Anul XLVII **2002**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
2

Redacţia: 3400 Cluj-Napoca, str. M. Kogalniceanu nr. 1 • Telefon: 405300

SUMAR – CONTENTS – SOMMAIRE

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 2, June 2002

SIMPLE SUBALGEBRAS OF GROUP GRADED ALGEBRAS

ILUŞCA BONTA

Abstract. We study the situation when the 1-component A_1 of a G-graded $\mathcal{O}$-algebra A has an $\mathcal{O}$-simple subalgebra $S \simeq M_n(\mathcal{O})$. We prove that the centralizer $C_A(S)$ of S is a graded subalgebra of A, and that there is a graded Morita equivalence between A and $C_A(S)$. This generalizes a theorem of L. Puig.

1. **Introduction**

Let G be a finite group and let $\mathcal{O}$ be a commutative local noetherian ring, complete with respect to the $J(\mathcal{O})$-adic topology, and such that the residue field $k = \mathcal{O}/J(\mathcal{O})$ is algebraically closed of characteristic $p > 0$. All $\mathcal{O}$-algebras are assumed to be finitely generated and free as $\mathcal{O}$-modules.

If $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ are two G-graded $\mathcal{O}$-algebras, then recall that the $\mathcal{O}$-algebra homomorphism $f\colon A \to B$ is called *G-graded* if $f(A_g) \subseteq B_g$ for all $g \in G$. A subalgebra C of A is a *graded subalgebra* if for any $c = \sum_{g\in G} c_g \in C$, the homogeneous component c_g also belongs to C for all $g \in G$. In this case we have that $C = \bigoplus_{g\in G} C_g$, where $C_g = C \cap A_g$.

An $\mathcal{O}$-algebra S is called *$\mathcal{O}$-simple* if is isomorphic to $\mathrm{End}_O(V)$ for some free $\mathcal{O}$-module V, that is, if S isomorphic to a matrix algebra $M_n(\mathcal{O})$ over $\mathcal{O}$ (where n is the dimension of V).

The centralizer of the subalgebra S in A is, by definition, the subalgebra

$$C_A(S) = \{a \in A \mid as = sa \text{ for all } s \in S\}.$$

If B is a G-graded $\mathcal{O}$-algebra, then the matrix algebra $A = M_n(B)$ is a G-graded algebra, where for each $g \in G$, A_g consists of matrices with entries in B_g. The A_1 has a subalgebra S isomorphic to $M_n(\mathcal{O})$, and there is an isomorphism $C_A(S) \simeq B$ of G-graded algebras, mapping an element $a \in C_A(S)$ to $eae = ea = ae$, where e is the matrix having 1 in the top left corner and 0 elsewhere. Moreover, there is an isomorphism $A \simeq S \otimes_{\mathcal{O}} C_A(S)$ of G-graded algebras, and there is a graded Morita equivalence between A and B (see Section 3 below).

In this note we consider the converse situation. We assume that $A = \bigoplus_{g\in G} A_g$ is a G-graded algebra and $S \simeq M_n(\mathcal{O})$ is an $\mathcal{O}$-simple subalgebra of A_1, and we show that there is a graded Morita equivalence between A and $C_A(S)$. This generalizes a theorem of L. Puig [2] (see also [3, Sections 1.7 and 1.9]. For notions and results on graded algebras and graded Morita equivalences we refer to [1].

2. Simple subalgebras

In this section $A = \bigoplus_{g\in G} A_g$ is a G-graded $\mathcal{O}$-algebra and $S \simeq \mathrm{End}_{\mathcal{O}}(L)$ be a G-graded $\mathcal{O}$-simple subalgebra of A_1 with $1_S = 1_A$. Let $C_A(S)$ be the centralizer of S and let e be a primitive idempotent of S. The next results are generalizations of [3, Propositions 7.5 and 7.6].

Proposition 2.1 *With the above notations and assumptions, the following statements hold.*

a) *$C_A(S)$ is a G-graded subalgebra of S.*

b) *There is an isomorphism of G-graded $\mathcal{O}$-algebras given by*

$$\phi : S \otimes_{\mathcal{O}} C_A(S) \to A, \qquad \phi(s \otimes a) = sa.$$

c) *There is an isomorphism of G-graded $\mathcal{O}$-algebras given by*

$$\eta : C_A(S) \to eAe, \qquad \eta(a) = ea = ae = eae.$$

Proof. a) We know that $C_A(S)$ is a subalgebra of A. We have to prove that $C_A(S)$ is G-graded subalgebra. For any $a = \sum_{g\in G} a_g \in A$, if $a \in C_A(S)$, then we have $as = sa$ for all $s \in S$. It follows that $\sum_{g\in G} a_g s = \sum_{g\in G} s a_g$. Since $S \subseteq A_1$, $a_g s = s a_g$ for all $s \in S$ and $g \in G$. This means that $a_g \in C_A(S)$ for all $g \in G$.

b) We know from the proof of [3, Proposition 7.5] that ϕ is an isomorphism of $\mathcal{O}$-algebras and that the map

$$\psi : A \to S \otimes_{\mathcal{O}} C_A(S), \quad \psi(a) = \sum_{u,v\in U} (u^{-1}ev \otimes \sum_{w\in U} (euav^{-1}e)^w)$$

is $\mathcal{O}$-algebra homomorphism, which is the inverse of ϕ. Here U denotes a finite set of invertible elements of S satisfying $1_S = \sum_{u\in U} e^u$ (recall that all the primitive idempotents of S are conjugate). We only have to verify that ϕ and ψ are grade-preserving.

Because A is a G-graded algebra, we have that $S \otimes_{\mathcal{O}} C_A(S)$ is also G-graded, with components $(S \otimes_{\mathcal{O}} C_A(S))_g = S \otimes_{\mathcal{O}} C_A(S)_g$. If $s \otimes a_g \in S \otimes_{\mathcal{O}} C_A(S)_g$, we have that $\phi(s \otimes a_g) = sa_g$ belongs to $SA_g \subseteq A_1A_g = A_g$, hence $\phi(S \otimes_{\mathcal{O}} C_A(S)_g) \subseteq A_g$. Finally, if $a_g \in A_g$ then

$$\psi(a_g) = \sum_{u,v\in U} (u^{-1}ev \otimes \sum_{w\in U} (eua_gv^{-1}e)^w) \in S \otimes_{\mathcal{O}} C_A(S)_g$$

since $U \subset A_1$, so $\psi(A_g) \subseteq S \otimes_{\mathcal{O}} C_A(S)_g$.

c) We know that $C_A(S)$ and eAe are isomorphic as $\mathcal{O}$-algebras. We have to prove they are isomorphic as G-graded algebras. For all $a_g \in C_A(S)_g$ we have $\eta(a_g) = ea_ge$ belongs to eA_ge, so $\eta(C_A(S)_g) \subseteq eA_ge$. Consequently η is G-graded. Similarly, the inverse of η, given by $eae \mapsto \sum_{u\in U}(eae)^u$ is a G-graded map, so the proposition is proved.

Proposition 2.2. Let M be a G-graded A-module. Then there is an isomorphism of G-graded A-modules given by

$$\phi : Se \otimes_{\mathcal{O}} eM \to M, \qquad \phi(s \otimes m) = sm.$$

Proof. Since M is a G-graded A-module and $e \in S \subseteq A_1$, we have that eM is a G-graded eAe-submodule of M, hence eM is a G-graded $C_A(S)$-module via the isomorphism η of Proposition 2.1 c). Consequently $Se \otimes_{\mathcal{O}} eM$ is a G-graded $S \otimes_{\mathcal{O}} C_A(S)$-module. We know that ϕ is homomorphism of A-modules. Letting $1_A = 1_S = \sum_{u\in U} e^u$ be a primitive decomposition of the identity in S, consider the map

$$\psi : M \to Se \otimes_{\mathcal{O}} eM, \qquad \psi(m) = \sum_{u\in U} u^{-1}e \otimes eum,$$

where U is a finite set of invertible elements of S.

We are going to show that ψ is the inverse of ϕ and that both maps are grade-preserving. First we have that

$$\begin{aligned}(\phi\circ\psi)(m) &= \phi(\sum_{u\in U} u^{-1}e\otimes eum)=\sum_{u\in U}\phi(u^{-1}e\otimes eum)\\ &= \sum_{u\in U} u^{-1}eum=\sum_{u\in U} e^u m=m,\end{aligned}$$

because $1_S = 1_A = \sum_{u\in U} e^u$.

On the other hand let $m\in M$ and let $s^{-1}et$ be a basis element of S, where $s,t\in U$. Then we have

$$\begin{aligned}(\psi\circ\phi)(s^{-1}ete\otimes em) &= \psi(s^{-1}etem)=\sum_{u\in U}u^{-1}e\otimes eus^{-1}etem\\ &= \sum_{u\in U}u^{-1}e\otimes u(u^{-1}eu)(s^{-1}es)s^{-1}tem\\ &= s^{-1}e\otimes etem = s^{-1}ete\otimes em,\end{aligned}$$

where we have used that $e^u e^s = 0$ unless $u=s$.

For all $s\otimes m_g \in Se\otimes_{\mathcal{O}} eM_g$, we have that $\phi(s\otimes m_g) = sm_g$ belongs to $SM_g\subseteq M_g$, so $\phi(Se\otimes_{\mathcal{O}} eM_g)\subseteq M_g$. Similarly, if $m_g\in M_g$, the $\psi(m_g)$ belongs to $Se\otimes_{\mathcal{O}} eM_g$ since $U\subset A_1$ and $e\in A_1$.

3. **A Morita equivalence**

We keep the notations and assumptions of the preceding section. The following result is a generalization to the case of G-graded algebras of [2, Theorem 3].

Theorem 3.1. *The algebras A and $C_A(S)$ are graded Morita equivalent.*

Proof. Since A is isomorphic to $S\otimes_{\mathcal{O}} C_A(S)$ as G-graded algebras, it is enough to prove the following statement. Let C be an $\mathcal{O}$-algebra and let $S\simeq \mathrm{End}_{\mathcal{O}}(L)$ be an $\mathcal{O}$-simple algebra. Then $S\otimes_{\mathcal{O}} C$ is graded Morita equivalent to C. Indeed, consider the functor

$$F: C-\mathrm{mod}\to S\otimes_{\mathcal{O}} C-\mathrm{mod},\quad F(M)=L\otimes_{\mathcal{O}} M.$$

Observe that if $M=\bigoplus_{x\in G} M_x$ is a G-graded C-module, then $F(M)$ is a G-graded $S\otimes_{\mathcal{O}} C$-module with components $F(M)_x = L\otimes_{\mathcal{O}} M_x$ for all $x\in G$. Moreover, if $M(g)$ is the g-th suspension of M (where $M(g)_x = M_{xg}$ for all $x\in G$), then

$F(M(g)) = F(M)(g)$. Therefore, the restriction of F gives a graded functor F^{gr} : $C-\mathrm{gr} \to S \otimes_{\mathcal{O}} C-\mathrm{gr}$, which clearly commutes with the grade forgetting functor. It remains to prove that F is an equivalence of categories. Observe that $L \simeq Se$, where e is a primitive idempotent of S. By replacing A with $A \otimes_{\mathcal{O}} C$ and e with $e \otimes 1$, Proposition 2.2 shows that any $S \otimes_{\mathcal{O}} C$-module is naturally isomorphic to a module of the form $L \otimes_{\mathcal{O}} M$, where M is a C-module. This immediately implies that F is an equivalence.

Remark 3.2. Alternatively, we could have used the isomorphism $C_A(S) \simeq eAe$ of G-graded algebras. Since $1_A = 1_S = \sum_{u \in U} e^u$, we have that $AeA = A$. Consequently, the G-graded bimodules Ae and e induce a graded Morita equivalence between A and eAe.

References

[1] A. Marcus, *Representation Theory of Group Graded Algebras*, Nova Science Publishers, Commack N.Y. 1999.

[2] L. Puig, *Pointed groups and construction of characters*, Math. Z. **176** (1981), 209–216.

[3] J. Thévenaz, *G-Algebras and Modular Representation Theory*, Clarendon Press, Oxford, 1995.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

Received: 28.06.2001

UNIVALENCE CONDITIONS FOR CERTAIN INTEGRAL OPERATORS

DANIEL BREAZ, NICOLETA BREAZ

Abstract. In this paper the result of V. Pescar and S. Owa, on univalence conditions of integral operators, is extended to the case of n univalent functions. New results are presented in theorems 1 and 3.

1. Introduction

Let A be the class of functions f, which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$ and let us denote with S the class of univalent functions.

Theorem A. [4] *If f is the univalent in U ,$\alpha \in C$ and $|\alpha| \leq \frac{1}{4}$ then the function*

$$G_\alpha(z) = \int_0^z [f'(t)]^\alpha \, dt$$

is univalent in U.

Theorem B. [3] *If the function $g \in S$,$\alpha \in C$, $|\alpha| \leq \frac{1}{4n}$ then the function defined by*

$$G_{\alpha,n}(z) = \int_0^z [f'(t^n)]^\alpha \, dt$$

is univalent in U for $n \in N^*$.

Theorem C. [2] *Let $\alpha, a \in C, \operatorname{Re}\alpha > 0$ and $f \in A$.*

If

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$$

$(\forall)\, z \in U$ *then* $(\forall)\, \beta \in U, \operatorname{Re}\beta \geq \operatorname{Re}\alpha$, *the function*

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t) \right]^{\frac{1}{\beta}}$$

is univalent.

Theorem D [1]. *If the function g is olomorphic in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$ the following inequalities hold*

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right| \qquad (*)$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}$$

the equalities hold in case $g(z) = \varepsilon\frac{z+u}{1+\overline{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark E [1]. For $z = 0$, from inequality (*) we obtain for every $\xi \in U$

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)|\,|\xi|}$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|}$$

for all $z \in U$.

Lemma F (Schwartz). *If the function g is olomorphic in $U, g(0) = 0$ and $|g(z)| \leq 1\,(\forall)\,z \in U$ then result:*

$|g(z)| \leq |z|, (\forall)\,z \in U$ *and* $|g'(0)| \leq 1$ *the equalities hold in case* $g(z) \leq \varepsilon z$ *where* $|\varepsilon| = 1$.

Theorem G [5]. *Let $\alpha, \gamma \in C, \operatorname{Re}\alpha = a > 0, g \in A, g(z) = z + b_2z^2 + \ldots$.*

If

$$\left|\frac{g''(z)}{g'(z)}\right| \leq \frac{1}{n}, (\forall)\,z \in U$$

and

$$|\gamma| \leq \frac{n + 2a}{2} \cdot \left(\frac{n + 2a}{n}\right)^{\frac{n}{2a}}$$

then $(\forall), \beta \in C, \operatorname{Re}\beta \geq a$, the function

$$G_{\beta,\gamma,n}(z) = \left\{\beta\int_0^z t^{\beta-1} \cdot [g'(t^n)]^\gamma\, dt\right\}^{\frac{1}{\beta}}$$

is univalent $(\forall)\,n \in N^* \setminus \{1\}$.

Theorem H [5]. *Let* $\alpha, \gamma \in C, \operatorname{Re}\alpha = b > 0, g \in A, g(z) = z + a_2 z^2 + \dots$.

If

$$\left|\frac{g''(z)}{g'(z)}\right| \leq 1, (\forall)\, z \in U$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2c}}{c} \cdot |z| \cdot \frac{|z|+2|a_2|}{1+2|a_2||z|}\right]}$$

then $(\forall)\, \beta \in C, \operatorname{Re}\beta \geq b$, *the function*

$$G_{\beta,\gamma}(z) = \left\{\beta \int_0^z t^{\beta-1} \cdot [g'(t)]^{\gamma}\, dt\right\}^{\frac{1}{\beta}}$$

is univalent.

2. Main results

Theorem 1. *Let* $\alpha, \gamma_i \in C, (\forall)\, i = \overline{1,p}, \operatorname{Re}\alpha = a \geq 0, f_i \in A, f_i(z) = z + a_2^i z^2 + \dots, (\forall)\, i = \overline{1,p}$.

If

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq \frac{1}{n}, (\forall)\, z \in U, i = \overline{1,p} \tag{1}$$

$$\frac{|\gamma_1| + \dots + |\gamma_p|}{|\gamma_1 \cdot \dots \cdot \gamma_p|} < 1 \tag{2}$$

and

$$|\gamma_1 \cdot \dots \cdot \gamma_p| \leq \frac{n+2a}{2} \cdot \left(\frac{n+2a}{n}\right)^{\frac{n}{2a}} \tag{3}$$

then $(\forall)\, \beta \in C, \operatorname{Re}\beta \geq a$,*the function*

$$G(z) = \left\{\beta \int_0^z t^{\beta-1} \cdot [f_1'(t^n)]^{\gamma_1} \cdot \dots \cdot [f_p'(t^n)]^{\gamma_p}\, dt\right\}^{\frac{1}{\beta}}$$

is univalent $(\forall)\, n \in N^* \setminus \{1\}$.

Proof. Let

$$h(z) \int_0^z [f_1'(t^n)]^{\gamma_1} \cdot \dots \cdot [f_p'(t^n)]^{\gamma_p}\, dt$$

$$p(z) = \frac{1}{|\gamma_1 \cdot \dots \cdot \gamma_p|} \cdot \frac{h''(z)}{h'(z)}$$

where $|\gamma_1 \cdot \dots \cdot \gamma_p|$ satisfies (3).

We have

$$p(z)=\frac{\gamma_1}{|\gamma_1\cdot...\cdot\gamma_p|}\cdot\frac{n\cdot z^{n-1}\cdot f_1''(z^n)}{f_1'(z^n)}+...+\frac{\gamma_p}{|\gamma_1\cdot...\cdot\gamma_p|}\cdot\frac{n\cdot z^{n-1}\cdot f_p''(z^n)}{f_p'(z^n)}$$

Applying the relations (1) and (2) we obtain:

$$|p(z)|\leq\frac{|\gamma_1|}{|\gamma_1\cdot...\cdot\gamma_p|}\cdot\left|\frac{n\cdot z^{n-1}\cdot f_1''(z^n)}{f_1'(z^n)}\right|+...+\frac{|\gamma_p|}{|\gamma_1\cdot...\cdot\gamma_p|}\cdot\left|\frac{n\cdot z^{n-1}\cdot f_p''(z^n)}{f_p'(z^n)}\right|\leq$$

$$\leq\frac{|\gamma_1|+...+|\gamma_p|}{|\gamma_1\cdot...\cdot\gamma_p|}<1$$

Considering Schwartz's lemma we have:

$$\frac{1}{|\gamma_1\cdot...\cdot\gamma_p|}\cdot\left|\frac{h''(z)}{h'(z)}\right|\leq\left|z^{n-1}\right|\leq|z|\Leftrightarrow\left|\frac{h''(z)}{h'(z)}\right|\leq|\gamma_1\cdot...\cdot\gamma_p|\cdot\left|z^{n-1}\right|\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{1-|z|^{2a}}{a}\right)\cdot\left|\frac{zh''(z)}{h'(z)}\right|\leq|\gamma_1\cdot...\cdot\gamma_p|\cdot\left(\frac{1-|z|^{2a}}{a}\right)\cdot|z^n| \qquad (4)$$

Let′s the function $Q:[0,1]\to R, Q(x)=\left(\frac{1-x^{2a}}{a}\right)\cdot x^n, x=|z|.$

We have

$$Q(x)\leq\frac{n+2a}{2}\cdot\left(\frac{n+2a}{n}\right)^{\frac{n}{2a}}\ (\forall)\,x\in[0,1]$$

According to the conditions (3) and (4) we obtain:

$$\left(\frac{1-|z|^{2a}}{a}\right)\cdot\left|\frac{zh''(z)}{h'(z)}\right|\leq 1$$

so, according to Theorem C, G is univalent.

Corollary 2. *Let* $\alpha,\beta,\gamma,\delta\in C, \operatorname{Re}\delta=a>0, f,g\in A, f(z)=z+a_2z^2+...,g(z)=z+b_2z^2+...$.

If

$$\left|\frac{f''(z)}{f'(z)}\right|\leq\frac{1}{n},(\forall)\,z\in U$$

$$\left|\frac{g''(z)}{g'(z)}\right|\leq\frac{1}{n},(\forall)\,z\in U$$

$$\frac{1}{|\alpha|}+\frac{1}{|\beta|}<1$$

and

$$|\alpha\beta|\leq\frac{n+2a}{2}\cdot\left(\frac{n+2a}{n}\right)^{\frac{n}{2a}}$$

then $(\forall)\,\gamma \in C, \operatorname{Re}\gamma \geq a$,*the function*

$$D_{\alpha,\beta,\gamma,n}(z) = \left\{ \gamma \int_0^z t^{\gamma-1} \cdot \left[f'(t^n)\right]^{\alpha} \cdot \left[g'(t^n)\right]^{\beta} dt \right\}^{\frac{1}{\gamma}}$$

is univalent $(\forall)\, n \in N^* \setminus \{1\}$.

Proof. In Theorem 1, we consider $p = 2, f_1 = f, f_2 = g, \gamma_1 = \alpha, \gamma_2 = \beta, \gamma = \beta$.

Remark. If in Theorem 1, we consider $p =, f_1 = g, \gamma_1 = \gamma, \gamma = \beta$, we obtained Theorem G.

Theorem 3. *Let* $\alpha, \gamma_i \in C, (\forall)\, i = \overline{1,n}, \operatorname{Re}\alpha = b > 0, f_i \in A, f_i(z) = z + a_2^i z^2 + ..., (\forall)\, i = \overline{1,n}$.

If

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq 1, (\forall)\, z \in U, i = \overline{1,n} \tag{5}$$

$$\frac{|\gamma_1| + ... + |\gamma_n|}{|\gamma_1 \cdot ... \cdot \gamma_n|} < 1 \tag{6}$$

and

$$|\gamma_1 \cdot ... \cdot \gamma_n| \leq \frac{1}{\max\limits_{|z|\leq 1} \left[\frac{1-|z|^{2b}}{b} \cdot |z| \cdot \frac{|z|+2|c|}{1+2|c||z|} \right]} \tag{7}$$

then $(\forall)\,\beta \in C, \operatorname{Re}\beta \geq b$,*the function*

$$H(z) = \left\{ \beta \int_0^z t^{\beta-1} \cdot \left[f_1'(t)\right]^{\gamma_1} \cdot ... \cdot \left[f_n'(t)\right]^{\gamma_n} dt \right\}^{\frac{1}{\beta}}$$

is univalent $(\forall)\, n \in N$.

Proof. Let

$$h(z) = \int_0^z \left[f_1'(t)\right]^{\gamma_1} \cdot ... \cdot \left[f_n'(t)\right]^{\gamma_n} dt$$

$$p(z) = \frac{1}{|\gamma_1 \cdot ... \cdot \gamma_n|} \cdot \frac{h''(z)}{h'(z)}$$

where $|\gamma_1 \cdot ... \cdot \gamma_n|$ satisfies (7).

We have

$$p(z) = \frac{\gamma_1}{|\gamma_1 \cdot ... \cdot \gamma_n|} \cdot \frac{f_1''(z)}{f_1'(z)} + ... + \frac{\gamma_n}{|\gamma_1 \cdot ... \cdot \gamma_n|} \cdot \frac{f_n''(z)}{f_n'(z)}$$

p is olomorph and $|\gamma_1 \cdot ... \cdot \gamma_n|$ satisfies the relation (7) implies $|p(z)| < 1$ according to (5) and (6).

$$p(0) = \frac{\gamma_1 a_2^1 + ... + \gamma_n a_2^n}{|\gamma_1 \cdot ... \cdot \gamma_n|} = c$$

$$|p(z)| \leq \frac{|z| + 2|c|}{1 + 2|c||z|}, (\forall) z \in U \Leftrightarrow \frac{1}{|\gamma_1 \cdot ... \cdot \gamma_n|} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{|z| + 2|c|}{1 + 2|c||z|}, (\forall) z \in U \Leftrightarrow$$

$$\left(\frac{1 - |z|^{2b}}{b} \right) \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq |\gamma_1 \cdot ... \cdot \gamma_n| \cdot \max_{|z| \leq 1} \left[\frac{1 - |z|^{2b}}{b} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c||z|} \right] \leq 1, (\forall) z \in U$$

so, according to Theorem C, H is univalent $(\forall), n \in N$.

Corollary 4. *Let* $\alpha, \beta, \gamma, \delta \in C, \operatorname{Re} \delta = c > 0, f, g \in A, f(z) = z + a_2 z^2 + ..., g(z) = z + b_2 z^2 + ...$.

If

$$\left| \frac{f''(z)}{f'(z)} \right| \leq 1, (\forall) z \in U$$

$$\left| \frac{g''(z)}{g'(z)} \right| \leq 1, (\forall) z \in U$$

$$\frac{1}{|\alpha|} + \frac{1}{|\beta|} < 1$$

and

$$|\alpha\beta| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1 - |z|^{2c}}{c} \cdot |z| \cdot \frac{|z| + 2\frac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}}{1 + 2\frac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}|z|} \right]}$$

then $(\forall) \gamma \in C, \operatorname{Re} \gamma \geq c$, *the function*

$$F_{\alpha,\beta,\gamma}(z) = \left\{ \gamma \int_0^z t^{\gamma - 1} \cdot [f'(t)]^\alpha \cdot [g'(t)]^\beta \, dt \right\}^{\frac{1}{\gamma}}$$

is univalent.

Proof. In Theorem 3, we consider $p = 2, f_1 = f, f_2 = g, \gamma_1 = \alpha, \gamma_2 = \beta$.

Remark. If in Theorem 3, we consider $p = 1, f_1 = g, \gamma_1 = \gamma, \gamma = \beta$, we obtained Theorem H.

References

[1] Z. Nehari, Conformal Mapping, McGraw-Hill Book Co., Inc., New York, Toronto, London, 1952.

[2] N.N. Pascu, An improvement of Becker's univalence criterion, Proceedings of the Commemorative Session Simion Stoilow, Braşov, (1987), 43-48.

[3] N.N. Pascu, V. Pescar, On the integral operators of Kim-Merkens and Pfaltzgraff, Studia(Mathematica), Univ. Babeş-Bolyai, Cluj-Napoca, 32, 2(1990), 185-192.

[4] J. A. Pfaltzgraff, Univalence of the integral of , Bull. London Math. Soc. 7(1975), no 3.254-256.

[5] V. Pescar and Shigeyoshi Owa, Univalence of certain integral operators, International J. Math. & Math. Sci. Vol. 23.No.10(2000), 697-701.

Acknowledgements. The author would like to thank to Prof. Petru T. Mocanu, from Department of Mathematics, University of Cluj, for all his encouragement and help in performing this research work.

1 Decembrie 1918 University of Alba Iulia, Departament of Mathematics

Received: 20.12.2001

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 2, June 2002

ON A QUOTIENT CATEGORY

SIMION BREAZ AND CIPRIAN MODOI

Abstract. We construct a suitable quotient category, in order to give natural interpretations of the notions almost projective module and almost injective module.

Introduction

The study of finite rank torsion free abelian groups using quasi-notions (see [2]) imposed in module theory the notions as "almost projective module", "almost injective module" and "almost flat module" ([1], [6], [9], [10]). In [11], E. Walker give a natural setting for the study of quasi-homomorphisms of abelian group, constructing the quotient category of the category of the abelian groups modulo the Serre class of all bounded groups, and he shows, that this quotient category is equivalent to the category constructed by Reid in [7], in order to give a categorial interpretation of the B. Jónson's quasi-decomposition theorem (see [2, Corollary 7.9]).

In this paper we show, using a analogous construction to the E. Walker's one, that the mentioned "almost-notions" have natural interpretations in a suitable quotient category.

1. **The basic construction**

Let $\mathcal{A}$ be an additive category and $S \subseteq \mathbb{N}^*$ be a multiplicative system such that $1 \in S$. We consider the class of all homomorphisms of the form $n_A = n1_A$, with $A \in \mathcal{A}$ and $n \in S$, and we denote it by Σ. Then, for each $A \in \mathcal{A}$, the class of all homomorphisms belonging to Σ having the domain or the codomain A are sets, being subclasses of the set $\mathrm{End}_{\mathcal{A}}(A)$. Thus the left, respectively right, fractional category of $\mathcal{A}$ with respect Σ is defined [5, chap. 1, 14.6]. Moreover, it is straightforward

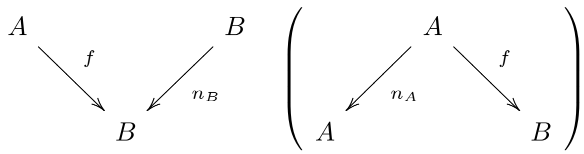
2000 *Mathematics Subject Classification.* 16E40.

Key words and phrases. quotient category, almost projective module, almost injective module.

to check that Σ is a bicalculable multiplicative system of homomorphisms in $\mathcal{A}$ in the sense of [5, chap. 1, 1.14] so, the notions of left fractional category, the right fractional category and the category of additive fraction of $\mathcal{A}$ with respect Σ coincide [5, chap. 1, 14.5 and chap. 4, 7.5]. We shall denote by $\mathcal{A}[\Sigma^{-1}]$ this category, and by $\mathbf{q}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$ (or simply $\mathbf{q}$ if there is no danger of confusion) the canonical functor. Recall that, this functor makes invertible the homomorphisms belonging to Σ, and satisfies the following universal property: for every category $\mathcal{B}$ and for every functor $F : \mathcal{A} \to \mathcal{B}$ making invertible all homomorphisms of Σ, there is a unique functor $\bar{F} : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}$ such that $\bar{F}\mathbf{q} = F$. Note that we may consider, the objects of $\mathcal{A}[\Sigma^{-1}]$ are the same as the objects of $\mathcal{A}$, in which case $\mathbf{q}(A) = A$ for all $A \in \mathcal{A}$, and

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}[\Sigma^{-1}]}(A,B) = \{\mathbf{q}(n_B)^{-1}\mathbf{q}(f)\mathbf{q}(m_A)^{-1} = \mathbf{q}(nm_B)^{-1}\mathbf{q}(f) = \\ \mathbf{q}(f)\mathbf{q}(nm_A)^{-1} \mid f \in \mathrm{Hom}_{\mathcal{A}}(A,B) \text{ and } n, m \in S\}.\end{aligned}$$

Since the category $\mathcal{A}[\Sigma^{-1}]$ may be seen as the category of left (right) fractions, the homomorphism $\mathbf{q}(n_B)^{-1}\mathbf{q}(f) : A \to B$ in this category may be visualized as a diagram of homomorphisms in $\mathcal{A}$:



Keeping in the mind that $f = 1_B^{-1} f$, we shall denote sometimes the above homomorphism by $n^{-1}f$ or, how we shall see, by $\frac{1}{n} \otimes f$.

Note that the functor $\mathbf{q}$ is left and right exact (that is, it commutes with finite limits and colimits), hence if $\mathcal{A}$ is finitely complete or finitely cocomplete then $\mathcal{A}[\Sigma^{-1}]$ has the same property [5, chap. 1, 14.5]. Moreover, if $\mathcal{A}$ is an abelian category, then $\mathcal{A}[\Sigma^{-1}]$ is abelian too [5, chap. 4, 7.6].

Let $\mathcal{A}$ be a full subcategory of an additive category $\mathcal{B}$. Let $\mathcal{A}'$ be the full subcategory of $\mathcal{B}[\Sigma^{-1}]$ consisting of those objects $\mathbf{q}(A)$, with $A \in \mathcal{A}$. The inclusion functor $\mathbf{i} : \mathcal{A} \to \mathcal{B}$ induce a fully faithful functor $\bar{\mathbf{i}} : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}[\Sigma^{-1}]$ which factors through $\mathcal{A}'$, the induced functor being representative, hence it is an equivalence.

In the sequel, we assume that *the category $\mathcal{A}$ is a locally small (that is, the subobjects of an object form a set) abelian category.* A full subcategory $\mathcal{T}$ of the category $\mathcal{A}$ is called *thick* or *Serre class* if, for every exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, $B \in \mathcal{T}$ if and only if $A \in \mathcal{T}$ and $C \in \mathcal{T}$. Then, the class of homomorphisms in $\mathcal{A}$

$$\Sigma_{\mathcal{T}} = \{f|\ \operatorname{Ker}(f) \in \mathcal{T},\ \operatorname{Coker}(f) \in \mathcal{T}\}$$

is a bicalculable multiplicative system and the category $\mathcal{A}[\Sigma^{-1}]$ is defined [5, chap. 4, 7.7 and 7.8], and it is denoted by $\mathcal{A}/\mathcal{T}$. In addition, the canonical functor carries all object from $\mathcal{T}$ into 0, and we can give the universal property with functors satisfying this condition [5, chap 4, exercise 7.3, f)] (see also [4, Section III]).

Remark 1.1 Using the construction of the fractional category of $\mathcal{A}$ with respect to $\Sigma_{\mathcal{T}}$, [5, Theorem 1.14.1], we observe that in this case the category $\mathcal{A}/\mathcal{T}$ is exactly the category constructed by Gabriel in [4]. We recall that in this construction the groups of homomorphisms in the quotient category $\mathcal{A}/\mathcal{T}$ which is induced by a Serre class $\mathcal{T}$ are the limit

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{T}}(A, B) = \varinjlim_{A/A' \in \mathcal{T}, B' \in \mathcal{T}} \operatorname{Hom}(A', B/B')$$

and the operations are canonical.

Returning to our case, we say that an object $A \in \mathcal{A}$ is *S-bounded* if there is $n \in S$ such that $n_A = 0$. If more precision is required, then the object A is called *bounded by $n \in S$*. It is straightforward to check that the class of all S-bounded objects of $\mathcal{A}$ forms a thick subcategory, the extremes of an extension being bounded by the same integer as the middle term, and this by the product of the integers which bound the extremes. We denote by $\mathcal{S}$ this subcategory. Clearly, the categories $\mathcal{A}[\Sigma^{-1}]$ and $\mathcal{A}/\mathcal{S}$ are isomorphic.

Lemma 1.2. *Let $\mathcal{A}$ be an abelian category, $1 \in S \subseteq \mathbb{N}^*$ and $\Sigma = \{n_A | A \in \mathcal{A}, n \in S\}$. Then the following hold for the category $\mathcal{A}[\Sigma^{-1}]$:*

a) *The homomorphism $\mathbf{q}(f)$ is a monomorphism in $\mathcal{A}[\Sigma^{-1}]$ if and only if* $\ker f$ *is S-bounded, for every homomorphism f in $\mathcal{A}$. The homomorphism $n^{-1}f$ is a monomorphism in $\mathcal{A}[\Sigma^{-1}]$ if and only if $\mathbf{q}(f)$ is a monomorphism in $\mathcal{A}[\Sigma^{-1}]$.*

b) *The homomorphism* $\mathbf{q}(f)$ *is an epimorphism in* $\mathcal{A}[\Sigma^{-1}]$ *if and only if* $\operatorname{coker} f$ *is* S*-bounded, for every homomorphism* f *in* $\mathcal{A}$*. The homomorphism* $n^{-1}f$ *is an epimorphism in* $\mathcal{A}[\Sigma^{-1}]$ *if and only if* $\mathbf{q}(f)$ *is an epimorphism in* $\mathcal{A}[\Sigma^{-1}]$*.*

c) *The homomorphism* $\mathbf{q}(f)$ *is an isomorphism in* $\mathcal{A}[\Sigma^{-1}]$ *if and only if* $\ker f$ *and* $\operatorname{coker} f$ *are* S*-bounded, for every homomorphism* f *in* $\mathcal{A}$*. The homomorphism* $n^{-1}f$ *is an isomorphism in* $\mathcal{A}[\Sigma^{-1}]$ *if and only if* $\mathbf{q}(f)$ *is an isomorphism in* $\mathcal{A}[\Sigma^{-1}]$*.*

Proof. The sentences relative to f are easy consequences of exactness of $\mathbf{q}$. But n_A^{-1} are isomorphisms in $\mathcal{A}[\Sigma^{-1}]$, for all $A \in \mathcal{A}$, and this completes the proof.

As is [11] we consider the category $\mathbb{Z}[S^{-1}]\mathcal{A}$, whose objects are the same as the objects of $\mathcal{A}$, and the homomorphisms sets are

$$\operatorname{Hom}_{\mathbb{Z}[S^{-1}]\mathcal{A}}(A,B) \cong \mathbb{Z}[S^{-1}] \otimes \operatorname{Hom}_{\mathcal{A}}(A,B),$$

for all $A, B \in \mathcal{A}$. The proof of the following results is inspired by the E. Walker's proof of [11, Theorem 3.1].

Proposition 1.3. *The categories* $\mathcal{A}[\Sigma^{-1}]$ *and* $\mathbb{Z}[S^{-1}]\mathcal{A}$ *are isomorphic.*

Proof. We shall view the quotient category $\mathcal{A}/\mathcal{S}$ as in Remark 1.1. If $n \in S$ and $A \in \mathcal{A}$, then we will denote $A[n] = \operatorname{Ker}(n_A)$ and $nA = \operatorname{Im}(n_A)$. If $n'_A : A \to nA$ is the canonical epimorphism n_A and $i_n = i_n^A : A[n] \to A$ is the canonical monomorphism, we have the exact sequence

$$0 \to A[n] \overset{i_n^A}{\to} A \overset{n'_A}{\to} nA \to 0.$$

Moreover, denote by $\alpha_n = \alpha_n^A : nA \to A/A[n]$ the canonical isomorphism and by $p_n = p_n^A = \alpha_n^A n'_A$ the canonical projection.

Let $f : A \to C$ be a homomorphism in $\mathcal{A}$ and $n \in S$. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[n] & \xrightarrow{i_n^A} & A & \xrightarrow{p_n^A} & A/A[n] & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle f} & & & & \\ 0 & \longrightarrow & C[n] & \xrightarrow{i_n^C} & C & \xrightarrow{p_n^C} & C/C[n] & \longrightarrow & 0 \end{array}$$

and the equalities $n_C f i_n^A = f n_A i_n^A = 0$ imply that there exists a homomorphism $f_n : A/A[n] \to C/C[n]$ such that $f_n p_A^n = p_C^n f$. Then $f_n \alpha_n : nA \to C/C[n]$. The objects A/nA and $C[n]$ are in the class $\mathcal{S}$, so $f_n\alpha_n$ represents a homomorphism $\overline{f_n\alpha_n} \in \operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(A,C)$.

We consider the additive functor $F : \mathbb{Z}[S^{-1}]\mathcal{A} \to \mathcal{A}/\mathcal{B}$ with $F(A) = A$ and $F(\frac{1}{n}f) = \overline{f_n\alpha_n}$ for every homomorphism $\frac{1}{n}f$ in $\mathbb{Z}[S^{-1}]\mathcal{A}$.

In order to construct the inverse of F, we define a functor $G : \mathcal{A}/\mathcal{B} \to \mathbb{Z}[S^{-1}]\mathcal{A}$, putting $G(A) = A$ for every object A, and for homomorphisms constructing the image by G in the following way: Let $\overline{g} \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A, B)$ be a homomorphism which is represented by $g' : A' \to B/B'$ and $n \in S$ such that $n_{A/A'} = 0$ and $n_{B'} = 0$. Then we can suppose that $\overline{g}$ is represented by $g : nA \to B/B[n]$. We put $G(\overline{g}) = \frac{1}{n^2}((\alpha_n^B)^{-1}gn'_A)$.

We denote by g' the composition between the restriction of g to n^2A with the canonical projection $\pi : B/B[n] \to B/B[n^2]$ and we obtain

$$g'(\alpha_{n^2}^A)^{-1}p_{n^2}^B = \pi gn^2 = \pi ngn = \pi p_n^B(\alpha_n^B)^{-1}gn = p_{n^2}^B(\alpha_n^B)^{-1}gn.$$

It follows, using the universal property of cokernels,

$$((\alpha_n^B)^{-1}gn)_{n^2} = g'(\alpha_{n^2}^A)^{-1}.$$

Thus

$$F(G(\overline{g})) = \overline{(\alpha_n^B)^{-1}gn)_{n^2}\alpha_{n^2}} = \overline{g'} = \overline{g}.$$

If $\frac{1}{n}f \in \mathbb{Z}[S^{-1}]\,\mathrm{Hom}(A, B)$, then

$$G(F(\frac{1}{n}f)) = G(\overline{f_n\alpha_n}) = \frac{1}{n^2}((\alpha_n^B)^{-1}f_n\alpha_n^Bn)) = \frac{1}{n^2}nf = \frac{1}{n}f,$$

and the proof is complete.

Proposition 1.4. *The canonical functor* $\mathbf{q} : \mathcal{A} \to \mathbb{Z}[S^{-1}]\mathcal{A}$ *preserves the injective objects and the projective objects.*

Proof. Let I be an injective object in $\mathcal{A}$ and let $0 \to A \overset{\frac{1}{n}\alpha}{\to} B$ be a monomorphism in $\mathbb{Z}[S^{-1}]\mathcal{A}$. Then $\alpha : A \to B$ is a homomorphism in $\mathcal{A}$ for which exists $n \in S$ with $n\,\mathrm{Ker}(\alpha) = 0$. Because I is an injective object, we obtain the exact sequence of abelian groups

$$\mathrm{Hom}(B, I) \to \mathrm{Hom}(A, I) \to \mathrm{Hom}(\mathrm{Ker}(\alpha), I) \to 0.$$

Applying the tensor product with S, which is an exact functor, because the group $\mathbb{Z}[S^{-1}]$ is torsion free, we find the exact sequence

$$\mathbb{Z}[S^{-1}]\,\mathrm{Hom}(B, I) \to \mathbb{Z}[S^{-1}]\,\mathrm{Hom}(A, I) \to \mathbb{Z}[S^{-1}]\,\mathrm{Hom}(\mathrm{Ker}(\alpha), I) \to 0$$

in which $\mathbb{Z}[S^{-1}]\operatorname{Hom}(\operatorname{Ker}(\alpha), I) = 0$ because for every $f \in \operatorname{Hom}(\operatorname{Ker}(\alpha), I)$ we have $nf = fn_{\operatorname{Ker}(\alpha)} = 0$, hence $n\operatorname{Hom}(\operatorname{Ker}(\alpha), I) = 0$. It follows that $\mathbf{q}(I)$ is an injective object in $\mathbb{Z}[S^{-1}]\mathcal{A}$.

Dual it may be proved that S preserves the projective objects.

Corollary 1.5. *If $\mathcal{A}$ is an abelian category with enough injective objects, then:*

a) $\mathbb{Z}[S^{-1}]\mathcal{A}$ has enough injective objects,

b) for all $A, C \in \mathcal{A}$ and $n \in S$ there exists the canonical isomorphisms $\operatorname{Ext}^n_{\mathbb{Z}[S^{-1}]\mathcal{A}}(C, A) \cong \mathbb{Z}[S^{-1}] \otimes \operatorname{Ext}^n_{\mathcal{A}}(C, A)$.

Proof. a) We choose an injective resolution for A in $\mathcal{A}$:

$$0 \to A \to I \to I/A \to 0,$$

and Proposition 1.4 proves that

$$0 \to \mathbf{q}(A) \to \mathbf{q}(I) \to \mathbf{q}(I/A) \to 0$$

represents an injective resolution in $\mathbb{Z}[S^{-1}]\mathcal{A}$.

b) We proceed by induction, using an injective resolution as in a). For the first step of the induction, observe that $\mathbb{Z}[S^{-1}] \otimes \operatorname{Ext}^1_{\mathcal{A}}(C, A)$ and $\operatorname{Ext}^1_{\mathbb{Z}[S^{-1}]\mathcal{A}}(C, A)$ are isomorphic, being both the cokernels of the homomorphism $\mathbb{Z}[S^{-1}]\operatorname{Hom}_{\mathcal{A}}(C, I) \to \mathbb{Z}[S^{-1}]\operatorname{Hom}_{\mathcal{A}}(C, A)$. Furthermore, for every $1 \le n \in S$ we have the isomorphisms

$$\operatorname{Ext}^n_{\mathbb{Z}[S^{-1}]\mathcal{A}}(C, I/A) \cong \operatorname{Ext}^{n+1}_{\mathbb{Z}[S^{-1}]\mathcal{A}}(C, A),$$

respectively

$$\operatorname{Ext}^n_{\mathcal{A}}(C, I/A) \cong \operatorname{Ext}^{n+1}_{\mathcal{A}}(C, A).$$

Corollary 1.6. *If $\mathcal{A}$ has enough projective objects, then $\mathbb{Z}[S^{-1}]\mathcal{A}$ has the same property.*

2. Almost projective and almost injective objects

Throughout of this section, $\mathcal{A}$ *will be a locally small abelian category with enough projective and enough injective objects.* An object P of $\mathcal{A}$ is called *S-almost projective (injective)*, if $\mathbf{q}(P)$ is an projective (respectively injective) object in $\mathbb{Z}[S^{-1}]A$.

Lemma 2.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then there exists a unique functor $\bar{F} : \mathbb{Z}[S^{-1}]\mathcal{A} \to \mathbb{Z}[S^{-1}]\mathcal{B}$ such that $\bar{F}\mathbf{q}_{\mathcal{A}} = \mathbf{q}_{\mathcal{B}}F$*

Proof. The statement is a consequence of the universal property of a quotient category modulo a Serre class and of the fact that, F being additive, for every $n \in S$ and for every $A \in \mathcal{A}$ we have $F(n_A) = n_{F(A)}$.

If A is an object in $\mathcal{A}$, then we shall denote by

$$\mathbf{H}_A = \mathrm{Hom}(A, -) : \mathcal{A} \to \mathcal{A}b$$

the canonical covariant functor, and by

$$\bar{\mathbf{H}}_A : \mathbb{Z}[S^{-1}]\mathcal{A} \to \mathbb{Z}[S^{-1}]\mathcal{A}b$$

the functor, which is induced by $\mathbf{H}$.

We recall that an abelian group G is a S-torsion group if the order of every element $g \in G$ is in S and G is S-bounded if there exists $n \in S$ such that $nG = 0$. We record the following characterization of the almost projective objects:

Proposition 2.2. *For a $P \in \mathcal{A}$, the following conditions are equivalent:*

(i) *P is S-almost projective;*

(ii) *The group $\mathrm{Ext}^1_{\mathcal{A}}(P, A)$ is S-torsion, for all $A \in \mathcal{A}$;*

(iii) *The group $\mathrm{Ext}^1_{\mathcal{A}}(P, A)$ is S-bounded, for all $A \in \mathcal{A}$;*

(iv) *There is an integer $n = n(P) \in S$, such that $n\,\mathrm{Ext}^1_{\mathcal{A}}(P, A) = 0$ for all $A \in \mathcal{A}$;*

(v) *There is an integer $n = n(P) \in S$, such that $n\,\mathrm{coker}\,\mathbf{H}_P(\alpha) = 0$, for all epimorphisms α in $\mathcal{A}$;*

(vi) *The functor $\bar{\mathbf{H}}_P$ is exact.*

Proof. As we have seen before in 1.5, $\mathrm{Ext}^1_{\mathbb{Z}[S^{-1}]A}(P, A) \cong \mathbb{Z}[S^{-1}] \otimes \mathrm{Ext}^1_{\mathcal{A}}(A, B)$, so (i)⇔(ii) is immediate. Moreover, the implications (iv)⇒(iii), (iii)⇒(ii) and (iv)⇒(v) are obvious.

(ii)⇒(iv). We suppose that, for every $n \in S$, there exists $A_n \in \mathcal{A}$ such that $n\,\mathrm{Ext}_{\mathcal{A}}(P, A_n) \neq 0$. Then

$$\mathrm{Ext}^1_{\mathcal{A}}(P, \prod_{n \in S} A_n) \cong \prod_{n \in S} \mathrm{Ext}^1_{\mathcal{A}}(P, A_n)$$

is not S-torsion.

(v)⇒(vi) follows by 1.2.

(vi)⇒(iii) The exactness of $\bar{\mathbf{H}}_P$ implies that the group $\operatorname{coker} \mathbf{H}_P(\alpha)$ is bounded by an integer $n_\alpha > 0$ for any epimorphism α in $\mathcal{A}$. By the Ker-Coker Lemma, we deduce that $\operatorname{Ext}^1_{\mathcal{A}}(P, A)$ are bounded, for all $A \in \mathcal{A}$

We will say that the exact sequence in $\mathcal{A}$:

$$E : 0 \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0$$

S-splits if it represents a splitting exact sequence in $\mathbb{Z}[S^{-1}]\mathcal{A}$ and, consequently, a S-monomorphism or a S-epimorphism *S-splits* if it splits in $\mathbb{Z}[S^{-1}]$Mod-R.

Lemma 2.3. *Let* $E : 0 \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0$ *be an exact sequence in* Mod-R. *The following are equivalent:*

a) E *S-splits;*

b) There exists $\beta' : M \to L$ *such that* $\beta\beta' = n1_M$ *for some integer* $n \in S$*;*

c) There exists $\alpha' : L \to K$ *such that* $\alpha'\alpha = n1_L$ *for some integer* $n \in S$.

Proof. This statement is a consequence of the characterization of the split short exact sequences [5, Exercise 4.3.13], using the fact that, for some integer $n \in S$, the homomorphism n_A represents the identity of A in $\mathbb{Z}[S^{-1}]$Mod-R.

Corollary 2.4. *The following are equivalent for an* R*-module* P*:*

a) P *is an almost projective;*

b) Every epimorphism $M \to P \to 0$ *in* $\mathcal{A}$ *splits in* $\mathbb{Z}[S^{-1}]\mathcal{A}$*;*

c) P *is isomorphic in* $\mathbb{Z}[S^{-1}]$Mod-R *to a* S*-direct summand in a free module;*

d) There exist an integer $n \in S$*, a family of* R*-homomorphisms* $\varphi_i : P \to R$*,* $i \in I$*, and* $x_i \in P$*,* $i \in I$*, such that* $na = \sum_{i \in I} \varphi_i(a)x_i$ *for all* $a \in P$.

Proof. *a)* ⇒ *b)* If $\alpha : M \to P$ is an epimorphism and P is almost projective, then the kernel of $\bar{\mathbf{H}}_P(\alpha) : \operatorname{Hom}_{\mathcal{A}}(P, M) \to \operatorname{Hom}_{\mathcal{A}}(P, P)$ is bounded by an integer $n \in S$ and this shows that there exists $\alpha' : P \to M$ such that $\alpha\alpha' = n1_P$.

b) ⇒ *c)* is obvious.

c) ⇒ *d)* If P is S-isomorphic to a S-direct summand in $R^{(I)}$, let $\beta : R^{(I)} \to P$ be a S-epimorphism which splits in $\mathbb{Z}[S^{-1}]$Mod-R, and let $\varphi : P \to R^{(I)}$ be a homomorphism such that $\beta\phi = n1_P$. We choose a basis $(e_i)_{i \in I}$ in $R^{(I)}$. Denote by $\varphi_i : P \to R$ the composite homomorphisms $\pi_i\varphi$, where $\pi_i : R^{(I)} \to R$ are the canonical projections, and put $x_i = \beta(e_i)$, where $(e_i)_{i \in I}$ is the canonical basis in

$R^{(I)}$. Since, for every $a \in P$, $\varphi_i(a) = 0$ for almost all i, we may write $\varphi : P \to R$, $\varphi = \sum_{i \in I} \varphi_i$ and we obtain

$$na = \beta\varphi(a) = \sum_{i \in I} (\beta\varphi_i)(a) = \sum_{i \in I} (\varphi_i(a)\beta(e_i)) = \sum_{i \in I} (\varphi_i(a)x_i).$$

d) $\Rightarrow$ *c*) The family $\varphi_i : P \to R$ induces a homomorphism $\varphi : P \to R^{(I)}$, while the correspondences $e_i \mapsto x_i$ give a homomorphism $\beta : R^{(I)} \to P$ which is an epimorphism in $\mathbb{Z}[S^{-1}]$Mod-R. Moreover, $\beta\varphi = n1_P$ showing that β splits in $\mathbb{Z}[S^{-1}]$Mod-R.

The implication *c*) $\Rightarrow$ *a*) is a consequence of the following observation: the class of projective objects in an abelian category is closed with respect the direct summands and the canonical functor preserves the projective objects.

We shall say that the pair $((\varphi_i)_{i \in I}, (x_i)_{i \in I})$ is a *S-dual basis for P*. Observe that for a finite generated module we may suppose that I is a finite set.

Corollary 2.5. *If P is a S-almost projective right R-module, then $\mathbb{Z}[S^{-1}] \otimes P$ is a projective $\mathbb{Q} \otimes R$-module.*

Proof. We consider $((\varphi_i)_{i \in I}, (x_i)_{i \in I})$ a S-dual basis for P. Then there exists an integer $n \in S$ such that for every $m \in M$, we have $nm = \sum_{i \in I} \varphi_i(m)x_i$. Thus

$$(1 \otimes \varphi_i, \frac{1}{n} \otimes x_i) \in \mathbb{Z}[S^{-1}] \otimes \mathrm{Hom}_{\mathbb{Z}[S^{-1}] \otimes R}(\mathbb{Z}[S^{-1}] \otimes M, \mathbb{Z}[S^{-1}] \otimes R) \times \mathbb{Z}[S^{-1}] \otimes M,$$

with $i \in I$, is a dual basis for the $\mathbb{Z}[S^{-1}] \otimes R$-module $\mathbb{Z}[S^{-1}] \otimes M$.

Remark 2.6. The converse of the previous statement is not true. Indeed, if A is a torsion abelian group which is not a bounded group, then it is not almost projective over $\mathbb{Z}$ (see corollary 2.8), but $0 = \mathbb{Z}[S^{-1}] \otimes A$ is a projective $\mathbb{Z}[S^{-1}] \otimes \mathbb{Z}$-module.

Proposition 2.7. *If R is a hereditary ring, then an R-module is S-almost projective if and only if it is a direct sum between a projective R-module and a R-module which is bounded as an abelian group.*

Proof. Let P be a S-almost projective R-module. If F is a free R-module and $\beta : F \to P$ is a R-epimorphism then β splits in $\mathbb{Z}[S^{-1}]$Mod-R, hence there exists $\beta' : P \to F$ such that $\beta\beta' = n1_P$ for some integer $n \in S$. Hence $\mathrm{Ker}(\beta')$ is bounded as an abelian group and it follows that P and $\mathrm{Im}(\beta')$ are isomorphic in $\mathbb{Z}[S^{-1}]$Mod-R. Observe that $\mathrm{Im}(\beta')$ is projective and it follows that the exact sequence $0 \to \mathrm{Ker}(\beta') \to P \to \mathrm{Im}(\beta') \to 0$ splits. The converse it obvious.

Corollary 2.8. *An abelian group is S-almost projective as $\mathbb{Z}$-module if and only if it is a direct sum between a free abelian group and a S-bounded group.*

In an analogous way we may prove

Proposition 2.9. *The following are equivalent for an object $I \in \mathcal{A}$:*

(i) *I is S-almost injective;*

(ii) *The group $\mathrm{Ext}_{\mathcal{A}}(A, I)$ is S-torsion for all $A \in \mathcal{A}$;*

(iii) *The group $\mathrm{Ext}^1_{\mathcal{A}}(A, I)$ is S-bounded, for all $A \in \mathcal{A}$;*

(iv) *There is an integer $n = n(I) \in S$, such that $n\,\mathrm{Ext}^1_{\mathcal{A}}(A, I) = 0$ for all $A \in \mathcal{A}$;*

(v) *There is an integer $n = n(I) \in S$, such that $n\,\mathrm{coker}\,\mathbf{H}^I(\alpha) = 0$, for all monomorphisms α in $\mathcal{A}$;*

(vi) *The functor $\bar{\mathbf{H}}^I$ is exact.*

Corollary 2.10. *An abelian group A is almost injective as an $\mathbb{Z}$-module if and only if $A \cong B \oplus D$ with B a bounded group and D a divisible group.*

For almost injective R-modules, as in the standard case we obtain an analogous statement to the Baer's criterion [8, Proposition I.6.5].

Proposition 2.11. *The R-module I is almost injective if and only if there exists an integer $n \in S$ such that for every right ideal U of R and for every R-homomorphism $\alpha : U \to R$, there exists $r \in R$ such that $n\alpha(x) = rx$ for all $x \in U$.*

3. **Almost flat modules**

In the end we give the interpretation for the almost-flat modules, introduced by Albrecht and Goeters in [1].

If A is a left R-module, and $\mathbf{T}_A = - \otimes_R A : \text{Mod-}R \to \mathcal{A}b$ is the tensor product functor, we shall denote by $\overline{\mathbf{T}}_A : \mathbb{Z}[S^{-1}]\text{Mod-}R \to \mathbb{Z}[S^{-1}]\mathcal{A}b$ the induced functor. We say that A is *S-almost flat* if and only if the functor $\overline{\mathbf{T}}_A$ is exact. We obtain the following characterization which shows that our definition is compatible with the definition of almost flat modules given in [1].

Proposition 3.1. *Let A be a left R-module. Then the following are equivalents:*

a) A is S-almost flat;

b) If $f : M \to N$ is a monomorphism in Mod-R*, then the kernel of the canonical homomorphism $f \otimes_R 1_A : M \otimes_R A \to N \otimes_R A$ is S-bounded;*

c) There exists $n \in S$ such that for every monomorphism $f : M \to N$ in Mod-R *we have $n\,\mathrm{Ker}(f \otimes_R 1_A) = 0$;*

d) If $M \in$ Mod-R*, then $\mathrm{Tor}^1_R(M, A)$ is S-bounded;*

e) There exists $n \in S$ such that $n\,\mathrm{Tor}^1_R(M, A) = 0$ for all $M \in$ Mod-R.

Proof. $a) \Rightarrow b)$ If the sequence $0 \to M \xrightarrow{f} N$ is exact in Mod-R, then $\mathbf{q}(f)$ is a monomorphism, and it follows that $\bar{\mathbf{T}}_A(\mathbf{q}(f)) = \mathbf{q}(\mathbf{T}_A(f))$ is a monomorphism in $\mathbb{Z}[S^{-1}]\mathcal{A}b$ showing that $\mathrm{Ker}(\mathbf{T}_A(f))$ is S-bounded.

$b) \Rightarrow a)$ Let $0 \to \mathbf{q}(L) \to \mathbf{q}(M) \to \mathbf{q}(N) \to 0$ be an exact sequence in $\mathbb{Z}[S^{-1}]$Mod-R. Then, from [4, Corollaire III.1], there exists an exact sequence $0 \to L' \to M' \to N' \to 0$ in Mod-R such that we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{q}(L) & \longrightarrow & \mathbf{q}(M) & \longrightarrow & \mathbf{q}(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{q}(L') & \longrightarrow & \mathbf{q}(M') & \longrightarrow & \mathbf{q}(N') & \longrightarrow & 0 \end{array}$$

in $\mathbb{Z}[S^{-1}]$Mod-R, the vertical arrows being isomorphisms. The short exact sequence $0 \to L' \to M' \to N' \to 0$ induces by hypothesis the exact sequence in $\mathcal{A}b$

$$0 \to B \to \mathbf{T}_A(L') \to \mathbf{T}_A(M') \to \mathbf{T}_A(N') \to 0,$$

with B a S-bounded group. This shows that in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{\mathbf{T}}_A(L) & \longrightarrow & \bar{\mathbf{T}}_A(M) & \longrightarrow & \bar{\mathbf{T}}_A(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{\mathbf{T}}_A(L') & \longrightarrow & \bar{\mathbf{T}}_A(M') & \longrightarrow & \bar{\mathbf{T}}_A(N') & \longrightarrow & 0 \end{array}$$

the rows are exact, and the vertical arrows are isomorphisms. It follows that the sequence $0 \to \bar{\mathbf{T}}_A(L) \to \bar{\mathbf{T}}_A(M) \to \bar{\mathbf{T}}_A(N) \to 0$ is exact, hence A is S-almost flat.

$b) \Rightarrow d)$ Let M be a right R-module, and let $0 \to K \xrightarrow{f} P \to M \to 0$ be an exact sequence with P projective. We apply $\mathbf{T}_A$ to obtain $Tor^1_R(M, A) \cong \mathrm{Ker}(\mathbf{T}_A(f))$. Observe that the last group is S-bounded, because f is a monomorphism.

$d) \Rightarrow e)$ The proof is similar with the proof of [1, Proposition 2.1]

$e) \Rightarrow c)$ follows from the fact that for every monomorphism $f : M \to N$, the group $\mathrm{Ker}(\mathbf{T}_A(f))$ is a homomorphic image of $\mathrm{Tor}^1_R(N/f(M), A)$. The implication $c) \Rightarrow b)$ is obvious.

Corollary 3.2. *An abelian group G is S-almost flat as an $\mathbb{Z}$-module if and only if $G = B \oplus H$ with B a S-bounded group and H a torsion free group*

Proof. If $G = B \oplus H$ with B a S-bounded group and H a torsion free group, then for every abelian group K we obtain $\mathrm{Tor}(K, G) \cong \mathrm{Tor}(K, B)$ and the last group is S-bounded.

Conversely, if $\mathrm{Tor}(K, G)$ is bounded by $n \in S$ for all $K \in \mathcal{A}b$ then then for every integer $m > 1$, using the isomorphism $\mathrm{Tor}(\mathbb{Z}(m), G) \cong G[m]$ ([3, Property 62(H)]), we obtain $nG[m] = 0$, proving that the torsion part of G is bounded by n. Therefore, G splits and its torsion part is a S-bounded group.

References

[1] Albrecht, U., Goeters, P. : Almost Flat Abelian Groups , Rocky Mount. J. of Math.25(3) (1995), 827-842.
[2] Arnold, D. M.: Finite Rank Torsion Free Abelian Groups and Rings, Lecture Notes in Mathematics, 931, Springer-Verlag, (1982).
[3] Fuchs, L.: Infinite abelian groups I, Cambridge Univ. Press (1970).
[4] Gabriel, P: Des catégories abelienes, Bull. Soc. Math. France, 90, (1962), 323-448.
[5] N. Popescu, L. Popescu, *Theory of categories*, Editura Academiei, Bucureşti, România and Sijthoff & Noordhoff International Publishers, 1979.
[6] Reid, J. D.: Abelian groups finitely generated over their endomorphism rings, Lect. Notes in Math., 874, Springer-Verlag, (1981), 41-52.
[7] Reid, J. D.: On the ring of quasi-homomorphisms of a torsion free group, Topics in Abelian Groups, Scott-Foresman, (1963), 51-68.
[8] Strenstrom, B., Rings of Quotiens, Springer-Verlag, (1975).
[9] Visonhaler, C.: Torsion free abelian groups quasi-projective over their endomorphism rings II, Pacific J. Math., 74, (1978), 261-265.
[10] Visonhaler, C. and Wickless, W.: Torsion free abelian groups quasi-projective over their endomorphism rings, Pacific J. Math., 68, (1977), 527-535.
[11] Walker, E.: Quotient categories and quasi-isomorphisms of abelian groups, Proc. Colloq. Abelian Groups, Budapest, (1964), 147-162.

"Babeş-Bolyai" University, 3400 Cluj-Napoca, Romania
E-mail address, Simion Breaz: bodo@math.ubbcluj.ro
E-mail address, Ciprian Modoi: cip7@math.ubbcluj.ro

Received: 10.04.2002

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 2, June 2002

CATEGORICAL SEQUENCES AND APPLICATIONS

GRAŢIELA CICORTAŞ

Abstract. Ralph Fox characterized the Lusternik-Schnirelmann category using the categorical sequences. In this paper we define the notion of G-categorical sequence, where G is a compact Lie group, and we prove that the result of Fox remains true for the equivariant Lusternik-Schnirelmann category.

1. **Introduction**

In the study of some problems of differential geometry, L. Lusternik and L. Schnirelmann introduced a new numerical topological invariant, defined for every closed subset A of a manifold M, called the category (Lusternik- Schnirelmann category) of A in M. This number is the minimum cardinality of a categorical covering of A in M, where "categorical covering" means a covering by categorical sets (see [5]).

This is a well-known and much studied homotopy invariant (see [3],[4],[5]). It gives important informations about the existence of critical points: when M is a smooth manifold, the Lusternik- Schnirelmann category of M is a lower bound for the number of critical points of a smooth function on M.

2. **Categorical sequences**

Let M be a topological space. A subset $A \subset M$ is called categorical in M if there exists an open subset $U \subset M$ such that $A \subset U$ and U is contractible in M.

Following Fox [3], we define the category of $X \subset M$ in M by the minimal number k such that X can be covered by k categorical subsets in M. We denote $cat(X, M) = k$.

2000 *Mathematics Subject Classification.* 58E05.

Key words and phrases. Lusternik-Schnirelmann category, G-category, G-categorical sequence.

Because every subset of a contractible set in M is also contractible in M, we obtain that every categorical subset of M is contractible in M; the converse is not true.

A covering of X by categorical subsets of M is called a categorical covering of X in M; a categorical covering which verifies the minimal condition from definition is called minimal categorical covering.

Definition 2.1. A finite sequence $\{A_1, A_2, \ldots, A_k = X\}$ of closed subsets of X is called a categorical sequence of X in M if:

(i) $A_1 \subset A_2 \subset \ldots \subset A_k$

(ii) $A_1, A_2 - A_1, \ldots, A_k - A_{k-1}$ are categorial subsets in M.

The number k is called the length of categorical sequence $\{A_1, A_2, \ldots, A_k\}$.

Ralph Fox [3] established the following characterization of category in terms of categorical sequences:

Theorem 2.1. *Let M be a separable, arcwise connected, metric space, and let $X \subset M$ be a subspace such that $cat(X, M) < \infty$.*

Then the category of X in M, $cat(X, M)$, is the minimum of the lengths of the categorical sequences of X in M.

3. Categorical sequences method for equivariant Lusternik-Schnirelmann category

For the definition and the properties of equivariant category we follow Fadell [2].

Let M be a topological space and let G be a compact Lie group which acts on M. Let A be an invariant subspace of M. A homotopy $H : A \times I \longrightarrow M$ is called equivariant if $H(gx, t) = gH(x, t), \forall x \in A, \forall g \in G$.

Definition 3.1. The set A is called G-categorical in M if there exists an equivariant homotopy $H : A \times I \longrightarrow M$ such that $H_0 = H(\cdot, 0)$ is the inclusion and $H_1 = H(\cdot, 1)$ has the image in a single orbit $Orb(x)$.

Here $Orb(x) = \{gx|g \in G\} = Gx$ is the orbit of the point x. The G-orbits should be considered as "equivariant points". (A is G-categorical if it can be deformed equivariant in an orbit Gx.)

Definition 3.2. Let X be an invariant subspace of M. We say that X has G-category k in M and we denote $Gcat(X, M) = k$ if X can be covered by k G-categorical open subsets in M, and k is the minimal number with this property. If X cannot be covered by a finite number of such G-categorical open subsets in M, we say that $Gcat(X, M) = \infty$.

We define $Gcat(X, M) = 0$ if $X = \emptyset$.

If G acts trivially on M, then the G-category is exactly the Lusternik-Schnirelmann category.

In general, $Gcat(X, M) \geq cat(X/G, M/G)$. If the action of G on X is free, then $Gcat(X, M) = cat(X/G, M/G)$.

For G-category we know some properties, which are contained in the following proposition (see Fadell [2]):

Proposition 3.1. *(i) (normalisation) If X is an invariant subspace of M, G-categorical (open or closed), then*

$$Gcat(X, M) = 1$$

(ii) (monotonicity)If X, Y are two invariant subspaces of M and $X \subseteq Y$, then

$$Gcat(X, M) \leq Gcat(Y, M)$$

(iii) (subadditivity) If X, Y are two invariant subspaces of M, then

$$Gcat(X \cup Y, M) \leq Gcat(X, M) + Gcat(Y, M)$$

(iv) (invariance) If $\phi : M \longrightarrow M$ is an equivariant homeomorphism and X is an invariant subspace of M, then

$$Gcat(X, M) = Gcat(\phi(X), M)$$

(v) (continuity) If M is a G-ANR and X is an invariant subspace of M, then there is an open, invariant subset $U \subseteq M$ such that $X \subseteq U$ and

$$Gcat(X, M) = Gcat(U, M)$$

(vi) If $Gcat(X, M) = k$, then X has k orbits.

Now, we define the corresponding notion of categorical sequence in equivariant context:

Definition 3.3. We say that a finite sequence $\{A_1, A_2, \ldots, A_k = X\}$ of closed, invariant subsets of X is a G-categorical sequence of X in M if:

(i) $A_1 \subset A_2 \subset \ldots \subset A_k$

(ii) $A_1, A_2 - A_1, \ldots, A_k - A_{k-1}$ are G-categorical subsets in M.

k is called the length of G-categorical sequence $\{A_1, A_2, \ldots, A_k\}$.

The main result is contained in the following theorem:

Theorem 3.1. *Let M be a separable, arcwise connected, metric space and let G be a compact Lie group which acts on M. Let X be a invariant subspace of M such that $Gcat(X, M) < \infty$.*

In these conditions $Gcat(X, M)$ is the minimum of the lengths of the G-categorical sequences of X in M.

For the proof of this theorem, we need the following lemma:

Lemma 3.1. *Let M be a separable, arcwise connected, metric space and let G be a compact Lie group which acts on M. Let X and Y be two invariant subspaces of M such that X, Y are disjoint and open in their union $X \cup Y$.*

Then

$$Gcat(X \cup Y, M) = max\{Gcat(X, M), Gcat(Y, M)\}.$$

Proof. Let $X = \bigcup_{i \in I} X_i, Y = \bigcup_{i \in I} Y_j$, where the open subsets X_i and Y_j are G-categorical in M and these coverings of X and Y are minimal.

The covering $\{X_i \cup Y_j\}_{(i,j) \in I \times J}$ is open and G-categorical for $X \cup Y$ in M;it contains a subcovering by s sets such that $s = max\{\mid I \mid, \mid J \mid\}$. Then $Gcat(X \cup Y, M) \geq max\{Gcat(X, M), Gcat(Y, M)\}$.

From Proposition 3.1.(ii) we obtain $Gcat(X, M) \leq Gcat(X \cup Y, M)$ and $Gcat(Y, M) \leq Gcat(X \cup Y, M)$.

Then the above inequality holds.□

The proof of Theorem 3.1. We follow the method established by Fox in [3].

First, we will prove that if $\{A_1, A_2, \ldots, A_k = X\}$ is a G-categorical sequence of X in M, then $Gcat(X, M) \leq k$.

If $k = 1$, then $Gcat(X, M) \leq 1$.

Suppose this statement true for $k \leq r-1$; let $\{A_1, A_2, \ldots, A_k = X\}$ be a G-categorical sequence for X in M. Because A is G-categorical in M, by Proposition 3.1(v) there is an open, invariant subset $U \subset M$ such that $A_1 \subset U$ and U is G-categorical in M.

We prove that $\{A_2 - U, A_3 - U, \ldots, A_r - U\}$ is a G-categorical sequence of $X - U$ in M (with the length $r-1$). The sequence $\{A_1, A_2, \ldots, A_k = X\}$ is G-categorical; then $A_1 \subset A_2 \subset \ldots \subset A_k = X$ and $A_2 - U \subset A_3 - U \subset \ldots \subset A_r - U = X - U$. The set $A_2 - A_1$ is G-categorical in M and $A_2 - A_1 \subset A_2 - U$; then the set $A_2 - A_1$ is G-categorical in M. A_2 and U being invariant sets, we prove easily that $A_2 - U$ is invariant:

$\forall x \in A_2 - U, \forall g \in G \Leftrightarrow x \in A_2$ and $x \notin U$ and $g \in G \Leftrightarrow (x \in A_2$ and $g \in G)$ and $(x \notin U$ and $g \in G)$

We know that $gx \in A_2$; suppose that $gx \in U$. But U is invariant, so $g^{-1}gx \in U$ and we obtain that $x \in U$; this statement is a contradiction.

In the same way, we show that all the sets $A_k - U$ are invariants, for $k = \overline{2, r}$.

Also, the sets $(A_3 - U) - (A_2 - U) = A_3 - A_2, \ldots, (A_r - U) - (A_{r-1} - U) = A_r - A_{r-1}$ are G-categorical in M.

We just must justify that all these sets are closed, but this is very easy: $\overline{A_k - U} = \overline{A_k \cap (CU)} = \overline{A_k} \cap \overline{CU} = A_k \cap (CU) = A_k - U, k = \overline{2, r}$.

We conclude that the sequence $\{A_2 - U, A_3 - U, \ldots, A_k - U = X - U\}$ is a G-categorical sequence of $X - U$ in M; from the induction hypothesis, we obtain:

$$Gcat(X - U, M) \leq r - 1.$$

By using the subadditivity property of Proposition 3.1, we obtain:

$$Gcat(X, M) \leq Gcat(X - U, M) + Gcat(U, M) \leq (r - 1) + 1 = r$$

Now, we will prove that there is a G-categorical sequence of X in M, such that its length is $\leq Gcat(X, M)$.

For $Gcat(X, M) = 1$ this statement is true.

Suppose that this is true also for $Gcat(X, M) \leq r - 1$ and let $\{B_1, B_2, \ldots, B_r\}$ be a minimal, G-categorical, open covering of X in M.

We define the sets:

$$C_i = \{x \in X | x \in B_j, \forall j \le i, x \notin B_j, \forall j > i\}, i = \overline{1,r};$$

these sets are closed in X.

We consider the sets C_1 and $X - B_1$; they are closed and disjoint in the (metric, so) normal space X. Then there is an open subset $D_1 \subset X$ such that

$$C_1 \subset D_1$$

$$\overline{D_1} \cap (X - B_1) = \emptyset$$

We suppose that we have $j-1$ open subsets $D_1, D_2, \ldots, D_{j-1}$ of X such that for $i \le j-1$ the following relations are true:

$$C_i - D_1 \cup D_2 \cup \ldots \cup D_{i-1} \subset D_i$$

$$\overline{D_i} \cap (X - B_i) = \emptyset$$

The subsets $X - B_j$ and $C_j - \bigcup_{i<j} D_i$ are closed in X and disjoint:

$$(X - B_j) \cap (C_j - \bigcup_{i<j} D_i) \subset (X - B_j) \cap (C_j - C_{j-1}) \subset (X - B_j) \cap B_j = \emptyset$$

Then there is $D_j \subset X$ open such that:

$$C_j - \bigcup_{i<j} D_i \subset D_j$$

$$\overline{D_j} \cap (X - B_j) = \emptyset$$

For the subsets $D_1, D_2, \ldots, D_r$ as above, the following relations are true:

$$\overline{D_1} - D_1 \subset B_1 - C_1 \subset B_2 \cup B_3 \cup \ldots \cup B_r$$

$$\bigcup_{i \le r} (\overline{D_i} - D_i) \subset B_2 \cup B_3 \cup \ldots \cup B_r$$

We obtain that

$$Gcat(\bigcup_{i \le r} (\overline{D_i} - D_i), M) \le r - 1$$

From the induction hypothesis, there is a G-categorical sequence $\{A_1, A_2, \ldots, A_{k-1} = \bigcup_{i \le r} (\overline{D_i} - D_i)\}$ of the set $\bigcup_{i \le r} (\overline{D_i} - D_i)$ in M, and its length is $k - 1 \le r - 1$.

We prove that $\{X \cap A_1, X \cap A_2, \ldots, X \cap A_{k-1}, X\}$ is a G-categorical sequence of X in M.

All these sets are closed in X. From $A_1 \subset A_2 \subset \ldots \subset A_{k-1}$ we obtain that $X \cap A_1 \subset X \cap A_2 \subset \ldots \subset X \cap A_{k-1} \subset X$. The subsets $A_1, A_2, \ldots, A_{k-1}$ are G-invariant, so $X \cap A_1, X \cap A_2, \ldots, X \cap A_{k-1}$ are G-invariant. The subset A_1 is G-categorical in M and $X \cap A_1$ will be also G-categorical in M. The subsets $X \cap A_2 - X \cap A_1 = X \cap (A_2 - A_1), \ldots, X \cap A_{k-1} - X \cap A_{k-2} = X \cap (A_{k-1} - A_{k-2})$ are G-categorical in M, because $A_2 - A_1, \ldots, A_{k-1} - A_{k-2}$ are G-categorical in M.

We just must justify that $X - X \cap A_{k-1}$ is a G-categorical subset in M. It is easy to see that $X - X \cap A_{k-1} = X - X \cap (\cup_{i \leq r} (\overline{D_i} - D_i))$ is open in X ($\cup_{i \leq r} (\overline{D_i} - D_i)$ is closed in X) and it is invariant. Every component of $X - X \cap A_{k-1}$ is contained in one of the sets $D_i \subset B_i$; every B_i is G-categorical in M. By using Proposition 3.1(ii) and Lemma 3.1 we obtain that $X - X \cap A_{k-1}$ is G-categorical in M. □

G-categorical sequences can be used for the proof of product inequality; for nonequivariant case, the reader can see [3] and [4].

For two G-spaces X, Y, we define the action of G on the product space $X \times Y$ by

$$G \times (X \times Y) \longrightarrow X \times Y$$

$$g(x, y) = (gx, gy).$$

Proposition 3.2. *Let X, Y two separable, arcwise connected, metric G-spaces. If X and Y are G-invariant, then*

$$Gcat(X \times Y) \leq Gcat(X) + Gcat(Y) - 1$$

Proof. Let $\{A_1, A_2, \ldots, A_m = X\}$ be a G-categorical sequence of X in X and let $\{B_1, B_2, \ldots, B_n = Y\}$ be a G-categorical sequence of Y in Y. We consider the sets

$$C_k = \bigcup_{i+j=k+1} A_i \times B_j.$$

All these sets are closed and G-invariant (because $A_i, 1 \leq i \leq m$, $B_j, 1 \leq j \leq n$ are G-invariant).

From $A_1 \subset \ldots \subset A_m = X$ and $B_1 \subset \ldots \subset B_n = Y$, we obtain that $C_1 \subset \ldots \subset C_{m+n-1} = X \times Y$. We only must show that $\{C_1, C_2 - C_1, \ldots, C_{m+n-1} - C_{m+n-2}\}$ are G-categorical in $X \times Y$.

A_1 is G-categorical in X; then there is an equivariant homotopy $H_A : A_1 \times I \longrightarrow X$ such that $H_{X,0} = H_X(\cdot, 0)$ is the inclusion and $H_{X,1} = H_X(\cdot, 1)$ has the image in a single orbit $Orb(x_{A_1})$. The same holds for B_1 and the equivariant homotopy $H_Y : B_1 \times I \longrightarrow Y$, with corresponding orbit $Orb(y_{B_1})$. Then

$$H : (A_1 \times B_1) \times I \longrightarrow X \times Y$$

defined by

$$H((x, y), t) = (H_X(x, t), H_Y(y, t))$$

is G-invariant: $H(g(x, y), t) = H((gx, gy), t) = (H_X(gx, t), H_Y(gy, t)) = (gH_X(x, t), gH_Y(y, t)) = gH((x, y), t), \forall (x, y) \in X \times Y, \forall g \in G$. Also, $H(\cdot, 0)$ is the inclusion and $H(\cdot, 1)$ has the image in a single orbit $Orb(x_{A_1}, y_{B_1})$. We conclude that C_1 is G-categorical in $X \times Y$.

Writing $C_{k+1} - C_k = \bigcup_{i+j=k+2} (A_i - A_{i-1}) \times (B_j - B_{j-1})$, $1 \leq k \leq m + n - 2$, ($A_0 = \emptyset$ and $B_0 = \emptyset$ for convenience), it is easy to see that $(A_i - A_{i-1}) \times (B_j - B_{j-1})$ is G-categorical in $X \times Y$ and the sets $(A_i - A_{i-1}) \times (B_j - B_{j-1})$, $(A_{i'} - A_{i'-1}) \times (B_{j'} - B_{j'-1})$, $i + j = i' + j'; i \neq i', j \neq j'$, satisfy the assumption of Lemma 3.1.

Then $C_{k+1} - C_k$ is G-categorical in $X \times Y$. □

References

[1] Th. Bartsch, *Borsuk-Ulam theorems, Lusternik-Schnirelmann category and index theories,* World Congres of Nonlinear Analysts'92, (ed. V. Lahshmikantham), vol.III, de Gruyter, 1996, pp. 2127–2134.

[2] E. Fadell, *The equivariant Lusternik-Schnirelmann method for invariant functionals and relative cohomological index theories,* Méthodes topologiques en analyse non linéaire, Montreal, 1985 (ed. A. Granas), pp. 41–70

[3] R. H. Fox, *On the Lusternik-Schnirelmann category,* Annals of Mathematics, vol. 42, 1941, pp. 333–370

[4] I. M. James, *On category, in the sense of Lusternik-Schnirelmann* , Topology, vol. 17, 1978, pp. 331–348

[5] L. Lusternik, L. Schnirelmann, *Methodes topologiques dans les problemes variationells,* Herman, Paris, 1934

University of Oradea, Faculty of Sciences, Armatei Romane 5,
3700, Oradea, Romania
E-mail address: cicortas@math.uoradea.ro

Received: 23.12.2001

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLVII, Number 2, June 2002

NUMERICAL SOLUTION OF THE KORTEWEG - DE VRIES BURGERS EQUATION BY USING QUINTIC SPLINE METHOD

TALAAT EL SAYED ALI EL DANAF

Abstract. In this work we will discuss the solution of the modified Burgers equation by using the collocation method with quintic splines. The test problem will be obtained discuss the accuracy of this problem. We make a comparison between the numerical and exact solution of the modified Burgers equation. The last section to discuss the stability analysis of this method.

1. **Introduction**

In this paper we will introduce a numerical solution for the Korteweg -de Vries Burgers equation (KdVB) which is a non-linear partial differential equation which involves both damping and dispersion take the following form

$$u_t + \epsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0 \tag{1}$$

This equation was derived by Su and Gardner [1] for a wide class of nonlinear system in the weak non-linearity and long wavelength approximation. The steady state solution of the KdVB equation has been shown to model [2] weak plasma shocks propagation perpendicularly to a magnetic field. When diffusion dominates dispersion the steady state solutions of the KdVB equation are monotonic shocks, and when dispersion dominates, the shocks are oscillatory. The KdVB equation has been obtained when including electron inertia effects in the description of weak nonlinear plasma waves [3]. The KdVB equation has also been used in a study of wave propagation through liquid field elastic tube [4] and for a description of shallow water waves on viscous fluid [5]. Canosa and Gazdag [6], who discussed the evolution of non-analytic initial data into a monotonic shock, have given brief details of a numerical solution for the KdVB equation using the accurate space derivative method. In this chapter we will use the finite element method with Quintic Spline interpolation function, and we will show the state of solution in variant times. Grad and Hu [3] showed that the dissipation effects dominate over dispersive effect when:

$$4\mu \leq \nu^2 \tag{2}$$

In this case the solution of (1) is a shock decreasing monotonically from the upstream to the downstream value of u. if

$$\nu^2 < 4\mu \tag{3}$$

The dispersive effects dominate over the dispative effects; in this case the shock becomes oscillatory upstream and monotonic downstream. In this work we introduce the Quintic Spline with finite element method to solve the KDVB equation, and discuss the stability and the accuracy of this solution comparing with the exact solution [7] with some initial and boundary conditions.

2. Exact Solution of the KdVB Equation

In this section we will introduce the exact solution of the KdvB equation which appeared at the first time for the two dimensional KdVB equation at [7]. We modify the solution to take the form:

$$\frac{12\nu^2}{\epsilon\mu}\left[1 - \frac{e^{\frac{2\nu}{\epsilon\mu}(x-\omega t)}}{(e^{\frac{\nu}{\epsilon\mu}(x-\omega t)} + E)^2}\right] \tag{4}$$

where E,is a positive constant, $\omega = \frac{12\nu^2}{25\mu}$, ϵ is the coefficient of the nonlinear term, ν is the viscosity coefficient and μ is the coefficient of the disperisive term. We note that the accuracy of the numerical solution depend on E.

3. Numerical Solution of the KdVB Equation with Collocation Quintic Spline Method

Consider the KdVB equation (1), where the ϵ is a positive parameter and the subscripts x, and t indicate to the differentiation with respect to x and t. The boundary conditions are chosen from:

$$u(a,t) = 1, u(b,t) = 0$$

$$u_x(a,t) = 0 = u_x(b,t)$$

$$u_{xx}(a,t) = 0 = u_{xx}(b,t)$$

Consider $x_i = a + ih, h = \frac{b-a}{N}$, $i = -3, -2, ..., N+3$. Then $\Pi := a = x_0 < x_1 < ... < x_n = b$ is an equal distance partition of the interval [a,b] by the knots x_i. Define the quintic B-spline function as

$$\phi_i(x) = \frac{1}{h^5}\begin{cases} (x-x_{i-3})^5 & x \in [x_{i-3}, x_{i-2}] \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 & x \in [x_{i-2}, x_{i-1}] \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 + 15(x-x_{i-1})^5 & x \in [x_{i-1}, x_i] \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 + 15(x-x_{i-1})^5 & \\ -20(x-x_i)^5 & x \in [x_i, x_{i+1}] \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 + 15(x-x_{i-1})^5 & \\ -20(x-x_i)^5 + 15(x-x_{i+1})^5 & x \in [x_{i+1}, x_{i+2}] \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 + 15(x-x_{i-1})^5 & \\ -20(x-x_i)^5 + 15(x-x_{i+1})^5 - (x-x_{i+2})^5 & x \in [x_{i+2}, x_{i+3}] \\ 0 & otherwise. \end{cases}$$

and let $\phi_i(x)$, be those quintic splines, for $i = 0, 1, , N$. Let

$$x_n = span\{\phi_{-2}, \phi_{-1}, \phi_0, \phi_1, ..., \phi_{N+1}, \phi_{N+2}\}$$

form a basis for the function defined over $[a, b]$, where the values of the quintic splines $\phi_i(x)$, and all its first, and second derivatives vanishes outside the interval $[x_{i-3}, x_{i+3}]$. We establish the value of $\phi_i(x)$ and its derivatives in the following table:

Table 1

x	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
ϕ	0	1	26	66	26	1	0
$\phi^{'}$	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$\frac{-50}{h}$	$\frac{-5}{h}$	0
$\phi^{''}$	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$\frac{-120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
$\phi^{'''}$	0	$\frac{60}{h^3}$	$\frac{-120}{h^3}$	0	$\frac{120}{h^3}$	$\frac{60}{h^3}$	0

Our task is to find an approximate solution $u_N(x, t)$ to the solution $u(x, t)$ in the form:

$$u_N(x, t) = \sum_{i=-2}^{N+2} \phi_i(x_j)\delta_i(t) \tag{5}$$

Where δ_i are unknowns dependent on time to be determined. Substitute from the values of $\phi_i(x)$ and its derivatives into (1), and suppose that δ_i are linearly interpolated between two levels n and $n+1$ by

$$\delta_i = \theta\delta_i^{n+1} + (1-\theta)\delta_i^n$$

Where $0 \leq \theta \leq 1$ is a parameter at the time $n\Delta t$. The time derivative descriptive using the finite difference formula

$$\frac{d\delta}{dt} = \frac{\delta_i^{n+1} - \delta_i^n}{\Delta t}$$

We get

$$\sum_{i=-2}^{N+2}(\phi_i + \frac{\theta\epsilon\Delta t}{h}\phi_i^{'} - \nu\frac{\theta\Delta t}{h^2}\phi_i^{''} + \mu\frac{\theta\Delta t}{h^3}\phi_i^{'''})\delta_i^{n+1} =$$

$$= \sum_{i=-2}^{N+2}(\phi_i + \frac{(1-\theta)\epsilon\Delta t}{h}\phi_i^{'} - \nu\frac{(1-\theta)\Delta t}{h^2}\phi_i^{''} + \mu\frac{(1-\theta)\Delta t}{h^3}\phi_i^{'''})\delta_i^n \tag{6}$$

Giving the parameter θ the value 1/2 we get the Crank-Nicolson formula which implies the recurrence relation

$$\sum_{i=-2}^{N+2}(\phi_i+\frac{\epsilon\Delta t}{2h}\phi_i^{'}-\nu\frac{\Delta t}{2h^2}\phi_i^{''}+\mu\frac{\Delta t}{2h^3}\phi_i^{'''})\delta_i^{n+1} = \sum_{i=-2}^{N+2}(\phi_i+\frac{\epsilon\Delta t}{2h}\phi_i^{'}-\nu\frac{\Delta t}{2h^2}\phi_i^{''}+\mu\frac{(\Delta t}{2h^3}\phi_i^{'''})\delta_i^n \tag{7}$$

Applying the boundary condition we can eliminate δ_{-2}, δ_{-1}, δ_{N+1} and δ_{N+2} to get the following system of non linear equations:

$$a_i\delta_{i-2}^{n+1}+b_i\delta_{i-1}^{n+1}+c_i\delta_i^{n+1}+d_i\delta_{i+1}^{n+1}+e_i\delta_{i+2}^{n+1} = a_i^{'}\delta_{i-2}^n+b_i^{'}\delta_{i-1}^n+c_i^{'}\delta_i^n+d_i^{'}\delta_{i+1}^n+e_i^{'}\delta_{i+2}^n \tag{8}$$

We can write this system of equations in the form

$$A\,[\delta]\,\delta^{n+1} = B\,[\delta]\,\delta^n \tag{9}$$

where the matrices and are Penta-diagonal matrices. The elements of the matrices and are given by:

$$a_i = 1 - r_1 z_{i-2} - r_2 + r_3, a_i^{'} = 1 + r_1 z_{i-2} + r_2 + r_3$$

$$b_i = 26 - 10r_1 z_{i-2} - 2r_2 + 2r_3, b_i^{'} = 26 + 10r_1 z_{i-2} + 2r_2 - 2r_3$$

$$c_i = 66 + 6r_2, c_i^{'} = 66 - 6r_2$$

$$d_i = 26 + 10r_1 z_{i-2} - 2r_2 - 2r_3, d_i^{'} = 26 - 10r_1 z_{i-2} + 2r_2 + 2r_3$$

$$e_i = 1 + r_1 z_{i-2} - r_2 + r_3, e_i^{'} = 1 - r_1 z_{i-2} + r_2 - r_3 \tag{10}$$

where

$$r_1 = \frac{5\epsilon\delta t}{2h}, r_2 = \frac{10\nu\delta t}{h^2},$$

$$r_3 = \frac{30\mu\delta t}{h^3},$$

$$z_{i-2} = \delta_{i-2} + 26\delta_{i-1} + 66\delta i + 26\delta_{i+1} + \delta_{i+2}$$

To solve this system we apply at first the initial condition to determine

$$\delta^0_{-2}, \delta_{-1}, \delta^0_0, ..., \delta^0_N, \delta^0_{N+1}, \delta^0_{N+2}$$

When $t = 0$, equation (6) takes the formula

$$u^0_N(x,t) = \sum_{i=-2}^{N+2} \phi_i(x_j)\delta_i(t)^0 \tag{11}$$

The approximate solution must satisfy the following:

(a). It must agree with the initial condition u (x, 0) at the knots and

(b). The first, second, and third derivatives of the approximate initial condition agree with those of the exact initial conditions at both ends of the range. So we get the system:

$$A\delta^0 = u_0(x) \tag{12}$$

Where A is $(N + 5)x(N + 5)$ square matrix which can be restored by the Penta -diagonal algorithm to $(N + 1)x5$. In the following we will give an illustration to point out how to as an example to compute the element of the matrix A. substitute from (8),(10) and (11) in (12)we have,

$$a'_0\delta^0_{-2} + b'_0\delta^0_{-1} + c'_0\delta^0_1 + d'_0\delta^0_1 + e'_0\delta^0_2 = u_0(x_0)$$

i.e.

$$(1 + r_1 z_{-2} + r_2 + r_3)\delta^0_{-2} + (26 + 10r_1 z_{-2} + 2r_2 - 2r_3)\delta^0_{-1} + (66 - 6r_2)\delta^0_0 +$$

$$+(26 - 10r_1 z_{-2} + 2r_2 + 2r_3)\delta^0_1 + (1 - r_1 z_{-2} + r_2 - r_3)\delta^0_2 = u_0(x_0) \tag{13}$$

Substitute the values of r_1, r_2, and z_{-2} in (13) and use the boundary conditions to eliminate δ_{-2} and δ_{-1} to get the first row in the matrix A and so on.

$$A = \begin{bmatrix} 54 & 60 & 6 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 25.25 & 67.5 & 26.25 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & & 1 & 26 & 66 & 26 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 6 & 60 & 54 \end{bmatrix},$$

By solving the system (13) we get $\{\delta^0_{-2}, \delta^0_{-1}, \delta^0_0, ..., \delta^0_N, \delta^0_{N+1}, \delta^0_{N+2}\}$. So we iterate by using the Pascal program, and hence the solution of equation (1) written as:

$$u(x,t) = \delta_{i-2} + 26\delta_{i-1} + 66\delta i + 26\delta_{i+1} + \delta_{i+2} \tag{14}$$

4. **Stability Analysis**

We apply the Von-Neumann stability for equation (9) so we must linearize this equation and put the nonlinear term as $z_{i-2} = d + 26d + 66d + 26d + d = 120d$, according to the Von-Neuman we have

$$\delta^n_j = \epsilon^n e^{ikx_j} \tag{15}$$

Hence after dividing by at both sides of equation (9) with the help of equation (16) we get

$$g = \frac{A + iB}{A - iB} \tag{16}$$

where

$$g = \frac{\epsilon^{n+1}}{\epsilon n},$$

$$A = 4\cos^2(kh) + 26\cos^2(\frac{kh}{2}) + 3 + r_2(\cos(kh) + 2)(\cos(kh) - 1)$$

$$A_1 = 4\cos^2(kh) + 26\cos^2(\frac{kh}{2}) + 3 - r_2(\cos(kh) + 2)(\cos(kh) - 1)$$

$$B = 4r_2 \sin(kh)(\cos^2(frackh2) + 5) + 4r_3 \sin(kh)(\cos(kh) - 1) \tag{17}$$

We note that $A_1 < A_2$, so

$$|\ g| \ = |\ \frac{A^2 + B^2}{A_1^2 + B^2}| \ \leq 1$$

Which means that the Quintic Splines method is unconditionally stable.

5. Test Problem

Canoza and Gazdag [4] have shown that the steady state solution for the KdvB equation with boundary conditions $u(a,t) = 1$ and $u(b,t) = 0$, exhibits different Behaviour depending on the relative values of ν and μ: (a)it is a shock wave decreasing monotonically from upstream to downstream if

$$\nu^2 \geq 4\mu$$

(b) it is a shock wave which becomes oscillatory upstream and monotonic downstream if

$$\nu^2 < 4\mu$$

These observations are confirmed in the following simulations take the initial condition as the step function

$$u(x,t) = \begin{cases} 1 & \text{if } 0 \leq x \leq 150, \\ 0 & \text{if } x > 150. \end{cases} \tag{18}$$

With μ and like Canoza and Gazdag [4] when $\nu = 6.0$, and 0.1 and 0.05. So we take the boundary conditions as:

$$u(0,t) = 1, u(220,t) = 0,$$

$$u_x(0,t) = 0 = u_x(220,t) \tag{19}$$

Now we make some comparison between the exact solution (4) with $\epsilon = 2$ and the parameter $E = 1000$. Note. The value of the constant E is large to be in the neighborhood of the boundary conditions.

6. Graphics

In this section we plot some graphics to note the behaviour of our numerical solution at some various values of the viscosity and dispressive coefficients, as follows.

Figures(1.a-1.f) show the behavior of the computed solution with, $\nu = 5, \mu = 6$ it means that ν^2 near to 4μ and at time step Δt $= 0.02$, and Δx $= 0.55$ It is confirmed that when the viscosity value is large (ν=5) then numerical solution of the KdVB equation is a shock wave decreasing monotonically from the upstream to the downstream value of the solution [5]. Similar shock wave solutions have been obtained for Burgers' equation [6,7].

Figures (2a-2f) show the behavior of the solution from t=0sec to t=50sec, with $\nu = 1$, $\mu = 20$,Δt = 0.02, and Δx = 0.55. We see that oscillations is increasing with respect to the time, but it is still stable.

Figures (3.a - 3.f) show the behavior of the numerical solution at $\nu = 2$ and $\mu = 4$, which means that $\nu^2 \equiv 4\mu$, which give a very smooth solution, and we will discuss the errors later.

Figures (4.a - 4.f), show the behaviour of the computed solution for $\nu = 0.05$ at times from $t = 0$ to $t = 50$. When viscosity value is small the numerical solution of the KdVB equation is a shock wave which becomes oscillatory upstream and monotonic downstream confirming the theoretical treatment [8,9,10]. These graphs also show that as n is decreased further the computed solutions become more oscillatory. The results are consistent with graphs presented by Vliegenthart for KdV equation, where for identical initial conditions similar behavior is observed [8].

7. Computational Results

In this section we compare between the numerical and exact solution for the KdVB equation and the errors of the Collocation method at each time step

Table 2

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at$\nu = 5, \mu = 6, \Delta t = 0.02 sec$ and $\epsilon = 2$for the time $t = 10 Sec.$ to $t = 60 Sec.$

Time(Sec)	10	20	30	40	50	60
$L_2 x 10^3$	0.0010	0.0012	0.0013	0.0012	0.0012	0.0011
$L_\infty x 10^3$	0.0002	0.0002	0.0003	0.0003	0.0003	0.0003

As we note from table 2 the numerical results are very close to the exact results at$nu = 5, \mu = 6 t = 10 sec$, and $\Delta t = 0.02 sec, \Delta x = 0.73 cm$. For the time increases the results are still close to the exact one which means that the method is very accurate. The errors are given in the following table.

Table 3

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines $\nu = 1, \mu = 2, \Delta t = 0.02 sec$ and $\epsilon = 2$for the time $t = 10 Sec.$ to $t = 60 Sec.$

Time (Sec)	10	20	30	40	50	60
$L_2 x 10^3$	0.0005	0.0006	0.0008	0.0010	0.0012	0.0016
$L_\infty x 10^3$	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001

Table 4

gives the relation between the numerical and exact solution of KdV-Burgers equation by using Collocation method at$\nu = 0.05 \mu = 1, t = 10 sec, \Delta t = 0.02 sec$ and $\epsilon = 2$

Time (Sec)	10	20	30	40	50	60
$L_2 x 10^3$	1.1723	1.2886	1.3618	1.4161	1.4598	1.4964
$L_\infty x 10^3$	0.6	0.6	0.6	0.6	0.6	0.6

As we said before the numerical solution to the KdVB equation by using the Collocation method depends on the ratio $\frac{\nu^2}{4\mu} << 1$, and when the ratio is very closed the solution is more accurate and the method is very good to examine

Table 5

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at$\mu = 10, \nu = 1, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to $t = 60Sec.$

Time(Sec)	10	20	30	40	50	60
$L_2 x10^3$	60.534	66.394	70.084	72.821	75.013	76.847
$L_\infty x10^3$	23.865	23.866	23.867	23.868	23.869	23.870

Table 6

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at$\mu = 1, \nu = 2, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to $t = 60Sec.$

Time (Sec)	10	20	30	40	50	60
L_2 x 10^3	0.0032	0.0032	0.0031	0.0031	0.0031	0.0031
L_∞ x 10^3	0.0013	0.0014	0.0014	0.0014	0.0014	0.00014

Table 7

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at $\mu = 0.1, \nu = 0.005, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to $t = 60Sec.$

Time (Sec)	10	20	30	40	50	60
L_2 x 10^3	0.0128	0.0144	0.0154	0.0160	0.0167	0.0171
L_∞ x 10^3	0.0059	0.0060	0.0063	0.0060	0.0065	0.0059

Table 8

The errors for the numerical solution of the KdvB equation by using Collocation with quintic Splines at $\mu = 0.1, \nu = 0.005, \Delta t = 0.02sec$ and $\epsilon = 2$ for the time $t = 10Sec.$ to $t = 60Sec.$

Time (Sec)	10	20	30	40	50	60
L_2 x 10^3	$8x10^{-5}$	$8x10^{-5}$	$8x10^{-5}$	$8x10^{-5}$	$8x10^{-5}$	$8x10^{-5}$
L_∞ x 10^3	$5x10^{-5}$	$5x10^{-5}$	$5x10^{-5}$	$5x10^{-5}$	$5x10^{-5}$	$5x10^{-5}$

8. **Conclusion**

The finite element method with the quintic spline is capable of producing an accurate and stable numerical solution for the Korteweg-de Vries-Burgers' equation even while the values of the viscosity coefficient are small [11]. The linear stability analysis shows that the numerical scheme is unconditionally stable. This is the first trail to compute numerically, the solution of the KDVB equation. So, this work compares the numerical solution of the KDVB equation with the exact one. But, there is no available other numerical example in the literatures to compare with.

References

[1] C.H. Su and C.s. Gardner,*" Drivation of the Korteweg-de Vries and Burgers' Equation", J. Math. Phy., 10, pp 536-539, (1969).*

[2] H. Gard and P. N. Hu, *" Unified shock profile in plasma", Phys. Fluids, 10, pp 2596-2602, (1967).*

[3] H. Grad and P. N. Hu, *" Collisional theory of shock and nonlinear waves in plasma", J. phys. Fluids, 15, 845-864, (1972).*

[4] R. S. johnson, *" A nonlinear equation incorporating damping and dispersion", J. Phys. Fluids Mech. 42, 94-60, (1970).*

[5] R. S. johnson, *" Shalllow water waves in a viscous fluid- the undular bore", Phy. Fluids, 15, 1693-1699,(1970).*

[6] J. Conosa and J. Gazdag, *" The Korteweg-de Vires - Burgers' Equation", J. Comp. Phys, 23, 393-403,(1977).*

[7] Wen-xiu Ma, *" An exact solution to two- dimenisional Korteweg-de Vires Burgers' Equation", J. Phys. A. Math. Gen., 26, L17-L20,(1993).*

[8] R. S. johnson, *" A nonlinear equation incorporating damping and dispersion II ", J. Fluid Mech. 42, pp 1693-1699,(1970).*

[9] D.H. Peregine, *" Calculations of the development of an undular bore". J. Fluid Mech. 25, pp 321-330,(1965).*

[10] G. B. Withman, *" Linear and nonlinear waves". John Wiley, New York. 2, pp 1555-1565, 1974.*

[11] A.H.A. Ali, L.R.T. Gardner and G.A. Gardner, *" Numerical Studies of the Korteweg-de Vires -Burgers Equation using B-spline finite Elements". J. Math. Phy. Sci. 27, No.1, pp 37-53, (1993).*

[12] D.H. Peregine, *" Calculations of the development of an undular bore". J. Fluid Mech. 25, pp 321-330,(1965).*

[13] G. B. Withman, *" Linear and nonlinear waves". John Wiley, New York. 2, pp 1555-1565, 1974.*

[14] A.H.A. Ali, L.R.T. Gardner and G.A. Gardner, *" Numerical Studies of the Korteweg-de Vires -Burgers Equation using B-spline finite Elements". J. Math. Phy. Sci. 27, No.1, pp 37-53, (1993).*

Mathematics Department, Faculty of Science,
Menoufyia University, Egypt

Received: 01.03.2002

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 2, June 2002

ON UNIFORMLY CONVEX MAPPINGS OF A BANACH SPACE INTO THE COMPLEX PLANE

DORINA RĂDUCANU

Abstract. Let E be a complex Banach space and let E be the unit ball in E, i.e. $B = \{x \in E : \|x\| < 1\}$. We introduce a new class of holomorphic functions in B and we obtain a few results concerning this new class.

1. **Introduction**

Let E^*be the dual space of E. For any $A \in E^*$we consider $\chi(A) = \{x \in E : A(x) \neq 0\}$ and $\gamma(A) = E \backslash \chi(A)$. If $A \neq 0$ then $\chi(A)$ is dense in E and $\chi(A) \cap \hat{B}$ is dense $\hat{B}$, where $\hat{B} = \{x \in E : \|x\| = 1\}$.

Let $H(B)$ be the family of all functions $f : B \to \mathbf{C}, f(0) = 0$ that are holomorphic in B, i.e. have the Fréchet derivative $f'(x)$ in each point $x \in B$. If $f \in H(B)$, then in some neighbourhood V of the origin, $f(x) = \sum\limits_{m=1}^{\infty} P_{m,f}(x)$, where the series is uniformly convergent on V and

$P_{m,f} : E \to \mathbf{C}$ are continuous and homogeneous polynomials of degree m.

Let $U = \{z \in \mathbf{C} : |z| < 1\}$. Denote by CV the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

that are convex in the unit disk U.

Goodman [1] defined the following subclass of CV.

Definition. A function f is called uniformly convex in U if f is in CV and has the property that for every circular arc γ contained in U, with center ζ also in U, the arc $f(\gamma)$ is a convex arc.

Goodman gave a two-variable analytic characterization of this class, denoted by UVC.

Theorem 1. *A function of the form* (1) *is in UCV if and only if*

$$\operatorname{Re}\left\{1+(z-\zeta)\frac{f''(z)}{f'(z)}\right\}\geq 0 \quad , \quad (z,\zeta)\in U\times U. \tag{2}$$

Also, Goodman proved that the best known bounds on the coefficients for the family UVC are $|a_n|\leq \frac{1}{n}, n\geq 2$.

Ma and Minda [3] and Ronning [4] independently found a more applicable one-variable characterization for UVC.

Theorem 2. *A function f of the form* (1) *is in UVC if and only if*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geq\left|\frac{zf''(z)}{f'(z)}\right| \quad , \quad z\in U. \tag{3}$$

2. **The class UCV_A**

Let $A\in E^*, A\neq 0$. For any $f\in H(B)$ of the form

$$f(x)=A(x)+\sum_{n=2}^{\infty}P_{n,f}(x) \quad , \quad x\in B \tag{4}$$

and for any $a\in\chi(A)\cap\hat{B}$ we set

$$f_a(z)=\frac{f(za)}{A(a)} \quad , \quad z\in U. \tag{5}$$

Obviously

$$f_a(z)=z+\sum_{n=2}^{\infty}\frac{P_{n,f}(a)}{A(a)}z^n \quad , \quad z\in U. \tag{6}$$

Moreover, it is easy to check that

$$f_a^{(n)}(z)=\frac{f^{(n)}(za)(a,...,a)}{A(a)} \quad , \quad n\in\mathbf{N}, z\in U. \tag{7}$$

We denote by UCV_A the family of all functions $f\in H(B)$ of the form (4) such that, for any $a\in\chi(A)\cap\hat{B}$ the function f_a belongs to the class UCV.

By using the properties of the functions in UCV, we obtain a few results concerning the family UCV_A.

Theorem 3. *If $f\in UCV_A$ and $a\in\hat{B}$, then*

$$|P_{n,f}(a)|\leq\frac{1}{n}|A(a)| \quad , \quad n\geq 2 \tag{8}$$

Proof. Suppose that $f \in UCV_A$. If $a \in \chi(A) \cap \hat{B}$, then $f_a \in UCV$ and hence we get (9). If $a \in \gamma(A) \cap \hat{B}$, evidently $a = \lim_{m \to \infty} a_m$, where $a_m \in X(A), m \in \mathbf{N}$. There exists $r_m \in \mathbf{R}_+$ such that $\frac{a_m}{r_m} \in \hat{B}$. Clearly $(r_m)_{m \geq 0}$ is bounded for the origin is an interior point of B. Since $\frac{a_m}{r_m} \in \chi(A) \cap \hat{B}, m \in \mathbf{N}$, by the first part of the proof we have

$$\left| P_{n,f}\left(\frac{a_m}{r_m}\right) \right| \leq \frac{1}{n} \left| A\left(\frac{a_m}{r_m}\right) \right| \quad , \quad m \in \mathbf{N}.$$

Hence

$$|P_{n,f}(a_m)| \leq \frac{r_m^{n-1}}{n} |A(a_m)| \quad , \quad m \in \mathbf{N}.$$

By taking the limit with $m \to \infty$, we obtain $P_{n,f}(a) = 0$.

Corollary 1. *All $f \in UCV_A$ vanish on $\gamma(A) \cap B$.*

Corollary 2. *If $f \in UCV_A$, then*

$$\|P_{n,f}\| \leq \frac{1}{n} \|A\| \quad , \quad n \geq 2$$

The following theorems provide necessary and sufficient conditions for functions in $H(B)$ to belong to the class UCV_A.

Theorem 4. *Let $f \in UCV_A$ and $f'(x) \neq 0$, for all $x \in B$. Then*

$$\operatorname{Re}\left\{1 + \frac{f''(x)(x,x)}{f'(x)}\right\} \geq \left|\frac{f''(x)(x,x)}{f'(x)}\right| \quad , \quad x \in \chi(A) \cap B. \tag{9}$$

Proof. Let $x \in \chi(A) \cap B, x \neq 0$. Then $a = \frac{x}{\|x\|} \in \chi(A) \cap \hat{B}$ and hence the function f_a belongs to the class UCV. From (3) we have

$$\operatorname{Re}\left\{1 + \frac{z f_a''(z)}{f_a'(z)}\right\} \geq \left|\frac{z f_a''(z)}{f_a'(z)}\right| \quad , \quad z \in U.$$

By using the equality

$$\frac{z f_a''(z)}{f_a'(z)} = \frac{f''(za)(za, za)}{f'(za)(za)} \quad , \quad z \in U$$

we obtain

$$\operatorname{Re}\left\{1 + \frac{f''(za)(za,za)}{f'(za)(za)}\right\} \geq \left|\frac{f''(za)(za,za)}{f'(za)(za)}\right| \quad , \quad z \in U.$$

By setting $z = \|x\|$, we get (9).

Theorem 5. *Let $f \in H(B)$, $f'(0) = A$ and $f'(x) \neq 0$, for all $x \in B$. If*

$$\operatorname{Re}\left\{1 + \frac{f''(x)(x,x)}{f'(x)}\right\} \geq \left|\frac{f''(x)(x,x)}{f'(x)}\right| \quad , \quad x \in B \tag{10}$$

then $f \in UCV_A$.

Proof. Let $a \in \chi(A) \cap \hat{B}$. Then $f_a'(z) = f'(za)(a) \neq 0, z \in U \setminus \{0\}$ and

$$\frac{zf_a''(z)}{f_a'(z)} = \frac{f''(za)(za, za)}{f'(za)} \quad , \quad z \in U.$$

From (10), we obtain $f_a \in UCV$, for all $a \in \chi(A) \cap \hat{B}$. Hence $f \in UCV_A$.

References

[1] Goodman, A.W., On uniformly convex functions, Ann. Polon. Math. 56(1991),87-92.

[2] Janiec,E., On starlike and convex maps of a Banach space into the complex plane, Demonstratio Math. vol. XXVII, 2(1994),511-516.

[3] Ma, W., Minda, D., Uniformly convex functions, Ann. Polon. Math. 57(1992), 166-175.

[4] Ronning, F., Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. 118, No.1,1993,190-196.

Department of Mathematics, Faculty of Sciences,
"Transilvania" University, Iuliu Maniu, 50, 2200 Braşov, Romania

Received: 26.03.2002

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 2, June 2002

A UNIVALENCE CONDITION

DORINA RĂDUCANU AND PAULA CURT

Abstract. In this paper we obtain a sufficient condition for univalence concerning holomorphic mappings of the unit ball in the space of n-complex variables.

1. Introduction

Let $\mathbf{C}^n$ be the space of n-complex variables $z = (z_1, ..., z_n)$ with the Euclidean inner product $\langle z.w \rangle = \sum_{k=1}^{n} z_k \bar{w}_k$ and norm $\|z\| = \langle z, z \rangle^{\frac{1}{2}}$.

Let B^n denote the open unit ball in $\mathbf{C}^n$,i.e. $B^n = \{z \in \mathbf{C}^n : \|z\| < 1\}$.We denote by $\mathcal{L}(\mathbf{C}^n)$ the space of continuous linear operators from $\mathbf{C}^n$ into $\mathbf{C}^n$,i.e .$n \times n$ complex matrices $A = (A_{jk})$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \|z\| < 1\}, \qquad A \in \mathcal{L}(\mathbf{C}^n)$$

$I = (I_{jk})$ denotes the identity in $\mathcal{L}(\mathbf{C}^n)$.

Let $H(B^n)$ be the class of holomorphic mappings

$$f(z) = (f_1(z), ..., f_n(z)), \qquad z \in B^n$$

from B^n into $\mathbf{C}^n$. We say that $f \in H(B^n)$ is *locally biholomorphic* in B^n if f has a local holomorphic inverse at each point in B^n or equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1 \le j,k \le n}$$

is nonsingular at each point $z \in B^n$.

A mapping $v \in H(B^n)$ is called a *Schwarz function* if $\|v(z)\| \le \|z\|$, for all $z \in B^n$.

If $f, g \in H(B^n)$ then f is *subordinate* to g $(f \prec g)$ in B^n if there exists a Schwarz function v such that $f(z) = g(v(z))$, $z \in B^n$.

A function $L : B^n \times [0,\infty) \to \mathbf{C}^n$ is a *subordination chain* if $L(\cdot,t)$ is holomorphic and univalent in $B^n, L(0,t) = 0$, for all $t \in [0,\infty)$ and $L(z,s) \prec L(z,t)$, whenever $0 \leq s \leq t < \infty$.

The subordination chain $L : B^n \times [0,\infty) \to \mathbf{C}^n$ is a *normalized* subordination chain if $DL(0,t) = e^t I$, for $t \in [0,\infty)$.

A basic result in the theory of n-complex variables subordination chains is due to J. A. Pfaltzgraff.

Theorem 1. [5] *Let $L(z,t) = e^t z + ...$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ such that:*

(i) $L(\cdot,t) \in H(B^n)$, for all $t \in [0,\infty)$

(ii) $L(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in B^n$.

Let $h(z,t)$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ which satisfies the following conditions:

(iii) $h(\cdot,t) \in H(B^n), h(0,t) = 0, Dh(0,t) = I$ and $\operatorname{Re}\langle h(z,t), z\rangle \geq 0$, for all $t \in [0,\infty)$ and $z \in B^n$.

(iv) For each $T > 0$ and $r \in (0,1)$ there is a number $K = K(r,T)$ such that $\|h(z,t)\| \leq K(r,T)$, when $\|z\| \leq r$ and $t \in [0,T]$.

(v) For each $z \in B^n, h(z,\cdot)$ is a measurable function on $[0,\infty)$.

Suppose $h(z,t)$ satisfies

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)\,h(z,t)\,,\ a.e\ \ t \in [0,\infty),\ for\ \ all\ \ z \in B^n \tag{1}$$

Further, suppose there is a sequence $(t_m)_{m\geq 0}, t_m > 0$ increasing to ∞ such that

$$\lim_{m\to\infty} e^{-t_m} L(z,t_m) = F(z) \tag{2}$$

locally uniformly in B^n.

Then for each $t \in [0,\infty), L(\cdot,t)$ is univalent in B^n.

P. Curt obtained a version of Theorem 1 for subordination chains which are not normalized .

Theorem 2. [2] *Let $L(z,t) = a_1(t)z + ..., a_1(t) \neq 0$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ such that:*

(i) $L(\cdot,t) \in H(B^n)$ for all $t \in [0,\infty)$

(ii) $L(z,t)$ *is a locally absolutely continuous function of* t*, locally uniformly with respect to* $z \in B^n$

(iii) $a_1(t) \in C^1[0,\infty)$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$.

Let $h(z,t)$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbf{C}^n$ *which satisfies the following conditions:*

(iv) $h(\cdot,t) \in H(B^n)$, $h(0,t) = 0$ *and* $\operatorname{Re}\langle h(z,t), z\rangle \geq 0$, *for all* $t \in [0,\infty)$ *and* $z \in B^n$

(v) For each $z \in B^n, h(z,\cdot)$ *is a measurable function on* $[0,\infty)$

(vi)For each $T > 0$ *and* $r \in (0,1)$, *there exists a number* $K = K(r,T)$ *such that* $\|h(z,t)\| \leq K(r,T)$, *when* $\|z\| \leq r$ *and* $t \in [0,T]$.

Suppose $h(z,t)$ *satisfies*

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)\,h(z,t)\,, a.e.\, t \in [0,\infty), \textit{ for all } z \in \mathbf{B}^n \tag{3}$$

Further suppose there is a sequence $(t_m)_{m\geq 0}, t_m > 0$ *increasing to* ∞ *such that*

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a_1(t_m)} = F(z) \tag{4}$$

locally uniformly in B^n.

Then for each $t \in [0,\infty), L(\cdot,t)$ *is univalent in* B^n.

2. Univalence conditions

By using Theorem 2, we obtain an univalence condition which generalize some n-dimensional univalence criteria [2], [3], [5].

Theorem 3. *Let* $f : B^n \to \mathbf{C}^n$ *be a locally biholomorphic function in* $B^n, f(0) = 0, Df(0) = I$ *and let* $a : [0,\infty) \to \mathbf{C}$ *be a function which satisfies the conditions:*

(i) $a \in C^1[0,\infty), a(0) = 1, a(t) \neq 0$, *for all* $t \in [0,\infty)$

(ii) $\lim_{t\to\infty} |a(t)| = \infty$

(iii) $\operatorname{Re} \frac{a'(t)}{a(t)} > 0$, *for all* $t \in [0,\infty)$.

If

$$\max_{\|z\|=e^{-t}} \left\| (a(t) - \|z\|)(Df(z))^{-1} D^2 f(z)(z,\cdot) + \frac{a(t) - a'(t)}{2} I \right\| <$$

$$< \frac{|a(t) + a'(t)|}{2} \tag{5}$$

for all $t \in [0,\infty)$, *then* f *is an univalent function in* B^n.

Remark

The second derivative of a function $f \in H(B^n)$ is a symmetric bilinear operator $D^2 f(z)(\cdot,\cdot)$ on $\mathbf{C}^n \times \mathbf{C}^n$ and $D^2 f(z)(w,\cdot)$ is the linear operator obtained by restricting $D^2 f(z)$ to $\{w\} \times \mathbf{C}^n$. The linear operator $D^2 f(z)(z,\cdot)$ has the matrix representation

$$D^2 f(z)(z,\cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 f_k(z)}{\partial z_j \partial z_m} z_m\right)_{1\le j,k\le n}$$

Proof. We define

$$L(z,t) = f\left(e^{-t}z\right) + \left(a(t)e^t - 1\right)e^{-t}Df\left(e^{-t}z\right)(z), \quad t \in [0,\infty), z \in B^n \tag{6}$$

We wish to show that $L(z,t)$ satisfies the conditions of Theorem 2 and hence $L(\cdot,t)$ is univalent in B^n, for all $t \in [0,\infty)$. Since $f(z) = L(z,0)$ we obtain that f is an univalent function in B^n.

It is easy to check that $a_1(t) = a(t)$ and hence $a_1(t) \neq 0, \lim t \to \infty |a_1(t)| = \infty$ and $a_1 \in C^1[0,\infty)$.

We have $L(z,t) = a_1(t)z + (holomorphicterm)$. Thus $\lim\limits_{t\to\infty} \frac{L(z,t)}{a_1(t)} = z$, locally uniform with respect to $B^n$and hence (4) holds with $F(z) = z$. Obviously $L(z,t)$ satisfies the absolute continuity requirements of Theorem2.

Straightforward calculations show that

$$DL(z,t) = \frac{a(t) + a'(t)}{2} Df\left(e^{-t}z\right)\left[I - E(z,t)\right], \tag{7}$$

where, for each fixed $(z,t) \in B^n \times [0,\infty), E(z,t)$ is the linear operator defined by

$$E(z,t) = -\frac{a(t) - a'(t)}{a(t) + a'(t)} I -$$
$$-2\frac{a(t) - e^{-t}}{a(t) + a'(t)} \left(Df\left(e^{-t}z\right)\right)^{-1} D^2 f\left(e^{-t}z\right)\left(e^{-t}z,\cdot\right) \tag{8}$$

For $t = 0$, we have

$$I - E(z,0) = \frac{2}{1 + a'(0)} I, \quad \text{for all } z \in B^n \tag{9}$$

Since $1 + a'(0) \neq 0$, we obtain that $I - E(z,0)$ is an invertible operator.

For $t > 0, E(\cdot,t) : \overline{B^n} \to \mathcal{L}(\mathbf{C}^n, \mathbf{C}^n)$ is holomorphic and from the weak maximum modulus theorem [4] it follows that $\|E(z,t)\|$ can have no maximum in B^n unless $\|E(z,t)\|$ is of constant value throughout B^n. If $z = 0$ and $t > 0$ we have

$$\|E(0,t)\| = \left|\frac{a(t) - a'(t)}{a(t) + a'(t)}\right| < 1.$$

We also have

$$\|E(z,t)\| \leq \max_{\|w\|=1} \|E(w,t)\|$$

If we let $u = e^{-t}w$ with $\|w\| = 1$, then $\|u\| = e^{-t}$ and by using (5) we obtain

$$\|E(z,t)\| \leq \max_{\|w\|=1} \|E(w,t)\| =$$

$$= \max \|u\| = e^{-t} \left\| \frac{2(a(t) - \|u\|)}{a(t) + a'(t)} (Df(u))^{-1} D^2 f(u)(u,\cdot) + \frac{a(t) - a'(t)}{a(t) + a'(t)} I \right\| < 1.$$

Since $\|E(z,t)\| < 1$ for all $z \in B^n$ and $t > 0$, it follows $I - E(z,t)$ is an invertible operator,too.

Further calculations show that

$$\frac{\partial L(z,t)}{\partial t} = \frac{a(t) + a'(t)}{2} Df\left(e^{-t}z\right)[I - E(z,t)](z) =$$

$$DL(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z).$$

Hence $L(z,t)$ satisfies the differential equation (3), for all $z \in B^n$ and $t \in [0,\infty)$, where

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z) \tag{10}$$

It remains to show that $h(z,t)$ satisfies the conditions $(iv), (v)$ and (vi) of Theorem 2. Clearly $h(z,t)$ satisfies the holomorphy and measurability requirements and $h(0,t) = 0$.

Since

$$\|h(z,t) - z\| = \|E(z,t)(h(z,t) + z)\| \leq \|E(z,t)\| \cdot \|h(z,t) + z\| < \|h(z,t) + z\|$$

We have $\langle \operatorname{Re} h(z,t), z\rangle \geq 0$, for all $(z,t) \in B^n \times [0,\infty)$.

By using the inequality

$$\left\|[I - E(z,t)]^{-1}\right\| \leq [1 - \|E(z,t)\|]^{-1}$$

we obtain

$$\|h(z,t)\| \leq \frac{1 + \|E(z,t)\|}{1 - \|E(z,t)\|} \|z\|.$$

The conditions of Theorem 2 being satisfied it follows that the functions $L(z,t), t \geq 0$ are univalent in B^n. In particular $f(z) = L(z,0)$ is univalent in B^n.

Remarks

1) If $a(t) = e^t, t \in [0,\infty)$, then Theorem 3 becomes the n-dimensional version of Becker's univalence criterion [4].

2) For $a(t) = \frac{e^t + ce^{-t}}{1+c}, t \geq 0, c \in \mathbf{C} \setminus \{-1\}, |c| \leq 1$, Theorem 3 becomes the n-dimensional version of Ahlfors and Becker's univalence criterion [2].

3) If $a(t) = \frac{e^{(\alpha-1)t} + ce^{-t}}{1+c}, t \geq 0, c \in \mathbf{C} \setminus \{-1\}, |c| \leq 1$ and $\alpha \in \mathbf{R}$ with $\alpha \geq 2$, we obtain the generalization of Ahlfors and Becker's n-dimensional criterion of univalence due to P. Curt [3].

References

[1] Becker, J., Loewnersche Differentialgleichung und Schichtheits - Kriterion, Math. Ann. 202 4(1973), 321-335.

[2] Curt, P., A generalization in n-dimensional complex spaces of Ahlfors and Becker's criterion for univalence, Studia, Math. XXXXIX, 1(1994), 31-38.

[3] Curt,P., Loewner chains and univalence criteria (to appear).

[4] Hille, E., Phillips, R.S., Functional analysis and semigroups, Amer. Math. Soc. Colloq. Publ. 31(1957).

[5] Pfaltzgraff, J. A., Subordination chains and univalence of holomorphic mappings in $\mathbf{C}^n$, Math. Ann. 210(1974), 55-68.

[6] Pfaltzgraff, J.A. K-Quasinconformal Extension Criteria in the disk, Complex Variables, 21(1993), 293-301.

Department of Mathematics, Faculty of Sciences,
"Transilvania" University, Iuliu Maniu, 50, 2200 Braşov, Romania

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

Received: 26.03.2002

CUBATURE FORMULAS ON TRIANGLE

MILENA SOLOMON

Abstract. In this article are presented some cubature formulas on triangle T which are obtained by the product of known quadrature formulas and some formulas obtaining by an approximation formula on triangle.

1. Introduction

Let $T = \{(x, y) \in \mathbb{R}^2 | \; x \geq 0, \; y \geq 0, \; x + y \leq 1\}$ the standard triangle from the Euclidean space $\mathbb{R}^2$, $f : T \to \mathbb{R}$ an integrable function on T, $\lambda_i f$, $i = \overline{0, m}$, some given information of f and w a nonnegative weight function on T_1.

Definition 1. The formula

$$\iint_T w(x, y) f(x, y) dx dy = \sum_{i=0}^{m} A_i \lambda_i f + R_m(f) \tag{1}$$

is called a cubature formula. The parameters A_i, $i = \overline{0, m}$, are the coefficients and $R_m(f)$ is the remainder term.

A way to construct cubature formulas on T is to use the quadrature formulas which are known from unidimensional case.

2. Cubature formulas

Consider

$$I(f) = \iint_T f(x, y) dx dy = \int_0^1 \int_0^{1-x} f(x, y) dx dy. \tag{2}$$

An efficient way to construct cubature formulas is to use the quadrature formulas after the integral (2) was transformed into integral on $D = [0, 1] \times [0, 1]$.

Thus, we introduce the substitution $y = t(1 - x)$ and yields

$$I(f) = \int_0^1 (1 - x) \left(\int_0^1 f(x, t(1 - x)) dt \right) dx. \tag{3}$$

In order to compute the integral on $[0,1]\times[0,1]$, it can be use the product of two quadrature formulas, for example:

$$\int_0^1 g(t)dt = \sum_{i=1}^{n_t} T_i g(t_i) + R(g),$$

where $R(g)=0,\ \forall\ g\in\mathcal{P}_{d_t}$ and

$$\int_0^1 (1-x)g(x)dx = \sum_{j=1}^{n_x} A_j(1-x_j)g(x_j) + R(g),$$

where $R(g)=0,\ \forall\ g\in\mathcal{P}_{d_x}$.

Thus, we obtained an approximation on the form:

$$Q(f) = \sum_{i=1}^{n_x}\sum_{j=1}^{n_x} A_j T_i(1-x_j)f(x_j, t_i(1-x_j)). \tag{4}$$

For example, if we use the Simpson's rule:

$$\int_0^1 g(t)dt \approx \frac{1}{6}\left[g(0)+4g\left(\frac{1}{2}\right)+g(1)\right]$$

it is obtained

$$\int_0^1 f(x,t(1-x))dt = \frac{1}{6}\left[f(x,0)+4f\left(x,\frac{1-x}{2}\right)+f(x,1-x)\right]+R_1(f)$$

and the Simpson's rule again

$$\int_0^1 (1-x)\frac{1}{6}\left[f(x,0)+4f\left(x,\frac{1-x}{2}\right)+f(x,1-x)\right]dx =$$

$$= \frac{1}{6}\left[\int_0^1 (1-x)f(x,0)dx + 4\int_0^1 (1-x)f\left(x,\frac{1-x}{2}\right)dx + \int_0^1 (1-x)f(x,1-x)dx\right] =$$

$$= \frac{1}{36}\left[f(0,0)+2f\left(\frac{1}{2},0\right)\right] + \frac{4}{36}\left[f\left(0,\frac{1}{2}\right)+4f\left(\frac{1}{2},\frac{1}{4}\right)\right] +$$

$$+\frac{1}{36}\left[f(0,1)+2f\left(\frac{1}{2},\frac{1}{2}\right)\right]+R(f)$$

it follows

Theorem 1. *If $f\in B_{12}(0,0)$, then*

$$\iint_T f(x,y)dxdy = \frac{1}{36}[f(0,0)+f(0,1)] + \frac{1}{18}\left[f\left(\frac{1}{2},0\right)+2f\left(0,\frac{1}{2}\right)\right] +$$

$$+\frac{2}{9}f\left(\frac{1}{2},\frac{1}{4}\right)+\frac{1}{18}f\left(\frac{1}{2},\frac{1}{2}\right)+R(f) \tag{5}$$

where

$$R(f)=\frac{1}{720}f^{(3,0)}(\xi_1,0)+\frac{1}{32}f^{(2,1)}(\xi_2,0)-\frac{25}{576}f^{(0,3)}(0,\eta_1)-\frac{7}{192}f^{(1,2)}(\xi_2,\eta_2)$$

with $\xi_1,\xi_2,\eta_1\in[0,1]$ *and* $(\xi_3,\eta_3)\in T_1$.

If we use the first level, trapezoidal's quadrature

$$\int_0^1 g(t)dt=\frac{1}{2}[g(0)+g(1)]-\frac{1}{2}g''(\xi)$$

it is obtained:

$$\int_0^1 f(x,t(1-x))dt\simeq\frac{1}{2}[f(x,0)+f(x,1-x)]$$

and, in the second level, the Simpson's quadrature:

$$\int_0^1 g(t)dt=\frac{1}{6}\left[g(0)+4g\left(\frac{1}{2}\right)+g(1)\right]-\frac{1}{2880}f^{(4)}(\xi)$$

$$\Rightarrow\int_0^1\frac{1-x}{2}[f(x,0)+f(x,1-x)]dx=\frac{1}{12}\left[f(0,0)+f(0,1)+\right.$$

$$\left.+4\left[f\left(\frac{1}{2},0\right)+f\left(\frac{1}{2},\frac{1}{2}\right)\right]+f(1,0)+f(1,0)\right]+R(f)$$

Thus

Theorem 2. *If* $f\in B_{12}(0,0)$, *then:*

$$\iint_T f(x,y)dxdy=\frac{1}{12}f(0,0)+\frac{1}{12}f(0,1)+\frac{1}{3}f\left(\frac{1}{2},0\right)+$$

$$+\frac{1}{3}f\left(\frac{1}{2},\frac{1}{2}\right)+\frac{1}{6}f(1,0)+R(f)\qquad(6)$$

where

$$R(f)=-\frac{1}{30}f^{(3,0)}(\xi_1,0)-\frac{1}{16}f^{(2,1)}(\xi_2,0)-\frac{1}{18}f^{(0,3)}(0,\eta_1)-\frac{13}{240}f^{(1,2)})(\xi_3,\eta_3)$$

with $\xi_1,\xi_2,\eta_1\in[0,1]$ *and* $(\xi_3,\eta_3)\in T_1$.

Another way to obtain the cubature formulas is to start from an approximation formula on T_1.

Let B_1 be the Birkhoff's operator:

$$(B_1f)(x,y)=f(1-y,y)+(x+y-1)f^{(1,0)}(0,y)$$

which generates the approximation formula

$$f=B_1f+Rf.$$

After integration on T it is obtained:

$$\iint_T f(x,y)dxdy = \int_0^1 (1-y)f(1-y,y)dy - \frac{1}{2}\int_0^1 (y-1)^2 f^{(1,0)}(0,y)dy.$$

Applying to each integrals the Simpson's quadrature, it is obtained:

$$\int_0^1 (1-y)f(1-y,y)dy \simeq \frac{1}{6}f(1,0) + \frac{1}{3}f\left(\frac{1}{2},\frac{1}{2}\right)$$

$$\int_0^1 (1-y)^2 f^{(1,0)}(0,y)dy \simeq \frac{1}{6}\left[f^{(1,0)}(0,0) + f^{(1,0)}\left(0,\frac{1}{2}\right)\right].$$

It follows:

Theorem 3. *If $f \in B_{11}(0,0)$, then:*

$$\iint_T f(x,y)dxdy = \frac{1}{6}\left[f(1,0) + 2f\left(\frac{1}{2},\frac{1}{2}\right)\right] -$$

$$-\frac{1}{12}\left[f^{(1,0)}(1,0) + f^{(1,0)}\left(0,\frac{1}{2}\right)\right] + R(f) \tag{7}$$

where:

$$R(f) = -\frac{1}{12}f^{(2,0)}(\xi,0), \quad \xi \in [0,1].$$

Starting from Lagrange's operator:

$$(L_1 f)(x,y) = \frac{1-x-y}{1-y}f(0,y) + \frac{x}{1-y}f(1-y,y)$$

and Hermite operator

$$(H_2 f)(x,y) = \frac{(1-x-y)(1-x+y)}{(1-x)^2}f(x,0)+$$

$$+\frac{y(1-x-y)}{1-x}f^{(0,1)}(x,0) + \frac{y^2}{(1-x)^2}f(x,1-x)$$

one obtains an interpolation formula

$$f = L_1 H_2 f + R_{12} f$$

where

$$(L_1 H_2 f)(x,y) = (1+y)(1-x-y)f(0,0) + x(1-x-y)f^{(0,1)}(0,0)+$$

$$+\frac{y^2(1-x-y)}{1-y}f(0,1) + \frac{x}{1-y}f(1-y,y).$$

After integration on T, one obtain:

$$\iint_T f(x,y)dxdy = f(0,0)\int_0^1\int_0^{1-x}(1+y)(1-x-y)dxdy+$$

$$+f^{(0,1)}(0,0)\int_0^1\int_0^{1-x}x(1-x-y)dxdy + f(0,1)\int_0^1\int_0^{1-x}\frac{y^2(1-x-y)}{1-y}dxdy+$$

$$+\int_0^1\int_0^{1-x}\frac{x}{1-y}f(1-y,y)dxdy =$$

$$= \frac{5}{24}f(0,0) + \frac{1}{24}f^{(0,1)}(0,0) + \frac{1}{24}f(0,1) + \int_0^1\int_0^{1-x}\frac{x}{1-y}f(1-y,y)dxdy.$$

In order to compute the integral $\int_0^1\int_0^{1-x}\frac{x}{1-y}f(1-y,y)dxdy$, we use the following cubature formulas:

$$\iint_T f(x,y)dxdy = \frac{1}{6}\left[g\left(\frac{1}{2},0\right) + g\left(0,\frac{1}{2}\right) + g\left(\frac{1}{2},\frac{1}{2}\right)\right] + R(f)$$

and we obtain

$$\int_0^1\int_0^1\frac{x}{1-y}f(1-y,y)dxdy \simeq \frac{1}{12}f(1,0) + \frac{1}{6}f\left(\frac{1}{2},\frac{1}{2}\right).$$

Thus, we arrive at the following result:

Theorem 4. *If $f \in B_{11}(0,0)$, then:*

$$\iint_T f(x,y)dxdy = \frac{5}{24}f(0,0) + \frac{1}{24}f^{(0,1)}(0,0) + \frac{1}{12}f(0,1) + \frac{1}{6}f\left(\frac{1}{2},\frac{1}{2}\right) + R(f) \quad (8)$$

where:

$$R(f) = -\frac{1}{24}f^{(2,0)}(\xi,0), \quad \xi\in[0,1].$$

References

[1] Barnhill, R.E., Birkhoff, G., Gordon, W.J., *Smooth interpolation in triangles*, J. Approx. Theory, 8(1973), 114-128.

[2] Barnhill, R.E., Mansfield, L., *Error bounds for smooth interpolation in triangles*, J. Approx. Theory, 11(1974), 306-318.

[3] Böhmer, K., Coman, Gh., *On some approximation schemes on triangles*, Mathematica (Cluj), 24(45), 1980, 231-235.

[4] Coman, Gh., *Analiză numerică*, Ed. Libris, Cluj-Napoca, 1995.

[5] Cools, R., Lyness, J.N., *A survey of numerical cubature over triangles*, Mathematics and Computer Science Divisio, Argonne National Laboratory, Argone III, 1994.

[6] Cools, R., *Monomial cubature rules since "Stroud" a compilation - part 2*, J. Comput. Appl. Math. 112(1999), 21-27.

[7] Cools, R., Mysovskikh, I.P., Schmid, H.J., *Cubature formulae and orthogonal polynomials*, Elsevier Preprint, January 2000.

[8] Stancu, D.D., *The remainder of certain linear approximation formulas in two variables*, SIAM J. Numer. Anal., Ser.B, 1(1964), 137-163.
[9] Stroud, A.H., *Approximate Calculation of Multiple Integrals*, Prentice-Hall, Englewood Cliffs, New York, 1971.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania
E-mail address: milenas73@yahoo.com

Received: 11.10.2001

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 2, June 2002

AN EXISTENCE UNIQUENESS THEOREM FOR AN INTEGRAL EQUATION MODELLING INFECTIOUS DISEASES

TIBERIU TRIF

Abstract. By using a global inversion theorem due to R. Plastock [3], we prove an existence uniqueness result concerning the initial-value problem for the delay nonlinear integral equation $x(t) = \psi(t) + \int_{t-\tau}^{t} f(s, x(s))ds$. We establish also the continuous dependence on ψ of the solution of this equation.

1. Introduction

To describe the spread of certain infectious diseases, K. L. Cooke and J. L. Kaplan [1] proposed the following delay integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds. \tag{1}$$

In this equation, $x(t)$ is the proportion of infectives in a population at time t, τ is the length of time an individual remains infectious, and $f(t, x(t))$ is the proportion of new infectives per unit time.

It should be mentioned that Eq. (1) can be also interpreted as an evolution equation for a single species population. In this case, $x(t)$ is the number of individuals at time t, τ is the lifetime, and $f(t, x(t))$ is the number of new births per unit time. It is assumed that each individual lives exactly to the age τ, and then dies.

In this paper we are concerned with the initial-value problem associated to the equation

$$x(t) = \psi(t) + \int_{t-\tau}^{t} f(s, x(s))ds. \tag{2}$$

2000 *Mathematics Subject Classification.* 45G10, 45M20, 47H99.

Key words and phrases. nonlinear integral equations, global inversion theorems.

More precisely, we look for positive continuous solutions of Eq. (2), when the proportion $\phi(t)$ of infectives is known for $t \in [-\tau, 0]$, i.e.

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau, 0]. \tag{3}$$

Obviously, we must assume that ϕ and ψ satisfy the equality

$$\phi(0) = \psi(0) + \int_{-\tau}^{0} f(s, \phi(s))ds. \tag{4}$$

Conditions ensuring the existence of at least one positive continuous solution of the initial-value problem (2)–(3) (with $\psi = 0$) have been given by R. Precup [4, 5, 6], E. Kirr [2], R. Precup and E. Kirr [7], and T. Trif [12]. It should be noted that, essentially, all these papers make use of different fixed point theorems. In the present paper we provide another approach of the problem (2)–(3). Namely, we obtain at once an existence uniqueness result, as well as the continuous dependence on ψ of the solution, for the initial-value problem (2)–(3) by using the following global inversion theorem due to R. Plastock [3] as the basic tool:

Theorem 1 ([3, Corollary 2.3]). *Let E and F be Banach spaces, let $A : E \to F$ be a local homeomorphism, and let $u : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous function. Assume that the following conditions are satisfied:*

(i) *$u(\cdot, s)$ is strictly increasing for every $s > 0$ and $u(0, s) = 0$ for all $s \in \mathbf{R}_+$;*

(ii) $\lim\limits_{\|x\| \to \infty} \|A(x)\| = \infty$;

(iii) *there exists a completely continuous operator $A_1 : E \to F$ such that the operator $A_2 := A + A_1$ satisfies*

$$\|A_2(x) - A_2(y)\| \geq u(\|x - y\|, r)$$

for every $r > 0$ and all $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$.

Then A is a (global) homeomorphism.

2. Main result

Concerning the initial-value problem (2)–(3) we will use the following hypotheses:

(H$_1$) $f : [-\tau, \infty[\times \mathbf{R} \to \mathbf{R}$ is a continuous function whose partial derivative with respect to the second argument, denoted by $f'_x(t,x)$, is continuous on $[-\tau, \infty[\times \mathbf{R}$;

(H$_2$) a is a positive real number, while $\phi : [-\tau, 0] \to [a, \infty[$ and $\psi : [0, \infty[\to \mathbf{R}$ are continuous functions satisfying (4);

(H$_3$) there exists a locally integrable function $b : [-\tau, \infty[\to \mathbf{R}$ such that

$$f(t,u) \geq b(t) \qquad \text{for all } (t,u) \in [-\tau, \infty[\times [a, \infty[$$

and

$$\psi(t) + \int_{t-\tau}^{t} b(s)ds > a \qquad \text{for all } t \in \mathbf{R}_+;$$

(H$_4$) there exist a continuous function $g : \mathbf{R}_+ \to \mathbf{R}_+$ and a continuous nondecreasing function $h : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$\int_1^\infty \frac{1}{h(u)} du = \infty, \tag{5}$$

such that

$$|f(t,u)| \leq g(t)h(|u|) \qquad \text{for all } (t,u) \in \mathbf{R}_+ \times \mathbf{R}.$$

Theorem 2. *Suppose that the hypotheses* (H$_1$)–(H$_4$) *are fulfilled. Let T be an arbitrary positive real number and let E be the Banach space consisting of all continuous functions from $[-\tau, T]$ to $\mathbf{R}$, endowed with the usual sup-norm. Then the operator $A : E \to E$, defined by*

$$\begin{array}{ll} A(x)(t) := x(t) - \phi(t) + \psi(0) & \text{if } t \in [-\tau, 0] \\ A(x)(t) := x(t) - \int_{t-\tau}^{t} f(s, x_\phi(s))ds & \text{if } t \in]0, T] \end{array}$$

for all $x \in E$ and all $t \in [-\tau, T]$, where $x_\phi : [-\tau, T] \to \mathbf{R}$ is the function defined by

$$x_\phi(t) := \begin{cases} \phi(t) & \text{if } t \in [-\tau, 0] \\ x(t) & \text{if } t \in]0, T], \end{cases}$$

is a global homeomorphism. In particular, there exists a unique continuous function $x : [-\tau, T] \to [a, \infty[$, satisfying (3) *and* (2) *for all $t \in]0, T]$.*

Proof. It is immediately seen that A is correctly defined because the hypotheses (H$_1$) and (H$_2$) guarantee that $A(x)$ is a continuous function for each $x \in E$.

Further, let $A_1 : E \to E$ be the operator defined by $A_1(x) := Id_E(x) - A(x)$, i.e.

$$\begin{array}{ll} A_1(x)(t) := \phi(t) - \psi(0) & \text{if } t \in [-\tau, 0] \\ A_1(x)(t) := \int_{t-\tau}^{t} f(s, x_\phi(s))ds & \text{if } t \in \,]0, T] \end{array}$$

for all $x \in E$ and all $t \in [-\tau, T]$.

From the definitions of A and A_1 it follows that for all $x, y \in E$ we have

$$|A(x)(t) - A(y)(t)| = |x(t) - y(t)|$$
$$|A_1(x)(t) - A_1(y)(t)| = 0$$

if $t \in [-\tau, 0]$, whilst

$$|A(x)(t) - A(y)(t)| \leq |x(t) - y(t)| + \int_0^t |f(s, x(s)) - f(s, y(s))|ds$$
$$|A_1(x)(t) - A_1(y)(t)| \leq \int_0^t |f(s, x(s)) - f(s, y(s))|ds$$

if $t \in \,]0, T]$. These inequalities ensure that A and A_1 are continuous and that A_1 is completely continuous, by virtue of the Arzelá–Ascoli theorem.

Now we prove that A is a local homeomorphism. In fact, we will prove a little bit more: for all $x \in E$ and all $r > 0$, the restriction of A to the ball $B(x, r)$ is injective. To see this, let $x \in E$ and $r > 0$ be arbitrarily chosen. Further, let $y, z \in B(x, r)$ be so that $A(y) = A(z)$. Since $A(y)(t) = y(t) - \phi(t) + \psi(0)$ and $A(z)(t) = z(t) - \phi(t) + \psi(0)$ for all $t \in [-\tau, 0]$, it follows that $y(t) = z(t)$ for all $t \in [-\tau, 0]$. On the other hand, if we set $m_x := \min\, x([0, T])$, $M_x := \max\, x([0, T])$, and

$$M := \max \{\, |f'_x(s, u)| \mid s \in [0, T],\ u \in [m_x - r, M_x + r] \,\},$$

then for each $t \in [0, T]$ it holds that

$$y(t) - \int_{t-\tau}^{t} f(s, y_\phi(s))ds = z(t) - \int_{t-\tau}^{t} f(s, z_\phi(s))ds,$$

hence

$$\begin{aligned} |y(t) - z(t)| &\leq \int_{t-\tau}^{t} |f(s, y_\phi(s)) - f(s, z_\phi(s))|ds \\ &\leq \int_0^t |f(s, y(s)) - f(s, z(s))|ds \\ &\leq M \int_0^t |y(s) - z(s)|ds. \end{aligned}$$

By the Gronwall inequality we conclude that $y(t) = z(t)$ for all $t \in [0, T]$. Hence $y = z$ and A is a local homeomorphism, as claimed.

Next, we prove that A satisfies condition (ii) in Theorem 1. To this end, remark that for each $x \in E$ and each $t \in [0, T]$ it holds that

$$x(t) = A(x)(t) + \int_{t-\tau}^{t} f(s, x_\phi(s))ds.$$

Taking into account the hypotheses (H_2) and (H_4), we deduce that

$$\begin{aligned} |x(t)| &\leq \|A(x)\| + \int_{-\tau}^{0} f(s, \phi(s))ds + \int_{0}^{t} |f(s, x(s))|ds \\ &\leq \|A(x)\| + \phi(0) + \int_{0}^{t} g(s)h(|x(s)|)ds. \end{aligned}$$

By a modified version of the Gronwall inequality (see M. Rădulescu and S. Rădulescu [9, p. 103]) we conclude that

$$\int_{\|A(x)\|+\phi(0)}^{|x(t)|} \frac{1}{h(u)}\,du \leq \int_{0}^{t} g(s)ds \leq \int_{0}^{T} g(s)ds$$

for all $x \in E$ and all $t \in [0, T]$. This inequality and (5) imply the validity of the condition (ii) in Theorem 1.

In conclusion, all the conditions in Theorem 1 are satisfied if the function $u : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is defined by $u(r, s) := r$. Therefore, A is a global homeomorphism, hence there exists a unique $x \in E$ such that $A(x) = \tilde{\psi}$, where $\tilde{\psi} : [-\tau, T] \to \mathbf{R}$ is the function defined by

$$\tilde{\psi}(t) := \begin{cases} \psi(0) & \text{if } t \in [-\tau, 0] \\ \psi(t) & \text{if } t \in]0, T]. \end{cases}$$

Clearly, x satisfies (3) and (2) for all $t \in]0, T]$. We claim that $x(t) \geq a$ for all $t \in [-\tau, T]$. To see this, set

$$T_0 := \inf\{\, t \in [-\tau, T] \;\mid\; \forall\; s \in [-\tau, t] \;:\; x(s) \geq a \,\}.$$

According to (H_2), we have $T_0 \geq 0$. Assume that $T_0 < T$. Since the function

$$\forall\;\, t \in \mathbf{R}_+ \;\longmapsto\; \psi(t) + \int_{t-\tau}^{t} b(s)ds \in \mathbf{R}$$

is continuous, it follows from (H_3) that the real number ε, defined by

$$\varepsilon := \min_{t \in [0,T]} \left(\psi(t) + \int_{t-\tau}^{t} b(s)ds \right) - a,$$

is positive. Set $\alpha := \min x([-\tau,T])$, $\beta := \max x([-\tau,T])$,

$$\gamma := \max\{\,|f(s,u)| \mid s \in [-\tau,T],\ u \in [\alpha,\beta]\,\},$$

and then choose $\delta > 0$ such that $4\gamma\delta \le \varepsilon$ and

$$|\psi(t) - \psi(T_0)| < \frac{\varepsilon}{2} \qquad \text{for all } t \in [T_0, T_0+\delta] \cap [0,T].$$

The assumption $T_0 < T$ implies the existence of a point $t \in [T_0, T_0+\delta] \cap [0,T]$ such that $x(t) < a$. But, on the other hand, we have

$$\begin{aligned} x(t) &= \psi(t) + \int_{T_0-\tau}^{T_0} f(s,x(s))ds - \int_{T_0-\tau}^{t-\tau} f(s,x(s))ds + \int_{T_0}^{t} f(s,x(s))ds \\ &\ge \psi(T_0) + \int_{T_0-\tau}^{T_0} b(s)ds - \int_{T_0-\tau}^{t-\tau} |f(s,x(s))|ds - \int_{T_0}^{t} |f(s,x(s))|ds \\ &\quad +\psi(t) - \psi(T_0) \\ &\ge a + \varepsilon - 2\gamma(t-T_0) - \frac{\varepsilon}{2} \ge a + \frac{\varepsilon}{2} - 2\gamma\delta \ge a. \end{aligned}$$

The obtained contradiction shows that $T_0 = T$. Consequently, $x(t) \ge a$ for all $t \in [-\tau, T]$. □

Suppose that the hypotheses (H_1)–(H_4) are satisfied. Let T and E be as in the statement of Theorem 2, and let $x : [-\tau,T] \to [a,\infty[$ be the unique continuous function satisfying (3) and (2) for all $t \in]0,T]$. Further, let $\psi_n : [0,\infty[\to \mathbf{R}$ ($n \in \mathbf{N}$) be a sequence of continuous functions satisfying

$$\psi_n(0) = \phi(0) - \int_{-\tau}^{0} f(s,\phi(s))ds$$

and

$$\psi_n(t) + \int_{t-\tau}^{t} b(s)ds > a \qquad \text{for all } t \in \mathbf{R}_+,$$

for each positive integer n. According to Theorem 2, for every n there exists a unique continuous function $x_n : [-\tau,T] \to [a,\infty[$ such that

$$\begin{aligned} &x_n(t) = \phi(t) && \text{for all } t \in [-\tau,0] \\ &x_n(t) = \psi_n(t) + \textstyle\int_{t-\tau}^{t} f(s,x_n(s))ds && \text{for all } t \in]0,T]. \end{aligned}$$

Corollary 3. *If $(\psi_n) \to \psi$ uniformly on $[0,T]$, then $(x_n) \to x$ uniformly on $[-\tau,T]$.*

Proof. For each positive integer n, let $\tilde{\psi}_n : [-\tau, T] \to \mathbf{R}$ be the function defined by

$$\tilde{\psi}_n(t) := \begin{cases} \psi_n(0) & \text{if } t \in [-\tau, 0] \\ \psi_n(t) & \text{if } t \in]0, T]. \end{cases}$$

Then for all n we have $x_n = A^{-1}(\tilde{\psi}_n)$. On the other hand, $x = A^{-1}(\tilde{\psi})$, where $\tilde{\psi}$ is defined as in the proof of Theorem 2. Since A^{-1} is continuous and $(\tilde{\psi}_n) \to \tilde{\psi}$ uniformly on $[-\tau, T]$, we conclude that $(x_n) \to x$ uniformly on $[-\tau, T]$. □

Corollary 4. *Suppose that the hypotheses* (H_1)–(H_4) *are fulfilled. Then there exists a unique continuous function* $x : [-\tau, \infty[\to [a, \infty[$, *satisfying* (3) *and* (2) *for all* $t \in]0, \infty[$.

Proof. According to Theorem 2, for each $T > 0$ there exists a unique continuous function $x_T : [-\tau, T] \to [a, \infty[$, satisfying (3) and (2) for all $t \in]0, T]$. Therefore, for all $T_1 > T_2 > 0$ and all $t \in]0, T_2]$ it holds that $x_{T_1}(t) = x_{T_2}(t)$. This remark enables us to define the function $x : [-\tau, \infty[\to [a, \infty[$ as follows: given $t \in [-\tau, \infty[$, select a real number $T \geq t$ and then set $x(t) := x_T(t)$. Clearly, x is the unique continuous function from $[-\tau, \infty[$ to $[a, \infty[$, satisfying (3) and (2) for all $t \in]0, \infty[$. □

Example. Let λ be a real number satisfying

$$\lambda \min_{t \in [0, \pi]} \int_{t-2}^{t} \ln(1 + \sin^2 s) ds > 1 \tag{6}$$

and let γ_0 be a root of the equation

$$\sqrt{\gamma} = \lambda \int_0^2 \ln(1 + \gamma^2 \sin^2 s) ds, \tag{7}$$

lying in $]1, \infty[$ (due to (6), Eq. (7) has at least one root in $]1, \infty[$). Then there exists a unique continuous function $x : [-2, \infty[\to [1, \infty[$, satisfying

$$\begin{aligned} &x(t) = \gamma_0 && \text{for all } t \in [-2, 0] \\ &x(t) = \textstyle\int_{t-2}^{t} (\lambda + s_+) \sqrt{x(s)} \ln\left(1 + x^2(s) \sin^2 s\right) ds && \text{for all } t \in]0, \infty[, \end{aligned}$$

where $s_+ := \max\{0, s\}$.

This follows by Corollary 4 because all the hypotheses (H_1)–(H_4) are fulfilled if we choose $\tau := 2$, $a := 1$,

$$\begin{aligned}
&f : [-2,\infty[\, \times \mathbf{R} \to \mathbf{R} && f(t,u) := (\lambda + t_+)\sqrt{|u|}\ln(1+u^2\sin^2 t),\\
&\phi : [-2,0] \to \mathbf{R} && \phi(t) := \gamma_0,\\
&\psi : [0,\infty[\, \to \mathbf{R} && \psi(t) := 0,\\
&b : [-2,\infty[\, \to \mathbf{R} && b(t) := \lambda\ln(1+\sin^2 t),\\
&g : \mathbf{R}_+ \to \mathbf{R}_+ && g(t) := \lambda + t,\\
&h : \mathbf{R}_+ \to \mathbf{R}_+ && h(u) := \sqrt{u}\ln(1+u^2).
\end{aligned}$$

References

[1] K. L. Cooke and J. L. Kaplan, *A periodicity threshold theorem for epidemics and population growth*, Math. Biosci. **31** (1976), 87–104.

[2] E. Kirr, *Existence and continuous dependence on data of the positive solutions of an integral equation from biomathematics*, Studia Univ. Babeş-Bolyai, Math. **41** (1996), no. 2, 59–72.

[3] R. Plastock, *Homeomorphisms between Banach spaces*, Trans. Amer. Math. Soc. **200** (1974), 169–183.

[4] R. Precup, *Positive solutions of the initial value problem for an integral equation modelling infectious disease*, in: Seminar on Fixed Point Theory, Preprint no. 3, I. A. Rus (ed.), Babeş-Bolyai University, Cluj-Napoca (1991), 25–30.

[5] R. Precup, *Nonlinear Integral Equations (Romanian)*, Babeş-Bolyai University, Cluj-Napoca, 1993.

[6] R. Precup, *Monotone technique to the initial value problem for a delay integral equation from biomathematics*, Studia Univ. Babeş-Bolyai, Math. **40** (1995), no. 2, 63–73.

[7] R. Precup and E. Kirr, *Analysis of a nonlinear integral equation modelling infectious diseases*, in: Analysis and Numerical Computation of Solutions of Nonlinear Systems Modelling Physical Phenomena, Ş. Balint (ed.), Timişoara (1997), 178–195.

[8] M. Rădulescu and S. Rădulescu, *Global inversion theorems and applications to differential equations*, Nonlinear Anal. **4** (1980), 951–965.

[9] M. Rădulescu and S. Rădulescu, *Theorems and Problems in Analysis (Romanian)*, Editura Didactică şi Pedagogică, Bucureşti, 1982.

[10] M. Rădulescu and S. Rădulescu, *An application of Hadamard-Lévy's theorem to an initial value problem for the differential equation $x^{(n)} = f(t,x)$*, Rev. Roumaine Math. Pures Appl. **33** (1988), 695–700.

[11] M. Rădulescu and S. Rădulescu, *An application of Hadamard-Lévy's theorem to a scalar initial value problem*, Proc. Amer. Math. Soc. **106** (1989), 139–143.

[12] T. Trif, *Positive solutions of a nonlinear integral equation from biomathematics*, Demonstratio Math. **32** (1999), 129–138.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania
E-mail address: ttrif@math.ubbcluj.ro

Received: 15.02.2002

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 2, June 2002

CRITICAL AND VECTOR CRITICAL SETS IN THE PLANE

LIANA ŢOPAN

Abstract. Given a non-empty set $C \subset \mathbb{R}^2$, is C the set of critical points for some smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ or vectorial map $f : \mathbb{R}^2 \to \mathbb{R}^2$? In this paper we give some results in this direction.

1. Introduction

A point $p \in \mathbb{R}^2$ is *critical* for a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ if its derivative at p is zero. $(df)_p = 0$. This means $\dfrac{\partial f}{\partial x}(p) = \dfrac{\partial f}{\partial y}(p) = 0$, in a smooth chart in p. The set of all critical points of f is denoted by $C(f)$. The image of $C(f)$ is the set of *critical values* $B(f) = f(C(f))$. If x is not critical, then it is *regular*. We say that $C \subset \mathbb{R}^2$ is *critical* if $C = C(f)$ for some smooth $f : \mathbb{R}^2 \to \mathbb{R}$. A *proper function* has the property that $f^{-1}(K)$ is compact for all compact sets K. Equivalently, when $f : \mathbb{R}^2 \to \mathbb{R}$, $|f(x)| \to \infty$ iff $|x| \to \infty$. We say that $C \subset \mathbb{R}^2$ is *properly critical* if f can be chosen to be proper. Clearly, a critical set is closed. What other properties does it have? In the compact case, there is just one other requirement.

Theorem. [No-Pu] *Let C be a compact non-empty subset of $\mathbb{R}^2$. The following assertions are equivalent:*

1. *C is critical*
2. *C is properly critical*
3. *The components of its complement are multiply connected.*

A *component* of a topological space is a maximal connected subset of the space. It is *multiply connected* if it is not simply connected. The condition on multiply connectivity is a topological condition on the complement, not on the space. If C is any finite set of points or a Cantor set in the plane, then it is properly critical. Their complements are multiply connected. On the other hand, a circle is not critical. If C

2000 *Mathematics Subject Classification.* 57R45, 58K05, 58E05, 58C25.

Key words and phrases. critical point, critical set, vector critical set, excellent mapping.

is the union of a circle and a point, then it is critical if and only if the point is inside the circle.

If its critical set is noncompact, it is unreasonable to expect properness of f. If $C = C(f)$ is closed, unbounded and connected, then by Sard's theorem, f is constant on C, $f(C) = c$, and $f^{-1}(c)$ is noncompact, so f is not proper.

Theorem. *If $C \subset \mathbb{R}^2$ is critical, compact and non-empty, then any bounded component of its complement has disconnected boundary. In particular, no compact curve in $\mathbb{R}^2$, smooth or not, is a critical set.*

Given a closed, noncompact set $K \subset \mathbb{R}^2$ when is there a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $K = C(f)$? We say that ∞ is *arcwise accessible* in $U \subset \mathbb{R}^2$ if there is an arc $\alpha : [0, \infty) \to U$ such that $\alpha(t) \to \infty$ as $t \to \infty$.

Theorem. [No-Pu] *A closed set $K \subset \mathbb{R}^2$ is critical if and only if ∞ is arcwise accessible in each simply connected component of $\mathbb{R}^2 \setminus K$.*

2. **Vector critical sets**

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map. The point $p \in \mathbb{R}^2$ is a *critical point* of f if $rank_p f \leq 1$. If f is given by $f = (f_1, f_2)$, then in some local chart around p, p is critical point of f if and only if the Jacobi matrix of f in p is singular, which means:

$$\det \begin{bmatrix} \dfrac{\partial f_1}{\partial x}(x_0, y_0) & \dfrac{\partial f_1}{\partial y}(x_0, y_0) \\ \dfrac{\partial f_2}{\partial x}(x_0, y_0) & \dfrac{\partial f_2}{\partial y}(x_0, y_0) \end{bmatrix} = 0$$

The set $C \subseteq \mathbb{R}^2$ is *vector critical* if it is the critical set of some smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$. In which conditions will a critical set $C \subset \mathbb{R}^2$ be vector critical? For a class of subsets of the plane, the answer is given by the following theorem:

Theorem. *Any critical set $C \subset \mathbb{R}^2$ is vector critical.*

Proof: Since C is critical, there is a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, so that $C = C(f)$, where

$$C(f) = \left\{ (x_0, y_0) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0 \right\}.$$

Define $F : \mathbb{R}^2 \to \mathbb{R}^2$, by $F(x, y) = (h(x, y), y)$, where $h : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$h(x,y) = \int_0^x \Big(\Big[\frac{\partial f}{\partial x}(x,y)\Big]^2 + \Big[\frac{\partial f}{\partial y}(x,y)\Big]^2\Big)dx.$$

Since h is smooth, so is F. We show that $C(f) = C(F)$.

The Jacobi matrix of f in some point $(x_0, y_0) \in \mathbb{R}^2$ is

$$J(F)(x_0,y_0) = \begin{bmatrix} \Big[\frac{\partial f}{\partial x}(x_0,y_0)\Big]^2 + \Big[\frac{\partial f}{\partial y}(x_0,y_0)\Big]^2 & \frac{\partial h}{\partial y}(x_0,y_0) \\ 0 & 1 \end{bmatrix}.$$

For $(x_0, y_0) \in C(f)$, we have $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$, so

$$J(F)(x_0,y_0) = \begin{bmatrix} 0 & \frac{\partial h}{\partial y}(x_0,y_0) \\ 0 & 1 \end{bmatrix}$$

and $(x_0, y_0) \in C(F)$. Conversely, if $(x_0, y_0) \in C(F)$, it follows that $\Big[\frac{\partial f}{\partial x}(x_0,y_0)\Big]^2 + \Big[\frac{\partial f}{\partial y}(x_0,y_0)\Big]^2 = 0$, and then $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$, so $(x_0, y_0) \in C(f)$. □

If, in theorem above f is supposed to be a harmonic function (this means that f has the property $\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0$), then F could be defined to be the map $F = (f, g)$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is the smooth map which is the solution of the system

$$\begin{cases} \frac{\partial g}{\partial x}(x,y) & = -\frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial y}(x,y) & = \quad \frac{\partial f}{\partial x}(x,y). \end{cases}$$

The converse of this theorem is not true. There are more vector critical sets than critical. A vector critical set which is not critical is the circle in the plane. The map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (\frac{x^3}{3} + xy - x, y)$ is critical exactly on the unit circle in $\mathbb{R}^2$.

3. The family of excellent mappings

An *excellent mapping* is a smooth function $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose critical points are all folds or cusps. A *fold* is a critical point such that, after smooth local changes of coordinates in the domain and image, the function is of the form

$$f(x, y) = (x^2, y),$$

the critical point being taken to the origin. For a *cusp*, after a change of coordinates, the function is of the form

$$f(x, y) = (xy - x^3, y),$$

where the critical point is taken to the origin.

For an excellent mapping, the set of critical points will consist of smooth curves; we call these *general folds* of the mapping. Also, the cusp points are isolated on the general fold. Let f be an excellent mapping and C a general fold of f through p. Thus p will be a fold point if the image of C near p is a smooth curve with non-zero tangent vector at p, and p will be a cusp point if the tangent vector is zero at p but it becoming non-zero at a positive rate as we move away from p on C.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an excellent mapping. The *derivative* of f with respect to V at p is the vector in $\mathbb{R}^2$

$$\nabla_V f(p) = \lim_{t \to 0_+} \frac{1}{t}[f(p + tV) - f(p)].$$

For each $p \in \mathbb{R}^2$, consider the vectors $V' = \nabla_V f(p)$ as a function of vectors V with $|V| = 1$. We shall use a certain system of curves defined by f in an open set $R \subset \mathbb{R}^2$. We let R contain p if the vectors V' are not all of the same length. For any $p \in R$, there will be a pair of opposite directions at p such that for V in these directions,

$|V'|$ is a minimum. (For V in the perpendicular direction, $|V'|$ will be a maximum.) Now R is filled up by smooth curves in these directions; we call these curves *curves of minimum* ∇f.

For any $p \in R$ and vector $V \neq 0$, $\nabla_V f(p) = 0$ if and only if p is a singular point and V is tangent to the curve of minimum ∇f.

Consider any general fold curve C. If a curve of minimum ∇f cuts C at a positive angle at p, then for the tangent vector $V(p)$, $\nabla_V f(p) \neq 0$, and hence p is a fold point. Suppose C is tangent to a curve of minimum ∇f at p. Then p is not a fold point, and hence is a cusp point, since f is excellent. Set $V^* = \nabla_V \nabla_V f(p)$; then $V^* \neq 0$. Since $\nabla_V f(p) = 0$, $\nabla_v f(p')$ is approximately in the direction of $\pm V^*$ for p' on C near p. It follows that $\nabla_W f(p)$ is a multiple of V^*, for all vectors W. As we move along the general fold through p, $\nabla_V f(p')$ changes from a negative to a positive multiple of V^* (approximately); hence $V(p')$ cutes the curves of minimum ∇f in opposite senses on the two sides of p. Therefore the curves of minimum ∇f lying on one side of C cut C on both sides of p. We call this side of C the *upper side* and the other the *lower side*.

The image of C has a cusp at $f(p)$, pointing in the direction of $-V^*$. For any vector W not tangent to C at p, $\nabla_W f(p)$ is a positive or negative multiple of V^*, according as W points into the upper or lower side of C.

Let f and g be mappings $\mathbb{R}^2 \to \mathbb{R}^2$ and $\varepsilon(p)$ a positive continuous function in $\mathbb{R}$. We say g is an *ε-approximation* to f if

$$|g(p) - f(p)| < \varepsilon(p), \quad \forall p \in \mathbb{R}.$$

If f and g are r-smooth, we say g is an *(r, ε)-approximation* to f if this inequality holds, and also the similar inequalities for all partial derivatives of orders $\leq r$, using fixed coordinate systems. We speak of *general approximations* and *r-approximations* in the two cases.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an excellent mapping. We describe certain approximations g to f which have the singularities of f and also further singularities.

(a) *Arbitrary approximations:* For any smooth curve C in the plane which touches no general fold, we may introduce two new folds, one at C and one near C.

For each $p \in C$, let p_t, $-1 \leq t \leq 1$, denote the points of a line segment S_p approximately perpendicular to C in p, with $p_0 = p$. We may choose these segments so that they cover a neighborhood U of C which touches no general fold of f. We change f to obtain g as follows: as t runs from -1 to 1, let $g(p_t)$ run along $f(S_p)$ from $f(p_{-1})$ to $f(p)$, then back a little, then on through $f(p)$ to $f(p_1)$. If f and C are smooth, we may construct g to be smooth. C is a fold for g and so is a curve C', consisting of the points $p_{1/2}$, for example. We may let $g = f$ in $\mathbb{R}^2 \setminus U$. With U small enough, g is an arbitrarily good approximation of f.

(b)*Approximations with first derivatives:* Let C_0 be a curve of fold points of f, without cusps. It may be the whole or a part of a complete general fold of f. We show that we may define g to be an arbitrarily good approximation of f together with first derivatives, so that there is a new pair of folds near C_0. If C_0 is closed, there will be no new cusps for g; otherwise, the new folds will meet in a pair of cusp points for g.

We may let p_t denote points of a neighborhood of C_0, as in (a), so that the image of each S_p under f is an arc folded over on itself, the fold occurring at p. Let $g(p_t) = f(p_t)$ for $-1 \leq t \leq 0$; as t runs from 0 to 1, let $g(p_t)$ move along $f(S_p)$ towards $f(p_1)$, then back a little, and then forward again to $f(p_1)$. So, we obtain two new folds.

We show that we may make g approximate to f near a given point p of C_0. Then, the approximation is possible near the all of C_0.

We may choose the coordinates so that f, near p, is given by

$$f(x, y) = (x^2, y).$$

We may define a smooth function $\phi : \mathbb{R} \to \mathbb{R}$, so that:

1. $\phi(-t) = \phi(t)$, for all $t \in \mathbb{R}$
2. $\phi(0) = 1$
3. $\phi(t) = 0$, for $|t| \geq 1$
4. $0 \leq \phi'(t) \leq \phi'(-\frac{1}{2}) = \alpha$, for $t < 0$.

For $\varepsilon > 0$, define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x, y) = \left(x^2 + \frac{10\varepsilon^2}{\alpha} \phi \left(\frac{x - 2\varepsilon}{\varepsilon} \right), y \right).$$

g is smooth and the Jacobian matrix of g has the form

$$J(g)(x, y) = \begin{bmatrix} 2x + \dfrac{10\varepsilon}{\alpha} \cdot \phi' \left(\dfrac{x - 2\varepsilon}{\varepsilon} \right) & 0 \\ 0 & 1 \end{bmatrix}$$

For $x \in (-\infty, \varepsilon] \cup [3\varepsilon, \infty)$, $\phi \left(\dfrac{x - 2\varepsilon}{\varepsilon} \right) = 0$. So, p is also a critical point of g. Moreover, as

$$\det J(g)(2\varepsilon, y) = 4\varepsilon + \frac{10\varepsilon}{\alpha} \cdot \phi'(0) = 4\varepsilon > 0$$

$$\det J(g) \left(\frac{5\varepsilon}{2}, y \right) = 5\varepsilon + \frac{10\varepsilon}{\alpha} \cdot \phi' \left(\frac{1}{2} \right) = 5\varepsilon + \frac{10\varepsilon}{\alpha} \cdot (-\alpha) = -5\varepsilon < 0$$

$$\det J(g)(3\varepsilon, y) = 6\varepsilon + \frac{10\varepsilon}{\alpha} \cdot \phi'(1) = 6\varepsilon > 0,$$

then there are two numbers $x_1 \in (2\varepsilon, \frac{5\varepsilon}{2})$ and $x_2 \in (\frac{5\varepsilon}{2}, 3\varepsilon)$, so that

$$\det J(g)(x_1, y) = \det J(g)(x_2, y) = 0 :$$

these define the points of the new folds.

Also, g is an approximation of f with first derivatives:

$$\left| 2x + \frac{10\varepsilon}{\alpha} \phi' \left(\frac{x - 2\varepsilon}{\varepsilon} \right) - 2x \right| = \left| \frac{10\varepsilon}{\alpha} \phi' \left(\frac{x - 2\varepsilon}{\varepsilon} \right) \right| \leq 10\varepsilon, \quad \forall x \in \mathbb{R}.$$

We show now how we may insert cusps. We consider several types of approximation.

(a) *Arbitrarily approximation:* We show that we may insert a pair of nearby arcs where the new function g will have fold points and run them together to give the new cusps.

We consider the smooth curve C, which touches no general fold of f and $p \in C$, as before. Suppose that near the regular point p, f is given by $f(x,y) = (x,y)$. Define ϕ as before and define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x,y) = \left(x + \frac{2\varepsilon}{\alpha}\phi\left(\frac{x}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right), y\right).$$

Then g is smooth, is an arbitrarily good approximation of f and $g = f$ outside a small neighborhood of p. The critical points of g are those of f and those given by

$$\det J(g)(x,y) = \det \begin{bmatrix} 1 + \frac{2}{\alpha}\phi\left(\frac{y}{\varepsilon}\right)\phi'\left(\frac{x}{\varepsilon}\right) & \frac{2}{\alpha}\phi\left(\frac{x}{\varepsilon}\right)\phi'\left(\frac{y}{\varepsilon}\right) \\ 0 & 1 \end{bmatrix} = 0,$$

or

$$1 + \frac{2}{\alpha}\phi\left(\frac{y}{\varepsilon}\right)\phi'\left(\frac{x}{\varepsilon}\right) = 0.$$

Since $\det J(g)(0,0) = 1 > 0$, $\det J(g)\left(\frac{\varepsilon}{2}, 0\right) = 1 + \frac{2}{\alpha} \cdot 1 \cdot (-\alpha) = -1 < 0$, and $\det J(g)(2\varepsilon, 0) = 1 > 0$, it is clear that there are two folds cutting the x-axis. If ϕ is sufficiently simple shape, these come together in two cusps.

(b)*Approximations with first derivatives:* Let p be a fold point of f, on a critical curve of f which contains no cusp points. Near p, f is given by $f(x,y) = (x^2, y)$. We define $g : \mathbb{R}^2 \to \mathbb{R}^2$, by

$$g(x,y) = \left(x^2 + \frac{10\varepsilon^2}{\alpha}\phi\left(\frac{x - 2\varepsilon}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right), y\right),$$

with ϕ chosen as before. Outside a little neighborhood of p, $g = f$. We have

$$J(g)(x,y) = \begin{bmatrix} 2x + \frac{10\varepsilon}{\alpha}\phi'\left(\frac{x-2\varepsilon}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right) & \frac{\partial g}{\partial y}(x,y) \\ 0 & 1 \end{bmatrix},$$

so $\det J(g)(0,0) = \frac{10\varepsilon}{\alpha}\phi'(-2)\phi(0) = 0$, which means p is a critical point of g. Since

$$\det J(g)(2\varepsilon, 0) = 4\varepsilon + \frac{10\varepsilon}{\alpha}\phi'(0)\phi(0) = 4\varepsilon > 0$$

$$\det J(g)\left(\frac{5\varepsilon}{2},0\right)=5\varepsilon+\frac{10\varepsilon}{\alpha}(-\alpha)\phi(0)=-5\varepsilon<0$$

$$\det J(g)(3\varepsilon,0)=6\varepsilon+\frac{10\varepsilon}{\alpha}\phi'(1)\phi(0)=6\varepsilon>0,$$

$\det J(g)$ becomes zero for two points of the x-axis. We obtain two new folds, joined at two cusp points, and g is an arbitrarily good approximation of f, together with first derivatives:

$$\left|2x+\frac{10\varepsilon}{\alpha}\phi'\left(\frac{x-2\varepsilon}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right)-2x\right|=\left|\frac{10\varepsilon}{\alpha}\phi'\left(\frac{x-2\varepsilon}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right)\right|<$$

$$<\frac{10\varepsilon}{\alpha}\cdot\alpha\cdot 1=10\varepsilon,\quad \forall\,(x,y)\in\mathbb{R}^2.$$

(c) *Approximations with first and second derivatives:* Let p be a cusp point of f. Near p, f is given by $f(x,y)=(xy-x^3,y)$. Define g near p by setting

$$g(x,y)=\left(xy-x^3\left[1-2\phi\left(\frac{x}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right)\right],y\right).$$

Then

$$J(g)(x,y)=\begin{bmatrix} y-3x^2\left[1-2\phi\left(\frac{x}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right)\right]+2x^3\cdot\frac{1}{\varepsilon}\phi'\left(\frac{x}{\varepsilon}\right)\phi\left(\frac{y}{\varepsilon}\right) & \frac{\partial g}{\partial y}(x,y) \\ 0 & 1 \end{bmatrix}$$

The curve C of general fold of g coincides with the original critical curve C_0: $y=3x^2$ of f for $|x|\geq\varepsilon$, it contains p and, by symmetry, is in the x-direction. Since

$$\frac{\partial f_1}{\partial x}(p)=\frac{\partial f_1}{\partial y}(p)=0,\quad \frac{\partial^2 f_1}{\partial x^2}\partial x\partial y(p)=1 \quad \text{şi} \quad \frac{\partial^3 f_1}{\partial x^3}(p)=6,$$

p is a cusp point for g [Wh]. At points of C where $x\leq-\varepsilon$, $g=f$ and $J(g)(x,y)\begin{bmatrix} y-3x^2 & x \\ 0 & 1\end{bmatrix}$, so $\frac{\partial^2 f_1}{\partial x^2}(x,y)=-6x>0$. For $x\geq\varepsilon$, $g=f$ and $\frac{\partial^2 f_1}{\partial x^2}(x,y)=-6x<0$. On the other hand, since $\frac{\partial^2 f_1}{\partial x^2}(p)=0$ and $\frac{\partial^3 f_1}{\partial x^3}(p)>0$, we have that $\frac{\partial^2 f_1}{\partial x^2}(x,y)$ has the same sign as x for $x\neq 0$ and $|x|$ small enough.

Therefore, as x runs from $-\varepsilon$ to ε, if we run along C, $\dfrac{\partial J}{\partial x} = \dfrac{\partial^2 f_1}{\partial x^2}$ changes sign at least three times. With the function ϕ of simple shape, it will change sign exactly three times; that is g will have three cusp points. We have thus introduced two new cusps, the three cusps lying on a single general fold curve.

Differentiating g, it follows that g is an arbitrarily good approximation of f, together with first and second derivatives.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an excellent mapping and p a cusp point on the general fold C. Suppose there is a smooth curve A which moves from p to ∞ into the lower side of C and which touches no general fold. Then there is arbitrarily good approximation g to f which agrees with f outside a neighborhood U of A, and for which the part of the fold near p is replaced by a pair of folds going near A, to ∞, without cusp points.

This may be seen as follows. Around p, f is given by $f(x, y) = (xy - x^3, y)$. Each line $y = a > 0$ is mapped by f so as to fold over on itself twice. The lines $y = a \leq 0$ have no such folds. We need merely insert such folds near the negative y-axis, to join the above folds. These can be extended down along all of A.

We saw that cusps may be eliminated from regions by arbitrarily good approximations. This is not true for folds.

Theorem 3.1. *Let p be a fold point of the excellent mapping f. Then for any neighborhood U of p, each sufficiently good approximation g to f which is excellent has a fold point in U.*

Proof: Since p is a fold point, there are two points p_1 and p_2 in U where the Jacobian has opposite signs. Let U_i be a circular neighborhood of p_i $(i = 1, 2)$ which touches no fold, and let U_i' be an interior circular neighborhood. For a sufficiently good approximation g to f, if g_t is the deformation of g into f,

$$g_t(q) = g(q) + t[f(q) - g(q)] \qquad (0 \leq t \leq 1),$$

then the image of the boundary ∂U_i does not touch the image of U_i' under f:

$$g_t(q) \neq f(q'), \quad q \in \partial U_i, \quad q' \in U_i', \quad 0 \leq t \leq 1.$$

Hence $g(U_i)$ and $f(U_i)$ cover $f(U_i')$ the same algebraic number of times. For f, this number is ± 1. Hence there is a point p_i' in U_i' such that the Jacobian of g at p_i' is of the same sign as the Jacobian of f in U_i. But the Jacobians of g at p_1' and at p_2' are of opposite sign. Then the segment $p_1'\ p_2'$ contains a singular point of g, and since g is excellent, there is a fold point of g in U. □

Theorem 3.2. *If Q is a bounded closed set in which f is non-singular, then any sufficiently good approximation g to f is non-singular in Q.*

Proof: It follows since the Jacobian involves only first derivatives. □

Theorem 3.3. *Let the arc A have end points p_1 and p_2 where f is non-singular. Then, for any sufficiently good 1-approximation g to f which is excellent, any arc A' from p_1 to p_2 which cuts only fold points of f and g cuts the same number of folds (mod 2) for each.*

Proof: This is clear, since the Jacobian of f and of g have the same sign at each p_i. □

Theorem 3.4. *Let p be a cusp point of f. Then for any neighborhood U of p, each sufficiently good 1-approximation g of f which is excellent has a cusp point in U.*

Proof: There is a curve $A = p_1\, p_2\, p_3\, p_4$ of minimum ∇f in U, which cuts the fold C through p at the points p_2 and p_3. The open arc $p_2\, p_3$ lies in the upper part of C and the open arcs $p_1\, p_2$ and $p_3\, p_4$ lie in the lower part. There is an arc B from p_1 to p_4 in the lower part of C, lying in U, such that A and B bound a region R' filled by curves of minimum ∇f. For any sufficiently good 1-approximation g to f, there will be an arc A^* of minimum ∇g, near A, which will bound, with part of B, a region R^* filled by curves of minimum ∇g. Also, g will be non-singular in B, and there will be fold points of g in R^*. The set Q of fold and cusp points of g in the closure $\overline{R^*}$ is a closed set. There is a lowest curve D of minimum ∇g in $\overline{R^*}$ which touches Q in a point p^*. Since p^* is not in B, $p^* \in R^*$. p^* is a singular point of g. Also, by definition of D, the general fold of g through p^* does not cross the curve D, and hence is tangent to D. Therefore, p^* is not a fold point of g and it follows that p^* is a cusp point of g. □

Theorem 3.5. *For any bounded closed set Q in which the only singularities of f are fold points, any sufficiently good 2-approximation g of f which is excellent has only folds in Q.*

Proof: Let p be a fold point of f in Q and let A be a short segment perpendicular to the fold, centered at p. Since $J(f)$ is of opposite signs at the two ends of A, so $F(g)$ will be. Hence $J(g)$ will vanish somewhere on A. Since f is excellent, the directional derivative of $J(f)$ in the direction of A is non-zero, hence the same is true for g and g has just one general fold cutting A. Thus the general folds of g are like those of f in Q, if the 2-approximation is good enough. Since the directions of curves of minimum ∇g and of general folds for g are nearly parallel to the similar curves for f, the conditions for fold points will be satisfied at all general fold points of g in Q, for a good approximation. Hence g will have no cusp points in Q. □

Theorem 3.6. *Let U be a neighborhood of the cusp point p of f. Then for any sufficiently good 2-approximation g to f which is excellent, there will be a cusp point p' of g in U, on a general fold C'; there will be no other general folds of g in U, and the number of critical points of g on C' in U will be odd.*

Proof: There will be a unique general fold C' of g in U. At two points p_1, p_2 of the general fold C of f, on opposite sides of p, the curves of minimum ∇f cut C in opposite senses; the same will be true, using g, for similar points p_1', p_2' of C'. Hence there will be an odd number of cusps of g between these points. There will be no cusps in $C' \cap U$ outside these points. □

Theorem 3.7. *With U, p and f as in the last theorem, any sufficiently good 3-approximation g to f has a unique general fold in U, with a unique cusp point on it.*

Proof: There is a unique C' as in the last theorem, with a cusp point p'. Since $\nabla_v \nabla_v f(p) \neq 0$, the similar relation $\nabla_{v'} \nabla_{v'} g(p') \neq 0$ holds. We see that $\nabla_{v'} g$ is in opposite directions on opposite sides of p' on C', and hence p' is the only cusp of g in U. □

References

[FT] Francis G.K., Troyer S.F., Excellent maps with given folds and cusps, Houston Journal of Math, vol 3. nr 2 (1977), 165-192.

[M,S,T] Malta I., Saldanha C.N., Tomei C., Critical Sets of Proper Whitney Functions in the Plane, Matemática Contemporanea, vol. 13, 1997, 181-228.

[No-Pu] Norton A., Pugh Ch., Critical Sets in the Plane, Michigan Journal of Math., 38, 1991, 441-459.

[Th1] Thom R., Quelques propriétés globales des varétés différentiables, Comment. Math. Helv., 28(1), 1954, 17-86.

[Th2] Thom R., Les singularités des applications différentiables, Ann. Inst. Fourier (Grenoble), 6(1955-1956), 43-87.

[Wh] Whitney H., On Singularities of Mappings of Euclidean Spaces, I: Mappings of the Plane into the Plane, Ann. of Math. 62(1955), 374-410.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

Received: 23.12.2001

BOOK REVIEWS

Haakan Hedenmalm, Boris Korenblum and Kehe Zhu, *Theory of Bergman Spaces,* Graduate Texts in Mathematics Vol. 199, Springer-Verlag, New York Berlin Heidelberg, 2000, ix+286 pp., ISBN: 0-387-98791-6.

Along with Hardy spaces, Bergman spaces constitute the most important spaces of analytic functions. While the function theory and and operator theory connected with Hardy spaces (zeros, interpolation, invariant subspaces, Toeplitz and Hankel operators) were well understood fifteen years ago, the study of their close relatives, the Bergman spaces, turned to be much more difficult. Significant breakthroughs, both function theoretic and operator theoretic, were done in the 1990's, and the present book concentrates on these latest developments. Some of them not achieved the final form so that the reader is brought to the frontier of current research in the area. The exercise sections at the end of each chapter includes, beside routine problems which can be used as homework assignments, also nontrivial ones (with references) or even open problems.

The Bergman spaces were introduced by the Polish mathematician Stefan Bergman in his book *The kernel function and conformal mapping,* (a second revised edition was published by the American Mathematical Society in 1970). Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. For $-1 < \alpha < \infty$ and $0 < p \leq \infty$, the Bergman space $A^p_\alpha = A^p_\alpha(\mathbb{D})$ is the space of analytic functions in $L^p(\mathbb{D}, dA_\alpha)$, for $dA_\alpha = (\alpha+1)(1-|z|^2)dA(z)$, where $dA(z) = \pi^{-1}dxdy$ is the normalized Lebesgue measure on $\mathbb{D}$. One puts $A^p = A^p_0$. They are closed subspaces in L^p, so that A^p_α is a complete linear metric space for $0 < p < 1$, respectively a Banach space for $p \geq 1$. For $1 < p < \infty$ the duality relation $(A^p_\alpha)* = A^q_\alpha$, with $p^{-1} + q^{-1} = 1$, holds, while for $0 < p \leq 1$ the dual of A^p_α is the Bloch space $\mathcal{B}$, which plays in the theory of Bergman spaces the same role as does the space BMOA in the theory of Hardy spaces. The Bloch space can be identified with the space of all analytic functions on $\mathbb{D}$ which are Lipschitz with respect to the Bergman metric.

The analogue of the Poisson transform in the context of Bergman spaces is the Berezin transform, which leads to the definition of a space of BMO type on the disk, whose analytic part is the Bloch space. The fixed points of the Berezin transform are exactly the harmonic functions.

The study of invariant subspaces of Bergman spaces is one of the central topic of the book. In fact, the old famous open problem of the existence of nontrivial invariant subspaces in separable Hilbert spaces is equivalent to a question of existence of some special z-invariant subspaces in the Hilbert-Bergman space A^2, explaining the growing interest in the study of Bergman spaces. With every invariant subspace $I \subset A^p_\alpha$ one associates an extremal problem – find $G \in A^p_\alpha$ which solves the extremal problem $\sup\{\mathrm{Re} f^{(n)}(0) : f \in I, \|f\|_{p,\alpha} \le 1\}$. It turns that G must be an A^p_α-inner function. One proves the existence of invariant subspaces I of arbitrary finite index $n = \dim(I/zI)$.

Other important topics treated in the book are: interpolation and sampling, characterizations of zero sets in A^p_α, cyclicity. One doesn't know geometric characterizations of zero sets as well as characterizations of cyclic vectors in Bergman spaces.

The prerequisites for the reading of the book are elementary real, complex, and functional analysis, and some familiarity with Hardy spaces H^p.

The authors are well known specialists in the field, and the book incorporates a lot of their original results.

Introducing the reader to an area of investigation of major interest, situated at the intersection between complex and functional analysis, the book appeals to graduate students and new researchers in these fields.

Ştefan Cobzaş

Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos Sanatalucia, Jan Pelant, Václav Zizler, *Functional Analysis and Infinite-Dimensional Geometry,* Canadian Mathematical Society (CMS) Books in Mathematics, Vol. 8, Springer-Verlag, New York Berlin Heidelberg, 2001, ix+451 pp., ISBN: 0-387-95219-5.

Banach spaces are the natural framework for many branches of mathematics as operator theory, nonlinear functional analysis, abstract analysis, optimization theory, probability theory. The last years were marked by an intense research activity in this area with spectacular discoveries solving old standing problems or leading to new area of investigation. In fact the interplay between theory and its applications is more dialectical – for instance, in the case of probability theory, Banach space theory furnishes powerful tools and far reaching generalizations for probability theory but, at the same time, a lot of deep results in Banach space theory are proved by probabilistic methods. The situation is the same with abstract analysis – the study of the differentiability of vector valued functions led from the beginning to the introduction of some geometric concepts in Banach space theory (smoothness, rotundity), and still continue to generate new important classes of Banach spaces as Radon-Nikodym spaces, Asplund spaces, etc. A good account on the current state of affairs in this field is given in the books of R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman, New York 1993, and M. FÁbian, *Differentiability of Convex Functions and Topology– Weak Asplund Spaces,* J. Wiley&Sons, New York 1997.

The specific of the present book is that it brings the reader from the fundamental results of the theory and leads him/her to the frontier of current research. The book by P. Habala, P. Hájek and V. Zizler, *Introduction to Banach Spaces*, could be considered as a preliminary version , or a skeleton, of the book, but the present one is considerably revised, updated and completed.

The book is based on graduated courses taught at the University of Alberta in Edmonton in the years 1984-1997, were the principal part of the text was prepared. In fact, each author spent some time at this university.

The book is divided into twelve chapters headed as follows: 1. *Basic Concepts in Banach Spaces,* 2. *Hahn-Banach and Banach Open Mapping Theorems,* 3. *Weak*

Topologies, 4. *Locally Convex Spaces,* 5. *Structure of Banach Spaces,* 6. *Schauder Bases,* 7. *Compact Operators on Banach Spaces,* 8. *Differentiability of Norms, Uniform Convexity,* 10. *Smoothness and Structure,* 11. *Weakly Compactly Generated Banach Spaces,* 12. *Topics in Weak Topology.* By its organization, the book can be used as a textbook for various types of courses in functional analysis: undergraduate first (Chapters 1-3 and 7) or second (Chapters 4-6, 8 and 10), graduate two-semester (Chapters 1-9), one semester (Chapters 1-3, 5 and 6 or 7), or graduate advanced one-semester (Chapters 8-10, or 11 and 12).

Beside classical material, the book contains also some recent and more specialized results as smooth variational principles, Lipschitz and uniform classification of Banach spaces, Asplund and weak Asplund spaces, Borel and analytic structures in Banach spaces, including original results of the authors.

The book is fairly self-contained, the prerequisites being basic courses in real analysis and topology (at the level of, e.g., Royden's book on real analysis). To make the text more accessible, the authors included the proofs of many facts considered as folklore by the specialists but which may look not such obvious for the newcomer and difficult to find. The book contain a large number of exercises with detailed hints, completing the main text with many important results.

The book is a valuable contribution to Banach space literature and can be used as a solid introduction to functional analysis, smoothing the way to more specialized books or research papers.

Stefan Cobzaş

Frank Deutsch, *Best Approximation in Inner Product Spaces,* Canadian Mathematical Society (CMS) Books in Mathematics, Vol. 7, Springer-Verlag, New York Berlin Heidelberg, 2001, xv+337 pp, ISBN:0-387-98940-4.

The book is based on a graduate course on Best Approximation taught by the author for over than twenty five years at the Pennsylvania State University. The course was attended by various categories of students - engineers, computer scientists, statisticians and mathematicians - who did not own the basic facts of functional and real analysis (e.g. L^p-spaces), necessary for the treatment of the subject in the context of normed linear spaces. In order to save the time necessary for these prerequisites and to concentrate on best approximation problems, the author decided to restrict the exposition to the framework of inner product spaces. These are the closest to the Euclidean space, such that the intuition and drawings help the reader to better understand the origins and the motivation of many considered notions and tools, without any references to other sources (excepting some linear algebra and advanced calculus).

The main innovation of the author is to work with incomplete inner product spaces rather than with Hilbert ones. This approach involves some technicalities but one gains in generality. For instance, Riesz representation theorem for the dual of a Hilbert space X is not true for inner product spaces, but the author finds a generalized representation for a functional $x^* \in X^*$ by a sequence (x_n) in X such that $x^*(x) = \lim_n < y, x_n >,\ y \in X$, and $\|x^*\| = \lim_n \|x_n\|$. The sequence (x_n) is Cauchy so that, if X is complete, $x = \lim_n x_n$ represents the functional x^*, and one obtains the Riesz representation theorem. This result allows to obtain a proof of the Hahn-Banach extension theorem in the case of inner product spaces without appealing to the Axiom of Choice. The author shows that a functional $x^* \in X^*$ is represented by an element of X if and only if x^* attains its norm on the unit ball of X. The representable functionals are important tools in the study of proximinality of various subsets of inner product spaces - hyperplanes, half-spaces, polyhedral sets, cones - as well as in the characterizations of best approximation elements. This is done in Chapters 4, *Characterizations of best approximation,* and 6, *Bounded linear* *functionals and best approximation from hyperplanes and half-spaces*, which are largely based on original results of the author.

As applications, we mention the study of generalized solutions (least square method) of linear equations and of generalized inverses of matrices and linear operators. A new proof of Weierstrass approximation theorem is also obtained.

A special attention is paid to algorithms for best approximation treated in Ch. 9, *The method of alternating projections.* This one, and Chapters 10, *Constrained interpolation from a convex set,* and 11, *Interpolation and approximation,* incorporate again a lot of original results of the author.

The last chapter of the book, Ch. 12, *Convexity of Chebyshev sets*, is concerned with the still unsolved problem of convexity of Chebyshev sets in Hilbert space.

Each chapter ends with a set of exercises and very interesting historical notes.

Written by a well-known specialist in best approximation theory, the book contains a good treatment of best approximation in inner product spaces and can be used as a textbook for graduate courses or for self-study.

Stefan Cobzaş

Operator Theory and Analysis, The M. A. Kaashoek Anniversary Volume, H. Bart, I. Gohberg and A.C.M. Ran - Editors, Operator Theory, Advances and Applications, Vol. 122, Birkhäuser Verlag, Boston-Basel-Berlin 2001, xxxix + 425 pp., ISBN 3-7643-6499-8.

The present volume contains the proceedings of the workshop organized at the Vrije Universiteit Amsterdam on November 12-14, 1997, on the occasion of the sixtieth birthday of Marinus (Rien) Adrianus Kaashoek. Professor M.A. Kaashoek is one of the leading experts in operator theory and its applications (especially to electrical engineering), the founder and the head of the Analysis and Operator Theory Group in Amsterdam. He published 6 books and over than 140 papers, many in cooperation with I. Gohberg. The worksop was attended by 44 participants from all over the world which presented 21 plenary lectures followed by lively discussions. An opening address, written by I. Gohberg and red by S. Goldberg, presents the charming personality and the remarkable scientific achievements of Professor M. A.

Kaashoek. Some personal reminiscences are presented by three of his PhD students: H. Bart, A.C.M. Ran and H.J. Woerdman. A photo, a Curriculum Vitae and a list of publications of M.A. Kaashoek are also included.

Beside these addresses and biographical material, the volume contains 16 contributed papers covering a wide range of topics in functional analysis and operator theory, centered around domains where the ideas and results of M.A. Kaashoek played an important role: factorization of matrix valued functions, Nevanlinna-Pick interpolation theory, spectral theory, Toeplitz operators, Jordan chains. Among the contributors to the volume we mention: V. Adamyan, R. Mennicken, D. Alpay, A. Dijksma, Y. Peretz, D.Z. Arov, H. Dym, R.L. Ellis, I. Gohberg, B. Nagy, A.E. Frazho, P. Lancaster, A. Markus, H. Langer.

Bringing together important new contributions to operator theory and its applications, written by leading experts in the field, the volume will be of interest to a wide range of readers in pure and applied mathematics and engineering.

S. Cobzaş

Daniel Beltiţă and Mihai Şabac, *Lie Algebras of Bounded Operators*, Operator Theory, Advances and Applications, Vol. 120, Birkhäuser Verlag, Boston-Basel-Berlin 2001, viii + 217 pp., ISBN 3-7643-6404-1.

The importance in the theory of finite dimensional Lie algebras of the Jordan canonical structure of linear map acting on finite-dimensional vector spaces is well known and well understood. The aim of the present book is to study the infinite dimensional case, emphasizing the role played by bounded operators on Banach spaces in the study of infinite dimensional Lie algebras. In fact, there is an interaction between operator theory and Lie algebra theory, the last offering solutions to some long-standing questions in operator theory related to the construction of joint spectral theory for non-commuting tuples of operators. Although in the infinite dimensional case one cannot speak about a plane Jordan canonical structure, like in the case of matrices, there are some classes of operators (Dunford spectral, Foiaş decomposable, scalar generalized and Colojoară scalar generalized operators) which admit a kind of Jordan decomposition.

The first chapter of the book, I. *Preliminaries,* containing three sections: A. *Lie Algebras,* B. *Complexes*, C. *Spectral Theory*, surveys the basic of Lie algebra theory, Koszul complexes in Banach spaces, and spectral theory for bounded operators. In this part, the proofs of the results which can be found in already existing books are omitted, with exact references to the corresponding books.

The rest is devoted to the exposition of the main theme of the book: the interplay between Lie algebra theory and spectral theory of bounded operators. A good idea on the topics the authors are dealing with is given by the headings of the chapters: II. *The Commutators and Nilpotence Criteria*, III. *Infinite Dimensional Variants of Lie and Engel Theorems,* IV. *Spectral Theory for Solvable Lie Algebras of Operators,* V. *Semisimple Lie Algebras of Operators.*

Modulo some basic results on Lie algebras and spectral theory, the book is self-contained. Original results of the authors, some of them published for the first time, are included. A rich bibliography, counting 173 items and covering practically all that was published in the field up to the present book, is included at the end of the book.

Exposing in a clear and accessible manner deep results on the interplay between Lie algebra theory and spectral theory of bounded operators on Banach spaces, the book will appeal to researchers working in both of these two areas. It can be used also as a base text for advanced graduate or postgraduate courses.

S. Cobzaş

Robert E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics Vol. 183, Springer-Verlag, New York Berlin Heidelberg, 1998, xix+596 pp., ISBN: 0-387-98431-3.

Many books on Banach spaces as, e.g., M. Day, *Normed Linear Spaces*, 3rd Edition, Springer Verlag 1973, or J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Vols. I(1977) and II(1979), also published by Springer Verlag, can be used by graduate students wishing to do research in Banach space theory, but can be too difficult for a student at his first contact with functional analysis. The aim of the present book is to provide this student with detailed proofs and a careful presentation of the fundamental results in Banach space theory. The only prerequisites for its reading are some measure theory and topology as presented, for instance, in W. Rudin's book *Real and Complex Analysis,* McGraw Hill 1987. Measure theory is used only for the applications of Banach space theory to the spaces L_p, and not as an essential tool in the development of the subject. Restricting to sequence spaces and treating only metric theory of Banach spaces, it is possible to use the book for an undergraduate course. Nets, which are extensively used in the study of weak topologies, are presented in detail in the first section of the second chapter. Appendix D is devoted to ultranets and Tihonov's compactness theorem. Other Appendices are A. *Prerequisites,* B. *Metric Spaces* and C. *The spaces* ℓ_p *and* ℓ_p^n.

The book contains five chapters and four appendices, as presented above. Ch. 1, *Basic concepts*, includes norms, linear operators, Baire category and three fundamental theorems, quotients, direct sums, Hahn-Banach theorem, dual spaces and reflexivity. Ch. 2, *The weak and weak* topologies*, contains some results on topology, topological vector spaces and locally convex spaces needed for the study on weak and weak* topologies on Banach spaces (including weak compactness and James' theorem, extreme points and Krein-Milman's theorem, support points, support functionals and Bishop-Phelps subreflexivity theorem). Ch. 3, *Linear operators,* is concerned with linear operators and their adjoints, compact and weakly compact operators (including Schauder and Gantmacher theorems and Riesz' theory). Ch. 4, *Schauder bases*, contains some basic results on Schauder bases in Banach spaces. The last section of this chapter is devoted to a presentation of James space J. The last chapter of the book, Ch. 5, *Rotundity and smoothness*, presents some results from the geometry of banach spaces – rotundity, uniform rotundity and generalizations, smoothness, uniform smoothness and generalizations.

Each section is followed by a set of exercises completing the main text. A lot of historical notes and comments are spread through the book, mentioning the original sources or tracing the development of the ideas. The bibliography at the end of the book counts 249 items.

The result is an excellent book on the basics of Banach spaces, which can be warmly recommended as a textbook for the introduction to the subject.

Stefan Cobzaş

Theodore W. Gamelin, *Complex Analysis*, Springer New York, Berlin, Heidelberg, 2001, 478 pp., ISBN 0-387-95069-9.

This is a beautiful book which provides a very good introduction to complex analysis for students with some familiarity with complex numbers. It is based on lectures given over the years by the author at several places, particularly the Interuniversity Summer School at Perugia (Italy) (the present reviewer was one of those students that took his wonderful course in Perugia in 1992), also UCLA, Brown University, Valencia (Spain), and La Plata (Argentina). The book consists of three parts. The first part includes Chapters I-VII. It presents a basic material about the complex plane and elementary functions, analytic functions, line integrals and harmonic functions, complex integration and analyticity, power series, Laurent series and isolated singularities, and the residue calculus.

The second part contains chapters VIII-XI and includes certain special topics such as the logarithmic integral (the argument principle, Rouché's theorem, Hurwitz's theorem, etc), the Schwarz lemma and hyperbolic geometry, harmonic functions and the reflection principle, and conformal mappings (the Riemann Mapping Theorem, the Schwarz-Christoffel formula, compactness of families of functions, etc).

The third part contains chapters XII-XVI. This part consists of a careful selection of several topics which certainly serve to complete the coverage of all background necessary for passing PhD qualifying exams in complex analysis, such as compact families of meromorphic functions, approximation theorems, some special functions (the Gamma function, Laplace transform, the Zeta function, Dirichlet series), the Dirichlet problem and Riemann surfaces.

The book is clearly written, with rigorous proofs, in a pleasant and accessible style. It is warmly recommended to students and all researchers in complex analysis.

Gabriela Kohr

Approaches to Singular Analysis, Juan Gil, Daniel Grieser, Matthias Lesch - Editors, Operator Theory, Advances and Applications, Vol. 125, Subseries "Advances in Partial Differential Equations", Birkhäuser Verlag, Boston-Basel-Berlin 2001, vi + 256 pp., ISBN 3-7643-6518-8.

The book is based on the workshop "Approaches to Singular Analysis", held at the Humboldt University Berlin in April 8-10, 1999, and contains articles by the participants at the workshop as well as some invited contributions. The aim of the workshop was to bring together young mathematicians interested in partial differential equations on singular configurations. Two main approaches to these problems can be emphasized: (1) the pseudodifferential approach, meaning to set up a pseudodifferential calculus adapted to the underlying configuration (the schools of R. Melrose at MIT, of B.-W. Schulze at Potsdam, and the results of B.A. Plamenevski and his coworkers), and (2) the direct approach, meaning the analysis of the geometric differential operators (Dirac, Laplace, etc.) in specific situations (there is a vast literature on these topics as, e.g., the papers by Brüning and Seeley, Cheeger, Lesch, Müller, a.o.).

There are included 5 papers by the participants and 3 invited contributions. The contributed papers deal with Boutet de Monvel's calculus for pseudodifferential operators (E. Schrohe, pp. 85-116), the *b*-calculus (D. Grieser, pp. 30-84),completed by a paper by R. Lauter and J. Seiler on a comparison between cone algebra nad *b*-calculus (pp. 117-130). A paper by J. Seiler (pp. 1-29) is dealing with cone algebra and kernel characterization of Green operators, and one by D. Grieser and M. Gruber with singular asymtotics (pp. 117-130).

The three invited papers are by B.-W. Schulze on operator algebras with symbol hierarchies on manifolds with singularities (pp. 167-207), J. Brüning on the resolvent expansion on singular spaces (pp. 208-233), and the last by B. Fedosov, B.-W. Schulze and N. Tarkhanov on general index formula on toric manifolds with conucal points (pp. 234-256).

Bringing together important contributions in the field of partial differential and pseudodifferential operators, this collection of papers will be of interest for researchers and scholars working in this area, as well as for those interested in applications to mathematical physics.

Radu Precup

Problems and Methods in Mathematical Physics - The Siegfried Prössdorf Memorial Volume, J. Elschner, I. Gohberg and B. Silbermann - Editors, Operator Theory, Advances and Applications, Vol. 121, Birkhäuser Verlag, Boston-Basel-Berlin 2001, viii+523 pp., ISBN 3-7643-6477-7.

In the Spring of 1997 preparations had begun for a conference in honor of Siegfried Prössdorf's 60th birthday, but his sudden and untimely death stopped for a while these plans. Nevertheless, many of his friends and colleagues decided that the conference, the 11th TMP, be organized and dedicated to honor the life and work of S. Prössdorf. The Conference took place in Chemnitz, Germany, from March 25 to 28, 1999, and the present volume contains its proceedings. The volume starts with three contributions, by Bernd Silbermann, V. Maz'ya and Jürgen Sprenkels, evoking the life and the charming personality of S. Prössdorf as well as his outstanding contributions to integral and pseudodifferential equations, numerical analysis, operator theory, boundary value problems, boundary element and approximation theory. The lists of Prössdorf's publications (134 papers and 6 books) and of dissertations conducted by him are also included.

Beside these three papers, the volume contains 24 original papers, most written by friends an coworkers of S. Prössdorf, dealing with topics which were close to the broad spectrum of his scientific preoccupations. There is a joint paper by S. Prössdorf and M. Yamamoto, started during Yamamoto's visit in September 1997 at the Weierstrass Institut für Angewandte Analysis und Stochastik in Berlin, and finished by Yamamoto alone.

Containing important contributions to integral and pseudodifferential equations, boundary value problems, operator theory and applications in physics and engineering, the volume is addressed to a wide audience in the mathematical and engineering science.

Paul Szilágyi

Karlheinz Gröchenig, *Foundations of Time-Frequency Analysis,* Applied and Numerical Harmonic Analysis, Birkhäuser Verlag, Boston-Basel-Berlin 2001, xv+359 pp, ISBN 3-7643-4022-3 and 0-8176-4022-3.

Time-frequency analysis is a form of local Fourier analysis that treats time and frequency simultaneously and symmetrically. Classical Fourier analysis employs two complementary representations to describe functions - the function f and its Fourier transform $\hat{f}$. The study of the relations between f and $\hat{f}$ is governed by two principles: (1) the smoothness-and-decay principle (if f is smooth then $\hat{f}$ decays quickly, and if f decays quickly then $\hat{f}$ is smooth), and (2) the uncertainty principle (f and $\hat{f}$ cannot be simultaneously small). In applications, for instance, the variable $x \in \mathbb{R}$ may signify "time" and $f(x)$ is the amplitude or electric field , while the Fourier transform $\hat{f}(\omega)$ is understood as the amplitude of the frequency ω.

Time-frequency analysis has its roots in the early development of quantum mechanics by H. Weyl, E. Wigner and J. von Neumann around 1930, and in the theoretical foundation of information theory by D. Gabor in 1946. Time-frequency analysis as an independent mathematical field was established by A.J.E.M Janssen around 1980. Its characteristic features consist in the richness and beauty of the involved mathematical structures and applications, ranging from the theory of short-time Fourier transform and classical results about the Wigner distribution, via the recent theory of Gabor frames, to quantitative methods in time-frequency analysis and the theory of pseudodifferential operators.

Although its contents is intimately related to applications in signal analysis and quantum mechanics, the book, written by a mathematician, is primarily devoted to mathematicians, its aim being a detailed mathematical investigation of the rich and elegant structures underlying time-frequency analysis. It is also accessible to engineers and physicists with a more theoretical orientation. The book is written at an introductory level, with detailed calculations whenever necessary, the main prerequisites being a solid course in analysis and some Hilbert space theory.

The book starts with an introductory chapter containing an exposition (without proofs) of the basic principles of Fourier analysis. Other necessary results, from functional analysis and Lebesgue integration, are mentioned in an Appendix at the end of the book. The mathematical theory of time-frequency analysis is developed in the rest of the chapters: 2. *Time-frequency analysis and the uncertainty principle* , 3. *The short-time Fourier transform,* 4. *Quadratic time-frequency representations,* 5. *Discrete time-frequency representations: Gabor frames,* 6. *Existence of Gabor frames,* 7. *The structure of Gabor frames,* 8. *Zak transform methods,* 9. *The Heisenberg group: A different point of view,* 10. *Wavelet tranforms,* 11. *Modulation spaces,* 12. *Gabor analysis of modulation spaces,* 13. *Window design and Wigner's lemma,* 14. *Pseudodifferential operators.*

Supplying a unified and systematic introduction to the mathematical foundations of time-frequency analysis and emphasizing the interdiscliplinary aspects of the subject, the book is of great interest for mathematicians, physicists, engineers in signal and image analysis, researchers and professionals in wavelet and mathematical signal analysis. By the detailed and careful presentation of the subject, the book can be used by graduate students too.

Damian Trif