Anul XLVI **2001**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
4

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

A STUDY OF FUNCTORS ASSOCIATED WITH TOPOLOGICAL GROUPS

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Abstract. The aim of this paper is to construct functors associated with topological groups as well as to investigate these functors. More precisely, we prove that for a given topological groups G there always exists a contravariant functor $F(G)$ from the homotopy category of pointed topological spaces and homotopy classes of base point preserving continuous maps to the category of groups and homomorphisms. We also prove that

(i) the functor $F(G)$ is natural in G in the sense that if the topological groups G and H have the same homotopy type then the groups $F(G)(X)$ and $F(H)(X)$ are isomorphic, for every pointed topological space X; and

(ii) the functor $F(G)$ is homotopy type invariant in the sense that if X and Y are two pointed spaces having the same homotopy type then the groups $F(G)(X)$ are $F(G)(Y)$ are isomorphic.

Moreover, given two topological groups G and H and a continuous homomorphism $\alpha : G \to H$, we show that there always exists a natural transformation between the functors $F(G)$ and $F(H)$ associated with topological groups G and H respectively.

1. **Introduction**

Throughout this paper we assume that (X, x_0) is pointed topological space and maps are base point preserving continuous maps. For simplicity, we write X in place of (X, x_0).

Now we recall following definitions and statements:

Definition 1.1. A pointed topological space is a nonempty topological space with a distinguished element.

Definition 1.2. A pointed topological group is a group G whose underlying set is equipped with a topology such that:

(i) The multiplication map $\mu : G \times G \to G$, given by $(x, y) \mapsto xy$, is continuous if $G \times G$ has the product topology;

(ii) The inversion map $i : G \to G$, given by $x \mapsto x^{-1}$, is continuous.

Then (G, e) is a pointed topological space where e is the identity element.

Definition 1.3. Let $A \subset X$ and let $f_0, f_1 : X \to Y$ be base point preserving continuous maps with $f_0|A = f_1|A$. We write $f_0 \simeq f_1 rel.A$, if there is a continuous map $F : X \times I \to Y$ with $F : f_0 \simeq f_1$ and $F(a,t) = f_0(a) = f_1(a)$, $\forall\ a \in A$ and all $t \in I$. Such a map F is called a homotopy relative to A from f_0 and f_1 and is denoted by $F : f_0 \simeq f_1 rel.A$.

Definition 1.4. If $f : X \to Y$ is base point preserving continuous maps, its homotopy class is the equivalence class $[f] = \{g \in C(X,Y) :\ f \simeq g\}$, where $C(X,Y)$ denotes the set all base point preserving continuous maps from X to Y.

The family of all such homotopy classes is denoted by $[X; Y]$.

Definition 1.5. A base point preserving continuous map $f : X \to Y$ is a homotopy equivalence if there is a base point preserving continuous map $g : Y \to X$ with $g \circ f \simeq I_X$ and $f \circ g \simeq I_Y$. Two spaces X and Y have the same homotopy type denoted by $X \approx Y$ if there is a homotopy equivalence $f : X \to Y$.

Definition 1.6. A category $\mathcal{C}$ consists of

(a) a class of objects $X, Y, Z, \ldots$ denoted by $Ob(\mathcal{C})$;

(b) for each ordered pair of objects X, Y a set of morphisms with domain X and range Y denoted by $\mathcal{C}(X,Y)$;

(c) for each ordered triple of objects X, Y and Z and a pair of morphisms $f : X \to Y$ and $g : Y \to Z$, their composite is denoted by $gf : X \to Z$, satisfying the following two axioms:

(i) *associativity*: if $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(Y,Z)$ and $h \in \mathcal{C}(Z,W)$, then $h(gf) = (hg)f \in \mathcal{C}(X,W)$;

(ii) *identity*: for each object Y in $\mathcal{C}$ there is a morphism $I_Y \in \mathcal{C}(Y,Y)$ such that if $f \in \mathcal{C}(X,Y)$, then $I_Y f = f$ and if $h \in \mathcal{C}(Y,Z)$, then $hI_Y = h$.

Definition 1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A contravariant functor T from $\mathcal{C}$ to $\mathcal{D}$ consists of

(i) an object function which assigns to every object X of $\mathcal{C}$ an object $T(X)$ of $\mathcal{D}$; and

(ii) a morphism function which assigns to every morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $T(f) : T(Y) \to T(X)$ in $\mathcal{D}$ such that

(a) $T(I_X) = I_{T(X)}$;

(b) $T(gf) = T(f)T(g)$, for $g : Y \to W$ in $\mathcal{C}$.

Definition 1.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose T_1 and T_2 are both contravariant functors from $\mathcal{C}$ and $\mathcal{D}$. A natural transformation ϕ from T_1 to T_2 is a function from the objects of $\mathcal{C}$ to the morphisms of $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$ the following condition hold:

$$\phi(X)T_1(f) = T_2(f)\phi(Y).$$

Lemma 1.9. *Homotopy is an equivalence relation on the set $C(X, Y)$ of all base point preserving continuous maps from X to Y.*

Lemma 1.10. *Let $f_i : X \to Y$ and $g_i : Y \to Z$, for $i = 0, 1$, be continuous. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$; that is $[g_0 \circ f_0] = [g_1 \circ f_1]$.*

In section 2, we construct and investigate functors associated with topological groups.

2. **Functors associated with topological groups**

We now construct functors associated with topological groups.

Let (X, x_0) be a topological space with base point x_0 and (G, e) be a topological group with identity e and $f : X \to G$ be a continuous map such that $f(x_0) = e$.

Now we construct the set M = set of all base preserving continuous maps from (X, x_0) to (G, e).

Then we have the following Proposition:

Proposition 2.1. *Let (X, x_0) be a pointed topological space. The set M of all base point preserving continuous maps from X to G, forms a group under the* *composition '$\Box$' on M defined by*

$$(f_1 \Box f_2)(x) = f_1(x) \cdot f_2(x), \ \forall \ x \in X, \ f_1, f_2 \in M,$$

where the right hand side multiplication '$\cdot$' is the multiplication defined on the topological group G.

Proof. First we show that M is nonempty.

Let $C : X \to G$ be defined by $C(x) = e, \ \forall \ x \in X$. Then C is a constant map such that $C \in M \ \Rightarrow \ M \neq \emptyset$.

Let $f_1, f_2 \in M$. Then

$$(f_1 \Box f_2)(x_0) = f_1(x_0) \cdot f_2(x_0) = e \cdot e = e,$$

by definition.

Thus $f_1 \Box f_2$ is a base preserving map. Since G be a topological group and the map, $M \times M \to M$,

$$(f_1, f_2) \mapsto f_1 \Box f_2, \ \forall \ f_1, f_2 \in M,$$

is continuous and hence $f_1 \Box f_2$ is a base point preserving continuous map from X to G. Hence $f_1 \Box f_2 \in M$.

Let $f_1, f_2, f_3 \in M$. Then

$$((f_1 \Box f_2) \Box f_3)(x) = (f_1 \Box f_2)(x) \cdot f_3(x) =$$

$$= (f_1(x) \cdot f_2(x)) \cdot f_3(x) == f_1(x) \cdot (f_2(x) \cdot f_3(x)) =$$

$$= f_1(x) \cdot (f_2 \Box f_3)(x) = (f_1 \Box (f_2 \Box f_3))(x).$$

Thus $((f_1 \Box f_2) \Box f_3)(x) = (f_1 \Box (f_2 \Box f_3))(x), \ \forall \ x \in X$.

Hence $(f_1 \Box f_2) \Box f_3 = f_1 \Box (f_2 \Box f_3)$.

$\Rightarrow$ '$\Box$' associative.

Now

$$(f_1 \Box C)(x) = f_1(x) \cdot C(x) = f_1(x) \cdot e = f_1(x)$$

and

$$(C \Box f_1)(x) = C(x) f_1(x) = e \cdot f_1(x) = f_1(x).$$

Thus $(f_1 \square C)(x) = (C \square f_1)(x),\ \forall\ x \in X\ \Rightarrow\ f_1 \square C = C \square f_1$.

$\Rightarrow$ C is a identity map from X to G.

Since C is a base point preserving continuous map from X to G and hence $C \in M$.

Let $f_1, f_2 \in M$ such that $(f_1 \square f_2)(x) = C(x)$

$$\Rightarrow\ f_1(x) \cdot f_2(x) = C(x)\ \Rightarrow\ f_1(x) \cdot f_2(x) = e.$$

Also $f_2(x) \cdot f_1(x) = e$.

Thus $f_1(x) \cdot f_2(x) = f_2(x) \cdot f_1(x) = e$ i.e.

$$(f_1 \square f_2)(x) = (f_2 \square f_1)(x) = e,\ \forall\ x \in X.$$

This shows that for each base point preserving continuous map there exists its inverse in M and hence $(M, \square)$ is a group.

We now carries over the composition '$\square$' on M to give an operation '$*$' on homotopy classes such that

$$[f] * [g] = [f \square g],\ \forall\ f, g \in M$$

where $f \square g$ is defined in Proposition 2.1.

Theorem 2.2. *If X be a pointed topological space and G is a topological group with base point e, then $[X; G]$ is a group.*

Proof. Let X be an arbitrary pointed topological space and G be a topological group.

Let $[X; G]$ = set of all homotopy classes of base point preserving continuous maps from X to G i.e. $[X; G] = \{[f]$ such that $f : X \to G$ is a base point preserving continuous map$\}$.

Now we define a composition '$*$' on $[X; G]$ by the rule:

$$[f] * [g] = [f \square g],\ \forall\ f, g \in M.$$

$f_1 \in [f]$ and $g_1 \in [g]\ \Rightarrow\ f_1 \simeq f$ and $g_1 \simeq g$ respectively.

$\Rightarrow$ $f_1 \square g_1 \simeq f \square g$, as the composite of two homotopic maps are homotopic.

$\Rightarrow$ $[f_1 \square g_1] = [f \square g]$, by Lemma 1.10.

$\Rightarrow\ [f_1] * [g_1] = [f] * [g] \ \Rightarrow$ '$*$' is well defined.

Then by using proposition 2.1, $[X;G]$ is a group under the composition '$*$'.

Theorem 2.3. *If $f : X \to Y$ is a base point preserving continuous map, then f induces a homomorphism $f^* : [Y;G] \to [X;G]$, for each topological group G.*

Proof. Define $f^* : [Y;G] \to [X;G]$ by

$$f^*([h]) = [h \circ f],\ \forall\ [h] \in [Y;G].$$

$h_0, h_1 : Y \to G$ and $h_0 \simeq h_1 \ \Rightarrow\ h_0 \circ f \simeq h_1 \circ f \ \Rightarrow\ [h_0 \circ f] = [h_1 \circ f]$, by Lemma 1.10 i.e. $[h_0] = [h_1] \ \Rightarrow\ f^*([h_0]) = f^*([h_1])$. $\Rightarrow$ This map is well defined.

Let $[h_1], [h_2] \in [Y;G]$.

Now $f^*([h_1] * [h_2]) = f^*([h_1 \Box h_2]) = [(h_1 \Box h_2] \circ f]$, by definition. Thus $\forall\ x \in X$,

$$[((h_1 \Box h_2) \circ f)(x)] = [(h_1 \Box h_2)(f(x))] = [h_1(f(x)) \cdot h_2(f(x))],$$

by definition of the product in $[Y;G]$

$$= [(h_1 \circ f)(x) \cdot (h_2 \circ f)(x)] = [((h_1 \circ f) \Box (h_2 \circ f))(x)]$$

$$\Rightarrow\ [(h_1 \Box h_2) \circ f] = [(h_1 \circ f) \Box (h_2 \circ f)] = [h_1 \circ f] * [h_2 \circ f]$$

$$= f^*([h_1] * f^*([h_2]).$$

Thus $f^*([h_1] * [h_2]) = f^*([h_1]) * f^*([h_2]) \ \Rightarrow\ f^*$ is a group homomorphism.

Theorem 2.4. *Let $\alpha : G \to H$ is a continuous group homomorphism between topological groups, then α induces a group homomorphism, $\alpha_* : [X;G] \to [X;H]$.*

Proof. Define $\alpha_* : [X;G] \to [X;H]$ by

$$\alpha_*([f]) = [\alpha \circ f],\ \forall\ f : X \to G.$$

Let $f_1, f_2 : X \to G$ and $f_1 \simeq f_2 \ \Rightarrow\ \alpha \circ f_1 \simeq \alpha \circ f_2$ i.e. $[f_1] = [f_2] \ \Rightarrow\ [\alpha \circ f_1] = [\alpha \circ f_2] \ \Rightarrow\ \alpha_*([f_1]) = \alpha_*([f_2])$.

Thus this map is well defined.

Let $[f_1], [f_2] \in [X;G]$.

Then $\alpha_*([f_1] * [f_2]) = \alpha_*[(f_1 \Box f_2)] = [\alpha \circ (f_1 \Box f_2)]$, by definition.

Thus $\forall\, x \in X$,

$$[(\alpha \circ (f_1 \Box f_2))(x)] = [\alpha((f_1 \Box f_2)(x))] = [\alpha(f_1(x) \cdot f_2(x))]$$

$$= [\alpha(f_1(x)) \cdot \alpha(f_2(x))] = [(\alpha \circ f_1)(x) \cdot (\alpha \circ f_2)(x)] = [((\alpha \circ f_1) \Box (\alpha \circ f_2))(x)]$$

$$\Rightarrow\ [\alpha \circ (f_1 \Box f_2)] = [(\alpha \circ f_1) \Box (\alpha \circ f_2)] = [\alpha \circ f_1] * [\alpha \circ f_2] = \alpha_*([f_1]) * \alpha_*([f_2]).$$

Thus $\alpha_*([f_1] * [f_2]) = \alpha_*([f_1]) * \alpha_*([f_2]) \ \Rightarrow\ \alpha_*$ is a group homomorphism.

Let Htp denote the category of pointed topological spaces and homotopy classes of their base point preserving continuous maps and Grp be the category of groups and their homomorphisms. Then we have the following theorems:

Theorem 2.5. *For a given topological group G, there exists a contravariant functor*

$$F(G) : Htp \rightarrow Grp.$$

Proof. Using Theorems 2.2-2.3, define $F(G)(X) = [X; G]$ which is a group and also for $\alpha : X \rightarrow Y$ in Htp, $\alpha^* = F(G)(\alpha) : [Y, G] \rightarrow [X; G]$ by

$$\alpha^*([g]) = [g \circ \alpha],\ \forall\ [g] \in [Y; G].$$

Let $\alpha : X \rightarrow Y$ and $\beta : Y \rightarrow Z$ be base point preserving continuous maps, then $\beta \circ \alpha : X \rightarrow Z$ is also a base point preserving continuous map.

Thus $(\beta \circ \alpha)^* = F(G)(\beta \circ \alpha) : [Z; G] \rightarrow [X; G]$ by

$$(\beta \circ \alpha)^*([g]) = [g \circ (\beta \circ \alpha)],\ \forall\ [g] \in [Z; G].$$

Thus $\forall\, x \in X$,

$$[(g \circ (\beta \circ \alpha))(x)] = [g((\beta \circ \alpha)(x))]$$

$$= [g(\beta(\alpha(x)))] = [(g \circ \beta)(\alpha(x))] = [((g \circ \beta) \circ \alpha)(x)]$$

$$\Rightarrow\ [g \circ (\beta \circ \alpha)] = [(g \circ \beta) \circ \alpha] = \alpha^*([(g \circ \beta)]) = \alpha^*(\beta^*([g])) = (\alpha^* \circ \beta^*)([g]).$$

Thus $\forall\ [g] \in [Z; G]$, $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

Also, for identity map $I_X : X \to X$, $I_X^* = F(G)(I_X) : [X;G] \to [X;G]$ defined by

$$I_X^*([g]) = [g \circ I_X] = [g].$$

Hence $F(G)$ is a contravariant functor.

Given topological groups G and H, $\exists$ two contravariant functors $F(G)$ and $F(H)$. Then $F(G)$ and $F(H)$ have the following relation:

Theorem 2.6. *Given topological groups G, H and a continuous homomorphism $\alpha : G \to H$ there exists a natural transformation*

$$\alpha_* : F(G) \to F(H).$$

Proof. For $[g] \in [Y;G]$ and $f : X \to Y$,

$$F(H)(f)(\alpha_*([g])) = F(H)(f)([\alpha \circ g]) = [(\alpha \circ g) \circ f]$$

i.e. $f^*(\alpha_*([g])) = [(\alpha \circ g) \circ f] \ \Rightarrow\ (f^* \circ \alpha_*)([g]) = [(\alpha \circ g) \circ f]$ and

$$\alpha_*(f^*([g])) = \alpha_*([g \circ f]) = [\alpha \circ (g \circ f)]$$

i.e. $(\alpha_* \circ f^*)([g]) = [\alpha \circ (g \circ f)]$.

Thus $f^* \circ \alpha_* = \alpha_* \circ f^* \ \Rightarrow\ \alpha_*$ is a natural transformation.

Lemma 2.7. *If two topological groups G and H have the same homotopy type, then the homotpy equivalence is a homomorphism.*

Proof. Since G and H have the same homotopy type then there exist continuous maps $f : G \to H$, $g : H \to G$ such that $f(e) = e'$, $g(e') = e$, $g \circ f \simeq I_G$ and $f \circ g \simeq I_H$, where $I_G : G \to G$ and $I_H : H \to H$ are identity maps. Then f and g are both homotopy equivalences.

Since G and H are topological groups, $\exists$ continuous multiplications $\mu : G \times G \to G$ and $\mu' : H \times H \to H$ such that the square

$$\begin{array}{ccc} G\times G & \xrightarrow{\ \mu\ } & G \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ H\times H & \xrightarrow[\ \mu'\ ]{} & H \end{array}$$

is commutative i.e. $\mu' \circ (f \times f) = f \circ \mu$.

Now $(f \circ \mu)(x, y) = f(\mu(x, y)) = f(xy)$ and

$$(\mu' \circ (f \times f))(x, y) = \mu'((f \times f)(x, y))$$

$$= \mu'(f(x), f(y)) = f(x) \cdot f(y).$$

Thus $f(xy) = f(x) \cdot f(y),\ \forall\ x, y \in G\ \Rightarrow\ f$ is a homomorphism.

Also, g is a homomorphism.

Thus we prove that the homotopy equivalences f and g are continuous group homomorphisms from G to H and H to G respectively.

Theorem 2.8. *If two topological groups G and H are such that G and H have the same homotopy type, then the groups $F(G)(X)$ and $F(H)(X)$ are isomorphic, for every pointed topological space X.*

Proof. Since the topological groups G and H have the same homotopy type, then there exist base point preserving continuous maps $f : G \to H$, $g : H \to G$ such that $g \circ f \simeq I_G$ and $f \circ g \simeq I_H$, where $I_G : G \to G$ and $I_H : H \to H$ are identity maps.

Let $f_* : F(G)(X) \to F(H)(X)$ be defined by

$$f_*([\alpha]) = [f \circ \alpha],\ \forall\ [\alpha] \in F(G)(X).$$

Using Theorem 2.4 and Lemma 2.7, f_* is a homomorphism from $F(G)(X)$ to $F(H)(X)$.

Then f_* satisfies the following properties:

(i) if $f \simeq g\ \Rightarrow\ f_* = g_*$;

(ii) $I_G : G \to G\ \Rightarrow\ I_{G*} = Id_{F(G)(X)}$;

(iii) $(g \circ f)_* = g_* \circ f_*$

for $(g \circ f)_* : F(G)(X) \to F(G)(X)$ defined by

$$(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha], \ \forall \ [\alpha] \in F(G)(X)$$

$$= [g \circ (f \circ \alpha)] = g_*([f \circ \alpha]) = g_*(f_*([\alpha])) = (g_* \circ f_*)([\alpha]).$$

Thus $\forall \ [\alpha] \in F(G)(X)$, $(g \circ f)_* = g_* \circ f_*$.

Since $g \circ f \simeq I_G$, we have $(g \circ f)_* = I_{G*}$, by (i) $\Rightarrow \ g_* \circ f_* = Id_{F(G)(X)}$, by (ii) and (iii) i.e. $g_* \circ f_* = Id$.

Again since $f \circ g \simeq I_H$, we have similarly

$$f_* \circ g_* = Id.$$

Since f_* is a homomorphism and $g_* \circ f_* = Id \ \Rightarrow \ f_*$ is a monomorphism. Again since f_* is a homomorphism and $g_* \circ f_* = Id \ \Rightarrow \ f_*$ is a epimorphism. Thus f_* is an isomorphism and g_* as its inverse.

Therefore the groups $F(G)(X)$ and $F(H)(X)$ are isomorphic.

Lemma 2.9. *Let G be a topological group and X, Y be two pointed topological spaces such that X and Y belong to the same homotopy type. Then the groups $F(G)(X)$ and $F(G)(Y)$ are isomorphic, where $F(G)$ is a contravariant functor from Htp to Grp given in Theorem 2.5.*

Proof. Let X, Y be two pointed topological spaces having the same homotopy type, then $\exists$ base point preserving continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq I_Y$ and $g \circ f \simeq I_X$, where $I_X : X \to X$ and $I_Y : Y \to Y$ are identity maps.

Define $f^* : F(G)(Y) \to F(G)(X)$ by

$$g^*([\alpha]) = [\alpha \circ f], \ \forall \ [\alpha] \in F(G)(Y).$$

Using Theorem 2.3 and Theorem 2.5, f^* is a homomorphism from $F(G)(Y)$ to $F(G)(X)$.

Then f^* satisfies the following properties:

(i) if $f \simeq g \ \Rightarrow \ f^* = g^*$;

(ii) $I_X : X \to X \;\Rightarrow\; I_X^* = Id_{F(G)(X)}$,

for $I_X^* : F(G)(X) \to F(G)(X)$ defined by

$$I_X^*([\alpha]) = [\alpha \circ I_X] = [\alpha],\ \forall\ [\alpha] \in F(G)(X)$$

i.e. $I_X^* = Id_{F(G)(X)}$

(iii) $(g \circ f)^* = f^* \circ g^*$,

for $(g \circ f)^* : F(G)(X) \to F(G)(X)$, defined by

$$(g \circ f)^*([\alpha]) = [\alpha \circ (g \circ f)],\ \forall\ [\alpha] \in F(G)(X) = [(\alpha \circ g) \circ f] =$$

$$= f^*([\alpha \circ g]) = f^*(g^*([\alpha])) = (f^* \circ g^*)([\alpha]).$$

Thus $\forall\ [\alpha] \in F(G)(X)$, $(g \circ f)^* = f^* \circ g^*$.

Since $g \circ f \simeq I_X$, we have $(g \circ f)^* = I_X^*$, by (i) $\Rightarrow$ $f^* \circ g^* = Id_{F(G)(X)}$, by (ii) and (iii) i.e. $f^* \circ g^* = Id$.

Again since $f \circ g \simeq I_Y$, we have similarly

$$g^* \circ f^* = Id.$$

Since f^* is a homomorphism and $g^* \circ f^* = Id \;\Rightarrow\; f^*$ is a monomorphism.

Again since f^* is a homomorphism and $f^* \circ g^* = Id \;\Rightarrow\; f^*$ is a epimorphism.

Therefore f^* is an isomorphism and g^* as its inverse.

Thus the groups $F(G)(X)$ and $F(G)(Y)$ are isomorphic.

Theorem 2.10. *For a given topological group G there always exists a contravariant functor $F(G) : Htp \to Grp$ such that $F(G)$ is homotopy type invariant.*

Proof. Using Lemma 2.9, it follows that $F(G)$ is a homotopy type invariant functor in th sense that if X and Y are the same homotopy type then the groups $F(G)(X)$ and $D(G)(Y)$ are isomorphic.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

SOME HOMEOMORPHISM THEOREMS

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Abstract. In this paper we give homeomorfism result for operators that satisfies Borsuk condition.

1. Introduction

Let X be a Banach space and $f : X \to X$ be an operator such that $F_f \neq \emptyset$. There are many papers in which using the fixed point theory we obtain the surjectivity of $\mathbf{1}_X - f$ (see: Aldea [1, 2], Browder [4], Danes [8], Danes-Kolomy [9], Deimling [10], Rus [14, 15, 16].
The aim of this paper is to give an answer to the following question. What conditions must satisfy f such that $\mathbf{1}_X - f$ be a homeomorphism?
Rus proved in [15] that if f is a φ contraction then $\mathbf{1}_X - f$ is a homeomorphism. In order to prove this he used a bijectivity and a data dependence results.

Also, it is possible to obtain homeomorphism result using domain invariance result respective closing range theorem (see: Cramer-Ray [6], Crandall-Pazzy [7], Dowing-Kirk [11], Zeidler [17]).
Following a similar technique we will give an answer to the mention question in case that operator f satisfy Borsuk condition.

Definition 1.1. Let X be a Banach space and $f : X \to X$ an operator. We say that f satisfies Borsuk condition (shortly (B)), if there exists $\eta > 0$ and $\varepsilon > 0$ such that for all $x_1,\ x_2 \in X$, inequality

$$||f(x_1) - f(x_2)|| < \eta$$

1991 *Mathematics Subject Classification.* 47H10.
Key words and phrases. homeomorphism, fixed points, Borsuk condition.

implies

$$||x_1 - x_2|| < \varepsilon.$$

Now we will give some operators' classes which satisfy (B) condition.

Remark 1.1. Let X be a Banach space. If $f : X \to X$ is near identity (in Campanato sense [5]), then f satisfies condition (B).

Proof. Because f is near $\mathbf{1}_X$ there exists constants $\lambda,\ k \in (0,1)$ such that

$$||x_1 - x_2 - \lambda(f(x_1) - f(x_2))|| \leq k \cdot ||x_1 - x_2||, \text{ for all } x_1,\ x_2 \in X \tag{1}$$

or

$$(1-k)||x_1 - x_2|| \leq \lambda||f(x_1) - f(x_2)||, \text{ for all } x_1,\ x_2 \in X.$$

So there are $\eta > 0$ and $\varepsilon \left(= \dfrac{\lambda}{1-k}\eta\right) > 0$ such that from $||f(x_1) - f(x_2)|| < \eta$ we have $||x_1 - x_2|| < \varepsilon$. We obtain that f verifies condition (B).

Remark 1.2. Let X be Banach space. If $f : X \to X$ is dilatation, then f satisfies (B) condition.

Proof. Because f is dilatation there exists $c > 1$ such that

$$c||x_1 - x_2|| \leq ||f(x_1) - f(x_2)||, \text{ for all } x_1,\ x_2 \in X$$

So there are $\eta > 0$ and $\varepsilon \left(= \dfrac{\eta}{c}\right) > 0$ such that from $||f(x_1) - f(x_2)|| < \eta$ we have $||x_1 - x_2|| < \varepsilon$. We obtain that f verifies condition (B).

Remark 1.3. Let X Banach space. If $f : X \to X$ is strong accretive, then f satisfies condition (B).

Proof. Because f is strong accretive there is $k > 1$ such that

$$k||x_1 - x_2|| \leq ||f(x_1) - f(x_2)||, \text{ for all } x_1,\ x_2 \in X$$

So there are $\eta > 0$ and $\varepsilon \left(= \dfrac{\eta}{k}\right) > 0$ such that from $||f(x_1) - f(x_2)|| < \eta$ we have $||x_1 - x_2|| < \varepsilon$. We obtain that f verifies (B) condition.

Definition 1.2. (Rus, [15]) A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if φ is increasing and $\varphi^n(t) \to 0$ when $n \to \infty$ for all $t \in \mathbb{R}_+$.

2. **Main result**

In what follows, we solve the problem for case of an operator which is sum of two operators and one of them satisfies condition (B).

Theorem 2.1. (Granas, [12]) *Let X be a Banach space and operator $F : X \to X$ be a complete continuous . If operator $f : X \to X$ satisfies condition (B) (with $f(x) = x - F(x)$ for all $x \in X$), then f is surjective.*

Theorem 2.2. *Let X be a Banach space, $F,\ L : X \to X$ be two continuous operators with F compact and functions $\varphi,\ \psi : [0, \infty) \to [0, \infty)$. Suppose that:*

(i)

$$\varphi(||x_1 - x_2||) \leq ||f(x_1) - f(x_2)|| \tag{2}$$

for all $x_1,\ x_2 \in X$ with $f(x) = \mathbf{1}_X(x) - F(x)$, for all $x \in X$;

(ii)

$$||L(x_1) - L(x_2)|| \leq \psi(||x_1 - x_2||) \tag{3}$$

for all $x_1,\ x_2 \in X$;

(iii) $\varphi(0) = 0,\ \varphi$ bijective and φ^{-1} comparison function;

(iv) $\psi(0) = 0$ and ψ comparison function.

Then $\mathbf{1}_X - f$ is bijective.

Proof. First, we prove that $F_{F+L} = \emptyset$. In order to apply Theorem 2.1 we will prove that f verifies condition (B). Let $x_1,\ x_2$ from X such that $||f(x_1) - f(x_2)|| < \eta$. From (2) and φ bijective we have

$$\begin{aligned} \varphi(||x_1 - x_2||) &\leq ||f(x_1) - f(x_2)|| < \eta \\ ||x_1 - x_2|| &\leq \varphi^{-1}(\eta) < \varphi^{-1}(\eta) + 1 = \varepsilon \end{aligned}$$

so f verifies condition (B).

From Theorem 2.1 we have that f is surjective. From (2) and (iii) we obtain that f is injective. Operator f is continuous from hypothesis and continuity of inverse operator results from inequality (2); so f is homeomorphism.

Let $x \in X$, because f is homeomorphism we define operator

$$R : X \to X;\ x \longmapsto R(x)$$

such that

$$f(R(x)) = L(x) \text{ for all } x \in X.$$

From (2) and (3) we have that

$$\begin{aligned}\varphi(||R(x_1) - R(x_2)||) &\leq ||f(R(x_1)) - f(R(x_2))|| = ||L(x_1) - L(x_2)|| \\ &\leq \psi(||x_1 - x_2||)\end{aligned}$$

for all $x_1,\ x_2 \in X$. Because φ is invertible

$$||R(x_1) - R(x_2)|| \leq (\varphi^{-1} \circ \psi)(||x_1 - x_2||) \quad (4)$$

for all $x_1,\ x_2 \in X$.

Because $\varphi^{-1},\ \psi$ are comparison functions we obtain that

$$||R(x_1) - R(x_2)|| \leq \varphi^{-1}(||x_1 - x_2||) \quad (5)$$

for all $x_1,\ x_2 \in X$. But φ^{-1} is comparison function. From the last statement and (4) we apply fixed point theorem for φ-contractions (see Rus [16]) we have $F_R = \{x^*\}$. From the definition of R results

$$(\mathbf{1}_X - F)(x^*) = L(x^*) \iff F_{F+L} = \{x^*\}.$$

Second, we prove that $\mathbf{1}_X - (F + L)$ is bijective.

Let $y \in X$. We denote by L_y operator $L+y$. It is easy to prove that operator L_y verifies inequality (3), so applying first part of our proof we have that $F_{F+L_y} = \{x^*\} \iff$ equation $F(x) + L(x) + y = x$ has only one solution. So $\mathbf{1}_X - (F + L)$ is bijective.

Theorem 2.3. *If we add to the hypotheses of Theorem 2.2 the following:*

(v) $\varphi(t) \geq \psi(t)$ *for all* $t \geq 0$;

(vi) there is the inverse of $\mathbf{1}_{[0,\infty)} - (\varphi^{-1} \circ \psi)$ *and it is continuous.*

Then $\mathbf{1}_X - (F + L)$ *is homeomorphism.*

Proof. From Theorem 2.2 we have that operator $\mathbf{1}_X - (F + G)$ is bijective, continuity of its results from the continuity of F and L.

Let x_i the unique solution of equations $x - F(x) - L(x) = y_i$, for $i = 1,\ 2$.

From (2) and (3) we have

$$\begin{aligned}\varphi(||x_1 - x_2||) &\leq ||f(x_1) - f(x_2)|| = ||L(x_1) - L(x_2) + y_1 - y_2|| \\ &\leq ||L(x_1) - L(x_2)|| + ||y_1 - y_2|| \\ &\leq \psi(||x_1 - x_2||) + ||y_1 - y_2|| \Longrightarrow\end{aligned}$$

From (iii) resuts

$$\begin{aligned}||x_1 - x_2|| &\leq (\varphi^{-1} \circ \psi)(||x_1 - x_2||) + \varphi^{-1}(||y_1 - y_2||) \\ &\leq (\varphi^{-1} \circ \psi)(||x_1 - x_2||) + ||y_1 - y_2||) \iff \\ (\mathbf{1}_{[0,\infty)} - (\varphi^{-1} \circ \psi))(||x_1 - x_2||) &\leq ||y_1 - y_2|| \Longrightarrow \\ ||x_1 - x_2|| &\leq (\mathbf{1}_{[0,\infty)} - (\varphi^{-1} \circ \psi))^{-1}(||y_1 - y_2||) \qquad (6)\end{aligned}$$

From last inequality and (vi) we have that

$$||(\mathbf{1}_X - (F+L))^{-1}(y_1) - (\mathbf{1}_X - (F+L))^{-1}(y_2)|| \leq (\mathbf{1}_{[0,\infty)} - (\varphi^{-1} \circ \psi))^{-1}(||y_1 - y_2||)$$

Which means that $(\mathbf{1}_X - (F+L))^{-1}$ is continuous operator, so $\mathbf{1}_X - (F+L)$ homeomorphism.

Remark 2.1. If X is finite dimensional Banach space, then Theorems 2.1, 2.2 are true without assumption of compactness on operator F.

Theorem 2.4. (Altman, [3]) *Let X be a finite dimensional Banach space, $F,\ L : X \to X$ two continuous operators and constants $c > 0$ and $k > 0$. Suppose that:*

(i)

$$c \cdot ||x_1 - x_2|| \leq ||f(x_1) - f(x_2)|| \qquad (7)$$

for all $x_1,\ x_2 \in X$ with $f(x) = \mathbf{1}_X(x) - F(x)$, for all $x \in X$;

(ii)

$$||L(x_1) - L(x_2)|| \leq k \cdot ||x_1 - x_2|| \qquad (8)$$

for all $x_1,\ x_2 \in X$;

(iii)

$$K < c.$$

Then

(a) $F_{F+L} = \{x^*\}$;

(b) $\mathbf{1}_X - (F + L) : X \to X$ *is homeomorphism;*

(c) Operator $(\mathbf{1}_X - (F + L))^{-1} : X \to X$ *is Lipschitz continuous.*

Proof. In order to prove theorem, we apply Theorem 2.2 and 2.3 considering $\varphi(t) = c \cdot t$ with $c > 1$ and $\psi(t) = k \cdot t$ with $k < 1$.

These functions verify assumption (i)-(v) from mentioned theorems.

Function $(\mathbf{1}_{[0,\infty)} - (\varphi^{-1} \circ \psi))(t) = \frac{c-k}{c} t$ verifies (vi).

Conclusion (c) of Altman's theorem results from inequality (6).

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

INTERPOLATION RESULTS FOR SOME CLASSES OF ABSOLUTELY SUMMING OPERATORS

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Abstract. K. Miyazaki, [9], has introduced the class of $(p, q; r)$-absolutely summing operators, which generalize the class of (p, q)-absolutely summing operators, introduced by Mitiagin and Pełczynski in 1966.

We establish an interpolation result for $(p, q; r)$ −absolutely summing operators and also for some other operator classes which generalize Miyazaki"s classes.

1. **Introduction**

The interpolation properties of the $p-$summable and the (p, q)-absolutely summing operators are well known. Miyazaki has extended the result concerning the interpolation stability for $(p, q)-$absolutely summing operators to the more general ideal of $(p, q; r)$-absolutely summing operators, which he introduced [9]. In this paper we will look at his result, because it relies on the presumption that the ideal of $(p, q; r)$-absolutely summing operators is normed, which in general does not happen, this ideal being only quasi-normed. N. Tita [11], [12] has introduced and studied ideals of operators which are (Φ, Ψ)-absolutely summing, where Φ and Ψ are symmetric norming functions, and which are more general than the (p, q)-absolutely summing operators and the largest part of the ideals studied by Miyazaki. Due to the non-linearity of the symmetric norming functions, nothing could be ascertained regarding the interpolation properties of these ideals of operators. For this reason we ask the question of existence of ideals of operators more general than those of Miyazaki, and which still satisfy the stability result proved by him. In order to answer to the above question we construct a class of absolutely summing opetators, which is based on the Lorentz-Zygmund spaces of sequences.

Key words and phrases. p-absolutely summing operators, p-q absolutely summing operators, symmetric norming function, Lorentz-Zygmund sequence ideals.

The present paper is a revised and extended version of [1]. This revision became necessary as we had not, at the time of writing [1], been aware of Myazaki"s work, and we realized that the class we had introduced was not satisfactorily motivated nor exhaustively treated.

2. **Preliminaries**

We first introduce some notation and recall a few known results. Throughout the paper $\mathbb{N}$ denotes the set of all positive integers, while E, F are Banach spaces over Γ, where Γ is the real or the complex field. By $\mathcal{F}(E)$ we denote a finite set of vectors $x_1, ..., x_n$ in E. We denote

$$L(E,F) := \{T : E \to F : \; T \text{ is linear and bounded}\},$$

and we let E^* be the dual space, $E^* = L(E,\Gamma)$. By U_E we denote the unit ball $\{x \in E : \|x\| \leq 1\}$. For $a \in E^*$ and $x \in E$, let $\langle x, a\rangle := a(x)$. We denote by l_∞ the set of all scalar sequences, $\{x_n\}_n$, with the property $\|x\|_\infty := \sup\limits_{n\in\mathbb{N}} |x_n| < \infty$, and by c_0 the set of all scalar sequences, $\{x_n\}_n$, with the property $\lim\limits_{n\to\infty} |x_n| = 0$. For $0 < p < \infty$, we let l_p denote the set of all scalar sequences $\{x_n\}_n$ such that $\|x\|_p := \left(\sum\limits_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty$.

The operator classes, which are the subject of this article, are closely related to some vector-valued sequence spaces. For this reason we shall recall here a few definitions and results about these spaces.

Definition 1. ([5]) Let $1 \leq p \leq \infty$. The vector sequence $\{x_n\}_n$ in E is strongly $p-$summable if the corresponding scalar sequence $\{\|x_n\|\}_n$ is in l_p. We denote by $l_p^{strong}(E)$ the set of all such sequences in E.

It is clearly a vector space under pointwise operations, and a natural norm is given by $\|\{x_n\}\|_p^{strong} := \left(\sum\limits_{n=1}^{\infty} \|x_n\|^p\right)^{\frac{1}{p}}$, respectively $\|\{x_n\}\|_\infty^{strong} := \sup\limits_n \|x_n\|$.

Definition 2. ([5]) Let $1 \leq p \leq \infty$. The vector sequence $\{x_n\}_n$ in E is weakly $p-$summable if the scalar sequences $\{|\langle x^*, x_n\rangle|\}_n$ are in l_p for every $x^* \in E^*$. We denote by $l_p^{weak}(E)$ the set of all such sequences in E.

It is clearly a vector space under pointwise operations, and a norm is given by $\|\{x_n\}\|_p^{weak} := \sup_{x^* \in U_{E^*}} \left(\sum_{n=1}^{\infty} |\langle x^*, x_n \rangle|^p \right)^{\frac{1}{p}}$, respectively $\|\{x_n\}\|_\infty^{weak} := \sup_{x^* \in U_{E^*}} \sup_n |\langle x^*, x_n \rangle| = \sup_n \|x_n\| = \|\{x_n\}\|_\infty^{strong}$.

Definition 3.([13]) For $x = \{x_n\}_n \in l_\infty$, let

$$s_n(x) := \inf \{\sigma \geq 0 : card\{i : |x_i| \geq \sigma\} < n\}.$$

Proposition 4. ([13]) The numbers $s_n(x)$ have the following properties:

(1). $\|x\|_\infty = s_1(x) \geq s_2(x) \geq ... \geq 0$, for all $x = \{x_n\}_n \in l_\infty$;

(2). $s_{n+m-1}(x+y) \leq s_n(x) + s_m(y)$, for all $x = \{x_i\}_i \in l_\infty, y = \{y_i\}_i \in l_\infty$, and $n, m \in \{1, 2, ...\}$, where $x + y = \{x_i + y_i\}_i$;

(3). $s_{n+m-1}(x \cdot y) \leq s_n(x) \cdot s_m(y)$, for all $x = \{x_i\}_i \in l_\infty, y = \{y_i\}_i \in l_\infty$, and $n, m \in \{1, 2, ...\}$, where $x \cdot y = \{x_i \cdot y_i\}_i$;

(4). If $x = \{x_m\}_m \in l_\infty$ and $card\{m : x_m \neq 0\} < n$ then $s_n(x) = 0$.

If the sequence $x = \{x_n\}_n \in l_\infty$ is ordered such that $|x_n| \geq |x_{n+1}|$, for any natural n, then $s_n(x) = |x_n|$, [13].

Definition 5. (Lorentz sequence spaces) ([9]) Let $1 \leq p \leq \infty$, $1 \leq q < \infty$, or $1 \leq p \leq \infty$, $q = \infty$. The vector sequence $\{x_n\}_n$ in E is strongly $(p,q)-$summable if $\sum_{n=1}^{\infty} \left[i^{\frac{1}{p}-\frac{1}{q}} \cdot s_n(\|x\|) \right]^q$ is finite, respectively $\sup_n i^{\frac{1}{p}} \cdot s_n(\|x\|)$ is finite, where

$$s_n(\|x\|) := s_n\left(\{\|x_i\|_E\}_i\right).$$

The space of all such sequences in E will be called the Lorentz sequence space and will be denoted by $l_{p,q}^{strong}(E)$. In particular, if $E = \Gamma$, then $l_{p,q}^{strong}(\Gamma)$ is denoted $l_{p,q}$.

It is clear that $l_{p,q}^{strong}(E)$ is a vector space under pointwise operations, and a natural quasi-norm is given by

$$\|\{x_n\}\|_{p,q}^{strong} := \left(\sum_{n=1}^{\infty} \left[i^{\frac{1}{p}-\frac{1}{q}} \cdot s_n(\|x\|) \right]^q \right)^{\frac{1}{q}},$$

respectively

$$\|\{x_n\}\|_{p,\infty}^{strong} := \sup_n i^{\frac{1}{p}} \cdot s_n(\|x\|).$$

It is important for our future considerations to recall the **lexicografic order** of the Lorentz spaces.

Proposition 6. ([7], [9]) (1) Let $1 \leq p < \infty$, $1 \leq q < q_1 \leq \infty$. Then $l_{p,q}^{strong}(E) \subset l_{p,q_1}^{strong}(E)$ and for every $\{x_i\}_i \in l_{p,q}^{strong}(E)$,

$$\|\{x_n\}\|_{p,q_1}^{strong} \leq c(p,q,q_1) \cdot \|\{x_n\}\|_{p,q}^{strong}.$$

(2) Let $1 \leq p < p_1 \leq \infty$, $1 \leq q, q_1 \leq \infty$. Then $l_{p,q}^{strong}(E) \subset l_{p_1,q_1}^{strong}(E)$ and, for every $\{x_i\}_i \in l_{p,q}^{strong}(E)$,

$$\|\{x_n\}\|_{p_1,q_1}^{strong} \leq \bar{c}(p,p_1,q,q_1) \cdot \|\{x_n\}\|_{p,q}^{strong}.$$

We now recall, from [6], some basic facts about the classical real interpolation method, called the K-method. An interpolation method is a method of constructing interpolation spaces from a given couple of spaces. For the reader interested in finding an introduction to interpolation theory we recommend, for example, [2], [6], [15].

We consider couples (A_0, A_1) of topological vector spaces A_0, A_1, which are both continuously embedded in a topological vector space $\mathcal{A}$. We denote this by $A_i \hookrightarrow \mathcal{A}$, $i = 1, 2$ and we say that (A_0, A_1) is an interpolation couple.

If (A_0, A_1), (B_0, B_1) are two such couples with $A_0, A_1 \hookrightarrow \mathcal{A}$, $B_0, B_1 \hookrightarrow \mathcal{B}$ and if A and B are two other spaces with $A \hookrightarrow \mathcal{A}$ and $B \hookrightarrow \mathcal{B}$ we say that A and B are interpolation spaces with respect to the couples (A_0, A_1) and (B_0, B_1) if the following interpolation property is fulfilled:

For every linear operator T such that $T : A_0 \to B_0$, $T : A_1 \to B_1$ it follows that $T : A \to B$.

Here we let the symbol $T : A \to B$ denote that the restriction to A of the linear operator T is continuous.

Let (A_0, A_1) be an interpolation couple of quasi-normed spaces. For every $a \in A_0 + A_1$ we define the functional

$$K(t, a, A_0, A_1) = K(t, a) = \inf_{a = a_0 + a_1} \left(\|a_0\|_{A_0} + t \cdot \|a_1\|_{A_1} \right),$$

where $a_i \in A_i, i = 0, 1$, and $0 < t < \infty$.

For $0 < \theta < 1$ and $0 < q \leq \infty$ the spaces

$$(A_0, A_1)_{\theta,q} := \left\{ a; a \in A_0 + A_1 : \left(\int_0^\infty \left[t^{-\theta} \cdot K(t,a) \right]^q \frac{dt}{t} \right) < \infty \right\},$$

if $q < \infty$, and

$$(A_0, A_1)_{\theta,\infty} := \left\{ a; a \in A_0 + A_1 : \sup_{t>0} \sup t^{-\theta} \cdot K(t,x) < \infty \right\}$$

with the quasi-norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} := \left(\int_0^\infty \left[t^{-\theta} \cdot K(t,a) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

respectively

$$\|a\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{t>0} \sup t^{-\theta} \cdot K(t,x),$$

are interpolation spaces. We have the following fundamental interpolation theorem.

Theorem 7. ([6]) *If (A_0, A_1), (B_0, B_1) are two interpolation couples of quasi-normed spaces and if T is a linear operator such that $T : A_0 \to B_0$, $T : A_1 \to B_1$ are both bounded, having the quasi-norms bounded from above by M_0 and M_1 respectively, then $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is also bounded, and its quasi-norm is bounded from above by M for which we have the so called convexity inequality $M \leq M_0^{1-\theta} \cdot M_1^{\theta}$.*

Theorem 8. ([13]) *Let $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1, q \leq \infty$, $0 < \theta < 1$. If $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ then $\left(l_{p_0,q_0}^{strong}(E), l_{p_1,q_1}^{strong}(E) \right)_{\theta,q} = l_{p,q}^{strong}(E)$. Moreover, the quasi-norms on both sides are equivalent.*

We can now introduce some classes of absolutely summing operators.

Definition 9. ([5]) Let $1 \leq p < \infty$. An operator $T \in L(E,F)$ is called absolutely p-summing, we write $T \in \Pi_p(E,F)$, if there is a constant $c \geq 0$ such that

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq c \cdot \sup_{a \in U_{E^*}} \left(\sum_{i=1}^n |\langle x_i, a \rangle|^p \right)^{\frac{1}{p}},$$

for every finite family of elements $x_1, ... x_n \in E$.

For $T \in \Pi_p(E,F)$ we define $\pi_p(T) := \inf c$, the infimum being taken over all constants $c \geq 0$ for which the above inequality holds.

Note that $\pi_p(\cdot)$ is a norm on the space of absolutely $p-$summing operators, [5], [10].

The most deep result concerning absolutely $p-$summing operators is given by the following statement called **the domination theorem.**

Theorem 10. ([5], [10]) *Let $1 \leq p < \infty$, $T \in L(E,F)$ and K be a weak*-compact norming subset of U_{E^*}. Then $T \in \Pi_p(E,F)$ if and only if there is a constant c and a* *regular probability measure* μ *on* K *such that*

$$\|Tx\| \leq c \cdot \left(\int_{U_{E^*}} (|\langle x, x^* \rangle|)^p \, d\mu(x^*) \right)^{\frac{1}{p}},$$

for every $x \in E$, *and* $\pi_p(T) = \inf c$.

We conclude this section by recalling the definition of the $(p, q; r)$ −absolutely summing operators

Definition 11. ([9]) For $1 \leq p, q, r \leq \infty$ an operator $T \in L(E, F)$ is called $(p, q; r)$ −absolutely summing provided there exists a constant $c > 0$ such that

$$\|\{Tx_i\}\|_{p,q}^{strong} \leq c \cdot \|\{x_i\}\|_r^{weak}$$

for every $\{x_i\} \in \mathcal{F}(E)$. We denote by $\Pi_{p,q;r}(E, F)$ the space of such operators acting between E and F.

The smallest number c for which the above inequality holds is denoted by $\pi_{p,q;r}(T)$.

It is observed in [7] that $\pi_{p,q;r}(\cdot)$ is a quasi-norm on the space of $(p, q; r)$-absolutely summing operators.

3. **Results**

We are first concerned with the interpolation result for $(p, q; r)$ −absolutely summing operators established in [9], as we also indicated in the introduction.

Theorem 12. *Let* $1 \leq p_1 < p_2 < \infty$, $1 \leq q_1, q_2, q, r < \infty$ *and* $0 < \theta < 1$. *If* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *then* $(\Pi_{p_1,q_1;r}(E, F), \Pi_{p_2,q_2;r}(E, F))_{\theta,q} \subset \Pi_{p,q;r}(E, F)$.

Proof. We shall use an idea owed to H. König, see Proposition 3 from [8], but first we must prove that $(\Pi_{p_1,q_1;r}(E, F), \Pi_{p_2,q_2;r}(E, F))$ is an interpolation couple.

Let $T \in \Pi_{p_1,q_1;r}(E, F)$ and $\{x_i\}_i \in \mathcal{F}(E)$. It follows that there exists a constant $c > 0$ such that $\|\{Tx_i\}\|_{p_1,q_1}^{strong} \leq c \cdot \|\{x_i\}\|_r^{weak}$.

But we know that $\|\{Tx_n\}\|_{p_2,q_2}^{strong} \leq \overline{c} \cdot \|\{Tx_n\}\|_{p_1,q_1}^{strong}$. Thus we obtain $\|\{Tx_n\}\|_{p_2,q_2}^{strong} \leq \overline{c} \cdot \|\{Tx_n\}\|_{p_1,q_1}^{strong} \leq \widetilde{c} \cdot \|\{x_i\}\|_r^{weak}$. In conclusion $T \in \Pi_{p_2,q_2;r}(E, F)$ and $\Pi_{p_1,q_1;r}(E, F) \subset \Pi_{p_2,q_2;r}(E, F)$.

Let now $T \in \Pi_{p_2,q_2;r}(E, F)$ and take $\{x_i\}_{i=1}^n \in \mathcal{F}(E)$ with $\|\{x_i\}\|_r^{weak} = 1$. The estimate of the K−functional

$K(t, T, \Pi_{p_1,q_1;r}(E, F), \Pi_{p_2,q_2;r}(E, F)) =$

$$= \inf \left\{ \pi_{p_1,q_1;r}(S) + t \cdot \pi_{p_2,q_2;r}(T-S) : S \in \Pi_{p_1,q_1;r}(E,F) \right\} \geq$$
$$\inf \left\{ \|\{Sx_i\}\|_{p_1,q_1}^{strong} + t \cdot \|\{(T-S)x_i\}\|_{p_2,q_2}^{strong} : S \in \Pi_{p_1,q_1;r}(E,F) \right\} \geq$$
$$\inf \left\{ \|\{y_i\}\|_{p_1,q_1}^{strong} + t \cdot \|\{Tx_i - y_i\}\|_{p_2,q_2}^{strong} : y_1, ..., y_n \in F \right\} =$$
$$= K\left(t, \{Tx_i\}_i, l_{p_1,q_1}^{strong}(F), l_{p_2,q_2}^{strong}(F)\right)$$

implies that

$$\|T\|_{\left(\Pi_{p_1,q_1;r}(E,F),\Pi_{p_2,q_2;r}(E,F)\right)_{\theta,q}} \geq \widehat{c} \cdot \|\{Tx_i\}\|_{\left(l_{p_1,q_1}^{strong}(F),l_{p_2,q_2}^{strong}(F)\right)_{\theta,q}}.$$

But we know that $\|\{Tx_i\}\|_{\left(l_{p_1,q_1}^{strong}(F),l_{p_2,q_2}^{strong}(F)\right)_{\theta,q}} \geq \widetilde{\widetilde{c}} \cdot \|\{Tx_i\}\|_{l_{p,q}^{strong}(F)}$. Therefore, by taking the supremum over all $\{x_i\}_{i=1}^n \in \mathcal{F}(E)$ with $\|\{x_i\}\|_r^{weak} = 1$, we get that

$$\|T\|_{\left(\Pi_{p_1,q_1;r}(E,F),\Pi_{p_2,q_2;r}(E,F)\right)_{\theta,q}} \geq c \cdot \pi_{p,q;r}(T).$$

In conclusion $\left(\Pi_{p_1,q_1;r}(E,F), \Pi_{p_2,q_2;r}(E,F)\right)_{\theta,q} \subset \Pi_{p,q;r}(E,F)$, as wanted. □

We now recall some results concerning the Lorentz-Zygmund sequence spaces, which were introduced by C.Bennet and K. Rudnick, [3], and generalize the Lorentz sequence spaces.

Definition 13. ([3], [4]) Let $1 \leq p, q \leq \infty$ and $-\infty < \gamma < \infty$. The Lorentz-Zygmund sequence spaces are defined as follows

$$l_{p,q,\gamma} = \left\{ \xi = \{\xi_n\}_n \in c_0 : \sum_{n=1}^{\infty} \left[n^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log n)^{\gamma} \cdot s_n(\xi) \right]^q < \infty \right\},$$

if $q < \infty$, and

$$l_{p,\infty,\gamma} = \left\{ \xi = \{\xi_n\}_n \in c_0 : \sup_n \left[n^{\frac{1}{p}} \cdot (1+\log n)^{\gamma} \cdot s_n(\xi) \right] < \infty \right\}.$$

Remark 1. ([4]) The formulas

$$\|\cdot\|_{p,q,\gamma} := \left(\sum_{n=1}^{\infty} \left[n^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log n)^{\gamma} \cdot s_n(\cdot) \right]^q \right)^{\frac{1}{q}},$$

respectively

$$\|\cdot\|_{p,\infty,\gamma} := \sup_n \left[n^{\frac{1}{p}} \cdot (1+\log n)^{\gamma} \cdot s_n(\cdot) \right],$$

define quasi-norms on $l_{p,q,\gamma}$, respectively on $l_{p,\infty,\gamma}$.

The **lexicografic order** of the Lorentz-Zygmund sequence spaces is important for our proofs so we establish it here.

Proposition 14. The following inclusions hold:

1. $l_{p_0,q,\gamma_0} \subseteq l_{p_1,q,\gamma_1}$, for $1 \leq p_0 < p_1 < \infty$, $1 \leq q \leq \infty$, $-\infty < \gamma_0, \gamma_1 < \infty$;
2. $l_{p,q_0,\gamma} \subseteq l_{p,q_1,\gamma}$, for $1 \leq p < \infty$, $1 \leq q_0 < q_1 \leq \infty$, $\gamma > 0$.

Moreover, in the first case, there is a constant c_1 such that $\|x\|_{p_1,q,\gamma_1} \leq c_1 \cdot \|x\|_{p_0,q,\gamma_0}$ for every $x \in l_{p_0,q,\gamma_0}$ and in the second case there is a constant c_2 such that $\|x\|_{p,q_1,\gamma} \leq c_2 \cdot \|x\|_{p,q_0,\gamma}$ for every $x \in l_{p,q_0,\gamma}$.

To prove this proposition, we shall need the following results.

Theorem 15. ([4]) *Let $0 < q \leq \infty$ and let $\varphi, \rho \in \mathcal{B}$ $\alpha_{\overline{\rho}} < \beta_{\overline{\varphi}}$. Then $\lambda^q(\varphi)$is continuously embedded in $\lambda^q(\rho)$, where*

$$\lambda^q(\varphi) = \left\{ \xi = \{\xi_n\}_n \in c_0 : \sum_{n=1}^{\infty} [\varphi(n) \cdot s_n(\xi)]^q \cdot n^{-1} < \infty \right\},$$

if $q < \infty$, and $\lambda^\infty(\varphi) = \left\{ \xi = \{\xi_n\}_n \in c_0 : \sup_n [\varphi(n) \cdot s_n(\xi)] < \infty \right\}$.

In [14] N. Tita has established a relation between Lorentz spaces and Lorentz-Zygmund spaces, which is content of the next result.

Theorem 16. *Let $1 \leq p, q \leq \infty$, $0 < \gamma < \infty$ and $\xi = \{\xi_n\}_n \in c_0$. Then $\xi \in l_{p,q,\gamma} \Leftrightarrow \left\{ 2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi) \right\}_n \in l_{r,q}$ where $\gamma = \frac{1}{r} - \frac{1}{q}$. Moreover, there are constants $\widetilde{c}(p,q,\gamma)$ and $\overline{c}(p,q,\gamma)$ such that $\widetilde{c}(p,q,\gamma) \cdot \left\| \left\{ 2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi) \right\}_n \right\|_{r,q} \leq \|\xi\|_{p,q,\gamma} \leq$*

$$\leq \overline{c}(p,q,\gamma) \cdot \left\| \left\{ 2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi) \right\}_n \right\|_{r,q}.$$

Proof of Proposition 14. 1. Consider $\varphi : (0,\infty) \to (0,\infty)$ defined by $\varphi(t) = t^{\frac{1}{p_0}} \cdot (1 + \log|t|)^{\gamma_0}$ and $\rho : (0,\infty) \to (0,\infty)$ defined by $\rho(t) = t^{\frac{1}{p_1}} \cdot (1 + \log|t|)^{\gamma_1}$. Then $\varphi, \rho \in \mathcal{B}$ and $\beta_{\overline{\varphi}} = \frac{1}{p_0}$, $\alpha_{\overline{\rho}} = \frac{1}{p_1}$, [4]. Hence if $0 < p_0 < p_1 < \infty$, then $\alpha_{\overline{\rho}} < \beta_{\overline{\varphi}}$, and Theorem 15 applies to give the desired inclusion.

To prove 2., note that by Theorem 16 $\xi \in l_{p,q_0,\gamma} \Leftrightarrow \left\{ 2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi) \right\}_n \in l_{r_0,q_0}$ where $\gamma = \frac{1}{r_0} - \frac{1}{q_0}$. Let $q_1 > q_0$ and r_1 such that $\gamma = \frac{1}{r_1} - \frac{1}{q_1}$. It follows that $r_0 < r_1$ and further on $l_{r_0,q_0} \subseteq l_{r_1,q_1}$. So we obtain that $\left\{ 2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi) \right\}_n \in l_{r_1,q_1}$ which then implies that $\xi \in l_{p,q_1,\gamma}$.

Remark 2. We must give here an explanation. In [14] there were given results for the operator ideals $L^{(s)}_{p,q,\gamma}$, where s is an additive and multiplicative s-scale, an $s-$scale being a rule $s : T \in L(E,F) \to \{s_n(T)\} \in l_\infty$ which assigns to every linear and bounded operator a bounded scalar sequence with the following properties:

1. $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$, for all $T \in L(E,F)$;

2. $s_{n+m-1}(T+S) \leq s_n(T) + s_m(S)$, for all $T, S \in L(E,F)$ and $n, m \in \{1, 2, ...\}$;
3. $s_{n+m-1}(T \circ S) \leq s_n(T) \cdot s_m(S)$, for all $T \in L(F, F_0), S \in L(E,F)$ and $n, m \in \{1, 2, ...\}$;
4. $s_n(T) = 0$, $\dim T < n$;
5. $s_n(I_E) = 1$, if $\dim E \geq n$, where $I_E(x) = x$, for all $x \in E$.

We call $s_n(T)$ the n-th $s-$number of the operator T. For properties, examples of $s-$numbers and relations between different s-numbers we refer the reader to [10], [12], [13].

If we take account of the similarity between the axioms of the sequence $\{s_n(T)\}_n$, where s is an additive s-scale, $T \in L(E,F)$, and the properties of $\{s_n(x)\}_n$, where $x = \{x_n\}_n \in l_\infty$, we can transfer the result obtained in [14] by N. Tita from $L^{(s)}_{p,q,\gamma}$ to $l_{p,q,\gamma}$.

In [14], an interpolation result for the Lorentz-Zygmund operator ideals $L^{(s)}_{p,q,\gamma}$ is also established. We can also transfer this to the sequence spaces case, as follows.

Theorem 17. *Let* $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0 \leq q_1 \leq \infty$, $1 \leq q \leq \infty$, $0 < \gamma_0, \gamma_1 < \infty$ *and* $0 < \theta < 1$. *Then*

$$\left(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1}\right)_{\theta,q} \subseteq l_{p,q,\gamma},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\gamma = (1-\theta) \cdot \gamma_0 + \theta \cdot \gamma_1$.

Moreover for every $x \in \left(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1}\right)_{\theta,q}$ *the following inequality is true*

$$\|x\|_{p,q,\gamma} \leq c(p,q,\gamma) \cdot \|x\|_{\left(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1}\right)_{\theta,q}}.$$

We start now our construction which generalizes Miyazaki's spaces..

Definition 18. Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $-\infty < \gamma < \infty$. The vector sequence $\{x_n\}_n$ in E is strongly $(p,q,\gamma)-$summable if $\{\|x_n\|\}_n \in l_{p,q,\gamma}$. We denote by $l^{strong}_{p,q,\gamma}(E)$ the set of all such sequences in E. It is easy to see that $l^{strong}_{p,q,\gamma}(E)$ is a vector space under pointwise operations, and a natural quasi-norm is given by $\|\{x_n\}\|^{strong}_{p,q,\gamma} := \|\{\|x_n\|\}\|_{p,q,\gamma}$.

Remark 3. It is not hard to verify that all the above results for $l_{p,q,\gamma}$ can be transferred to $l^{strong}_{p,q,\gamma}(E)$.

Definition 19. Suppose that $1 \leq p, q, r \leq \infty$ and $-\infty < \gamma < \infty$. An operator $T \in L(E, F)$ is called $(p, q, \gamma; r)$ −absolutely summing provided there exists a constant $c > 0$ such that $\|\{Tx_m\}\|_{p,q,\gamma}^{strong} \leq c \cdot \|\{x_m\}\|_r^{weak}$, for every $\{x_m\}_m \in \mathcal{F}(E)$. We denote by $\Pi_{p,q,\gamma;r}(E, F)$ the space of such operators acting between E and F.

The smallest number c for which the above inequality holds is denoted by $\pi_{p,q,\gamma;r}(T)$.

Remark 4. It is routine to verify that the constant coming from $\|\cdot\|_{p,q,\gamma}^{strong}$ can be used to prove the triangle inequality, and thus $\pi_{p,q,\gamma;r}(\cdot)$ is a quasi-norm on the space of $(p, q, \gamma; r)$ −absolutely summing operators.

Theorem 20. *Let* $1 \leq p_1 < p_2 < \infty$, $1 \leq q_1 \leq q_2 \leq \infty$, $1 \leq q, r \leq \infty$, $0 < \gamma_1, \gamma_2 < \infty$ *and* $0 < \theta < 1$. *If* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *and* $\gamma = (1-\theta) \cdot \gamma_1 + \theta \cdot \gamma_2$ *then*

$$\left(\Pi_{p_1,q_1,\gamma_1;r}(E, F), \Pi_{p_2,q_2,\gamma_2;r}(E, F)\right)_{\theta,q} \subset \Pi_{p,q,\gamma;r}(E, F).$$

Proof. We shall use the idea from the case of $(p, q; r)$ −absolutely summing operators. First we must prove that $\left(\Pi_{p_1,q_1,\gamma_1;r}(E, F), \Pi_{p_2,q_2,\gamma_2;r}(E, F)\right)$ is an interpolation couple.

Let $T \in \Pi_{p_1,q_1,\gamma_1;r}(E, F)$ and $\{x_i\}_i \in \mathcal{F}(E)$. It follows that there exists a constant $c > 0$ such that $\|\{Tx_i\}\|_{p_1,q_1,\gamma_1}^{strong} \leq c \cdot \|\{x_i\}\|_r^{weak}$. But $\|\{Tx_n\}\|_{p_2,q_2,\gamma_2}^{strong} \leq \overline{c} \cdot \|\{Tx_n\}\|_{p_1,q_1,\gamma_1}^{strong}$. Hence we obtain

$$\|\{Tx_n\}\|_{p_2,q_2,\gamma_2}^{strong} \leq \overline{c} \cdot \|\{Tx_n\}\|_{p_1,q_1,\gamma_1}^{strong} \leq \widetilde{c} \cdot \|\{x_i\}\|_r^{weak}.$$

¿From this it follows that $T \in \Pi_{p_2,q_2,\gamma_2;r}(E, F)$, and therefore $\Pi_{p_1,q_1,\gamma_1;r}(E, F) \subset \Pi_{p_2,q_2,\gamma_2;r}(E, F)$.

Let now $T \in \Pi_{p_2,q_2,\gamma_2;r}(E, F)$ and pick $\{x_i\}_{i=1}^n \in \mathcal{F}(E)$ with $\|\{x_i\}\|_r^{weak} = 1$. The estimate of the K−functional $K\left(t, T, \Pi_{p_1,q_1,\gamma_1;r}(E, F), \Pi_{p_2,q_2,\gamma_2;r}(E, F)\right) =$
$\inf\left\{\pi_{p_1,q_1,\gamma_1;r}(S) + t \cdot \pi_{p_2,q_2,\gamma_2;r}(T - S) : S \in \Pi_{p_1,q_1,\gamma_1;r}(E, F)\right\} \geq$
$\inf\left\{\|\{Sx_i\}\|_{p_1,q_1,\gamma_1}^{strong} + t \cdot \|\{(T-S)x_i\}\|_{p_2,q_2,\gamma_2}^{strong} : S \in \Pi_{p_1,q_1,\gamma_1;r}(E, F)\right\} \geq$
$\inf\left\{\|\{y_i\}\|_{p_1,q_1,\gamma_1}^{strong} + t \cdot \|\{Tx_i - y_i\}\|_{p_2,q_2,\gamma_2}^{strong} : y_1, ..., y_n \in F\right\} =$
$= K\left(t, \{Tx_i\}_i, l_{p_1,q_1,\gamma_1}^{strong}(F), l_{p_2,q_2,\gamma_2}^{strong}(F)\right)$, implies that

$$T_{\left(\Pi_{p_1,q_1,\gamma_1;r}(E,F),\Pi_{p_2,q_2,\gamma_2;r}(E,F)\right)_{\theta,q}} \geq \widehat{c} \cdot \|\{Tx_i\}\|_{\left(l_{p_1,q_1,\gamma_1}^{strong}(F), l_{p_2,q_2,\gamma_2}^{strong}(F)\right)_{\theta,q}}.$$

But we know that $\|\{Tx_i\}\|_{\left(l^{strong}_{p_1,q_1}(F),l^{strong}_{p_2,q_2}(F)\right)_{\theta,q}} \geq \widetilde{\widetilde{c}} \cdot \|\{Tx_i\}\|^{strong}_{p,q,\gamma}$. Taking the supremum, over all these $\{x_i\}_{i=1}^{n} \in \mathcal{F}(E)$, we get

$$\|T\|_{\left(\Pi_{p_1,q_1,\gamma_1;r}(E,F),\Pi_{p_2,q_2,\gamma_2;r}(E,F)\right)_{\theta,q}} \geq c \cdot \pi_{p,q;r}(T).$$

In conclusion $\left(\Pi_{p_1,q_1,\gamma_1;r}(E,F), \Pi_{p_2,q_2,\gamma_2;r}(E,F)\right)_{\theta,q} \subset \Pi_{p,q,\gamma;r}(E,F)$. □

We can further on generalize the Miyasaki operator classes. First we introduce some vector-valued sequence spaces.

Definition 21. Let $1 \leq p,q < \infty$, $-\infty < \gamma < \infty$. The vector sequence $\{x_n\}_n$ in E is weakly (p,q,γ) −summable if the scalar sequences $\{|\langle x^*, x_n\rangle|\}_n$ are in $l_{p,q,\gamma}$ for every $x^* \in E^*$. We denote by $l^{weak}_{p,q,\gamma}(E)$ the set of all such sequences in E.

Proposition 22. Suppose that $1 \leq q < p < \infty$ and $\gamma < 0$, or $1 \leq q < p < \infty$ and $0 < \gamma$ such that $\frac{1}{q} - \frac{1}{p} \geq \gamma$. Then $l^{weak}_{p,q,\gamma}(E)$ is a vector space under pointwise operations, and the formula

$$\|\{x_n\}\|^{weak}_{p,q,\gamma} := \sup_{x^* \in U_{E^*}} \left(\sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}}(1+\log n)^{\gamma}|\langle x^*, x_n\rangle|\right]^q\right)^{\frac{1}{q}}$$

defines a quasi-norm $\|\cdot\|^{weak}_{p,q,\gamma} : l^{weak}_{p,q,\gamma}(E) \to \mathbb{R}_+$.

Proof. The first step is to show that the quantity in the right side of the formula is finite. We shall apply the closed graph theorem like in the case of absolutely p−summing operators, cf. [5]. Let $x = \{x_n\}_n \in l^{weak}_{p,q,\gamma}(E)$ and associate with it the map $u : E^* \to l_{p,q,\gamma}$ given by $u(x^*) = \{\langle x^*, x_n\rangle\}_n$. Note that u is a well-defined linear map. Consider now a sequence $\{x^*_k\}_k$ which converges to x^*_0 in E^*.Then for each n, the scalar sequence $\{\langle x^*_k, x_n\rangle\}_k$ converges to $\langle x^*_0, x_n\rangle$. Thus, if we take into account the fact that $\left\{n^{\frac{1}{p}-\frac{1}{q}}(1+\log n)^{\gamma}\right\} \in c_0$, for which purpose we have made the choice of p,q and γ, we obtain as a consequence, that u has closed graph. Therefore, u is bounded. In other words

$$\|u\| = \sup_{x^* \in U_{E^*}} \left(\sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}}(1+\log n)^{\gamma}|\langle x^*, x_n\rangle|\right]^q\right)^{\frac{1}{q}} < \infty.$$

Now it is easy to check that $\|\cdot\|^{weak}_{p,q,\gamma}$ is a quasi-norm on $l^{weak}_{p,q,\gamma}(E)$. □

Definition 23. Let $1 \leq p,q < \infty$ and $-\infty < \gamma < \infty$. Suppose that $1 \leq s < r < \infty$ and $\alpha < 0$, or $1 \leq s < r < \infty$ and $0 < \alpha$ are such that $\frac{1}{s} - \frac{1}{r} \geq \alpha$. An operator

$T \in L(E,F)$ is called $(p,q,\gamma;r,s,\alpha)$ $-$absolutely summing if there exists a constant $c \geq 0$ such that $\|\{Tx_i\}\|_{p,q,\gamma}^{strong} \leq c \cdot \|\{x_i\}\|_{r,s,\alpha}^{weak}$ for every $\{x_i\} \in \mathcal{F}(E)$. We denote by $\Pi_{p,q,\gamma;r,s,\alpha}(E,F)$ the space of such operators acting between E and F.

The smallest number c for which the above inequality holds is denoted by $\pi_{p,q,\gamma;r,s,\alpha}(T)$.

Remark 5. It is straightforward to verify that the constant coming from $\|\cdot\|_{p,q,\gamma}^{strong}$ can be used to prove the triangle inequality and thus $\pi_{p,q,\gamma;r,s,\alpha}(\cdot)$ is a quasi-norm on the space of $(p,q,\gamma;r,s,\alpha)$ $-$absolutely summing operators.

Remark 6. Using the domination theorem it is routine to prove that

$$\Pi_{p,q,\gamma;p,q,\gamma}(E,F) \supseteq \Pi_q(E,F).$$

Moreover $\pi_q(T) \geq \pi_{p,q,\gamma;p,q,\gamma}(T)$ for every $T \in \Pi_q(E,F)$.

If the sequence $\alpha_n = \left\{n^{\frac{q}{p}-1} \cdot (1+\log n)^{\gamma \cdot q}\right\}$ is a decreasing one then $\Pi_{p,q,\gamma;p,q,\gamma}(E,F)$ is of the type $\Pi_{\Phi,\Psi}(E,F)$, where Φ, Ψ are symmetric norming function.

The following theorem, which is a representation result for our class of operators, will be the essential ingredient in our main theorem.

Theorem 24. *Let $1 \leq p,q < \infty$ and $-\infty < \gamma < \infty$. Suppose that $1 \leq s < r < \infty$ and $\alpha < 0$, or $1 \leq s < r < \infty$ and $0 < \alpha$ are such that $\frac{1}{s} - \frac{1}{r} \geq \alpha$. Then an operator $T \in L(E,F)$ is $(p,q,\gamma;r,s,\alpha)$ $-$absolutely summing if and only if $\widehat{T}\left(l_{r,s,\alpha}^{weak}(E)\right)$ is contained in $l_{p,q,\gamma}^{strong}(F)$, where $\widehat{T} : \{x_i\}_i \to \{Tx_i\}_i$. In this case*

$$\left\|\widehat{T} : l_{r,s,\alpha}^{weak}(E) \to l_{p,q,\gamma}^{strong}(F)\right\| = \pi_{p,q,\gamma;r,s,\alpha}(T).$$

The proof is similar to the case of $p-$absolutely summing operators, cf. [5], so we omit it.

We are now ready to state our main result.

Theorem 25. *Let $1 \leq p,q < \infty$ and $-\infty < \gamma < \infty$. Suppose that $1 \leq s < r < \infty$ and $\alpha < 0$, or $1 \leq s < r < \infty$ and $0 < \alpha$ are such that $\frac{1}{s} - \frac{1}{r} \geq \alpha$. Let also $0 < \theta < 1$. Then*

$$\left(\Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F), \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)\right)_{\theta,q} \subseteq \Pi_{p,q,\gamma;r,s,\alpha}(E,F),$$

where $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *and* $\gamma = (1-\theta) \cdot \gamma_1 + \theta \cdot \gamma_2$.

Proof. We shall use an idea owed to A. Pietsch, see [10], Proposition 1.2.6. First we must prove that $\left(\Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F), \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)\right)$ is an interpolation couple.

Let $T \in \Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F)$ and $\{x_i\}_i \in \mathcal{F}(E)$. It follows that there exists a constant $\widetilde{c} > 0$ such that $\|\{Tx_i\}\|^{strong}_{p_1,q_1,\gamma_1} \leq \widetilde{c} \cdot \|\{x_i\}\|^{weak}_{r,s,\alpha}$. But we know that $\|\{Tx_n\}\|^{strong}_{p_2,q_2,\gamma_2} \leq \overline{c} \cdot \|\{Tx_n\}\|^{strong}_{p_1,q_1,\gamma_1}$. Therefore

$$\|\{Tx_i\}\|^{strong}_{p_2,q_2,\gamma_2} \leq \overline{c} \cdot \|\{Tx_i\}\|^{strong}_{p_1,q_1,\gamma_1} \leq c \cdot \|\{x_i\}\|^{weak}_{r,s,\alpha}.$$

In conclusion $T \in \Pi_{p_2,q_2;r,s,\alpha}(E,F)$ and $\Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F) \subset \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)$.

Let now $\{x_i\}_i \in l^{weak}_{r,s,\alpha}(E)$. We define the operator $X : T \in L(E,F) \to \{Tx_i\}_i$. It follows from the preceding representation theorem that $\{Tx_i\}_i \in l^{strong}_{p_1,q_1,\gamma_1}(F)$, if $T \in \Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F)$ and $\{Tx_i\}_i \in l^{strong}_{p_2,q_2,\gamma_2}(F)$, if $T \in \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)$. Thus

$$\begin{aligned} X &: \Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F) \to l^{strong}_{p_1,q_1,\gamma_1}(F), \\ X &: \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F) \to l^{strong}_{p_2,q_2,\gamma_2}(F), \end{aligned}$$

are linear and bounded.

It now follows from the interpolation Theorems 6 and 8 that

$$X : \left(\Pi_{p_1,q_1,\gamma_1;r,s,\alpha}(E,F), \Pi_{p_2,q_2,\gamma_2;r,s,\alpha}(E,F)\right)_{\theta,q} \to$$

$$\left(l^{strong}_{p_1,q_1,\gamma_1}(E), l^{strong}_{p_2,q_2,\gamma_2}(E)\right)_{\theta,q} \subseteq l^{strong}_{p,q,\gamma}(E)$$

Hence the assertion follows from the representation theorem. □

Acknowledgements. I would like to express my gratitude to N. Tita for many fruitfull discussions.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

LACUNARY STRONG A-CONVERGENCE WITH RESPECT TO A MODULUS

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Abstract. The definition of lacunary strong convergence with respect to a modulus is extended to a definition of lacunary strong A-convergence with respect to a modulus when $A = (a_{ik})$ is an infinite matrix of complex numbers. We study some connections between lacunary strong A-convergence with respect to a modulus and lacunary A-statistical convergence.

1. **Introduction**

The notion of modulus function was introduced by Nakano [11]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,

(iii) f is increasing and

(iv) f is continuous from the right at 0. It follows that f must be continuous on $[0, \infty)$.

Connor [2], Esi [3], Kolk [8], Maddox [9], [10], Öztürk and Bilgin [12], Pehlivan and Fisher [13], Ruckle [14] and others used a modulus function to construct sequence spaces.

Following Freedman et al. [4], we call the sequence $\theta = (k_r)$ lacunary if it is an increasing sequence of integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r/k_{r-1}$. These notations will be used throughout the paper. The sequence space of lacunary

1991 *Mathematics Subject Classification.* 40A05, 40F05.

Key words and phrases. lacunary sequence, modulus function, statistical convergence.

strongly convergent sequences N_θ was defined by Freedman et al. [4], as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |x_i - s| = 0 \text{ for some } s \right\}.$$

Recently, the concept of lacunary strongly convergence was generalized by Pehlivan and Fisher [13] as below:

$$N_\theta(f) = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f(|x_i - s|) = 0 \text{ for some } s \right\}.$$

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write $Ax = (A_i(x))$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik}x_k$ converges for each i.

The purpose of this paper is to introduce and study a concept of lacunary strong A-convergence with respect to a modulus.

2. $N_\theta(A,f)$ Convergence

Definition. Let $A = (a_{ik})$ be an infinite matrix of complex numbers and f be a modulus. We define

$$N_\theta(A,f) = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) = 0 \text{ for some } s \right\},$$

$$N_\theta^0(A,f) = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f(|A_i(x)|) = 0 \right\}.$$

A sequence $x = (x_k)$ is said to be lacunary strong A-convergent to a number s with respect to a modulus if there is a complex number s such that $x \in N_\theta(A,f)$. Note that, if we put $f(x) = x$, then $N_\theta(A,f) = N_\theta(A)$ and $N_\theta^0(A,f) = N_\theta^0(A)$. If $x \in N_\theta(A)$, we say that x is lacunary strong A-convergent to s. If x is lacunary strong A-convergent to the value s with respect to a modulus f, then we write $x_i \to s(N_\theta(A,f))$. If $A = I$ unit matrix, we write $N_\theta(f)$ and $N_\theta^0(f)$ for $N_\theta(A,f)$ and $N_\theta^0(A,f)$, respectively. Hence $N_\theta(f)$ is the same as the space $N_\theta(f)$ of Pehlivan and Fisher [13].

$N_\theta(A,f)$ and $N_\theta^0(A,f)$ are linear spaces. We consider only $N_\theta^0(A,f)$. Suppose that $x, y \in N_\theta^0(A,f)$ and a, b are in C, the complex numbers. Then there exist integers

T_a and T_b such that $|a| \leq T_a$ and $|b| \leq T_b$. We therefore have

$$h_r^{-1} \sum_{i \in I_r} f(|aA_i(x) + bA_i(y)|) \leq T_a h_r^{-1} \sum_{i \in I_r} f(|A_i(x)|) + T_b h_r^{-1} \sum_{i \in I_r} f(|A_i(y)|).$$

This implies that $ax + by \in N_\theta^0(A, f)$.

Now we give relation between lacunary strong A-convergence and lacunary strong A-convergence with respect to a modulus.

Theorem 1. *Let f be any modulus. Then $N_\theta(A) \subseteq N_\theta(A, f)$ and $N_\theta^0(A) \subseteq N_\theta^0(A, f)$.*

Proof. We consider $N_\theta(A) \subseteq N_\theta(A, f)$ only. Let $x \in N_\theta(A)$ and $\varepsilon \succ 0$. We choose $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every u with $0 \leq u \leq \delta$. We can write

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) = h_r^{-1} \sum_1 f(|A_i(x) - s|) + h_r^{-1} \sum_2 f(|A_i(x) - s|)$$

where the first summation is over $|A_i(x) - s| \leq \delta$ and the second over $|A_i(x) - s| \succ \delta$. By definition of f, we have

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \leq \varepsilon + 2f(1)\delta^{-1} h_r^{-1} \sum_{i \in I_r} |A_i(x) - s|.$$

Therefore $x \in N_\theta(A, f)$.

Theorem 2. *Let f be any modulus. If $\lim_{t \to \infty} \dfrac{f(t)}{t} = \beta \succ 0$, then $N_\theta(A) = N_\theta(A, f)$.*

Proof. If $\lim_{t \to \infty} \dfrac{f(t)}{t} = \beta \succ 0$, then $f(t) \geq \beta t$ for all $t \succ 0$. Let $x \in N_\theta(A, f)$. Clearly,

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \geq h_r^{-1} \sum_{i \in I_r} \beta |A_i(x) - s| = \beta h_r^{-1} \sum_{i \in I} |A_i(x) - s|,$$

therefore $x \in N_\theta(A)$. By using Theorem 1 the proof is complete.

We now give an example to show that $N_\theta(A) \neq N_\theta(A, f)$ in the case when $\beta = 0$. Consider $A = I$ and the modulus $f(x) = \sqrt{x}$. In the case $\beta = 0$, define x_i to be h_r at the first term in I_r for every r and $x_i = 0$ otherwise. Then we have

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x)|) = h_r^{-1} \sum_{i \in I_r} \sqrt{|x_i|} = h_r^{-1} \sqrt{|h_r|} \to 0 \text{ as } r \to \infty$$

and so $x \in N_\theta(A, f)$. But $h_r^{-1} \sum_{i \in I_r} |A_i(x)| = h_r^{-1} \sum_{i \in I_r} |x_i| = h_r^{-1} h_r \to 1$ as $r \to \infty$ and so $x \notin N_\theta(A)$.

Theorem 3. *Let f be any modulus. Then* *(i) For* $\liminf q_r \succ 1$ *we have* $w(A,f) \subseteq N_\theta(A,f)$.

(ii) For $\limsup q_r \prec \infty$ *we have* $N_\theta(A,f) \subseteq w(A,f)$.

(iii) $w(A,f) = N_\theta(A,f)$ *is* $1 \succ \liminf_r q_r \leq \limsup_r q_r \prec \infty$,

where $w(A,f) = \left\{ x = (x_i) : \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} f(|A_i(x) - s|) = 0 \text{ for some } s \right\}$ *(see, Esi [3]).*

Proof. (i) Let $x \in w(A,f)$ and $\liminf q_r \succ 1$. There exist $\delta \succ 0$ such that $q_r = (k_r/k_{r-1}) \geq 1 + \delta$ for sufficiently large r. We have, for sufficiently large r, that $(h_r/k_r) \geq \delta/(1+\delta)$ and $(k_r/h_r) \leq (1+\delta)/\delta$. Then

$$\begin{aligned} k_r^{-1} \sum_{i-1}^{k_r} f(|A_i(x) - s|) &\geq k_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) \\ &= (h_r/k_r) h_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) \\ &\geq \delta/(1+\delta) h_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) \end{aligned}$$

which yields that $x \in N_\theta(A,f)$.

(ii) If $\limsup q_r \prec \infty$ then there exists $K \succ 0$ such that $q_r \prec K$ for every r. Now suppose that $x \in N_\theta(A,f)$ and $\varepsilon \succ 0$. There exists m_0 such that for every $m \geq m_0$,

$$H_m = h_m^{-1} \sum_{i\in I_m} f(|A_i(x) - x|) \prec \varepsilon.$$

We can also find $T \succ 0$ such that $H_m \leq T$ for all m. Let n be any integer with $k_r \geq n \succ k_{r-1}$. Now write

$$n^{-1} \sum_{i=1}^{n} f(|A_i(x) - s|) \leq k_r^{-1} \sum_{i=1}^{k_r} f(|A_i(x) - s|)$$

$$= k_{r-1}^{-1} \left(\sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{k_r} \right) \sum_{i\in I_m} f(|A_i(x) - s|)$$

$$= k_{r-1}^{-1} \sum_{m=1}^{m_0} \sum_{i\in I_m} f(|A_i(x) - s|) + k_{r-1}^{-1} \sum_{m=m_0+1}^{k_r} \sum_{i\in I_m} f(|A_i(x) - s|)$$

$$\leq k_{r-1}^{-1} \sum_{m=1}^{m_0} \sum_{i\in I_m} f(|A_i(x) - s|) + \varepsilon (k_r - k_{m_0}) k_{r-1}^{-1}$$

$$= k_{r-1}^{-1} (h_1 H_1 + h_2 H_2 + \cdots + h_{m_0} H_{m_0}) + \varepsilon (k_r - k_{m_0}) k_{r-1}^{-1}$$

$$\leq k_{r-1}^{-1} \left(\sup_{1\leq i\leq m_0} H_i k_{m_0} \right) + \varepsilon K \prec k_{r-1}^{-1} k_{m_0} T + \varepsilon K$$

from which we deduce that $x \in w(A, f)$. (iii) follows from (i) and (ii).

The next result follows from Theorem 2 and 3.

Theorem 4. *Let f be any modulus. If* $\lim_{t\to\infty} \frac{f(t)}{t} = \beta \succ 0$ *and* $l \prec \liminf_r q_r \leq \limsup_r q_r \prec \infty$, *then* $N_\theta(A) = w(A, f)$.

3. Lacunary A-statistical convergence

The notation of statistical convergence was given in earlier works [1], [4], [6], [15] and [16]. Recently, Fridy and Orhan [7] introduced the concept of lacunary statistical convergence:

Let θ be a lacunary sequence. Then a sequence $x = (x_k)$ is said to be lacunary statistically convergent to a number s if for every $\varepsilon \succ 0$, $\lim_{r\to\infty} h_r^{-1}|K_\theta(\varepsilon)| = 0$, where $|K_\theta(\varepsilon)|$ denotes the number of elements in $K_\theta(\varepsilon) = \{i \in I_r : \ |x_i - s| \geq \varepsilon\}$. The set of all lacunary statistical convergent sequences is denoted by S_θ.

Let $A = (a_{ik})$ be an infinire matrix of complex numbers. Then a sequence $x = (x_k)$ is said to be lacunary A-statistically convergent to a number s if for every $\varepsilon \succ 0$, $\lim_{r\to\infty} h_r^{-1}|KA_\theta(\varepsilon)| = 0$, where $|KA_\theta(\varepsilon)|$ denotes the number of element in $KA_\theta(\varepsilon) = \{i \in I : \ |A_i(x) - s| \geq \varepsilon\}$. The set of all lacunary A-statistical convergent sequences is denoted by $S_\theta(A)$.

The following Theorem gives the relation between of the lacunary A-statistical convergence and lacunary strongly A-convergence.

Let $I_r^1 = \{i \in I_r : \ |A_i(x) - s| \geq \varepsilon\} = KA_\theta(\varepsilon)$ and $I_r^2 = \{i \in I_r : \ |A_i(x) - s| \prec \varepsilon\}$.

Theorem 5. *Let A be a limitation method, then*

(i) $x_i \to s(N_\theta(A))$ implies $x_i \to s(S_\theta(A))$.

(ii) x is bounded and $x_i \to s(S_\theta(A))$ implys $x_i \to s(N_\theta(A))$.

(iii) $S_\theta(A) = N_\theta(A)$ is x is bounded.

Proof. (i) If $\varepsilon \succ 0$ and $x_i \to s(N_\theta(A))$ we can write

$$h_r^{-1} \sum_{i \in I_r} |A_i(x) - s| \geq h_r^{-1} |KA_\theta(\varepsilon)|\varepsilon.$$

It follows that $x_i \to s(S_\theta(A))$. Note that in this part of the proof we do not need the limitation method of A.

(ii) Suppose that x is lacunary A-statistical convergent to s. Since x is bounded and A is limitation method, there is a constant $T > 0$ such that $|A_i(x) - s| \leq T$ for all i. Therefore we have, for every $\varepsilon \succ 0$, that

$$h_r^{-1} \sum_{i \in I_r} |A_i(x) - s| \leq h_r^{-1} \sum_{i \in I_r^1} |A_i(x) - s| + h_r^{-1} \sum_{i \in I_r^2} |A_i(x) - s| \leq T h_r^{-1} |K A_\theta(\varepsilon)| + \varepsilon.$$

Taking the limit as $\varepsilon \to 0$, the result follows. (iii) follows from (i) and (ii).

Now we give the relation between of the lacunary A-statistical convergence and lacunary strongly A-convergence with respect to modulus.

Theorem 6. *(i) For any modulus f, $x_i \to s(N_\theta(A,f))$ implies $x_i \to s(S_\theta(A))$.*

(ii) f is bounded and $x_i \to s(S_\theta(A))$ imply $x_i \to s(N_\theta(A,f))$.

(iii) $S_\theta(A) = N_\theta(A,f)$ if f is bounded.

Proof. (i) Let f be any modulus. If $\varepsilon \succ 0$ and $x_i \to s(N_\theta(A,f))$ we can write

$$h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \geq h_r^{-1} \sum_{i \in I_r^1} f(|A_i(x) - s|) \succ h_r^{-1} |K A_\theta(\varepsilon)| f(\varepsilon).$$

It follows that $x_i \to s(S_\theta(A))$.

(ii) Suppose that f is bounded. Since f is bounded, there exists an integer T such that $f(x) \leq T$ for all $x \geq 0$. We see that

$$\begin{aligned} h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) &\leq h_r^{-1} \sum_{i \in I_r^1} f(|A_i(x) - s|) + h_r^{-1} \sum_{i \in I_r^2} f(|A_i(x) - s|) \\ &\leq T h_r^{-1} |K A_\theta(\varepsilon)| + f(\varepsilon). \end{aligned}$$

Since f is continuous and $x_i \to s(S_\theta(A))$, it follows from $\varepsilon \to 0$ that $x_i \to s(N_\theta(A,f))$. (ii) follows from (i) and (ii).

As an example to show that $S_\theta(A) \neq N_\theta(A,f)$ when f is unbounded, consider $A = I$. Since f is unbounded, there exists a positive sequence $0 \prec y_1 \prec y_2 \prec \ldots$ such that $f(y_i) \geq h_i$. Define the sequence $x = (x_i)$ by putting $x_{k_i} = y_i$ for $i = 1, 2, \ldots$ and $x_i = 0$ otherwise. We have $x \in S_\theta(A)$, but $x \notin N_\theta(A,f)$.

Finally we consider the case when $x_k \to s$ implies $x_k \to s(N_\theta(A,f))$.

Lemma 7. ([6]) *If* $\liminf q_r \succ 1$ *then $x_i \to s(S)$ implies $x_i \to s(S_\theta)$.*

Theorem 8. *Let* $\liminf q_r \succ 1$*, A is regular and f is bounded. Then* $x_i \to s$ *implies* $x_i \to s(N_\theta(A, f))$.

Proof. Let $x_i \to s$. By regularity of A and definition of statistical convergence we have $A_i(x) \to s(S)$. Since $\liminf q_r \succ 1$ it follows lemma 7 that $A_i(x) \to s(S_\theta)$ i.e. $x_i \to s(S_\theta(A))$. Thus, using Theorem 6, we have $x_i \to s(N_\theta(A, f))$.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

A GENERALIZED INVERSION FORMULA AND SOME APPLICATIONS

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Abstract. In this paper we shall establish a general result involving Dirichlet product of arithmetical functions, which provides information on the subtle properties of the integers.

1. Introduction and preliminaries

The Möbius function $\mu(n)$ is defined as follows:

$$\mu(1) = 1, \quad \mu(q_1 \cdot q_2 \cdot \dots \cdot q_k) = (-1)^k$$

if all the primes $q_1, q_2, \dots, q_k$ are different; $\mu(n) = 0$ if n has a squared factor. The Möbius inversion formula is a remarkable tool in numerous problems involving integers and there are other inversion formulas involving $\mu(n)$. In particular, we obtain the following well-known theorem:

If

$$G(x) = \sum_{n=1}^{\lfloor x \rfloor} F\left(\frac{x}{n}\right)$$

for all positive x, $(x \geq 1)$, then

$$F(x) = \sum_{n=1}^{\lfloor x \rfloor} \mu(n) G\left(\frac{x}{n}\right)$$

and conversely.

Many of these inversion formulas can be written in the form of a single formula which generalizes them all.

2. The main result

First of all, we establish the following theorem:

Theorem 1. *Given arithmetical functions* $\alpha, \beta, u : \mathbb{N}^* \to \mathbb{C}$ *such that*

$$\alpha * \beta = u, \quad u(n) = \begin{cases} 1, & \text{if} \quad n = 1 \\ 0, & \text{if} \quad n \geq 2 \end{cases}$$

let $h : A \times \mathbb{N}^* \to A$ *be a function, such that:*

a) $h(x, 1) = x$ *for all* $x \in A$, *where* $A \subset \mathbb{C}$, $A \neq \emptyset$.

b) $h(h(x, n), k)$ *is constant for* x *constant and* $nk =$ *constant, where* $x \in A$ *and* $n, k \in \mathbb{N}^*$.

Let $F, G : \mathbb{C} \to \mathbb{C}$ *be functions such that* $F(x) = G(x) = 0$ *for all* $x \in \mathbb{C} \setminus A$. *Suppose that the both series:*

$$\sum_{n,k \in \mathbb{N}^*} \alpha(n)\beta(k)G(h(h(x, n), k)), \qquad \sum_{n,k \in \mathbb{N}^*} \beta(n)\alpha(k)F(h(h(x, n), k))$$

converge absolutely.

Then, for all $x \in A$, *we have*

$$F(x) = \sum_{n \in \mathbb{N}^*} \beta(n)G(h(x, n)) \tag{1}$$

if and only if

$$G(x) = \sum_{n \in \mathbb{N}^*} \alpha(n)F(h(x, n)). \tag{2}$$

Proof. Suppose that (1) is true. It follows that

$$\sum_{n \in \mathbb{N}^*} \alpha(n)F(h(x, n)) = \sum_{n \in \mathbb{N}^*} \alpha(n) \sum_{k \in \mathbb{N}^*} \beta(k)G(h(h(x, n), k)) =$$

$$= \sum_{n \in \mathbb{N}^*} \sum_{k \in \mathbb{N}^*} \alpha(n)\beta(k)G(h(h(x, n), k)).$$

An absolutely convergent series can be rearranged in an arbitrary way without affecting the sum. We have

$$\alpha(1)\beta(1) = 1, \quad G(h(h(x, 1), 1)) = G(h(x, 1)) = G(x).$$

We can arrange the terms as follows:

$$\sum_{\substack{n,k,d \in \mathbb{N}^* \\ nk = d \neq 1}} \alpha(n)\beta(k)G(h(h(x, n), k)) = \sum_{d \in \mathbb{N}^*, d \neq 1} G(h(h(x, n), k)) \sum_{\substack{n,k \in \mathbb{N}^* \\ nk = d, d \neq 1}} \alpha(n)\beta(k) = 0,$$

because $\alpha * \beta = u$.

Therefore

$$\sum_{n\in\mathbb{N}^*} \alpha(n)F(h(x,n)) = G(x).$$

Conversely, (2) implies (1) and hence the theorem is proved.

3. **Examples**

1) Letting $A = [0,\infty)$, $h : A \times \mathbb{N}^* \to A$,

$$h(x,n) = \frac{x}{n}, \quad h(x,1) = \frac{x}{1} = x \text{ for all } x \in A;$$

$$h(h(x,n),k) = h\left(\frac{x}{n},k\right) = \frac{x}{nk} = \text{constant}$$

for $nk = \text{constant}$ and $x = \text{constant}$.

Consider the mappings $F, G : [0,\infty) \to \mathbb{C}$ such that $F(x) = G(x) = 0$ for all $x \in [0,1)$. We deduce from theorem 1, that

$$F(x) = \sum_{n\le x} \beta(n)G\left(\frac{x}{n}\right)$$

and

$$G(x) = \sum_{n\le x} \alpha(n)F\left(\frac{x}{n}\right)$$

are equivalent. Moreover, if we let $\alpha(n) = 1$ for all $n \in \mathbb{N}^*$, we deduce that

$$F(x) = \sum_{n\le x} \mu(n)G\left(\frac{x}{n}\right)$$

and

$$G(x) = \sum_{n\le x} F\left(\frac{x}{n}\right)$$

are equivalent for all positive x, $(x \ge 1)$.

2) Let us denote by $\overline{P}(x)$ the number of the integers $k \in \mathbb{N}^*$ such that $k \le x$, $k \ne a^b$ for all $a, b \in \mathbb{N}^*$, $b \ge 2$. It is known that

$$\sum_{2^n\le x} \overline{P}(x^{1/n}) = \lfloor x-1 \rfloor.$$

We deduce from theorem 1 that

$$\sum_{2^n\le x} \mu(n)\lfloor x^{1/n} - 1 \rfloor = \overline{P}(x).$$

3) The number $Q(x)$ of squarefree numbers not exceeding x satisfies

$$\sum_{x/n^2 \geq 1} Q(x/n^2) = \lfloor x \rfloor.$$

If we use theorem 1, we have

$$\sum_{x/n^2 \geq 1} \mu(n) \left\lfloor \frac{x}{n^2} \right\rfloor = Q(x).$$

4) If $|z| < 1$, we have

$$\frac{z}{1-z} = \sum_{n \in \mathbb{N}^*} z^n.$$

Letting $A = U(0,1)$, $h(z,n) = z^n$, $F(z) = z$, $G(z) = \dfrac{z}{1-z}$, $\alpha(n) = 1$, $\beta(n) = \mu(n)$ for all $n \in \mathbb{N}^*$, we have:

$$\sum_{n,k \in \mathbb{N}^*} \beta(n)\alpha(k)F(h(h(z,n),k)) = \sum_{n,k \in \mathbb{N}^*} \mu(n) z^{nk}$$

$$\sum_{\substack{n,k,d \in \mathbb{N}^* \\ nk=d=const}} |\mu(n) z^{nk}| \leq \sum_{\substack{n,k,d \in \mathbb{N}^* \\ nk=d=const}} |z^{nk}| = \sum_{d \in \mathbb{N}^*} \sum_{nk=d} |z|^{nk} \leq \sum_{d \in \mathbb{N}^*} d|z|^d$$

(because $\sum_{nk=d} |z|^{nk} \leq d|z|^d$).

It is possible to apply Cauchy's test:

$$\lim_{d \to \infty} \sqrt[d]{d|z|^d} = \lim_{d \to \infty} \sqrt[d]{d} \cdot |z| = |z| < 1.$$

It follows that series $\sum_{n,k \in \mathbb{N}^*} \beta(n)\alpha(k)F(h(h(z,n),k))$ converges absolutely for all $z \in U(0,1)$. We can also show that $\sum_{n,k \in \mathbb{N}^*} \alpha(n)\beta(k)G(h(h(z,n),k))$ converges absolutely. We deduce from theorem 1 that

$$\frac{z}{1-z} = \sum_{n \in \mathbb{N}^*} z^n$$

and

$$z = \sum_{n \in \mathbb{N}^*} \mu(n) \frac{z^n}{1-z^n},$$

are equivalent for all $z \in U(0,1)$.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

PARALLEL NUMERICAL METHODS FOR SOLVING NONLINEAR EQUATIONS

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1. Introduction

The basis for constructing a parallel algorithm is either a serial algorithm or the problem itself. In trying to parallelize a serial algorithm a pragmatic approach would seem reasonable. Serial algorithm are analysed for frequently occurring basic elements which are then put into parallel form. These parallelization principles rely on definite serial algorithms. This corresponds to the serial way of thinking normally encountered in numerical analysis. What is needed is a parallel way of thinking.

In the following, we shall apply these principles to some numerical methods for solving single non-linear equations.

First we shall consider the one-dimensional case and assume that the real function $f(x)$ has only one zero in the interval $[a, b]$.

The methods for the determination of zeros can be subdivided into two types:

(i) search methods

(ii) locally iterative methods.

2. Search methods

2.1. The Bisection Method. The simplest search method is the bisection method, with the following code:

Repeat
  $c := (a + b)/2$;
  If $f(a) * f(c) > 0$ then $a := c$
    else $b := c$
Until $abs(b - a) < \varepsilon$;

Considering f given as a function, continuous over the interval $[a, b]$, the serial bisection method needs $\log_2((b-a)/\varepsilon)$ function evaluations, additions and multiplications to enclose the zero in an interval of length ε.

We could adapt this serial algorithm at a parallel execution with 3 processors, by means of three sequences, a_n, b_n and c_n, every processor generating one of them:

```
a[0] := a;
b[0] := b;
c[0] := (a + b)/2;
n := 0;
Repeat in parallel
    n := n + 1;
    If f(a[n − 1]) ∗ f(c[n − 1]) > 0 then begin
        a[n] := c[n − 1];
        b[n] := b[n − 1];
        c[n] := (a[n] + b[n])/2
        end
    else begin
        a[n] := a[n − 1];
        b[n] := c[n − 1];
        c[n] := (a[n] + b[n])/2
    end
Until abs(b[n] − a[n]) < ε
```

Unfortunately, this parallel version of the bisection method does not bring a speed improvement, because, mainly, the number of operations is still of $\log_2((b-a)/\varepsilon)$ order, and we have to take into account the time spend for the processors communications.

But, if we think in parallel, the bisection method can clearly be performed on a computer consisting of r processors: for each iteration step the function is simultaneously evaluated at r points which thereby subdivide the actual interval in $r+1$ equidistant subintervals. The new interval points are chosen on the basis of the signs of the function values. So, this parallel bisection method requires $\log_{r+1}((b-a)/\varepsilon)$ evaluations. The speed-up ratio is therefore

$$S = \frac{\log_2((b-a)/\varepsilon)}{\log_{r+1}((b-a)/\varepsilon)} = \log_2(r+1).$$

2.2. **Other Search Methods.** As we saw, in applying the bisection method it is sufficient to have a function which is continuous over the interval $[a, b]$. With functions having a high degree of smoothness, high order search methods can be constructed, which will converge yet faster.

The example given by Miranker [5] demonstrates this principle.

Let $f(x)$ be a function which is differentiable over $[0, 1]$ and let $f'(x) \in [m, M]$ $(m, M > 0)$ for all $x \in [0, 1]$. An algorithm is developed for a computer able to evaluate this function values $y_i = f(x_i)$, $x_i \in [0, 1]$, $(i = 1, 2)$, in parallel.

Initially, two piecewise linear functions $\underline{S}(x), \overline{S}(x)$, are defined which enclose $f(x)$ in $[x_1, x_2]$ (see Fig.1).

We see that $\underline{S}(x) \leq f(x) \leq \overline{S}(x)$, where

$$\overline{S}(x) := \begin{cases} y_1 + M(x - x_1), & x \leq \dfrac{(M-s)x_1 + (x-m)x_2}{M-m} \\ y_2 + m(x - x_2), & x \geq \dfrac{(M-x)x_1 + (s-m)x_2}{M-m} \end{cases}$$

where $s := \dfrac{y_2 - y_1}{x_2 - x_1}$. In a similar manner we define $\underline{S}(x)$.

The zero z of f is on the right-hand side of the zero x_1^* of $\overline{S}(x)$:

$$z \geq x_1^* = \max\left\{x_1 - \frac{y_1}{M}, x_2 - \frac{y_2}{m}\right\}$$

and

$$z \leq x_2^* = \min\left\{x - \frac{y_1}{m}, x_2 - \frac{y_2}{M}\right\}.$$

Considering each possible case, it can be shown that the inequality

$$\frac{x_2^* - x_1^*}{x_2 - x_1} \leq 1 - \frac{m}{M}$$

applies.

Comparing this method with the parallelized bisection method it is possible to obtain a speed-up, provided that

$$\frac{m}{M} \geq \frac{2}{3}$$

applies.

Remark. Miranker shows, also, that is $f \in C^2[0,1]$, the algorithm even has quadratic convergence.

3. Locally iterative methods

The best known locally iterative methods are the Newton's method and the secant method. Because the second one is a discrete version of the first one (the derivative is replaced by the difference quotient), we shall discuss only the secant method.

For x_0 an initial approximation and supposing, without restriction of generality, that $f \in C^d[a,b]$, $f'(x) \neq 0$, for all $x \in (a,b)$ and $f(a) < 0$ and $f(b) > 0$.

The the serial secant method is the following:

if $f(a) \cdot f''(a) > 0$ then $x_0 = b$ else $x_0 = a$; $n := 0$;

Repeat

$n := n + 1;$

$x[n] := a - \dfrac{b - a}{f(b) - f(a)} f(a)$

If $f(x[n]) < 0$ then $a := x[n]$;

else $b := x[n]$

Until $abs(x[n] - x[n-1]) < \varepsilon$

Denoting

$$\delta_k := \max |z - x[k]|,$$

where z is the zero of f, we speak of convergence of order λ, if

$$\lim_{k \to \infty} \frac{\delta_{k+1}}{(\delta_k)^\lambda} = c > 0$$

applies.

It is known that the serial secant method has the order of convergence $\dfrac{1+\sqrt{5}}{2} \simeq 1.618\ldots$. Trying to parallelize this serial algorithm, we can use 3 processors to generate the sequences a_n, b_n and c_n, according to the following code:

$a[0] := a;$

$b[0] := b;$

$c[0] := (a[0] * f(b[0]) - b[0] * f(a[0]))/(f(b[0]) - f(a[0]));$

$n := 0;$

Repeat in parallel

$n := n + 1;$

if $f(c[n-1]) < 0$ then begin

$a[n] := c[n-1];$

$b[n] := b[n-1];$

$c[n] := (a[n] * f(b[n]) - b[n] * f(a[n]))/(f(b[n]) - f(a[n]))$

end

else if $f(c[n-1]) > 0$ then

begin

$a[n] := a[n-1];$

$b[n] := c[n-1];$

$c[n] := (a[n] * f(b[n]) - b[n] * f(a[n]))/(f(b[n]) - f(a[n]))$

end

Until $f(c[n-1]) = 0$;

Unfortunately, this algorithm does not bring an important improvement in speed-up, because of the time of communication between processors. But we may think the secant method directly in parallel, as follows.

We imagine an SIMD computer with r processors (see Chiorean [1]). Starting with approximations $x_{0,1}; x_{0,2}; \ldots, x_{0,r}$ of z, it is requires to determine r improuved approximations

$$x_{k+1,i} = \phi_{k,i}(x_{k,1}; x_{k-1,1}; \ldots, x_{0,r})$$

at every iteration step. Here,

$$\phi_{k,i} : \mathbb{R}^{(k+1)r} \to \mathbb{R}^r, \quad k \geq 0.$$

According to Corliss [3], the iteration series $x_{k,i}$ remains close to the zero z if the starting approximation is suitable, and that it will thus finally converge to the zero.

Taking into account all this, and considering an SIMD parallel computer with $r = 3$ processors, the serie $x_{k,i}$, $i = 1, 2, 3$ for the secant parallel method is generated in the following way:

$$x_{k+1,1} = x_{k,1} - \frac{x_{k,1} - x_{k,2}}{f(x_{k,1}) - f(x_{k,2})} f(x_{k,1})$$

$$x_{k+1,2} = x_{k,2} - \frac{x_{k,2} - x_{k,3}}{f(x_{k,2}) - f(x_{k,3})} f(x_{k,2})$$

$$x_{k+1,3} = x_{k,3} - \frac{x_{k,3} - x_{k,1}}{f(x_{k,3}) - f(x_{k,1})} f(x_{k,3}),$$

where $x_{k,i}$ are those in the Fig.2.

It can be proved that the order of convergence for this parallel secant method is 2, compared with 1,618. . . for the serial secant method.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

POINTWISE APPROXIMATION BY GENERALIZED SZÁSZ-MIRAKJAN OPERATORS

ZOLTÁN FINTA

Abstract. In this paper we establish some local approximation properties for a generalized Szász - Mirakjan - type operator.

1. **Introduction**

It is well - known the operator of Szász - Mirakjan [11] defined by

$$(S_n f)(x) \;=\; e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!}\, f\left(\frac{k}{n}\right), \tag{1}$$

where f is any function defined on $[0,\infty)$ such that $(S_n|f|)(x) < \infty$. The operator S_n was generalized by Pethe and Jain in [10], by Stancu in [12] and by Mastroianni in [7], obtaining S_n^α operators

$$(S_n^\alpha f)(x) \;=\; (1+n\alpha)^{-x/\alpha} \cdot \sum_{k=0}^{\infty} \left(\alpha + \frac{1}{n}\right)^{-k} \cdot \frac{x(x+\alpha)\ldots(x+(k-1)\alpha)}{k!}\, f\left(\frac{k}{n}\right), \tag{2}$$

where α is a nonnegative parameter depending on the natural number n and f is any real function defined on $[0,\infty)$ with $(S_n^\alpha|f|)(x) < \infty$. This operator has been also considered by Della Vecchia and Kocic' [3]. It was studied extensively the uniform convergency in compact interval, monotonicity, convexity, evaluation of the remainder in approximation formula and degeneracy property of the operators (2), respectively.

In [6] Lupaş has introduced the following operator:

$$(L_n f)(x) \;=\; 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k\; k!}\, f\left(\frac{k}{n}\right), \tag{3}$$

where $f : [0,\infty) \to R$, $(nx)_0 = 1$ and $(nx)_k = nx(nx+1)\ldots(nx+k-1),\ k \geq 1$. This operator was studied by Agratini [1] and Miheşan [8]. In fact we have $S_n^{1/n} = L_n$ [7, p. 250].

1991 *Mathematics Subject Classification.* 41A36, 41A40.

Key words and phrases. Szász - Mirakjan operator, modulus of smoothness.

The purpose of this paper is to establish pointwise approximation properties for the Szász - Mirakjan - type operator defined by (2).

In what follows we denote by $C_B[0,\infty)$ the set of all bounded and continuous functions on $[0,\infty)$ endowed with the norm $\|f\| = \sup\{\ |f(x)|\ :\ x \in [0,\infty)\ \}$. Let $\Delta_h^2(f,x) = f(x-h) - 2f(x) + f(x+h)$ $(x \geq h)$ be the usual symmetric second difference of f and $\omega^2(f,\delta) = \sup_{0<h\leq\delta,\ x\geq h} |\Delta_h^2(f,x)|$ the modulus of smoothness of f.

2. **Main results**

The following results give some local approximation properties for S_n^α :

Theorem 1. *For every function $f \in C[0,\infty)$ we have*

$$|(S_n^\alpha f)(x) - f(x)| \ \leq\ 2\ \omega^2\left(\ f\ ,\ \sqrt{(\alpha+\frac{1}{n})\frac{x}{2}}\ \right). \tag{4}$$

Proof. Let $e_0(x) = 1$ and $e_1(x) = x$ $(x \geq 0)$. In view of [7, p. 239, Theorem 2.3] we obtain that S_n^α reproduces every linear function and $(S_n^\alpha(e_1 - xe_0)^2)(x) = (\alpha + \frac{1}{n})x$, $x \geq 0$. Then, by [9, p. 255, Theorem 2.1] we have

$$|(S_n^\alpha f)(x) - f(x)| \ \leq\ \left(1 + \frac{1}{2}\left(\alpha + \frac{1}{n}\right)\cdot\frac{x}{h^2}\right)\ \omega^2\left(f,h\right).$$

Putting here $h = \sqrt{(\alpha + 1/n)\ x/2}$ we obtain (4). Specifically we have

$$|(L_n f)(x) - f(x)| \ \leq\ 2\ \omega^2(f, \sqrt{x/n}).$$

Thus the theorem is proved.

Let $f \in C_B[0,\infty)$ and $\beta \in (0,1]$. Then the Lipschitz - type maximal function of order β of f is defined as

$$f_\beta^\sim(x) \ =\ \sup_{\substack{t\neq x\\ t\in[0,\infty)}} \frac{|f(x) - f(t)|}{|x-t|^\beta}, \qquad x \in [0,\infty).$$

Moreover, we define for $f \in C_B[0,\infty)$, $\beta \in (0,1]$ and $h > 0$ the following kind of generalized Lipschitz - type maximal function of order β and step - size h,

$$f_{\beta,h}^\sim(x) \ =\ \sup_{\substack{t\neq x\\ t\in[0,\infty)}} \frac{|\Delta_h^1(f,x) - \Delta_h^1(f,t)|}{|x-t|^\beta}, \qquad x \in [0,\infty),$$

where $\Delta_h^1(f,x) = f(x+h) - f(x)$, $x \in [0,\infty)$, $h > 0$. Then, by standard method [5] we obtain the following result:

Theorem 2. *Let $f \in C_B[0,\infty)$ and $\beta \in (0,1]$. Then for all $x \in [0,\infty)$ and all $h > 0$ we have the inequalities*

a) $|(S_n^\alpha f)(x) - f(x)| \;\le\; f_\beta^\sim(x) \cdot (S_n^\alpha(\cdot - x)^\beta)(x)$;

b) $|(S_n^\alpha f)(x) - f(x)| \;\le\; f_\beta^\sim(x) \cdot (2x/n)^{\beta/2}$;

c) $|(S_n^\alpha f)(x) - f(x)| \;\le\; \left\{\frac{1}{h}\int_0^h f_{\beta,s}^\sim(x)\, ds\right\} (S_n^\alpha(\cdot - x)^\beta)(x) \;+\; \left\{\frac{1}{h}\, f_{\beta,h}^\sim(x)\right\} \cdot \frac{1}{1+\beta} \cdot$
$\cdot\, (S_n^\alpha(\cdot - x)^{1+\beta}, x)$;

d) $|(S_n^\alpha f)(x) - f(x)| \;\le\; \left\{\frac{1}{h}\int_0^h f_{\beta,s}^\sim(x)\, ds\right\} (2x/n)^{\beta/2} \;+\; \left\{\frac{1}{h}\, f_{\beta,h}^\sim(x)\right\} \cdot \frac{1}{1+\beta} \cdot$
$\cdot\, (2x/n)^{(1+\beta)/2}$.

To establish the saturation result for S_n^α we use the following Voronovskaja - type formula:

Theorem 3. *Let $f \in C[0,\infty)$ be twice differentiable at some point $x > 0$ and let us assume that $f(t) = O(t^2)$. If $\alpha = \alpha(n)$ then*

$$\lim_{n\to\infty} n((S_n^\alpha f)(x) - f(x)) \;=\; \begin{cases} \frac{x}{2}\, f''(x), & \text{for} \quad \alpha = o(n^{-1}) \\ x\, f''(x), & \text{for} \quad \alpha = n^{-1}. \end{cases} \tag{5}$$

Proof. We obtain formula (5) following the proof of [1, Theorem 4]. Indeed, by Taylor's expansion

$$f\left(\frac{k}{n}\right) - f(x) \;=\; \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)^2 \left(\frac{1}{2} f''(x) + \varepsilon\left(\frac{k}{n} - x\right)\right)$$

we obtain

$$\begin{aligned}(S_n^\alpha f)(x) - f(x) \;&=\; f'(x)(S_n^\alpha(e_1 - xe_0))(x) + \frac{1}{2} f''(x)(S_n^\alpha(e_1 - xe_0)^2)(x) + \\ &\quad + (S_n^\alpha((e_1 - xe_0)^2\varepsilon))(x),\end{aligned}$$

where ε is bounded and $\lim_{t\to 0} \varepsilon(t) = 0$. Because S_n^α leaves linear functions invariant we have

$$(S_n^\alpha f)(x) - f(x) \;=\; \frac{1}{2} f''(x)(S_n^\alpha(e_1 - xe_0)^2)(x) + (S_n^\alpha((e_1 - xe_0)^2\varepsilon))(x). \tag{6}$$

Recalling the Cauchy - Schwarz inequality we obtain

$$\begin{aligned}(S_n^\alpha((e_1 - xe_0)^2\varepsilon))(x) &\leq (S_n^\alpha(e_1 - xe_0)^2)(x)\ (S_n^\alpha((e_1 - xe_0)^2\varepsilon))(x)\\ &\leq \|\varepsilon^2\| \left(\alpha + \frac{1}{n}\right)^2 x^2.\end{aligned}$$

But $\alpha = o(n^{-1})$ or $\alpha = n^{-1}$ therefore

$$\lim_{n\to\infty} n(S_n^\alpha((e_1 - xe_0)^2\varepsilon))(x) = 0.$$

Hence we conclude that (6), $\alpha = o(n^{-1})$ or $\alpha = n^{-1}$ and $(S_n^\alpha(e_1 - xe_0)^2)(x) = \left(\alpha + \frac{1}{n}\right) x$ lead us to the asymptotic formula (5).

The saturation result is as follows:

Theorem 4. *Let $f \in C[0,\infty)$, $f(x) = O(x^2)$ and $\alpha = \alpha(n)$ such that $\alpha = o(n^{-1})$ or $\alpha = n^{-1}$. If $(S_n^\alpha f)(x) - f(x) = o_x(x/n)$ $(x \geq 0,\ n \to \infty)$ then f is a linear function; furthermore*

$$|(S_n^\alpha f)(x) - f(x)| \leq M \cdot \frac{x}{n} \qquad (x \geq 0,\ n = 1, 2, \ldots)$$

holds if and only if f has a derivative belonging to Lip 1, where

$$Lip\ 1 = \{\ f\ :\ |f(x+h) - f(x)| \leq K_f h,\ x \geq 0,\ h > 0\ \}.$$

Proof. By [4, Theorem 5.1] we have that f is locally and hence globally linear. Furthermore, we have $(S_n^\alpha(e_1 - xe_0)^2)(x) = (\alpha + \frac{1}{n})x$ and the proofs of [4, Theorem 5.1] and [4, Theorem 5.4] hold for S_n^α on every finite interval $[a, b] \subseteq [0,\infty)$ and $(S_n^\alpha f)(x) - f(x) = O(x/n)$ implies that f has a derivative which is absolutely continuous on every interval $(a, b) \subseteq [0,\infty)$. But, in view of Theorem 3 we have $\lim_{n\to\infty} (n/x)\ ((S_n^\alpha f)(x) - f(x)) = f''(x)/2$ or $\lim_{n\to\infty} (n/x)\ ((S_n^\alpha f)(x) - f(x)) = f''(x)$ at every point x, where $f''(x)$ exists. So $(S_n^\alpha f)(x) - f(x) = O(x/n)$ implies $f''(x) = O(1)$ and this is the same as $f' \in Lip$ 1.

The reverse statement follows from Theorem 1 since $f' \in Lip$ 1 implies $\omega^2(f,\delta) = O(\delta^2)$. Thus the theorem is proved.

In [7, p. 244, Theorem 4.2] is established the inequality $f(x) \leq (S_n^\alpha f)(x)$, $x \geq 0$ for a convex function $f \in C_B[0,\infty)$. The next theorem gives a similar result without use the evaluation of the remainder term.

Theorem 5. *Let* $f \in C_B[0,\infty)$ *be a convex function.*

Then $f(x) \le (S_n f)(x) \le (S_n^\alpha f)(x)$ *for all* $x \ge 0$.

Proof. The first inequality is known [2]. For the second inequality we consider the following Taylor's expansion:

$$(S_n f)(t) = (S_n f)(x) + (t-x)(S_n f)'(x) + \int_x^t (t-u)(S_n f)''(u)\,du.$$

Hence, by [7, p. 240, Theorem 2.8] we obtain

$$\begin{aligned}
&(S_n^\alpha f)(x) - (S_n f)(x) = \\
&= \frac{1}{\Gamma\left(\frac{x}{\alpha}\right)} \cdot \int_0^\infty e^{-t}\, t^{\frac{x}{\alpha}-1}\,(S_n f)(\alpha t)\,dt - (S_n f)(x) \\
&= \frac{1}{\alpha\,\Gamma\left(\frac{x}{\alpha}\right)} \cdot \int_0^\infty e^{-\frac{t}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{x}{\alpha}-1} [\,(S_n f)(t) - (S_n f)(x)\,]\,dt \\
&= \frac{1}{\alpha\,\Gamma\left(\frac{x}{\alpha}\right)} \cdot \int_0^\infty e^{-\frac{t}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{x}{\alpha}-1} \Big[(t-x)(S_n f)'(x) + \\
&\qquad + \int_x^t (t-u)(S_n f)''(u)\,du\Big]\,dt.
\end{aligned}$$

But $(S_n^\alpha e_1)(x) = e_1(x)$, therefore

$$\begin{aligned}
&(S_n^\alpha f)(x) - (S_n f)(x) = \\
&= \frac{1}{\alpha\,\Gamma\left(\frac{x}{\alpha}\right)} \cdot \left\{ \int_0^x e^{-\frac{t}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{x}{\alpha}-1} \cdot \left[\int_t^x (u-t)(S_n f)''(u)\,du\right] dt \,+ \right. \\
&\qquad \left. + \int_x^\infty e^{-\frac{t}{\alpha}} \left(\frac{t}{\alpha}\right)^{\frac{x}{\alpha}-1} \cdot \left[\int_x^t (t-u)(S_n f)''(u)\,du\right] dt \right\}.
\end{aligned}$$

On the other hand

$$(S_n f)''(x) = e^{-nx} \cdot n^2 \sum_{k=0}^\infty \frac{(nx)^k}{k!} \cdot \Delta^2_{1/n}(f, \frac{k}{n}) \ge 0$$

for the convex function f [2, *p.* 136]. So $(S_n^\alpha f)(x) \ge (S_n f)(x)$.

Remark. The inequality $f(x) \le (S_n^\alpha f)(x)$ can be proved by Jensen's inequality as well.

Indeed, let

$$s_{n,k}(x,\alpha) = (1+n\alpha)^{-x/\alpha} \cdot \left(\alpha + \frac{1}{n}\right)^{-k} \cdot \frac{x(x+\alpha)\dots(x+(k-1)\alpha)}{k!}, \qquad k \ge 0.$$

Then, by [7, p. 239, Theorem 2.3] we have $\sum_{k=0}^{\infty} s_{n,k}(x,\alpha) = 1$ and

$$\sum_{k=0}^{\infty} \frac{k}{n} s_{n,k}(x,\alpha) = x, \qquad x \geq 0. \tag{7}$$

Using Jensen's inequality we obtain

$$\sum_{k=0}^{m} s_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) + \left[\sum_{k\geq m+1} s_{n,k}(x,\alpha)\right] f(0) \geq f\left(\sum_{k=0}^{m} s_{n,k}(x,\alpha)\cdot\frac{k}{n}\right).$$

Hence, by continuity of f and (7) we obtain

$$(S_n^{\alpha} f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) \geq f\left(\sum_{k=0}^{\infty} s_{n,k}(x,\alpha)\cdot\frac{k}{n}\right) = f(x).$$

This completes the proof.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

SOME DUALITY THEOREMS FOR LINEAR-FRACTIONAL PROGRAMMING HAVING THE COEFFICIENTS IN A SUBFIELD K OF REAL NUMBERS

DOINA IONAC

Abstract. In this paper we obtain some duality results for linear-fractional programming having the coefficients in a field K of real numbers, having the property that the rational numbers set $\mathbb{Q} \subseteq K$.

1. We consider the following linear-fractional programming problem:

(PF). Find

$$Max \left\{ f\left(x\right) \equiv \frac{cx + c_0}{dx + d_0} | Ax \leq b, x \geq 0 \right\},$$

where $A = (a_{ij})$ is a $m \times n$ matrix $(m < n)$ with $rankA = m$ and with the elements in K, c and d are n-vectors with components in K, b is a m-vector with components in K and c_0 and d_0 are constants in K.

Let X be the feasible set in $\mathbb{R}^n$ of the problem PF, and let XK be the feasible set of the problem PF in K^n defined by $XK = X \cap K^n$.

Next, we suppose that:

(a) $dx + d_0 > 0$, $\forall x \in X$,

and also we will denote by E the set

$$E = \{x \in \mathbb{R}^n | x \geq 0, dx + d_0 > 0\}.$$

Obviously, $X \subseteq E$ and E is a convex set.

Definition 1. i) The problem (PF) is called **regular** if X is nonempty, f is not constant and it exists $M > 0$ such that: $0 < dx + d_0 < M$, $\forall x \in X$.

ii) The problem (PF) is called **pseudo-regular** if X is non-empty, f is not constant and verifies the condition (a).

Key words and phrases. Fractional programming, duality, regular and pseudo-regular programs.

Next we study some duality properties for problem (PF) under the hypothesis of regularity and pseudo-regularity which is larger than considered, for instance, by Shesham [5]. Some examples [2'] show that there are regular linear-fractional problems with unbounded feasible solution set X having an infinite optimum value.

2. A way to construct a dual for problem (PF) is by transforming it into a linear programming problem by the variable change $y = tx$:

(PL). Find

$$Max\{(cy + c_0 t | Ay - bt \leq 0, dy + d_0 t = 1, y \geq 0, t \geq 0\}.$$

Let the objective function of problem (PL) be denoted by $g(y,t) = cy + c_0 t$.

Also let XL be the feasible set of the problem (PL).

The following property extends to the case when (PF) is regular a similar result obtained by Charnes-Cooper [1] (see, [3], [6], [7]) under the supposition that X is a bounded nonempty set.

Theorem 2. *(i) If the problem (PF) is regular then for every feasible solution (y,t) of the problem (PL), we have $t > 0$.*

(ii) If problem (PF) satisfy the condition (a), then for any $x' \in X$, there exists $(y',t') \in XL$, such that $x' = \frac{y'}{t'}$ and $f(x') = g(y',t')$.

(iii) If problem (PF) is regular then for any feasible solution (y',t') of the problem (PL) there exists a feasible solution $x' \in X$ of (PF) such that $x' = \frac{y'}{t'}$ and $f(x') = g(y',t')$.

Next we need an auxiliary results which establishes the relationship between problems (PF) and (PL) that generalizes for regular linear-fractional programming a result obtained by Charnes-Cooper [1] (see, also [3], [6]) in the case when the feasible set is bounded and nonempty:

Theorem 3. *If the problem (PF) is regular, then only one of the following statements holds:*

(i) The problems (PF) and (PL) have both optimal solutions and its optimal values are equal and finite. Moreover, if (y^*,t^*) is an optimal solution of (PL) then $x^* = \frac{y^*}{t^*}$ is an optimal solution for (PF) and conversely if x'' is an optimal solution of (PF) there exists an optimal solution (y'',t'') of (PL) such that $y'' = t''x''$.

(ii) The problems (PF) and (PL) have both infinite optimal values, that is $\sup\{f(x)|x \in X\} = \sup\{g(y,t)|(y,t) \in XL\} = +\infty$.

3. Let XK be the feasible set of the problem (PFK),

(PFK). Find

$$Max \left\{ \frac{cx + c_0}{dx + d_0} | Ax \leq b,\, x \geq 0,\, x \in K^n \right\},$$

that is $XK = X \cap K^n$.

We consider also the auxiliary linear programming problem (PLK) on the field K associated to the problem (PL):

(PLK). Find

$$Max\{(cy + c_0 t)|Ay - bt \leq 0, dy + d_0 t = 1, y \geq 0, t \geq 0, (y,t) \in K^{n+1}\}.$$

Remark 1. The feasible set XK is a polytop (i.e. the intersection of a finite number of closed semi-spaces) and since a_{ij}, c_j, $b_i \in K$, ($i = 1, 2, ..., m$ and $j = 1, 2, ..., n$), any extremal points of X is an element in K^n.

The following result establishes a relationship between (PF) and (PFK):

Theorem 4. *If the condition (a) holds then the following statements are true:*

(i) Problem (PF) is infeasible (i.e. $X = \emptyset$) if and only if the problem (PFK) is infeasible (i.e. $XK = \emptyset$).

(ii) Problem (PF) has no optimal solution, if and only if (PFK) has no optimal solution.

(iii) Problem (PF) has an optimal solution if and only if the problem (PFK) has an optimal solution.

(iv) If x' is an optimal solution of the problem (PF) and x'' is an optimal solution for problem (PFK), then $f(x') = f(x'')$.

Proof. The proof of this theorem has the same main idea that a similar result for rational programming problems from [4]. □

4. We can associate to problem (PF) a linear dual (see, [6]) which is the dual of the auxiliary linear programming problem (PL):

DL. Find

$$Min\ z,$$

subject to:

$A^t u + dz \geq c,$

$bu - d_0 z \leq -c_0,$

$u \geq 0,\ u \in \mathbb{R}^m,\ z \in \mathbb{R}.$

Let consider the problem:

DLK. Find

$$Min\ z,$$

subject to

$A^t u + dz \geq c,$

$bu - d_0 z \leq -c_0$, and

$u \geq 0,\ z \in K,\ u \in K^m.$

Theorem 5. *If the problem (PF) is regular only one of the statement holds:*

(i) Both problems (PF) and (DL) (primal and dual) has feasible solutions. In this case, both problems have optimal solutions and its optimal values are equal.

(ii) problem (PF) has feasible solutions and (DL) has not feasible solutions. In this case, the problem (PF) has an infinite optimum.

Proof. Since by hypothesis (PF) is regular, it has feasible solutions, and by Theorem 2, problem (PL) has feasible solutions too. Then by linear programming duality theorem only one of the following two cases holds:

a) the dual problem (DL) is a feasible problem;

b) the dual (DL) is an unfeasible problem.

(i) If the dual problem (DL) is a feasible problem then the optimal values of the primal and dual problems are equal. But then, by Theorem 3, the optimal values of the problems (PF) and (PL) are equal.

(ii) If the dual problem (DL) is an unfeasible problem then, by the linear programming duality theorem, it follows that problem (PL) has an infinite optimal value. But then, by Theorem 3, the problem (PF) has an infinite optimal value too. □

Theorem 6. *If the problem (PFK) is regular only one of the following statements holds:*

(i) Both primal and dual problems (PFK) and (DLK) have feasible solutions. In this case, both problems have optimal solutions and its optimal values are equal.

(ii) Problem (PFK) has feasible solutions and (DLK) has not. In this case, the problem (PFK) has an infinite optimum.

Proof. By duality Theorem 5, only one of the following cases holds:

a) Both primal and dual problems (PF) and (DL) are feasible problems. In this case, both problems have optimal solutions and its optimal values are equal.

b) Problem (PF) is a feasible problem and (DL) is an unfeasible problem. In this case the problem (PF) has an infinite optimum.

(i) Let consider the case when both problems (PF) and (DL) have feasible solutions. Then, by Theorem 4 and by Theorem 3.2 [2], it follows that the problems (PFK) and (DLK) have optimal solutions and its optimal values are equal with that of (PF) and (DL), respectively. Therefore, under the theorem hypotheses the problems (PFK) and (DLK) have optimal solutions and its optimal values are equal.

(ii) On the other part, when (DL) is unfeasible, from Theorem 5 it follows that problem (PF) has an infinite optimum. But then by Theorem 3.2 [2], (DLK) is an unfeasible problem and by Theorem 4 (PFK) has an infinite optimum. □

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

ON GAUSS TYPE FUNCTIONAL EQUATIONS AND MEAN VALUES BY H. HARUKI AND TH. M. RASSIAS

ZHENG LIU

Abstract. In this paper we give a concise summary of some recent results on Gauss type functional equations and mean values by H. Haruki and Th. M. Rassias.

1. **Introduction**

Ten years ago, in [5] Haruki reconsidered the Gauss' functional equation

$$f\left(\frac{a+b}{2}, \sqrt{ab}\right) = f(a,b) \quad (a,b>0), \tag{1.1}$$

where $f : R^+ \times R^+ \to R$ is an unknown function.

It is well known that $f(a,b) = AG(a,b)$ satisfies (1.1) where $AG(a,b)$ is the airhtmetic-geometric mean of Gauss of a,b defined as the common limit of the sequences (a_n), (b_n) given recurrently by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = (a_n + b_n)/2, \quad b_{n+1} = \sqrt{a_n b_n}.$$

The result given by Haruki may be stated as follows.

Theorem 1.1. *Let $f : R^+ \times R^+ \to R$. If f can be represented by the form, containing some function p, in $R^+ \times R^+$*

$$f(a,b) = \frac{1}{2\pi}\int_0^{2\pi} p(r)d\theta,$$

where $p : R^+ \to R$ and $p''(x)$ is continuous in R^+, then the only solution of (1.1) is given by

$$f(a,b) = c_1 \frac{1}{AG(a,b)} + c_2, \tag{1.2}$$

where c_1 and c_2 are arbitrary real numbers.

1991 *Mathematics Subject Classification.* 39B22.

Key words and phrases. Gauss' functional equation, monotonic function, power mean.

It should be noted that Gauss established an integral representation of $AG(a,b)$ as

$$AG(a,b)=\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}\right)^{-1}. \tag{1.3}$$

So, (1.2) can be represented by using (1.3) as

$$f(a,b)=\frac{c_1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}+c_2.$$

May be motivated by this fact, in [5] Haruki considered the following type mean value of a,b

$$M(a,b;p(r)):=p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi}p(r)d\theta\right),$$

where $p: R^+\to R$, $p''(x)$ is a continuous function in R^+, $p=p(x)$ is strictly monotonic in R^+, and denote $\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$ by r.

The following theorem was proved in [5].

Theorem 1.2. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;p(r))=AG(a,b)$ holds for all positive real numbers a,b if and only if $p(r)=c_1(1/r)+c_2$.

(ii) $M(a,b;p(r))=G(a,b)$ holds for all positive real numbers a,b if and only if $p(r)=c_1(1/r^2)+c_2$.

(iii) $M(a,b;p(r))=A(a,b)$ holds for all positive numbers a,b if and only if $p(r)=c_1\log r+c_2$.

(iv) $M(a,b;p(r))=\sqrt{\frac{a^2+b^2}{2}}$ holds for all positive real numbers a,b if and only if $p(r)=c_1r^2+c_2$.

(v) There exists no $p(r)$ such that $M(a,b;p(r))=H(a,b)$ holds for all positive real numbers a,b.

Since then, around the above two theorems, a series of new generalization appeared one after another.

We would like to make a survey in this paper.

Throughout this paper, let a and b be two any positive real numbers. A mean value of a,b, denoted by $M(a,b)$ is defined to be a real-valued function M, which satisfies the following postulates:

(P_1) $M: R^+\times R^+\to R$;

(P_2) $M(a,b)=M(b,a)$ (symmetry property);

(P_3) $M(a,a) = a$ (reflexivity property).

The arithmetic, geometric, and harmonic mean values of a, b are denoted by $A(a,b)$, $G(A,b)$ and $H(a,b)$, respectively.

In what follows, we also use the power means defined by

$$P_q(a,b) = \left(\frac{a^q + b^q}{2}\right)^{\frac{1}{q}}$$

for $q \neq 0$, while, for $q = 0$,

$$P_0(a,b) = G(a,b).$$

We denote also the power function

$$e_n(x) = x^n \text{ for } n \neq 0$$

and

$$e_0(x) = \log x.$$

2. **Gauss Type Functional Equations**

$$f\left(\frac{a+b}{2}, \frac{2ab}{a+b}\right) = f(a,b) \quad (a, b > 0), \tag{2.1}$$

where $f : R^+ \times R^+ \to R$ is an unknown function of the above equation. By following the theory on Gauss' functional equation (cf. [1], [2], [3], [4]), a new result on this functional equation is given as

Theorem 2.1. *Let $f : R^+ \times R^+ \to R$ be a function. If f can be represented by*

$$f(a,b) = \frac{1}{2\pi}\int_0^{2\pi} q(s)d\theta \quad (a, b > 0),$$

where $s = a\cos^2\theta + b\sin^2\theta$, $q : R^+ \to R$ is a function such that $q''(x)$ is continuous in R^+, then the only solution of (2.1) is given by

$$f(a,b) = c_1 \frac{1}{\sqrt{ab}} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

An open problem for the functional equation (2.1) is given as follows:

Let $f : R^+ \times R^+ \to R$ be a continuous function in $R^+ \times R^+$. Is the only continuous solution of the functional equation (2.1) given by

$$f(a,b) = F(ab),$$

where $F : R^+ \to R$ is an arbitrary continuous function of a real variable x?

In [13], the author treat the functional equation

$$f\left(\sqrt{ab}, \frac{2ab}{a+b}\right) = f(a,b) \quad (a,b>0), \tag{2.2}$$

where $f : R^+ \times R^+ \to R$ is an unknown function of the above equation.

By following the theory on Gauss' functional equation, we obtained

Theorem 2.2. *Let $f : R^+ \times R^+ \to R$ be a function. If f can be represented by*

$$f(a,b) = \frac{1}{2\pi}\int_0^{2\pi} u(t)d\theta \quad (a,b>0),$$

where $t = \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)^{-\frac{1}{2}}$, $u : R^+ \to R$ is a function such that $u''(x)$ is continuous in R^+, then the only solution of (2.2) is given by

$$f(a,b) = c_1 GH(a,b) + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

$GH(a,b)$ is the geometric-harmonic mean of a and b defined as the common limit of the sequences (a_n), (b_n) given recurrently by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{2a_n b_n}{a_n + b_n}.$$

Also, an open problem for the functional equation (2.2) is given as follows:

Let $f : R^+ \times R^+ \to R$ be a continuous function in $R^+ \times R^+$. Is the only continuous solution of the functional equation (2.2) given by

$$f(a,b) = F(GH(a,b)),$$

where $F : R^+ \to R$ is an arbitrary continuous function of a real variable x?

In [16], G. Toader considered a more general functional equation

$$f(P_q(a,b), P_s(a,b)) = f(a,b). \tag{2.3}$$

Denote

$$r_n(\theta) = (a^n \cos^2\theta + b^n \sin^2\theta)^{1/n}, \quad n \neq 0$$

and

$$r_0(\theta) = \lim_{n\to 0} r_n(\theta) = a^{\cos^2\theta} b^{\sin^2\theta}.$$

For a strictly monotonic function $p : R^+ \to R$, consider the function

$$f(a,b;p,n) = \frac{1}{2\pi}\int_0^{2\pi} p(r_n(\theta))d\theta. \tag{2.4}$$

G. Toader proved the following theorem.

Theorem 2.3. *If the function f is a solution of (2.3) which can be represented by (2.4), where p has a continuous second-order derivative in R^+, then*

$$p = c_1 e_{q+s-n} + c_2, \tag{2.5}$$

where c_1 and c_2 are arbitrary real numbers.

Remark. For $n = 2$, $q = 1$ and $s = 0$, we get the necessity part of Theorem 1.1. For $n = 1$, $q = 1$ and $s = -1$, we get the necessity part of Theorem 2.1. For $n = -2$, $q = 0$ and $s = -1$, we get the necessity part of Theorem 2.2. In all these three cases, as we have already mentioned, the condition is also sufficient.

In [17], the following theorem was proved.

Theorem 2.4. *If $n \neq 0$, $q = n$ and $s = -n$, then the function f given by (2.4) and p given by (2.5), verifies the relation (2.3).*

In [10], Kim and Rassias considered a generalized functional equation, namely

$$f(P_q^k(a,b), P_s^k(a,b)) = f(a,b) \tag{2.6}$$

where

$$P_q^k(a,b) = (ab)^{(1-k)/2}\left(\frac{a^q + b^q}{2}\right)^{\frac{k}{q}}.$$

The following theorem was proved.

Theorem 2.5. *If the function f is a solution of (2.6) which can be represented by (2.4), where p has a continuous second-order derivative in R^+, then*

$$p = c_1 e_{-n+kq+ks} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

Clearly, Theorem 2.3 is a special case of Theorem 2.5.

In [18], S. Toader, Rassias and G. Toader consider a more general functional equation

$$f(M(a,b), N(a,b)) = f(a,b), \tag{2.7}$$

where M and N are two given means.

It is not difficult to prove the following theorem.

Theorem 2.6. *If the function f defined by (2.4) in case $n = 1$ is a solution of (2.6), where p has a continuous second-order derivative in R^+, then the function p is a solution of the differential equation*

$$p''(c) + 4p'(x)[M''_{ab}(c,c) + N''_{ab}(c,c)] = 0.$$

Remark. In case $n = 1$, Theorem 2.3 and Theorem 2.5 can be deduced from Theorem 2.6.

3. Mean Values by H. Haruki and Th.M. Rassias

In [7], Haruki and Rassias considered the following two mean values of a, b:

$$M(a,b;q(s)) := q^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} q(s)d\theta\right),$$

where $q : R^+ \to R$, $q''(x)$ is a continuous function in R^+, $q = q(x)$ is strictly monotonic in R^+, and denote $a\cos^2\theta + b\sin^2\theta$ by s; and

$$M(a,b;u(t)) := u^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} u(t)d\theta\right),$$

where $u : R^+ \to R$, $u''(x)$ is a continuous function in R^+, $u = u(x)$ is strictly monotonic in R^+, and denote $(\cos^2\theta/a + \sin^\theta/b)^{-1}$ by t.

The following two theorems are proved.

Theorem 3.1. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;q(s)) = A(a,b)$ holds for all positive real numbers a, b if and only if $q(s) = c_1 s + c_2$.

(ii) $M(a,b;q(s)) = G(a,b)$ holds for all positive real numbers a, b if and only if $q(s) = c_1(1/s) + c_2$.

(iii) $M(a,b;q(s)) = P_{\frac{1}{2}}(a,b)$ holds for all positive real numbers a, b if and only if $q(s) = c_1 \log s + c_2$.

(iv) $M(a,b;q(s)) = \sqrt{H(a,b)G(a,b)}$ holds for all positive real numbers a, b if and only if $q(s) = c_1(1/s^2) + c_2$.

Theorem 3.2. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b,u(t)) = G(a,b)$ holds for all positive real numbers a, b if and only if $u(t) = c_1 t + c_2$.

(ii) $M(a,b,u(t)) = H(a,b)$ *holds for all positive real numbers* a,b *if and only if* $u(t) = c_1(1/t) + c_2$.

(iii) $M(a,b,u(t)) = P_{-\frac{1}{2}}(a,b)$ *holds for all positive real numbers* a,b *if and only if* $u(t) = c_1 \log s + c_2$.

(iv) $M(a,b,u(t)) = \sqrt{A(a,b)G(a,b)}$ *holds for all positive real numbers* a,b *if and only if* $u(t) = c_1 t^2 + c_2$.

Noticed that the geometric-harmonic mean $GH(a,b)$ can be represented by a first complete elliptic integral as

$$GH(a,b) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}, \tag{3.1}$$

the author in [12] considered the mean value of a,b

$$M(a,b;v(z)) = v^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} v(z)d\theta\right),$$

where $v : R^+ \to R$, $v''(x)$ is a continuous function in R^+, $v = v(x)$ is strictly monotonic in R^+, and denote $(\cos^2\theta/a^2 + \sin^2\theta/b^2)^{-\frac{1}{2}}$ by z.

The following theorem is proved.

Theorem 3.3. *Let* $c_1(\neq 0)$ *and* c_2 *be arbitrary real constants.*

(i) $M(a,b;v(z)) = GH(a,b)$ *holds for all positive real numbers* a,b *if and only if* $v(z) = c_1 z + c_2$.

(ii) $M(a,b;v(z)) = G(a,b)$ *holds for all positive real numbers* a,b *if and only if* $v(z) = c_1 z^2 + c_2$.

(iii) $M(a,b;v(z)) = H(a,b)$ *holds for all positive real numbers* a,b *if and only if* $v(z) = c_1 \log z + c_2$.

(iv) $M(a,b;v(z)) = (H(a^2,b^2))^{1/2}$ *holds for all positive real numbers* a,b *if and only if* $v(z) = c_1(1/z^2) + c_2$.

(v) There exists no $v(z)$ *such that* $M(a,b;v(z)) = A(a,b)$ *holds for all positive real numbers* a,b.

It should be noted that in [8] Kim also considered the mean value $M(a,b;v(z))$ and got the results (ii), (iii), (iv) of Theorem 3.3.

In [16] and [17], G. Toader and Rassias considered a generalization of the above mentioned four mean values $M(a, b; p(r))$, $M(a, b; q(s))$, $M(a, b; u(t))$ and $M(a, b; v(z))$ as follows:

Denote

$$r_n(\theta) = (a^n \cos^2 \theta + b^n \sin^2 \theta)^{1/n}, \quad n \neq 0,$$

and

$$r_0(\theta) = \lim_{n \to 0} r_n(\theta) = a^{\cos^2 \theta} b^{\sin^2 \theta}.$$

For a strictly monotonic function $p : R^+ \to R$, set

$$M(a, b; p, r_n) = p^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} p(r_n(\theta)) d\theta \right).$$

It is easy to prove that $M(a, b; p, r_n)$ is a mean value.

As was stated in Theorem 1.2, Theorem 3.1, Theorem 3.2 and Theorem 3.3, the means $M(a, b; p, r_n)$ can represent some known means for special choice of p and n. In [10], the following theorem was proved.

Theorem 3.4. *If for some twice continuously differentiable function p the mean $M(a, b; p, r_n)$ reduces at the power mean $P_q(a, b)$, then*

$$p = c_1 e_{2q-n} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

In [17], the following theorem was proved.

Theorem 3.5. *The mean $M(a, b; p, r_n)$ reduces to the power mean $P_q(a, b)$ for arbitrary n if*

$$p = c_1 e_{2q-n} + c_2, \quad c_1, c_2 \in R$$

and q takes one of following values; (i) $q = 0$, (ii) $q = n$; or (iii) $q = n/2$.

In [9], Kim considered some further extensions of values by H. Haruki and Th.M. Rassias as follows:

$$M(a, b; h(s)) := \frac{1}{H(a, b)} h^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} h(s) d\theta \right), \tag{3.2}$$

where $h : R^+ \to R$, $h''(x)$ is a continuous function in R^+, $h = h(x)$ is strictly monotonic in R^+, and denote $(\cos^2 \theta / a^2 + \sin^2 \theta / b^2)^{-1}$ by s,

$$M(a, b; k(s)) := \frac{1}{H(a, b)} k^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} k(s) d\theta \right), \tag{3.3}$$

where $k : R^+ \to R$, $k''(x)$ is a continuous function in R^+, $k = k(x)$ is strictly monotonic in R^+, and denote $(a\cos\theta)^2 + (b\sin\theta)^2$ by s.

The following theorems are proved:

Theorem 3.6. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;h(s)) = A(a,b)$ holds for all positive real numbers a,b if and only if $h(s) = c_1 s + c_2$.

(ii) $M(a,b;h(s)) = ab(a+b)/(a^2+b^2)$ holds for all positive real numbers a,b if and only if $h(s) = c_1(1/s) + c_2$.

(iii) $M(a,b;h(s)) = H(a,b)$ holds for all positive real numbers a,b if and only if $h(s) = c_1 \log s + c_2$.

(iv) $M(a,b;h(s)) = \sqrt{2(a+b)^2(ab)^2/(3a^4+3b^4+2(ab)^2)}$ holds for all positive real numbers a,b if and only if $h(s) = c_1(1/s^2) + c_2$.

(v) $M(a,b;h(s)) = \sqrt{(a^2+b^2)(a+b)^2/8ab}$ holds for all positive real numbers a,b if and only if $h(s) = c_1 s^2 + c_2$.

Theorem 3.7. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;k(s)) = (a^2+b^2)(a+b)/4ab$ holds for all positive real numbers a,b if and only if $k(s) = c_1 s + c_2$.

(ii) $M(a,b;k(s)) = A(a,b)$ holds for all positive real numbers a,b if and only if $k(s) = c_1(1/s) + c_2$.

(iii) $M(a,b;k(s)) = (a+b)^3/8ab$ holds for all positive real numbers a,b if and only if $k(s) = c_1 \log s + c_2$.

(iv) $M(a,b;k(s)) = \sqrt{(ab)(a+b)^2/2(a^2+b^2)}$ holds for all positive real numbers a,b if and only if $k(s) = c_1(1/s^2) + c_2$.

(v) $M(a,b;k(s)) = \sqrt{(a+b)^2(3a^4+3b^4+2(ab)^2)/32(ab)^2}$ holds for all positive real numbers a,b if and only if $k(s) = c_1 s^2 + c_2$.

Instead of (3.2) and (3.3), in [14] the author considered in general, the following two mean values of a,b:

$$M(a,b;h(s),q) := \frac{1}{P_q(a,b)} h^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} h(s)d\theta\right), \tag{3.4}$$

and

$$M(a,b;k(s),q) := \frac{1}{P_q(a,b)} k^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} k(s)d\theta\right), \tag{3.5}$$

where $h(s)$ and $k(s)$ are just the same as in (3.2) and (3.3).

Moreover, denote

$$s_n(\theta) = (a^{2n}\cos^2\theta + b^{2n}\sin^2\theta)^{\frac{1}{n}}, \quad n \neq 0,$$

and

$$s_0(\theta) = \lim_{n\to 0} s_n(\theta) = a^{2\cos^2\theta} b^{2\sin^2\theta}.$$

If $p : R^+ \to R$ is a strictly monotonic function, then

$$M(a, b; p, s_n; q) = \frac{1}{P_q(a, b)} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(s_n(\theta))d\theta\right)$$

defines a mean value of a, b. Clearly, (3.4) is given for $n = -1$ and (3.5) is given for $n = 1$.

We have the following two theorems.

Theorem 3.8. *If for some twice continuously differentiable function p the mean $M(a, b; p, s_n; q)$ reduces at the power mean $P_r(a, b)$, then*

$$p = c_1 e_{(q+r)/2-n} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

Theorem 3.9. *The mean $M(a, b; p, s_n; q)$ reduces to the power mean $P_r(a, b)$ for arbitrary n if*

$$p = c_1 e_{(q+r)/2-n} + c_2, \quad c_1, c_2 \in R$$

and r takes one of the following values: (i) $r = -q$ or (ii) $r = q = n$.

In [10], Kim and Rassias considered a new mean value

$$M(a, b; p, r_{n,k}) := (ab)^{(1-k)/2} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{n,k}(\theta))d\theta\right) \tag{3.6}$$

where $p : R^+ \to R$ is a strictly monotonic function, n and k are real numbers,

$$r_{n,k}(\theta) = (a^{kn}\cos^2\theta + b^{kn}\sin^2\theta)^{\frac{1}{n}}, \quad n, k \neq 0,$$

and

$$r_{0,k}(\theta) = \lim_{n\to 0} r_{n,k}(\theta) = a^{k\cos^2\theta} b^{k\sin^2\theta}, \quad k \neq 0.$$

The mean can represent some known means for special choice of p, k and n. Two well-known examples are given for $n = 2$, $k = 1$, $p(x) = x^{-1}$ and $n = -2$, $k = 1$, $p(x) = x$ respectively. They correspond to the arithmetic-geometric mean of Gauss (1.3) and geometric-harmonic mean (3.1) respectively.

Kim and Rassias in [10] also considered the following generalization of the power means defined by

$$H_q^k(a,b) = (ab)^{(1-k)/2}\left(\frac{2a^qb^q}{a^q+b^q}\right)^{k/q}, \quad k \neq 0$$

for $q \neq 0$, while $H_0^k(a,b) = \lim_{q\to 0} H_q^k(a,b) = \sqrt{ab}$ for $q = 0$.

It is not difficult to prove the following theorems.

Theorem 3.10. *If the mean $M(a,b;p,r_{n,k})$ reduces to the power mean $P_q^k(a,b) = H_{-q}^k(a,b)$ for some twice continuously differentiable function p, then*

$$p = c_1 e_{(2kq-nk^2)/k^2} + c_2,$$

where c_1 and c_2 are arbitrary real numbers.

Theorem 3.11. *The mean $M(a,b;p,r_{n,k})$ reduces to the power mean $P_q^k(a,b)$ for some arbitrary n if*

$$P = c_1 e_{(2kq-nk^2)/k^2} + c_2, \quad c_1, c_2 \in R$$

and q takes one of the following values: (i) $q = 0$, (ii) $q = nk$; or (iii) $q = nk/2$.

Theorem 3.12. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;p,r_{1,k}) = \dfrac{1}{2}(a^k+b^k)(ab)^{(1-k)/2}$ holds for all positive real numbers a,b if and only if $p(s) = c_1 s + c_2$.

(ii) $M(a,b;p,r_{1,k}) = G(a,b)$ holds for all positive real numbers a,b if and only if $p(s) = c_1(1/s) + c_2$.

(iii) $M(a,b;p,r_{1,k}) = \dfrac{1}{4}(ab)^{(1-k)/2}(a^{k/2}+b^{k/2})^2$ holds for all positive real numbers a,b if and only if $p(s) = c_1 \log s + c_2$.

(iv) $M(a,b;p,r_{1,k}) = \dfrac{\sqrt{2}(ab)^{(k+2)/4}}{(a^k+b^k)^{1/2}}$ holds for all positive real numbers a,b if and only if $p(s) = c_1(1/s^2) + c_2$.

(v) $M(a,b;p,r_{1,k}) = \dfrac{[3(a^{2k}+b^{2k})+2(ab)^k]^{1/2}}{[8(ab)^{(k-1)}]^{1/2}}$ holds for all positive real numbers a,b if and only if $p(s) = c_1 s^2 + c_2$.

Theorem 3.13. *Let $c_1(\neq 0)$ and c_2 be arbitrary real constants.*

(i) $M(a,b;p,r_{-1,k}) = G(a,b)$ holds for all positive real numbers a,b if and only if $p(s) = c_1 s + c_2$.

(ii) $M(a,b;p,r_{-1,k}) = 2(ab)^{(k+1)/2}(a^k+b^k)^{-1}$ holds for all positive real numbers a,b if and only if $p(s) = c_1(1/s) + c_2$.

(iii) $M(a,b;p,r_{-1,k}) = 4(ab)^{(1+k)/2}(a^{k/2}+b^{k/2})^{-2}$ *holds for all positive real numbers* a, b *if and only if* $p(s) = c_1 \log s + c_2$.

(iv) $M(a,b;p,r_{-1,k}) = \frac{1}{\sqrt{2}}(a^k+b^k)^{1/2}(ab)^{(2-k)/4}$ *holds for all positive real numbers* a, b *if and only if* $p(s) = c_1(1/s^2) + c_2$.

(v) $M(a,b;p,r_{-1,k}) = \frac{[8(ab)^{k+1}]^{1/2}}{[3(a^{2k}+b^{2k})+2(ab)^k]^{1/2}}$ *holds for all positive real numbers* a, b *if and only if* $p(s) = c_1 s^2 + c_2$.

Instead of (3.6), Rassias and Kim in [15] introduce in general, the following mean values of a, b:

$$M(a,b;p,r_{n,k};q) := [P_q(a,b)]^{(1-k)}p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{n,k}(\theta))d\theta\right)$$

where $p(r_{n,k}(\theta))$ is just the same as in (3.6).

The following theorems are proved.

Theorem 3.14. *If the mean* $M(a,b;p,r_{n,k};q)$ *reduces to the power mean* $P_s(a,b)$ *for some twice continuously differentiable function* p*, then*

$$p = c_1 e_{\frac{2q(k-1)+2s}{k^2} - n} + c_2,$$

where c_1 *and* c_2 *are arbitrary real numbers.*

Theorem 3.15. *The mean* $M(a,b;p,r_{n,k};q)$ *reduces to the power mean* $P_s(a,b)$ *for some arbitrary* n *if*

$$p = c_1 e_{\frac{2q(k-1)+2s}{k^2} - n} + c_2, \quad c_1, c_2 \in R$$

and s *takes one of the following values: (i)* $s = q = 0$*, (ii)* $s = -q$*,* $k = 2$*, (iii)* $s = q = nk$*; or (iv)* $s = q = nk/2$*.*

Acknowledgment. The author is greatly indebted to Professor Th.M. Rassias for his encouragement of doing this work and presentation of references [10], [11], [15] and [18].

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

SEMILINEAR EQUATIONS IN HILBERT SPACES WITH QUASI-POSITIVE NONLINEARITY

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Abstract. The problem is to show that $Ax + F(x) = 0$ has a solution, where A is linear, maximal monotone and the nonlinearity F is a quasi-positive operator of Leray-Schauder type. The existence result is obtained as a consequence of the properties of the Leray-Schauder degree. Finally, some applications are given.

1. **Introduction**

Let H be a real Hilbert space with the inner product denoted by $< \cdot, \cdot >$ and the corresponding norm

$$\|x\| = \sqrt{< x, x >} \quad , \qquad x \in H.$$

Let us consider the semilinear equation

$$Ax + F(x) = 0, \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a densely defined linear operator and $N : H \to H$ is nonlinear. We establish an existence and uniqueness result for the equation (1.1) under some monotonicity conditions. Moreover, assume that A is maximal monotone. Equations of the form (1.1) arise in natural way in the theory of elliptic equations or integro-differential equations.

An operator $F : H \to H$ is called quasi-positive if there exists $\alpha \in \mathbf{R}$ such that

$$< F(x), x > \geq \alpha \|F(x)\|^2 \quad , \qquad \forall x \in H, \ x \neq 0. \tag{1.2}$$

This notion is close related with the angle-bounded operators. First, the angle-boundedness concept is defined for linear operators acting from a Banach space into its dual, then the definition can be extended to nonlinear operators. For details, see [7].

2. The Results

We give the following:

Lemma 2.1. *If* $F : H \to H$ *is a quasi-positive operator with* $\alpha > 1/2$, *then*

$$\|x - F(x)\| \leq \|x\| \quad , \quad \forall\, x \in H,\; x \neq 0.$$

Proof. We have:

$$\|x - F(x)\|^2 = < x - F(x), x - F(x) > =$$

$$= \|x\|^2 - 2 < F(x), x > + \|F(x)\|^2 \leq$$

$$\leq \|x\|^2 - (2\alpha - 1)\|F(x)\|^2 \leq \|x\|^2 . \quad \square$$

If A is linear, maximal monotone, then for all $\lambda > 0$, the operator $I + \lambda A$ is invertible with continuous inverse $(I + \lambda A)^{-1} : H \to H$ and

$$\left\|(I + \lambda A)^{-1}\right\| \leq 1.$$

For proof and further properties, see [3].

Now, the equation (1) can be written as

$$(I + A)x = x - F(x) \Leftrightarrow x = (I + A)^{-1}(x - F(x)),$$

or

$$x = T(x) \Leftrightarrow (I - T)(x) = 0, \tag{2.1}$$

where $T = (I + A)^{-1}(I - F)$.

If F is an operator of Leray-Schauder type, then $I - F$ is compact and consequently, T is compact, as the product of a continuous operator with a compact one.

Indeed, if $D \subset H$ is bounded and $(x_n)_{n \geq 1} \subset D$, then there exists x such that $(I - F)(x_{k_n}) \to (I - F)(x)$, at least on a subsequence. Further, $(I + A)^{-1}$ is continuous, so $Tx_{k_n} \to Tx$.

In conclusion, the operator $I - T$ is compact perturbation of the identity map and consequently, the Leray-Schauder degree can be considered.

Roughly speaking, the degree of ϕ at y, relative to D, denoted $d(\phi, D, y)$, is a measure of the number of the solutions of the equation $\phi(x) = y$ in D.

In an infinite dimensional Banach space X, the Leray-Schauder degree is defined for compact perturbations of the identity map, also named Leray-Schauder operators, $\phi \in (LS)$. Some properties of the Leray-Schauder degree are of interest in our work.

Proposition 2.1. *Let $\phi : D \subset X \to X$ be such that $I - \phi$ is compact and let $y \in X \setminus \phi(\partial D)$. Then the Leray-Schauder degree $d(\phi, D, y)$ satisfies the following properties:*

(a) *If $d(\phi, D, y) \neq 0$, then $y \in \phi(D)$.*

(b) *If $H \in C([0,1] \times D, X)$ is such that $I - H(t, \cdot)$ is compact, for all $t \in [0,1]$ and $y \in X \setminus H([0,1] \times \partial D)$, then the degree*

$$d(H(t,\cdot), D, y) = constant \quad , \quad \forall\, t \in [0,1].$$

(c) *The degree for the identity map $I : X \to X$ is*

$$d(I, D, y) = \begin{cases} 1 & , \quad y \in D \\ 0 & , \quad y \notin D \end{cases} .$$

For more details, see [4], [5].

Now, we can establish the following existence result:

Theorem 2.1. *Let $A : D(A) \subset H \to H$, $0 \in IntD(A)$, linear, maximal monotone and $F : H \to H$ be an (LS) - operator such that*

$$< F(x), x > \geq \alpha \, \|F(x)\|^2 \quad , \quad \forall\, x \in H, \; x \neq 0,$$

for some $\alpha > 1/2$. Then the equation $Ax + F(x) = 0$ has at least one solution $x \in D(A)$.

Proof. Let $B = B(0, r)$ be such that $\overline{B} \subset D(A)$. We have seen that the equation $Ax + F(x) = 0$ is equivalent with

$$(I - T)(x) = 0,$$

where $T = (I + A)^{-1}(I - F)$ is compact.

Let us consider the Leray-Schauder homotopy

$$H(t, x) = x - tT(x) \quad , \quad x \in \overline{B}, \; t \in [0,1].$$

If $0 \in H(1, \partial B)$, the conclusion follows immediately. In order to use the invariance to homotopy of the Leray-Schauder degree, we prove that $0 \notin H([0,1), \partial B)$. Let us suppose by contrary that $H(t,x) = 0$, for some $x \in \partial B$ and $t \in [0,1)$. It results

$$\|x\| = t\,\|T(x)\| \leq \|T(x)\| = \left\|(I+A)^{-1}(I-F)\right\| \leq$$

$$\leq \left\|(I+A)^{-1}\right\| \cdot \|x - F(x)\| \leq \|x - F(x)\| \leq \|x\|\,.$$

We must have equalities all over, in particular $T(x) = 0$. Hence $x = 0 \in \partial B$, contradiction. This means that $0 \notin H([0,1], \partial B)$ and further,

$$d(H(1,\cdot), B, 0) = d(H(0,\cdot), B, 0) \Rightarrow$$

$$\Rightarrow d(I - T, B, 0) = d(I, B, 0) = 1.$$

In conclusion, $d(I - T, B, 0) \neq 0$, thus the equation $(I - T)(x) = 0$ and equivalent, the equation $Ax + F(x) = 0$ has at least one solution in $D(A)$. $\square$

3. **An Application**

Now, we are in position to show how the theoretical results from the previous section can be applied to the elliptic boundary value problems.

Let $\Omega \subset \mathbf{R}^n$ be open, bounded and let $a_{ij} \in C^1(\overline{\Omega})$, $1 \leq i,j \leq n$ be real valued functions satisfying the ellipticity property

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq 0 \quad , \quad \forall\, \xi = (\xi_1, ..., \xi_n) \in \mathbf{R}^n.$$

Let us consider the following elliptic problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(t)\frac{\partial x}{\partial x_i}\right) + g(t,x) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

The particular case $g(t,x) = a_0(t)x$, with $a_0 \in C(\overline{\Omega})$, $a_0 > p > 0$, is studied in [3], using Lax-Milgram theorem. Some existence results are also obtained in [1] and [2], as a consequence of some general considerations about saddle points. The general case of problem (3.1) is studied in [6], under the assumption that the nonlinear part is strongly monotone.

Here we assume that g satisfies

$$\int_{\Omega} g(t,x(t)) \cdot x(t)\, dt \geq \alpha \int_{\Omega} g^2(t,x(t))\, dt, \tag{3.2}$$

for some $\alpha > 1/2$. Remark that in case $g(t,x) = a_0(t)x$, the condition (3.2) is fulled with $\alpha < 1/\|a_0\|$.

Under the condition (3.2), the problem (3.1) has at least one solution in weak sense. Indeed, we can apply theorem 2.1 in the following functional background:

$$H = L^2(\Omega)\ ,\ Ax = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(t)\frac{\partial x}{\partial x_i}\right)\quad ,\quad D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

and $(Fx)(t) = g(t,x)$. The problem (3.1) can be written in the abstract form

$$Ax + F(x) = 0\quad ,\quad x \in D(A) \subset L^2(\Omega).$$

We have:

$$< Ax, x >= \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} \frac{\partial x}{\partial x_j} \cdot \frac{\partial x}{\partial x_i} \geq 0,$$

and $I + A$ is surjective, e.g. [2], therefore A is maximal monotone.

Finally, if g is compact perturbation of the identity, then the assertion is proved.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

DYNAMICS ON $(P_{cp}(X), H_d)$ GENERATED BY A SET OF DYNAMICS ON (X, d)

IOAN A. RUS, BOGDAN RUS

Abstract. In this paper we study the following problem: Let (X, d) be a complete metric space. Let $f_1, \ldots, f_m : X \to X$ be some continuous weakly Picard operators. These operators generates the following operator

$$T_f : P_{cp}(X) \to P_{cp}(X), \quad A \mapsto f_1(A) \cup \cdots \cup f_m(A).$$

Is the operator $T_f : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ weakly Picard operator?

1. **Introduction**

Let X be a nonempty set and $f_1, \ldots, f_m : X \to X$ some operators. These operators generate the following operator on $P(X)$

$$T_f : P(X) \to P(X), \quad T_f(A) := f_1(A) \cup \cdots \cup f_m(A).$$

The problem is to study the operator T_f depending on the properties of the operators $f_1, \ldots, f_m$. In what follow we shall study this problem from the point of view of the Picard operators theory.

Throughout this paper we follow terminologies and notations in [27] and [36]. See also [31], [32] and [34]. For the multivalued operator theory see [36], [2], [21], [23].

2. **Iterated Picard operator systems**

We begin our study with the following open problem

Problem 1. (see [32] and [34]) Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ continuous Picard operators. Is the operator $T_f : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ Picard operator?

For the Problem 1 we have the following partial results:

1991 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. iterated Picard operator system, fractal operator, Bessaga operator, Janos operator, attractor.

Theorem 2.1. (see [1], [13], [6], [42]) *If the operators $f_1, \ldots, f_m$ are a-contraction, then the operator*

$$T_f : P_{cp}(X) \to P_{cp}(X)$$

is an a-contraction.

Remark 2.1. By definition, the unique fixed point of T_f is the attractor of the iterated operator systems (IOS) $f_1, \ldots, f_m$.

Theorem 2.2. (see [33]) *If the operators $f_1, \ldots, f_m$ are φ-contractions, then the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ is a φ-contraction.*

Theorem 2.3. (see [24]) *If the operators $f_1, \ldots, f_m$ are of Meir-Keeler type, then the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ is a Meir-Keeler type operator.*

The following open problems are in connection with the Problem 1.

Problem 2. Let X be a nonempty set and $f_1, \ldots, f_m$ Bessaga operators. Does there exist $Y \subset P(X)$ such that

(a) $T_f(Y) \subset Y$,

(b) $T_f : Y \to Y$ is Bessaga operator?

Problem 3. Let X be a nonempty set and $f_1, \ldots, f_m$ Janos operators. Does there exist $Y \subset P(X)$ such that

(a) $T_f(T) \subset Y$,

(b) $T_f : Y \to Y$ is Janos operator?

Problem 4. Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ continuous Bessaga operators. Is the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ Bessaga operator?

Problem 5. Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ continuous Janos operators. Is the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ Janos operator?

In the case $m = 1$ we have

Example 2.1. Let $f : R \to R$, $f(x) = \frac{1}{2}x$ and $T_f : P(R) \to P(R)$, $T_f(A) = f(A)$. We remark that f is Bessaga operator (f is $\frac{1}{2}$-contraction), but $cardF_{T_f} > 1$. For example $\{0\}, R, R_+, R_-, R_+^*, R_-^*, \{2^k | k \in Z\}$, are fixed points of T_f.

Theorem 2.4. *Let X be a nonempty set and $f : X \to X$ a Bessaga operator. Then there exists $Y \subset P(X)$ such that* *(a)* $T_f(Y) \subset Y$

(b) $T_f : Y \to Y$ *is Bessaga operator.*

If $cardX > 1$*, then there exists* $Y \subset P(X)$ *such that* $cardY > 1$*.*

Proof. Here T_f is the following operator, $T_f : P(X) \to P(X), T_f(A) = f(A)$. By a theorem of Bessaga ([27]) there exists a metric d on X such that (X, d) is a complete metric space and $f : (X, d) \to (X, d)$ is an a-contraction. By a theorem of Nadler ([22]) the operator $T_f : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ is an a-contraction. By the contraction principle $T_f|_{P_{cp}(X)}$ is Picard operator. So, $T_f|_{P_{cp}(X)}$ is Bessaga operator ($Y = P_{cp}(X, d)$).

Theorem 2.5. *Let* (X, d) *be a compact metric space and* $f : X \to X$ *continuous Janos operator. Then the operator* $T_f : P_{cp}(X) \to P_{cp}(X)$ *is Janos operator.*

Proof. By a theorem of Janos ([27]) there exists an equivalent metric (with d) ρ on X such that $f : (X, \rho) \to (X, \rho)$ is an a-contraction. By a theorem of Nadler ([22]) the operator $T_f : (P_{cp}(X), H_\rho) \to (P_{cp}(X), H_\rho)$ is an a-contraction, These imply that

$$\delta_{H_\rho} : (T_f(P_{cp}(X))) \leq a\delta_{H_\rho}(P_{cp}(X))$$

and

$$\delta_{H_\rho} : (T_f^n(P_{cp}(X))) \leq a^n\delta_{H_\rho}(P_{cp}(X)).$$

So

$$\bigcap_{n \in N} T_f^n(P_{cp}(X)) = \{\{x^*\}\}$$

where x^* is the unique fixed point of f.

3. Iterated weakly Picard operator systems

The basic problem of this paper is the following

Problem 6. Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ continuous WPOs. Is the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ WPO?

The following open problems are in connection with the Problem 6.

Problem 7. Let (X, d) be a complete metric space and $f_1, \ldots, f_m \in C(X, X)$. We suppose that

$$F_{f_i} = F_{f_i^n} \neq \emptyset, \quad i = \overline{1, m},\ n \in N^*.$$

We ask if

$$F_{T_f} = F_{T_f^n} \neq \emptyset, \quad n \in N^*.$$

Problem 8. Let (X,d) be a compact metric space and $f_1,\ldots,f_m \in C(X,X)$. We suppose that

$$\bigcap_{n\in N} f_i^n(X) = F_{f_i}, \quad i = \overline{1,m}.$$

Does the operator T_f satisfy the condition

$$\bigcap_{n\in N} T_f^n(P_{cp}(X)) = F_{T_f}?$$

Problem 9. (see [4], [26]) Let (X,d) be a complete metric space and $f_i \in C(X,X)$, $i = \overline{1,m}$. We suppose that

$$\omega_{f_i}(x) \neq \emptyset,\ \forall\, x \in X,\ \forall\, i = \overline{1,m}.$$

Does this imply that

$$\omega_{T_f}(A) \neq \emptyset,\ \forall\, A \in P_{cp}(X)?$$

Problem 10. (see [4], [26]) Let (X,d) be a complete metric space and $f_i \in C(X,X)$, $i = \overline{1,m}$. If there exists $x \in X$ such that the recurrent point set of f_i,

$$R_{f_i}^{(x)} \neq \emptyset, \quad i = \overline{1,m},$$

does exist $A \in P_{cp}(X)$ such that

$$R_{T_f}(A) \neq \emptyset?$$

In the case $m = 1$, we have

Example 3.1. Let X be a Banach space, $K \in C([a,b]\times[a,b]\times X, X)$, $K(t,s,\cdot) : X \to X$ a L_K-Lipschitz operator, for all $t,s \in [a,b]$. Let $f : C([a,b],X) \to C([a,b],X)$ be defined by

$$f(x)(t) = x(a) + \int_a^t K(t,s,x(s))ds.$$

Let $X_\alpha := \{x \in C([a,b],X) | x(a) = \alpha\},\ \alpha \in X$. Then

- $X = \bigcup X_\alpha$ is a partition of X,
- f is continuous,
- $X_\alpha \in I_{cl}(f)$,

- $f|_{X_\alpha}$ is a Picard operator, $\alpha \in X$,
- $T_f : P_{cp}(X_\alpha) \to P_{cp}(X_\alpha)$ is Picard operator, $\alpha \in X$,
- $T_f : \bigcup_{\alpha \in X} P_{cp}(X_\alpha) \to \bigcup_{\alpha \in X} P_{cp}(X_\alpha)$ is WPO with respect to the generalized Hausdorff-Pompeiu metric.

More general we have

Theorem 3.1. *Let (X, d) be a complete metric space, $X = \bigcup_{\alpha \in J} X_\alpha$ a partition of X, $f : X \to X$ a continuous operator such that:*

(i) $X_\alpha \in I_{cl}(f)$,

(ii) $f : X_\alpha \to X_\alpha$ is a-contraction, for all $\alpha \in J$.

Then there exists $S(X) \subset P(X)$ such that:

(i) $S(X) \in I(T_f)$,

(ii) $T_f : S(X) \to S(X)$ is WPO with respect to the generalized Hausdorff-Pompeiu metric on $S(X)$.

Proof. By a theorem of Nadler $T_f : P_{cp}(X_\alpha) \to P_{cp}(X_\alpha)$ is a-contraction for all $\alpha \in J$. Let $S(X) := \bigcup_{\alpha \in J} P_{cp}(X_\alpha)$. Then for all $A \in S(X)$, $T_f^n(A)$ converges to $T_f^\infty(A)$. If $A \in P_{cp}(A_\alpha)$, then $T_f^\infty(A) \in P_{cp}(X_\alpha)$, and is the unique fixed point of T_f in $P_{cp}(X_\alpha)$.

4. Attractor and sequences of contractions

Let (X, d) be a complete metric space, $f_1, \ldots, f_m : X \to X$ a-contractions. Then $T_f : P_{cp}(X) \to P_{cp}(X)$ is a-contraction. By definition the unique fixed point of T_f, A^*, is the attractor of the iterated operator system $f_1, \ldots, f_m$. The attractor A^* has the following properties (see [13], [43], [1],...):

a)

(i) $\emptyset \neq A^*$ is compact,

(ii) $f_i(A^*) \subset A^*$, for $1 \leq i \leq m$,

(iii) A^* is minimal with respect to (i) and (ii).

b) for all $x \in A^*$, there exists a sequence $i_1, \ldots, i_s, \ldots$ such that

$$f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_s}(y) \to x \text{ as } s \to \infty,$$

for all $y \in X$.

The above properties of the attractor give rise to the following problems:

Problem 11. Let $(X < d)$ be a complete metric space and $f, f_n : X \to X$, $n \in N$. We suppose that

(i) f and f_n are a-contractions, $n \in N$,

(ii) $f_n \xrightarrow{d} f$.

Does f_n^∞ converges to f^∞?

Problem 12. Let (X, d) be a complete metric space and $f, f_n : X \to X$ WPOs, $n \in N$. If $(f_n)_{n\in N}$ converges to f, does $(f_n^\infty)_{n\in N}$ converges to f^∞?

Problem 13. Let (X, d) be a complete metric space and $f_1, \dots, f_m : X \to X$ φ-contractions. Let $(g_n)_{n\in N}$ a sequence in $\{f_1, \dots, f_m\}$. Does converge the sequences

$$x_n := (g_0 \circ \cdots \circ g_n)(x)$$

and

$$y_n := (g_n \circ \cdots \circ g_0)(x)?$$

Problem 14. Let (X, d) be a complete metric space and $f_n : X \to X$ a r_n-contraction, $n \in N$. If $r_n \to 0$ as $n \to \infty$, does f_n converges to a constant operator?

We have the following result for the above problems

Theorem 4.1. (see [28]) *Let (X, d) be a complete metric space and $f, f_n : X \to X$, $n \in N$. We suppose that:*

(a) f is Picard operator ($F_f = \{x^*\}$);

(b) the sequence $(f_n)_{n\in N}$ is asymptotical uniform convergent to f;

(c) $F_{f_n} \neq \emptyset$, for all $n \in N$.

If $x_n^* \in F_{f_n}$, then $x_n^* \to x^*$ as $n \to \infty$.

Proof. By definition the sequence $(f_n)_{n\in N}$ is asymptotical uniform convergent to f if for all $\varepsilon > 0$ there exist $n_0(\varepsilon), m_0(\varepsilon)$ such that

$$d(f_n^m(x), f^m(x)) < \varepsilon$$

for all $n \geq n_0(\varepsilon)$, $m \geq m_0(\varepsilon)$ and all $x \in X$.

We have

$$d(x_n^*, x^*) = d(f_n^m(x_n^*), f^m(x^*)) \leq$$

$$\leq d(f_n^m(x_n^*), f^m(x_n^*)) + d(f^m(x_n^*), f^m(x^*)).$$

Let $\varepsilon > 0$ and $n_0(\varepsilon), m_0(\varepsilon)$ such that

$$d(f_n^m(x_n^*), f^m(x_n^*)) \leq \frac{\varepsilon}{2},$$

for all $n \geq n_0(\varepsilon),\ m \geq m_0(\varepsilon)$.

On the other hand for each $n \geq n_0(\varepsilon)$ there exists $m_n(\varepsilon)$ such that

$$d(f^{m_n(\varepsilon)}(x_n^*), x^*) < \frac{\varepsilon}{2}.$$

Theorem 4.2. (see [1], [5], [18]) *Let (X, d) be a complete metric space and $f_n : X \to X$ a α_n-contraction, such that $\alpha_n \to 0$ as $n \to \infty$. Let $x^* \in X$. Then the following statements are equivalent:*

(i) there exists $x_0 \in X$ such that $f_n(x_0) \to x^*$ as $n \to \infty$;

(ii) $f_n(x) \to x^*$ as $n \to \infty$, for all $x \in X$;

(iii) $x_n^* \to x^*$ as $n \to \infty$, where x_n^* is the unique fixed point of f_n.

Proof. (i) $\Rightarrow$ (ii). From the condition (i) we have

$$d(f_n(x), x^*) \leq d(f_n(x), f_n(x_0)) + d(f_n(x_0), x^*) \leq$$

$$\leq a_n d(x, x_0) + d(f_n(x_0), x^*).$$

(ii) $\Rightarrow$ (ii). We have

$$d(x_n^*, x^*) \leq d(f_n(x_n^*), f_n(x^*)) + d(f_n(x^*), x^*) \leq$$

$$\leq \alpha_n d(x_n^*, x^*) + d(f_n(x^*), x^*).$$

So

$$d(x_n^*, x^*) \leq \frac{1}{1-\alpha_n} d(f_n(x^*), x^*) \to 0 \text{ as } n \to \infty.$$

(iii) $\Rightarrow$ (i). It follows from

$$d(f_n(x^*), x^*) \leq d(f_n(x^*), f_n(x_n^*)) + d(f_n(x_n^*), x^*) \leq$$

$$\leq (\alpha_n + 1) d(x_n^*, x^*).$$

Remark 4.1. For other results for the Problem 11-14 see [1], [5], [10], [22], [18], [19], [28], [36].

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 4, December 2001

ON CERTAIN INEQUALITIES INVOLVING THE IDENTRIC MEAN IN n VARIABLES

TIBERIU TRIF

Abstract. In this paper we establish one Chebyshev type and two Ky Fan type inequalities for the weighted identric mean in n variables.

1. Introduction and notation

Let $n \geq 2$ be a given integer, let

$$A_{n-1} = \{ (\lambda_1, \dots, \lambda_{n-1}) \mid \lambda_i \geq 0,\ i = 1, \dots, n-1,\ \lambda_1 + \cdots + \lambda_{n-1} \leq 1 \}$$

be the Euclidean simplex, and let μ be a probability measure on A_{n-1}. For each $i \in \{ 1, \dots, n \}$, the ith weight w_i associated to μ is defined by

$$\begin{aligned} w_i &= \int_{A_{n-1}} \lambda_i d\mu(\lambda) \qquad \text{if} \quad 1 \leq i \leq n-1, \\ w_n &= \int_{A_{n-1}} (1 - \lambda_1 - \cdots - \lambda_{n-1}) d\mu(\lambda), \end{aligned}$$

where $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in A_{n-1}$. Obviously, $w_i > 0$ for all $i \in \{ 1, \dots, n \}$, and $w_1 + \cdots + w_n = 1$. We also define

$$\begin{aligned} w_{ij} &= \int_{A_{n-1}} \lambda_i \lambda_j d\mu(\lambda) \qquad \text{if} \quad 1 \leq i, j \leq n-1, \\ w_{in} &= w_{ni} = \int_{A_{n-1}} \lambda_i (1 - \lambda_1 - \cdots - \lambda_{n-1}) d\mu(\lambda) \qquad \text{if} \quad 1 \leq i \leq n-1, \\ w_{nn} &= \int_{A_{n-1}} (1 - \lambda_1 - \cdots - \lambda_{n-1})^2 d\mu(\lambda). \end{aligned}$$

Taking into account the Liouville formula (see, for instance, [1])

$$\int_{A_{n-1}} \lambda_1^{p_1 - 1} \cdots \lambda_{n-1}^{p_{n-1} - 1} f(\lambda_1 + \cdots + \lambda_{n-1}) d\lambda_1 \cdots d\lambda_{n-1} = \frac{\Gamma(p_1) \cdots \Gamma(p_{n-1})}{\Gamma(p_1 + \cdots + p_{n-1})} \int_0^1 x^{p_1 + \cdots + p_{n-1} - 1} f(x) dx,$$

1991 *Mathematics Subject Classification.* 26D15.

in the special case $\mu = (n-1)!$ we get $w_i = 1/n$ for all $i \in \{1, \dots, n\}$ and

$$\begin{aligned} w_{ii} &= \frac{2}{n(n+1)} \qquad \text{for all} \quad i \in \{1, \dots, n\}, \\ w_{ij} &= \frac{1}{n(n+1)} \qquad \text{for all} \quad i, j \in \{1, \dots, n\},\ i \neq j. \end{aligned}$$

Next, recall that the *identric mean* $I(x_1, x_2)$ of the positive real numbers x_1 and x_2 is defined by

$$\begin{aligned} &I(x_1, x_2) = \frac{1}{e}\left(\frac{x_2^{x_2}}{x_1^{x_1}}\right)^{1/(x_2 - x_1)} \qquad \text{if} \quad x_1 \neq x_2, \\ &I(x_1, x_1) = x_1. \end{aligned}$$

It is easily seen that the following integral representation holds:

$$I(x_1, x_2) = \exp\left(\int_0^1 \log(tx_1 + (1-t)x_2)dt\right). \tag{1.1}$$

Given $X = (x_1, \dots, x_n) \in \,]0, \infty[^n$, we set

$$\lambda \cdot X := \lambda_1 x_1 + \cdots + \lambda_{n-1}x_{n-1} + (1 - \lambda_1 - \cdots - \lambda_{n-1})x_n$$

for all $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in A_{n-1}$. Starting from (1.1), in [7] it was pointed out that

$$I(X; \mu) := \exp\left(\int_{A_{n-1}} \log(\lambda \cdot X) d\mu(\lambda)\right)$$

can be considered as the weighted identric mean of $x_1, \dots, x_n$. For $\mu = (n-1)!$ we obtain the unweighted and symmetric identric mean of $x_1, \dots, x_n$

$$I(X) = I(x_1, \dots, x_n) = \exp\left((n-1)! \int_{A_{n-1}} \log(\lambda \cdot X) d\lambda_1 \cdots d\lambda_{n-1}\right).$$

As in the case of other means, $I(X; \mu)$ can be generalized as follows: for each real number r we set $X^r := (x_1^r, \dots, x_n^r)$, and then define

$$\begin{aligned} I_r(X; \mu) &:= (I(X^r; \mu))^{1/r} \qquad \text{if} \quad r \neq 0, \\ I_0(X; \mu) &:= \lim_{r \to 0} I_r(X; \mu) = x_1^{w_1} \cdots x_n^{w_n} \qquad \text{(see [5])}. \end{aligned}$$

The means $I_r(X; \mu)$ are special cases of the so-called Stolarsky-Tobey means introduced in [5]: namely $I_r(X; \mu) = E_{r,r}(X; \mu)$. Consequently, several inequalities (of the Jensen, Minkowski, Hölder, Rennie, and Kantorovich type, respectively) involving the means I_r can be obtained as special cases of the results listed in [5]. In Section 2 of this paper we complete these inequalities by proving a Chebyshev type inequality for I_r.

Let

$$A(X;\mu) := w_1x_1 + \cdots + w_nx_n \qquad \text{and} \qquad G(X;\mu) := x_1^{w_1}\cdots x_n^{w_n}$$

be the weighted arithmetic and geometric mean, respectively, of $x_1, \ldots, x_n$. For $\mu = (n-1)!$ we obtain the usual arithmetic and geometric mean of $x_1, \ldots, x_n$

$$\begin{aligned} A(X) &= A(x_1,\ldots,x_n) = \frac{x_1+\cdots+x_n}{n}, \\ G(X) &= G(x_1,\ldots,x_n) = (x_1\cdots x_n)^{1/n}. \end{aligned}$$

A famous result due to Ky Fan asserts that if $0 < x_i \le 1/2$ for all $i \in \{1,\ldots,n\}$, then

$$\frac{G(X;\mu)}{G(\mathbf{1}-X;\mu)} \le \frac{A(X;\mu)}{A(\mathbf{1}-X;\mu)}, \tag{1.2}$$

where $\mathbf{1} - X := (1-x_1,\ldots,1-x_n)$. The following refinement of (1.2) has been recently obtained in [7]:

$$\frac{G(X;\mu)}{G(\mathbf{1}-X;\mu)} \le \frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)} \le \frac{A(X;\mu)}{A(\mathbf{1}-X;\mu)}. \tag{1.3}$$

In Section 3 of this paper we establish a converse of the left inequality in (1.3) as well as an improvement of the right inequality in (1.3).

2. Chebyshev's inequality for the identric mean in n variables

Theorem 2.1. *Let $X = (x_1,\ldots,x_n) \in \mathbf{R}^n$ and $Y = (y_1,\ldots,y_n) \in \mathbf{R}^n$ such that $0 < x_1 \le \cdots \le x_n$ and $0 < y_1 \le \cdots \le y_n$, and let $X \cdot Y := (x_1y_1,\ldots,x_ny_n)$. Then*

$$\begin{aligned} I_r(X;\mu)I_r(Y;\mu) &\le I_r(X\cdot Y;\mu) \qquad \textit{for all} \quad r>0, \\ I_r(X;\mu)I_r(Y;\mu) &\ge I_r(X\cdot Y;\mu) \qquad \textit{for all} \quad r<0. \end{aligned}$$

Proof. According to Chebyshev's inequality, we have

$$(\lambda\cdot X^r)(\lambda\cdot Y^r) \le \lambda\cdot(X\cdot Y)^r$$

for all $r \in \mathbf{R}$ and all $\lambda \in A_{n-1}$, hence

$$\int_{A_{n-1}} \log(\lambda\cdot X^r)d\mu(\lambda) + \int_{A_{n-1}} \log(\lambda\cdot Y^r)d\mu(\lambda) \le \int_{A_{n-1}} \log(\lambda\cdot(X\cdot Y)^r)d\mu(\lambda)$$

for all $r \in \mathbf{R}$. Exponentiating both sides yields

$$I(X^r;\mu)I(Y^r;\mu) \le I((X\cdot Y)^r;\mu) \qquad \text{for all} \quad r \in \mathbf{R}.$$

This inequality implies the conclusion of the theorem. □

Besides the identric mean $I(x_1, x_2)$ of the positive real numbers x_1 and x_2, the logarithmic mean of x_1 and x_2 is another important special case of the Stolarsky mean of x_1 and x_2. Recall that the *logarithmic mean* of x_1 and x_2 is defined by

$$L(x_1, x_2) = \frac{x_1 - x_2}{\log x_1 - \log x_2} \qquad \text{if} \quad x_1 \ne x_2,$$
$$L(x_1, x_1) = x_1.$$

Theorem 2.2. *Let x_1, x_2, y_1, y_2 be positive real numbers.*
If $(x_1 - x_2)(y_1 - y_2) > 0$, then

$$L(x_1, x_2)L(y_1, y_2) < L(x_1y_1, x_2y_2), \tag{2.1}$$

while if $(x_1 - x_2)(y_1 - y_2) < 0$, then

$$L(x_1, x_2)L(y_1, y_2) > L(x_1y_1, x_2y_2). \tag{2.2}$$

In the proof we shall use the elementary

Lemma 2.3. *The following assertions are true:*

a) $f_1(v) = v\log v - v + 1$ is strictly decreasing from $]0,1[$ onto $]0,1[$, and strictly increasing from $]1,\infty[$ onto $]0,\infty[$.

b) $f_2(v) = v\log v - 2v + \log v + 2$ is strictly increasing from $]0,\infty[$ onto $]-\infty,\infty[$.

c) $f_3(v) = v^2 - 2v\log v - 1$ is strictly increasing from $]0,1[$ onto $]-1,0[$.

d) $f_4(v) = v\log^2 v - (v-1)^2$ is strictly increasing from $]0,1[$ onto $]-1,0[$.

Proof of the Theorem 2.2. Suppose first that $(x_1 - x_2)(y_1 - y_2) > 0$. Due to the symmetry, we may assume that $x_1 > x_2$ and $y_1 > y_2$, so $u := \frac{x_1}{x_2} > 1$, $v := \frac{y_1}{y_2} > 1$. Taking into account the homogeneity of L, inequality (2.1) is equivalent to

$$\frac{u-1}{\log u}\cdot\frac{v-1}{\log v} < \frac{uv-1}{\log u + \log v},$$

i. e. to

$$(u-1)(v-1)(\log u + \log v) - (uv-1)\log u \log v < 0. \tag{2.3}$$

Let $v \in]1, \infty[$ be fixed, and let $f :]0, \infty[\to \mathbf{R}$ be the function defined by

$$f(u) := (u-1)(v-1)(\log u + \log v) - (uv-1)\log u \log v. \tag{2.4}$$

Then we have

$$f'(u) = (v - 1 - v\log v)\log u + \frac{u-1}{u}(v - 1 - \log v),$$
$$f''(u) = \frac{v - 1 - \log v - u(v\log v - v + 1)}{u^2}.$$

Since $v > 1$, by virtue of Lemma 2.3 a) and b) we obtain

$$f''(u) < \frac{v - 1 - \log v - (v\log v - v + 1)}{u^2} = -\frac{v\log v - 2v + \log v + 2}{u^2} < 0$$

for all $u \in]1, \infty[$, hence f' must be strictly decreasing on $]1, \infty[$. Therefore $f'(u) < 0$ for $u > 1$, because $f'(1) = 0$. This implies that f is also strictly decreasing on $]1, \infty[$. Consequently, $f(u) < 0$ for $u > 1$, because $f(1) = 0$. This proves the validity of (2.3).

Suppose now that $(x_1 - x_2)(y_1 - y_2) < 0$, and assume that $x_1 > x_2$ and $y_1 < y_2$. Then we have $u := \frac{x_1}{x_2} > 1$ and $v := \frac{y_1}{y_2} < 1$. Depending on u and v, we distinguish the following possible cases:

Case I. $uv = 1$.

Then inequality (2.2) is equivalent to $L(u, 1)L(1/u, 1) > 1$. Since $L(1/u, 1) = L(u, 1)/u$, this transforms into the well-known inequality $L(u, 1) > \sqrt{u} = G(u, 1)$ (see [8]).

Case II. $uv > 1$.

Then inequality (2.2) is equivalent to (2.3). Let $v \in]0, 1[$ be fixed, and let $f :]0, \infty[\to \mathbf{R}$ be the function defined by (2.4). By virtue of Lemma 2.3 a) and c), for all $u \in]1/v, \infty[$ we have

$$f''(u) < \frac{v - 1 - \log v - \frac{1}{v}(v\log v - v + 1)}{u^2} = \frac{v^2 - 2v\log v - 1}{u^2 v} < 0,$$

hence f' must be strictly decreasing on $]1/v, \infty[$. But $f'(1/v) = v\log^2 v - (v-1)^2 < 0$, according to Lemma 2.3 d), so $f'(u) < 0$ for $u > 1/v$. This implies that f is also strictly decreasing on $]1/v, \infty[$. Consequently, $f(u) < 0$ for $u > 1/v$, because $f(1/v) = 0$. This proves the validity of (2.3).

Case III. $uv < 1$.

Then inequality (2.2) is equivalent to

$$(u-1)(v-1)(\log u + \log v) - (uv-1)\log u \log v > 0. \tag{2.5}$$

Let again $v \in]0,1[$ be fixed, and let $f :]0,\infty[\to \mathbf{R}$ be the function defined by (2.4). Set

$$\tilde{v} := \frac{v - 1 - \log v}{v \log v - v + 1}.$$

By Lemma 2.3 a), b), and c) we have $1 < \tilde{v} < 1/v$. It is immediately seen that $f''(u) > 0$ for $u \in]1, \tilde{v}[$ and $f''(u) < 0$ for $u \in]\tilde{v}, 1/v[$. Consequently, f' is strictly increasing on $]1, \tilde{v}[$ and strictly decreasing on $]\tilde{v}, 1/v[$. Since $f'(1) = 0$ and $f'(1/v) = v \log^2 v - (v-1)^2 < 0$, it follows that there exists a unique $\bar{v} \in]\tilde{v}, 1/v[$ such that $f'(\bar{v}) = 0$, $f'(u) > 0$ for $u \in]1, \bar{v}[$, and $f'(u) < 0$ for $u \in]\bar{v}, 1/v[$. Therefore f is strictly increasing on $]1, \bar{v}[$ and strictly decreasing on $]\bar{v}, 1/v[$. Since $f(1) = f(1/v) = 0$, we can conclude that $f(u) > 0$ for all $u \in]1, 1/v[$. This completes the proof of (2.5). □

Remark. It would be interesting to study whether Theorem 2.2 can be generalized for n variables (the author does not know the answer).

3. **Two inequalities related to (1.3)**

In this section, both a converse of the left inequality in (1.3) and a refinement of the right inequality in (1.3) are obtained. They are contained in the following two theorems.

Theorem 3.1. *If $X = (x_1, \ldots, x_n) \in]0, 1/2]^n$, then it holds that*

$$\log \frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)} - \log \frac{G(X;\mu)}{G(\mathbf{1}-X;\mu)} \le \left(\sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} \frac{w_i}{x_i(1-x_i)}\right) - \sum_{i=1}^{n} \frac{w_i}{1-x_i}. \tag{3.1}$$

Theorem 3.2. *If $X = (x_1, \ldots, x_n) \in \,]0, 1/2]^n$, then it holds that*

$$\log \frac{A(X;\mu)}{A(\mathbf{1}-X;\mu)} - \log \frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)} \tag{3.2}$$
$$\geq \frac{1-2\bar{x}}{2\bar{x}^2(1-\bar{x})^2} \sum_{i,j=1}^{n} (w_{ij} - w_i w_j) x_i x_j,$$

where $\bar{x} := \max\{x_1, \ldots, x_n\}$.

In the proofs of Theorem 3.1 and Theorem 3.2 we shall use the following lemmas.

Lemma 3.3. *Let $J \subseteq \mathbf{R}$ be a nonempty interval, let $X = (x_1, \ldots, x_n) \in J^n$, and let $\phi : J \to \mathbf{R}$ be a twice differentiable function such that $\phi''(x) \geq 0$ for all $x \in J$. Then it holds that*

$$\sum_{i=1}^{n} w_i \phi(x_i) - \int_{A_{n-1}} \phi(\lambda \cdot X) d\mu(\lambda) \tag{3.3}$$
$$\leq \sum_{i=1}^{n} w_i x_i \phi'(x_i) - \left(\sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} w_i \phi'(x_i)\right).$$

Proof. The nonnegativity of ϕ'' ensures that

$$\phi(\lambda \cdot X) \geq \phi(x_i) + \phi'(x_i)(\lambda \cdot X - x_i)$$

for all $i \in \{1, \ldots, n\}$ and all $\lambda \in A_{n-1}$. Integrating over A_{n-1} with respect to μ yields

$$\phi(x_i) - \int_{A_{n-1}} \phi(\lambda \cdot X) d\mu(\lambda) \leq x_i \phi'(x_i) - \phi'(x_i)(w_1 x_1 + \cdots + w_n x_n)$$

for all $i \in \{1, \ldots, n\}$. Multiplying both sides by w_i and then summing the obtained inequalities, we get (3.3). □

Given the nonempty interval $J \subseteq \mathbf{R}$, to each function $\phi : J \to \mathbf{R}$ we associate the function $L\phi : J^n \to \mathbf{R}$ defined by

$$L\phi(X) := \int_{A_{n-1}} \phi(\lambda \cdot X) d\mu(\lambda) - \phi\left(\sum_{i=1}^{n} w_i x_i\right) \qquad X = (x_1, \ldots, x_n) \in J^n.$$

Lemma 3.4. *Suppose that ϕ has a continuous second derivative in J, and let $X = (x_1, \ldots, x_n) \in J^n$, $\underline{x} := \min\{x_1, \ldots, x_n\}$, $\bar{x} := \max\{x_1, \ldots, x_n\}$. Then there* *exists a point* $\tilde{x} \in [\underline{x}, \bar{x}]$ *such that*

$$L\phi(X) = \frac{1}{2}\phi''(\tilde{x})Le_2(X),$$

where $e_2(x) = x^2$.

Proof. Set $\lambda^0 := (w_1, \ldots, w_{n-1}) \in A_{n-1}$ and $x_0 := w_1x_1 + \cdots + w_nx_n$. Obviously, $x_0 = \lambda^0 \cdot X$. Next, let $\varphi : A_{n-1} \to \mathbf{R}$ be the function defined by $\varphi(\lambda) := \phi(\lambda \cdot X)$. For each $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1}$ there exists $\xi \in]0, 1[$ such that

$$\varphi(\lambda) = \varphi(\lambda^0) + d\varphi(\lambda^0)(\lambda - \lambda^0) + \frac{1}{2}d^2\varphi(\lambda^0 + \xi(\lambda - \lambda^0))(\lambda - \lambda^0),$$

hence

$$\phi(\lambda \cdot X) = \phi(x_0) + \phi'(x_0)\sum_{i=1}^{n-1}(x_i - x_n)(\lambda_i - w_i) \tag{3.4}$$

$$+\frac{1}{2}\phi''(x_\xi)\sum_{i,j=1}^{n-1}(x_i - x_n)(x_j - x_n)(\lambda_i - w_i)(\lambda_j - w_j),$$

where $x_\xi := (\lambda^0 + \xi(\lambda - \lambda^0)) \cdot X$. Further, let

$$m := \inf \phi''([\underline{x}, \bar{x}]) \qquad \text{and} \qquad M := \sup \phi''([\underline{x}, \bar{x}]).$$

Taking into account that

$$\sum_{i,j=1}^{n-1}(x_i - x_n)(x_j - x_n)(\lambda_i - w_i)(\lambda_j - w_j) = \left(\sum_{i=1}^{n-1}(x_i - x_n)(\lambda_i - w_i)\right)^2 \geq 0,$$

from (3.4) we get

$$\frac{1}{2}m\sum_{i,j=1}^{n-1}(x_i - x_n)(x_j - x_n)(\lambda_i - w_i)(\lambda_j - w_j)$$

$$\leq \phi(\lambda \cdot X) - \phi(x_0) - \phi'(x_0)\sum_{i=1}^{n-1}(x_i - x_n)(\lambda_i - w_i)$$

$$\leq \frac{1}{2}M\sum_{i,j=1}^{n-1}(x_i - x_n)(x_j - x_n)(\lambda_i - w_i)(\lambda_j - w_j)$$

for all $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1}$. Integrating over A_{n-1} with respect to μ yields

$$\frac{1}{2}m\sum_{i,j=1}^{n-1}(w_{ij} - w_iw_j)(x_i - x_n)(x_j - x_n) \leq L\phi(X)$$

$$\leq \frac{1}{2}M\sum_{i,j=1}^{n-1}(w_{ij} - w_iw_j)(x_i - x_n)(x_j - x_n).$$

As a simple computation shows, we have

$$\sum_{i,j=1}^{n-1}(w_{ij}-w_iw_j)(x_i-x_n)(x_j-x_n)=Le_2(X),$$

hence $\frac{1}{2}mLe_2(X)\le L\phi(X)\le\frac{1}{2}MLe_2(X)$. Now, the continuity of ϕ'' ensures the existence of a point $\tilde{x}\in[\underline{x},\bar{x}]$ such that $L\phi(X)=\frac{1}{2}\phi''(\tilde{x})Le_2(X)$. □

Proof of the Theorem 3.1. Inequality (3.1) follows at once from (3.3) if we take $J:=]0,1/2]$ and $\phi:J\to\mathbf{R}$ to be the function $\phi(x)=\log(1-x)-\log x$, whose second derivative is

$$\phi''(x)=\frac{1-2x}{x^2(1-x)^2}\ge 0\qquad\text{for all } x\in J.$$

□

Proof of the Theorem 3.2. With the same choices for J and ϕ, from Lemma 3.4 we conclude the existence of a point $\tilde{x}\in[\underline{x},\bar{x}]$ such that

$$\begin{aligned}\log\frac{A(X;\mu)}{A(\mathbf{1}-X;\mu)}-\log\frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)}&=\frac{1-2\tilde{x}}{2\tilde{x}^2(1-\tilde{x})^2}Le_2(X)\\&=\frac{1-2\tilde{x}}{2\tilde{x}^2(1-\tilde{x})^2}\sum_{i,j=1}^{n}(w_{ij}-w_iw_j)x_ix_j\\&\ge\frac{1-2\bar{x}}{2\bar{x}^2(1-\bar{x})^2}\sum_{i,j=1}^{n}(w_{ij}-w_iw_j)x_ix_j,\end{aligned}$$

because ϕ'' is decreasing on J. □

Remark. For $\mu=(n-1)!$, inequalities (3.1) and (3.2) reduce to

$$\log\frac{I(X)}{I(\mathbf{1}-X)}-\log\frac{G(X)}{G(\mathbf{1}-X)}\le\frac{1}{n^2}\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}\frac{1}{x_i(1-x_i)}\right)-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1-x_i}$$

and

$$\log\frac{A(X)}{A(\mathbf{1}-X)}-\log\frac{I(X)}{I(\mathbf{1}-X)}\ge\frac{1-2\bar{x}}{2n^2(n+1)\bar{x}^2(1-\bar{x})^2}\left(n\sum_{i=1}^{n}x_i^2-\left(\sum_{i=1}^{n}x_i\right)^2\right),$$

respectively.

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BOOK REVIEWS

T. Banakh, T. Radul, M. Zarichnyi, *Absorbing Sets in Infinite-Dimensional Manifolds* Mathematical Studies Monograph Series, Volume 1, VNTL Publishers, Ukraine, 1996, 232 pp.

The book is devoted to the theory of absorbing sets and its applications, most of them consisting in beautiful and short characterizations of many remarkable spaces. The term *space*, stands for *separable, metrizable topological space.*

The first chapter of the book contains an exposition of the basic theory of absorbing sets.

A space X is said to be $\mathcal{C}$-*absorbing* (with respect to a given class $\mathcal{C}$ of spaces) if:

a) X is strongly $\mathcal{C}$-*universal ANR* (absolute neighborhood retract) satisfying $SDAP$ (strong discrete approximation property)

b) $X \in \sigma\mathcal{C}$ (i.e. X is a countable union of spaces from $\mathcal{C}$)

c) X is a Z_σ space (i.e. X is a countable union of Z-spaces in $\mathcal{C}$.

[recall that a set $A \subseteq X$ is a Z-space if for every open cover $\mathcal{U}$ of X, there exists a continuous map $f : X \to X$ such that $f(X) \cap A = \emptyset$ and f, id_X are $\mathcal{U}$-close, i.e. for each $x \in X$ with $f(x) \neq x$, there exists $U \in \mathcal{U}$ such that $x \in U, f(x) \in U$].

The second chapter, "Construction of absorbing sets", contains examples of absorbing sets with respect to several classes of sets, sometimes defined by dimensional conditions.

The third chapter contains some even more technical results concerning strong universality for pairs and for spaces.

The last two chapters include applications to infinite products, topological groups, convex sets, spaces of probability measures.

The results included in the 232 pages of the book integrate the work of the authors with work of many other mathematicians such as K. Borsuk, C. Bessaga, A. Pełczińsky, H. Toruńczyk, M. Bestvina, J. Mogilski, T. Dobrowolski, O. Keller.

The book contains many exercises and notes and comments at the end of each chapter.

V. Anisiu

G.P. Galdi, J.G. Heywood and R. Rannecher Eds., *Fundamental Directions in Mathematical Fluid Mechanics*, Birkhäuser, Boston-Basel-Berlin 2000, 293 pp., ISBN 3-7643-6414-9.

This volume consists of six research articles, written by excellent experts in fluid mechanics. Each of these articles treats an important topic in the theory of Navier-Stokes equations in a rigorous mathematical manner. A very important problem in this area is to go beyond the presently known global existence of weak solutions, to the global existence of smooth solutions, for which uniqueness result and continuous dependence on the data can be provided. For this reason, Galdi's article *An introduction to the Navier-Stokes initial-boundary value problem* gives an overview of this topic. The article of Gervasio, Quarteroni and Saleri *Spectral approximation of Navier-Stokes equations* is devoted to extension of spectral Galerkin methods to domains with complicated geometries by using the techniques of domain decomposition. It is well known that the rigorous explanation of bifurcation phenomena in fluid mechanics has been a main topic in the theory of the Navier-Stokes equations. The article of Heywood and Nagata *Simple proofs of bifurcation theorems* introduces bifurcation theory in a general setting that is convenient for application to the Navier-Stokes equations. The two articles of Heywood and Padula *On the steady transport equation* and *On the existence and uniqueness theory for the steady compressible viscous flow* give a simplified approach to the theory of steady compressible viscous flows. Finally the article of Rannacker *Finite element methods for the incompressible Navier-Stokes equations* combines the theory and implementation of the finite element method, with an emphasis on a priori and a posteriori error estimation and adaptive mesh refinement.

The book is an important addition to the literature and it will be a very good investment for interested researchers.

M. Kohr

Andreescu, T., Gelca, R., *Mathematical Olympiad Challenges,* Birkhauser, Boston - Basel - Berlin, 2000, 260 pp + xv, ISBN 0-8176-4155-6.

This beautiful book of T. Andreescu and R. Gelca is a comprehensive collection of high level and non-standard problems for mathematical olympiads and competitions. The both authors were educated at the strong Romanian mathematics school and therefore their present work contains a part of this past experience. At present, Titu Andreescu is the Executive Director of the American Mathematics Competitions. Under his guidance the US Olympic Team has obtained very high scores (first place with perfect score in 1994), and was situated on the third place at the least World Mathematical Olympiad in Korea. USA will be the host of the 2001 Mathematical Olympiad and T. Andreescu is the director of the organizing committee. I have to note that the book begins with the following words of the famous Romanian mathematician Grigore Moisil : "Matematică, matematică, matematică, atâta matematică? Nu, mai multă! ("Mathematics, mathematics, mathematics, so much mathematics? No, even more! "). In fact these words give us a good expression of the spirit of this book : create and solve more and more problems since this is the best way to learn and to understand mathematics.

The included problems are clustered into three self-contained chapters : Geometry and Trigonometry, Algebra and Analysis, Number Theory and Combinatorics, each of them containing ten sections. For instance, the topics of sections in Geometry and Trigonometry are the following : A property of equilateral triangles, Dissections of polygonal surfaces, Regular polygons, Cyclic quadrilaterals, Power of a point, Geometric constructions and transformations, Problems with physical flower, Tetrahedra inscribed in parallelepipeds, Telescopic sums and products in trigonometry, Trigonometric substitutions. A background material, some representative examples, and beautiful diagrams are included to complete each section. Most of the proposed problems were successfully tested in classrooms as well as in national and international mathematical competitions. From this point of view the Romanian experience is very well represented. The second part of the book contains the completed and detailed solutions of the proposed problems. These are presented in a very didactic way, encouraging the readers to move away from routine exercises and memorized algorithms toward creative solutions and non-standard problem-solving techniques. The name of author and source of most of the proposed problems are mentioned in this part. At the end of the book a glossary of used definitions and fundamental properties is included.

In the authors‘ preface one can read that this work "... is written as a textbook to be used in advanced problem solving courses, or as a reference source for people interested in tackling challenging mathematical problems". I have to conclude that in this respect all the purposes of the book are successfully fulfilled.

Dorin Andrica

T.M. Atanackovic, A. Guran, *Theory of Elasticity for Scientists and Engineers*, Birkäuser, Boston-Basel-Berlin, 2000, 374 pp., ISBN 0-8176-4072-X.

The present book is intended to be an introduction to theory of elasticity. It is a new and comprehensive text, as well as a good reference work providing an excellent introduction to theory of elasticity and its applications. The book contains ten chapters. The first chapter has an introductory character, containing the theory of stress. The second chapter begins with the description of deformation at a point. Further, the nonlinear strain tensor is obtained and the geometrical meaning of its components are studied. On the other hand, the strain tensor in the case of small deformations is derived by linearization and its properties are examined. We remark that these preliminary chapters treat the basic concepts of stress and strain, using only Cartesian vector and tensor notation. Chapter three treats the relation between stress and strain. This chapter introduces constitutive equations for an elastic body and the thermoelastic stress-strain relation. Chapter four is devoted to some boundary value problems of elasticity theory. The authors give a summary of equations of linear elasticity theory and use the scalar and vector potential theory to solve several problems in this field (the Lamé potential, the Galerkin vector, the Love function, the method of Papkovich and Neuber, etc.). The aim of chapter five is to give a presentation of a large number of some important problems of elasticity theory for which solutions are available (torsion, bending and rotation of a prismatic rod). This chapter is a very good reference source for researchers in the field. Chapter six is concerned with the plane strain, plane stress, and generalized plane stress problems and presents several methods to solve these problems (the complex variable method, Fourier transform method). Chapter seven is devoted to the energy method in elasticity theory. The aim of chapter eight is to derive the von Kármán theory of plates and in chapter nine is treated the contact and elastic impact problems for elastic bodies. The last chapter of the book refers to the stability of elastic bodies. Some examples illustrating stability analysis are included. At the end of each chapter a selected number of problems are given.

To conclude, I think this book is very important for introducing readers in mechanical engineering, mechanics and applied mathematics to a modern view of theory of elasticity. The book is clearly written, contains a wealth of information, introducing the reader to a modern and active area of investigations.

I recommend this book to all specialists in this area.

M. Kohr

William G. Litvinov, *Optimization in Elliptic Problems with Applications to Mechanics of Deformable Bodies and Fluid Mechanics,* Operator Theory Advances and Applications, Vol. 119, Birkhäuser Verlag, Basel-Boston-Berlin 2000, 522 pages, ISBN 3-7643-6199-9.

The author offers the reader a thorough introduction to contemporary research in optimization theory for elliptic systems with its numerous applications, and a textbook at the undergraduate and graduate levels for courses in pure and applied mathematics.

The mathematical models of the modern technology and production contain elliptic equations and systems. Optimization of the processes from these models is reduced to optimization problems for elliptic equations and systems. The numerical solution of such problems is associated with some questions. Some of them are the following:

- The setting of the optimization problem ensures the existence of a solution on a set of admissible controls, which is a subset of some infinite-dimensional vector space.
- Reduction of the infinite-dimensional problem to a sequence of finite dimensional problems such that the solutions of the finite-dimensional problems converge to the solution of the infinite-dimensional problem.
- Numerical solution of the finite dimensional problems.

The book is devoted to these questions. Attention is focused on the settings of the problems, on the proof of existence theorems, and on the method of approximate solution of optimization problems. For elliptic equations and systems the author investigates optimization problems in which the coefficients of equations, the shape of domains, and right-hand sides of equations are considered to be controls. The results are applied to various optimization problems of mechanics of deformable bodies, plates, shells, composite materials, and structures made of them, as well as to the optimization problems of mechanics of viscous fluids.

The book is written in an accessible and self-contained manner. It will be of interest to research mathematicians and science engineers working in solid and fluid mechanics, and in optimization theory of partial differential equations.

P. Szilágyi

Vladimir Maz'ya, Serguei Nazarov, Boris Plamenevskij, *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains,* Volume I. 435 pages, ISBN 3-7643-6397-5, Volume II. 323 pages, ISBN 3-7643-6398-3, Operator Theory Advances and Applications, Vols. 111 and 112, Birkhäuser Verlag, Basel-Boston-Berlin, 2000.

The book is devoted to the development and applications of asymptotic methods to boundary value problems for elliptic equations in singularly perturbed domains Ω, which can have corners, edges, small holes, small slits, thin ligaments etc. The boundary value problems are considered first in domains $\Omega(\epsilon)$ that depend on a small parameter ϵ, where $\Omega(0) = \Omega$. So the boundary of the domain $\Omega(0)$ is not smooth and contains a number of singular points, contours or surfaces. A transition from $\Omega(0)$ to $\Omega(\epsilon)$ results in the fact that isolated points convert into small cavities, contours convert into thin tubes and surfaces into flat holes, or the boundary of the domain near a conical point or an edge becomes smooth and so on. These perturbations of the domain are said to be singular, in the contrast to regular perturbations, when the boundaries of the domains $\Omega(0)$ and $\Omega(\epsilon)$ are closed smooth surfaces. The authors investigate the behaviour of solutions u_ϵ of the boundary value problems, eigenvalues of the corresponding operator, and the behaviour of different functionals as $\epsilon \to 0$.

A lot of attention is paid to particular problems of mathematical physics. Most of the problems considered in the two volumes emerged from problems in hydrodynamics and aerodynamics, the theory of elasticity, fracture mechanics, electrostatics and others. A substantial body of results has been accumulated on the applications of asymptotic methods to physical problems. This knowledge has been particularly useful in a broad range of engineering problems.

The authors offer in the 20 chapters of the book a complete theory of boundary value problems in domains, which have corners, edges or other singularities. Most of the material presented in the book is based on results of the authors, which have been partly published in scientific journals. This book can be considered as unique in the mathematical literature, because it presents for the first time a profound and complete mathematical analysis of the asymptotic theory of elliptic boundary value.

Finally we remark, that the book originally was published by Akademic Verlag GmbH, Germany, under the title "Asymptotic Theorie Elliptischer Radwertaufgaben in singulär gestörten Gebieten" in 1991.

P. Szilágyi

Ring Theory and Algebraic Geometry, Editors: Ágnel Granja, Hose Ágnel Hermida, Alain Verschoren, Lecture Notes in Pure and Applied Mathematics, vol. 221, Marcel Dekker (2001), xv+339pp, ISBN 0-8247-0559-9.

The volume under review presents papers presented at the "Fift International Conference on Algebra and Algebraic Geometry (SAGA V)", held at the University of León, Spain.

The aim of this book is to exhibit some interaction between algebra and algebraic geometry and it contains 20 research papers and surveys. The contributors are important specialists in some actual domains in mathematics: modules and lattices, algebras and representation theories, affine and projective algebraic varieties, simplicial and cellular complexes, cones, polytipes, arithmetics, etc.

Brzeziński, Caenepeel, Militaru and *Zhu* study condition when induction functors and their adjuncts are separable; *Bueso, Gómez-Torecillas* and *Lobillo* characterize the solvable polinomial algebras and present an algorithm to compute the Gelfand-Kirillov dimension for f.g. modules over these algebras; *Campillo* and *Pisón* "show how mathematics in toric geometry can be understood of appropriate classes of commutative semigroups"; *Cuadra* and *Van Oystaeyen* present properties for some invariants of coalgebras (the Picard group and the Brauer group); *Escoriza and Torrecillas* give the concept of multiplication object in monoidal categories; *Facchini* describes some connections between "semi-local endomorphism ring" and "Krull-Schmidt theorems"; *Hartillo-Hermoso* presents an algoritm which compute a global Bernstein polynomial; *Morey* and *Vasconcelos* study the divisors of Rees algebras of ideals. Mention that the others contributors of this volume are: *Cabezas, Camacho, Gómez,* *Jiménez-Merhan, Pastor, Reyes, Rodrigeuz, Calderón-Martin, Martin-Gonzáles, Corriegos, Sáncez-Giralda, Castro-Jiménes, Moreno-Frias, Gago-Vargas, Gonzáles, Idelhadj, Yahya, S. Gonzáles, Matinez, Malliavin, Núnez, Pisaborro, Smet, Verchoren, Ucha-Enriquez, Verchoren, Vidal.*

The authors are well-knows experts from quite different schools.

The book permits an easy access to the present state of knowledge. Students and researchers interested in Ring Theory and in Algebraic Geometry will take a full benefit and they find here a good source of inspiration.

Simion Breaz

Constantin Udrişte, *Geometric Dynamics*, Mathematics and Its Applications, Vol. 513, Kluwer Academic Publishers, Dordrecht-Boston-London 2000, xvi+395 pp., ISBN: 0-7923-5277-7.

A field line is a curve $\alpha : I \to D$ of class C^1, satisfying the differential equation $\alpha'(t) = X(\alpha(t))$ (or the equivalent integral equation $\alpha(t) = \alpha(t_0) + \int_{t_0}^{t} X(\alpha(s))ds$) where D is an open connected subset of R^n and X is a vector field of class C^1 on D. Geometric dynamics is a tool for developing a mathematical representation of real world phenomena, based on the notion of field line. The author systematically exemplifies the theoretical mathematical concepts on examples from the applied sciences: theoretical mechanics, physics, thermodynamics, biology, chemistry etc. The basic idea of the author is to emphasize that a field line is a geodesic of a suitable geometrical structure on a given space (the so called Lorentz-Udrişte world-force law). That means that creating wider classes of Riemann-Jacobi, Riemann-Jacobi-Lagrange, or Finsler-Jacobi manifolds, one obtains that all trajectories of a given vector field are geodesics.

The book is divided into 11 chapters headed as follows: 1. *Vector fields,* 2. *Particular vector fields,* 3. *Field lines*, 4. *Stability of equilibrium points,* 5. *Potential differential systems of order one and catastrophe theory*, 6. *Field hypersurfaces,* 7. *Bifurcation theory,* 8. *Submanifolds orthogonal to field lines,* 9. *Dynamics induced by a vector field,* 10. *Magnetic dynamical systems and Sabba Ştefănescu conjectures,*

11. *Bifurcation in the mechanics of hypoelastic granular material* (this last chapter is written by Lucia Drăguşin).

The characteristic feature of the book is the strong interplay between mathematics and its applications to other areas, which makes it of interest to a large audience, including first years graduates, teachers, and researchers whose work involves mathematics, mechanics, physics, engineering, biology, and economics.

Stefan Cobzaş