Anul XLVII **2002**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
3

Redacţia: 3400 Cluj-Napoca, str. M. Kogălniceanu nr. 1 • Telefon: 405300

SUMAR – CONTENTS – SOMMAIRE

M. Banaru, On Nearly-Cosymplectic Hypersurfaces in Nearly-Kählerian Manifolds ... 3

Daniel Breaz and Nicoleta Breaz, Two Integral Operators ... 13

Adriana Buică, Quasilinearization for the Forced Düffing Equation ... 21

Endre Buzogány, Ildikó Ilona Mezei and Viorica Varga, Two-Variable Variational-Hemivariational Inequalities ... 31

Cristian Chifu-Oros, Functionals which Satisfy a Maximum Principle ... 43

Rodica Covaci, $\mathcal{X}$-Maximal Subgroups in Finite π-Solvable Groups with Respect to a Schunck Class $\mathcal{X}$ ... 53

Iuliu Crivei, On Some Ω-Pure Exact Sequences of Modules ... 63

Zoltán Finta, Quantitative Estimates for Some Linear and Positive Operators ... 71

Vasile Pop, Maximal Sets on a Hypersphere ... 84

Liana Ţopan, Critical Sets of 1-Dimensional Manifolds ... 91

Book Reviews ... 99

STUDIA UNIV. “BABEŞ–BOLYAI”, MATHEMATICA, Volume **XLVII**, Number 3, September 2002

ON NEARLY-COSYMPLECTIC HYPERSURFACES IN NEARLY-KÄHLERIAN MANIFOLDS

M. BANARU

Abstract. It is proved that the type number of a nearly-cosymplectic hypersurface in a nearly-Kählerian manifold is at most one. It is also proved that such a hypersurface is minimal if and only if it is totally geodesic.

1. **Introduction**

The theory of almost contact metric structures occupies one of the leading places in modern differential-geometrical researches. It is due to a number of its applications in modern mathematical physics (e.g. in classical mechanics [1] and in theory of geometrical quantization [10]) and to the riches of the internal contents of the theory as well, and also to its close connection with other sections of geometry.

One of the most important examples of almost contact metric structures, wich appreciably determines their role in differential geometry, is the structure induced on an oriented hypersurface in an almost Hermitian manifold. Well known scientists such as D.E. Blair, S. Goldberg, V.F. Kirichenko, S. Sasaki, S. Tanno were engaged in studying almost contact metric hypersurfaces in almost Hermitian manifolds.

In the present note, nearly-cosymplectic hypersurfaces in nearly-Kählerian manifolds are considered. We can mention that the class of nearly-Kählerian manifolds is one of the most important classes of almost Hermitian manifolds [9]. A great number of significant works is devoted to its studying. For not going in details of such an extensive subject, we remark only, that the six-dimensional sphere with a nearly-Kählerian structure is considered in [7], [8], [11], [15] etc.

Received by the editors: 24.04.2002.

2000 *Mathematics Subject Classification.* 53C55, 53C40.

Key words and phrases. nearly-Kählerian manifold, almost contact metric structure, totally geodesic submanifold.

The present work is a continuation of researches of the author, who studied cosymplectic hypersurfaces in six-dimensional submanifolds of Cayley algebra before (see [3], [4]).

2. Preliminaries

We consider an almost Hermitian (AH) manifold, i.e. a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^∞. We recall that the fundamental (or Kählerian [14]) form of an almost Hermitian manifold is determined by

$$F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

Let $\left(M^{2n}, J, g = \langle \cdot, \cdot \rangle\right)$ be an arbitrary almost Hermitian manifold. We fix a point $p \in M^{2n}$. As $T_p(M^{2n})$ we denote the tangent space at the point p, $\{J_p, g_p = \langle \cdot, \cdot \rangle\}$ is the almost Hermitian structure at the point p induced by the structure $\{J, g = \langle \cdot, \cdot \rangle\}$. The frames adapted to the structure (or A-frames) look as follows [2]

$$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}}),$$

where ε_a are the eigenvectors corresponded to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$, $\varepsilon_{\hat{a}} = \overline{\varepsilon_a}$. Here the indice a ranges from 1 to n, and we state $\widehat{a} = a + n$.

The matrix of the operator of the almost complex structure written in an A-frame looks as follows:

$$(J_j^k) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$. By direct computing, it is easy to obtain that the matrixes of the metric g and of the fundamental form F in an A-frame look as follows, respectively:

$$(g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right), \quad (F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right).$$

As it is well-known [9], an almost Hermitian manifold is called nearly-Kählerian (NK), if

$$\nabla_X(J)Y + \nabla_Y(J)X = 0, \quad X, Y \in \aleph(M^{2n}),$$

where ∇ is the Levi-Civita connection of the metric.

Let N be an oriented hypersurface in an almost Hermitian manifold M^{2n}, and let σ be the second fundamental form of the immersion of N into M^{2n}. As it is well-known [16], the almost Hermitian structure on M^{2n} induces an almost contact metric structure on N. We recall [16], that an almost contact metric structure on an odd-dimensional manifold N is defined by the system $\{\Phi, \xi, \eta, g\}$ of tensor fields on this manifold, where ξ is a vector, η is a covector, Φ is a tensor of the type $(1,1)$ and g is a Riemannian metric on N such that

$$\eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N).$$

The almost contact metric structure is called nearly-cosymplectic, if

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0, \quad \nabla_X(\eta)Y + \nabla_Y(\eta)X = 0, \quad X, Y \in \aleph(N).$$

At the end of this section, note that when we give a Riemannian manifold and its submanifold, the rank of the determined second fundamental form is called the type number (see, for example, [12]).

3. Three theorems

Now, we can state the main results of this work.

THEOREM A. *The type number of a nearly-cosymplectic hypersurface in a nearly-Kählerian manifold is at most one.*

Proof.

Let N be an oriented hypersurface in a nearly-Kählerian manifold M^{2n}. We use the first group of Cartan structural equations of an almost contact metric structure induced on a hypersurface in an almost Hermitian manifold [16]:

$$\begin{gathered}
d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \omega^\gamma \wedge \omega_\beta + B^{\alpha\beta\gamma}\omega_\beta \wedge \omega_\gamma + \left(\sqrt{2}B^{\alpha n}{}_\beta + i\sigma^\alpha_\beta\right)\omega^\beta \wedge \omega + \\
+ \left(-\sqrt{2}\widetilde{B}^{n\alpha\beta} - \frac{1}{\sqrt{2}}B^{\alpha\beta n} - \frac{1}{\sqrt{2}}B^{\alpha\beta}{}_n + i\sigma^{\alpha\beta}\right)\omega_\beta \wedge \omega, \\
d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \omega_\gamma \wedge \omega^\beta + B_{\alpha\beta\gamma}\omega^\beta \wedge \omega^\gamma + \left(\sqrt{2}B_{\alpha n}{}^\beta - i\sigma^\beta_\alpha\right)\omega_\beta \wedge \omega + \\
+ \left(-\sqrt{2}\widetilde{B}_{n\alpha\beta} - \frac{1}{\sqrt{2}}B_{\alpha\beta n} - \frac{1}{\sqrt{2}}B_{\alpha\beta}{}^n - i\sigma_{\alpha\beta}\right)\omega^\beta \wedge \omega, \\
d\omega = \sqrt{2}B_{n\alpha\beta}\omega^\alpha \wedge \omega^\beta + \sqrt{2}B^{n\alpha\beta}\omega_\alpha \wedge \omega_\beta + \\
+ \left(\sqrt{2}B^{n\alpha}{}_\beta - \sqrt{2}B_{n\beta}{}^\alpha - 2i\sigma^\alpha_\beta\right)\omega^\beta \wedge \omega_\alpha + \\
+ \left(\widetilde{B}_{n\beta n} + B_{n\beta}{}^n + i\sigma_{n\beta}\right)\omega \wedge \omega^\beta + \left(\widetilde{B}^{n\beta n} + B^{n\beta}{}_n - i\sigma^\beta_n\right)\omega \wedge \omega_\beta,
\end{gathered} \tag{1}$$

where

$$\begin{gathered}
\widetilde{B}^{abc} = -\frac{i}{2}J^a_{\hat{b},\hat{c}}, \quad \widetilde{B}_{abc} = \frac{i}{2}J^{\hat{a}}_{b,c}; \\
B^{abc} = -\widetilde{B}^{a[bc]}, \quad B_{abc} = -\widetilde{B}_{a[bc]}; \\
B^{ab}{}_c = -\frac{i}{2}J^a_{\hat{b},c}, \quad B_{ab}{}^c = \frac{i}{2}J^{\hat{a}}_{b,\hat{c}}.
\end{gathered}$$

Here and further, the indices a, b, c range from 1 to n and the indices α, β, γ range from 1 to $n-1$; $\hat{a} = a + n$. $\{B^{abc}\}$, $\{B_{abc}\}$ and $\{B^{ab}{}_c\}$, $\{B_{ab}{}^c\}$ are the components of Kirichenko virtual (KV) and Kirichenko structural (KS) tensors, respectively [5].

Taking into account that an almost Hermitian structure is nearly-Kählerian if and only if [6]

$$B^{abc} + B^{acb} = 0, \quad B_{abc} + B_{acb} = 0, \quad B^{ab}{}_c = 0, \quad B_{ab}{}^c = 0,$$

we can rewrite the Cartan structural equations (1) in the following form:

$$\begin{gathered}
d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta\gamma}\omega_\beta \wedge \omega_\gamma + i\sigma^\alpha_\beta \omega^\beta \wedge \omega + \\
+ \left(-\sqrt{2}\widetilde{B}^{n\alpha\beta} - \frac{1}{\sqrt{2}}B^{\alpha\beta n} + i\sigma^{\alpha\beta}\right)\omega_\beta \wedge \omega, \\
d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta\gamma}\omega^\beta \wedge \omega^\gamma - i\sigma^\beta_\alpha \omega_\beta \wedge \omega + \\
+ \left(-\sqrt{2}\widetilde{B}_{n\alpha\beta} - \frac{1}{\sqrt{2}}B_{\alpha\beta n} - i\sigma_{\alpha\beta}\right)\omega^\beta \wedge \omega, \\
d\omega = \sqrt{2}B_{n\alpha\beta}\omega^\alpha \wedge \omega^\beta + \sqrt{2}B^{n\alpha\beta}\omega_\alpha \wedge \omega_\beta -
\end{gathered} \tag{2}$$

$$-2i\sigma^{\alpha}_{\beta}\omega^{\beta}\wedge\omega_{\alpha}+\left(\widetilde{B}_{n\beta n}+i\sigma_{n\beta}\right)\omega\wedge\omega^{\beta}+\left(\widetilde{B}^{n\beta n}-i\sigma^{\beta}_{n}\right)\omega\wedge\omega_{\beta}.$$

Compareing (2) with the Cartan structural equations of a nearly-cosymplectic structure [16]:

$$\begin{aligned} d\omega^{\alpha}&=\omega^{\alpha}_{\beta}\wedge\omega^{\beta}+D^{\alpha\beta\gamma}\omega_{\beta}\wedge\omega_{\gamma}+D^{\alpha\beta}\omega_{\beta}\wedge\omega,\\ d\omega_{\alpha}&=-\omega^{\beta}_{\alpha}\wedge\omega_{\beta}+D_{\alpha\beta\gamma}\omega^{\beta}\wedge\omega^{\gamma}+D_{\alpha\beta}\omega^{\beta}\wedge\omega,\\ d\omega&=-\frac{2}{3}D_{\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}-\frac{2}{3}D^{\alpha\beta}\omega_{\alpha}\wedge\omega_{\beta},\end{aligned}\tag{3}$$

where

$$\begin{aligned} D^{\alpha\beta\gamma}&=\frac{i}{2}\Phi^{\alpha}_{[\widehat{\beta},\widehat{\gamma}]},\quad D_{\alpha\beta\gamma}=-\frac{i}{2}\Phi^{\widehat{\alpha}}_{[\beta,\gamma]},\\ D^{\alpha\beta}&=\frac{3}{2}i\Phi^{\alpha}_{\widehat{\beta},n},\quad D_{\alpha\beta}=-\frac{3}{2}i\Phi^{\widehat{\alpha}}_{\beta,n},\end{aligned}$$

we get the conditions, whose simultaneous fulfilment is a criterion for the structure on N to be nearly-cosymplectic:

$$1)\ B^{\alpha\beta\gamma}=D^{\alpha\beta\gamma},\quad 2)\ -\frac{3}{\sqrt{2}}\widetilde{B}^{n\alpha\beta}+i\sigma^{\alpha\beta}=-D^{\alpha\beta},\quad 3)\ \sqrt{2}B^{n\alpha\beta}=-\frac{2}{3}D^{\alpha\beta},$$

$$4)\ \sigma^{\alpha}_{\beta}=0,\quad 5)\ \sigma^{\beta}_{n}=0\tag{4}$$

and the formulae, obtained by complex conjugation (no need to write them down explicitly).

From $(4)_3$ we have

$$D^{\alpha\beta}=-\frac{3}{\sqrt{2}}B^{n\alpha\beta}.$$

We substitute this value in $(4)_2$:

$$-\frac{3}{\sqrt{2}}\widetilde{B}^{n\alpha\beta}+i\sigma^{\alpha\beta}=\frac{3}{\sqrt{2}}B^{n\alpha\beta}.$$

Since

$$B^{n\alpha\beta}=-\widetilde{B}^{n[\alpha\beta]}=-\frac{1}{2}\left(\widetilde{B}^{n\alpha\beta}-\widetilde{B}^{n\beta\alpha}\right)=-\widetilde{B}^{n\alpha\beta},$$

we obtain $\sigma_{\alpha\beta}=0$. That is why we can rewrite the conditions (4) as follows:

$$1)\ B^{\alpha\beta\gamma}=D^{\alpha\beta\gamma},\ \ 2)\ B^{n\alpha\beta}=-\frac{\sqrt{2}}{3}D^{\alpha\beta},\ \ 3)\ \sigma^{\alpha\beta}=0,$$

$$4)\ \sigma^{\alpha}_{\beta}=0,\ \ 5)\ \sigma^{\beta}_{n}=0.\tag{5}$$

We have that the conditions

$$\sigma^{\alpha\beta}=\sigma^{\alpha}_{\beta}=\sigma^{\beta}_{n}=0$$

are necessary for the structure, induced on an oriented hypersurface in a nearly-Kählerian manifold M^{2n}, to be nearly-cosymplectic. So, the matrix of the second fundamental form of the immersion of a nearly-cosymplectic hypersurface into a nearly-Kählerian manifold looks as follows:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c} & 0 & \\ 0 & \vdots & 0 \\ & 0 & \\ \hline 0\dots0 & \sigma_{nn} & 0\dots0 \\ \hline & 0 & \\ 0 & \vdots & 0 \\ & 0 & \end{array}\right), \quad p,s = 1,\dots,2n-1. \tag{6}$$

As it is evident, *rank* $\sigma \leq 1$, i.e. the type number of the hypersurface is at most one, Q.E.D.

$\square$

Considering the matrix of the second fundamental form of the immersion of N into M^{2n}, we come to another result.

THEOREM B. *A nearly cosymplectic hypersurface N in a nearly-Kählerian manifold M^{2n} is minimal if and only if*

$$\sigma(\xi, \xi) = 0.$$

Proof.

Let us use a criterion of minimality for an arbitrary hypersurface [13]

$$g^{ps}\sigma_{ps} = 0, \quad p,s = 1,\dots,2n-1.$$

Knowing how the matrix of the contravariant metric tensor looks [16]

$$(g^{ps}) = \left(\begin{array}{c|c|c} 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_\alpha \\ \hline 0\dots 0 & 1 & 0\dots 0 \\ \hline I_\alpha & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & 0 \end{array}\right),$$

we have

$$g^{ps}\sigma_{ps} = g^{\alpha\beta}\sigma_{\alpha\beta} + g^{\widehat{\alpha}\widehat{\beta}}\sigma_{\widehat{\alpha}\widehat{\beta}} + g^{\widehat{\alpha}\beta}\sigma_{\widehat{\alpha}\beta} + g^{\alpha\widehat{\beta}}\sigma_{\alpha\widehat{\beta}} + g^{\alpha n}\sigma_{\alpha n} + g^{\widehat{\alpha}n}\sigma_{\widehat{\alpha}n} + g^{nn}\sigma_{nn} = \sigma_{nn}.$$

Therefore $g^{ps}\sigma_{ps} = 0 \Leftrightarrow \sigma_{nn} = 0$. The equality $\sigma_{nn} = 0$ means that $\sigma(\xi, \xi) = 0$, Q.E.D.

$\square$

THEOREM C. *If N is a nearly-cosymplectic hypersurface in a nearly-Kählerian manifold M^{2n} and t is its type number, then the following statements are equivalent:*

1) N is a minimal hypersurface in M^{2n},

2) N is a totally geodesic hypersurface in M^{2n},

3) $t \equiv 0$.

Proof.

If a nearly-cosymplectic hypersurface is minimal, then in view of THEOREM B $\sigma_{nn} = \sigma(\xi, \xi) = 0$, and consequently the matrix (6) vanishes. This indicates that the hypersurface is totally geodesic. It is evident that type number vanishes at its every point.

Conversely, if $t \equiv 0$, then the matrix of the second fundamental form vanishes, i.e. the hypersurface is totally geodesic. As $\sigma_{nn} = 0$, N is a minimal hypersurface in M^{2n}.

$\square$

4. Some additional results

Taking into account that the class of nearly-Kählerian manifolds contains all Kählerian manifolds [9] as well as the class of nearly-cosymplectic manifolds contains all cosymplectic manifolds [16], by force of THEOREM A we come to the following results:

Corollary 1A. The type number of a nearly-cosymplectic hypersurface in a Kählerian manifold is at most one.

Corollary 2A. The type number of a cosymplectic hypersurface in a nearly-Kählerian manifold is at most one.

Corollary 3A. The type number of a cosymplectic hypersurface in a Kählerian manifold is at most one.

Similarly, by force of THEOREM B and THEOREM C we have:

Corollary 1B (2B, 3B). A nearly-cosymplectic (cosymplectic, cosymplectic) hypersurface in a Kählerian (nearly-Kählerian, Kählerian) manifold is minimal if and only if $\sigma(\xi, \xi) = 0$.

Corollary 1C (2C, 3C). If N is a nearly-cosymplectic (cosymplectic, cosymplectic) hypersurface in a Kählerian (nearly-Kählerian, Kählerian) manifold M^{2n} and t is its type number, then the following statements are equivalent:

1) N is a minimal hypersurface in M^{2n},

2) N is a totally geodesic hypersurface in M^{2n},

3) $t \equiv 0$.

References

[1] Arnol'd V.I. Mathematical methods of classical mechanics, Springer-Verlag, Berlin–Heidelberg–New-York, 1989.

[2] Arsen'eva O.E., Kirichenko V.F. Self-dual geometry of generalized Hermitian surfaces, Mat. Sbornik, 189(1998), N1, 21–44.

[3] Banaru M. On six-dimensional Hermitian submanifolds of Cayley algebra satisfying the G-cosymplectic hypersurfaces axiom, Annuaire de l'Université de Sofia "St. Kl. OHRIDSKI", 94(2000), 91–96.

[4] Banaru M. Two theorems on cosymplectic hypersurfaces of six-dimensional Hermitian submanifolds of Cayley algebra, Journal Harbin Inst. Techn., 8(2001), N1, 38–40.

[5] Banaru M. Kirichenko tensors, Problems of Complex Analysis, Smolensk, 2(2000), 42–48.

[6] Banaru M. A new characterization of the Cray-Hervella classes of almost Hermitian manifolds, 8th International Conference on Differential Geometry and Its Applications, Opava. Czech Rèpublic, 2001, Abstracts, p. 4.
[7] Ejiri N. Totally real submanifolds in a 6-sphere, Proc. Amer. Math. Soc., 83(1981), 759–763.
[8] Gray A. The structure of nearly Kähler manifolds, Math. Ann., 223(1976), 223–248.
[9] Gray A., Hervella L.M. The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Math. Pura Appl., 123(1980), N4, 35–58.
[10] Hurt N.E. Geometric quantization in action, Reidel Publishing Company, Dordrecht–Boston-London, 1983.
[11] Kirichenko V.F. On nearly-Kählerian structures induced by means of 3-vector cross products on six-dimensional submanifolds of Cayley algebra, Vestnik MGU, 3(1973), 70–75.
[12] Kurihara H., Takagi R. A note on the type number of real hypersurfaces in $P_n(C)$, Tsukuba J. Math., 22(1998), 793–802.
[13] Norden A.P. Theory of surfaces, GITTL, Moscow, 1956.
[14] Sato T. An example of an almost Kähler manifold with pointwise constant holomorphic sectional curvature, Tokyo J. Math., 23(2000), N2, 387–401.
[15] Sekigawa K. Almost complex submanifolds of a 6-dimensional sphere, Kodai Math. J., 6(1983), 174–185.
[16] Stepanova L.V. Contact geometry of hypersurfaces of quasi-Kählerian manifolds, MSPU, Moscow, 1995.

Smolensk University of Humanities, Gertsen str., 2, Smolensk-214014, Russia
E-mail address: banaru@keytown.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 3, September 2002

TWO INTEGRAL OPERATORS

DANIEL BREAZ, NICOLETA BREAZ

Abstract. The aim of this work is to prove the univalence criteria for some integral operators.

1. **Introduction**

In this paper an equivalence criterion obtained by V. Pescar on integral operators, see [5], is extended to the case of more $S-$class functions.

Theorem A [2]. If the function $f(z)$ belongs to the class S then, for any complex number $\gamma, |\gamma| \leq \frac{1}{4}$ the function

$$F_{\gamma}(z) = \int_{0}^{z} \left(\frac{f(t)}{t} \right)^{\gamma} dt$$

is in S.

Theorem B [3]. If the function f is regular in unit disc U, $f(z) = z + a_2 z^2 + ...$ and

$$\left(1 - |z|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1$$

for all $z \in U$, then the function f is univalent in U.

Theorem C [1]. Let α be a complex number, $\mathrm{Re}\alpha > 0$, and $f(z) = z + a_2 z^2 + ...$ be a regular function in U. If

$$\frac{1 - |z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1$$

for all $z \in U$, then for any complex number $\beta, \mathrm{Re}\,\beta \geq \mathrm{Re}\,\alpha$,the function

$$F_{\beta}(z) = \left[\int_{0}^{z} t^{\beta - 1} f'(t)\, dt \right]^{\frac{1}{\beta}}$$

is in the class S.

Received by the editors: 21.12.2001.

Theorem D [6]. If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$ the following inequalities hold

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right| \tag{1}$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}$$

the equalities hold in case $g(z) = \varepsilon\frac{z+u}{1+\overline{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark E [7]. For $z = 0$, from inequality (1) we obtain for every $\xi \in U$

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|}$$

for all $z \in U$.

Theorem F [5]. Let $\gamma \in C, f \in S, f(z) = z + a_2z^2 + \ldots$.

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1, (\forall)\, z \in U$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z|+|a_2|}{1+|a_2|\cdot|z|}\right]}$$

then

$$F_\gamma(z) = \int\limits_0^z \left(\frac{f(t)}{t}\right)^\gamma dt \in S$$

Theorem G [5]. Let $\alpha, \beta, \gamma \in C, f \in S, f(z) = z + a_2z^2 + \ldots$.

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1, (\forall)\, z \in U$$
$$Re\gamma \geq \operatorname{Re}\delta > 0$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot |z| \cdot \frac{|z|+|a_2|}{1+|a_2|\cdot|z|}\right]}$$

then

$$G_{\beta,\gamma}(z) = \left[\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\gamma} dt\right]^{\frac{1}{\beta}} \in S$$

2. Main results

Theorem 1. Let $\alpha_n \in C, f_n \in S, f_n(z) = z + a_2^n z^2 + ..., n \in N^*$.

If

$$\left|\frac{zf_n'(z) - f_n(z)}{zf_n(z)}\right| \leq 1, (\forall)\, n \in N^*, (\forall)\, z \in U \tag{2}$$

$$\frac{|\alpha_1| + |\alpha_2| + ... + |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} < 1 \tag{3}$$

$$|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\left(1-|z|^2\right)\cdot |z| \cdot \frac{|z|+|c|}{1+|c|\cdot|z|}\right]} \tag{4}$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_n a_2^n\right|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|}$$

then

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \in S$$

Proof. We have $f_n \in S, (\forall)\, n \in N^*$ and $\frac{f_n(z)}{z} \neq 0, (\forall)\, n \in N^*$.

For $z = 0$ we have $\left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n} = 1$.

Consider the function

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \frac{F''(z)}{F'(z)}$$

The function $h(z)$ has the form:

$$h(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot \frac{zf_1'(z) - f_1(z)}{zf_1(z)} + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot \frac{zf_n'(z) - f_n(z)}{zf_n(z)}$$

We have:

$$h(0) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot a_2^1 + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot a_2^n$$

By using the relations (2) and (3) we obtain

$$|h(z)| < 1$$

and

$$|h(0)| = \frac{\left|\alpha_1 \cdot a_2^1 + ... + \alpha_n \cdot a_2^n\right|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} = |c|$$

Applying Remark E for the function h we obtain

$$\frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \left| \frac{F''(z)}{F'(z)} \right| \leq \frac{|z| + |c|}{1 + |c|\,|z|}\ (\forall)\, z \in U \Leftrightarrow$$

$$\Leftrightarrow \left| \left(1 - |z|^2\right) \cdot z \cdot \frac{F''(z)}{F'(z)} \right| \leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|}, (\forall)\, z \in U \tag{5}$$

Let′s consider the function $H : [0, 1] \to R$

$$H(x) = \left(1 - x^2\right) x \frac{x + |c|}{1 + |c|\, x}; x = |z|\,.$$

$$H\left(\frac{1}{2}\right) = \frac{3}{8} \cdot \frac{1 + |c|}{2 + |c|} > 0 \Rightarrow \max_{x \in [0,1]} H(x) > 0$$

Using this result and the form (5) we have:

$$\left| \left(1 - |z|^2\right) \cdot z \cdot \frac{F''(z)}{F'(z)} \right| \leq$$

$$\leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \max_{|z|<1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|} \right], (\forall)\, z \in U \tag{6}$$

Applying the condition (4) in the form (6) we obtain:

$$\left(1 - |z|^2\right) \left| \frac{zF''(z)}{F'(z)} \right| \leq 1, (\forall)\, z \in U,$$

and from Theorem B $F \in S$.

Corollary 2. Let $\alpha, \beta \in C, f, g \in S, f(z) = z + a_2 z^2 + ..., g(z) = z + b_2 z^2 + ...,$.

If

$$\left| \frac{zf'(z) - f(z)}{zf(z)} \right| \leq 1, (\forall)\, z \in U$$

$$\left| \frac{zg'(z) - g(z)}{zg(z)} \right| \leq 1, (\forall)\, z \in U$$

$$\frac{1}{\alpha} + \frac{1}{\beta} < 1$$

$$|\alpha\beta| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|} \right]}$$

where

$$|c| = \frac{|\alpha a_2 + \beta b_2|}{|\alpha\beta|}$$

then

$$F_{\alpha\beta}(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha \cdot \left(\frac{g(t)}{t} \right)^\beta dt \in S$$

Proof. In Theorem 1, we consider $n = 2, f_1 = f, f_2 = g, \alpha_1 = \alpha, \alpha_2 = \beta$.

Remark. If in Theorem 1, we consider $n = 1, f_1 = f, \alpha_1 = \gamma$, we obtained Theorem F.

Theorem 3. Let $\alpha_n, \gamma, \delta \in C, f_n \in S, f_n(z) = z + a_2^n z^2 + ..., n \in N^*$.

If

$$\left| \frac{z f_n'(z) - f_n(z)}{z f_n(z)} \right| \leq 1, (\forall)\, n \in N^*, (\forall)\, z \in U \tag{7}$$

$$\frac{|\alpha_1| + |\alpha_2| + ... + |\alpha_n|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} < 1 \tag{8}$$

$$Re\gamma \geq \operatorname{Re}\delta > 0$$

$$|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1 - |z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|} \right]} \tag{9}$$

where

$$|c| = \frac{\left| \alpha_1 a_2^1 + ... + \alpha_n a_2^n \right|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|}$$

then

$$G(z) = \left[\gamma \int_0^z t^{\gamma - 1} \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(t)}{t} \right)^{\alpha_n} dt \right]^{\frac{1}{\gamma}} \in S$$

Proof. We consider the function

$$h(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(t)}{t} \right)^{\alpha_n}$$

$$p(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \frac{h''(z)}{h'(z)}$$

$$p(z) = \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_1 \cdot \frac{z f_1'(z) - f_1(z)}{z f_1(z)} + ... + \frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \alpha_n \cdot \frac{z f_n'(z) - f_n(z)}{z f_n(z)}$$

By using the relations (7) and (8) we obtain

$$|p(z)| < 1$$

and

$$|p(0)| = \frac{\left| \alpha_1 \cdot a_2^1 + ... + \alpha_n \cdot a_2^n \right|}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} = |c|$$

Applying Remark E for the function p we obtain

$$\frac{1}{|\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n|} \cdot \left| \frac{h''(z)}{h'(z)} \right| \leq \frac{|z| + |c|}{1 + |c|\,|z|} (\forall)\, z \in U \Leftrightarrow$$

$$\Leftrightarrow \left| \frac{1 - |z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot z \cdot \frac{h''(z)}{h'(z)} \right| \leq |\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n| \cdot \frac{1 - |z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c| \cdot |z|}, (\forall)\, z \in U \tag{10}$$

Let′s consider the function $Q:[0,1]\to R$

$$Q(x)=\frac{1-x^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}x\frac{x+|a_2|}{1+|a_2|x};x=|z|.$$

$$Q\left(\frac{1}{2}\right)>0\Rightarrow\max_{x\in[0,1]}Q(x)>0$$

Using this result and the relation (10) we have:

$$\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right|\le$$

$$\le|\alpha_1\cdot\alpha_2\cdot...\cdot\alpha_n|\cdot\max_{|z|<1}\left[\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\cdot|z|\cdot\frac{|z|+|c|}{1+|c|\cdot|z|}\right],(\forall)\,z\in U \tag{11}$$

Applying the condition (9) in the relation (11) we obtain:

$$\left(1-|z|^2\right)\left|\frac{zh''(z)}{h'(z)}\right|\le 1,(\forall)\,z\in U,$$

and from Theorem C, $G\in S$.

Remark. If we consider $\gamma=1,\operatorname{Re}\delta=1$ we obtain Theorem 1.

Corollary 4. Let $\alpha,\beta,\gamma,\delta\in C,f,g\in S,f(z)=z+a_2z^2+...,g(z)=z+b_2z^2+...,$.

If

$$\left|\frac{zf'(z)-f(z)}{zf(z)}\right|\le 1,(\forall)\,z\in U$$

$$\left|\frac{zg'(z)-g(z)}{zg(z)}\right|\le 1,(\forall)\,z\in U$$

$$Re\gamma\ge\operatorname{Re}\delta>0$$

$$\frac{1}{\alpha}+\frac{1}{\beta}<1$$

$$|\alpha\beta|\le\frac{1}{\max\limits_{|z|\le 1}\left[\frac{1-|z|^{2\operatorname{Re}\delta}}{\operatorname{Re}\delta}\cdot|z|\cdot\frac{|z|+|c|}{1+|c|\cdot|z|}\right]}$$

where

$$|c|=\frac{|\alpha a_2+\beta b_2|}{|\alpha\beta|}$$

then

$$G_{\alpha\beta,\gamma}(z)=\left[\gamma\int_0^z t^{\gamma-1}\left(\frac{f(t)}{t}\right)^{\alpha}\cdot\left(\frac{g(t)}{t}\right)^{\beta}dt\right]^{\frac{1}{\gamma}}\in S$$

Proof. In Theorem 3, we consider $n=2,f_1=f,f_2=g,\alpha_1=\alpha,\alpha_2=\beta$.

Remark. If in Theorem 3, we consider $n=1,f_1=f,\gamma=\beta$, we obtained Theorem F.

Acknowledgements: The author would like to thank to Prof. Petru T. Mocanu, from Department of Mathematics, University of Cluj, for all his encouragement and help in performing this research work.

References

[1] N.N. Pascu, An improvement of Becker′s univalence criterion, Proceedings of the Commerative Session Simion Stoilow, Braşov, (1987), 43-48.

[2] I.J. Kim. E.P. Merkes, On an integral of powewrs of a spiral-like function, Kyungpook. Math. J.,12,2,December 1972, 249-253.

[3] J.Becker, Löwnersche Differentialgleichung und quasikonform fortsetzbare schichte Functionen, J. Reine Angew. Math. 255 (1972), 23-43.

[4] N.N. Pascu, V. Pescar, On the integral operators of Kim-Merkes and Pfaltzgraff, Studia (Mathematica), Univ. Babeş-Bolyai, Cluj-Napoca, 32, 2(1990), 185-192.

[5] V. Pescar, On some integral operations which preserve the univalence,Journal of Mathematics, Vol. xxx (1997) pp.1-10, Punjab University.

[6] Z. Nehari, Conformal mapping, McGraw-Hill Book Comp., New York, 1952 (Dover. Publ. Inc., 1975).

[7] G.M. Goluzin, Geometriceskaia teoria funktii kompleksnogo peremenogo, Moscova, 1966.

1 Decembrie 1918 University of Alba Iulia, Departament of Mathematics

QUASILINEARIZATION FOR THE FORCED DÜFFING EQUATION

ADRIANA BUICĂ

Abstract. In this paper we present the quasilinearization method for the periodic problem related to the forced Düffing equation. We obtain two monotone sequences of approximate solutions, with quadratic order of convergence. We work in the presence of lower and upper solutions. The approximate problems are linear.

1. Introduction

In this paper we apply the quasilinearization method to the periodic problem for the forced Düffing equation

$$\begin{cases} x'' + kx' + f(t,x) = 0 \\ x(0) = x(T),\ x'(0) = x'(T) \end{cases}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $k \in \mathbb{R}$. Existence of a lower and an upper solution is assumed. We say that α_0 is a lower solution of the problem (1.1) if $\alpha_0 \in C^2[0,T]$ and

$$\begin{cases} \alpha_0'' + k\alpha_0' + f(t,\alpha_0) \geq 0 \\ \alpha_0(0) = \alpha_0(T),\ \alpha_0'(0) = \alpha_0'(T) \end{cases}$$

Whenever the reversed inequality holds for some function $\beta_0 \in C^2[0,T]$, we say that β_0 is an upper solution.

Received by the editors: 30.05.2002.

2000 *Mathematics Subject Classification.* 34A45, 34B15.

Key words and phrases. periodic problem, Düffing equation, quasilinearization, quadratic convergence.

We consider the following iterative schemes

$$\begin{cases} \alpha''_{n+1} + k\alpha'_{n+1} + f\left(t, \alpha_n\right) + \frac{\partial f}{\partial x}(t, \alpha_n)\left(\alpha_{n+1} - \alpha_n\right) = 0, \\ \alpha_{n+1}(0) = \alpha_{n+1}(T),\ \alpha'_{n+1}(0) = \alpha'_{n+1}(T) \end{cases} \tag{1.1}$$

$$\begin{cases} \beta''_{n+1} + k\beta'_{n+1} + f\left(t, \beta_n\right) + \frac{\partial f}{\partial x}(t, \alpha_n)\left(\beta_{n+1} - \beta_n\right) = 0, \\ \beta_{n+1}(0) = \beta_{n+1}(T),\ \beta'_{n+1}(0) = \beta'_{n+1}(T). \end{cases} \tag{1.2}$$

The sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ obtained as solutions of the linear problems (1.1) and (1.2) are monotone and converge quadratically to the solution of (1.1). In addition, we require, roughly speaking, that the nonlinear function f is decreasing and convex.

We say that a sequence $(\alpha_n)_{n\geq 0}$ converges quadratically to x^* in $C[0,T]$ (with the supremum norm), whenever there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that

$$||x^* - \alpha_{n+1}|| \leq c||x^* - \alpha_n||^2, \quad \text{for all } n \geq n_0.$$

The type of problems which is the object of our work is extensively studied in the literature. Let us remind only some references which are related to the technique used in our paper. The method of lower and upper solutions for (1.1) is presented by Wang-Cabada-Nieto in [11], together with a monotone iterative method . C. Wang [10] studied the case of reversedly lower and upper solutions.

The quasilinearization method is a tool for obtaining approximate solutions to nonlinear equations with rapide convergence. It was applied to a variety of problems (see the monograph [8] by Lakshmikantham-Vatsala and the references therein), and even some very efficient abstract schemes were given in [2, 3, 4]. Some boundary value problems were studied with the quasilinearization method in [5, 6, 8, 9]. Our approach is closely to [6] and some examples in [8], since we prefer to assume convexity for the nonlinear part and obtain the approximations as solutions of corresponding linear problems, rather than do not impose convexity but consider nonlinear approximate problems (like in [5, 9]). Anyway, our results can be easily extended to the case of nonlinearities of DC-type (i.e. $f = f_1 - f_2$, where f_1 and f_2 are convex), as in [8].

2. Preliminaries

The aim of this section is to establish some comparison and existence results for the linear problem of the form (1.1), which will be needed later on.

Lemma 2.1. *Let $g, l : [0,T] \to \mathbb{R}$ be two continuous functions with $l(t) < 0$ for every $t \in [0,T]$. Let $x \in C^2[0,T]$ be such that*

$$\begin{cases} -(x'' + kx' + l(t)x) = g(t) \\ x(0) = x(T),\ x'(0) = x'(T). \end{cases}$$

If $g(t) \geq 0$ for all $t \in [0,T]$ then $x(t) \geq 0$ for all $t \in [0,T]$.

Proof. First we prove by contradiction that $x(0) \geq 0$. Let us assume that $x(0) < 0$. We distinguish three cases: $x'(0) = 0$; $x'(0) < 0$ and $x'(0) > 0$. Every case lead to

(S) *there exists $t_1 \in (0,T)$ such that $x(t_1) < 0$ and $x'(t_1) = 0$.*

Then t_1 is a local minimum for x, which also implies that $x''(t_1) > 0$. When we replace these in the following relation

$$-[x''(t_1) + kx'(t_1) + l(t_1)x(t_1)] = g(t_1)$$

we get a contradiction.

Let us prove now the above statement (S).

Case 1. Whenever $x'(0) = 0$, if we replace in the differential equation of x, we obtain $x''(0) \leq -l(0)x(0) < 0$. Then x' is strictly decreasing in some neighborhood of 0, V. But $x'(0) = 0$. Thus $x'(t) < 0$ for all $t \in V$. Hence x is strictly decreasing in V. Relation $x(0) = x(T)$ assures that (S) is valid.

Case 2. Whenever $x'(0) < 0$ we have that $x'(t) < 0$ in some neighborhood of 0. The rest is like in Case 1.

Case 3. Whenever $x'(0) > 0$ we have that, also, $x'(T) > 0$. Then x is strictly increasing in some neighborhood of T. Relation $x(0) = x(T)$ guarantees (S).

Hence we know that $x(0) = x(T) \geq 0$. It is easy to see that the existence of some $t^* \in (0,T)$ with $x(t^*) < 0$ assures that (S) hold. But this lead to a contradiction, as we have already proved. Then $x(t) \geq 0$ for all $t \in [0,T]$. $\square$

Lemma 2.2. *Let* $l : [0, T] \to \mathbb{R}$ *be a continuous function with* $l(t) < 0$ *for all* $t \in [0, T]$*. Then the problem (2.3) has a unique solution for every* $g \in C[0, T]$*.*

Proof. We apply Theorem 3.1, page 214 from [7] and deduce that it is sufficient if we prove that the only solution of the corresponding homogeneous equation with $x(0) = x(T)$ and $x'(0) = x'(T)$ is the null solution. It is easy to see that this is valid on the base of Lemma 2.1. □

Throughout this paper let us consider

$$D = \left\{ x \in C^2[0, T] : \ x(0) = x(T), \ x'(0) = x'(T) \right\}.$$

Lemma 2.3. *Let* $l : [0, T] \to \mathbb{R}$ *be a continuous function with* $l(t) < 0$ *for all* $t \in [0, T]$*. The linear operator* $L : D \to C[0, T]$*,* $Lx = -\left(x'' + kx' + l(t)x\right)$ *is bijective and its inverse is positive and completely continuous between* $C[0, T]$ *to itself.*

Proof. The bijectivity of L is assured by Lemma 2.2. It is easy to see that L is continuous from D endowed with C^2 norm

$$||x||_{C^2} = ||x|| + ||x'|| + ||x''||,$$

to $C[0, T]$ with the supremum norm, denoted here $|| \cdot ||$. Then L^{-1} exists and is continuous between $C[0, T]$ and D. Of course, is continuous between $C[0, T]$ to itself. Complete continuity of L^{-1} is assured because, in addition, D is compactly imbedded in $C[0, T]$. The positivity of L^{-1}, i.e. $y \geq 0$ implies $L^{-1}y \geq 0$, follows by Lemma 2.1. □

3. **Main results**

Throughout this section let us denote

$$\Omega = \{(t, u) \in [0, T] \times \mathbb{R} : \ \alpha_0(t) \leq u \leq \beta_0(t)\}$$

and consider the order interval in the space $C[0, T]$,

$$[\alpha_0, \beta_0] = \{x \in C[0, T], \ \alpha_0(t) \leq x(t) \leq \beta_0(t) \text{ for all } t \in [0, T]\},$$

where $\alpha_0, \beta_0 \in C[0, T]$ with $\alpha_0(t) \leq \beta_0(t)$ for all $t \in [0, T]$. The following Lemma is a unicity result for the nonlinear problem (1.1).

Lemma 3.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous and $\alpha_0, \beta_0 \in D$, be a lower and, respectively, an upper solution of (1.1), such that*

$$\alpha_0(t) \le \beta_0(t) \text{ for all } t \in [0,T].$$

Assume that $f(t,\cdot)$ is C^1 on $\mathbb{R}$ and $\frac{\partial f}{\partial x}(t,u) < 0$ for all $(t,u) \in \Omega$. Then (1.1) has at most one solution in $[\alpha_0, \beta_0]$.

Proof. Whenever x and y are two solutions of (1.1) in $[\alpha_0, \beta_0]$, we have that $z = x - y$ satisfies the following relations

$$-(z'' + kz') = f(t, x(t)) - f(t, y(t)) = l(t)z,$$

where

$$l(t) = \begin{cases} \frac{f(t,x(t)) - f(t,y(t))}{x(t) - y(t)}, & x(t) \ne y(t) \\ \frac{\partial f}{\partial x}(t, x(t)), & x(t) = y(t). \end{cases}$$

It easy to see that $l(t) < 0$ for all $t \in [0,T]$ and that $z \in D$. We apply Lemma 2.2 and obtain that $z = 0$, i.e. $x = y$. $\square$

The next theorem is our main result.

Theorem 3.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous and $\alpha_0, \beta_0 \in D$, be a lower and, respectively, an upper solution of (1.1), such that*

$$\alpha_0(t) \le \beta_0(t) \text{ for all } t \in [0,T].$$

Assume that $f(t,\cdot)$ is C^2 on $\mathbb{R}$ and convex on $[\alpha_0(t), \beta_0(t)]$ for all $t \in [0,T]$, and that $\frac{\partial f}{\partial x}(t,u) < 0$ for all $(t,u) \in \Omega$. Then the sequences (α_n) and (β_n) given by the iterative schemes (1.1) and (1.2) are well and uniquely defined in D, and converge monotonically and quadratically in $C[0,T]$ to the unique solution of (1.1) in $[\alpha_0, \beta_0]$.

Proof. The fact that α_n and β_n are well and uniquely defined in D is assured by Lemma 2.2.

The differentiability and convexity of $f(t,\cdot)$ on $[\alpha_0(t), \beta_0(t)]$ imply the following relations

$$\frac{\partial f}{\partial x}(t,u)(v-u) \le f(t,v) - f(t,u) \le \frac{\partial f}{\partial x}(t,v)(v-u), \tag{3.3}$$

for all $\alpha_0(t) \leq u \leq v \leq \beta_0(t)$.

We shall prove by induction that the following proposition is valid for all $n \geq 0$.

$$(P_n)\begin{cases} \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n \\ \alpha_{n+1} \text{ is a lower solution of (1.1)} \\ \beta_{n+1} \text{ is an upper solution of (1.1)} \end{cases}$$

Let us verify first for $n = 0$. In order to avoid some complicated formulas, let us denote $L_0 x = -\left(x'' + kx' + \frac{\partial f}{\partial x}(t, \alpha_0)x\right)$. Using this notation, we can write (1.1) for $n = 0$ in the form

$$L_0\alpha_1 = f(t, \alpha_0) - \frac{\partial f}{\partial x}(t, \alpha_0)\alpha_0.$$

Then, using also the fact that α_0 is a lower solution, we obtain

$$L_0(\alpha_1 - \alpha_0) = L_0\alpha_1 + \alpha_0'' + k\alpha_0' + \frac{\partial f}{\partial x}(t, \alpha_0)\alpha_0 = \alpha_0'' + k\alpha_0' + f(t, \alpha_0) \geq 0.$$

By Lemma 2.1, it follows that

$$\alpha_0 \leq \alpha_1.$$

Analogously one can prove that $\beta_1 \leq \beta_0$.

Using one of the inequalities (3.3) we have

$$L_0(\beta_1 - \alpha_1) = f(t, \beta_0) - \frac{\partial f}{\partial x}(t, \alpha_0)\beta_0 - f(t, \alpha_0) + \frac{\partial f}{\partial x}(t, \alpha_0)\alpha_0 \geq 0.$$

Thus, by Lemma 2.1,

$$\alpha_1 \leq \beta_1.$$

Let us prove now that α_1 is a lower solution of (1.1). We have

$$\alpha_1'' + k\alpha_1' + f(t, \alpha_1) = f(t, \alpha_1) - f(t, \alpha_0) - \frac{\partial f}{\partial x}(t, \alpha_0)(\alpha_1 - \alpha_0) \geq 0,$$

where we have used (1.1) and (3.3) for $\alpha_0 \leq \alpha_1$.

Analogously, β_1 is an upper solution for (1.1).

The proof of the fact that, if (P_n) is valid then (P_{n+1}) is true, can be done in the same manner as above. In order to avoid the repetion, let us skip it.

At this moment we have that for every $n \geq 0$, $\alpha_{n+1} \in D$ is a solution of the linear differential equation (1.1) and that

$$\alpha_0(t) \leq \alpha_1(t) \leq ... \leq \alpha_n(t) \leq ... \leq \beta_0(t) \ \text{ for all } t \in [0, T].$$

We shall prove that the sequence (α_n) converges uniformly on $[0,T]$ and its limit is a solution of (1.1).

For each $t \in [0,T]$, let us denote by $x^*(t)$ the limit of the numerical sequence $(\alpha_n(t))$ and $\sigma_n(t) = L\alpha_{n+1}(t)$, where L is the linear operator between D and $C[0,T]$ given by $Lx = -(x'' + kx' - x)$. Using (1.1) we get that

$$\sigma_n(t) = f(t, \alpha_n) + \alpha_{n+1}(t) + \frac{\partial f}{\partial x}(t, \alpha_n(t)). \tag{3.4}$$

Because the functions f and $\frac{\partial f}{\partial x}$ are continuous and the sequence (α_n) is bounded in $C[0,T]$, we have that (σ_n) is bounded in $C[0,T]$. Also, we can write

$$\alpha_{n+1} = L^{-1}\sigma_n. \tag{3.5}$$

By Lemma 2.3, L^{-1} is completely continuous. Hence the sequence (α_n) is compact in $C[0,T]$. It is also monotone. Then it is uniformly convergent to x^*. When we pass to the limit for $n \to \infty$ in (3.5) and (3.4) we get that $x^* = L^{-1}\left[f(t,x^*) + x^*\right]$. Thus $x^* \in D$ and $Lx^* = f(t,x^*) + x^*$, which is equivalent to the fact that x^* is a solution of the problem (1.1).

Analogously, the sequence (β_n) converges uniformly on $[0,T]$, and its limit is a solution of (1.1). By Lemma 3.1, the solution is unique in $[\alpha_0, \beta_0]$.

In order to justify that the order of convergence of the sequence (α_n) to x^* is 2, we denote

$$p_n = x^* - \alpha_n$$

and consider the linear operator $L_* x = -\left[x'' + kx' + \frac{\partial f}{\partial x}(t,x^*)x\right]$. Let us remember that, by convexity of f, $\frac{\partial f}{\partial x}(t,x^*) \geq \frac{\partial f}{\partial x}(t,\alpha_n)$, since $x^* \geq \alpha_n$. The following inequalities hold.

$$\begin{aligned}
L_* p_{n+1} &\leq -\left[p''_{n+1} + kp'_{n+1} + \frac{\partial f}{\partial x}(t,\alpha_n)p_{n+1}\right] \\
&= -(x'' + kx') - \frac{\partial f}{\partial x}(t,\alpha_n)x^* + \left[\alpha''_{n+1} + k\alpha'_{n+1} + \frac{\partial f}{\partial x}(t,\alpha_n)\alpha_{n+1}\right] \\
&= f(t,x^*) - \frac{\partial f}{\partial x}(t,\alpha_n)p_n - f(t,\alpha_n) \\
&\leq \left[\frac{\partial f}{\partial x}(t,x^*) - \frac{\partial f}{\partial x}(t,\alpha_n)\right] p_n \\
&\leq a \cdot p_n^2.
\end{aligned}$$

We have used relation (3.3) for $\alpha_n \le x^*$. The last inequality is true because the function $\frac{\partial f}{\partial x}(t,\cdot)$ is monotone increasing and Lipschitz on the compact interval $[\alpha_0(t), \beta_0(t)]$ for each $t \in [0,T]$. Using the positivity of L_*^{-1}, assured by Lemma 2.3, we obtain

$$0 \le p_{n+1} \le aL_*^{-1}\left(p_n^2\right),$$

and than, continuity of L_*^{-1} gives that there exists $c > 0$ with

$$||p_{n+1}|| \le c||p_n||^2.$$

In the same manner one can prove the quadratic convergence of (β_n). □

References

[1] B. Ahmad, J.J. Nieto, N. Shahzad, The Bellman-Kalaba-Lakshmikantham quasilinearization method for Neumann problems, J. Math. Anal. Appl. 257(2001) 356-363.
[2] A. Buică, Monotone iterations for the initial value problem, Seminar on Fixed Point Theory Cluj-Napoca 3(2002), 137-149.
[3] A. Buică, R. Precup, Monotone Newton-type iterations for nonlinear equations, Proc. Tiberiu Popoviciu Itinerant Seminar on Functional Equations, Approximation and Convexity, E. Popoviciu ed., Srima, Cluj, 2002, to appear.
[4] A. Buică, R. Precup, Abstract generalized quasilinearization method for coincidences, submitted.
[5] A. Cabada, J.J. Nieto, Quasilinearization and rate of convergence for higher-order nonlinear periodic boundary-value problem, J. Optim. Theory Appl. 108(2001) 97-107.
[6] P.W. Eloe, Y. Gao, The method of quasilinearization and a three-point boundary value problem, J. Korean Math. Soc. 39(2002), 319-330.
[7] A. Halanay, Teoria calitativă a ecuaţiilor diferenţiale (Qualitative Theory of Differential Equations, in romanian), Editura Academiei R.P. Romıane, 1963.
[8] V. Lakshmikantham, A.S. Vatsala, Generalized Quasilinearization for Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1998.
[9] J.J. Nieto, Generalized quasilinearization for a second order ordinary differential equation with Dirichlet boundary conditions, Proc. Amer. Math. Soc. 125(1997) 2599-2604.
[10] C. Wang, Generalized upper and lower solution method for the forced Duffing equation, Proc. Amer. Math. Soc. 125(1997) 397-406.
[11] M.-X. Wang, A. Cabada, J.J. Nieto, Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions, Ann. Pol. Math. 58.3(1993) 221-235.

Department of Applied Mathematics, Babeş-Bolyai University,
1 Kogălniceanu str., 3400 Cluj-Napoca, Romania
E-mail address: abuica@math.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 3, September 2002

TWO-VARIABLE VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

ENDRE BUZOGÁNY, ILDIKÓ ILONA MEZEI, VIORICA VARGA

Abstract. In this paper we guarantee the solution for two-variable variational-hemivariational inequalities and we give some applications.

1. Introduction

The aim of this paper is to establish a two-variable result concerning the hemivariational inequalities. These inequalities appear as a generalisation of variational inequalitis, but they are more general than these ones, having applications in several branches of mathematics, mechanics, economy engineering.

The paper is organized as follows. In the Section 2 we formulate the problem and give some notions and results which will be used later. In Section 3 we establish the main results of this paper, i.e. we guarantee solution for hemivariational inequality. Finally in Section 4 we give some applications. More precisely, we obtain a Brouwer's type variational inequality, the Schauder fixed point theorem (and Brouwer fixed point theorem), a hemivariational inequality of Panagiotopoulos-Fundo-Rădulescu type, and a result concerning the Nash equilibrium theory.

2. Preliminaries

Let X be a Banach space, X^* its dual. We consider the following hypotheses:

(H_T) $T : X \to L^p(\Omega, \mathbb{R}^k)$ is a linear, continuous operator, where $p \in [1, \infty)$, $k \geq 1$ and Ω is a bounded open set in $\mathbb{R}^N$.

(H_j) $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz with respect to the second variable and there exist $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R})$ and $h_2 \in L^{\infty}(\Omega, \mathbb{R})$

Received by the editors: 10.04.2002

This paper was supported by Sapientia Foundation.

such that

$$|w| \leq h_1(x) + h_2(x)|y|^{p-1}$$

for a.e. $x \in \Omega$, every $y \in \mathbb{R}^k$ and $w \in \partial j(x,y)$, where $\partial j(x,y)$ is the Clarke generalized gradient of j, see [4], i.e. $\partial j(x,y) = \{w \in \mathbb{R}^k : \langle w, z\rangle \leq j_y^0(x,y;z),\ \text{for all } z \in \mathbb{R}^k\}$ where $j_y^0(x,y;z)$ is the partial Clarke derivative of the locally Lipschitz mapping $j(x,\cdot)$ at the point $y \in \mathbb{R}^k$ with respect to the direction $z \in \mathbb{R}^k$, where $x \in \Omega$, that is

$$j_y^0(x,y,z) = \operatorname{limsup}_{\substack{y' \to y \\ t \to 0^+}} \frac{j(x, y'+tz) - j(x,y')}{t}.$$

Let K be a subset of X, $\mathcal{A} : K \times K \rightsquigarrow X^*$, $G : K \times X \rightsquigarrow \mathbb{R}$ two set-valued mappings with nonempty values. Under hypotheses (H_T) and (H_j) the main problem of this paper is the following

(P) Find $u \in K$ such that, for every $v \in K$

$$\sigma(\mathcal{A}(u,u), v-u) + G(u, v-u) + \int_\Omega j_y^0(x, Tu(x), Tv(x) - Tu(x))dx \subseteq \mathbb{R}_+.$$

Here $\sigma(\mathcal{A}(w,u), v-u) = \sup\limits_{x^* \in \mathcal{A}(w,u)} \langle x^*, v-u\rangle$. The (P) is equivalent with

(P') Find $u \in K$ such that, for every $v \in K$

$$\sigma(\mathcal{A}(u,u), v-u) + \inf G(u, v-u) + \int_\Omega j_y^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0.$$

The euclidean norm in $\mathbb{R}^k$ and the duality pairing between the Banach space and its dual will be denoted by $|\cdot|$, resp. $\langle \cdot, \cdot \rangle$.

In order to state existence results for (P), we need some notions and preliminary results.

Definition 2.1. *Let K be convex.*

(i) *A set-valued mapping $\mathcal{F} : K \rightsquigarrow X^*$ is said to be* upper demicontinuous at $x_0 \in K$ *(udc at $x_0 \in K$) if for any $h \in X$, the real-valued function $x \mapsto \sigma(\mathcal{F}(x), h) = \sup\limits_{x^* \in \mathcal{F}(x)} \langle x^*, h\rangle$ is upper semicontinuous at x_0. $\mathcal{F}$ is upper demicontinuous on K (udc on K) if it is udc in every $x \in K$.*

(ii) *$\mathcal{F} : K \rightsquigarrow X^*$ is said to be* upper demicontinuous from the line segments in K *if the application $t \mapsto \sigma(\mathcal{F}(tx + (1-t)y), h)$ is upper semicontinuous on the interval* $[0,1]$, $\forall x, y \in K, h \in X$.

(iii) $F : K \to X^*$ *is said to be* w^*-demicontinuous in u_0 *if for any sequence* $\{u_n\} \subset K$ *converging to* u_0 *(in the strong topology), the image sequence* $\{F(u_n)\}$ *converges to* $F(u_0)$ *in the weak***-topology in* X^**.*

Remark 2.1. (i) If $\mathcal{F}(x) = \{F(x)\}$, that is, if $\mathcal{F}$ is a single valued map, then $\mathcal{F}$ is udc at $u_0 \in K$ if and only if the operator $F : K \to X^*$ is w^*-demicontinuous at $u_0 \in K$.

(ii) If $\mathcal{F}(x) = \{F(x)\}$ is hemicontinuous, (see for example [8]), then $\mathcal{F}$ is udc from the line segments in K.

The $h \mapsto \sigma(\mathcal{F}(x), h)$ is a lower semicontinuous sublinear function.

Lemma 2.1. [11, Lemma 2.2] *Let* $\mathcal{F} : K \rightsquigarrow X^*$ *be an udc set-valued map with bounded values, i.e.* $\sup\limits_{x^* \in \mathcal{F}(x)} \|x^*\| < \infty,\ \forall x \in K$. *Then the function* $u \mapsto \sigma(\mathcal{F}(u), v-u)$ *is upper semicontinuous,* $\forall\ v \in K$*.*

Now, we recall some notions from [1]. Let Y, Z be two metric spaces and a set-valued map (with nonempty values) $F : Y \rightsquigarrow Z$.

Definition 2.2. *F is called* lower semicontinuous at $y \in Y$ *(lsc at y) if and only if for any $z \in F(y)$ and for any sequence $\{y_n\}$, converging to y, there exists a sequence $\{z_n\}$, $z_n \in F(y_n)$ converging to z.*

It is said to be lower semicontinuous (lsc) if it is lsc at every point $y \in Y$.

Let us consider a function $f : Graph(F) \to \mathbb{R}$. We define the *marginal function* $g : Y \to \mathbf{R} \cup \{+\infty\}$ by $g(y) = \sup_{z \in F(y)} f(y, z)$. We have the Maximum Theorem, see [1, Theorem 1.4.16, p.48].

Lemma 2.2. *If f and F are lower semicontinuous on Y, then the marginal function is also lower semicontinuous.*

Definition 2.3. *Let K be a convex subset of X and let Z be a topological vector space. The set-valued map $F : K \rightsquigarrow Z$ (with nonempty values) is* convex *if and only if* $\forall\ x_1, x_2 \in K,\ \forall\ \lambda \in [0,1] :\ \lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2)$.

Remark 2.2. $F : K \rightsquigarrow Z$ is convex if and only if $\forall\ x_i \in K,\ \forall\ \lambda_i \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1, n \in \mathbb{N}^*,$ we have $\sum_{i=1}^n \lambda_i F(x_i) \subseteq F\left(\sum_{i=1}^n \lambda_i x_i\right).$

Definition 2.4. *The mapping* $F : K \subseteq X \rightsquigarrow X^*$ *is* monotone *if* $\langle f_1 - f_2, u - v\rangle \geq 0,\ \forall u, v \in K,\ \forall f_1 \in F(u), f_2 \in F(v)$.

Lemma 2.3.([12, Lemma 1.]) *If T and j satisfy the (H_T) and (H_j) respectively and V_1, V_2 are non-empty subsets of X, then the mapping defined by*

$$(u,v) \mapsto \int_\Omega j_y^0(x, Tu(x), Tv(x))dx,\ (u,v) \in V_1 \times V_2$$

is upper semicontinuous.

Lemma 2.4. [7] *Let X be a Hausdorff topological vector space, K a subset of X and for each $x \in K$, let $S(x)$ be a closed subset of X, such that*

(i) *there exists $x_0 \in K$ such that the set $S(x_0)$ is compact*;

(ii) *S is KKM-mapping, i.e. for each $x_1, x_2, \ldots, x_n \in K$, $co\{x_1, x_2, \ldots, x_n\} \subseteq \cup_{i=1}^n S(x_i)$, where co stands for the convex hull operator.*

Then $\bigcap\limits_{x\in K} S(x) \neq \emptyset$.

3. Main results on Existence of Solutions for (P)

We need some additional hypotheses to obtain solution for (P).

(H_G) (1) $G(u,0) \subseteq \mathbb{R}_+,\ \forall\ u \in K$;

(2) $G(u,\cdot)$ is convex, $\forall u \in K$;

(3) $G(\cdot,\cdot)$ is lsc on $K \times X$;

(4) $G(u,\cdot)$ is subhomogenous, i.e. $tG(u,y) \subseteq G(u,ty)$, $\forall t \in [0,1]$, $u \in K$, $y \in X$.

$(H_{\mathcal{A}})$ (1) $\mathcal{A}$ has bounded values, i.e. $\sup\limits_{x^* \in \mathcal{A}(u,v)} \|x^*\| < \infty$, $\forall u,v \in K$;

(2) $\mathcal{A}(v,\cdot) : K \rightsquigarrow X^*$ is udc on K, $\forall v \in K$;

(3) $\mathcal{A}(\cdot,u) : K \rightsquigarrow X^*$ is udc from the line segments in K, $\forall u \in K$.

(4) $\mathcal{A}(\cdot,u)$ has the monotonicity property

$$\sigma(\mathcal{A}(v,u), v-u) \geq \sigma(\mathcal{A}(u,u), v-u), \forall u,v \in K.$$

The main result of this paper is the following

Theorem 3.1. *Let K be a convex, closed subset of a Banach space X and $\mathcal{A} : K \times K \rightsquigarrow X^*$, $G : K \times X \rightsquigarrow \mathbb{R}$, $T : X \to L^p(\Omega, \mathbb{R}^k)$ and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ satisfying $(H_{\mathcal{A}})$, (H_G), (H_T) and (H_j) respectively. In addition, if*

(H_{coer}) *there exists a compact subset K_0 of K and $u_0 \in K$ such that*

$$\{\sigma(\mathcal{A}(u,u), u_0 - u) + G(u, u_0 - u) + \int_\Omega j_y^0(x, Tu(x), Tu_0(x) - Tu(x))dx\} \cap \mathbb{R}_-^* \neq \emptyset,$$

for all $u \in K \setminus K_0$. Then (P) has at least a solution.

Proof. For $w \in K$, let

$$T_1(w) = \{u \in K : \sigma(\mathcal{A}(u,u), w-u) + \inf\, G(u, w-u) + \\ + \int_\Omega j_y^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0\};$$

$$T_2(w) = \{u \in K_0 : \sigma(\mathcal{A}(w,u), w-u) + \inf\, G(u, w-u) + \\ + \int_\Omega j_y^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0\}.$$

Step 1. $T_1(u_0) \subseteq K_0$, where u_0 is from (H_{coer}). Suppose that there exists $u \in T_1(u_0) \subset K$ such that $u \notin K_0$. from the definition of $T_1(u_0)$, we have that

$$\sigma(\mathcal{A}(u,u), u_0 - u) + \inf G(u, u_0 - u) + \int_\Omega j^0(x, Tu(x), Tu_0(x) - Tu(x))dx \geq 0.$$

But this contradicts the (H_{coer}). Therefore $T_1(u_0) \subseteq K_0$.

Step 2. We prove that $T_1 : K \rightsquigarrow K$ is KKM-mapping, i.e.

$$\forall w_1, ..., w_n \in K : co\{w_1, ..., w_n\} \subseteq \bigcup_{i=1}^n T_1(w_i).$$

Contrary, we suppose that there exist $\lambda_1, \ldots, \lambda_n \geq 0$, $\sum\limits_{i=1}^n \lambda_i = 1$ such that $\overline{w} = \sum\limits_{i=1}^n \lambda_i w_i \notin T_1(w_i)$, for $i = \overline{1,n}$. Therefore

$$\sigma(\mathcal{A}(\overline{w},\overline{w}), w_i - \overline{w}) + \inf\, G(\overline{w}, w_i - \overline{w}) + \\ + \int_\Omega j^0(x, T\overline{w}(x), -Tw_i(x) - T\overline{w}(x))dx < 0,\ i = \overline{1,n}.$$

Let $\mathcal{I} = \{i = \overline{1,n} : \lambda_i \neq 0\}$. Multiplying the above inequalities by λ_i for $i \in \mathcal{I}$ and using the homogenity of T, we have

$$\sigma(\mathcal{A}(\overline{w},\overline{w}), \lambda_i w_i - \lambda_i \overline{w}) + \lambda_i \inf\, G(\overline{w}, w_i - \overline{w}) + \\ + \int_\Omega j^0(x, T\overline{w}(x), -T(\lambda_i w_i)(x) - T(\lambda_i \overline{w})(x))dx < 0, \forall i \in \mathcal{I}.$$

Adding the above relations for $i \in \mathcal{I}$ and using that $h \mapsto \sigma(\mathcal{A}(\overline{w},\overline{w}), h)$ and $h \mapsto j^0(x, T\overline{w}(x), h)$ are subadditive, $z \mapsto \inf\, G(\overline{w}, z)$ is convex for all $\overline{w} \in K$, T is additive and $(H_G)(1)$ we get

$$0 \leq \sigma(\mathcal{A}(\overline{w},\overline{w}), \sum_{i\in\mathcal{I}} \lambda_i w_i - \sum_{i\in\mathcal{I}} \lambda_i \overline{w}) + \sum_{i\in\mathcal{I}} \lambda_i \inf\, G(\overline{w}, w_i - \overline{w}) +$$

$$+\int_\Omega j^0(x, T\overline{w}(x), \sum_{i\in\mathcal{I}} T(\lambda_i w_i)(x) - \sum_{i\in\mathcal{I}} T(\lambda_i \overline{w})(x))dx < 0,$$

which is absurd. Therefore, T_1 is KKM-mapping.

Step 3. We prove that $\bigcap\limits_{w\in K} \overline{T_1(w)} \neq \emptyset$. Here, $\overline{T_1(w)}$ is the closure of $T_1(w)$. Indeed, from Step 1, we have that $T_1(u) \subseteq K_0$. Since K_0 is compact, $\overline{T_1(u_0)}$ is also compact. Using the Step 2 and applying Lemma 2.4., we obtain that $\bigcap\limits_{w\in K} \overline{T_1(w)} \neq \emptyset$.

Step 4. $\bigcap\limits_{w\in K} T_1(w) = \bigcap\limits_{w\in K} T_2(w)$.

(a) Let $u \in \bigcap\limits_{w\in K} T_1(w)$ i.e. $\sigma(\mathcal{A}(u,u), w-u) + \inf\, G(u, w-u) + \int\limits_\Omega j^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0,\ \forall w \in K$. From (H_{coer}), we have that $u \in K_0$. From the $(H_\mathcal{A})(4)$, we can write that

$$\sigma(\mathcal{A}(w,u), w-u) + \inf\, G(u, w-u) + \int\limits_\Omega j^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0,\ \forall w \in K,$$

i.e. $u \in \bigcap\limits_{w\in K} T_2(w)$.

(b) Let $u \in \bigcap\limits_{w\in K} T_2(w)$, i.e. $\sigma(\mathcal{A}(v,u), v-u) + \inf\, G(u, v-u) + \int\limits_\Omega j^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0,\ \forall v \in K$. Let $v \in K$ be an arbitrary element. Let $v_t = tv + (1-t)u,\ t \in [0,1]$. Clearly, $v_t \in K$. We have

$$\sigma(\mathcal{A}(v_t,u), v_t-u) + \inf\, G(u, v_t-u) + \int_\Omega j^0(x, Tu(x), Tv_t(x) - Tu(x))dx \geq 0,\ \forall t \in [0,1].$$

From the linearity of T, we have that

$$\sigma(\mathcal{A}(v_t,u), t(v-u)) + \inf\, G(u, t(v-u)) +$$

$$+\int_\Omega j^0(x, Tu(x), t(Tv(x) - Tu(x)))dx \geq 0,\ \forall t \in [0,1].$$

From the (H_G) (4) and from the fact that $j^0_y(x, Tu(x), \cdot)$ is positive homogeneous, we obtain

$$\sigma(\mathcal{A}(v_t,u), v-u) + \inf\, G(u, v-u) + \int_\Omega j^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0,\ \forall t \in (0,1].$$

Using $(H_\mathcal{A})$ (3), we have that $\limsup\limits_{t\to 0^+} \sigma(\mathcal{A}(v_t,u), v-u) \leq \sigma(\mathcal{A}(u,u), v-u)$. Therefore,

$$\sigma(\mathcal{A}(u,u), v-u) + \inf\, G(u, v-u) + \int_\Omega j^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0.$$

Since $v \in K$ was arbitrary, u is a solution for (P). $\square$

Step 5. $\bigcap\limits_{w\in K} \overline{T_1(w)} = \bigcap\limits_{w\in K} T_2(w)$.

Clearly, $\bigcap_{w\in K} T_2(w) \subseteq \bigcap_{w\in K} \overline{T_1(w)}$ from Step 4. Conversely, let $v \in \bigcap_{w\in K} \overline{T_1(w)}$. We prove that $v \in \bigcap_{w\in K} T_2(w)$. Since $\overline{T_1(u_0)} \subset K_0$, we have that $\bigcap_{w\in K} \overline{T_1(w)} \subseteq K_0$. Therefore, $v \in K_0 \cap \overline{T_1(w)},\ \forall w \in K$.

Now, let $u \in K$ be a fixed element. Since $v \in \overline{T_1(u)}$, there exists a sequence $\{v_n\}$ from $T_1(u)$ such that $v_n \to v$. Since $v_n \in T_1(u)$, we have

$$\sigma(\mathcal{A}(v_n, v_n), u - v_n) + \inf\ G(v_n, u - v_n) + \int_\Omega j^0(x, Tv_n(x), Tu(x) - Tv_n(x))dx \geq 0.$$

From $(H_\mathcal{A})(4)$, we have

$$\sigma(\mathcal{A}(u, v_n), u - v_n) + \inf\ G(v_n, u - v_n) + \int_\Omega j^0(x, Tv_n(x), Tu(x) - Tv_n(x))dx \geq 0.$$

From $(H_\mathcal{A})(1)$ and (2), applying Lemma 2.1 we obtain that $v \mapsto \sigma(\mathcal{A}(u, v), u - v)$ is usc, therefore

$$\limsup_{n\to\infty} \sigma(\mathcal{A}(u, v_n), u - v_n) \leq \sigma(\mathcal{A}(u, v), u - v).$$

From Lemma 2.2 (with $F = G,\ Y := K \times X,\ Z := \mathbb{R},\ f((y_1, y_2), z) = -z$, where $z \in G(y_1, y_2)$) and $(H_G)(3)$ we have that $v \mapsto \inf\ G(v, u - v)$ is usc, therefore

$$\limsup_{n\to\infty} \inf\ G(v_n, u - v_n) \leq \inf\ G(v, u - v).$$

Using the Lemma 2.3 we get the following inequality

$$\limsup_{n\to\infty} \int_\Omega j^0(x, Tv_n(x), Tu(x) - Tv_n(x))dx \leq \int_\Omega j^0(x, Tv(x), Tu(x) - Tv(x))dx.$$

Summarizing the above relations, we get

$$\sigma(\mathcal{A}(u, v), u - v) + \inf\ G(v, u - v) +$$

$$+ \int_\Omega j^0(x, Tv(x), Tu(x) - Tv(x))dx \geq 0,$$

i.e. $v \in T_2(u)$. Since u was arbitrary, we have that $v \in \bigcap_{u\in K} T_2(u)$.

Step 6. From Steps 3, 4 and 5, we have that $\bigcap_{w\in K} T_1(w) \neq \emptyset$, which means that u is a solution for (P).□

Remark 3.1 If K is compact in the above theorem, the hypothesis (H_{coer}) can be omitted.

4. Applications

As a first application, we can deduce easily the Schauder fixed point theorem from Theorem 3.1. on Banach spaces. For the completeness, we give the proof.

Corollary 4.1 *Let K be a compact, convex subset of a Banach space X and $f : K \to K$ be a continuous function. Then f has a fixed point.*

Proof. Let $\mathcal{A} \equiv 0$, $j \equiv 0$, $T \equiv 0$ and $G : K \times X \rightsquigarrow \mathbb{R}$ defined by $G(u, v) = [\|u + v - f(u)\| - \|u - f(u)\|, \infty)$.

We verify (H_G). Clearly, $G(u, 0) = [0, \infty) = \mathbb{R}_+$ and $v \rightsquigarrow G(u, v)$ is convex, $\forall\ v \in K$. Since f is continuous, the function $(u, x) \mapsto \|u + x - f(u)\| - \|u - f(u)\|$ is continuous also. Therefore, it's easy to prove that $(u, x) \rightsquigarrow G(u, x)$ is lsc on $K \times X$. The subhomogeneity of $G(u, \cdot)$ for $t = 0$ and $t = 1$ is trivial. Otherwise, this follows from the triangle inequality. Therefore, from Theorem 3.1 it follows that there exists $u_0 \in K$ such that

$$[\|v - f(u_0)\| - \|u_0 - f(u_0)\|, \infty) = G(u_0, v - u_0) \subseteq \mathbb{R}_+,\ \forall\ v \in K.$$

In particular, we have $\|v - f(u_0)\| - \|u_0 - f(u_0)\| \geq 0,\ \forall\ v \in K$. Let $v := f(u_0)$. We have $-\|u_0 - f(u_0)\| \geq 0$, i.e. $u_0 = f(u_0)$. □

Corollary 4.2 (Brouwer fixed point theorem) *Let $f : K \to K$ be a continuous function, K being a compact, convex subset of $\mathbb{R}^n$. Then f has a fixed point.*

Corollary 4.3 [12, Theorem 1.] *Let K be a compact and convex subset of a Banach space X and j and T satisfying (H_j) and (H_T) respectively. If the operator $A : K \to X^*$ is w^*-demicontinuous, then there exists $u \in K$ such that*

$$(PPFR) \qquad \langle Au, v - u\rangle + \int_\Omega j_y^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0,\ \forall v \in K.$$

Proof. Let $\mathcal{A} : K \times K \rightsquigarrow X^*$ defined by $\mathcal{A}(v, u) = \{A(u)\},\ \forall u, v \in K$ and $G \equiv 0$. Let $v \in K$ be fixed. From Remark 2.1, $\mathcal{A}(v, \cdot)$ is udc on K (with bounded values). Therefore, $(H_\mathcal{A})$ holds. Since $\sigma(\mathcal{A}(u, u), v - u) = \langle Au, v - u\rangle$, the assertion follows easily from Theorem 3.1. □

The following result is of Browder's type, see [2].

Corollary 4.4 *Let K be a convex, closed subset of a Banach space, $\mathcal{A} : K \times K \rightsquigarrow X^*$ be an operator satisfying $(H_\mathcal{A})$. Suppose that there exists a compact subset $K_0 \subset K$ and $u_0 \in K$ such that $\sigma(\mathcal{A}(u, u), u_0 - u) < 0,\ \forall u \in K \setminus K_0$. Then* *there exists* $u \in K$ *such that*

$$\sigma(\mathcal{A}(u,u), v-u) \geq 0, \ \forall v \in K.$$

Proof. We apply Theorem 3.1 for $G \equiv 0$, $j \equiv 0$ and $T \equiv 0$. □

Remark 4.1 Similar results were obtained by Y-Q. Chen in [3] and by A. M. Croicu and I. Kolumbán in [5].

Finally let X_1 and X_2 two Banach spaces, $K_1 \subseteq X_1$, $K_2 \subseteq X_2$ two nonempty closed, convex sets. Let $F_i : K_1 \times K_2 \to X_i^*$, $i = 1, 2$ two operators. Our aim is to give existence result for the following problem:

Find $(u_1, u_2) \in K_1 \times K_2$ such that

$$(NP) \qquad \langle F_1(u_1,u_2), x-u_1\rangle \geq 0, \ \forall x \in K_1$$

$$\langle F_2(u_1,u_2), y-u_2\rangle \geq 0, \ \forall y \in K_2.$$

The above problem is originated from the Nash equilibrium points, see [10] and [9].

Theorem 4.1 *Suppose that*

(i) *for every* $x_i \in K_i$, $i = 1, 2$ *the mappings* $F_1(\cdot, x_2) : K_1 \to X_1^*$ *and* $F_2(x_1, \cdot) : K_2 \to X_2^*$ *are monotones and udc on the line segments in* K_1 *respective* K_2 (*in particular hemicontinuous*);

(ii) *for every* $x_i \in K_i$, $i = 1, 2$ *the mappings* $F_1(\cdot, x_2) : K_1 \to X_1^*$ *and* $F_2(x_1, \cdot) : K_2 \to X_2^*$ *are* w^**-demicontinuous*;

(iii) *there exist* $K_i^0 \subseteq K_i$, $i = 1, 2$ *compact sets and* $x_i^0 \in K_i^0$ *such that for every* $(x_1, x_2) \in (K_1 \times K_2) \setminus (K_1^0 \times K_2^0)$

$$\langle F_1(x_1,x_2), x_1^0 - x_1\rangle + \langle F_2(x_1,x_2), x_2^0 - x_2\rangle < 0.$$

Then (NP) *has at least a solution.*

Proof. First let $j \equiv 0$, $G \equiv 0$ and $T \equiv 0$ in Theorem 3.1. Moreover, let $X := X_1 \times X_2$, $K := K_1 \times K_2$ and $\mathcal{A} : K \times K \rightsquigarrow X^*$ be a single-valued map, defined by

$$\mathcal{A}((x,y),(z,t)) = (F_1(x,t), F_2(z,y)), \ \forall (x,y), \ (z,t) \in K.$$

Clearly, $\mathcal{A}$ satisfies $(H_{\mathcal{A}})$. Let $K_0 := K_1^0 \times K_2^0$ and $u_0 := (x_1^0, x_2^0) \in K_0$. The K_0 and u_0 satisfy the (H_{coer}) condition from Theorem 3.1. Therefore, there exists $u = (u_1, u_2) \in K$ such that $\langle \mathcal{A}(u, u), w - u \rangle \geq 0,\ \forall w \in K$. This is equivalent with

$$\langle F_1(u_1, u_2), w_1 - u_1 \rangle + \langle F_2(u_1, u_2), w_2 - u_2 \rangle \geq 0,\ \forall w_i \in K_i,\ i = \overline{1, 2}.$$

Substituting $w_2 := u_2$ and $w_1 := u_1$ respectively, we obtain that $u = (u_1, u_2) \in K$ is a solution for (NP). $\square$

Remark 4.2 If K_1 and K_2 are compact sets, the hypothesis (iii) from the above theorem can be omitted.

Remark 4.3 From the above theorem we obtain also the Brouwer fixed point theorem (see Corollary 4.2) choosing $K = K_1 = K_2$, $F_1(u_1, u_2) = -f(u_1) + u_2$ and $F_2(u_1, u_2) = u_2 - u_1$.

References

[1] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston-Basel-Berlin, 1990.
[2] F.E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces,* Math. Ann.,177(1968), 283-301.
[3] Y.-Q. Chen, *On the semi-monotone operator theory and applications,* Jour. Math. Anal. Appl., **231**(1999), 177-192.
[4] F. Clarke, *Optimization and Nonsmooth Analysis,* John Wiley&Sons, New York, 1983.
[5] A. M. Croicu and J. Kolumbán, *On a generalized hemivariational inequality on Banach spaces,* in preparation.
[6] K. Fan, *A generalization of Tychonoff's fixed point theorem,* Math. Ann., **142**(1961), 305-310.
[7] K. Fan, *A minimax inequality and its application,* in O.Shisha (Ed.), Inequalities vol. **3**, Academic Press, New York, 1972, 103-113.
[8] G. Kassay, *The equilibrium problem & Related Topics*, Cluj Napoca, 2000.
[9] G. Kassay and J. Kolumbán, *System of multi-valued variational inequalities,* Publ. Math. Debrecen, **56**(2000), 185-195.
[10] G. Kassay, J. Kolumbán and Zs. Páles, *On Nash Stationary Points,* Publ. Math. Debrecen, **54**(1999), 267-279.
[11] A. Kristály and Cs. Varga, *Hemivariational inclusions and applications,* submitted for publication.
[12] P. D. Panagiotopoulos, M. Fundo and V. Rădulescu, *Existence Theorems of Hartman-Stampacchia Type for Hemivariational Inequalities and Applications,* Journal of Global Optimization, **15**(1999), 41-54.

Babeş-Bolyai University, 1 Kogălniceanu str., 3400 Cluj-Napoca, Romania
E-mail address: endreb@cs.ubbcluj.ro,
ikulcsar@math.ubbcluj.ro,
vvarga@cs.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 3, September 2002

FUNCTIONALS WHICH SATISFY A MAXIMUM PRINCIPLE

CRISTIAN CHIFU-OROS

Abstract. The purpose of this paper is to present some examples of functionals, defined on the solutions of an elliptic equation, which satisfy a maximum principle.

1. **Introduction**

Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. Let us consider the following differential operator:

$$Lu := \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_i \frac{\partial}{\partial x_i} + c \tag{1}$$

We assume that L satisfies the following maximum principles ([1]):

MP: There is a subset $\Gamma \subset \partial\Omega$ such that, if:

1. $u \in C(\overline{\Omega})$
2. the derivatives of u occurring in L are continuous in $\overline{\Omega} \backslash \Gamma$
3. $Lu \geq 0$,in $\overline{\Omega} \backslash \Gamma$

then $\sup\limits_{\overline{\Omega}} \varphi(u) = \sup\limits_{\Gamma} \varphi(u)$

Let us consider the following system:

$$Lu_k + f_k(x, u) = 0,\ k = \overline{1, m},\ x \in \Omega. \tag{2}$$

Let $\varphi \in C^2(\mathbb{R}^m)$. The following result is given in [3] (see also [1]):

Theorem 1.1. *Let u be a solution of (2). If:*

(i) the hessian of φ is positive semidefinite,

(ii) $-\sum\limits_{k=1}^{m} \frac{\partial\varphi(y)}{\partial y_k} f_k(x, y) + c(x) \left[\varphi(y) - \sum\limits_{k=1}^{m} \frac{\partial\varphi(y)}{y_k} y_k\right] \geq 0, \forall y \in \mathbb{R}^m,$

then $\sup\limits_{\overline{\Omega}} \varphi(u) = \sup\limits_{\Gamma} \varphi(u)$

Received by the editors: 06.06.2002.

The purpose of this paper is to use the Theorem 1.1 for constructing functionals defined on the solution of system (2), which satisfy a maximum principle.

If $c = 0$, then the condition (ii) from Theorem 1.1, becomes:

$$-\sum_{k=1}^{m} \frac{\partial \varphi(y)}{y_k} f_k(x, y) \geq 0, \forall y \in \mathbb{R}^m$$

$$\sum_{k=1}^{m} \frac{\partial \varphi(y)}{y_k} f_k(x, y) \leq 0, \forall y \in \mathbb{R}^m$$

$$\frac{\partial \varphi}{\partial y_1} f_1 + ... + \frac{\partial \varphi}{\partial y_m} f_m \leq 0, \forall y \in \mathbb{R}^m$$

We assume $f_k(x, y) = f_k(y)$, and we can choose φ by solving the partial differential equation:

$$\frac{dy_1}{f_1} = \frac{dy_2}{f_2} = ... = \frac{dy_m}{f_m} \tag{3}$$

in the form $\varphi(y) = k$, where k is a constant.

2. Examples of functionals which satisfies MP

We will consider the system given in [1]

$$\begin{cases} \Delta u + f(u, v) = 0 \\ \Delta v + g(u, v) = 0 \end{cases} \tag{4}$$

1. Let $f(u, v) = -\frac{1}{\beta} v$, $g(u, v) = \alpha u$. We have:

$$\begin{cases} \Delta u - \frac{1}{\beta} v = 0 \\ \Delta v + \alpha u = 0 \end{cases} \tag{5}$$

The functional corresponding to this system is $\varphi(u, v) = \alpha u^2 + \beta v^2$. Hence, since $\alpha \geq 0$, $\beta > 0$, φ satisfies Theorem 1.1. We have:

Theorem 2.1. *If (u,v) is a solution of (5) and $\alpha \geq 0$, $\beta > 0$, then $\alpha u^2 + \beta v^2$ verifies MP.*

Remark 2.1. This result represent a generalization of example 1, given in [1].

2. Let $f(u, v) = -\alpha u - \beta v,\ g(u, v) = \delta u + \gamma v$.We have:

$$\begin{cases} \Delta u - \alpha u - \beta v = 0 \\ \Delta v + \delta u + \gamma v = 0 \end{cases} \tag{6}$$

The equation corresponding to this system is:

$$\frac{du}{-\alpha u - \beta v} = \frac{dv}{\delta u + \gamma v}$$

If $u = zv$, we obtain:

$$\frac{\gamma z + \delta}{\gamma z^2 + (\alpha + \delta)z + \beta} = -\frac{1}{v}dv$$

and if we put $\int \frac{\gamma z + \delta}{\gamma z^2 + (\alpha + \delta)z + \beta}dz = \ln F(z)$, we will have:

$$\varphi(u, v) = \Phi\left[vF\left(\frac{u}{v}\right)\right], \Phi \in C^1(\mathbb{R})$$

We can consider $\varphi(u, v) = vF\left(\frac{u}{v}\right)$, but because of F, the properties of such functional are very hard to study.

What we can observe is that if $\alpha = \delta$ we have:

$$\frac{\gamma z + \alpha}{\gamma z^2 + 2\alpha z + \beta}dz = -\frac{1}{v}dv$$

In this way we will obtain:

$$\varphi(u, v) = \Phi\left(\sqrt{\gamma u^2 + \alpha uv + \beta v^2}\right)$$

where $\Phi \in C^1(\mathbb{R})$.

If we put $\Phi(t) = t^2$, then:

$$\varphi(u, v) = \gamma u^2 + \alpha uv + \beta v^2.$$

Theorem 2.2. *If (u,v) is a solution of (6), and the matrix* $\begin{pmatrix} 2\gamma & \alpha \\ \alpha & 2\beta \end{pmatrix}$ *is positive semidefinite i.e.* $\gamma \geq 0$, $\alpha^2 \leq 4\beta\gamma$, *then* $\gamma u^2 + \alpha uv + \beta v^2$ *verifies MP.*

Remark 2.2. This result represents a generalization of 1.

3. In the general case of system (4)

$$\begin{cases} \Delta u + f(u, v) = 0 \\ \Delta v + g(u, v) = 0 \end{cases}$$

the corresponding equation is $\frac{du}{dv} = \frac{f(u, v)}{g(u, v)}$:

$$g(u, v)du - f(u, v)dv = 0. \tag{7}$$

We consider the differential form $\omega = g(u,v)du - f(u,v)dv$. ω is a total differential if:

$$\frac{\partial g}{\partial v} = -\frac{\partial f}{\partial u}. \tag{8}$$

We will choose φ in the form:

$$\varphi(u,v) = \int\limits_{(0,0)}^{(u,v)} g(u,v)du - f(u,v)dv + C \tag{9}$$

The conditions of Theorem 1.1, becomes:

$$g(u,v)\int\limits_0^u \frac{\partial g(u,v)}{\partial v}du - f(u,v)\int\limits_0^v \frac{\partial f(u,v)}{\partial u}dv \leq 0 \tag{10}$$

$$\frac{\partial g}{\partial u} - \int\limits_0^v \frac{\partial f^2(u,v)}{\partial u^2}dv \geq 0 \tag{11}$$

$$\left(\frac{\partial g}{\partial u} - \int\limits_0^v \frac{\partial^2 f(u,v)}{\partial u^2}dv\right)\left(\int\limits_0^u \frac{\partial^2 g(u,v)}{\partial v^2}du - \frac{\partial f}{\partial v}\right) \geq \left(\frac{\partial g}{\partial v} - \frac{\partial f}{\partial u}\right)^2 \tag{12}$$

Because of (8) we have:

$$-\frac{\partial^2 f}{\partial u^2} = \frac{\partial^2 g}{\partial u \partial v}; \frac{\partial^2 g}{\partial v^2} = -\frac{\partial^2 f}{\partial u \partial v}; -\int\limits_0^v \frac{\partial^2 f}{\partial u^2}dv = \frac{\partial g}{\partial u}; \int\limits_0^u \frac{\partial^2 g}{\partial v^2}du = -\frac{\partial f}{\partial v}$$

In these conditions, (10), (11), (12), becomes

$$0 \leq 0$$

$$\frac{\partial g}{\partial u} \geq 0 \tag{13}$$

$$\left(\frac{\partial g}{\partial v}\right)^2 \leq -\frac{\partial g}{\partial u}\frac{\partial f}{\partial v} \tag{14}$$

$$\left(\frac{\partial f}{\partial u}\right)^2 \leq -\frac{\partial g}{\partial u}\frac{\partial f}{\partial v} \tag{15}$$

It is obvious that (14) or (15) are satisfied if:

$$\frac{\partial f}{\partial v} \leq 0. \tag{16}$$

Theorem 2.3. *In conditions (8), (13), (14/15), (16), if (u,v) is a solution of (4), then (9) verifies MP.*

Remark 2.3. If $f(u,v) = f(v)$, $g(u,v) = g(u)$, the conditions from above are: $g'(u) \geq 0$, $f'(v) \leq 0$. This case appears in [1].

As an example if $f(u,v) = u - 2v,\ g(u,v) = 2u - v$, then the conditions of Theorem 2.3 are satisfied and this implies that $(u-v)^2$ verifies MP.

Let us consider now the functions from 2, i.e.

$$f(u,v) = -\alpha u - \beta v$$
$$g(u,v) = \gamma u + \delta v$$

From (8) we have $\delta = \alpha$, and so g is $g(u,v) = \gamma u + \alpha v$.

Conditions (13), (14/15), (16) are: $\gamma \geq 0$, $\beta \leq 0$, $\alpha^2 \leq \beta\gamma$. In conclusion if (u,v) is a solution of (4), with f and g as above, and $\gamma \geq 0$, $\beta \leq 0$, $\alpha^2 \leq \beta\gamma$, then $\frac{1}{2}\gamma u^2 + 2\alpha uv + \frac{1}{2}\beta v^2$ verifies MP.

Remark 2.4. In this way (but choosing another method) we have obtained a functional as in 2, and the condition are the same.

4. Let us consider the system

$$\begin{cases} -\Delta u = \lambda f(x,u) - v \\ -\Delta v = \delta u - \gamma v \end{cases} \tag{17}$$

This system appears in [2] and the authors are looking for the existence of a positive solution. We will try to find a functional with the properties from Theorem 1.1.

Let $f(x,u) = f(u)$. We have:

$$\begin{cases} -\Delta u = \lambda f(u) - v \\ -\Delta v = \delta u - \gamma v \end{cases} \tag{18}$$

We will put $f_1(u,v) = \lambda f(u) - v$, $g_1(u,v) = \delta u - \gamma v$, and obtain:

$$\begin{cases} \Delta u + f_1(u,v) = 0 \\ \Delta v + g_1(u,v) = 0 \end{cases} \tag{19}$$

From (8) we have $-\gamma = -\lambda f\'(u)$, *i.e.* $f(u) = \dfrac{\gamma}{\lambda} u$

(19) becomes:

$$\begin{cases} \Delta u + \gamma u - v = 0 \\ \Delta v + \delta u - \gamma v = 0 \end{cases} \tag{20}$$

The conditions (13), (14/15),(16), are satisfied if $\delta \geq 0$, $\gamma^2 \leq \delta$.

So, if $\delta \geq 0$, $\gamma^2 \leq \delta$, , and (u,v) is a solution of (20), then $\frac{1}{2}\delta u^2 + 2\gamma uv + \frac{1}{2}v^2$, verifies MP.

Remark 2.5. This is a particular case of example given at 3.

Remark 2.6. If we try to find φ, in the classical way, we´ll obtain $\delta u^2 + 2\gamma uv + v^2$, which, in the same conditions, verifies MP.

Remark 2.7. We can try to find a integrating factor for

$$(\delta u - \gamma v)du + (v - \lambda f(u))dv = 0$$

from:

$$(\delta u - \gamma v)\frac{\partial \mu}{\partial v} - (v - \lambda f(u))\frac{\partial \mu}{\partial u} + (-\gamma + \lambda f\'(u))\mu = 0.$$

Let us consider now the system:

$$\begin{cases} \Delta u + f(u,v,w) = 0 \\ \Delta v + g(u,v,w) = \\ \Delta w + h(u,v,w) = 0 \end{cases} \tag{21}$$

5. Let $f(u,v,w) = -v - w$, $g(u,v,w) = u - w$, $h(u,v,w) = u + v$, (21) becomes:

$$\begin{cases} \Delta u - v - w = 0 \\ \Delta v + u - w = 0 \\ \Delta w + u + v = 0 \end{cases} \tag{22}$$

Let $\varphi(u,v) = u^2 + v^2 + w^2$. Condition (ii) from Theorem 1.1 becomes:

$$\frac{\partial \varphi}{\partial u} f(u,v,w) + \frac{\partial \varphi}{\partial v} g(u,v,w) + \frac{\partial \varphi}{\partial w} h(u,v,w) \leq 0.$$

φ satisfies this condition, and the hessian of φ is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ which is positive definite. We have the following result:

Theorem 2.4. *If (u,v,w) is a solution of (22) then $u^2+v^2+w^2$ verifies MP.*

6. Let $f(u,v,w) = -\beta v - \gamma w$, $g(u,v,w) = \alpha u - \gamma w$, $h(u,v,w) = \alpha u + \beta v$, (21) becomes:

$$\begin{cases} \Delta u - \beta v - \gamma w = 0 \\ \Delta v + \alpha u - \gamma w = 0 \\ \Delta w + \alpha u + \beta v = 0 \end{cases} \tag{23}$$

Let $\varphi(u,v) = \alpha u^2 + \beta v^2 + \gamma w^2$. Condition (ii) from Theorem 1.1 is verified by φ. The hessian of φ is $\begin{pmatrix} 2\alpha & 0 & 0 \\ 0 & 2\beta & 0 \\ 0 & 0 & 2\gamma \end{pmatrix}$ which is positive definite if $\alpha, \beta, \gamma \geq 0$.

Theorem 2.5. *If (u,v,w) is a solution of (24), and $\alpha, \beta, \gamma \geq 0$, then $\alpha u^2 + \beta v^2 + \gamma w^2$ verifies MP.*

7. Let $f(u,v,w) = w - v$, $g(u,v,w) = u - w$, $h(u,v,w) = v - u$, (21) becomes:

$$\begin{cases} \Delta u + w - v = 0 \\ \Delta v + u - w = 0 \\ \Delta w + v - u = 0 \end{cases} \tag{24}$$

Let $\varphi_1(u,v,w) = u^2 + v^2 + w^2$. (ii) from Theorem 1.1 is verified by φ, and the hessian of φ is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ which is positive definite.

Theorem 2.6. *If (u,v,w) is a solution of (24) then $u^2 + v^2 + w^2$ verifies MP.*

Let now $\varphi_2(u,v) = u^2 + v^2 + w^2 + uv + uw + vw$. φ verifies the condition (ii) from Theorem 1.1. The hessian of φ is:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{7}{6} \end{pmatrix}$$

and it is positive definite. In this way we obtain the following result:

Theorem 2.7. *If (u,v,w) is a solution of (230 then $u^2 + v^2 + w^2 + uv + uw + vw$ verifies MP.*

Remark 2.8. It is obvious that the example from above prove the fact that the functional corresponding to a system, and which satisfy am maximum principle, is not unique.

8. Let $f(u,v,w) = -u + v - w$, $g(u,v,w) = -u - v + w$, $h(u,v,w) = u - v - w$, (21) becomes:

$$\begin{cases} \Delta u - u + v - w = 0 \\ \Delta v - u - v + w = 0 \\ \Delta w + u - v - w = 0 \end{cases} \tag{25}$$

Let $\varphi(u,v) = u^2 + v^2 + w^2$. (ii) becomes $-2(u^2 + v^2 + w^2) \leq 0$, and the hessian of φ, as we saw, is positive definite. We have:

Theorem 2.8. *If (u,v,w) is a solution of (25) then $u^2+v^2+w^2$ verifies MP.*

Remark 2.9. If $f(u,v,w) = -\alpha u + \beta v - \gamma w$, $g(u,v,w) = -\alpha u - \beta v + \gamma w$, $h(u,v,w) = \alpha u - \beta v - \gamma w$, with $\alpha, \beta, \gamma \geq 0$, and (u,v,w) is a solution of the corresponding system, then $\alpha u^2 + \beta v^2 + \gamma w^2$ verifies MP.

Let us suppose now that $c \neq 0$. Condition (ii) from Theorem 1.1 becomes:

$$-\frac{\partial \varphi}{\partial y_1} f_1 - ... - \frac{\partial \varphi}{\partial y_m} f_m + c\left(\varphi - \frac{\partial \varphi}{\partial y_1} y_1 - ... - \frac{\partial \varphi}{\partial y_m} y_m\right) \geq 0$$

$$(f_1 + cy_1)\frac{\partial \varphi}{\partial y_1} + ... + (f_m + cy_m)\frac{\partial \varphi}{\partial y_m} \leq c\varphi$$

Let $\varphi = y_1^2 + ... + y_m^2$. We have:

$$2y_1 f_1 + ... + 2y_m f_m + c\left(y_1^2 + ... + y_m^2\right) \leq 0 \tag{26}$$

If $m = 2$ then (26) becomes:

$$2uf(u,v) + 2vg(u,v) + c(u^2 + v^2) \leq 0 \tag{27}$$

Remark 2.10. If $f = cu$ and $g = cv$, condition (27) is verified for $c \leq 0$, and so $u^2 + v^2$ verifies MP.

Remark 2.11. If $f(u,v) = -\alpha u + \beta v$, $g(u,v) = -\beta u - \gamma v$, then condition (27) is verified for $\alpha, \beta \geq 0$, and $c \leq 0$

Remark 2.12. In general case if $c \leq 0$ and $tf(t_1, ..., t_{k-1}, t, t_{k+1}, ..., t_m) \leq 0$, then $\sum\limits_{i=1}^{m} u_i^2$ satisfies MP.

Remark 2.13. If we take $\varphi(u,v) = \alpha u^2 + \beta v^2 + \gamma w^2$ (for the system with f and g like in remark 11), then condition (27) take place if $c \leq 0$, $\alpha, \gamma > 0$, $\beta^2 < \alpha\gamma$.

References

[1] Cosner C., Schaefer P., *On the development of functionals which satisfy a maximum principle*, Applicable Analysis, 26(1987), 45 - 60.

[2] Figueiredo D., Mitidieri E., *A maximum principle for an elliptic system and applications to semiliniar problem*, SIAM J. MATH. ANAL., 17(4), 1986, 836 -849.

[3] Rus A. I., *Maximum principles for elliptic system*, International series of numerical mathematics, 107, Birkhnauser Verlag, Basel, 1992.

Babeş-Bolyai University, Faculty of Business

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 3, September 2002

$\mathcal{X}$-MAXIMAL SUBGROUPS IN FINITE π-SOLVABLE GROUPS WITH RESPECT TO A SCHUNCK CLASS $\mathcal{X}$

RODICA COVACI

Abstract. Let π be an arbitrary set of primes and $\mathcal{X}$ be a π-Schunck class, i.e. $\mathcal{X}$ is a π-closed Schunck class. The paper establishes an existence and conjugacy theorem on $\mathcal{X}$-maximal subgroups in finite π-solvable groups. For the proof of the main result are used some theorems given in [4] generalizing Ore's theorems from [8]. Finally, some applications on $\mathcal{X}$-projectors in finite π-solvable groups are given.

1. **Preliminaries**

All groups considered in the paper are finite. We denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.

Some definitions will be reminded here:

Definition 1.1. A group G is *primitive* if G has a stabilizer W, i.e. a maximal subgroup W of G such that $core_G W = \{1\}$, where

$$core_G W = \cap\{W^g / g \in G\}.$$

Definition 1.2. a) A group G is π-*solvable* if any chief factor of G is either a solvable π-group or a π'-group. If π is the set of all primes, we obtain the notion of solvable group.

b) A class $\mathcal{X}$ of groups is π-*closed* if:

$$G/O_{\pi'}(G) \in \mathcal{X} \Rightarrow G \in \mathcal{X},$$

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G.

Definition 1.3. a) A class $\mathcal{X}$ of groups is a *homomorph* if $\mathcal{X}$ is closed under homomorphism, i.e. if $G \in \mathcal{X}$ and N is a normal subgroup of G, then $G/N \in \mathcal{X}$.

Received by the editors: 15.04.2002.

2000 *Mathematics Subject Classification.* 20D10.

Key words and phrases. Schunck class, projector, π-solvable group.

b) A homomorph $\mathcal{X}$ is a *Schunck class* if $\mathcal{X}$ is primitively closed, i.e. if any group G, all of whose primitive factor groups are in $\mathcal{X}$, is itself in $\mathcal{X}$.

c) We shall call π-*homomorph*, respectively π-*Schunck class*, a π-closed homomorph, respectively a π-closed Schunck class.

Definition 1.4. Let $\mathcal{X}$ be a class of groups, G a group and H a subgroup of G.

a) H is an $\mathcal{X}$-*maximal subgroup* of G if: (i) $H \in \mathcal{X}$; (ii) $H \leq H^* \leq G, H^* \in \mathcal{X}$ imply $H = H^*$.

b) H is an $\mathcal{X}$-*projector* of G if for any normal subgroup N of G, HN/N is $\mathcal{X}$-maximal in G/N.

c) H is an $\mathcal{X}$-*covering subgroup* of G if: (i) $H \in \mathcal{X}$; (ii) $H \leq K \leq G$, $K_0 \lhd K$, $K/K_0 \in \mathcal{X}$ imply $K = HK_0$.

The following results will be used in the paper:

Proposition 1.5. ([1]) A solvable minimal subgroup of a finite group is abelian.

Proposition 1.6. ([6]) Let G be a group and N a subgroup of G. The following two conditions are equivalent:

(1) N is normal in G and G/N is primitive;

(2) there is a maximal subgroup W of G such that $N = core_G W$.

2. Ore's generalized theorems

In [4] are given some theorems generalizing Ore's theorems from [8]. In order to be used in the present paper, we remind them:

Theorem 2.1. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Corollary 2.2. *If G is a primitive π-solvable group, then G has at most one minimal normal subgroup which is a solvable π-group.*

Corollary 2.3. *If a primitive π-solvable group G has a minimal normal subgroup which is a solvable π-group, then G has no minimal normal subgroups which are π'-groups.*

Theorem 2.4. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then $C_G(N) = N$.*

Theorem 2.5. *Let G be a π-solvable group such that:*

(i) there is a minimal subgroup M of G which is a solvable π-group and $C_G(M) = M$;

(ii) there is a minimal normal subgroup L/M of G/M such that L/M is a π'-group. Then G is primitive.

Theorem 2.6. *If G is a π-solvable group satisfying (i) and (ii) from 2.5., then any two stabilizers W_1 and W_2 of G are conjugate in G.*

Theorem 2.7. *If G is a primitive π-solvable group, $V < G$, such that there is a minimal normal subgroup M of G which is a solvable π-group and $MV = G$, then V is a stabilizer of G.*

3. An existence and conjugacy theorem on $\mathcal{X}$-maximal subgroups in finite π-solvable groups

In preparation for the main theorem we give the following lemma:

Lemma 3.1. *If G is a finite group, W is a maximal subgroup of G and $A \neq \{1\}$ is a normal subgroup of G, such that $AW = G$ and $A \cap W = \{1\}$, then A is a minimal normal subgroup of G.*

Proof. We have that $A \neq \{1\}$ is a normal subgroup of G. Let now $A^* \neq \{1\}$ be a normal subgroup of G, such that $A^* \leq A$. We shall prove that $A^* = A$. Since

$$W \leq A^*W \leq AW = G,$$

it follows that $A^*W = W$ or $A^*W = G$. But, if we suppose that $A^*W = W$, we have

$$A^* \subseteq A \cap W = \{1\},$$

hence the contradiction $A^* = \{1\}$. So $A^*W = G$. In order to prove that $A^* = A$, suppose that $A^* < A$. This means that there is an element $a \in A \setminus A^* \subseteq G = A^*W$. Then $a = a^*w$, with $a^* \in A^*$, $w \in W$. It follows that

$$w = (a^*)^{-1}a \in A \cap W = \{1\},$$

hence $w = 1$, which implies the contradiction $a = a^* \in A^*$. So $A^* = A$. $\square$

The main theorem of this paper is the following:

Theorem 3.2. *Let $\mathcal{X}$ be a π-Schunck class, G a π-solvable group and A an abelian normal subgroup of G with $G/A \in \mathcal{X}$. Then:*

(1) there is a subgroup S of G with $S \in \mathcal{X}$ and $AS = G$;

(2) there is an $\mathcal{X}$-maximal subgroup S of G with $AS = G$;

(3) if S_1 and S_2 are $\mathcal{X}$-maximal subgroups of G with $AS_1 = G = AS_2$, then S_1 and S_2 are conjugate in G.

Proof.

(1) Let

$$\mathcal{S} = \{S^*/S^* \leq G, AS^* = G\}.$$

Since $G \in \mathcal{S}$, we have $\mathcal{S} \neq \emptyset$. Considering $\mathcal{S}$ ordered by inclusion and applying Zorn's lemma, $\mathcal{S}$ has a minimal element S. Obviously, $AS = G$.

We shall prove that $S \in \mathcal{X}$.

Put $D = S \cap A$. Let us notice that D is a normal subgroup of G. Indeed, if $g \in G$ and $d \in D$, we have $g = as$, with $a \in A$, $s \in S$, and so, A being abelian and D being normal in S,

$$g^{-1}dg = (as)^{-1}d(as) = s^{-1}a^{-1}das = s^{-1}a^{-1}ads = s^{-1}ds \in D.$$

Let W be a maximal subgroup of S. Then $D \leq W$, else $DW \neq W$, hence

$$W < DW \leq S$$

and so $DW = S$. But this implies

$$G = AS = ADW = AW,$$

which means that $W \in \mathcal{S}$, in contradiction with the minimality of S in $\mathcal{S}$.

Put $N = core_S W$. We have $D \leq N$. Indeed, from $D \leq W$ it follows that $D = D^g \leq W^g$ for any $g \in S$, hence $D \leq core_S W = N$. Then

$$S/N \cong (S/D)/(N/D).$$

But

$$S/D = S/S \cap A \cong AS/A = G/A \in \mathcal{X}.$$

$\mathcal{X}$ being a homomorph, it follows that $S/N \in \mathcal{X}$.

For any primitive factor group S/N of S, we have $S/N \in \mathcal{X}$. Indeed, S/N being primitive, it follows from 1.6. that there is a maximal subgroup W of S such that $N = core_S W$. But we proved that in this case we have $S/N \in \mathcal{X}$. This means that any primitive factor group S/N of S is in $\mathcal{X}$. The primitive closure of $\mathcal{X}$ leads now to $S \in \mathcal{X}$. Thus (1) is proved.

(2) Let now

$$\mathcal{S}^* = \{S/S \leq G, S \in \mathcal{X}, AS = G\}$$

ordered by inclusion. Because of (1), $\mathcal{S}^* \neq \emptyset$. By Zorn's lemma, $\mathcal{S}^*$ has a maximal element $S \in \mathcal{S}^*$. Obviously, $S \leq G$, $S \in \mathcal{X}$, $AS = G$. We shall prove that S in an $\mathcal{X}$-maximal subgroup of G. Let $S \leq S^* \leq G$, with $S^* \in \mathcal{X}$. Then $S = S^*$, as the following considerations show: from $AS = G$ it follows that $AS^* = G$ and so $S^* \in \mathcal{S}^*$; but $S \leq S^*$, $S^* \in \mathcal{S}^*$ imply by the maximality of S that $S = S^*$.

(3) Let S_1 and S_2 be $\mathcal{X}$-maximal subgroups of G with $AS_1 = G = AS_2$. We shall prove by induction on $|G|$ that S_1 and S_2 are conjugate in G.

Let us distinguish two cases:

a) $G \in \mathcal{X}$. S_1 and S_2 being $\mathcal{X}$-maximal subgroups of G, we have $S_1 = G = S_2$ and so S_1 and S_2 are conjugate in G.

b) $G \notin \mathcal{X}$. It means that there is a primitive factor group G/N with $G/N \notin \mathcal{X}$, else the primitive closure of $\mathcal{X}$ leads to the contradiction $G \in \mathcal{X}$. We also have $NS_1 \neq G$ and $NS_2 \neq G$. Indeed, if we suppose, for example, that $NS_1 = G$, we obtain

$$NS_1/N = G/N \notin \mathcal{X}$$

and on the other side

$$NS_1/N \cong S_1/S_1 \cap N \in \mathcal{X}.$$

Let us prove that AN/N is minimal normal subgroup of G/N. The factor group G/N being primitive, we apply 1.6. and there is a minimal subgroup W of G with $N = core_G W$. We have $A \nleq W$, because supposing that $A \leq W$ it follows that for any $g \in G$, $A = A^g \leq W^g$, hence

$$A \leq \cap\{W^g/g \in G\} = core_G W = N$$

and

$$G/A = AS_1/A \cong S_1/A \cap S_1 \in \mathcal{X}$$

and so

$$G/N \cong (G/A)/(N/A) \in \mathcal{X},$$

in contradiction with $G/N \notin \mathcal{X}$. Put $A_1 = A \cap W$. Since W is a maximal subgroup of G, we have $AW = W$ or $AW = G$. But $AW = W$ implies $A \leq W$, a contradiction. So $AW = G$. It is easy to prove that A_1 is a normal subgroup of G. Indeed, if $g \in G = AW$ and $a_1 \in A_1$, put $g = aw$, with $a \in A$, $w \in W$ and, A being abelian and A_1 being normal in W, we have:

$$g^{-1}a_1 g = (aw)^{-1}a_1(aw) = w^{-1}a^{-1}a_1 aw = w^{-1}a_1 a^{-1}aw = w^{-1}a_1 w \in A_1.$$

We are now in the hypotheses of lemma 3.1. Indeed, W/A_1 is a maximal subgroup of G/A_1, A/A_1 is a normal $\neq \{1\}$ subgroup of G/A_1 satisfying:

$$A/A_1 \cdot W/A_1 = G/A_1 \text{ and } A/A_1 \cap W/A_1 = \{1\}.$$

It follows that A/A_1 is a minimal normal subgroup of G/A_1. From this and from the isomorphism

$$AN/N \cong A/A_1$$

we obtain that AN/N is minimal normal subgroup of G/N.

Denote by $M = AN$. It follows that for $i = 1, 2$, we have

$$(NS_i)M = (NS_i)(AN) = G.$$

Furthermore, NS_i/N is a stabilizer of G/N, for $i = 1, 2$. In order to prove this, we use theorem 2.7. In the primitive π-solvable group G/N, we consider $NS_i/N < G/N$ and $M/N = AN/N$ minimal normal subgroup of G/N. Obviously, $M/N \cdot NS_i/N = G/N$. It remains to prove that M/N is a solvable π-group. Being a minimal normal subgroup of the π-solvable group G/N, M/N is either a solvable π-group or a π'-group. If we suppose that M/N is a π'-group, we obtain

$$M/N \leq O_{\pi'}(G/N) \leq G/N$$

and

$$(G/N)/O_{\pi'}(G/N) \cong ((G/N)/(M/N))/(O_{\pi'}(G/N)/(M/N)).$$

But $\mathcal{X}$ being a homomorph, we have

$$(G/N)/(M/N) \cong G/M \cong G/AN \cong (G/A)/(AN/A) \in \mathcal{X}$$

and

$$(G/N)/O_{\pi'}(G/N) \in \mathcal{X},$$

hence by the π-closure of the class $\mathcal{X}$ we obtain the contradiction $G/N \in \mathcal{X}$. It follows that M/N is a solvable π-group. Applying 2.7., NS_i/N is a stabilizer of G/N.

The next step in our proof is to show that NS_1/N and NS_2/N are conjugate in G/N. For this, we apply theorem 2.6. to the π-solvable group G/N. Indeed, G/N satisfies the conditions (i) and (ii) from theorem 2.5., as we prove below:

(i) $M/N = AN/N$ is minimal normal subgroup of G/N, such that M/N is a solvable π-group and $C_{G/N}(M/N) = M/N$. The last condition follows from theorem 2.4. applied to the primitive π-solvable group G/N and its minimal normal subgroup M/N which is a solvable π-group.

(ii) There is a minimal normal subgroup $(L/N)/(M/N)$ of $(G/N)/(M/N)$, such that $(L/N)/(M/N)$ is a π'-group. Indeed, if we suppose the contrary, then any minimal normal subgroup $(L/N)/(M/N)$ of the π-solvable group $(G/N)/(M/N)$ is a solvable π-group. But M/N being a solvable π-group, it follows that L/N is also a solvable π-group. Theorem 2.1. applied to the primitive π-solvable group G/N, which has the minimal normal subgroup M/N such that M/N is a solvable π-group, leads to the conclusion that G/N has one and only one minimal subgroup. Since L/N is $a \neq \{1\}$ normal subgroup of G/N, two possibilities can happen:

1) L/N is a minimal normal subgroup of G/N. It follows that $M/N = L/N$, in contradiction with the assumption that $(L/N)/(M/N)$ is a minimal normal subgroup of $(G/N)/(M/N)$.

2) L/N is not a minimal normal subgroup of G/N and so $M/N < L/N$. But this also leads to a contradiction, as the following shows:

$$G/N = M/N \cdot NS_1/N < L/N \cdot NS_1/N = G/N.$$

We are now in the hypotheses of theorem 2.6., hence NS_1/N and NS_2/N are conjugate in G/N. It follows that

$$NS_1 = (NS_2)^g = NS_2^g,$$

where $g \in G$.

Denote by

$$G^* = NS_1 = NS_2^g$$

and by

$$A^* = A \cap G^*.$$

We can now apply the induction for G^*, where $G^* = NS_1 < G$. Indeed, A^* is an abelian normal subgroup of G^*, with

$$G^*/A^* = G^*/A \cap G^* \cong AG^*/A = ANS_1/A = G/A \in \mathcal{X}$$

and S_1 and S_2^g are $\mathcal{X}$-maximal subgroups in G^*. We also have:

$$A^*S_1 = (A \cap G^*)S_1 = S_1(A \cap G^*) = (S_1 A) \cap G^* = G \cap G^* = G^*$$

and

$$A^*S_2^g = (A \cap G^*)S_2^g = S_2^g(A \cap G^*) = (S_2^g A) \cap G^* = (S_2 A)^g \cap G^* = G \cap G^* = G^*.$$

By the induction, S_1 and S_2^g are conjugate in G^*. It follows that S_1 and S_2 are conjugate in G. $\square$

Remarks. a) Theorem 3.2. was earlier establishes in [2], but the proof was based on some of R. Baer's theorems from [1]. In the present paper, we give a new proof, based on Ore's generalized theorems given in [4].

b) Particularly, for π the set of all primes, we obtain from theorem 3.2. a theorem given in [6] by W. Gaschütz.

4. **Projectors in finite π-solvable groups**

Theorem 3.2 is important for the study of projectors in finite π-solvable groups, as the following result (given in [3]) shows:

Theorem 4.1. *If $\mathcal{X}$ is a π-Schunck class, then any two $\mathcal{X}$-projectors of a π-solvable group G are conjugate in G.*

Proof. By induction on $|G|$. We remind the proof from [3]:

Let S_1 and S_2 be two $\mathcal{X}$-projectors of G. Let M be a minimal normal subgroup of G. Put $S_1^* = MS_1$ and $S_2^* = MS_2$. Applying the induction for G/M, we obtain that S_1^*/M and S_2^*/M are conjugate in G/M, hence S_1^* and S_2^* are conjugate in G, i.e. $S_1^* = (S_2^*)^g$, with $g \in G$.

We prove that S_1 and S_2 are conjugate in G, considering the two cases which are possible for the minimal normal subgroup M of the π-solvable group G:

1) M is a solvable π-group. Then by 1.5., M is abelian. We are now in the hypotheses of theorem 3.2.: S_1^* is π-solvable, where

$$S_1^* = MS_1 = MS_2^g,$$

M is a normal abelian subgroup of S_1^*, with $S_1^*/M \in \mathcal{X}$ and S_1 and S_2^g are $\mathcal{X}$-maximal subgroups in S_1^*. Applying theorem 3.2., we deduce that S_1 and S_2^g are conjugate in S_1^*. It follows that S_1 and S_2 are conjugate in G.

2) M is a π'-group. Then

$$M \leq O_{\pi'}(S_1^*)$$

and

$$S_1^*/O_{\pi'}(S_1^*) \cong (S_1^*/M)/(O_{\pi'}(S_1^*)/M) \in \mathcal{X},$$

which imply by the π-closure of $\mathcal{X}$ that $S_1^* \in \mathcal{X}$. Hence by the fact that S_1 and S_2^g are $\mathcal{X}$-maximal in S_1^*, we obtain $S_1 = S_1^* = S_2^g$. $\square$

The conjugacy theorem 4.1. on projectors can be completed with an existence theorem. In [5], we proved by means of Ore's generalized theorems ([4]) the following result:

Lemma 4.2. *Let $\mathcal{X}$ be a π-homomorph. $\mathcal{X}$ is a Schunck class is and only if any finite π-solvable group G has $\mathcal{X}$-covering subgroups.*

It is well-known that for a homomorph $\mathcal{X}$ and a finite group G, any $\mathcal{X}$-covering subgroup of G is also an $\mathcal{X}$-projector of G. Thus lemma 4.2. leads to the following existence theorem on projectors:

Theorem 4.3. *If $\mathcal{X}$ is a π-Schunck class, then any finite π-solvable group G has $\mathcal{X}$-projectors.*

In [3] we proved the following result:

Lemma 4.4. *A π-homomorph $\mathcal{X}$ with the property that any finite π-solvable group has $\mathcal{X}$-projectors is a Schunck class.*

Theorem 4.5. *Let $\mathcal{X}$ be a π-homomorph. $\mathcal{X}$ is a Schunck class if and only if any finite π-solvable group has $\mathcal{X}$-projectors.*

References

[1] Baer, R., *Classes of finite groups and their properties,* Illinois J. Math., **1**, No. 4, 1957, 115-187.
[2] Covaci, R., *Some properties of projectors in finite π-solvable groups,* Studia Univ. "Babeş-Bolyai", Math., **XXVI**, No. 1, 1981, 5-8
[3] Covaci, R., *Projectors and covering subgroups,* Studia Univ. "Babeş-Bolyai", Math., **XXVII**, 1982, 33-36.
[4] Covaci, R., *A generalization of some of Ore's theorems,* Studia Univ. "Babeş-Bolyai", Math., **XLIII**, No. 1, 1998, 15-21.
[5] Covaci, R., *Some applications of Ore's generalized theorems in the formation theory,* Studia Univ. "Babeş-Bolyai", Math., **XLIV**, No. 4, 1999, 3-9.
[6] Gaschütz, W., *Selected topics in the theory of soluble groups,* Australian National University, Canberra, Jan.-Feb. 1969.
[7] Huppert, B., *Endliche Gruppen I,* Springer Verlag, Berlin-New York, 1967.
[8] Ore, O., *Contributions to the theory of groups of finite order,* Duke Math. J., 5, 1939, 431-450.

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, Cluj-Napoca

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLVII, Number 3, September 2002

ON SOME Ω-PURE EXACT SEQUENCES OF MODULES

IULIU CRIVEI

Abstract. Let R be an associative ring with non-zero identity. We shall consider a family Ω of left R-modules of the form R/Rr^n, where $r \in R$ and $n \geq 1$ is a natural number depending on r such that $r^n \neq 0$ for each $r \neq 0$. We shall characterize Ω-pure exact sequences of right R-modules and absolutely Ω-pure right R-modules. We shall also establish the structure of Ω-pure-projective right R-modules.

1. Introduction

In this paper we denote by R an associative ring with non-zero identity and all R-modules are unital. By Mod-R we denote the category of right R-modules. By a homomorphism we understand an R-homomorphism. The injective hull of an R-module A is denoted by $E(A)$.

Let Ω be a class of left R-modules and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1}$$

be a short exact sequence of right R-modules, where f and g are homomorphisms. If the tensor product $f \otimes_R 1_D : A \otimes_R D \to B \otimes_R D$ is a monomorphism for every $D \in \Omega$, it is said that the sequence (1) is Ω-pure. If A is a submodule of B, f is the inclusion monomorphism and the sequence (1) is Ω-pure, then A is said to be an Ω-pure submodule of B.

A right R-module M is called projective with respect to the sequence (1) if the natural homomorphism $Hom_R(M, B) \to Hom_R(M, C)$ is surjective. A right R-module is called injective with respect to the sequence (1) (or with respect to the monomorphism f) if the natural homomorphism $Hom_R(B, M) \to Hom_R(A, M)$ is

Received by the editors: 21.05.2002.
2000 *Mathematics Subject Classification.* 16D80.
Key words and phrases. tensor product, Ω-pure submodule, short exact sequence.

surjective. A right R-module P is said to be Ω-pure-projective if P is projective with respect to every Ω-pure short exact sequence of right R-modules.

Following Maddox [2], a right R-module M is said to be absolutely pure if M is pure in every right R-module which contains M as a submodule.

If $\Omega = \{R/Rr \mid r \in R\}$, then an Ω-pure exact sequence (1) is called RD-pure [5].

Denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{Z}$ the ring of integers, $R^* = R \setminus \{0\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and by $\mathcal{P}(\mathbb{N}^*)$ the set of all subsets of $\mathbb{N}^*$.

Let $\varphi : R \to \mathcal{P}(\mathbb{N}^*)$ be a function such that for every $r \in R^*$ and every $n \in \varphi(r)$, $r^n \neq 0$.

In this paper we shall consider the family of left R-modules

$$\Omega = \{R/Rr^n \mid r \in R^*, n \in \varphi(r)\}.$$

Notice that if the exact sequence (1) is RD-pure, then it is Ω-pure. Also, if $\varphi(r) = \{1\}$ for every $r \in R$, then Ω-purity is the same as RD-purity.

We shall characterize Ω-pure short exact sequences and we shall determine the structure of Ω-pure-projective right R-modules. Also, we introduce the notion of absolutely Ω-pure right R-module and we establish some properties for such modules.

2. **Basic results**

We shall recall two results which will be used later in the paper.

Theorem 2.1. [4, Proposition 2.3] *Let T be a set of right R-modules which contains a family of generators for Mod-R and let $p^{-1}(T)$ be the class of all short exact sequences in Mod-R with the property that every R-module in T is projective with respect to them. Then:*

(i) For every right R-module L there exists a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

in $p^{-1}(T)$ with $M \in T$.

(ii) Every right R-module which is projective with respect to each sequence in $p^{-1}(T)$ is a direct summand of a direct sum of R-modules in T.

Lemma 2.2. [6, Lemma 7.16] *Consider the commutative diagram with exact rows in Mod-R*

$$\begin{tikzcd}
 & M_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"] & M_3 \arrow[r] \arrow[d, "\varphi_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 &
\end{tikzcd}$$

The following statements are equivalent:

(i) There exists $\alpha : M_3 \to N_2$ with $g_2\alpha = \varphi_3$;

(ii) There exists $\beta : M_2 \to N_1$ with $\beta f_1 = \varphi_1$.

Now we can characterize Ω-pure submodules.

Theorem 2.3. *Let A be a submodule of a right R-module B. Then the following statements are equivalent:*

(i) A is Ω-pure in B.

(ii) For every $r \in R^*$ and every $n \in \varphi(r)$,

$$Ar^n = A \cap Br^n\,.$$

(iii) For every $r \in R^*$ and every $n \in \varphi(r)$, $c = br^n \in A$ for some $b \in B$ implies $c = ar^n$ for some $a \in A$.

Proof. $(i) \iff (ii)$ A is Ω-pure in B if and only if for every $r \in R^*$ and every $n \in \varphi(r)$ the sequence of $\mathbb{Z}$-modules

$$0 \to A \otimes_R R/Rr^n \overset{f\otimes 1_{R/Rr^n}}{\longrightarrow} B \otimes_R R/Rr^n \overset{g\otimes 1_{R/Rr^n}}{\longrightarrow} C \otimes_R R/Rr^n \to 0 \qquad (2)$$

is exact, where $f : A \to B$ is the inclusion homomorphism. It is known the isomorphism of $\mathbb{Z}$-modules

$$D \otimes_R R/K \cong D/DK\,,$$

where D is a right R-module and K is a left ideal of R. Then the sequence (2) is exact if and only if the sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow A/Ar^n \overset{f_1}{\longrightarrow} B/Br^n \overset{g_1}{\longrightarrow} C/Cr^n \longrightarrow 0 \qquad (3)$$

is exact, where $f_1(a + Ar^n) = a + Br^n$ for every $a \in A$. But f_1 is injective if and only if $A \cap Br^n \subseteq Ar^n$. Since the converse inclusion is clear, it follows that A is Ω-pure in B if and only if for every $r \in R^*$ and every $n \in \varphi(r)$, we have $Ar^n = A \cap Br^n$.

$(ii) \Longrightarrow (iii)$ Assume that (ii) holds. Let $r \in R^*$ and $n \in \varphi(r)$. Then $Ar^n = A \cap Br^n$. Let $c = br^n \in A$ for some $b \in B$. Then $c \in A \cap Br^n = Ar^n$. Hence there exists $a \in A$ such that $c = ar^n$.

$(iii) \Longrightarrow (ii)$ Assume that (iii) holds. Let $r \in R^*$ and $n \in \varphi(r)$. Let $c \in A \cap Br^n$. Then there exists $a \in A$ such that $c = ar^n$. Then $c \in Ar^n$. It follows that $A \cap Br^n \subseteq Ar^n$. Therefore, $A \cap Br^n = Ar^n$. □

Theorem 2.4. *The following statements are equivalent:*

(i) The exact sequence (1) of right R-modules is Ω-pure.

(ii) For every $r \in R^*$, for every $n \in \varphi(r)$ and for every commutative diagram of right R-modules:

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ r^nR \arrow[u, "k"] \arrow[r, "v"'] & R \arrow[u, "h"'] \end{tikzcd} \tag{4}$$

where k and h are homomorphisms and v is the inclusion homomorphism, there exists a homomorphism $w : R \to A$ such that $k = wv$.

(iii) For every $r \in R^*$ and for every $n \in \varphi(r)$, the right R-module R/r^nR is projective with respect to the exact sequence (1) of right R-modules.

Proof. We may suppose without loss of generality that A is an Ω-pure submodule of B and f is the inclusion homomorphism.

$(i) \Longrightarrow (ii)$ Assume that (i) holds. Let $r \in R^*$ and $n \in \varphi(r)$. Now consider the commutative diagram (4) of right R-modules, where v is the inclusion homomorphism. Denote $b = h(1)$ and $c = k(r^n)$. Then

$$c = fk(r^n) = hv(r^n) = h(r^n) = br^n \,.$$

By Theorem 2.3, there exists $a \in A$ such that $c = ar^n$. Define the homomorphism $w : R \to A$ by $w(1) = a$. Then

$$wv(r^n) = w(r^n) = ar^n = c = k(r^n)\,,$$

hence $k = wv$.

$(ii) \Longrightarrow (i)$ Assume that (ii) holds. Let $r \in R^*$, $n \in \varphi(r)$ and suppose that $c = br^n \in A$ for some $b \in B$. Define the homomorphisms $h : R \to B$ by $h(1) = b$ and

$k : r^nR \to A$ by $k(r^ns) = cs$ for every $s \in R$. If $r^ns = r^nt$ for some $s, t \in R$, then

$$cs - ct = c(s - t) = br^n(s - t) = 0\,,$$

hence k is well defined. Let $v : r^nR \to R$ be the inclusion homomorphism. We have

$$hv(r^n) = h(r^n) = br^n = c = fk(r^n)\,,$$

that is, $hv = fk$. Thus we obtain a commutative diagram (4). Hence there exists an homomorphism $w : R \to A$ such that $k = wv$. Denote $a = w(1)$. Then

$$c = k(r^n) = wv(r^n) = w(r^n) = ar^n\,.$$

By Theorem 2.3, the exact sequence (1) is Ω-pure.

(ii) $\Longrightarrow$ *(iii)* Assume that *(ii)* holds. Let $r \in R^*$ and $n \in \varphi(r)$. Consider the exact sequence of right R-modules

$$0 \longrightarrow r^nR \xrightarrow{v} R \xrightarrow{q} R/r^nR \longrightarrow 0 \tag{5}$$

where v is the inclusion homomorphism and q is the natural projection. Let $p : R/r^nR \to C$ be a homomorphism. Since R is projective, there exists a homomorphism $h : R \to B$ such that $gh = pq$. We have $ghv = pqv = 0$, hence there exists a homomorphism $k : r^nR \to A$ such that $hv = fk$. Hence there exists a homomorphism $w : R \to A$ such that $wv = k$. Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\
0 \arrow[r] & r^nR \arrow[r, "v"'] \arrow[u, "k"] & R \arrow[r, "q"'] \arrow[u, "h"'] & R/r^nR \arrow[r] \arrow[u, "p"'] & 0
\end{tikzcd} \tag{6}$$

with exact rows. By Lemma 2.2, there exists a homomorphism $u : R/r^nR \to B$ such that $p = gu$. Therefore R/r^nR is projective with respect to the exact sequence (1).

(iii) $\Longrightarrow$ *(ii)* Assume that *(iii)* holds. Let $r \in R^*$ and $n \in \varphi(r)$. Consider the commutative diagram of right R-modules (4), where v is the inclusion homomorphism. We construct the exact sequence (5), where q is the natural projection. Since $ghv = gfk = 0$, there exists a homomorphism $p : R/r^nR \to C$ such that $pq = gh$. Thus we obtain a commutative diagram (6) with exact rows. Since R/r^nR is projective with respect to the sequence (1), there exists a homomorphism $u : R/r^nR \to B$ such that

$p = gu$. By Lemma 2.2, there exists a homomorphism $w : R \to A$ such that $k = wv$. $\square$

By Theorems 2.1 and 2.4, we deduce the following two corollaries, giving the structure of Ω-pure-projective R-modules.

Corollary 2.5. *For every right R-module L there exists a short exact sequence of right R-modules*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

where M is Ω-pure-projective and N is an Ω-pure submodule of M.

Corollary 2.6. *Every Ω-pure-projective right R-module is a direct summand of a direct sum of R-modules of the form R/r^nR, where $r \in R$ and $n \in \varphi(r)$.*

Corollary 2.7. *Let $r \in R^*$ and $n \in \varphi(r)$. Then the following statements are equivalent:*

(i) The right ideal r^nR is Ω-pure in R.

(ii) The right ideal r^nR is a direct summand of R.

Proof. $(i) \Longrightarrow (ii)$ Assume that (i) holds. Consider the exact sequence (5) of right R-modules. By Theorem 2.4, R/r^nR is projective with respect to the sequence (5). Then the sequence (5) splits, that is, r^nR is a direct summand of R.

$(ii) \Longrightarrow (i)$ Clear. $\square$

3. **Absolutely Ω-pure modules**

We shall give the following definition.

Definition. A right R-module A is called *absolutely Ω-pure* if A is Ω-pure in each right R-module which contains it as a submodule.

In the sequel we shall denote by $\mathcal{A}$ the class of absolutely Ω-pure right R-modules.

Theorem 3.1. *Let A be a right R-module. Then the following statements are equivalent:*

(i) $A \in \mathcal{A}$.

(ii) A is Ω-pure in $E(A)$.

(iii) For every $r \in R^*$ and $n \in \varphi(r)$, A is injective with respect to the inclusion homomorphism $v : r^n R \to R$.

Proof. $(i) \Longrightarrow (ii)$ Clear.

$(ii) \Longrightarrow (iii)$ Assume that (ii) holds. Denote $B = E(A)$ and let $r \in R^*$ and $n \in \varphi(r)$. Let $k : r^n R \to A$ be a homomorphism. Since B is injective, there exists a homomorphism $h : R \to B$ such that $hv = fk$. By Theorem 2.4, there exists a homomorphism $w : R \to A$ such that $k = wv$. Hence A is injective with respect to v.

$(iii) \Longrightarrow (i)$ Assume that (iii) holds. Let $r \in R^*$ and $n \in \varphi(r)$. Let B be a right R-module which contains A as a submodule. Consider the commutative diagram (4) of right R-modules, where f is the inclusion homomorphism. Then there exists a homomorphism $w : R \to A$ such that $wv = k$, because A is injective with respect to v. By Theorem 2.4, A is Ω-pure in B, that is, A is absolutely Ω-pure. □

Remark. Every injective right R-module is absolutely Ω-pure.

Corollary 3.2. *The class $\mathcal{A}$ is closed under taking direct products and direct summands.*

Proof. It follows as for injectivity [3, Proposition 2.2]. □

Lemma 3.3. *The class $\mathcal{A}$ is closed under taking direct sums.*

Proof. Let $(A_i)_{i\in I}$ be a family of absolutely Ω-pure right R-modules and let $A = \oplus_{i\in I} A_i$. Let $r \in R^*$ and $n \in \varphi(r)$ and let $k : r^n R \to A$ be a homomorphism. Since $k(r^n R)$ is generated by $k(r^n)$, there exists a finite subset $J \subseteq I$ such that $k(r^n R) \subseteq \oplus_{i\in J} A_i = D$. By Corollary 3.2, $D \in \mathcal{A}$. Therefore by Theorem 3.1, there exists a homomorphism $q : R \to D$ such that $qv = u$, where $u : r^n R \to D$ is the homomorphism defined by $u(r^n s) = k(r^n s)$ for every $s \in R$. Let $\alpha : D \to A$ be the inclusion homomorphism. Then $\alpha qv = \alpha u = k$. By Theorem 3.1, $A \in \mathcal{A}$. □

Theorem 3.4. *Let (1) be a short exact sequence of right R-modules and let $A, C \in \mathcal{A}$. Then $B \in \mathcal{A}$.*

Proof. Similar to the proof given for absolutely F/U-pure modules [1, Theorem 2.7]. □

References

[1] Crivei, I., *Absolutely F/U-pure modules*, Studia Univ. "Babeş-Bolyai", Mathematica **44**, no.4 (1999), 15-21.
[2] Maddox, B., *Absolutely pure modules*, Proc. Amer. Math. Soc., **18** (1967), 207-234.
[3] Sharpe, D.W., Vámos, P., *Injective modules*, Cambridge Univ. Press, 1972.

[4] Stenström, B., *Pure submodules*, Ark. Math. **7** (1967), 151-171.
[5] Warfield, R.B. Jr., *Purity and algebraic compactness for modules*, Pacific Journal of Math. **28** (1969), 699-719.
[6] Wisbauer, R., *Foundations of module and ring theory*, Gordon and Breach, Reading, 1991.

Department of Mathematics, Technical University,
R0-3400 Cluj-Napoca, Romania

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 3, September 2002

QUANTITATIVE ESTIMATES FOR SOME LINEAR AND POSITIVE OPERATORS

ZOLTÁN FINTA

Abstract. The purpose of this paper is to establish quantitative estimates for the rate of convergence of some linear and positive operators. The most of them are generated by special functions.

1. Introduction

For the Bernstein operator

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad f \in C[0,1], \quad \varphi(x) = \sqrt{x(1-x)}$$

it is well - known that there exists an absolute constant $C > 0$ such that

$$|B_n(f,x) - f(x)| \;\leq\; C\;\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right), \; x \in [0,1] \tag{1}$$

and

$$\|B_n(f) - f\| \;\leq\; C\;\omega_2^{\varphi}\left(f, \sqrt{\frac{1}{n}}\right). \tag{2}$$

(see [2, p. 308, Theorem 3.2] and [3, p. 117, Theorem 9.3.2], respectively). Here

$$\omega_2(f,\delta) = \sup_{0<h\leq\delta}\; \sup_{x, x\pm h \in [0,1]} |f(x+h) - 2f(x) + f(x-h)|$$

is the usual second moduli of smoothness and

$$\omega_2^{\varphi}(f,\delta) = \sup_{0<h\leq\delta}\; \sup_{x\pm h\varphi(x) \in [0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|,$$

Received by the editors: 04.06.2002.

2000 *Mathematics Subject Classification.* 41A10, 41A36.

Key words and phrases. the first and the second moduli of smoothness, the first and the second modulus of smoothness of Ditzian - Totik.

$\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ is the second modulus of smoothness of Ditzian - Totik. Furthermore, we shall use the first and second moduli of smoothness of a function $g : I \to \Re$ as defined by

$$\omega_1(g,\delta) = \sup_{0<h\leq\delta} \sup_{x,x+h\in I} |g(x+h) - g(x)|,$$

$$\omega_2(g,\delta) = \sup_{0<h\leq\delta} \sup_{x,x\pm h\in I} |g(x+h) - 2g(x) + g(x-h)|,$$

and the following Ditzian - Totik type moduluses of smoothness:

$$\omega_1^{\varphi}(g,\delta) = \sup_{0<h\leq\delta} \sup_{x\pm h\varphi(x)\in[0,1]} \left| g\left(x + \frac{h}{2}\varphi(x)\right) - g\left(x - \frac{h}{2}\varphi(x)\right) \right|,$$
$$g \in C[0,1], \varphi(x) = \sqrt{x(1-x)},$$

$$\omega_2^{\varphi}(g,\delta)_\infty = \sup_{0<h\leq\delta} \sup_{x\pm h\varphi(x)\in[0,\infty)} |g(x+h\varphi(x)) - 2g(x) + g(x-h\varphi(x))|,$$
$$g \in C_B[0,\infty), \varphi(x) = \sqrt{x},$$

where $C_B[0,\infty)$ denotes the set of all bounded and continuous functions on $[0,\infty)$.

The aim of this paper is to establish pointwise and global uniform quantitative estimates for some linear and positive operators using the above mentioned moduluses of smoothness, obtaining estimates similar to (1) and (2). These operators are the following:

1. *Stancu's operator* [9]:

$$S_n^{\alpha}(f,x) = \sum_{k=0}^{n} w_{n,k}(x,\alpha) f\left(\frac{k}{n}\right), \ f \in C[0,1], \ \alpha \geq 0$$

and

$$w_{n,k}(x,\alpha) = \begin{pmatrix} n \\ k \end{pmatrix} \frac{\prod_{i=o}^{k-1} (x+i\alpha) \prod_{j=0}^{n-k-1} (1-x+j\alpha)}{\prod_{r=0}^{n-1} (1+r\alpha)};$$

2. *Lupaş' operator* [5]:

$$\bar{B}_n(f,x) = \frac{1}{B(nx, n-nx)} \cdot \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)\, dt, \qquad x \in (0,1)$$

and $\quad \bar{B}_n(f,0) = f(0), \bar{B}_n(f,1) = f(1);$

3. *Miheşan's operators* [7]:

a) if ${}_2F_1(a,b,c,z)$ is the hypergeometric function and in the integral form

$$ {}_2F_1(a,b,c,z) = \frac{1}{B(a,c-a)} \cdot \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}\, dt, $$

$a,b,c,z \in \Re$, $|z|<1$, $c \neq 0,-1,-2,\dots$ and $c>a>0$ then

$$ F_n^*(f,x) = \sum_{k=0}^{n} \frac{{}_2F_1\left(\frac{x}{\alpha}+k, b, \frac{1}{\alpha}+n, z\right)}{{}_2F_1\left(\frac{x}{\alpha}, b, \frac{1}{\alpha}, z\right)} \cdot w_{n,k}(x,\alpha) \cdot f\left(\frac{k}{n}\right), $$

$f \in C[0,1]$, $x \in [0,1]$, $\alpha > 0$, $b \geq 0$, $0 \leq z < 1$;

b) if ${}_1F_1(a,c,z)$ is the confluent hypergeometric function of the first kind and in the integral form

$$ {}_1F_1(a,c,z) = \frac{1}{B(a,c-a)} \cdot \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}\, dt, $$

$a,c,z \in \Re$, $c \neq 0,-1,-2,\dots$ and $c>a>0$ then

$$ \mathcal{F}_n^*(f,x) = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{\int_0^1 t^{\frac{x}{\alpha}+k-1}(1-t)^{\frac{1-x}{\alpha}+n-k-1}e^{zt}\, f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}e^{zt}\, dt}, $$

$f \in C[0,1]$, $x \in [0,1]$, $\alpha > 0$, $z \geq 0$;

c)

$$ \begin{aligned} L_n^*(f,x) &= e^{-na} \sum_{k=0}^{\infty} \frac{(na)^k}{k!} \cdot \frac{nx(nx+1)\dots(nx+k-1)}{na(na+1)\dots(na+k-1)} \cdot \\ &\quad \cdot {}_1F_1(na-nx, na+k, na) \cdot f\left(\frac{k}{n}\right), \end{aligned} $$

$f \in C[0,\infty)$, $x \in [0,a]$;

d)

$$ \begin{aligned} \tilde{L}_n(f,x) &= \left(\frac{b+c}{c}\right)^{-nx} \sum_{k=0}^{\infty} \frac{b(b+1)\dots(b+k-1)}{c(c+1)\dots(c+k-1)} \cdot \\ &\quad \cdot \frac{nx(nx+1)\dots(nx+k-1)}{k!} \cdot \left(\frac{b}{b+c}\right)^k \cdot \\ &\quad \cdot {}_2F_1\left(nx+k, c-b, c+k, \frac{b}{b+c}\right) \cdot f\left(\frac{k}{n}\right), \end{aligned} $$

$f \in C[0,\infty)$, $x \in [0,\infty)$ $0<b<c$.

4. Furthermore, we define a *generalization of Goodmann and Sharma's operator* as follows:

$$ U_n^{\alpha}(f,x) = f(0)w_{n,0}(x,\alpha) + f(1)w_{n,n}(x,\alpha) + $$

$$+\sum_{k=1}^{n-1} w_{n,k}(x,\alpha) \int_0^1 (n-1)\left(\begin{array}{c} n-2 \\ k-1 \end{array}\right) t^{k-1}(1-t)^{n-1-k} f(t)\,dt,$$

$f \in C[0,1]$, $\alpha \geq 0$.

Remark 1. *a*) For $b = c$ we obtain

$$\tilde{L}_n(f,x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{nx(nx+1)\dots(nx+k-1)}{2^k\ k!}\, f\left(\frac{k}{n}\right).$$

This operator was introduced by Lupaş in [6].

b) Here we mention that throughout this paper C denotes absolute constant and not necessarily the same at each occurrence.

2. **Theorems**

Before we state our results let us observe that the operators introduced in 1), 2), 3*a*), 3*b*) and 4) are generated by special functions. Indeed, if $\mathcal{B}_\alpha : C[0,1] \to C[0,1]$,

$$\mathcal{B}_\alpha(f,x) = \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)\,dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\,dt};$$

$F_{b,z}^\alpha : C[0,1] \to C[0,1]$,

$$F_{b,z}^\alpha(f,x) = \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-b} f(t)\,dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-b}\,dt}$$

and $\quad \mathcal{F}_z^\alpha : C[0,1] \to C[0,1]$,

$$\mathcal{F}_z^\alpha(f,x) = \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} e^{zt} f(t)\,dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} e^{zt}\,dt}$$

then, in view of [9, Theorem 3.1] and [7, Propoziţia 2.18 and Propoziţia 2.19] we have

$$S_n^\alpha(f,x) = \mathcal{B}_\alpha(B_n(f),x); \tag{3}$$

$$U_n^\alpha(f,x) = \mathcal{B}_\alpha(U_n(f),x), \tag{4}$$

where

$$U_n(f,x) = f(0)(1-x)^n + f(1)x^n +$$

$$+\sum_{k=1}^{n-1}\left(\begin{array}{c} n \\ k \end{array}\right) x^k(1-x)^{n-k} \cdot \int_0^1 (n-1)\left(\begin{array}{c} n-2 \\ k-1 \end{array}\right) t^{k-1}(1-t)^{n-1-k} f(t)\,dt,$$

$f \in C[0,1]$, is the *Goodman - Sharma's operator* [8];

$$\bar{B}_n(f,x) = \mathcal{B}_{\frac{1}{n}}(f,x); \tag{5}$$

$$F_n^*(f,x) = F_{b,z}^{\alpha}(B_n(f),x) \tag{6}$$

and

$$\mathcal{F}_n^*(f,x) = \mathcal{F}_z^{\alpha}(B_n(f),x). \tag{7}$$

Furthermore, let us consider the following notations

$$\begin{aligned}\beta(n,x,\alpha,b,z) &= \frac{1}{n}\ (1-z)^{-(b+1)} \cdot \frac{x(1-x)}{1+\alpha} + (1-z)^{-(b+1)} \cdot \frac{\alpha x(1-x)}{1+\alpha} + \\ &+ 2\ (1-z)^{-(b+1)} \cdot \left(1-(1-z)^{2(b+1)}\right) x^2,\end{aligned}$$

$x \in [0,1]$, $\alpha > 0$, $b \geq 0$, $0 \leq z < 1$;

$$\gamma(n,x,\alpha,z) = \frac{1}{n}\ e^z \cdot \frac{x(1-x)}{1+\alpha} + e^z \cdot \frac{\alpha x(1-x)}{1+\alpha} + 2\ e^z\ (1-e^{-2z})\ x^2,$$

$x \in [0,1]$, $\alpha > 0$, $z \geq 0$;

$$\begin{aligned}\beta'(n,\alpha,b,z) &= \frac{1}{4n}\ (1-z)^{-(b+1)} \cdot \frac{1}{1+\alpha} + \frac{1}{4}\ (1-z)^{-(b+1)} \cdot \frac{\alpha}{1+\alpha} + \\ &+ 2\ (1-z)^{-(b+1)} \left(1-(1-z)^{2(b+1)},\right)\end{aligned}$$

$\alpha > 0$, $b \geq 0$, $0 \leq z < 1$ and

$$\gamma'(n,\alpha,z) = \frac{1}{4n}\ e^z \cdot \frac{1}{1+\alpha} + \frac{1}{4}\ e^z \cdot \frac{\alpha}{1+\alpha} + 2\ e^z(1-e^{-2z}),$$

$\alpha > 0$, $z \geq 0$, respectively.

The next theorem contains the local approximation results for the above mentioned operators:

Theorem 1. *For all $f \in C[0,1]$ we have*

a) $|S_n^{\alpha}(f,x) - f(x)| \ \leq\ C\ \omega_2\left(f, \sqrt{\frac{1+n\alpha}{n(1+\alpha)} \cdot x(1-x)}\ \right)$, $\quad x \in [0,1]$;

b) $|U_n^{\alpha}(f,x) - f(x)| \ \leq\ C\ \omega_2\left(f, \sqrt{\left(\frac{2}{n+1} + \alpha\right) \cdot \frac{x(1-x)}{1+\alpha}}\ \right)$, $\quad x \in [0,1]$;

c) $|\bar{B}_n(f,x) - f(x)| \ \leq\ C\ \omega_2\left(f, \sqrt{\frac{x(1-x)}{n+1}}\ \right)$, $\quad x \in [0,1]$;

d) $|F_n^*(f,x) - f(x)| \ \leq\ C\ \omega_1\left(f, \sqrt{\beta(n,x,\alpha,b,z)}\ \right)$, $\quad x \in [0,1]$;

e) $|\mathcal{F}_n^*(f,x) - f(x)| \ \leq\ C\ \omega_1\left(f, \sqrt{\gamma(n,x,\alpha,z)}\ \right)$, $\quad x \in [0,1]$.

For all $f \in C[0,\infty)$ we have

$f)$ $|L_n^*(f,x)-f(x)| \;\le\; C\,\omega_2\left(f,\sqrt{\frac{x}{n}+\frac{x(a-x)}{na+1}}\,\right), \quad x\in[0,a];$

$g)$ $|\tilde{L}_n(f,x)-f(x)| \;\le\; C\,\omega_2\left(f,\sqrt{\frac{x}{n}+\frac{nx^2(c-b)+c(b+1)x}{nb(c+1)}}\,\right), \quad x\in[0,\infty).$

With the notations $\|f\| = \sup\,\{|f(x)| : x\in[0,1]\}$ for $f\in C[0,1]$ and $\|f\|_\infty = \sup\,\{|f(x)| : x\ge 0\}$ for $f\in C_B[0,\infty)$, the global approximation results can be included in the following theorem:

Theorem 2. *For all $f\in C[0,1]$ and $\varphi(x)=\sqrt{x(1-x)}$ we have*

$a)$ $\|S_n^\alpha(f)-f\| \;\le\; C\,\omega_2^\varphi\left(f,\sqrt{\frac{1+n\alpha}{n(1+\alpha)}}\,\right);$

$b)$ $\|U_n^\alpha(f)-f\| \;\le\; C\,\omega_2^\varphi\left(f,\sqrt{\frac{1}{1+\alpha}\left(\frac{2}{n+1}+\alpha\right)}\,\right);$

$c)$ $\|\bar{B}_n(f)-f\| \;\le\; C\,\left\{\omega_2^\varphi\left(f,\sqrt{\frac{1}{n}}\,\right)+\omega_2^\varphi\left(f,\sqrt{\frac{2}{n+1}}\,\right)\right\};$

$d)$ $\|F_n^*(f)-f\| \;\le\; C\,\omega_1^\varphi\left(f,\sqrt[4]{\beta'(n,\alpha,b,z)}\,\right),$

$e)$ $\|\mathcal{F}_n^*(f)-f\| \;\le\; C\,\omega_1^\varphi\left(f,\sqrt[4]{\gamma'(n,\alpha,z)}\,\right).$

For all $f\in C_B[0,\infty)$ and $\varphi(x)=\sqrt{x}$ we have

$f)$ $\|\tilde{L}_n(f)-f\|_\infty \;\le\; C\,\omega_2^\varphi\left(f,\sqrt{\frac{1}{n}}\,\right)_\infty,$ *when $b=c$.*

3. Proofs

Proof of Theorem 1. The statements $a)$, $b)$, $c)$ can be proved with the same method, therefore we shall give the proof for $b)$. In fact $a)$ was proved in [4, Lemma 4], when $0<\alpha(n)\cdot n\le 1$ $(n=1,2,\ldots)$, obtaining the estimate (1) for S_n^α.

At first, let us observe that U_n^α preserves the linear functions. Indeed, by (4), [8, (2.2)] and definition of $\mathcal{B}_\alpha$ we get

$$\begin{aligned} U_n^\alpha(u-x,x) &= \mathcal{B}_\alpha\left(U_n(u-x,t),x\right)\\ &= \mathcal{B}_\alpha\left(U_n((u-t)+(t-x),t),x\right)\\ &= \mathcal{B}_\alpha\left(t-x,x\right)=\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}-x=0. \end{aligned} \tag{8}$$

Moreover, by (4) and [8, (2.2) - (2.3)] we obtain

$$\begin{aligned} U_n^\alpha\left((u-x)^2,x\right) &= \mathcal{B}_\alpha\left(U_n((u-x)^2,t),x\right)\\ &= \mathcal{B}_\alpha\left(U_n((u-t)^2+2(u-t)(t-x)+(t-x)^2,t),x\right)\\ &= \mathcal{B}_\alpha\left(U_n((u-t)^2,t)+(t-x)^2,x\right)\\ &= \mathcal{B}_\alpha\left(\frac{2t(1-t)}{n+1}+t^2-2xt+x^2,x\right)\\ &= \frac{2}{n+1}\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}+1\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+\frac{B\left(\frac{x}{\alpha}+2,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}- \end{aligned}$$

$$-\,2x\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+x^2\cdot\frac{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}=\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}. \tag{9}$$

Finally, by (4) and [8, (2.4)] we get

$$\begin{aligned} |U_n^\alpha(f,x)| &\le \mathcal{B}_\alpha\left(|U_n(f,t)|,x\right)\\ &\le \|U_n(f)\|\cdot\mathcal{B}_\alpha(1,x)=\|U_n(f)\|\le\|f\|. \end{aligned}$$

Thus

$$\|U_n^\alpha(f)\| \le \|f\|. \tag{10}$$

Now, let $g\in C^2[0,1]$. By Taylor's formula we have

$$g(u)=g(x)+(u-x)g'(x)+\int_x^u (u-v)g''(v)\,dv. \tag{11}$$

Hence, by (8) we have

$$U_n^\alpha(g,x)-g(x)=U_n^\alpha\left(\int_x^u (u-v)g''(v)\,dv\ ,\ x\right).$$

Then, by (9)

$$\begin{aligned} |U_n^\alpha(g,x)-g(x)| &\le U_n^\alpha\left(\left|\int_x^u |u-v|\cdot|g''(v)|\,dv\right|,x\right)\\ &\le U_n^\alpha\left((u-x)^2,x\right)\cdot\|g''\|=\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}\cdot\|g''\|. \end{aligned}$$

Hence, by (10)

$$\begin{aligned} |U_n^\alpha(f,x)-f(x)| &\le |U_n^\alpha(f-g,x)-(f-g)(x)|+|U_n^\alpha(g,x)-g(x)|\\ &\le 2\,\|f-g\|+\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}\cdot\|g''\|. \end{aligned}$$

Thus

$$\begin{aligned} |U_n^\alpha(f,x)-f(x)| &\le 2\ \inf_g \left\{ \|f-g\| + \left(\frac{2}{n+1}+\alpha\right)\cdot \frac{x(1-x)}{1+\alpha}\cdot \|g''\| \right\} \\ &= 2\ K_2\left(f, \left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}\right). \end{aligned}$$

Using the equivalence between $K_2(f,\delta)$ and $\omega_2(f,\sqrt{\delta}\ )$ (see [2, p. 177, Theorem 2.4]) we obtain that

$$|U_n^\alpha(f,x)-f(x)| \le C\ \omega_2\left(f, \sqrt{\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}}\right).$$

In view of [7, Lemma 2.22] and [7, (2.50)] we have that L_n^* and $\tilde{L}_n$ preserve the linear functions and

$$L_n^*((u-x)^2,x) = \frac{x}{n}+\frac{x(a-x)}{na+1}$$

and

$$\tilde{L}_n((u-x)^2,x) = \frac{x}{n}+\frac{nx^2(c-b)+c(b+1)x}{nb(c+1)},$$

respectively. Using the same idea as above, we get

$$|L_n^*(f,x)-f(x)| \le C\ \omega_2\left(f, \sqrt{\frac{x}{n}+\frac{x(a-x)}{na+1}}\right)$$

and

$$|\tilde{L}_n(f,x)-f(x)| \le C\ \omega_2\left(f, \sqrt{\frac{x}{n}+\frac{nx^2(c-b)+c(b+1)x}{nb(c+1)}}\right).$$

Thus we have proved the f) and g) statements.

For d) and e) we use the standard method:

$$|f(u)-f(x)| \le \omega_1(f,|u-x|) \le (1+\delta^{-2}(u-x)^2)\ \omega_1(f,\delta),$$

where $u,x\in[0,1]$ and $\delta>0$. Hence

$$|F_n^*(f,x)-f(x)| \le \left[1+\delta^{-2}\cdot F_n^*((u-x)^2,x)\right]\cdot\omega_1(f,\delta) \tag{12}$$

and

$$|\mathcal{F}_n^*(f,x)-f(x)| \le \left[1+\delta^{-2}\cdot \mathcal{F}_n^*((u-x)^2,x)\right]\cdot\omega_1(f,\delta), \tag{13}$$

respectively. Therefore we have to estimate $F_n^*((u-x)^2,x)$ and $\mathcal{F}_n^*((u-x)^2,x)$. These estimates can be found by (6) and (7), if we determine an upper and lower bound for $F_{b,z}^\alpha(f,x)$ and $\mathcal{F}_z^\alpha(f,x)$, respectively.

Let $b > 0$ and $f \geq 0$ on $[0,1]$ (for $b = 0$ we receive back the Stancu's operator using the definition of F_n^*). Then there exists a natural number m such that $m \leq b < m+1$. From $0 < 1-z \leq 1-zt \leq 1$ $(0 \leq z < 1,\ 0 \leq t \leq 1)$ we obtain $(1-zt)^{m+1} < (1-zt)^b \leq (1-zt)^m$. Hence

$$\begin{aligned} F_{b,z}^{\alpha}(f,x) &\leq \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m-1}\, f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m}\, dt} \\ &= \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m+1}\cdot(1-zt)^{-2}\cdot f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m+1}\cdot(1-zt)^{-1}\, dt} \\ &\leq (1-z)^{-2}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m+1}\cdot f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m+1}\, dt}. \end{aligned}$$

Using $(m-1)-$ times the last inequality, we obtain

$$\begin{aligned} F_{b,z}^{\alpha}(f,x) &\leq (1-z)^{-m-1}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt} \\ &\leq (1-z)^{-(b+1)}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt} \end{aligned} \tag{14}$$

In similar way

$$F_{b,z}^{\alpha}(f,x) \geq (1-z)^{b+1}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt}. \tag{15}$$

Analogously, from $1 \leq e^{zt} \leq e^z$ $(z \geq 0,\ 0 \leq t \leq 1\ )$ and $f \geq 0$ on $[0,1]$, we get

$$\mathcal{F}_z^{\alpha}(f,x) \leq e^z\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt} \tag{16}$$

and

$$\mathcal{F}_z^{\alpha}(f,x) \geq e^{-z}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt}, \tag{17}$$

respectively. Now, by (6), (14) and (15) we have

$$
\begin{aligned}
&F_n^*((u-x)^2,x) = \\
&\quad = F_{b,z}^{\alpha}\left(B_n((u-x)^2,t),x\right) \\
&\quad = F_{b,z}^{\alpha}\left(B_n((u-t)^2+2(u-t)(t-x)+(t-x)^2,t),x\right) \\
&\quad = F_{b,z}^{\alpha}\left(B_n((u-t)^2,t)+(t-x)^2,x\right) \\
&\quad = F_{b,z}^{\alpha}\left(\frac{t(1-t)}{n}+t^2-2xt+x^2,x\right) \\
&\quad \le \frac{1}{n}\cdot(1-z)^{-(b+1)}\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}+1\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+(1-z)^{-(b+1)}\cdot\frac{B\left(\frac{x}{\alpha}+2,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}- \\
&\qquad -(1-z)^{b+1}\cdot 2x\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+(1-z)^{-(b+1)}\cdot x^2\cdot\frac{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)} \\
&\quad = \frac{1}{n}\,(1-z)^{-(b+1)}\cdot\frac{x(1-x)}{1+\alpha}+(1-z)^{-(b+1)}\cdot\frac{\alpha x(1-x)}{1+\alpha}+ \\
&\qquad +2\,(1-z)^{-(b+1)}\cdot\left(1-(1-z)^{2(b+1)}\right)\,x^2 \\
&\quad = \beta(n,x,\alpha,b,z). \qquad (18)
\end{aligned}
$$

Hence, by (12) and choosing $\delta^2=\beta(n,x,\alpha,b,z)$ we get for $C=2$

$$
|F_n^*(f,x)-f(x)| \le C\,\omega_1\left(f,\sqrt{\beta(n,x,\alpha,b,z)}\right).
$$

Analogously, by (7), (16) and (17) we have

$$
\begin{aligned}
\mathcal{F}_n^*\left((u-x)^2,x\right) &= \mathcal{F}_z^{\alpha}\left(B_n((u-x)^2,t),x\right) \\
&= \mathcal{F}_z^{\alpha}\left(B_n((u-t)^2,t)+(t-x)^2,x\right) \\
&= \mathcal{F}_z^{\alpha}\left(\frac{t(1-t)}{n}+t^2-2xt+x^2,x\right) \\
&\le \frac{1}{n}\cdot e^z\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}+1\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+e^z\cdot\frac{B\left(\frac{x}{\alpha}+2,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}- \\
&\quad -e^{-z}\cdot 2x\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+e^z\cdot x^2\cdot\frac{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)} \\
&= \frac{1}{n}\cdot e^z\cdot\frac{x(1-x)}{1+\alpha}+e^z\cdot\frac{\alpha x(1-x)}{1+\alpha}+2e^z(1-e^{-2z})x^2 \\
&= \gamma(n,x,\alpha,z). \qquad (19)
\end{aligned}
$$

Hence, by (13) and choosing $\delta^2=\gamma(n,x,\alpha,z)$ we get for $C=2$

$$
|\mathcal{F}_n^*(f,x)-f(x)| \le C\,\omega_1\left(f,\sqrt{\gamma(n,x,\alpha,z)}\right),
$$

which completes the proof of the theorem.

Proof of Theorem 2. For the proof of $a)$ see [1, Theorem A]. The proof of $b)$ is a standard one [3, Chapter 9]: using (11), (8), [3, p. 141, (9.6.1)] and (9), we obtain for $g \in C^2[0,1]$:

$$\begin{aligned} |U_n^\alpha(g,x) - g(x)| &\leq U_n^\alpha \left(\left| \int_x^u \frac{|u-v|}{\varphi^2(v)} \cdot |\varphi^2(v) g''(v)| \, dv \; \right|, x \right) \\ &\leq \left(\frac{2}{n+1} + \alpha \right) \cdot \frac{1}{1+\alpha} \cdot \|\varphi^2 g''\|. \end{aligned}$$

Hence, by (10), we have

$$\begin{aligned} |U_n^\alpha(f,x) - f(x)| &\leq |U_n^\alpha(f-g,x) - (f-g)(x)| + |U_n^\alpha(g,x) - g(x)| \\ &\leq 2\,\|f-g\| + \left(\frac{2}{n+1} + \alpha \right) \cdot \frac{1}{1+\alpha} \cdot \|\varphi^2 g''\|. \end{aligned}$$

Using [3, p. 11, Theorem 2.1.1] we obtain

$$\|U_n^\alpha(f) - f\| \leq C\, \omega_2^\varphi \left(f, \sqrt{\frac{1}{1+\alpha}\left(\frac{2}{n+1} + \alpha \right)} \right).$$

For $c)$ we can write:

$$\|\bar{B}_n(f) - f\| \leq \|\bar{B}_n(f) - S_n^{\frac{1}{n}}(f)\| + \|S_n^{\frac{1}{n}}(f) - f\|.$$

On the other hand, by (3), (5), (2) and $a)$, we have

$$\begin{aligned} &|\bar{B}_n(f,x) - S_n^{\frac{1}{n}}(f,x)| \leq \\ &\leq \frac{1}{\int_0^1 \; t^{nx-1}(1-t)^{n(1-x)-1}\, dt} \cdot \int_0^1 \; |f(t) - B_n(f,t)| \cdot t^{nx-1}(1-t)^{n(1-x)-1}\, dt \\ &\leq \|f - B_n(f)\| \leq C\, \omega_2^\varphi \left(f, \sqrt{\frac{1}{n}} \right) \end{aligned}$$

and

$$\|S_n^{\frac{1}{n}}(f) - f\| \leq C\, \omega_2^\varphi \left(f, \sqrt{\frac{2}{n+1}} \right).$$

In conclusion

$$\|\bar{B}_n(f) - f\| \leq C \left\{ \omega_2^\varphi \left(f, \sqrt{\frac{1}{n}} \right) + \omega_2^\varphi \left(f, \sqrt{\frac{2}{n+1}} \right) \right\}.$$

For the proof of $d)$ and $e)$ we use

$$|f(u) - f(x)| \leq \omega_1(f, |u-x|) \leq \left(1 + \delta^{-4}(u-x)^2\right) \omega_1(f, \delta^2),$$

where $u, x \in [0,1]$ and $\delta > 0$. But, in view of [3, p. 25, Corollary 3.1.3] we have

$$\omega_1(f,\delta^2) \ \le \ C\ \omega_1^{\varphi}(f,\delta),$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. So

$$|f(u) - f(x)| \ \le \ C\ \left(1 + \delta^{-4}(u-x)^2\right) \omega_1^{\varphi}(f,\delta).$$

Hence

$$|F_n^*(f,x) - f(x)| \ \le \ C\ \left[1 + \delta^{-4} F_n^*((u-x)^2, x)\right] \omega_1^{\varphi}(f,\delta) \tag{20}$$

and

$$|\mathcal{F}_n^*(f,x) - f(x)| \ \le \ C\ \left[1 + \delta^{-4} \mathcal{F}_n^*((u-x)^2, x)\right] \omega_1^{\varphi}(f,\delta). \tag{21}$$

By (18), (19) and $\beta(n,x,\alpha,b,z) \le \beta'(n,\alpha,b,z)$, $\gamma(n,x,\alpha,z) \le \gamma'(n,\alpha,z)$ for all $x \in [0,1]$, we obtain

$$F_n^*((u-x)^2, x) \ \le \ \beta'(n,\alpha,b,z)$$

and

$$\mathcal{F}_n^*((u-x)^2, x) \ \le \ \gamma'(n,\alpha,z).$$

In conclusion, by (20) and choosing $\delta^4 = \beta'(n,\alpha,b,z)$ we get for $C = 2$ the assertion $d)$ of Theorem 2, and, by (21) and $\delta^4 = \gamma'(n,\alpha,z)$ we obtain for $C = 2$ the assertion $e)$ of Theorem 2.

For $f)$ we use again the standard method [3, Chapter 9]: if $g \in C_B[0,\infty)$ is twice differentiable such that $g', \varphi^2 g'' \in C_B[0,\infty)$ then, by [3, p. 141, (9.6.1)] and [7, (2.50)] for $b = c$, we have

$$\begin{aligned} |\tilde{L}_n(g,x) - g(x)| \ &\le \ \tilde{L}_n\left(\left| \int_x^u \frac{|u-v|}{v} \cdot |v g''(v)|\, dv \right|, x\right) \\ &\le \ \tilde{L}_n\left(\frac{(u-x)^2}{x}, x\right) \cdot \|\varphi^2 g''\|_\infty = \frac{2}{n} \cdot \|\varphi^2 g''\|_\infty. \end{aligned}$$

Because $\tilde{L}_n(1,x) = 1$ (see [7, (2.50)]) we get $\|\tilde{L}_n(f)\|_\infty \le \|f\|_\infty$, $f \in C_B[0,\infty)$. Thus

$$\begin{aligned} \|\tilde{L}_n(f) - f\|_\infty \ &\le \ \|\tilde{L}_n(f-g) - (f-g)\|_\infty + \|\tilde{L}_n(g) - g\|_\infty \\ &\le \ 2\ \|f-g\|_\infty + \frac{2}{n}\ \|\varphi^2 g''\|_\infty. \end{aligned}$$

Hence

$$\|\tilde{L}_n(f) - f\|_\infty \ \le \ 2\ K_2^{\varphi}\left(f, \frac{1}{n}\right)_\infty \ \le \ C\ \omega_2^{\varphi}\left(f, \sqrt{\frac{1}{n}}\right)_\infty$$

(see [3, p. 11, Theorem 2.1.1] for the equivalence between $K_2^{\varphi}\left(f, \frac{1}{n}\right)_{\infty}$ and $\omega_2^{\varphi}\left(f, \sqrt{\frac{1}{n}}\right)_{\infty}$).

References

[1] Della Vecchia, B. and Mache, D. H. : *On approximation properties of Stancu - Kantorovich operators*, Rev. Anal. Numér. Théorie Approximation, 27 (1) (1998), 71 - 80.
[2] DeVore, R. A. and Lorentz, G. G. : *Constructive Approximation*, Springer - Verlag, Berlin Heidelberg New York, 1993.
[3] Ditzian, Z. and Totik, V. : *Moduli of Smoothness*, Springer - Verlag, Berlin Heidelberg New York London, 1987.
[4] Finta, Z. : *On some properties of Stancu operator*, Rev. Anal. Numér. Théorie Approximation, 27 (1)(1998), 99 - 106.
[5] Lupaş, A. : *Contribuţii la teoria aproximării prin operatori liniari*, Ph. D. Thesis, Cluj - Napoca, 1975.
[6] Lupaş, A. : *The approximation by some positive linear operators*, Proceedings of the International Dortmund Meeting on Approximation Theory (Editors M. W. Müller et al.), Akademie Verlag, Berlin, 1995, 201 - 229.
[7] Miheşan, V. : *Aproximarea funcţiilor continue prin operatori liniari şi pozitivi*, Ph. D. Thesis, Cluj - Napoca, 1997.
[8] Parvanov, P. E. and Popov, B. D. : *The limit case of Bernstein's operators with Jacobi - weights*, Mathematica Balkanica, 8 (2 - 3) (1994), 165 - 177.
[9] Stancu, D. D. : *Approximation of functions by a new class of linear polynomial operators*, Rev. Roumaine Math. Pures Appl., 13 (8) (1968), 1173 - 1194.

Babeş-Bolyai University, Department of Mathematics,
1, M. Kogălniceanu st., 3400 Cluj, Romania
E-mail address: fzoltan@math.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 3, September 2002

MAXIMAL SETS ON A HYPERSPHERE

VASILE POP

Abstract. It is studied the problem of the maximum number of points situated on a hypersphere of radius 1 with the property that the distances between any two points is at least r. It is solved the case $r = \sqrt{2}$.

1. **Introduction**

The goal of this paper is to find the maximum number of points of hypersphere, such that the distances between every two points is great that a given number. The solution of the problem in the general case is very difficult. we solved the problem in a remarcable particular case.

Let $S_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n |\ x_1^2 + \cdots + x_n^2 = 1\}$ be the unit sphere in $\mathbb{R}^n$. For every real number $r \in [0, 2]$ we define the natural numbers $N(n, r)$ and $\overline{N}(n, r)$ by: $N(n, r)$ is the maximum number of elements of a set $M \subseteq S_{n-1}$ with the property that the distance $d(A, B)$ between every two points $A, B \in M$ satisfies the relation $d(A, B) > r$.

$\overline{N}(n, r)$ is the maximum numbers of elements of a set $M \subseteq S_{n-1}$ with the property $d(A, B) \geq r$ for every $A, B \in M$.

We think that the determination of a general expression for the functions $N, \overline{N} : \mathbb{N}^* \times [0, 2] \to \mathbb{N}$ is not possible. We solve the problem for $r = \sqrt{2}$. The following properties of N and $\overline{N}$ are easy to verify.

1. $N(n, r) \leq \overline{N}(n, r)$;
2. $N(n, r) \leq N(n + 1, r)$;
3. $\overline{N}(n, r) \leq \overline{N}(n + 1, r)$;
4. $N(n, r_1) \leq N(n, r_2)$ for $n_1 > n_2$;
5. $\overline{N}(n, r_1) \leq \overline{N}(n, r_2)$ for $r_1 > r_2$;
6. $N(1, r) = 2$ for $r \in [0, 2)$.

Received by the editors: 05.02.2002.

7. $N(1,2)=0$;

8. $\overline{N}(1,r)=2$;

9. $\overline{N}(2,r)=\left[\dfrac{\pi}{\arcsin\dfrac{r}{2}}\right]$;

10. $N(2,r)=\overline{N}(2,r)$, if $\dfrac{\pi}{\arcsin\dfrac{r}{2}}\notin\mathbb{N}$;

11. $N(2,r)=\overline{N}(2,r)-1$, if $\dfrac{\pi}{\arcsin\dfrac{r}{2}}\in\mathbb{N}$;

12. $N(n,2)=0$, $\overline{N}(n,2)=2$.

Theorem 1. *For every natural number $n\geq 1$ we have*

$$N(n,\sqrt{2})=n+1 \ \textit{and}\ \overline{N}(n,\sqrt{2})=2n.$$

Proof. The distance between the points $X=(x_1,\dots,x_n)$ and $Y=(y_1,\dots,y_n)$ is:

$$d^2(X,Y)=\sum_{k=1}^{n}(x_k-y_k)^2.$$

We have

$$\begin{aligned}
&d(X,Y)>\sqrt{2}\Leftrightarrow d^2(X,Y)>2\\
&\Leftrightarrow\sum_{k=1}^{n}x_k^2+\sum_{k=1}^{n}y_k^2+2\sum_{k=1}^{n}x_ky_k>2\\
&\Leftrightarrow\sum_{k=1}^{n}x_ky_k<0 \qquad (1)
\end{aligned}$$

Taking account of the symmetry of the sphere, we can suppose that

$$A_1=(-1,0,\dots,0).$$

For $X=A_1$, condition (1) for implies $y_1>0$, $\forall\ Y\in M_n$.

Let $X=(x_1,\overline{X})$, $Y=(y_1,\overline{Y})\in M_n\setminus\{A_1\}$, $\overline{X},\overline{Y}\in\mathbb{R}^{n-1}$.

We have

$$\sum_{k=1}^{n}x_ky_k<0\Rightarrow x_1y_1+\sum_{k=1}^{n-1}\overline{x}_k\overline{y}_k<0\Leftrightarrow\sum_{k=1}^{n-1}x'_ky'_k<0,$$

where

$$x'_k=\frac{\overline{x}_k}{\sqrt{\sum\overline{x}_k^2}},\quad y'_k=\frac{\overline{y}_k}{\sqrt{\sum\overline{y}_k^2}},$$

therefore

$$(x'_1, \ldots, x'_{n-1}), (y'_1, \ldots, y'_{n-1}) \in S_{n-2}$$

and verify condition (1).

If a_n is the search number of points in $\mathbb{R}^n$, we obtain $a_n \leq 1 + a_{n-1}$ and $a_1 = 2$ implies that $a_n \leq n + 1$.

We show that $a_n = n+1$, giving an example of a set M_n with $(n+1)$ elements satisfying the conditions of the problem.

$$A_1 = (-1, 0, 0, 0, \ldots, 0, 0)$$

$$A_2 = \left(\frac{1}{n}, -c_1, 0, 0, \ldots, 0, 0\right)$$

$$A_3 = \left(\frac{1}{n}, \frac{1}{n-1}c_1, -c_2, 0, \ldots, 0, 0\right)$$

$$A_4 = \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-1}c_2, -c_3, \ldots, 0, 0\right)$$

$$A_{n-1} = \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, -c_{n-2}, 0\right)$$

$$A_n = \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, \frac{1}{2}c_{n-2}, -c_{n-1}\right)$$

$$A_{n+1} = \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, \frac{1}{2}c_{n-2}, c_{n-1}\right)$$

where

$$c_k = \sqrt{\left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n-k+1}\right)}, \quad k = \overline{1, n-1}.$$

We have

$$\sum_{k=1}^{n} x_k y_k = -\frac{1}{n} < 0 \text{ and } \sum_{k=1}^{n} x_k^2 = 1, \ \forall\ X, Y \in \{A_1, \ldots, A_{n+1}\}.$$

This points are on the unit sphere in $\mathbb{R}^n$ and the distance between any two points are equal to

$$d = \sqrt{2}\sqrt{1 + \frac{1}{n}} > \sqrt{2}.$$

Remark. For $n = 2$ the points form an equilateral triangle in the unit circle; for $n = 3$ the four points from a regular tetrahedron and in $\mathbb{R}^n$ the points from an n dimensional regular simplex.

For the function $\overline{N}$ we have $\overline{N}(1, \sqrt{2}) = 2$.

$$(M = \{-1, 1\}, \quad \overline{N} = (2, \sqrt{2}) = 4), \quad (M = \{(-1, 0), (1, 0), (0, -1), (0, 1)\})$$

By induction, intersecting the hypersphere S_n from $\mathbb{R}^{n+1}$ with the hyperplane $x_{n+1} = 0$ we obtain the hypersphere S_n, which contains a maximal set with $2n$ points and considering the points $(0, \ldots, 0, -1)$ and $(0, \ldots, 0, 1)$ on S_n we obtain a maximal set with $2(n+1)$ points, hence $\overline{N}(n+1, \sqrt{2}) = 2(n+1)$.

We remark that a maximal set for $\overline{N}(n, \sqrt{2})$ is

$$M = \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1), (-1, 0, \ldots, 0),$$

$$(0, -1, 0, \ldots, 0), \ldots, (0, \ldots, 0, -1)\}$$

with n distances equal with 2 and the rest of $C_{2n}^2 - n = 2n(n-1)$ distances equal with $\sqrt{2}$.

It is known that every real euclidean n-dimensional space is isomorphic with $\mathbb{R}^n$ and we can transpose the results by the isomorphism.

If $(V, \langle \cdot, \cdot \rangle)$ is an real euclidean space, by the theorem 1, we have the following consequences.

Proposition 1. If the dimension of V is n, then for any $(n+2)$ vectors with norm 1, there exists two with the distances is at most $\sqrt{2}$.

Proposition 2. If the dimension of V is n, then for any $(n+2)$ nonzero vectors there exists two vectors with an acute angle

$$\left(d(X, Y) \leq \sqrt{2}, \quad \|X\| = \|Y\| = 1 \Leftrightarrow \langle X, Y \rangle \geq 0 \Leftrightarrow \widehat{X, Y} \leq \frac{\pi}{2} \right)$$

Proposition 3. If in euclidean space V there exists a set of $(n+1)$ vectors $\{X_1, \ldots, X_n, X_{n+1}\}$ with the property $\langle X_i, X_j \rangle < 0$, for any $i \neq j$, $i, j = \overline{1, n}$, then the dimension of V is $\dim V \geq n$.

2. **Applications**

Problem 1. Let $n \in \mathbb{N}^*$ be a natural number. Find all $m \in \mathbb{N}^*$ so that there exists a matrix $A \in \mathcal{M}(m, n)(\mathbb{R})$ with the property that all elements of the matrix $A \cdot A^t$, outside the main diagonal to be negative numbers.

Solution. Denote by $L_1, \ldots, L_m \in \mathbb{R}^n$ the lines of matrix A. The element b_{ij} of the matrix $B = A \cdot A^t$ is the inner product $\langle L_i, L_j \rangle$, so the condition is that for $i \neq j$ to have $\langle L_i, L_j \rangle < 0$. From proposition 3 we obtain $m \leq n+1$.

Problem 2. If $\{P_1, \dots, P_n, P_{n+1}\}$ is a set of polynomials with real coefficients so that:

$$\int_0^1 P_i(x)P_j(x)dx < 0, \text{ for any } i \neq j,$$

then at least one polynomial has the degree at least $(n-1)$.

Solution. Denote by V the real vector space generated by the polynomials $P_1, \dots, P_n, P_{n+1}$, and on V define the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Using the proposition 3, it result that $\dim V \geq n$. If $degP_k < n-1$ for all $k = \overline{1, n+1}$, then V is a subspace in the space of polynomials with degree $\leq n-2$, with the dimension $(n-1)$. We obtain the contradiction $n \leq n-1$.

Problem 3. Show that the inequalities

$$a_1a_2 + b_1b_2 < 0$$

$$a_1a_3 + b_1b_3 < 0$$

$$a_1a_4 + b_1b_4 < 0$$

$$a_2a_3 + b_2b_3 < 0$$

$$a_2a_4 + b_2b_4 < 0$$

$$a_3a_4 + b_3b_4 < 0$$

does not hold simultaneously.

Solution. In euclidean plane $\mathbb{R}^2$ consider the points $A_i(a_i, b_i)$, $i = \overline{1, 4}$. The condition $a_ia_j + b_ib_j < 0$ is equivalent with the angle $\widehat{A_iOA_j} > \dfrac{\pi}{2}$, which is impossible for every $i \neq j$.

Remark. Another remarkable value for r is $r = 1$. We have not succeed to find $\overline{N}(n, 1)$ but we suppose that $\overline{N}(n, 1) = n(n+1)$.

References

[1] I. Tomescu, *Introduction a la combinatorique*, Ed. Tehnică, Bucureşti, 1972.

[2] I.M. Glaman, Iu.I. Liubici, *Analiză liniară pe spaţii finit dimensionale*, Ed. Şt. şi Enciclopedică, Bucureşti, 1980.

Department of Mathematics, Technical University,
Str. C. Daicoviciu 15, Cluj-Napoca, Romania

CRITICAL SETS OF 1-DIMENSIONAL MANIFOLDS

LIANA ŢOPAN

Abstract. In this paper we give characterizations for the critical sets of the 1-dimensional manifolds. Given a non-empty set $K \subset M$, with M a smooth manifold of dimension 1, is K the set of critical points for some smooth function $f : M \to \mathbb{R}$?

1. Introduction

Let M be a smooth 1-dimensional manifold and $f : M \to \mathbb{R}$ a smooth function. The point $p \in M$ is a *critical point* of f if, for some chart (U, φ) around p, $\varphi(p)$ is a critical point of the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$, i.e. $\text{rang}_{\varphi(p)} f \circ \varphi^{-1} = 0$, or $(f \circ \varphi^{-1})'(\varphi(p)) = 0$. Otherwise, p will be a *regular* point of f. The set of all critical points of f is called the *critical set* of f and will be denoted by $C(f)$. The number $y_0 \in \mathbb{R}$ is a *critical value* of f if it is the image of a critical point and a *regular value* if it is the image of a regular point. The set of critical values of f is called the *bifurcation set* of f and is denoted by $B(f)$. A set $C \subset M$ is called *critical* if it is the critical set of some smooth function $f : C \to \mathbb{R}$; $C = C(f)$. C is *properly critical* if f can be chosen to be proper.

If $M = \mathbb{R}$, the atlas which gives the structure of M has one single chart $(\mathbb{R}, 1_{\mathbb{R}})$. In this case, $x \in C(f)$ if and only if $f'(x) = 0$. The following theorem [To-An] characterizes the critical sets of $\mathbb{R}$.

Theorem 1.1. *$C \subset \mathbb{R}$ is critical if and only if C is closed.*

It follows that any finite union of closed bounded intervals (some of them might be degenerated to a point), together with two closed unbounded intervals, one

Received by the editors: 06.06.2002.

2000 *Mathematics Subject Classification.* 57R45, 58K05, 58E05, 58C25.

Key words and phrases. critical point, critical set

This paper is partially supported by the European Research Training Network "GEOMETRIC ANALYSIS".

of them to $-\infty$ and the other to $+\infty$, is a critical set. Also, any Cantor (real) set, beeing closed, will be critical.

For the case $M = \mathbb{R}$, there are no other requirements for the set C to be critical, except to be closed. This is, in fact, the minimal condition for a set to be critical (it is easy to see that any critical set is closed). If we impose some supplementary conditions on C, it will become properly critical.

Theorem 1.2. *Let C be a subset of $\mathbb{R}$. If C is compact, C is properly critical.*

Proof. C being compact, it is closed, so critical. C is bounded, and there is some $r > 0$ with $C \subset (-r, r)$. Choose $R > r$. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth positive function which satisfies

1. $g(x) = 1, \quad \forall\, x \in (-r, r)$
2. $g(x) = 0, \quad \forall\, x \in (-\infty, -R) \cup (R, +\infty)$
3. $0 \leq g(x) \leq 1, \quad \forall\, x \in \mathbb{R}$. (see [To-An]).

A theorem of Whitney provides that any closed subset of $\mathbb{R}$ is the set of the zeros of a smooth positive real function (see [An-To]) and let $f : \mathbb{R} \to \mathbb{R}$ have this property : $C = f^{-1}(0)$. Define $h : \mathbb{R} \to \mathbb{R}$, by

$$h(x) = f(x)g(x) + e^{|x|}(1 - g(x)).$$

h is smooth on $\mathbb{R} \setminus \{0\}$. For $x \in (-r, r)$, since $g(x) = 1$, then $h(x) = f(x)$ and h is smooth on $(-r, r)$, which is an open neighborhood of 0. It follows that h is smooth on the entire $\mathbb{R}$.

It is easy to verify that $h^{-1}(0) = C$. For $x_0 \in C$, since $x_0 \in (-r, r)$, then $g(x_0) = 1$ and $h(x_0) = f(x_0) = 0$. For $h(x_0) = 0$, since $f(x) \geq 0$, $e^{|x|} > 0$ and $0 \leq g(x) \leq 1$ for all x, then $f(x_0)g(x_0) = e^{|x_0|}(1 - g(x_0)) = 0$, so $f(x_0) = 0$ and $g(x_0) = 1$, which means that $x_0 \in f^{-1}(0) = C$.

Let $H : \mathbb{R} \to \mathbb{R}$ be the function given by $H(x) = \int\limits_0^x h(t)dt$. Obviously, $C(H) = C$. To prove that H is a proper function, it is enough to verify that $|H(x)| \to \infty$ as $|x| \to \infty$ (see [Ra]).

For $x > R$, we have

$$H(x) = \int_0^x h(t)dt = \int_0^R h(t)dt + \int_R^x h(t)dt = \int_0^R h(t)dt + \int_R^x e^t dt =$$

$$= \int_0^R h(t)dt + e^x - e^R = e^x + \int_0^R h(t)dt - e^R$$

so $\lim\limits_{x\to\infty} H(x) = \infty$.

For $x < -R$, we have

$$H(x) = -\int_x^0 h(t)dt = -\int_x^{-R} h(t)dt - \int_{-R}^0 h(t)dt =$$

$$= -\int_{-R}^0 h(t)dt - \int_{-R}^0 e^{-t}dt = -\int_{-R}^0 h(t)dt + e^R - e^{-x}$$

so $\lim\limits_{x\to-\infty} H(x) = -\infty$.

It follows that C is the critical set of the smooth and proper function H, so C is properly critical. $\square$

The converse of the above theorem is not true. There are smooth proper functions $f : \mathbb{R} \to \mathbb{R}$, whose critical sets are not compact. For example, $f(x) = x + \sin x$, whose critical set is $C(f) = \{(2k+1)|\ k \in \mathbb{Z}\}$, discrete and unbounded in $\mathbb{R}$, so non-compact.

2. Critical Sets on 1-Dimensional Manifolds

Using the characterization of the connected and compact 1-dimensional manifolds, it follows that it is enough to study the critical sets of the interval $[0, 1]$ on the real axis and of the circle S^1 on the plane.

Let M be a smooth 1-dimensional manifold, connected and compact (with or without boundary). According to a theorem of Whitney , M can be properly embedded in $\mathbb{R}^3$ (i.e. there exists an injective and proper immersion $i : M \hookrightarrow \mathbb{R}^3$). Also, there exists $f : M \to \mathbb{R}$ smooth, which is a Morse function (f is said to be a *Morse function* if its critical points are all non-degenerated. The critical points of a Morse function are, also, isolated in M).

Let $S = C(f) \cup \partial M$. As M is of dimension 1, ∂M will be either a smooth compact 0-manifold without boundary, or the empty set. Anyway, ∂M will be at the most a finite union of points. Also, from the compactness of M it follows that $C(f)$ is finite, too, $C(f)$ beeing a discrete subset of a compact. So S is finite and $M \setminus S$ has a finite number of components $L_1, \ldots, L_N$, which are smooth 1-dimensional manifolds.

Proposition 2.1. *f is a diffeomorphism between each L_i and an open interval of $\mathbb{R}$.*

Proof. Let L be one of the manifolds L_i. For all $x \in L$, we have $(df)_x = (df_{|L})_x \neq 0$, so f is a local diffeomorphism on L. Since L is connected, it follows that $f(L)$ is a connected open set. But $f(L)$ is contained in the compact $f(M)$, so $f(L)$ is an open interval (a, b).

We prove now that f is injective on L, and then $f_{|L}$ will be a diffeomorphism. Let $p \in L$ and $c = f(p) \in (a, b)$. Let Q be the set of all points $q \in L$ with the property that there is an arc $\gamma : [c, d] \to L$ joining q and p, $\gamma(c) = p$, $\gamma(d) = q$ and $(f \circ \gamma)(t) = t$, for all $t \in [c, d]$. Since $p \in Q$, then Q is non-empty.

Q is an open set of L: Let $q \in Q$. There is an arc $\gamma : [c, d] \to L$ such that $\gamma(c) = p$, $\gamma(d) = q$ and $(f \circ \gamma)(s) = s$, for s in the interval $[c, d]$. But f beeing a local diffeomorphism in q, there exists a neighborhood V_q of q for which $f_{|V_q} : V_q \to f(V_q)$ is a diffeomorphism. We may choose V_q to be an open connected subset of M. Then $f(V_q) = (c', c'')$, with $a < c' < d < c'' < b$. It follows that γ and $(f_{|V_q})^{-1}$ coincide on

$(c', d]$ and γ can be extended on $[d, c'')$ such that it coincides with $(f_{|V_q})^{-1}$. It follows that any point of V_q can be joined to p, so $V_q \subset Q$ and Q is open in L.

Q is closed in L: It is enough to show that $L \setminus Q$ is open. Let $l \in L \setminus Q$. Then l cannot be joined to p with the requiered conditions. As before, there exists a neighborhood V_l of l with $f_{|V_l} : V_l \to f(V_l)$ diffeomorphism, V_l open and connected and $f(V_l) = (c', c'')$. Suppose there exists a point $q \in V_l$ which can be joined to p. Take $V_q \subset Q$ a neighborhood of q. Every point in $V_l \cap V_q$ can be joined to p, because of V_q and, the same time, cannot be joined to p, beeing on V_l. So, in fact, no point of V_l can be joined to q, which means that $V_l \subset L \setminus Q$, and $L \setminus Q$ is open.

Since L is connected, then $Q = L$. Let $p \neq q$, $p, q \in L$. We showed that there is an arc $\gamma : [c, d] \to L$, with $\gamma(c) = p$, $\gamma(d) = q$ and $(f \circ \gamma)(t) = t$, for all $t \in [c, d]$. We have:

$$f(p) = f(\gamma(c)) = (f \circ \gamma)(c) = c \quad \text{and}$$

$$f(q) = f(\gamma(d)) = (f \circ \gamma)(d) = d,$$

so $f(p) \neq f(q)$, which shows that f is non-injective, so f is a diffeomorphism between L and the open interval (a, b). $\square$

Since every L_i is diffeomorphic to an open interval, then $\overline{L}_i \setminus L_i$ has at the most two points, $\forall\ i = \overline{1, N}$. We can suppose that for all $i = \overline{1, N}$, $\overline{L}_i \setminus L_i$ has exactly two points. Indeed, since L_i is diffeomorphic to an open interval, then $\overline{L}_i \setminus L_i$ has at least one point, and if $\overline{L}_i \setminus L_i$ has exactly one point, it could be only for the case when $N = 1$ and $M = S^1$.

A point $p \in S$ is either a point of the boundary of M, or the intersection point of the boundaries of two sets $\overline{L}_i$ and $\overline{L}_j$. It cannot be the intersection point of three sets $\overline{L}_i$, $\overline{L}_j$ and $\overline{L}_k$, since M is 1-dimensional and the situation below cannot happen.

We call $L_1, \ldots, L_k$ a *chain* if for all $j = \overline{1, k-1}$, $\overline{L}_j$ and $\overline{L}_{j+1}$ have exactly one single intersection point p_j (which belongs to both boudaries). Denote by p_0 the other boundary point of L_1 and by p_k the other boundary point of L_k. Since we have a finite number of L_i, there is a maximal chain, to which we cannot add an other L_i.

Proposition 2.2. *If $L_1, \ldots, L_k$ is a maximal chain, it contains all L_i, $i = \overline{1, N}$. If $\overline{L}_0$ and $\overline{L}_k$ have an intersection point (which will belong to both boundaries), then M is diffeomorphic to a circle. Otherwise, M is diffeomorphic to a closed interval of $\mathbb{R}$.*

Proof. Let us suppose that there exists some L_i which does not belong to the maximal chain. We denote it by L. $\overline{L}$ cannot contain p_0 or p_k, since the chain cannot be extended. $\overline{L}$ contains none of the points p_i, $i = \overline{1, k-1}$, since L_i, L_{i+1} and L would have a common boundary point. It follows that $\overline{L}$ does not intersect $\bigcup\limits_{i=1}^{k} \overline{L}_i$, which is a contradiction to the connectivity of M.

We prove now the second part of the proposition. We construct the requiered diffeomorphisms by using the following lemma:

Lemma 2.3. *Let $g : [a, b] \to \mathbb{R}$ be continuous, smooth on $[a, b] \setminus \{c\}$ and such that $g' > 0$, for all $x \in [a, b] \setminus \{c\}$. Then there exists a smooth map $\breve{g} : [a, b] \to \mathbb{R}$ which agrees to g in a neighborhood of the points a and b and whose derivative is positive on $[a, b]$.*

Sketch of the proof: Let g be a smooth non-negative function, which vanishes outside (a, b), is equal to 1 in a neighbourhood of c and satisfies $\int\limits_a^b g(t)dt = 1$. Define

$\tilde{g} : [a, b] \to \mathbb{R}$, by

$$\tilde{g}(x) = g(a) + \int_a^x [kg(t) + g'(t)(1 - g(t))]dt,$$

with

$$k = g(b) - g(a) - \int_a^b g'(t)(1 - g(t))dt$$

a strictly positive constant. □

The restriction of f to any L_i is a diffeomorphism. The monotony of f could change when f passes through a boundary point of $\overline{L}_i$. To avoid this inconvenient, we use a technical trick. Let $f(p_j) = a_j$. Then $f_{|L_j}$ is a diffeomorphism between L_j and the interval (a_{j-1}, a_j) (or (a_j, a_{j-1})). For each $j = \overline{1, k}$, choose an affine map $\tau_j : \mathbb{R} \to \mathbb{R}$ such that $\tau_j(a_{j-1}) = j - 1$ and $\tau_j(a_j) = j$ (the map τ_j is of the form $t \to \alpha t + \beta$, $\alpha, \beta \in \mathbb{R}$). Let $f_j : \overline{L}_j \to [j - 1, j]$ be the maps given by $f_j = \tau_j \circ f$.

If $a_0 \neq a_k$, the maps f_j will agree on every common point of their domains. We may construct the map $F : M \to [0, k]$, having the following properties:

1. $F_{|\overline{L}_j} = f_j$
2. F is continuous on M
3. F is a diffeomorphism on $M \setminus \{p_1, \dots, p_{k-1}\}$

By using Lemma 2.3, f can be chosen to be a diffeomorphism on M.

If $a_0 = a_k$, let $g_j = \exp [i\frac{2\pi}{k}f_j]$. We may define now $G : M \to S^1$, such that:

1. $G_{|\overline{L}_j} = g_j$
2. G is continuous on M
3. G is a diffeomorphism on $M \setminus \{p_1, \dots, p_{k-1}\}$

Again, G can be made to be a global diffeomorphism. □

We obtained

Theorem 2.4. (the classification of connected compact 1-manifolds) *Any smooth connected and compact* 1*-dimensional manifold is diffeomorphic either to* S^1*, or to the interval* $[0, 1]$.

The last theorem provides that it is enough to find the critical sets of S^1 and of $[0, 1]$.

Theorem 2.5. *Let* $K \subset [0,1]$*. Then* K *is critical in* $[0,1]$ *if and only if* K *is closed in* $[0,1]$*.*

Proof. Any critical set is closed. Conversely, let K be a closed subset of $[0,1]$. Since $[0,1]$ is closed in $\mathbb{R}$, then K is closed in $\mathbb{R}$. According to Theorem 1.1, there is a smooth function $f : \mathbb{R} \to \mathbb{R}$ with $C(f) = K$. Let $g : [0,1] \to \mathbb{R}$, $g = f_{|[0,1]}$. g is smooth and $C(g) = K$. $\square$

Theorem 2.6. *Let* $K \subset S^1$*. Then* K *is critical in* S^1 *if and only if* K *is closed in* S^1*.*

Proof. If K is critical, K is closed. Conversely, let K be a closed subset of S^1. Suppose that $K \neq S^1$ (S^1 is the critical set of any constant function defined on S^1). K is a compact subset of the plane and the only component of its complement is multiply connected. Using the characterisation of the critical sets of the plane given by Norton and Pugh [No-Pu], it follows that K is the critical set of a smooth map $f : \mathbb{R}^2 \to \mathbb{R}$. $C(f) = K$. Then K will be the critical set of $f_{|S^2} : S^2 \to \mathbb{R}$. $\square$

References

[Gu-Po] Guillemin V., Pollack A., Differential Topology, Prentice Hall, Inc., Englewood Cliffs New-Jersey, 1977.

[No-Pu] Norton A., Pugh Ch., Critical Sets in the Plane, Michigan Journal of Math., 38, 1991, 441-459.

[Ra] Rădulescu S., Rădulescu M., Teoreme şi probleme de analiză matematică, E.D.P., Bucureşti (1982), 104.

[To-An] Ţopan L., Andrica D., Caracterizarea mulţimilor critice pe axa reală, Gazeta Matematică seria A, nr.2/2001, 73-79.

"Babeş-Bolyai" University, Cluj-Napoca

BOOK REVIEWS

European Congress of Mathematics: Barcelona, July 10-14, 2000, Vol. I, l+582 pp., ISBN: 3-7643-6417-3, Vol. II, xii+641 pp., ISBN:3-7643-6418-1, Carles Casacuberta, Rosa Maria Miró-Roig, Sebastià Xambó-Descamps- Editors, Progress in Mathematics, Birkhäuser Boston-Basel-Berlin 2001.

These are the proceedings of the Third European Congress of Mathematics (3ecm), held from July 10th to July 14th, 2000, in Barcelona. The congress was organized by the Societat Catalana de Matématiques, under the auspices of the European Mathematical Society (EMS). The initiative of the organizations of ECMs belongs to Max Karoubi (France) and was et on course shortly after the creation of EMS in 1990. The first ECM took place in Paris in 1992 and the second in Budapest in 1996.

The 3ecm was attended by over than 1300 people coming from 87 countries. There were awarded the EMS Prizes to 8 young mathematicians: S. Alekser (Israel), R. Cerf (France), D. Gartsgory (U.S.A.), E. Grenier (France), D. Joyce (U.K.), V. Lafforgue (France), S.Yu. Nemirovski (Russia), P. Seidel (France). The Felix Klein Prize was attributed to D.C. Dobson (U.S.A.). The first volume contains short presentations of the winners. The volume contains also the speeches delivered at the opening and at the closing ceremonies, including two addresses by Rolf Jeltch, President of the EMS, on the aims and perspectives of the EMS.

Beside this introductory material, the first volume contains the articles written by plenary (8) and parallel speakers (30).

The second volume contains the the articles by prize winners and those presented and the mini-symposia organized during the Congress. There nine mini-symposian dealing with the following topics: Computer Algebra, Curves over Finite Fields and Codes, Free Boundary Problems, Mathematical Finance, Quantum Chaology, Quantum Computing, String and M-Theory, Contact Geometry and Hamiltonian

Dynamics, Wavelet Applications in Signal Processing. Seven round tables on topics of general interest were also organized. A third volume, containing material from these round tables, will be published jointly by Societat Catalana de Matématiques and Centre Internacional de Mètodes Numèrics en Enginyeria (CIMNE), Barcelona.

The ECMs are major events in the life of the European mathematical community, particularly this one, organized within the World Mathematical Year 2000 and having as motto *Shaping the 21st Century.*

The volumes are a must for any mathematics library.

S. Cobzaş

Roland Hagen, Steffen Roch and Bernd Silbermann, *C^*-algebras and Numerical Analysis,* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 236, M. Dekker, Inc., Basel - New York 2001, 376 pages, ISBN: 0-8247-0460-6.

The book is concerned with two apparently unrelated fields–numerical analysis and Banach and C^*-algebras. Its aim is to emphasize how tools and results from Banach and C^*-algebras (e.g. Gelfand's theory, Fredholm theory, states and ideals) shed a new light on numerical methods for solving operator equations or eigenvalue problems. These methods are adequate mainly for the study of the stability of these methods. Other questions, such as, e.g., the rapidity of convergence, can't be treated within this framework. The general idea is the following: one considers an operator equation $Ax = y$, where A is a continuous linear operator on a normed space X, and a sequence (A_n) of continuous linear operators on X (the approximation operators), and the approximate equations $A_nx_n = y_n,\ n \in \mathbb{N}$. One supposes that the sequence (A_n) converges strongly to A, i.e. $A_nz \to Az$ in the norm of X, for every $z \in X$. If further, starting with some n_0 the equations $A_nx_n = y_n$ have unique solutions x_n for all sequences (y_n) converging to y, and the sequence (x_n) converges to x (the solution of $Ax = y$), then the approximation method (A_n) is called applicable. By a result of N.I. Polski (1963), if A is invertible then the approximation method (A_n) is applicable if and only if there is an n_0 such that the operators A_n are invertible for $n \geq n_0$ and the norms of their inverses are uniformly bounded. Such approximation sequences are called stable. Typical examples of approximation methods are the Galerkin type methods or, more generally, finite section methods, which are largely studied in the book, with emphasis on equations with Toeplitz and Hankel operators and their finite sections. If the operator A is not invertible then one works with Moore-Penrose generalized inverses for matrices and for operators on Hilbert space or for elements of C^*-algebras.

One denotes by $\mathcal{F}$ the set of all uniformly bounded sequences (A_n) of operators on a Banach space X. With respect to the operations of addition, multiplication, multiplication by scalars, and the sup-norm $\|(A_n)\| = \sup_n \|A_n\|$, $\mathcal{F}$ becomes a Banach algebra, and the set $\mathcal{G}$ of all sequences (A_n) in $\mathcal{F}$ with $\|A_n\| \to 0$ is a closed ideal in $\mathcal{F}$. The main goal of the book is to prove that the quotient algebra $\mathcal{F}/\mathcal{G}$ is the adequate frame for the study of many problems of numerical analysis. For instance, the sequence (A_n) of approximation operators is stable if and only if the coset $(A_n) + \mathcal{G}$ is invertible in the algebra $\mathcal{F}/\mathcal{G}$ (a result of A.V Kozak (1973)). A finer and deeper study of numerical methods requires to work in some C^*-subalgebras of the Banach algebra $\mathcal{F}$. Beside the numerical solutions of operator equations, the algebraic approach proposed by the authors allows to treat problems concerning the approximation of eigenvalues, computation and stability of spectra or pseudospectra, the study of Rayleigh quotients of eigenvalues, of numerical ranges and of the asymptotic behavior of the determinants of the matrices A_n.

A good idea on the contents of the book is given by the headings of its chapters: 1. *The algebraic language of numerical analysis*; 2. *Regularization of approximation methods*; 3. *Approximation of spectra*; 4. *Stability analysis for concrete approximation methods*; 5. *Representation theory*; 6. *Fredholm sequences*; 7. *Self-adjoint sequences.*

Containing fine results from analysis and functional analysis applied to numerical methods, the book is addressed to a wide audience, first students who want to see applications of functional analysis and to learn numerical analysis, but also to mathematicians and engineers interested in theoretical aspects of numerical analysis. The value of the book is raised by the wealth of nontrivial examples illustrating the theoretical concepts. The authors are well known specialists in functional analysis, and the book incorporates many of their recent results, some of them published here for the first time.

S. Cobzaş

V. V. Beletsky, *Essays on the Motion of Celestial Bodies*, Transl. from the Russ. by Andrei Iacob, Birkhäuser Verlag, Basel-Boston-Berlin, 2001, XVIII+372 pp, ISBN 3-7643-5866-1.

The book under review is the story of some interesting theoretical investigations in the mechanics of space flight, i.e., in the theory of motion of spacecraft. Some new problems of celestial mechanics are discussed as well.

..."Dear Fagot, show us something simple for the start" – it is with this epigraph from M. Bulgakov's *The Master and Margarita* that the first essay of the book under review begins. And the reader is not being lied to – the book begins with some well known, and for that reason simple, classical results about unperturbed and perturbed motion of a satellite, the problem of two fixed centers, the influence of the radiation pressure on the orbit of a satellite, the "Laplace Theorem", the restricted three-body problem, etc.

The book uses results published by other scientists, but essentially contains the outhor's own research. The problems treated in this essay (first Russian edition in 1972) continue to be investigated and developed by many authors. Interesting later developments was made by *A. P. Ivanov* in the theory of impact-free motions and by the author in the problem of the dynamics of a system of linked bodies.

In order to reveal the beauty of the research process leading to the results, the emphasis is put on the analysis that can be carried out on the level of graphs and drawings, and sometimes numbers. Whenever possible, the investigation relies on maximum intuitive, elegant geometric tools. The book can be read profitably by anyone with the mathematical background typically offered in the first few years of undergraduate studies in mathematics, physics and engineering, including students, teachers, scientists and engineers.

As *V. I. Arnold* and *Ya. B. Zeldovich* remark in they review of the first edition of Beletsky's "*Essays*" (*Priroda*, No. 10, 1973, 115-117), this book "marks the affirmation of a new style in the scientific literature. The author explains in a frank and detailed manner the reasons behind each calculation, its difficulties, and the psychological side of the research. The book contains no attempts to inflate the importance of results or to give results while hiding the methods used to obtain them. The book is adorned by humorous illustrations by *I. V. Novozhilov*, Doctor in Physico-Mathematical Sciences. ... The general impression that the "*Essays*" make is not that this is a boring lesson, but rather a discussion with brilliant, knowledgeable and wise interlocutor. Even people with little interests in space problems will go through the book with satisfactions, perhaps omitting the calculations."

The Russian edition of this book was awarded the 1999 *F. A. Zander Prize* of the *Russian Academy of Science.*

Ferenc Szenkovits

Functional Analysis, Lecture Notes in Pure and Applied Mathematics, Vol. 150, Edited by Klaus D. Bierstedt, Albrecht Pietsch, Wolfgang M. Ruess and Dietmar Vogt, M. Dekker, Inc., New York 1994, xviii + 526 pp, ISBN 0-8247-9066-9.

These are the Proceedings of an International Symposium on Functional Analysis, held in Essen, Germany, November 24-30, 1991. The first goal of the conference was to emphasize and deep the interaction between three branches of functional analysis: (i) the geometry of Banach spaces; (ii) the theory of Fréchet spaces with applications to analysis and partial differential equations (PDE); (iii) semigroups of operators and evolution equations. The second one was to vitalize the scientific contacts between functional analysts in East and West, by taking advantage on the political opening which occurred that time in Eastern Europe.

The conference was structured into three main lecture series delivered by S. Heinrich (the topic (i)), R. Meise (ii), and Ph. Bénilan (iii), and about 30 further contributions. The volume contains 29 articles by 39 authors from 12 countries all around the world, and cover nearly all the topics presented at the conference, plus some additional ones.

To be more specific we mention some of them. Ph. Bénilan and P. Wittbold wrote a survey on nonlinear evolution equations in Banach spaces. Another survey, by S. Heinrich, is concerned with random approximation in numerical analysis. An interesting paper by A. Pelczynski surveys properties of function spaces depending on the dimension of their domains of definition. Applications of orthonormal trigonometric systems to the geometry of Banach spaces are presented in a paper by A. Pietsch and J. Wenzel. There are several papers dealing with the analysis of vector-valued functions or with spaces of vector-valued functions as, for instance, vector-valued versions of some representation theorems in analysis (W. Arendt), vector-valued Lagrange interpolation and convergence of Hermite series (H. König), spaces of Lipschitz functions on Banach spaces (Ch. Stegall), Freét spaces of continuous vector-valued functions (P. Domanski and L. Drewnowski). Ultradistributions and with applications to PDE are discussed in two papers, one by R. Meise, B.A. Taylor and D. Vogt, and the other one by M. Langenbruch. Extensions of Josefson-Nissenzweig and Pitt's theorems to Fréchet setting are done by J. Bonet and M. Lindström, and M.S. Ramanujan and D. Vogt, respectively. Other topics are: spaces of harmonic functions (M. Zahariuta), Liouville theorem for coherent analytic sheaves (V. P. Palamodov), interpolation for Hardy spaces on disk and on bidisc (V. Kisliakov).

In spite of the time passed since the publication of the volume, the included topics are still valuable references for the working mathematician interested in functional analysis, mainly in its applications to the analysis of vector-valued functions, to spaces of vector-valued functions and to PDE.

S. Cobzaş

Ronald Cross, *Multivalued Linear Operators,* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 213, M. Dekker, Inc., Basel - New York 1998, ix+335 pages, ISBN: 0-8247-0219-0.

Let X, Y be linear spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A relation is a multivalued mapping $T : D(T) \subset X \to 2^Y \setminus \{\emptyset\}$. The relation T is called linear if $D(T)$ is a subspace of X and $T(\alpha x+\beta y) = \alpha Tx+\beta Ty$, for all $x, y \in D(T)$ and scalars α, β. The simplest example of a linear relation is the inverse T^{-1} of a linear operator $T : X \to Y$, defined by $T^{-1}y = \{x \in X : Tx = y\}$. In this case $D(T^{-1}) = R(T) = T(X)$. Linear relations were considered in the early thirties of the last century by J. von Neumann in order to define the adjoints of non-densely defined linear differential operators.

The aim of the present book is to develop a systematic study of linear relations, especially in the framework of normed spaces. Beside giving some elegant and transparent formulations and proofs of some theorems in classical operator theory in Banach spaces as, e.g., the closed graph and the closed range theorems, the study of linear relations contributes to the enrichment and clarification of many aspects of the operator theory, mainly those concerned with non-closable and non-densely-defined linear operators.

The first three chapters of the book, I. *Linear relations: Algebraic properties*, II. *Normed linear relations,* and III. *Adjoints of linear relations,* provide a self-contained foundation course on linear relations. Ch. IV, *Operational quantities of linear relations*, is concerned with various numerical functions defined on some classes of linear relations, most of them being generalizations of some well known quantities in the theory of bounded linear operators. Some important classes of linear relations, such as compact, precompact, strictly singular, strictly cosingular, upper and lower semi-Fredholm, are introduced and studied in Ch. V, *Semi-Fredholm linear relations.* The spectral theory for linear relations is developed in Chapters VI, *Spectral theory* and VII, *The essential spectrum.* The emphasis in Ch. VIII, *The second adjoints of linear relations* is on weakly compact and weakly completely continuous linear relations with applications to Tauberian theorems.

The book, largely based on the results obtained by the author and his collaborators or doctoral students, presents for the first time in book form a systematic treatment of various aspects of multivalued linear relations.

For these reasons, the book is of interest to a large audience, including researchers in functional analysis and operator theory, differential equations (ordinary or partial), mathematical economics and other domains. Its first three chapters can be used for advanced graduate, or post-graduate, courses in functional analysis or operator theory.

Tiberiu Trif

Solomon Leader, *the Kurzweil-Henstock Integral and Its Differentials–A Unified Theory of Integration on* $\mathbb{R}$ *and* $\mathbb{R}^n$, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 242, M. Dekker, Inc., Basel - New York 2001, viii+335 pages,

ISBN: 0-8247-0535-1.

The main defects of the Riemann integral are the restriction to bounded integrands and feeble convergence properties. These defects were remedied by the Lebesgue integral, and its development led to general measure theory and integration on measure spaces, with many applications in functional analysis and probability theory. Beside requiring tedious preliminaries, the Lebesgue integral involve absolute integrability, so that the semiconvergent improper Riemann integrals can not be treated within this theory. A significant breakthrough was done around 1960 independently by J. Kurzweil and R. Henstock. Their main idea was to replace the number δ measuring the finesse of a division by a function $\delta(\ )$, called a gauge function. This simple modification yields the so called generalized Riemann integral, whose properties overcome the defects of both Riemann and Lebesgue integrals.

The present book is essentially based on the results published by the author between the years 1985 and 1995, mainly in the journal Real Analysis Exchange. Inspired by some ideas of Kurzweil and Henstock, he develops a process of integration based on "summants", which are functions S defined on the set of all tagged intervals contained in an elementary figure (a finite union of closed intervals) K. A tagged cell is a pair (I, t), where I is an interval in $[-\infty, \infty]$ and t is an endpoint of I. This definition includes equivalents of Lebesgue, Stieltjes, Denjoy-Perron integrals, considered on bounded as well as on unbounded intervals. Another important innovation is the definition of differentials based on the integration of summants–a differential is an equivalence class on an interval K with respect to the equivalence relation $\int_K |S-S'| = 0$. In this approach, every function f on K induces an integrable differential df and every integrable differential is the differential of a function. Also, the fundamental theorem of calculus can be proved under very general hypotheses.

The book is divided into eleven chapters headed as follows: 1. *Integration of summants;* 2. *Differentials and their integrals;* 3. *Differentials with special properties;* 4. *Measurable sets and functions;* 5. *The Vitali Covering Theorem applied to differentials;* 6. *Derivatives and differentials;* 7. *Essential properties of functions;* 8. *Absolute continuity;* 9. *Conversion of Lebesgue-Stieltjes integrals into Lebesgue integrals;* 10. *Some results on higher dimensions;* 11. *Mathematical background.* Each section ends with a set of exercises, some of them being research topics, deserving further investigation.

The book can be used as a textbook for a graduate course on special topics in real analysis, or as a supplementary text for first year graduate courses in real analysis. It can be used also as a monograph by people interested in the foundation of integration theory and calculus.

S. Cobzaş

Stephen Lynch, *Dynamical Systems with Applications using MAPLE.*

The book is a good introduction to dynamical systems theory. In the first part of the text, differential equations are used to model examples taken from mechanical systems, chemical kinetics, electric circuits, interacting species and economics. In the second part real and complex discrete dynamical systems are considered, with examples taken from economics, population dynamics, nonlinear optics and material science.

The theory and applications are presented with the aid of the MAPLE algebraic manipulation package. Throughout the book, MAPLE is viewed as a tool for solving systems or producing exciting graphics. The author suggests that the reader should save the relevant example programs. These programs can then be edited accordingly when attempting the exercises at the end of each chapter.

The text is aimed at graduate students and working scientists in various branches of applied mathematics, natural sciences and engineering. The material is intelligible to readers with a general mathematical background. Fine details and theorems with proof are kept at a minimum. This book is informed by the research interests of the author which are nonlinear ordinary differential equations, nonlinear optics and fractals. Some chapters include recently published research articles and provide a useful resource for open problems in nonlinear dynamical systems.

An efficient tutorial guide to MAPLE is included. The knowledge of a computer language would be beneficial but not essential. The MAPLE programs are kept as simple as possible and the author's experience has shown that this method of teaching using MAPLE works well with computer laboratory class of small sizes.

I recommend "Dynamical Systems with Applications using MAPLE" as a good handbook for a diverse readership, for graduates and professionals in mathematics, physics, science and engineering.

Damian Trif

Jon H. Davis, *Differential Equations with MAPLE*, Birkhäuser Verlag 2000, xiv + 409 pp, ISBN 0-8176-4181-5.

Differential Equations is an important subject in pure and applied mathematics. MAPLE is a program for symbolic manipulation of mathematical expressions, numerical computations and graphics. The ability of Maple to handle complicated calculations makes it possible to deal with much more interesting and substantial problems than are possible if only hand calculations are allowed.

Some differential equations are susceptible to analytic means of solution while others require the generation of numerical solution trajectories to see the behavior of the equation under study. Maple can be used for both situations. The student does not understand an algorithm unless he can code it and, of course, the solution curve plots are more informative than columns of numbers when numerical methods are used.

The first part of the text introduces MAPLE, by a self contained discussion. The second part covers conventional differential equation topics: first order equations, n-th order equations and systems, periodic solutions, stability, boundary value problems, Laplace transform methods and numerical methods. The last part of the text consists of MAPLE differential equations applications and some hard programming projects.

The book integrates MAPLE with differential equations by using it to investigate topics that are inaccessible without computational aid. There are routines for recognizing and solving a variety of differential equation problems but more important is the experience of what sort of problems have simple solutions and what the form of those solutions should look like.

MAPLE exercises are part of the learning process. It is important that students undertake learning MAPLE as a programming tool. This enables them to use MAPLE to solve their own problems.

As universities throughout the world move to incorporate a programming package into the differential equations curricula, I recommend this book as an excellent combination of basic theory of differential equations and MAPLE.

Prof. Damian Trif

Cristian E. Gutiérrez, *The Monge-Ampére Equation,* Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser Boston-Basel-Berlin, 2001.

The classical Monge-Ampére equation has been the center of considerable interest in recent years because of its important role in several areas of applied mathematics. In reflecting these developments, this works stresses the geometric aspects of this theory, using some techniques from harmonic analysis - covering lemmas and set decompositions. Moreover, Monge-Ampére type equations have applications in the areas of differential geometry, the calculus of variations, and several optimization problems, such that Monge-Kantorovitch mass transfer problem.

This book is an essentially self-contained exposition of the theory of weak solutions, including the regularity results of L.A. Caffarelli. The presentation unfolds systematically from introductory chapters, and an effort is made to present complete proofs of all theorems. There are included examples, illustrations, bibliographical references at the end of each chapter, and a comprehensive index.

The topics covered in the book include: Generalized solutions, Non-divergence equations, The cross-sections of Monge-Ampére, Convex solutions of $D^2u = 1$ in $\mathbb{R}^n$, Regularity theory, $W^{2,p}$ estimates.

The Monge-Ampére equation is a concise and useful book for students and researchers in the field of nonlinear equations.

Adriana Buică

Raghavan Narasimhan and Yves Noevergelt, *Complex Analysis in One Variable*, Second Edition, Birkhäuser Verlag, Basel-Boston-Berlin 2001, xiv + 381 pp, ISBN 3-7643-4164-5 and 0-8176-4164-5.

The book is a presentation of complex analysis in one variable with connections to other branches of mathematics (several complex variables, real analysis, de Rham theory etc.). It has two parts. The first part, due to Raghavan Narasimhan, is essentially just a reprint of the first edition and contains the theory of complex analysis. The second part, due to Yves Noevergelt, is a collection of exercises, problems, examples and relevant references.

The first three chapters of the first part (Elementary theory of holomorphic functions, Covering spaces and the Monodromy theorem and The winding number and the Residue theorem) deal with classical material. They also include the Looman-Menchoff theorem. Chapter 4 presents Picard's theorem. Chapters 5 and 6 are devoted to inhomogenous Cauchy-Riemann equation, Runge's theorem and its various application. The Riemann Mpping theorem is presented in the next chapter. Chapter 8 (Functions of several complex variables) is meant to contrast the behavior in higher dimensions with that in the complex plane. Chapter 9 is an introduction on Riemann surfaces. Chapter 10 contains Tom Wolff's proof of the Corona theorem. The last chapter, Chapter 11, deals with subharmonic functions and their generalizations to several variables.

The book is addressed to graduate students who intend to specialize in mathematics. It can also be useful in doctoral work in mathematics, teaching careers in colleges or technical activities. The book requires knowledge of multivariable calculus, set theory, Lebesgue integration and elementary functional analysis.

Grigore Şt. Sălăgean

p-Adic Functional Analysis, Lecture Notes in Pure and Applied Mathematics:
Vol. 192, W.H. Schikhof. C. Perez-Garcia, J. Kakol - Editors, M. Dekker, New York 1997, 399 pp, ISBN 0-8247-0038-4.
Vol. 207, J.Kakol, N. De Grande-De Kimpe, V. Perez-Garcia - Editors, M. Dekker, Inc., New York 1999, 331 pp, ISBN 0-8247-8254-2.

p-Adic (ultrametric or non-archimedean) analysis is the analysis over a field with an ultrametric valuation, i.e. a valuation | | satisfying the strong (or ultrametric) triangle inequality

$$|a+b| \leq \max\{|a|, |b|\},$$

which is essential for the entire theory. Classical examples of non-archimedean (n.a.) valued fields are the fields $\mathbb{Q}_p$, for p a prime natural number, which entail the name "p-adic analysis" attributed to the domain. It was developed by A.F. Monna in a series of papers published in Proceedings of the Dutch Royal Academy of Sciences, starting with 1943. He collected the results up to 1973 in a book, A.F. Monna, *Analyse Non-Archimédienne,* Springer Verlag, Berlin 1973. Another book on the same topic is A.C.M. van Rooij, *Non-Archimedean Functional Analysis,* M. Dekker, New York 1978.

Although, at the beginning, the theory looked a little strange and useless, the efforts of a permanently increasing number of mathematicians transformed it into a well established mathematical discipline. Beside interesting and nontrivial results, most of them drastically contrasting with those in classical real or complex analysis, the theory has found recently some spectacular applications to mathematical physics and probability theory. Two recent books, V.S. Vladimirov, I.V. Volovich, E.J. Zelenov, *p-adic numbers in mathematical physics*, World Scientific, Singapore 1994, and A.Yu. Khrennikov, *p-adic valued distributions in mathematical physics,* Kluwer AP, Dordrecht 1994, are good sources for these applications .

Motivated by the growing interest in n.a. functional analysis, a conference on this topic was organized in 1990 at Laredo, Spain, by Jose M. Bayod, N. De Grande-De Kimpe and J. Martinez-Maurica. Its Proceedings were published by M. Dekker in 1992 as volume 137 in the series Lecture Notes in Pure and Applied Mathematics (LNPAM).

The present two volumes contain the Proceedings of the fourth conference, held in 1996 at the University of Nijmegen, The Netherlands, and of the fifth conference held in 1998 at the University Adam Mickiewicz of Poznan, Poland. They reveal the state of the art in the realm of n.a. analysis and contain research articles presented at the conference in 30-minute talks.

The fourth conference was attended by over than 40 researchers from 15 countries. The 1996 Proceedings volume (vol. 192 in LNPAM) contain 29 papers dealing with topics as spaces of p-adic analytic or continuous functions, functional and differential p-adic equations, uniform approximation (Stone-Weierstrass type theorems), almost periodic functions, Euclidean models for p-adic spaces, inductive limits of locally convex spaces and closed graph theorems, hypergeometric series, Tauberian theorems. N.a. convexity has a more algebraic character than the classical (real) convexity and is developed in a paper on locally convex modules.

The fifth conference was attended by mathematicians from Europe, North and South America, Africa and Japan, and its Proceedings, the volume 207 in LNPAM, contain 21 contributed papers. Various topics discussed by the participants were inspired by recent designs for p-adic models in modern physics and probability theory. Again, p-adic analytic functions and the properties of spaces of analytic functions are discussed in several papers. Other topics included in the volume are: Fourier transform for p-adic tempered distributions, spectral properties of p-adic Banach algebras, Banach-Dieudonné theorem, orthonormal bases, Mahler bases .

These conferences on p-adic functional analysis, and the corresponding Proceedings volumes, are the most authoritative sources in this relatively new area of investigation which is p-adic analysis. The volumes are addressed first to researchers in p-adic analysis, but researchers in mathematical physics and probability theory will find new and unexpected approaches to their field. The young researchers can find here a fertile land, with a lot of open problems deserving further investigation.

S. Cobzaş

Victor P. Pikulin and Stanislav Pohozaev, *Equations of Mathematical Physics*–A practical course, Birkhäuser Verlag, Basel-Boston-Berlin 2001, viii+207 pp., ISBN: 3-7643-6501-3.

The aim of this book is to present the main methods and tools for solving the basic problems from mathematical physics. This course is addressed especially to students for the study of the main equations from the mathematical physics, but it is also a valuable book for all those interested in the theory of partial differential equations, by means of the superposition method. Regarding the structure of the book, let me list the titles of the three chapters: I. Elliptic Problems (including the Green function method and the method of conformal mappings) II. Hyperbolic Problems (including the Fourier, Laplace and Hankel integral transforms) III. Parabolic problems (including also the Fourier and Laplace integral transform methods and the method of separation of variables). Let us remark that each chapter contain several examples, as well as, problems for independent study and answers to them.

Because of the importance of the domain, the very good quality of the paper and the writing style of the authors, I must recommend this well written book as a textbook for students and a mini-handbook for other scientists from applied mathematics.

Adrian Petruşel

Categorical Perspectives, Editors: Jürgen Koslowski and Austin Metlon, Trends in Mathematics, Birkhäuser (2001), x+281pp, SBN 0-8176-4186-6 SPIN 10761690 ISBN 3-7643-4186-6.

The volume under review contains papers presented at the conference held in honor of Professor George E. Strecker's 60th birthday which was held in August 1998 on Kent State University.

The aim of the editors was to exhibit some fundamental facts in the category theory and some interaction between this and other domains (topology, computer science, etc.). The volume contains 15 teaching, expository and research papers.

As teaching papers we mention the papers of Y. T. Rhineghost *The Functor that Wouldn't be* and *The Emergence of Functors* as well as the George E. Strecker's paper *10 Rules for Surviving as a Mathematician and Teacher* and the Alois Zmrzlina's paper *Too Many Functors*. Expository papers are *Categories: A Free Tour* by Letz Schróder, *Contributions and Importance of Professor George E. Strecker's Research* by Jürgen Koslowski, *Connections and Polarities* by Austin Melton and *Categorical Closure Operators* by G. Castellini. As research papers the reader can find *Extension of Maps from Dense Subspaces* by H. L. Bentley, *Characterisation of subspaces of Important Types of Convergence Spaces in the Realm of Convenient Topology* by Gerhard Preu, *The Naturals are Lndelöf iff Ascoli Holds* by Y. T. Rhineghost, *Revisiting the Celebrated Thesis of J. de Groot: "Everything is Linear"* by Ludvik Janos, *Finite Ultrametric Spaces and Computer Science* by Vladimir A. Lemin, *The Copnumber of a Graph is Bounded by [3/2 genus (G)]+3* by Bernd S. W. Schroeder and *Abelian Groups: Simultaneosly Reflective and Coreflective Subcategories versus Modules* by Robert El Bashir, Horst Herrlich and Miroslav Hušek.

The authors are experts from quite different well known schools.

The book permits an easy access to the current information in the field. Graduate students and researchers interested in category theory and related areas will take a full benefit and they find here a good source of inspiration.

Simion Breaz

Stefan Caenepeel and Freddy Van Oystaeyen Editors, *Hopf Algebras and Quantum Groups*. Lecture notes in pure and applied mathematics 209, Marcel Dekker, New York-Basel, 2000, xii+309 pp., Softcover, ISBN 0-8247-0395-2.

The volume under review is based on the proceedings of the colloquium on Hopf Algebras and Quantum Groups held at the Free University of Brussels, Belgium. It contains high quality refereed research papers and survey papers covering topics like Nichols algebras and pointed Hopf algebras, cross product algebras, graded coalgebras, coalgebra-Galois extensions, Doi-Hopf modules, cyclic cohomology, Schur-Weyl categories, classical Lie superalgebras and finite-dimensional quantum groupoids.

The authors and their contributions are the following. *N. Andruskievitsch and H.-J. Schneider*, Lifting of Nichols algebras of type A_2 and pointed Hopf algebras of order p^4; *Y. Bespalov and B. Drabant*, Survey of cross product algebras; *C. Boboc*, A Morita-Takeuchi context for graded coalgebras; *T. Brzeziński*, Coalgebra-Galois extension from the extension theory point of view; ıS. Caenepeel, B. Ion, G. Militaru and S. Zhu, Separable functors for the category of Doi-Hopf modules II; *M.A. Farinati and A. Solotar*, Cyclic cohomology of coalgebras, coderivations and de Rham cohomology; *D. Gurevich and Z. Mriss*, Schur-Weyl categories and non-quasiclassical Weyl-type formula; *Y. Kashina*, A generalized power map for Hopf algebras; *I.M. Musson*, Associated varieties for classical lie superalgebras; D. Nikshych and L. Vainerman, Algebraic version of a finite-dimnesional quantum groupoid; *F. Panaite and F. Van Oystaeyen*, Quasi-Hopf algebras and the centre of a tensor category; *Ş. Raianu*, An easy proof for the uniqueness of integrals; *M. Takeuchi*, The coquasitriangular Hopf algebra associated to a rigid Yang-Baxter Coalgebra; *A. Tyc*, On the regularity of the algebra of covariants for actions of pointed Hopf algebras on regular commutative algebras; *A. Van Daele and Y. Zhang*, A survey on multiplier Hopf algebras.

The book is an important addition to the literature on this subject which had a tremendous development in the last 15 years. It will be a useful source of information and ideas for researchers in algebra, number theory and mathematical physics, and for all those interested in Hopf algebras.

Andrei Marcus

Gary F. Birkenmaier, Jae Keol Park, Young Soo Park (Eds), *International symposium on ring theory*, Birkhäuser Verlag, Boston, 2001, xviii+446 pp., Hardcover, ISBN 0-8176-4158-0.

The present volume is the Proceedings of the Third Korea-China-Japan International Symposium on Ring Theory held jointly with the Second Korea-Japan Joint Ring Theory Seminar, which took place at Kyongju, Korea, between June 28 and July 3, 1999.

It contains more than 30 both survey and research articles of mathematicians from Korea-China-Japan area, but also from Europe and the United States.

The articles covering various actual topics of Ring Theory may be classified in several main branches: Classical Ring Theory, Module Theory, Representation Theory, Hopf Algebras Theory and some other special subjects.

In the papers on classical part of Ring Theory, the results refer to stability properties of exchange rings, generalized principally injective maximal ideals, Auslander-Gorenstein rings, skew polynomial rings, non-commutative valuation rings, generalized Jordan derivations, theories of Harada in artinian rings, quasi-Frobenius or finitely pseudo-Frobenius rings.

Among the topics connected to Module Theory we mention generalized deviations of posets and applications, good conditions for the total, generalizations of injectivity, a short history of the flat cover conjecture, CS-properties, dual bimodules and Nakayama permutations, maximal t-corational extensions, torsion-free modules over valuation domains, generalized Matlis duality, hopfian modules or linkage maps.

Representation Theory is present through semicentral reduced algebras, derived equivalences and tilting theory, generalized Jordan derivations, Hecke orders, cellular orders and quasi-hereditary orders, infinite quivers and cohomology groups.

Topics of Hopf Algebras included in the articles are the coinduced functor and homological properties of Hopf modules, Hopf algebra coaction and group-graded rings or QcF-algebras.

The final section presents several open problems, especially on Classical Ring Theory and Module Theory, offering ideas for future research.

The well-known mathematicians which contributed to this book, touching a rather wide range of important topics of the nowadays research in Ring Theory, make the volume a valuable tool and source of inspiration for an algebraist working on a high level in this field.

Septimiu Crivei

Schwartz, Laurent – *A Mathematician Grappling with His Century*, Birkhäuser, 2001, 504 pp., Softcover, ISBN 3-7643-6052-6.

"I am a mathematician". It is the first sentence of this autobiography of Laurent Schwartz. A great Romanian mathematician, Gr. C. Moisil, used to say that one of the biggest temptations of a mathematician is to be *only* a mathematician. Without any doubt, Laurent Schwartz is a good illustration of how someone could be determined enough to resist such a temptation.

It would be pointless to try to describe the contents of the book. What could be said in just a few lines about an entire life of one of the greatest mathematicians of the last century, who was, at the same time, one of its greatest consciences?! Let me only mention that he did not stay away from any important problem or idea. He was involved in the communist movement, as a supporter of the ideas of Trotzki, he experienced the problems of the Jewish people during the second World War, being himself a Jew. After the war, he became involved in a lot of committees, fighting for the rights of people from Algeria, Vietnam or Afghanistan. He was one of the founders of the International Committee of Mathematicians, an international organization which managed to help some of the Soviet mathematicians that were subjects to persecution in their country, back in the communist period.

Of course, in any (auto)biography of a scientist, the personal and professional matters do interfere, they cannot be treated separately. This is no exception. If I didn't say a word about the mathematics of Laurent Schwartz, it is because this is, by now, well known to any mathematician. It is, nevertheless, instructing to find out about the mathematical discoveries and inventions of Schwartz came into being.

A great scientist is not necessarily a good writer. I would rather say that the writing skill is the exception, not the rule, but the book of Schwartz is more fascinating than a novel. It really keeps you awake at night. Usually, we are smiling unconfidentely when someone tells us: "My life was a novel". The life of Schwartz *was*, indeed.

Let me mention that the book is, also, a valuable (and personal, of course) contribution to the history of mathematics of the twentieth century. Many important figures of contemporary mathematics are present in the pages of Schwartz autobiography, not only as colleagues and friends, but also as relatives (he is the nephew of Jacques Hadamard and the son in law of Paul Lévy).

A final word of appreciation is due to the photographic material present in the book, which is very interesting and inedit.

Paul A. Blaga

I. John Cagnol, Michael P. Polis, Jean-Paul Zolesio (Eds.)- *Shape Optimization and Optimal Design,* Lecture Notes in Pure and applied Mathematics, vol. 216, Marcel Dekker, New York-Basel, 2001, ISBN: 0-8247-0556-4.

II. Giuseppe Da Prato, Jean-Paul Zolesio (Eds.)-*Partial Differential Equation Methods in Control Analysis,* Lecture Notes in Pure and Applied Mathematics, vol. 188, Marcel Dekker, New York-Basel-Hong Kong, 1997, ISBN: 0-8247-9837-6.

The first volume mentioned comprises papers from the sessions "Distributed Parameter Systems" and "Optimization Methods and Engineering Design" held within the 19th conference System Modeling and Optimization in Cambridge, England.

The second volume presents papers from the Conference on Control and Shape Optimization held at Scuola Normale Superiore di Pisa, Italy. Both the conferences were organized by the International Federation for Information Processing (IFIP).

The papers present the latest developments and major advances in the fields of active and passive control for systems governed by partial differential equations- in particular in shape analysis and optimal shape design.

Traditionally, optimal shape design has been treated as a branch of the calculus of variations, more specifically of optimal control. The subject interfaces with at least four fields: optimization, optimal control, PDEs and their numerical solutions.

The main question that optimal shape design tries to answer is: "What is the best shape for a physical system?".

Many problems that arise in technical and industrial applications can be formulated as the minimization of functionals with respect to a geometrical domain which must belong to an admissible family. Optimal shape design is used in various fields, like those mentioned in the books: fluid mechanics, linear elasticity, thermo-elasticity, soil mechanics, electricity, aircraft industry, material sciences, biodynamics.

The authors of the articles are well known for important results in this field of research.

Some of the aspects treated are:

- shape sensitivity analysis (that is the sensitivity of the solutions with respect to the shape of the domain) for the Navier-Stokes equation, Maxwell's equation, for some problems with singularities (I)

- the study of the material derivative, the shape derivative on a fractured manifold (I), the shape derivative for the Laplace-Beltrami equation (II), the shape hessian for a nondifferentiable variational free boundary problem (II),the shape gradients for mixed finite element formulation (II),the eulerian derivative for non-cylindrical functionals (I)
- numerical aspects (using finite element approximation and other methods, some of them original) for: shape problems in linear elasticity (I), parallel solution of contact problems (I), modeling of oxygen sensors (I), control of a periodic flow around a cylinder (I), shape identification problems associated with the stationary heat conduction in 2D(II)
- boundary controllability of thermo-elastic plates (I)
- regularity properties for the weak solutions to certain parabolic equations(II)
- homogenization and continuous dependence for Dirichlet problems, asymptotic analysis on singular perturbations (II), asymptotic analysis of aircraft wing model in subsonic flow (I)
- mapping method in problems governed by hemivariational inequalities (I)
- feedback laws for the optimal control of parabolic variational inequalities

Many more subjects are treated in the 41 papers by 50 authors, which allow the reader to get a good idea about the latest research directions in this very active field of applied mathematics.

Daniela Inoan

Stephanie Frank Singer, *Symmetry in Mechanics: A Gentle, Modern Introduction*, Birkhäuser, Boston-Basel-Berlin, 2001, VII+193 pp, ISBN 0-8176-4145-9.

This book is aimed at anyone who has observed that symmetry yields simplification and wants to know why. The author eschews density of topics and efficiency of presentation in favor of a gentler tone, a coherent story, digressions on mathematicians, physicists and their notations, simple examples worked out in detail, and reinforcement of the basics.

This text introduces some basic constructs of modern symplectic geometry in the context of an old celestial mechanics problem, the two-body problem. The derivation of Kepler's laws of planetary motion from Newton's laws of gravitation are presented, first in the style of an undergraduate physics course, and then again in the language of symplectic geometry. All necessary constructs of symplectic geometry are introduced and illustrated in text.

Chapter 0 covers some preliminary material. Here are presented basic notations and conventions, the physical and mathematical background.

Chapter 1 presents the two-body problem, i.e., the derivation of Kepler's laws of planetary motion from Newton's laws of gravitation in the classical language of vector calculus.

Chapters 2-7 develope the concepts and terminology necessary for the final chapter, providing a detailed translation between the quite different languages of mathematics and physics. In this part are presented: the symplectic structure of the phase space of mechanical systems (chapter 2), a bridge to differential geometry (chapter 3), the importance of total energy (chapter 4), symmetries as Lie group actions (chapter 5), the Lie algebras of infinitesimal symmetries (chapter 6), and relationship between conserved quantities and momentum maps. This part of the monograph contains many examples, illustrations and exercices.

Chapter 8 presents the derivation it started with (chapter 1), but in the more sophisticated language of modern symplectic and differential geometry, presented in the previous chapters.

Readers desiring broader or more sophisticated texts should consult the Recommended Reading sections.

For the student, mathematician or physicist, this gentle introduction to symplectic reduction via mechanics will be a rewarding experience. This book can be used as a supplement to courses on differential geometry or Lie theory, or could be a major component of a course on symplectic geometry or classical mechanics, providing motivation for a more standard exposition of the mathematics. It would also be appropriate at the end of an example-driven semester course on classical mechanics, in which case students should be encouraged to work out the symplectic versions of examples treated earlier. *Symmetry in Mechanics* requires only competency in multivariable calculus, linear algebra and introductory physics.

Ferenc Szenkovits

Recent Advances in Operator Theory and Related topics – The Béla Szőkefalvi-Nagy Memorial Volume, László Kérchy, Ciprian Foias, Israel Gohberg, Heinz Langer - Editors, l+669 pp., Operator Theory: Advances and Applications, Vol. 127, Birkhäuser Verlag,

Boston-Basel-Berlin 2001, ISBN 3-7643-6607-9.

Béla Szőkefalvi-Nagy was born in Kolozsvár, Transylvania, (now Cluj-Napoca, Romania), in 1913. In 1929 his family moved to Szeged, Hungary, where he followed the university, having as teachers great mathematical personalities as F. Riesz and A. Haar. Soon he became a collaborator of F. Riesz and their collaboration culminated in the monograph "Leǫns d'analyse fonctionelle" published in 1952, a standard reference in functional analysis, translated into six languages. He passed away in 1998 and the present volume contains the proceedings of Szőkefalvi-Nagy Memorial Conference held in Szeged in August 1999. The conference was attended by 91 mathematicians all over the world, who delivered 19 plenary talks in the morning and 63 talks in two parallel sections in the afternoon. The volume contains 35 contributed talks by participants at the conference or by experts who were unable to attend the conference. The included papers deal with various topics in operator theory, a field which owes so much to Szőkefalvi-Nagy, written by friends, former students or collaborators. Among the contributors we mention: D. Alpay, I. Gohberg, H. Bercovici, C. Foias, A. E. Frazho, J. B. Conway, L. Zsido, R G. Douglas, J. Eschmeier, J. Esterle, D. Gaşpar, N. Suciu, Z. Sebestyén, L. Kérchy, H. Langer.

Beside these research papers, the volume contains the farewell speech given by Ciprian Foias at the grave site in Szeged, some reminiscences by Israel Gohberg, and a presentation of the life and work of Szőkefalvi-Nagy by L. Kérchy and H. Langer. A list of publications of Szőkefalvi-Nagy and some photos from the family album provided by Erszébet Szőkefalvi-Nagy, Béla's daughter, are also included.

Giving tribute to one of the founders of modern operator theory and bringing together important contributions of leading experts in operator theory, this valuable volume will be of interest first to operator theorists, but also to researchers in functional analysis and mathematical physics.

S. Cobzaş

Turaev, V. – *Introduction to Combinatorial Torsions*, Birkhäuser (Lectures in Mathematics, ETH Zürich), 2000, 123 pp., Softcover, ISBN 3-7643-6403-3.

In the recent period, the various kind of torsions became an important tool in low dimensional topology. The book under review, written by one of the best experts in the field, aims to provide a systematic introduction to combinatorial torsions of cellular spaces and manifolds (especially the three dimensional case).

The first notion of torsion was introduced by Reidemeister in 1935 and the theory was later developed mainly by Whitehead and Milnor.

The book is divided in three chapters. The first two are devoted to an exposition of the algebraic theory of torsions as well as to various geometrical realisations due to Reidemeister, Franz, Whitehead, Milnor. There is presented, also, a notion due to the author, the so-called "maximal abelian torsion" and it is examined the connection between different torsions and and the Alexander polynomial of links and 3-manifolds.

The final chapter deals with more special subjects, namely some other notions introduced by the author: sign-refined torsions and other structures on manifolds (homological orientations, Euler structures) with an application to the construction of the Conway link function for homology 3-spheres. Finally, there is described the connection between the sign-refined torsions and the Seiberg-Witten invariant of 3-manifolds.

The intended audience includes graduate students and researchers in mathematics and physics, intersted in low dimensional topology, with a background in combinatorial topology and homological algebra.

Paul A. Blaga

Jonathan M. Borwein and Adrian S. Lewis, *Convex Analysis and Nonlinear Optimization, Theory and Examples,* Canadian Mathematical Society (CMS) Books in Mathematics, Vol. 3, Springer-Verlag, New York Berlin Heidelberg, 2000, ISBN:0-387-98940-4.

The book is a concise account of convex analysis, its applications and extensions. It is aimed primarily at first-year graduate students, so that the treatment is restricted to Euclidean space, a framework equivalent, in fact, to the space $\mathbb{R}^n$, but the coordinate free notation, adopted by the authors, is more flexible and elegant. The proof techniques are chosen, whenever possible, in such a way that the extension to infinite dimensions be obvious for readers familiar with functional analysis (Banach space theory). Some of the challenges arising in infinite dimensions are discussed in Chapter 9, *Postscript: Infinite versus finite dimensions,* in which case the results involve deeper geometric properties of Banach spaces. The last section of this chapter contains notes on previous chapters, explaining which results extend to infinite dimension and which not, as well as sources where these extensions can be found.

The authors adopted a succint style, avoiding as much as possible complicated technical details, their goal being "to showcase a few memorable principles rather than to develop the theory to its limits". The book consists of short, self-contained sections, each followed by a rather extensive set of exercises grouped into three categories: examples that illustrate the ideas in the text or easy expansions of sketched proofs (no mark); important pieces of additional theory or more testing examples (marked by one asterisk); and longer, harder examples or peripheral theory (marked by two asterisks). Some bibliographical comments are also included along with these exercises, an approach which allow the authors to cover a large variety of topics. A good idea on the included material is given by the headings of the chapters and the presentation of some topics included in the main text or in exercises.

Ch. 1, *Background* - Euclidean spaces, symmetric matrices, in the main text, and Radstrom cancellation, recession cones, affine sets, inequalities for matrices, in exercises.

Ch. 2, *Inequality constraints* - optimality conditions, theorems of alternative, max-functions, in the main text, and nearest points, coercivity, Carathéodory's theorem, Kirchoff's law, Schur convexity, steepest descent, in exercises.

Ch. 3, *Fenchel duality* - subgradients and convex functions, the value function, the Fenchel conjugate, in the main text, and normal cones, Bregman distances, Log-convexity, Duffin's duality gap, Psenichnii-Rockafellar condition, order-convexity and order subgradients, symmetric Fenchel duality, in exercises.

Ch. 4, *Convex analysis* - continuity of convex functions, Fenchel biconjugation, Lagrangian duality, in the main text, and polars and polar calculus, extreme and exposed points, Pareto minimization, von Neumann minimax theorem, Kakutani's saddle point theorems, Fisher information function, in exercises.

Ch. 5, *Special cases* - polyhedral convex sets and functions, functions of eigenvalues, duality, convex process duality, in the main text, and polyhedral algebra, polyhedral cones, convex spectral functions, DC functions, normal cones, order epigraphs, multifunctions, in exercises.

Ch. 6, *Nonsmooth optimization* - generalized derivatives, regularity and strict differentiability, tangent cones, the limiting subdifferential, in the main text, and Dini derivatives and subdifferentials, mean value theorem, regularity and nonsmooth calculus, subdifferentials of eigenvalues, contingent and Clarke cones, Clarke's subdifferentials, in exercises.

Ch. 7, *Karush-Kuhn-Tucker theory* - metric regularity, the KKT theorem, metric regularity and the limiting subdifferential, second order conditions,in the main text, and

Lipschitz extension, closure and Ekeland's principle, Liusternik theorem, Slater condition, Hadamard's inequality, Guignard optimality conditions, higher order conditions, in exercises.

Ch. 8, *Fixed points* - the Brouwer fixed point theorem, selection and the Kakutani-Fan fixed point theorem, variational inequalities, in the main text, and nonexpansive mappings and Browder-Kirk fixed point theorem, Knaster-Kuratowski-Mazurkiewicz principle, hairy ball theorem, hedgehod theorem, Borsuk-Ulam theorem, Michael's selection theorem, Hahn-Katetov-Dowker sandwich theorem, single-valuedness and maximal monotonicity, cuscos and variational inequalities, Fan minimax inequality, Nash equilibrium, Bolzano-Poincaré-Miranda intermediate value theorem, in exercises.

There is a chapter, Chapter 10, containing a list of named results and notation, organized by sections. Beside this, the book contains also an Index.

The bibliography counts 168 items.

Written by two experts in optimization theory and functional analysis, the book is an ideal introductory teaching text for first-year graduate students. By the wealth of highly non-trivial exercises, many of which are guided, it can serve for self-study too.

Stefan Cobzaş

Andreas Juhl, *Cohomological Theory of Dynamical Zeta Functions,* Progress in Mathematics, Vol. 194 Birkhäuser Verlag, Boston-Basel-Berlin 2001, x+709 pp., ISBN 3-7643-6405-X.

Dynamical zeta functions are associated to dynamical systems with a countable set of periodic orbits. The dynamical zeta functions of the geodesic flow of locally symmetric spaces of rank one are known as the generalized Selberg zeta functions.

The present book is concerned with these zeta functions from a cohomological point of view. Originally, the Selberg zeta functions appeared in the spectral theory of automorphic forms and were suggested by an analogy between Weil's explicit formula for the Riemann zeta function and Selberg's trace formula. The purpose of the cohomological theory is to understand the analytical properties of the zeta functions on the basis of suitable analogs of the Lefschetz fixed point formula in which periodic orbits of the geodesic flow take the place of fixed points. According to geometric quantization the Anosov foliations of the sphere bundle provide a natural source for the definition of the cohomological data in the Lefschetz formula. The Lefschetz formula method can be considered as a link between the automorphic approach (Selberg trace formula) and Ruelle's approach (transfer operators). It yields a uniform cohomological characterization of the zeros and poles of the zeta functions and a new understanding of the functional equations from an index theoretical point of view. The divisors of the Selberg zeta functions also admit characterizations in terms of harmonic currents on the sphere bundle which represent the cohomology classes in the Lefschetz formulas in the sense of Hodge theory. The concept of harmonic currents to be used for that purpose is introduced here for the first time. Harmonic currents for the geodesic flow of a noncompact hyperbolic space with a compact convex core generalize the Patterson-Sullivam measure on the limit set and are responsible for the zeros and poles of the corresponding zeta function.

The book is not a textbook but describes the present state of the art of the research in a new field on the cutting edge of global analysis, harmonic analysis and dynamical systems. The majority of results suggest generalizations and raise new questions, some open problems being emphasized explicitly throughout the text. It should be appealing not only to specialists on zeta functions which will find their objects of favorite interest connected in new ways with index theory, geometric quantization methods, foliation theory and representation theory. In this way the book will attract specialists in geometric quantization methods. From the point of view of smooth hyperbolic dynamics the Lefschetz formula method is a link between the automorphic method and the method of Perron-Frobenius operators, relations which are far from being fully understood.

Paul A. Blaga

Flávio Ulhoa Coelho, Héctor A. Merklen (editors): *Representations of algebras, proceedings of the conference held in São Paulo*, Lecture notes in pure and applied mathematics, Volume 224, Marcel Dekker, 2001, xvii+282 pp., ISBN 0-8247-0733-8.

Seventy-two researchers from 17 different countries attended the Conference on Representations of Algebras-São Paulo (CRASP), held at the Instituto de Matemática e Estatística of the Universidade de São Paulo. There were 14 invited talks and 32 contributions.

This book is a valuable collection of these contributions covering almost every research topic belonging to the large domain called Representation Theory of Algebras. We can find new results related with Hopf, derived tubular, tame tilted, symmetric quasi-Schurian, wild hereditary, concealed-canonical, Koszul, coil, quasitilted and Brauer star algebras. A complete classification of the representation-infinite connected tame tilted algebras with almost regular connecting component is given. The existence of almost split morphisms and sequences in some special categories is discussed and we have a combinatorial characterization of hereditary categories containing simple objects. The concept of twisted Hopf algebra is introduced following the constructions which appeared in the theory of Ringel-Hall algebras and quantum groups.

The collection proves to be an excellent guide for getting familiarized with the newest developments in Algebra Representations Theory.

Andrei Mărcuş