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SOME REMARKS ON GROUPS OF POINTWISE SYMMETRIES OF THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

GALINA BANARU

Abstract. A necessary and sufficient condition for a third-order ordinary differential equation to possess a five-dimensional group of pointwise symmetries is established.

1. **Introduction**

The investigation of symmetries groups of differential equations in general (and of ordinary differential equations in particular) is one of the most important problems of differential equations geometry. The author of the present article studies third-order ordinary differential equations. Before that [1] the author obtained a complete solution of the problem in the case when such an equation has a seven-dimensional or a six-dimensional group of pointwise symmetries: the corresponding criteria have been obtained. (We recall [2] that seven is the maximum of the possible dimension of the pointwise symmetries group of a third-order ordinary differential equation). The present work is devoted to the analysis of the problem in the case when the dimension of the pointwise symmetries group is equal to five.

2. **Preliminaries**

We consider a third-order ordinary differential equation

$$y''' = f(x, y, y', y'') \tag{1}$$

given on a plane where the pseudo-group of point analytical transformations of coordinates acts:

$$\tilde{x} = \varphi_1(x, y);\ \tilde{y} = \varphi_2(x, y).$$

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The equation (1) is bound in an invariant way (concerning the given transformations) with such a geometrical object as a fiber space with a connection. The Cartan structural equations of the above-mentioned fiber space looks as follows:

$$
\begin{aligned}
D\omega^1 &= \omega^1 \wedge \omega^1_1 + \Omega^1 \\
D\omega^2 &= \omega^1 \wedge \omega^2_1 + \omega^2 \wedge \omega^2_2 \\
D\omega^2_1 &= \omega^2_1 \wedge (\omega^2_2 - \omega^1_1) + \omega^1 \wedge \omega^2_{11} + \omega^2 \wedge \omega^1_{11} \\
D\omega^2_{11} &= \omega^2_{11} \wedge (\omega^2_2 - 2\omega^1_1) + \omega^2_1 \wedge \omega^1_{11} + \Omega^2_{11} \\
D\omega^1_1 &= \omega^1 \wedge \omega^1_{11} + \Omega^1_1 \\
D\omega^2_2 &= \omega^1 \wedge \omega^1_{11} + \Omega^2_2 \\
D\omega^1_{11} &= \omega^1_1 \wedge \omega^1_{11} + \Omega^1_{11}.
\end{aligned} \tag{2}
$$

The torsion-curvature forms of the equations (2) looks as follows:

$$
\begin{aligned}
\Omega^1 &= \frac{1}{2}(a\omega^2_1 + b\omega^2_{11}) \wedge \omega^2 \\
\Omega^2_{11} &= \frac{1}{2}(c\omega^1 - e\omega^2_1) \wedge \omega^2 \\
\Omega^1_1 &= \frac{1}{2}(g\omega^1 + h\omega^2_1 + k\omega^2_{11}) \wedge \omega^2 + \frac{1}{2}b\omega^2_{11} \wedge \omega^2_1 \\
\Omega^2_2 &= \frac{1}{2}[3g\omega^1 + (3h - 2m)\omega^2_1 + (3k - 2a)\omega^2_{11}] \wedge \omega^2 + \frac{1}{2}b\omega^2_1 \wedge \omega^2_{11} \\
\Omega^1_{11} &= (\frac{1}{2}e + g)\omega^1 \wedge \omega^2_1 + \frac{1}{2}[n\omega^1 + r\omega^2_1 + (h + m)\omega^2_{11}] \wedge \omega^2 + (\frac{1}{2}a - k)\omega^2_1 \wedge \omega^2_{11}.
\end{aligned} \tag{3}
$$

The coefficients

$$a, b, c, e, g, h, k, m, n, r \tag{4}$$

being present in the torsion-curvature forms make up a complete system of differential invariants of the equation (1). They completely characterize the equation (1)and, thus, determine its geometry. The differentials of the invariants are as follows:

$$
\begin{aligned}
da + 2a(\omega^1_1 - \omega^2_2) - b\omega^1_{11} &= h\omega^1 + ... \\
db + b(3\omega^1_1 - 2\omega^2_2) &= (k - a)\omega^1 + ... \\
dc - 3c\omega^1_1 &= \sigma_1 \\
de - e(\omega^1_1 + \omega^2_2) &= \sigma_2 \\
dg - g(\omega^1_1 + \omega^2_2) &= \sigma_3
\end{aligned} \tag{5}
$$

$$dh + h(\omega_1^1 - 2\omega_2^2) + (a-k)\omega_{11}^1 = \sigma_4$$

$$dk + 2k(\omega_1^1 - \omega_2^2) = \sigma_5$$

$$dm + m(\omega_1^1 - 2\omega_2^2) = (r+bc)\omega^1 + ...$$

$$dn - n(2\omega_1^1 + \omega_2^2) - (g+e)\omega_{11}^1 = \sigma_6$$

$$dr - 2r\omega_2^2 - m\omega_{11}^1 = \sigma_7.$$

The right parts of all equalities are linear combinations of the main forms of the second-order tangent element: $\omega^1, \omega^2, \omega_1^2, \omega_{11}^2$. We denote such combinations by the symbols $\sigma_i, \sigma, \tilde{\sigma}$. From the given relations it is seen that the differential invariants of the equation (1) are either relative invariants or become relative invariants when some relative invariants vanish.

3. **The main result**

Now, we consider the invariant c. According to (5),

$$dc - 3c\omega_1^1 = \sigma_1. \tag{6}$$

Thus, c is one of the invariants that is relative from the beginning. For this reason for c, as well as for any relative invariant, two different cases are possible: $c = 0$ and $c \neq 0$. From the multitude of third-order ordinary differential equations we select those for which the invariant c is different from zero and all others differential invariants vanish:

$$a = b = e = g = h = k = m = n = r = 0. \tag{7}$$

Let us do the canonization $c \overset{k}{=} 1$. In extracted particular case according to (6), the differential form ω_1^1 will be a linear combination of the main forms of the second-order tangent element:

$$\omega_1^1 = t\omega^1 + t_1\omega^2 + t_2\omega_1^2 + t_3\omega_{11}^2, \tag{8}$$

where t, t_1, t_2, t_3 are some new invariants. Having an exterior differentiation of the equality (8), we shall find the relations for differentials of these invariants:

$$dt = \sigma_8;\ dt_1 - t_1\omega_2^2 = \sigma_9;$$

$$dt_2 - t_2\omega_2^2 = \sigma_{10};\ dt_3 - t_3\omega_2^2 = \sigma_{11}. \tag{9}$$

The obtained relations show that the coefficients t_1, t_2 t_3 are relative invariants and t is an absolute invariant.

The exterior differentiation of (8) will give us another useful equality:

$$\omega_{11}^1 = p\omega^1 + p_1\omega^2 + p_2\omega_1^2 + p_3\omega_{11}^2. \tag{10}$$

(Here p, p_1, p_2 p_3 are new invariants of the equation (1) we are interested in). Having an exterior differentiation of (10), we get:

$$dp = \sigma_{12};\ dp_1 - p_1\omega_2^2 = \sigma_{13};$$

$$dp_2 - p_2\omega_2^2 = \sigma_{14};\ dp_3 - p_3\omega_2^2 = \sigma_{15}. \tag{11}$$

Therefore, p_1, p_2, p_3 are relative invariants and p is an absolute invariant.

Let us select the case when all the "new" relative invariants vanish:

$$t_1 = t_2 = t_3 = p_1 = p_2 = p_3 = 0. \tag{12}$$

In this case the equalities (8) and (10) will look as follows: $\omega_1^1 = t\omega^1$; $\omega_{11}^1 = p\omega_1$, and the Cartan structural equations (2) will be written down as follows:

$$D\omega^1 = D\omega_2^2 = 0$$

$$D\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^2 \wedge \omega_2^2$$

$$D\omega_1^2 = \omega_1^2 \wedge (\omega_2^2 - t\omega^1) + \omega^1 \wedge (\omega_{11}^2 - p\omega^2) \tag{13}$$

$$D\omega_2^2 = \omega_{11}^2 \wedge (\omega_2^2 - 2t\omega^1) + \omega^1 \wedge (\frac{1}{2}\omega^2 - p\omega_1^2).$$

Having an exterior differentiation of (13), we shall be convinced that (13) are structure equations of a some transformations group. The dimension of this group is equal to five. For the equation (1) that we are interested in the group mentioned is a group of pointwise symmetries (in the selected particular case). So, we have proved

Proposition I. *If $c \neq 0$ and equalities (7) and (12) are fulfilled, then the equation (1) has a five-dimensional group of pointwise symmetries. The structure equations of this group looks as (13).*

It turns out that the inverse statement is also true.

Proposition II. *If the equation (1) has a five-dimensional group of pointwise symmetries, then $c \neq 0$ and equalities (7) and (12) are fulfilled.*

Proof. Assume the equation (1) possesses a five-dimensional group of pointwise symmetries. Then among its differential invariants in (4) there is at least one that is different from zero. Otherwise [1], the symmetries group is the seven-dimensional group $g_{2,6}(3)$ (using the Cartan's terminology [3]).

Let I be one of the relative invariants of the equation (1). Then its differential satisfies the equality:

$$dI + I(s_1\omega_1^1 + s_2\omega_2^2) = r_1\omega^1 + r_2\omega^2 + r_3\omega_1^2 + r_4\omega_{11}^2. \tag{14}$$

We assume that $I \neq 0$. Canonizing $I \overset{k}{=} 1$, from (14) we obtain:

$$s_1\omega_1^1 + s_2\omega_2^2 = r_1\omega^1 + r_2\omega^2 + r_3\omega_1^2 + r_4\omega_{11}^2. \tag{15}$$

Having an exterior differentiation of this equality, as one of differential results we obtain the relation:

$$dr_1 - r_1\omega_1^1 + (s_1 + s_2)\omega_{11}^1 = \sigma.$$

The invariant r_1 (like the others invariants of the equation (1)) must be a constant in the case when (2) are the structure equations of the symmetries group of (1). Under this condition the last equality looks as follows:

$$-r_1\omega_1^1 + (s_1 + s_2)\omega_{11}^1 = \sigma. \tag{16}$$

Now, we assume that the invariant s_2 is different from zero and express the differential form ω_2^2. Substituting the relation for ω_2^2 in (2), we obtain:

$$D\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^2 \wedge (-\frac{s_1}{s_2}\omega^1 + \tilde{\sigma}).$$

We have an exterior differentiation of this equality. Among others we obtain the following relation:

$$-\frac{s_1}{s_2} = 1, \text{ or } s_1 + s_2 = 0.$$

As seen from (5), among the relative invariants only the invariant k satisfies this condition. But being different from zero, the invariant k does not suit us for the reason that if we admit that the equation (1) has not any other invariants different from zero, then the symmetries group will be the six-dimensional group $g_{4,2}$. If at least one invariant is different from zero, then according to (15) and (16) the forms

$\omega_1^1, \omega_2^2, \omega_{11}^1$ will turn out to be dependent on $\omega^1, \omega^2, \omega_1^2, \omega_{11}^2$ and, thus, the symmetries group can not have a dimension more than four.

Hence, all the relative invariants from (4) for which $s_2 \neq 0$ must vanish. Therefore

$$b = e = g = k = m = 0 \Rightarrow a = k = 0.$$

Moreover, the coefficients $h, n,$ become relative invariants for which $s_2 \neq 0$. If we consider this fact as mentioned before, we can come to conclusion that $h = n = r = 0$.

Only the invariant $I = c$ satisfies the condition $s_2 = 0$. Therefore, $c \neq 0$.

In this case, as it is mentioned before, the forms ω_1^1 and ω_{11}^1 are expressed in a linear way though the main forms of the second order tangent element. If we admit that any of invariants $t_1, t_2, t_3, p_1, p_2, p_3,$ are different from zero, then owing to (9) and (11) the form ω_2^2 will also be dependent on $\omega^1, \omega^2, \omega_1^2, \omega_{11}^2$, and so the symmetries group can not have a dimension more then four.

That's why, in the case we are interested in $t_1 = t_2 = t_3 = p_1 = p_2 = p_3 = 0$. The Proposition II is proved completely.

Proposition III. *The structure equations (13) determine the transformations group* $g_{5,5}$.

Proof. We substitute

$$\omega^1 = \Theta^2;\ \omega^2 = \Theta^1;\ \omega_2^2 = \Theta_1^1 + t\Theta^2;$$

$$\omega_1^2 = \Theta_2^1 + t\Theta^1;\ \omega_{11}^2 = -\Theta_{22}^1 + t\Theta_2^1 + p\Theta^1.$$

According to the substitution, the equations (13) may be written down as follows:

$$\begin{gathered} D\Theta^2 = D\Theta_1^1 = 0 \\ D\Theta^1 = \Theta^1 \wedge \Theta_1^1 + \Theta^2 \wedge \Theta_2^1 \\ D\Theta_2^1 = \Theta_{22}^1 \wedge \Theta^2 + \Theta_2^1 \wedge \Theta_1^1 \\ D\Theta_{22}^1 = \Theta^2 \wedge ((2p - t^2)\Theta_2^1 - \frac{1}{2}\Theta^1) + \Theta_{22}^1 \wedge \Theta_1^1. \end{gathered} \tag{17}$$

Now, we use the structural equations of the group $g_{5,5}$ for the third-order ordinary differential equation [3]:

$$D\Theta^1 = \Theta^1 \wedge \Theta_1^1 + \Theta^2 \wedge \Theta_2^1$$

$$D\Theta^2 = 0$$

$$D\Theta^1_1 = 0$$

$$D\Theta^1_2 = \Theta^1_{22} \wedge \Theta^2 + \Theta^1_2 \wedge \Theta^1_1$$

$$D\Theta^1_{22} = \Theta^2 \wedge (m_2\Theta^1_2 + m_3\Theta^1) + \Theta^1_{22} \wedge \Theta^1_1.$$

It is quite evident that (17) are structure equations of $g_{5,5}$, in addition, $m_2 = 2p - t^2; m_3 = -\frac{1}{2}$. The Proposition III have been proved.

Remark. In our case the finite transformations of the group look as follows:

$$\tilde{x} = c_1 x + \psi(y);$$

$$\tilde{y} = y + c_2,$$

where $\psi(y)$ is the general solution of the equation

$$\psi''' - (t^2 - 2p)\psi' + \frac{1}{2}\psi = 0.$$

Taking together all the proved statements, we state the following result:

Theorem. *Third-order ordinary differential equations have a five-dimensional group of pointwise symmetries if and only if* $c \neq 0$, *and conditions (7) and (12) are fulfilled. In addition, the only possible group of pointwise symmetries is (with the precision to an isomorphism) the group* $g_{5,5}$.

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Smolensk State Pedagogical University, Prjevalsky str., 4,
Smolensk, 214000, RUSSIA
E-mail address: banaru@keytown.com

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ON SPECTRA OF SOME TENSORS OF SIX–DIMENSIONAL KÄHLERIAN SUBMANIFOLDS OF CAYLEY ALGEBRA

M.B. BANARU

Abstract. The spectra of the metric tensor, of the almost complex structure, of the fundamental form, of the Riemannian curvature tensor, of the Ricci tensor and of the Weyl tensor of six–dimensional Kählerian submanifolds of Cayley algebra are computed.

It is proved that six–dimensional Kählerian submanifolds of Cayley algebra are CRK–manifolds, i.e. their Weyl tensor of conformal curvature is J–invariant.

1. **Preliminaries**

We consider an almost Hermitian manifold, i.e. a $2n$–dimensional manifold M^{2n} with Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad \forall X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^{∞}.

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G–structure, where G is the unitary group $U(n)$ [1]. Its elements are the frames adapted to the structure (A–frames). They look as follows:

$$(p,\, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}}),$$

where $p \in M^{2n}$, ε_a are the eigenvectors corresponded to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$. Here $a = 1, \ldots, n$; $\widehat{a} = a + n$.

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Therefore, the matrices of the operator of the almost complex structure and of the Riemannian metric written in an A–frame look as follows, respectively:

$$(J^k_j) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right); \quad (g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right); \tag{1}$$

where I_n is the identity matrix; $k, j = 1, \dots, n$.

We recall that the fundamental form of an almost Hermitian manifold is determined by

$$F(X,Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

By direct computing it is easy to obtain that in an A–frame the fundamental form matrix looks as follows:

$$(F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right) \tag{2}$$

It is expedient to consider the other tensors written in an A–frame. This corresponds to the problems of the study of almost Hermitian manifolds. We remark that the notion of the tensor spectrum was introduced by V.F. Kirichenko [1].

2. Kählerian structure on $M^6 \subset \mathbf{O}$

Let $\mathbf{O} \equiv R^8$ be the Cayley algebra. As it well–known [2], two non–isomorphic three fold vector cross products are defined on it by means of the relations

$$P_1(X,Y,Z) = -X(\overline{Y}Z) + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

$$P_2(X,Y,Z) = -(X\overline{Y})Z + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

where $X, Y, Z \in \mathbf{O}$, $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbf{O}$, $X \to \overline{X}$ is the conjugation operator. Moreover, any other three fold vector cross product in the octave algebra is isomorphic to one of the above–mentioned.

If $M^6 \subset \mathbf{O}$ is a six–dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle \cdot, \cdot \rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \quad \alpha = 1, 2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$ [2]. The submanifold $M^6 \subset \mathbf{O}$ is called Kählerian, if the following condition is fulfilled

$$\nabla J = 0,$$

where ∇ is the Levi–Civita connection of the metric. The point $p \in M^6$ is called general [3], if

$$e_0 \notin T_p(M^6) \text{ and } T_p(M^6) \subseteq L(e_0)^{\perp},$$

where e_0 is the unit of Cayley algebra and $L(e_0)^{\perp}$ is its orthogonal supplement. A submanifold $M^6 \subset \mathbf{O}$, consisting only of general points, is called a general–type submanifold [3]. In what follows all submanifolds M^6 to be considered are assumed to be of general–type.

3. Riemannian curvature tensor spectrum

The tensor of the Riemannian curvature (or Riemann–Christoffel tensor) plays an important role in the geometry of manifolds. The outstanding American mathematician Alfred Gray noted [4] that the identities the Riemannian curvature tensor satisfies are very important for the study of almost Hermitian manifolds. Taking into account the properties of the symmetry and of the reality of this tensor as well as the Ricci identity [5], [6], it is sufficient to obtain four (out of sixteen) types of components, that determine completely its spectrum.

Now, let $M^6 \subset \mathbf{O}$ be a six–dimensional Kählerian submanifold of the octave algebra. In [7] the Cartan structure equations of Kählerian have been obtained:

$$d\omega^a = \omega^a_b \wedge \omega^b;$$

$$d\omega_a = -\omega^b_a \wedge \omega_b;$$

$$d\omega^a_b = \omega^a_c \wedge \omega^c_b - T^7_{\widehat{a}\widehat{h}} T^7_{bg} \omega_h \wedge \omega^g,$$

where $\{T^7_{kj}\}$ are the components of the configuration tensor of M^6 [8] (or the Euler curvature tensor [9]). Here and further $a, b, c, d, g, h = 1, 2, 3$; $\widehat{a} = a + 3$; $k, j, m, l = 1, 2, 3, 4, 5, 6$.

Taking into account the fact that the Cartan structure equations must look as follows:

$$d\omega^k = \omega^k_j \wedge \omega^j;$$

$$d\omega^k_j = \omega^k_m \wedge \omega^m_j + \frac{1}{2} R^k_{jml} \omega^m \wedge \omega^l,$$

we compute the spectrum of the Riemannian curvature tensor of six–dimensional Kählerian submanifolds of Cayley algebra. We get such values

$$R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0, \tag{3}$$

$$R_{\hat{a}b\hat{c}d} = -2T^7_{\hat{a}\hat{c}}T^7_{bd}.$$

We remark that the condition

$$R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0 \tag{4}$$

is a criterion [10] for an arbitrary almost Hermitian $M^6 \subset \mathbf{O}$ to belong to the class of $R1$–manifolds (in A. Gray's terminology [4], or parakählerian manifolds [11], or f–spaces [12]). But, however, A. Gray has proved [4] that every Kählerian manifold is parakählerian. That's why (4) could be obtained from the above–mentioned result [10].

4. **Ricci tensor spectrum**

We recall that the Ricci tensor of a Riemannian manifold [5], [6] is determined by the relation

$$ric_{kj} = R^l_{kjl}.$$

In view of the reality of the Ricci tensor for determining of its spectrum it is sufficient to find the components ric_{ab} and $ric_{\hat{a}b}$. From (3) we get:

$$ric_{ab} = 0, \quad ric_{\hat{a}b} = -2T^7_{\hat{a}\hat{c}}T^7_{cb}.$$

Since the condition $ric_{ab} = 0$ is a criterion for an arbitrary almost Hermitian manifold to possess a J–invariant Ricci tensor [13], we have the following Theorem.

Theorem I. *Every six–dimensional Kählerian submanifold of Cayley algebra possesses a J–invariant Ricci tensor.*

5. **Weyl tensor spectrum**

Now, we give the values of Weyl tensor spectrum of six–dimensional Kählerian submanifolds of the octave algebra. This tensor is determined by

$$W_{ijkl} = R_{ijkl} + \frac{1}{n-2}(ric_{ik}g_{jl} + ric_{jl}g_{ik} - ric_{il}g_{jk} - ric_{jk}g_{il}) +$$

$$+\frac{\mathcal{K}}{(n-1)(n-2)}(g_{jk}g_{il}-g_{jl}g_{ik}),$$

where $\mathcal{K}$ is the scalar curvature of M^6 [6]. Like in the case of the Riemannian curvature tensor, proceeding from the properties of the Weyl tensor, it is sufficient to find the components

$$W_{abcd}, \quad W_{\hat{a}bcd}, \quad W_{\hat{a}\hat{b}cd}, \quad W_{\hat{a}b\hat{c}d},$$

that determine completely the Weyl tensor spectrum. We obtain such values

$$W_{abcd}=0, \quad W_{\hat{a}bcd}=0,$$

$$W_{\hat{a}\hat{b}cd}=-\frac{1}{2}(T^7_{\hat{a}\hat{h}}T^7_{hc}\delta^b_d+T^7_{\hat{b}\hat{h}}T^7_{hd}\delta^a_c-T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^b_c-T^7_{\hat{b}\hat{h}}T^7_{hc}\delta^a_d)+\frac{\mathcal{K}}{20}\delta^{ba}_{cd},$$

$$W_{\hat{a}b\hat{c}d}=-2T^7_{\hat{a}\hat{c}}T^7_{bd}+\frac{1}{2}(T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^c_b+T^7_{\hat{c}\hat{h}}T^7_{hb}\delta^a_d)+\frac{\mathcal{K}}{20}\delta^c_b\delta^a_d.$$

As the condition

$$W_{\hat{a}bcd}=0$$

is a criterion for an arbitrary almost Hermitian manifold to belong to the CRK–class (or $cR3$–class [14]), we obtain the following Theorem.

Theorem II. *Every six–dimensional Kählerian submanifold of Cayley algebra is a CRK–manifold.*

6. Table of classical tensors of six-dimensional Kählerian submanifolds of Cayley algebra

Let us put together the obtained results. The spectra of the structure tensors and of the fundamental form are found from (1) and (2). We remark that all these data define more exactly the result [15] obtained on six–dimensional Hermitian submanifolds of Cayley algebra.

Tensor	Tensor spectrum
Almost complex structure	$J^a_b = i\delta^a_b, \quad J^{\hat a}_b = 0, \quad J^a_{\hat b} = 0, \quad J^{\hat a}_{\hat b} = -i\delta^b_a$
Riemannian metric	$g_{ab}, \quad g_{\hat a b} = \delta^a_b, \quad g_{a\hat b} = \delta^b_a, \quad g_{\hat a\hat b} = 0$
Fundamental form	$F_{ab}, \quad F_{\hat a b} = -i\delta^a_b, \quad F_{a\hat b} = i\delta^b_a, \quad F_{\hat a\hat b} = 0$
Riemannian curvature tensor	$R_{abcd} = 0, \quad R_{\hat a bcd} = 0, \quad R_{\hat a\hat b cd} = 0,$ $R_{\hat a b\hat c d} = -2T^7_{\hat a\hat c}T^7_{bd}$
Ricci tensor	$ric_{ab} = 0, \quad ric_{\hat a b} = -2T^7_{\hat a\hat c}T^7_{cb}$
Weyl tensor	$W_{abcd} = 0, \quad W_{\hat a bcd} = 0,$ $W_{\hat a\hat b cd} = -\frac{1}{2}(T^7_{\hat a\hat h}T^7_{hc}\delta^b_d + T^7_{\hat b\hat h}T^7_{hd}\delta^a_c - T^7_{\hat a\hat h}T^7_{hd}\delta^b_c - -T^7_{\hat b\hat h}T^7_{hc}\delta^a_d) + \frac{\mathcal{K}}{20}\delta^{ba}_{cd},$ $W_{\hat a b\hat c d} = -2T^7_{\hat a\hat c}T^7_{bd} + \frac{1}{2}(T^7_{\hat a\hat h}T^7_{hd}\delta^c_b + T^7_{\hat c\hat h}T^7_{hb}\delta^a_d) + +\frac{\mathcal{K}}{20}\delta^c_b\delta^a_d$

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Smolensk State Pedagogical University, Prjevalsky str., 4,
Smolensk, 214000, RUSSIA

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ON SEPARABLE EXTENSIONS OF GROUP GRADED ALGEBRAS

ILUŞCA BONTA

Abstract. We study extension $A \to B$ of G-graded $\mathcal{O}$-algebra, when G is a finite group. Such extensions occur when we consider blocks of normal subgroups and the associated graded source algebra, and we prove a refinement of a lifting theorem by B. Külshammer, T. Okuyama and A. Watanabe.

1. G-graded interior algebras

1.1. Let G be a finite group and let $\mathcal{O}$ be a complete discrete valuation ring with residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic $p > 0$. Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ be two G-graded $\mathcal{O}$-algebras, and recall that the $\mathcal{O}$-algebra homomorphism $f\colon A \to B$ is called G-graded if $f(A_g) \subseteq B_g$ for all $g \in G$.

If $M = \bigoplus_{x\in G} M_x$ is a G-graded (A, A)-bimodule, we shall consider the $\mathcal{O}$-submodule

$$M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

We say that B is a G-graded A-interior $\mathcal{O}$-algebra if B is a G-graded (A, A)-bimodule and $(xa)y = x(ay)$ for all $a \in A$, $x, y \in B$. Observe that in this case, the map $\beta : A \to B$, $\beta(a) = a1_B = 1_B a$ is a unitary homomorphism of G-graded $\mathcal{O}$-algebras. Conversely, given a unitary homomorphism $\beta : A \to B$ of G-graded $\mathcal{O}$-algebras, then B becomes a G-graded A-interior $\mathcal{O}$-algebra in an obvious way.

Let B be a G-graded A-interior $\mathcal{O}$-algebra. It is well-known that $B \otimes_A B$ is a G-graded (B, B)-bimodule, where if $x \in B_g$ and $y \in B_h$ then, by definition $x \otimes_A y \in (B \otimes_A B)_{gh}$. Remark that the multiplication map

$$\mu\colon B \otimes_A B \to B, \qquad \mu(x \otimes_A y) = xy$$

for $x, y \in B$ is a homomorphism of G-graded (B, B)-bimodules.

Lemma 1.2. *Let B be a G-graded A-interior $\mathcal{O}$-algebra and let $\mu: B\otimes_A B\to B$ denote the map of G-graded (B,B)-bimodules satisfying $\mu(x\otimes y)=xy$ for $x,y\in G$. Then the following statements are equivalent:*

(1) There exists a homomorphism $\nu: B\to B\otimes_A B$ of G-graded (B,B)-bimodules such that $\mu\circ\nu=1_B$.

(2) There exists an element $w=\sum_{j=1}^k x_j\otimes y_j\in(B\otimes_A B)^B$ where x_j,y_j are homogeneous elements such that $\sum_{j=1}^k x_jy_j=1_B$.

Proof. If (1) holds, let $\nu\colon B\to B\otimes_A B$ be a map of G-graded (B,B)-bimodules such that $\mu\circ\nu=\mathrm{id_B}$, and let $w=\nu(1_B)$.

Then $w\in(B\otimes_A B)^B$ and $\mu(w)=\mu(\nu(1_B))=1_B$. Since w has the form $w=\sum_{j=1}^k x_j\otimes y_j$ and each x_j, y_j is a sum of homogeneous elements, we may clearly assume that x_j,y_j are homogeneous, so (2) holds.

Conversely, assume that (2) holds. Since $\mu(w)=\sum_{j=1}^k x_jy_j=1\in B_1$, it follows that if $x_j\in B_g$, then $y_j\in B_{g^{-1}}$, so $w\in(B\otimes_A B)_1$. Then the map

$$\nu\colon B\to B\otimes_A B,\quad \nu(x)=xw=wx$$

is a homomorphism of G-graded (B,B)-bimodules, as for $x_g\in B_g$, we have $\nu(x_g)=x_gw\in(B\otimes_A B)_g$. Moreover, for all $x\in B$ $\mu(\nu(x))=x\mu(\nu(1))=x\mu(w)=x1_B=x$.

1.3. We say B is a *separable* G-graded A-interior $\mathcal{O}$-algebra, if the equivalent condition of Lemma 1.2 are satisfied.

This discussion is motivated by the following situation considered in [3] and [4].

Let H be a finite group, N a normal subgroup of H and let $G=H/N$. Then the group algebra $\mathcal{O}H$ can be regarded as a G-graded algebra, and $\mathcal{O}N$ is also an H-algebra. Let $b\in Z(\mathcal{O}N)$ be a block independent, and assume that b is G-invariant, that is, $b\in Z(\mathcal{O}H)$. Then the algebra $B=b\mathcal{O}H$ is a strongly G-graded $\mathcal{O}$-algebra. Note that $\beta=\{b\}$ is a point of N on B_1 and $\alpha=\{b\}$ is a point of H on B_1.

Let P_γ be a defect pointed group of H_α. Recall that if

$$\mathrm{Br}_P^{B_1} : B_1^P \to B_1^P/(\sum_{Q<P} \mathrm{Tr} B_1^Q + J(\mathcal{O})B_1^P)$$

is the Brauer map, then there is a primitive idempotent $i \in B_1^P$ such that $Br_P^{B_1}(i) \neq 0$, and γ is the point of B_1^P containing i. The interior $\mathcal{O}P$-algebra $A := iBi = i\mathcal{O}Hi$ is called a *source algebra* of B. By [4, Proposition 3.2], A is a strongly G-graded algebra and the structural map $\mathcal{O}P \to A$, $u \mapsto iu = ui$ is a homomorphism of G-graded algebras in a natural way (the degree of $u \in P$ is $uN \in G$).

Lemma 1.4. *With the above notations, the algebra A is a separable G-graded $\mathcal{O}P$-interior $\mathcal{O}$-algebra.*

Proof. By [5, Lemma 14.1] there are elements $a, b \in B_1^P$ such that $1_B = \mathrm{Tr}_P^H(aib)$. Consider the element

$$v = \sum_{h \in [H/P]} hai \otimes ibh^{-1} \in \mathcal{O}H \otimes_{\mathcal{O}P} \mathcal{O}H,$$

where $[H/P]$ is a set of representatives for the left cosets of P in H. Then as in [2, Lemma 4], $vh = hv$ for all $h \in G$ and $\sum_{h\in[H/P]} haibt^{-1} = \mathrm{Tr}_P^H(aib) = 1_B$. It follows that the element $w = ivi \in (A \otimes_{\mathcal{O}P} A)^A$ satisfies $\mu(w) = i = 1_A$, hence by Lemma 1.2, A is a separable G-graded $\mathcal{O}P$-interior algebra.

2. **The lifting theorem**

The following result is a generalization to the case of G-graded algebras of [2, Theorem 3].

Theorem 2.1. *Let B be a separable G-graded A-interior $\mathcal{O}$-algebra, let I be an G-graded ideal in an arbitrary G-graded A-interior $\mathcal{O}$-algebra C such that $I \subseteq J(I)$, and let $\rho\colon B \to C/I$ be a unitary homomorphism of G-graded A-interior $\mathcal{O}$-algebras.*

Suppose that there exist a map of G-graded (A, A)-bimodules $\tau_0\colon B \to C$ such that $\tau_0(x) + I = \rho(x)$ for $x \in B$. Then there exists a homomorphism of G-graded A-interior $\mathcal{O}$-algebras $\tau\colon B \to C$ such that $\tau(x) + I = \rho(x)$ for $x \in B$.

Moreover τ is unitary and unique up to conjugation with elements in $1 + I_1^A$.

Proof. Since B is a separable G-graded A-interior $\mathcal{O}$-algebra there exists an element $w = \sum_{j=1}^{k} x_j \otimes y_j \in (B \otimes_A B)^B$ with $x_j, y_j \in B$ homogeneous elements such that $\sum_{j=1}^{k} x_j y_j = 1_B$.

The construction τ is given in [1, Theorem 3]. We only have to verify that τ is grade-preserving.

Consider the map of (A, A)-bimodules

$$\theta\colon B \otimes_A B \to I^{2^n}, \quad \theta(x \otimes y) = \tau_n(xy) - \tau_n(x)\tau_n(y)$$

for $x, y \in B$. If $x_g \in B_g$ and $y_h \in B_h$ we have $x_g y_h \in B_{gh}$ because B is a G-graded A-algebra. We know that τ_n is a G-graded map of (A, A)-bimodules, so $\tau(x_g y_h) \in C_{gh}$ and $\tau(x_g) \in C_g, \tau(y_h) \in C_h$ (hence $\tau(x_g)\tau(y_h) \in C_g C_h \subseteq C_{gh}$). Finally, $\tau_n(x_g y_h) - \tau_n(x_g)\tau_n(y_h) \in C_{gh}$, so $\theta(x \otimes y) \in I^{2^n} \cap C_{gh}$. This means that θ G-graded.

Consider the map of (A, A)-bimodules

$$\lambda\colon B \otimes_A B \otimes_A B \to I^{2^n}, \qquad \lambda(x \otimes y \otimes z) = \theta(x \otimes y)\tau_n(z).$$

If $x_g \in B_g$, $y_h \in B_h$ and $z_l \in B_l$ we have $x_g y_h \in B_{gh}$ and $z_l \in B_l$. Since θ and τ_n are G-graded, we have $\theta(x_g \otimes y_h) \in I^{2^n} \cap C_{gh}$ and $\tau_n(z_l) \in I^{2^n} \cap C_l$. It follows that $\theta(x_g \otimes y_h)\tau_n(z_l)$ belongs to $(I^{2^n} \cap C_{gh})(I^{2^n} \cap C_l) \subseteq I^{2^n} \cap C_{ghl}$, hence $\lambda(x_g \otimes y_h \otimes z_l) \in I^{2^n} \cap C_{ghl}$. Then

$$\eta\colon B \to I^{2^n}, \qquad \eta(x) = \lambda(x \otimes w) = \sum_{j=1}^{k} \theta(x \otimes x_j)\tau_n(y_j)$$

is a map of G-graded (A, A)-bimodules with

$$\tau_n(x)\eta(y) - \eta(xy) + \eta(x)\tau_n(y) + I^{2^{n+1}} = \tau_n(xy) - \tau_n(x)\tau_n(y) + I^{2^{n+1}}.$$

If $x_g \in B_g$ then $x_g \otimes w \in B \otimes_A B$, and since λ is G-graded, we obtain $\lambda(x_g \otimes w) \in I^{2^n} \cap C_g$. It follows that η is G-graded. We get a map of (A, A)-bimodules

$$\tau_{n+1}\colon B \to C, \quad \tau_{n+1}(x) = \tau_n(x) + \eta(x)$$

with $\tau_{n+1}(x) + I^{2^n} = \tau_n(x) + I^{2^n}$ and $\tau_{n+1}(x)\tau_{n+1}(y) + I^{2^{n+1}} = \tau_{n+1}(xy) + I^{2^{n+1}}$ for $x, y \in B$. Since τ_n and η are G-graded, if $x_g \in B_g$ we have $\tau_n(x_g) \in C_g$ and $\eta(x_g) \in I^{2^n} \cap C_g$, so $\tau_n(x_g) + \eta(x_g) \in C_g$. Consequently τ_{n+1} is G-graded.

We have constructed the sequence $(\tau_n)_{n=0}^{\infty}$. Since $I \subseteq J(C)$ the map

$$\tau \colon B \to C, \quad \tau(x) = \lim_{n\to\infty} \tau_n(x)$$

is a well-defined unitary homomorphism of G-graded A-interior $\mathcal{O}$-algebras such that $\tau(x) + I = \tau_0(x) + I = \rho(x)$ for $x \in B$.

Finally, suppose that $\tau' : B \to C$ is another homomorphism of G-graded A-interior $\mathcal{O}$-algebras such that $\tau'(x) + I = \rho(x)$ for $x \in B$. Then

$$\delta \colon B \to I, \quad \delta(x) = \tau(x) - \tau'(x)$$

is a map of G-graded (A, A)-bimodules such that $\delta(xy) = \tau(x)\delta(y) + \delta(x)\tau'(y)$ for $x, y \in B$ and clearly δ is grade-preserving.

We consider the map of (A, A)-bimodules

$$\Phi \colon B \otimes_A B \to I, \quad \Phi(x \otimes y) = \tau(x)\delta(y)$$

for $x, y \in B$ and let $a = \Phi(w) = \sum_{j=1}^{k} \tau(x_j)\delta(y_j) \in I_1^A$. If $x_g \in B_g$ and $y_h \in B_h$ then, since τ and δ are G-graded we have $\tau(x_g)\delta(y_h) \in C_g(I \cap C_h) \subseteq I \cap C_{gh}$, hence Φ is G-graded too. Because B is a separable G-graded A-interior $\mathcal{O}$-algebra, there exists an element $w = \sum_{j=1}^{k} x_j \otimes y_j \in (B \otimes_A B)^B$ where x_j, y_j are homogeneous elements such that $\sum_{j=1}^{k} x_j y_j = 1_B$ imply $w \in (B \otimes_A B)_1$. We have that Φ is a map of G-graded (A, A)-bimodules. Therefore, if $x_j \in B_g$ and $y_j \in B_{g^{-1}}$ then $\tau(x_j) \in C_g$ and $\delta(y_j) \in I \cap C_{g-1}$, so $\tau(x_j)\delta(y_j) \in I^A \cap C_1 = I_1^A$.

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Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

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ON THE EIGENVALUE PROBLEM FOR A GENERALIZED HEMIVARIATIONAL INEQUALITY

ANA-MARIA CROICU

Abstract. In this paper the eigenvalue problem for a generalized hemivariational inequality is studied. Some general existence results are obtained. Applications from Engineering illustrate the theory.

1. Introduction

The mathematical theory of hemivariational inequalities and their applications in Mechanics, Engineering or Economics, were introduced and developed by P.D. Panagiotopoulos ([38], [39], [40], [41], [42], [44]). This theory has been developed in order to fill the gap existing in the variational formulations of boundary value problems (B. V. P.s) when nonsmooth and generally nonconvex energy functions are involved in the formulations of the problem. In fact, this theory of hemivariational inequalities may be considered as an extension of the theory of variational inequalities ([16], [23], [27], [26]). For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs ([39], [44], [36], [35]).

Until now many hemivariational inequalities have been formulated and studied ([36], [37], [43], [39], [14], [2], [17], [45], [35], [48], [31], [19], [3], [29], [30], [15], [1], [28], [18], [8]), and eigenvalue problems for hemivariational inequalities have been presented ([22], [33], [34], [20], [46], [7], [10], [6], [21]).

The study of eigenvalue problems for hemivariational inequalities has a deep practical motivation. For instance, the loading-unloading problems and thus also the hysteresis problems are typical examples for the theory of hemivariational inequalities and can be reduced to the study of the eigenvalue problem. Indeed, D. Motreanu

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and P. D. Panagiotopoulos ([44], [32]) proved that the global behaviour of a loading-unloading problem of a deformable body is governed by a sequence of hemivariational inequality expressions, one for each branch. They proved that the changing of branch leads to an eigenvalue problem. The stability of a Von Karman plate in adhesive contact with a rigid support or of Von Karman plates adhesively connected in sandwich form is another motivation for the study of eigenvalue problems for hemivariational inequalities ([24], [25]). Recent papers deal with eigenvalue hemivariational inequalities on a sphere-like type manifold ([6], [7]), with nonsymmetric perturbed eigenvalue hemivariational inequalities ([10], [46]), which imply applications in adhesively connected plates, etc.

In this paper we deal with a type of eigenvalue problem for a hemivariational inequality governed by two variable operators. The hemivariational inequality, which gave rise to the problem studied here, was introduced in [12], [11] as an extension to several hemivariational-variational problems. The aim of the present paper is to provide general existence results of the solutions on real Banach spaces and real reflexive Banach spaces. Finally, we illustrate our theoretical results by an application to Engineering.

2. The abstract framework

We assume that the following statements are valid:

(H1) V is a real Banach space endowed with the norm topology, and V^* is its dual endowed with the weak*-topology. Throughout the paper the duality pairing between a Banach space and its dual is denoted by $< .,. >$;

(H2) $T : V \to L^p\left(\Omega, \Re^k\right)$ is a linear and continuous operator, where $1 \leq p < \infty, k \geq 1$ and $\Omega \subseteq \Re^n$ is a bounded open set in n-dimensional Euclidean space;

(H3) $A : V \times V \rightsquigarrow V^*$ is a set-valued mapping;

The properties of the set-valued mapping A will be given later.

(H4) $j = j\left(x, y\right) : \Omega \times \Re^k \to \Re$ is a Caratheodory function, which is locally Lipschitz with respect to the second variable and satisfies the following assumption:

$$\exists h_1 \in L^{\frac{p}{p-1}}\left(\Omega, \Re\right) \text{ and } h_2 \in L^{\infty}\left(\Omega, \Re\right)$$

such that

$$|z| \leq h_1(x) + h_2(x)|y|^{p-1} \text{ a.e. } x \in \Omega, \forall y \in \Re^k, \forall z \in \partial j(x,y)$$

where,

$$j^0(x,y)(h) = \lim \sup_{\substack{y' \to y \\ t \to 0^+}} \frac{j(x, y' + th) - j(x, y')}{t}$$

is the (partial) Clarke derivative of the locally Lipschitz mapping $j(x,.)$, $x \in \Omega$ fixed, at the point $y \in \Re^k$ with respect to the direction $h \in \Re^k$, and

$$\partial j(x,y) = \left\{ z \in \Re^k : \langle z, h \rangle \leq j^0(x,y)(h) \quad , \forall h \in \Re^k \right\}$$

is the Clarke generalized gradient of the mapping $j(x,.)$ at the point $y \in \Re^k$.

We recall some basic concepts, which are needed to formulate the problem under consideration.

Definition 1. *We say that the set-valued mapping $A : V \rightsquigarrow V^*$* ***is monotone*** *if it satisfies the relation*

$$\langle f - g, u - v \rangle \geq 0 \quad , \forall u, v \in V, \forall f \in A(u), \forall g \in A(v).$$

Definition 2. *We say that the set-valued mapping $A(.,v) : V \rightsquigarrow V^*$, where $v \in V$ fixed,* ***has the monotone property (M)*** *if it verifies the relation*

$$\sup_{f \in A(u,v)} \langle f, u - v \rangle \geq \sup_{g \in A(v,v)} \langle g, u - v \rangle \quad , \forall u \in V. \tag{M}$$

Remark 1. *Every set-valued mapping $A(.,v) : V \rightsquigarrow V^*$ (where $v \in V$ is fixed) which is monotone has the monotone property (M), but the inverse is not always true.*

Definition 3. *The set-valued mapping $A : V \rightsquigarrow V^*$ is said to be* ***concave*** *if*

$$(1-\alpha) A(x_1) + \alpha A(x_2) \supseteq A((1-\alpha) x_1 + \alpha x_2) \quad , \forall \alpha \in [0,1], \forall x_1, x_2 \in V.$$

Definition 4. *The set-valued mapping $\Im : V \rightsquigarrow V^*$ defined by*

$$\Im u := \left\{ f \in V^* : \|f\| = \|u\|, \langle f, u \rangle = \|u\|^2 \right\}, \forall u \in V$$

is called the ***duality map of V.***

The duality map has the following representation:

Proposition 1. *(see [4]) For every* $u \in V$, $\Im u = \partial \left(\frac{1}{2} \|u\|^2 \right)$.

Because the Banach space V is endowed with the norm topology and its dual V^* is endowed with the weak*-topology then, according to [35], [9], [5], [13], we can state some properties of the duality map:

Theorem 2. *Duality map* $\Im$ *has the following properties:*

(i) for every $u \in V$*, the set* $\Im u$ *is convex and*

for every $\lambda \in \Re$*, for every* $u \in V$, $\Im (\lambda u) = \lambda \Im (u)$*;*

(ii) the set $\Im (u)$ *is weakly*-compact, for every* $u \in V$ *;*

(iii) the duality map $\Im$ *is weakly*-upper semicontinuous.*

The duality map $\Im$ is successfully involved in the representation of the semi-inner products.

The **semi-inner products** $(.,.)_\pm : V \times V \to \Re$ are defined (according to [13]) by

$$\begin{aligned} (x,y)_+ &= \|y\| \lim_{t \to 0^+} \frac{\|y+tx\| - \|y\|}{t} \\ (x,y)_- &= \|y\| \lim_{t \to 0^+} \frac{\|y\| - \|y-tx\|}{t}. \end{aligned}$$

Remark 2. *If* V *is a Hilbert space endowed with the inner product* $(.,.)_V$ *, then*

$$(x,y)_+ = (x,y)_- = (x,y)_V \ , \forall x,y \in V.$$

Thus, let us note the representations of the semi-inner products:

Proposition 3. *(see [13]): The following estimations hold:*

$$\begin{aligned} (x,y)_+ &= \max \{ \langle f,x \rangle : f \in \Im y \} \\ (x,y)_- &= \min \{ \langle f,x \rangle : f \in \Im y \}. \end{aligned}$$

Our goal is to study the following problem (EP):

Find $u \in V, \lambda \in \Re \setminus \{0\}$ such that

$$\sup_{f \in A(u,u)} \langle f, v-u \rangle + \int_\Omega j^0 (x, Tu(x)) (Tv(x) - Tu(x)) \, dx \geq \lambda (v-u,u)_+ , \forall v \in V \tag{EP}$$

which is the eigenvalue problem corresponding to the hemivariational inequality problem (P):

Find $u \in V$ such that

$$\sup_{f \in A(u,u)} \langle f, v-u \rangle + \int_\Omega j^0 \left(x, Tu\left(x\right)\right) \left(Tv\left(x\right) - Tu\left(x\right)\right) dx \geq 0, \forall v \in V. \qquad \text{(P)}$$

Remark 3. *In fact, the eigenvalue problem (EP) is equivalent with the following problem:*

Find $u \in V, \lambda \in \Re \setminus \{0\}$ such that

$$\sup_{f \in A(u,u)} \langle f, v \rangle + \int_\Omega j^0 \left(x, Tu\left(x\right)\right) \left(Tv\left(x\right)\right) dx \geq \lambda \left(v, u\right)_+ \ , \forall v \in V.$$

Because our approach is based on the results obtained for the problem (P),we will take into account the earliest formulation of the eigenvalue problem (EP).

For the general study of this eigenvalue problem (EP) we need some to recall results about the existence of solutions of the problem (P).

Theorem 4. (see [12]) *Assume that all the hypotheses (H1)-(H4) are satisfied. Moreover, the following assumptions hold:*

(i) for each $v \in V$, the set-valued mapping $A\left(., v\right) : V \rightsquigarrow V^*$ has the monotone property (M) and it is weakly-upper semicontinuous from the line segments of V in V^* ;*

(ii) for each $u \in V$, the set-valued mapping $A\left(u, .\right) : V \rightsquigarrow V^*$ is weakly-upper semicontinuous;*

(iii) there exists a compact subset $K \subseteq V$, and an element $u_0 \in V$ such that the coercivity condition

$$\sup_{f \in A(u,u)} \langle f, u_0 - u \rangle + \int_\Omega j^0 \left(x, Tu\left(x\right)\right) \left(Tu_0\left(x\right) - Tu\left(x\right)\right) dx < 0 \quad , \forall u \in V \diagdown K$$

holds;

(iv) for each $u, v \in V$, the set $A\left(u, v\right)$ is weakly-compact.*

Then the problem (P) admits a solution $u \in V$.

If in addition $A\left(u, u\right)$ is a convex set, then u is also a solution of the following problem (Pc):

Find $u \in V$, $f \in A\left(u, u\right)$ such that

$$\langle f, v-u \rangle + \int_\Omega j^0 \left(x, Tu\left(x\right)\right) \left(Tv\left(x\right) - Tu\left(x\right)\right) dx \geq 0 \quad , \quad \forall v \in V. \qquad \text{(Pc)}$$

We define the set R(A,j,V) of **asymptotic directions** by

$$R(A,j,V)=\left\{\begin{array}{c} w\in V \ \mid \exists\left(u_n\right)\subseteq V, t_n:=\left\|u_n\right\|\rightarrow\infty, w_n:=\frac{u_n}{\left\|u_n\right\|}\rightharpoonup w, \\ \\ \inf_{f\in A(u_n,u_n)}\left\langle f,u_n\right\rangle-\int_\Omega j^0\left(x,Tu_n\left(x\right)\right)\left(-Tu_n\left(x\right)\right)dx\leq 0 \end{array}\right\}.$$

Theorem 5. *(see* [11]*) Assume that all the hypotheses (H1)-(H4) are satisfied, and V is a real reflexive Banach space. Moreover,*

(i) for each $v\in V$*, the set-valued mapping* $A(.,v):V\rightsquigarrow V^*$ *is weakly-upper semicontinuous from the line segments of* V *into* V^* *, concave and monotone;*

(ii) for each $u\in V$*, the set-valued mapping* $A(u,.):V\rightsquigarrow V^*$ *is weakly-upper semicontinuous;*

(iii) $R(A,j,V)=\emptyset$*;*

(iv) for each $u,v\in V$ *, the set* $A(u,v)$ *is weakly-compact.*

Then the problem (P) admits a solution.

If in addition the set A(u,u) is convex, then the problem (Pc) admits solution also.

3. The main results

The aim of our study is to provide verifiable conditions ensuring the existence of solutions to problem (EP). Our existence results concerning problem (EP) are the following.

Theorem 6. *Assume that all the hypotheses (H1)-(H4) are satisfied. Moreover, the following assumptions hold:*

(i) for each $v\in V$*, the set-valued mapping* $A(.,v):V\rightsquigarrow V^*$ *has the monotone property (M) and it is weakly*-upper semicontinuous from the line segments of* V *in* V^* *;*

(ii) for each $u\in V$*, the set-valued mapping* $A(u,.):V\rightsquigarrow V^*$ *is weakly*-upper semicontinuous;*

(iii) there exists a compact subset $K\subseteq V$*, and an element* $u_0\in V$ *such that*

$$\left\|u_0\right\|\leq\left\|u\right\|,\ \forall u\in V\backslash K$$

and

$$\sup_{f\in A(u,u)} \langle f, u_0 - u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tu_0\left(x\right) - Tu\left(x\right)\right)dx < 0, \forall u \in V \diagdown K;$$

(iv) for each $u, v \in V$*, the set* $A(u,v)$ *is weakly*-compact.*

Then for every $\lambda < 0$*, the problem (EP) admits a solution* $u \in V$*.*

If in addition $A(u,u)$ *is a convex set, then the following problem (EPc):*

Find $u \in V, \lambda \in \Re \setminus \{0\}, \ f \in A(u,u)$ *such that*

$$\langle f, v-u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tv\left(x\right) - Tu\left(x\right)\right)dx \geq \lambda\left(v-u,u\right)_+ \quad , \quad \forall v \in V \tag{EPc}$$

admits a solution $u \in V, f \in A(u,u)$ *for every* $\lambda < 0$*.*

Theorem 7. *Assume that all the hypotheses (H1)-(H4) are satisfied, and* V *is a real reflexive Banach space. Moreover,*

(i) for each $v \in V$*, the set-valued mapping* $A(.,v) : V \rightsquigarrow V^*$ *is weakly-upper semicontinuous from the line segments of* V *into* V^* *, concave and monotone;*

(ii) for each $u \in V$*, the set-valued mapping* $A(u,.) : V \rightsquigarrow V^*$ *is weakly-upper semicontinuous;*

(iii)$R(A, j, V) = \emptyset$*;*

(iv) for each $u, v \in V$ *, the set* $A(u,v)$ *is weakly-compact.*

Then the problem (EP) admits a solution.

If in addition the set A(u,u) is convex, then the problem (EPc) admits solution also.

Remark 4. *Under the assumptions of the Theorems 6, 7 not only the eigenvalue problem (EP) but also the hemivariational inequality (P) admits solution.*

4. **Proofs of the theorems**

4.1. **Proof of the first theorem.**

The assumptions of the Theorem 6 allow to apply Theorem 4.

First, let us note that the eigenvalue inequality of problem (Ep) can be rewritten, according to the Proposition 3, as

$$\sup_{f\in A(u,u)} \langle f, v-u\rangle + \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tv\left(x\right) - Tu\left(x\right)\right)dx \quad \geq \quad \lambda \sup_{g\in\Im u}\langle g, v-u\rangle, \quad \forall v \quad \in \quad V.$$

So,

$$\sup_{f\in A(u,u)}\langle f,v-u\rangle-\lambda\sup_{g\in\Im u}\langle g,v-u\rangle$$

$$+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq 0,\forall v\in V.$$

Consider $\lambda<0$. Hence, $(-\lambda)>0$ and in this case we can obtain

$$\sup_{f\in A(u,u)}\langle f,v-u\rangle+\sup_{g\in\Im u}\langle(-\lambda)\,g,v-u\rangle$$

$$+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq 0,\forall v\in V.$$

By the Theorem 2(i), we can note that

$$\sup_{f\in A(u,u)}\langle f,v-u\rangle+\sup_{g\in\Im(-\lambda u)}\langle g,v-u\rangle$$

$$+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq 0,\forall v\in V. \tag{1}$$

Knowing that $\sup_{a\in A,b\in B}\left(\phi\left(a\right)+\psi\left(b\right)\right)=\sup_{a\in A}\phi\left(a\right)+\sup_{b\in B}\psi\left(b\right)$, problem (EP) and the inequality (1) lead us to the following problem:

Find $u\in V$ such that

$$\sup_{f\in F(u,u)}\langle f,v-u\rangle+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq 0,\forall v\in V \tag{EPn}$$

where, we denoted by F the set-valued mapping defined by $F:V\times V\rightsquigarrow V^{*}, F\left(u,v\right)=A\left(u,v\right)+\Im\left(-\lambda v\right)$.

We show that all the hypotheses of the Theorem 4 are verified in the case of the problem (EPn).

'Hypothesis (i)':

Let $v\in V$ be a fixed element. Then, using the monotone property (M) of $A\left(.,v\right)$, we have

$$\begin{aligned}\sup_{f\in F(u,v)}\langle f,u-v\rangle &= \sup_{f\in A(u,v)+\Im(-\lambda v)}\langle f,u-v\rangle=\sup_{\substack{f\in A(u,v)\\ g\in\Im(-\lambda v)}}\left(\langle f,u-v\rangle+\langle g,u-v\rangle\right)\\ &= \sup_{f\in A(u,v)}\langle f,u-v\rangle+\sup_{g\in\Im(-\lambda v)}\langle g,u-v\rangle\\ &\geq \sup_{f\in A(v,v)}\langle f,u-v\rangle+\sup_{g\in\Im(-\lambda v)}\langle g,u-v\rangle=\sup_{f\in F(v.v)}\langle f,u-v\rangle\,.\end{aligned}$$

This proves that the set-valued mapping $F\left(.,v\right)$ has the monotone property (M).

Moreover, the definition of the mapping F and the assumption (i) on the operator A(.,v) imply that the mapping $F(.,v)$ is weakly*-upper semicontinuous from the line segments of V in V^*.

'Hypothesis (ii)':

Because $A(u,.)$ is weakly*-upper semicontinuous, by the assumption (ii), and because $\Im(.)$ is weakly*-upper semicontinuous, according to the Theorem 2(iii), it follows that $F(u,.)$ is weakly*-upper semicontinuous.

'Hypothesis (iii)':

Let both $K \subseteq V$ and $u_0 \in V$ be the elements from the assumption (iii). The question we need to ask is if:

$$\sup_{f\in F(u,u)} \langle f, u_0 - u\rangle + \int_\Omega j^0(x, Tu(x))(Tu_0(x) - Tu(x))\,dx < 0 \quad ,\forall u \in V\diagdown K$$

i.e.

$$\sup_{f\in A(u,u)} \langle f, u_0 - u\rangle + \sup_{g\in\Im(-\lambda u)} \langle g, u_0 - u\rangle$$

$$+ \int_\Omega j^0(x, Tu(x))(Tu_0(x) - Tu(x))\,dx < 0 \quad ,\forall u \in V\diagdown K$$

which leads us to the

$$\sup_{f\in A(u,u)} \langle f, u_0 - u\rangle + \int_\Omega j^0(x, Tu(x))(Tu_0(x) - Tu(x))\,dx$$

$$< \lambda \sup_{g\in\Im u} \langle g, u_0 - u\rangle, \forall u \in V\diagdown K. \tag{2}$$

We note that the left hand side of the relation (2) is less than zero, by the assumption (iii). Moreover, the right hand side of the relation (2) is greater than zero, for $\lambda < 0$, because of the Proposition 1 and assumption (iii). Precisely, for $\forall g \in \Im u = \partial\left(\frac{1}{2}\|u\|^2\right)$, $\forall u \in V\backslash K$,

$$\langle g, u_0 - u\rangle \le \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u\|^2 \le 0,$$

which implies that

$$\sup_{g\in\Im u} \langle g, u_0 - u\rangle \le 0. \tag{3}$$

If we multiply the inequality (3) by λ ($\lambda < 0$), we obtain

$$\lambda \sup_{g\in\Im u} \langle g, u_0 - u\rangle \ge 0.$$

As a conclusion, the 'hypothesis (iii)' is verified.

'Hypothesis (iv)':

By the assumption (iv), as well as by the Theorem 2(ii), we can infer that the set $F(u,v)$ is weakly*-compact, for every $u, v \in V$.

Finally, according to the Theorem 4 the eigenvalue problem (EP) admits a solution $u \in V$, when $\lambda < 0$.

In addition, if $A(u,u)$ is convex, it follows from the Theorem 2(i) that $F(u,u)$ is also convex. So, by the second part of the Theorem 4, we infer that the eigenvalue problem (EPc) admits solution for every $\lambda < 0$.

4.2. **Proof of the second theorem.** For the proof of the Theorem 7, we proceed in the same way. Again, for $\lambda < 0$, we note that the eigenvalue problem (Ep) is equivalent to the hemivariational inequality problem

Find $u \in V$ such that

$$\sup_{f \in F(u,u)} \langle f, v-u \rangle + \int_{\Omega} j^0 (x, Tu(x)) (Tv(x) - Tu(x)) \, dx \geq 0 \quad , \forall v \in V \qquad \text{(EPn)}$$

where, we denoted by F the set-valued mapping defined by $F : V \times V \rightsquigarrow V^*, F(u,v) = A(u,v) + \Im(-\lambda v)$.

We show that all the hypotheses of the Theorem 5 are verified in the case of the problem (EPn).

'Hypothesis (i)':

First, let us emphasize that, because V is a reflexive Banach space, there exists an equivalent norm on V, such that under this new norm, the duality map is a single-valued monotone demicontinuous function. Having this, let $v \in V$ be a fixed element. Then using the fact that $A(.,v)$ is monotone, we have

$$\langle f_1 + \Im(-\lambda v) - f_2 - \Im(-\lambda v), u_1 - u_2 \rangle = \langle f_1 - f_2, u_1 - u_2 \rangle \geq 0,$$

for every $f_1 \in A(u_1, v), f_2 \in A(u_2, v)$.

This proves that the set-valued mapping $F(.,v)$ is monotone.

By the definition of the operator F, and the assumption (i), it follows that $F(.,v)$ is concave.

Moreover, the definition of the mapping F and the assumption (i) on the operator A(.,v) imply that the mapping $F(.,v)$ is weakly-upper semicontinuous from the line segments of V in V^*.

'Hypothesis (ii)':

Because $A(u,.)$ is weakly-upper semicontinuous, by the assumption (ii), and because $\Im(.)$ is demicontinuous, it follows that $F(u,.)$ is weakly-upper semicontinuous.

'Hypothesis (iii)':

Assume, by contradiction, that there exists $w \in R(F,j,V)$. This means that

$$\exists (u_n) \subseteq V, t_n := \|u_n\| \to \infty, w_n := \frac{u_n}{\|u_n\|} \rightharpoonup w \text{ such that}$$

$$\inf_{f \in F(u_n,u_n)} \langle f, u_n \rangle - \int_\Omega j^0(x, Tu_n(x))(-Tu_n(x))\,dx \leq 0. \tag{4}$$

Taking into account the definition of the operator F, inequality (4) becomes:

$$\inf_{f \in A(u_n,u_n)} \langle f, u_n \rangle - \lambda \langle \Im u_n, u_n \rangle - \int_\Omega j^0(x, Tu_n(x))(-Tu_n(x))\,dx \leq 0. \tag{5}$$

Knowing that

$$-\lambda \langle \Im u_n, u_n \rangle > 0$$

the relation (5) may be true if and only if the next inequality holds:

$$\inf_{f \in A(u_n,u_n)} \langle f, u_n \rangle - \int_\Omega j^0(x, Tu_n(x))(-Tu_n(x))\,dx \leq 0.$$

We can conclude that $w \in R(A,j,V)$, which is a contradiction with our assumption (iii).

'Hypothesis (iv)':

By the assumption (iv), as well as by the definition of the operator F, we can infer that the set $F(u,v)$ is weakly-compct, for every $u, v \in V$.

Finally, according to the Theorem 7, the eigenvalue problem (EP) admits a solution $u \in V$, when $\lambda < 0$.

In addition, if $A(u,u)$ is convex, it follows that $F(u,u)$ is also convex. So, by the second part of the Theorem 5, we infer that the eigenvalue problem (EPc) admits solution, for every $\lambda < 0$.

5. Applications to Engineering

Our results can be applied directly to the study of B. V. P.s in Engineering. Let us analyze a very general situation which leads us to the hemivariational inequality problem (EP). For instance, let us consider an open, bounded, connected subset

$\Omega \subseteq \Re^3$ referred to a fixed Cartesian coordinate system $Ox_1x_2x_3$ and we formulate the problem

$$-\Delta u + h(u) + cu = f \text{ in } \Omega \tag{6}$$

$$u = 0 \text{ on } \Gamma. \tag{7}$$

Here Γ is the boundary of Ω and we assume that Γ is sufficiently smooth ($C^{1,1}$-boundary is sufficient), c is a given constant, and h is a continuous function, which has the property

$$u(x)\, h(u(x)) \geq 0, \forall x \in \Omega. \tag{8}$$

In order to physically motivate problem (6),(7) in a simple way, we interpret u as the temperature of a medium in a region Ω. The differential equation in (6) describes a stationary temperature state with the heat source $f - h(u) - cu$ that depends on temperature (see [47]).

We seek a function u such that to verify (6), (7) with

$$-f \in \partial j(x, u) \tag{9}$$

where $j(x, .)$ is a locally Lipschitz function.

Let us consider the Sobolev space $V = H_0^1(\Omega)$, which can be viewed as a Hilbert space endowed with the inner-product

$$(u, v) = \int_\Omega uv dx, \ \forall u, v \in V.$$

Let us denote by $C(\Omega)$ the constant of the Poincaré-Friedrichs inequality

$$\int_\Omega v^2 dx \leq C(\Omega) \int_\Omega (\nabla v)^2 dx, \ \forall v \in V. \tag{10}$$

Moreover, let us assume that the following directional growth condition holds:

$$j^0(x, \xi)(-\xi) \leq \alpha(x) \mid \xi \mid, \ \forall x \in \Omega, \forall \xi \in \Re \tag{11}$$

for some nonnegative function $\alpha \in L^2(\Omega)$, with

$$\|\alpha\|_{L^2(\Omega)} \leq \frac{1}{C(\Omega)} \ . \tag{12}$$

Now, we multiply (6) by $(v - u)$ and integrate over Ω. This gives us the following relation

$$\int_\Omega -\Delta u (v-u) dx + \int_\Omega h(u)(v-u) dx + c \int_\Omega u(v-u) dx = \int_\Omega f(v-u) dx. \tag{13}$$

Then from the Gauss-Green Theorem applied to (13) we are led to the equality

$$\int_\Omega \nabla u \nabla (v-u)\,dx + \int_\Omega h(u)(v-u)\,dx + c\int_\Omega u(v-u)\,dx$$
$$= \int_\Gamma \frac{\partial u}{\partial n}(v-u)\,d\Gamma + \int_\Omega f(v-u)\,dx. \tag{14}$$

Because $u, v \in H_0^1(\Omega)$ the surface integral vanishes.

Relation (9) implies that

$$-f(v-u) \le j^0(x,u)(v-u). \tag{15}$$

If we introduce the notation

$$a(u,v) = \int_\Omega \nabla u \nabla v dx$$

then the relations (14) and (15) give us the inequality

$$a(u,v) + \int_\Omega h(u)(v-u)\,dx + c\int_\Omega u(v-u)\,dx$$
$$+ \int_\Omega j^0(x,u)(v-u)\,dx \ge 0, \forall v \in V. \tag{16}$$

Let us note that there exists a linear monotone continuous operator $B : V \to V^*$ such that

$$\langle B(u), v\rangle = a(u,v), \quad \forall u, v \in V.$$

Consider $\Im : V \to V^*$ the duality isomorphism

$$\langle \Im u, v\rangle = (u,v), \ \forall u, v \in V$$

Thus, if we consider the following multivalued mapping

$$\begin{aligned} A &: V \times V \rightsquigarrow V^* \\ A(u,v) &= B(u) + \Im(h(v)) \end{aligned}$$

then the hemivariational inequality (16) lead us to the following problem:

find $u \in V$ such that for any $v \in V$

$$\sup_{f \in A(u,u)} \langle f, v-u\rangle + \int_\Omega j^0(x,u)(v-u)\,dx \ge (-c)\langle \Im u, v-u\rangle \tag{EPeng}$$

First, let us remark that the operator A satisfies the assumptions (i), (ii), (iv) of the Theorem 7. All we have to do now is to verify if the assumption (iii) is satisfied. For this goal, let us assume that there exists $w \in R(A, j, V)$. So,

$$\exists (u_n) \subseteq V, t_n := \|u_n\|_{L^2(\Omega)} \to \infty, w_n := \frac{u_n}{\|u_n\|_{L^2(\Omega)}} \rightharpoonup w \text{ such that}$$

$$\int_\Omega (\nabla u_n)^2 dx + \int_\Omega h(u_n) u_n dx - \int_\Omega j^0(x, u_n(x))(-u_n(x)) dx \leq 0. \qquad (17)$$

There exists a rank m such that $\|u_n\|_{L^2(\Omega)} > 1$, for every $n \geq m$. By the Holder inequality and because of the relations (10), (11), (12), the following evaluation holds for every $u_n, n \geq m$:

$$\int_\Omega (\nabla u_n)^2 dx \geq \frac{1}{C(\Omega)} \int_\Omega (u_n)^2 dx > \frac{1}{C(\Omega)} \left(\int_\Omega (u_n)^2 dx \right)^{\frac{1}{2}}$$

$$= \frac{\|u_n\|_{L^2(\Omega)}}{C(\Omega)} \geq \|\alpha\|_{L^2(\Omega)} \cdot \|u_n\|_{L^2(\Omega)} \geq \int_\Omega \alpha(x) \cdot |\, u(x) \,|\, dx$$

$$\geq \left| \int_\Omega j^0(x, u(x))(-u(x)) dx \right| \geq \int_\Omega j^0(x, u(x))(-u(x)) dx.$$

The last evaluation and the property (8) of the function h show us that the relation (17) is impossible. This contradiction guarantees that the assumption (iii) of the Theorem 7 is also satisfied.

Since all the assumptions of the Theorem 7 are ensured and the embedding $V \subseteq L^2(\Omega)$ is linear and continuous, we can prove the existence of solutions of (EPeng) for all $c > 0$.

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Technical University of Cluj-Napoca, Romania;
School of Computational Science and Information Technology,
Florida State University, USA

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WHEELER-FEYNMAN PROBLEM ON A COMPACT INTERVAL

VERONICA ANA DÂRZU

Abstract. In this paper the problem (1)+(2) is studied.

1. Introduction

In the paper [1] and [3] the author study the Weeler-Feynman problem on R. In this paper we consider the following Weeler-Feynman problem:

$$x'(t) = f(t, x(t), x(t-h), x(t+h)), \quad t \in [a,b], \tag{1}$$

$$x(t) = \varphi(t), \ t \in [t_0 - h, t_0 + h], \tag{2}$$

where $t_0 \in [a, b], \quad a \leq t_0 - h, t_0 + h \leq b$ and $\varphi \in C^1[t_0 - h, t_0 + h]$

2. Remarks and examples

2.1. By a solution of (1) we understand a function $x \in C[a - h, b + h] \cap C^1[a, b]$ which satisfies the relation (1) for all $t \in [a, b]$.

2.2. Let $\alpha, \ \beta, \ \gamma \in R, \ \beta \neq 0, \ \gamma \neq 0, \ t_0 \in [a, b]$. We consider the following problem:

$$x'(t) = \alpha x(t) + \beta x(t-h) + \gamma x(t+h)), \quad t \in [a,b], \tag{3}$$

$$x(t) = \varphi(t), \ t \in [t_0 - h, t_0 + h], \tag{4}$$

where $t_0 \in [a, b], \ a \leq [t_0 - h, t_0 + h] \leq b$.

We shall apply the method of steps on intervals $[t_0, b]$ and $[a, t_0]$ to find some "if and only" conditions for the existence of a solution of problem (3)+(4).

Let $t \in [t_0, t_0 + h]$

$$\varphi'(t) = \alpha\varphi(t) + \beta\varphi(t-h) + \gamma x(t+h)$$

Then:

$$x(t) := x_1(t) = \frac{1}{\gamma}[\alpha\varphi(t-h) + \beta\varphi(t-2h) - \varphi'(t-h)],\ t \in [t_0+h, t_0+2h]$$

Let $t \in [t_0+h, t_0+2h]$

$$x_1'(t) = \alpha x_1(t) + \beta\varphi(t-h) + \gamma x(t+h)$$

Then:

$$x(t) := x_2(t) = \frac{1}{\gamma}[\alpha x_1(t-h) + \beta\varphi(t-2h) - x_1'(t-h)],\ t \in [t_0+2h, t_0+3h]$$

By the same way the final step on $[t_0, b]$:

$$x_{n_b}(t) = \frac{1}{\gamma}[\alpha x_{n_b-1}(t-h) + \beta x_{n_b-2}(t-2h) - x_{n_b-1}'(t-h)],\ t \in [t_0+n_b h, b]$$

where $n_b = [\frac{b-t_0}{h}]$.

By the same way on $[a, t_0]$ we find $n_a = [\frac{t_0-a}{h}]$.

Let $n := max\{n_a, n_b\}$.

Let $\varphi \in C^{n+1}[t_0-h, t_0+h]$.

Let $x \in C^n[a-h, b+h] \cap C^{n+1}[a, b]$ be a solution of problem (3)+(4).

We have:

$$x^{(k+1)}(t) = \alpha x^{(k)}(t) + \beta x^{(k)}(t-h) + \gamma x^{(k)}(t+h),\ k \in 0, 1, \ldots, n$$

For $t = t_0$, we have:

$$\varphi^{(k+1)}(t_0) = \alpha\varphi^{(k)}(t_0) + \beta\varphi^{(k)}(t_0-h) + \gamma\varphi^{(k)}(t_0+h),\ k \in \{0, 1, \ldots, n\}$$

Then the problem (3)+(4) has a solution if and only if:

$$\varphi^{(k+1)}(t_0) = \alpha\varphi^{(k)}(t_0) + \beta\varphi^{(k)}(t_0-h) + \gamma\varphi^{(k)}(t_0+h),\ k \in \{0, 1, \ldots, n\}.$$

2.3. For the case in which $\beta = 0$ or $\gamma = 0$ see [2].

3. The main result

In what follow we consider the problem (1)+(2). We need the following conditions.

Let $n_a := [\frac{t_0-a}{h}]$, $n_b := [\frac{b-t_0}{h}]$, $n := max\{n_a, n_b\}$.

Let $f \in C^{n+1}([a,b] \times R^3)$.

(C1):For all $u_1 \in [a,b]$, $u_2, u_4, u_5 \in R$, there exist a unique $u_3 \in R$, $u_3 = f_1(u_1, u_2, u_4, u_5)$, $f_1 \in C^{n+1}([a,b] \times R^3)$, such that, $u_5 = f(u_1, u_2, u_3, u_4)$.

(C2):For all $u_1 \in [a,b]$, $u_2, u_3, u_5 \in R$, there exist a unique $u_4 \in R$, $u_4 = f_2(u_1, u_2, u_3, u_5)$, $f_2 \in C^{n+1}([a,b] \times R^3)$, such that, $u_5 = f(u_1, u_2, u_3, u_4)$.

We have

Theorem 1. *Let $f \in C^{n+1}([a,b] \times R^3)$ satisfies (C1) and (C2). If $\varphi \in C^{n+1}[t_0 - h, t_0 + h]$, then the problem (1)+(2) has a unique solution if and only if φ satisfies the following condition:*

$$\varphi^{(k+1)}(t_0) = [f(t, \varphi(t), \varphi(t-h), \varphi(t+h)]^{(k)}_{t=t_0}, \ , \ k \in \{0, 1, \dots, n\}. \tag{5}$$

Proof. By the method of steps we construct the solution of (1) +(2) as follows. Let $t \in [t_0, t_0 + h]$

$$\varphi'(t) = f(t, \varphi(t), \varphi(t-h), x(t+h))$$

From (C2) we have

$$x(t) := x_1(t) = f_2(t-h, \varphi(t-h), \varphi(t-2h), \varphi'(t-h)), \ t \in [t_0 + h, t_0 + 2h] \ .$$

By the same method we find the final step:

$$x_{n_b}(t) = f(t-h, x_{n_b-1}(t-h), x_{n_b-1}(t-2h), x\prime_{n_b-1}(t-h)), \ t \in [t_0 + n_b h, b]$$

where $n_b = [\frac{b-t_0}{h}]$.

We must have:

$$\varphi(t_0 + h) = x_1(t_0 + h)$$

$$x_p(t_0 + (p+1)h) = x_{p+1}(t_0 + (p+1)h), \quad p \leq n_b - 1$$

By the same way we have the solution on $[a, t_0]$ with the condition

$$\varphi(t_0 - h) = x_{-1}(t_0 - h)$$

$$x_{-p}(t_0 - (p+1)h) = x_{-(p+1)}(t_0 - (p+1)h), \quad p \le n_a - 1$$

where $n_a = [\frac{t_0 - a}{h}]$.

So the solution is:

$$x(t) = \begin{cases} x_{-n_a}(t) & \text{dacă} \quad t \in [a, t_0 - n_a h] \\ x_{-k}(t) & \text{dacă} \quad t \in [t_0 - (k+1)h, t_0 - kh], 1 \le k \le n_a - 1 \\ \varphi(t) & \text{dacă} \quad t \in [t_0 - h, t_0 + h] \\ x_k(t) & \text{dacă} \quad t \in [t_0 + kh, t_0 + (k+1)h], \quad 1 \le k \le n_b - 1 \\ x_{n_b}(t) & \text{dacă} \quad t \in [t_0 + n_b h, b] \end{cases}$$

Let $n = max\{n_a, n_b\}$.

Now we prove the necessity of the condition (5). Let $x \in C[a-h, b+h] \cap C^1[a, b]$ a solution of the problem (1)+(2).

Then $x \in C^n[a-h, b+h] \cap C^{n+1}[a, b]$ is a solution.

We have:

$$x^{(k+1)}(t) = [f(t, x(t), x(t-h), x(t+h))]^{(k)}, \; t \in [a, b], \; k \in \{0, 1, \dots, n\}.$$

For $t = t_0$, we have (5).

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Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

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CERTAIN SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS WITH MISSING AND TWO FIXED POINTS

S.R. KULKARNI AND MRS. S.S. JOSHI

Abstract. The systematic study of some novel subclasses $\Omega^*_{pi}(\alpha, \beta, \mu, z_0)$, $(i = 0, 1)$ consisting functions of the type

$$f(z) = a_0 z^{-1} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n},\ a_0 > 0, a_{p+n} \geq 0, p \in N$$

which are meromorphic and univalent in $U^* = \{z : 0 < |z| < 1\}$ is presented here. The various results for example coefficient estimates, radius of convexity, distortion theorem are obtained for $f(z)$ to be in the above mentioned classes.

1. Introduction and Definitions

Let Ω denote the class of functions of the form

$$f(z) = a_0 z^{-1} + \sum_{n=1}^{\infty} a_n z^n,\ a_0 > 0 \tag{1.1}$$

which are analytic in the punctured disk $U^* = \{z : 0 < |z| < 1\}$. Further, Ω^* is the class of all functions in Ω which are univalent in U^*. We denote by Ω^*_p, a subclass of Ω^* consisting functions of the form

$$f(z) = a_0 z^{-1} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n},\ a_0 > 0, a_{p+n} > 0, p \in N, \tag{1.2}$$

$$N = \{1, 2, 3, \cdots\}.$$

Definition. A function $f(z)$ belonging to the class Ω^*_p is in the class $\Omega^*_p(\alpha, \beta, \mu)$ if it satisfies the condition

$$\left| \frac{z^2 f'(z) + a_0}{\mu z^2 f'(z) - a_0 + (1+\mu)\alpha a_0} \right| < \beta, \tag{1.3}$$

for some $0 \leq \alpha < 1, 0 < \beta \leq 1$ and $0 \leq \mu \leq 1$.

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For a given real number $z_0(0 < z_0 < 1)$. Let $\Omega_{pi}(i = 0, 1)$ be a subclass of Ω_p^* satisfying the condition $z_0 f(z_0) = 1$ and $-z_0^2 f'(z_0) = 1$ respectively.

Let

$$\Omega_{pi}^*(\alpha, \beta, \mu, z_0) = \Omega_p^*(\alpha, \beta, \mu) \cap \Omega_{pi} \ (i = 0, 1). \tag{1.4}$$

In our systematic investigation of the various properties and characteristics of the class $\Omega_{pi}^*(\alpha, \beta, \mu)$, we shall require use of number of other classes of functions associated with Ω_p^*. First of all, a function $f \in \Omega_p^*$ is said to be meromorphic starlike of order α in U^* if it satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > -\alpha, \ z \in U^*, 0 \leq \alpha < 1. \tag{1.5}$$

On the other hand, a function $f \in \Omega_p^*$ is said to convex of order α in U, if it satisfies the inequality

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\alpha, \ z \in U^*, 0 \leq \alpha < 1. \tag{1.6}$$

For other subclasses of meromorphic univalent function, one may refer to the recent work of Aouf [1], Aouf and Darwish [2], Cho *et al* [3], Joshi *et al* [4], Srivastava and Owa [5]. In the present paper we obtain coefficient estimates, distortion theorems, closure theorems and radius of convexity of order $\delta(0 \leq \delta < 1)$ for the classes $\Omega_{pi}^*(\alpha, \beta, \mu, z_0)(i = 0, 1)$. Further, we look for necessary and sufficient condition that a subset B of the real interval $[0, 1]$ should satisfy the property $\cup_{z_r \in B}\Omega_{p0}^*(\alpha, \beta, \mu, z_r)$ and $\cup_{z_r \in B}\Omega_{p1}(\alpha, \beta, \mu, z_r)$ each forms a convex family. The techniques used are similar to Uralegaddi and Ganigi [6].

2. **Main Results**

Coefficient Estimates

Theorem 1. Let the function $f(z)$ be defined by (1.2) is in the class $\Omega_p^*(\alpha, \beta, \mu)$ if and only if

$$\sum_{n=0}^{\infty}(p + n)(1 + \mu\beta)a_{p+n} \leq \beta a_0(1 - \alpha)(1 + \mu). \tag{2.1}$$

The result is sharp and is given by

$$f(z) = \frac{a_0}{z} + \frac{\beta(1 - \alpha)(1 + \mu)a_0 z^{p+n}}{(p + n)(1 + \mu\beta)}, n \geq 1. \tag{2.2}$$

Proof. The proof of Theorem 1 is straightforward, hence omitted.

Theorem 2. Let the function $f(z)$ be defined by (1.2). Then $f(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ if and only if

$$\sum_{n=0}^{\infty} \left[\frac{(p+n)(1+\mu\beta)}{\beta(1-\alpha)(1+\mu)} + z_0^{p+n+1} \right] a_{p+n} \leq 1. \tag{2.3}$$

Proof. Since $f(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$, we have

$$z_0 f(z_0) = a_0 + \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1}, \ a_0 \geq 0, \ a_{p+n} \geq 0,$$

which gives

$$a_0 = 1 - \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1} \tag{2.4}$$

substituting this value of a_0 (given by (2.4)) in Theorem 1, we get the desire assertion.

Theorem 3. Let the function $f(z)$ be defined (1.2). Then $f(z) \in \Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ if and only if

$$\sum_{n=0}^{\infty} (p+n) \left[\frac{(1+\mu\beta)}{\beta(1-\alpha)(1+\mu)} - z_0^{p+n+1} \right] a_{p+n} \leq 1. \tag{2.5}$$

Proof. Since $-z_0^2 f'(z_0) = 1$, we have

$$a_0 = 1 + \sum_{n=0}^{\infty} (p+n) a_{p+n} z_0^{p+n+1} \tag{2.6}$$

Eliminating a_0 from (2.1) and (2.6) we get the required result.

An immediate consequence of Theorem 2 and Theorem 3 may be stated as the following.

Corollary 1. Let, $f(z)$ given by (1.2) be in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ then

$$a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{(p+n)(1+\mu\beta) + \beta(1+\mu)(1-\alpha) z_0^{p+n+1}}. \tag{2.7}$$

The equality in the (2.7) is attained for the function $f(z)$ given by

$$f(z) = \frac{(p+n)(1+\mu\beta) + \beta(1+\mu)(1-\alpha) z^{p+n+1}}{z[(p+n)(1+\mu\beta) + \beta(1+\mu)(1-\alpha) z_0^{p+n+1}]}, \tag{2.8}$$

$$p \in N, n \geq 0.$$

Corollary 2. Let the function $f(z)$ given by (1.2) in the class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ then

$$a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{(p+n)[(1+\mu\beta) - \beta(1+\mu)(1-\alpha) z_0^{p+n+1}]}. \tag{2.9}$$

The equality holds for the function $f(z)$ given by

$$f(z) = \frac{(p+n)(1+\mu\beta) + \beta(1+\mu)(1-\alpha) z^{p+n+1}}{z(p+n)[(1+\mu\beta) - \beta(1+\mu)(1-\alpha) z_0^{p+n+1}]}. \tag{2.10}$$

3. Distortion Theorem

In this section, we prove distortion theorem associated with the classes introduced in section 1, we first state the following theorem.

Theorem 4. Let $f(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ then,

$$|f(z)| \geq \frac{p(1+\mu\beta) - \beta(1+\mu)(1-\alpha)r^{p+1}}{r[p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}]}, \tag{3.1}$$

for $0 < |z| = r < 1$. The result is sharp.

Proof. Since $f \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$, by applying assertion (2.3) of Theorem 2, we obtain

$$\sum_{n=0}^{\infty} a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}}. \tag{3.2}$$

Further from (2.4), we have

$$a_0 = 1 - \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1}$$

$$\geq \frac{(1+\mu\beta)p}{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}}. \tag{3.3}$$

Hence we have

$$|f(z)| \geq a_0 r^{-1} - r^p \sum_{n=0}^{\infty} a_{p+n}$$

$$\geq \frac{p(1+\mu\beta) - \beta(1+\mu)(1-\alpha)r^{p+1}}{r[p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}]}, \tag{3.4}$$

by using (3.2) and (3.3). Further, the result is sharp for the function $f(z)$ given by

$$f(z) = \frac{p(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z^{p+1}}{z[p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z_0^{p+1}]}. \tag{3.5}$$

Theorem 5. If $f(z) \in \Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ then

$$|f(z)| \leq \frac{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)r^{p+1}}{r[p(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]} \tag{3.6}$$

for $0 < |z| = r < 1$. The result is sharp.

Proof. It follows from assertion (2.5) of Theorem 3, that

$$\sum_{n=0}^{\infty} a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{p[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]} \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} (p+n) a_{p+n} \leq \frac{\beta(1+\mu)(1-\alpha)}{[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]}. \tag{3.8}$$

From (2.6) we have

$$a_0 = 1 + \sum_{n=0}^{\infty}(p+n)a_{p+n}z_0^{p+n+1} \tag{3.9}$$
$$\leq \frac{(1+\mu\beta)}{[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]}.$$

Hence we have

$$|f(z)| \leq a_0 r^{-1} + r^{p+1}\sum_{n=0}^{\infty} a_{p+n}$$
$$\leq \frac{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)r^{p+1}}{rp[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]}, \tag{3.10}$$

by using (3.7) and (3.9). Further the result is sharp for the function given by

$$f(z) = \frac{p(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z^{p+1}}{zp[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+1}]}. \tag{3.11}$$

4. **Closure Theorems**

Let the functions $f_j(z)$ be defined, for $j = 1, 2, \cdots, m$ by

$$f_j(z) = \frac{a_{0,j}}{z} + \sum_{n=0}^{\infty} a_{p+n,j}z^{p+n} \ (a_{0,j} > 0, a_{p+n,j} \geq 0) \ z \in U^*. \tag{4.1}$$

Theorem 6. Let $f_j(z)$ defined by (4.1) be in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=0}^{m} d_j f_j(z), \ (d_j \geq 0) \tag{4.2}$$

is also in the same class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$, where

$$\sum_{j=0}^{m} d_j = 1. \tag{4.3}$$

Proof. According to the definition (4.2) we have

$$h(z) = \frac{b_0}{z} + \sum_{n=0}^{\infty} b_{p+n}z^{p+n}, \tag{4.4}$$

where

$$b_0 = \sum_{j=0}^{m} d_j a_{0,j} \text{ and } b_{p+n} = \sum_{j=0}^{m} d_j a_{p+n,j}, (n = 0, 1, 2, \cdots, m).$$

Since $f_j(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ $(j = 0, 1, 2, \cdots, m)$, using Theorem 2 we have

$$\sum_{n=0}^{\infty}\{(p+n)(1+\mu\beta) + \beta(1-\alpha)(1+\mu)z_0^{p+n+1}\} \leq \beta(1-\alpha)(1+\mu)$$

for every $j = 0, 1, \cdots, m$. Therefore we have

$$\begin{aligned} & \sum_{n=0}^{\infty}\{(p+n)(1+\mu\beta)+\beta(1-\alpha)(1+\mu)z_0^{p+n+1}\}(\sum_{j=0}^{m} d_j a_{p+n,j}) \\ = & \sum_{j=0}^{m} d_j\{\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta)+\beta(1-\alpha)(1+\mu)z_0^{p+n+1}]a_{p+n,j}\} \\ \leq & (\sum_{j=0}^{m} d_j)\beta(1-\alpha)(1+\mu) \\ = & \beta(1-\alpha)(1+\mu) \end{aligned}$$

which shows that $h(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$.

Theorem 7. Let the functions $f_j(z)(j = 0, 1, \cdots, m)$ defined by (4.1) be in the class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ for every $j = 0, 1, \cdots, m$. Then the function $h(z)$ defined by (4.2) is also in the same class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$, under the assumption (4.3).

Proof. The proof of Theorem 7, can be given on using the same techniques as in the proof of Theorem 6, using Theorem 3.

Theorem 8. The class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ is closed under convex linear combination.

Proof. Let $f_j(z)(j = 0, 1, \cdots, m)$ defined by (4.1) be in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$, it is sufficient to show that the function $H(z)$ defined by

$$H(z) = \lambda f_1(z) + (1-\lambda)f_2(z), \; 0 \leq \lambda \leq 1, \tag{4.5}$$

is also in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$. Since

$$H(z) = \frac{\lambda a_{0,1} + (1-\lambda)a_{0,2}}{z} + \sum_{n=0}^{\infty}\{\lambda a_{p+n,1} + (1-\lambda)a_{p+n,2}\}z^{p+n}$$

with the aid of Theorem 2, we have

$$\sum_{n=0}^{\infty}\{(p+n)(1+\mu\beta)+\beta(1-\alpha)(1+\mu)z_0^{p+n+1}\}[\lambda a_{p+n,1} + (1-\lambda)a_{p+n,2}]$$
$$\leq \beta(1-\alpha)(1+\mu) \tag{4.6}$$

which implies that $H(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$. In a similar manner, by using Theorem 3, we can prove the following Theorem.

Theorem 9. The class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ is closed under convex linear combination.

Theorem 10. Let

$$f_0(z) = 1/z \tag{4.7}$$

and

$$f_{p+n}(z)=\frac{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z^{p+n+1}}{z[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}\ ,n\geq 0 \tag{4.8}$$

then $f(z)$ is in the class $\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$, if and only if it can be expressed in the form:

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z),\ \text{where } \lambda_n\geq 0, \tag{4.9}$$

$$\lambda_i=0(i=1,2,\cdots,p-1,p\geq 2)\ \text{and}\ \sum_{n=0}^{\infty}\lambda_n=1. \tag{4.10}$$

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\lambda_n f_n(z)\\ &= \lambda_0/z+\sum_{n=0}^{\infty}\frac{[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z^{p+n+1}]\lambda_{p+n}}{z[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)]z_0^{p+n+1}}\\ &= \frac{1}{z}\left[\lambda_0+\sum_{n=0}^{\infty}\frac{(p+n)(1+\mu\beta)\lambda_{p+n}}{[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}\right]\\ &\quad+\sum_{n=0}^{\infty}\frac{\beta(1+\mu)(1-\alpha)\lambda_{n+p}z^{p+n}}{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}} \end{aligned}$$

Then it follows from theorem 2, that

$$\sum_{n=0}^{\infty}\frac{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}}{\beta(1+\mu)(1-\alpha)}\ \frac{\beta(1+\mu)(1-\alpha)\lambda_{p+n}}{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}}$$

$$=\sum_{n=0}^{\infty}\lambda_{p+n}=1-\lambda_0\leq 1.$$

Also by definition we have $z_0 f_{p+n}(z_0)=1$. Therefore

$$z_0 f(z_0)=\sum_{n=0}^{\infty}\lambda_{p+n}z_0 f_{p+n}(z_0)=\sum_{n=0}^{\infty}\lambda_{p+n}=1.$$

This implies $f\in\Omega_{p0}$, so by theorem 2, $f(z)\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$.

Conversely, assume that the function $f(z)$ given by (1.2) belongs to the class $\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$. Then

$$a_{p+n}\leq\frac{\beta(1+\mu)(1-\alpha)}{(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}},n\geq 0. \tag{4.11}$$

Setting

$$\lambda_{p+n}=\frac{[(p+n)(1+\mu\beta)+\beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}{\beta(1+\mu)(1-\alpha)}\ a_{p+n},\ n\geq 0$$

and

$$\lambda_0 = 1 - \sum_{n=0}^{\infty} \lambda_{p+n}.$$

Hence, it is observed that $f(z)$ can be expressed in the form (4.9). This completes the proof of Theorem 10.

In a similar manner, we can prove the following Theorem.

Theorem 11. Define

$$f_0(z) = \frac{1}{z} \tag{4.12}$$

and

$$f_{p+n}(z) = \frac{(p+n)(1+\mu\beta) + \beta(1+\mu)(1-\alpha)z^{p+n+1}}{z(p+n)[(1+\mu\beta) - \beta(1+\mu)(1-\alpha)z_0^{p+n+1}]}, \; n \geq 0 \tag{4.13}$$

then $f(z)$ is in the class $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$ if and only if it can be expressed in the form (4.9) where $\lambda_n \geq 0$ and (4.10).

5. **Radius of Convexity**

In this section we determine the radius of convexity of order $\delta(0 \leq \delta < 1)$ for the class $\Omega^*_{pi}(\alpha, \beta, \mu, z_0)(i = 0, 1)$.

Theorem 12. Let the function defined by (1.2) be in the class $\Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ or $\Omega^*_{p1}(\alpha, \beta, \mu, z_0)$, then $f(z)$ is convex of order $\delta(0 \leq \delta < 1)$ in $0 < |z| < R^*(\alpha, \beta, \mu, \delta)$ where

$$R^*(\alpha, \beta, \mu, \delta) = \inf_n \left[\frac{(1-\delta)(1+\mu\beta)}{(1-\alpha)\beta(1+\mu)(p+n+2-\delta)} \right]^{1/(p+n+1)}, n \geq 0. \tag{5.1}$$

The result (5.1) is sharp.

Proof. It is sufficient to show that

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| \leq (1-\delta), \; 0 \leq \delta < 1,$$

for $0 < |z| < R^*(\alpha, \beta, \mu, \delta)$.

We have

$$\left| \frac{f'(z) + [zf'(z)]'}{f'(z)} \right| \leq \sum_{n=0}^{\infty} \frac{(p+n)(p+n+1)a_{p+n}|z|^{p+n+1}}{a_0 - \sum_{n=0}^{\infty}(p+n)a_{p+n}|z|^{p+n+1}}.$$

Thus

$$\left| \frac{f'(z) + [zf'(z)]'}{f'(z)} \right| \leq (1-\delta)$$

if

$$\sum_{n=0}^{\infty}(p+n)(p+n+2-\delta)a_{p+n}|z|^{p+n+1} \leq (1-\delta)a_0 \tag{5.2}$$

when $f(z) \in \Omega^*_{p0}(\alpha,\beta,\mu,z_0)$, using (2.4) we find that inequality (5.2) is equivalent to

$$\sum_{n=0}^{\infty}\{(p+n)(p+n+2-\delta)|z|^{p+n+1} + (1-\delta)z_0^{p+n+1}\}a_{p+n} \leq (1-\delta). \tag{5.3}$$

But Theorem 2 ensures

$$\sum_{n=0}^{\infty}(1-\delta)\left[\frac{(p+n)(1+\mu\beta)}{\beta(1-\alpha)(1+\mu)} + z_0^{p+n+1}\right]a_{p+n} \leq (1-\delta). \tag{5.4}$$

Hence (5.3) holds if

$$\{(p+n)(n+p+2-\delta)|z|^{p+n+1} + (1-\delta)z_0^{p+n+1}\}a_{p+n}$$
$$\leq \left\{(1-\delta)\left[\frac{(p+n)(1+\mu\beta)}{\beta(1-\alpha)(1+\mu)} + z_0^{p+n+1}\right]\right\}a_{p+n}, n \geq 0,$$

or if

$$|z| \leq \left[\frac{(1-\delta)(1+\mu\beta)}{(1-\alpha)\beta(1+\mu)(p+n+2-\delta)}\right]^{1/(p+n+1)},\ n \geq 0.$$

Thus $f(z)$ is convex of order $\delta(0 \leq \delta < 1)$ in $0 < |z| < R^*(\alpha,\beta,\mu,\delta)$.

In other case when $f(z) \in \Omega^*_{p1}(\alpha,\beta,\mu,z_0)$ using (2.6) we find that the inequality (5.2) is equivalent to

$$\sum_{n=0}^{\infty}(p+n)[(p+n+2-\delta)|z|^{p+n+1} - (1-\delta)z_0^{p+n+1}]a_{p+n} \leq (1-\delta). \tag{5.5}$$

Therefore, in view of Theorem 3, the inequality (5.5) holds if

$$(p+n)[(p+n+2-\delta)|z|^{p+n+1} - (1-\delta)z_0^{p+n+1}]a_{p+n}$$
$$\leq (1-\delta)(p+n)\left[\frac{(1+\mu\beta)}{(1-\alpha)\beta(1+\mu)} - z_0^{p+n+1}\right]a_{p+n}$$

or if

$$|z| \leq \left[\frac{(1-\delta)(1+\mu\beta)}{(1-\alpha)\beta(1+\mu)(p+n+2-\delta)}\right]^{1/(p+n+1)},\ n \geq 0.$$

This completes the proof of theorem 12.

Sharpness for the class $\Omega^*_{p0}(\alpha,\beta,\mu,z_0)$ follows by taking the functions $f(z)$ given by (2.8), whereas for the class $\Omega^*_{p1}(\alpha,\beta,\mu,z_0)$, sharpness follows if we take the function given by (2.10).

Remark. The conclusion of Theorem 12 is independent of z_0.

6. **Convex Family**

Let B be a nonempty subset of a real interval $[0,1]$. We define a family $\Omega^*_{p0}(\alpha,\beta,\mu,B)$ by

$$\Omega^*_{p0}(\alpha,\beta,\mu,B)=\cup_{z_r\in B}\Omega^*_{p0}(\alpha,\beta,\mu,z_r).$$

If B has only one element, then $\Omega^*_{p0}(\alpha,\beta,\mu,B)$ is known to be a convex family by Theorems 6 and 8. It is interesting to investigate this class for other subset B.

We shall make use of the following

Lemma 1. If $f(z)\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0)\cap\Omega^*_{p0}(\alpha,\beta,\mu,z_1)$ where z_0 and z_1 are distinct positive numbers then $f(z)=1/z$.

Proof. If $f(z)\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0)\cap\Omega^*_{p0}(\alpha,\beta,\mu,z_1)$ and let

$$f(z)=a_0z^{-1}+\sum_{n=0}^{\infty}a_{p+n}z^{p+n},\ a_0>0,a_{p+n}>0,p\in N,$$

then

$$a_0=1-\sum_{n=0}^{\infty}a_{p+n}z_0^{p+n+1}=1-\sum_{n=0}^{\infty}a_{p+n}z_1^{p+n+1}$$

since $a_{p+n}\geq 0, z_0>0$ and $z_1>0$, this implies $a_{p+n}\equiv 0$ for each $n\geq 0$ and $f(z)=1/z$. Hence the proof of lemma is complete.

Theorem 13. If B is contained in the interval $[0,1]$, then $\Omega^*_{p0}(\alpha,\beta,\mu,B)$ is a convex family if and only if B is connected.

Proof. Suppose B is connected and $z_0,z_1\in B$ with $z_0\leq z_1$. To prove $\Omega^*_{p0}(\alpha,\beta,\mu,B)$ is a convex family it suffices to show, for

$$\begin{aligned} f(z) &= a_0z^{-1}+\sum_{n=0}^{\infty}a_{p+n}z^{p+n}\in\Omega^*_{p0}(\alpha,\beta,\mu,z_0),\\ g(z) &= b_0z^{-1}+\sum_{n=0}^{\infty}b_{p+n}z^{p+n}\in\Omega^*_{p0}(\alpha,\beta,\mu,z_1),\end{aligned}$$

and $0\leq\lambda\leq 1$, that there exists a $z_2(z_0\leq z_2\leq z_1)$ such that

$$h(z)=\lambda f(z)+(1-\lambda)g(z)$$

is in the $\Omega^*_{p0}(\alpha, \beta, \mu, z_2)$. Since $f \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ and $g(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_1)$. We have

$$\begin{aligned} a_0 &= 1 - \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1} \\ b_0 &= 1 - \sum_{n=0}^{\infty} b_{p+n} z_1^{p+n+1}. \end{aligned}$$

Therefore we have

$$\begin{aligned} t(z) &= zh(z) \\ &= \lambda a_0 + (1-\lambda) b_0 + \lambda \sum_{n=0}^{\infty} a_{p+n} z^{p+n} + (1-\lambda) \sum_{n=0}^{\infty} b_{p+n} z^{p+n} \\ &= 1 + \lambda \sum_{n=0}^{\infty} (z^{p+n} - z_0^{p+n+1}) a_{p+n} + (1-\lambda) \sum_{n=0}^{\infty} (z^{p+n+1} - z_1^{p+n+1}) b_{p+n} \quad (6.1) \end{aligned}$$

$t(z)$ being real when z is real with $t(z_0) \leq 1$ and $t(z_1) \geq 1$, there exists $z_2 \in [z_0, z_1]$, such that $t(z_2) = 1$. This implies that

$$z_2 h(z_2) = 1 \text{ for some } z_2, z_0 \leq z_2 \leq z_1, \text{ that is } h(z) \in \Omega_{p0}.$$

Now, in view of (6.1) and $z_2h(z_2) = 1$, we have

$$\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) - \beta(1-\alpha)(1+\mu)z_2^{p+n+1}]\{\lambda a_{p+n} + (1-\lambda)b_{p+n}\}$$

$$= \lambda \sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) - \beta(1-\alpha)(1+\mu)z_0^{p+n+1}]a_{p+n}$$

$$+(1-\lambda)\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) - \beta(1-\alpha)(1+\mu)z_1^{p+n+1}]b_{p+n}$$

$$+\beta(1-\alpha)(1+\mu)\lambda \sum_{n=0}^{\infty}[z_2^{p+n+1} - z_0^{p+n+1}]a_{p+n}$$

$$+\beta(1-\alpha)(1+\mu)(1-\lambda)\sum_{n=0}^{\infty}[z_2^{p+n+1} - z_1^{p+n+1}]b_{p+n}$$

$$= \lambda \sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) + \beta(1-\alpha)(1+\mu)z_0^{p+n+1}]a_{p+n}$$

$$+(1-\lambda)\sum_{n=0}^{\infty}[(p+n)(1+\mu\beta) + \beta(1-\alpha)(1+\mu)z_1^{p+n+1}]b_{n+p}$$

$$\leq \lambda\beta(1-\alpha)(1+\mu) + (1-\lambda)\beta(1-\alpha)(1+\mu)$$

$$= \beta(1-\alpha)(1+\mu)$$

by Theorem 2, since $f(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ and $g(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_1)$. Hence we have $h(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_2)$, by Theorem 2. Since z_0, z_1 and z_2 are arbitrary, the family $\Omega^*_{p0}(\alpha, \beta, \mu, B)$ is convex.

Conversely, if B is not connected, then there exists z_0, z_1 and z_2 such that $z_0, z_1 \in B$ and $z_2 \notin B$ and $z_0 < z_2 < z_1$. Assume that $f(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_0)$ and $g(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_1)$ are not both equal to $1/z$. Then, for fixed z_2 and $0 \leq \lambda \leq 1$, we have from (6.1)

$$t(\lambda) = t(z_2, \lambda) = 1 + \lambda \sum_{n=0}^{\infty} a_{p+n}(z_2^{p+n+1} - z_0^{p+n+1}) + (1-\lambda)\sum_{n=0}^{\infty} b_{p+n}(z_2^{p+n+1} - z_1^{p+n+1}).$$

Since $t(z_2, 0) < 1$ and $t(z_2, 1) > 1$, there must exists; $\lambda_0, 0 < \lambda_0 < 1$, such that $t(z_2, \lambda_0) = 1$ or $z_2h(z_2) = 1$, where $h(z) = \lambda_0 f(z) + (1-\lambda_0)g(z)$. Thus $h(z) \in \Omega^*_{p0}(\alpha, \beta, \mu, z_2)$. From Lemma 1, we have $h(z) \notin \Omega^*_{p0}(\alpha, \beta, \mu, B)$. Since $z_2 \in B$ and $h(z) \neq z$. This implies that the family $\Omega^*_{p0}(\alpha, \beta, \mu, B)$ is not convex which is a contradiction.

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Department of Mathematics, Fergusson College, Pune 411001, India
E-mail address: drsr@mailijol.com

Flat No.7, Shivam Appt., Waranali Road, Vishrambag,
Sangli 416415, India
E-mail address: sayali_75@yahoo.com

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CONTINUITY AND SUPERSTABILITY OF JORDAN MAPPINGS

YOUNG WHAN LEE AND GWANG HUI KIM

Abstract. We show that every strong approximate one-to-one Jordan functional on an algebra is a Jordan functional and every approximate one-to-one Jordan functional on a Banach algebra is continuous.

1. **Introduction**

A linear mapping f from a normed algebra A into a normed algebra B is an ε-homomorphism if for every a, b in A

$$\|f(ab) - f(a)f(b)\| \leq \varepsilon \|a\| \|b\| .$$

In [7, Proposition 5.5], Jarosz proved that every ε-homomorphism from a Banach algebra into a continuous function space $C(S)$ is necessarily continuous, where S is a compact Hausdorff space. A Jordan functional on a Banach algebra A is a nonzero linear functional ϕ such that $\phi(a^2) = \phi(a)^2$ for every a in A. Every Jordan functional ϕ on A is multiplicative [2]. We are concerned with linear mappings f on Banach algebras which are approximate Jordan mappings. A linear mapping f from a normed algebra A into a normed algebra B is called an ε-approximate Jordan mapping if for all a in A

$$\left\|f(a^2) - f(a)^2\right\| \leq \varepsilon \|a\|^2 .$$

If B is the complex field, then f is called an ε-appoximate Jordan functional. For ε-appoximate mappings the reader is referred to [3],[4],[5],[6],[9],[10],[11].

A linear mapping f is a strong ε-approximate Jordan mapping if $||f(a^2) - f(a)^2|| < \varepsilon$. Also a continuous linear mapping f between normed algebras is an ε-near Jordan mapping if $\|f - J\| \leq \varepsilon$ for some continuous Jordan mapping J. In this paper, we prove that every strong ε-approximate one-to-one Jordan functional on an

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algebra is a Jordan functional and every ε-approximate one-to-one Jordan functional on a Banach algebra is continuous.

2. **Main Results**

Theorem 1. *If f is a strong ε-approximate one-to-one Jordan functional on an algebra A, then f is a Jordan functional. In particular if A is a Banach algebra, then f is continuous.*

Proof. Since, for every $x, y \in A$, $|f\left((x+y)^2\right) - f(x+y)^2| \le \varepsilon$, we have $|f(xy + yx) - 2f(x)f(y)| \le 3\varepsilon$. If x and y are commute, $|f(xy) - f(x)f(y)| \le \frac{3\varepsilon}{2}$. Now we use the method of the proof in [1]. Let $c(\varepsilon) = \frac{1+\sqrt{1+4\varepsilon}}{2}$. Note that $c(\varepsilon)^2 - c(\varepsilon) = \varepsilon$ and $c(\varepsilon) > 1$. Let $a \in A$. If $a \ne 0$ we may assume that $|f(a)| > c(\varepsilon)$ because $|f(ta)| > c(\varepsilon)$ for some $t \in R$ and $f((ta)^2) = f(ta)^2$ implies $f(a^2) = f(a)^2$. Say $|f(a)| = c(\varepsilon) + p$ for some $p > 0$. Then

$$\begin{aligned} |f(a^2)| &= |f(a)^2 - (f(a)^2 - f(a^2))| \ge |f(a)^2| - |(f(a)^2 - f(a^2))| \\ &\ge (c(\varepsilon)+p)^2 - \varepsilon > c(\varepsilon) + 2p. \end{aligned}$$

By induction, $|f(a^{2^n})| > c(\varepsilon) + (n+1)p$ for all $n = 1, 2, 3, \cdots$. For every $x, y, z \in A$ which they are commute, $|f(xyz) - f(xy)f(z)| \le \frac{3\varepsilon}{2}$ and $|f(xyz) - f(x)f(yz)| \le \frac{3\varepsilon}{2}$. So $|f(xy)f(z) - f(x)f(yz)| \le 3\varepsilon$. Hence

$$|f(xy)f(z) - f(x)f(y)f(z)|$$

$$\le |f(xy)f(z) - f(x)f(yz)| + |f(x)f(yz) - f(x)f(y)f(z)| \le 3\varepsilon + |f(x)|\frac{3\varepsilon}{2}.$$

By letting $x = a, y = a$ and $z = a^{2^n}$, we have

$$|f(a^2) - f(a)^2| \le \frac{3\varepsilon + |f(a)|\frac{3\varepsilon}{2}}{|f(a^{2^n})|}.$$

Letting $n \longrightarrow +\infty$ shows that $f(a^2) = f(a)^2$.

Theorem 2. *Let f be an ε-approximate Jordan functional on a normed algebra A with the multiplicative norm. Then for each $a \in A$, either $|f(a)| \le \frac{1+\sqrt{1+4\varepsilon}}{2}\|a\|$ or $f(a^2) = f(a)^2$.*

Proof. Let $a \in A$ and $c = \frac{a}{\|a\|}$. If $|f(a)| > \frac{1+\sqrt{1+4\varepsilon}}{2}\|a\|$ then $|f(c^{2^n})| > c(\varepsilon) + (n+1)p$ for all $n = 1, 2, 3$ and for some p, where $c(\varepsilon) = \frac{1+\sqrt{1+4\varepsilon}}{2}$, by the proof of Theorem 1.

For any natural number m, n,

$$|f(c^n c^m) - f(c^n)f(c^m)|$$

$$\leq |f((c^n + c^m)^2) - f(c^n + c^m)^2| + |f((c^n)^2) - f(c^n)^2| + |f((c^m)^2) - f(c^m)^2|$$

$$\leq \frac{\varepsilon}{2}\left(\|c^n + c^m\|^2 + \|c^n\|^2 + \|c^m\|^2\right) = 3\varepsilon.$$

Then we have

$$|f(c^2) - f(c)^2| \leq \frac{1}{|f(c^{2^n})|}\big(|f((c^2)f(c^{2^n}) - f(c^2 + c^{2^n})|$$

$$+|f(c^2 \cdot c^{2^n}) - f(c^2)f(c^{2^n})| + |f(c)||f(c \cdot c^{2^n}) - f(c)f(c^{2^n})|\big)$$

$$\leq \frac{6\varepsilon + 3|f(c)|\varepsilon}{|f(c^{2^n})|} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

This shows that $f(a^2) = f(a)^2$.

Corollary 3. *Let S be a compact Hausdorff space and $C(S)$ the set of all continuous complex valued functions. If f is an ε-approximate Jordan mapping from a Banach algebra A with the multiplicative norm into $C(S)$, then for each $a \in A$, either $\|f(a)\| \leq \frac{1+\sqrt{1+4\varepsilon}}{2}\|a\|$ or $f(a^2) = f(a)^2$.*

Proof. For every $x \in S$, we can define a linear functional $f_x : A \longrightarrow C$ by $f_x(a) = f(a)(x)$ for all $a \in A$. Then for every $a \in A$,

$$|f_x(a^2) - f_x(a)^2| \leq \left\|f(a^2) - f(a)^2\right\| \leq \varepsilon \|a\|^2.$$

By Theorem 2, either $\|f_x(a)\| \leq \frac{1+\sqrt{1+4\varepsilon}}{2}$ or $f_x(a^2) = f_x(a)^2$ for any $a \in A$. Then we complte the proof.

In Theorem 2 and Corollary 3 we used the assumption that an algebra A has the multiplicative norm. It is not known that whether they hold or not without such condition. With another condition we obtain the following theorem.

Theorem 4. *Let f be an ε-approximate Jordan functional on a Banach algebra A such that $f(a) = 0$ implies $f(a^2) = 0$ for each $a \in A$. Then f is continuous and $\|f\| \leq \frac{1+\sqrt{1+4\varepsilon}}{2}$.*

Proof. If A does not posses a unit, then we can extened f to $A \oplus (\lambda 1)$ by putting $f(a \bigoplus \lambda 1) = f(a) + \lambda$, and the extended f is still an ε -approximate Jordan functional. Thus without loss of generality we may assume that A has a unit. Suppose that f is discontinuous. Then the kernel $Ker(f)$ of f is a dense subset of A. Since the unit element 1 is the closure of $Ker(f)$, we can choose $c \in Ker(f)$ such that $\|c-1\| \leq \frac{1}{3}$. Then c is invertible, and $c^{-1} = 1 + \sum_{n=1}^{\infty}(1-c)^n$. And so $\left\|c^{-1}\right\| \leq \frac{1}{1-\|c-1\|} \leq \frac{3}{2}$. Let $b = \frac{c}{\|c\|} \in Ker(f)$. Then $b^{-1} = \|c\|\, c^{-1}$and $\left\|b^{-1}\right\| \leq 2$. Put $|f(b^{-1})| = \alpha$. Note that for every $x, y \in A$

$$|f(xy+yz) - 2f(x)f(y)| \leq |f((x+y)^2) - (f(x+y))^2|$$

$$+|f(x^2) - f(x)^2| + |f(y^2) - f(y)^2| \leq 2\varepsilon(\|x\|^2 + \|y\|^2 + \|x\|\,\|y\|).$$

If b^{-1} is not in $Ker(f)$, then for every a in A with $\|a\| = 1$,

$$\begin{aligned} |f(a)| &= \frac{1}{2\alpha}|2f(a)f(b^{-1})| \\ &\leq \frac{1}{2\alpha}(|2f(a)f(b^{-1}) - f(ab^{-1} + b^{-1}a)| \\ &+ |f(bb^{-1}ab^{-1} + b^{-1}ab^{-1}b) - 2f(b^{-1}ab^{-1})f(b)| \leq \frac{28\varepsilon}{\alpha}. \end{aligned}$$

Thus f is bounded and it is a contradiction. Therefore b^{-1} is in $Ker(f)$. By assumption, b^{-2} is in $Ker(f)$. Then for every a in A with $\|a\| = 1$,

$$|f(a)| = \frac{1}{2}(|f(a + b^{-1}ab)| + |f(a + bab^{-1})| + |f(b^{-1}ab + bab^{-1})|)$$

$$= \frac{1}{2}(|f(a + b^{-1}ab) - 2f(b^{-1}a)f(b)| + |f(a + bab^{-1}) - 2f(ab^{-1})f(b)|$$

$$+|f(b^{-1}ab + bab^{-1}) - 2f(bab)f(b^{-2})|) \leq 35\varepsilon.$$

Thus f is continuous. Since $|f(a^2) - f(a)^2| < \varepsilon$ for every $a \in A$ with $\|a\| = 1$, $|f(a^2)| - \varepsilon \leq |f(a^2)| \leq \|f\|$ and consequently $\|f\| \geq \|f\|^2 - \varepsilon$. This proves $\|f\| \leq \frac{1+\sqrt{1+4\varepsilon}}{2}$.

Corollary 5. *Every ε-approximate one-to-one Jordan functional on a Banach algebra is continuous and its norm is less than or equal to* $\frac{1+\sqrt{1+4\varepsilon}}{2}$.

Let f be an ε-near Jordan mapping from a Banach algebra A into a Banach algebra B. Then there exists a Jordan mapping J such that $\|f - J\| \leq \epsilon$. For every a in A,

$$\begin{aligned} \left\|f(a^2) - f(a)^2\right\| &\leq \left\|f(a^2) - J(a^2)\right\| + \left\|f(a)^2 - J(a)^2\right\| \\ &\leq \varepsilon\|a\|^2 + \|f(a) - J(a)\|\,\|f(a)\| + \|J(a)\|\,\|f(a) - J(a)\| \\ &\leq (\varepsilon + \varepsilon\|f\| + \varepsilon\|J\|)\,\|a\|^2. \end{aligned}$$

Therefore f is a $\varepsilon(1 + \|f\| + \|J\|)$-approximate Jordan mapping. We are concerned with it's converse. By the method of the proof in [8] we obtain the following theorem.

Theorem 6. *For every $\varepsilon > 0$ and $K > 0$, there exists a positive integer m such that every $\frac{\varepsilon}{m}$-approximate Jordan mapping with norm less than or equal to K on a finite demensional Banach algebra A is an ε- near Jordan mapping.*

Proof. Let $J(A)$ be the set of all bounded Jordan mapping on a finite dimensional Banach algebra $A, BL(A)$ the set of all bounded linear mappings on A, and let for each f in $BL(A)$

$$N(f) = \inf\{\|f - J\| : J \in J(A)\},$$

$$M = \{f \in BL(A) : N(f) \geq \varepsilon \text{ and } \|f\| \leq k\}$$

and

$$G_n = \left\{ f \in BL(A) : \sup_{\|a\| \leq 1} \left\|f(a^2) - f(a)^2\right\| \geq \frac{\varepsilon}{n} \right\}.$$

Since M is a closed and bounded subset of a finite dimensional space $BL(A)$, M is compact. Since G_n is open for each n and

$$M \subset BL(A) \setminus J(A) \subset \bigcup_{n=1}^{\infty} G_n,$$

there is m such that $M \subset G_m$. If $f \in BL(A) \setminus G_m$, then $f \in BL(A) \setminus M$. Therefore if f is an $\frac{\varepsilon}{m}$- approximate Jordan mapping then f is an ε- near Jordan mapping.

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Department of Mathematics, Taejon University, Taejon, 300-716, Korea
E-mail address: ywlee@dragon.taejon.ac.kr

Department of Mathematics, Kangnam University, Suwon, 449-702, Korea
E-mail address: ghkim@kns.kangnam.ac.kr

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RELATION BETWEEN THE PALAIS-SMALE CONDITION AND COERCIVENESS FOR MULTIVALUED MAPPINGS

MEZEI ILDIKÓ ILONA

Abstract. The aim of this paper is to study the coerciveness property of a class of multivalued mappings satisfying the Palais-Smale condition.

1. **Introduction**

Many papers has been devoted to show that the Palais-Smale condition implies the coerciveness. In the differentiable case this property is studied by L. Caklovici, S.Li, and M. Willem [2], for the locally Lipschitz functionals by Cs. Varga and V. Varga [11]. For the class of functions introduced by A. Szulkin [10], which is lower semicontinuous, this property has been proved by D. Goeleven in the paper [7]. For continuous functionals this result is proved by Fang [6]. These results are generalized by J.-N. Corvellec, see [4].

In a recent paper D. Motreanu and V.V. Motreanu [8] studied this problem for a class of functional of type $\Phi + \gamma$, where Φ is a locally Lipschitz function and γ is a proper, convex, lower semicontinuous functional.

In this paper we study the coerciveness of the function $\gamma + \sigma$, where σ is a locally Lipschitz function and γ is a convex lower semicontinuous function. The main tool used in the proof the coerciveness property is the classical Ekeland's variational principle [5].

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Key words and phrases. locally Lipschitz function, critical point, Palais-Smale condition, coercive.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and let $A : X \rightsquigarrow X$ be a multivalued map with $A(x) \neq \emptyset$, $\forall x \in X$, i.e $DomA = X$. Let X^* be the dual of X.

Definition 2.1 [1] $A : X \rightsquigarrow X$ is *Lipschitz around* $x \in X$ if there exists a positive constant l and a neighborhood U of x such that

$$\forall x_1, x_2 \in U,\ \|y_1 - y_2\| \leq l\|x_1 - x_2\|, \quad \forall y_1 \in A(x_1),\ y_2 \in A(x_2).$$

If A is Lipschitz around all $x \in X$, we say that A is *locally Lipschitz.*

Definition 2.2[9] *The generalized directional derivative* of the locally Lipschitz function $f : X \to \mathbb{R}$ at the point $x_0 \in X$ in the direction $h \in X$ is defined by

$$f^0(x_0, h) = \limsup_{x \to x_0\ t \searrow 0} \frac{f(x + th) - f(x)}{t}.$$

Let $p \in X^*$ such that $\|p\|_* < \infty$, where $\|p\|_* = sup\{\langle p, x\rangle : \|x\| \leq 1, x \in X\}$.

Lemma 2.1 *If* $A : X \rightsquigarrow X$ *is locally Lipschitz, then the function* $x \mapsto \sigma(A(x), p)$ *is locally Lipschitz, where*

$$\sigma(A(x), p) = sup\{\langle p, y\rangle : y \in A(x)\}, p \in X^*.$$

Proof. We consider an arbitrary $x_0 \in X$. Since A is locally Lipschitz, there exist $l > 0$ and an U neighborhood of x_0 such that:

$$\forall x_1, x_2 \in U, \forall y_1 \in A(x_1), y_2 \in A(x_2) : \|y_1 - y_2\| \leq l\|x_1 - x_2\|.$$

We can suppose that $\sigma(Ax_1, p) \geq \sigma(Ax_2, p)$. It's easy to verify that

$$0 \leq \sigma(Ax_1, p) - \sigma(Ax_2, p) \leq \sup_{y_1 \in Ax_1, y_2 \in Ax_2} \langle p, y_1 - y_2\rangle.$$

But

$$\sup_{y_i \in Ax_i} \langle p, y_1 - y_2\rangle = \sup_{y_i \in Ax_i} \langle p, \frac{y_1 - y_2}{\|y_1 - y_2\|}\|y_1 - y_2\|\rangle =$$

$$= \sup_{y_i \in Ax_i} \langle p, \frac{y_1 - y_2}{\|y_1 - y_2\|}\rangle \cdot \|y_1 - y_2\| \leq \|p\|_* \cdot l \cdot \|x_1 - x_2\|,$$

providing that $y_1 \neq y_2$ .The case $y_1 = y_2$ is trivial.

Therefore $x \mapsto \sigma(Ax, p)$ is locally Lipschitz. □

We consider an appropriate class of function as [9, chapter3].

Let $J : X \to R$ be a function given by

$$(H) \qquad J(x) = \psi(x) + \sigma(A(x), p),$$

where $\psi : X \to R$ is a convex lower semicontinuous function, $A : X \rightsquigarrow X$ is a locally Lipschitz multivalued map and $p \in X^*$.

Definition 2.3 A point $u \in X$ is said to be *critical point of* J *for* $p \in X^*$ if it satisfies the following variational inequality

$$\psi(v) - \psi(u) + (\sigma(A(\cdot), p))^0(u, v - u) \geq 0, \ \ \forall v \in X.$$

Definition 2.4 The function J satisfies the *Palais-Smale condition at level* c (briefly $(\mathrm{PS})_c$) if for each sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $\psi(v) - \psi(u_n) - (\sigma(A(\cdot), p))^0(u_n, v - u_n) \geq -\varepsilon_n \|v - u_n\|$, $\forall v \in X$, where $\varepsilon_n \to 0$, $\{u_n\}$ contains a convergent subsequence.

Definition 2.5 We say that J is *coercive*, if for $\|u\| \to \infty$ we have $J(u) \to \infty$.

As we said above our main tool is the Ekeland's principle, which we recall now.

Theorem 2.1 *Let X be a complete metric space and let $f : X \to (-\infty, \infty]$ be a lower semicontiuous function such that $inf_X f \in \mathbb{R}$. Let $\varepsilon > 0$ and $u \in X$ be given such that $f(u) \leq inf_X f + \varepsilon$. Then for every $\lambda > 0$, there exists an element $v \in X$, such that*

i) $f(v) < f(u)$;

ii) $f(v) < f(w) + \frac{\varepsilon}{\lambda} \cdot d(v, w)$, *for every* $w \neq v$;

iii) $d(u, v) \leq \lambda$.

3. Main result

Theorem 3.1 *Let X be a Banach space,J a bounded bellow function satisfying* (H) *and $p \in X^*$ such that $\|p\|_* < \infty$. Define*

$$c := \liminf_{\|u\| \to \infty} J(u).$$

Then, if $c \in \mathbb{R}$*, there exists a sequence* $\{v_n\} \subset X$ *such that:*

(i) $\|v_n\| \to \infty$;

(ii) $J(v_n) \to c$;

(iii) $\psi(v) - \psi(v_n) + (\sigma(A(\cdot), p))^0(v_n, v - v_n) \geq -\varepsilon_n \cdot \|v - v_n\|$, *where* $\varepsilon_n \to 0$, $\forall v \in X$.

Proof. From the definition of c there exists a sequence u_n such that $J(u_n) \leq c + \frac{1}{n}$ and $\|u_n\| \geq 2n$, for $n \in N \setminus \{0\}$ sufficiently large. Evidently J is lower semicontinuous and so we can apply the Theorem 2.1, with $f = J$, $\varepsilon = c + \frac{1}{n} - \inf_X J$ and $\lambda = n$.

Thus there exists $v_n \in X$ such that:

$$J(v_n) \leq J(u_n) \leq c + \frac{1}{n}; \tag{1}$$

$$J(w) > J(v_n) - \frac{1}{n}\big(c + \frac{1}{n} - inf_X J\big)\|v_n - w\|, \ \ \forall w \neq v_n;$$

$$\|u_n - v_n\| \leq n. \tag{2}$$

Thus, for each $w \in X$ we have

$$J(w) - J(v_n) \geq -\frac{1}{n}\big(c + \frac{1}{n} - inf_X J\big)\|w - v_n\|.$$

Let $w = (1-t)v_n + tv$, where v is fixed in X and $t \in [0,1]$. Replacing w in the last inequality we obtain

$$\psi(v_n + t(v - v_n)) - \psi(v_n) + \sigma(A((1-t)v_n + tv), p) - \sigma(A(v_n), p) \geq -\varepsilon_n t\|v - v_n\|,$$

where $\varepsilon_n = \big(c + \frac{1}{n} - \inf_X J\big)\frac{1}{n}$.

Since ψ is convex, we have

$$t(\psi(v) - \psi(v_n)) + \sigma(A((1-t)v_n + tv), p) - \sigma(A(v_n), p) \geq -\varepsilon_n t\|v - v_n\|.$$

Dividing this relation by t we get

$$\psi(v) - \psi(v_n) + \frac{1}{t}\Big[\sigma(A(v_n + t(v - v_n)), p) - \sigma(A(v_n), p)\Big] \geq -\varepsilon_n\|v - v_n\|. \tag{3}$$

Taking the limit as $t \searrow 0$ and using that

$$\sigma(A(\cdot,p))^0(v_n, v-v_n) = \limsup_{w_n \to v_n \ t \searrow 0} \frac{\sigma(A(w_n + t(v-v_n)),p) - \sigma(A(w_n),p)}{t} \geq$$

$$\geq \lim_{t \searrow 0} \frac{\sigma(A(v_n + t(v-v_n)),p) - \sigma(A(v_n),p)}{t}$$

we obtain

$$\psi(v) - \psi(v_n) + (\sigma(A(\cdot),p))^0 (v_n, v-v_n) \geq -\varepsilon_n \|v - v_n\|, \ \ \varepsilon_n \to 0,$$

$\forall v \in X$ i.e. exactly the (iii).

From (2) and (1) we have $\|v_n\| \geq \|u_n\| - \|u_n - v_n\| \geq 2n - n = n$, and $J(v_n) \to c$ respectively thus we have constructed a sequence such that (i), (ii) and (iii) are satisfied. □

Corollary 3.1 *Let X be a Banach space and let $J : X \to \mathbb{R}$ be a function of the form $J(x) = \psi(x) + \sigma(Ax,p)$, with $\|p\|_* < \infty$ satisfying* (H) *and the* (PS) *condition. If J is bounded bellow, then J is coercive.*

Proof. We proceed by contradiction. Assume that

$$c = \liminf_{\|u\| \to \infty} J(u) \in \mathbb{R}.$$

Then by the main theorem, there exists a sequence v_n such that $\|v_n\| \to \infty$, $J(v_n) \to c$ and $\psi(v) - \psi(v_n) + (\sigma(A(\cdot),p))^0 (v_n, v-v_n) \geq -\varepsilon_n \|v - v_n\|$, $\forall v \in X$, where $\varepsilon_n \to 0$. Since J satisfies the (PS) condition, we can choose a convergent subsequence of $\{v_n\}$, which is in contradiction with $\|v_n\| \to \infty$. □

Remark 3.1 The Corollary 3.1 generalize some results from the papers [2], [11], [7] and [8].

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Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania
E-mail address: ikulcsar@math.ubbcluj.ro

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COMMON FIXED POINT THEOREMS FOR MULTIVALUED OPERATORS ON COMPLETE METRIC SPACES

AUREL MUNTEAN

1. Introduction

The purpose of this paper is to prove a common fixed point theorem for multivalued operators defined on a complete metric space. Then, as consequences, we obtain some generalizations of several results proved in [6] for singlevalued operators.

For other results of this type see [1], [2], [3] and [5]. The metric conditions which appears in Theorem 3.1 generalize some conditions given in [6].

2. Preliminaries

Let X be a nonempty set. We denote:

$$P(X) := \{A \subset X \mid A \neq \varnothing\} \quad \text{and} \quad P_{cl}(X) := \{A \in P(X) \mid A = \bar{A}\}.$$

If (X, d) is a metric space, $B \in P(X)$ and $a \in A$, then

$$D(a, B) := \inf\{d(a, b) \mid b \in B\}.$$

Definition 2.1. If $T : X \multimap X$ is a multivalued operator, then an element $x \in X$ is a ***fixed point of T***, iff $x \in T(x)$.

We denote by $F_T := \{x \in X \mid x \in T(x)\}$ ***the fixed points set of T***.

Definition 2.2. Let $(T_n)_{n\in\mathbb{N}^*}$ be a sequence of multivalued operators $T_n : X \to P(X)$, $(\forall)\ n \in \mathbb{N}^*$. Then we denote by

$$Com(T) := \{x \in X \mid x \in T_n(x), \quad (\forall)\, n \in \mathbb{N}^*\} = \bigcap_{n\in\mathbb{N}^*} F_{T_n}$$

the ***common fixed points*** set of the sequence $(T_n)_{n\in\mathbb{N}^*}$.

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Lemma 2.3. (I.A.Rus [4]). *Let* $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+ \quad (k \in \mathbb{N}^*)$ *be a function and denote by* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$*, the mapping given by* $\psi(t) = \varphi(t, t, ..., t), \quad (\forall)\ t \in \mathbb{R}_+$.

Suppose that the following conditions are satisfied:

i) $(r \leqslant s, \quad r, s \in \mathbb{R}_+^k) \quad \Rightarrow \quad \varphi(r) \leqslant \varphi(s)$;

ii) φ *is upper semi-continuous;*

iii) $\psi(t) < t$*, for each* $t > 0$.

Then $\lim\limits_{n\to\infty} \psi^n(t) = 0$*, for each* $t \geqslant 0$.

In [6], T.Veerapandi and S.A.Kumar gave the following result:

Theorem 2.4. *Let* X *be a Hilbert space,* $Y \in P_{cl}(X)$ *and* $T_n : Y \to Y$*, for* $n \in \mathbb{N}$*, be a sequence of mappings.*

We suppose that at least one of the following conditions is satisfied:

i) there exist real numbers $a, b, c,$ *satisfying* $0 \leqslant a, b, c < 1$ *and* $a + 2b + 2c < 1$ *such that for each* $x, y \in Y$ *and* $x \neq y$,

$$\|T_i(x) - T_j(y)\|^2 \leqslant a \cdot \|x - y\|^2 + b\Big(\|x - T_i(x)\|^2 + \|y - T_j(y)\|^2\Big) +$$

$$+\frac{c}{2}\Big(\|x - T_j(y)\|^2 + \|y - T_i(x)\|^2\Big), \text{ for } i, j;$$

ii) there exist a real number h *satisfying* $0 \leqslant h < 1$ *such that for all* $x, y \in Y$ *and* $x \neq y$,

$$\|T_i(x) - T_j(y)\|^2 \leqslant h \cdot \max\Big\{\|x - y\|^2, \frac{1}{2}\Big(\|x - T_i(x)\|^2 + \|y - T_j(y)\|^2\Big),$$

$$\frac{1}{4}\Big(\|x - T_j(y)\|^2 + \|y - T_i(x)\|^2\Big)\Big\}, \text{ for } i, j.$$

Then, $(T_n)_{n\in\mathbb{N}^*}$ *has a unique common fixed point.*

3. The main results

The first result of this section improve and generalize Theorem 2.4 in the multivoque case.

Theorem 3.1. *Let (X, d) be a complete metric space and* $S, T : X \to P_{cl}(X)$ *multivalued operators.*

We suppose that there exists a function $\varphi : \mathbb{R}_+^3 \to \mathbb{R}_+$ *such that:*

i) $(r \leqslant s, \quad r, s \in \mathbb{R}_+^3) \quad \Rightarrow \quad \varphi(r) \leqslant \varphi(s)$;

ii) $\varphi(t, t, t) < t$ *for each* $t > 0$;

iii) φ *is continuous;*

iv) for each $x \in X$, *any* $u_x \in S(x)$ *and for all* $y \in X$, *there exists* $u_y \in T(y)$ *so that we have*

$$d^2(u_x, u_y) \leqslant \varphi\left(d^2(x,y)\,, \quad \frac{d^2(x,u_x)+d^2(y,u_y)}{2}\,, \quad \frac{d^2(x,u_y)+d^2(y,u_x)}{4}\right).$$

In these conditions, $F_S = F_T = \{x^*\}$.

Proof. Let $x_0 \in X$ arbitrarily. Then we can construct a sequence $(x_n) \subset X$ such that

$$\begin{cases} x_{2n+1} \in S(x_{2n}) \\ x_{2n+2} \in T(x_{2n+1}) \end{cases} \quad (\forall)\, n \in \mathbb{N}.$$

Denote by $d_n := d(x_n\,, x_{n+1})$, $n \in \mathbb{N}$. We have several steps in our proof.

Step I. Let us prove that the sequence (d_n) is monotone decreasing. Indeed, we have successively:

$$d^2_{2n+1} = d^2(x_{2n+1}, x_{2n+2}) \leqslant$$

$$\leqslant \varphi\Big(d^2(x_{2n}, x_{2n+1}), \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x_{2n+1}, x_{2n+2})}{2},$$

$$\frac{d^2(x_{2n}, x_{2n+2}) + d^2(x_{2n+1}, x_{2n+1})}{4}\Big) \leqslant$$

$$\leqslant \varphi\left(d^2_{2n}\,, \frac{d^2_{2n}+d^2_{2n+1}}{2}\,, \frac{(d_{2n}+d_{2n+1})^2}{4}\right) < \max\left\{d^2_{2n}\,, \frac{d^2_{2n}+d^2_{2n+1}}{2}\right\} = d^2_{2n},$$

from where it follows $d_{2n+1} < d_{2n}$. By an analogous method we have $d_{2n+2} < d_{2n+1}$.

Step II. We prove that $\lim\limits_{n\to\infty} d_n = 0$.

For this purpose, let us define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, by $\psi(t) = \varphi(t,t,t)$. Obviously, ψ is monotone increasing and $\psi(t) < t$, $(\forall)\, t > 0$.

By induction, we can prove that $d^2_n \leqslant \psi^n(d^2_0)$, $(\forall)\, n \geqslant 1$.

Indeed, we have

$$d^2_1 \leqslant \varphi\left(d^2_0\,, \frac{d^2_1+d^2_0}{2}\,, \frac{(d_0+d_1)^2}{4}\right) \leqslant \varphi(d^2_0,\ d^2_0,\ d^2_0) = \psi(d^2_0).$$

If inequality $d^2_{2n} \leqslant \psi^{2n}(d^2_0)$ is true, then we get successively:

$$d^2_{2n+1} \leqslant \varphi\left(d^2_{2n}\,, \frac{d^2_{2n}+d^2_{2n+1}}{2}\,, \frac{(d_{2n}+d_{2n+1})^2}{4}\right) \leqslant \varphi(d^2_{2n}, d^2_{2n}, d^2_{2n}) = \psi(d^2_{2n}) \leqslant$$

$$\leq \psi(\psi^{2n}(d^2_0)) = \psi^{2n+1}(d^2_0).$$

By passing to limit as $n \to \infty$, if $d_0 > 0$ it follows

$$\lim_{n\to\infty} d^2_n \leqslant \lim_{n\to\infty} \psi^n(d^2_0) = 0, \quad \text{and hence} \quad \lim_{n\to\infty} d_n = 0.$$

For $d_0 = 0$, the sequence (d_n) being decreasing it is obviously that $\lim\limits_{n\to\infty} d_n = 0$.

Step III. We'll prove that the sequence (x_n) is Cauchy in X, i.e. for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for each $m, n \geqslant k$, $d(x_m, x_n) < \varepsilon$.

Suppose, by contradiction, that (x_{2n}) is not Cauchy sequence. Then, there exists $\varepsilon > 0$ such that for each $2k \in \mathbb{N}$ there exist $2m_k, 2n_k \in \mathbb{N}$, $2m_k > 2n_k \geqslant 2k$, with the property $d(x_{2m_k}, x_{2n_k}) > \varepsilon$.

In what follows, let us suppose the numbers $2m(k)$ and $2n(k)$ as follows:

$$2m(k) := \inf\{2m_k \in \mathbb{N} \mid 2m_k > 2n_k \geqslant 2k,\ d(x_{2n_k}, x_{2m_k-2}) \leqslant \varepsilon,\ d(x_{2n_k}, x_{2m_k}) > \varepsilon\}$$

and $2n(k) := 2n_k$. Then, $(\forall)\, 2k \in \mathbb{N}$ we have:

$$\varepsilon < d(x_{2n(k)}, x_{2m(k)}) \leqslant d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) +$$
$$+ d(x_{2m(k)-1}, x_{2m(k)}).$$

Using step II, we deduce that

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}) = \varepsilon. \tag{1}$$

From the triangle inquality, we get:

$$|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leqslant d(x_{2m(k)-1}, x_{2m(k)})$$

and

$$|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leqslant d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2n(k)}, x_{2n(k)+1}).$$

Using again step III and the relation (1), it follows

$$\begin{cases} \lim\limits_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon \\ \lim\limits_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)-1}) = \varepsilon. \end{cases} \tag{2}$$

Then, we have successively:

$$d(x_{2n(k)}, x_{2m(k)}) \leqslant d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}) \leqslant d(x_{2n(k)}, x_{2n(k)+1}) +$$
$$+ \Big[\varphi(d^2(x_{2n(k)}, x_{2m(k)-1}),\ \frac{d^2(x_{2n(k)}, x_{2n(k)+1}) + d^2(x_{2m(k)-1}, x_{2m(k)})}{2},$$

$$\left.\frac{d^2(x_{2n(k)}, x_{2m(k)}) + d^2(x_{2m(k)-1}, x_{2n(k)+1})}{4}\right]^{\frac{1}{2}}.$$

Because φ is continuous, passing to the limit as $k \to \infty$, we have:

$$\varepsilon \leqslant \left[\varphi\left(\varepsilon^2,\, 0,\, \frac{\varepsilon^2}{2}\right)\right]^{\frac{1}{2}} \leqslant \left[\psi(\varepsilon^2)\right]^{\frac{1}{2}} < \varepsilon, \quad \text{a contradiction.}$$

Step IV. We prove that $F_T \neq \varnothing$.

Because (x_n) is Cauchy sequence in the complete metric space (X, d) we obtain that there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$.

From $x_{2n+1} \in S(x_{2n})$ we have that there exists $u_n \in T(x^*)$ such that:

$$d^2(x_{2n+1}, u_n) \leqslant$$

$$\leq \varphi\left(d^2(x_{2n}, x^*)\,,\ \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x^*, u_n)}{2}\,,\ \frac{d^2(x_{2n}, u_n) + d^2(x^*, x_{2n+1})}{4}\right) <$$

$$< \max\left\{d^2(x_{2n}, x^*)\,,\ \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x^*, u_n)}{2}\,,\ \frac{d^2(x_{2n}, u_n) + d^2(x^*, x_{2n+1})}{4}\right\}$$

$$:= M.$$

Consequently, we have the following situations:

a. Case $M = d^2(x_{2n}, x^*)$. In this case, we have

$$d^2(x_{2n+1}, u_n) \leqslant d^2(x_{2n}, x^*),$$

from where

$$\lim_{n\to\infty} d(x_{2n+1}, u_n) \leqslant \lim_{n\to\infty} d(x_{2n}, x^*) = 0,$$

i.e.

$$\lim_{n\to\infty} d(x_{2n+1}, u_n) = 0.$$

b. Case $M = \dfrac{d^2(x_{2n}, x_{2n+1}) + d^2(x^*, u_n)}{2}$. We deduce successively:

$$d^2(x_{2n+1}, u_n) \leqslant \frac{d^2(x_{2n}, x_{2n+1}) + d^2(x^*, u_n)}{2} \leqslant$$

$$\leq \frac{d^2(x_{2n}, x_{2n+1}) + [d(x^*, x_{2n+1}) + d(x_{2n+1}, u_n)]^2}{2},$$

i.e. $d^2(x_{2n+1}, u_n) - 2\cdot d(x^*, x_{2n+1})\cdot d(x_{2n+1}, u_n) - [d^2(x_{2n}, x_{2n+1}) + d^2(x^*, x_{2n+1})] \leqslant 0$, therefore

$$d(x_{2n+1}, u_n) \leqslant d(x^*, x_{2n+1}) + \sqrt{2\cdot d^2(x^*, x_{2n+1}) + d^2(x_{2n}, x_{2n+1})}.$$

Passing to the limit in this inequality, as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_{2n+1}, u_n) = 0.$$

c. Case $M = \dfrac{d^2(x_{2n}, u_n) + d^2(x^*, x_{2n+1})}{4}$. In this case, from the inequality

$$d^2(x_{2n+1}, u_n) \leqslant \frac{d^2(x_{2n}, u_n) + d^2(x^*, x_{2n+1})}{4},$$

we have, again,

$$\lim_{n\to\infty} d(x_{2n+1}, u_n) = 0.$$

Passing to the limit, as $n \to \infty$, in inequality

$$d(x^*, u_n) \leqslant d(x^*, x_{2n+1}) + d(x_{2n+1}, u_n),$$

on the basis of the limit $\lim\limits_{n\to\infty} d(x_{2n+1}, u_n) = 0$, we obtain $d(x^*, u_n) \to 0$ as $n \to 0$.

Since $u_n \in T(x^*)$, $(\forall)\, n \in \mathbb{N}$ and $T(x^*)$ is a closed set, it follows that $x^* \in T(x^*)$, i.e. $x^* \in F_T$.

Step V. We'll obtain, now, the conclusion of our theorem. We first prove that $F_S \subset F_T$.

Let $x^* \in F_S$. From $x^* \in S(x^*)$ we have that there exists $u \in T(x^*)$ such that

$$d^2(x^*, u) \leqslant \varphi\left(d^2(x^*, x^*)\,,\ \frac{d^2(x^*, x^*) + d^2(x^*, u)}{2}\,,\ \frac{d^2(x^*, u) + d^2(x^*, x^*)}{4}\right).$$

If we suppose that $d(x^*, u) > 0$, then we obtain

$$d^2(x^*, u) \leqslant \varphi\left(0,\ \frac{d^2(x^*, u)}{2}\,,\ \frac{d^2(x^*, u)}{4}\right) < \frac{d^2(x^*, u)}{2},$$

a contradiction. Thus, $d(x^*, u) = 0$, which means that $u = x^*$. It follows that $x^* \in T(x^*)$ and so $F_S \subset F_T$.

We shall prove now the equality $F_S = F_T$ betwen the fixed points set for S and T.

If we assume that there exists $y^* \in F_T$ such that $y^* \neq x^* \in F_S$, then we have

$$d^2(x^*, y^*) \leqslant \varphi\left(d^2(x^*, y^*)\,,\ \frac{d^2(x^*, x^*) + d^2(y^*, y^*)}{2}\,,\ \frac{d^2(x^*, y^*) + d^2(y^*, x^*)}{4}\right) =$$

$$= \varphi\left(d^2(x^*, y^*)\,,\ 0\,,\ \frac{d^2(x^*, y^*)}{2}\right) \leqslant \psi(d^2(x^*, y^*)) < d^2(x^*, y^*),$$

a contradiction, proving the fact that $F_S = F_T \in P(X)$.

In fact, we have obtained, even more, namely that $F_S = F_T = \{x^*\}$. $\square$

Corollary 3.2. *Let (X, d) be a complete metric space and* $S, T : X \to P_{cl}(X)$ *multivalued operators .*

We suppose that there exist $a, b, c \in \mathbb{R}_+$, $a + 2b + 2c < 1$, *such that for each* $x \in X$, *each* $u_x \in S(x)$ *and for all* $y \in X$, *there exists* $u_y \in T(y)$ *so that we have*

$$d^2(u_x, u_y) \leqslant a \cdot d^2(x, y) + b \cdot [d^2(x, u_x) + d^2(y, u_y)] + \frac{c}{2} \cdot [d^2(x, u_y) + d^2(y, u_x)].$$

Then, $F_S = F_T = \{x^*\}$.

Proof. Applying Theorem 3.1 for the function $\varphi : \mathbb{R}_+^3 \to \mathbb{R}_+$, $\varphi(t_1, t_2, t_3) = at_1 + 2bt_2 + 2ct_3$, which satisfies the conditions i), ii) and iii) of this theorem, we obtain the conclusion. $\square$

Remark 3.3. If T and S are singlevalued operators, then Corollary 3.2 is Theorem 3 from [6].

Corollary 3.4. *Let (X, d) be a complete metric space and* $S, T : X \to P_{cl}(X)$ *multivalued operators.*

We suppose that there exists $h \in]0, 1[$ *such that for each* $x \in X$, *any* $u_x \in S(x)$ *and for all* $y \in X$, *there exists* $u_y \in T(y)$ *so that we have*

$$d^2(u_x, u_y) \leqslant h \cdot \max\left\{d^2(x, y)\,,\ \frac{d^2(x, u_x) + d^2(y, u_y)}{2}\,,\ \frac{d^2(x, u_y) + d^2(y, u_x)}{4}\right\}.$$

In these conditions, $F_S = F_T = \{x^*\}$.

Proof. We apply Theorem 3.1 for the function $\varphi : \mathbb{R}_+^3 \to \mathbb{R}_+$, $\varphi(t_1, t_2, t_3) = h \cdot \max\{t_1, t_2, t_3\}$, which satisfies the conditions i), ii) and iii) of this theorem. $\square$

Remark 3.5. Corollary 3.4 is a generalization for multivalued operators of Theorem 4 from [6], theorem proved for singlevalued operators in Hilbert spaces.

Remark 3.6. Let (X, d) be a complete metric space and $(T_n)_{n \in \mathbb{N}}$ be a sequence of multivalued operators $T_n : X \to P_{cl}(X)$, $(\forall)\, n \in \mathbb{N}$.

If each pair of multivalued operators (T_0, T_n), for $n \in \mathbb{N}^*$, satisfies similar conditions as in Theorem 3.1, then $F_{T_n} = F_{T_0} = \{x^*\}$, for all $n \in \mathbb{N}^*$.

We next give a generalization of Theorem 1 of N.Negoescu [2].

Theorem 3.7. *Let (X, d) be a compact metric space,* $S, T : X \to P_{cl}(X)$ *and* $\varphi : \mathbb{R}^3_+ \to \mathbb{R}_+$. *Suppose that the following conditions are satisfies:*

i) $(r \leqslant s;\ r, s \in \mathbb{R}^3_+) \quad \Rightarrow \quad \varphi(r) \leqslant \varphi(s)$;

ii) $\varphi(t, t, t) < t, \quad (\forall)\, t > 0$;

iii) *S or T be continuous;*

iv) $d^2(u_x, u_y) < \varphi\Big(d^2(x, y)\,,\ d(x, u_x)\cdot d(y, u_y)\,,\ d(x, u_y)\cdot d(y, u_x)\Big)$, *for all* $x, y \in X,\ x \neq y$ *and for all* $(u_x, u_y) \in S(x) \times T(x)$.

In these conditions:

a. *S or T has a strict fixed point;*

b. *if both S and T have such fixed points, then the pair (S, T) has a common fixed point.*

Proof. **a.** Let S be continuous and we consider the function $f(x) := D(x, S(x))$. Because f is continuous on X, it follows that f takes its minimum value, i.e. there exists $x_0 \in X$ such that $f(x_0) = \inf\{f(x) \mid x \in X\}$.

We prove that x_0 is a fixed point of S or some $x_1 \in S(x_0)$ is a fixed point of T.

Indeed, we choose:

$x_1 \in S(x_0)$ be such that $d(x_0, x_1) = D(x_0, S(x_0))$;

$x_2 \in T(x_1)$ be such that $d(x_1, x_2) = D(x_1, T(x_1))$;

$x_3 \in S(x_2)$ be such that $d(x_2, x_3) = D(x_2, S(x_2))$.

We shall prove that $D(x_0, S(x_0)) = 0$ or $D(x_1, T(x_1)) = 0$, i.e. $x_0 \in S(x_0)$ or $x_1 \in T(x_1)$. We suppose that $D(x_0, S(x_0)) > 0$ and $D(x_1, T(x_1)) > 0$. Hence, using the inequality iv), we have:

$$d^2(x_1, x_2) < \varphi\Big(d^2(x_0, x_1)\,,\ d(x_0, x_1)\cdot d(x_1, x_2)\,,\ d(x_0, x_2)\cdot d(x_1, x_1)\Big) \leqslant$$

$$\leqslant \max\Big\{d^2(x_0, x_1)\,,\ d(x_0, x_1)\cdot d(x_1, x_2)\Big\} := M.$$

Consequently, we distinguish the following situations:

I. Case $M = d^2(x_0, x_1)$**.** In this case, we deduce $d(x_1, x_2) < d(x_0, x_1)$.

II. Case $M = d(x_0, x_1)\cdot d(x_1, x_2)$**.** In this case, we have $d^2(x_1, x_2) < d(x_0, x_1)\cdot d(x_1, x_2)$. Since $d(x_1, x_2) = D(x_1, T(x_1)) > 0$, it follows that $d(x_1, x_2) <$

$d(x_0, x_1)$. Now,

$$d^2(x_3, x_2) < \varphi\bigg(d^2(x_2, x_1)\,,\ d(x_2, x_3)\cdot d(x_1, x_2)\,,\ d(x_2, x_2)\cdot d(x_1, x_3)\bigg) \leqslant$$

$$\leqslant \max\bigg\{d^2(x_1, x_2)\,,\ d(x_2, x_3))\cdot d(x_1, x_2)\bigg\}.$$

Analogously, it follows that $d^2(x_2, x_3) < d^2(x_1, x_2)$ or $d^2(x_2, x_3) < d(x_2, x_3)\cdot d(x_1, x_2)$.

In the second situations, if $d(x_2, x_3) = 0$, we obtain a contradiction. Thus, it follows that $d(x_2, x_3) < d(x_1, x_2)$.

Similarly, we deduce successively:

$$D(x_2, S(x_2)) = d(x_2, x_3) < d(x_1, x_2) < d(x_0, x_1) = f(x_0),$$

which contradict the minimality of $f(x_0)$. Therefore, $D(x_0, S(x_0)) = 0$ or $D(x_1, T(x_1)) = 0$. So, $x_0 \in S(x_0)$ or $x_1 \in T(x_1)$.

b. We assume that there exist $u \in S(u)$ and $v \in T(v)$, such that $u \neq v$. Then, using the hypothesis iv) we get, again, a contradiction:

$$d^2(u, v) < \varphi\bigg(d^2(u, v)\,,\ d(u, u)\cdot d(v, v)\,,\ d^2(u, v)\bigg) \leqslant d^2(u, v).$$

So, u=v, meaning that u is a common fixed point of S and T. □

Remark 3.8. If $\varphi : \mathbb{R}^3_+ \to \mathbb{R}_+$, $\varphi(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$, from Theorem 3.7, we get a result of Negoescu [2, Theorem 1].

Remark 3.9. We note that Theorem 3.7 is true for $S = T : X \to P_{cl}(X)$.

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Carol I High School Sibiu, 2400 Sibiu, ROMANIA
E-mail address: aurelmuntean@yahoo.com

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THE φ-CATEGORY OF SOME PAIRS OF PRODUCTS OF MANIFOLDS

CORNEL PINTEA

Abstract. In this paper we will show that in certain topological conditions on the manifold M, the φ-category of the pairs

$(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}), (P_n(\mathbf{R}) \times M, T^a \times S^{m-a})$

is infinite for suitable choices of the numbers m, n, a.

1. Introduction

Let us first recall that the φ-category of a pair (M, N) of smooth manifolds is defined as

$$\varphi(M, N) = \min\{\#C(f) \mid f \in C^\infty(M, N)\},$$

where $C(f)$ denotes the critical set of the smooth mapping $f : M \to N$ and $\#C(f)$ its cardinality. For more details, see for instance [AnPi].

In the previous papers [Pi1], [Pi3] is studied the φ-category of the pairs $(P_n(\mathbf{R}), \mathbf{R}^m), (P_n(\mathbf{R}), S^m), (P_n(\mathbf{R}), T^a \times \mathbf{R}^{m-a})$ and is proved that it is infinite for suitable choices of the numbers m, n, a.

Using those results as well as some others, in this paper we will show the same think for some pairs of the form $(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}), (P_n(\mathbf{R}) \times M, T^a \times S^{m-a})$.

2. Some useful results

In this section we will recall some results proved in some various previous papers and which we are going to use in the next sections.

Theorem 2.1. ([Pi1]) *Let M, N be compact connected differentiable manifolds having the same dimension m. In these conditions the following statements are true:*

(i) If $m \geq 3$ and $\pi_1(N)$ has no subgroup isomorphic with $\pi_1(M)$, then $\varphi(M, N) \geq \aleph_0$;

(ii) If $m \geq 4$ and $\pi_q(M) \not\cong \pi_q(N)$ for some $q \in \{2, 3, \ldots, m-2\}$, then $\varphi(M, N) \geq \aleph_0$.

If G, H are two groups, then the *algebraic φ- category*of the pair (G, H) is defined as

$$\varphi_{alg}(G,H) = \min\{[H : \operatorname{Im} f] \mid f \in \operatorname{Hom}(G,H)\}.$$

Recall that for an abelian group G the subset $t(G)$ of all elements of finite order forms a subgroup of G called the *torsion subgroup.*

Proposition 2.2. ([Pi2]) *If G,H are finitely generated abelian groups such that* rank*[G/t(G)]*<rank*[H/t(H)], then* $\varphi_{alg}(G,H) \geq \aleph_0$

Theorem 2.3. ([Pi2]) *Let M^m, N^n be compact connected differential manifolds such that $m \geq n \geq 2$. If $\varphi_{alg}(\pi(M), \pi(N)) \geq \aleph_0$, then $\varphi(M,N) \geq \aleph_0$.*

Theorem 2.4. ([Pi3]) *If M is a smooth manifold and n is a natural number such that* $\dim M < n$*, then $\varphi(M, \mathbf{R}^n) = \varphi(M, S^n)$.*

Theorem 2.5. ([Pi3]) *If n is a natural number such that n+1 and n+2 are not powers of 2, then we have*

$$\varphi(P_n(\mathbf{R}), S^m) = \varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0 \quad \textit{if } n < m \leq 2^{[log_2 n]+1} - 2$$
$$\varphi(P_n(\mathbf{R}), S^m) = \varphi(P_n(\mathbf{R}), \mathbf{R}^m) = 0 \quad \text{if } m \geq 2n-1.$$

Theorem 2.6. ([Pi3]) *If M^n, N^n, P are differentiable manifolds such that $\pi(P)$ is a torsion group and $\pi(N)$ is a free torsion group and $p : M \to N$ is a differentiable covering mapping, then $\varphi(P, M) = \varphi(P, N)$.*

Corollary 2.7. ([Pi3]) *If M is a differentiable mapping such that $\pi(M)$ is a torsion group, then $\varphi(M, \mathbf{R}^n) = \varphi(M, T^a \times \mathbf{R}^{n-a})$, for any $a \in \{1, \ldots, n-1\}$. In particular, for $a = n$ we get that $\varphi(M, \mathbf{R}^n) = \varphi(M, T^n)$.*

Theorem 2.8. ([Pi4]) *If n is a natural number such that n+1 and n+2 are not powers of 2, then we have*

$$\varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0 \quad \textit{if } n < m \leq 2^{[log_2 n]+1} - 2$$
$$\varphi(P_n(\mathbf{R}), \mathbf{R}^m) = 0 \quad \text{if } m \geq 2n-1.$$

3. Main results

In this section we will see the announced topological conditions on the manifold M in order that the φ-category of the pairs $(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}), (P_n(\mathbf{R}) \times M, T^a \times S^{m-a})$ to be infinite.

Theorem 3.1. *If M,N,P are differentiable manifolds such that* $\dim M \leq \dim N \leq \dim P$ *and M is injectively immersable in N, then* $\varphi(M,P) \leq \varphi(N,P)$.

Proof. Let $j : M \to N$ be an injective immersion and $f : N \to P$ be a differential mapping. Recall that if $\alpha : X \to Y$ is a morphism of vector spaces (linear mapping) then $\dim X = \dim Ker\alpha + \dim Im\alpha$. Further on we have successively:

$$x \in C(f \circ j) \Leftrightarrow rank_x(f \circ j) < \dim M \Leftrightarrow \dim Imd(f \circ j)_x < \dim M \Leftrightarrow$$

$$\Leftrightarrow \dim M - \dim Kerd(f \circ j)_x < \dim M \Leftrightarrow \dim Ker[(df)_{j(x)} \circ (dj)_x] > 0 \Rightarrow$$

$$\Rightarrow \dim Ker(df)_{j(x)} > 0 \Leftrightarrow \dim N - \dim Im(df)_{j(x)} > 0 \Leftrightarrow$$

$$\Leftrightarrow \dim Im(df)_{j(x)} < \dim N \Leftrightarrow rank_{j(x)} f < \dim N \Leftrightarrow j(x) \in C(f).$$

Therefore we showed that $j[C(f \circ j)] \subseteq C(f)$, which implies that

$$\#C(f \circ j) = \#j[C(f \circ j)] \leq \#C(f),$$

that is $\varphi(M,P) \leq \#C(f \circ j) \leq \#C(f)$. The last inequalities holds for any differential mapping $f : N \to P$, which means that

$$\varphi(M,P) \leq \varphi(N,P).\Box$$

Theorem 3.2. *If n is a natural number such that n+1, n+2 are not powers of 2 and M is a differential manifold such that* $\pi(M)$ *is a torsion group, then we have*

(i) $\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) \geq \aleph_0$ *if* $n + \dim M \leq m \leq 2^{[log_2 n]+1} - 2$, $\forall\, a \in \{1, \dots, m-1\}$*;*

(ii) $\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) = 0$ *if* $m \geq 2(n + \dim M)$ *and M is a compact manifold.*

Proof. (i) First of all observe that, according to corollary 2.7, $\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m)$. Because $P_n(\mathbf{R})$ can be embedded in $P_n(\mathbf{R}) \times M$ it follows, according to theorem 3.1, that $\varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \varphi(P_n(\mathbf{R}), \mathbf{R}^m)$. But in the given hypothesis we get that $\varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0$, because of theorem 2.8, that is we have

$$\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0.$$

(ii) Follows easily from the equality $\varphi(P_n(\mathbf{R}) \times M, T^a \times \mathbf{R}^{m-a}) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m)$ and from the Whitney's embedding theorem.$\Box$

Theorem 3.3. *If n is a natural number such that $n+1$, $n+2$ are not powers of 2 and M is a differential manifold, then we have*

(*i*) $\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \aleph_0$
if $n + \dim M < m \leq 2^{[log_2 n]+1} - 2$;

(*ii*) $\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) = 0$ *if* $m \geq 2(n + \dim M)$

and M is a compact manifold.

Proof. (*i*) First of all observe that, according to theorem 2.1, $\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m)$. Because $P_n(\mathbf{R})$ can be embedded in $P_n(\mathbf{R}) \times M$ it follows, according to 3.1, that $\varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \varphi(P_n(\mathbf{R}), \mathbf{R}^m)$. But in the given hypothesis we get that $\varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0$, because of theorem 2.8, that is we have

$$\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m) \geq \varphi(P_n(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0.$$

(*ii*) Follows easily from the equality $\varphi(P_n(\mathbf{R}) \times M, S^m) = \varphi(P_n(\mathbf{R}) \times M, \mathbf{R}^m)$ and from the Whitney's embedding theorem.□

Theorem 3.4. *If $m \geq 3, n \geq 2$ are natural numbers and M is a compact connected differentiable manifold such that $n + \dim M = m$, then*

(*i*) $\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0$; $\forall\, a \in \{1, \ldots, m-2\}$
(*ii*) $\varphi(P_n(\mathbf{R}) \times M, T^m) \geq \aleph_0$
(*iii*) $\varphi(P_n(\mathbf{R}) \times M, S^m) \geq \aleph_0$ *if* $m \geq 4$.

Proof. (*i*) Because $\pi(T^a \times S^{m-a}) \simeq \pi(T^a) \times \pi(S^{m-a}) \simeq \underbrace{(\mathbf{Z} \times \cdots \times \mathbf{Z})}_{\text{a times}}$, it follows that $\pi(T^a \times S^{m-a})$ has no subgroup isomorphic with $\pi(P_n(\mathbf{R}) \times M) \simeq \pi(P_n(\mathbf{R})) \times \pi(M) \simeq \mathbf{Z}_2 \times \pi(M)$. Therefore, according to theorem 2.1 (*i*), it follows that $\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0$. The inequality $\varphi(P_n(\mathbf{R}) \times M, T^m) \geq \aleph_0$ can be proved in the same way.

(*iii*) Because $\pi_n(P_n(\mathbf{R}) \times M) \simeq \pi_n(P_n(\mathbf{R})) \times \pi_n(M) \simeq \pi_n(S^n) \times \pi_n(M) \simeq \mathbf{Z} \times \pi_n(M)$ and $\pi_n(S^m) = 0$ it follows that $\pi_n(P_n(\mathbf{R}) \times M) \not\simeq \pi_n(S^m)$. Therefore, according to theorem 2.1 (*ii*), it follows that $\varphi(P_n(\mathbf{R}) \times M, S^m) \geq \aleph_0$.□

Theorem 3.5. *If m,n are natural numbers and M a compact connected differential manifold such that $n + \dim M \geq m \geq 2$ and $\pi(M)$ is a torsion group, then*

$$\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0,\ \forall\, a \in \{1, \ldots, m-1\}.$$

Proof. Because $\pi(P_n(\mathbf{R}) \simeq \mathbf{Z}_2$ and $\pi(M)$ are torsion groups, it follows that $\pi\big(P_n(\mathbf{R}) \times M\big) \simeq \pi(P_n(\mathbf{R}) \times \pi(M) \simeq \mathbf{Z}_2 \times \pi(M)$ is a torsion group too. Because $\pi(T^a \times S^{m-a}) \simeq \pi(T^a) \times \pi(S^{m-a}) \simeq \underbrace{(\mathbf{Z} \times \cdots \times \mathbf{Z})}_{\text{a times}} \times \pi(S^{m-a})$ is a free torsion group, it follows that $\mathrm{Hom}\Big(\pi\big(P_n(\mathbf{R}) \times M\big), \pi(T^a \times S^{m-a})\Big) = 0$, that is

$$\varphi_{alg}\Big(\pi\big(P_n(\mathbf{R}) \times M\big), \pi(T^a \times S^{m-a})\Big) \geq \aleph_0,$$

which means, according to theorem 2.3, that

$$\varphi\big(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}\big) \geq \aleph_0.\square$$

Theorem 3.6. *If m,n are natural numbers and M a compact connected differential manifold such that $n + \dim M \geq m \geq 2$ and $\pi(M)$ is a free abelian group with rank$\pi(M) < m - 1$, then*

$$\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0 \; \forall\, a \in \{\mathrm{rank}\, \pi(M) + 1, \ldots, m - 1\}.$$

Proof. Because $\pi\big(P_n(\mathbf{R}) \times M\big) \simeq \mathbf{Z}_2 \times \pi(M)$ it follows that

$$\frac{\pi\big(P_n(\mathbf{R}) \times M\big)}{t\Big(\pi\big(P_n(\mathbf{R}) \times M\big)\Big)} \simeq \pi(M).$$

Therefore rank$\dfrac{\pi\big(P_n(\mathbf{R})\times M\big)}{t\Big(\pi\big(P_n(\mathbf{R})\times M\big)\Big)}$ =rank $\pi(M) < a$ =rank $\pi(T^a \times S^{m-a})$. Using proposition 2.2, it follows that $\varphi_{alg}(\pi\big(P_n(\mathbf{R}) \times M\big), \pi(T^a \times S^{m-a})) \geq \aleph_0$, that is, according to theorem 2.3, one can conclude that $\varphi(P_n(\mathbf{R}) \times M, T^a \times S^{m-a}) \geq \aleph_0.\square$

4. **Applications**

Example 4.1. *Let $n_1, \ldots, n_p$ be natural numbers such that $n_i + 1, n_i + 2$ are not powers of 2, for some $i \in \{1, \ldots, p\}$.*

(i) If $n_1 + \ldots + n_p < m \leq 2^{[\log_2 n_i]+1} - 2$, then

$$\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), T^a \times \mathbf{R}^{m-a}) \geq \aleph_0 \; (\forall)\, a \in \{1, \ldots, m - 1\} \textit{ and}$$

$$\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), S^m) = \varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), \mathbf{R}^m) \geq \aleph_0$$

(ii) If $m \geq 2(n_1 + \ldots + n_p)$, then

$$\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), T^a \times \mathbf{R}^{m-a}) = 0 \; \forall\, a \in \{1, \ldots, m - 1\},$$

and $\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), S^m) = \varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), \mathbf{R}^m) = 0$

Proof. It is enough to take in the theorems 3.2, 3.3

$$M = P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_{i-1}}(\mathbf{R}) \times P_{n_{i+1}}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}).\square$$

Example 4.2. (i) *If* $m, n_1, \ldots, n_p \geq 2$ *are natural numbers such that* $n_1 + \ldots + n_p \geq m \geq 2$, *then* $\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}), T^a \times S^{m-a}) \geq \aleph_0$, $(\forall)\, a \in \{1, \ldots, m-1\}$.

(ii) *If* $a, b, m, n_1 \ldots n_p \geq 2$ *are natural numbers such that* $a < b$ *and* $a + n_1 + \ldots + n_p \geq m \geq 2$, *then* $\varphi(P_{n_1}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}) \times T^a, T^b \times S^{m-b}) \geq \aleph_0$.

Proof. (i) It is enough to take in the theorem 3.5 $M = P_{n_2}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R})$.

(ii) It is enough to take in the theorem 3.6 $M = P_{n_2}(\mathbf{R}) \times \ldots \times P_{n_p}(\mathbf{R}) \times T^b.\square$

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Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania
E-mail address: `cpintea@math.ubbcluj.ro`

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PICARD PAIRS AND WEAKLY PICARD PAIRS OF OPERATORS

ALINA SÎNTĂMĂRIAN

Abstract. The purpose of this paper is to introduce the notions of Picard pair, c-Picard pair, weakly Picard pair and c-weakly Picard pair of operators and to present examples for these notions. We also study the data dependence of the common fixed points set of c-weakly Picard pairs of operators.

1. **Introduction**

Let (X, d) be a metric space. Further on we shall need the following notations

$$P(X) := \{ \ Y \mid \emptyset \neq Y \subseteq X \ \}$$

$$P_{cl}(X) := \{ \ Y \mid Y \in P(X) \text{ and } Y \text{ is a closed set } \}$$

and the following functionals

$$D : P(X) \times P(X) \to \mathbb{R}_+, \ D(A, B) = \inf \{ \ d(a, b) \mid a \in A, \ b \in B \ \},$$

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \ H(A, B) = \max \left\{ \sup_{a \in A} D(a, B), \ \sup_{b \in B} D(b, A) \right\}.$$

Let $f_1, f_2 : X \to X$ be two operators. We denote by G_{f_1} the graph of f_1, by F_{f_1} the fixed points set of f_1 and by $(CF)_{f_1,f_2}$ the common fixed points set of f_1 and f_2.

The purpose of this paper is to study the following problems:

Problem 1.1. *Let (X, d) be a metric space and $f_1, f_2 : X \to X$ be two operators. Determine the metric conditions which imply that (f_1, f_2) is a (weakly) Picard pair of operators or (and) f_1, f_2 are (weakly) Picard operators.*

Problem 1.2. *Let (X, d) be a metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators such that $(CF)_{f_1,f_2}, (CF)_{g_1,g_2} \neq \emptyset$. We suppose that there exists $\eta > 0$ with*

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the property that for each $x \in X$, there are $i_x, j_x \in \{1,2\}$ so that $d(f_{i_x}(x), g_{j_x}(x)) \leq \eta$. In these conditions estimate the Pompeiu-Hausdorff distance $H((CF)_{f_1,f_2}, (CF)_{g_1,g_2})$.

Throughout the paper we follow the terminology and the notations from Rus [7], [8] and Rus-Mureşan [9], [10].

2. **Picard pairs and weakly Picard pairs of operators**

Definition 2.1. [Rus [6], [7], [8]] *Let (X,d) be a metric space. An operator $f : X \to X$ is a* Picard operator *(briefly* P. o.*) iff there exists $x^* \in X$ such that $F_f = \{x^*\}$ and $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.*

Let (X,d) be a metric space. We say that a P. o. $f : X \to X$ is a *c-Picard operator* ($c \in [0,+\infty[$) (briefly *c-P. o.*) iff the following condition is satisfied

$$d(x,x^*) \leq c\, d(x, f(x)),$$

for each $x \in X$, where x^* is the unique fixed point of f.

Definition 2.2. [Rus [6], [7], [8]] *Let (X,d) be a metric space. An operator $f : X \to X$ is a* weakly Picard operator *(briefly* w. P. o.*) iff for each $x_0 \in X$, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges and its limit is a fixed point of f.*

For examples of P. o. and w. P. o. see for instance Rus [6], [7], [8].

Let (X,d) be a metric space and $f : X \to X$ be a w. P. o.. We consider the operator $f^\infty : X \to F_f$, defined as follows

$$f^\infty(x) = \lim_{n\to\infty} f^n(x),$$

for each $x \in X$.

Definition 2.3. [Rus-Mureşan [10]] *Let (X,d) be a metric space and $f : X \to X$ be a w. P. o.. We say that f is a* c-weakly Picard operator ($c \in [0,+\infty[$) *(briefly* c-w. P. o.*) iff the following condition is satisfied*

$$d(x, f^\infty(x)) \leq c\, d(x, f(x)),$$

for each $x \in X$.

Examples of c-w. P. o. are given in Rus-Mureşan [10].

Definition 2.4. *Let (X,d) be a metric space and $f_1, f_2 : X \to X$ be two operators. We say that the pair of operators (f_1, f_2) is a* Picard pair of operators *(briefly* P. p. o.*) iff there exists $x^* \in X$ such that $(CF)_{f_1,f_2} = \{x^*\}$ and for each $x \in X$ and for* *every* $y \in \{f_1(x), f_2(x)\}$, *the sequence* $(x_n)_{n\in\mathbb{N}}$ *defined as follows*: $x_0 = x$, $x_1 = y$ *and* $x_{2n-1} = f_i(x_{2n-2})$, $x_{2n} = f_j(x_{2n-1})$, *for each* $n \in \mathbb{N}^*$, *where* $i, j \in \{1, 2\}$, *with* $i \neq j$, *converges to* x^*.

The sequence $(x_n)_{n\in\mathbb{N}}$ *is, by definition, a* sequence of successive approximations for the pair (f_1, f_2), starting from (x_0, x_1).

Definition 2.5. *Let* (X, d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators which form a P. p. o.. We say that* (f_1, f_2) *is a* c-Picard pair of operators ($c \in [0, +\infty[$) (*briefly* c-P. p. o.) *iff the following condition is satisfied*

$$d(x, x^*) \leq c\ d(x, y),$$

for each $(x, y) \in G_{f_1} \cup G_{f_2}$, *where* x^* *is the unique common fixed point of* f_1 *and* f_2.

Definition 2.6. *Let* (X, d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators. We say that the pair of operators* (f_1, f_2) *is a* weakly Picard pair of operators (*briefly* w. P. p. o.) *iff for each* $x \in X$ *and for every* $y \in \{f_1(x), f_2(x)\}$, *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *such that*:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{2n-1} = f_i(x_{2n-2})$ *and* $x_{2n} = f_j(x_{2n-1})$, *for each* $n \in \mathbb{N}^*$, *where* $i, j \in \{1, 2\}$, *with* $i \neq j$;

(iii) *the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent and its limit is a common fixed point of* f_1 *and* f_2.

Definition 2.7. *Let* (X, d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators which form a w. P. p. o.. Then we consider the multivalued operator* $(f_1, f_2)^\infty : G_{f_1} \cup G_{f_2} \to P((CF)_{f_1,f_2})$ *as follows*: *for each* $(x, y) \in G_{f_1} \cup G_{f_2}$, *we define* $(f_1, f_2)^\infty(x, y) = \{\ z \in (CF)_{f_1,f_2} \mid$ *there exists a sequence of successive approximations for the pair* (f_1, f_2), *starting from* (x, y), *that converges to* $z\ \}$.

Definition 2.8. *Let* (X, d) *be a metric space and* $f_1, f_2 : X \to X$ *be two operators which form a w. P. p. o.. We say that* (f_1, f_2) *is a* c-weakly Picard pair of operators ($c \in [0, +\infty[$) (*briefly* c-w. P. p. o.) *iff there exists a selection* $f_{1,2}^\infty$ *of* $(f_1, f_2)^\infty$ *such that*

$$d(x, f_{1,2}^\infty(x, y)) \leq c\ d(x, y),$$

for each $(x, y) \in G_{f_1} \cup G_{f_2}$.

Remark 2.1. *It is obvious that a P. p. o. is a w. P. p. o. and a c-P. p. o. is a c-w. P. p. o..*

Further on we shall give some examples of c-P. p. o. and c-w. P. p. o..

Theorem 2.1. *Let (X,d) be a complete metric space and $f_1, f_2 : X \to X$ be two operators for which there exists $a \in [0, 1/2[$ such that*

$$d(f_1(x), f_2(y)) \le a\ [d(x, f_1(x)) + d(y, f_2(y))],$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$, (f_1, f_2) is c-P. p. o. and f_1 and f_2 are c-P. o., with $c = (1-a)/(1-2a)$.

Proof. The conclusion follows immediately from Kannan's theorem [3] and from the Theorem 2 given by Rus in [5]. □

Theorem 2.2. *Let (X,d) be a complete metric space and $f_1, f_2 : X \to X$ be two operators for which there exist $a, b \in \mathbb{R}_+$, with $a + b < 1$ such that*

$$d(f_1(x), f_2(y)) \le a\ d(x, f_1(x)) + b\ d(y, f_2(y)),$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ and (f_1, f_2) is c-P. p. o., with $c = (1 - \min\{a,b\})/[1-(a+b)]$.

Theorem 2.3. *Let (X,d) be a complete metric space and $f_1, f_2 : X \to X$ be two operators. We suppose that there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$d(f_1(x), f_2(y)) \le \alpha\ d(x,y) + \beta\ [d(x, f_1(x)) + d(y, f_2(y))] + \gamma\ [d(x, f_2(y)) + d(y, f_1(x))],$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ and (f_1, f_2) is c-P. p. o., with $c = [1-(\beta+\gamma)]/[1-(\alpha+2\beta+2\gamma)]$.

Proof. The fact that $F_{f_1} = F_{f_2} = \{x^*\}$ follows from a theorem given by Rus in [4].

In order to prove the second part of the conclusion we shall take again the proof.

Let $i, j \in \{1, 2\}$, with $i \neq j$. Let $x_0 \in X$ and we take $x_{2n-1} = f_i(x_{2n-2})$, $x_{2n} = f_j(x_{2n-1})$, for each $n \in \mathbb{N}^*$.

We have

$$d(x_1, x_2) = d(f_i(x_0), f_j(x_1)) \leq$$

$$\leq \alpha d(x_0, x_1) + \beta[d(x_0, f_i(x_0)) + d(x_1, f_j(x_1))] + \gamma[d(x_0, f_j(x_1)) + d(x_1, f_i(x_0))] =$$

$$= \alpha d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma d(x_0, x_2) \leq$$

$$\leq \alpha d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma[d(x_0, x_1) + d(x_1, x_2)]$$

and hence

$$d(x_1, x_2) \leq (\alpha + \beta + \gamma)/[1 - (\beta + \gamma)]\ d(x_0, x_1).$$

Similarly, we have that

$$d(x_2, x_3) \leq (\alpha + \beta + \gamma)/[1 - (\beta + \gamma)]\ d(x_1, x_2).$$

By induction we get that

$$d(x_n, x_{n+1}) \leq \left[\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}\right]^n\ d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

This implies that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, because (X, d) is a complete metric space. The limit of the sequence $(x_n)_{n\in\mathbb{N}}$ is the unique common fixed point x^* of f_1 and f_2.

We have

$$d(x_n, x^*) \leq \left[\frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)}\right]^n\ \frac{1 - (\beta + \gamma)}{1 - (\alpha + 2\beta + 2\gamma)}\ d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

For $n = 0$, we obtain

$$d(x_0, x^*) \leq [1 - (\beta + \gamma)]/[1 - (\alpha + 2\beta + 2\gamma)]\ d(x_0, f_i(x_0)).$$

So, we can assert that (f_1, f_2) is a c-P. p. o., with $c = [1-(\beta+\gamma)]/[1-(\alpha+2\beta+2\gamma)]$.
□

Remark 2.2. *If we take* $\alpha = \beta = 0$ *in the metric condition of the Theorem* 2.3*, then the part which affirms that* $F_{f_1} = F_{f_2} = \{x^*\}$ *is a result given by Chatterjea in* [1] *and we have that* (f_1, f_2) *is* c*-P. p. o., with* $c = (1 - \gamma)/(1 - 2\gamma)$*.*

Theorem 2.4. *Let (X, d) be a complete metric space and $f_1, f_2 : X \to X$ be two operators for which there exist $a_1, \dots, a_5 \in \mathbb{R}_+$, with $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < 1$ such that*

$$d(f_1(x), f_2(y)) \leq a_1\ d(x,y) + a_2\ d(x, f_1(x)) + a_3\ d(y, f_2(y)) +$$

$$+ a_4\ d(x, f_2(y)) + a_5\ d(y, f_1(x)),$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ and (f_1, f_2) is c-P. p. o., with $c = (1-l)^{-1}$, where $l = \max\{(a_1 + a_2 + a_4)/[1 - (a_3 + a_4)], (a_1 + a_3 + a_5)/[1 - (a_2 + a_5)]\}$.

Proof. The proof is made similarly with that of the Theorem 2.3. □

Theorem 2.5. *Let (X, d) be a complete metric space and $f_1, f_2 : X \to X$ be two operators. We suppose that there exists $a \in [0, 1[$ such that*

$$d(f_1(x), f_2(y)) \leq a\ \max\{d(x,y), d(x, f_1(x)), d(y, f_2(y)),$$

$$1/2\ [d(x, f_2(y)) + d(y, f_1(x))]\},$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ and (f_1, f_2) is c-P. p. o., with $c = (1-a)^{-1}$.

Proof. The fact that $F_{f_1} = F_{f_2} = \{x^*\}$ follows from a theorem given by Ćirić in [2]. For the second part of the conclusion, the proof is made similarly with that of the Theorem 2.3. □

Theorem 2.6. *Let (X, d) be a complete metric space and $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ be a continuous function which satisfies the following two conditions:*

(i_φ) φ is monoton increasing in each variable;

(ii_φ) $\varphi(t,t,t,2t,0) \leq t$, $\varphi(t,t,t,0,2t) \leq t$ and $\varphi(t,0,0,t,t) \leq t$, for each $t > 0$.

Let $f_1, f_2 : X \to X$ be two operators for which there exists $a \in [0, 1[$ such that

$$d(f_1(x), f_2(y)) \leq a\ \varphi(d(x,y), d(x, f_1(x)), d(y, f_2(y)), d(x, f_2(y)), d(y, f_1(x))),$$

for each $x, y \in X$.

Then $F_{f_1} = F_{f_2} = \{x^*\}$ and (f_1, f_2) is c-P. p. o., with $c = (1-a)^{-1}$.

Proof. The proof is made similarly with that of the Theorem 2.3, taking into account the properties of the function φ. □

Remark 2.3. *It is an open question if the operators* f_1 *and* f_2 *from the Remark* 2.2, *the Theorems* 2.2, 2.3, 2.4, 2.5 *or* 2.6 *are P. o..*

Theorem 2.7. *Let* (X, d) *be a complete metric space and* $f_1, f_2 : X \to X$ *be two continuous operators. We suppose that there exist* $a_1, a_2 \in [0, 1[$ *such that for each* $i, j \in \{1, 2\}$, *with* $i \neq j$ *we have*

$$d(f_i(x), f_j(f_i(x))) \leq a_i\ d(x, f_i(x)),$$

for each $x \in X$.

Then $F_{f_1} = F_{f_2} \in P_{cl}(X)$ *and* (f_1, f_2) *is c-w. P. p. o., with* $c = (1 - \max\{a_1, a_2\})^{-1}$.

Proof. We show in the beginning that $F_{f_1} = F_{f_2}$. Let $x^* \in F_{f_1}$. Then we have

$$d(x^*, f_2(x^*)) = d(f_1(x^*), f_2(f_1(x^*))) \leq a_1\ d(x^*, f_1(x^*)) = 0.$$

So $x^* \in F_{f_2}$ and thus we are able to write that $F_{f_1} \subseteq F_{f_2}$. Analogously we get that $F_{f_2} \subseteq F_{f_1}$. Hence $F_{f_1} = F_{f_2}$.

It is not difficult to see that F_{f_1} and F_{f_2} are closed sets. In order to prove that let $i \in \{1, 2\}$ and $x_n \in F_{f_i}$, for each $n \in \mathbb{N}$, with the property that $x_n \to x^*$, as $n \to \infty$. From $x_n = f_i(x_n)$, for each $n \in \mathbb{N}$ and taking into account the fact that f_i is continuous we get, by letting n to tend to infinity, that $x^* = f_i(x^*)$, i. e. $x^* \in F_{f_i}$. So F_{f_i} is a closed set.

Further on we shall prove that $(CF)_{f_1, f_2} \neq \emptyset$. Let $i, j \in \{1, 2\}$, with $i \neq j$. Let $x_0 \in X$ and we put $x_{2n-1} = f_i(x_{2n-2})$, $x_{2n} = f_j(x_{2n-1})$, for each $n \in \mathbb{N}^*$. We have

$$d(x_1, x_2) = d(f_i(x_0), f_j(x_1)) = d(f_i(x_0), f_j(f_i(x_0))) \leq$$

$$\leq a_i\ d(x_0, f_i(x_0)) = a_i\ d(x_0, x_1).$$

Similarly, we have that

$$d(x_2, x_3) \leq a_j\ d(x_1, x_2).$$

We put $a = \max\{a_1, a_2\}$. By induction we get that

$$d(x_n, x_{n+1}) \leq a^n\ d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

This implies that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, because (X,d) is a complete metric space. Let $x^* = \lim_{n\to\infty} x_n$. From $x_{2n-1} = f_i(x_{2n-2})$, $x_{2n} = f_j(x_{2n-1})$, for each $n \in \mathbb{N}^*$ and taking into account the fact that f_1 and f_2 are continuous, it follows that $x^* \in (CF)_{f_1,f_2}$. So $(CF)_{f_1,f_2} = F_{f_1} = F_{f_2} \neq \emptyset$. By an easy calculation we have

$$d(x_n, x^*) \leq a^n/(1-a)\ d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

For $n = 0$ we get

$$d(x_0, x^*) \leq (1-a)^{-1}\ d(x_0, x_1).$$

Therefore (f_1, f_2) is a c-w. P. p. o., where $c = (1 - \max\{a_1, a_2\})^{-1}$. □

Remark 2.4. *It is an open question if the operators f_1 and f_2 from the Theorem* 2.7 *are w. P. o..*

3. Data dependence of the common fixed points set of c-weakly Picard pairs of operators

Theorem 3.1. *Let (X,d) be a metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that*:

*(*i*) (f_1, f_2) is a c_f-w. P. p. o. and (g_1, g_2) is a c_g-w. P. p. o.*;

*(*ii*) there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1, 2\}$ so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then

$$H\left((CF)_{f_1,f_2}, (CF)_{g_1,g_2}\right) \leq \eta\ \max\{c_f, c_g\}.$$

Proof. It is not difficult to see that

$$H\left((CF)_{f_1,f_2}, (CF)_{g_1,g_2}\right) \leq \max\left\{ \sup_{x\in(CF)_{g_1,g_2}} d(x, f_{1,2}^{\infty}(x, f_{i_x}(x))),\right.$$

$$\left.\sup_{x\in(CF)_{f_1,f_2}} d(x, g_{1,2}^{\infty}(x, g_{j_x}(x)))\right\}.$$

If $x \in (CF)_{g_1,g_2}$, then we have

$$d(x, f_{1,2}^{\infty}(x, f_{i_x}(x))) \leq c_f \ d(x, f_{i_x}(x)) = c_f \ d(g_{j_x}(x), f_{i_x}(x)) \leq c_f \ \eta.$$

If $x \in (CF)_{f_1,f_2}$, then we get

$$d(x, g_{1,2}^{\infty}(x, g_{j_x}(x))) \leq c_g \ d(x, g_{j_x}(x)) = c_g \ d(f_{i_x}(x), g_{j_x}(x)) \leq c_g \ \eta.$$

From these, using the following lemma (see [8])

Lemma 3.1. *Let (X, d) be a metric space and $A, B \in P(X)$. We suppose that there exists $\eta \in \mathbb{R}$, $\eta > 0$ such that*:

(i) *for each $a \in A$, there exists $b \in B$ so that $d(a, b) \leq \eta$,*

(ii) *for each $b \in B$, there exists $a \in A$ so that $d(b, a) \leq \eta$.*

Then $H(A, B) \leq \eta$.

We obtain the conclusion of the theorem. □

Further on we shall give some consequences of the abstract result given in Theorem 3.1.

Theorem 3.2. *Let (X, d) be a complete metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that*:

(i_f) *there exists $a_f \in [0, 1/2[$ such that*

$$d(f_1(x), f_2(y)) \leq a_f \ [d(x, f_1(x)) + d(y, f_2(y))],$$

for each $x, y \in X$;

(i_g) *there exists $a_g \in [0, 1/2[$ such that*

$$d(g_1(x), g_2(y)) \leq a_g \ [d(x, g_1(x)) + d(y, g_2(y))],$$

for each $x, y \in X$;

(ii) *there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1, 2\}$ so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ and

$$d(x_f^*, x_g^*) \leq \eta \ (1 - a)/(1 - 2a),$$

where $a = \max \{a_f, a_g\}$.

Proof. From the Theorem 2.1 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o., with $c_f = (1 - a_f)/(1 - 2a_f)$. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o., with $c_g = (1 - a_g)/(1 - 2a_g)$. The fact that $d(x_f^*, x_g^*) \leq \eta\ (1-a)/(1-2a)$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.3. *Let (X, d) be a complete metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that*:

(i$_f$) *there exist $a_f, b_f \in \mathbb{R}_+$, with $a_f + b_f < 1$ such that*

$$d(f_1(x), f_2(y)) \leq a_f\ d(x, f_1(x)) + b_f\ d(y, f_2(y)),$$

for each $x, y \in X$;

(i$_g$) *there exist $a_g, b_g \in \mathbb{R}_+$, with $a_g + b_g < 1$ such that*

$$d(g_1(x), g_2(y)) \leq a_g\ d(x, g_1(x)) + b_g\ d(y, g_2(y)),$$

for each $x, y \in X$;

(ii) *there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1, 2\}$ so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ and

$$d(x_f^*, x_g^*) \leq \eta\ \max\ \{c_f, c_g\},$$

where $c_f = (1 - \min\ \{a_f, b_f\})/[1 - (a_f + b_f)]$ and $c_g = (1 - \min\ \{a_g, b_g\})/[1 - (a_g + b_g)]$.

Proof. From the Theorem 2.2 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o.. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o.. Now, the fact that $d(x_f^*, x_g^*) \leq \eta\ \max\ \{c_f, c_g\}$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.4. *Let (X, d) be a complete metric space and $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that*:

(i$_f$) *there exist $\alpha_f, \beta_f, \gamma_f \in \mathbb{R}_+$, with $\alpha_f + 2\beta_f + 2\gamma_f < 1$ such that*

$$d(f_1(x), f_2(y)) \leq \alpha_f\ d(x, y) + \beta_f\ [d(x, f_1(x)) + d(y, f_2(y))] +$$

$$+ \gamma_f\ [d(x, f_2(y)) + d(y, f_1(x))],$$

for each $x, y \in X$;

(i_g) *there exist* $\alpha_g, \beta_g, \gamma_g \in \mathbb{R}_+$, *with* $\alpha_g + 2\beta_g + 2\gamma_g < 1$ *such that*

$$d(g_1(x), g_2(y)) \leq \alpha_g\ d(x, y) + \beta_g\ [d(x, g_1(x)) + d(y, g_2(y))]+$$

$$+\gamma_g\ [d(x, g_2(y)) + d(y, g_1(x))],$$

for each $x, y \in X$;

(ii) *there exists* $\eta > 0$ *such that for each* $x \in X$, *there are* $i_x, j_x \in \{1, 2\}$ *so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ *and*

$$d(x_f^*, x_g^*) \leq \eta\ \max\ \{c_f, c_g\},$$

where $c_f = [1 - (\beta_f + \gamma_f)]/[1 - (\alpha_f + 2\beta_f + 2\gamma_f)]$ *and* $c_g = [1 - (\beta_g + \gamma_g)]/[1 - (\alpha_g + 2\beta_g + 2\gamma_g)]$.

Proof. From the Theorem 2.3 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o.. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o.. The fact that $d(x_f^*, x_g^*) \leq \eta\ \max\ \{c_f, c_g\}$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.5. *Let* (X, d) *be a complete metric space and* $f_1, f_2, g_1, g_2 : X \to X$ *be four operators. We suppose that*:

(i_f) *there exist* $a_1^f, \ldots, a_5^f \in \mathbb{R}_+$, *with* $a_1^f + a_2^f + a_3^f + 2\max\ \{a_4^f, a_5^f\} < 1$ *such that*

$$d(f_1(x), f_2(y)) \leq a_1^f\ d(x, y) + a_2^f\ d(x, f_1(x)) + a_3^f\ d(y, f_2(y))+$$

$$+a_4^f\ d(x, f_2(y)) + a_5^f\ d(y, f_1(x)),$$

for each $x, y \in X$;

(i_g) *there exist* $a_1^g, \ldots, a_5^g \in \mathbb{R}_+$, *with* $a_1^g + a_2^g + a_3^g + 2\max\ \{a_4^g, a_5^g\} < 1$ *such that*

$$d(g_1(x), g_2(y)) \leq a_1^g\ d(x, y) + a_2^g\ d(x, g_1(x)) + a_3^g\ d(y, g_2(y))+$$

$$+a_4^g\ d(x, g_2(y)) + a_5^g\ d(y, g_1(x)),$$

for each $x, y \in X$;

(ii) there exists $\eta > 0$ *such that for each* $x \in X$*, there are* $i_x, j_x \in \{1,2\}$ *so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ *and*

$$d(x_f^*, x_g^*) \leq \eta \ \max\{c_f, c_g\},$$

where $c_f = (1 - l_f)^{-1}$*, with* $l_f = \max\{(a_1^f + a_2^f + a_4^f)/[1 - (a_3^f + a_4^f)], (a_1^f + a_3^f + a_5^f)/[1 - (a_2^f + a_5^f)]\}$ *and* $c_g = (1 - l_g)^{-1}$*, with* $l_g = \max\{(a_1^g + a_2^g + a_4^g)/[1 - (a_3^g + a_4^g)], (a_1^g + a_3^g + a_5^g)/[1 - (a_2^g + a_5^g)]\}$.

Proof. From the Theorem 2.4 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o.. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o.. Now, the fact that $d(x_f^*, x_g^*) \leq \eta \ \max\{c_f, c_g\}$ follows from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.6. *Let* (X, d) *be a complete metric space and* $f_1, f_2, g_1, g_2 : X \to X$ *be four operators. We suppose that:*

(i_f*) there exists* $a_f \in [0, 1[$ *such that*

$$d(f_1(x), f_2(y)) \leq a_f \ \max\{d(x,y), d(x, f_1(x)), d(y, f_2(y)),$$
$$1/2\ [d(x, f_2(y)) + d(y, f_1(x))]\},$$

for each $x, y \in X$;

(i_g*) there exists* $a_g \in [0, 1[$ *such that*

$$d(g_1(x), g_2(y)) \leq a_g \ \max\{d(x,y), d(x, g_1(x)), d(y, g_2(y)),$$
$$1/2\ [d(x, g_2(y)) + d(y, g_1(x))]\},$$

for each $x, y \in X$;

(ii) there exists $\eta > 0$ *such that for each* $x \in X$*, there are* $i_x, j_x \in \{1,2\}$ *so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ *and*

$$d(x_f^*, x_g^*) \leq \eta \ (1 - \max\{a_f, a_g\})^{-1}.$$

Proof. From the Theorem 2.5 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o., with $c_f = (1 - a_f)^{-1}$. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o., with $c_g = (1 - a_g)^{-1}$. The fact that $d(x_f^*, x_g^*) \leq \eta\ (1 - \max\ \{a_f, a_g\})^{-1}$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.7. *Let (X, d) be a complete metric space and $\varphi_f, \varphi_g : \mathbb{R}_+^5 \to \mathbb{R}_+$ be two continuous functions which satisfy the following conditions:*

($\mathrm{i}_{\varphi_{f,g}}$) *$\varphi_f$ and φ_g are monoton increasing in each variable;*

($\mathrm{ii}_{\varphi_{f,g}}$) *$\varphi_f(t, t, t, 2t, 0) \leq t$, $\varphi_f(t, t, t, 0, 2t) \leq t$ and $\varphi_f(t, 0, 0, t, t) \leq t$, for each $t > 0$ and $\varphi_g(t, t, t, 2t, 0) \leq t$, $\varphi_g(t, t, t, 0, 2t) \leq t$ and $\varphi_g(t, 0, 0, t, t) \leq t$, for each $t > 0$.*

Let $f_1, f_2, g_1, g_2 : X \to X$ be four operators. We suppose that:

(i_f) *there exists $a_f \in [0, 1[$ such that*

$$d(f_1(x), f_2(y)) \leq a_f\ \varphi_f(d(x, y), d(x, f_1(x)), d(y, f_2(y)), d(x, f_2(y)), d(y, f_1(x))),$$

for each $x, y \in X$;

(i_g) *there exists $a_g \in [0, 1[$ such that*

$$d(g_1(x), g_2(y)) \leq a_g\ \varphi_g(d(x, y), d(x, g_1(x)), d(y, g_2(y)), d(x, g_2(y)), d(y, g_1(x))),$$

for each $x, y \in X$;

(ii) *there exists $\eta > 0$ such that for each $x \in X$, there are $i_x, j_x \in \{1, 2\}$ so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then $F_{f_1} = F_{f_2} = \{x_f^*\}$, $F_{g_1} = F_{g_2} = \{x_g^*\}$ and

$$d(x_f^*, x_g^*) \leq \eta\ (1 - \max\ \{a_f, a_g\})^{-1}.$$

Proof. From the Theorem 2.6 we have that $F_{f_1} = F_{f_2} = \{x_f^*\}$ and that (f_1, f_2) is c_f-P. p. o., with $c_f = (1 - a_f)^{-1}$. From the same theorem we also have that $F_{g_1} = F_{g_2} = \{x_g^*\}$ and that (g_1, g_2) is c_g-P. p. o., with $c_g = (1 - a_g)^{-1}$. Now, the fact that $d(x_f^*, x_g^*) \leq \eta\ (1 - \max\ \{a_f, a_g\})^{-1}$ follows immediately from the Remark 2.1 and the Theorem 3.1. □

Theorem 3.8. *Let* (X, d) *be a complete metric space and* $f_1, f_2, g_1, g_2 : X \to X$ *be four operators. We suppose that*:

(i_f) *there exist* $a_1^f, a_2^f \in [0, 1[$ *such that for each* $k, l \in \{1, 2\}$, *with* $k \neq l$ *we have*

$$d(f_k(x), f_l(f_k(x))) \leq a_k^f \; d(x, f_k(x)),$$

for each $x \in X$;

(i_g) *there exist* $a_1^g, a_2^g \in [0, 1[$ *such that for each* $k, l \in \{1, 2\}$, *with* $k \neq l$ *we have*

$$d(g_k(x), g_l(g_k(x))) \leq a_k^g \; d(x, g_k(x)),$$

for each $x \in X$;

(ii) *there exists* $\eta > 0$ *such that for each* $x \in X$, *there are* $i_x, j_x \in \{1, 2\}$ *so that*

$$d(f_{i_x}(x), g_{j_x}(x)) \leq \eta.$$

Then

$$H((CF)_{f_1,f_2}, (CF)_{g_1,g_2}) \leq \eta \; (1 - \max \; \{a_1^f, a_2^f, a_1^g, a_2^g\})^{-1}.$$

Proof. From the Theorem 2.7 we have that (f_1, f_2) is c_f-w. P. p. o., with $c_f = (1 - \max \{a_1^f, a_2^f\})^{-1}$ and that (g_1, g_2) is c_g-w. P. p. o., with $c_g = (1 - \max \{a_1^g, a_2^g\})^{-1}$. The conclusion follows from the Theorem 3.1. □

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Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

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CHARACTERIZATIONS OF INJECTIVE MULTIPLIERS ON PARTIALLY ORDERED SETS

ÁRPÁD SZÁZ AND JÓZSEF TÚRI

Abstract. An ordered pair $(\mathcal{D}, \mathcal{E})$ of subsets of a partially ordered set $\mathcal{A}$ is called a pairing in $\mathcal{A}$ if the meet $D \wedge E = \inf\{D, E\}$ exists for all $D \in \mathcal{D}$ and $E \in \mathcal{E}$. Moreover, the set $\mathcal{D}$ is said to separate the points of $\mathcal{E}$ if for each $E_1, E_2 \in \mathcal{E}$ with $E_1 \neq E_2$ there exists $D \in \mathcal{D}$ such that $D \wedge E_1 \neq D \wedge E_2$.

A function F of $\mathcal{D}$ to $\mathcal{E}$ is called nonexpansive if $F(D) \leq D$ for all $D \in \mathcal{D}$. Moreover, the function F is called a multiplier if $F(D_1) \wedge D_2 = D_1 \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$. If in particular $\mathcal{D}$ is a meet semilattice in $\mathcal{A}$, then the function F is a nonexpansive multiplier if and only if $F(D_1 \wedge D_2) = F(D_1) \wedge D_2$ for all $D_1, D_2 \in \mathcal{D}$.

After summarizing some basic properties of pairings, nonexpansive functions and multipliers, it is shown that if F is a multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then $\mathcal{E}$ separates the points of $\mathcal{D}$ if and only if F is injective and $\mathcal{D}$ separates the points of $\mathcal{E}$. Moreover, some sufficient conditions are given in order that a nonexpansive function and a multiplier of $\mathcal{D}$ to $\mathcal{E}$ be the identity function of $\mathcal{D}$.

The results obtained naturally extends and supplement some former statements of G. Szász, J. Szendrei, Á. Száz and G. Pataki on multipliers on semilattices and partially ordered sets. Moreover, they are also closely related to the works of several mathematicians on the extensions of semilattices and semigroups by the module theoretic methods of R. E. Johnson, Y. Utumi, G. D. Findlay and J. Lambek.

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1. **Partially ordered sets**

According to Birkhoff [2, p.1] a nonvoid set $\mathcal{A}$ together with a reflexive, transitive and antisymmetric relation $\leq$ is briefly called a poset. The use of the script letter is mainly motivated by the fact that each poset $\mathcal{A}$ is isomorphic to a family of sets partially ordered by set inclusion. The isomorphism is established by the mapping $A \mapsto]\,A\,]$, where $A \in \mathcal{A}$ and $]\,A\,] = \{\, B \in \mathcal{A} : \quad B \leq A \,\}$.

As usual, a poset $\mathcal{A}$ is called (1) totally ordered if for each $A, B \in \mathcal{A}$ either $A \leq B$ or $B \leq A$ holds, (2) well-ordered if each nonvoid subset of $\mathcal{A}$ has a minimum (least element). Moreover, a subset $\mathcal{D}$ of $\mathcal{A}$ is called (1) descending if $A \in \mathcal{A}$, $D \in \mathcal{D}$ and $A \leq D$ imply $A \in \mathcal{D}$, and (2) cofinal if for each $A \in \mathcal{A}$ there exists $D \in \mathcal{D}$ such that $A \leq D$.

The infimum (greatest lower bound) and the supremum (least upper bound) of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ will be understood in the usual sense. However, instead of $\inf \mathcal{D}$ and $\sup \mathcal{D}$, we shall use the lattice theoretic notations $\bigwedge \mathcal{D}$ and $\bigvee \mathcal{D}$, respectively. Thus, for instance $E = \bigwedge \mathcal{D}$ if and only if $E \in \mathcal{A}$ such that for each $A \in \mathcal{A}$ we have $A \leq E$ if and only if $A \leq D$ for all $D \in \mathcal{D}$.

However, in the sequel, we shall only need some very particular cases of the above definitions whenever, for $A, B \in \mathcal{A}$, we write $A \wedge B = \inf \{ A, B\}$ and $A \vee B = \sup \{ A, B\}$. Concerning the operation $\wedge$, we shall frequently use the next simple theorems which, in their present forms, are usually not included in the standard books on lattices.

Theorem 1.1. *If $\mathcal{A}$ is a poset and $A, B, C, D \in \mathcal{A}$, then*

(1) $A \leq B$ if and only if $A = A \wedge B$; and thus $A = A \wedge A$;

(2) $A \leq B$ and $C \leq D$ imply $A \wedge C \leq B \wedge D$ whenever $A \wedge C$ and $B \wedge D$ exist.

Theorem 1.2. *If $\mathcal{A}$ is a poset and $A, B, C \in \mathcal{A}$, then*

(1) $A \wedge B = B \wedge A$ whenever either $B \wedge A$ or $A \wedge B$ exist;

(2) $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ whenever $A \wedge B$ and $B \wedge C$ and moreover either $(A \wedge B) \wedge C$ or $A \wedge (B \wedge C)$ exist.

Remark 1.3. A slightly weaker form of the assertion (2) can be found in Birkhoff [2, Theorem 1, p.8]. Moreover, a somewhat weaker form of the dual of this assertion can be found in Grätzer [7, Exercise 31, p.8].

Theorem 1.4. *If $\mathcal{A}$ is a poset and $\mathcal{D} \subset \mathcal{A}$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ is descending;

(2) $A \in \mathcal{A}$ and $D \in \mathcal{D}$ imply $A \wedge D \in \mathcal{D}$ whenever $A \wedge D$ exists.

Remark 1.5. From the above theorems, by using the dual $\mathcal{A}(\geq)$ of the poset $\mathcal{A}(\leq)$, one can easily get some analogous theorems for the operation $\vee$ and the ascending subsets of $\mathcal{A}(\leq)$. However, in the sequel, we shall mainly need the operation $\wedge$. Therefore, we shall assume here some rather particular terminology.

A nonvoid subset $\mathcal{B}$ of poset $\mathcal{A}$ is called a semilattice in $\mathcal{A}$ if $D \wedge E$ exists in $\mathcal{A}$ and belongs to $\mathcal{B}$ for all $D, E \in \mathcal{B}$. Moreover, a nonvoid subset $\mathcal{D}$ of a semilattice $\mathcal{B}$ in a poset $\mathcal{A}$ is called an ideal of $\mathcal{B}$ if $D \wedge E$ is in $\mathcal{D}$ for all $D \in \mathcal{D}$ and $E \in \mathcal{B}$. Note that, by Theorem 1.5, $\mathcal{D}$ is an ideal of $\mathcal{B}$ if and only if $\mathcal{D}$ is descending subset of $\mathcal{B}$.

If $\mathcal{D}$ and $\mathcal{E}$ are subsets of a poset $\mathcal{A}$ such that $D \wedge E$ exists for all $D \in \mathcal{D}$ and $E \in \mathcal{E}$, then we write $\mathcal{D} \wedge \mathcal{E} = \{ D \wedge E : \quad D \in \mathcal{D}, \quad E \in \mathcal{E} \}$. Note that if $\mathcal{B}$ is a semilattice in a poset $\mathcal{A}$, then $\mathcal{B} = \mathcal{B} \wedge \mathcal{B}$. Moreover, if $\mathcal{D}$ and $\mathcal{E}$ are ideals of $\mathcal{B}$, then $\mathcal{D} = \mathcal{D} \wedge \mathcal{B}$ and $\mathcal{D} \cap \mathcal{E} = \mathcal{D} \wedge \mathcal{E}$. Therefore, the ideal $\mathcal{D} \cap \mathcal{E}$ inherits some useful properties of $\mathcal{D}$ and $\mathcal{E}$.

2. **Separating pairings in posets**

Definition 2.1. For every subset $\mathcal{D}$ of a poset $\mathcal{A}$, we define

$$\mathcal{D}^* = \Big\{ A \in \mathcal{A} : \quad \forall \ D \in \mathcal{D} : \quad \exists \ A \wedge D \Big\}.$$

Concerning the mapping $*$ of $\mathcal{P}(\mathcal{A})$ to itself, we can easily establish the following

Theorem 2.2. *If $\mathcal{D}$ and $\mathcal{E}$ are subsets of a poset $\mathcal{A}$, then the following assertions are equivalent:*

(1) $\mathcal{E} \subset \mathcal{D}^*$; *(2) $\mathcal{D} \subset \mathcal{E}^*$.*

Proof. Suppose that the assertion (1) holds and $D \in \mathcal{D}$. Then, $E \in \mathcal{D}^*$ for all $E \in \mathcal{E}$. Therefore, $D \wedge E = E \wedge D$ exists for all $E \in \mathcal{E}$. Consequently, $D \in \mathcal{E}^*$, and thus the assertion (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ can now be immediately established by interchanging the roles of $\mathcal{D}$ and $\mathcal{E}$ in the implication $(1) \Longrightarrow (2)$.

Remark 2.3. From the above theorem, by [29, Lemma 2.3], it follows that the mappings $*$ and $*$ establish a Galois connection between the posets $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$.

Therefore, as an immediate consequence of [29, Theorem 2.4], we can also state

Theorem 2.4. *If $\mathcal{A}$ is a poset, then*

(1) $\mathcal{D}^* = \mathcal{D}^{***}$ for all $\mathcal{D} \subset \mathcal{A}$;

(2) the composite mapping $**$ is a closure operation on $\mathcal{P}(\mathcal{A})$ such that $\mathcal{P}(\mathcal{A})^* = \mathcal{P}(\mathcal{A})^{**}$;

(3) the restriction of the mapping $*$ to $\mathcal{P}(\mathcal{A})^*$ is an inversion invariant injection of $\mathcal{P}(\mathcal{A})^*$ onto itself.

Hence, by [29, Theorem 1.9], it is clear that in particular we also have

Corollary 2.5. *If $\mathcal{A}$ is a poset, then $\mathcal{P}(\mathcal{A})^*$ is a complete poset.*

Definition 2.6. If $\mathcal{D}$ and $\mathcal{E}$ are nonvoid subsets of $\mathcal{A}$ such that $\mathcal{E} \subset \mathcal{D}^*$, then we say that the ordered pair $(\mathcal{D}, \mathcal{E})$ is a pairing in $\mathcal{A}$.

Our prime example for pairings is described in the following

Theorem 2.7. *If $\mathcal{A}$ is a poset with $\mathcal{A}^* \neq \emptyset$, then $\mathcal{A}^*$ is the largest subset of $\mathcal{A}$ such that $(\mathcal{A}^*, \mathcal{A})$ is a pairing in $\mathcal{A}$. Moreover, $\mathcal{A}^*$ is a semilattice in $\mathcal{A}$.*

Proof. The first statement is immediate from Definition 2.6 and Theorem 2.2. To prove the second statement, note that if $A, B \in \mathcal{A}^*$ and $C \in \mathcal{A}$, then by Definition 2.1 $A \wedge B$ and $A \wedge (B \wedge C)$ exist. Therefore, by Theorem 1.2, $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ also exists. Thus, again by Definition 2.1, $A \wedge B \in \mathcal{A}^*$.

Definition 2.8. If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that for any $E_1, E_2 \in \mathcal{E}$, with $E_1 \neq E_2$, there exists $D \in \mathcal{D}$ such that $E_1 \wedge D \neq E_2 \wedge D$, then we say that $\mathcal{D}$ separates the points of $\mathcal{E}$.

Concerning the existence of separating pairings, we can only state the following generalization of [28, Theorem 2.9], and its immediate consequences.

Theorem 2.9. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D}$ *is a cofinal subset of* $\mathcal{E}$, *then* $\mathcal{D}$ *separates the points of* $\mathcal{E}$.

Corollary 2.10. *If* $\mathcal{A}$ *is a poset such that* $\mathcal{A}^*$ *is cofinal in* $\mathcal{A}$, *then* $\mathcal{A}^*$ *separates the points of* $\mathcal{A}$.

Corollary 2.11. *If* $\mathcal{A}$ *is a semilattice, then* $\mathcal{A}$ *separates the points of itself.*

In the sequel, we shall also need the following rather particular

Theorem 2.12. *Let* $(\mathcal{D}, \mathcal{E})$ *be a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D} \subset \mathcal{E}$. *Suppose that* $\mathcal{U} \subset \mathcal{D}$ *and* $\mathcal{V} \subset \mathcal{E}$ *such that* $\mathcal{U} \wedge \mathcal{D} \subset \mathcal{U}$ *and* $\mathcal{U} \wedge \mathcal{V} \subset \mathcal{V}$. *Moreover, suppose that* $\mathcal{U}$ *separates the points* $\mathcal{D}$ *and* $\mathcal{V}$ *separates the points of* $\mathcal{U}$. *Then* $\mathcal{V}$ *also separates the points of* $\mathcal{D}$.

Proof. Suppose that $D_1, D_2 \in \mathcal{D}$ such that $D_1 \neq D_2$. Then, since $\mathcal{U}$ separates the points of $\mathcal{D}$, there exists $U \in \mathcal{U}$ such that $D_1 \wedge U \neq D_2 \wedge U$. Moreover, since $\mathcal{U} \wedge \mathcal{D} \subset \mathcal{U}$, we also have $D_1 \wedge U, D_2 \wedge U \in \mathcal{U}$. Therefore, since $\mathcal{V}$ separates the points of $\mathcal{U}$, there exists $V \in \mathcal{V}$ such that $(D_1 \wedge U) \wedge V \neq (D_2 \wedge U) \wedge V$. Hence, by Theorem 1.2, it follows that $D_1 \wedge (U \wedge V) \neq D_2 \wedge (U \wedge V)$. Moreover, since $\mathcal{U} \wedge \mathcal{V} \subset \mathcal{V}$, we also have $U \wedge V \in \mathcal{V}$. Therefore, the required assertion is also true.

Corollary 2.13. *If* $\mathcal{A}$ *is a poset and* $\mathcal{D}$ *is an ideal of* $\mathcal{A}^*$ *such that* $\mathcal{D}$ *separates the points of* $\mathcal{A}^*$ *and* $\mathcal{A}$ *separates the points of* $\mathcal{D}$, *then* $\mathcal{A}$ *also separates the points of* $\mathcal{A}^*$.

3. Nonexpansive functions on posets

Definition 3.1. A function F of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ to $\mathcal{A}$ is called nonexpansive if $F(D) \leq D$ for all $D \in \mathcal{D}$.

Clearly, the identity function $\Delta_{\mathcal{D}}$ of $\mathcal{D}$ is nonexpansive. Moreover, to provide a less trivial example, we can also at once state

Example 3.2. If $\mathcal{T}$ is a subset of an upper complete poset $\mathcal{A}$, then the function $\circ$, defined by $A^{\circ} = \sup\{V \in \mathcal{T} : \quad V \leq A\}$ for all $A \in \mathcal{A}$, is nonexpansive. Note that, in particular, $\mathcal{A}$ may be the family of all subsets of a set X and $\mathcal{T}$ may be a topology on X.

Remark 3.3. To let the reader feel the importance of nonexpansive functions, it is also worth mentioning that if F is a nonexpansive function of a poset $\mathcal{A}$ to itself, then the minimal elements of $\mathcal{A}$ are fixed points of F. The dual statement has previously been stressed by Bronsted [4].

By the corresponding definitions, we evidently have the following two theorems.

Theorem 3.4. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *and* F *is a function of* $\mathcal{D}$ *to* $\mathcal{E}$, *then the function* F', *defined by* $F'(D) = F(D) \wedge D$ *for all* $D \in \mathcal{D}$, *is nonexpansive. Moreover,* F *is nonexpansive if and only if* $F' = F$.

Corollary 3.5. *If* F *is a nonexpansive function of an ideal* $\mathcal{D}$ *of a semilattice* $\mathcal{A}$ *onto a subset* $\mathcal{E}$ *of* $\mathcal{A}$, *then* $\mathcal{E} \subset \mathcal{D}$.

Theorem 3.6. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *and* F *is a function of* $\mathcal{D}$ *to* $\mathcal{E}$, *then the following assertions are equivalent:*

(1) F *is nonexpansive;*

(2) $F(D_1) = F(D_1) \wedge D_2$ *for all* $D_1 \in \mathcal{D}$ *and* $D_2 \in \mathcal{A}$ *with* $D_1 \leq D_2$.

Remark 3.7. In this respect, it is also worth noticing that a function F of a poset $\mathcal{D}$ to another $\mathcal{E}$ is nondecreasing if and only if $F(D_1) = F(D_1) \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$ with $D_1 \leq D_2$.

Therefore, in addition to Theorem 3.6, we may also naturally state the following theorem of [18].

Theorem 3.8. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *and* F *is a function of* $\mathcal{D}$ *to* $\mathcal{E}$, *then the following assertions are equivalent:*

(1) F *is nonexpansive and nondecreasing;*

(2) $F(D_1) \wedge D_2 = F(D_1) \wedge F(D_2)$ *for all* $D_1, D_2 \in \mathcal{D}$ *with* $D_1 \leq D_2$.

In contrast to the injective nondecreasing functions, the inverse of an injective nonexpansive function need not be nonexpansive. Namely, we have the following natural extension of an observation of Szász [22, p.449].

Theorem 3.9. *If* F *is an injective function of a subset* $\mathcal{D}$ *of a poset* $\mathcal{A}$ *to* $\mathcal{A}$ *such that both* F *and* F^{-1} *are nonexpensive, then* $F = \Delta_{\mathcal{D}}$.

Proof. Note that, in this case, we have $D = F^{-1}\big(F(D)\big) \leq F(D) \leq D$, and hence $F(D) = D$ for all $D \in \mathcal{D}$.

Analogously to [8, (4.43) Theorem], we can also prove the following

Theorem 3.10. *If* F *is an injective nonexpansive function of an ideal* $\mathcal{D}$ *of a well-ordered set* $\mathcal{A}$ *to* $\mathcal{A}$, *then* $F = \Delta_{\mathcal{D}}$.

Proof. If this is not the case, then by the well-orderedness of $\mathcal{A}$ there exists a smallest element D of $\mathcal{D}$ such that $F(D) \neq D$. Hence, by the nonexpansibility of F, it follows that $F(D) < D$. Moreover, by using Corollary 3.5 and the injectivity of F, we can see that $F(D) \in \mathcal{D}$ and $F\big(F(D)\big) \neq F(D)$. But, this contradicts the minimality of D.

Moreover, as a dual of [8, (4.43) Theorem], we can also state

Theorem 3.11. *If F is an injective nondecreasing function of a dually well-ordered set $\mathcal{A}$ to itself, then F is nonexpansive.*

Hence, by using Theorem 3.9, we can easily derive

Corollary 3.12. *If F is an injective nondecreasing function of a dually well-ordered set $\mathcal{A}$ onto itself, then $F = \Delta_{\mathcal{A}}$.*

Proof. In this case, F^{-1} is also an injective nondecreasing function of $\mathcal{A}$ onto itself. Therefore, by Theorem 3.11, not only F, but also F^{-1} is nonexpansive. Therefore, Theorem 3.9 can be applied.

Moreover, as an immediate consequence of Theorems 3.11 and 3.10, we can also state

Corollary 3.13. *If F is an injective nondecreasing function of a well-ordered and dually well-ordered set $\mathcal{A}$ to itself, then $F = \Delta_{\mathcal{A}}$.*

4. **Nonexpansive multipliers on posets**

Definition 4.1. If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$, then a function F of $\mathcal{D}$ to $\mathcal{E}$ is called a multiplier if $F(D_1) \wedge D_2 = D_1 \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$.

The above definition can be illustrated with the following examples of [18].

Example 4.2. If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{E}$, then the identity function $\Delta_{\mathcal{D}}$ of $\mathcal{D}$ is a nonexpansive multiplier.

Example 4.3. If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D}$ is a semilattice in $\mathcal{A}$, then for each $A \in \mathcal{E}$ the function F, defined by $F(D) = A \wedge D$ for all $D \in \mathcal{D}$, is a nonexpansive multiplier.

Example 4.4. Let $\mathcal{A}$ be a distributive lattice [2, p.12] with a least element O and a greatest element X such that $X \neq O$. Choose $A \in \mathcal{A}$ such that $A \neq O$, and define $\mathcal{D} = \{ D \in \mathcal{A} : \quad A \wedge D = O \}$ and $F(D) = A \vee D$ for all $D \in \mathcal{D}$. Then

$\mathcal{D}$ is an ideal of $\mathcal{A}$ such that $\mathcal{D}$ does not separate the points of $\mathcal{A}$, and F is a nonextendable multiplier such that $D < F(D)$ for all $D \in \mathcal{D}$.

Remark 4.5. Moreover, it is also worth noticing that F is meet-preserving and $\mathcal{D} \cap F(\mathcal{D}) = \emptyset$.

The importance of nonexpansive multipliers is also apparent from the following theorems of [18].

Theorem 4.6. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ and F is a function of $\mathcal{D}$ to $\mathcal{E}$, then the following assertions are equivalent:*

(1) F is a nonexpansive multiplier;

(2) $F(D_1) \wedge D_2 = F(D_1) \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$.

Corollary 4.7. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ and F is a nonexpansive multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then $(\mathcal{E}, \mathcal{E})$ is also a pairing in $\mathcal{A}$.*

Theorem 4.8. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ and F is a function of $\mathcal{D}$ to $\mathcal{E}$, then each of the following assertions implies the subsequent one:*

(1) F is a nonexpansive multiplier;

(2) $F(D_1) = D_1 \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$ with $D_1 \leq D_2$;

(3) $F(D_1 \wedge D_2) = F(D_1) \wedge D_2$ for all $D_1 \in \mathcal{D}$ and $D_2 \in \mathcal{A}$ with $D_1 \wedge D_2 \in \mathcal{D}$.

Corollary 4.9. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ and F is a nonexpansive multiplier of $\mathcal{D}$ to $\mathcal{E}$, then F is nondecreasing.*

Theorem 4.10. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D}$ is a semilattice in $\mathcal{A}$, and F is a function of $\mathcal{D}$ to $\mathcal{E}$, then the following assertions are equivalent:*

(1) F is a nonexpansive multiplier;

(2) $F(D_1 \wedge D_2) = F(D_1) \wedge D_2$ for all $D_1, D_2 \in \mathcal{D}$;

(3) $F(D_1) = D_1 \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$ with $D_1 \leq D_2$.

Corollay 4.11. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D}$ is a semilattice in $\mathcal{A}$, and F is a nonexpansive multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then F is meet-preserving and $\mathcal{E}$ is also a semilattice in $\mathcal{A}$.*

Corollay 4.12. *If F is a nonexpansive multiplier of an ideal $\mathcal{D}$ of a semilattice $\mathcal{A}$ onto a subset $\mathcal{E}$ of $\mathcal{A}$, then F is idempotent and $\mathcal{E}$ is also an ideal of $\mathcal{A}$.*

Moreover, as some straightforward generalizations of [28, Theorems 6.2 and 6.3], we can also prove the following two theorems.

Theorem 4.13. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D}$ *separates the points of* $\mathcal{E}$, *and* F *is a multiplier of* $\mathcal{D}$ *to* $\mathcal{E}$ *such that* $F'(\mathcal{D}) \subset \mathcal{E}$, *then* F *is nonexpansive.*

Proof. If $D \in \mathcal{D}$, then by the above assumption on F' we have $F'(D) \in \mathcal{E}$. Hence, by the corresponding definitions and Theorem 1.2, it is clear that

$$F'(D) \wedge Q = \big(F(D) \wedge D\big) \wedge Q = Q \wedge \big(F(D) \wedge D\big) = \big(Q \wedge F(D)\big) \wedge D =$$

$$\big(F(Q) \wedge D\big) \wedge D = F(Q) \wedge (D \wedge D) = F(Q) \wedge D = Q \wedge F(D) = F(D) \wedge Q$$

for all $Q \in \mathcal{D}$. Hence, since $\mathcal{D}$ separates the points of $\mathcal{E}$, it follows that $F'(D) = F(D)$. Therefore, $F' = F$, and thus by Theorem 3.3 F is nonexpansive.

Theorem 4.14. *Let* $(\mathcal{D}, \mathcal{E})$ *be a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{E} \wedge \mathcal{D} \subset \mathcal{E}$. *Suppose that* F *is a multiplier of a subset* $\mathcal{D}_F$ *to* $\mathcal{E}$ *such that* $\mathcal{D}_F$ *separates the points of* $\mathcal{E}$. *Define*

$$F^{-} = \big\{ (D, E) \in \mathcal{D} \times \mathcal{E} : \quad \forall\ Q \in \mathcal{D}_F : \quad E \wedge Q = D \wedge F(Q) \big\}.$$

Then F^{-} *is the largest multiplier of a subset* $\mathcal{D}_{F^{-}}$ *of* $\mathcal{D}$ *to* $\mathcal{E}$ *such that* $F \subset F^{-}$. *Moreover, if in particular* $\mathcal{D}$ *is a semilattice in* $\mathcal{A}$, *then* $\mathcal{D}_{F^{-}}$ *is already an ideal of* $\mathcal{D}$.

5. **Injective multipliers on posets**

The following theorem has mainly been suggested by Máté [14, Proposition 4]. For a generalization, see also [26, Theorem 2.3].

Theorem 5.1. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *and* F *is a multiplier of* $\mathcal{D}$ *onto* $\mathcal{E}$, *then the following assertions are equivalent:*

(1) $\mathcal{E}$ *separates the points of* $\mathcal{D}$;

(2) F *is injective and* $\mathcal{D}$ *separates the points of* $\mathcal{E}$.

Proof. Suppose that the assertion (1) holds. Then, for any $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$, there exists $E \in \mathcal{E}$ such that $D_1 \wedge E \neq D_2 \wedge E$. Hence, by choosing $D \in \mathcal{D}$ such that $E = F(D)$, we can see that

$$F(D_1) \wedge D = D_1 \wedge F(D) = D_1 \wedge E \neq D_2 \wedge E = D_2 \wedge F(D) = F(D_2) \wedge D.$$

Therefore, $F(D_1) \neq F(D_2)$, and thus the first part of the assertion (2) also holds.

Moreover, since for any $E_1, E_2 \in \mathcal{E}$ with $E_1 \neq E_2$ there exist $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$ such that $E_1 = F(D_1)$ and $E_2 = F(D_2)$, it is clear that the second part of the assertion (2) is also true.

Suppose now that the assertion (2) holds. Then, by the first part of the assertion (2), for any $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$, we have $F(D_1) \neq F(D_2)$. Therefore, by the second part of the assertion (2), there exists $D \in \mathcal{D}$ such that $F(D_1) \wedge D \neq F(D_2) \wedge D$. Hence, by defining $E = F(D)$, we can see that $E \in \mathcal{E}$ such that

$$D_1 \wedge E = D_1 \wedge F(D) = F(D_1) \wedge D \neq F(D_2) \wedge D = D_2 \wedge F(D) = D_2 \wedge E.$$

Therefore, the assertion (1) also holds.

Now, as some immediate consequences of Theorem 5.1, we can also state

Corollary 5.2. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{E}$ separates the points of $\mathcal{D}$, then every multiplier F of $\mathcal{D}$ onto $\mathcal{E}$ is injective.*

Corollary 5.3. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that there exists an injective multiplier F of $\mathcal{D}$ onto $\mathcal{E}$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ separates the points of $\mathcal{E}$; *(2) $\mathcal{E}$ separates the points of $\mathcal{D}$.*

Corollary 5.4. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D}$ separates the points of $\mathcal{E}$, and F is a multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then the following assertions are equivalent:*

(1) F is injective; *(2) $\mathcal{E}$ separates the points of $\mathcal{D}$.*

Moreover, by using Theorems 5.1 and 2.12, we can also prove the following

Theorem 5.5. *Let $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{E}$. Suppose that F is an injective multiplier of a subset $\mathcal{D}_F$ of $\mathcal{D}$ onto a subset $\mathcal{E}_F$ of $\mathcal{E}$ such that $\mathcal{D}_F \wedge \mathcal{D} \subset \mathcal{D}_F$ and $F'(\mathcal{D}_F) \subset \mathcal{E}$, and $\mathcal{D}_F$ separates the points of $\mathcal{E}$. Then $\mathcal{E}_F$ separates the points of $\mathcal{D}$.*

Proof. In this case, by Theorem 5.1, $\mathcal{E}_F$ separates the points of $\mathcal{D}_F$. Moreover, by Theorem 4.13, F is nonexpansive. Therefore, if $E \in \mathcal{E}_F$, then by choosing $D \in \mathcal{D}_F$ such that $E = F(D)$ and using Theorem 4.8, we can see that $E \wedge Q = F(D) \wedge Q = F(D \wedge Q) \in \mathcal{E}_F$ for all $Q \in \mathcal{D}$. Thus, in particular, $\mathcal{E}_F \wedge \mathcal{D}_F \subset \mathcal{E}_F$ also holds. Hence, by Theorem 2.12, it is clear that the required assertion is also true.

By the above theorem, it is clear that in particular we also have

Corollary 5.6. *Let $\mathcal{A}$ be a poset and suppose that F is an injective multiplier of an ideal $\mathcal{D}$ of $\mathcal{A}^*$ onto a subset $\mathcal{E}$ of $\mathcal{A}$ such that $\mathcal{D}$ separates the points of $\mathcal{A}$. Then $\mathcal{E}$ separates the points of $\mathcal{A}^*$.*

Moreover, by using Theorem 5.5, we can see that in some particular cases the maximal extension F^- of an injective multiplier F is also injective.

Theorem 5.7. *Let $(\mathcal{D}, \mathcal{E})$ be a pairing in a poset $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{E}$ and $\mathcal{E} \wedge \mathcal{D} \subset \mathcal{E}$. Suppose that F is an injective multiplier of a subset $\mathcal{D}_F$ of $\mathcal{D}$ to $\mathcal{E}$ such that $\mathcal{D}_F \wedge \mathcal{D} \subset \mathcal{D}_F$ and $\mathcal{D}_F$ separates the points of $\mathcal{E}$. Then F^- is also injective.*

Proof. In this case, by Theorem 5.5, the range $\mathcal{E}_F$ of F separates the points of $\mathcal{D}$. Hence, since F^- is an extension of F, it is clear that the range $\mathcal{E}_{F^-}$ of F^- separates the points of the domain $\mathcal{D}_{F^-}$ of F^-. Therefore, by Theorem 5.1, the required assertion is also true.

Corollary 5.8. *Let $\mathcal{A}$ be a poset and suppose that F is an injective multiplier of an ideal $\mathcal{D}$ of $\mathcal{A}^*$ to $\mathcal{A}$ such that $\mathcal{D}$ separates the points of $\mathcal{A}$. Then F^- is also injective.*

6. Some further results on injective multipliers

A counterpart of the following theorem is attributed to Devinatz and Hirschman by Wang [31, p.1134].

Theorem 6.1. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$, and F is an injective multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then F^{-1} is an injective multiplier of $\mathcal{E}$ onto $\mathcal{D}$.*

Proof. In this case, we evidently have

$$F^{-1}(E_1) \wedge E_2 = F^{-1}(E_1) \wedge F\left(F^{-1}(E_2)\right) =$$

$$F\left(F^{-1}(E_1)\right) \wedge F^{-1}(E_2) = E_1 \wedge F^{-1}(E_2)$$

for all $E_1, E_2 \in \mathcal{E}$.

Now, we are ready to prove the following counterpart of Theorem 3.9.

Theorem 6.2. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{E}$ separates the points of $\mathcal{D}$, and F is a multiplier of a subset $\mathcal{D}_F$ of $\mathcal{D}$ onto $\mathcal{E}$ such that $F'(\mathcal{D}_F) \subset \mathcal{D} \cap \mathcal{E}$, then $F = \Delta_{\mathcal{D}_F}$.*

Proof. In this case, by Theorem 5.1, F is injective and $\mathcal{D}_F$ separates the points of $\mathcal{E}$. Hence, by Theorem 6.1, it follows that F^{-1} is a multiplier of $\mathcal{E}$ onto $\mathcal{D}_F$. Moreover, by Theorem 4.13, it is clear that not only F, but also F^{-1} is nonexpansive. Namely, we have

$$\big(F^{-1}\big)'(E) = F^{-1}(E) \wedge E = F\big(F^{-1}(E)\big) \wedge F^{-1}(E) = F'\big(F^{-1}(E)\big) \in \mathcal{D}$$

for all $E \in \mathcal{E}$. Therefore, by Theorem 3.9, the required assertion is also true.

Hence, it is clear that in particular we also have

Corollary 6.3. *If $\mathcal{A}$ is a poset and F is a multiplier of a subset $\mathcal{D}$ of $\mathcal{A}^*$ onto a subset $\mathcal{E}$ of $\mathcal{A}$ such that $F'(\mathcal{D}) \subset \mathcal{A}^* \cap \mathcal{E}$ and $\mathcal{E}$ separates the points of $\mathcal{A}^*$, then $F = \Delta_{\mathcal{D}}$.*

Corollary 6.4. *If F is a multiplier of a subset $\mathcal{D}$ of $\mathcal{A}$ onto a subset $\mathcal{E}$ of $\mathcal{A}$ such that $F'(\mathcal{D}) \subset \mathcal{E}$ and $\mathcal{E}$ separates the points of $\mathcal{A}^*$, then $F = \Delta_{\mathcal{D}}$.*

Moreover, by using Corollary 6.3, we can also prove the following

Theorem 6.5. *If $\mathcal{A}$ is a poset and F is a multiplier of a subset $\mathcal{D}$ of $\mathcal{A}^*$ to $\mathcal{A}$ such that the sets $F^{-1}(\mathcal{A}^*)$ and $F\big(F^{-1}(\mathcal{A}^*)\big)$ separates the points of $\mathcal{A}$ and $\mathcal{A}^*$, respectively, then $F = \Delta_{\mathcal{D}}$.*

Proof. Define $\mathcal{D}_0 = F^{-1}(\mathcal{A}^*)$ and $\mathcal{E}_0 = F\big(F^{-1}(\mathcal{A}^*)\big)$, and denote by F_0 the restriction of F to $\mathcal{D}_0$. Then, by the corresponding definitions, it is clear that $\mathcal{D}_0 \subset \mathcal{D}$ and $\mathcal{E}_0 \subset \mathcal{A}^*$ $\big($in fact, $\mathcal{E}_0 = F(\mathcal{D}) \cap \mathcal{A}^*\big)$, and F_0 is a multiplier of $\mathcal{D}_0$ onto $\mathcal{E}_0$. Moreover, from Theorem 4.13 we can see F_0 is nonexpansive. Therefore, $F_0{}' = F_0$, and thus $F_0{}'(\mathcal{D}_0) = \mathcal{E}_0$. Hence, by using Corollary 6.3, we can infer that $F_0 = \Delta_{\mathcal{D}_0}$. On the other hand, from Theorem 4.14, we know that F_0 has a unique maximal extension F_0^-. Therefore, we necessarily have $F_0^- = \Delta_{\mathcal{A}^*}$, and thus the required assertion is also true.

Now, as an immediate consequence of this theorem, we can also state

Corollary 6.6. *If F is a multiplier of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ onto a subset $\mathcal{E}$ of $\mathcal{A}$ such that both $\mathcal{D}$ and $\mathcal{E}$ separate the points of $\mathcal{A}$, then $F = \Delta_{\mathcal{D}}$.*

From Theorem 6.2, by Theorem 5.1, it is clear that the following theorem is also true.

Theorem 6.7. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D}$ *separates the points of* $\mathcal{E}$, *and* F *is an injective multiplier of* $\mathcal{D}$ *onto* $\mathcal{E}$ *such that* $F'(\mathcal{D}) \subset \mathcal{D} \cap \mathcal{E}$, *then* $F = \Delta_{\mathcal{D}}$.

Moreover, by using Theorem 5.5 instead of Theorem 5.1, we can also prove the following

Theorem 6.8. *Let* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D} \subset \mathcal{E}$. *Suppose that* F *is an injective multiplier of a subset* $\mathcal{D}_F$ *of* $\mathcal{D}$ *to* $\mathcal{E}$ *such that* $\mathcal{D}_F \wedge \mathcal{D} \subset \mathcal{D}_F$ *and* $F'(\mathcal{D}_F) \subset \mathcal{D} \cap \mathcal{E}$, *and* $\mathcal{D}_F$ *separates the points of* $\mathcal{E}$. *Then* $F = \Delta_{\mathcal{D}_F}$.

Proof. In this case, by Theorem 4.13, F is nonexpansive. On the other hand, by Theorem 6.1, F^{-1} is a multiplier of the range $\mathcal{E}_F$ of F onto $\mathcal{D}_F$. Moreover, by Theorem 5.5, $\mathcal{E}_F$ separates the points of $\mathcal{D}$. Therefore, again by Theorem 4.13, F^{-1} is also nonexpansive. Namely, we again have $\left(F^{-1}\right)'(E) \in \mathcal{D}$ for all $E \in \mathcal{E}_F$. Therefore, by Theorem 3.9, the required assertion is also true.

Hence, it is clear that in particular we also have

Corollary 6.9. *If* $\mathcal{A}$ *is a poset and* F *is an injective multiplier of an ideal* $\mathcal{D}$ *of* $\mathcal{A}^*$ *to* $\mathcal{A}$ *such that* $F'(\mathcal{D}) \subset \mathcal{A}^*$ *and* $\mathcal{D}$ *separates the points of* $\mathcal{A}$, *then* $F = \Delta_{\mathcal{D}}$.

Corollary 6.10. *If* F *is an injective multiplier of an ideal* $\mathcal{D}$ *of a semilattice* $\mathcal{A}$ *to* $\mathcal{A}$ *such that* $\mathcal{D}$ *separates the points of* $\mathcal{A}$, *then* $F = \Delta_{\mathcal{D}}$.

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Institute of Mathematics and Informatics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
E-mail address: szaz@math.klte.hu

Institute of Mathematics and Computer Science, College of Nyiregyháza, H-4400 Nyiregyháza, Pf. 166, Hungary
E-mail address: turij@zeus.nyf.hu

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ON SOME INEQUALITIES FOR THE ε-ENTROPY NUMBERS

NICOLAE TIŢA

Abstract. We prove the inequalities:

$$\sum_{n=1}^{k} \alpha_n \epsilon_n \left(S_1 + \ldots + S_r\right) \leq \left(2^r - 1\right) c \sum_{n=1}^{k} \alpha_n \left(\epsilon_n(S_1) + \ldots + \epsilon_n(S_r)\right)$$

and

$$\sum_{n=1}^{k} \alpha_n \, \epsilon_n(S_1 \ \ldots \ S_r) \leq \left(2^r - 1\right) c \sum_{n=1}^{k} \alpha_n \epsilon_n(S_1) \ldots \epsilon_n(S_r),$$

$k = 1, 2, \ldots,$ $r \geq 2$, where $(\epsilon_n(S))$ is the sequence of ϵ - entropy numbers of the linear and bounded operator $S : X \to X\,(S \in L(X))$ and (α_n) is such that $1 = \alpha_1 \geq \ldots \geq 0$ and $\alpha_{n^r} \leq \frac{c}{n^{r-1}} \alpha_n$, $\forall n \in N$. X is a Banach space.

1. **Introduction**

Let X be a Banach space and let $T : X \to X$ be a linear and bounded operator $(T \in L(X))$. The ϵ - entropy numbers of the operator T are defined, [1],[2],[4],[6], as follows:

$$\epsilon_n(T) = \inf\{\sigma > 0 \, : \, \exists y_1, \ldots y_n \in X \text{ s.t. } TU_X \subseteq \cup_{i=1}^{n} \{y_i + \sigma U_X\}\}, \ n = 1, 2, \ldots,$$

where $U_X = \{x \in X : \| x \| \leq 1\}$.

It is well known [1],[4],[6] that: $\| T \| = \epsilon_1(T) \geq \epsilon_2(T) \geq \ldots \geq 0$ and $\epsilon_{n_1 n_2}(S + T) \leq \epsilon_{n_1}(S) + \epsilon_{n_2}(T), \epsilon_{n_1 n_2}(ST) \leq \epsilon_{n1}(S)\epsilon_{n_2}(T),\ n_1, n_2 = 1, 2, \ldots.$

In the papers [5],[6] are presented the inequalities:

$$\sum_{n=1}^{k} \frac{\epsilon_n(S + T)}{n} \leq 3 \sum_{n=1}^{k} \frac{\epsilon_n(S) + \epsilon_n(T)}{n} \qquad (a)$$

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$$\sum_{n=1}^{k} \frac{\epsilon_n(ST)}{n} \leq 3 \sum_{n=1}^{k} \frac{\epsilon_n(S) \cdot \epsilon_n(T)}{n}, \qquad k = 1, 2, \ldots \tag{b}$$

By reiteration we obtain:

$$\sum_{n=1}^{k} \frac{\epsilon_n\left(S_1 + \ldots + S_r\right)}{n} \leq 3^{r-1} \sum_{n=1}^{k} \frac{\epsilon_n(S_1) + \ldots + \epsilon_n(S_r)}{n} \tag{a'}$$

and an analog inequality (b') for the product of r operators.

In this paper we prove, in a simple way, that the factor 3^{r-1} can be replaced by $(2^r - 1)$.

More generally, is [6], the sequence $\left(\frac{1}{n}\right)$ is replaced by (α_n), where $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ and $\alpha_{n^2} \leq \frac{c}{n}\alpha_n,\ \forall n \in N$

2. **Results**

Firstly we remark that, from the inequalities of ϵ-entropy numbers for the sum and product of two operators we obtain:

Proposition 1.1 *The ϵ- entropy numbers verify the following inequalities:*

$$\epsilon_{n^r}\left(S_1 + \ldots + S_r\right) \leq \epsilon_n\left(S_1\right) + \ldots + \epsilon_n\left(S_r\right) \tag{1}$$

$$\epsilon_{n^r}\left(S_1 \ldots S_r\right) \leq \epsilon_n\left(S_1\right) \ldots \epsilon_n\left(S_r\right) \tag{2}$$

Now we prove:

Theorem 1.2. *The ϵ-entropy numbers verify the inequalities:*

$$\sum_{n=1}^{k} \alpha_n\, \epsilon_n\left(S_1 + \ldots + S_r\right) \leq \left(2^r - 1\right) c \sum_{n=1}^{k} \alpha_n\left(\epsilon_n\left(S_1\right) + \ldots + \epsilon_n\left(S_r\right)\right) \tag{3}$$

$$\sum_{n=1}^{k} \alpha_n \epsilon_n\left(S_1\ \ldots\ S_r\right) \leq \left(2^r - 1\right) c \sum_{n=1}^{k} \alpha_n \epsilon_n\left(S_1\right)\ \ldots\ \epsilon_n\left(S_r\right), \tag{4}$$

where (α_n) is a sequence such that $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ and $\alpha_{n^r} \leq \frac{c}{n^{r-1}}\alpha_n,\ \forall n \in N;\ \ k = 1, 2, \ldots$

Proof. We prove only (4). Tthe proof for (3) is similar. By using the inequality (2) and the fact that the sequence $(\epsilon_n(S))$ is non increasing we obtain:

$$\sum_{n=1}^{k} \alpha_n \epsilon_n (S_1 \dots S_r) \le \sum_{n=1}^{(k+1)^r-1} \alpha_n \epsilon_n (S_1 \dots S_r) =$$

$$= \sum_{n=1}^{k} \sum_{i=n^r}^{(n+1)^r-1} \alpha_i \epsilon_i (S_1 \dots S_r) \le$$

$$\le \sum_{n=1}^{k} [(n+1)^r - n^r] \, \alpha_{n^r} \epsilon_{n^r} (S_1 \dots S_r) \le$$

$$\le \sum_{n=1}^{k} \left(2^k - 1\right) n^{r-1} \frac{c}{n^{r-1}} \alpha_n \epsilon_{n^r} (S_1 \dots S_r) \le$$

$$\le (2^r - 1) \; c \sum_{n=1}^{k} \alpha_n \epsilon_n(S_1) \dots \epsilon_n(S_r).$$

The proof is fulfiled.

3. **Application**

Let l_∞ be the normed space of all bounded sequence, where

$$\| x \|_\infty = \sup_n |x_n|.$$

For all $x \in l_\infty$, $card(x) = card\{n \in \mathcal{N} \; : \; x_n \neq 0\}$. We denote by K the set of all sequences $x \in l_\infty$ such that $card(x) \le n$ and $x_1 \ge x_2 \ge \dots \ge 0$.
A function $\phi : K \to R$ is called symmetric norming function, [3],[4],[6], if:

1. $\phi(x) > 0,$ for $x \in K, \; x \neq 0;$
2. $\phi(\alpha_n) = \alpha\phi(x), \; \alpha \ge 0, x \in K;$
3. $\phi(x+y) \le \phi(x) + \phi(y);$
4. $\phi(1, 0, 0, \dots) = 1;$
5. If $x, y \in K$ are such that

$$\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i, \; k = 1, 2, \dots$$

then $\phi(x) \le \phi(y)$.

Example of such functions are: $\phi_\infty : \; x \in K \to x_1, \; \phi_1 : \; x \in K \to \sum_1^n x_i$ and $\phi_\omega : \; x \in K \to \sum_{i=1}^n \frac{x_i}{i}$.

It is known, [3],[6],[7], that, for all symmetric norming function ϕ, the functions: $\phi_{(p)} : (x_i) \in K \to (\phi(x_i^p))^{\frac{1}{p}}$, $1 \leq p < \infty$ and $\overline{\phi} : (x_i) \in K \to \phi(\{\alpha_i x_i\})$ are symmetric norming functions.

If $x \in l_\infty$ are such that $x_1 \geq x_2 \geq \ldots \geq 0$, we consider

$$\phi(x) = \lim_{n\to\infty} \phi(x_1, \ldots, x_n, 0, 0, \ldots).$$

In [4], [7], the classes $L^{(\epsilon)}_{\phi_{(p)}}(X)$ are considered, where $L^{(\epsilon)}_{\phi_{(p)}}(X) = \{T \in L(X) : \phi_{(p)}(\{\epsilon_n(T)\}) < \infty\}$, $1 \leq p < \infty$. If ϕ is replaced by $\overline{\phi}$, from the inequality (a) and the Minkowski inequality (for $\phi_{(p)}$,[3],[4],[7]) in [5], [7] is proved that

$\| T \|^{(\epsilon)}_{\overline{\phi}_{(p)}} = \overline{\phi}_{(p)}(\epsilon_n(T)) = (\phi(\{\alpha_n \epsilon_n^p(T)\}))^{\frac{1}{p}}$ is a quasi-norm.

From the above inequality (a') and the properties (2) and (5) of the functions ϕ, it results that:

$$\| \sum_{n=1}^{r} S_n \|^{(\epsilon)}_{\overline{\phi}} \leq 3^{r-1} \sum_{n=1}^{r} \| S_n \|^{(\epsilon)}_{\overline{\phi}},$$

but from the theorem 1.2 we obtain that the factor 3^{r-1} can be replaced by $(2^r - 1)$ if $\alpha_n = \frac{1}{n}$, $n = 1, 2, \ldots$. A similar result is also true for all sequences (α_n) as above.

Remarks: For the dyadic entropy numbers $e_n(T) = \epsilon_{2^{n-1}}(T)$, $n = 1, 2, \ldots$, are known, [4] , [7], the inequalities:

$$\sum_{n=1}^{k} e_n(S \star T) \leq 2 \sum_{n=1}^{k} e_n(S) \star e_n(T),$$

where $\star$ is $+$ or $\bullet$.

For the case of r operators $r > 2$ it results:

$$\sum_{n=1}^{k} e_n(S_1 \star \ldots \star S_r) \leq r \sum_{n=1}^{k} e_n(S_1) \star \ldots \star e_n(S_r),\ k = 1, 2, \ldots$$

This results from the fact that $e_{(n-1)r+1}(S_1 \star \ldots \star S_r) \leq e_n(S_1) \star \ldots \star e_n(S_r)$ as follows:

$$\sum_{n=1}^{k} e_n(S_1 \star \ldots \star S_r) \leq \sum_{n=1}^{rk} e_n(S_1 \star \ldots \star S_r) = \sum_{n=1}^{k} \sum_{i=(n-1)r+1}^{rn} e_i(S_1 \star \ldots \star S_r)$$

$$\leq r \sum_{n=1}^{k} e_{(n-1)r+1}(S_1 \star \ldots \star S_r) \leq r \sum_{n=1}^{k} e_n(S_1) \star \ldots \star e_n(S_r).$$

We can also prove the inequality

$$\prod_{n=1}^{k} e_n \left(\prod_{i=1}^{r} S_i \right) \leq \prod_{n=1}^{k} \prod_{i=1}^{r} e_n^r (S_i), \quad k = 1, 2, \ldots; \ r \geq 2.$$

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"Transilvania" University of Braşov, Faculty of Sciences,
Department of Mathematics, 2200 Braşov, Romania
E-mail address: tita@info.unitbv.ro

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