_2) \mid g_2 \in R_2^{-1} g_3, m_2 \in S_2^{-1} m_3\} \\ &\qquad \le \inf\{d_2(g_2, m_2) \mid g_2 \in R_2^{-1} g_3, m_2 \in S_2^{-1} m_3\} \\ &\qquad = d_2(R_2^{-1} g_3, S_2^{-1} m_3) \le d_3(g_3, m_3).\end{aligned}$$

□

If we split again the pseudometric $d : G \times M \to [0, +\infty]$ in the family of relations $(P_\varepsilon)_{\varepsilon \ge 0}$, compatibility condition (C') for $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon \ge 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon \ge 0}$ changes to

$$(C') \qquad Q_\varepsilon(g_2, m_2) \Rightarrow \exists g_1 \in R^{-1} g_2 \ \exists m_1 \in S^{-1} m_2 : P_\varepsilon(g_1, m_1).$$

Lemma 3.7. *The class of multicontexts $(\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon \ge 0}$, where $\mathbb{K}^{\mathcal{T}}$ is a standard topological context and P_ε is a binary relation between the object and the attribute* *set of $\mathbb{K}^{\mathcal{T}}$ satisfying axioms (M'), (M_0) and (M_∞) is the object class of a category denoted $\mathbf{TopCon}'_\varepsilon$, whose morphisms are the multivalued standard morphisms which are satisfying (C').*

Proof. The identity (R_e, S_e) is obviously a morphism in $\mathbf{TopCon}'_\varepsilon$. Let $(R_1, S_1) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0}$ and $(R_2, S_2) : (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_3^{\mathcal{T}}, R_\varepsilon)_{\varepsilon\geq 0}$ be morphisms in $\mathbf{TopCon}'_\varepsilon$. Their composition is again in $\mathbf{TopCon}'_\varepsilon$. To see this, let $g_3 \in G_3$ and $m_3 \in M_3$ with $R_\varepsilon(g_3, m_3)$. Then there are $g_2 \in R_2^{-1} g_3$ and $m_2 \in S_2^{-1} m_3$ with $Q_\varepsilon(g_2, m_2)$, hence there are $g_1 \in R_1^{-1} g_2$ and $m_1 \in S_1^{-1} m_2$ with $P_\varepsilon(g_1, m_1)$. Now,

$$g_1 \in R_1^{-1} g_2 \subseteq (R_2 \circ R_1)^{-1} g_3 \subseteq (R_2 \Box R_1)^{-1} g_3,$$

$$m_1 \in S_1^{-1} m_2 \subseteq (S_2 \circ S - 1)^{-1} m_3 \subseteq (S_2 \Box S_1)' m_3$$

which completes the proof. □

Proceeding in a similar manner as before, we can prove the following Lemma:

Lemma 3.8. *The categories $\mathbf{TopCon}'_{\mathbf{d}}$ and $\mathbf{TopCon}'_\varepsilon$ are equivalent.*

Let us denote the functors from the Hartung duality by $\mathbf{T}$ and $\mathbf{S}$. The functor $\mathbf{T} : Lat \to TopCon$ is defined on objects by $\mathbf{T}(L) = \mathbb{K}^{\mathcal{T}}(L)$ and for any morphism $f : L_1 \to L_2$, the image of f by $\mathbf{T}$ is a multivalued standard morphism $\mathbf{T}f : \mathbb{K}^{\mathcal{T}}(L_2) \to \mathbb{K}^{\mathcal{T}}(L_1)$ defined by $\mathbf{T}f = (R_f, S_f)$ where

$$R_f \subseteq \mathcal{F}_0(L_2) \times \mathcal{F}_0(L_1),\ (F_2, F_1) \in R_f \Leftrightarrow f^{-1}(F_2) \subseteq F_1,$$

$$S_f \subseteq \mathcal{I}_0(L_2) \times \mathcal{I}_0(L_1),\ (I_2, I_1) \in S_f \Leftrightarrow f^{-1}(I_2) \subseteq I_1.$$

The functor $\mathbf{S}$ is defined on objects by $\mathbf{S}(\mathbb{K}^{\mathcal{T}}) := \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ and for every multivalued standard morphism $(R, S) : \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$, the image of (R, S) by $\mathbf{S}$ is a 0-1-lattice homomorphism $\mathbf{S} : \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}) \to \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$ is defined by $\mathbf{S}(R, S) := f_{RS}$ where $f_{RS}(A, B) := (R^{[-1]}A, S^{[-1]}B)$ for all closed concepts (A, B) in $\mathbb{K}_2^{\mathcal{T}}$.

As we have seen before, the restriction of $\mathbf{S}$ to the metric case, $\mathbf{S} : TopCon_d \to Lat_d$, is well-defined, the morphisms in $\mathbf{L}at_d$ being the expansive mappings with respect to the correspondent pseudometric of a lattice L in $\mathbf{L}at_d$.

Consider now $f : (L_1, \rho_1) \to (L_2, \rho_2)$ satisfying $\rho_2(f(x), f(y)) \geq \rho_1(x, y)$ for every $x, y \in L_1$. Then $d_1(F_1, I_1) \leq d_2(f(F_1), f(I_1)) \leq d_2(F_2, I_2)$ for all $F_2 \in R_f^{-1} F_1$ and $I_2 \in S_f^{-1} I_1$ since $(F_2, F_1) \in R_f$ is equivalent to $F_2 \subseteq f(F_1)$, and $(I_2, I_1) \in S_f$ is equivalent to $I_2 \subseteq f(I_1)$. It follows that $d_1(F_1, I_1) \leq d_2(R_f^{-1} F_1, S_f^{-1} I_1)$. Moreover,

$d_1(R_f F_2, S_f I_2) = \inf d_1(F_1, I_1) \leq \inf d_2(f(F_1), f(I_1)) \leq d_2(F_2, I_2)$ which proves that if we consider expansive mappings as morphisms between metric lattices, both (iii) and the dual of (iv) are satisfied, i.e., the restriction of $\mathbf{T}$ to the metric case $\mathbf{T} : Lat_d \to TopCon_d$ is also well-defined.

Example:

Let us consider the following lattice the metric being labeled on its Hesse-diagram, the morphism f being given by arrows:

As one can easily check, f is an expansion. Now

$$\begin{aligned} R_f^{-1} &= \{F_2 \in \mathcal{F}_0(L_2) \mid (F_2, F_1) \in R_f\} \\ &= \{F_2 \in \mathcal{F}_0(L_2) \mid f^{-1}(F_2) \subseteq F_1\} \\ &= \{F_2, F_3, [1)\} \end{aligned}$$

and, dually, $S_f^{-1}(I_1) = \{I_2, I_3, (0]\}$. As we can easily see, $d_2(F_2, I_2) = 4, d_1(F_1, I_1) = 3$ i.e., the dual of (iv) (and so (iii)) is satisfied.

Remark 5. While dealing with mappings between (pseudo)metric spaces, contractions are more often used as expansive maps. We are considering expansions in this section in order to give a necessary condition that the isomorphisms $\iota : (L, \rho) \to$

$(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), d)$ and $(R_\alpha, S_\beta) : (\mathbb{K}^{\mathcal{T}}, d) \to (\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})), \rho)$ belong to the considered categories.

Unfortunately, the map $\iota : (L, \rho) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), d)$ even if an isomorphism in **Lat**, fails to be an isomorphism in $\mathbf{L}at_d$ (i.e., a bijective isometry). Indeed, $d(\iota a, \iota b) = d((F_a, I_a), (F_b, I_b)) = \max\{\rho(F_a, I_b), \rho(F_b, I_b)\} \leq \rho(a, b)$ for every $a, b \in L$. Obviously, ι can not generally be an isometry and so the categories $\mathbf{L}at_d$ and $\mathbf{T}opCon_d$ fails to be dual equivalent.

On the other hand, consider $(\mathbb{K}^{\mathcal{T}}, d)$ a standard topological context in $\mathbf{T}opCon_d$. Then $(R_{\alpha_1}, S_{\beta_1}) : (\mathbb{K}^{\mathcal{T}}, d) \to (\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})), \rho)$ is a multivalued pseudometric morphism. Indeed, consider $F \in \mathcal{F}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ and $I \in \mathcal{I}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$. Then for every $g \in R_\alpha^{-1}F$ and $m \in S_\beta^{-1}I$,

$$\rho(F, I) = \rho(\alpha(g)'', \beta(m)'') \leq \rho(\alpha(g), \beta(m)) \leq \sigma((A, B), (C, D))$$

for every $(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with $g \in A$, and every $(C, D) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with $m \in D$. Since $\sigma((A, B), (C, D)) = \max\{d(A, D), d(C, B)\}$, choose for $(A, B) := (G, \emptyset)$ and for $(C, D) := (\emptyset, M)$. Then $d(C, B) = 0$ and $d(A, D) \leq d(g, m)$. It follows that $\rho(F, I) \leq d(R_\alpha^{-1}F, S_\beta^{-1}I)$, i.e., the dual of (iv) which then implies (iii).

Remark 6. Generally, (R_α, S_β) can not be an isomorphism in $\mathbf{T}opCon_d$ since from the above calculus we deduce that $\rho(F, I) = 0$ for every $F \in \mathcal{F}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ and $I \in \mathcal{I}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$, and so **T** and **S** are failing even to be adjoint.

Even if contractions or expansions between pseudometric spaces have proved their usefulness several times, for example in establishing a duality between pseudometric complete lattices and pseudometric contexts, the condition that a map is expansive is too strong in order to obtain a categorical duality or an adjoint situation between the restrictions of the two functors **T** and **S** to the (pseudo)metric case. More general, the most used morphisms between pseudometric spaces are continuous maps. In the following we shall define the analogous concept in the case of standard topological contexts with pseudometric.

Definition 3.3. The multivalued standard morphism $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ between two standard topological contexts with pseudometric is called **pseudometric continuous** if for every $\varepsilon > 0$ and every $g_2 \in G_2$, there is a $\delta > 0$ so that for every $m_2 \in M_2$, with $d_2(g_2, m_2) < \delta$, we have $d_1(R^{-1}g_2, S^{-1}m_2) < \varepsilon$.

The morphism (R, S) is called **pseudometric uniformously continuous** if for every $\varepsilon > 0$, every $g_2 \in G_2$ and $m_2 \in M_2$ there is a $\delta > 0$ such that $d_2(g_2, m_2) < \delta$ implies $d_1(R^{-1}g_2, S^{-1}m_2) < \varepsilon$.

We shall denote the category of standard topological contexts with pseudometric with pseudometric continuous morphisms by $\mathbf{T}C_d$ and that of pseudometric lattices with continuous lattice homomorphism by $\mathbf{L}_d$ and we shall prove that the restrictions of the well-known functors $\mathbf{T}$ and $\mathbf{S}$ of the Hartung duality, $\mathbf{T} : L_d \to TC_d$ and $\mathbf{S} : TC_d \to L_d$, respectively, are well-defined. We have seen before that the object maps of $\mathbf{T}$ and $\mathbf{S}$, respectively, are well-defined.

Proposition 3.9. *For every pseudometric continuous standard multivalued morphism* $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$*, the induced lattice morphism* $\mathbf{S}(R, S) := f_{RS} : (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}), \rho_2) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}), \rho_1)$ *defined by* $f_{RS}(A, B) := (R^{[-1]}A, S^{[-1]}B)$ *is a continuous mapping with respect to the metric topology of the correspondent concept lattices.*

Proof. Consider $\varepsilon > 0$ and $(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$. Then, for every $a \in A$, there is a $\delta > 0$ such that for every $m_2 \in M_2$ with $d_2(a, m_2) < \delta$, we have that $d_1(R^{-1}a, S^{-1}m_2) < \varepsilon$. Take a closed concept $(C, D) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$ whose distance to (A, B) is less than δ, i.e., $d_2(A, D) < \delta$ and $d_2(C, B) < \delta$. Then, for the chosen $a \in A$, we shall find a $d \in D$ with $d_2(a, d) < \delta$, hence $d_1(R^{-1}a, S^{-1}d) < \varepsilon$. Since $d_1(R^{[-1]}A, S^{[-1]}D) \leq d_1(R^{-1}a, S^{-1}d)$, it follows that $d_1(R^{[-1]}A, S^{[-1]}D) < \varepsilon$. The same holds for $d_1(R^{[-1]}C, S^{[-1]}B)$ concluding that $\rho_1(f_{RS}(A, B), f_{RS}(C, D)) < \varepsilon$, i.e., f_{RS} is continuous. □

Remark 7. Analogous arguments shows that if (R, S) is a pseudometric uniformously continuous morphism, then the induced 0-1-lattice homomorphism f_{RS} is uniformously continuous too.

Proposition 3.10. *For every continuous pseudometric 0-1-lattice homomorphism* $f : (L_1, \rho_1) \to (L_2, \rho_2)$*, the induced multivalued standard morphism* $(R_f, S_f) : (\mathbb{K}^{\mathcal{T}}(L_2), d_2) \to (\mathbb{K}^{\mathcal{T}}(L_1), d_1)$ *is pseudometric continuous.*

Proof. Consider $F_1 \in \mathcal{F}_0(L_1)$ and $I_1 \in \mathcal{I}_0(L_1)$. Then, for every $x \in F_1$, there is a $\delta > 0$ such that for every $y \in L_1$ with $\rho_1(x, y) < \delta$, we have $\rho_2(f(x), f(y)) < \varepsilon$. Then, for every $y \in I_1$ with $\rho_1(x, y) < \delta$, we have $d_1(F_1, I_1) < \delta$. By the definition of $R_f^{-1}F_1$ and $S_f^{-1}I_1$, we obtain that $d_2(R_f^{-1}F_1, S_f^{-1}I_1) < \varepsilon$. □

Remark 8. If f is uniformously continuous then (R_f, S_f) is pseudometric uniformously continuous too.

The above results say nothing else than the restriction of the two functors to the pseudometric continuous case are well-defined.

Consider now $\iota : (L, \rho) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), d)$ defined by $\iota a := (F_a, I_a)$. In order to prove the continuity of ι we have to show that for every $\varepsilon > 0$ and every $a \in L$, there is a $\delta > 0$ such that for every $b \in L$ with $\rho(a, b) < \delta$, we have $d((F_a, I_a), (F_b, I_b)) < \varepsilon$.

By definition, $d((F_a, I_a), (F_b, I_b)) := \max\{\sigma(F_a, I_b), \sigma(F_b, I_b)\} < \varepsilon$ if and only if $\sigma(F_a, I_b) = \inf \rho(F, I) < \varepsilon$. It follows that there is an $F \in F_a$ and an $I \in I_b$ with $\rho(F, I) < \varepsilon$.

On the other hand, since $a \in F$ and $b \in I$, we conclude that $\rho(F, I) \leq \rho(a, b)$. Choose $\delta := \varepsilon$, hence ι is continuous. Moreover, since δ do not depend on $a \in L$, we can conclude that ι is uniformously continuous. As one can easily see, ι is not a homeomorphism, hence the categories $\mathbf{T}C_d$ and $\mathbf{L}_d$ are not dual equivalent.

The same holds for $(R_\alpha, S_\beta) : (\mathbb{K}^{\mathcal{T}}, d) \to (\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})), \rho)$. Since the pseudometric ρ on $\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ is the trivial one, and since there are several examples of pseudometrics on $\mathbb{K}^{\mathcal{T}}$ which are not trivial, we conclude that $(R_{\alpha^{-1}}, S_{\beta^{-1}})$, i.e., the inverse of (R_α, S_β) in the category $\mathbf{T}opCon_d$ is pseudometric continuous (and even more, pseudometric uniformously continuous), but (R_α, S_β) itself is generally not pseudometric continuous.

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KELVIN-HELMHOLTZ INSTABILITY OF RIVLIN-ERICKSEN VISCOELASTIC FLUID IN POROUS MEDIUM

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Abstract. Kelvin-Helmholtz instability of Rivlin-Ericksen elastico-viscous fluid in porous medium is considered. The case of two uniform streaming fluids separated by a horizontal boundary is considered. It is found that for the special case when perturbations in the direction of streaming are ignored, perturbation transverse to the direction of streaming are found to be unnafected by the presence of streaming. In every other direction, a minimum value of wave-number has been found and the system is unstable for all wave-numbers greater than this minimum wave number.

1. **Introduction**

When two superposed fluids flow one over the other with a relative horizontal velocity, the instability of the plane interface between the two fluids, when it occurs in this instance, is known as 'Kelvin-Helmholtz instability'. The instability of the plane interface separating two uniform superposed streaming fluids, under varying assumptions of hydrodynamics, has been discussed in the celebrated monograph by Chandrasekhar [1]. The experimental observation of the Kelvin-Helmholtz instability has been given by Francis [2]. The medium has been assumed to be non-porous.

With the growing importance of viscoelastic fluids in modern technology and industries and the investigations on such fluids are desirable. The Rivlin-Ericksen fluid is one such viscoelastic fluid. Many research workers have paid their attention towards the study of Rivlin-Ericksen fluid. Johri [3] has discussed the viscoelastic Rivlin-Ericksen incompressible fluid under time-dependent pressure gradient. Sisodia and Gupta [4] and Srivastava and Singh [5] have studied the unsteady flow of a dusty

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elastico-viscous Rivlin-Ericksen fluid through channel of different cross-sections in the present of the time dependent pressure gradient. Recently, Sharma and Kumar [6] have studied the thermal instability of a layer of Rivlin-Ericksen elastico-viscous fluid acted on by a uniform rotation and found that rotation has a stabilizing effect and introduces oscillatory modes in the system.

The flow through a porous medium has been of considerable interest in recent years particularly among geophysical fluid dynamicists. An example in the geophysical context is the recovery of crude oil from the pores of reservoir rocks. A great number of applications in geophysics may be found in a recent book by Phillips [7]. The gross effect when the fluid slowly percolates through the pores of the rock is given by Darcy's law. As a result, the usual viscous term in the equation of motion of Rivlin-Ericksen fluid is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\overrightarrow{q}\right]$, where μ and μ' are the viscosity and viscoelasticity of the Rivlin-Ericksen fluid, k_1 is the medium permeability and $\overrightarrow{q}$ is the Darcian (filter) velocity of the fluid. Generally, it is accepted that comets consists of a dusty 'snowball' of a mixture of frozen gases which, in the process of their journey, changes from solid to gas and vice-versa. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical contex (McDonnel [8]). The instability of the plane interface between two uniform superposed and streaming fluids through porous medium has been investigated by Sharma and Spanos [9]. More recently, Sharma et al. [10] have studied the thermosolutal convection in Rivlin-Ericksen fluid in porous medium in the presence of uniform vertical magnetic field.

Keeping in mind the importance of non-Newtonian fluids in modern technology and industries and various applications mentioned above, Kelvin-Helmholtz instability of Rivlin-Ericksen viscoelastic fluid in porous medium has been considered in the present paper.

2. **Formulation of the problem and perturbation equations**

The initial stationary state, whose stability we wish to examine is that of an incompressible elastico-viscous Rivlin-Ericksen fluid in which there is a horizontal streaming in the x-direction with velocity $U(z)$ through a homogeneous, isotropic porous medium. The character of the equilibrium of this initial state is determined by supposing that the system is slightly disturbed and then following its further evolution.

Let $p, \rho, g, v, v', \overrightarrow{q}(U(z), 0, 0)$ denote, respectively, the pressure, density, acceleration due to gravity, kinematic viscosity, kinematic viscolasticity, and velocity of Rivlin-Ericksen viscoelastic fluid. This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity ε and medium permeability k_1 and interfacial tension effect is ignored. Then the equations of motion, continuity and incompressibility for the Rivlin-Ericksen elastico-viscous fluid through a porous medium are given by

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \overrightarrow{q}}{\partial t} + \frac{1}{\varepsilon}(\overrightarrow{q}\cdot\nabla)\overrightarrow{q}\right] = -\nabla p + \rho\overrightarrow{g} - \frac{\rho}{k_1}\left(v + v'\frac{\partial}{\partial t}\right)\overrightarrow{q}, \tag{1}$$

$$\nabla\cdot\overrightarrow{q} = 0, \tag{2}$$

$$\varepsilon\frac{\partial\rho}{\partial t} + (\overrightarrow{q}\cdot\nabla)\rho = 0. \tag{3}$$

Let $\delta p, \delta\rho$ and $\overrightarrow{u}(u, v, w)$ denote the perturbations in pressure p, density ρ and velocity $\overrightarrow{q}(U(z), 0, 0)$ respectively. Then, the linearized perturbation equations of fluid layer become

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \overrightarrow{u}}{\partial t} + \frac{1}{\varepsilon}(\overrightarrow{q}\cdot\nabla)\overrightarrow{u} + \frac{1}{\varepsilon}(\overrightarrow{u}\cdot\nabla)\overrightarrow{q}\,\widehat{i}\right] = -\nabla\delta p + \overrightarrow{g}\,\delta p - \frac{\rho}{k_1}\left(v + v'\frac{\partial}{\partial t}\right)\overrightarrow{u}, \tag{4}$$

$$\nabla\cdot\overrightarrow{u} = 0, \tag{5}$$

$$\left[\varepsilon\frac{\partial}{\partial t} + (\overrightarrow{q}\cdot\nabla)\right]\delta p = -w\frac{d\rho}{dz}. \tag{6}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is of the form

$$\exp[i(k_x x + k_y y + nt)], \tag{7}$$

where n is the growth rate, $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number and k_x, k_y are horizontal wave numbers.

Substituting for $\delta\rho$, Eq.(4) with the help of Eqs.(5),(6) and expression (7) yields

$$\left[\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right]\overrightarrow{u} + \frac{\rho}{\varepsilon^2}w(DU)\widehat{i} = -\nabla\delta p + i\overrightarrow{g}\frac{w(D\rho)}{\varepsilon n + k_x U}, \tag{8}$$

where $\widehat{i}$ is unit vector in the x-direction and $D = d/dz$.

Writing the three component equations of (8) and eliminating u, v and δp with the help of (5), we obtain

$$D\left[\left\{\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right\} Dw - \frac{ik_x\rho}{\varepsilon^2}(DU)w\right] -$$

$$-k^2\left[\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right] w = igk^2(D\rho)\frac{w}{\varepsilon n + k_x U}. \qquad (9)$$

3. Two uniform streaming fluids separated by a horizontal boundary

Consider the case when two superposed streaming fluids of uniform densities ρ_1 and ρ_2, uniform viscosities μ_1 and μ_2 and uniform viscoelasticities μ'_1 and μ'_2 are separated by a horizontal boundary at $z = 0$. The subscript 1 and 2 distinguish the lower and the upper fluids respectively.

The density ρ_2 of the upper fluid is taken to be less than the density ρ_1 of the lower fluid so that, in the absence of streaming, the configuration is stable, and the porous medium throughout is assumed to be isotropic and homogeneous. Let the two fluids be streaming with constant velocities U_1 and U_2. Then in each of the two regions of constant ρ, μ, μ' and U, Eq.(9) reduces to

$$(D^2 - k^2)w = 0. \qquad (10)$$

The boundary conditions to be satisfied here are:

(a) Since U is discontinuous at $z = 0$, the uniqueness of the normal displacement of any point on the interface, according to Eq.(8), implies that

$$\frac{w}{\varepsilon n + k_x U}, \qquad (11)$$

must be continuous at an interface.

(b) Integrating Eq.(9) between $0 - \eta$ and $0 + \eta$ and passing to the limit $\eta = 0$, we obtain, in view of (11), the jump condition

$$\Delta_0\left[\left\{\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right\} Dw - \frac{ik_x\rho}{\varepsilon^2}(DU)w\right] = igk^2\Delta_0(\rho)\left(\frac{w}{\varepsilon n + k_x U}\right)_0 \qquad (12)$$

(for $z = 0$) while the equation valid everywhere else ($z \neq 0$) is

$$D\left[\left\{\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right\} Dw - \frac{ik_x\rho}{\varepsilon^2}(DU)w\right] -$$

$$-k^2\left[\frac{i\rho}{\varepsilon^2}(\varepsilon n+k_xU)+\frac{\rho}{k_1}(v+inv')\right]w=igk^2(D\rho)\frac{w}{\varepsilon n+k_xU}. \tag{13}$$

Here $\Delta_0(f)=f(z_0+0)-f(z_0-0)$ is the jump which a quantity experiences at the interface $z=z_0$; and the subscript 0 distinguish the value a quantity, known to be continuous at an interface, takes at the interface $z=z_0$.

The general solution of Eq.(10) is a linear combination of the integrals e^{+kz} and e^{-kz}. Since $\dfrac{w}{\varepsilon n+k_xU}$ must be continuous on the surface $z=0$ and w cannot increase exponentially on either side of the interface, the solutions appropriate for two regions are

$$w_1=A(\varepsilon n+k_xU_1)e^{+kz},\quad (z<0) \tag{14}$$

$$w_2=A(\varepsilon n+k_xU_2)e^{-kz},\quad (z>0). \tag{15}$$

Applying the boundary condition (12) to the solutions (14)-(15), we obtain the dispersion relation

$$\left[1+\frac{\varepsilon}{k_1}(\alpha_1v_1'+\alpha_2v_2')\right]n^2+$$
$$+\left[\frac{2k_x}{\varepsilon}(\alpha_1U_1+\alpha_2U_2)+\frac{k_x}{k_1}(\alpha_1v_1'U_1+\alpha_2v_2'U_2)-\frac{i\varepsilon}{k_1}(\alpha_1v_1+\alpha_2v_2)\right]n+$$
$$+\left[\frac{k^2}{\varepsilon^2}(\alpha_1U_1^2+\alpha_2U_2^2)-\frac{ik_x}{k_1}(\alpha_1v_1U_1+\alpha_2v_2U_2)-gk(\alpha_1-\alpha_2)\right]=0, \tag{16}$$

where

$$\alpha_{1,2}=\frac{\rho_{1,2}}{\rho_1+\rho_2},\quad v_{1,2}=\frac{\mu_{1,2}}{\rho_{1,2}},\quad v_{1,2}'=\frac{\mu_{1,2}'}{\rho_{1,2}}.$$

$v_1\left(=\dfrac{\mu_1}{\rho_1}\right)$, $v_1'\left(=\dfrac{\mu_1'}{\rho_1}\right)$, $v_2\left(=\dfrac{\mu_2}{\rho_2}\right)$ and $v_2'\left(=\dfrac{\mu_2'}{\rho_2}\right)$ are the kinematic viscosities and kinematic viscoelasticities of the lower and upper fluids respectively.

Equation (16) yields

$$in=-\left[+\frac{\varepsilon}{k_1}(\alpha_1v_1+\alpha_2v_2)+\frac{2ik_x}{\varepsilon}(\alpha_1U_1+\alpha_2U_2)+\frac{ik_x}{k_1}(\alpha_1v_1'U_1+\alpha_2v_2'U_2)\right]\pm$$
$$\pm\left\{\left[\frac{\varepsilon}{k_1}(\alpha_1v_1+\alpha_2v_2)\right]^2-\frac{4ik_x\alpha_1\alpha_2}{k_1}(v_1-v_2)(U_1-U_2)+\right.$$
$$+\frac{4k_x^2\alpha_1\alpha_2}{\varepsilon k_1}(v_2'U_1-v_1'U_2)(U_1-U_2)-\frac{2i\varepsilon k_x}{k_1^2}[(\alpha_1^2v_1v_1'U_1+\alpha_2^2v_2v_2'U_2)+$$
$$+\alpha_1\alpha_2(v_1v_2'U_1+v_1'v_2U_2)+\alpha_1\alpha_2(U_1-U_2)(v_1v_2'-v_1'v_2)]+$$
$$+\left[\frac{k_x}{k_1}(\alpha_1v_1'U_1+\alpha_2v_2'U_2)\right]^2+\frac{4\alpha_1\alpha_2k_x^2}{\varepsilon^2}(U_1-U_2)^2-$$

$$-4gk(\alpha_1-\alpha_2)\left[1+\frac{\varepsilon}{k_1}(\alpha_1 v_1'+\alpha_2 v_2')\right]\Bigg\}^{\frac{1}{2}}. \tag{17}$$

Some cases of interest are now considered.

(a) When $k_x = 0$, equation (17) yields

$$in = -\frac{\varepsilon}{k_1}(\alpha_1 v_1+\alpha_2 v_2) \pm \left\{\left[\frac{\varepsilon}{k_1}(\alpha_1 v_1+\alpha_2 v_2)\right]^2 -\right.$$

$$\left.-4gk(\alpha_1-\alpha_2)\left[1+\frac{\varepsilon}{k_1}(\alpha_1 v_1'+\alpha_2 v_2')\right]\right\}^{\frac{1}{2}}. \tag{18}$$

Here we assume kinematic viscosities v_1, v_2 and kinematic viscoelasticities v_1', v_2' of the two fluids to be equal i.e., $v_1 = v_2 = v$, $v_1' = v_2' = v'$. However, any of the essential features of the problem are not obscured by this simplifying assumption. Eq.(18), then, becomes

$$in = -\frac{\varepsilon v}{k_1} \pm \left[\left(\frac{\varepsilon v}{k_1}\right)^2 + 4gk(\alpha_2-\alpha_1)\left\{1+\frac{\varepsilon v'}{k_1}\right\}\right]^{\frac{1}{2}}. \tag{19}$$

(i) **Unstable case**

For the potentially unstable configuration ($\rho_2 > \rho_1$), it is evident from Eq.(19) that one of the values of in is positive which means that the perturbations grow with time and so the system is unstable.

(ii) **Stable case**

For the potentially stable configuration ($\rho_2 < \rho_1$), Eq.(19) yields that both the values of in are either real, negative or complex conjugates with negative real parts implying stability of the system.

It is interesting to note from above that for the special case when perturbations in the direction of streaming are ignored ($k_x = 0$), the system is unstable for potentially unstable configuration and the system is stable for potentially stable configuration and not depending upon kinematic viscoelasticity, medium porosity and medium permeability. This is in contrast to the case of Walters' viscoelastic fluid B', where the system can be stable or unstable depending upon kinematic viscoelaticity, medium porosity and medium permeability (Sharma et al. [11]).

It is also clear from Eq.(18), that for the special case when perturbations in the direction of streaming are ignored ($k_x = 0$), the perturbation transverse to the direction of streaming ($k_y \neq 0$) are unaffected by the presence of streaming.

(b) In every other direction, instability occurs when

$$\frac{\alpha_1\alpha_2 k_x^2}{\varepsilon^2}(U_1 - U_2)^2 > gk(\alpha_1 - \alpha_2). \tag{20}$$

The kinematic viscosities v_1 and v_2 and the kinematic viscoelasticies v_1' and v_2' of two fluids here are assumed to be equal (let $v_1 = v_2 = v$, $v_1' = v_2' = v'$), but this simplifying assumption does not obscure any of the essential features of the problem.

Thus for a given difference in velocity $(U_1 - U_2)$ and for a given direction of the wave-vector $\overrightarrow{k}$, instability occurs for all wave numbers.

$$k > \left[\frac{g\varepsilon^2(\alpha_1 - \alpha_2)}{\alpha_1\alpha_2(U_1 - U_2)^2\cos^2\theta}\right], \tag{21}$$

where θ is the angle between the direction of $\overrightarrow{k}(k_x, k_y, 0)$ and $\overrightarrow{U}(U, 0, 0)$, i.e. $k_x = k\cos\theta$. Hence, for a given velocity differences $(U_1 - U_2)$, instability occurs for the least wave number when $\overrightarrow{k}$ is in the direction of $\overrightarrow{U}$ and this minimum wave number; k_{min}, is given by

$$k_{min} = \left[\frac{g\varepsilon^2(\alpha_1 - \alpha_2)}{\alpha_1\alpha_2(U_1 - U_2)^2}\right]. \tag{22}$$

For $k > k_{min}$, the system is unstable.

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BOOK REVIEWS

John J. Benedetto, *Harmonic Analysis and Applications*, Studies in Advanced Mathematics, CRC Press, Boca Raton-New York-London-Tokyo 1997, xix+336 pp., ISBN 0-8493-7879-6.

The present book is a textbook and an essay, the author goal being "to present harmonic analysis at level that exhibits its vitality, intricacy and simplicity, power, elegance, and usefulness" (from the Preface). The author restricts to classical harmonic analysis, the fundamental components being the the trigonometric functions, with emphasis on *analysis,* meaning determination of harmonics or components of a given function, and *synthesis*, meaning the reconstruction of this function in terms of its components. The methods are primarily those of real analysis with very little complex analysis, the development being done within the framework of spaces L^1 and L^2. The prerequisites for the reading of the book are a basic course in real analysis as, e.g., J. Benedetto, "Real Variable and Integration", B.G. Teubner, Stuttgart 1976. Although abstract harmonic analysis (invariant measures on locally compact groups, Banach algebras, representation theory) are not considered, the treatment has a Banach algebra flavor, and is a substantial part of the harmonic analysis on a commutative locally compact group.

A selection of the book (the corresponding numbers of definitions and propositions are listed in Prologue I) was used by the author as material for upper undergraduate courses, taught for many years to students in engineering, physics, computer science, and mathematics. The exercises at the end of each chapter range from elementary to difficult and from theoretical to computational and/or computed oriented (using MATLAB programs). The first 30 exercises of each chapter are appropriate for Course I.

The book contains many examples from engineering and physics and very interesting historical comments on the evolution of the ideas in this very fertile areas of mathematics, which shaped the development of mathematics in the 20th century (measure theory, topology, set theory, functional analysis).

The book is, in essence, on classical harmonic analysis, including careful proofs of the basic theorems, but the exposition is done in a way to provide perspectives of many topics, some of them (e.g. Wiener's Generalized Harmonic Analysis) being extensively treated. Due to these perspectives, of lengthy historical comments and exercises, the book can serve also as a textbook for more advanced courses than Course I. Also, the limitation to classical harmonic analysis is compensated to some extent by a serious bibliography, referenced at appropriate junctures in the text.

Written by a leading specialist in harmonic analysis, with over than 100 published papers (including 9 books), the present book is a very good text on harmonic analysis, its applications and evolution, and can be used as a textbook as well as an essay for students and as general reference for engineers, mathematicians, physicists, and other people using harmonic analysis.

S. Cobzaş

Joseph A. Cima and William T. Ross, *The Backward Shift on the Hardy Space*, Mathematical Surveys and Monographs Vol. 79, xi+ 199 pp., American Mathematical Society 2000, ISBN: 0-8218-2083-4.

The book is devoted to the study of invariant subspaces of the backward shift operator on the Hardy space H^p of analytic functions on the open unit disc $\mathbb{D} = \{|z| < 1\}$. The backward shift operator B is defined by

$$Bf = \frac{f - f(0)}{z} = a_1 + a_2 z + a_3 z^2 + ...$$

for $f = a_0 + a_1 z + a_2 z^2 + ... \in H^p$.

As the backward shift operator on H^2 (the Hilbert case) is presented in detail in Nikolskiĭ's book *Treatise on the Shift Operator*, Springer Verlag, Berlin-New York 1986, the authors of the present book focus on the Banach case ($p \in [1, \infty)$) and the Fréchet case ($p \in (0, 1)$). The characterization of the invariant subspaces of the backward operator on H^p for $1 \leq p < \infty$ was settled down by R. Douglas, H. S. Shapiro and A. Shields, Annale Institut Fourier (Grenoble) **20** (1970), 37-76. The case

$p \in (0, 1)$ was solved by A.B. Aleksandrov, Investigations on Linear Operators and the Theory of Functions IX, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI), **92** (1979), 7-29, in a paper which was never translated from its original Russian and using a quite complicated technique–distribution theory and Coiffman's atomic decomposition for the Hardy space. The authors gather up these results together with the necessary background material which is surveyed in appropriate places. The reader is supposed to be acquainted with the basic of functional analysis (at the level of Rudin's book), complex function theory and H^p spaces (Duren's and Garnett's books), and harmonic analysis (Stein's book).

The main results and the technique used for their proofs are briefly, but in a very clear manner, explained in the first chapter of the book entitled *Overview*.

A good idea on the organization of the book is given by the headings of the rest of its chapters: 2. *Classical boundary value results*; 3. *The Hardy space on the disk*; 4. *The Hardy spaces on the upper-half plane*; 5. *The backward shift on H^p for $p \in [1, \infty)$*; 6. *The backward shift on H^p for $p \in (0, 1)$*.

Written by two eminent specialists and combining techniques from functional analysis, operator theory, harmonic analysis, real and complex analysis, this beautiful book appeals to a large audience, meaning people interested in the topics listed above. It can be used also as a textbook for advanced graduate or post-graduate courses.

Stefan Cobzaş

David L. Jagerman, *Difference Equations with Applications to Queues*, Pure and Applied Mathematics Series, Vol. 233, M. Dekker, Inc., Basel - New York 2000, xi+241 pages, ISBN: 0-8247-9007-3.

This monograph presents a theory of difference and functional equations with continuous argument, based on a generalization of the Riemann integral introduced by N.E. Nörlund in his famous monograph published in 1924. This approach permits greater flexibility in constructing solutions and approximate solving nonlinear first order equations by a variety of of methods, including an adaptation of the Lie-Gröbner theory.

Ch. 1, *Operators and Functions*, is a general overview of the operators and functions which are important in the difference calculus. Ch. 2, *Generalities on Difference Equations*, considers the genesis of difference and gives a number of exercises. Casorati's determinant is introduced, and Heyman's theorem and a theorem of of Milne-Thompson on the asymptotic behavior of the linear independence of solutions are proved.

Chapters 3 and 4, *Nörlund Sums: Part one* and *Part two*, respectively, contain the basic properties of Nörlund sums as well as representations obtained by means of Euler-Maclaurin expansions. Fourier expansions and the extension to the complex plane of the Euler-Maclaurin representation are also studied. Some examples are included.

Ch. 5, *The First Order Difference Equation*, as the title shows, deal with first order difference equations, both linear and nonlinear. The method of Truesdell for differential-difference equations is discussed and applied to a queuing model. Simultaneous first-order nonlinear equations are solved approximately.

In Ch. 6, *The Linear Equation with Constant Coefficients*, beside the study of linear equations with constant coefficients, some methods of solving partial difference equations are also included. Application is made to the probability $P(t)$ that an $M/M/1$ queue be empty, given that it is initially empty. An asymptotic development for $P(t)$ is obtained for large t and a practical approximation is constructed.

The final chapter, Ch. 7,*Linear Difference Equations with Polynomial Coefficients*, describes the linear difference equations with polynomial coefficients. The method of depression of the order, the Casorati's determinant and Heyman's theorem, are some of the tools used in this chapter. However, the main technique for solution is based on the π, ρ operator method of Boole and Milne-Thompson, which constructs the solution in terms of factorial series. Application is made to the last-come-first-served queue with exponential reneging; in particular, the Laplace transform is obtained for the waiting time distribution.

J. Sándor

Kenneth L. Kuttler, *Modern Analysis,* Studies in Advanced Mathematics, CRC Press, Boca Raton-New York-London-Tokyo 1998, 572 pp., ISBN 0-8493-7166-X.

This is an advanced course on real and abstract analysis, meaning topology, functional analysis, measure theory and integration, and applications.

The first two chapters, 1. *Set theory and topology* and 2. *Compactness and continuous functions,* contain the basic of general topology including Urysohn's lemma, Stone-Weierstrass theorem and Arzela-Ascoli compactness criterium. Tichonoff's theorem on the compactness of the product is proved in the chapter on locally convex spaces (Chapter 6). Functional analysis is developed in three chapters: 3.*Banach spaces,* 4. *Hilbert spaces,* and 6. *Locally convex topological vector spaces* (separation theorems, weak and weak topologies, Tychonoff's fixed point theorem). Brouwer fixed point theorem is proved in Appendix 4 at the end of the book. There is also a chapter, 5. *Calculus in Banach space,* exposing the basic results on Fréchet differentiability, including the inverse function theorem and applications to ordinary differential equations.

A good part of the book is devoted to measure theory and integration, with emphasis on Lebesgue measure and integral, and on Radon measures. This is done in the chapters: 7. *Measures and measurable functions* (monotone classes and algebras, Egoroff's convergence theorem), 8. *The abstract Lebesgue integral,* 9. *The construction of measures* (outer measures and Caratheodory's definition of measurable sets, Radon measures and Riesz representation theorem for positive functionals on $C_c(\Omega)$), 10. *Lebesgue measure* (Lebesgue measure in $\mathbb{R}^n$, change of variables by linear transformations, polar coordinates), 11. *Product measures* (Fubini and Tonelli theorems, completion of a product measure), 12. *The L^p spaces* (completeness, density of simple functions, continuity of translation operator, separability, convolution, mollifiers, and density of smooth functions), 13. *Representation theorems* (Radon-Nikodym theorem, Clarkson inequality, the duals of L^p, $1 \leq p < \infty$, and $C(T)$, for T compact), 14. *Fundamental theorem of calculus* (Vitali covering theorem, differentiation with respect to Lebesgue measure, the change of variables for multiple integrals), 15. *General Radon measures* (Besicovitch covering theorem, differentiation with respect to Radon measures, Young measures). A chapter, 23. *Integration of vector valued functions*, presents the Bochner integral and Riesz representation theorem for the dual of $L^p(\Omega, X)$, X a Banach space).

Three chapters, 19. *Hausdorff measures,* 20. *The area formula,* and 21 *The coarea formula,* deal with Hausdorff measures and very general change of variable formulas for surface integrals in $\mathbb{R}^n$.

Among the applications, we mention Chapter 17. *Probability,* containing a short but thorough exposition of basic results in probability theory. Other applications are to Fourier analysis and distribution theory given in chapters 16. *Fourier transforms* (based on Schwartz class of rapidly decreasing smooth functions and on tempered distributions), 18 *Weak derivatives* (Morrey's inequality and Rademacher theorem on a.e. differentiability of Lipschitz functions), 22. *Fourier analysis in* $\mathbb{R}^n$ (Marcinckiewicz interpolation theorem, Calderon-Zygmund decomposition, Michlin's generalization to L^p of Plancherel theorem, Calderon-Zygmund theory of singular integrals).

The last chapter of the book, 24. *Convex functions,* presents some of the most important results on convex functions, culminating with a proof of Alexandrov's theorem on a.e. twice differentiability of convex functions.

Three appendices: 1. *The Hausdorff maximal theorem,* 2. *Stone's theorem and partitions of unity,* 3. *Taylor series and analytic functions,* and 4. *The Brouwer fixed point theorem*, complete the main text. There are also a set of well chosen exercises at the end of each chapter, some of them routine, others containing more advanced topics and results which were not included in the main body of the book.

The book is an ideal text for graduate-level real analysis courses and basic courses on measure theory, using a modern approach. Its specific feature is the presentation, with complete proofs and in an accessible but rigorous way, of some deep results in modern analysis, available only in more specialized texts and needing a lot of technicalities for their understanding.

We warmly recommend the book to all people desiring to teach or to learn some fundamental results in modern analysis, in a reasonable period of time.

S. Cobzaş

Rafael H. Villarreal, *Monomial Algebras*, Pure and Applied Mathematics 238, Marcel Dekker 2010, ix+455pp, ISBN 0-8247-0524-6.

The volume under review presents methods which can be used to study monomial algebras and their presentation ideals including computational methods.

The book is divided in 11 chapters. Chapter 1 contains the basic facts and methods on commutative algebra and homological algebra. In order to present the basic properties of monomial algebras, the author presents in Chapter 2 the affine and graded algebras and in Chapter 3 he exhibits the importance of Rees algebras and associated graded algebras. Chapters 4 and 5 present the Hilbert series of graduates modules and Stanley-Reisner rings which are used in the Sanley's proof of the upper bound conjecture for simplicial spheres. In Chapters 6, 8 and 9 the connections between monomial algebras, graph theory and polyhedral theory are presented. The author presents in Chapter 7, 9 and 10 some features of toric ideals the monomial curves, the affine toric varieties and their toric ideals.

The book contains 280 exercises and numerous examples and graphs. Therefore, graduate students and researchers interested in commutative algebra and in its connections with computational issues in algebraic geometry and combinatorics will find this volume very useful.

S. Breaz

Sorin Dăscălescu, Constantin Năstăsescu, Şerban Raianu, *Hopf Algebras. An Introduction.* Monographs and textbooks in pure and applied mathematics 235,

Marcel Dekker, New York-Basel, 2001, ix+401 pp., Hardcover, ISBN 0-8247-0481-9.

The volume under review is aimed to introduce the reader to modern results on Hopf algebras. The material, presented from a ring theoretical point of view, has grown out of courses given over several years by the authors at the University of Bucharest.

The book is divided into 7 chapters. Chapter 1 presents basic facts on algebras and coalgebras, while Chapter 2 studies categories of comodules over a coalgebra. Chapter 3 examines in some depth cosemisimple, semiperfect and co-Frobenius coalgebras. Chapter 4 introduces bialgebras, Hopf algebras and Hopf modules, and Chapter 5 is devoted to integrals, the case of Hopf algebras obtained by Ore extensions being thoroughly treated. Chapter 5 discusses actions and coactions of Hopf algebras on algebras, and Hopf-Galois extensions. The last chapter presents various results on finite dimensional Hopf algebras, such as the order of the antipode, the Nichols-Zoeller theorem, character theory, the Taft-Wilson theorem, pointed Hopf algebras of dimension p^n. Appendices on the language of category theory and on C-groups and C-cogroups are also included. Each section contains many exercises accompanied by detailed solutions.

The authors are among the most important contributors to the field, and the above choice of topics reflects their interests. The presentation is very clear and reasonably self contained for a graduate student. The book is one of the best choices for a graduate course on Hopf algebras, and it will definitely be a valuable investment for any student and researcher interested in algebra.

Andrei Marcus

Martin Väth, *Volterra and Integral Equations of Vector Functions,* Pure and Applied Mathematics, Vol. 224, M. Dekker, Inc., Basel - New York 2000, vi+349 pages, ISBN: 0-8247-0342-1.

The book is dealing with Volterra-type integral equations of the form

$$x(t) = \int_0^t f(t, s, x(s))ds + g(t) \tag{1}$$

where f, g are given functions and x is the unknown function, all taking values in a Banach space (usually infinite dimensional). In fact the author considers a more general situation of some operator equations assumed to satisfy some "Volterra-typical" conditions. The emphasis is on the well-posedness of the problems, meaning existence, uniqueness and continuous dependence on the data, however the main part of the book is concerned with the existence of the solutions. A specific feature of the book is the extensive use of methods based on measures on noncompactnes, on fixed point theorems of Darbo type, and on quasinormed preideal spaces of vector functions.

The first chapter of the book, Ch. 1, *Preliminaries,* is concerned with fixed point theorems (mainly for operators which are condensing with respect to a measure of noncompactness), Bochner measurable functions and integrals, Lebesgue-Bochner function spaces, and ideal spaces. The framework is that of functions with values in a pseudometric space (more general than a Banach space), which is more appropriate for the subsequent development.

Ch. 2, *General Existence Results*, based mainly on author's original results, deals with existence results for abstract Volterra operators satisfying some boundedness and compactness conditions, containing as particular cases many types of Volterra operators.

The main tool used in the third chapter, *Integral Operators in Banach Spaces*, for defining integral operators and studying their properties (boundedness and compactness) is that of Carathéodori functions. To prove compactness results for integral operators, the author uses merely equimeasurability conditions rather than equicontinuity ones, leading to the notion of strict Carathéodori function. The general framework for the study of integral operators is that of ideal spaces. In fact, the author have written a book on this topic - "Ideal Spaces", Lect. Notes in Math. Vol. 1664, Springer Verlag, Berlin 1997.

The last chapter of the book, Ch. 4, *Dependence on Parameters*, is concerned with continuous dependence on the data, the averaging principle in nonlinear mechanics and Bogoljubov type theorems.

Developing general principles and results for Volterra type integral equations, most based on author's original results, and specifying them to particular equations arising in models from physics, mechanics and biology, the book will be of great interest for researchers in applied functional analysis, differential and integral equations, and their applications in other areas of human knowledge.

S. Cobzaş