Anul XLVII **2002**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
4

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

PROFESSOR DIMITRIE D. STANCU, AT HIS 75th BIRTHDAY ANNIVERSARY

GHEORGHE COMAN, IOAN A. RUS AND LEON ŢÂMBULEA

Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

1. We present briefly a short vita, the career and some of the research activity of professor D.D. Stancu in approximation theory, numerical analysis and probability theory.

He was born on February 11, 1927 in the township Călacea, Timiş District, Romania, in a farmer's family. In the first years he had many difficulties, but with the help of his mathematics teacher he succeeded to study in the prestigious Lyceum "Moise Nicoară", from the city Arad.

In the period 1947-1951 he studied at University "Victor Babeş", from Cluj, Romania. When he was a student he was mostly under the influence of academician professor Tiberiu Popoviciu (1906-1975), a great master of numerical analysis and approximation theory. This has strongly influenced and stimulated his research work.

After his graduation, in 1951, he was named assistant at the Department of Mathematics, University "Victor Babeş", Cluj.

He has obtained the Ph. D. in Mathematics in 1956. His advisor for the doctoral dissertation was professor Tiberiu Popoviciu.

In a normal succession, he advanced up to the rank of full professor, in 1969.

He was happy to benefite from fellowship at the University of Wisconsin, at Madison, Numerical Analysis Department, conducted by the late professor Preston C. Hammer. He spent at the University of Wisconsin, in Madison, the academic year 1961-1962.

Received by the editors: 01.09.2002.

Professor D.D. Stancu has participated in different events in the USA; among them we mention that he has presented contributed papers at several regional meetings of the American Mathematical Society held in Chicago, Milwakee and New York.

He has a nice family. His wife, dr. Felicia Stancu was a lecturer of mathematics at the same University. They have two daughters: Angela (1957) and Mirela (1958), both teaching mathematics at secondary schools in Cluj-Napoca. From them they have three grandchildren: Alexandru (1983) and George (1992) (the sons of Mirela) and Ştefana (1991) (the daughter of Angela).

Professor D.D. Stancu has taught several courses at the University "Babeş-Bolyai", Cluj-Napoca: Mathematical Analysis, Numerical Analysis, Approximation Theory, Probability Theory and Constructive Theory of Functions.

He has used probabilistic methods in Approximation Theory of Functions. He had a large number of doctoral students, from Romania and abroad.

Besides the United States, Professor D.D. Stancu, he has participated in many scientific events in Germany (Stuttgart, Hannover, Hamburg, Goettingen, Dortmund, Münster, Siegen, Würzburg, Berlin, Oberwolfach), Italy (Roma, Napoli, Potenza, L'Aquila), England (Lancaster, Durham), Hungary (Budapest), France (Paris), Bulgaria (Sofia, Varna), Czech Republic (Brno).

His publication lists about 120 items.

There are more than 50 papers where his name is included in their titles.

Since 1961 he is a member of American Mathematical Society. He is also a member of the society: "Gesellschaft für Angewandte Mathematik und Mechanik" (Germany).

He is for many years a reviewer of the journals: "Mathematical Reviews" (USA) and "Zentralblatt für Mathematik" (Germany).

He is Editor in Chief of Revue d'Analyse Numérique et de Théorie de l'Approximation" (Cluj-Napoca, Romania) and a member of the Editorial Board of the Italian journal Calcolo, published by Springer-Verlag.

In 1968 he has obtained one of the Research Awards of the Department of Education in Bucharest, for his research work in Numerical Analysis and Approximation Theory.

University "Lucian Blaga", from Sibiu, has accorded him, in 1995, the scientific title of Doctor Honoris Causa.

In 1999 Professor D.D. Stancu has been elected a Honorary Member of the Romanian Academy of Sciences.

In August 1999 he has participated at the "Alexits Memorial Conference" in Budapest.

In May, 2000 he has participated at the International Simposium on "Trends in Approximation Theory", dedicated to the 60^{th} birthday anniversary of Professor Larry L. Schumacher. He has presented a contributed paper, in collaboration with Professor J. Wanzer Drane, from University of South Carolina, Columbia, S.C.

In June 2000 Professor D.D. Stancu was invited to present colloquium talks at several American Universities: Ohio State University, Columbus, OH., University of South Carolina, Columbia, S.C., Vanderbilt University, Nasville, TN., PACE University, Pleasantville, N.Y.

The main contributions in research work of D.D. Stancu fall into the following list of topics: Interpolation Theory, Numerical differentiation, Orthogonal Polynomials, Numerical quadreatures and cubatures, Taylor-type expansions, Approximation of functions by linear positive operators, Representation of remainders in linear approximation formulas, Probabilistic methods for construction and investigation of linear positive operators of approximation, Use of interpolation and calculus of finite differences in probability theory and mathematical statistics.

In 1996, Professor D.D. Stancu has organized in Cluj-Napoca, an "International Conference on Approximation and Optimization", in conjonction with the Second European Congress of Mathematics, held in Budapest.

At present he and his colleagues are organizing an "International Symposium on Numerical Analysis and Approximation Theory" (May 9-11, University Babeş-Bolyai, Cluj-Napoca, Romania).

His intensive research work and his important results obtained in Numerical Analysis and Approximation Theory has brought him recognition in his country and abroad.

We conclude by wishing to DiDi Stancu many fruitfull and happy years, with health and satisfactions in his research work.

3. SCIENTIFIC PAPERS CONTAINING IN THE TITLE THE NAME OF D. D. STANCU

1. Abel, U., Asymptotic Approximation with *Stancu Beta Operators*, Rev. Analyse Numer. et Theorie del'Approximation, vol.27(1998).
2. Acu, D., Boncut, M., Cismaşiu, C., Evaluation of the remainder in an approximation formula of *D.D. Stancu*, Calcolo, vol. 35(1998), 191-196.
3. Acu, D., On the *D.D. Stancu* method of parameters, Studia Univ. Babeş-Bolyai, Mathematica, vol. 47(1997), 1-7.
4. Agratini, O., Approximation Properties of a Class of Operators of *Stancu*-Kantorovich Type, Babeş-Bolyai Univ. Preprint nr.1, Fac. Math., Cluj-Napoca, pp. 3-12, 1994.
5. Agratini, O., On the monotonicity of a sequence of *Stancu*-Bernstein type operators, Studia Univ. Babeş-Bolyai, Mathematica, vol. 41(1996), 17-23.
6. Agratini, O., Application of divided differences to the study of monotonicity of a sequences of *D.D. Stancu polynomials,* Rev. Analyse Numerique et Theorie de l'Approximation, vol. 25(1996), pp. 3-10.
7. Agratini, O., On Simultaneous Approximation by *Stancu-Bernstein Operators,* Proceedings of ICAOR, vol. I, Transilvania Press, Cluj-Napoca, 1997, pp. 157-162.
8. Agratini, O., Linear Combinations of *D.D. Stancu polynomials,* Rev. Analyse Numer. et Theorie de l'Approximation, vol. 27(1998).
9. Agratini, O., *Stancu* modified operators revisited, Rev. Analyse Numer. Theorie de l'Approximation, vol. **31**(2002).
10. Altomare, F., Campiti, M., Korovkin-type Approximation Theory and its Applications & 5.2.7: *Stancu Operators* (301-306); & 6.1: *Stancu-Schnabl operators* (385-399, 419-420, 423-479); *Bernstein-Stancu operators* (117-123, 220-221), Walter de Gruyter, Berlin, New-York, 1994.
11. Bărbosu, D., On a blending operator of Bernstein-*Stancu* type, Proc. Annual Conference Romanian Soc. Math. Sci., Bucharest, May 29-June 1, 1997, pp. 217-222.

12. Bărbosu, D., The approximation of the Bögel-continuous functions using the Bernstein-*Stancu* polynomials, Bul. Sti. Univ. Baia Mare, vol. 8(1992), 11-18.

13. Bărbosu, D., On the approximation of three variate B-continuous functions using Bernstein-*Stancu* type operators, Bul. Stiint. Univ. Baia Mare, Ser. B, vol 9(1993), pp. 9-18.

14. Bărbosu, D., L'approximation par des polynomes du type Bernstein-*Stancu* des fonctions Boegel-continues, Bul. Stiint. Univ. Baia Mare, Ser. B, vol. 8(1992), pp. 11-18.

15. Birkhoff, G., Schultz, M.H., Varga, R.S., Piecewise Hermite Interpolation in One and Two Variables with Applications to Partial Differential Equations, Numerische Mathematik, vol. 11(1968), pp. 232-236 & 6. Piecewise Hermite Interpolation: *Method of Stancu* and Simonsen.

16. Campiti, M., Limit semigroup of *Stancu*-Muhlbach operators associated with positive projections, Ann. Sc. Norm. Pisa, Cl. Sci. IV, Ser. 19(1992), nr.1, pp. 51-67.

17. Campiti, M., A generalization of *Stancu*-Muhlbach operators, Constructive Approximation, vol. 7(1991), nr. 1, pp. 1-18.

18. Campiti, M., Metafune, G., Solutions of abstract Cauchy problems approximated by *Stancu*-Schnabl-type operators, Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Natur. vol. 130(1996), pp. 81-93.

19. Chen, W.Z., Tian, J.S., On approximation Properties of *Stancu Operators of Integral Type,* Acta Sci. Natur. Univ. Amoien vol. 26(1987), pp. 270-276.

20. Chen, W.Z., Gu, S.M., Approximation Theorems *for Stancu-type operators,* Xiamen Daxue Xuebno Ziran Kexue Ban vol. 32(1993), pp. 679-684.

21. Chen, W., On two dimensional *Stancu*-Mühlbach operator, Approximation Theory Appl. vol. 13(1997), nr. 3, pp. 70-77.

22. Della Vecchia, B., On *Stancu operator* and its Generalizations, Inst. Applied Mathematics, Napoli, Rapp. Tecnico nr. 47(1988).

23. Della Vecchia, B., On the Approximation of Functions by means of the *Operators of D.D. Stancu,* Studia Univ. Babeş-Bolyai, Mathematica, vol. 37(1992), pp. 3-36.
24. Della Vecchia, B., Mache, D.H., On Approximation Properties of *Stancu-Kantorovich Operators,* Rev. Analyse Numer. et Theorie de l'Approximation, vol. 27(1998).
25. De la Cal, J., On *Stancu*-Mühlbach operators and some connected problems concerning probability distributions, J. Approx. Theory vol. 74(1993), pp.56-68.
26. Di Lorenzo, A., Occorsio, M.R., *Polinomi di Stancu,* Ins. Applicazioni della Matematica, Napoli, Rapp. Tecnico nr. 121(1995), 42 pag.
27. Finta, Z., On some *Propertes of Stancu Operator,* Rev. Analyse Numer. et Theorie de l'Approximation vol. 27(1998).
28. Felbecker, G., Ueber Verallgemeinerte, *Stancu*-Mühlbach Operatoren, Z. Angew. Math. Mech. 53, Sonderheft, T188-T189(1973).
29. Frenţiu, M., On the Asymptotic Aspect of the Approximation of Functions by means of the *D.D. Stancu operators,* Babeş-Bolyai University, Preprint nr. 8(1987), pp. 57-64.
30. Gonska, H.H., Meier, J., Quantitative Theorems and Approximation by *Bernstein-Stancu Operators,* Calcolo vol. 21(1984), pp. 317-335.
31. Gonska, H.H., On approximation in spaces of continuous functions (& 4.2 Approximation by bivariate Bernstein-*Stancu operators*), Bull. Austral. Math. Soc. vol. 28(1983), 411-432.
32. Gori, L., *Stancu*, D.D., Mean Value Formulae for Integrals Involving Generalized Orthogonal Polynomials, Rev. Analyse Numer. et Theorie de l'Approximation, vol. 27(1998).
33. Gunttner, R., Beitrage zur Theorie der Operatoren vom Bernstein schen Typ (& 5.2 *Die Operatoren von Stancu*), Doctoral Dissertation, Techn. Univ. Clausthal, 1974.
34. Horova, I., Budikova, M., A note on *D.D. Stancu Operators,* Ricerche di Matematica vol. 44(1995), pp. 397-407.

35. Horova, I., Gegenbauer Polynomials, optimal Kernels and *Stancu Operators,* Approximation Theory and Function Series, Bolyai Math. Society, Math. Studies, 5, Budapest, 1996, 227-235.
36. Jiang, T., On the rate of convergence of *Stancu*-Sikkema-Bernstein operator and *Stancu*-Sikkema-Kantorovich operator for functions of bounded variation, Math. Appl. vol. 3(1990), nr. 2, pp. 89-90.
37. Jiang, G.J., Approximation properties of *Stancu*-Kantorovich operators, Journal of Shandong-Mining-Institute, vol. 9(1990), nr. 2, pp. 193-196.
38. Lorentz, G.G., Approximation of Functions, Chelsea Publ. Comp. New-York, 1986, (In Preface to the Second Edition: "I am grateful to *D.D. Stancu,* Cluj, Romania, for many corrections").
39. Mastroianni, G., Occorsio, M.R., Sulle derivete dei *Polinomi di Stancu,* Rend. Accad. Sci. M.F.N., Ser. IV, vol. 45(1978), pp. 273-281.
40. Mastroianni, G., Occorsio, M.R., Una Generalizzazione dell'*Operatore di Stancu,* Rend. Accad. Sci. M.F.N., Ser. IV, vol 45(1978), 495-511.
41. Mühlbach, G., Verallgemeinerung der Bernstein und Lagrange Polynome. Bemerkungen zu einer Klasse linearer *Polynom-operatoren von D.D. Stancu,* Rev. Roumanie Math. Pres Appl., vol. 15(1970). pp. 1235-1252.
42. Occorsio, D., Simoncelli, A.C., Generalized *Stancu*-Polya Curves, Rev. Analyse Numer. et Theorie de l'Approximation, vol. 27(1998).
43. Raşa, I., Vladislav, T., Some properties of Bernstein and *Stancu Operators,* Approximation and Optimization, Proceedings of ICAOR, vol. I, Transilvania Press, Cluj-Napoca, 1997, pp. 345-350.
44. Rus, I.A., Iterates of *Stancu* operators, via contraction principle, Studia Babeş-Bolyai, Mathematica, vol.47, No.4/2002.
45. Vlaic, G., *Bivariate Approximation by D.D. Stancu type Polynomials* on a Triangle, Approximation and Optimization, Proceedings of ICAOR, vol. II, Transilvania Press, Cluj-Napoca, 1997, pp. 247-252.
46. Vlaic, G., Multivariate Approximation by *Stancu type operators* on a simplex, Bul. St. "Politehnica", Timişoara, vol. 41(55)(1996), pp. 247-252.

47. Vlaic, G., On a bivariate Multiparameter *Approximation Operator of D.D. Stancu,* Studii şi Cercetări Matematice, Academia Română (sub tipar).

48. Walz, G., Trigonometric Bezier and *Stancu polynomials* over intervals and triangles, Comput. Aided Geom. Design, vol. 14(1997), pp. 393-397.

49. Zhang, C., Asymptotic formulae for approximation error for smooth functions by *Stancu* operator on Simplex, J. Math, Res. Expo, vol. 17(1997), nr. 4, pp. 573-577.

50. Zhao, J., L_{sp} saturation for *Stancu*-Kantorovich operators, J. Math. Wuhan Univ., vol. 8(1988), nr. 3, pp. 257-262.

51. Xue, Y., L- approximation by the iterates of *Stancu*-Kantorovich operators, J. Math. Res. Expo. vol. 17, nr. 4, 1997, pp. 570-572.

52. Xiong, J.; Yang. R.; Cao, F., Approximation theorems by *Stancu*-Kantorovich polynomials on a simplex, J. Qufu Norm. Univ. Nat. Sci., vol. 19(1993), nr. 4, pp. 29-34.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

APPROXIMATION PROPERTIES OF A BIVARIATE STANCU TYPE OPERATOR

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Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

Abstract. An extension of Stancu's operator $P_m^{(\alpha,\beta)}$ to the case of bivariate functions is presented and some approximation properties of this operator are discussed.

1. **Preliminaries**

In 1969 (see[8]), D.D. Stancu constructed and studied a linear and positive operator, depending on two positive parameters α and β which satisfy the condition $0 \le \alpha \le \beta$. This operator, denoted by $P_m^{(\alpha,\beta)}$, associates to any function $f \in C([0,1])$ the polynomial $P_m^{(\alpha,\beta)} f$, defined by:

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{mk}(x) \ f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.1}$$

where $p_{mk}(x)$ are the fundamental Bernstein polynomials. In the monograph by F. Altomare and M. Campiti ([1]) this operator is called "the operator of Bernstein-Stancu".

A first extensions of the operator (1.1) to the case of bivariate functions was given by F. Stancu in her doctoral thesis (see [9]). The aim of the present paper is to extend the operator (1.1) to the case of B-continuous (Bőgel continuous functions). More exactly, we shall present a GBS (Generalized Boolean Sum) operator of Stancu type and some properties of this operator.

Received by the editors: 17.09.2002.

2000 *Mathematics Subject Classification.* 41A10, 41A36, 41A63, 41A80.

Key words and phrases. Stancu's operators, Korovkin theorem, bivariate function, modulus of smoothness, Voronovskaia theorem.

The terminus of "B-continuous function" was introduced by K. Bőgel ([5],[6]). A first result concerning the approximation of this kind of functions is due to E. Dobrescu and I. Matei ([7]).

An important "test function theorem", (the analogous of the well known Korovkin theorem), for the approximation of B-continuous functions by GBS operators was introduced by C. Badea and C. Cottin ([3)]. Approximation properties of the GBS operators were studied by C. Badea, C. Cottin, H.H. Gonska, D. Kacsó and many others.

2. **The GBS operator of Stancu type**

Let be $I = [0,1]$ and let $I^2 = [0,1] \times [0,1]$ be the unit square. The space of all B-continuous functions on I^2 will be denoted by $C_b(I^2)$.

Next, we consider four non-negative parameters $\alpha_1, \beta_1, \alpha_{2,}\beta_2$,satisfying the conditions $0 \leq \alpha_1 \leq \beta_1, 0 \leq \alpha_2 \leq \beta_2$. If $f \in C_b(I^2)$, the parametric extensions of the operator $P_m^{(\alpha,\beta)}$are defined respectively by:

$$\left({}_xP_m^{(\alpha_1,\beta_1)} f\right)(x,y) = \sum\nolimits_{k=0}^{m} p_{mk}(x) f\left(\frac{k+\alpha_1}{m+\beta_1}, y\right), \tag{2.1}$$

$$\left({}_yP_n^{(\alpha_2,\beta_2)} f\right)(x,y) = \sum\nolimits_{l=0}^{n} p_{nl}(y) f\left(x, \frac{l+\alpha_2}{n+\beta_2}\right). \tag{2.2}$$

It is easy to see that ${}_xP_m^{(\alpha_1,\beta_1)}$and ${}_yP_n^{(\alpha_2,\beta_2)}$are linear and positive operators, well defined on $C_b(I^2)$.

Let $L_{m,n} : C_b(I^2) \to C_b(I^2)$ be the tensorial product of ${}_xP_m^{(\alpha_1,\beta_1)}$ and ${}_yP_n^{(\alpha_2,\beta_2)}$, i.e.

$$L_{m,n} =_x P_{my}^{(\alpha_1,\beta_1)} \circ P_n^{(\alpha_2,\beta_2)}. \tag{2.3}$$

Then, $L_{m,n} : C_b(I^2) \to C_b(I^2)$ associates to any $f \in C_b(I^2)$ the bivariate polynomial

$$L_{m,n}\, f(x,y) = \sum\nolimits_{k=0}^{m} \sum\nolimits_{l=0}^{n} p_{mk}(x)p_{n,l}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{l+\alpha_2}{n+\beta_2}\right) \tag{2.4}$$

It is well known (see for example [4] or [10]) that the operator (2.4) has the following properties:

Lemma 2.1. If $e_{ij} : I^2 \to \mathbb{R}$ $(i, j \in \mathbb{N}, 0 \le i + j \le 2)$ are the test functions the following equalities hold

(i) $(L_{m,n}e_{00})(x, y) = 1$;

(ii) $(L_{m,n}e_{10})(x, y) = x + \frac{\alpha_1-\beta_1 x}{m+\beta_1}$;

(iii) $(L_{m,n}e_{01})(x, y) = y + \frac{\alpha_2-\beta_2 y}{n+\beta_2}$;

(iv) $(L_{m,n}e_{20})(x, y) = x^2 + \frac{mx(1-x)+(\alpha_1-\beta_1 x)(2mx+\beta_1 x+\alpha_1)}{(m+\beta_1)^2}$;

(v) $(L_{m,n}e_{02})(x, y) = y^2 + \frac{ny(1-y)+(\alpha_2-\beta_2 y)(2ny+\beta_2 y+\alpha_2)}{(m+\beta_2)^2}$;

for any $(x, y) \in I^2$.

Lemma 2.2 *The operator (2.4) is linear and positive.*

Definition 2.1. Let $S_{m,n} : C_b(I^2) \to C_b(I^2)$ be the boolean sum of ${}_xP_m^{(\alpha_1,\beta_1)}$ and ${}_yP_n^{(\alpha_2,\beta_2)}$, i.e.

$$S_{m,n} =_x P_m^{(\alpha_1,\beta_1)} +_y P_n^{(\alpha_2,\beta_2)} -_x P_m^{(\alpha_1,\beta_1)} \circ_y P_n^{(\alpha_2,\beta_2)} \tag{2.5}$$

The operator $S_{m,n}$ will be called GBS operator of Stancu type.

By direct computation, one obtains:

Lemma 2.3. If $S_{m,n} : C_b(I^2) \to C_b(I^2)$ is the GBS operator of Stancu type, then

$$\begin{aligned} &(S_{m,n}f)(x,y) = \\ &\sum\nolimits_{k=0}^{m} \sum\nolimits_{l=0}^{n} p_{mk}(x)\, p_{nl}(y) \times \left\{ f\left(\tfrac{k+\alpha_1}{m+\beta_1}, y\right) \right. \\ &\left. + f\left(x, \tfrac{l+\alpha_2}{n+\beta_2}, y\right) - f\left(\tfrac{k+\alpha_1}{m+\beta_1}, \tfrac{l+\alpha_2}{n+\beta_2}\right) \right\} \end{aligned} \tag{2.6}$$

for any $f \in C_b(I^2)$ and any $(x, y) \in I^2$.

Remark 2.1. For $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$, the GBS operator of Stancu type is reduced to the GBS operator of Bernstein type, which interpolates any function $f \in C_b(I^2)$ on the boundary of the unit square I^2. If $\alpha_1 = \beta_1 = 0$ and $\alpha_2 \ne 0, \beta_2 \ne 0$, the corresponding operator interpolates any $f \in C_b(I^2)$ on the left and respectively on the right side of the boundary of unit square I^2. Others particular cases of the GBS operator of Stancu type can be discussed in a similar way.

Theorem 2.1. *For any $f \in C_b(I^2)$,the sequence $\{S_{m,n}f\}_{m,n\in\mathbb{N}}$ converges to f, uniformly on I^2 as m and n tend to infinity*

Proof. Let us to introduce the following notations

$$\begin{aligned} u_m(x) &= \frac{\alpha_1 - \beta_1 x}{m+\beta_1}, \\ v_n(y) &= \frac{\alpha_2 - \beta_2 y}{n+\beta_2}, \\ w_m, n(x,y) &= x^2 + y^2 + \frac{mx(1-x) + (\alpha_1 - \beta_1 x)(2mx + \beta_1 + \alpha_1)}{(m+\beta_1)^2} \\ &+ \frac{ny(1-y) + (\alpha_2 - \beta_2 y)(2ny + \beta_2 + \alpha_2)}{(n+\beta_2)^2}. \end{aligned}$$

Then the results contained in Lemma 2.1 can be written in the form

$$\begin{aligned} &(L_{m,n}e_{00})(x,y) = 1; \\ &(L_{m,n}e_{10})(x,y) = x + u_m(x); \\ &(L_{m,n}e_{01})(x,y) = y + v_n(y); \\ &(L_{m,n}(e_{20}+e_{02}))(x,y) = x^2 + y^2 + w_{m,n}(x,y), \text{ for any } (x,y) \in I^2. \end{aligned}$$

Because the sequences $\{u_m(x)\}_{m\in\mathbb{N}}$, $\{v_n(x)\}_{n\in\mathbb{N}}$ and $\{w_{m,n}(x)\}_{m,n\in\mathbb{N}}$ tend to zero, uniformly on I^2 as m and n tend to infinity, we can apply the Korovkin - type theorem for the approximation of B-continuous functions due C.Badea, I.Badea and H.H.Gonska (see [2]. Applying this theorem, it follows that $S_{m,n}f$ tend to f, uniformly on I^2, for any $f \in C_b(I^2)$ as m and n tend to infinity.

Next the approximation order of any function $f \in C_b(I^2)$ by $S_{m,n}f$ will be established, using the mixed modulus of smoothness (see [3]). We need the following result, due to C. Badea and C. Cottin [see [3]).

Theorem 2.2. *Let X and Y be compact real intervals. Furthermore, let $L : C_b(X,Y) \to C_b(X,Y)$ be a positive linear operator and U the associated GBS operator. Then, for all $f \in C_b(X,Y)$, $(x,y) \in X \times Y$ and $\delta_{1,}\delta_2 > 0$ the inequality*

$$\begin{aligned} &|(f - Uf)(x,y)| \le |f(x,y)| \cdot |1 - L(x;x,y| + \\ &\{L(1;x,y) + \tfrac{1}{\delta_1}\sqrt{L((x-\circ)^2;x,y)} + + \tfrac{1}{\delta_2}\sqrt{L((y-*)^2;x,y)} + \\ &+ \tfrac{1}{\delta_1\delta_2}\sqrt{L((x-\circ)^2(y-*)^2;x,y)}\ \}\,\omega_{mixed}(\delta_1,\delta_2) \end{aligned} \tag{2.7}$$

holds.

Lemma 2.4. The bivariate operator of Stancu verifies the following equalities:

(i) $L_{m,n}((x-\circ)^2;x,y)=\frac{mx(1-x)+(\alpha_1-\beta_1 x)^2}{(m+\beta_1)^2}$;

(ii) $L_{m,n}((y-*)^2;x,y)=\frac{ny(1-y)+(\alpha_2-\beta_2 y)^2}{(n+\beta_2)^2}$;

(iii) $L_{m,n}((x-\circ)^2(y-*)^2=\frac{1}{(m+\beta_1)^2(n+\beta_2)^2}\times$
$\left\{mx(1-x)+(\alpha_1-\beta_1 x)^2\right\}\times\left\{ny(1-y)+(\alpha_2-\beta_2 y)^2\right\}.$

Proof. The equalities follow from the linearity of L_{mn} and Lemma 2.1. □

Theorem 2.3. *The GBS operators of Stancu S_{mn} verify the inequality:*

$$\begin{aligned}&|S_{m,n}f(x,y)-f(x,y)|\le\\&\left\{\frac{1}{\delta 1}\cdot\frac{1}{m+\beta_1}\sqrt{\frac{m}{4}+(\alpha_1-\beta_1 x)^2}+\frac{1}{\delta_2}\sqrt{\frac{n}{4}+(\alpha_2-\beta_2 y)^2}+\right.\\&\left.+\frac{1}{\delta_1\delta_2}\cdot\frac{1}{(m+\beta_1)(n+\beta_2)}\sqrt{\{\frac{m}{4}+(\alpha_1-\beta_1 x)^2\}\{\frac{n}{4}+(\alpha_2-\beta_2 y)^2\}}\ \right\}\times\\&\times\omega_{mixed}(\delta_1\delta_2),\end{aligned}\tag{2.8}$$

f*or any* $\delta_1,\delta_2>0$ *and any* $(x,y)\in I^2$.

Proof. We apply the Lemma 2.4 and the inequalities $x(1-x)\le\frac{1}{4},\ y(1-y)\le\frac{1}{4}$ for any$(x,y)\in I^2$.□

Remark 2.2. The inequality (2.8) give us the order of the local approximation of f by $S_{m,n}f$.

The order of the global approximation of $f\in C_b(I^2)$ by $S_{m,n}f$ is expressed in

Theorem 2.4. The GBS operator of Stancu verify the following inequality:

$$|S_{m,n}f(x,y)-f(x,y)|\le\frac{9}{4}\omega_{mixed}\left(\frac{\sqrt{m+4\alpha_1^2}}{m+\beta_1},\frac{\sqrt{n+4\alpha_2^2}}{n+\beta_2}\right)\tag{2.9}$$

Proof. Taking into account that $(\alpha_1-\beta_1 x)^2\le\alpha_1^2$ and $(\alpha_2-\beta_2 y)^2\le\alpha_1^2$ for any $(x,y)\in I^2$, from Theorem 2.3, we get:

$$\begin{aligned}&|S_{m,n}f(x,y)-f(x,y)|\le\\&\left\{\frac{1}{2\delta_1}\frac{\sqrt{m+4\alpha_1^2}}{m+\beta_1}+\frac{1}{2\delta_2}\frac{\sqrt{n+4\alpha_2^2}}{n+\beta_2}+\frac{\sqrt{(m+4\alpha_1^2)(n+4\alpha_2^2)}}{4\delta_1\delta_2(m+\beta_1)(m+\beta_2)}\right\}\omega_{mixed}(\delta_1\delta_2).\end{aligned}$$

Choosing then

$$\delta_1 = \frac{\sqrt{m+4\alpha_1^2}}{m+\beta_1}; \qquad \delta_2 = \frac{\sqrt{n+4\alpha_2^2}}{n+\beta_2};$$

it follows (2.9) and the proof ends □.

Remark 2.3. The inequality (2.9) can be more rafinated, taking into account of the values of $\alpha_{1,}\alpha_2$ with respect β_1 and β_2.

References

[1] Altomare, F., Campiti, M., *Korovkin -type Approximation Theory and its Application*, de Gruyter Series Studies in Mathematics, Vol.17, Walter de Gruyter &Co, Berlin, New-York, 1994.

[2] Badea, C., Badea, I., Gonska, H.H., *A test function theorem and approximation by pseudopolynomials*, Bull. Austral. Math. Soc. 34(1986), 53-64.

[3] Badea, C., Cottin., *Korovkin-type theorems for Generalided Boolean Sum Operators*, Colloquia Mathematica Societatis Janos Bolyai, 58, Approximation Theory, Kecskemet(Hungary)(1990), 51-68.

[4] Barbosu, D., *A Voronovskaia-type theorem for the bivariate operator of Stancu* (submitted).

[5] Bőgel, K., *Mehrdimensionale Differentiation von Funktionen mehrer Veränderlicher*, J. Reine Angew. Math.170(1934), 197-217.

[6] Bőgel, K., *Ubër die mehrdimensionale Differentiation Integration und beschrankte Variation*, J. Reine Angew. Math.173(1935), 5-29.

[7] Dobrescu, E., Matei, I., *Aproximarea prin polinoame de tip Bernstein a funcţiilor bidimensional continue*, Anal. Univ. Timişoara, Seria Ştiinţe matematice fizice, Vol. IV(1966), 85-90.

[8] Stancu, D.D., *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Universitatis Babes-Bolyai, 14(1969), 31-45(in Romanian).

[9] Stancu, D. D., Coman, Gh., Agratini, O., Trâmbiţaş, R., *Analiză numerică şi teoria aproximării*, Vol. I, Presa Universitară Clujeană, 2001.

[10] Stancu, F., *Aproximarea funcţiilor de două şi mai multe variabile cu ajutorul operatorilor liniari pozitivi*, Ph. D. Thesis, Cluj-Napoca (1984) (in Romanian).

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FIXED POINTS OF R-CONTRACTIONS

ANTAL BEGE

Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. Let X be a set and $R = (R_n)_{n\geq 0}$, $R_n \subset X \times X$ a sequence of binary relations on X. The operator $f : X \longrightarrow X$ is R-contraction if

$$(x, y) \in R_n \Longrightarrow (f(x), f(y)) \in R_{n+1}.$$

The first theorem concerning R-contraction is due to Eilenberg [2]. Further I. A. Rus [7] and Grudzinski [3] generalize this concept. We prove some results which generalize the theorems in [7] and [3] under certain conditions.

1. Introduction

Let X be a set, $f : X \longrightarrow X$ an operator and F_f be a fixed point set of f:

$$F_f := \{x \in X \mid f(x) = x\}.$$

We introduce the following notations:

$$\Delta(X) := \{(x, x) \mid x \in X\},$$

$$f^0 = 1_X,\ f^1 = f,\ f^n(x) := f(f^{n-1}(x)), \quad n \geq 2.$$

Let X be a nonempty set, $R_n \subset X \times X$ a sequence of symmetric binary relations on X. Throughout this paper we suppose that:

a)

$$X \times X = R_0 \supset R_1 \supset ... \supset R_n \supset ...$$

b)

$$\bigcap_{n=0}^{\infty} R_n = \Delta(x) = \{(x, x) \,|\, x \in X\}.$$

Received by the editors: 01.09.2002.

2000 *Mathematics Subject Classification.* 06A10, 47H10.

Key words and phrases. R-contraction, fixed point.

Firstly Eilenberg [2] proved the discrete version of Banach fixed point theorem. Later I. A. Rus [7] introduced the concept of R-contractions:

Definition 1. The operator $f : X \longrightarrow X$ is R-contraction if

$$(x, y) \in R_n \Longrightarrow (f(x), f(y)) \in R_{n+1}.$$

I. A. Rus [7], [6], [8] and indepently I. A. Grudzinsky [3] proved fixed point theorems for R-contractions (see Bege [1]).

In this paper we generalize the concept of R-contractions and we prove some fixed point theorems for this contractions.

2. **Generalized R-contractions**

In this section we introduce the concept of generalized R-contraction and give some examples.

Definition 2. Let $X \neq \emptyset$, $R_n \subset X \times X$, $\quad n \in \mathbb{N}$. We say that $f : X \longrightarrow X$ **generalized R-contraction of the type** d_i if

d$_1$) $(x, f(x)) \in R_n, \quad (y, f(y)) \in R_n$

$$\Rightarrow (f(x), f(y)) \in R_{n+1}$$

d$_2$) $(x, y) \in R_n,\ (x, f(x)) \in R_n, \quad (y, f(x)) \in R_n$

$$\Rightarrow (f(x), f(y)) \in R_{n+1}$$

d$_3$) $(x, y) \in R_n, \quad (x, f(y)) \in R_n$

$$\Rightarrow (f(x), f(y)) \in R_{n+1}$$

d$_4$)

$$(x, f(x)) \in R_n \Rightarrow (f(x), f^2(x)) \in R_{n+1}$$

We remark that if an operator is R-contraction then it is a generalized d_4 contraction.

In the following part of this section we present some examples concerning R-contractions and generalized contractions.

Example 1 (S. Reich [5])
Let (X,d) be a metric space and $a,b,c \in \mathbb{R}_+,\ a+b+c<1$ such that

$$d\left(f(x),f(y)\right) \leq a \cdot d\left(x,y\right) + b \cdot d\left(x,f(x)\right) + c \cdot d\left(y,f(y)\right), \quad \forall x,y \in X.$$

If

$$R_n = \left\{(x,y) \in X \times X \mid d(x,y) \leq \frac{a+b}{1-c} \cdot (a+b+c)^n \cdot \delta(X)\right\},$$

$$Y = \left\{x \in X \mid d(x,f(x)) \leq \frac{a+b}{1-c} \cdot \delta\left(X\right)\right\} \neq \emptyset$$

then R_n satisfies the conditions (a) and (b) and f generalized contraction of the type d_2.

Example 2 (R. Kannan [4])
Let (X,d) be a metric space, and $f: X \longrightarrow X$ one operator for which exist $a \in \mathbb{R},\ a<\frac{1}{2}$, such that:

$$d(f(x),f(y)) \leq a \cdot [d(x,f(x)) + d(y,f(y))], \quad \forall x,y \in X.$$

If

$$R_n = \left\{(x,y) \in X \times X \mid d(x,y) \leq \frac{a}{1-a} \cdot (2a)^n \cdot \delta(X)\right\},$$

$$Y = \left\{x \in X \mid d(x,f(x)) \leq \frac{a}{1-a} \cdot \delta\left(X\right)\right\} \neq \emptyset,$$

then R_n satisfies the conditions $a)$ şi $b)$ and f generalized R-contraction of the type d_1.

3. Main results

Theorem 3. *Let X be a nonempty set, $R_n \subset X \times X$ a sequence of symmetrical binary relations on X, satisfying the conditions* **a) b)** *and*
c) *If $(x_n)_{n\geq 0}$ is a sequence in X such that $(x_n, x_{n+k}) \in R_n, \quad \forall n,k \geq 0$, then there exist unique $x \in X$ satisfying the condition $(x_n, x) \in R_n, \quad \forall n \geq 0$.*
Let $f: X \longrightarrow X$ be a generalized R-contraction of type d_3. Then f has an unique fixed point.

Proof.
Let $x_0 \in X$, $x_n = f\left(x_{n-1}\right),\ \forall n \geq 1$.

From the form of R_0 and definition 2 we have:

$$(x_0, x_1) \in R_0,\ (x_0, x_2) \in R_0 \Longrightarrow (x_1, x_2) = (f(x_0), f(x_1)) \in R_1,$$

$$(x_0, x_2) \in R_0,\ (x_0, x_3) \in R_0 \Longrightarrow (x_1, x_3) = (f(x_0), f(x_2)) \in R_1.$$

From mathematical induction follows that: $(x_1, x_{n+1}) \in R_1,\ \forall n \geq 0.$

But

$$(x_1, x_n) \in R_1,\ (x_1, x_{n+1}) \in R_1 \Longrightarrow (x_2, x_{n+1}) \in R_2, \quad \forall n \geq 1$$

and generally

$$(x_k, x_{k+n}) \in R_k, \quad \forall k \geq 0, \forall n \geq 0.$$

Condition $\mathbf{c}_1$**)** implies the existence of unique $x^* \in X$ such that $(x^*, x_n) \in R_n,\ \forall n \geq 0.$

But

$$(x^*, x_n) \in R_n,\ (x^*, x_{n+1}) \in R_{n+1} \subset R_n \Longrightarrow (f(x^*), x_{n+1}) \in R_{n+1},\ \forall n \geq 0.$$

Because x^* is unique, $x^* = f(x^*)$.

If we have $y^* \in X$, for which $y^* = f(y^*)$, then

$$(x^*, y^*) = (x^*, f(y^*)) \in R_0 \Longrightarrow (f(x^*), f(y^*)) = (x^*, y^*) \in R_1$$

Similarly $(x^*, y^*) \in R_n$ for all n.

From **b)** we have $x^* = y^*$.

Corollary 4. ([7], Theorem 2.1) *If $f : X \longrightarrow X$ is a R-contraction, and $R_n \subset X \times X$, $n \in N$, a sequence of binary symmetrical relations, satisfying the conditions* **a) b)** *and* **c)** *, then:*

$$F_f = \{x^*\}$$

and

$$(f^n(x_0), x^*) \in R_n, \quad \forall x_0 \in X,\ n \in N.$$

Theorem 5. *Let X be a nonempty set and $R_n \subset X \times X$, $n \in N$ a sequence of symmetrical binary relations on X, satisfying the conditions* **a)**, **b)**,

$\mathbf{c}_1$*)*

If $(x_n)_{n\geq 0}$ is a sequence in X such that $(x_n, x_{n+k}) \in R_n$ for all n, $k \in N$ then there *exist unique* $x \in X$ *for which* $(x_n, x) \in R_n$, $\forall n \in N$.

If $f : X \longrightarrow X$ *is a generalized R-contraction of type* $\mathbf{d}_1$*) and satisfies the following condition:*

e)

For every $x_0 \in X$

$$(f^n(x_0), x) \in R_n \Longrightarrow \left(f^{n+1}(x_0), f(x)\right) \in R_{n+1} \quad (n \in N).$$

Then f has an unique fixed point.

Proof.

In same way (see the proof of theorem 1) we have that if $x_0 \in X$, $x_n = f(x_{n-1})$, $\forall n \geq 1$ then:

$$(x_k, x_{k+n}) \in R_k,\ \forall k \geq 0,\ \forall n \geq 0.$$

The condition $\mathbf{c}_1$) implies the existence of the unique $x^* \in X$ such that $(x^*, x_n) \in R_n,\ \forall n \geq 0$.

But from **e)** :

$$(x_n, x^*) = (f^n(x_0), x^*) \in R_n \Longrightarrow \left(f^{n+1}(x_0), f(x^*)\right) = (x_{n+1}, f(x^*)) \in R_{n+1}.$$

We have $(x_0, f(x^*)) \in R_0$ so $(x_n, f(x^*)) \in R_n$ for all n. The uniqueness of x^* implies $x^* = f(x^*)$.

In the next we prove the uniqueness of the fixed point:

Let $x^*, y^* \in F_f$. From **b)** $(x^*, f(x^*)) \in R_n$ and $(y^*, f(y^*)) \in R_n$ for all $n \geq 0$. This implies that $(x^*, y^*) \in R_n$ (f generalized R-contraction of type $\mathbf{d}_1$)). So $x^* = y^*$.

Theorem 6. *Let X be a nonempty set and* $R_n \subset X \times X$, $n \in N$ *a sequence of symmetrical binary relations on X, satisfying the conditions* **a)**, **b)**,

$\mathbf{c}_2$*)*

If $(x_n)_{n \geq 0}$ *is a sequence in X such that* $(x_n, x_{n+k}) \in R_n$ *for all* n, $k \in N$ *then there exist* $x \in X$ *(not necessary unique) for which* $(x_n, x) \in R_n$, $\forall n \in N$.

f) *For all* $x, y, z \in X$, $n \in N$

$$(x, y) \in R_{n+k}, \quad (y, z) \in R_{n+k} \Longrightarrow (x, z) \in R_n.$$

If $f : X \longrightarrow X$ *is a generalized R-contraction of type* $\mathbf{d}_3$*) then* $F_f = \{x^*\}$.

Proof.

We consider the iterares of f in x_0: $x_n = f(x_{n-1})$, $\forall n \geq 1$.

From the first part of the proof of Theorem 1, there exist $x^* \in X$ such that

$$(x^*, x_{n+k}) \in R_n \quad \forall n \geq 0.$$

f generalized R-contraction of type $\mathbf{d}_3$**)** which implies:

$$(x^*, x_{n+2k}) \in R_{n+k},\ (x^*, x_{n+2k+1}) \in R_{n+k+1} \subset R_{n+k} \Longrightarrow$$

$$\Longrightarrow (f(x^*), x_{n+2k+1}) \in R_{n+k+1} \subset R_{n+k}.$$

From condition **f)**:

$$(x^*, x_{n+2k+1}) \in R_{n+k}, \quad (f(x^*), x_{n+2k+1}) \in R_{n+k} \Longrightarrow (x^*, f(x^*)) \in R_n,$$

$$(x^*, f(x^*)) \in \bigcap_{n \in N} R_n = \Delta(x)$$

which implies $x^* = f(x^*)$.

The proof of uniqueness is same with the proof in Theorem 1.

Corollary 7. (Grudzinski [3]) *Let X be a nonempty set and $R_n \subset X \times X$, $n \in N$ a sequence of reflexive and symmetrical binary relations on X, satisfying the conditions* **a), b), c$_2$), f)** . *Let $f : X \longrightarrow X$ be R-contraction.*

Then f has an unique fixed point.

References

[1] A. Bege, *Discrete fixed point theory and applications* (Romanian), Presa Univ. Clujeană, 2002.

[2] J. Dugundji, A. Granas, *Fixed point theory*, Vol. 1, P. W. N., Warsawa, 1982.

[3] W. Grudzinski, On the discrete Banach principle, *Zeszyty Nauk. Politech. Lodz Mat.*, **26** (1994), 81–88.

[4] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71–76.

[5] S. Reich, Some remarks concerning contraction mappings, *Canad Math. Bull.*, **14** (1971), 121–124.

[6] I. A. Rus, Discrete fixed point theorems, *Studia Univ. Babeş-Bolyai, Math.*, **33** (1988), 61–64.

[7] I. A. Rus, R-contractions, *Studia Univ. Babeş-Bolyai, Math.*, **34** (1989), 58–62.

[8] I. A. Rus, A. Petruşel, G. Petruşel, *Fixed point theory 1950-2000, Romanian contributions*, House of the Book of Science, Cluj-Napoca, 2002.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

An adaptive cubature on triangle

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Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. Starting from an elementary cubature formula on triangle which is exact on $\mathbb{P}_2^2$ (bivariate polynomials having global degree 2) an adaptive cubature method is devised. Also a MATLAB implementation is given.

1. Introduction

Let us consider the triangle Δ with the vertices V_i, $i = \overline{1,3}$, and the cubature formula (see figure 1):

$$\begin{aligned} P_i &= \frac{1}{2}\left(V_j + V_k\right), \qquad \{i,j,k\} = \{1,2,3\}, \\ I &= \int_T f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{\mathrm{area}(\Delta)}{6}\sum_{i=1}^{3} f(P_i). \end{aligned} \tag{1}$$

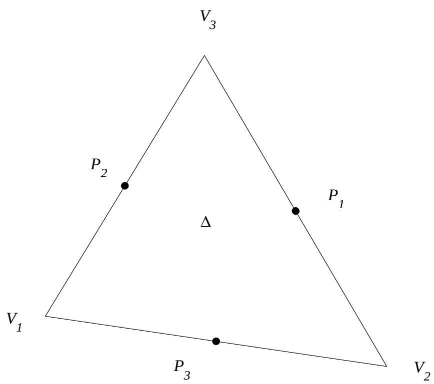

FIGURE 1. The triangle and edges' midpoints.

It can be easily seen that P_i, $i = \overline{1,3}$, are the midpoints of the triangle edges and the formula is exact for each $f \in \mathbb{P}_2^2$ (bivariate polynomials having total degree

Received by the editors: 01.09.2002.

equal to 2). We shall try to turn this formula into and adaptive cubature algorithm. The idea is to combine two elementary cubature formula, one which subevaluates I and one which overestimates it. If the absolute value of the difference of the results provided by these formulae is less than a given tolerance ε, we stop, and the result is the value given by the second formula. Otherwise, we proceed with a subdivision of the triangle, and apply the same method to each triangle of the subdivision.

2. The algorithm

For a detailed description of an adaptive numerical integration algorithm see [2, pp. 166–170]. We can decompose our triangle, denoted by Δ, into four triangles, Δ_1, Δ_2, Δ_3 and Δ_4 determined by vertices and the middle points (a Delaunay triangulation, [1], see Figure 2). The first step will be the formula given by (1) applied to Δ and the second step will be the same, but applied to each of the four triangle of the triangulation, Δ_i, $i = \overline{1,4}$. Let I_1 be the value provided by the elementary formula

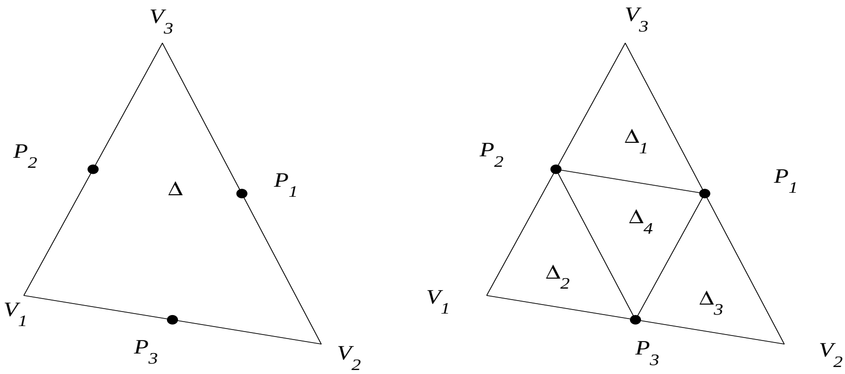

FIGURE 2. Initial triangle and the subdivision

(1), and I_2 the value obtained summing the four values obtained applying (1) to each triangle of the subdivision. A stopping criterion could be

$$|I_1 - I_2| < \varepsilon,$$

where ε is the desired tolerance. If the criterion is not fulfilled, then we apply the same procedure recursively to each triangle of the subdivision.

The detailed description is given in Algorithm 1.

Algorithm 1 Adaptive cubature algorithm on triangle; call $result := \text{adapt}(f, \Delta, \varepsilon)$, where f is the function, Δ is the triangle and ε is the desired tolerance; elem_formula implements the elementary cubature, given by (1)

Let $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ be the triangles determined by vertices and midpoints

$I_1 := \text{elem\_formula}(f, \Delta)$;

$I_2 := \text{elem\_formula}(f, \Delta_1) + \text{elem\_formula}(f, \Delta_2) +$
$\quad \text{elem\_formula}(f, \Delta_3) + \text{elem\_formula}(f, \Delta_4)$;

if $|I_1 - I_2| < \varepsilon$ **then**

$\quad result := I_2$;

else

$\quad result := \text{adapt}(f, \Delta_1, \varepsilon) + \text{adapt}(f, \Delta_2, \varepsilon) +$
$\quad\quad \text{adapt}(f, \Delta_3, \varepsilon) + \text{adapt}(f, \Delta_4, \varepsilon)$;

end if

3. The implementation

For adaptive cubature on triangle implementation see [3, 4].

We have implemented this algorithm in MATLAB[1]. The implementation follows the description given by algorithm 1. We introduced an auxiliary input parameter, `trace`, which (when is nonzero) allows us to obtain information about the execution and to represent graphically the process of computing. When trace is set, the additional output parameter, `stat` gives us the number of function evaluation and the number of triangles. Some optimizations which save several function evaluations are possible. Since the value I_1 is the value of the integral on the triangle Δ_4, we compute it once and give it as an input parameter further. We can do the same thing with the values of function in midpoints. Here is the MATLAB source code:

```
function [vi,stat]=mpcubatd2vb(f,x,y,err,trace)
%cubature with middle points, exact for P_2^2%
%call vi=mpcubatd2(f,x,y,err,trace)
%f - the function
%x,y - coordinates of vertices
```

[1] MATLAB©is a trademark of MathWorks, Inc., Natick, MA 01760-2098

```
%err - the error
%trace - tracing  indicator

global FEN TRIN

if nargin<5
    trace=0;
else
    if trace
        clf
        FEN=0; TRIN=0;
    end
end
if nargin < 4
    err=1e-3;
end
[xp,yp]=midpoints(x,y); %subdivision
fp=feval(f,xp,yp);
area=1/2*abs(det([x(:),y(:),ones(3,1)]));
I1=area/3*sum(fp);
vi=quadrg2(f,x,y,xp,yp,fp,err,area,I1,trace);
if trace & (nargout==2)
    FEN=FEN+3;
    stat=struct('nev',FEN,'ntri',TRIN);
end

function vi=quadrg2(f,x,y,xp,yp,fp,err,area,I1,trace)
%cubature with midpoints, internal use
%call vi=quadrg2(f,x,y,xp,yp,fp,err,trace)
%f - the function
%x,y - coordinates of vertices
%xp,yp - midpoints  coordinates
```

```
%fp - value of f in barycenter
%err - the error
%area - area of the triangle
%I1 - the first estimation (elementary formula)
%trace - tracing  indicator

global FEN TRIN
if trace
    fill(x,y,'r','FaceColor','none','EdgeColor','k'); hold on;
    axis equal
    plot(xp,yp,'ok');
    %pause
end
v1x=[xp(2),xp(1),x(3)]; v1y=[yp(2),yp(1),y(3)];
[P1Mx,P1My]=midpoints(v1x,v1y);
v2x=[x(1),xp(3),xp(2)]; v2y=[y(1),yp(3),yp(2)];
[P2Mx,P2My]=midpoints(v2x,v2y);
v3x=[xp(3),x(2),xp(1)]; v3y=[yp(3),y(2),yp(1)];
[P3Mx,P3My]=midpoints(v3x,v3y);
[P4Mx,P4My]=midpoints(xp,yp);
fP1M=feval(f, P1Mx, P1My);
fP2M=feval(f, P2Mx, P2My);
fP3M=feval(f, P3Mx, P3My);
if trace
    FEN=FEN+9;
    TRIN=TRIN+4;
end
zz=area/12;
arean=area/4;
I11=zz*sum(fP1M);
I12=zz*sum(fP2M);
I13=zz*sum(fP3M);
```

```
I14=zz*(fP1M(3)+fP2M(1)+fP3M(2));
I2=I11+I12+I13+I14;
if abs(I2-I1)<err
  vi=I2;
else
 vi=quadrg2(f,v1x,v1y,P1Mx,P1My,fP1M, err,arean,I11,trace)+...
     quadrg2(f,v2x,v2y,P2Mx,P2My,fP2M,err,arean,I12,trace)+...
     quadrg2(f,v3x,v3y,P3Mx,P3My,fP3M,err,arean,I13,trace)+...
     quadrg2(f,xp,yp,P4Mx, P4My,[fP1M(3),fP2M(1),fP3M(2)],+...
       err,arean,I14,trace);
end %if

function [bx,by]=midpoints(x,y)
bx=[x(2)+x(3),x(1)+x(3),x(1)+x(2)]/2;
by=[y(2)+y(3),y(1)+y(3),y(1)+y(2)]/2;
```

4. **A numerical example**

We wish to approximate

$$\int_{\Delta} f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

where Δ is the triangle with vertices $V_1(0,0)$, $V_2(1,0)$, $V_3(0,1)$ and f is

$$f(x,y) = y\sin x.$$

The graph of f is given in Figure 3.

The exact value is

$$\cos(1) - 1/2 \approx 0.04030230586814,$$

obtained with the following Maple session:

```
> Digits:=20;
```

$$Digits := 20$$

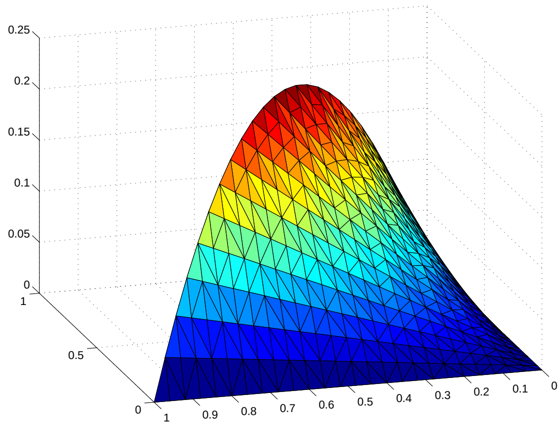

FIGURE 3. the graph of f

```
> w:=int(int(sin(x)*y,y=0..1-x),x=0..1);
```

$$w := \cos(1) - \frac{1}{2}$$

```
> evalf(w);
```

$$.04030230586813971740$$

For $\varepsilon = 10^{-3}$ we need no subdivision, as we can see from the following MATLAB session fragment:

```
>> [vi2,stat]=mpcubatd2vb(@fintegr,x,y,1e-3,1)


vi2 =


   0.04028255698461




stat =


     nev: 12
    ntri: 4
```

For $\varepsilon = 10^{-4}$ a subdivision was done (see Figure 4). The results are as follows:

```
>> [vi2,stat]=mpcubatd2vb(@fintegr,x,y,1e-4,1)


vi2 =


   0.04030110314738




stat =


      nev: 48
     ntri: 20
```

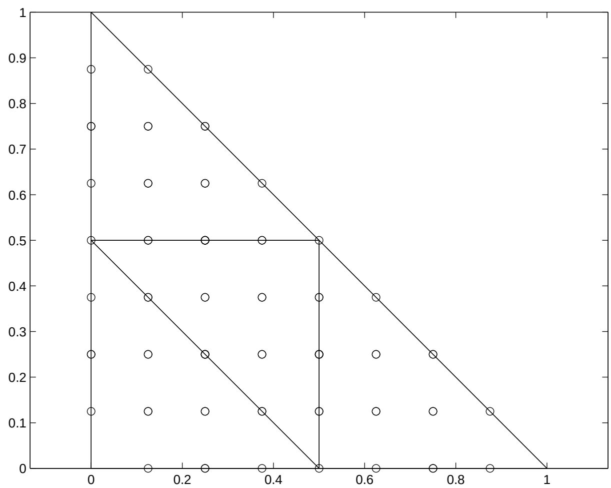

Figure 4. Computing the integral for $\varepsilon = 10^{-4}$. One subdivision and 48 function evaluation needed

Finally, we try for $\varepsilon = 10^{-6}$ (see Figure 5)

```
>> [vi2,stat]=mpcubatd2vb(@fintegr,x,y,1e-6,1)
```

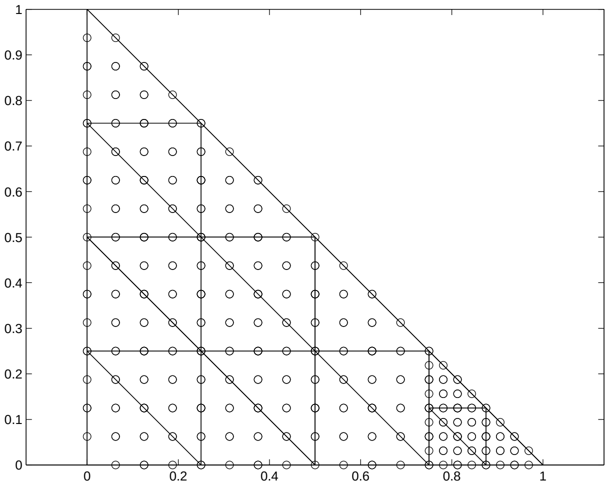

FIGURE 5. Computing the integral for $\varepsilon = 10^{-6}$. 228 function evaluation needed

```
vi2 =

   0.04030231573315

stat =

      nev: 228
     ntri: 100
```

References

[1] M. de Berg, M. van Kreveld, M. Overmars, O. Schwarzkopf – *Computational Geometry. Algorithms and Applications*, Springer Verlag, Berlin, Heidelberg, New York, 1997.

[2] C. Überhuber – *Computer Numerik*, Band 2, Springer Verlag, Berlin, Heidelberg, New York, 1995.

[3] D. P. Laurie – Algorithm 584: CUBTRI: Automatic Cubature over a Triangle, ACM TOMS, vol. 8, No. 2, 1982, pp. 210–218.

[4] R. Cools, D. Laurie and L. Pluym – Cubpack++: A C++ Package for Automatic Two-Dimensional Cubature, ACM TOMS Vol. 23, No. 1, 1997, pp. 1–15.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLVII, Number 4, December 2002

ON BLEIMANN, BUTZER AND HAHN TYPE GENERALIZATION OF BALÁZS OPERATORS

OGÜN DOĞRU

Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. In this paper we introduced a generalization of Balázs operators [4] which includes the Bleimann, Butzer and Hahn operators [6]. We define a space of general Lipschitz type maximal functions and obtain the approximation properties of these operators. Also we estimate the rate of convergence of these operators. In the last section, we obtain derivative and bounded variation properties of these generalized operators.

1. **Introduction**

In [4], K. Balázs introduced the discrete linear positive operators defined by

$$(R_n f)(x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f(\frac{k}{b_n}) \binom{n}{k} (a_n x)^k, \quad x \geq 0,\ n \in \mathbb{N} \tag{1}$$

where a_n and b_n are positive numbers, independent of x.

After simple computation, we have

$$(R_n e_0)(x) = 1$$

$$(R_n e_1)(x) = \frac{n}{b_n} \frac{a_n x}{1+a_n x}$$

$$(R_n e_2)(x) = \frac{n(n-1)}{b_n^2} \left(\frac{a_n x}{1+a_n x} \right)^2 + \frac{n}{b_n^2} \frac{a_n x}{1+a_n x}$$

where e_n represents the monomial $e_n(x) = x^n$ for $n = 0, 1, 2$.

These equalities show that both of classical Bohman-Korovkin theorems in [7], [13] and weighted Korovkin type theorems in [10] and [9] do not valid.

Received by the editors: 01.10.2002.

2000 *Mathematics Subject Classification.* 41A25, 41A36.

Key words and phrases. Generalized linear positive operators, Korovkin type theorem, Lipschitz type maximal functions, difference operator, bounded variation.

In [4], Voronoskaja type formula was given for operators (1), under the some restriction of sequences a_n and b_n.

In [1] and [2], O. Agratini introduced a Kantorovich type integral form of operators (1) and obtained the degree of approximation in polynomial weighted function spaces.

By choosing $a_n = n^{\beta-1}$, $b_n = n^{\beta}$ for $n \in \mathbb{N}$ and $0 < \beta < 1$, the operator (1) was denoted by the symbol $R_n^{[\beta]}$. Also, for some $0 < \beta < 1$ values in [4], [5] and [17], convergence, derivative and saturation properties of $R_n^{[2/3]}$ were investigated by K. Balázs, J. Szabados and V. Totik respectively.

A recent paper is given by O. Agratini in [2] about Voronovskaja type theorem for Kantorovich type generalization of the $R_n^{[\beta]}$.

On the other hand in [6], G. Bleimann, P.L. Butzer and L. Hahn introduced the Bernstein type sequence of linear positive operator defined as

$$(L_n f)(x) = (1+x)^{-n} \sum_{k=0}^{n} f(\frac{k}{n-k+1}) \binom{n}{k} x^k, \quad x \geq 0,\ n \in \mathbb{N}. \tag{2}$$

In [6], pointwise convergence properties of operators (2) are investigated on compact subinterval $[0,b]$ of $[0,\infty)$. In [11], T. Hermann investigated the behavior of operators (2) when the growth condition for f is weaker than polynomial one. In [12], C. Jayasri and Y. Sitaraman proved direct and inverse theorems of operators (2) in the some subspaces on positive real axis. In [8], by using the test functions $\left(\frac{x}{1+x}\right)^{\nu}$ for $\nu = 0, 1, 2$, a Korovkin type theorem was given by Ö. Çakar and A.D. Gadjiev and they obtained some approximation properties of (2) in a subclass of continuous and bounded functions on all positive semi-axis.

The aim of this paper is to investigate the approximation properties of a generalization of K. Balázs's operators R_n in Bleimann, Butzer and Hahn operators type on the all positive semi-axis.

2. **Construction of the operators**

We consider the sequence of linear positive operators

$$(A_n f)(x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_{n,k}}\right) \binom{n}{k} (a_n x)^k, \quad x \geq 0,\ n \in \mathbb{N} \tag{3}$$

where a_n and $b_{n,k}$ satisfy the following conditions for every n and k;

$$a_n k + b_{n,k} = c_n \text{ and } \frac{n}{c_n} \to 1 \text{ for } n \to \infty. \tag{4}$$

Since replacing b_n by $b_{n,k}$, these operators different from the operators R_n.

Clearly, if we choose $a_n = 1$, $b_{n,k} = n - k + 1$ for every n and k then $c_n = n + 1$ the conditions (4) are satisfied. These operators turn out into Bleimann, Butzer and Hahn operators. Therefore, these operators are a Bleimann, Butzer and Hahn operators type generalization of Balázs operators.

3. **Approximation properties**

In this section, we will give a Korovkin type theorem in order to obtain approximation properties of operators (3).

In [14], B. Lenze introduced a Lipschitz type maximal function as

$$f_\alpha^\sim(x) = \sup_{\substack{t>0 \\ t\neq x}} \frac{|f(t) - f(x)|}{|x-t|^\alpha}.$$

Firstly, we define a space of general Lipschitz type maximal functions.

Let $W_\alpha^\sim$ be the space of functions defined as

$$W_\alpha^\sim = \left\{ f : \sup(1 + a_n t)^\alpha \ f_\alpha(x,t) \leq M \left(\frac{a_n}{1 + a_n x} \right)^\alpha, x \geq 0 \right\} \tag{5}$$

where f is bounded and continuous on $[0, \infty)$, M is a positive constant, $0 < \alpha \leq 1$ and f_α is the following function

$$f_\alpha(x,t) = \frac{|f(t) - f(x)|}{|x-t|^\alpha}.$$

Example 1. For any $M_1 > 1$, let the sequence of functions f_n be

$$f_n(x) = \frac{1 + M_1 a_n x}{1 + a_n x}.$$

Then for all $x, t \geq 0$, $x \neq t$, we have

$$|f(t) - f(x)| = \frac{(M_1 - 1)a_n |x-t|}{(1 + a_n x)(1 + a_n t)}.$$

By choosing $M_1 - 1 \leq M$, one obtains $f_n \in W_1^\sim$.

Also, if $\dfrac{a_n}{1 + a_n x}$ is bounded then $W_\alpha^\sim \subset Lip_{M_1}(\alpha)$ where M_1 is a positive constant which satisfies the following inequality

$$M \left(\frac{a_n}{1 + a_n x} \right)^\alpha \left(\frac{1}{1 + a_n t} \right)^\alpha \leq M_1.$$

Really, if $f \in W_\alpha^\sim$ then for all $x, t \geq 0$, $x \neq t$ we can write

$$|f(t) - f(x)| \leq M \left(\frac{a_n}{1 + a_n x} \right)^\alpha \left(\frac{1}{1 + a_n t} \right)^\alpha |x - t|^\alpha$$

and $f \in Lip_{M_1}(\alpha)$. Clearly that, if $a_n \leq 1$ or $x \geq 1$ then $\dfrac{a_n}{1 + a_n x}$ is bounded.

Theorem 2. *If L_n is the sequence of positive linear operators acting from $W_\alpha^\sim$ to $C_B[0, \infty)$ and satisfying the following conditions for $\nu = 0, 1, 2$*

$$\left\| \left(L_n \left(\frac{a_n t}{1 + a_n t} \right)^\nu \right)(x) - \left(\frac{a_n x}{1 + a_n x} \right)^\nu \right\|_{C_B} \to 0 \text{ for } n \to \infty \tag{6}$$

then, for any function f in $W_\alpha^\sim$ one has

$$\|L_n f - f\|_{C_B} \to 0 \quad \text{for} \quad n \to \infty.$$

where $C_B[0, \infty)$ denotes the space of functions which is bounded and continuous on $[0, \infty)$.

Proof. This proof is similar to the proof of Korovkin theorem.

Let $f \in W_\alpha^\sim$,. Since f is continuous on $[0, \infty)$, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(t) - f(x)| < \epsilon \text{ for } \left| \frac{a_n t}{1 + a_n t} - \frac{a_n x}{1 + a_n x} \right| < \delta$$

and since f is bounded on $[0, \infty)$, there is a positive constant M such that

$$|f(t) - f(x)| < \frac{2M}{\delta^2} \left[\frac{a_n(t - x)}{(1 + a_n t)(1 + a_n x)} \right]^2 \text{ for } \left| \frac{a_n t}{1 + a_n t} - \frac{a_n x}{1 + a_n x} \right| \geq \delta.$$

Thus, for all $t, x \in [0, \infty)$ one has

$$|f(t) - f(x)| < \epsilon + \frac{2M}{\delta^2} \left[\frac{a_n(t - x)}{(1 + a_n t)(1 + a_n x)} \right]^2. \tag{7}$$

By using basic properties of positive linear operators, we have

$$\begin{aligned} \|L_n f - f\|_{C_B} &\leq \|(L_n |f - f(x)|)\|_{C_B} + \\ &+ \|f\|_{C_B} \|(L_n 1) - 1\|_{C_B} \end{aligned} \tag{8}$$

By using the inequality (7) and conditions (6) in (8), the proof is complete.

Now, we will give the first main result about approximation properties of operators (3) with the help of Theorem 2.

Theorem 3. *If A_n is the sequence of positive linear operators defined by (3), then for each $f \in W_\alpha^\sim$*

$$\|(A_n f) - f\|_{C_B} \to 0 \quad \textit{for} \quad n \to \infty.$$

Proof. For the operators in (3), it is easily to verify that

$$(A_n 1)(x) = 1$$

$$\left(A_n\left(\frac{a_n t}{1+a_n t}\right)\right)(x) = \frac{n}{c_n}\frac{a_n x}{1+a_n x}$$

$$\left(A_n\left(\frac{a_n t}{1+a_n t}\right)^2\right)(x) = \left(\frac{n}{c_n}\right)^2\left(\frac{a_n x}{1+a_n x}\right)^2 + \frac{1}{c_n}\frac{n}{c_n}\frac{a_n x}{1+a_n x}.$$

By using the conditions (4) and Theorem 2, the proof is obvious.

4. **Approximation order**

In this section, we give a result about rate of convergence of operators (3).

Theorem 4. *If $f \in W_\alpha^\sim$ then for all $x \geq 0$ we have*

$$|(A_n f)(x) - f(x)| \leq M\left(\frac{n}{c_n}a_n - 1\right)^\alpha \tag{9}$$

where the constants M and $0 < \alpha \leq 1$ are defined in the definition of the space $W_\alpha^\sim$ and the operators A_n are defined in (3).

Proof. If $f \in W_\alpha^\sim$, we can write

$$|(A_n f)(x) - f(x)|$$

$$\leq M\left(\frac{a_n}{1+a_n x}\right)^\alpha \frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}\left|\frac{k}{b_{n,k}} - x\right|^\alpha \left(\frac{1}{1+a_n\frac{k}{b_{n,k}}}\right)^\alpha \binom{n}{k}(a_n x)^k.$$

From the conditions (4), we get

$$\frac{1}{1+a_n\dfrac{k}{b_{n,k}}} = \frac{b_{n,k}}{c_n}.$$

If we use this equality in the last inequality, we obtain

$$|(A_n f)(x) - f(x)|$$

$$\leq M\left(\frac{a_n}{1+a_n x}\right)^\alpha \frac{1}{c_n^\alpha}\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}|k - x(c_n - a_n k)|^\alpha \binom{n}{k}(a_n x)^k.$$

By using the Hölder inequality for $p=\dfrac{2}{\alpha}$, $q=\dfrac{2}{2-\alpha}$ and considering $((A_ne_0)(x))^{\frac{2-\alpha}{2}}=1$ we have

$$|(A_nf)(x)-f(x)| \tag{10}$$

$$\leq M\left(\frac{a_n}{c_n(1+a_nx)}\right)^{\alpha}\left(\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}(k-x(c_n-a_nk))^2\binom{n}{k}(a_nx)^k\right)^{\frac{\alpha}{2}}.$$

On the other hand, it is obvious that

$$\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k=(1+a_nx)^n,$$

$$\sum_{k=1}^{n}k\binom{n}{k}(a_nx)^k=na_nx(1+a_nx)^{n-1},$$

$$\sum_{k=1}^{n}k^2\binom{n}{k}(a_nx)^k=(a_nx)^2n(n-1)(1+a_nx)^{n-2}+a_nxn(1+a_nx)^{n-1}.$$

By using these equalities, after simplifications, we obtain

$$\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}(k-x(c_n-a_nk))^2\binom{n}{k}(a_nx)^k\leq$$

$$\leq x^2c_n^2\left[\left(\frac{n}{c_n}a_n-1\right)^2-\frac{n}{c_n^2}a_n^2\right]\leq x^2c_n^2\left(\frac{n}{c_n}a_n-1\right)^2.$$

If we use last inequality in (10), we have

$$\begin{aligned}|(A_nf)(x)-f(x)| &\leq M\left(\frac{a_n}{c_n(1+a_nx)}\right)^{\alpha}x^{\alpha}c_n^{\alpha}\left(\frac{n}{c_n}a_n-1\right)^{\alpha}\\ &= M\left(\frac{a_nx}{1+a_nx}\right)^{\alpha}\left(\frac{n}{c_n}a_n-1\right)^{\alpha}.\end{aligned}$$

Since $\left(\dfrac{a_nx}{1+a_nx}\right)^{\alpha}\leq 1$, the proof is complete.

Since Theorem 4 is valid for all $x\geq 0$, this proof gives uniform convergence of the operators A_n to f without using Korovkin type theorem.

5. **Derivative properties**

Firstly, explicit formula for derivatives of Bernstein polynomials with difference operators is obtained by G.G. Lorentz in [15, p.12]. A lot of studies have included derivative properties of positive linear operators. In [16], D.D. Stancu obtained the monotonicity properties from different orders of the derivatives of Bernstein polynomials with the help of divided differences.

In this part, we will give some derivative properties of operators A_n defined in (3) with the help of difference operators.

We can easily compute:

$$\frac{d}{dx}(A_nf)(x) = na_n(1+a_nx)^{-n-1}\sum_{k=0}^{n-1}\left[f(\frac{k+1}{b_{n,k}}) - f(\frac{k}{b_{n,k}})\right]\binom{n-1}{k}(a_nx)^k \quad (11)$$

and by using induction method for derivatives of $k-$order, we have

$$\frac{d^k}{dx^k}(A_nf)(x) \quad (12)$$

$$= n(n-1)\dots(n-k+1)a_n^k(1+a_nx)^{-n-k}\sum_{\nu=0}^{n-k}\Delta^k f(\frac{\nu}{b_{n,k}})\binom{n-k}{\nu}(a_nx)^\nu,$$

where $\Delta^k f\left(\frac{\nu}{b_{n,k}}\right)$ is difference operator defined as

$$\Delta^k f(\frac{\nu}{b_{n,k}}) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f\left(\frac{\nu+i}{b_{n,k}}\right).$$

Theorem 5. *Let $f \in C^1[0,\infty)$. Then the operators A_n have the monotonicity properties.*

Proof. If $f \in C^1[0,\infty)$ then $f \in C^1\left[\frac{k}{b_{n,k}}, \frac{k+1}{b_{n,k}}\right]$. Therefore, from (11) we can write

$$\frac{d}{dx}(A_nf)(x) = na_n(1+a_nx)^{-n-1}\sum_{k=0}^{n-1}\int_{\frac{k}{b_{n,k}}}^{\frac{k+1}{b_{n,k}}} f'(\xi)d\xi\binom{n-1}{k}(a_nx)^k.$$

Since $\int_{\frac{k}{b_{n,k}}}^{\frac{k+1}{b_{n,k}}} f'(\xi)d\xi \geq 0 \ (\leq 0)$ for $f'(x) \geq 0 \ (\leq 0)$, we have

$$\frac{d}{dx}(A_nf)(x) \geq 0 (\leq 0) \quad \text{for} \quad f'(x) \geq 0 (\leq 0)$$

and this completes the proof.

In [15, p.23], G.G. Lorentz gives an estimate related to the total variation of Bernstein polynomials. Similarly, in the following theorem, we give an estimate of bounded variation between the operators A_n and f.

Theorem 6. *The operators A_n preserve the functions of bounded variation on $[0,\infty)$.*

Proof. By using formula (11), we get

$$V_n(A_nf) = \int_0^\infty \left|\frac{d}{dx}(A_nf)(x)\right| dx \tag{13}$$
$$\leq \sum_{k=0}^{n-1} \left|\Delta f(\frac{k}{b_{n,k}})\right| na_n \binom{n-1}{k} \int_0^\infty (a_nx)^k(1+a_nx)^{-n-1}dx.$$

Since $k > -1$ and $-k+n > 0$, we can write

$$\int_0^\infty (a_nx)^k(1+a_nx)^{-n-1}dx = \frac{\Gamma(1+k)\Gamma(-k+n)}{a_n\Gamma(1+n)}.$$

If we use properties of Gamma function in this equality, we have

$$\int_0^\infty (a_nx)^k(1+a_nx)^{-n-1}dx = \frac{k!(n-k-1)!}{a_nn!}.$$

By using this equality in (13), we obtain

$$V_n(A_nf) \leq V(f)$$

which gives the proof.

References

[1] Agratini, O., *An approximation process of Kantorovich type*, Math. Notes, **2**(1), (2001), 3-10.

[2] Agratini, O., *On approximation properties of Balázs-Szabados operators and their Kantorovich extension*, Korean J. Comput. Appl. Math., **9**(2), (2002), 361-372.

[3] Altomare, F. and Campiti, M., *Korovkin-Type Approximation Theory and its Applications*, De Gruyter Series Studies in Mathematics, Vol.17, Walter de Gruyter, Berlin-New York, 1994.

[4] Balázs, C., *Approximation by Bernstein type rational functions*, Acta Math. Acad. Sci. Hungar., **26**(1-2), (1975), 123-134.

[5] Balázs, C. and Szabados, J., *Approximation by Bernstein type rational functions II*, Acta Math. Acad. Sci. Hungar., **40**, (1982), 331-337.

[6] Bleimann, G., Butzer, P.L. and Hahn, L., *A Bernstein-type operator approximating continuous functions on the semi-axis*, Math. Proc. A, **83**(3), (1980), 255-262.

[7] Bohman, H., *On approximation of continuous and analytic functions*, Arkif für Matematik, **2**(3), (1951), 43-56.

[8] Çakar, Ö., and Gadjiev, A.D., *On uniform approximation by Bleimann, Butzer and Hahn operators on all positive semi-axis*, Trans. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci. **19**(5), (1999), 21-26.

[9] Doğru, O., *On weighted approximation of continuous functions by linear positive operators on infinite intervals*, Math. Cluj, **41**(64), (1999), 39-46.

[10] Gadjiev, A.D., *On P.P. Korovkin type theorems*, Math. Zametki, **20**, (1976), 781-786 (in Russian).

[11] Hermann, T., *On the operator of Bleimann, Butzer and Hahn*, Colloq. Math. Soc. János Bolyai, 58. Approx. Th., Kecskemét (Hungary), (1990), 355-360.

[12] Jayasri, C. and Sitaraman, Y., *Direct and inverse theorems for certain Bernstein-type operators*, Indian J. Pure Appl. Math. **16**, (1985), 1495-1511.

[13] Korovkin, P.P., *Linear Operators and Approximation Theory*, Delhi, 1960.
[14] Lenze, B., *Bernstein-Baskakov-Kantorovich operators and Lipschitz-type Maximal Functions*, Colloq. Math. Soc. János Bolyai, 58. Approx. Th., Kecskemét (Hungary), (1990), 469-496.
[15] Lorentz, G.G., *Bernstein Polynomials*, Toronto, 1953.
[16] Stancu, D.D., *Application of divided differences to the study of monotonocity of the derivatives of the sequence of Bernstein polynomials*, Calcolo, **16**(4), (1979), 431-445.
[17] Totik, V., *Saturation for Bernstein type rational functions*, Acta Math. Hungar., **43**(3-4), (1984), 219-250.

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ON APPROXIMATION PROPERTIES OF STANCU'S OPERATORS

ZOLTÁN FINTA

Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. The purpose of the paper is to present pointwise and uniform approximation theorems for some Stancu's operators using the classical moduli of smoothness and the second modulus of smoothness of Ditzian - Totik.

1. **Introduction**

One of the most studied operator (see e.g. the bibliography of [1]) is $B_n^\alpha : C[0,1] \to C[0,1]$,

$$B_n^\alpha(f,x) = \sum_{k=0}^{n} w_{n,k}(x,\alpha) \cdot f\left(\frac{k}{n}\right), \quad n = 1,2,\dots, \quad x \in [0,1], \quad \alpha \geq 0, \tag{1}$$

where

$$w_{n,k}(x,\alpha) = \binom{n}{k} \cdot \frac{\prod_{i=0}^{k-1} (x+i\alpha) \prod_{j=0}^{n-k-1} (1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)} \tag{2}$$

and α is a parameter which may depend only on the natural number n. This positive linear polynomial operator was introduced by D. D. Stancu in [15]. In the case $\alpha = 0$, B_n^α is the Bernstein operator B_n given by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \cdot f\left(\frac{k}{n}\right). \tag{3}$$

The Stancu - Kantorovich polynomial operator was defined in [14] as follows: $K_n^\alpha : L^p[0,1] \to L^p[0,1], \quad 1 \leq p \leq \infty$,

$$K_n^\alpha(f,x) = (n+1) \sum_{k=0}^{n} w_{n,k}(x,\alpha) \cdot \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\, du, \quad n = 1,2,\dots, \quad x \in [0,1] \tag{4}$$

Received by the editors: 20.09.2002.

2000 *Mathematics Subject Classification.* 41A10, 41A25, 41A36.

Key words and phrases. Stancu operator, the first and the second modulus of smoothness, the second modulus of smoothness of Ditzian - Totik.

and α and $w_{n,k}(x,\alpha)$ have the same meaning as above. For $\alpha = 0$, K_n^α is the Kantorovich operator K_n given by

$$K_n(f,x) = (n+1) \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\, du. \tag{5}$$

The spaces $L^p[0,1]$, $1 \le p \le \infty$, are endowed with the norm

$$\|f\|_p = \left\{ \int_0^1 |f(x)|^p \, dx \right\}^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ we consider $C[0,1]$ instead of $L^\infty[0,1]$ with

$$\|f\| = \|f\|_\infty = \sup\left\{ |f(x)| \;:\; x \in [0,1] \right\}.$$

The corresponding operator to Bernstein operator on the positive semiaxis is the so - called Szász - Mirakjan operator defined by $S_n : C_B[0,\infty) \to C_B[0,\infty)$,

$$S_n(f,x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \cdot f\left(\frac{k}{n}\right), \quad n = 1,2,\ldots, x \in [0,\infty), \tag{6}$$

where $C_B[0,\infty)$ denotes the set of all bounded and continuous functions on $[0,\infty)$ endowed with the norm

$$\|f\|_* = \sup\left\{ |f(x)| \;:\; x \in [0,\infty) \right\}.$$

The operator S_n was generalized by Stancu in [16], obtaining S_n^β operators

$$S_n^\beta(f,x) = (1+n\beta)^{-x/\beta} \cdot \sum_{k=0}^{\infty} \left(\beta + \frac{1}{n}\right)^{-k} \cdot \frac{x(x+\beta)\ldots(x+(k-1)\beta)}{k!} \cdot f\left(\frac{k}{n}\right), \tag{7}$$

where $\beta > 0$ is a parameter depending on the natural number n.

Furthermore, in the paper [17], Stancu has introduced a generalization of the well - known Baskakov operator $V_n : C_B[0,\infty) \to C_B[0,\infty)$,

$$V_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k(1+x)^{-n-k} \cdot f\left(\frac{k}{n}\right), n = 1,2,\ldots, x \in [0,\infty), \tag{8}$$

defined by

$$V_n^\gamma(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x,\gamma) \cdot f\left(\frac{k}{n}\right), \tag{9}$$

where

$$v_{n,k}(x,\gamma) = \binom{n+k-1}{k} \cdot \frac{\prod_{i=0}^{k-1} (x+i\gamma) \prod_{j=0}^{n-1} (1+j\gamma)}{\prod_{r=0}^{n+k-1} (1+x+r\gamma)} \tag{10}$$

and $\gamma \geq 0$ depends on the natural number n.

The purpose of this paper is to establish pointwise and uniform approximation properties for the operators (1) – (2), (4), (7) and (9) – (10). On the other hand the paper will be a survey of some results given by the author regarding the above mentioned Stancu's operators.

To establish these results we shall use the following notations:

$$\omega(g,t)_p = \sup_{0<h\leq t} \left\{ \int_0^1 |g(x+h)-g(x)|^p \, dx \right\}^{1/p},$$
$$g \in L^p[0,1], \quad 1 \leq p < \infty, \quad x, x+h \in [0,1];$$

$$\omega_2(g,t) = \sup_{0<h\leq t} \sup_{x,x\pm h\in I} |g(x+h)-2g(x)+g(x-h)|,$$
$$g \in C(I), \ I=[0,1] \ \text{ or } \ I=[0,\infty);$$

$$\omega_2^{\varphi}(g,t) = \sup_{0<h\leq t} \sup_{x\pm h\varphi(x)\in I} |g(x+h\varphi(x))-2g(x)+g(x-h\varphi(x))|,$$
$$g \in C[0,1] \quad \text{and} \quad \varphi(x)=\sqrt{x(1-x)},$$
$$g \in C_B[0,\infty) \quad \text{and} \quad \varphi(x)=\sqrt{x} \quad \text{or}$$
$$g \in C_B[0,\infty) \quad \text{and} \quad \varphi(x)=\sqrt{x(1+x)};$$

$$\omega_2^{\varphi}(g,t)_p = \sup_{0<h\leq t} \left\{ \int_0^1 |g(x+h\varphi(x))-2g(x)+g(x-h\varphi(x))|^p \, dx \right\}^{1/p},$$
$$g \in L^p[0,1], \ 1 \leq p < \infty, \ x \pm h\varphi(x) \in [0,1]$$
$$\text{and} \quad \varphi(x)=\sqrt{x(1-x)}, \ x\in[0,1];$$

$$\omega_2^{\phi}(g,t) = \sup_{0<h\leq t} \sup_{x\pm h\phi(x)\in[0,\infty)} |g(x+h\phi(x))-2g(x)+g(x-h\phi(x))|,$$
$g \in C_B[0,\infty)$ and $\phi : [0,\infty) \to \Re$ is an admissible step - weight function (see [3]).

Here we mention that throughout this paper C and C_0 denote absolute constants and not necessarily the same at each occurrence.

2. Theorems

In [5, Theorem 1] we have proved the following

Theorem 1. *For* $f \in C[0,1]$ *and* $x \in [0,1]$ *we have*

$$|B_n^\alpha(f,x) - f(x)| \;\le\; C\,\omega_2\left(f, \sqrt{\frac{1+n\alpha}{n(1+\alpha)}\cdot x(1-x)}\right).$$

Remark 1. We can obtain the estimate of Theorem 1 with $C = 2$ using [13, p. 255, Theorem 2.1].

Furthermore, by [6, p. 100, Theorem 1], we have

Theorem 2. *Let* $f \in C[0,1]$ *and* $\alpha = \alpha(n) = o(n^{-1})$, $\alpha\, n \le 1$, $n = 1,2,\ldots$. *Then*

$$|B_n^\alpha(f,x) - f(x)| \;\le\; C\cdot\frac{x(1-x)}{n}, \quad x\in[0,1], \quad n = 1,2,\ldots$$

holds exactly when $\omega_2(f,h) \;\le\; C\,h^2,\; h>0.$

Using [2, p. 79, Theorem A] or [6, p. 100, Theorem 3], we get

Theorem 3. *For* $f \in C[0,1]$ *and* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *we have*

$$\|B_n^\alpha(f) - f\| \;\le\; C\,\omega_2^\varphi\left(f, \sqrt{\frac{1+n\alpha}{n(1+\alpha)}}\right).$$

The next result requires the following lemma (see [12, p. 317, (2.1)] or [19]):

Lemma 1. *Let* $f \in C[0,1]$ *and* $\varphi(x) = \sqrt{x(1-x)}$, $x\in[0,1]$. *Then*

$$\frac{1}{n}\;\|\varphi^2(B_n(f))''\| \;\le\; C_0\;\|B_n(f) - f\|,$$

where C_0 *is an absolute constant.*

Then our result is (see [7, p. 2, Theorem 3]) :

Theorem 4. *Let* $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *and* $\alpha = \alpha(n)$, $2\,C_0\,\alpha\,n \le 1$, $n = 1,2,\ldots$, *where* C_0 *denotes the absolute constant of Lemma 1 above. Then there exists an absolute constant* $C > 0$ *such that*

$$C^{-1}\,\|B_n(f) - f\| \;\le\; \|B_n^\alpha(f) - f\| \;\le\; C\,\|B_n(f) - f\|$$

and

$$C^{-1}\,\omega_2^\varphi(f, n^{-1/2}) \;\le\; \|B_n^\alpha(f) - f\| \;\le\; C\,\omega_2^\varphi(f, n^{-1/2}).$$

Hence, in view of [3, p. 177, (9.3.3)], we obtain immediately

Corollary 1. *Let* $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, $\alpha = \alpha(n)$ *with* $2\ C_0\ \alpha\ n \leq 1$, $n = 1, 2, \ldots$ *and* $0 < \delta < 2$. *Then*

$$\|B_n^\alpha(f) - f\| \ = \ O(n^{-\delta/2}) \ \text{ iff } \ \omega_2^\varphi(f,h) \ = \ O(h^\delta), \quad h > 0.$$

The following results will be in connection with the operator K_n^α. More precisely, we have (see [8, Theorem 1, Lemma 2 and Theorem 3]) :

Theorem 5. *Let* $f \in L^p[0,1]$, $1 \leq p \leq \infty$ *and* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. *Then there exists* $C > 0$ *such that*

(*i*) $\|K_n^\alpha(f) - f\|_p \ \leq \ C \ \left\{\omega_2^\varphi(f, n^{-1/2})_p + n^{-1}\ \|f\|_p\right\},$
where $\alpha = \alpha(n) = O(n^{-1})$ *and* $1 < p \leq \infty$;

(*ii*) $\|K_n^\alpha(f) - f\|_1 \ \leq \ C \ \left\{\omega_2^\varphi(f, n^{-1/2})_1 + n^{-1}\ \|f\|_1\right\},$
where $\alpha = \alpha(n) = O(n^{-4})$.

For the converse result we need a lemma :

Lemma 2. *For* $f \in L^p[0,1]$, $1 < p \leq \infty$ *and* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *we have*

$$\frac{1}{n}\ \|\varphi^2(K_n(f))''\|_p \ \leq \ C_0\ \|K_n(f) - f\|_p,$$

where C_0 *is an absolute constant.*

Remark 2. The above Lemma does not hold for $p = 1$ (see [8, Remark 2]).

Our result is

Theorem 6. *Let* $f \in L^p[0,1]$, $1 < p \leq \infty$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *and* $\alpha = \alpha(n)$, $p/(p-1)\ C_0\ \alpha\ n \leq \delta < 1$, $n = 1, 2, \ldots$, *where* C_0 *denotes the absolute constant of Lemma 2. Then*

$$(1-\delta)\ \|K_n(f) - f\|_p \ \leq \ \|K_n^\alpha(f) - f\|_p \ \leq \ (1+\delta)\ \|K_n(f) - f\|_p$$

and there exists an absolute constant $C > 0$ *such that*

$$C^{-1}\left[\omega_2^\varphi(f, n^{-1/2})_p + \omega(f, n^{-1})_p\right] \leq \|K_n^\alpha(f) - f\|_p \leq C\left[\omega_2^\varphi(f, n^{-1/2})_p + \omega(f, n^{-1})_p\right].$$

In what follows we give the theorems concerning to the operator S_n^β using [9, p. 62, Theorem 1] and [10] :

Theorem 7. *For $f \in C[0,\infty)$ and $x \in [0,\infty)$ we have*

$$|S_n^\beta(f,x) - f(x)| \;\le\; 2\ \omega_2\left(f, \sqrt{\left(\beta + \frac{1}{n}\right)\frac{x}{2}}\,\right).$$

Theorem 8. *Let $f \in C_B[0,\infty)$ and $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$. Then*

$$\|S_n^\beta(f) - f\|_* \;\le\; C\ \omega_2^\varphi\left(f, \sqrt{\frac{1}{n} + \beta}\,\right).$$

Theorem 9. *Let $f \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$ and $\beta = \beta(n)$, $2\ C_0\ \beta\ n \le \delta < 1$, $n = 1, 2, \dots,$ where C_0 denotes the absolute constant of Lemma 3 below. Then*

$$(1-\delta)\ \|S_n(f) - f\|_* \;\le\; \|S_n^\beta(f) - f\|_* \;\le\; (1+\delta)\ \|S_n(f) - f\|_*$$

and there exists an absolute constant $C > 0$ such that

$$C^{-1}\ \omega_2^\varphi(f, n^{-1/2}) \;\le\; \|S_n^\beta(f) - f\|_* \;\le\; C\ \omega_2^\varphi(f, n^{-1/2}).$$

Lemma 3. [19] *Let $f \in C_B[0,\infty)$ and $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$. Then*

$$\frac{1}{n}\ \|\varphi^2 (S_n(f))''\|_* \;\le\; C_0\ \|S_n(f) - f\|_*,$$

where C_0 is an absolute constant.

Finally, we give the results about the operator V_n^γ. This operator is linear, positive and bounded, but it does not preserve the linear functions. Therefore we consider the following two cases :

a)

$$L_n^\gamma(f,x) \;=\; a_0(n)\cdot V_{n_0}^\gamma(f,x) \;+\; a_1(n)\cdot V_{n_1}^\gamma(f,x), \tag{11}$$

where

$$n = n_0 < n_1 \le A\ n, \qquad |a_0(n)| + |a_1(n)| \;\le\; A,$$
$$a_0(n) + a_1(n) = 1, \qquad a_0(n)\cdot n_0^{-1} + a_1(n)\cdot n_1^{-1} = 0$$

and $\gamma = \gamma(n) \leq B/(4n)$, $n = 1, 2, \ldots$, $0 < B < 1$. Here A and B are given absolute constants. Following [11] (see also [4]), we have

Theorem 10. *Let $L_n^\gamma : C_B[0,\infty) \to C_B[0,\infty)$ be given by (11), $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$ and $\phi : [0,\infty) \to \Re$ be an admissible step - weight function of the Ditzian - Totik modulus and $\gamma = \gamma(n) \leq B/(4n)$, $n = 1, 2, \ldots$, $0 < B < 1$. Then*

$$|L_n^\beta(f,x) - f(x)| \;\leq\; C\ \omega_2^\phi\left(f, n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right), \quad x \in [0,\infty).$$

In particular, we obtain a local estimation of the approximation error for $\phi = 1:$

$$|L_n^\beta(f,x) - f(x)| \;\leq\; C\ \omega_2^\phi\left(f, \sqrt{\frac{x(1+x)}{n}}\right)$$

and we get a uniform (global) estimation of the approximation error for $\phi = \varphi:$

$$\|L_n^\beta(f) - f\|_* \;\leq\; C\ \omega_2^\varphi(f, n^{-1/2}).$$

b)

$$\tilde{V}_n^\gamma(f,x) = \sum_{k=0}^{\infty} \tilde{v}_{n,k}(x,\gamma) \cdot f\left(\frac{k}{n}\right), \tag{12}$$

where

$$\tilde{v}_{n,k}(x,\gamma) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \cdot \frac{\prod_{i=0}^{k-1}\,(x+i\gamma) \cdot \prod_{j=1}^{n}\,(1+j\gamma)}{\prod_{r=1}^{n+k}\,(1+x+r\gamma)} \tag{13}$$

(see also [18]). By [10], we have

Theorem 11. *For $\tilde{V}_n^\gamma : C_B[0,\infty) \to C_B[0,\infty)$ given by (12) - (13), $f \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$ and $0 < \gamma < 1$ we have*

$$\|\tilde{V}_n^\gamma(f) - f\|_* \;\leq\; C\ \omega_2^\varphi\left(f, \sqrt{\frac{1}{n} + \frac{\gamma}{1-\gamma}}\,\right).$$

Theorem 12. *Let $f \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$ and $\gamma = \gamma(n)$, $2\ C_0 \cdot (\gamma/(1-\gamma)) \cdot n \leq \delta < 1$, $n = 1, 2, \ldots,$ where C_0 denotes the absolute constant of Lemma 4 below. Then*

$$(1-\delta)\ \|V_n(f) - f\|_* \;\leq\; \|\tilde{V}_n^\gamma(f) - f\|_* \;\leq\; (1+\delta)\ \|V_n(f) - f\|_*$$

and there exists an absolute constant $C > 0$ such that

$$C^{-1}\ \omega_2^{\varphi}(f, n^{-1/2}) \ \leq\ \|\tilde{V}_n^{\gamma}(f) - f\|_* \ \leq\ C\ \omega_2^{\varphi}(f, n^{-1/2}).$$

Lemma 4. [19] *Let $f \in C_B[0,\infty)$ and $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$. Then*

$$\frac{1}{n}\ \|\varphi^2(V_n(f))''\|_* \ \leq\ C_0\ \|V_n(f) - f\|_*,$$

where C_0 is an absolute constant.

References

[1] Della Vecchia, B. : *On the approximation of functions by means of the operators of D. D. Stancu,* Studia Univ. Babeş - Bolyai, Mathematica, 37 (1)(1992), 3 - 36.

[2] Della Vecchia, B. and Mache, D. H. : *On approximation properties of Stancu - Kantorovich operators,* Rev. Anal. Numér. Théorie Approximation, 27 (1)(1998), 71-80.

[3] Ditzian, Z. and Totik, V. : *Moduli of Smoothness,* Springer - Verlag, Berlin Heidelberg New York London, 1987.

[4] Felten, M. : *Local and Global Approximation Theorems for Positive Linear Operators,* J. Approx. Theory, 94(1998), 396-419.

[5] Finta, Z. : *Quantitative estimates for some linear and positive operators* (to appear in Studia Univ. Babeş - Bolyai, Mathematica).

[6] Finta, Z. : *On some properties of Stancu operator,* Rev. Anal. Numér. Théorie Approximation, 27(1)(1998), 99-106.

[7] Finta, Z. : *Direct and converse results for Stancu operator,* Period. Math. Hungar., 44(1)(2002), 1-6.

[8] Finta, Z. : *$L^p(p \geq 1)-$ approximation by Stancu - Kantorovich polynomials* (to appear in Rev. Anal. Numér. Théorie Approximation).

[9] Finta, Z. : *Pointwise approximation by generalized Szász - Mirakjan operators,* Studia Univ. Babeş - Bolyai, Mathematica, 46(4)(2001), 61-67.

[10] Finta, Z. : *Direct and converse theorems for integral - type operators* (to appear in Demonstratio Mathematica).

[11] Finta, Z. : *Direct local and global approximation theorems* (submitted).

[12] Knoop, H. B. and Zhou, X. L. : *The lower estimate for linear positive operators (II),* Results in Mathematics, 25(1994), 315-330.

[13] Păltănea, R. : *Best constants in estimates with second order moduli of continuity,* Proceedings of the International Dortmund Meeting on Approximation Theory (Edited by M. W. Müller, M. Felten, D. H. Mache), Akademie Verlag, Berlin, 1995, 251 - 275.

[14] Razi, Q. : *Approximation of a function by Kantorovich type operators,* Mat. Vesnic., 41(1989), 183-192.

[15] Stancu, D. D. : *Approximation of functions by a new class of linear polynomial operators,* Rev. Roumaine Math. Pures Appl., 13(8)(1968), 1173-1194.

[16] Stancu, D. D. : *A study of the remainder in an approximation formula using a Favard - Szász type operators,* Studia Univ. Babeş - Bolyai, Mathematica, 25(1980), 70-76.

[17] Stancu, D. D. : *Two classes of positive linear operators,* Anal. Univ. Timişoara, Ser. Ştin. Matem., 8(1970), 213-220.

[18] Stancu, D. D. : *On the beta approximating operators of second kind,* Rev. Anal. Numér. Théorie Approximation, 24(1-2)(1995), 231-239.

[19] Totik, V. : *Approximation by Bernstein polynomials,* Amer. J. Math., 116(4)(1994), 995-1018.

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ON GENERATION OF FAMILIES OF SURFACES

IOAN GÂNSCĂ, DELIA DRĂGAN, AND RADU T. TRÎMBIŢAŞ

Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

Abstract. In this paper we define families of interpolation surfaces of Hermite and Birkhoff type. Particular cases of these surfaces are illustrated graphically.

1. Introduction

The modelling and remodelling of surfaces come up in different activities as civil engineering, industries of airplanes, ships, automobiles, industrial and artistic objects, scientific research and others.

There exists a large number of classical and modern methods for generating surfaces. As modern methods for generating surfaces we mention those of Bézier, Coons, Shepard and others [5], frequently encountered in Computer Aided Design (CAD) and Computer Aided Geometric Design (CAGD).

In our paper we present two procedures for defining surfaces.

2. Surfaces with two support curves and tangent ribbons

In this section we define a family of surfaces each one containing the same two opposite space curves, say (C1) and (C2) and different tangent ribbons (across-boundary derivatives), see Figure 1.

Suppose the curves (C_1) and (C_2) are represented by the equations:

$$(C_1)\begin{cases} y = 0 \\ z = h_0(x) \end{cases} \quad \textit{and} \quad (C_2)\begin{cases} y = y_1(x) \\ z = h_1(x), \end{cases} \quad x \in [0, a]. \tag{2.1}$$

For the given functions $h_0(x)$, $h_1(x)$, $m_0(x)$ and $y_1(x)$, $x \in [0, a]$, let us find the surface (S), having the equation $z = f(x, y)$, $x \in [0, a]$, $y \in [0, y_1(x)]$, which satisfies

Received by the editors: 01.11.2002.

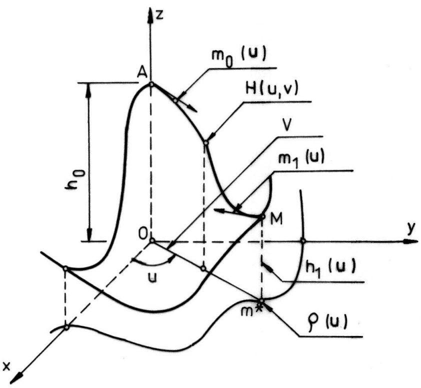

FIGURE 1. A surface with two support curves and tangent ribbons

the following conditions:

$$\begin{aligned} &f(x,0)=h_0(x), \quad && f(x,y_1(x))=h_1(x), \\ &f'_y(x,0)=m_0(x), \quad && f''_{y^2}(x,y_1(x))=m_1(x), \quad x\in[0,a]. \end{aligned} \tag{2.2}$$

The unique solution of the problem (2.2) is the third-degree Hermite interpolation polynomial with respect to the y-variable. By direct calculus one obtains:

$$\begin{aligned} &f\left(x,y;h_0(x),h_1(x),m_0(x),m_1(x)\right)= \\ &\frac{\left[m_0(x)+m_1(x)\right]y_1(x)-2\left[h_1(x)-h_0(x)\right]}{y_1^3(x)}y^3+ \\ &\frac{3\left[h_1(x)-h_0(x)\right]-y_1(x)\left[2m_0(x)+m_1(x)\right]}{y_1^2(x)}y^2+m_0(x)y+h_0(x). \end{aligned} \tag{2.3}$$

In cardinal form [5], the polynomial f is:

$$\begin{aligned} &f\left(x,y;h_0(x),h_1(x),m_0(x),m_1(x)\right)=H_3^0\left(y;y_1(x)\right)h_0(x)+ \\ &H_3^1\left(y;y_1(x)\right)m_0(x)+H_3^2\left(y;y_1(x)\right)m_1(x)+H_3^3\left(y;y_1(x)\right)h_1(x), \end{aligned} \tag{2.4}$$

where the cardinal (blending) functions $H_3^i(y; y_1(x))$, $i = \overline{0,3}$ are

$$\begin{aligned} H_3^0(y; y_1(x)) &= \frac{[y_1(x) - y]^2 [2y + y_1(x)]}{y_1^3(x)}, \\ H_3^1(y; y_1(x)) &= \frac{[y_1(x) - y]^2 y}{y_1^2(x)}, \\ H_3^2(y; y_1(x)) &= \frac{[y - y_1(x)] y^2}{y_1^2(x)}, \\ H_3^3(y; y_1(x)) &= \frac{y^2 [3y_1(x) - 2y]}{y_1^3(x)}, \quad x \in [0, a],\ y \in [0, y_1(x)]. \end{aligned} \tag{2.5}$$

The equation:

$$z = f(x, y; y_1(x), h_0(x), h_1(x), m_0(x), m_1(x)), \quad x \in [0, a],\ y \in [0, y_1(x)],$$

where f is given by (2.3) or (2.4), represents a family of surfaces which depend on $y_1(x)$, $h_0(x)$, $h_1(x)$, $m_0(x)$, $m_1(x)$. The functions $m_0(x)$ and $m_1(x)$ determine the shape of each surface.

Remark 2.1.

a) A surface (S) and its symmetric with respect to xOz plane has the equation:

$$z = f(x, |y|, y_1(x), h_0(x), h_1(x), m_0(x), m_1(x)), \tag{2.6}$$

where $y \in [0, a]$, $|y| \leq y_1(x)$.

b) A surface (S) and its symmetric with respect to yOz plane is represented by the equation:

$$z = f(|x|, y, y_1(|x|), h_0(|x|), h_1(|x|), m_0(|x|), m_1(|x|)), \tag{2.7}$$

where $|x| \leq a$, $y \in [0, y_1(x)]$.

c) The equation of the surface (S) and its symmetric with respect to xOz and yOz planes is:

$$z = f(|x|, |y|; y_1(|x|), h_0(|x|)), m_0(|x|), m_1(|x|), \tag{2.8}$$

$|x| \leq a$, $|y| \leq y_1(x)$.

Figure 2 shows the surface from the family (2.8), where f is given by (2.3) corresponding to the following data:

$$h_0(x) = \frac{4}{75}(x-15)(x-20) - \frac{4}{125}x(x-20) + \frac{4}{125}x(x-15),$$
$$h_1(x) = 1 + \frac{1}{2}\sin\frac{2\pi}{15}\left(x+\frac{3}{2}\right),$$
$$m_0(x) = \frac{1}{6}\cos\frac{\pi}{5}(x+1),$$
$$m_1(x) = -\frac{1}{4}, \quad a = 20, \quad y_1(x) = 20.$$

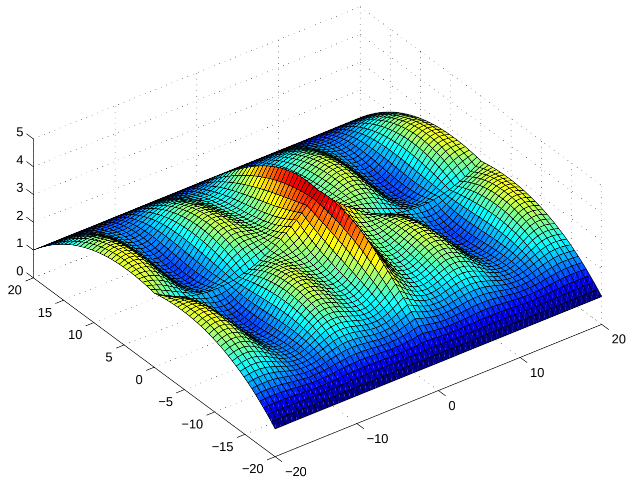

FIGURE 2. A surfaces from the family (2.8)

3. Surfaces having a point and a curve as supports

Let us consider the point $A(0,0,h_0)$ and a curve (C) represented, in cylindrical coordinate system, by the following equations:

$$(C)\begin{cases} x = \rho_1(u)\cos u, \\ y = \rho_1(u)\sin(u), \\ z = h_1(u), \qquad u\in[0,2\pi]. \end{cases} \tag{3.1}$$

(see figure 3).

1. For the beginning we determine a curve (C^*) which passes through the point A and a fixed point of the curve (C), which will be denoted by

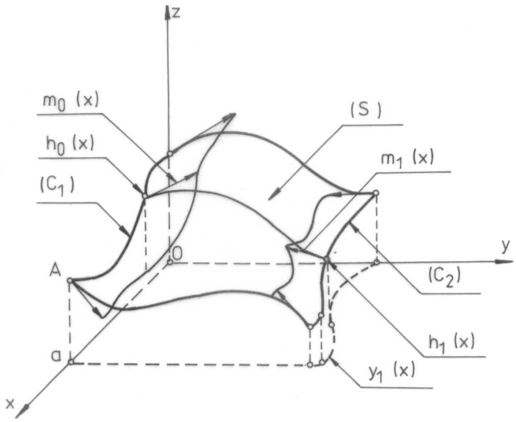

FIGURE 3. A surface having a point and a curve as supports

$B(\rho_1(u)\cos u, \rho_1(u)\sin u, h_1(u))$, u - fixed for the moment and having in A and b the slopes $m_0(u)$ and $m_1(u)$ respectively.

The curve (C^*) is uniquely represented by a third degree Hermite interpolation polynomial in v-variable, similar to (2.3):

$$(C^*)\begin{cases} x = v\cos u \\ y = v\sin u \\ z = h\left(v; h_0, h_1(u), m_0(u), m_1(u)\rho_1(u)\right), \end{cases}$$

$v \in [0, \rho_1(u)]$, $u \in [0, 2\pi]$, u fixed, where

$$\begin{aligned} & h\left(v; h_0, h_1(u), m_0(u), m_1(u), \rho_1(u)\right) = \\ & \frac{[m_0(u) + m_1(u)]\,\rho_1(u) 2\,[h_1(u) - h_0]}{\rho_1^3(u)} v^3 \\ & + \frac{3\,[h_1(u) - h_0] - \rho_1(u)\,[2m_0(u) + m_1(u)]}{\rho_1^2(u)} v^2 + m_0(u) + h_0, \end{aligned} \tag{3.2}$$

or in canonical form

$$\begin{aligned} & h\left(v; h_0, h_1(u), m_0(u), m_1(u)\rho_1(u)\right) = H_3^0\left(v; \rho_1(u)\right) h_0 + \\ & + H_3^1\left(v; \rho_1(u)\right) m_0(u) + H_3^2\left(v; \rho_1(u)\right) m_1(u) + H_3^3\left(v; \rho_1(u)\right) h_1(u) \end{aligned} \tag{3.3}$$

with

$$\begin{aligned} H_3^0\left(v;\rho_1(u)\right) &= \frac{\left[\rho_1(u)-v\right]^2\left[2v+\rho_1(u)\right]}{\rho_1^3(u)}, \\ H_3^1\left(v;\rho_1(u)\right) &= \frac{\left[\rho_1(u)-v\right]^2 v}{\rho_1^2(u)}, \\ H_3^2\left(v;\rho_1(u)\right) &= \frac{\left[v-\rho_1(u)\right] v^2}{\rho_1^2(u)}, \\ H_3^3\left(v;\rho_1(u)\right) &= \frac{v^2\left[3\rho_1(u)-v\right]}{\rho_1^3(u)}, \end{aligned} \tag{3.4}$$

$v \in [0,\rho_1(u)],\ u \in [0,2\pi],\ u$ fixed.

The surface (Γ) generated by the curve (C^*), for u-variable is represented by the equations:

$$(\Gamma)\begin{cases} x = v\cos u, \\ y = v\sin u, \\ z = h\left(v;h_0,h_1(u),m_0(u),m_1(u),\rho_1(u)\right), \end{cases} \tag{3.5}$$

where $u \in [0,2\pi]$, $v \in [0,\rho_1(u)]$, and h is given by (3.2) or (3.3).

A surface from the family (3.5) corresponding to the data

$$m_0(u) = -\frac{1}{5}, \quad m_1(u) = \frac{1}{2}, \quad h_0 = 10,$$
$$h_1(u) = 5 + \sin 5u - |\sin 5u|, \quad \rho_1(u) = 15.$$

is represented in Figure 4.

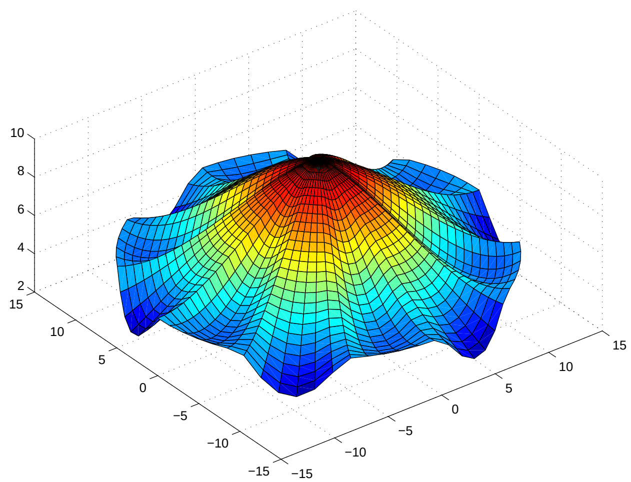

FIGURE 4. A surface from family (3.5)

2. Next, we consider that the generating curve (C^*) passes through the points A and B, in A has the slope $m_0(u)$ and for $v = \rho_0(u)$, u-fixed, it has an inflexion point. Such a curve is unique represented with the aid of a third degree Birkhoff interpolation polynomial [4].

The surface (Σ) generated by the curve (C^*) when u is variable, $u \in [0, 2\pi]$ has the following parametric equations:

$$\begin{aligned} x &= v \cos u \\ y &= v \sin u \\ z &= B\left(v; h_0, h_1(u), m_0(u), \rho_0(u), \rho_1(u)\right), \end{aligned} \tag{3.6}$$

where

$$B\left(v; h_0, h_1(u), m_0(u), \rho_0(u), \rho_1(u)\right) = \frac{h_0 - h_1(u) + m_0(u)\rho_1(u)}{\rho_1^2(u)\left[3\rho_0(u) - \rho_1(u)\right]} v^2 \left[\left[v - 3\rho_0(u)\right] + m_0(u)v + h_0,\right.$$

$u \in [0, 2\pi]$, $v \in [0, \rho_1(u)]$.

We note that the equations (3.6) represents a family of surfaces. The shape of each member of this family depends on h_0, $h_1(u)$, $m_0(u)$, $\rho_0(u)$ and $\rho_1(u)$. The surface from family (3.6) corresponding to the particular case

$$\begin{aligned} &h_0 = 8, \quad h_1(u) = 1 + \frac{1}{4}\sin 12u, \quad m_0(u) = -1, \\ &\rho_0(u) = 15, \quad \rho_1(u) = 20 \end{aligned}$$

is given in Figure 5.

References

[1] Barnhill, R.E., (1977), Representation and Approximation of Surfaces, Mathematical Software III, Academic Press, Inc.NewYork, San Francisco, London, 1977, 69-120.

[2] Coman G., Gânscă I. and Ţâmbulea L., (1989), Some Practical Applications of Blending Approximation, Itinerant Seminar of Functional Equations, Approximation and Convexity, Cluj-Napoca, Preprint nr.7, 5-22.

[3] Coman G., Gânscă I. and Ţâmbulea L., (1991), Some New Roof-surfaces Generated by Blending Interpolation Technique, STUDIA, Univ. "Babeş-Bolyai", Mathematica, XXXVI, 1, 119-130.

[4] Davis, Ph. (1975), Interpolation and Approximation, Dover, New York.

[5] Farin, G., (1990) Curves and Surfaces for Computer Aided Geometric Design, Academic Press, Inc. Harcourt Brace Jovanovich, Publishers, Boston, San Diego, New York, London, Sydney, Tokyo, Toronto.

[6] Mihailescu, M. and Horvath, I., (1977), Velaroidal Shell for Covering Universal Industrial Halls, Acta Technica Academiae Scientiarum Hungarica, 85, 1-2, 135-145.

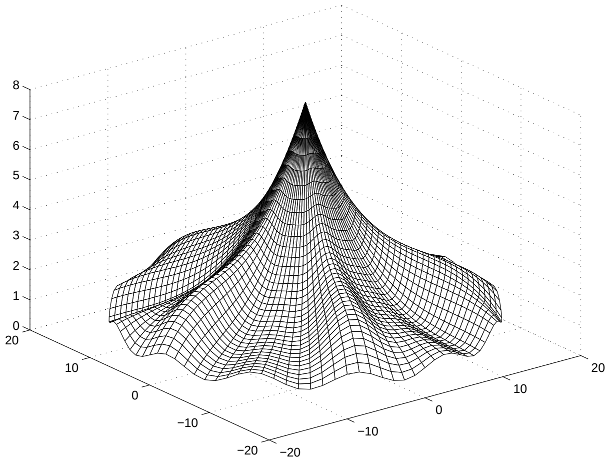

FIGURE 5. A surface from the family (3.6)

[7] Zetea E., Gânscă I. and Hristea I., (2000), Surfaces with Controlled Parabolic Points Generated by Curves of the Third Order, Acta Technica Napocensis, Section Civil Engineering – Architecture, 43, 123-130.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

FIXED POINT THEOREM IN ΛE-SPACES

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Dedicated to Professor D.D. Stancu on his 75^{th} birthday

The theory of metric spaces is a very useful tool in applied mathematics. However, by some practical problems this theory can not be applied. For this reason the concept of probabilistic metric space was introduced in 1942 by Menger [7]. It was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [10]. Menger proposed to replace the distance $d(x, y)$ by a distribution function $F_{x,y}$ whose value $F_{x,y}(t)$, for any real number t, is interpreted as the probability that the distance between x and y is less than t. The study of contraction mappings for probabilistic metric spaces was initiated by Sehgal [12],[13], Sherwood [16] and Bharucha-Reid [14]. Radu in [8] and [9] introduced other types of contractions in probabilistic metric spaces. The notion of E-space was introduced by Sherwood [16] in 1969 as a generalization of Menger space for random variables. For new results and applications of probabilistic analysis one can consult Constantin and Istrăţescu's book [2]. New results in fixed point theory in probabilistic metric spaces can be find in [4] and in Hadzic's book [3].

Hutchinson and Rüschendorf [5] showed that the Brownian motion can be characterized as a fixed point of a special stochastic process. They proved a fixed point theorem using a first moment condition. Our goal is to generalize this idea and to replace the first moment condition by a more less restrictive hypothesis. Using a generalization of the notion of E-space to the so called ΛE-space we will prove a new fixed point theorem. As application Brownian bridge-type stochastic fractal interpolation functions will be constructed.

In the first section we recall the notions of probabilistic metric space and E-space. The next section contains the definition and some properties of ΛE-space. The main result of this paper is the fixed point theorem in section 3. The last section contain an application of our main theorem to the stochastic fractal interpolation.

1. Probabilistic metric spaces

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

A mapping $F : \mathbb{R} \to [0, 1]$ is called a **distribution function** if it is non-decreasing and left continuous.

By Δ we shall denote the set of all distribution functions F. Let Δ be ordered by the relation "$\leq$", i.e. $F \leq G$ if and only if $F(t) \leq G(t)$ for all real t. Also $F < G$ if and only if $F \leq G$ but $F \neq G$.

We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

Received by the editors: 08.10.2002.

Key words and phrases. probabilistic metric space, fixed point, fractal interpolation.

Let H denote the Heviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases} \tag{1.1}$$

Let X be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x, y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbb{R}$ by $F_{x,y}(t)$, respectively.

The pair $(X, \mathcal{F})$ is a **probabilistic metric space** (briefly **PM space**) if X is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1^0 $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;

2^0 $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3^0 if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s + t) = 1$.

A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a **t-norm** if the following conditions are satisfied:

4^0 $T(a, 1) = a$ for every $a \in [0, 1]$;

5^0 $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$;

6^0 if $a \geq c$ and $b \geq d$ then $T(a, b) \geq T(c, d)$;

7^0 $T(a, T(b, c)) = T(T(a, b), c)$ for every $a, b, c \in [0, 1]$.

We list here the simplest:

$T_1(a, b) = max\{a + b - 1, 0\}$,

$T_2(a, b) = ab$,

$T_3(a, b) = Min(a, b) = min\{a, b\}$,

A **Menger space** is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a probabilistic metric space, T is a t-norm, and instead of 3^0 we have the stronger condition:

8^0 $F_{x,y}(s + t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}_+$.

If the t-norm T satisfies the condition

$$sup\{T(a, a) : a \in [0, 1[\} = 1,$$

then the (t, ϵ) -topology is metrizable (see [11]).

In 1966, V.M. Sehgal [13] introduced the notion of a contraction mapping in probabilistic metric spaces.

The mapping $f : X \to X$ is said to be a **contraction** if there exists $r \in]0, 1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbb{R}_+$.

A sequence $(x_n)_{n\in\mathbb{N}}$ from X is said to be **fundamental** if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1$$

for all $t > 0$.

The element $x \in X$ is called **limit** of the sequence $(x_n)_{n\in\mathbb{N}}$, and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$.

A probabilistic metric (Menger) space is said to be **complete** if every fundamental sequence in that space is convergent.

For example, if (X, d) is a metric space, then the metric d induces a mapping $\mathcal{F} : X \times X \to \Delta^+$, where $\mathcal{F}(x, y) = F_{x,y}$ is defined by

$$F_{x,y}(t) = H(t - d(x, y)),\ t \in \mathbb{R}.$$

Moreover $(X, \mathcal{F}, Min)$ is a Menger space. Bharucha-Reid and Sehgal show that $(X, \mathcal{F}, Min)$ is complete if the metric d is complete (see [14]). The space $(X, \mathcal{F}, Min)$ thus obtained is called the **induced Menger space**.

The notion of E-space was introduced by Sherwood [16] in 1969. Next we recall this definition and we show that if (X, d) is a complete metric space then the E-space is also complete.

Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space.

The ordered pair $(\mathcal{E}, F)$ is an **E-space over the metric space** (Y, ρ) (briefly, an E-space) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F}$ is the mapping from $\mathcal{E} \times \mathcal{E}$ into Δ^+ defined via $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega | \ \rho(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbb{R}$.

If $\mathcal{F}$ satisfies the condition

$$\mathcal{F}(x, y) \neq H, \text{ if } x \neq y,$$

then $(\mathcal{E}, \mathcal{F})$ is said to be a **canonical E-space.** Sherwood [16] proved that every canonical $\mathcal{E}$-space is a Menger space under $T = T_m$, where $T_m(a, b) = \max\{a + b - 1, 0\}$. In the following we suppose that $\mathcal{E}$ is a canonical E-space.

The convergence in an $\mathcal{E}$-space is exactly the probability convergence.

The E-space $(\mathcal{E}, \mathcal{F})$ is said to be **complete** if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete.

If we start with a complete metric space (X, d) then we obtain a complete E-space.

Proposition 1.1. ([6]) *If (X, d) is a complete metric space then the E-space $(\mathcal{E}, F)$ is also complete.*

2. ΛE-spaces

Let Λ be a nonempty set and, for $\lambda \in \Lambda$, let (Y^λ, d^λ) be metric space. Denote $\mathcal{E}^\lambda$ the set of random variables from Ω into Y^λ and let

$$\mathcal{F}^\lambda : \mathcal{E}^\lambda \times \mathcal{E}^\lambda \to \Delta^+$$

be defined via $\mathcal{F}^\lambda(x, y) := F^\lambda_{x,y}$, where

$$F^\lambda_{x,y}(t) := P(\{\omega \in \Omega | d^\lambda(x^\lambda(\omega), y^\lambda(\omega)) < t\})$$

for all $t \in \mathbb{R}$. Denote

$$F_{x,y}(t) := \inf_{\lambda \in \Lambda} F^\lambda_{x,y}(t)$$

and

$$\mathcal{F}(x, y) := F_{x,y}.$$

The ordered pair $(\mathcal{E}^\lambda, \mathcal{F}^\lambda)$ is an E-space over the metric space Y^λ.

Let $Y := \prod_{\lambda \in \Lambda} Y^\lambda$, $e \in Y$ and define

$$\mathcal{E} := \{x \in \prod_{\lambda \in \Lambda} \mathcal{E}^\lambda | \quad \lim_{t \to \infty} \inf_{\lambda \in \Lambda} P(\{\omega \in \Omega | d^\lambda(x^\lambda(\omega), e^\lambda(\omega)) < t\}) = 1\}.$$

Remark. $\mathcal{E}$ is the set of bounded random functions. The convergence in $\mathcal{E}$ is similar to the uniform convergence in metric space.

The triplet $(\mathcal{E}, \mathcal{F}, T)$ is called ΛE-space.

In the following let $T := T_m$.

Proposition 2.1. $(\mathcal{E}, \mathcal{F}, T)$ *is a Menger space.*

Proof. Conditions 1^o and 2^o are satisfied by definition. Since $F^{\lambda}_{x,y}$ satisfies 8^o for all $\lambda \in \Lambda$, we can write

$$\begin{aligned} F^{\lambda}_{x,y}(t+s) &\geq T(F^{\lambda}_{x,z}(t), F^{\lambda}_{z,y}(s)) \geq \\ &\geq \inf_{\lambda} \max(F^{\lambda}_{x,z}(t) + F^{\lambda}_{z,y}(s) - 1, 0) \geq \\ &\geq \max(\inf_{\lambda} F^{\lambda}_{x,z}(t) + \inf_{\lambda} F^{\lambda}_{z,y}(s) - 1, 0) = \\ &= T(F_{x,z}(t), F_{z,y}(s)) \end{aligned}$$

for all $t, s \in \mathbb{R}_+$. Taking the infimum over λ we obtain the triangle inequality:

$$F_{x,y}(t+s) = \inf_{\lambda \in \Lambda} F^{\lambda}_{x,y}(t+s) \geq T(F_{x,z}(t), F_{z,y}(s))$$

for all $t, s \in \mathbb{R}_+$. □

Proposition 2.2. *If $(Y^{\lambda}, d^{\lambda})$ are complete metric spaces for all $\lambda \in \Lambda$, then $(\mathcal{E}, \mathcal{F}, T)$ is a complete Menger space.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $\mathcal{E}$, i.e.

$$\lim_{n,m\to\infty} F_{x_n,x_m}(t) = \lim_{n,m\to\infty} \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < t\}) = 1 \qquad (2.2)$$

for all $t > 0$ and

$$\lim_{t\to\infty} \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), e^{\lambda}(\omega)) < t\}) = 1. \qquad (2.3)$$

Since, for $\lambda \in \Lambda$

$$P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < t\}) \geq F_{x_n,x_m}(t),$$

it follows that for $\epsilon > 0$, exists $n_\epsilon \in \mathbb{N}$ such that, if $n > n_\epsilon$ and $m > n_\epsilon$ then

$$P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < t\}) > 1 - \epsilon.$$

So, $(x^{\lambda}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the E-space $(\mathcal{E}^{\lambda}, \mathcal{F}^{\lambda})$. According to Proposition 1.1 $(\mathcal{E}^{\lambda}, \mathcal{F}^{\lambda})$ is complete for all $\lambda \in \Lambda$. Denote $x^{\lambda} := \lim_{n\to\infty} x^{\lambda}_n$, and $x := (x^{\lambda} | \lambda \in \Lambda)$.

Now we have to show that

(i) $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$,

and

(ii) $x \in \mathcal{E}$.

By the relation (2.2) for all $t > 0$ and $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for $n, m > n_\epsilon$ and $\lambda \in \Lambda$

$$P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < \frac{t}{2}\}) > 1 - \frac{\epsilon}{2}.$$

Since

$$P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}(\omega)) < t\}) \geq$$

$$\geq P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < \frac{t}{2}\}) + P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_m(\omega), x^{\lambda}(\omega)) < \frac{t}{2}\}) - 1 > 1 - \epsilon$$

we have

$$\inf_{\lambda\in\Lambda} F^{\lambda}_{x_n,x}(t) > 1 - \epsilon$$

for all $t > 0$ and $\epsilon > 0$. So

$$\lim_{n\to\infty} F_{x_n,x}(t) = 1, \quad \text{for all} \quad t > 0.$$

In order to show (ii) we use relation (2.3). For $\epsilon > 0$ there exists $t_\epsilon > 0$ such that for all $t \geq t_\epsilon$ the following inequalities hold

$$F_{x,e}(2t) \geq T(F_{x_n,x}(t), F_{x_n,e}(t)) \geq T(F_{x_n,x}(1), F_{x_n,e}(t)) >$$
$$> 1 - \frac{\epsilon}{2} + F_{x_n,e}(t) - 1 > 1 - \epsilon.$$

□

3. The main result

The main result of this paper is the following fixed point theorem:

Theorem 3.1. *Let $(\mathcal{E}, \mathcal{F}, T)$ be a complete ΛE- space, and let $f : \mathcal{E} \to \mathcal{E}$ be a contraction with ratio r. Suppose there exists $z \in \mathcal{E}$ and a real number γ such that*

$$\sup_{\lambda\in\Lambda} P(\{\omega \in \Omega |\ d^\lambda(z^\lambda(\omega), f(z^\lambda)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \text{ for all } t > 0.$$

Then there exists a unique $x_0 \in \mathcal{E}$ such that $f(x_0) = x_0$.

Proof. Let $a_0 = z$ and $a_n = f(a_{n-1})$ for $n \geq 1$.

First we show that $(a_n)_{n\in\mathbb{N}}$ is a fundamental sequence in $(\mathcal{E}, \mathcal{F}, T)$.

Let $f_n = f \circ \cdots \circ f$ n-times.

Since $a_{n+k} = f_n(a_k)$ and $a_n = f_n(a_0)$, we have

$$F_{a_n,a_{n+k}}(s) = F_{f_n(z),f_n(a_k)}(s) \geq \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega |\ r^n d^\lambda(z^\lambda(\omega), a_k^\lambda(\omega)) < s\}) =$$

$$\geq \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega |\ r^n d^\lambda(z^\lambda(\omega), a_k^\lambda(\omega)) < s \cdot (1 + \sqrt{r} + \cdots + \sqrt{r}^{k-1})(1 - \sqrt{r})\}) \geq$$
$$\geq P(\{\omega \in \Omega |\ r^n[d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) + d^\lambda(f(z^\lambda(\omega)), f_2(z^\lambda(\omega))) + \cdots +$$
$$+ d^\lambda(f_{k-1}(z^\lambda(\omega)), f_k(z^\lambda(\omega)))] < s \cdot (1 + \sqrt{r} + \cdots + \sqrt{r}^{k-1})(1 - \sqrt{r})\}) \geq$$
$$\geq \inf_{\lambda\in\Lambda}[P(\{\omega \in \Omega |\ d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) < \frac{s(1-\sqrt{r})}{r^n}\}) +$$
$$+ P(\{\omega \in \Omega |\ d^\lambda(f(z^\lambda(\omega)), f_2(z^\lambda(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}}{r^n}\}) + \cdots +$$
$$+ P(\{\omega \in \Omega |\ d^\lambda(f_{k-1}(z^\lambda(\omega)), f_k(z^\lambda(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^n}\})] - (k-1) \geq$$
$$\geq \inf_{\lambda\in\Lambda}[P(\{\omega \in \Omega |\ d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) < \frac{s(1-\sqrt{r})}{r^n}\}) +$$

$$+ P(\{\omega \in \Omega |\ r d^\lambda(z^\lambda(\omega)), f(z^\lambda(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}}{r^n}\}) + \cdots +$$
$$+ P(\{\omega \in \Omega |\ r^{k-1} d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^n}\})] - (k-1) =$$
$$= 1 - \sup_{\lambda\in\Lambda}[P(\{\omega \in \Omega |\ d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) \geq \frac{s(1-\sqrt{r})}{r^n}\}) +$$
$$+ P(\{\omega \in \Omega |\ d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) \geq \frac{s(1-\sqrt{r})\sqrt{r}}{r^{n+1}}\}) + \cdots +$$
$$+ P(\{\omega \in \Omega |\ d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) < \frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^{n+k-1}}\})] \geq$$
$$\geq 1 - \gamma \cdot r^n \left(\frac{1}{s(1-\sqrt{r})} + \frac{r^{1/2}}{s(1-\sqrt{r})} + \ldots + \frac{r^{(k-1)/2}}{s(1-\sqrt{r})} \right) >$$

$$> 1 - \gamma \frac{r^n}{s(1-\sqrt{r})^2}.$$

Since

$$\lim_{n\to\infty} \left(1 - \gamma \frac{r^n}{s(1-\sqrt{r})^2}\right) = 1,$$

we have, for $t > 0$,

$$\lim_{n\to\infty} F_{a_n, a_{n+k}}(t) = 1,$$

uniformly with respect to k. The space $(\mathcal{E}, \mathcal{F}, T)$ being complete, (a_n) is convergent. Let x_0 be its limit.

Next we show that x_0 is a fixed point of f.

For we have

$$F_{a_n, f(x_0)}(\frac{t}{2}) \geq F_{a_{n-1}, x_0}(\frac{t}{2}) \text{ for all } t > 0.$$

Using 8^0 it follows

$$F_{x_0, f(x_0)}(t) \geq T(F_{x_0, a_n}(\frac{t}{2}), F_{a_n, f(x_0)}(\frac{t}{2})) \geq T(F_{x_0, a_n}(\frac{t}{2}), F_{a_{n-1}, x_0}(\frac{t}{2})).$$

Since $\lim_{n\to\infty} a_n = x_0$, we have

$$F_{x_0, f(x_0)}(t) = 1 \;\; for\ all \;\; t > 0,$$

therefore

$$f(x_0) = x_0.$$

For the uniqueness we suppose that there exists an other element $x' \in \mathcal{E}$ such that $f(x') = x'$. For $n \in \mathbb{N}$ and $t > 0$, we have

$$F_{x_0, x'}(t) = F_{f^n(x_0), f^n(x')}(t) \geq F_{x_0, x'}\left(\frac{t}{r^n}\right).$$

Since $\lim_{n\to\infty} r^n = 0$, we have

$$F_{x_0, x'}(t) = 1 \text{ for all } t > 0,$$

therefore $x_0 = x'$. □

4. **Application: stochastic fractal interpolation**

In [5] Hutchinson and Rüschendorf showed that the Brownian bridge can be characterized as the fixed point of a "scaling" function. Indeed, let $(\Omega, \mathcal{K}, P)$ be a probability space and let $\Lambda = \mathbb{R}_+$, the set of positive real numbers. Define the Brownian bridge as the stochastic process $(X_t^\lambda)_{t\in\mathbb{R}_+}$ with the following properties:

$$P(\{\omega \in \Omega | \quad t \mapsto X^\lambda(t, \omega) \quad \text{is continuous}\}) = 1,$$

and, for every $t \geq 0$ and every $h > 0$,

$$X^\lambda(t+h) - X^\lambda(t) \stackrel{d}{=} N(0, \lambda h),$$

thus

$$P(\{\omega \in \Omega | X^\lambda(t+h, \omega) - X^\lambda(t, \omega) < x\}) = \frac{1}{\sqrt{2\pi} h\lambda} \int_{-\infty}^{x} e^{-\frac{t^2}{2\lambda^2 h^2}} dt.$$

$N(a, b)$ denote the normal distribution with mean a and variance b.

We suppose

$$X^\lambda(0, \omega) = 0 \quad a.s. \quad \text{and} \quad X^\lambda(1, \omega) = 1 \quad a.s..$$

Denote $I = [0,1]$, and define the functions

$$\Phi_1 : I \to [0, \frac{1}{2}], \quad \Phi_1(s) = \frac{s}{2},$$

and

$$\Phi_2 : I \to [\frac{1}{2}, 1], \quad \Phi_1(s) = \frac{s+1}{2}.$$

Let $\lambda \in \Lambda$ and denote p^λ the random point with distribution $N(0, \frac{\lambda}{2})$.

Let $\varphi_1^\lambda, \varphi_2^\lambda : \mathbb{R} \times \Lambda \to \mathbb{R}$ be the affine transformations characterized by $\varphi_1^\lambda(0,\lambda) = 0$, $\varphi_1^\lambda(1,\lambda) = \varphi_2^\lambda(0,\lambda) = p^\lambda$, $\varphi_2^\lambda(1,\lambda) = 1$ for all $\lambda \in \Lambda$. Denote $r_1^\lambda = Lip\varphi_1^\lambda = |p^\lambda|$, $\quad r_2^\lambda = Lip\varphi_2^\lambda = |1 - p^\lambda|$. For φ_1^λ, φ_2^λ we obtain

$$\varphi_1^\lambda(a,\lambda) = p^\lambda a \quad \text{and} \quad \varphi_2^\lambda(a,\lambda) = (1-p^\lambda)a + p^\lambda.$$

Denote $\mathbf{L}$ the set of functions from $\mathbb{R} \times \Lambda$ to $\mathbb{R}$,

$$\mathbf{L} := \{u : \mathbb{R} \times \Lambda \to \mathbb{R}\}.$$

Let $\psi_1, \psi_2 : \mathbf{L} \to \mathbf{L}$ be mappings satisfying the following property:

$$\psi_i(u)(a,\lambda) = u(a, \frac{\lambda}{2r_i^2}), \qquad i = 1,2.$$

Let

$$S_i^\lambda = \varphi_i^\lambda \circ \psi_i.$$

Using the definition of the process, we have

$$X^\lambda|_{X^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} S_1^\lambda \circ X^\lambda(2t), \quad t \in [0, \frac{1}{2}].$$

Similarly

$$X^\lambda|_{X^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} S_2^\lambda \circ X^\lambda(2t-1), \quad t \in [\frac{1}{2}, 1].$$

This relations can be written as follows

$$X^\lambda|_{X^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} \sqcup_i S_i^\lambda \circ X^\lambda \circ \Phi_i^{-1}(t), \quad t \in [0,1].$$

For each $\lambda > 0$, we have

$$X^\lambda \stackrel{d}{=} \sqcup_i S_i^\lambda \circ X^{\lambda(i)} \circ \Phi_i^{-1},$$

where $X^{\lambda(i)} \stackrel{d}{=} X^\lambda$ are chosen independently of one another.

Let $Y^\lambda = L_1([0,1])$ and d^λ the Euclidean metric in $\mathbb{R}$, for all $\lambda \in \Lambda$. In this case $\mathcal{E}^\lambda$ is the space of real random variables and $\mathcal{E}$ is their product space. By Theorem 2.2 $(\mathcal{E}, \mathcal{F}, T)$ is a complete ΛE- space. Consider the function $f : \mathcal{E} \to \mathcal{E}$, defined by $f := (f^\lambda | \lambda \in \Lambda)$ where

$$f^\lambda(X) := \sqcup_i S_i^\lambda \circ X^{\lambda(i)} \circ \Phi_i^{-1}$$

for all $X \in \mathcal{E}$. If X_0 is a fixed point of f then, for all $\lambda \in \Lambda$,

$$X_0^\lambda \stackrel{d}{=} f^\lambda(X_0^\lambda).$$

Hutchinson and Rüschendorf [5] proved that, if the set of all functions $Z \in \mathcal{E}$ such that

$$\sup_{\lambda \in \Lambda} \lambda^{-\frac{1}{2}} E_\omega \int_I |Z(t,\lambda,\omega)| dt < \infty$$

there exists a fixed point of f. Motivated by this result, we consider the following problem.

Let Λ be a nonempty set and let $0 = t_0 < t_1 < ... < t_N = 1$, $t_i \in \mathbb{R}$, $i \in \{0, ..., N\}$ be $N + 1$ given points. Consider N bijections

$$\Phi_i : I \to [t_{i-1}, t_i] = I_i$$

for $i \in \{1, ..., N\}$, with Lipschitz constant α_i.

Let $Y^\lambda := L_1(I)$ and let $\beta(\lambda) > 0$ for all $\lambda \in \Lambda$. For $u, v \in Y^\lambda$, define

$$d^\lambda(u, v) := \beta(\lambda) \left(\int_I |u(a) - v(a)| da \right).$$

Let $\mathcal{E}$ be defined as in previous section with $e = 0$.

For all $\lambda \in \Lambda$ and $i \in \{1, ..., N\}$ define the random function $\varphi_i^\lambda : \mathbb{R} \to \mathbb{R}$, $\varphi_i^\lambda \in Lip^{(<1)}$, and r_i^λ denote its Lipschitz constant. Let $\gamma_i : \Lambda \to \mathbb{R}$ be real functions. Consider the mappings $\psi_i : \mathbf{L} \to \mathbf{L}$ such that

$$\psi_i(u)(a, \lambda) := u(a, \gamma_i(\lambda)),$$

and S_i^λ be defined as above, i.e. $S_i^\lambda := \varphi_i^\lambda \circ \psi_i$. Suppose for $\lambda \in \Lambda$ there exists $\delta(\lambda) > 0$ such that the following Lipschitz condition will be satisfied:

$$\inf_\lambda P(\{\omega \in \Omega | \delta(\lambda) \int_I |u(a, \gamma_i(\lambda), \omega) - v(a, \gamma_i(\lambda), \omega)| da < s\}) \geq$$
$$\geq \inf_\lambda P(\{\omega \in \Omega | \beta(\lambda) \int_I |u(a, \lambda, \omega) - v(a, \lambda, \omega)| da < s\})$$

for all $u, v \in \mathcal{E}$.

Let p_i^λ be given random variable ($i \in \{0, ..., N\}$). Suppose the next interpolation properties are fulfilled:

for $u \in \mathcal{E}$, $\lambda \in \Lambda$ and $i \in \{1, ..., N-1\}$

$$\varphi_1^\lambda(u(0, \lambda, \omega)) = p_0^\lambda(\omega) \quad a.s. \tag{4.4}$$
$$\varphi_{i+1}^\lambda(u(0, \lambda, \omega)) = \varphi_i^\lambda(u(1, \lambda, \omega)) = p_i^\lambda(\omega) \quad a.s. \tag{4.5}$$
$$\varphi_N^\lambda(u(1, \lambda, \omega)) = p_N^\lambda(\omega) \quad a.s. \tag{4.6}$$

If $x \in \mathcal{E}$ then the random function $f(x)$ is defined by

$$f^\lambda(x) = \sqcup_i S_i^\lambda \circ x \circ \Phi_i^{-1}, \tag{4.7}$$

Theorem 4.1. *Suppose*

$$ess \sup_\omega \sup_{\lambda \in \Lambda} \sum_{i=1}^N \frac{r_i^\lambda(\omega) \alpha_i \beta(\lambda)}{\delta(\lambda)} < 1 \tag{4.8}$$

and there exists a real number γ such that

$$\sup_{\lambda \in \Lambda} P(\{\omega \in \Omega | \sum \alpha_i |\varphi_i^\lambda(0)| \geq t\}) \leq \frac{\gamma}{t} \text{ for all } t > 0. \tag{4.9}$$

Then there exists a random fractal interpolation function $x^* \in \mathcal{E}$ such that

$$f(x^*) = x^*$$

and

$$x^*(t_i, \lambda, \omega) = p_i^\lambda(\omega) \quad a.s., \quad i \in \{0, ..., N\}, \lambda \in \Lambda. \tag{4.10}$$

Proof. For the random functions $x, z : I \times \Lambda \times \Omega \to \mathbb{R}$, $i \in \{1, ..., n\}$ let as define

$$F_{x,z}(t) := \inf_{\lambda} P(\{\omega \in \Omega | \beta(\lambda) \left(\int_I |x(a,\lambda,\omega) - z(a,\lambda,\omega)| da \right) < t\}).$$

Assuming this has been done, in order to show that f is a contraction map we compute

$$F_{f(x),f(z)}(t) = \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | d^\lambda(f(x), f(z)) < t\}) =$$

$$= \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | \beta(\lambda)(\sum_{i=1}^{N} \int_{I_i} |\varphi_i^\lambda(\psi_i(x(\Phi_i^{-1}(a),\lambda,\omega)) -$$

$$-\varphi_i^\lambda(\psi_i(z(\Phi_i^{-1}(a),\lambda,\omega))|da) < t\}) \geq$$

$$\geq \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | \beta(\lambda)(\sum_{i=1}^{N} r_i^\lambda(\omega)\alpha_i \int_I |\psi_i(x(a,\lambda,\omega)) - \psi_i(z(a,\lambda,\omega))|da) < t\}) \geq$$

$$\geq \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | \left(\sum_{i=1}^{N} \frac{\alpha_i r_i^\lambda(\omega)\beta(\lambda)}{\delta(\lambda)} \right) \delta(\lambda) \cdot$$

$$\cdot \left(\int_I |\psi_i(x(a,\lambda,\omega)) - \psi_i(z(a,\lambda,\omega))|da \right) < t\}) \geq$$

$$\geq \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | r \left(\int_I |x(a,\lambda,\omega) - z(a,\lambda,\omega)|da \right) < t\}).$$

So we have

$$F_{f(x),f(z)}(t) \geq F_{x,z}(\frac{t}{r}).$$

Using Theorem 3.1 for the contraction f there exists a fractal interpolation function x^*.

Next we have to show the interpolation property of x^*. For $i \in \{1, ..., N\}$ we have the following equalities

$$x^*(t_i,\lambda,\omega) = f(x^*(t_i,\lambda,\omega)) = S_i^\lambda(x^*(t_N,\lambda,\omega)) = p_i^\lambda(\omega).$$

□

This fractal interpolation function x^* can be considered a generalized Brownian motion.

Remark: If

$$\sup_{\lambda\in\Lambda} \beta(\lambda) E_\omega \sum \alpha_i |\varphi_i^\lambda(0)| < \infty$$

then, by Tchebysev inequality, (4.9) is fulfilled.

References

[1] A. T. Bharucha-Reid, *Fixed point theorems in probabilistic analysis*, Bull. Amer. Math. Soc. 82(1976), 641-657.

[2] Gh. Constantin, I. Istraţescu, *Elements of Probabilistic Analysis,* Kluwer Academic Publishers, 1989.

[3] O. Hadžić, E. Pap, *Fixed point theory in probabilistic metric spaces*, Kluwer Acad. Publ., 2001.

[4] O. Hadzić, R. Meciar, E. Pap, *A Fixed Point Theorem in Probabilistic Metric Spaces and Application,* Journal of Mathematical Analysis and Application, **202** (1996), 433-449.

[5] J. E. Hutchinson, L. Rüschendorf, *Selfsimilar Fractals and Selfsimilar Random Fractals,* Progress in Probability, 46, (2000), 109-123.
[6] J. Kolumbán, A. Soós, *Invariant sets of random variables in complete metric spaces,* Studia Univ. Babes-Bolyai, Mathematica, XLVII, 3 (2001), 49-66.
[7] K. Menger, *Statistical Metrics,* Proc.Nat. Acad. of Sci.,U.S.A. 28 (1942), 535-537.
[8] V. Radu, *On some contraction type mappings in Menger spaces,* Sem. Teor. Prob. Apl. Univ. Timisoara 78, 1985.
[9] V. Radu, *On some fixed point theorem in PM-spaces,* Lectures Notes in Mathematics, Springer Verlag, 1233(1987), 125-133.
[10] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces,* North Holland, New-York, Amsterdam, Oxford, 1983.
[11] B. Schweizer, A. Sklar, E. Thorp, *The Metrization of Statistical Metric Spaces,* Pacific Journal of Mathematics, 10 (1960), no. 2, 673-675.
[12] V. M. Sehgal, *Some Fixed Point Theorems in Functional Analysis and Probability*, Ph.D.Thesis, Wayne State University, 1966.
[13] V. M. Sehgal, *A Fixed Point Theorem for Mappings with a Contractive Iterate,* Proc. Amer. Math. Soc.,23 (1969), 631-634.
[14] V. M. Sehgal, A. T. Bharucha-Reid, *Fixed Points of Contraction Mappings on Probabilistic Metric Spaces,* Math. Systems Theory, 6 (1972), 92-102.
[15] H. Sherwood, *Complete Probabilistic Metric Spaces,* Z.Wahrsch.verw. Geb., 20 (1971), 117-128.
[16] H. Sherwood, *E-spaces and their relation to other classes of probabilistic metric spaces,* J.London Math. Soc., 44 (1969), 441-448.

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RELATION BETWEEN THE AMOUNT OF INFORMATION AND THE LIKELIHOOD FUNCTION

ION MIHOC AND CRISTINA IOANA FĂTU

Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

Abstract. The objective of this paper is to give some properties for the Fisher information measure and as well as some relations and informational characterizations.

1. **Introduction**

The notion of information plays a central role both in the life of the person and of society, as well as in all kinds of scientific research. The notion of information is so universal, it penetrates our everyday life so much that from this point of view, it can be compared only with the notion of energy [5],[6].

The information theory is an important branch of probability theory and it has very much applications in mathematical statistics. The notion of information plays a central role in the fundamental statistical works of R.A.Fisher. Thus, e.g., Fisher characterized a sufficient statistical function by the fact that it exhausts all the information on the estimated parameter, contained by the sample.

Let X be a random variable on the probability space (Ω, K, P). A statistical problem arises when the distribution of X is not known and we want to draw some inference concerning the unknown distribution of X on the basis of a limited number of observations on X. A general situation may be described as follows: The functional form of the distribution function is known and merely the values of a finite number of parameters,involved in the distribution function, are unknown; i.e., the probability density function of the random variable X is known except for the value of a finite number of parameters. In general, the parameters $\theta_1, \theta_2, ..., \theta_k$ will not be subject to

Received by the editors: 29.10.2002.

2000 *Mathematics Subject Classification.* 94A17.

Key words and phrases. gain of information, Fisher information, local parameter.

any a priori restrictions; i.e., they may take any values. However, the parameters may in some cases be restricted to certain intervals. In the next we shell restrict ourselves to the case of a single parameter θ.

2. **Fisher's information measure**

Let X be a continuous random variable and its probability density function $f(x;\theta)$ depends on a parameter θ which values in a specified parameter space $D_\theta, D_\theta \subseteq \mathbf{R}$.Thus we are confronted, not with one distribution of probability, but with a family of distributions. To each value of θ, $\theta \in D_\theta$,there corresponds one member of the family. A family of probability density functions will be denoted by the symbol $\{f(x;\theta); \theta \in D_\theta\}$.Any member of this family of probability density functions will be denoted by the symbol $f(x;\theta), \theta \in D_\theta$.

Let $S_n(X) = (X_1, X_2, ..., X_n)$ denote a random sample from a distribution that has a probability density function which is one member (but which member we do not known) of the family $\{f(x;\theta); \theta \in D_\theta\}$ of the probability density functions. That is, our sample arises from a distribution that has the probability distribution $f(x;\theta), \theta \in D_\theta$.Our problem is that of defining a statistic $T = T(X_1, X_2, ..., X_n)$, so that if $x_1, x_2, ..., x_n$ are the observed experimental values of $X_1 X_2, ..., X_n$,then the number $t = t(x_1, x_2, ..., x_n)$ will be a good point estimate of θ.

In the next we suppose that the parameter θ is unknown and we estimate a specified function of $\theta, g(\theta)$ with the help of statistic $T = T(X_1, X_2, ..., X_n)$ which is based on a random sample $S_n(X) = (X_1, X_2, ..., X_n)$, where X_i are independent and identically distributed $(i.i.d.)$ random variable with density $f(x;\theta), \theta \in D_\theta$.

A well known means of measuring the quality of the statistic

$$T = T(X_1, X_2, ..., X_n)$$

is to use the inequality of Cramér-Rao which states that, under certain regularity conditions for $f(x;\theta)$ (more particularly, it requires the possibility of differentiating under the integral sign) any unbiased estimator of $g(\theta)$ has variance which satisfies the following inequality [4]

$$VarT \geq \frac{[g^{'}(\theta)]^2}{n.I_X(\theta)} = \tag{2.1}$$

$$= \frac{[g^{'}(\theta)]^2}{I_n(\theta)}, \tag{2.1a}$$

where

$$I_X(\theta) = \int_\Omega \left(\frac{\partial \ln f(x;\theta)}{\partial\theta}\right)^2 f(x;\theta)dx = \tag{2.2}$$

$$= \int_\Omega \frac{1}{f(x;\theta)} \left(\frac{\partial f(x;\theta)}{\partial\theta}\right)^2 dx, \tag{2.3}$$

and

$$I_n(\theta) = E\left[\left(\frac{\partial \ln L(x_1, x_2, ..., x_n;\theta)}{\partial\theta}\right)^2\right] = \tag{2.4}$$

$$= \int_\Omega ... \int_\Omega \left(\frac{\partial L(x_1, x_2, ..., x_n;\theta)}{\partial\theta}\right)^2 L(x_1, x_2, ..., x_n;\theta)dx_1...dx_n = \tag{2.5}$$

$$= nE\left[\left(\frac{\partial \ln f(x;\theta)}{\partial\theta}\right)^2\right] = n\int_\Omega \left(\frac{\partial \ln f(x;\theta)}{\partial\theta}\right)^2 f(x;\theta)dx, \tag{2.6}$$

$$f(x;\theta) = f(x_i;\theta), i = \overline{1,n}, \tag{2.7}$$

$$L(x_1, x_2, ..., x_n;\theta) = \prod_{i=1}^{n} f(x_i;\theta) \tag{2.8}$$

is the joint probability density function of $X_1, X_2, ..., X_n$.

This joint probability density function of $X_1, X_2, ..., X_n$ may be regarded as a function of θ and it is called the likelihood function of the random sample $S_n(X) = (X_1 X_2, ..., X_n)$.

The quantity $I_X(\theta)$ is known as Fisher$^{'}s$ information measure and it measures the information about $g(\theta)$ which is contained in an observation of X.Also, the quantity $I_n(\theta) = n.I_X(\theta)$ measures the information about $g(\theta)$ contained in a random sample $S_n(X) = (X_1 X_2, ..., X_n)$,than then X_i, $i = \overline{1,n}$ are independent and identically distributed random variables with density $f(x;\theta), \theta \in D_\theta$. An unbiased estimator of $g(\theta)$ that achieves this minimum from (2.1) is known as an efficient estimator.

3. Some properties of Fisher's information measure

Let $f(x;\theta), \theta \in D_\theta$ be a positive probability density function in the interval $[a,b]$ depending on the continuous parameter θ in a continuously differentiable way.

Definition 1. [6], [3]. The gain of information when a distribution with probability density function $f(x;\theta_0)$ is replaced by another one with probability density function $f(x;\theta_1)$ has the form

$$I[f(x;\theta_1)\|f(x;\theta_0)] = I(\theta_1\|\theta_0) = \tag{3.1}$$

$$= \int_a^b f(x;\theta_1)\log_2 \frac{f(x;\theta_1)}{f(x;\theta_0)}dx. \tag{3.2}$$

Theorem 1. *Let X be a continuous random variable with probability density function $f(x;\theta), \theta \in D_\theta$. Then we have the following relation*

$$k\left.\frac{d^2 I(\theta_1\|\theta_0)}{d\theta_1^2}\right|_{\theta_1=\theta_0} = I_F[f(x;\theta_0)], \tag{3.3}$$

where

$$I_F[f(x;\theta_0)] = \int_a^b \left(\frac{\partial \ln f(x;\theta_0)}{\partial\theta_0}\right)^2 f(x;\theta_0)dx, \tag{3.4}$$

$$k = \ln 2. \tag{3.6}$$

Proof. Indeed, if we have in view the form (3.1) of the gain of information and we compute the derivative, we obtain

$$\frac{dI(\theta_1\|\theta_0)}{d\theta_1} = \frac{d}{d\theta_1}\left(\int_a^b f(x;\theta_1)\log_2\frac{f(x;\theta_1)}{f(x;\theta_0)}dx\right) =$$

$$= \int_a^b \left(\frac{df(x;\theta_1)}{d\theta_1}\log_2\frac{f(x;\theta_1)}{f(x;\theta_0)} + f(x;\theta_1)\frac{d}{d\theta_1}\log_2\frac{f(x;\theta_1)}{f(x;\theta_0)}\right)dx =$$

$$= \frac{1}{k}\int_a^b \left(\frac{d\ln f(x;\theta_1)}{d\theta_1}\ln\frac{f(x;\theta_1)}{f(x;\theta_0)} + \frac{df(x;\theta_1)}{d\theta_1}\right)dx =$$

$$= \frac{1}{k}\int_a^b \left(1+\ln\frac{f(x;\theta_1)}{f(x;\theta_0)}\right)\frac{df(x;\theta_1)}{d\theta_1}dx,$$

respectively,

$$\frac{dI(\theta_1\|\theta_0)}{d\theta_1} = \frac{1}{k}\int_a^b \left(1 + \ln\frac{f(x;\theta_1)}{f(x;\theta_0)}\right)\frac{df(x;\theta_1)}{d\theta_1}dx. \tag{3.7}$$

Now, if we compute the second derivative of $I(\theta_1\|\theta_0)$, we get

$$\frac{d^2I(\theta_1\|\theta_0)}{d\theta_1^2} = \frac{1}{k}\int_a^b \left[\frac{1}{f(x;\theta_1)}\left(\frac{df(x;\theta_1)}{d\theta_1}\right)^2 + \right.$$
$$\left. + \left(1 + \ln\frac{f(x;\theta_1)}{f(x;\theta_0)}\right)\frac{d^2f(x;\theta_1)}{d\theta_1^2}\right] dx, \tag{3.8}$$

and, hence, if we consider $\theta_1 = \theta_0$,we obtain

$$\left.\frac{d^2I(\theta_1\|\theta_0)}{d\theta_1^2}\right|_{\theta_1=\theta_0} = \frac{1}{k}\int_a^b \frac{1}{f(x;\theta_0)}\left(\frac{df(x;\theta_0)}{d\theta_0}\right)^2 dx + \frac{1}{k}\int_a^b \frac{d^2f(x;\theta_0)}{d\theta_0^2}dx =$$
$$= \frac{1}{k}I_F[f(x;\theta_0)], \tag{3.9}$$

because from the relation

$$\int_a^b f(x;\theta_0)dx = 1, \tag{3.10}$$

we obtain

$$\int_a^b \frac{df(x;\theta)}{d\theta_0}dx = 0, \int_a^b \frac{d^2f(x;\theta)}{d\theta_0^2}dx = 0. \tag{3.11}$$

Remark 1. From this theorem it follows that the gain of information can be considered as a generating-function of the Fisher information measure [2].

Theorem 2. *Let X be a continuous random variables and $f(x;\theta)$ its probability density function which depends on a parameter θ with values in the specified parameter space D_θ and , more $f(x;\theta)$ is absolutely continuous in θ. If θ is a local parameter for X, i.e.,*

$$f(x;\theta) = f_1(x-\theta), \theta \in D_\theta, \tag{3.12}$$

then

$$I_F[f(x;\theta)] = I_F[f_1(x-\theta)], \tag{3.13}$$

where

$$I_F[f(x;\theta)] = \int_{\mathbf{R}} \left[\frac{1}{f(x;\theta)} \frac{\partial f(x;\theta)}{\partial \theta}\right]^2 f(x;\theta)dx, \tag{3.14}$$

$$I_F[f_1(x-\theta)] = \int_{\mathbf{R}} \left[\frac{1}{f_1(x-\theta)} \frac{\partial f_1(x-\theta)}{\partial \theta}\right]^2 f_1(x-\theta)dx, \tag{3.15}$$

are Fisher's information measures.

Proof. Indeed, from (3.12) and (3.14), we obtain

$$\begin{aligned} I_F[f(x;\theta)] &= \int_{-\infty}^{\infty} \left[\frac{1}{f(x;\theta)} \frac{\partial f(x;\theta)}{\partial \theta}\right]^2 f(x;\theta)dx = \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{f_1(x-\theta)} \frac{\partial f_1(x-\theta)}{\partial \theta}\right]^2 f_1(x-\theta)dx = \\ &= \int_{-\infty}^{\infty} \left[-\frac{f_1'(x-\theta)}{f_1(x-\theta)}\right]^2 f_1(x-\theta)dx = \\ &= \int_{-\infty}^{\infty} \left[\frac{f_1'(u)}{f_1(u)}\right]^2 f_1(u)du = \\ &= I_F[f_1(u)], \end{aligned}$$

if we have in view the change of variables

$$u = x - \theta. \tag{3.16}$$

Corollary 3. *If the parameter θ is a scale parameter for X with center m as follows*

$$f(x;\theta) = e^{-\theta} f_2[(x-m)e^{-\theta}], -\infty < \theta < \infty, \tag{3.17}$$

then

$$I_F[f(x;\theta)] = I_F(f_2), \tag{3.18}$$

when

$$I_F(f_2) = \int_{-\infty}^{\infty} \left[1 - x\frac{f_2'(x)}{f_2(x)}\right]^2 f_2(x)dx, \tag{3.19}$$

constantly in θ and m,-$\infty < \theta, m < \infty$.

Proof. From (3.17), we obtain

$$\frac{\partial f(x;\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left\{e^{-\theta} f_2[(x-m)e^{-\theta}]\right\} = \\ = -f(x;\theta) - (x-m)e^{-2\theta} f_2'[(x-m)e^{-\theta}], \tag{3.20}$$

where

$$f_2'(v) = \frac{df_2(v)}{dv}, v = (x-m)e^{-\theta}. \tag{3.21}$$

Then

$$I_F[f(x;\theta)] = \int_{-\infty}^{\infty} \left[\frac{1}{f(x;\theta)}\frac{\partial f(x;\theta)}{\partial\theta}\right]^2 f(x;\theta)dx = \\ = \int_{-\infty}^{\infty} \left[\frac{-f(x;\theta) - (x-m)e^{-2\theta} f_2'[(x-m)e^{-\theta}]}{f(x;\theta)}\right]^2 f(x;\theta)dx = \\ = \int_{-\infty}^{\infty} \left\{-1 - \frac{(x-m)e^{-2\theta} f_2'[(x-m)e^{-\theta}]}{f(x;\theta)}\right\}^2 f(x;\theta)dx. \tag{3.22}$$

If we make the following change of variables

$$v = (x-m)e^{-\theta}, \tag{3.23}$$

then we obtain

$$I_F[f(x;\theta)] = \int_{-\infty}^{\infty} \left[-1 - v\frac{f_2'(v)}{f_2(v)}\right]^2 f_2(v)dv, \tag{3.24}$$

because we have

$$\left\{-1 - \frac{(x-m)e^{-2\theta} f_2'[(x-m)e^{-\theta}]}{f(x;\theta)}\right\}^2 f(x;\theta)dx = \left[-1 - v\frac{f_2'(v)}{f_2(v)}\right]^2 f_2(v)dv. \tag{3.25}$$

4. **Application**

Let X be a continuous random variable which follows a normal distribution, that is, its probability density function is defined by

$$f(x;m,\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\},x\in\mathbf{R}, \tag{4.1}$$

where $\sigma>0$ and $m\in\mathbf{R}$ are the two parameters of the distribution, namely, m is a location parameter and σ^2 is a scale parameter.

Then for the function

$$g(x;m,\sigma^2)=-\ln f(x;m,\sigma^2)= \tag{4.2}$$

$$=\ln\sqrt{2\pi}\ +\frac{1}{2}\ln\sigma^2+\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2, \tag{4.3}$$

we obtain

$$\frac{\partial g(x;m,\sigma^2)}{\partial x}=\frac{x-m}{\sigma}, \tag{4.4}$$

$$\frac{\partial^2 g(x;m,\sigma^2)}{\partial x^2}=\frac{1}{\sigma^2}>0,\forall x\in\mathbf{R}, \tag{4.5}$$

and from here it follows that the probability density (4.1) is a strongly unimodal function and more it is an absolute continuous function, if we have in view the following remark.

Remark 2. [1] Let X be a continuous random variable on the probability space (Ω,K,P) and $f(x)$, $x\in(a,b),a<b,(a,b)\subset\mathbf{R}$ its probability density function. If the function g,defined as

$$g(x)=-\ln f(x),x\in(a,b) \tag{4.6}$$

is a convex function, than f is called strongly unimodal.

Such strongly unimodal probability density function is absolutely continuous within (a,b) and more

$$g'(x)=-\frac{f'(x)}{f(x)},(f(x)\neq o,x\in(a,b)) \tag{4.7}$$

is a non-decreasing function.

Also, we say that X is absolutely continuous random variable if its probability density $f(x)$ is an absolutely continuous function.

Then, according to the relation (2.2), when $\theta = m$, we obtain

$$I_F(x;\theta) = I_F(x;m) = \frac{1}{\sigma^2}. \tag{4.8}$$

Now, we consider the relation

$$f(x;\theta) = e^{-\theta} f_2[(x-m)e^{-\theta}], -\infty < \theta < +\infty \tag{4.9}$$

and if

$$e^{\theta} = \sigma, \tag{4.10}$$

then

$$e^{-\theta} = \frac{1}{\sigma}, \theta = \ln\sigma \tag{4.11}$$

and from (4.9), we obtain

$$f(x;\theta) = e^{-\theta} f(v) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\} = f(x;m,\sigma^2), \tag{4.12}$$

where

$$v = \frac{x-m}{\sigma}. \tag{4.13}$$

Also, according to the relation (2.2), when $\theta = \sigma^2$,we obtain

$$I_F[f(x;\theta)] = I_F[f(x;\sigma^2)] = \frac{1}{2\sigma^4}. \tag{4.14}$$

References

[1] Ibragimov, J. A., *On the composition on unimodal distributions,* Teorija veroyatnostey, 1 (1956), 283-288 (In Russian).
[2] Golomb, S. W., *The information – generating function of a probability distribution*, Transactions of the Fourth Prague Conference on Information Theory, Prague, 1967,291-295.
[3] Kullback, S., *Information Theory and Statistics*,Wiley, New York, 1959.
[4] Rao, C. R., Linear Statistical Inference and Its Applications, John Wiley and Sons, Inc., New York, 1965.
[5] Rényi, A., *Some fundamental questions of information theory*, MTA III, Oszt. Közl., 10 (1960), 251-282.
[6] Rényi, A., *Probability Theory*, Akadémiai Kiado, Budapest, 1970.

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ON THE CLASSIFICATION OF THE NOMOGRAPHIC FUNCTIONS OF FOUR VARIABLES (II)

MARIA MIHOC

Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. In this article we analysed the determinants Massau for some functions of four variables. We constructed also the nomograms in space with coplanar points and compound nomograms consisting of plane nomograms with alignment points.

In [3] the focus is on the study of the functions of four variables and their classification according to the rank of the functions with respect to each variable they depend on.

Further proceeding with this study implies the analysis of the nomograms in space with coplanar points (the nomograms on which the function can be nomographically represented) for some of those function classes.

A lot of authors beginning with R. Soreau, then J. Wojtowicz [5, 6], M. Warmus [4] have dealt with the correct definition of the rank of the functions of three variables with respect to one of its variables (and respectivelly, to all variables). They defined this rank as being equal to the minimum number of linear independent functions from the expression of $F(z_1, z_2, z_3)$. This expression consists of a sum of products where every product term consists of two factors; one of them is a function of one variable (i.e. to one with respect to which we define the rank), the second factor is a function of the other two variables.

We have extended [3] this definition to the case of the functions of four variables $F(z_1, z_2, z_3, z_4)$.

Definition 1. [3] The function of four variables $F(z_1, z_2, z_3, z_4)$ is said to be of rank n with respect to z_1, if there exist the functions $U_i(z_1)$, $V_i(z_2, z_3, z_4)$, $i = \overline{1, n}$,

Received by the editors: 01.09.2002.

2000 *Mathematics Subject Classification.* 65S05.

so that:

$$F(z_1, z_2, z_3, z_4) \equiv \sum_{i=1}^{n} U_i(z_1) V_i(z_2, z_3, z_4) \tag{1}$$

where n is the greatest possible natural number for which (1) occurs.

The functions $U_i(z_1)$, $i = \overline{1, n}$ are linear independent, $V_i(z_2, z_3, z_4)$, $i = \overline{1, n}$ are linear independent too.

Definition 2. [3] The function $F \equiv F(z_1, z_2, z_3, z_4)$ is called nomographic in space if:

a) the rank of the function F with respect to each of its variables is greater than one,

b) there exist the functions $X_i(z_1)$, $Y_i(z_2)$, $Z_i(z_3)$, $T_i(z_4)$, $i = \overline{1, 4}$, so that:

$$F(z_1, z_2, z_3, z_4) \equiv \begin{vmatrix} X_1(z_1) & X_2(z_1) & X_3(z_1) & X_4(z_1) \\ Y_1(z_2) & Y_2(z_2) & Y_3(z_2) & Y_4(z_2) \\ Z_1(z_3) & Z_2(z_3) & Z_3(z_3) & Z_4(z_3) \\ T_1(z_4) & T_2(z_4) & T_3(z_4) & T_4(z_4) \end{vmatrix}. \tag{2}$$

The determinant of type (2) will be called a Massau form (or determinant Massau) of the function F.

Theorem 3. [3] *If the function of four variables $F(z_1, z_2, z_3, z_4)$ is nomographic in space, then it is of rank two, three or four with respect to each of the variables z_i, $i = \overline{1, 4}$ i.e. it has one of the forms:*

$$F \equiv X_1 G_1 + X_2 G_2 \tag{3}$$

$$F \equiv X_1 G_1 + X_2 G_2 + X_3 G_3 \tag{4}$$

$$F \equiv X_1 G_1 + X_2 G_2 + X_3 G_3 + X_4 G_4 \tag{5}$$

with respect to variable z_1. The functions G_i, $i = \overline{1, 4}$ are the rank one, two or three with respect to their variables.

We have introduced the following abbreviations:

$$F = F(z_1, z_2, z_3, z_4); \qquad X_i = X_i(z_1), \qquad Y_i = Y_i(z_2),$$

$$Z_i = Z_i(z_3), \qquad T_i = T_i(z_4), \qquad G_i = G_i(z_2, z_3, z_4), \qquad i = \overline{1, 4}.$$

Definition 4. The nomographic representation of the function F (that has been brought to the form (2)) is equivalent to the nomographic representation of the equation Soreau associated to this function.

The equation Soreau has been obtained by equalisation with zero of the determinant Massau from (2).

The functions F, which have the forms (3)-(5) (or can be brought to these forms) can be nomographically represented by nomograms with coplanar points, because the determinant (2) equated with zero leads to the condition of coplanarity of four points in space, $P_i(x, y, z)$, $i = \overline{1,4}$ (i.e. four points situated in the same plane). The coordinates of these points are (in the system of cartesian coordinates in space $XOYZ$):

$$P_i : \; x = \frac{A_1(z_i)}{A_4(z_i)}, \qquad y = \frac{A_2(z_i)}{A_4(z_i)}, \qquad z = \frac{A_3(z_i)}{A_4(z_i)}, \qquad i = \overline{1,4} \tag{6}$$

and $A_j(z_i)$, $i, j = \overline{1,4}$ successively take the values $X_j(z_1)$, $Y_j(z_2)$, $Z_j(z_3)$, $T_j(z_4)$, $j = \overline{1,4}$.

The formulas (6) are obtained by division of the elements of determinant (2) by those of the fourth column. If at least one element of the last column of the determinant is equal to zero, we can apply an elementary transformation in order to obtain at least one column with all elements different from zero.

Each of point P_i is situated on the curves C_i (of the parameter z_i), where (6) are their parametric equations. By elimination of the parameter z_i from (6) we obtain the equations of two cylindrical surfaces

$$S_1^i(x, y) = 0, \qquad S_2^i(x, z) = 0, \qquad i = \overline{1,4}. \tag{7}$$

The intersection of these surfaces gives exactly the distort curve C_i in space. Therefore, the function of four variables F brought to the form (2) can be nomographically represented by a nomogram in space with coplanar points (see fig. 1). The nomogram consists of four scales (z_i), $i = \overline{1,4}$. These scales are situated on the distort curves in space, C_i, $i = \overline{1,4}$.

The nomogram in figure 1 is employed as follows: We provide the values of the three variables of the equation $F(z_1, z_2, z_3, z_4) = 0$. These values are also the marks of the scales of the variables z_i, situated on the curve C_i, $i = \overline{1,3}$; three points

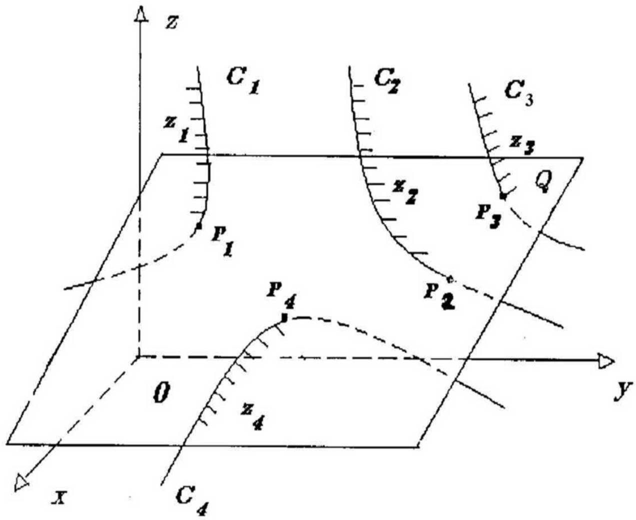

FIGURE 1. The nomogram in space with coplanar points

correspond to them on this curves. These points determine a plane that intersects the fourth curve at another point. The mark of the respective point will give the required value of the variable to arrive at (the fourth variable of the equation).

We proceed to the analysis of some classes of functions $F(z_1, z_2, z_3, z_4)$, which are of rank r_{z_i} with respect to variables z_i, $i = \overline{1,4}$, where $2 \le r_{z_i} \le 4$. We will write the forms Massau, to which they can be brought, and we will construct the nomogram by which these functions are represented.

I. The function F is of rank two with respect to each of its variables z_i, $i = \overline{1,4}$ (having form (3))

$$F \equiv X_1 G_1 + X_2 G_2$$

where the functions G_i, $i = 1, 2$ are of rank one each with respect to variable z_2, i.e.

$$G_i(z_2, z_3, z_4) \equiv Y_i(z_2) H_i(z_3, z_4) \tag{8}$$

the functions $H_i(z_3, z_4)$, $i = 1, 2$ are also of rank one each with respect to their variables

$$H_i(z_3, z_4) \equiv Z_i(z_3) T_i(z_4). \tag{9}$$

According to the remarks above, the function F take the form:

$$F \equiv X_1 Y_1 Z_1 T_1 + X_2 Y_2 Z_2 T_2. \tag{10}$$

Six determinants Massau (respectively six equations Soreau) correspond to this function i.e.:

$$a)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ 0 & Y_1 & 0 & Y_2 \\ 0 & 0 & Z_2 & -Z_1 \\ T_2 & 0 & T_1 & 0 \end{vmatrix} = 0 \qquad b)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ 0 & Y_1 & 0 & Y_2 \\ Z_2 & 0 & Z_1 & 0 \\ 0 & 0 & -T_2 & T_1 \end{vmatrix} = 0$$

$$c)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ Y_2 & 0 & Y_1 & 0 \\ 0 & Z_1 & 0 & Z_2 \\ 0 & 0 & T_2 & -T_1 \end{vmatrix} = 0 \qquad d)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ 0 & 0 & -Y_2 & Y_1 \\ 0 & Z_1 & 0 & Z_2 \\ T_2 & 0 & T_1 & 0 \end{vmatrix} = 0 \qquad (11)$$

$$e)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ 0 & 0 & Y_2 & -Y_1 \\ Z_2 & 0 & Z_1 & 0 \\ 0 & T_1 & 0 & T_2 \end{vmatrix} = 0 \qquad f)\ F \equiv \begin{vmatrix} X_1 & X_2 & 0 & 0 \\ Y_2 & 0 & Y_1 & 0 \\ 0 & 0 & -Z_2 & Z_1 \\ 0 & T_1 & 0 & T_2 \end{vmatrix} = 0$$

All these forms Massau are distinctly projective, because there cannot exist a square matrix of fourth order, whose determinant is different from zero, by which one of the forms (11) a)-f) can be brought to any of the remaining forms.

Only the form Massau (and the equation Soreau) (11)a) will be analized below. By multiplying first column of the determinant from the equation (11)a) with the positive factor a, adding it to the last column and then dividing each of its lines by the elements of the last column we obtain the equation Soreau:

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a}\frac{X_2}{X_1} & 0 & 1 \\ 0 & \frac{Y_1}{Y_2} & 0 & 1 \\ 0 & 0 & -\frac{Z_2}{Z_1} & 1 \\ \frac{1}{a} & 0 & \frac{1}{a}\frac{T_1}{T_2} & 1 \end{vmatrix} = 0 \qquad (12)$$

The equation (12) also gives the parametric equations of the curve C_i, $i = \overline{1,4}$; in this case they are the straight lines D_i, i.e. D_i are the supports for the scales of the variables z_i of the nomogram in space.

$$\begin{array}{llllllll} D_1: & x=\frac{1}{a} & y=\frac{1}{a}\frac{X_2}{X_1} & z=0; & D_3: & x=0 & y=0 & z=-\frac{Z_2}{Z_1} \\ D_2: & x=0 & y=\frac{Y_1}{Y_2} & z=0; & D_4: & x=\frac{1}{a} & y=0 & z=\frac{1}{a}\frac{T_1}{T_2} \end{array} \tag{13}$$

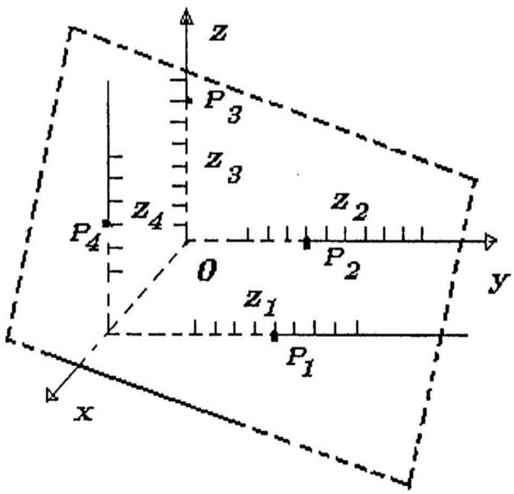

FIGURE 2. The nomogram in space with straight lines

The scales of the variables z_1 and z_2 are situated on the plan XOY and those of the variables z_3 and z_4 on the plan XOZ (see figure 2).

The use of the nomogram is that in the general case of a nomogram in space with coplanar points.

Due to the particular position of the straight lines D_i, $i = \overline{1,4}$, we can also imagine another nomogram for the function (3), brought to the form (11)a), more convenient for the user. This is a compound nomogram consisting of nomograms with alignment points; each constitutive nomogram has three scales in the same plan.

The equation Soreau (11)a) can be decomposed into four equations, i.e.

$$z = 0 \qquad\qquad y = 0$$

$$\begin{vmatrix} w & 0 & 1 \\ \frac{1}{a} & \frac{1}{a}\frac{X_2}{X_1} & 1 \\ 0 & \frac{Y_1}{Y_2} & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} w & 0 & 1 \\ 0 & -\frac{Z_2}{Z_1} & 1 \\ \frac{1}{a} & \frac{1}{a}\frac{T_1}{T_2} & 1 \end{vmatrix} = 0 \tag{14}$$

The first two equations of (14) are the equations of the plans XOY and XOZ; and the others two are the equations Soreau, which represent the conditions of alignment of three points in the respective plans. The first line of each determinant Massau from (14) include the parametric equations of the axis OX, which is the support of a mute scale (a scale without marks). The other lines of the determinants give the parametric equations of the scales z_i, $i = \overline{1,4}$ of the function (10).

Each of the last two equations (14) is nomographically represented by a plan nomogram with alignment points with straight line scales for the variables z_1 and z_2 (respectively z_3 and z_4) and a mute scale.

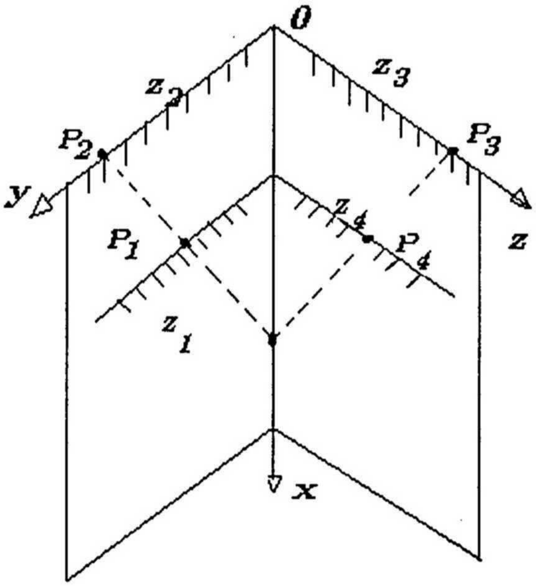

FIGURE 3

The use of the compound nomogram is the following (see figure 3): A straight line that crosses the axis OX (mute scale) in one point is plotted through two of the points of the given mark (which correspond to the given values of variables of (10)). The last point is also joined by a straight line with the third point, of given mark situated on the third scale. The point where the alignment line crosses the fourth scale gives the values of the fourth variable.

We mention the fact that each of four variables of the function (10) can be found if the three other variables are known.

We recall the fact that the genus of one nomogram of the equation of three variables is the number of curve scales of the nomogram consists of. We define now the genus of the nomogram in space.

Definition 5. The genus of a nomogram in space with coplanar points is equal with the number of curve scales the nomogram consists of.

According to this definition the genus of a nomogram in space can be zero, one, two, three or four.

Therefore, the nomograms in space built for the function (10), both the nomogram with coplanar points and the compound nomogram consisting of nomograms with alignment points, are of genus zero; its sales have parallel supports.

The function (10) can be represented and by another nomogram of genus zero if we subject the determinant Massau from (11)a) to the following transformations

$$\begin{vmatrix} \dfrac{X_1}{aX_1+bX_2} & \dfrac{X_2}{aX_1+bX_2} & 0 & 1 \\ 0 & \dfrac{Y_1}{bY_1+dY_2} & 0 & 1 \\ 0 & 0 & \dfrac{Z_2}{cZ_2-dZ_1} & 1 \\ \dfrac{T_2}{aT_2+cT_1} & 0 & \dfrac{T_1}{aT_2+cT_1} & 1 \end{vmatrix} = 0 \qquad (15)$$

and two those nomograms are (see figure 4).

In this case, the function (10) is represented by a nomogram of genus two with straight line scales for the variables z_2 and z_3, and curve scales for the variables z_1 and z_4.

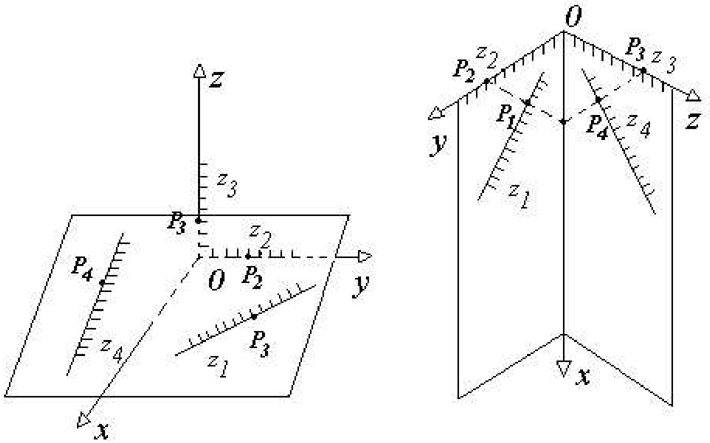

FIGURE 4. The nomogram of genus zero

Other equations (11) can also be represented by similar nomograms with coplanar points, or by a compound nomogram consisting of two plane nomograms with alignment points (like those above). The difference only consisting in the change of the variables of scales of the nomogram, while their supports stay fixed.

References

[1] L. Bal, M. Mihoc, *Sur la représentation nomographique des équations à quatre variables*, Revue d'analyse num. et de la théorie de l'approx., 3, nr.2(1974), 125-142.
[2] N.A. Glagolev, *Kurs nomografii*, Izd. 2-e, Moskva, Vysš. Škola, 1961.
[3] M. Mihoc, *On the classification of the nomographic functions of four variables (I)*, PU.M.A., 9, 1-2(1998), 129-137.
[4] M. Warmus, *Nomographic functions*, Rozpravy Matematyczne, 16(1959), 1-151.
[5] J. Wojtowicz, *Über die korrekte Definition des Ranges eines nomographischen Polynoms und über die Stetigkeit und die Differenzierbarkeit der verallgemeinerten nomographischen Polynome*, Ann. Polon. Math., 8(1960), 177-184.
[6] J. Wojtowicz, *Nomogramovalnošč wielomianów nomograficznych czwartego rzedu nomograficznego o czterech zmiennych* (dissertation), Politehnika Warszawa, 1961.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

APPLICATION OF CLOSE TO CONVEXITY CRITERION TO FILTRATION THEORY

PETRU T. MOCANU

Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. We give new and more simple proofs for the univalence of certain functions related to an inverse boundary problem in the theory of filtration. These proofs are based on the criterion of close to convexity.

1. **Introduction and statement of the results**

Let U be the unit disc of the complex plane $\mathbb{C}$ and let $H(U)$ denote the class of holomorphic functions in U.

A function $f \in H(U)$, with $f(0) = 0$ is called starlike if it is univalent and $f(U)$ is starlike (with respect to the origin). A necessary and sufficient condition for f to be starlike is given by $f'(0) \neq 0$ and

$$\text{Re}\,\frac{zf'(z)}{f(z)} > 0, \quad z \in U.$$

A function $f \in H(U)$ is called convex if it is univalent and $f(U)$ is convex.

A necessary and sufficient condition for f to be convex is given by $f'(0) \neq 0$ and

$$\text{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U.$$

It is easy to show that the function f is convex if and only if the function $g(z) = zf'(z)$ is starlike (Alexander duality theorem).

The function $f \in H(U)$ is called close-to-convex if there exists a convex function φ such that

$$\text{Re}\,\frac{f'(z)}{\varphi'(z)} > 0, \quad z \in U.$$

Received by the editors: 14.10.2002.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. close to convex functions, filtration theory.

According to Alexander duality theorem the function f is close-to-convex if there exists a starlike function g such that

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \quad z \in U. \tag{1}$$

It is well known that all close to convex functions are univalent [4], [6].

In a mathematical model of the theory of filtration [2], [3], [5] it occurs the problem of finding conditions for univalence of the function F defined in the lower half-plane $\Omega = \{\zeta \in \mathbb{C} : \ \operatorname{Im}\zeta < 0\}$ by

$$F(\zeta) = G(\zeta) + H(\zeta), \tag{2}$$

with

$$G(\zeta) = \frac{i}{\pi}\sqrt{1-\zeta^2}\int_{-1}^{1}\frac{\varphi(t)dt}{(t-\zeta)\sqrt{1-t^2}} \tag{3}$$

and

$$H(\zeta) = -\frac{2Ti}{\pi}\arctan\frac{\lambda\sqrt{1-\zeta^2}}{\lambda'} = -\frac{T}{\pi}\log\frac{\lambda' + i\lambda\sqrt{1-\zeta^2}}{\lambda' - i\lambda\sqrt{1-\zeta^2}},$$

where $T > 0$, $\lambda, \lambda' \in [0,1]$ with $\lambda^2 + {\lambda'}^2 = 1$ and

$$0 \le \arg\frac{\lambda' + i\lambda\sqrt{1-\zeta^2}}{\lambda' - i\lambda\sqrt{1-\zeta^2}} \le \pi$$

L. A. Aksentiev in [1] proved the following result by using the argument principle.

Theorem 1. *If the function φ is increasing on $[-1,1]$, then the function F given by (2) is univalent in the half-plane Ω.*

We shall give a more simple proof of this theorem by using the criterion of close to convexity.

In addition we shall prove the following result.

Theorem 2. *If the function φ is increasing on $[-1,1]$, then the function G given by (3) is univalent in the domain $D = \mathbb{C} \setminus [-1,1]$.*

2. **Proof of Theorem 1**

For $z \in U$, let consider the transform

$$\zeta = -i\frac{1+z}{1-z},$$

which maps the unit disc U on the lower half-plane Ω.

The function F becomes

$$f(z)=F[\zeta(z)]=\frac{i}{\pi}\sqrt{1+z^2}\int_{-1}^{1}\frac{1-tz}{(t-\zeta)\sqrt{1-t^2}}\varphi(t)dt$$

$$-i\frac{2T}{\pi}\arctan\left[\frac{\sqrt{2}\lambda}{\lambda'}\frac{\sqrt{1+z^2}}{1-z}\right],\quad z\in U.$$

Since

$$f'(z)=G'(\zeta)\zeta'(z)+H'(\zeta)\zeta'(z),$$

where

$$G'(\zeta)=\frac{i}{\pi\sqrt{1-\zeta^2}}\int_{-1}^{1}\frac{1-t\zeta}{(t-\zeta)^2\sqrt{1-t^2}}\varphi(t)dt$$

$$=-\frac{i}{\pi\sqrt{1-\zeta^2}}\int_{-1}^{1}\frac{d}{dt}\left[\frac{\sqrt{1-t^2}}{t-\zeta}\right]\varphi(t)dt=\frac{i}{\pi\sqrt{1-\zeta^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-\zeta}d\varphi(t),$$

$$H'(\zeta)=\frac{2iT\lambda\lambda'}{\pi}\cdot\frac{\zeta}{(1-\lambda^2\zeta^2)\sqrt{1-\zeta^2}}$$

and

$$\zeta'(z)=-\frac{2i}{(1-z)^2},$$

we deduce

$$f'(z)=\frac{\sqrt{2}}{\pi\sqrt{1+z^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t+i-(t-i)z}d\varphi(t)$$

$$-\frac{i2\sqrt{2}T\lambda\lambda'}{\pi}\frac{1+z}{[(1-z)^2+\lambda^2(1+z)^2]\sqrt{1+z^2}}.$$

Since

$$(1-z)^2+\lambda^2(1+z)^2=-i[1-z+i\lambda(1+z)][\lambda+i+(\lambda-i)z],$$

we deduce

$$[\lambda+i(\lambda-i)z]\sqrt{1+z^2}f'(z)=\frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{\lambda+i+(\lambda-i)z}{t+i-(t-i)z}\sqrt{1-t^2}d\varphi(t)$$

$$+\frac{2\sqrt{2}T\lambda\lambda'}{\pi}\frac{1+z}{[1-z+\lambda(1+z)]}. \tag{4}$$

For $\lambda\in[0,1]$ and $t\in[-1,1]$, we have

$$\operatorname{Re}\frac{\lambda+i+(\lambda-i)z}{t+i-(t-i)z}>0,\quad z\in U.$$

On the other hand we have

$$\operatorname{Re}\frac{1+z}{1-z+\lambda(1+z)}=\operatorname{Re}\frac{1}{\dfrac{1-z}{1+z}+i\lambda}>0,\quad z\in U.$$

Hence for $\lambda \in [0,1]$, from (4) we deduce

$$\text{Re}\,\{[\lambda + i + (\lambda - i)z]\sqrt{1+z^2}f'(z)\} > 0, \quad z \in U. \tag{5}$$

Let the function g be defined by

$$g(z) = \frac{z}{[\lambda + i + (\lambda - i)z]\sqrt{1+z^2}}.$$

Since

$$\frac{zg'(z)}{g(z)} = 1 - \frac{(\lambda - i)z}{\lambda + i + (\lambda - i)z} - \frac{z^2}{1+z^2},$$

and

$$\text{Re}\,\frac{z^2}{1+z^2} < \frac{1}{2}, \quad z \in U$$

$$\text{Re}\,\frac{(\lambda - i)z}{\lambda + i + (\lambda - i)z} = \text{Re}\,\frac{kz}{1+kz} < \frac{1}{2}, \quad z \in U,$$

with $|k| = |(\lambda - i)/(\lambda + i)| = 1$, we deduce

$$\text{Re}\,\frac{zg'(z)}{g(z)} > 0, \quad z \in U,$$

which shows that g is starlike. Since (5) can be rewritten as the inequality (1), we deduce that f is close to convex, hence f is univalent in U and this implies that F is univalent in Ω.

3. **Proof of Theorem 2**

For $z \in U$, let consider the Jukowski transform

$$\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

which maps the unit disc U onto the domain $D = \mathbb{C} \setminus [-1,1]$.

The function G, given by (3) becomes

$$g(z) = G[\zeta(z)] = \frac{1}{\pi}(1-z^2)\int_{-1}^{1} \frac{\varphi(t)dt}{(1-2tz+z^2)\sqrt{1-t^2}}.$$

Since for $t = \cos\theta$, we have

$$\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} = \frac{1-z^2-2iz\sin\theta}{1+z^2-2z\cos\theta},$$

and we deduce

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\cos\theta)\left[\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + 2i\frac{z\sin\theta}{1+z^2-2z\cos\theta}\right]d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \varphi(\cos\theta) d\theta.$$

Since the function $\varphi(\cos\theta)$ is increasing on $[-\pi, 0]$ and decreasing on $[0, \pi]$, by applying the well known theorem of Kaplan concerning the Poisson integral [4], we deduce that g is univalent in U and this implies that G is univalent on D.

References

[1] L. A. Aksentiev, *An application of the principle of the argument to the study of univalence conditions* I, (Russian) Izv. Vyssh. Uchebn. Zaved Mat., 12(79)(1968), 3-15.
[2] Şt. I. Gheorghiţă, *Metode matematice în hidrigazodinamica subterană*, Ed. Academiei, Bucureşti, 1966.
[3] N. B. Ilinskii, *An inverse mixed problem in the theory of filtration*, (Russian) Trudy Sem. Kraev. Zad. 1(1964), 19-28.
[4] W. Kaplan, *Close to convex schlicht functions*, Michig. Math. J., 1, 2(1952), 169-185.
[5] M. T. Nuzhin, N. B. Ilinskii, *Methods of constructing an undergroung contour of hidrotechnical structures. Inverse boundary value problems in the theory of filtration*, (Russian), Uzg. Kazansk. Univ., Kazan, 1963.
[6] S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku, A, 2, 40(1935), 167-188.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

ITERATES OF STANCU OPERATORS, VIA CONTRACTION PRINCIPLE

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Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. In this note we prove that some Stancu operators are weakly Picard operators.

Let $\alpha, \beta \in R$, $0 \leq \alpha \leq \beta$ and let $n \in N^*$. We consider the Stancu operators ([7], [2])

$$P_{n,\alpha,\beta} : C[0,1] \to C[0,1]$$

$$f \mapsto P_{n,\alpha,\beta}(f)$$

where

$$P_{n,\alpha,\beta}(f)(x) := \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k}x^k(1-x)^{n-k}. \tag{1}$$

Let $P^m_{n,\alpha,\beta}$ be the m^{th} iterate of the operator $P_{n,\alpha,\beta}$. We have

Theorem 1. *Let $n \in N^*$ and $\beta > 0$. Then for all $f \in C[0,1]$,*

$$P^m_{n,0,\beta}(f)(x) \to f(0) \text{ as } m \to \infty,$$

uniformly with respect to $x \in \left[0, \frac{n}{n+\beta}\right]$.

Proof. Consider the Banach space $\left(C\left[0, \frac{n}{n+\beta}\right], \|\cdot\|_C\right)$ where $\|\cdot\|_C$ is the Chebyshev norm. Let

$$X_\gamma := \left\{f \in C\left[0, \frac{n}{n+\beta}\right] \mid f(0) = \gamma\right\}, \quad \gamma \in R.$$

We remark that

(a) X_γ is a closed subset of $C\left[0, \frac{n}{n+\beta}\right]$, $\gamma \in R$;

(b) X_γ is an invariant subset of $P_{n,0,\beta}$ for all $\beta > 0$, $n \in N^*$, $\gamma \in R$;

Received by the editors: 06.02.2002.

2000 *Mathematics Subject Classification.* 41A10, 47H10.

Key words and phrases. Stancu operators, iteration operators, contraction principle, weakly Picard operators.

(c) $C\left[0, \dfrac{n}{n+\beta}\right] = \bigcup_{\gamma\in R} X_\gamma$ is a partition of $C\left[0, \dfrac{n}{n+\beta}\right]$.

Now we prove that

$$P_{n,0,\beta} : X_\gamma \to X_\gamma$$

is a contraction, for all $n \in N^*$, $\beta > 0$ and $\gamma \in R$.

Let $f, g \in X_\gamma$. From (1) we have

$$|P_{n,0,\beta}(f)(x) - P_{n,0,\beta}(g)(x)| = |P_{n,0,\beta}(f-g)(x)| \leq$$
$$\leq \left(\sum_{k=1}^{n} \binom{n}{k} x^k (1-x)^{n-k}\right) \|f-g\|_C =$$
$$= (1-(1-x)^n)\|f-g\|_C \leq \left(1-\left(1-\frac{n}{n+\beta}\right)^n\right)\|f-g\|_C.$$

From this we have that

$$\|P_{n,0,\beta}(f) - P_{n,0,\beta}(g)\|_C \leq \left(1-\left(1-\frac{n}{n+\beta}\right)^n\right)\|f-g\|_C,$$

for all $f, g \in X_\gamma$.

We remark that $1-\left(1-\dfrac{n}{n+\beta}\right)^n < 1$.

On the other hand the constant function $\gamma \in X_\gamma$ and is a fixed point of $P_{n,0,\beta}$.

Let $f \in C\left[0, \dfrac{n}{n+\beta}\right]$. Then $f \in X_{f(0)}$ and from the contraction principle ([5]) it follows that

$$P^m_{n,0,\beta}(f)(x) \to f(0) \text{ as } m \to \infty.$$

Theorem 2. *Let $n \in N^*$ and $\alpha > 0$. Then for all $f \in C[0,1]$,*

$$P^m_{n,\alpha,\alpha}(f)(x) \to f(1) \text{ as } m \to \infty,$$

uniformly with respect to $x \in \left[\dfrac{\alpha}{n+\alpha}, 1\right]$.

Proof. Let $X_\gamma := \left\{f \in C\left[\dfrac{\alpha}{n+\alpha}, 1\right] \mid f(1) = \gamma\right\}$, $\gamma \in R$. Then

(a) X_γ is a closed subset of $C\left[\dfrac{\alpha}{n+\alpha}, 1\right]$, for all $\gamma \in R$;

(b) X_γ is an invariant subset of the operator $P_{n,\alpha,\alpha}$, for all $\gamma \in R$, $\alpha > 0$ and $n \in N^*$;

(c) $C\left[\dfrac{\alpha}{n+\alpha}, 1\right] = \bigcup_{\gamma\in R} X_\gamma$ is a partition of $C\left[\dfrac{\alpha}{n+\alpha}, 1\right]$.

Let us prove that

$$P_{n,\alpha,\alpha}|_{X_\gamma} : X_\gamma \to X_\gamma$$

is a contraction, for all $n \in N^*$, $\alpha > 0$ and $\gamma \in R$.

Let $f, g \in X_\gamma$. From (1) we have

$$\|P_{n,\alpha,\alpha}(f) - P_{n,\alpha,\alpha}(g)\|_C \leq \left(1 - \left(\frac{\alpha}{n+\alpha}\right)^n\right) \|f-g\|_C.$$

On the other hand the constant function γ is a fixed point of $P_{n,\alpha,\alpha}$ and $\gamma \in X_\gamma$.

Now the proof follows from the contraction principle.

Remark 1. For the case $\alpha = \beta = 0$, see [4] and [6].

Remark 2. Let (X, d) be a complete metric space. By definition an operator $A : X \to X$ is weakly Picard operator (briefly, WPO) if the sequences $(A^m(x))_{m\in N}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

For an WPO we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{m\to\infty} A^m(x).$$

In the terms of WPOs we can formulate the Theorem 1 and 2 as follow

Theorem 1'. *Let $n \in N^*$ and $\beta > 0$. Then the Stancu operators $P_{n,0,\beta}$ are WPOs on $C\left[0, \frac{n}{n+\beta}\right]$.*

Theorem 2'. *Let $n \in N^*$ and $\alpha > 0$. Then the Stancu operators $P_{n,\alpha,\alpha}$ are WPOs on $C\left[\frac{\alpha}{n+\alpha}, 1\right]$.*

Remark 3. The applications of the contraction principle to study the iterations of other approximation operators ([1]-[3]) will be presented elsewhere.

References

[1] J.A. Adell, F.G. Badia, J. de la Cal, *On the iterates on some Bernstein-type operators*, J. Math. Anal. Appl., 209(1997), 529-541.

[2] O. Agratini, *Aproximare prin operatori liniari*, Presa Universitară Clujeană, Cluj-Napoca, 2000.

[3] S. Karlin, Z. Ziegler, *Iteration of positive approximation operators*, J. Approx. Theory, 3(1970), 310-339.

[4] R.P. Kelisky, T.J. Rivlin, *Iterates of Bernstein polynomials*, Pacific J. Math., 21(1967), 511-520.

[5] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[6] I.A. Rus, *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl. (to appear).

[7] D.D. Stancu, *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Univ. Babeş-Bolyai, 14(1969), 2, 31-45.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

APPPROXIMATION OF BIVARIATE FUNCTIONS BY MEANS OF THE OPERATORS $S_{m,n}^{\alpha,\beta;a,b}$

DIMITRIE D. STANCU, LUCIA A. CĂBULEA, AND DANIELA POP

Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. By starting from the Steffensen theta operator $\theta^{\alpha,\beta}$, defined at (2.1), one constructs the bivariate operator given at (2.2), which depends on the parameters α, β, a, b. In the case $\beta = b = 0$ one obtains the Stancu operators $S_{m,n}^{\alpha;a}$, investigated anterior in the paper [10]. In the case $\alpha = a = 0$ we get a bivariate operator of Cheney-Sharma. For the remainder of the approximation formula (3.1) we present three representations: (3.2), (3.3) and (3.4). In the final part of the paper we give estimations of the order of approximation of a bivariate function f by means of the operators introduced at (2.2).

1. Introduction

It is known that the **omega operators** Ω, considered in 1902 by Jensen [3], include the **shift operator** E^a, defined by $(E^a f)(x) = f(x+a)$, the **central mean operator** μ, defined by

$$(\mu_h f)(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

and the **integration operator**.

An operator T which commutes with all shift operators is called a **shift invariant operator**, that is $TE^a = E^a T$.

A special case of an omega operator is represented by the **theta operator** θ, introduced in 1927 in his book [11] by J.F. Steffensen. Such an operator is sometime called **delta operator** and is denoted by Q in the book of F.B. Hildebrand [2],

Received by the editors: 01.09.2002.

2000 *Mathematics Subject Classification.* 41A10, 41A25, 41A36, 41A63, 41A80.

Key words and phrases. Bernstein operators, umbral calculus, remainders, order of approximation.

published in 1956. This last term was used very often by specialists in **umbral calculus**: G.-C. Rota [6], S. Roman [5] and others.

A **theta operator** θ is a **shift-invariant operator** for which θe_1 is a constant different from zero, where $e_1(t) = t$.

Typical examples of theta operators are represented by the **forward, backward** and **central differences** operators $\Delta_h, \nabla_h, \delta_h$, as well as by the **prederivative operator** $D_h = \Delta_h/h$. We consider that D_0 is the derivative operator D.

Another, very interesting example is represented by the **Abel operator** $A_a = DE^a = E^a D$, which in the case of $p_m(x; a) = x(x - ma)^{m-1}$ leads to the formula:

$$A_a p_m(x; a) = mx(x - (m - 1)a)^{m-2}.$$

It is known that a θ operator can be expressed as a power series in the derivative operator.

One can see that: (i) for every theta operator θ we have $\theta c = 0$, where c is a constant; (ii) if p_m is a polynomial of degree m, then θp_m is of degree $m - 1$. This is the reason that the θ operators are called **reductive operators**.

A sequence of polynomials (p_m) is called by I.M. Sheffer [7] and Gian-Carlo Rota [6], as well by his collaborators, the sequence of **basic polynomials** if we have: $p_0(x) = 1$, $p_m(0) = 0$ $(m \geq 1)$, $\theta p_m = m p_{m-1}$. These polynomials were called by Steffensen [12] **poweroids**, considering that they represent an extension of the mathematical notion of power.

It is easy to see that: (i) if (p_m) is a basic sequence of polynomials for a theta operator, then it is a basic sequence; (ii) if (p_m) is a sequence of basic polynomials, then it is a basic sequence for a theta operator.

By induction can be proved that every theta operator has a unique sequence of basic polynomials associated with it.

J.F. Steffensen [12] observed that the property of the polynomial sequence $e_m(x) = x^m$ to be of binomial type, can be extended to any sequence of basic polynomials associated to a theta operator.

Illustrative examples: (i) if θ is the derivative operator D, then $p_m(x) = x^m$; (ii) if θ is the prederivative operator $D_h = \Delta_h/h$, then we obtain the factorial power:

$$p_m(x) = x^{[m,h]} = x(x - h) \dots (x - (m - 1)h).$$

2. Use of the Steffensen theta operator $\theta^{\alpha,\beta}$ for construction the approximating operators $S_{m,n}^{\alpha,\beta;a,b}$

Now let us consider the **theta operator of Steffensen** [12]:

$$\theta^{\alpha,\beta} = \frac{1}{\alpha}[1 - E^{-\alpha}]E^{\beta}, \tag{2.1}$$

where α and β are nonnegative parameters.

In this case the basic polynomials are

$$p_m(x;\alpha,\beta) = p_m^{\alpha,\beta}(x) = x(x+\alpha+m\beta)^{[m-1,-\alpha]} = \frac{x}{x+m\beta}(x+m\beta)^{[m,-\alpha]}.$$

These are polynomials of binomial type.

By using them we can give a generalized Abel-Jensen combinatorial formula

$$(x+y)(x+y+m\beta)^{[m-1,-\alpha]} =$$
$$= \sum_{k=0}^{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]} y(y+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

Selecting $y = 1-x$ we can write the identity

$$(1+\alpha+m\beta)^{[m-1,-\alpha]} =$$
$$= \sum_{k=0}^{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

We introduce the polynomials $p_{m,k}^{\alpha,\beta}(x)$, defined by the relation

$$(1+\alpha+m\beta)^{[m-1,-\alpha]} p_{m,k}^{\alpha,\beta}(x) =$$
$$= \sum_{k=0}^{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

Let f be a real-valued bivariate function defined on the square $D = [0,1] \times [0,1]$.

We define the bivariate operator $S_{m,n}^{\alpha,\beta;a,b}$ by means of the formula

$$(S_{m,n}^{\alpha,\beta;a,b} f)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}^{\alpha,\beta}(x) q_{n,j}^{a,b}(y) f\left(\frac{i}{m}, \frac{j}{n}\right), \tag{2.2}$$

where

$$(1+a+nb)^{[n-1,-a]} q_{n,j}^{a,b}(y) = \binom{n}{j} y(y+a+jb)^{[j-1,-a]}(1-y)(1-y+a+(n-j)b)^{[n-1-j,-a]}.$$

Now we present two special cases of this operator:

(i) In the case $\beta = b = 0$ we have

$$(S_{m,n}^{\alpha;a} f)(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}^{\alpha}(x) q_{n,j}^{a}(y) f\left(\frac{i}{m}, \frac{j}{n}\right),$$

where

$$p_{m,k}^{\alpha}(x) = \binom{m}{k} x^{k,-\alpha}(1-x)^{[m-k,-\alpha]}/1^{[m,-\alpha]},$$

$$q_{n,j}^{a}(y) = \binom{n}{j} y^{[j,-\alpha]}(1-y)^{[n-j,-a]}/1^{[n,-a]}.$$

The approximation properties of this operator have been studied in the paper [10].

(ii) If $\alpha = a = 0$ we obtain

$$(S_{m,n} f)(x,y;\beta,b) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x;\beta) q_{n,j}(y;b) f\left(\frac{i}{m}, \frac{j}{n}\right),$$

where

$$p_{m,k}(x;\beta) = \frac{\binom{m}{k} x(x+k\beta)^{k-1}(1-x+(m-k)\beta)^{m-k-1}}{(1+m\beta)^{m-1}}$$

and

$$q_{n,j}(y;b) = \frac{\binom{n}{j} y(y+jb)^{j-1}(1-y+(n-j)b)^{n-j-1}}{(1+nb)^{n-1}}.$$

This operator represents **an extension to two variables of the second operator of Cheney-Sharma** [1].

We can see that

$$(S_{m,n} e_{0,0})(x,y) = 1, \quad (S_{m,n} e_{1,0})(x,y) = x,$$

$$(S_{m,n} e_{0,1})(x,y) = y, \quad (S_{m,n} e_{1,1})(x,y) = xy.$$

For $e_{2,0}(x,y) = x^2$ and $e_{0,2}(x,y) = y^2$ we have

$$(S_{m,n} e_{2,0})(x,y) = (S_m e_2)(x),$$

$$(S_{m,n} e_{0,2})(x,y) = (S_n e_2)(y)$$

and we can write [1]:

$$\lim_{m\to\infty} (S_m e_2)(x) = x^2, \quad \lim_{n\to\infty} (S_n e_2)(y) = y^2,$$

uniformly on the interval $[0,1]$.

According to the bivariate criterion of Bohman-Korovkin, we can state

Theorem 2.1. *If $f \in C(D)$ and $\alpha = \alpha(m) \to 0$, $m\beta(m) \to 0$ for $m \to \infty$, while $b = b(n) \to 0$ and $n\beta(n) \to 0$ when $n \to \infty$, then we have*

$$\lim_{m,n\to\infty} (S_{m,n}f)(x, y) = f(x, y),$$

uniformly on the square D.

3. Evaluation of the remainder

Since the approximation formula

$$f(x, y) = (S_{m,n}^{\alpha,\beta;a,b} f)(x, y) + (R_{m,n}^{\alpha,\beta;a,b} f)(x, y) \tag{3.1}$$

has the degree of exactness (1,1), by applying an extension of the Peano theorem (see [8]) we are able to find an integral representation of the remainder.

We now formulate

Theorem 3.1. *If $f \in C^{2,2}(D)$, then we can give the following integral representation for the remainder of formula (3.1):*

$$(R_{m,n}^{\alpha,\beta;a,b} f)(x, y) = \tag{3.2}$$

$$= \int_0^1 G_m^{\alpha,\beta}(t; x) f^{(2,0)}(t, y)dt + \int_0^1 H_n^{a,b}(z, y) f^{(0,2)}(x, z)dz-$$

$$-\int_0^1 \int_0^1 G_m^{\alpha,\beta}(t; x) H_n^{a,b}(z, y) f^{(2,2)}(t, z)dtdz,$$

where

$$G_m^{\alpha,\beta}(t, x) = (R_{m,n}^{\alpha,\beta;a,b} \varphi_x)(t),$$

$$H_n^{a,b}(z, y) = (R_{m,n}^{\alpha,\beta;a,b} \psi_y)(z),$$

with

$$\varphi_x(t) = \frac{1}{2}[x - t + |x - t|], \quad \psi_y(z) = \frac{1}{2}[y - z + |y - z|]$$

and the use of the notation

$$f^{(n,s)}(u, v) = \frac{\partial^{r+s} f(u, v)}{\partial u^r \partial v^s} \quad (r, s = 0, 1, 2).$$

Proof. Formula (3.2) can be obtained if we use a representation of Peano-Milne type, given in the paper [8], for the remainder of a bivariate linear approximation formula having a certain degree of exactness.

If we assume that $x \in \left[\dfrac{r-1}{m}, \dfrac{r}{m}\right]$, we can give for the Peano kernel $G_m^{\alpha,\beta}(t,x)$ the following expression

$$G_m^{\alpha,\beta}(t;x) = \begin{cases} -\sum\limits_{k=0}^{i=1} p_{m,k}^{\alpha,\beta}(x)\left(t-\dfrac{k}{m}\right) & \text{if} \quad t \in \left[\dfrac{i-1}{m}, \dfrac{i}{m}\right] \\ & (1 \le i \le r-1) \\ -\sum\limits_{k=0}^{r-1} p_{m,k}^{\alpha,\beta}(x)\left(t-\dfrac{k}{m}\right) & \text{if} \quad t \in \left[\dfrac{r-1}{m}, x\right] \\ & \\ -\sum\limits_{k=}^{m} p_{m,k}^{\alpha,\beta}(x)\left(\dfrac{k}{m}-t\right) & \text{if} \quad t \in \left[x, \dfrac{r}{m}\right] \\ & \\ -\sum\limits_{k=i}^{m} p_{m,k}^{\alpha,\beta}\left(\dfrac{k}{m}-t\right) & \text{if} \quad t \in \left[\dfrac{i-1}{m}, \dfrac{i}{m}\right] \\ & (r \le i \le m) \end{cases}$$

The dual Peano kernel $H_n^{a,b}(z,y)$ has a similar expression.

If we take into account that on the square D we have $G_m^{\alpha,\beta}(t,x) \le 0$ and $H_n^{a,b}(z,y) \le 0$, we can apply the first law of the mean to the integrals and we find that

$$(R_{m,n}^{\alpha,\beta;a,b}f)(x,y) =$$
$$= f^{(2,0)}(\xi,y)\int_0^1 G_m^{\alpha,\beta}(t,x)dt + f^{(0,2)}(x,\eta)\int_0^1 H_n^{a,b}(z,y)dz -$$
$$-f^{(2,2)}(\xi,\eta)\left[\int_0^1 G_m^{\alpha,\beta}(t,x)dt\right]\left[\int_0^1 H_n^{a,b}(z,y)dz\right],$$

where ξ and η are certain points from the interval $(0,1)$.

It is easy to see that we have

$$\int_0^1 G_m^{\alpha,\beta}(t,x)dt = \frac{1}{2}(R_m^{\alpha,\beta}e_{2,0})(x),$$

$$\int_0^1 H_n^{a,b}(z,y)dz = \frac{1}{2}(R_n^{a,b}e_{0,2})(y),$$

where $R_m^{\alpha,\beta}$ and $R_n^{a,b}$ are the univariate remainders:

$$R_m^{\alpha,\beta} = I - S_m^{\alpha,\beta}, \quad R_n^{a,b} = I - S_n^{a,b}.$$

Now we can state the following

Corollary 3.1. *If $f \in C^{2,2}(D)$, then the remainder of the approximation formula (3.1) can be represented under the following Cauchy form*

$$(R_{m,n}^{\alpha,\beta;a,b}f)(x,y) = \tag{3.3}$$

$$= \frac{1}{2}(R_m^{\alpha,\beta}e_2)(x)f^{(2,0)}(\xi,y) + \frac{1}{2}(R_n^{a,b}e_2)f^{(0,2)}(x,\eta) -$$

$$-\frac{1}{4}(R_m^{\alpha,\beta}e_2)(x)(R_n^{a,b}e_2)(y)f^{(2,2)}(\xi,\eta).$$

Because $(S_m^{\alpha,\beta}f)(x)$ and $(S_n^{a,b}f)(y)$ are interpolatory at both sides of the interval $[0,1]$, we can conclude that $(R_m^{\alpha,\beta}e_2)(x)$ contains the factor $x(x-1)$, while $(R_n^{a,b}e_2)(y)$ has the factor $y(y-1)$.

Since $R_m^{\alpha,\beta}e_0 = 0$, $R_n^{a,b}e_0 = 0$ and the remainder is different from zero for any convex function f of the first order, we can apply a criterion of T. Popoviciu [4] and we find that the remainder is of simple form. Consequently we can state the following

Theorem 3.2. *If the second-order divided differences of the function f are bounded on the square D, we can give an expression of the remainder of the formula (3.1) in terms of divided differences*

$$(R_{m,n}^{\alpha,\beta;a,b}f)(x,y) = (R_m^{\alpha,\beta}e_{2,0})(x)[x_{m,1},x_{m,2},x_{m,3};f(t,y)] =$$

$$+(R_n^{a,b}e_{0,2})(y)[y_{n,1},y_{n,2},y_{n,3};f(x,z)] -$$

$$-(R_m^{\alpha,\beta}e_{2,0})(x)(R_n^{a,b}e_{0,2})(y)\begin{bmatrix} x_{m,1},x_{m,2},x_{m,3} & \\ & ;f(t,z) \\ y_{n,1},y_{n,2},y_{n,3} & \end{bmatrix}, \tag{3.4}$$

where $x_{m,1},x_{m,2},x_{m,3},y_{n,1},y_{n,2},y_{n,3}$ are certain points in the interval $[0,1]$.

If we apply the mean-value theorem to the divided differences, we arrive at the Corollary 3.1.

4. Estimation of the order of approximation

We will use the **bivariate modulus of continuity**

$$\omega(f;\delta_1,\delta_2) = \sup\{|f(x,y) - f(x',y')| :\ |x-x'| \le \delta_1,\ |y-y'| \le \delta_2\},$$

where (x,y) and (x',y') are points of the square D and $\delta_1,\delta_2 \in \mathbb{R}_+$.

Because the constants are reproduced by our operator and $p_{m,k}^{\alpha,\beta}(x) \geq 0$, $q_{n,j}^{a,b}(y) \geq 0$, when $x, y \in [0,1]$, we can write

$$|f(x,y) - (S_{m,n}^{\alpha,\beta;a,b} f)(x,y)| \leq$$
$$\leq \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}^{\alpha,\beta}(x) q_{n,j}^{a,b}(y) \left| f(x,y) - f\left(\frac{k}{m}, \frac{j}{n}\right)\right|.$$

By using a basic property of the modulus of continuity, we can write

$$|f(x,y) - (S_{m,n}^{\alpha,\beta;a,b} f)(x,y)| \leq$$
$$\leq \left[1 + \frac{1}{\delta_1^2} \sum_{k=0}^{m} p_{m,k}^{\alpha,\beta}(x) \left(x - \frac{k}{m}\right)^2 + \frac{1}{\delta_2^2} \sum_{j=0}^{n} q_{n,j}^{a,b}(y) \left(y - \frac{j}{n}\right)^2\right] \omega(f; \delta_1, \delta_2).$$

Since our partial operators are interpolatory in 0 and 1, we can write

$$\sum_{k=0}^{m} p_{m,k}^{\alpha,\beta}(x) \left(x - \frac{k}{m}\right)^2 = (S_m^{\alpha,\beta} e_2)(x) - x^2 = -(R_m^{\alpha,\beta} e_2)(x) = \frac{x(1-x)}{m} A_m^{\alpha,\beta}.$$

By selecting

$$\delta_1 = c\sqrt{\frac{x(1-x)}{m}}, \quad \delta_2 = d\sqrt{\frac{y(1-y)}{n}} \quad (c > 0,\ d > 0),$$

we get

$$|f(x,y) - (S_{m,n}^{\alpha,\beta;a,b} f)(x,y)| \leq$$
$$\leq \left[1 + \frac{1}{c^2} A_m^{\alpha,\beta} + \frac{1}{a^2} B_n^{a,b}\right] \omega\left(f; c\sqrt{\frac{x(1-x)}{m}}, d\sqrt{\frac{y(1-y)}{n}}\right).$$

If we choose $c = d = 2$ and take into consideration that $t(1-t) \leq \frac{1}{4}$ on $[0,1]$, we can state

Theorem 4.1. *The order of approximation of the function $f \in C(D)$ is evaluated by the following inequality*

$$\|f - S_{m,n}^{\alpha,\beta;a,b} f\| \leq \left[1 + \frac{1}{4}(A_m^{\alpha,\beta} + B_n^{a,b})\right] \omega\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right),$$

where $A_m^{\alpha,\beta} = o\left(\frac{1}{m}\right)$, $B_n^{a,b} = o\left(\frac{1}{n}\right)$.

In the particular case $\alpha = \beta = a = b = 0$, we obtain the inequality

$$\|f - B_{m,n} f\| \leq \frac{3}{2}\omega\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right),$$

corresponding to the approximation by the bidimensional Bernstein polynomial $B_{m,n}$.

References

[1] Cheney, E.W. and Sharma, A., *On a generalization of Bernstein polynomials*, Riv. Mat. Univ. Parma **5**(1964), 77-84.

[2] Hildebrand, F.B., *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.

[3] Jensen, L.W., *Sur une identité d'Abel et sur d'autres formules analogues*, Acta Mathematica, **26**(1902), 307-318.

[4] Popoviciu, T., *Sur le reste dans certaines formules linéaires d'approximation de l'analyse*, Mathematica **1**(**24**)(1959), 95-142.

[5] Roman, S., *The Umbral Calculus*, Academic Press, Orlando, Florida, 1984.

[6] Rota, G.-C., *Finite Operator Calculus*, Academic Press, New York, 1975.

[7] Sheffer, I.M., *Some properties of polynomial sets of type zero*, Duke Math. J. **5**(1939), 590-622.

[8] Stancu, D.D., *The remainder of certain linear approximation formulas in two variables*, J. SIAM Numer. Anal. Ser. B, **1**(1964), 137-163.

[9] Stancu, D.D., *Approximation of functions by a new class of linear polynomial operators*, Rev. Roum. Math. Pures Appl. **13**(1968), 1173-1194.

[10] Stancu, D.D., *Aproximarea funcţiilor de două şi mai multe variabile printr-o clasă de polinoame de tip Bernstein*, Stud. Cerc. Mat. Buc., **22**(1970), 335-345.

[11] Steffensen, J.F., *Interpolation*, Williams, Baltimore, 1927.

[12] Steffensen, J.F., *The poweroid, an extension of the mathematical notion of power*, Acta Math. **73**(1941), 333-336.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLVII, Number 4, December 2002

ON THE ORTHOGONAL POLYNOMIALS OF PARETO AND APPLICATIONS ON MATHEMATICAL MODELS

FABIAN TODOR

Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

Abstract. The advantage of determining the differential stochastic equations with applications in Economy, through an orthogonal polynomial system unfolds from the fact that we can approximate the solution, improve the solution, and determine the degree of precision of the approximation.

We shall make such a determination through the orthogonal polynomials associated with the law of Pareto.

The advantage of determining the differential stochastic equations with applications in Economy, through an orthogonal polynomial system unfolds from the fact that we can approximate the solution, improve the solution, and determine the degree of precision of the approximation.

We shall make such a determination through the orthogonal polynomials associated with the law of Pareto. By these means we will step into Numerical Analysis.

I) We have seen the stochastic equations applied in Economy such that (see [4])

$$\frac{dS}{S} = \mu dt + \sigma dX \tag{1.1}$$

or the Black & Scholem variante, that is to say:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1.2}$$

where r and σ are constants -if we have a simplyfied version of the problem - but in a more general case of the problem, these constants are dependent on t.

Received by the editors: 17.09.2002.

This solution may be expressed by orthogonal polynomials, that is to say we found the formula:

$$V(x,\tau) = V(x,0)\exp(A,\tau) = \sum_{n=0}^{\infty} a_n(\tau) P_n(x) \tag{1.3}$$

where $P_n(x)$ for $n = 0, 1, 2, 3...$ are the orthogonal polynomial associated for the Pareto distribution, and $a_n(\tau)$ are the coefficients (see [4]).

To obtain (1.3), we have the following hypothesis: $\frac{r(t)}{\sigma^2(t)} = c$ and $\alpha = c - 3 > 0$, then the found Pareto law had the following probability density:

$$f(x-\lambda) = \frac{\alpha\lambda^\alpha}{x^{\alpha+1}} \quad \text{where} \quad x > \lambda \ \text{ and } \ \lambda = 1. \tag{1.4}$$

In order to construct the orthogonal polynomial system associated to the weights $\rho(x) = f(x-1)$ given by (1.4) - we must use the following formula, (see [3])

$$P_{n(x)} = b_n \begin{vmatrix} M_0 & M_1 & .... & M_n \\ M_1 & M_2 & .... & M_{n+1} \\ .... & .... & .... & .... \\ 1 & x & .... & x^n \end{vmatrix} \tag{1.5}$$

where $M_n = \int\limits_1^\infty x^n \frac{\alpha}{x^{\alpha+1}} dx$ and $b_n =$ the normalization constant.

M_n is the moment of order n of the density $\rho(x) = f(x-1) = \frac{\alpha}{x^{\alpha+1}}$ for $x > 1$ as $\lambda = 1$

The M_n moments for $n = 0, 1, 2, 3...$are estimated by improper integrals as the superior limit is ∞, which exist if $\alpha > n$.

Clearly, by the following transformation: $u = \frac{1}{x}$ in this integral $\int\limits_1^\infty \alpha x^{n-\alpha-1} dx$ we have the results

$\int\limits_0^1 \alpha u^{-n+\alpha-1} du = \frac{\alpha}{\alpha-n}$ if $\alpha > n$ as if $\alpha < 1$ or if $\alpha = 1$ the integral does not exist. The results are: $M_n = \frac{\alpha}{\alpha-n}$ for $\alpha > n$ that is to say: $M_0 = 1$, $M_1 = \frac{\alpha}{\alpha-1}, ..., M_k = \frac{\alpha}{\alpha-k}, ..., M_n = \frac{\alpha}{\alpha-n}$.

The orthogonal polynomials associated with the Pareto law, that is to say $P_n(x)$ given by (1.5) are expressed by the following relation:

$$P_0 = b_0, \quad P_1 = b_1 \left[x - \frac{\alpha}{\alpha-1} \right],$$

$$P_2 = b_2 \left[\begin{array}{c} x^2 \frac{\alpha}{(\alpha-1)^2(\alpha-2)} + x \frac{\alpha}{(\alpha-1)(\alpha-2)(\alpha-3)} + \\ + \frac{\alpha^2(2\alpha+1)}{(\alpha-1)(\alpha-3)(a^2-2)^2} \end{array} \right], \ldots$$

$$P_n\left(x\right) = b_n \left[x^n........\right] \tag{1.6}$$

for $\alpha > n$.

(II)

1) In applications we may conclude early that formula (1.3) which renders the value V of the option will be trucated by the orthogonal polynomials of Pareto, having the condition $\alpha > n$ for the existence of these polynomials.

In order to know in which manner the above mentioned condition is implied in the applications, we must revise the definition of $\alpha = c - 3 = \frac{r(t)}{\sigma^2(t)} - 3 > 0$.

For the $r(t)$ interest rate of the active S, which is the stochastic processes in the capital market, we may say there are few mentioned models(see [1]).

Also, for the $\sigma^2\left(t\right)$ variance, or the $\sigma\left(t\right)$ volatility of active S, there are several mathematical models, inclusively by Hermite and Laguerre orthogonal polynomials(see [2]).

Therefore in a given time period $t \in (a, b)$, the strategy over the $r(t)$ rate and its relation with $\sigma^2\left(t\right)$, will be optimized. By these means, the value of α will be determined knowingly, through knowledge of mathematical models, and then we will see how the (1,3) formula will be truncated.

2) Another application of Pareto's law will appear from the following proposition:

a) *"If x obeys to the Pareto's law with α and λ, then the variable $y = \log(1 + \frac{x}{\lambda})$ will obey to the exponential law of $\frac{1}{\alpha}$mean.*

Therefrom the following statement

b)*"If $x_{(1)}, x_{(2)}, ...x_{(n)}$ is a sample of ordinated values of a distribution x of Pareto, then the variables $x_{(k)}$ will be expressed such as:*

$$x_{(k)} = \lambda \left[\prod_{j=1}^{k} (1 + v_j) - 1 \right]$$

for $k = 1, 2...n$; where v_j are independent Pareto variables".

Certainly, the b) statement results from a) especially due to variable y obeying to an exponential law and we apply the exponential law theorems concerning the ordinated variables extracted from an n volume sample.

In conclusion, the independent Pareto variables will be used in order statistics.

References

[1] Cox J.C., Ingersoll J.et Ross S., *"A theory of the term structure of interest rates"* Econometrica (1985) 53 p. 385-467.

[2] Meddahi N., Renault E., *Temporal Aggregation of Volatility Models*" CIRANO DP,2000, 22.

[3] Szegö G., *"Orthogonal polynomials*" NewYork 1959

[4] Todor F., "*Sur les polynômes orthogonaux associés aux modèles mathématiques appliquées en pratique"* Proceedings of the 26th ARA Congress, Polytechnic International Press 2001 p531-534.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

COMBINED SHEPARD-LEAST SQUARE OPERATORS – COMPUTING THEM USING SPATIAL DATA STRUCTURES

MARIA GABRIELA TRÎMBIŢAŞ

Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

Abstract. The low degree of exactness and large number of computation required are well-known drawbacks of classical Shepard operator. They can be overcome using combined Shepard operators and local interpolation schemes. Spatial data structures could support efficient evaluation of such operators.

1. Introduction

Unfortunately, the classical Shepard operator (see [1]) defined by

$$\left(S_{n,\mu}f\right)(x)=\sum_{k=0}^{n}w_k(x)f(x_k) \tag{1}$$

$$w_k(x)=\frac{|x-x_k|^{-\mu}}{\sum\limits_{k=0}^{n}|x-x_k|^{-\mu}}, \tag{2}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^s$, and $X=\{x_0,x_1,\ldots,x_n\}\subset\mathbb{R}^s$ is a set of $n+1$ pairwise distinct points, has a low degree of exactness (i.e. 0) and requires a large amount of computation. The solution is to replace the values of f by a suitable polynomial interpolation operator $(L_kf)(x;x_k)$, which can depend on k (see [2, 3], and the references therein) and the weight functions given by (2) with the so called Franke-Little weights:

$$\bar{w}_k(x)=\frac{\dfrac{(R-|x-x_k|)_+^{\mu}}{R^{\mu}|x-x_k|^{\mu}}}{\sum\limits_{i=0}^{n}\dfrac{(R-|x-x_i|)_+^{\mu}}{R^{\mu}|x-x_i|^{\mu}}} \tag{3}$$

Received by the editors: 08.10.2002.

(see [7, 6, 5]). In (3), R is a given positive real constant, and the + subscript denotes the positive part.

The local variant of (1) is

$$\left(\bar{S}^L_{n,\mu} f\right)(x) = \sum_{k=0}^{n} \bar{w}_k(x)(L_k f)(x; x_k), \tag{4}$$

called the *combined local Shepard-type operator.*

In order to compute the various local Shepard-type interpolants we are interested to report efficiently the point located into the ball $B(x, R)$. The naive approach (computing $d_k = |x - x_k|$ and checking $d_k < R$) needs a time $O(n)$ for each point x. Computational geometry techniques and data structures allow us to perform this task in polylogarithmic time [4].

In this paper $L_m f$ will be a least square approximation polynomial.

2. Combined Shepard least-square local operators

We shall consider two kind of discrete least-square approximation polynomials:

1. polynomials which reproduces the values of f in x_k, $k = \overline{0, n}$;
2. polynomials which reproduces the values of f and of the first order partial derivatives of f in x_k, $k = \overline{0, n}$.

Only the bivariate operators will be considered.

Proposition 1. *The following relation hold*

$$\left(\bar{S}^L_{n,\mu} f\right)(x_j, y_j) = (L_j f)(x_j, y_j)$$

and

$$\frac{\partial}{\partial x}\left(\bar{S}^L_{n,\mu} f\right)(x_j, y_j) = \frac{\partial}{\partial x}(L_j f)(x_j, y_j)$$
$$\frac{\partial}{\partial y}\left(\bar{S}^L_{n,\mu} f\right)(x_j, y_j) = \frac{\partial}{\partial y}(L_j f)(x_j, y_j)$$

Proof. It can be shown that, for all k and j, the weights (3) satisfy

$$\bar{w}_k(x_j, y_j) = \delta_{kj} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}, \tag{5}$$

$$\sum_{k=0}^{n} \bar{w}_k(x, y) = 1, \tag{6}$$

and

$$\frac{\partial}{\partial x}\bar{w}_k(x_j, y_j) = \frac{\partial}{\partial y}\bar{w}_k(x_j, y_j) = 0. \tag{7}$$

(5) implies

$$\left(\bar{S}^L_{n,\mu} f\right)(x_j, y_j) = \sum_{k=0}^{n} \bar{w}_k(x_j, y_j)(L_k f)(x_j, y_j) = (L_j f)(x_j, y_j);$$

From (5) and (7), one obtains

$$\frac{\partial}{\partial x}\left(\bar{S}^L_{n,\mu} f\right)(x_j, y_j) = \sum_{k=0}^{n}\left[\frac{\partial}{\partial x}\bar{w}_k(x_j, y_j)(L_k f)(x_j, y_j) + \bar{w}_k(x_j, y_j)\frac{\partial}{\partial x}(L_k f)(x_j, y_j)\right] = \frac{\partial}{\partial y}(L_j f)(x_j, y_j),$$

and analogously in y. □

Thus $\left(\bar{S}^L_{n,\mu} f\right)$ maintains the local shape properties of $L_k f$.

Let f_k be $f(x_k, y_k)$, $f_{x,k}$ be $\frac{\partial}{\partial x}f(x_k, y_k)$ and $f_{y,k}$ be $\frac{\partial}{\partial y}f(x_k, y_k)$ respectively.

For the first case $L_k f$ is defined by

$$\begin{aligned}(L_k f)(x, y) := {} & c^*_{k1}(x - x_k)^2 + c^*_{k2}(x - x_k)(y - y_k) + c^*_{k3}(y - y_k)^2 \\ & + c^*_{k4}(x - x_k) + c^*_{k5}(y - y_k) + f_k.\end{aligned} \tag{8}$$

The coefficients are the solution of the following discrete least-square problem

$$\begin{aligned}\sum_{\substack{i=0 \\ i\neq k}}^{n} \omega_i(x_k, y_k)\,[& c_{k1}(x - x_k)^2 + c_{k2}(x - x_k)(y - y_k) + c_{k3}(y - y_k)^2 \\ & + c_{k4}(x - x_k) + c_{k5}(y - y_k) + f_k - f_i]^2 \longrightarrow \min,\end{aligned} \tag{9}$$

where

$$\omega_i(x, y) = \left[\frac{(R_q - d_i)_+}{R_q d_i}\right]^2,$$

R_q is a radius of influence about node (x_i, y_j) (in general not equal to R) and d_i is the Euclidean distance between (x, y) and (x_i, y_i). The problem (9) leads us to a 5×5 system of linear equations.

Another possible choice for L_k is

$$(L_k f)(x, y) := c^*_{k1}(x - x_k) + c^*_{k2}(y - y_k) + f_k; \tag{10}$$

analogously, in this case, we obtain a 2×2 system of linear equation. It is easy to show, using Proposition 1, that the combined local Shepard operators obtained in this way reproduce the values of f in x_k, $k = \overline{0, n}$.

For the second case we choose

$$\begin{aligned}(L_k f)(x, y) := c^*_{k1}(x - x_k)^2 + c^*_{k2}(x - x_k)(y - y_k) + c^*_{k3}(y - y_k)^2 \\ + f_{x,k}(x - x_k) + f_{y,k}(y - y_k) + f_k.\end{aligned} \tag{11}$$

The corresponding least square problem is

$$\begin{aligned}\sum_{\substack{i=0\\ i\neq k}}^{n} \omega_i(x_k, y_k) \left[c_{k1}(x - x_k)^2 + c_{k2}(x - x_k)(y - y_k) + c_{k3}(y - y_k)^2\right. \\ \left. + f_{x,k}(x - x_k) + f_{y,k}(y - y_k) + f_k - f_i\right]^2 \longrightarrow \min,\end{aligned} \tag{12}$$

and it leads us to a 3×3 system of linear equations.

Another possibility is given by

$$\begin{aligned}(L_k f)(x, y) := {} & c^*_{k1}(x - x_k)^3 + c^*_{k2}(x - x_k)^2(y - y_k) \\ & + c^*_{k3}(x - x_k)(y - y_k)^2 + c^*_{k,4}(y - y_k)^3 \\ & + c^*_{k1}(x - x_k)^2 + c^*_{k2}(x - x_k)(y - y_k) \\ & + c^*_{k3}(y - y_k)^2 + f_{x,k}(x - x_k) + f_{y,k}(y - y_k) + f_k.\end{aligned} \tag{13}$$

The choice (8) appears in [6, 7, 5, 9, 8, 10, 11], (10) in [11], but (11) and (13) are original.

The efficient computation of the operator given by (4) requires the efficient solving of a circular range searching problem.

Let $P := \{p_1, \ldots, p_n\}$ be a set of point from $\mathbb{R}^s$ and R a region from the same space. A s-dimensional *range searching problem* asks for the points from P lying inside the query region R. If the region is a hyperparallelepiped, i.e. $R = [x_1, x_1'] \times \cdots \times [x_s, x_s']$, then we have an *orthogonal range-searching problem.* If R

is a ball from $\mathbb{R}^s$, we have a *circular range searching problem.* Our approach is to solve a simpler orthogonal range searching problem instead of the circular range searching (since this approach eliminates a large number of points) and then to check the reported points.

3. Implementation

One of the most used data structure for orthogonal range query is the *range tree*[4]. A solution based on range tree is given in [12].

Another solution is inspired from a paper of Renka[8]. The smallest bounding box containing the nodes $\prod_{k=1}^{s}\left[x_{\min}^k, x_{\max}^k\right]$ is partitioned into an uniform grid of cells, having NR cells on each dimension. Each cell points to the list of point indices contained in that cell. Such an example for the 2D case is given in Figure 1.

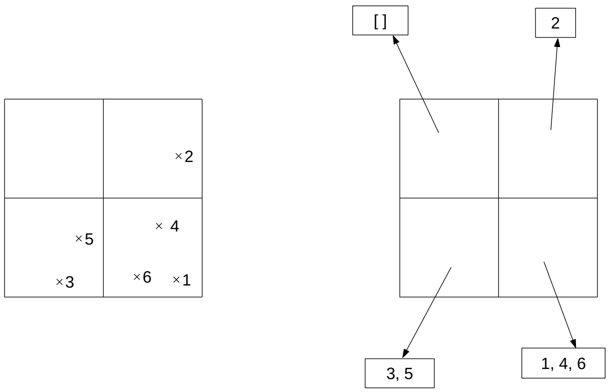


FIGURE 1. A 2D grid of cell and its representation

The algorithm 1 describes the creation of the data structure. If the second argument NR is not provided we can initialize it with a default value; Renka suggests in [9]

$$NR = \lfloor (N/3)^{1/\dim} \rfloor.$$

The orthogonal range searching is easy to implement using this data structure (the algorithm 2): first the cell which must be scanned are determined (i.e. the cell which intersects the searching domain), and then the list of points corresponding to that cell are concatenated. The points from the outer cells which lie outside the searching range must be eliminated.

Algorithm 1 Creating the cell grid

Input: the set of N points P, the number of cells, NR (optional);

Output: a grid of cell LCELL, each containing the list of points in the cell

set all cells to **nil**;

{compute the cell sizes}

$dc_1 := \min(NR, \lfloor x^1_{\max} - x^1_{\min} \rfloor + 1);$

$\vdots$

$dc_s := \min(NR, \lfloor x^s_{\max} - x^s_{\min} \rfloor + 1);$

for $K := N$ **downto** 1 **do**

{find the cell}

$i_1 := \min(NR, \lfloor x^k_1 - x^1_{\min} \rfloor + 1);$

$\vdots$

$i_s := \min(NR, \lfloor x^k_s - x^s_{\min} \rfloor + 1);$

add K to the list $LCELL(i_1, \ldots, i_s)$;

end for

Now we are able to compute the local Shepard interpolant on a set of points X:

- build the spatial data structure;
- for each point x in X
  - perform the orthogonal range searching into the hypercube centered in x and with the radius R
  - apply formulas (3) and (4).

This approach has a drawback: the accuracy tends to decrease into the areas where the interpolation nodes are sparse. We can avoid this situation, allowing R_q and R to vary with k: the radii are choosen such that the ball $B(x_j, R_q)$ contains at least N_q nodes and the ball $B(x, R)$ contains at least N_w nodes. Thus, instead of an orthogonal range searching we perform a N_q-th (or a N_w-th) nearest neighbor search of x_j and x, respectively. This can be done scanning the grid in a circular fashion starting with the cell containing x. In order to facilitate the scanning we can associate a Boolean indicator to each cell, which is true when the cell was already scanned.

Algorithm 2 The orthogonal range searching

$PTLIST :=$ **nil**;

{determine the outer cells, i. e. the scan limits}

$imin_1 := \max(1, \lfloor (liminf_1 - x^1_{\min})/dc_1 \rfloor + 1)$;

$imax_1 := \min(NR, \lfloor (limsup_1 - x^1_{\min})/dc_1 \rfloor + 1)$

$\vdots$

$imin_s := \max(1, \lfloor (liminf_s - x^s_{\min})/dc_s \rfloor + 1)$;

$imax_s := \min(NR, \lfloor (limsup_s - x^s_{\min})/dc_s \rfloor + 1)$

for $i_1 := imin_1$ **to** $imax_1$ **do**

$\vdots$

for $i_s := imin_s$ **to** $imax_s$ **do**

$JL := LCELL(i_1, \ldots, i_s)$;

if the cell $(i_1, \ldots, i_s)$ is peripheral **then**

remove the points which lay cell outside the searching range from JL;

end if

concatenate $PTLIST$ and JL

end for

$\vdots$

end for

The algorithms described above are implemented in MATLAB[1]. The cell grid is represented as a structure which contains information about the grid: dimension, number of cell over each coordinates, the size of a cell, minimum and maximum in each coordinate and a cell array, where each cell contains an array with point indices; this representation allows easy location of cell and concatenation of point lists.

The linear system which gives the solution of least square system can be ill-conditioned. For this reason the system is solved using a QR factorization. If the results are not satisfactory (system too ill-conditioned) more points are added, and the solving process is redone.

[1] MATLAB©is a trademark of the MathWorks, Inc. Natick, MA 01760-2098

4. Examples and graphs

One of the most frequent function used as example to illustrate Shepard interpolation is the Franke's function [6, 8], $f_1 : [0,1] \times [0,1] \to \mathbb{R}$, given by:

$$\begin{aligned} f_1(x,y) = &0.75 \exp\left(-\left((9x-2)^2 + (9y-2)^2\right)/4\right) \\ &+ 0.75 \exp\left(-(9x+1)^2/49 - (9y+1)/10\right) \\ &+ 0.5 \exp\left(-\left((9x-7)^2 + (9y-3)^2\right)/4\right) \\ &- 0.2 \exp\left(-(9x-4)^2 - (9y-7)^2\right). \end{aligned}$$

Its graph appears in Figure 2(a). Figures 2(b) and 2(c) give the graphs of local Shepard operator combined with a least square polynomial having the degree 1 (given by formula (10)) and 2 (formula (8)), respectively. The graph of the local Shepard interpolant combined with a 2nd degree least square polynomial, considering first order partial derivatives (formula (11)) is given in Figure 2(d). All the interpolants were obtained taking $\mu = 2$, $N_q = 17$ and $N_w = 23$. The best result is obtained for the 2nd degree least square polynomial, without derivatives. This phenomenon deserves further investigations.

References

[1] D. Shepard – A Two Dimensional Interpolation Function for Irregularly Spaced Data, *Proc. 23rd Nat. Conf. ACM*, 517-523, 1968.

[2] Gh. Coman, R. Trîmbiţaş – Bivariate Shepard interpolation, "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, *Seminar on Numerical and Statistical Calculus, Preprint* No. 1, 41–83, 1999.

[3] R. Trîmbiţaş – Local bivariate Shepard interpolation, Proceedings of the 4-th International Conference on Functional Analysis and Approximation Theory, Acquafedda di Maratea, Italy, September 2000, *Supplemento ai Rendiconti del Circolo Matematico di Palermo*, Serie II, No. 68, 865–878, 2002.

[4] M. de Berg, M. van Kreveld, M. Overmars, O. Schwarzkopf – Computational Geometry. Algorithms and Applications, Springer Verlag, Berlin, Heidelberg, New York, 1997.

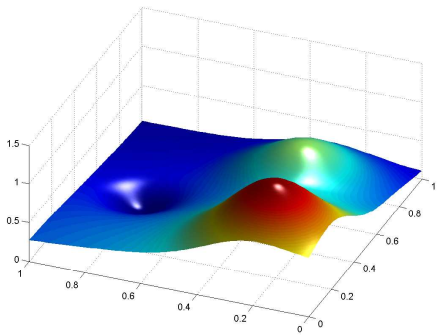
[5] R. Franke, G. M. Nielson – Smooth interpolation of large sets of scattered date, *International Journal for Numerical Methods in Engineering*, 15, 1691–1704, 1980.

[6] R. Franke – Scattered data Interpolation: Tests of Some Methods, *Math. Comp.* 38, 181–200, 1982.

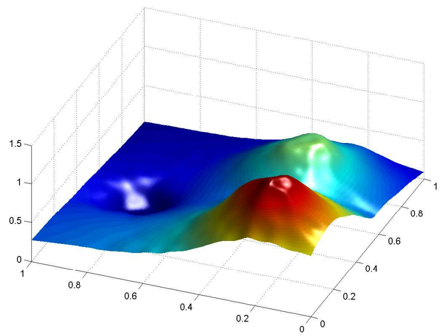
[7] R. Franke, G. M. Nielson – Scattered Data Interpolation and Applications: A tutorial and Survey, in *Geometric Modeling - Methods and Applications*, H. Hagen, D. Roller (eds), Springer Verlag, 1990.

[8] R. J. Renka – Multivariate Interpolation of Large Sets of Scattered Data, *ACM Transactions on Mathematical Software*, Vol. 14, No. 2, 139-148, 1988.

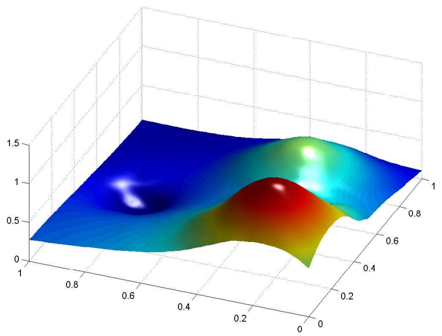
[9] R. J. Renka – Algorithm 660, QSHEP2D: Quadratic Shepard Method for Bivariate Interpolation of Scattered Data, *ACM Transactions on Mathematical Software*, Vol. 14, No. 2, 149, 1988.



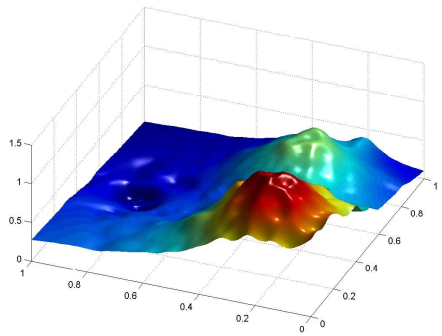
(a) The graph of f_1



(b) Local Shepard-least square, degree 1



(c) Local Shepard-least square, degree 2



(d) Local Shepard-least square, degree 2, with derivatives

FIGURE 2. f_1 and various Shepard-least square interpolants for $\mu = 2$

[10] R. J. Renka – Algorithm 661, QSHEP3D: Quadratic Shepard Method for Trivariate Interpolation of Scattered Data, *ACM Transactions on Mathematical Software*, Vol. 14, No. 2, 151, 1988.
[11] K. Basso, P. R. A. Zingano, C. M. S. Freitas, Interpolation of Scattered Data: Investigating Alternatives for the Modified Shepard Method, Proceedings of SIBGRAPI ' 98, Rio de Janeiro, 1998.
[12] M. G. Trîmbiţaş, R. Trîmbiţaş, Range searching and local Shepard interpolation, Proceedings of Roger 2002, to appear.

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BOOK REVIEWS

William Arveson, *A Short Course on Spectral Theory*, Graduate Texts in Mathematics, Vol. 209, Springer, New York, Berlin, Heidelberg, 2002, x+135 pp., ISBN 0-387-95300-0.

The fundamental problem of operator theory is the calculation of spectra of operators on infinite dimensional spaces, especially on Hilbert spaces. The theory has deep applications to the study of partial differential and integral operators, to the mathematical foundation of quantum mechanics, noncommutative K-theory and the classification of simple C^*-algebras.

The aim of the present book, based on a fifteen-week course taught for several times by the author at the University of Berkeley, is to make the reader acquainted with the basic results in spectral theory, needed for the study of more advanced topics listed above. The prerequisites are elementary functional analysis and measure theory.

In the first chapter, *Spectral theory and Banach algebras,* the theory is developed in the natural framework of Banach algebras and includes spectral radius, regular representation, the spectral permanence theorem, and an introduction to analytic functional calculus. The abstract notions are illustrated on concrete examples of operators.

Ch. 2, *Operators on Hilbert space,* is concerned with spectral theory for operators on Hilbert space and their C^*-algebras, normal operators, compact operators, spectral measures. For the sake of clarity the treatment is restricted to separable Hilbert spaces. A good companion in reading this part could be another book by the same author: An invitation to C^*-algebras, Springer 1998.

Ch. 3, *Asymptotics: Compact perturbations and Fredholm theory,* contains the Calkin algebra, Riesz theory for compact operators, Fredholm operators and Fredholm index.

In the last chapter, Ch. 4, *Methods and applications,* a variety of operator theoretic methods are applied to determine the spectra of Toeplitz operators, the results being definitive only for Toeplitz operators with continuous symbol. An elementary theory of Hardy spaces H^2 is also developed. The book ends with the study of states on C^*-algebras and a proof of Gelfand-Naimark representation theorem.

The book is a clear, short and thorough introduction to spectral theory, accessible to first or second year graduate students. As the author points out in the Preface: "this material is the essential beginning for any serious student in modern analysis".

S. Cobzaş

Raymond A. Ryan, *Introduction to Tensor Products of Banach Spaces,* Springer Monographs in Mathematics, Springer, New York Berlin Heidelberg, 2002, xiv+225

pp., ISBN: 1-85233-437-1.

Quoting from the Preface: "tensor products are a natural and productive way to understand many of the themes of modern Banach space theory and "tensorial thinking" yields insight into many mysterious phenomena". The first book on normed tensor products was written by R. Schatten in 1950, followed by the fundamental works of A. Grothendieck – Produits tensoriels topologiques et espaces nucléaires, Memoirs AMS 1955, and the famous "Resumé" published at Sao Paolo in 1953 (reprinted 1996).

The aim of the present book is to give a thorough and relatively short introduction to tensor products of Banach spaces, starting from the algebraic theory, presented in the first chapter, and bringing the reader to the frontier of current research in the area.

The two basic constructions of tensor norms on the tensor product of two Banach spaces are presented in Chapters 2, *The projective tensor product,* and 3. *The inductive tensor product.* The relevance of the approximation property for the theory of tensor products (e.g. the study of reflexivity), and the efficiency of the tools furnished by tensor products in the study of approximation property are presented in Chapter 4, *The approximation property.*

Chapter 5, *The Radon-Nikodým property,* is concerned with vector measures and vector integration (Pettis and Bochner) and with representability of various types of operators on $C(K)$ and $L_1(\mu)$ spaces.

The study of cross-norms is done in Chapters 6, *The Chevet-Saphard tensor products,* and 7, *Tensor norms,* and includes Grothendieck's inequality, p-summing and p-integral operators.

The last chapter of the book, Chapter 8, *Operator ideals,* contains a brief introduction to this related topic. For a full treatment of the interrelations between tensor products and operator ideals, the reader is referred to the recent book of A. Defant and K. Floret, Tensor norms and operator ideals, North Holland 1993.

The book has also three appendices: A. *Suggestions for further reading,* B. *Summability in Banach spaces,* and C. *Spaces of measures.*

The book is an excellent introduction to the theory of tensor products and it is highly recommended to graduate students in analysis and to researchers in other areas needing results from this field.

S. Cobzaş