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PROFESSOR WOLFGANG W. BRECKNER AT HIS 60TH ANNIVERSARY

ŞTEFAN COBZAŞ

Professor Wolfgang Werner Breckner was born in Sibiu, Romania, on October 6, 1942. After finishing the high school in 1960, he went to Cluj-Napoca and enrolled the Faculty of Mathematics and Mechanics of the Babeş-Bolyai University. During the studies he was one of the best students, so that in 1965, after graduating, he was retained at this faculty as an assistant at the Chair of Mathematical Analysis, headed by Professor Tiberiu Popoviciu, member of the Romanian Academy. In 1971 he obtained the Ph.D. degree with the thesis "Characterization theorems for the solutions of certain optimization problems", elaborated under the guidance of Tiberiu Popoviciu. In 1972 he was promoted Lecturer and in 1990 Associate Professor. Since 1993 he is full Professor at the Chair of Analysis and Optimization of the Faculty of Mathematics and Computer Science at present, and since 1992 he is the head of this chair.

He married in 1965 Maria Erzsébet Corvin. They have two daughters Brigitte Erika (born in 1970) and Hannelore Inge (born 1971). They graduated both the Faculty of Mathematics and Computer Science of the Babeş-Bolyai University, earned Ph.D.'s in Germany, and now are affiliated as lecturers with our faculty.

The managerial and professional skills of Professor Breckner determined his election in 1997 as a vice-rector of the Babeş-Bolyai University. Since then, he acted in this position.

As a recognition of the value of his research he was invited to spend several research stages at some universities in Germany: in 1991 at the Gerhard Mercator University Duisburg, in 1994 and 1998 at the Technical University Munich, and in 1995 and 2001 at the Martin Luther University Halle.

He was member of the Organizing Committees of several symposia and colloquia held in Cluj-Napoca and member of the Editorial Board of their proceedings as well. Among these I do mention the International Conference on Approximation and Optimization (ICAOR), a satellite conference of the European Congress of Mathematics, Budapest, 1996.

Over the years he taught courses and conducted seminars on mathematical analysis, functional analysis, optimization, operations research, convex analysis. All these were, and still are, characterized by the clarity of the exposure, and by the novelty and richness of the included topics, as can be seen also from the five textbooks he published at the University.

The research activity of Professor Breckner, as reflected by over than 60 published papers, covers three main directions: functional analysis, applications of functional analysis to best approximation and optimization, and applications of functional analysis to convex analysis. In all of these areas he obtained significant results as: very general principles of condensation of singularities for families of nonlinear functions, extensions of the uniform boundedness principle of Banach and Steinhaus, Hahn-Banach theorems for modules, duality theorems for optimization problems in ordered topological vector spaces, characterizations of the solutions of nonlinear best approximation problems, Lagrange multiplier rules, continuity and equicontinuity results for generalized convex functions and for set-valued functions, respectively for families of such functions. Beside these research papers he published a monograph "Introduction to the theory of convex optimization problems with restrictions", Dacia Publishers, Cluj-Napoca 1979.

The impact of his research on the mathematical community is reflected by over than 200 quotations of his papers, including some having in title "Breckner s-convex functions", nominating a class of functions introduced and studied by W. Breckner. Professor Breckner is a reviewer for Zentralblatt für Mathematik and for Mathematical Reviews, and member of the Editorial Boards of the journals Mathematica Pannonica (Hungary), Studia Universitatis Babes-Bolyai (Series Mathematica), Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie.

I tried to emphasize in this short presentation some of the highlights of the scientific, didactic and social achievements of Professor W. W. Breckner. Of course, many things remained untold, some of them being presented at the official celebration of the 60th birthday of Professor Breckner organized by the faculty on November 8, 2002.

On my part and on the behalf of my colleagues, I wish Professor Breckner a long life, good health and all the best for many years to come.

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2. *Théorèmes de caractérisation des éléments de la meilleure approximation.* C. R. Acad. Sc. Paris, Sér. A, **266**, 206-208 (1968) (with I. Kolumbán)

3. *Über die Charakterisierung von Minimallösungen in linearen normierten Räumen.* Mathematica, **10(33)**, 33-46 (1968) (with I. Kolumbán)

4. *Bemerkungen über die Existenz von Minimallösungen in normierten linearen Räumen.* Mathematica, **10(33)**, 223-228 (1968)

5. *Dualität bei Optimierungsaufgaben in topologischen Vektorräumen.* Mathematica, **10(33)**, 229-244 (1968) (with I. Kolumbán)

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7. *Zur Charakterisierung von Minimallösungen in normierten linearen Räumen.* Mathematica, **11(34)**, 49-52 (1969) (with B. Brosowski)

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15. *On certain ordered topological vector spaces occurring in optimization theory* (Romanian). Rev. Anal. Numer. Teoria Aproximaţiei, **2**, 45-50 (1973)

16. *On teaching of the congruence of triangles to sixth form pupils* (Romanian). In: Chircev A., Lăscuş V., Fodor T. (eds.), *School and Pupils* (Romanian). Casa Corpului Didactic a Judeţului Cluj, Cluj-Napoca, 1974, 325-333 (with M. Breckner)

17. *Charakterisierung der Minimallösungen bei Optimierungsaufgaben mit vektorwertigen Funktionen.* I. Operations Research Verfahren, **21**, 39-47 (1975)

18. *On the continuity of convex mappings.* Mathematica - Rev. Anal. Numér. Théorie Approx., Ser. L'Analyse Numér. Théorie Approx., **6**, 117-123 (1977) (with G. Orbán)

19. *A Hahn-Banach type extension theorem for linear mappings into ordered modules.* Mathematica - Rev. Anal. Numér. Théorie Approx., Ser. Mathematica, **19(42)**, 13-27 (1977) (with E. Scheiber)

20. *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen.* Publ. Inst. Math. (Beograd), **23(37)**, 13-20 (1978)

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31. *Multiplier rules for optimization problems with a finite number of constraints.* Studia Univ. Babeş-Bolyai, Ser. Math., **33**, No. 1, 15-37 (1988) (with I. Kolumbán)

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34. *Generalized quasiconvex functions.* Babeş-Bolyai University Cluj-Napoca, Seminar on Optimization Theory, Report No. 8, 13-26 (1989)

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44. *On the characterization of the continuity of generalized convex functions by means of hyponorms.* Mathematica, **36(59)**, 5-13 (1994)

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57. *Equicontinuity and Hölder equicontinuity of families of generalized convex mappings.* New Zealand Math. J., **28**, 155-170 (1999) (with T. Trif)

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3. *Introduction to the Theory of Convex Constrained Optimization Problems* (Romanian). Editura Dacia, Cluj, 1974, 220 pages

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6. Jointly with I. Maruşciac editor of the volume *Proceedings of the Colloquium on Approximation and Optimization.* University of Cluj-Napoca, Department of Mathematics, 1985, 352 pages

7. Jointly with D. D. Stancu, G. Coman and P. Blaga editor of the volumes *Approximation and Optimization*,Transilvania Press, Cluj-Napoca,1997, Vol. 1: xiv+374 pages; Vol. 2: vii+252 pages

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2. *Problem Book in Mathematical Analysis* (Romanian). Universitatea Babeş-Bolyai Cluj-Napoca, Fac. de Matematică, Catedra de Analiză, 1977, ii+365 pages (in collaboration)

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4. *Problem Book in Operations Research* (Romanian). Universitatea din Cluj-Napoca, Fac. de Matematică, 1983, ii+201 pages (with D. I. Duca)

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IV. Miscellanea

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2. *Professor József Kolumbán at his 60th anniversary.* Studia Univ. Babeş-Bolyai, Ser. Math., 41, No. 1, 109-116 (1996)

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4. *The multicultural Babeş-Bolyai University of Cluj-Napoca.* Higher Education in Europe, **26**, 391-398 (2001)

STARLIKENESS CONDITIONS FOR THE BERNARDI OPERATOR

DANIEL BREAZ AND NICOLETA BREAZ

Dedicated to Professor Wolfgang W. Breckner at his 60^{th} anniversary

Abstract. Let U be the unit disc of the complex plane: $U=\{z\in C, |z|<1\}$ and $A_n=\left\{f\in H(U), f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+..., z\in U\right\}$, and the class of starlike functions in U, $S^*(\alpha)=\left\{f\in A, \mathrm{Re}\dfrac{zf'(z)}{f(z)}>\alpha, z\in U\right\}$ the class of starlike functions of order α. We consider the integral operator $F(z)=\dfrac{1+\gamma}{z^\gamma}\int\limits_0^z f(t)\,t^{\gamma-1}dt$ and we study its starlikeness properties.

1. Introduction

In this paper a α order starlikeness condition for Bernardi operator is obtained. This condition is on extension of the results of Gh. Oros, see [1], which is obtained from our result for $\alpha=1$.

Lemma A. [2] *Let q the univalent function in U and let θ and ϕ be analytic functions in the domain $D\subset q(U)$ with $\phi(w)\neq 0$, when $w\in q(U)$.*

Set

$$Q(z)=nzq'(z)\,\phi[q(z)]$$

$$h(z)=\theta[q(z)]+Q(z)$$

and suppose that:

i) Q is starlike

and

ii) $\mathrm{Re}\dfrac{zh'(z)}{Q(z)}=\mathrm{Re}\left[\dfrac{\theta'[q(z)]}{\phi[q(z)]}+\dfrac{zQ'(z)}{Q(z)}\right]>0.$

If p is analytic in U, with

$$p(0)=q(0), p'(0)=...=p^{(n-1)}(0)=0, p(U)\subset D$$

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and

$$\theta\left[p\left(z\right)\right]+zp'\left(z\right)\phi\left[p\left(z\right)\right]\prec\theta\left[q\left(z\right)\right]+zq'\left(z\right)\phi\left[q\left(z\right)\right]$$

then $p\prec q$, *and* q *is the best dominant.*

2. **Main results**

Theorem 1. *Let* $\gamma\geq 0,\alpha>0$ *and*

$$h\left(z\right)=\frac{1}{1-\alpha z}+\frac{n\alpha z}{\left(1-\alpha z\right)\left(1+\gamma-\alpha\gamma z\right)}\qquad(1)$$

If $f\in A_n$ *and*

$$\frac{zf'\left(z\right)}{f\left(z\right)}\prec h\left(z\right)$$

then

$$Re\frac{zF'\left(z\right)}{F\left(z\right)}>\frac{1}{1+\alpha}$$

where

$$F\left(z\right)=\frac{1+\gamma}{z^{\gamma}}\int_{0}^{z}f\left(t\right)t^{\gamma-1}dt\qquad(2)$$

Proof. From 2 we deduce

$$\gamma F\left(z\right)+zF'\left(z\right)=\left(\gamma+1\right)f\left(z\right)\qquad(3)$$

If we consider

$$p\left(z\right)=\frac{zF'\left(z\right)}{F\left(z\right)}$$

then (3) becomes

$$\frac{zp'\left(z\right)}{p\left(z\right)+\gamma}+p\left(z\right)=\frac{zf'\left(z\right)}{f\left(z\right)}$$

But

$$\frac{zf'\left(z\right)}{f\left(z\right)}\prec h\left(z\right)$$

implies

$$\frac{zp'\left(z\right)}{p\left(z\right)+\gamma}+p\left(z\right)\prec h\left(z\right)$$

We apply Lemma 1 to prove that:

$$Re\frac{zF'\left(z\right)}{F\left(z\right)}>\frac{1}{1+\alpha}$$

We have:

$$q\left(z\right)=\frac{1}{1-\alpha z}$$

$$\theta\left(w\right)=w$$

$$\phi\left(w\right)=\frac{1}{w+\gamma}$$

$$\theta\left[q\left(z\right)\right]=\frac{1}{1-\alpha z}$$

$$\phi\left[q\left(z\right)\right]=\frac{1-\alpha z}{1+\gamma-\alpha\gamma z}$$

$$Q\left(z\right)=nzq'\left(z\right)\phi\left[q\left(z\right)\right]=\frac{n\alpha z}{\left(1-\alpha z\right)\left(1+\gamma-\alpha\gamma z\right)}.$$

$$h\left(z\right)=\theta\left[q\left(z\right)\right]+Q\left(z\right)=\frac{1}{1-\alpha z}+\frac{n\alpha z}{\left(1-\alpha z\right)\left(1+\gamma-\alpha\gamma z\right)}$$

Because Q is starlike and $\operatorname{Re}\phi\left[q\left(z\right)\right]>0$,from Lemma 1 we deduce

$$p\prec q\Leftrightarrow\frac{zF'\left(z\right)}{F\left(z\right)}\prec\frac{1}{1+\alpha z}\Rightarrow Re\frac{zF'\left(z\right)}{F\left(z\right)}>\frac{1}{1+\alpha}$$

The last relation is equivalent to

$$F\in S^{*}\left(\frac{1}{1+\alpha}\right)$$

Remark. For $\alpha=1$ we obtain the result of Gh. Oros [1].

Corollary 1. *Let*

$$h\left(z\right)=\frac{1}{1-z}+\frac{n\alpha z}{\left(1-z\right)\left(2-z\right)}$$

If $f\in A$ *and*

$$\frac{zf'\left(z\right)}{f\left(z\right)}\prec h\left(z\right)$$

then

$$Re\frac{zF'\left(z\right)}{F\left(z\right)}>\frac{1}{2}$$

where

$$F\left(z\right)=\frac{2}{z}\int_{0}^{z}f\left(t\right)dt$$

Proof. In Theorem 1 we put $\alpha=1,\gamma=1,n=1.$

Corollary 2. *Let*

$$h\left(z\right)=\frac{1}{1-2z}+\frac{n\alpha z}{\left(1-2z\right)\left(1+\gamma-2\gamma z\right)}$$

If $f\in A_{n}$ *and*

$$\frac{zf'\left(z\right)}{f\left(z\right)}\prec h\left(z\right)$$

then

$$Re\frac{zF'(z)}{F(z)} > \frac{1}{3}$$

where

$$F(z) = \frac{1+\gamma}{z^{\gamma}}\int_{0}^{z} f(t)\, t^{\gamma-1}dt$$

Proof. In Theorem 1 we put $\alpha = 2$.

Theorem 2. *Let* $\gamma \geq 0, \alpha > 0$ *and*

$$h(z) = \frac{1+\alpha z}{1-\alpha z} + \frac{2n\alpha z}{(1-\alpha z)(1+\gamma-(1-\gamma)\alpha z)} \tag{4}$$

If $f \in A_n$ *and*

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

then

$$Re\frac{zF'(z)}{F(z)} > \frac{1-\alpha}{1+\alpha}$$

where

$$F(z) = \frac{1+\gamma}{z^{\gamma}}\int_{0}^{z} f(t)\, t^{\gamma-1}dt \tag{5}$$

Proof. From (5) we deduce:

$$\gamma F(z) + zF'(z) = (\gamma+1) f(z) \tag{6}$$

Let

$$p(z) = \frac{zF'(z)}{F(z)}$$

Then (3) becomes

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) = \frac{zf'(z)}{f(z)}$$

But

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

implies

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) \prec h(z)$$

We use Lemma 1 to prove that:

$$Re\frac{zF'(z)}{F(z)} > \frac{1-\alpha}{1+\alpha}$$

We have:

$$q(z)=\frac{1+\alpha z}{1-\alpha z}$$

$$\theta(w)=w$$

$$\phi(w)=\frac{1}{w+\gamma}$$

$$\theta[q(z)]=\frac{1+\alpha z}{1-\alpha z}$$

$$\phi[q(z)]=\frac{1-\alpha z}{1+\gamma-(1-\gamma)\alpha z}$$

$$Q(z)=nzq'(z)\phi[q(z)]=\frac{2n\alpha z}{(1-\alpha z)(1+\gamma-(1-\gamma)\alpha z)}$$

$$h(z)=\theta[q(z)]+Q(z)=\frac{1+\alpha z}{1-\alpha z}+\frac{2n\alpha z}{(1-\alpha z)(1+\gamma-(1-\gamma)\alpha z)}$$

Because Q is starlike and Re $\phi[q(z)]>0$ from Lemma 1 we deduce :

$$p\prec q\Leftrightarrow\frac{zF'(z)}{F(z)}\prec\frac{1+\alpha z}{1-\alpha z}\Rightarrow Re\frac{zF'(z)}{F(z)}>\frac{1-\alpha}{1+\alpha}$$

The last relation is equivalent to

$$F\in S^{*}\left(\frac{1-\alpha}{1+\alpha}\right)$$

Remark. For $\alpha=1$ we obtain the result of Gh. Oros [1].

Corollary 3. *Let*

$$h(z)=\frac{1+2z}{1-z}$$

If $f\in A$ and

$$\frac{zf'(z)}{f(z)}\prec h(z)$$

then

$$Re\frac{zF'(z)}{F(z)}>0$$

where

$$F(z)=\frac{2}{z}\int_{0}^{z}f(t)\,dt$$

Proof. In Theorem 2 we put $\alpha=1,\gamma=1,n=1$.

Corollary 2. *Let*

$$h(z)=\frac{1+2z}{1-2z}+\frac{4nz}{(1-2z)(1+\gamma+2(1-\gamma)z)}$$

If $f \in A_n$ *and*

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

then

$$Re\frac{zF'(z)}{F(z)} > -\frac{1}{3}$$

where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t)\, t^{\gamma-1} dt$$

Proof. In Theorem 2 we put $\alpha = 2$.

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A NOTE ON THE DIVISIBILITY OF SOME COMPRESSION SEMIGROUPS IN $\mathrm{Sl}(2,\mathbb{R})$

BRIGITTE E. BRECKNER

Dedicated to my father Wolfgang W. Breckner on the occasion of his 60th birthday

Abstract. We give elementary proofs (avoiding, as much as possible, any machinery of Lie theory) for the divisibility of those compression semigroups in $\mathrm{Sl}(2,\mathbb{R})^+$ who are known to be the prototypes of the three dimensional exponential Lie subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$.

Why this note has been written. The natural nonabelian analogues of cones in real vector spaces are the divisible closed subsemigroups of connected Lie groups, these are exactly the exponential Lie semigroups. In [4] K.H. Hofmann and W.A.F. Ruppert classify the reduced exponential Lie semigroups and show that these semigroups are built up from a few building blocks, the so-called *Master Examples*. In 1999 B.E. Breckner and W.A.F. Ruppert started a project devoted to the study of the topological semigroup compactifications of divisible subsemigroups of Lie groups. A first step for carrying out this project is to investigate the topological semigroup compactifications of the *Master Examples*. So, Breckner and Ruppert focused for the beginning on one of the *Master Examples*, namely the exponential Lie subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$. It has turned out, however, that for the study of the compactifications of these semigroups one needs a very detailed knowledge of general structural features of $\mathrm{Sl}(2,\mathbb{R})$ (see [1]). We remark in passing that, using the tools introduced in [1], Breckner and Ruppert offer in [2] a fairly comprehensive study of the topological semigroup compactifications of certain subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$ (including the exponential ones). A main result of [1], with important consequences for the investigations

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in [2], is the determination of the conjugacy classes of exponential subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$ (see 7.14 of [1]):

Let S be a three dimensional exponential subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. *Then S is conjugate to exactly one of the following semigroups:*

1. $\mathrm{Sl}(2,\mathbb{R})^+$,
2. $S_1 = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+b \geq c+d\}$,
3. $S^1 = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+c \geq b+d\}$,
4. $S^1_\lambda = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+c \geq b+d$ *and* $a + \frac{1}{\lambda} b \geq \lambda c + d\}$, *for some real* $\lambda > 0$.

In [1] the exponentiality of the semigroups S_1, S^1, and S^1_λ is shown by a typical Lie theoretical argument, involving the determination of the Lie wedges of the semigroups. Nevertheless, the exponentiality of these semigroups is of interest also from a pure algebraical point of view. To see this, recall that a closed submonoid of a connected Lie group is divisible if and only if it is an exponential Lie semigroup (cf, eg, 2.7 of [4]). Thus, a problem of own interest is to prove the divisibility of the semigroups S_1, S^1, and S^1_λ by a direct, algebraical argument. The present paper offers such a proof.

Divisible semigroups. A semigroup S is called *divisible* if $\forall s \in S$, $\forall n \in \mathbb{N}^*$ $\exists x \in S$ such that $x^n = s$.

Notations. Following [3], we write $\mathrm{Sl}(2,\mathbb{R})^+$ for the semigroup of matrices with nonnegative entries in $\mathrm{Sl}(2,\mathbb{R})$. For fixed positive reals $\lambda, \mu > 0$ we define the following subsets of $\mathrm{Sl}(2,\mathbb{R})^+$:

$$S_\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \middle| \, a + \frac{1}{\lambda} b \geq \lambda c + d \right\},$$

$$S^\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \middle| \, a + \frac{1}{\lambda} c \geq \lambda b + d \right\}, \text{ and } S^\mu_\lambda = S_\lambda \cap S^\mu.$$

The main statement. *The sets S_λ, S^λ, and S^μ_λ are divisible semigroups for every* $\lambda, \mu > 0$.

Remark. The first step in the proof of the main statement is to show that S_λ, S^λ, and S_λ^μ are indeed semigroups. For this it suffices to show that S_λ is a semigroup, because S^λ is the image of S_λ under the anti-isomorphism sending every matrix to its transpose. That S_λ is a semigroup is not obvious, since it cannot be seen immediately that the product of two arbitrary elements of S_λ belongs to S_λ. So, it turned out to be very convenient to follow [1] and to represent S_λ as a compression semigroup.

Compression semigroups. Let S be a semigroup which acts on some space X. Then for every subset M of X, we define the *compression semigroup of M in S* as the set

$$\mathrm{compr}_S(M) = \{s \in S \mid sM \subseteq M\}.$$

It is obvious that $\mathrm{compr}_S(M)$ is either empty or a subsemigroup of S.

The set S_λ as a compression semigroup. (cf 6.8 of [1]) Consider the natural action of $\mathrm{Sl}(2,\mathbb{R})^+$ (as a semigroup of endomorphisms of $\mathbb{R}^2$) on $\mathbb{R}^2$ and define for a fixed real $\lambda > 0$ the cone

$$C_\lambda = \{(x, y) \in \mathbb{R}^2 \mid x \geq \lambda y \geq 0\}.$$

The reader is invited to check by a straightforward computation that S_λ is the compression semigroup of C_λ in $\mathrm{Sl}(2,\mathbb{R})^+$ (see also 6.8 of [1]).

The following notion, similar to that of a compression semigroup, will be crucial for the proof of the main statement.

Almost compression semigroups. Let S be a semigroup which acts on some space X and consider M, M' subsets of X such that $M' \subseteq M$. We define the *almost compression semigroup of the pair (M, M') in S* to be the set

$$\mathrm{alcompr}_S(M, M') = \{s \in S \mid sM \subseteq M'\}.$$

It follows readily from its definition that $\mathrm{alcompr}_S(M, M')$ is either empty or a subsemigroup of S.

We collect now some facts needed for the proof of the main statement.

Fact 1: *The semigroup* $\mathrm{Sl}(2,\mathbb{R})^+$ *is divisible.*

For those who are familiar with Lie theory this is a well-known result. It can be proved by direct calculation involving the formula for the exponential function $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{Sl}(2,\mathbb{R})$ (cf, eg, p. 416 ff. of [3]).

Fact 2: *Let S be a divisible semigroup and $(S_i)_{i\in I}$ a family of subsemigroups of S such that $S \setminus S_i$ is a semigroup for every $i \in I$. Then the intersection $\cap_{i\in I} S_i$ is either empty or a divisible semigroup.*

Proof: Put $T = \cap_{i\in I} S_i$ and choose $s \in T$ and $n \in \mathbb{N}^*$ arbitrarily. Since S is divisible there exists $x \in S$ such that $x^n = s$. Then x belongs to T. Otherwise the fact that $x \notin S_i$ for some $i \in I$ would imply that $s = x^n \in S \setminus S_i$, a contradiction. Thus T is a divisible subsemigroup of S, if it is not empty. □

Fact 3: *Let $\lambda > 0$. The set*

$$\mathrm{Sl}(2,\mathbb{R})^+ \setminus S_\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a + \frac{1}{\lambda} b < \lambda c + d \right\}$$

is a semigroup.

Proof: Put $\tilde{S}_\lambda = \mathrm{Sl}(2,\mathbb{R})^+ \setminus S_\lambda$. We prove that $\tilde{S}_\lambda$ is an almost compression semigroup. For this consider again the natural action of $\mathrm{Sl}(2,\mathbb{R})^+$ on $\mathbb{R}^2$ and define the sets

$$\tilde{C}_\lambda = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x \le \lambda y\} \setminus \{(0,0)\}, \quad \tilde{W}_\lambda = \{(x,y) \in \tilde{C}_\lambda \mid x < \lambda y\}.$$

We show that

$$(*) \qquad \tilde{S}_\lambda = \mathrm{alcompr}_{\mathrm{Sl}(2,\mathbb{R})^+}(\tilde{C}_\lambda, \tilde{W}_\lambda).$$

If $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+$ is such that $s\tilde{C}_\lambda \subseteq \tilde{W}_\lambda$ then $s \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \in \tilde{W}_\lambda$. Hence $a\lambda + b < \lambda(\lambda c + d)$ or, equivalently, $a + \frac{1}{\lambda} b < \lambda c + d$.

Conversely, if $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+$ is such that $a + \frac{1}{\lambda} b < \lambda c + d$ then we observe first that $\lambda d > b$, since multiplying the first inequality with $d > 0$ yields (note that $ad = 1 + bc$)

$$ad + \frac{1}{\lambda} bd < \lambda cd + d^2 \implies 1 + bc + \frac{1}{\lambda} bd < \lambda cd + d^2 \implies 1 < (\lambda d - b)(c + \frac{1}{\lambda} d).$$

Pick an arbitrary $(x,y) \in \tilde{C}_\lambda$. Then there exists $\alpha, \beta \in \mathbb{R}_+$ with $\alpha^2+\beta^2 \neq 0$ such that $(x,y) = \alpha(0,1) + \beta(\lambda,1)$. Now

$$s\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \in \tilde{W}_\lambda \quad \text{as well as} \quad s\begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} a\lambda+b \\ c\lambda+d \end{pmatrix} \in \tilde{W}_\lambda.$$

Since $\alpha^2+\beta^2 \neq 0$ we conclude that $s\begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{W}_\lambda$. This proves $(*)$, so $\tilde{S}_\lambda$ is a semigroup. □

Proof of the main statement: Fact 1, Fact 2, and Fact 3 imply that S_λ is divisible. Since the anti-isomorphism sending every matrix to its transpose maps S_λ onto S^λ, it follows that S^λ is also divisible and that $\mathrm{Sl}(2,\mathbb{R})^+ \setminus S^\lambda$ is a semigroup. Using once again Fact 2, it finally follows that $S_\lambda^\mu = S_\lambda \cap S^\mu$ is divisible. □

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A MAXIMUM PRINCIPLE FOR A MULTIOBJECTIVE OPTIMAL CONTROL PROBLEM

WOLFGANG W. BRECKNER

Rezumat. Un principiu de maxim pentru o problemă vectorială de control optimal. Ca aplicaţie a unei reguli abstracte a multiplicatorilor s-a stabilit în lucrarea [1] un principiu de maxim pentru o problemă vectorială de control optimal guvernată de o ecuaţie integrală de tip Fredholm. Pentru a nu mări excesiv lungimea lucrării [1], demonstraţia acestui principiu a fost acolo doar schiţată. În prezenta lucrare se dă acum demonstraţia completă.

1. **Introduction**

In the paper [1] we have established multiplier rules for so-called weak dynamic multiobjective optimization problems by using a suitable generalization of the derived sets introduced by M. R. Hestenes [2], [3], [4] for scalar optimization problems. Also in that paper we have used the obtained multiplier rules to state necessary conditions for the local solutions of an abstract multiobjective optimal control problem. Furthermore, we have noticed that these very general optimality conditions can yield a maximum principle for a multiobjective optimal control problem governed by an integral equation of Fredholm type (Theorem 5.1 in [1]). But, in order to avoid an excessive length of the paper, in [1] we have limited ourselves only to a sketch of this application. The goal of the present paper is to give the complete proof of this specific maximum principle.

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2. Notations

Throughout this paper, N is the set of all positive integers, R is the set of all real numbers, and for every $m \in N$, R^m is the usual m-dimensional Euclidean space of all m-tuples $v = (v_1, \dots, v_m)$ of real numbers. The subset of R^m, consisting of all vectors $v = (v_1, \dots, v_m)$ with $v_j \geq 0$ for each $j \in \{1, \dots, m\}$, is denoted by R^m_+. The inner product of two vectors $v, w \in R^m$ is denoted by $\langle v, w \rangle$. If $v \in R^m$, then $\|v\|$ marks its Euclidean norm. Given any number $r > 0$, we put

$$B^m_+(r) = \{v \in R^m_+ \mid \|v\| \leq r\}.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces over the same field, then $(\mathcal{X}, \mathcal{Y})^*$ denotes the normed linear space of all continuous linear mappings $A : \mathcal{X} \to \mathcal{Y}$. Given a point x_0 in a normed linear space and a number $r > 0$, we denote by $B(x_0, r)$ the closed ball centered at x_0 with radius r.

If M is a subset of a normed linear space, then int M designates the interior of M and cl M the closure of M.

Finally, we mention some notations regarding functions. The Fréchet derivative of a function f of a single variable is denoted by df, while the partial Fréchet derivative with respect to the nth variable of a function f of several variables is denoted by $d_n f$. If x is a point in a linear space $\mathcal{X}$ and A is a linear mapping from $\mathcal{X}$ into another linear space, then Ax denotes the value of A at x.

3. A Necessary Optimality Condition

Let $\mathcal{X}$ be a Banach space, which does not reduce to its zero-vector, let X be a nonempty open subset of $\mathcal{X}$, let U be a nonempty set, let m_1, m_2 and m_3 be positive integers, and let

$$f_1 : X \times U \to R^{m_1}, \quad f_2 : X \times U \to R^{m_2}, \quad f_3 : X \times U \to R^{m_3}$$

be vector-valued functions which are Fréchet differentiable at each point (x, u) in $X \times U$ with respect to the first variable. Further, let K_1, K_2 and K_3 be convex cones in the spaces R^{m_1}, R^{m_2} and R^{m_3}, respectively, satisfying the following assumptions:

$$\text{int } K_1 \neq \emptyset, \ \text{int } K_2 \neq \emptyset, \ K_2 \text{ and } K_3 \text{ are closed.} \tag{1}$$

For each $i \in \{1, 2, 3\}$, we define by

$$K_i^* = \{w \in R^{m_i} | \ \forall \ v \in K_i : \ \langle v, w \rangle \geq 0\}$$

the dual cone of K_i.

Let $F : X \times U \to \mathcal{X}$ be a function which is Fréchet differentiable at each point $(x, u) \in X \times U$ with respect to the first variable, and let S be the set defined by

$$S = \{(x, u) \in X \times U \mid F(x, u) = 0, \ f_2(x, u) \in K_2, \ f_3(x, u) \in K_3\}.$$

A point $(x_0, u_0) \in \mathcal{X} \times U$ is said to be a:

(i) *weakly K_1-maximal point* of f_1 over S if $(x_0, u_0) \in S$ and

$$[f_1(x_0, u_0) + \text{int } K_1] \cap f_1(S) = \emptyset;$$

(ii) *local weakly K_1-maximal point* of f_1 over S if $(x_0, u_0) \in S$ and if there is a neighbourhood V of x_0 such that

$$[f_1(x_0, u_0) + \text{int } K_1] \cap f_1(S \cap (V \times U)) = \emptyset.$$

The problem of finding the weakly K_1-maximal points of f_1 over S is called a *weak multiobjective optimal control problem* and is expressed in short as

(CP) $\quad f_1(x, u) \ \longrightarrow_{K_1}$ max weakly

subject to $(x, u) \in X \times U, \ F(x, u) = 0, \ f_2(x, u) \in K_2, \ f_3(x, u) \in K_3$.

The introduction of problem (CP) allows one to call the weakly K_1-maximal points of f_1 over S *solutions* to problem (CP). By analogy, the local weakly K_1-maximal points of f_1 over S can be named *local solutions* to problem (CP).

As an application of multiplier rules stated for arbitrary weak dynamic multiobjective optimization problems, in Section 4 of the paper [1] we have derived necessary optimality conditions for the local solutions to problem (CP). One of the theorems given there will be recalled here. In order to formulate shorter this theorem, we put $m = m_1 + m_2 + m_3$ and conceive the corresponding space R^m as the product space $R^{m_1} \times R^{m_2} \times R^{m_3}$, i.e. any vector $v \in R^m$ is identified with a certain triple $(v_1, v_2, v_3) \in R^{m_1} \times R^{m_2} \times R^{m_3}$. In particular, the zero-vector in R^m

is $0 = (0_1, 0_2, 0_3)$, where 0_i $(i \in \{1,2,3\})$ is the zero-vector in R^{m_i}. Further, we consider the vector-valued function $f : X \times U \to R^m$ defined by

$$f(x,u) = (f_1(x,u), f_2(x,u), f_3(x,u)).$$

By using these notations, the following theorem is valid.

THEOREM 1 [1, Theorem 4.6]. Let $(x_0, u_0) \in \mathcal{X} \times U$ be a local solution to problem (CP) for which the operator $A = d_1F(x_0, u_0)$ is bijective, and let $D \subseteq R^m$ be a non-empty set such that, for all $n \in N$ and all n-tuples $(d^1, \ldots, d^n)$ of points belonging to D, there exist a number $r_0 > 0$ and a function $\omega_2 : B^n_+(r_0) \to U$ satisfying the following conditions:

(i) $\omega_2(0) = u_0$;

(ii) for each $x \in X$, the function $t \in B^n_+(r_0) \longmapsto F(x, \omega_2(t)) \in \mathcal{X}$ is continuous on $B^n_+(r_0)$;

(iii) the function $t \in B^n_+(r_0) \longmapsto d_1F(x_0, \omega_2(t)) \in (\mathcal{X}, \mathcal{X})^*$ is continuous at 0;

(iv) $\lim_{x \to x_0} \sup \left\{ \|d_1F(x, \omega_2(t)) - d_1F(x_0, \omega_2(t))\| \;\middle|\; t \in B^n_+(r_0) \right\} = 0$;

(v) for each $x \in X$, the function $t \in B^n_+(r_0) \longmapsto f(x, \omega_2(t)) \in R^m$ is continuous on $B^n_+(r_0)$;

(vi) there is a number $a > 0$ such that $B(x_0, a) \subseteq X$ and such that

$$\sup\left\{ \|d_1f(x, \omega_2(t))\| \;\middle|\; x \in B(x_0, a),\ t \in B^n_+(r_0) \right\} < \infty;$$

(vii) $\sup \left\{ \|F(x_0, \omega_2(t))\| \,/\, \|t\| \;\middle|\; t \in B^n_+(r_0),\ t \neq 0 \right\} < \infty$;

(viii) $\sup \left\{ \|d_1f(x_0, \omega_2(t)) - d_1f(x_0, u_0)\| \,/\, \|t\| \;\middle|\; t \in B^n_+(r_0),\ t \neq 0 \right\} < \infty$;

(ix) $\lim_{x \to x_0} \sup \left\{ \|d_1f(x, \omega_2(t)) - d_1f(x_0, \omega_2(t))\| \;\middle|\; t \in B^n_+(r_0) \right\} = 0$;

(x) $\lim_{t \to 0} \frac{1}{\|t\|} \left[f(x_0, \omega_2(t)) - f(x_0, u_0) - Pt - d_1f(x_0, u_0)\omega_0(t) \right] = 0$, where

$$Pt = t_1 d^1 + \ldots + t_n d^n \quad \text{for all } t = (t_1, \ldots, t_n) \in R^n$$

and

$$\omega_0(t) = A^{-1}F(x_0, \omega_2(t)) \quad \text{for all } t \in B^n_+(r_0).$$

Then there exists a vector

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) \in K_1^* \times K_2^* \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$$

such that

$$\sup\big\{\langle d_1, \lambda_1^*\rangle + \langle d_2, \lambda_2^*\rangle + \langle d_3, \lambda_3^*\rangle \;\big|\; (d_1, d_2, d_3) \in D\big\} \leq 0$$

and

$$\langle f_2(x_0, u_0), \lambda_2^*\rangle = 0.$$

hold.

4. **The Maximum Principle**

In this section we apply Theorem 1 to derive a maximum principle for a multiobjective optimal control problem governed by an integral equation of Fredholm type.

In what follows we suppose that T is a positive number, V is a non-empty subset of a real Banach space $\mathcal{V}$, and $\mathcal{W}$ is a real Banach space which does not reduce to its zero-vector. Let I denote the interval $[0, T]$, let $C(I, \mathcal{W})$ be the linear space of all continuous functions $x : I \to \mathcal{W}$ endowed with the norm

$$\|x\| \;=\; \max\,\{\|x(\tau)\| \mid \tau \in I\},$$

and let $PC(I, V)$ be the set of all piecewise continuous functions $u : I \to V$ that are continuous at 0 and continuous on the left at each point belonging to the interval $]0, T]$.

Further, let

$$\varphi_i : I \times \mathcal{W} \times \operatorname{cl} V \to R^{m_i} \quad (i \in \{1, 2, 3\})$$

be functions that are continuous, Fréchet differentiable with respect to the second variable and such that the mappings

$$d_2\varphi_i : I \times \mathcal{W} \times \operatorname{cl} V \to (\mathcal{W}, R^{m_i})^* \quad (i \in \{1, 2, 3\})$$

are continuous, and let

$$\phi : I \times I \times \mathcal{W} \times \operatorname{cl} V \to \mathcal{W}$$

be a function which is continuous, Fréchet differentiable with respect to the third variable, and for which the mapping

$$d_3\phi : I \times I \times \mathcal{W} \times \mathrm{cl}\, V \to (\mathcal{W}, \mathcal{W})^*$$

is continuous and has the property that the family

$$\left\{ d_3\phi(\sigma, \tau, \cdot, v) : \mathcal{W} \to (\mathcal{W}, \mathcal{W})^* \;\middle|\; (\sigma, \tau, v) \in I \times I \times V \right\}$$

is uniformly equicontinuous on each closed bounded subset of $\mathcal{W}$.

As in Section 3, let K_1, K_2 and K_3 be convex cones in the spaces R^{m_1}, R^{m_2} and R^{m_3}, respectively, satisfying the assumptions specified in (1).

The problem we will discuss in this section is:

$$\text{(ECP)} \qquad \int_0^T \varphi_1(\tau, x(\tau), u(\tau))\, d\tau \;\longrightarrow_{K_1}\; \max \text{ weakly}$$

subject to

$$x \in C(I, \mathcal{W}), \;\; u \in PC(I, V),$$

$$x(\sigma) \;=\; \int_0^T \phi(\sigma, \tau, x(\tau), u(\tau))\, d\tau \quad (\sigma \in I),$$

$$\int_0^T \varphi_2(\tau, x(\tau), u(\tau))\, d\tau \in K_2, \quad \int_0^T \varphi_3(\tau, x(\tau), u(\tau))\, d\tau \in K_3.$$

This problem is a special case of the problem (CP) investigated in the preceding section. To see this, it suffices to define the functions

$$f_i : C(I, \mathcal{W}) \times PC(I, V) \to R^{m_i} \quad (i \in \{1, 2, 3\})$$

by

$$f_i(x, u) = \int_0^T \varphi_i(\tau, x(\tau), u(\tau))\, d\tau \quad (i \in \{1, 2, 3\}),$$

on the one hand, and

$$F : C(I, \mathcal{W}) \times PC(I, V) \to C(I, \mathcal{W})$$

by

$$F(x,u)(\sigma) = x(\sigma) - \int_0^T \phi(\sigma,\tau,x(\tau),u(\tau))\,d\tau \quad (\sigma \in I),$$

on the other hand, as well as to take $\mathcal{X} = X = C(I,\mathcal{W})$ and $U = PC(I,V)$.

Furthermore, it should be emphasized that the functions f_i $(i \in \{1,2,3\})$ and F introduced above are Fréchet differentiable with respect to the first variable. The corresponding partial Fréchet derivatives are given by

$$d_1 f_i(x,u)y \;=\; \int_0^T d_2\varphi_i(\tau,x(\tau),u(\tau))y(\tau)\,d\tau \quad (i \in \{1,2,3\}),$$

$$(d_1 F(x,u)y)(\sigma) \;=\; y(\sigma) - \int_0^T d_3\phi(\sigma,\tau,x(\tau),u(\tau))y(\tau)\,d\tau \quad (\sigma \in I),$$

for all $(x,u) \in C(I,\mathcal{W}) \times PC(I,V)$ and all $y \in C(I,\mathcal{W})$. Thus it makes sense to try to specialize Theorem 1 to problem (ECP).

To this end we define the functions

$$\varphi : I \times \mathcal{W} \times \operatorname{cl} V \to R^m \quad \text{and} \quad f : C(I,\mathcal{W}) \times PC(I,V) \to R^m$$

by

$$\varphi(\tau,w,v) \;=\; (\varphi_1(\tau,w,v),\varphi_2(\tau,w,v),\varphi_3(\tau,w,v)),$$
$$f(x,u) \;=\; (f_1(x,u),f_2(x,u),f_3(x,u)),$$

respectively. Then we have

$$f(x,u) \;=\; \int_0^T \varphi(\tau,x(\tau),u(\tau))\,d\tau, \quad d_1 f(x,u)y \;=\; \int_0^T d_2\varphi(\tau,x(\tau),u(\tau))y(\tau)\,d\tau$$

for all $(x,u) \in C(I,\mathcal{W}) \times PC(I,V)$ and all $y \in C(I,\mathcal{W})$.

Taking into account all these assumptions and considerations concerning the problem (ECP), we get from Theorem 1 the following result.

THEOREM 2 [1, Theorem 5.1]. Let $(x_0,u_0) \in C(I,\mathcal{W}) \times PC(I,V)$ be a local solution to problem (ECP) satisfying the following conditions:

(j) for each $y \in C(I, \mathcal{W})$ the integral equation

$$x \;=\; y + \int_0^T d_3\phi(\cdot, \tau, x_0(\tau), u_0(\tau)) x(\tau)\, d\tau$$

has a unique solution $x \in C(I, \mathcal{W})$;

(jj) there is a number $a > 0$ such that

$$\sup\, \big\{ \|d_2\varphi(\tau, x(\tau), v)\| \;\big|\; (\tau, x, v) \in I \times C(I, \mathcal{W}) \times V,\ \|x - x_0\| \le a \big\} < \infty.$$

Then there exists a vector

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in K_1^* \times K_2^* \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$$

such that

$$\max\, \{H(\tau, v) \mid v \in V\} = H(\tau, u_0(\tau)) \;\text{ for all } \tau \in I_0 \tag{2}$$

and

$$\langle \int_0^T \varphi_2(\tau, x_0(\tau), u_0(\tau))\, d\tau, \lambda_2^* \,\rangle = 0, \tag{3}$$

where I_0 is the set of all points $\tau \in \,]0, T]$ at which u_0 is continuous, $H(\tau, \cdot) : V \to R$ is the function defined by

$$H(\tau, v) = \langle\, \varphi(\tau, x_0(\tau), v) + \int_0^T d_2\varphi(\sigma, x_0(\sigma), u_0(\sigma)) h(\sigma; \tau, v)\, d\sigma, \lambda^* \,\rangle,$$

and $h(\cdot\, ; \tau, v) : I \to \mathcal{W}$ denotes the solution of the variational equation

$$x = \phi(\cdot, \tau, x_0(\tau), v) + \int_0^T d_3\phi(\cdot, t, x_0(t), u_0(t)) x(t)\, dt.$$

Proof. At first we notice that the operator $A = d_1F(x_0, u_0)$ is bijective because of condition (j). Next we construct a subset D of the space R^m which satisfies the hypotheses of Theorem 1. For this purpose we associate with each pair $(\tau, v) \in I_0 \times V$ the following expressions:

$$\begin{aligned} \alpha(\tau, v) &= \varphi(\tau, x_0(\tau), v) - \varphi(\tau, x_0(\tau), u_0(\tau)), \\ \beta(\tau, v) &= \phi(\cdot, \tau, x_0(\tau), v) - \phi(\cdot, \tau, x_0(\tau), u_0(\tau)), \\ d(\tau, v) &= \alpha(\tau, v) + d_1 f(x_0, u_0) \circ A^{-1}\beta(\tau, v). \end{aligned}$$

After that we put

$$D = \{d(\tau, v) \mid (\tau, v) \in I_0 \times V\}.$$

Now, let n be any positive integer, and let $d^j = d(\tau_j, v_j)$ $(j \in \{1, \dots, n\})$ be points belonging to D. For each $j \in \{1, \dots, n\}$ we set $\alpha^j = \alpha(\tau_j, v_j)$ and $\beta^j = \beta(\tau_j, v_j)$. Then we have

$$d^j = \alpha^j + d_1 f(x_0, u_0) \circ A^{-1} \beta^j \quad \text{for all } j \in \{1, \dots, n\}.$$

Without loss of the generality we can assume that the points $d^1, \dots, d^n$ are in such a manner numbered that

$$0 < \tau_1 \leq \tau_2 \leq \dots \leq \tau_n \leq T.$$

Put $\tau_0 = 0$. Then choose a number $r > 0$ satisfying

$$r < \tau_{j+1} - \tau_j \quad \text{whenever } j \in \{0, \dots, n-1\} \text{ and } \tau_j < \tau_{j+1} \tag{4}$$

and

$$[\tau_j - r, \tau_j] \subseteq I_0 \quad \text{for all } j \in \{1, \dots, n\}.$$

Set $r_0 = r/n$.

Next we define a function $\omega_2 : B^n_+(r_0) \to PC(I, V)$. Fix any point $t = (t_1, \dots, t_n)$ in $B^n_+(r_0)$, Then we have

$$t_1 + \dots + t_n \leq n\|t\| \leq r. \tag{5}$$

For each $j \in \{1, \dots, n\}$ we denote

$$N_j = \big\{k \in N \bigm| j < k \leq n \text{ and } \tau_k = \tau_j\big\}$$

and

$$a_j = \begin{cases} t_j & \text{if } N_j = \emptyset \\ t_j + \sum_{k \in N_j} t_k & \text{if } N_j \neq \emptyset. \end{cases}$$

It is easily seen that (4) and (5) imply

$$0 < \tau_j - a_j \leq \tau_j - a_j + t_j \leq T \quad \text{for all } j \in \{1, \dots, n\}. \tag{6}$$

When $n > 1$, then we additionally have

$$\tau_j - a_j + t_j \leq \tau_{j+1} - a_{j+1} \quad \text{for all } j \in \{1, \dots, n-1\}. \tag{7}$$

From (6) and (7) it follows that the intervals I_j ($j \in \{1, \dots, n\}$), defined by

$$I_j =]\tau_j - a_j, \tau_j - a_j + t_j] \text{ for every } j \in \{1, \dots, n\},$$

satisfy

$$I_j \subseteq I \text{ for all } j \in \{1, \dots, n\},$$

and

$$I_j \cap I_k = \emptyset \text{ for all } j, k \in \{1, \dots, n\}, j \neq k.$$

These properties of the intervals I_j ($j \in \{1, \dots, n\}$) enable us to define the function $\omega_2(t) : I \to V$ by

$$\omega_2(t)(\tau) = \begin{cases} v_j & \text{if } \tau \in I_j \text{ for some } j \in \{1, \dots, n\} \\ u_0(\tau) & \text{if } \tau \in I \setminus (I_1 \cup \ldots \cup I_n). \end{cases}$$

In view of this definition we obviously have $\omega_2(t) \in PC(I, V)$.

In what follows we prove that the number r_0 and the function ω_2 defined above satisfy the conditions (i) – (x) of Theorem 1. In the proofs of some of these conditions we shall use the compact set

$$L = [\tau_1 - r, \tau_1] \cup \ldots \cup [\tau_n - r, \tau_n],$$

which is enclosed in I_0. Besides, given any $t = (t_1, \dots, t_n) \in B_+^n(r_0)$, we shall need the intervals

$$L_j = [\tau_j - a_j, \tau_j - a_j + t_j], \text{ where } j \in \{1, \dots, n\}.$$

They satisfy

$$L_j \subseteq [\tau_j - r, \tau_j] \subseteq L \text{ for all } j \in \{1, \dots, n\}.$$

Indeed, let j be any index in $\{1, \dots, n\}$. Since $t_j \leq a_j$, we have $L_j \subseteq [\tau_j - a_j, \tau_j]$. On the other hand, the inequality $a_j \leq t_1 + \ldots + t_n$ holds. Consequently, (5) implies $a_j \leq r$, whence $[\tau_j - a_j, \tau_j] \subseteq [\tau_j - r, \tau_j]$. Thus we have $L_j \subseteq [\tau_j - r, \tau_j] \subseteq L$, as claimed.

Now we consecutively prove that the conditions (i) – (x) occurring in Theorem 1 are satisfied.

Condition (i): If $t = 0$, then $I_j = \emptyset$ for every $j \in \{1, \dots, n\}$. Thus we have $\omega_2(0) = u_0$.

Condition (ii): We fix a function $x \in C(I, \mathcal{W})$. Since the functions

$$(\sigma, \tau) \in I \times L \longmapsto \phi(\sigma, \tau, x(\tau), v_j) \in \mathcal{W} \quad (j \in \{1, \dots, n\})$$

and

$$(\sigma, \tau) \in I \times L \longmapsto \phi(\sigma, \tau, x(\tau), u_0(\tau)) \in \mathcal{W}$$

are continuous on the compact set $I \times L$, there exists a number $c > 0$ such that

$$\|\phi(\sigma, \tau, x(\tau), v_j)\| + \|\phi(\sigma, \tau, x(\tau), u_0(\tau))\| \leq c \tag{8}$$

for all $(\sigma, \tau) \in I \times L$ and all $j \in \{1, \dots, n\}$.

Let $t^1 = (t_1^1, \dots, t_n^1)$ and $t^2 = (t_1^2, \dots, t_n^2)$ be points in $B_+^n(r_0)$. For every $j \in \{1, \dots, n\}$ we put

$$L_{j1} = [\tau_j - a_{j1}, \tau_j - a_{j1} + t_j^1], \quad L_{j2} = [\tau_j - a_{j2}, \tau_j - a_{j2} + t_j^2],$$

$$M_j = \{\tau_j - (1 - \tau)a_{j1} - \tau a_{j2} \mid \tau \in [0, 1]\},$$

where a_{j1} and a_{j2} are the numbers used in the definition of the function $\omega_2(t^1)$ and $\omega_2(t^2)$, respectively. Obviously, we have

$$|a_{j1} - a_{j2}| \leq |t_j^1 - t_j^2| + \sum_{k \in N_j} |t_k^1 - t_k^2| \leq n \, \|t^1 - t^2\| \tag{9}$$

for every $j \in \{1, \dots, n\}$. Fix any $\sigma \in I$. In virtue of (8) and (9) it follows that

$$\begin{aligned} \Big\| \int_{L_{j1}} \phi(\sigma, \tau, x(\tau), v_j) \, d\tau - \int_{L_{j2}} \phi(\sigma, \tau, x(\tau), v_j) \, d\tau \Big\| &\leq c(2|a_{j1} - a_{j2}| + |t_j^1 - t_j^2|) \\ &\leq c(2n + 1) \, \|t^1 - t^2\| \end{aligned}$$

and

$$\Big\| \int_{M_j} \phi(\sigma, \tau, x(\tau), u_0(\tau)) \, d\tau \Big\| \leq c|a_{j1} - a_{j2}| \leq cn \, \|t^1 - t^2\|$$

for all $j \in \{1, \dots, n\}$. Accordingly, we have

$$\Big\| \int_{\tau_{j-1}}^{\tau_j} \phi(\sigma, \tau, x(\tau), \omega_2(t^1)(\tau)) \, d\tau - \int_{\tau_{j-1}}^{\tau_j} \phi(\sigma, \tau, x(\tau), \omega_2(t^2)(\tau)) \, d\tau \Big\|$$

$$\leq \Big\| \int_{M_j} \phi(\sigma, \tau, x(\tau), u_0(\tau)) \, d\tau \Big\| + \Big\| \int_{L_{j1}} \phi(\sigma, \tau, x(\tau), v_j) \, d\tau - \int_{L_{j2}} \phi(\sigma, \tau, x(\tau), v_j) \, d\tau \Big\|$$

$$+\sum_{k\in N_j}\Big\|\int\limits_{L_{k1}}\phi(\sigma,\tau,x(\tau),v_k)\,d\tau-\int\limits_{L_{k2}}\phi(\sigma,\tau,x(\tau),v_k)\,d\tau\Big\|\le 2cn(n+1)\,\|t^1-t^2\|$$

for every $j\in\{1,\dots,n\}$ such that $\tau_{j-1}<\tau_j$. Taking into account that

$$\Big\|\int\limits_0^T\phi(\sigma,\tau,x(\tau),\omega_2(t^1)(\tau))\,d\tau-\int\limits_0^T\phi(\sigma,\tau,x(\tau),\omega_2(t^2)(\tau))\,d\tau\Big\|$$

$$\le\sum_{j=1}^n\Big\|\int\limits_{\tau_{j-1}}^{\tau_j}\phi(\sigma,\tau,x(\tau),\omega_2(t^1)(\tau))\,d\tau-\int\limits_{\tau_{j-1}}^{\tau_j}\phi(\sigma,\tau,x(\tau),\omega_2(t^2)(\tau))\,d\tau\Big\|,$$

we obtain

$$\Big\|\int\limits_0^T\phi(\sigma,\tau,x(\tau),\omega_2(t^1)(\tau))\,d\tau-\int\limits_0^T\phi(\sigma,\tau,x(\tau),\omega_2(t^2)(\tau))\,d\tau\Big\|$$
$$\le 2cn^2(n+1)\,\|t^1-t^2\|.$$

Since $\sigma\in I$ was arbitrarily chosen, this result implies

$$\|F(x,\omega_2(t^1))-F(x,\omega_2(t^2))\|\le 2cn^2(n+1)\,\|t^1-t^2\|.$$

Thus the function $t\in B_+^n(r_0)\longmapsto F(x,\omega_2(t))\in C(I,\mathcal{W})$ is continuous on $B_+^n(r_0)$.

Condition (iii): Since the functions

$$(\sigma,\tau)\in I\times L\longmapsto d_3\phi(\sigma,\tau,x_0(\tau),v_j)\in(\mathcal{W},\mathcal{W})^*\quad(j\in\{1,\dots,n\})$$

and

$$(\sigma,\tau)\in I\times L\longmapsto d_3\phi(\sigma,\tau,x_0(\tau),u_0(\tau))\in(\mathcal{W},\mathcal{W})^*$$

are continuous on the compact set $I\times L$, there exists a number $c>0$ such that

$$\|d_3\phi(\sigma,\tau,x_0(\tau),v_j)-d_3\phi(\sigma,\tau,x_0(\tau),u_0(\tau))\|\le c \tag{10}$$

for all $(\sigma,\tau)\in I\times L$ and all $j\in\{1,\dots,n\}$.

Let the number $\varepsilon>0$ be arbitrarily given. Let $t\in B_+^n(r_0)$ be such that $\|t\|<\varepsilon/(cn)$. Fix any function $y\in C(I,\mathcal{W})$ for which $\|y\|\le 1$. In virtue of (10), the expression

$$g(\sigma)=\Big\|\int\limits_0^T\big[d_3\phi(\sigma,\tau,x_0(\tau),\omega_2(t)(\tau))-d_3\phi(\sigma,\tau,x_0(\tau),u_0(\tau))\big]\,y(\tau)\,d\tau\Big\|$$

satisfies for all $\sigma \in I$

$$g(\sigma) \le \sum_{j=1}^{n} \int_{L_j} \|d_3\phi(\sigma,\tau,x_0(\tau),v_j) - d_3\phi(\sigma,\tau,x_0(\tau),u_0(\tau))\| \cdot \|y(\tau)\|\, d\tau$$

$$\le c(t_1 + \ldots + t_n) \le cn\,\|t\| < \varepsilon.$$

On the other hand we have

$$\big\| \big[d_1F(x_0,\omega_2(t)) - d_1F(x_0,u_0)\big]\, y \big\| = \max\,\{g(\sigma) \;\mid \sigma \in I\}.$$

Consequently, it follows that

$$\big\| \big[d_1F(x_0,\omega_2(t)) - d_1F(x_0,u_0)\big]\, y \big\| < \varepsilon.$$

Since y was arbitrarily chosen in $C(I,\mathcal{W})$ such that $\|y\| \le 1$, we get

$$\|d_1F(x_0,\omega_2(t)) - d_1F(x_0,u_0)\| \le \varepsilon.$$

So we have shown that the function

$$t \in B^n_+(r_0) \longmapsto d_1F(x_0,\omega_2(t)) \in (C(I,\mathcal{W}),C(I,\mathcal{W}))^*$$

is continuous at 0.

Condition (iv): Let the number $\varepsilon > 0$ be arbitrarily given. Since the family

$$\big\{d_3\phi(\sigma,\tau,\cdot,v) : \mathcal{W} \to (\mathcal{W},\mathcal{W})^* \;\big|\; (\sigma,\tau,v) \in I \times I \times V\big\}$$

is uniformly equicontinuous on the set

$$W = \big\{w \in \mathcal{W} \;\big|\; \|w\| \le \|x_0\| + 1\big\},$$

there is a number $\delta > 0$ such that

$$\|d_3\phi(\sigma,\tau,w_1,v) - d_3\phi(\sigma,\tau,w_2,v)\| < \varepsilon/T \tag{11}$$

for all $w_1, w_2 \in W$ with $\|w_1 - w_2\| < \delta$ and all $(\sigma,\tau,v) \in I \times I \times V$. Now fix any $x \in C(I,\mathcal{W})$ such that $\|x - x_0\| < \min\,\{1,\delta\}$. Then we have $x(\tau), x_0(\tau) \in W$ and $\|x(\tau) - x_0(\tau)\| < \delta$ for all $\tau \in I$. Next fix a point $t \in B^n_+(r_0)$ and, for short, denote $u = \omega_2(t)$. Then (11) implies

$$\big\| \big[d_1F(x,u) - d_1F(x_0,u)\big]\, y \big\| = \max\,\Big\{\Big\| \int_0^T G(\sigma,\tau) y(\tau)\, d\tau \Big\| \;\Big|\; \sigma \in I\Big\}$$

$$\leq \max \left\{ \int_0^T \left\| G(\sigma,\tau)\, d\tau \right\| \;\middle|\; \sigma \in I \right\} \leq \varepsilon$$

for all $y \in C(I, \mathcal{W})$ satisfying $\|y\| \leq 1$, where

$$G(\sigma,\tau) = d_3\phi(\sigma,\tau,x(\tau),u(\tau)) - d_3\phi(\sigma,\tau,x_0(\tau),u(\tau)).$$

Consequently, we have

$$\|d_1F(x,u) - d_1F(x_0,u)\| \leq \varepsilon.$$

Since t was arbitrarily chosen in $B_+^n(r_0)$, the following inequality is true:

$$\sup \left\{ \|d_1F(x,\omega_2(t)) - d_1F(x_0,\omega_2(t))\| \;\middle|\; t \in B_+^n(r_0) \right\} \leq \varepsilon.$$

Thus we have

$$\lim_{x \to x_0} \sup \left\{ \|d_1F(x,\omega_2(t)) - d_1F(x_0,\omega_2(t))\| \;\middle|\; t \in B_+^n(r_0) \right\} = 0.$$

Condition (v): We fix a function $x \in C(I, \mathcal{W})$. Since the functions

$$\tau \in L \longmapsto \varphi(\tau, x(\tau), v_j) \in R^m \quad (j \in \{1,\ldots,n\})$$

and

$$\tau \in L \longmapsto \varphi(\tau, x(\tau), u_0(\tau)) \in R^m$$

are continuous on the compact set L, there exists a number $c > 0$ such that

$$\|\varphi(\tau, x(\tau), v_j)\| + \|\varphi(\tau, x(\tau), u_0(\tau))\| \leq c \tag{12}$$

for all $\tau \in L$ and all $j \in \{1,\ldots,n\}$.

Let $t^1 = (t_1^1,\ldots,t_n^1)$ and $t^2 = (t_1^2,\ldots,t_n^2)$ be points in $B_+^n(r_0)$. By using the intervals L_{j1}, L_{j2} and M_j ($j \in \{1,\ldots,n\}$) that we previously employed to show that condition (ii) is satisfied, it follows from (9) and (12) that

$$\left\| \int_{L_{j1}} \varphi(\tau, x(\tau), v_j)\, d\tau - \int_{L_{j2}} \varphi(\tau, x(\tau), v_j)\, d\tau \right\|$$

$$\leq c(2|a_{j1} - a_{j2}| + |t_j^1 - t_j^2|) \leq c(2n+1)\, \|t^1 - t^2\|$$

and that

$$\left\| \int_{M_j} \varphi(\tau, x(\tau), u_0(\tau))\, d\tau \right\| \leq c|a_{j1} - a_{j2}| \leq cn\, \|t^1 - t^2\|$$

for all $j \in \{1, \dots, n\}$. Accordingly, we have

$$\Big\| \int_{\tau_{j-1}}^{\tau_j} \varphi(\tau, x(\tau), \omega_2(t^1)(\tau))\, d\tau - \int_{\tau_{j-1}}^{\tau_j} \varphi(\tau, x(\tau), \omega_2(t^2)(\tau))\, d\tau \Big\|$$

$$\leq \Big\| \int_{M_j} \varphi(\tau, x(\tau), u_0(\tau))\, d\tau \Big\| + \Big\| \int_{L_{j1}} \varphi(\tau, x(\tau), v_j)\, d\tau - \int_{L_{j2}} \varphi(\tau, x(\tau), v_j)\, d\tau \Big\|$$

$$+ \sum_{k \in N_j} \Big\| \int_{L_{k1}} \varphi(\tau, x(\tau), v_k)\, d\tau - \int_{L_{k2}} \varphi(\tau, x(\tau), v_k)\, d\tau \Big\| \leq 2cn(n+1)\, \|t^1 - t^2\|$$

for every $j \in \{1, \dots, n\}$ such that $\tau_{j-1} < \tau_j$. Taking into account that

$$\|f(x, \omega_2(t^1)) - f(x, \omega_2(t^2))\|$$

$$= \Big\| \int_0^T \varphi(\tau, x(\tau), \omega_2(t^1)(\tau))\, d\tau - \int_0^T \varphi(\tau, x(\tau), \omega_2(t^2)(\tau))\, d\tau \Big\|$$

$$\leq \sum_{j=1}^{n} \Big\| \int_{\tau_{j-1}}^{\tau_j} \varphi(\tau, x(\tau), \omega_2(t^1)(\tau))\, d\tau - \int_{\tau_{j-1}}^{\tau_j} \varphi(\tau, x(\tau), \omega_2(t^2)(\tau))\, d\tau \Big\|,$$

we obtain

$$\|f(x, \omega_2(t^1)) - f(x, \omega_2(t^2))\| \leq 2cn^2(n+1)\, \|t^1 - t^2\|.$$

Thus the function $t \in B_+^n(r_0) \longmapsto f(x, \omega_2(t)) \in R^m$ is continuous on $B_+^n(r_0)$.

Condition (vi): Set

$$B(x_0, a) = \big\{ x \in C(I, \mathcal{W}) \;\big|\; \|x - x_0\| \leq a \big\}$$

and

$$c = \sup\, \big\{ \|d_2\varphi(\tau, x(\tau), v)\| \;\big|\; \tau \in I, x \in B(x_0, a), v \in V \big\}.$$

Let x be in $B(x_0, a)$, and let t be in $B_+^n(r_0)$. Since the function $\omega_2(t)$ takes its values in V, we have

$$\|d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau))\, y(\tau)\| \leq \|d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau))\| \cdot \|y(\tau)\| \leq c\, \|y\|$$

for all $\tau \in I$ and all $y \in C(I, \mathcal{W})$. This result implies

$$\|d_1 f(x, \omega_2(t)) y\| = \Big\| \int_0^T d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau))\, y(\tau)\, d\tau \Big\|$$

$$\leq T\, \sup \big\{ \|d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau))\, y(\tau)\| \;\big|\; \tau \in I \big\} \leq cT\, \|y\|$$

for all $y \in C(I, \mathcal{W})$. Hence, we have $\|d_1 f(x, \omega_2(t))\| \leq cT$. Since x and t were arbitrarily chosen in $B(x_0, a)$ and $B^n_+(r_0)$, respectively, it is true that

$$\sup \left\{ \|d_1 f(x, \omega_2(t))\| \;\middle|\; x \in B(x_0, a), t \in B^n_+(r_0) \right\} \leq cT.$$

Condition (vii): Since the functions

$$(\sigma, \tau) \in I \times L \longmapsto \phi(\sigma, \tau, x_0(\tau), u_0(\tau)) \in \mathcal{W}$$

and

$$(\sigma, \tau) \in I \times L \longmapsto \phi(\sigma, \tau, x_0(\tau), v_j) \in \mathcal{W} \quad (j \in \{1, \ldots, n\})$$

are continuous on the compact set $I \times L$, there exists a number $c > 0$ such that

$$\|\phi(\sigma, \tau, x_0(\tau), u_0(\tau)) - \phi(\sigma, \tau, x_0(\tau), v_j)\| \leq c$$

for all $(\sigma, \tau) \in I \times L$ and all $j \in \{1, \ldots, n\}$. Then we have

$$\begin{aligned} &\|F(x_0, \omega_2(t))\| \\ &= \max \left\{ \left\| \int_0^T \left[\phi(\sigma, \tau, x_0(\tau), u_0(\tau)) - \phi(\sigma, \tau, x_0(\tau), \omega_2(t)(\tau))\right] d\tau \right\| \;\middle|\; \sigma \in I \right\} \\ &\leq \max \left\{ \sum_{j=1}^{n} \int_{L_j} \left\|\phi(\sigma, \tau, x_0(\tau), u_0(\tau)) - \phi(\sigma, \tau, x_0(\tau), v_j)\right\| d\tau \;\middle|\; \sigma \in I \right\} \\ &\leq c(t_1 + \ldots + t_n) \leq cn \, \|t\| \end{aligned}$$

for all $t \in B^n_+(r_0)$, and thus

$$\sup \left\{ \|F(x_0, \omega_2(t))\| \,/\, \|t\| \;\middle|\; t \in B^n_+(r_0), t \neq 0 \right\} \leq cn.$$

Condition (viii): Since the functions

$$\tau \in L \longmapsto d_2\varphi(\tau, x_0(\tau), v_j) \in (\mathcal{W}, R^m)^* \quad (j \in \{1, \ldots, n\})$$

and

$$\tau \in L \longmapsto d_2\varphi(\tau, x_0(\tau), u_0(\tau)) \in (\mathcal{W}, R^m)^*$$

are continuous on the compact set L, there exists a number $c > 0$ such that

$$\|d_2\varphi(\tau, x_0(\tau), v_j) - d_2\varphi(\tau, x_0(\tau), u_0(\tau))\| \leq c$$

for all $\tau \in L$ and all $j \in \{1, \ldots, n\}$. Fix any $t \in B^n_+(r_0)$. Then we have

$$\begin{aligned} &\left\| [d_1 f(x_0, \omega_2(t)) - d_1 f(x_0, u_0)]\, y \right\| \\ &= \Big\| \int_0^T \big[d_2\varphi(\tau, x_0(\tau), \omega_2(t)(\tau)) - d_2\varphi(\tau, x_0(\tau), u_0(\tau))\big]\, y(\tau)\, d\tau \Big\| \\ &\le \int_0^T \big\| d_2\varphi(\tau, x_0(\tau), \omega_2(t)(\tau)) - d_2\varphi(\tau, x_0(\tau), u_0(\tau)) \big\|\, d\tau \\ &= \sum_{j=1}^n \int_{L_j} \big\| d_2\varphi(\tau, x_0(\tau), v_j) - d_2\varphi(\tau, x_0(\tau), u_0(\tau)) \big\|\, d\tau \\ &\le c(t_1 + \ldots + t_n) \le cn\, \|t\| \end{aligned}$$

for all $y \in C(I, \mathcal{W})$ satisfying $\|y\| \le 1$. This result implies

$$\|d_1 f(x_0, \omega_2(t)) - d_1 f(x_0, u_0)\| \le cn\, \|t\|.$$

Since t was arbitrarily chosen in $B^n_+(r_0)$, we get

$$\sup \big\{ \|d_1 f(x_0, \omega_2(t)) - d_1 f(x_0, u_0)\| \;/\; \|t\| \;\big|\; t \in B^n_+(r_0),\ t \neq 0 \big\} \le cn.$$

Condition (ix): Let the number $\varepsilon > 0$ be arbitrarily given. For each $(\tau, w) \in I \times \mathcal{W}$ we denote

$$g_j(\tau, w) = \|d_2\varphi(\tau, x_0(\tau) + w, v_j) - d_2\varphi(\tau, x_0(\tau), v_j)\| \quad (j \in \{1, \ldots, n\}),$$

and

$$g(\tau, w) = \|d_2\varphi(\tau, x_0(\tau) + w, u_0(\tau)) - d_2\varphi(\tau, x_0(\tau), u_0(\tau))\|.$$

Since the function

$$(\tau, w, v) \in I \times \mathcal{W} \times \operatorname{cl} V \longmapsto d_2\varphi(\tau, w, v) \in (\mathcal{W}, R^m)^*$$

is continuous, we can apply Lemma 2, given in [5], and conclude that

$$\lim_{s \to 0}\ \sup \big\{ g_j(\tau, w) \;\big|\; \tau \in I, w \in \mathcal{W}, \|w\| \le s \big\} = 0 \text{ for each } \ j \in \{1, \ldots, n\}$$

and that

$$\lim_{s \to 0}\ \sup \big\{ g(\tau, w) \;\big|\; \tau \in I, w \in \mathcal{W}, \|w\| \le s \big\} = 0.$$

Consequently, there is a number $\delta > 0$ such that

$$\sum_{j=1}^{n} \sup \left\{ g_j(\tau, w) \;\middle|\; \tau \in I, w \in \mathcal{W}, \|w\| \le \delta \right\} < \varepsilon/(2T)$$

and

$$\sup \left\{ g(\tau, w) \;\middle|\; \tau \in I, w \in \mathcal{W}, \|w\| \le \delta \right\} < \varepsilon/(2T).$$

Now let $x \in C(I, \mathcal{W})$ be any function satisfying $\|x - x_0\| < \delta$. Fix any $t \in B_+^n(r_0)$. Then we have

$$\begin{aligned}
&\left\| \left[d_1 f(x, \omega_2(t)) - d_1 f(x_0, \omega_2(t)) \right] y \right\| \\
&= \left\| \int_0^T \left[d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau)) - d_2\varphi(\tau, x_0(\tau), \omega_2(t)(\tau)) \right] y(\tau)\, d\tau \right\| \\
&\le \int_0^T \left\| d_2\varphi(\tau, x(\tau), \omega_2(t)(\tau)) - d_2\varphi(\tau, x_0(\tau), \omega_2(t)(\tau)) \right\| d\tau \\
&\le \sum_{j=1}^{n} \int_0^T \left\| d_2\varphi(\tau, x(\tau), v_j) - d_2\varphi(\tau, x_0(\tau), v_j) \right\| d\tau \\
&+ \int_0^T \left\| d_2\varphi(\tau, x(\tau), u_0(\tau)) - d_2\varphi(\tau, x_0(\tau), u_0(\tau)) \right\| d\tau \\
&\le T \sum_{j=1}^{n} \sup \left\{ g_j(\tau, x(\tau) - x_0(\tau)) \;\middle|\; \tau \in I \right\} + T \sup \left\{ g(\tau, x(\tau) - x_0(\tau)) \;\middle|\; \tau \in I \right\} < \varepsilon
\end{aligned}$$

for all $y \in C(I, \mathcal{W})$ satisfying $\|y\| \le 1$. This result implies

$$\|d_1 f(x, \omega_2(t)) - d_1 f(x_0, \omega_2(t))\| \le \varepsilon.$$

Since t was arbitrarily chosen in $B_+^n(r_0)$, we have

$$\sup \left\{ \|d_1 f(x, \omega_2(t)) - d_1 f(x_0, \omega_2(t))\| \;\middle|\; t \in B_+^n(r_0) \right\} \le \varepsilon.$$

Consequently, it is true that

$$\lim_{x \to x_0} \sup \left\{ \|d_1 f(x, \omega_2(t)) - d_1 f(x_0, \omega_2(t))\| \;\middle|\; t \in B_+^n(r_0) \right\} = 0.$$

Condition (x): We denote

$$P_\alpha t = t_1 \alpha^1 + \ldots + t_n \alpha^n \quad \text{for all } t = (t_1, \ldots, t_n) \in B_+^n(r_0).$$

We claim that the function

$$t \in B_+^n(r_0) \longmapsto f(x_0, \omega_2(t)) - f(x_0, u_0) - P_\alpha t \in R^m$$

satisfies

$$\lim_{t\to 0} \frac{1}{\|t\|}\big[f(x_0, \omega_2(t)) - f(x_0, u_0) - P_\alpha t\big] = 0. \tag{13}$$

To prove this, let the number $\varepsilon > 0$ be arbitrarily given. Since the functions

$$\tau \in L \longmapsto \varphi(\tau, x_0(\tau), v_j) \in R^m \quad (j \in \{1, \dots, n\})$$

and

$$\tau \in L \longmapsto \varphi(\tau, x_0(\tau), u_0(\tau)) \in R^m$$

are continuous on the compact set L, they are uniformly continuous on this set. Thus there exists a number $\delta > 0$ such that for all $j \in \{1, \dots, n\}$ and all $\tau \in L$ satisfying $|\tau - \tau_j| < \delta$ the following inequalities hold:

$$\|\varphi(\tau, x_0(\tau), v_j) - \varphi(\tau_j, x_0(\tau_j), v_j)\| < \varepsilon/(2n);$$

$$\|\varphi(\tau, x_0(\tau), u_0(\tau)) - \varphi(\tau_j, x_0(\tau_j), u_0(\tau_j))\| < \varepsilon/(2n).$$

These inequalities imply

$$\|\varphi(\tau, x_0(\tau), v_j) - \varphi(\tau, x_0(\tau), u_0(\tau)) - \alpha^j\| < \varepsilon/n \tag{14}$$

for all $j \in \{1, \dots, n\}$ and all $\tau \in L$ satisfying $|\tau - \tau_j| < \delta$.

Now let $t \in B_+^n(r_0) \setminus \{0\}$ be any point such that $\|t\| < \delta/n$. Then we have

$$\begin{aligned} &\|f(x_0, \omega_2(t)) - f(x_0, u_0) - P_\alpha t\| \\ &= \Big\| \sum_{j=1}^n \int_{L_j} \big[\varphi(\tau, x_0(\tau), v_j) - \varphi(\tau, x_0(\tau), u_0(\tau)) - \alpha^j\big]\, d\tau \Big\| \\ &\le t_1 A_1 + \ldots + t_n A_n, \end{aligned} \tag{15}$$

where

$$A_j = \max\big\{\|\varphi(\tau, x_0(\tau), v_j) - \varphi(\tau, x_0(\tau), u_0(\tau)) - \alpha^j\| \bigm| \tau \in L_j\big\}$$

for $j \in \{1, \dots, n\}$. Next take into consideration that, if $\tau \in L_j$ for some $j \in \{1, \dots, n\}$, then τ lies in L and satisfies

$$|\tau - \tau_j| \le a_j \le t_1 + \ldots + t_n \le n\,\|t\| < \delta. \tag{16}$$

Consequently, (14) implies $A_j < \varepsilon/n$ for all $j \in \{1,\dots,n\}$. In view of this result, we get from (15) that

$$\|f(x_0,\omega_2(t)) - f(x_0,u_0) - P_\alpha t\| < \varepsilon\,(t_1+\ldots+t_n)/n \le \varepsilon\,\|t\|,$$

and hence

$$\Big\|\frac{1}{\|t\|}\big[f(x_0,\omega_2(t)) - f(x_0,u_0) - P_\alpha t\big]\Big\| < \varepsilon.$$

Thus (13) is true, as claimed.

Next, we denote

$$P_\beta t = t_1\beta^1 + \ldots + t_n\beta^n \quad \text{for all } t=(t_1,\dots,t_n)\in B^n_+(r_0).$$

A reasoning similar to that used in the proof of (13) reveals that the function

$$t \in B^n_+(r_0) \longmapsto F(x_0,\omega_2(t)) + P_\beta t \in C(I,\mathcal{W})$$

satisfies

$$\lim_{t\to 0}\frac{1}{\|t\|}\big[F(x_0,\omega_2(t)) + P_\beta t\big] = 0. \tag{17}$$

Indeed, let the number $\varepsilon > 0$ be arbitrarily given. Since the functions

$$(\sigma,\tau)\in I\times L \longmapsto \phi(\sigma,\tau,x_0(\tau),v_j)\in\mathcal{W} \quad (j\in\{1,\dots,n\})$$

and

$$(\sigma,\tau)\in I\times L \longmapsto \phi(\sigma,\tau,x_0(\tau),u_0(\tau))\in\mathcal{W}$$

are continuous on the compact set $I\times L$, they are uniformly continuous on this set. Thus there exists a number $\delta>0$ such that for all $j\in\{1,\dots,n\}$, all $\sigma\in I$, and all $\tau\in L$ satisfying $|\tau-\tau_j|<\delta$ the following inequalities hold:

$$\|\phi(\sigma,\tau,x_0(\tau),v_j) - \phi(\sigma,\tau_j,x_0(\tau_j),v_j)\| < \varepsilon/(2n);$$

$$\|\phi(\sigma,\tau,x_0(\tau),u_0(\tau)) - \phi(\sigma,\tau_j,x_0(\tau_j),u_0(\tau_j))\| < \varepsilon/(2n).$$

These inequalities imply

$$\|\phi(\sigma,\tau,x_0(\tau),u_0(\tau)) - \phi(\sigma,\tau,x_0(\tau),v_j) + \beta^j(\sigma)\| < \varepsilon/n \tag{18}$$

for all $j\in\{1,\dots,n\}$, all $\sigma\in I$, and all $\tau\in L$ satisfying $|\tau-\tau_j|<\delta$.

Now, let $t \in B^n_+(r_0) \setminus \{0\}$ be any point such that $\|t\| < \delta/n$. Then we have

$$\|F(x_0, \omega_2(t))(\sigma) + (P_\beta t)(\sigma)\|$$

$$= \Big\| \sum_{j=1}^{n} \int_{L_j} \big[\phi(\sigma, \tau, x_0(\tau), u_0(\tau)) - \phi(\sigma, \tau, x_0(\tau), v_j) + \beta^j(\sigma)\big]\, d\tau \Big\|$$

$$\leq t_1 B_1(\sigma) + \ldots + t_n B_n(\sigma) \tag{19}$$

for every $\sigma \in I$, where

$$B_j(\sigma) = \max\, \big\{ \|\phi(\sigma, \tau, x_0(\tau), u_0(\tau)) - \phi(\sigma, \tau, x_0(\tau), v_j) + \beta^j(\sigma)\| \;\big|\; \tau \in L_j \big\}$$

for $j \in \{1, \ldots, n\}$. As before, now take into consideration that if $\tau \in L_j$ for some index $j \in \{1, \ldots, n\}$, then τ lies in L and satisfies (16). Consequently, (18) implies

$$B_j(\sigma) < \varepsilon/n \;\text{ for all } j \in \{1, \ldots, n\} \text{ and all } \sigma \in I.$$

In view of this result, we get from (19) that

$$\|F(x_0, \omega_2(t))(\sigma) + (P_\beta t)(\sigma)\| < \varepsilon\,(t_1 + \ldots + t_n)/n \leq \varepsilon\, \|t\|$$

for all $\sigma \in I$. From this it follows that

$$\|F(x_0, \omega_2(t)) + P_\beta t\| < \varepsilon\, \|t\|,$$

and hence

$$\Big\| \frac{1}{\|t\|} \big[F(x_0, \omega_2(t)) + P_\beta t\big] \Big\| < \varepsilon.$$

Thus (17) is true, as claimed.

From (17) we obtain

$$\lim_{t \to 0} \frac{1}{\|t\|} \big[\omega_0(t) + \sum_{j=1}^{n} t_j A^{-1} \beta^j\big] = 0, \tag{20}$$

where

$$\omega_0(t) = A^{-1} F(x_0, \omega_2(t)) \;\text{ for all } t \in B^n_+(r_0).$$

Obviously, (20) yields

$$\lim_{t \to 0} \frac{1}{\|t\|} \big[d_1 f(x_0, u_0)\omega_0(t) + \sum_{j=1}^{n} t_j\, d_1 f(x_0, u_0) \circ A^{-1} \beta^j\big] = 0. \tag{21}$$

Finally, note that the point Pt defined by

$$Pt = t_1 d^1 + \ldots + t_n d^n \;\text{ for all } t = (t_1, \ldots, t_n) \in R^n,$$

in our case can be written under the form

$$Pt = P_\alpha t + \sum_{j=1}^{n} t_j \, d_1 f(x_0, u_0) \circ A^{-1}\beta^j.$$

Accordingly, we conclude from (13) and (21) that

$$\lim_{t\to 0} \frac{1}{\|t\|}\big[f(x_0, \omega_2(t)) - f(x_0, u_0) - Pt - d_1 f(x_0, u_0)\omega_0(t)\big] = 0.$$

Summing up, all the hypotheses of Theorem 1 are fulfilled. By applying this theorem, it follows that there is a vector

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) \in K_1^* \times K_2^* \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$$

satisfying the inequality

$$\langle d(\tau, v), \lambda^*\rangle \leq 0 \quad \text{whenever } (\tau, v) \in I_0 \times V \tag{22}$$

as well as the equality (3).

From (22) we obtain (2). Indeed, to see this, we fix any $\tau \in I_0$. Since we have

$$A^{-1}\phi(\cdot, \tau, x_0(\tau), v) = h(\cdot; \tau, v) \quad \text{for all } v \in V,$$

it follows that

$$d_1 f(x_0, u_0) \circ A^{-1}\phi(\cdot, \tau, x_0(\tau), v) = \int_0^T d_2\varphi(\sigma, x_0(\sigma), u_0(\sigma))h(\sigma; \tau, v)\, d\sigma.$$

In view of this result, $H(\tau, \cdot)$ can be rewritten as follows:

$$H(\tau, v) = \langle \varphi(\tau, x_0(\tau), v) + d_1 f(x_0, u_0) \circ A^{-1}\phi(\cdot, \tau, x_0(\tau), v), \lambda^*\rangle$$

for every $v \in V$. Therefore we have

$$H(\tau, v) - H(\tau, u_0(\tau)) = \langle d(\tau, v), \lambda^*\rangle \quad \text{for all } v \in V.$$

In virtue of (22) it follows that

$$H(\tau, v) \leq H(\tau, u_0(\tau)) \quad \text{for all } v \in V.$$

Consequently, the equality (2) holds, which completes the proof.

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ADJOINTS OF LIPSCHITZ MAPPINGS

ŞTEFAN COBZAŞ

Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

Abstract. The aim of this paper is to show that the Lipschitz adjoint of a Lipschitz mapping F, defined by I. Sawashima, Lecture Notes Ec. Math. Syst., Vol. 419, Springer Verlag, Berlin 1975, pp. 247-259, corresponds in a canonical way to the adjoint of a linear operator associated to F.

1. **Introduction**

Let X be a metric space with a distinguished point e (a fixed point in X which is taken to be the zero element if X is a normed space). A metric space X with a distinguished point e is called also a pointed metric space. For a Banach space Y denote by $\mathrm{Lip}_0(X,Y)$ the space of all Lipschitz mappings $F : X \to Y$ vanishing at e. Equipped with the norm

$$L(F) = \sup\{\|F(x_1) - F(x_2)\|/\|x_1 - x_2\| : x_1, x_2 \in X,\ x_1 \neq x_2\}$$

$\mathrm{Lip}_0(X,Y)$ becomes a Banach space. For $Y = \mathbb{R}$ one puts $\mathrm{Lip}_0(X) = \mathrm{Lip}_0(X,\mathbb{R})$. It was shown by Arens and Eels [5] (see also [19]) that $\mathrm{Lip}_0(X)$ is even a dual Banach space, i.e. there exists a Banach space Z such that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to Z^*.

Banach spaces of Lipschitz functions, called also Lipschitz duals, were used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case. For instance, Schnatz [18] used them to prove duality and characterization results in best approximation problems in a linear metric space X. In this case one could happen that the dual X^* of X be trivial, $X^* = \{0\}$, so that the methods of linear functional analysis doesn't work. Sawashima [17] defined

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Lipschitz duals of Lipschitz mappings and proved some nonlinear ergodic theorems (see also [14] and [16]).

Lipschitz mappings were also considered in the attempt to develop a nonlinear spectral theory, see [2], [4], [15]. The paper [8] contains a survey on extension results for Lipschitz mappings and their connections to some best approximation problems in spaces of Lipschitz functions.

The aim of this paper is to show that, for a normed space X, the realization of $\mathrm{Lip}_0(X)$ as a dual space can be pushed a little further to obtain a correspondence between Lipschitz duals of Lipschitz mappings and the adjoints of some linear operators. This fact allows to prove some results for Lipschitz mappings by reducing them to the linear case.

2. The Lipschitz adjoint of a Lipschitz mapping

We shall present first the construction of Arens and Eels [5] (see also [19, p.38]) of the space for which $\mathrm{Lip}_0(X)$ is the dual space. Remark that another, less explicit, realization of $\mathrm{Lip}_0(X)$ as a dual space was given by de Leeuw [10] (see also [19, p. 33]).

Let (X, ρ) be metric space. A *molecule* on X is a function $m : X \to \mathbb{R}$ with finite support $\sigma(m) = \{x \in X : m(x) \neq 0\}$, and such that $\sum_{x \in X} m(x) = 0$. Denote by $M(X)$ the space of molecules on X. For $x, y \in X$ put $m_{x,y} = h_x - h_y$, where h_x denotes the characteristic function of the set $\{x\}$. One can show that every $m \in M(X)$ can be written, in at least one way, in the form $m = \sum_{i=1}^{n} a_i m_{x_i,y_i}$. Put

$$\|m\|_{AE} = \inf\{\sum_{i=1}^{n} |a_i|\rho(x_i, y_i) : m = \sum_{i=1}^{n} a_i m_{x_i,y_i}\}.$$

It follows that $\| \ \|_{AE}$ is a norm on the vector space $M(X)$. Denote by $AE(X)$ the completion of the normed space $(M(X), \| \ \|_{AE})$. The application $i_X : X \to AE(X)$ defined by

$$i_X(x) = m_{x,e} \tag{1}$$

is an isometric embedding of X into $AE(X)$. Define $S : AE(X)^* \to \mathrm{Lip}_0(X)$ by

$$(S\varphi)(x) = \varphi(m_{x,e}), \qquad \varphi \in AE(X)^*. \tag{2}$$

It follows that S is a nonexpansive linear mapping

$$L(S\varphi) \leq \|\varphi\|, \qquad \varphi \in AE(X)^*.$$

Define now an application $R : \mathrm{Lip}_0(X) \to AE(X)^*$ in the following way. For $f \in \mathrm{Lip}_0(X)$ let first

$$(Rf)(m) = \sum_x m(x)f(x), \quad m \in M(X). \tag{3}$$

Since

$$|(Rf)(m)| \leq L(f)\|m\|_{AE}$$

it follows that Rf is a continuous linear functional on $M(X)$, which uniquely extends to a continuous linear functional on the completion $AE(X)$ of $M(X)$, denoted by the same symbol Rf. Therefore $Rf \in AE(X)^*$ and

$$\|Rf\| \leq L(f), \quad f \in \mathrm{Lip}_0(X).$$

Straightforward calculations show that R and S are inverses, so that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to $AE(X)^*$.

The Banach space $AE(X)$ has some remarkable properties, from which we mention the following one, where the application i_X is defined by (1).

Theorem 1. [19, Theorem 2.2.4] *Let X be a pointed metric space and Y a Banach space. For every $F \in \mathrm{Lip}_0(X,Y)$ there exists a unique continuous linear map $\Psi(F) : AE(X) \to Y$ such that $\Psi(F) \circ i_X = F$. Furthermore $\|\Psi(F)\| = L(F)$.*

From now on we shall suppose that X and Y are real normed spaces, so that the distinguished points are their null elements. Sawashima [17] defined the Lipschitz adjoint (or dual) $F^{\#} : \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$ of a Lipschitz map $F \in \mathrm{Lip}_0(X,Y)$ by the formula

$$F^{\#}g = g \circ F, \quad g \in \mathrm{Lip}_0(Y).$$

He showed that $F^{\#}$ is a continuous linear operator and that

$$\|F^{\#}\| = L(F) = \|F^{\#}|_{Y^*}\|.$$

We shall show that $F^{\#}$ corresponds in a canonical way to the usual adjoint of the linear operator attached to F by Theorem 1.

Let $F \in \mathrm{Lip}_0(X,Y)$ and let i_X, i_Y be the isometric embeddings of X, Y into $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}_0(Y)$, respectively (see (1)). Let $\Psi(F) : AE(X) \to Y$ be the bounded linear operator attached to F by Theorem 1, and let

$$\Phi = i_Y \circ \Psi.$$

Let also S_1, R_1, and S_2, R_2 be the linear isometries between the spaces $\mathrm{Lip}_0(X)$ and $AE(X)^*$, and $\mathrm{Lip}_0(X)$ and $AE(X)^*$, respectively (see the formulae (2) and (3)).

Theorem 2. *We have*

$$F^{\#} = S_1 \circ \Phi(F)^* \circ R_2 \quad \textit{or, equivalently,} \quad \Phi(F)^* = R_1 \circ F^{\#} \circ S_2$$

i.e. the following diagrams are commutative:

$$\begin{array}{ccc} AE(Y)^* & \xrightarrow{\Phi(F)^*} & AE(X)^* \\ {\scriptstyle R_2}\big\uparrow & & {\scriptstyle S_1}\big\downarrow \\ \mathrm{Lip}_0(Y) & \xrightarrow{F^{\#}} & \mathrm{Lip}_0(X) \end{array} \quad \textit{or, equivalently,} \quad \begin{array}{ccc} AE(Y)^* & \xrightarrow{\Phi(F)^*} & AE(X)^* \\ {\scriptstyle S_2}\big\downarrow & & {\scriptstyle R_1}\big\uparrow \\ \mathrm{Lip}_0(Y) & \xrightarrow{F^{\#}} & \mathrm{Lip}_0(X) \end{array}$$

Proof.

We have

$$\Phi(m_{x,0}) = i_Y(\Psi(F)(m_{x,0}) = i_Y(F(x)) = m_{F(x),0}. \tag{4}$$

Put

$$T = S_1 \circ \Phi(F)^* \circ R_2,$$

Therefore

$$\begin{aligned} &(S_1\varphi)(x) = \varphi(M_{x,0}), \quad x \in X,\ \varphi \in AE(X)^*. \\ &\Phi(F)^*(\psi) = \psi \circ \Phi(F), \quad \psi \in AE(Y)^*, \\ &(R_2 g)(m) = \sum_{y\in Y} m(y)g(y), \quad g \in \mathrm{Lip}_0(Y),\ m \in M(Y). \end{aligned}$$

Taking into account these formulae, the definitions of the operators R and S, and formula (4), we obtain successively:

$$\begin{aligned} (Tg)(x) &= (S_1 \circ \Phi(F)^* \circ R_2)(g)(x) = S_1(\Phi(F)^*(R_2 g))(x) \\ &= S_1((R_2 g) \circ \Phi(F))(x) = \\ &= ((R_2 g) \circ \Phi(F))(m_{x,0}) = \\ &= (R_2 g)(m_{x,0}) = g(F(x)) = (g \circ F)(x) = F^{\#}(g)(x). \end{aligned}$$

Theorem 2 is proved. □

We conclude by some open questions. Schauder theorem on the compactness of the adjoint of a compact linear operator between two Banach spaces is well known:

If $A : X \to Y$ is linear and compact then its adjoint $A^* : Y^* \to X^*$ is also compact. In connection with this property we raise the following problems.

Problem 1. Which conditions on a Lipschitz operator $F \in \mathrm{Lip}_0(X, Y)$ entail the compactness of the associated operator $\Phi(F) : AE(X) \to AE(Y)$?

Problem 2. Prove a Schauder type theorem for the Lipschitz adjoint $F^{\#}$ of a Lipschitz operator $F \in \mathrm{Lip}_0(X, Y)$.

Yamamuro [20] defined another kind of adjoint of a Fréchet differentiable mapping and proved a Schauder type theorems for such adjoints. Yamamuro defined the adjoint of a Fréchet differentiable mapping F of a Hilbert space X into itself as a mapping $G : X \to X$ such that $G' = (F')^*$, where A^* denotes the Hilbert adjoint of a continuous linear operator A on X. A thorough study of compactness for nonlinear mappings and their adjoints is done by Batt [6], but his results do not cover the Lipschitz case considered here. Lipschitz duals and duals of Lipschitz mappings were considered in [14, 16] too.

A natural hypothesis for Problem 2 would be the compactness of F, meaning that it sends bounded sets into relatively compact sets. To work with compact sets in $\mathrm{Lip}_0(X, Y)$ we need compactness criteria in spaces of Lipschitz functions. As pointed out J. Appel [1], there are no such criteria, and it turned out that some existing ones were false (e.g. those in [11] or [12]). In this context the following problem is apparently still open:

Problem 3. Find compactness criteria in the space $\mathrm{Lip}_0(X, Y)$.

In [9] we have proved a compactness criterium, but only for families of continuous Fréchet differentiable Lipschitz operators defined on an open subset of a Banach space X and with values in another Banach space Y.

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THE APPROXIMATION OF THE EQUATION'S SOLUTION IN LINEAR NORMED SPACES USING APPROXIMANT SEQUENCES (II)

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Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

Abstract. Considering a function between two linear normed spaces and a arbitrary approximant sequence, we will study the conditions for the convergence of this sequence towards one solution of the equation generated by this function. The speed of convergence should be of a big enough order, characterized by a number $p \in \mathbb{N}$.

1. Introduction

One of the most often used methods for the approximation of an equation's solutions is that of constructing a sequence that is convergent to that solution. In order to do that it is necessary to know this solution and maybe also its quality of being the only one existing near a determined point.

A sequence having the quality described above will be called an approximant sequence.

From the practical point of view, in order to make an approximation of the solution with an error that doesn't exceed the maximum admissible value, it is important not to use too many terms of the approximant sequence, that is to obtain a good speed of approximation.

In order to make the concepts above clear, let us consider X and Y two normed linear spaces, their norm $\|\cdot\|_X$ and respectively $\|\cdot\|_Y$ a set $D \subseteq X$, a function $f : D \longrightarrow Y$, θ_Y, the null element of the space Y and, using these elements, the equation:

$$f(x) = \theta_Y \tag{1}$$

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To clarify these notions, we will have:

Definition 1.1. *In addition to the data above, let us also consider* $p \in \mathbb{N}$, *and* $(x_n)_{n\in\mathbb{N}} \subseteq D$. *We say that the sequence is an approximant sequence of the order* p *of a solution of the equation* (1), *if there exist* $\alpha, \beta \geq 0$ *so that for any* $n \in \mathbb{N}$ *we have:*

$$\begin{aligned} &\|f(x_{n+1})\|_Y \leq \alpha \|f(x_n)\|_Y^p ; \\ &\|x_{n+1} - x_n\|_X \leq \beta \|f(x_n)\|_Y . \end{aligned} \tag{2}$$

As we showed in papers [3] and [4], if $(x_n)_{n\in\mathbb{N}}$ is an approximant sequence of the order p, $p \geq 2$; X is a Banach space; $f : D \to Y$ is continuous, and the constants α and β that verify **Definition 1.1** are chosen so that:

$$\begin{aligned} &\rho_0 = \alpha^{\frac{1}{p-1}} \parallel f(x_0) \parallel_Y, \\ &S(x_0, \delta) = \{x \in X / \parallel x - x_0 \parallel_X \leq \delta\} \subseteq D, \end{aligned} \tag{3}$$

with:

$$\delta = \frac{\beta\alpha^{\frac{1}{p-1}}}{1 - \rho_0^{p-1}},$$

then the approximant sequence is convergent towards the element x^* which, together with all the terms of the sequence $(x_n)_{n\in\mathbb{N}}$ is placed in the ball $S(x_0, \delta)$ and x^* is a solution of the equation (1). For any $n \in \mathbb{N}$ the following inequalities take place:

$$\begin{aligned} &\parallel x_{n+1} - x_n \parallel_X \leq \beta\alpha^{\frac{1}{p-1}} \rho_0^{p^n} \\ &\parallel x^* - x_n \parallel \leq \frac{\beta\alpha^{\frac{1}{p-1}} \rho_0^{p^n}}{1 - \rho_0^{p^n\,(p-1)}}. \end{aligned} \tag{4}$$

These inequalities justify the fact of calling it an approximant sequence of the order p ; the last inequality will also give an evaluation of a superior margin of the error through which x_n approximates x^*.

Above x_0 is the initial element of the sequence, the starting element of the approximation proceeding.

The convergence or the non-convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ as well as the convergence speed, materialized through the number p, depend on the fact of correctly choosing x_0.

In order to verify the inequalities (4) as well as the affirmations preceding them we have to make the inequalities (2) true. But this often proves to be difficult, and this is the reason for which we will try to replace them with more practical conditions. Nevertheless we will consider that the function $f : D \to Y$ admits Fréchet derivatives up to the order p included.

As a series of iterative methods known in practice use the inverse of the Fréchet derivative of the first order of the mapping $[f\ '(x_n)]^{-1}$, an unpractical condition, as the existence of this mapping implies solving the linear equation $f'(x_n)h = q$; $h \in X$, $q \in Y$, we will try to eliminate the conditions about the inverse of the Fréchet derivative from the hypothesis, but we will try to demonstrate this existence.

From the results that have inspired this work of research we will mention primarily the well-known theorem of **L. V. Kantorovich** for the case when the approximant sequence $(x_n)_{n\in N}$ is generated by the **Newton - Kantorovich** method [5], [6]. In this case the existence of the mapping $[f'(x)]^{-1} \in (Y, X)^*$ is supposed only for $x = x_0$, as this is the initial point of the iterative method . In what the convergence of the same method is concerned, we also mention the result obtained by **Misovski, I. P.** , [7], where from a certain point of view the conditions of the convergence are simpler, but the existence of the mapping $[f\ '(x)]^{-1}$ and of a constant $M > 0$ satisfying the inequality $\left\|[f'(x)]^{-1}\right\| \leq M$ for any x - an element of a certain ball centered in the initial element x_0 - is imposed. Then **Păvăloiu, I.**, in [8] , [9], generalizes these results for the convergence of a sequence generated by the relation of recurrence:

$$x_{n+1} = Q(x_n) \tag{5}$$

where $Q : X \to X$ verifies certain conditions. In the result obtained by **Păvăloiu, I.**,. **Misovski's** condition mentioned above does not appear explicitly, but the use of the result in concrete cases makes it necessary. Thus this general result can be applied in the case of the **Newton-Kantorovich** method to obtain **Misovski's** result and in the case of **Chebischev's** method, obtaining a corresponding result.

By changing one of the conditions our result is more easily applicable than that of **Păvăloiu, I.** for concrete methods. We also succeed to show that for any

$n \in \mathbb{N}$, $[f'(x_n)]^{-1}$ exists and these mappings taken for any $n \in \mathbb{N}$ form an equally margined set.

2. Main results

We will proceed in the same way as in our papers [1] , [2].

Let us now note by $(X^p, Y)^*$ the set of p -linear and continuous mappings defined on

$$X^p = \underbrace{X \times \cdots \times X}_{p\ times}$$

(the p times Cartesian product), taking values in Y.

The fact that the mapping $f^{\ (p)} : D \to (X^p, Y)^*$ verifies **Lipschitz's** condition is resumed to the existence of the constant $L > 0$, so that for any $x, y \in D$ we can have:

$$\left\| f^{(p)}(x) - f^{(p)}(y) \right\| \leq L \left\| x - y \right\|_X \tag{6}$$

so that L will be called **Lipschitz's** constant.

From the verification of such a condition with the constant $L > 0$ we can easily deduce that for any $x, y \in D$ the following inequality takes place:

$$\left\| f(x) - f(y) - \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(y)(x-y)^i \right\|_Y \leq \frac{L}{(p+1)!} \parallel x - y \parallel_X^{p+1} . \tag{7}$$

Then if we take $x_0 \in D$ and $\delta > 0$ so that:

$$S(x_0, \delta) = \{ x \in X / \left\| x - x_0 \right\| \leq \delta \} \subseteq D$$

and we define the numbers $L_0, ..., L_p > 0$ through:

$$L_k = \left\| f^{(k)}(x_0) \right\| + L_{k+1}\delta; \quad k = 0, 1, .., p \tag{8}$$

with $L_{p+1} = L$, then for any $x \in S(x_0, \delta)$ we have:

$$\left\| f^{(k)}(x) \right\| \leq L_{k+1}\delta \tag{9}$$

for any $k \in \{0, 1, ..., p\}$ and for any $x, y \in S(x_0, \delta)$ we have:

$$\left\| f^{(k-1)}(x) - f^{(k-1)}(y) \right\| \leq L_k \left\| x - y \right\|_X ,$$

for any $k \in \{1, 2, ...p + 1\}$.

Under the conditions mentioned above, the following takes place:

Theorem 2.1. *In addition to the data above we consider* $p \in N$, $\delta > 0$, $(x_n)_{n\in N} \subseteq D$.

If:

i) X *is a Banach space and* $S(x_0,\delta) \subseteq D$, $S(x_0,\delta)$ *representing the ball with the center* x_0 *and radius* δ;

ii) *the function* $f : D \to Y$ *admits Fréchet derivatives up to the order* p *including it , and, for* $f^{(p)} : D \to (X^p, Y)^*$ *the number* $L > 0$ *exists so that for any* $x, y \in D$ *the following inequality* (6) *is verifies:*

iii) $a, b \geq 0$ *exist so that for any* $n \in \mathbb{N}$ *we have the inequalities:*

$$\left\| f(x_n) + \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_n)(x_{n+1} - x_n)^i \right\|_Y \leq a \left\| f\left(x_n\right) \right\|_Y^{p+1} \tag{10}$$

and:

$$\left\| f'\left(x_n\right)\left(x_{n+1} - x_n\right) \right\|_Y \leq b \left\| f\left(x_n\right) \right\|_Y ; \tag{11}$$

iv) *the mapping* $f'(x_0) \in (X,Y)^*$ *is invertible;*

v) *if we note:*

$$\begin{gathered} \rho_0 = \left\| f\left(x_0\right) \right\|_Y , \qquad B_0 = \left\| \left[f'\left(x_0\right) \right]^{-1} \right\| , \qquad h_0 = bL_2 B_0^2 \rho_0 \\ M = \left\| \left[f'\left(x_0\right) \right]^{-1} \right\| e^{1+2^{-2p=3}} , \quad \alpha = a + L \frac{(bM)^{p+1}}{(p+1)!} \end{gathered} \tag{12}$$

the following inequalities are verified:

$$h_0 \leq \frac{1}{2}, \quad \alpha^{\frac{1}{p}} \rho_0 < \frac{1}{4}, \quad \delta \geq \frac{bM\rho_o}{1 - \alpha\rho_0^p} \tag{13}$$

then:

j) $x_n \in S(x_0,\delta)$, $[f'(x_n)]^{-1}$ *exists and* $\left\| [f'(x_n)]^{-1} \right\| \geq M$ *for any* $n \in \mathbb{N}$;

jj) *the equation* (1) *admits a solution* $x^* \in S(x_0,\delta)$;

jjj) *the sequence* $(x_n)_{n\in N}$ *is an approximant sequence of the order* $p+1$ *of this solution of the equation* (1);

jv) *the following estimates hold:*

$$\max \left\{ \left\| f(x_n) \right\|_Y , \frac{1}{Mb} \left\| x_{n+1} - x_n \right\|_X \right\} \leq \alpha^{\frac{(p+1)^n - 1}{p}} \left\| f(x_0) \right\|_Y^{(p+1)^n} \tag{14}$$

and:

$$\|x^* - x_n\|_X \leq \frac{bM\alpha^{-\frac{1}{p}}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}}{1-(\alpha\rho_0^p)^{(p+1)^n}} \tag{15}$$

for any $n \in \mathbb{N}$.

Proof. From the invertibility of the mapping $f'(x_0) \in (X,Y)^*$ we clearly deduce that:

$$\|f'(x_0)\|,\ \left\|[f'(x_0)]^{-1}\right\| > 0.$$

Let the sequences $(\rho_n)_{n\in N}$, $(B_n)_{n\in N}$ and $(h_n)_{n\in N}$ be so that:

$$\rho_0 = \|f(x_0)\|_Y,\ B_0 = \left\|[f'(x_0)]^{-1}\right\|$$

and for any $n \in \mathbb{N}$, we have:

$$h_n = bL_2B_n^2\rho_n,\ \rho_{n+1} = \alpha\rho_n^{p+1},\ B_{n+1} = \frac{B_n}{1-h_n}.$$

We will show that for any $n \in \mathbb{N}$ the following statements are true:

a) $x \in S(x_0, \delta)$,

b) $[f'(x_n)]^{-1} \in (Y,X)^*$ exists, and $\left\|[f'(x_n)]^{-1}\right\| \leq B_n$,

c) $\|f(x_n)\|_Y \leq \rho_n = \alpha^{-\frac{1}{p}}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}$, (16)

d) $h_n \leq \min\left\{\frac{1}{2},\ \beta^{-\frac{1}{p}}(\beta h_0)^{(p+1)^n}\right\}$, where $\beta = \frac{4}{(4h_0)^p}$,

e) $B_0 \leq B_n \leq M$.

Using mathematical induction we notice that for $n = 0$ the statements **a**)$-$**e**) are evidently true from the hypotheses of the theorem with the notations we have introduced.

Let us suppose that for any $n \leq k$ the assertions **a**) $-$ **e**) are true, and let us demonstrate them for $n = k+1$.

a)We notice that for any $n \in N,\ n \leq k$ we have:

$$\|x_{n+1} - x_n\|_X = \left\|[f'(x_n)]^{-1}f'(x_n)(x_{n+1}-x_n)\right\|_X \leq$$

$$\leq \left\|[f'(x_n)]^{-1}\right\| . \|f'(x_n)(x_{n+1}-x_n)\|_Y \leq Mb\|f(x_n)\|_Y \leq Mb\rho_n.$$

So:

$$\|x_{n+1}-x_n\|_X \leq Mb\alpha^{-\frac{1}{p}}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}, \tag{17}$$

from where:

$$\|x_{n+1}-x_0\|_X \leq \sum_{n=0}^{k}\|x_{n+1}-x_n\|_X \leq Mb\alpha^{-\frac{1}{p}}\sum_{n=0}^{k}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n-1}$$

From $p \geq 1$ we deduce that $(p+1)^n - 1 > np$ for any $n \in \mathbb{N},\ n > 0$ and as $\rho_0 < 1$ we deduce that:

$$\sum_{n=0}^{k}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n-1} < \sum_{n=0}^{k}(\alpha\rho_0^p)^n = \frac{1-(\alpha\rho_0^p)^{k+1}}{1-\alpha\rho_0^p} < \frac{1}{1-\alpha\rho_0^p}.$$

So:

$$\|x_{n+1}-x_0\|_X \leq Mb\frac{\rho_0}{1-\alpha\rho_0^p} \leq \delta$$

from where it results immediately that $x_{n+1} \in S(x_0,\delta)$.

b)Let:

$$H_k = \left[f'(x_k)\right]^{-1}\left(f'(x_k)-f'(x_{k+1})\right) \in (X,X)^*,$$

its existence and its belonging to $(X,X)^*$ are guaranteed by the hypothesis of the induction. It is obvious that:

$$\|H_k\| \leq \left\|[f'(x_k)]^{-1}\right\| \cdot \|f'(x_k)-f'(x_{k+1})\| \leq B_k L_2 \|x_{k+1}-x_k\|_X$$

But:

$$\|x_{k+1}-x_k\|_X \leq \left\|[f'(x_k)]^{-1}\right\| \cdot \|f'(x_k)(x_{k+1}-x_k)\|_Y \leq bB_k\|f(x_k)\|_Y \leq$$

$$\leq bB_k\rho_k,$$

from where:

$$\|H_k\| \leq bL_2B_k^2\rho_k = h_k \leq \frac{1}{2} < 1$$

and according to the well known **Banach's theorem** we deduce that:

$$(I_k - H_k)^{-1} \in (X,X)^*$$

and:

$$\left\|(I_k-H_k)^{-1}\right\| \leq \frac{1}{1-\|H_k\|} \leq \frac{1}{1-h_k}$$

(here $I_X : X \to X$ represents the identical mapping of the space X).

Obviously:

$$I_X - H_k = [f'(x_k)]^{-1} f'(x_{k+1}),$$

from where:

$$f'(x_{k+1}) = f'(x_k)(I_k - H_k).$$

The hypothesis of the induction guarantees the existence of the mapping $[f'(x_k)]^{-1} \in (Y,X)^*$, so, from the above, the mapping $(I_k - H_k)^{-1}$ will exist, so the mapping $[f'(x_{k+1})]^{-1} = (I_k - H_k)^{-1}\ [f'(x_k)]^{-1}$ will exist as well, and:

$$\left\| [f'(x_{k+1})]^{-1} \right\| \leq \left\| [f'(x_k)]^{-1} \right\| \cdot \left\| (I_X - H_k)^{-1} \right\| \leq \frac{B_k}{1-h_k} = B_{k+1}.$$

c) Clearly:

$$\|f(x_{k+1})\|_Y \leq \left\| f(x_{k+1}) - f(x_k) - \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_k)(x_{k+1}-x_k)^i \right\|_Y +$$

$$+ \left\| f(x_k) + \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_k)(x_{k+1}-x_k)^i \right\|_Y .$$

Because of the fact that $x_k, x_{k+1} \in S(x_0, \delta) \subseteq D$, of the hypothesis **ii**) and using the remark that precedes the text of the theorem we deduce that:

$$\left\| f(x_{k+1}) - f(x_k) - \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_k)(x_{k+1}-x_k)^i \right\|_Y \leq \frac{L}{(p+1)!} \|x_{k+1} - x_k\|_X^{p+1},$$

also using the first inequality from the hypothesis **iii**) we deduce that:

$$\|f(x_{k+1})\|_Y \leq \frac{L}{(p+1)!} \|x_{k+1} - x_k\|_X^{p+1} + a \|f(x_k)\|_Y^{p+1} \leq$$

$$\leq \left[a + \frac{L(Mb)^{p+1}}{(p+1)!} \right] \|f(x_k)\|_X^{p+1} \leq \alpha \rho_k^{p+1} = \rho_{k+1}.$$

As $\rho_{k+1} = \alpha \rho_k^{p+1}$ and $\rho_k = \alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}} \rho_0 \right)^{(p+1)^k}$ we deduce that:

$$\alpha^{\frac{1}{p}} \rho_{k+1} = \left(\alpha^{\frac{1}{p}} \rho_k \right)^{p+1} = \left(\alpha^{\frac{1}{p}} \rho_k \right)^{(p+1)^{k+1}},$$

so:

$$\rho_{k+1} = \alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}} \rho_k \right)^{(p+1)^{k+1}}.$$

d) We have the equalities:

$$h_{k+1} = L_2 b B_{k+1}^2 \rho_{k+1} = L_2 b \alpha \rho_k^{p+1} \left(\frac{B_k}{1-h_k} \right)^2 = \alpha h_k \frac{\rho_k^p}{(1-h_k)^2}.$$

From $h_k \le \frac{1}{2}$, we deduce that:

$$\frac{h_k}{(1-h_k)^2} \le 2$$

so:

$$h_{k+1} \le 2\alpha\rho_k^p.$$

We have:

$$\alpha^{\frac{1}{p}}\rho_0 < 1 \Rightarrow \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^k} < \alpha^{\frac{1}{p}}\rho_0 \Rightarrow \rho_k < \rho_0 \Rightarrow h_{k+1} \le 2\alpha\rho_0^p \Rightarrow$$

$$\Rightarrow h_{k+1} \le 2\left(\alpha^{\frac{1}{p}}\rho_0\right)^p < \frac{1}{2^{2p-1}} \le \frac{1}{2}.$$

Meanwhile:

$$h_{k+1} = \frac{\alpha h_k}{(1-h_k)^2} \cdot \frac{h_k^p}{(bL_2B_k^2)^p} = \frac{\alpha}{(bL_2)^p} \cdot \frac{1}{B_k^{2p}} \cdot \frac{h_k^{p+1}}{(1-h_k)^2}$$

From $B_k \ge B_0$ and:

$$\frac{1}{(1-h_k)^2} \le 4$$

we deduce that:

$$h_{k+1} \le \frac{4\alpha h_k^{p+1}}{(bL_2)^p\ B_0^{2p}} < \frac{4h_k^{p+1}}{(bL_2B_0^2)^p 4^p \rho_0^p} = \beta h_k^{p+1}$$

and then, it the same way as in the proof of **c**) we deduce that:

$$h_{k+1} = \beta^{-\frac{1}{p}}\left(\beta^{\frac{1}{p}}h_0\right)^{(p+1)^{k+1}}$$

e) Because $B_{k+1} = \frac{B_k}{1-h_k}$ and $h_k \in]0, \frac{1}{2}]$ we have $B_{k+1} \ge B_k$, so $B_{k+1} \ge B_0$.

The same initial relation implies:

$$B_{k+1} = \frac{B_k}{(1-h_0)(1-h_1)\ldots(1-h_k)}.$$

Using the inequality between the geometric mean and the arithmetic mean we deduce:

$$\frac{1}{(1-h_0)(1-h_1)\ldots(1-h_k)} \le \left[\frac{1}{k+1}\sum_{i=0}^{k}\frac{1}{1-h_i}\right]^{k+1} =$$

$$= \left[1 + \frac{1}{k+1} \sum_{i=0}^{k} \frac{h_i}{1 \ - \ h_i}\right]^{k+1} .$$

As $\beta^{\frac{1}{p}} h_0 = \dfrac{4^{\frac{1}{p}}}{4} \leq 1$ we deduce that:

$$\max \left\{ \beta^{-\frac{1}{p}} \left(\beta^{\frac{1}{p}} h_0 \right)^{(p+1)^n} \Big/ \ n \in \mathbb{N} \right\} = \beta^{-\frac{1}{p}} \left(\beta^{\frac{1}{p}} h_0 \right) = h_0$$

and:

$$\sum_{i=0}^{k} \frac{h_i}{1 - h_i} \leq \sum_{i=0}^{k} \frac{h_i}{1 - \beta^{-\frac{1}{p}} (\beta^{\frac{1}{p}} h_0)^{(p+1)^i}} \leq \frac{1}{1 - h_0} \sum_{i=0}^{k} h_i.$$

But for $k \in \mathbb{N}$ we have:

$$h_{k+1} = \frac{\alpha h_k \rho_k^p}{(1 - h_k)^2} \leq 2\alpha\alpha^{-1} \left(\alpha^{\frac{1}{p}} \rho_0 \right)^{p(p+1)^k} = 2 \left(\alpha \rho_0^p \right)^{(p+1)^k} ,$$

and so:

$$\sum_{i=0}^{k} h_i = h_0 + 2 \sum_{i=1}^{k} \left(\alpha \rho_0^p \right)^{(p+1)^{i-1}} = h_0 + 2\alpha\rho_0^p \sum_{i=1}^{k} \left(\alpha \rho_0^p \right)^{(p+1)^{i-1} - 1} .$$

For $i \geq 2$ we have:

$$(p+1)^{i-1} - 1 = p \left[1 + (p+1) + ... + (p+1)^{i-2} \right] \geq p \left(i - 1 \right),$$

so:

$$\sum_{i=0}^{k} h_i \leq h_0 + 2\alpha\rho_0^p \left[1 + \sum_{i=2}^{k} \left(\alpha^p \rho_0^{p^2} \right)^{i-1} \right] < h_0 + \frac{2\alpha\rho_0^p}{1 - \alpha^p \rho_0^{p^2}} < \frac{1}{2} + \frac{2^{2p^2 - 2p + 1}}{2^{2p^2} - 1}$$

But, as $p \geq 1$ we have :

$$2^{2p^2} - 1 = 1 + 2 + 2^2 + ... + 2^{2p^2 - 1} \geq 2^{2p^2 - 1}$$

so, evidently:

$$\sum_{i=0}^{k} h_i < \frac{1}{2} + \frac{2^{2p^2 - 2p + 1}}{2^{2p^2 - 1}} = \frac{1}{2} + 2^{-2p+2}$$

and:

$$\sum_{i=0}^{k} \frac{h_i}{1 - h_i} \leq \frac{1}{1 - h_0} (\frac{1}{2} + 2^{-2p+2}) \leq 1 + 2^{-2p+3},$$

from where:

$$\left(1 + \frac{1}{k+1} \sum_{i=0}^{k} \frac{h_i}{1 - h_i} \right)^{k+1} \leq \left(1 + \frac{1 + 2^{-2p+3}}{k+1} \right)^{k+1} \leq \exp \left(1 + 2^{-2p+3} \right)$$

and:

$$B_{k+1} \leq B_0 \exp\left(1 + 2^{-2p+3}\right).$$

From the above we deduce that the statements **a)-e)** from (16) are true for $n = k+1$. According to the principle of mathematical induction these statements are true for any $n \in \mathbb{N}$.

Now we will deduce that, that sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, because:

$$\|x_{n+m} - x_n\|_X < \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\|_X \leq \sum_{i=n}^{n+m-1} Mb\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^i} =$$

$$= bM\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n} \sum_{j=0}^{m-1} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^{n+j}-(p+1)^n}.$$

But, for any $j \in \{0, 1, ..., m-1\}$ we have:

$$(p+1)^{n+j} - (p+1)^n = (p+1)^n \left[(p+1)^j - 1\right] =$$

$$= p(p+1)^n \left[1 + (p+1) + ... + (p+1)^{j-1}\right] \geq jp(p+1)^n,$$

so:

$$\|x_{n+m} - x_n\|_X < bM\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n} \sum_{j=0}^{m-1} \left[(\alpha\rho_0^p)^{(p+1)^n}\right]^j$$

and so:

$$\|x_{n+m} - x_n\|_X < \frac{bM\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}}{1 - (\alpha\rho_0^p)^{(p+1)^n}} \tag{18}$$

The last inequality and the condition:

$$\alpha^{\frac{1}{p}}\rho_0 < \frac{1}{4} < 1$$

determine the fact that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space X, so $(x_n)_{n\in\mathbb{N}}$ is convergent. If we note:

$$x^* = \lim_{n\to\infty} x_n \in X$$

and if we make so that $m \to \infty$ in the inequality (18) we deduce that:

$$\|x^* - x_n\|_X \leq \frac{bM\alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}}{1 - (\alpha\rho_0^p)^{(p+1)^n}},$$

(this is the inequality (15)), from where for $n = 0$ we can deduce:

$$\|x^{*} - x_0\|_X \leq \frac{bM\rho_0}{1 - \alpha\rho_0^p} \leq \delta,$$

so $x^{*} \in S(x_0, \delta)$.

From:

$$\|f(x_n)\|_Y \leq \alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}$$

and the condition $\alpha^{\frac{1}{p}}\rho_0 < 1$ we deduce that:

$$\lim_{n\to\infty} \|f(x_n)\|_Y = 0,$$

from where:

$$f(x^{*}) = \theta_Y,$$

so x^{*} is a solution of the equation (1).

The inequalities:

$$\|x_{n+m} - x_n\|_X \leq Mb \|f(x_n)\|_Y, \quad \|f(x_{n+1})\|_Y \leq \alpha \|f(x_n)\|_Y^{p+1},$$

show that the sequence $(x_n)_{n\in N}$ is a approximant sequence of the order $p+1$ for the solution x^{*}.

Form the inequality **c**) from (16) together with (17) we deduce the inequality (14).

In this way the theorem is proven.

3. **Special cases**

Now we will see how **Theorem 2.1** is applied in the case of particular proceedings of approximation.

Let us first suppose that the function $f : D \to Y$ admits for any $x \in D$ a Fréchet derivative of the first order, an $L > 0$ exists so that:

$$\|f'(x) - f'(y)\| \leq L \|x - y\|_X$$

for any $x, y \in D$, and the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ verifies for any $n \in \mathbb{N}$ the equality:

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y. \tag{19}$$

Obviously, if for any $n \in \mathbb{N}$, $[f'(x_n)]^{-1}$ exists, the relation (19) is equivalent to:

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n), \tag{20}$$

form under which the **Newton-Kantorovich method** is well known. But the form (20)of the relation (19) will be one of the conclusions of the statement that will be established.

Because:

$$\|f(x_n) + f'(x_n)(x_{n+1} - x_n)\|_Y = 0 \le 0 \cdot \|f(x_n)\|_Y^2$$

and:

$$\|f'(x_n)(x_{n+1} - x_n)\|_Y = 1 \cdot \|f(x_n)\|_Y ,$$

we deduce that the inequalities (10) and (11) of the hypothesis **iii)** of **Theorem 2.1** are verified for $a = 0$ and $b = 1$. In this case:

$$p = 1,\ L_2 = L,\ h_0 = 2LB_0^2\rho_0,\ \alpha = \frac{LM^2}{2},\ M = \left\| [f'(x_0)]^{-1} \right\| \mathbf{e}^3$$

and thus the inequality of hypothesis **v**) of **Theorem 2.1** become:

$$\rho_0 < \frac{1}{4}.$$

As $\alpha\rho_0 = \dfrac{LM^2 h_0}{4LB_0^2} = \dfrac{\mathbf{e}^9 h_0}{4}$, we need the condition $h_0 < \dfrac{1}{\mathbf{e}^9}$ or $B_0^2\rho_0 < \dfrac{1}{2\mathbf{e}^9 L}$, condition that evidently also implies $h_0 \le \dfrac{1}{2}$.

In what the radius of the ball on which the properties take place is concerned, it verifies the inequality:

$$\delta \ge \frac{M\rho_0}{1 - \alpha\rho_0}.$$

As $\alpha\rho_0 < \dfrac{1}{4}$ we deduce that $\dfrac{1}{1-\alpha\rho_0} < \dfrac{4}{3}$ and so if $\delta \ge \dfrac{3M\rho_0}{4}$ the requirement is fulfilled. Also:

$$M = \left\| [f'(x_0)]^{-1} \right\| \mathbf{e}^3.$$

In this way we have the following:

Corollary 3.1. *We consider the same elements as in **Theorem** 2.1. If:*

i) *X is a Banach space, and* $S(x_0, \delta) \subseteq D$;

ii) *for any $x \in D$, there exists $f'(x) \in (X,Y)^*$, representing the Fréchet derivative of f in x and there exists $L > 0$ so that:*

$$\|f'(x) - f'(y)\| \le L \|x - y\|_X$$

for any $x, y \in D$;

iii)*the sequence verifies the equality:*

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y,$$

iv) *the mapping $f'(x_0) \in (X,Y)^*$ is invertible;*

v) *the initial point $x_0 \in D$ verifies the inequalities:*

$$\left(\left\|[f'(x_0)]^{-1}\right\|\right)^2 \|f(x_0)\|_Y < \frac{1}{2\mathbf{e}^9 L}, \quad \delta \ge \frac{3\mathbf{e}^3}{4} \left\|[f'(x_0)]^{-1}\right\| \cdot \left\|[f'(x_0)]^{-1}\right\|,$$

then:

j)*$x_n \in S(x_0, \delta)$ and $[f'(x_n)]^{-1} \in (Y,X)^*$ exists, having the relations:*

$$\left\|[f'(x_n)]^{-1}\right\| \le \left\|[f'(x_0)]^{-1}\right\| \mathbf{e}^3$$

and:

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n)$$

for any $n \in \mathbb{N}$.

jj) *the equation (1)admits a solution $x^* \in S(x_0, \delta)$;*

jjj) *the sequence $(x_n)_{n\in\mathbb{N}}$ is a approximant sequence of the second order of the solution x^* of this equation;*

jv) *the following estimates hold:*

$$\max\left\{\|f(x_n)\|_Y, \frac{1}{M}\|x_{n+1} - x_n\|_X\right\} \le \left(\frac{LM^2}{2}\right)^{2^n - 1} \|f(x_n)\|_Y^{2^n},$$

$$\|x^* - x_n\|_X \le \frac{M\rho_0 \left(\frac{\rho_0 LM^2}{2}\right)^{2^n}}{1 - \left(\frac{\rho_0 LM^2}{2}\right)^{2^n}}$$

where $M = \left\|[f'(x_0)]^{-1}\right\| \mathbf{e}^3$ and $\|f(x_0)\|_Y$.

Let us now consider the case of **Chebyshev's method**. In this case $f : D \to Y$ admits, for any $x \in X$, Fréchet derivatives of the first and the second order, and in addition to the main sequence $(x_n)_{n\in N} \subseteq D$, we consider another sequence $(y_n)_{n\in\mathbb{N}} \subseteq D$ so that for any $n \in \mathbb{N}$ the following is verified:

$$\begin{cases} f'(x_n)(x_{n+1} - x_n) + f(x_n) + \dfrac{1}{2}f''(x_n)y_n^2, \\ f'(x_n)y_n + f(x_n) = \theta_Y \end{cases} \tag{21}$$

If for any $n \in N$, $[f'(x_n)]^{-1}$ exists, we can deduce from the relation (21) that:

$$x_{n+1} = x_n - \left[f'\left(x_n\right)\right]^{-1} f\left(x_n\right) - \frac{1}{2}[f'(x_n)]^{-1}f''(x_n)\{[f'(x_n)]^{-1}f(x_n)\}^2 \tag{22}$$

the form under which **Chebychev's method** is known. We will show that in this case the conditions of **Theorem 2.1** will be verified for $p = 2$.

So we will have:

Theorem 3.2. *We consider the same data as in* ***theorem 2.1****. If:*

i) *X is a Banach space and $S(x_0, \delta) \subseteq D$, $S(x_0, \delta)$ representing the ball with the centre x_0 and the radius δ;*

ii) *the function admits Fréchet derivatives up to the second order included, and for $f'' : D \to (X^2, Y)^*$, the number $L > 0$ exists, so that for any $x, y \in D$ the following inequality is verified:*

$$\left\|f''\left(x\right) - f''\left(y\right)\right\| \leq L\left\|x - y\right\|_X; \tag{23}$$

iii) *the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$, together with an auxiliary sequence $(y_n)_{n\in\mathbb{N}} \subseteq D$, verifies the relations (21) for any $n \in \mathbb{N}$;*

iv) *the mapping $f'\left(x_0\right) \in (X, Y)^*$ is invertible;*

v) *if we note:*

$$\begin{gathered} \rho_0 = \left\|f\left(x_0\right)\right\|_Y, \; B_0 = \left\|\left[f'\left(x_0\right)\right]^{-1}\right\|, \; M = B_0\mathbf{e}^{\frac{3}{2}}, \; b = \frac{L_2M^2\rho_0}{2}, \\ a = (b+1)\frac{\left(L_2M^2\right)^2}{2}, \; \alpha = a + L\frac{(bM)^3}{6}; \end{gathered} \tag{24}$$

the following inequalities are verified:

$$\alpha^{\frac{1}{2}}\rho_0 < \frac{1}{4}, \; \frac{bM\rho_0}{1-\alpha\rho_0^2} \leq \delta \leq \frac{1}{L}\left(\frac{1}{2bB_0^2\rho_0} - \left\|f''\left(x_0\right)\right\|\right); \tag{25}$$

then:

j) $x_n \in S(x_0, \delta)$, the mapping $[f'(x_n)]^{-1} \in (Y,X)^*$ exists, we have the inequality $\left\|[f'(x_n)]^{-1}\right\| \leq M$ for any $n \in \mathbb{N}$ and the sequence $(x_n)_{n\in\mathbb{N}}$ is generated by the relation of recurrence (21) *or* (22) *is convergent;*

jj) the equation (1) *are the solution $x^* \in S(x_0, \delta)$;*

jjj) the sequence$(x_n)_{n\in\mathbb{N}}$ is an approximant sequence of the third order of this solution of the equation (1);

jv) the following estimates hold:

$$\max\left\{\|f(x_n)\|_Y, \ \frac{1}{Mb}\|x_{n+1}-x_n\|_X\right\} \leq \alpha^{\frac{3^n-1}{2}}\|f(x_0)\|_Y^{3^n}, \tag{26}$$

and:

$$\|x^*-x_n\|_X \leq Mb\frac{\alpha^{\frac{3^n-1}{2}}\|f(x_0)\|_Y^{3^n}}{1-\left(\alpha\|f(x_0)\|_Y^2\right)^{3^n}}, \tag{27}$$

for any $n \in \mathbb{N}$.

Proof. From the condition:

$$\delta \leq \frac{1}{L}\left(\frac{1}{2bB_0^2\rho_0} - \|f''(x_0)\|\right),$$

if we keep in mind that $L_2 = \| f''(x_0) \| + L\delta$, we deduce that:

$$h_0 = bL_2B_0^2\rho_0 \leq \frac{1}{2}.$$

We will introduce the same sequences as in the proof of **theorem 2.1.** We will show that for any $n \in \mathbb{N}$ the following properties are verified:

a) $x_n \in S(x_0, \delta)$;

b) $[f'(x_n)]^{-1} \in (Y,X)^*$ exists and $\left\|[f'(x_n)]^{-1}\right\| \leq B_n$;

c) $\|f(x_n)\|_Y \leq \rho_n \leq \dfrac{(\sqrt{\alpha}\rho_0)^{3^n}}{\sqrt{\alpha}}$;

d) $h_n \leq \min\left\{\dfrac{1}{2}, \dfrac{(\beta h_0)^{3^n}}{\sqrt{\beta}}\right\}$, where $\beta = \dfrac{1}{4h_0^2}$;

e) $B_n \leq B_0 \leq M$;

f) $\|f'(x_n)(x_{n+1}-x_n)\|_Y \leq b\|f(x_n)\|_Y$;

g) $\left\|f(x_n) + f'(x_n)(x_{n+1}-x_n) + \dfrac{1}{2}f''(x_n)(x_{n+1}-x_n)^2\right\|_Y \leq a\|f(x_n)\|_Y^3$.

To start with , let us suppose that the properties **a)-e)** are true for a certain number $n \in \mathbb{N}$. We will show that, for that number, the properties **f)** and **g**) are also verified.

Indeed, we first notice that from $x_n \in S(x_0, \delta)$ we deduce that:

$$\|f''(x_n)\| \leq L_2.$$

Then it is obvious that:

$$\|y_n\|_X \leq \left\|[f'(x_n)]^{-1}\right\| \cdot \|f(x_n)\|_Y$$

and:

$$\|x_{n+1} - x_n - y_n\|_X = \left\|[f'(x_n)]^{-1}[f'(x_n)(x_{n+1} - x_n) - f'(x_n)y_n]\right\|_X \leq$$
$$\leq \left\|[f'(x_n)]^{-1}\right\|_Y \cdot \left\|-f(x_n) - \frac{1}{2}f''(x_n)y_n^2 + f(x_n)\right\|_Y \leq \frac{1}{2}B_n L_2 \|y_n\|_X^2 \leq$$
$$\leq \frac{1}{2}M^3 L_2 \|f(x_n)\|_Y^2.$$

So:

$$\|f'(x_n)(x_{n+1} - x_n)\|_Y = \left\|-f(x_n) - \frac{1}{2}f''(x_n)y_n^2\right\|_Y \leq$$
$$\leq \left(1 + \frac{1}{2}M^2 L_2 \|f(x_n)\|_Y\right)\|f(x_n)\|_Y.$$

As $\sqrt{\alpha}\rho_0 < 1$ we deduce that:

$$\left(\sqrt{\alpha}\rho_0\right)^{3^n} \leq \sqrt{\alpha}\rho_0$$

and:

$$\|f(x_n)\|_Y \leq \rho_n \leq \frac{\left(\sqrt{\alpha}\rho_0\right)^{3^n}}{\sqrt{\alpha}} \leq \frac{\sqrt{\alpha}\rho_0}{\sqrt{\alpha}} = \rho_0$$

and thus:

$$\|f'(x_n)(x_{n+1} - x_n)\|_Y \leq \left(1 + \frac{1}{2}M^2 L_2 \rho_0\right)\|f(x_n)\|_Y = b\|f(x_n)\|_Y.$$

But, from the symmetry of $f''(x) \in \mathcal{L}_2(X, Y)$ for any $x \in D$, we have:

$$f''(x_n)(x_{n+1} - x_n)^2 - f''(x_n)y_n^2 = f''(x_n)(x_{n+1} - x_n)^2 -$$
$$-f''(x_n)(x_{n+1} - x_n, y_n) + f''(x_n)(y_n, x_{n+1} - x_n) - f''(x_n)y_n^2 =$$
$$= f''(x_n)(x_{n+1} - x_n, x_{n+1} - x_n - y_n) + f''(x_n)(y_n, x_{n+1} - x_n - y_n) =$$
$$= [f''(x_n)(x_{n+1} - x_n) + f''(x_n)y_n](x_{n+1} - x_n - y_n),$$

then it is obvious that:

$$\left\|f(x_n)+f'(x_n)(x_{n+1}-x_n)+\frac{1}{2}f''(x_n)(x_{n+1}-x_n)^2\right\|_Y=$$

$$=\frac{1}{2}\left\|f''(x_n)(x_{n+1}-x_n)^2-f''(x_n)y_n^2\right\|_Y\leq$$

$$\leq\frac{1}{2}\left[\|f''(x_n)(x_{n+1}-x_n)\|+\|f''(x_n)y_n\|\right]\cdot\|x_{n+1}-x_n-y_n\|_X\leq$$

$$\leq\frac{1}{2}\|f''(x_n)\|\cdot\|x_{n+1}-x_n-y_n\|_X\cdot\left(\|x_{n+1}-x_n\|_X+\|y_n\|_X\right).$$

It is obvious that:

$$\|x_{n+1}-x_n\|_X=\left\|[f'(x_n)]^{-1}f'(x_n)(x_{n+1}-x_n)\right\|_X\leq Mb\|f(x_n)\|_Y,$$

so:

$$\left\|f(x_n)+f'(x_n)(x_{n+1}-x_n)+\frac{1}{2}f''(x_n)(x_{n+1}-x_n)^2\right\|_Y\leq$$

$$\leq\frac{1}{2}M^3L_2^2(Mb+M)\|f(x_n)\|_Y^3=\frac{1}{2}(b+1)M^4L_2^2\|f(x_n)\|_Y^3=a\|f(x_n)\|_Y^3.$$

So indeed **f**) and **g**) are true for the $n\in\mathbb{N}$ we considered.

The statements **a)-e)** are proven similarly to the proof of **theorem 2.1**. This entitles us to assert that the properties **a)-g)** are true for any $n\in\mathbb{N}$. Also, the properties **f)** and **g)** , together with the hypothesis show that impossible to apply **theorem 2.1** with $p=2$. Using this theorem, we deduce that the conclusions of the theorem to be proved are true.

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CONTINUITY OF THE SOLUTION OF A NONLINEAR PDE WITH RESPECT TO THE DOMAIN

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Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

Abstract. In this paper we consider a nonlinear variational problem and study the continuity of a solution with respect to the domain. The topology on the set of domains is the Hausdorff complementary topology. In the end, the continuity is used to prove the existence of a solution for an optimal shape design problem.

1. Introduction

A very actual research field, shape optimization deals with problems in which the optimization variable is the shape of a geometric domain. The existence of solutions for such a problem has been studied in many works. For example, optimal shape design problems for PDEs were considered in [8], [6], [1]; for variational inequalities in [8], [1], [4], [5]; for hemivariational inequalities in [2], [3].

An essential point in the study of optimal shape design problems is the choice of the convergence of the domains. In this paper, following [1] we shall consider the Hausdorff complementary topology, also used in [6], [5].

The shape optimization problem that we study is given in a general form and the system is governed by a nonlinear variational equality. This is a more general setting than the one in [1], where the variational problem is linear. After introducing some preliminary notions, we prove the continuity of the solution of the variational equality with respect to the underlying domain. We formulate then a shape optimization problem and prove that it has at least a solution.

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2. Preliminaries

We present here some notions and results used in the paper, following [1]. Let D be a bounded, open, nonempty subset of $\mathbb{R}^N$.
Denote $\mathcal{G}(D) = \{\Omega \subset D \mid \Omega \text{ open }, \Omega \neq \mathbb{R}^N\}$. The Hausdorff complementary metric ρ_H^C is defined by:

$$\rho_H^C(\Omega_1, \Omega_2) = \|d_{C\Omega_2} - d_{C\Omega_1}\|_{C(D)},$$

where the distance function for a set $A \subset \mathbb{R}^N$ is:

$$d_A(x) = \begin{cases} \inf_{y \in A} |y - x|, & A \neq \emptyset \\ +\infty, & A = \emptyset \end{cases}$$

and $C\Omega$ is the complementary set of Ω.
The metric topology induced is complete and the Hausdorff complementary covergence is denoted by $\Omega_n \xrightarrow{H^C} \Omega$.

Theorem 1. *(i) The space $(\mathcal{G}(D), \rho_H^C)$ is a compact metric space.*
(ii) Let $\{\Omega_n\}$ be a sequence in $\mathcal{G}(D)$, Ω in $\mathcal{G}(D)$ such that $\Omega_n \xrightarrow{H^C} \Omega$. For any compact subset $K \subset \Omega$, there exists $N(K) \in \mathbb{N}$ such that for all $n \geq N(K)$, $K \subset \Omega_n$ (compactivorous property) .

The domains considered in this paper are of a special type, more precisely they satisfy the uniform cone property.
Given $\lambda > 0$, $0 < \omega \leq \pi/2$ and a direction $d \in \mathbb{R}^N$, $|d| = 1$, we denote $C(\lambda, \omega, d)$ the set

$$C(\lambda, \omega, d) = \{y \in \mathbb{R}^N \ : \ \frac{1}{\tan \omega}|P_H(y)| < y \cdot d < \lambda\}$$

where P_H is the orthogonal projection onto the hyperplane H through the origin and orthogonal to the direction d. The translated cone for $x \in \mathbb{R}^N$ is $C_x(\lambda, \omega, d) = x + C(\lambda, \omega, d)$.
Let $\Omega \subset \mathbb{R}^N$ with $\partial\Omega \neq 0$. Ω is said to satisfy the *uniform cone property* if

$$\exists \lambda > 0, \exists \omega > 0, \exists r > 0 \text{ such that } \forall x \in \partial\Omega, \exists d \in \mathbb{R}^N, |d| = 1$$

$$\text{such that } \forall y \in B(x, r) \cap \bar{\Omega} \text{ we have } \ C_y(\lambda, \omega, d) \subset \text{int}\Omega$$

It is proved in [1] that the family of open lipschitzian domains included in D, which satisfy the uniform cone property is compact with respect to the Hausdorff complementary topology. This family will be denoted with $L(D, r, \omega, \lambda)$.

Let Ω be an open subset of D and $\phi \in \mathcal{D}(\otimes)$, the space of infinitely smooth and compactly supported on Ω functions. Denoting by $e_0(\phi)$ the extension by zero of ϕ to D, we have that $e_0(\phi) \in \mathcal{D}(\mathcal{D})$. By definition, $\|\phi\|_{H^1(\Omega)} = \|e_0(\phi)\|_{H^1(D)}$ and e_0 extends by continuity and density to a linear isometric map between two Sobolev spaces, i.e. $e_0 : H_0^1(\Omega) \to H_0^1(D)$. Denote by $H_0^1(\Omega; D)$ the image of $H_0^1(\Omega)$ by e_0.

Theorem 2. *(i) The linear subspace $H_0^1(\Omega; D)$ of $H_0^1(D)$ is closed and isometrically isomorphic to $H_0^1(\Omega)$.*
(ii) If $\psi \in H_0^1(\Omega; D)$ then $\psi|_\Omega \in H_0^1(\Omega)$ and $\forall \alpha$, $|\alpha| \leq 1$ $\partial^\alpha \psi = 0$ a.e. in $D \setminus \Omega$.
(iii) If a sequence converges in $H_0^1(\Omega)$- weak then it converges in $L^2(\Omega)$- strong.

3. Main result

Let Ω and D be bounded, open subsets of $\mathbb{R}^N$, $\Omega \subset D$ and consider the variational equality

$$\begin{aligned}&\text{Find } u_\Omega \in H_0^1(\Omega) \text{ such that}\\ &\int_\Omega \{A(x, u_\Omega(x))\nabla u_\Omega(x) \cdot \nabla\phi(x) + a(x, u_\Omega(x))\phi(x)\}dx \\ &= \langle f|_\Omega, \phi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},\ \forall \phi \in H_0^1(\Omega)\end{aligned} \tag{1}$$

where $f \in H^{-1}(D)$, $f|_\Omega$ denotes the restriction of f belonging to $H^{-1}(\Omega)$, A and a are functions such that: $A = (a_{ij})_{i,j=1}^N$, $a_{ij} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, A_i is the i-th row of the matrix A. We suppose that these functions have the following properties:

(P1) a_{ij} and a are measurable with respect to the first variable, $A_i(x, \eta) \cdot \xi$ are continuous with respect to (η, ξ), for a.e. $x \in \mathbb{R}^N$ and for all $i, j = 1, \ldots, N$,

(P2) $|a(x, \eta) - a(x, \tilde{\eta})| \leq c_1|\eta - \tilde{\eta}|$ and $|a_{ij}(x, \eta) - a_{ij}(x, \tilde{\eta})| \leq c_2|\eta - \tilde{\eta}|$ for a.e. $x \in \mathbb{R}^N$ and for all $\eta, \tilde{\eta} \in \mathbb{R}$, with c_1, c_2 positive constants,

(P3) $\sum_{i,j=1}^N a_{ij}(x, \eta)\xi_i\xi_j \geq c_3\|\xi\|_N^2$ and $a(x, \eta)\eta \geq c_4|\eta|^2$, for a.e. $x \in \mathbb{R}^N$ and for all $\eta \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

(P4) $|A_i(x, \eta) \cdot \xi| \leq c_5(k_1(x) + |\eta| + \|\xi\|)$, $|a(x, \eta)| \leq c_6(k_2(x) + |\eta|)$ for a.e. $x \in \mathbb{R}^N$

and for all $\eta, \tilde{\eta} \in \mathbb{R}$, with $k_1, k_2 \in L^2(D)$ positive functions.

According to [7], pg. 76 we have:

Theorem 3. *In the conditions mentioned above, the variational problem (1) has at least a solution* u_Ω.

Lemma 4. *If* $u_\Omega \in H_0^1(\Omega)$ *is a solution of the variational problem (1), and* $u = e_0(u_\Omega)$*, then*

$$\|u\|_{H_0^1(D)} \leq \alpha \|f\|_{H^{-1}(D)}, \tag{2}$$

with α *a positive constant.*

Proof. $u = e_0(u_\Omega)$ is a solution of the variational problem

$$\begin{aligned} & u \in H_0^1(\Omega; D) \text{ such that} \\ & \int_D \{A(x, u(x))\nabla u(x) \cdot \nabla\phi(x) + a(x, u(x))\phi(x)\}dx \\ & = \langle f, \phi\rangle_{H^{-1}(D)\times H_0^1(D)}, \forall \phi \in H_0^1(\Omega; D) \end{aligned} \tag{3}$$

Using **(P3)**, Hölder and Poincaré inequalities we get:

$$\begin{aligned} \|u\|^2_{H_0^1(D)} &\leq \frac{1}{c_3}\int_D A(x, u(x))\nabla u(x) \cdot \nabla u(x)dx + \frac{1}{c_4}\int_D a(x, u(x))u(x)dx \\ &\leq \alpha \int_D \{A(x, u(x))\nabla u(x) \cdot \nabla u(x) + a(x, u(x))u(x)\}dx \\ &= \alpha\langle f, u\rangle_{H^{-1}(D)\times H_0^1(D)} \leq \alpha\|f\|_{H^{-1}(D)}\|u\|_{H_0^1(D)}. \quad \square \end{aligned}$$

We consider $\{\Omega_n\}$ a sequence of open subsets of D and the corresponding variational equalities:

$$\begin{aligned} & u_{\Omega_n} \in H_0^1(\Omega_n) \text{ such that} \\ & \int_{\Omega_n} \{A(x, u_{\Omega_n}(x))\nabla u_{\Omega_n}(x) \cdot \nabla\phi(x) + a(x, u_{\Omega_n}(x))\phi(x)\}dx \\ & = \langle f|_{\Omega_n}, \phi\rangle_{H^{-1}(\Omega_n)\times H_0^1(\Omega_n)}, \forall \phi \in H_0^1(\Omega_n) \end{aligned} \tag{4}$$

Denoting with $u_n = e_0(u_{\Omega_n})$ the extension by zero to D of u_{Ω_n}, this satisfies

$$\begin{aligned} & u_n \in H_0^1(\Omega_n; D) \text{ such that} \\ & \int_D \{A(x, u_n(x))\nabla u_n(x) \cdot \nabla\phi(x) + a(x, u_n(x))\phi(x)\}dx \\ & = \langle f, \phi\rangle_{H^{-1}(D)\times H_0^1(D)}, \forall \phi \in H_0^1(\Omega_n; D) \end{aligned} \tag{5}$$

It takes place:

Theorem 5. *Let $D \subset \mathbb{R}^N$ be a bounded, open, nonempty domain; $\{\Omega_n\}$ a sequence of open subsets of D with $\Omega_n \stackrel{H^C}{\to} \Omega$. Denote by u_n a solution of (5). Then there exists a subsequence (still denoted u_n) and $u \in H_0^1(D)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(D)$, $u = 0$ a.e. in $D \setminus \bar{\Omega}$ and*

$$\begin{aligned} &\int_D \{A(x, u(x))\nabla u(x) \cdot \nabla\phi(x) + a(x, u(x))\phi(x)\}dx \\ &= \langle f, \phi\rangle_{H^{-1}(D)\times H_0^1(D)}, \ \forall\phi \in H_0^1(\Omega; D) \end{aligned} \tag{6}$$

(or equivalently

$$\begin{aligned} &\int_\Omega \{A(x, u(x))\nabla u(x) \cdot \nabla\phi(x) + a(x, u(x))\phi(x)\}dx \\ &= \langle f|_\Omega, \phi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \ \forall\phi \in H_0^1(\Omega) \ ). \end{aligned}$$

If, in addition, the domain Ω is locally lipschitzian, then $u \in H_0^1(\Omega; D)$.

Proof. According to the Lemma 4, for each $n \in \mathbb{N}$ we have:

$$\|u_n\|_{H_0^1(D)} \leq \alpha\|f\|_{H^{-1}(D)}$$

which implies the existence of a subsequence, still denoted by u_n, and of an element $u \in H_0^1(D)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(D)$.
Let $\phi \in \mathcal{D}(\Omega)$ and $K = \text{supp}\phi$ a compact subset of Ω. Then, according to Theorem 1 there exists a rank $N(K) > 0$ such that for all $n \geq N(K)$, $K \subset \Omega_n$ and

$$\int_D \{A(x, u_n(x))\nabla u_n(x) \cdot \nabla\phi(x) + a(x, u_n(x))\phi(x)\}dx = \langle f, \phi\rangle_{H^{-1}(D)\times H_0^1(D)}$$

We want to pass to the limit in this equality. We have:

$$\begin{aligned} &|\int_D a(x, u_n(x))\phi(x)dx - \int_D a(x, u(x))\phi(x)dx| \\ &\leq \int_D |a(x, u_n(x)) - a(x, u(x))||\phi(x)|dx \\ &\leq c_1 \int_D |u_n(x) - u(x)||\phi(x)|dx \leq c_1\|u_n - u\|_{L^2(D)}\|\phi\|_{L^2(D)} \to 0, \end{aligned}$$

since the weak convergence in $H_0^1(D)$ implies the strong convergence in $L^2(D)$.
The mappings $x \mapsto A(x, u_n(x))$, $x \mapsto A(x, u(x))$, $x \mapsto A^T(x, u_n(x))$ and $x \mapsto$

$A^T(x,u(x))$ belong to $L^2(D)$:

$$\int_D |a_{ij}(x,u_n(x))|^2dx \le \int_D c_5^2(\bar{k}_1(x)+|u_n(x)|)^2dx$$
$$= c_5^2\int_D \{|\bar{k}_1(x)|^2+|u_n(x)|^2+2|\bar{k}_1(x)||u_n(x)|\}dx$$
$$\le c_5^2\{\int_D |\bar{k}_1(x)|^2dx+\int_D |u_n(x)|^2dx$$
$$+2\big(\int_D |\bar{k}_1(x)|^2dx\big)^{1/2}\big(\int_D |u_n(x)|^2dx\big)^{1/2}\}<\infty,$$

where $\bar{k}_1(x)=k_1(x)+1$.

From $\phi\in\mathcal{D}(\Omega)$ it follows that $\nabla\phi\in L^\infty(D)$ so the mapping $x\mapsto A^T(x,u_n(x))\nabla\phi(x)$ is also in $L^2(D)$ and converges strongly to the mapping $x\mapsto A^T(x,u(x))\nabla\phi(x)$. Indeed,

$$\|A^T(\cdot,u_n(\cdot))\nabla\phi(\cdot)-A^T(\cdot,u(\cdot))\nabla\phi(\cdot)\|^2_{L^2(D)}$$
$$=\int_D \|A^T(x,u_n(x))\nabla\phi(x)-A^T(x,u(x))\nabla\phi(x)\|_N^2dx$$
$$\le\int_D \|A^T(x,u_n(x))-A^T(x,u(x))\|^2_{N^2}\cdot\|\nabla\phi(x)\|^2_Ndx$$
$$\le\int_D c^2N^4|u_n(x)-u(x)|^2\|\nabla\phi(x)\|_N^2dx$$
$$\le\|\nabla\phi\|^2_{L^\infty(D)}c^2N^4\int_D|u_n(x)-u(x)|^2dx\to 0.$$

(We used here the fact that $\|A^T(x,\eta)-A^T(x,\tilde{\eta})\|_{N^2}\le cN^2|\eta-\tilde{\eta}|$ which follows immediately from **(P2)**).

We have now the convergences:

$$A^T(\cdot,u_n(\cdot))\nabla\phi(\cdot)\to A^T(\cdot,u(\cdot))\nabla\phi(\cdot) \text{ strongly in } L^2(D)$$
$$\nabla u_n(\cdot)\rightharpoonup\nabla u(\cdot) \text{ weakly in } L^2(D),$$

which implies that

$$\int_D A^T(x,u_n(x))\nabla\phi(x)\cdot\nabla u_n(x)dx\to\int_D A^T(x,u(x))\nabla\phi(x)\cdot\nabla u(x)dx$$

hence

$$\int_D A(x,u_n(x))\nabla u_n(x)\cdot\nabla\phi(x)dx\to\int_D A(x,u(x))\nabla u(x)\cdot\nabla\phi(x)dx.$$

So $u\in H_0^1(D)$ satisfies the variational equality (6) for every $\phi\in\mathcal{D}(\Omega)$. By density this extends to all ϕ in $H_0^1(\Omega;D)$.

The proof of the other statements in the theorem is as in [1]:

$u_n = 0$ almost everywhere in $D \setminus \Omega_n$. Then

$$\int_D \chi_{C\bar{\Omega}_n} |u(x)|^2 = \int_{D\setminus\bar{\Omega}_n} |u_n(x) - u(x)|^2 dx \leq \int_D |u_n(x) - u(x)|^2 dx \to 0$$

and so

$$0 = \liminf_{n\to 0} \int_D \chi_{C\bar{\Omega}_n} |u_n(x) - u(x)|^2 dx \geq \int_D \chi_{C\bar{\Omega}} |u(x)|^2 dx.$$

Therefore $u \in H_0^1(D)$, $u = 0$ a.e. in $D \setminus \bar{\Omega}$. For lipschitzian domains, the trace of u is well defined on $\partial\Omega$. It is zero since u and ∇u are zero a.e. in the locally lipschitz domain $C\bar{\Omega}$ by using the Gauss-Green formula. □

Remark 6. If the matrix function $A = (a_{i,j})_{i,j=1}^N$ is such that $a_{ij} : \mathbb{R}^N \to \mathbb{R}$, $A \in L^\infty(D; \mathcal{L}(\mathbb{R}^\mathcal{N}, \mathbb{R}^\mathcal{N}))$ with $a_{ij} = a_{ji}$, $\alpha I \leq A \leq \beta I$ ($0 < \alpha \leq \beta$ constants) and $a = 0$ then the variational problem (1) becomes a linear one :

$$\int_\Omega A(x)\nabla u_\Omega(x) \cdot \nabla\phi(x)dx = \langle f|_\Omega, \phi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \ \forall \phi \in H_0^1(\Omega).$$

for which the continuity of the solution with respect to the underlying domain is proved in [1], Th. 4.1.,p. 266.

We consider now the shape optimization problem:

$$\begin{aligned} &\text{Find } (\Omega^*, u^*) \in \bigcup_{\Omega \in L(D,r,\omega,\lambda)} \{\Omega \times S(\Omega)\} \text{ such that} \\ &J(\Omega^*, u^*) = \min_{\Omega\in L(D,r,\omega,\lambda)} \min_{v\in S(\Omega)} J(\Omega, v) \end{aligned} \tag{7}$$

We say that the pair (Ω_n, v_n) converges to (Ω, v) if

$$\begin{aligned} &\text{(i) } \Omega_n \overset{H^C}{\to} \Omega \text{ and} \\ &\text{(ii) } e_0(v_n) \to e_0(v) \text{ in } L^2(D) \end{aligned} \tag{8}$$

We make the hypothesis (see also [3]) that the cost functional J is lower semicontinuous with respect to the convergence: $(\Omega_n, v_n) \to (\Omega, v)$.

Theorem 7. *In the conditions stated above, the optimization problem (7) admits at least one solution.*

Proof. We shall use the same ideea as in the direct method of the calculus of variations.

Let (Ω_n, u_{Ω_n}) be a minimizing sequence for the problem (7). The family $L(D, r, \omega, \lambda)$ is compact with respect to the Hausdorff complementary topology, so there exists a subsequence of Ω_n, still denoted by Ω_n, and a set $\Omega \in L(D, r, \omega, \lambda)$ such that $\Omega_n \stackrel{H^C}{\to} \Omega$. Next, since $u_{\Omega_n} \in S(\Omega_n)$ we get, according to Theorem 5, that there exists a subsequence u_{Ω_n} and $u \in H_0^1(\Omega; D)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(D)$, $u_n \to u$ strongly in $L^2(D)$ $(u_n = e_0(u_{\Omega_n}))$. Also, u satisfies the variational equality (6), which means $u|_\Omega \in S(\Omega)$.

Finally, by the fact that the cost functional J is lower semicontinous, $(\Omega, u|_\Omega)$ is a solution for the optimization problem (7). □

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CONSTRAINT CONTROLLABILITY IN INFINITE DIMENSIONAL BANACH SPACES

MARIAN MUREŞAN

Dedicated to Professor Wolfgang W. Breckner at his 60^{th} anniversary

Abstract. Some well known criteria of controllability of linear and time invariant systems in $\mathbb{R}^n$ has been extended in various directions. First we review briefly this topic. Then we introduce a necessary and sufficient criterion of approximately locally null-controllability for a system of differential equations in infinite dimensional Banach spaces. Several comments end the paper.

Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Denote by W an open neighborhood of a point $x_0 \in \mathbb{R}^n$. Consider the following system of differential equations

$$x'(t) = f(t, x(t), u(t)), \qquad x(t_0) = x_0,\ t \in T \tag{1}$$

where T is an interval (bounded or not), $t_0 \in T$, $T \ni t \mapsto x(t) \in \mathbb{R}^n$ is the state trajectory, and $T \ni t \mapsto u(t) \in U \subset \mathbb{R}^m$ is the control function.

Example. If f is a linear functions and the dynamics of system (1) is time invariant, we get the simplest case

$$x'(t) = Ax(t) + Bu(t), \qquad A \in M_{n\times n},\ B \in M_{n\times m}. \tag{2}$$

Roughly speaking, (1) is said to be *controllable* if every state is accessible from every other state.

We mention some topics and works related to the idea of controllability

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- controllability in the time invariant case in finite dimensional spaces, [?], [?] and the references therein;
- controllability in the non-linear case in finite or infinite dimensional spaces, fixed point method, [?], [?], [?], [?], [?];
- controllability of convex processes in finite dimensional spaces, [?], [?], [?], [?];
- constraint controllability in Banach spaces, [?], [?], [?], [?], [?], [?], [?], [?], [?], [?];
- approximate null controllability of certain differential inclusions in infinite dimensional Banach spaces, [?].

1. **Linear case in finite dimensional spaces**

In this case we have system (2), i.e.,

$$x'(t) = Ax(t) + Bu(t), \qquad A \in M_{n\times n},\ B \in M_{n\times m}.$$

If the control function u is (at least) Lebesgue integrable, the general solution of the above system is

$$x(t) = e^{At}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau, \quad t \in T. \tag{3}$$

Following [?] we say that system (1) is *(completely) state*

(i) *approximately controllable* on the finite interval $[t_0, t_f] \subset T$ if given $\varepsilon > 0$ and two arbitrary initial and final points x_0 and x_f in the state space there is an admissible controller $u(\cdot)$ on $[t_0, t_f]$ steering x_0, along a solution curve of (1), to an ε-ball of x_1, that is such that $\|x(t_f, t_0, x_0, u) - x_1\| \le \varepsilon$.

(ii) *exactly controllable* on $[t_0, t_f]$ if $\varepsilon = 0$ in (i).

To system (2) we introduce the so-called *controllability Gramian*

$$G(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_f-\tau)}BB^Te^{A^T(t_f-\tau)}d\tau, \tag{4}$$

and the *controllability matrix*

$$Q = [B, AB, A^2B, \cdots, A^{n-1}B]. \tag{5}$$

It is well-known the next characterization theorem

Theorem 1.1. *For the linear time invariant system* (2) *the following statements are equivalent*

(a) (2) *is completely controllable;*

(b) *the controllability Gramian satisfies* $G(t_0, t) > 0$ *for all* $t > t_0$;

(c) *the controllability matrix* Q *has rank* n *(Kalman criterion);*

(d) *the rows of* $e^{At}B$ *are linearly independent functions of time;*

(e) *the rows of* $(sI - A)^{-1}B$ *are linearly independent functions of* s;

(f) *rank*$([A - \lambda I, B]) = n$, *for all* λ *(suffices to check only the eigenvalues of* A*);*

(g) $v^T B = 0$ *and* $v^T A = \lambda v^T \implies v = 0$ *(Popov-Belevich-Hautus test);*

(h) *given any set* Γ *of numbers in* $\mathbb{C}$ *there exists a matrix* K *such that the spectrum of* $A + BK$ *is equal to* Γ *(pole placement condition).*

2. **The result**

In order to present our result we introduce some notations. Let Z be a topological space and $Y \subset Z$. By $\operatorname{int} Y$ and $\operatorname{cl} Y$ we denote the set of interior points, and the closure of Y, respectively. Let Z be a linear space and $Y \subset Z$, then by $\operatorname{co} Y$ we denote the convex hull of Y. If X is a Banach space, then by $\mathcal{L}(X)$ we denote the space of linear and bounded operators from X in X. X^* is the Banach space of the linear and continuous functionals on X. Let F be a multifunction from a σ-algebra to a topological space. By S_F we denote the set of measurable selections from F. Under convenient assumptions, by S_F^1 we denote the set of Bochner integrable selections from F, see [**?**], [**?**], [**?**].

Consider a real interval $T := [t_0, t_f]$ with $t_0 < t_f$ and μ the Lebesgue measure on T. Let X and Y be separable real Banach spaces. Let $\mathrm{B}_\delta = \{x \in X \mid \|x\| \leq \delta\}$. We denote the closed unit ball by B, too. We consider further

(U) a weakly measurable multifunction $U : T \rightsquigarrow Y$ having nonempty and closed values;

(B) a Carathéodory mapping $B : T \times Y \to X$ (measurable in the first variable and continuous in the second one) such that there exits a positive integrable function

m defined on T satisfying

$$U(t,u) \subset m(t)\mathrm{B}, \quad \text{for all} \;\; t \in T, \quad u \in U(t). \tag{6}$$

(A) a family $\{A(t)\}_{t\in T}$ of linear and densely defined operators generating an evolution operator $S : \Delta = \{(t,s) \in T \times T \mid t_0 \le s \le t \le t_f\} \to \mathcal{L}(X)$, i.e.

$S(t,t) = I$, $\forall t \in T$, I is the identity,

$S(t,\tau)S(\tau,s) = S(t,s)$, $\forall\ t_0 \le s \le \tau \le\le t \le t_f$,

$S : \Delta \to \mathcal{L}(X)$ is continuous in the strong operator topology, [**?**].

Also, $B(t,U(t)) := \{x \in X \mid \exists\ u \in U(t) \text{ with } x = B(t,u)\}$. For $M \subset X$, $M \neq \emptyset$, the support function $\sigma_M(\cdot)$ of M is defined by

$$\sigma_M(x^*) = \sup_{x\in M}(x^*,x) = \sup_{x\in M} x^*(x) = \sigma(x^*(M)), \quad x^* \in X^*.$$

Under the above conditions our attention focuses on the following system

$$x'(t) = A(t)x(t) + B(t,u(t)), \quad t \in T,\ u \in S_U. \tag{7}$$

Throughout the present paper we are interested in some properties of the mild solutions of the system (7), i.e. given $x_0 \in X$ (as initial value) a mild solution of (7) is a continuous function $x \in C(T,X)$ which can be written as

$$x(t) = S(t,t_0)x_{t_0} + \int_{t_0}^{t} S(t,s)B(s,u(s))ds,\ t \in T, \tag{8}$$

where u is a measurable selection of the multifunction U such that $B(\cdot,u(\cdot)) \in L^1$.

The reachable set from x_0 at time $t \in T$ is defined as

$$R(t,x_0) = \{x(t) \in X \mid x(\cdot) \text{ is a mild solution of (7)}\}.$$

Different notions of controllability are investigated in [**?**] and [**?**]. We now recall here only one in [**?**]. System (7) is said to be *approximately locally null-controllable* if there exists an open neighborhood V of the origin such that for all $x_0 \in V$, $0 \in \mathrm{cl}(R(t_f,x_0))$.

Remarks 2.1.

(a) From (U) it follows that $S_U \neq \emptyset$; moreover, from the Castaing representation theorem, [**?**, theorem 5.6], [**?**, theorem 4.2.3], or [**?**, p. 76] it follows that there exists a countable family of measurable functions $\{u_n\}_{n\ge1}$ such that $U(t) = \mathrm{cl}\{u_n(t) \mid n \ge 1\}$, for all $t \in T$.

(b) The multifunction U has closed values. Then, by [**?**, theorem 6.5] the multifunction $T \ni t \mapsto B(t, U(t))$ is weakly measurable. Since $B(t, U(t)) \subset m(t)\mathrm{B}$, $t \in T$, and each mapping $B(\cdot, u_n(\cdot))$ is a measurable selection of $B(\cdot, U(\cdot))$, we conclude that the multifunction $B(\cdot, U(\cdot))$ has a family $(B(\cdot, u_n(\cdot)))_n$ of integrable selections. Thus the definition of mild solution in (8) makes sense and the reachable set is nonempty.

(c) The mapping $T \times Y \ni (t, u) \mapsto S(t_f, t)B(t, u) \in X$ is Carathéodory. As above we conclude that the multifunction

$$[t_0, t] \ni s \mapsto S(t, s)B(s, U(s))$$

is weakly measurable, for all $t \in [t_0, t_f]$.

Theorem 2.1. *Suppose the assumptions* (U), (B), and (A) *are satisfied. Then*

(a) *if $S(t_f, t)B(t, U(t)) \neq \{0\}$ on a set of positive Lebesgue measure and* (7) *is approximately locally null-controllable, then there exists $x^* \in X^* \setminus \{0\}$ and $E \subset T$ Lebesgue measurable such that*

$$\mu(E) > 0, \text{ and } 0 < \sigma(x^*(S(t_f, t)B(t, U(t)))), \quad \forall\, t \in E;$$

(b) *if $0 \in B(t, U(t))$ a.e. and for every $x^* \in X^* \setminus \{0\}$ there exists $E \subset T$ Lebesgue measurable with $\mu(E) > 0$ such that for all $t \in E$ $\sigma(x^*(S(t_f, t)B(t, U(t)))) > 0$, system* (7) *is approximately locally null-controllable.*

Proof. (a) From the definition of approximately locally null-controllability we have that there is a positive δ such that for all $x_0 \in \mathrm{int}(\mathrm{B}_\delta)$ it holds that $0 \in \mathrm{cl}(R(t_f, x_0))$. Then $0 \leq \sigma(x^*(\mathrm{cl}(R(t_f, x_0))))$. Also $0 \leq \sigma(x^*(R(t_f, x_0)))$. Using theorem 2.2 in [**?**], we have

$$\begin{aligned}
0 &\leq \sigma(x^*(R(t_f, x_0))) \\
&= \sigma(x^*(S(t_f, t_0)x_0)) + \sigma(x^*(\int_{t_0}^{t_f} S(t_f, t)B(t, u(t))))dt \\
&= \sigma(x^*(S(t_f, t_0)x_0)) + \int_{t_0}^{t_f} \sigma(x^*(S(t_f, t)B(t, u(t))))dt,
\end{aligned}$$

for any $x_0 \in \mathrm{int}(\mathrm{B}_\delta)$ and $x^* \in X^*$. Therefore we can write

$$0 \leq \int_{t_0}^{t_f} \sigma(x^*(S(t_f,t)B(t,u(t))))dt.$$

Since $S(t_f,t)B(t,U(t)) \neq \{0\}$ on a set of positive Lebesgue measure, we see that there exists $x^* \in X^* \setminus \{0\}$ and $E \subset T$ Lebesgue measurable, with $\mu(E) > 0$ such that $0 < \sigma(x^*(S(t_f,t)B(t,U(t))))$, for all $t \in E$.

(b) Choose $x^* \in X^* \setminus \{0\}$. Then choose $E \subset T$ Lebesgue measurable with $\mu(E) > 0$ such that for all $t \in E$ $\sigma(x^*(S(t_f,t)B(t,U(t)))) > 0$. Thus we can define the nonempty multifunction L as

$$E \ni t \rightsquigarrow L(t) := \{u \in U(t) \mid x^*(S(t_f,t)B(t,u)) > 0\}.$$

We consider the following mapping

$$E \times Y \ni (t,u) \mapsto g(t,u) := x^*(S(t_f,t)B(t,u))$$

and remark that it is Carathéodory. Then by theorem 6.5 in [**?**] the multifunction

$$E \ni t \rightsquigarrow H(t) := x^*(S(t_f,t)B(t,U(t)))$$

is weakly measurable, hence graph measurable. Recalling that g is Carathéodory and using corollary 6.3 in [**?**], we have that the set

$$\{(t,u) \mid x^*(S(t_f,t)B(t,u)) > 0\}$$

is measurable. Then the multifunction L is graph measurable since

$$graph(L) = graph(H) \cap \{(t,u) \mid x^*(S(t_f,t)B(t,u)) > 0\}.$$

Using the Aumann selection theorem, we get a measurable selection u_1 from L such that $u_1(t) \in L(t)$, a. e. on E.

Now as we mentioned in (c) of Remarks 2.1 the mapping

$$T \times Y \ni (t,u) \mapsto S(t_f,t)B(t,u)$$

is Carathéodory. U has complete values. Then by theorem 6.5 in [**?**] the multifunction

$$T \ni t \rightsquigarrow S(t_f,t)B(t,U(t))$$

is weakly measurable. Thus it is graph measurable. By hypothesis $0 \in S(t_f,t)B(t,U(t))$, for all $t \in T$. Then by theorem 7.2 in [?], we get a measurable selection $u_2(t) \in U(t)$, $t \in T$, such that

$$0 = S(t_f,t)B(t,u_2(t)), \quad \text{a.e.}$$

The selections u_1 and u_2 are integrable, too. Thus we can define

$$\hat{u} = \chi_E u_1 + \chi_{T\setminus E} u_2 \in S^1_U.$$

Let $\hat{x} \in C(T,X)$ be the (unique) mild solution generated by $\hat{u}$ and starting from the origin, i.e., $x_0 = 0$. Then we have

$$\begin{aligned} x^*(\hat{x}(t_f)) &= \int_{t_0}^{t_f} \sigma(x^*(S(t_f,t)B(t,\hat{u}(t))))dt \\ &= \int_E \sigma(x^*(S(t_f,t)B(t,u_1(t))))dt > 0. \end{aligned}$$

Thus

$$\sigma(x^*(R(t_f,0))) > 0.$$

Since $x \mapsto \sigma(x^*(R(t_f,x)))$ is continuous, we can find $\delta > 0$ such that for all $x \in \operatorname{int} \mathrm{B}_\delta$ we have $\sigma(x^*(R(t_f,x))) > 0$. Then $0 \in \mathrm{clco}R(t_f,x) = \mathrm{cl}R(t_f,x)$ for all $x \in \operatorname{int} \mathrm{B}_\delta$ and thus system (7) is approximately locally null-controllable.

Now the proof is complete.

Remarks 2.2.

(a) Our theorem 2.1 is related to theorem 2.2 in [?].

(b) In theorem 2.2 in [?] the multifunction U is considered having convex values and being on a weakly compact subset of Y. We need not such an assumption of convexity of U. Regarding the second assumption, we have required instead that U is integrably bounded.

(c) In theorem 2.2 in [?] the Carathéodory mapping B has linear growth. We need not such an assumption.

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ATTRIBUTIVE CAUSALITY

I. PURDEA AND N. BOTH

Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

1. Motivation

It is known that each of the notion is characterized by some basic properties and by a set of individuals, satisfying these properties. Both elements mentioned above are expressed by conventional terms.

The judgements, as relations between terms, are formally expressed by propositions as (binary) relations in the set of terms. But the pairing of two terms in a relation supposes new attributes.

Example 1. The proposition 'a is the son of b', near the fact that a, b are human individuals, suggests also new attributes concerning personnel properties and/or mutual obligations (see also Example 3).

2. Algebraical step

If M is a set, then each element $x \in M$ is characterized by a set $\mathcal{A}_x$ of attributes from the universe U of all the attributes. We accept that the set $\mathcal{A}_x$ distinguishes the element from any other element of M. This fact may be formulated by

Axiom 1. $x \neq t \Rightarrow \mathcal{A}_x \neq \mathcal{A}_t, \forall x, t \in M$.

Denote by $\mathcal{A}_M$ the set of all the attributes of all the elements of M and observe that

$$\mathcal{A}_M = \bigcup_{x \in M} \mathcal{A}_x.$$

On the other hand, the fact that the elements belong to the same set M offers some common attributes. Therefore we are able to formulate

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Axiom 2. $I_M = \bigcap_{x\in M} \mathcal{A}_x \neq \emptyset$.

Remark 1. $M \subseteq N \Rightarrow I_N \subseteq I_M$; in particular, $I_\emptyset = U$.

Corollary 1. $\forall x \in M :\ \mathcal{A}_x \neq \emptyset$.

Proposition 1. $I_M \cap I_N \subseteq I_{M\cap N}$.

Proof. $I_M \cap I_N = (\bigcap\limits_{x\in M} \mathcal{A}_x) \cap (\bigcap\limits_{y\in N} \mathcal{A}_y) = \bigcap\limits_{z\in M\cup N} \mathcal{A}_z \subseteq \bigcap\limits_{t\in M\cap N} \mathcal{A}_t = I_{M\cap N}$.

We also consider that the name itself of the element x is an attribute of the notion designated by x; this justifies

Axiom 3. $\forall x \in M :\ x \in \mathcal{A}_x$.

Corollary 2. $M \subseteq U$.

Remark 2. $\forall x \in M :\ |\mathcal{A}_x| \geq 2$.

This follows from Axiom 1, Corollary 1 and Axiom 3.

Proposition 2. $M \neq \emptyset \Leftrightarrow \mathcal{A}_M \neq \emptyset$.

Proof. $x \in M \neq \emptyset \Rightarrow x \in \mathcal{A}_x \subseteq \mathcal{A}_M \neq \emptyset$.

$\mathcal{A}_M \neq \emptyset \Rightarrow \exists x \in M :\ \mathcal{A}_x \neq \emptyset \Rightarrow M \neq \emptyset$.

Proposition 3. $\mathcal{A}_{M\cup N} = \mathcal{A}_M \cup \mathcal{A}_N$.

Corollary 3. $M \subseteq N \Rightarrow \mathcal{A}_M \subseteq \mathcal{A}_N$.

Proof. $M \subseteq N \Leftrightarrow M \cup N = N \Leftrightarrow \mathcal{A}_{M\cup N} = \mathcal{A}_N$; but $\mathcal{A}_{M\cup N} = \mathcal{A}_M \cup \mathcal{A}_N$ (Proposition 3), and so $\mathcal{A}_M \cup \mathcal{A}_N = \mathcal{A}_N \Leftrightarrow \mathcal{A}_M \subseteq \mathcal{A}_N$.

Corollary 4. $\mathcal{A}_{M\cap N} \subseteq \mathcal{A}_M \cap \mathcal{A}_N$.

Proof. As $M \cap N \subseteq M$ and $M \cap N \subseteq N$, with Corollary 3 it results that:

$$\mathcal{A}_{M\cap N} \subseteq \mathcal{A}_M \quad \text{and} \quad \mathcal{A}_{M\cap N} \subseteq \mathcal{A}_N \Rightarrow \mathcal{A}_{M\cap N} \subseteq \mathcal{A}_M \cap \mathcal{A}_N.$$

Remark 3. In Corollary 3, the equality is not true, as it results from:

Example 2. Let M be the set of all triangles in the plane and N the set of squares.

$\mathcal{A}_M = \{\text{triangle, convex, bounded}, \ldots\}$

$\mathcal{A}_N = \{\text{square, convex, bounded}, \ldots\}$

As $M \cap N = \emptyset$ (because there is not 'square–triangle') with Proposition 2 we have $\mathcal{A}_{M\cap N} = \emptyset$; but $\mathcal{A}_M \cap \mathcal{A}_N \neq \emptyset$ (it contains at least the convex and bounded plane figures).

3. Attributive extensions

Given the binary relation $r = (A, B, R)$, the statement $(a, b) \in R \subseteq A \times B$ offers a very dry information concerning individuals a, b as well as the pair (a, b).

Example 3. The relation $r = (A, A, R)$, where A is the set of human individuals, and

$$(x, y) \in R \Leftrightarrow \text{'x is the son of y'}$$

ignore essential attributes such as: the rights or the obligations of x relatively to y, the mutual affection and so on.

From this arises the necessity to consider the corresponding attributive sets $\mathcal{A}_A$, $\mathcal{A}_B$ the attributive extension.

Definition 1. The *attributive extension* of the relation $r = (A, B, R)$ is the relation $\mathfrak{r} = (\mathcal{A}_A, \mathcal{A}_B, \mathcal{R})$, where

$$(\lambda, \pi) \in \mathcal{R} \Leftrightarrow \text{ there is } (a, b) \in R \text{ such that } (\lambda, \pi) \in \mathcal{A}_a \times \mathcal{A}_b.$$

We recall that $s = (C, D, S)$ is a *natural extension* of $r = (A, B, R)$ if $r \subseteq s$, that is $A \subseteq C$, $B \subseteq D$, $R \subseteq S$. In this case, r is a *natural restriction* of s.

Remark 4. If s is a natural extension of r then $\mathfrak{s} = (\mathcal{A}_C, \mathcal{A}_D, \mathcal{S})$ is a natural extension of $\mathfrak{r}$, that is

$$r \subseteq s \Rightarrow \mathfrak{r} \subseteq \mathfrak{s}.$$

This results from Corollary 3.

Proposition 4. Any attributive extension is also a natural extension

$$r \subseteq \mathfrak{r}.$$

Proof. From the Axiom 3 we have:

$$A \subseteq \mathcal{A}_A, \; B \subseteq \mathcal{A}_B.$$

From the Definition 1 we obtain:

$$(a, b) \in R, \; a \in \mathcal{A}_a \text{ and } b \in \mathcal{A}_b \Rightarrow (a, b) \in \mathcal{R},$$

so $R \subseteq \mathcal{R}$.

The main purpose of this paper is to suggest a distinction between the 'formal' and the 'causative' relations.

Definition 2. The *pair* $(a, b) \in A \times B$ is *causative* if

$$\mathcal{A}_a \cap \mathcal{A}_b \setminus I_A \cap I_B \neq \emptyset \text{ (see also Axiom 2).}$$

Otherwise, the *pair* (a, b) is *formal.*

The *relation* $r = (A, B, R)$ is called *causative* if all the pairs $(a, b) \in R$ are causative. If all the pairs in R are formal, then the *relation* r is called *formal.*

From this point of view, two particular relations are disputed

$$\delta_A = (A, A, \Delta_A) \text{ and } o = (A, B, \emptyset).$$

The principle of identity impose the 'causativity' of the first and the common sense impose the 'formality' of the second. In this light, we formulate

Axiom 4. a) The identical relation δ_A is causative.

b) The empty relation o is formal.

4. Prospect

(1) The (two-valued) predicates on the set M may be considered as relations between predicative letters $\mathcal{P} \in \Pi$ and the individuals $x \in M$. The problem is to select these predicates $\mathcal{P}(x)$ for which the pair $(\mathcal{P}, x)$ is causative (see [5]).

(2) The causative relations suggest an 'algebraic refinement' of the social relations between individuals or (professional, confessional) groups (see [3]).

(3) Starting from the correspondence $x \mapsto \mathcal{A}_x$ we may define some 'attributive operations' between sets, which allows us to approach aesthetic problems (see [4]).

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SOME APPLICATIONS OF WEAKLY PICARD OPERATORS

IOAN A. RUS

Dedicated to Professor Wolfgang W. Breckner at his 60^{th} anniversary

Abstract. In this paper we give some applications of weakly Picard operators theory to linear positive approximation operators, to difference equations with deviating argument and to functional-integral equations.

1. **Introduction**

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations:

$$F_A := \{x \in X |\ A(x) = x\};$$

$$I(A) := \{Y \subset X |\ A(Y) \subset Y,\ Y \neq \emptyset\};$$

$$A^0 := 1_X,\ A^1 := A, \ldots, A^{n+1} := A \circ A^n, \quad n \in N.$$

By definition an operator A is weakly Picard operator (WPO) if the sequence of successive approximations, $(A^n(x))_{n\in N}$ converges for all $x \in X$ and the limit is a fixed point of A. If the operator A is WPO and $F_A = \{x^*\}$, then by definition the operator A is Picard operator (PO). For an WPO A we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

We have the following characterization of the WPOs.

Theorem 1.1 (I. A. Rus [6], [7], [12]). *An operator A is WPO if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A),\ \forall\ \lambda \in \Lambda$;

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(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is PO,* $\forall\ \lambda \in \Lambda$.

The aim of this paper is to give some applications of this theorem.

2. Iterates of two variables Bernstein operator

Let $\overline{D} = \{(x,y) \in R^2|\ x,y \in R_+,\ x+y \leq 1\}$ and $e_{ij} : \overline{D} \to R_+$ be defined by $e_{ij} := x^i y^j$, $i,j \in N$.

Let we denote by $\|\cdot\|_C$ the Chebyshev norm on $C(\overline{D})$.

In what follow we consider the two variables Bernstein operator (see D. D. Stancu [13])

$$B_n : C(\overline{D}) \to C(\overline{D}), \quad n \in N^*$$

defined by

$$B_n(f)(x,y) := \sum_{0\leq i+j\leq n} \frac{n!}{i!j!(n-i-j)!} x^i y^j (1-x-y)^{n-i-j} f\left(\frac{i}{n}, \frac{j}{n}\right). \tag{2.1}$$

It is well known that ([13]):

$$e_{00}, e_{01}, e_{10} \in F_{B_n}, \quad n \in N^*.$$

We have

Theorem 2.1. *The operator* B_n *is WPO and*

$$B_n^\infty(f)(x,y) = f(0,0) + [f(1,0) - f(0,0)]x + [f(0,1) - f(0,0)]y, \quad x,y \in \overline{D};\ f \in C(\overline{D}).$$

Proof. Let

$$X_{\alpha,\beta,\gamma} := \{f \in C(\overline{D})|\ f(0,0) = \alpha,\ f(1,0) = \beta,\ f(0,1) = \gamma\},$$

$$f_{\alpha,\beta,\gamma}(x,y) := \alpha + (\beta-\alpha)x + (\gamma-\alpha)y, \quad x,y \in \overline{D},$$

for all $\alpha, \beta, \gamma \in R$.

We remark that

(i) $X_{\alpha,\beta,\gamma}$ is a closed subset of $C(\overline{D})$;

(ii) $X_{\alpha,\beta,\gamma}$ is an invariant subset of B_n, for all $\alpha,\beta,\gamma \in R$ and $n \in N^*$;

(iii) $C(\overline{D}) = \bigcup_{\alpha,\beta,\gamma \in R} X_{\alpha,\beta,\gamma}$ is a partition of $C(\overline{D})$;

(iv) $f_{\alpha,\beta,\gamma} \in X_{\alpha,\beta,\gamma} \cap F_{B_n}$.

Now we prove that

$$B_n|_{X_{\alpha,\beta,\gamma}} : X_{\alpha,\beta,\gamma} \to X_{\alpha,\beta,\gamma}$$

is a contraction for all $\alpha, \beta, \gamma \in R$ and $n \in N^*$.

Let $f, g \in X_{\alpha,\beta,\gamma}$. From (2.1) we have

$$|B_n(f)(x,y) - B_n(g)(x,y)| = |B_n(f-g)(x,y)| \leq$$

$$\leq |1 - (1-x-y)^n - x^n - y^n| \cdot \|f-g\|_C \leq$$
$$\leq \left(1 - \frac{1}{2^{n-1}}\right) \|f-g\|_C, \ \forall\, x, y \in \overline{D}.$$

So,

$$\|B_n(f) - B_n(g)\|_C \leq \left(1 - \frac{1}{2^{n-1}}\right) \|f-g\|_C, \ \forall\, f, g \in X_{\alpha,\beta,\gamma};$$

i.e., $B_n|_{X_{\alpha,\beta,\gamma}}$ is a contraction for all $\alpha, \beta, \gamma \in R$.

From the contraction principle $f_{\alpha,\beta,\gamma}$ is the unique fixed point of B_n in $X_{\alpha,\beta,\gamma}$ and that $B_n|_{X_{\alpha,\beta,\gamma}}$ is a PO.

From the Theorem 1.1 the proof follows.

Remark 2.1. For the one dimensional case see I. A. Rus [10], [11], [12] and O. Agratini and I. A. Rus [1]. See also R.P. Kelisky and T.J. Rivlin [4].

Remark 2.2. The case $\overline{D} = [0,1] \times [0,1]$ (see P. L. Butzer [3]) will be presented elsewhere.

Remark 2.3. A similar result for Bernstein operators on a simplex we have.

3. Difference equations in $C([0,1], X)$

Let X be a Banach space. We denote by $\|\cdot\|_C$ the Chebyshev norm on $C([0,1], X)$. Let $h \in C([0,1] \times X \times X, X)$ and $g \in C([0,1] \times X, X)$ be two operators. In what follow we consider the following difference equation with deviating argument, in $C([0,1], X)$,

$$x_{n+1}(t) = h(t, x_n(t), x_n(0)) + g(t, x_n(t)), \quad t \in [0,1), \ n \in N^* \tag{3.1}$$

For to study this equation we consider the operator

$$A : C([0,1], X) \to C([0,1], X)$$

$$A(x)(t) := h(t, x(t), x(0)) + g(t, x(t)).$$

We have

Theorem 3.1. *We suppose that*

(i) $h(0,\lambda,\lambda)=\lambda,\ \forall\ \lambda\in X$;

(ii) $g(0,\lambda)=0,\ \forall\ \lambda\in X$;

(iii) $g(t,\cdot)$ *is an* α*-contraction for all* $t\in[0,1]$;

(iv) $h(t,\cdot,\lambda)$ *is a* β*-contraction for all* $t\in[0,1]$, $\lambda\in X$;

(v) $\alpha+\beta<1$.

Then the operator A *is WPO.*

Proof. Let

$$X_\lambda := \{x\in C([0,1],X) \mid x(0)=\lambda\},\quad \lambda\in X.$$

Then

(a) X_λ is a closed subset of $C([0,1],X)$;

(b) $X_\lambda\in I(A)$, for all $\lambda\in X$;

(c) $C([0,1],X)=\bigcup\limits_{\lambda\in\Lambda} X_\lambda$ is a partition of $C([0,1],X)$.

From (i)-(v) we have that the restriction of A to X_λ is an $(\alpha+\beta)$-contraction.

By the Theorem 1.1 we have that the operator A is WPO.

Let x^*_λ be the unique fixed point of the operator A in X_λ. It is clear that $cardF_A=cardX$, and that F_A is the equilibrium solution set of the equation (3.1).

From the Theorem 3.1 we have

Theorem 3.2. *In the conditions of the Theorem 3.1, let* $(x_n)_{n\in N}$ *be a solution of the equation (3.1). If* $x_0\in X_\lambda$*, then* $x_n\in X_\lambda$*, for all* $n\in N$*. Moreover*

$$x_n\to x^*_\lambda \text{ as } n\to\infty.$$

Remark 3.1. In the conditions of Theorem 3.1 the equilibrium solution x^*_λ is globally asymptotically stable relative to X_λ.

Remark 3.2. For the fixed point technique in the theory of difference equations see M. A. Şerban [14].

Remark 3.3. The following example is in the conditions of the Theorem 3.1:

$$x_{n+1}(t)=\frac{1}{2}t\sin x_n(t)+x_n(0),\quad n\in N$$

$$x_0 \in C[0,1]$$

4. Functional-integral equations

Let X be a Banach space $f \in C([a,b] \times X, X)$ and $K \in C([a,b] \times [a,b] \times X, X)$. Consider the following functional-integral equation

$$x(t) = x(a) + \int_a^t f(s, x(s))ds + \int_\alpha^t \int_a^s K(s, u, x(u))duds \tag{4.2}$$

$$t \in [a,b]; \quad x \in C([a,b], X)$$

Let $X_\lambda := \{x \in C([a,b], X) | \ x(a) = \lambda\}$, $\lambda \in X$ and $A : C([a,b], X) \to C([a,b], X)$ defined by $A(x)(t) :=$ second part of (4.1).

If we denote by S the solution set of the eq. (4.1) then $S = F_A$.

We remark that

(a) X_λ is a closed subset of $C([0,1], X)$ for all $\lambda \in X$;

(b) $X_\lambda \in I(A)$;

(c) $C([0,1], X) = \bigcup_{\lambda \in X} X_\lambda$ is partition of $C([0,1], X)$;

(d) if $f(s, \cdot)$ is L_f-Lipschitz and $K(s, u, \cdot)$ is L_K-Lipschitz for all $s, u \in [a,b]$ then the restriction of A to X_λ is a contraction with respect to a suitable Bielecki's norm. More exactly if we denote

$$\|x\|_B = \max_{a \le t \le b} (\|x(t)\| e^{-\tau(t-a)})$$

then we have

$$\|A(x) - A(y)\|_B \le \left(\frac{L_f}{\tau} + \frac{L_K}{\tau^2} \right) \|x - y\|_B, \ \forall \ x, y \in X_\lambda; \ \lambda \in X.$$

Let x^*_λ be the unique fixed point of A in X_λ. From the Theorem 1.1 it follows that the operator A is WPO and $cardF_A = cardX$.

So, we have

Theorem 4.1. *In the above conditions*

(1) $cardS = cardX$

(2) the solution x^*_λ is globally asimptotically stable with respect to X_λ.

Remark 4.1. For other types of functional integral equations see R. Precup [5], I. A. Rus [8] and [9].

Remark 4.2. For other applications of the WPO see A. Buică [2], I. A. Rus [6], [7].

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ON CERTAIN CLASSES OF GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS, II

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Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

In the first part [8] we have studied the η-invex functions first introduced by the author in 1988. We have also introduced and studied η-invexity, η-pseudo-invexity, Jensen-invexity (and the underlying invex and Jensen-invex sets), almost-invexity, as well as almost-cvazi-invexity.

In this second part we shall introduce and study the notions of A-convexity; resp. Λ-invexity ($\Lambda \subset [0,1]$, dense).

1. A-convex functions

Definition 1.1. ([5]) Let X be a real linear space, and $B : X \times X \to \mathbb{R}$ a given application. We say that a function $f : X \to \mathbb{R}$ is **B-subadditive** (superadditive) if one has

$$f(x+y) \leq (\geq) f(x) + f(y) + B(x,y) \text{ for all } x, y \in X. \tag{1}$$

An immediate property related to this definition is:

Proposition 1.1. *If B is an antisymmetric application and f is B-subadditive (superadditive), then f is subadditive (superadditive).*

Proof. One can write

$$f(x+y) \leq f(x) + f(y) + B(x,y) \text{ and } f(x+y) \leq f(y) + f(x) + B(y,x)$$

By addition, it follows

$$f(x+y) \leq f(x) + f(y) + \frac{1}{2}[B(x,y) + B(y,x)] = f(x) + f(y),$$

since $B(x,y) = -B(y,x)$, B being antisymmetric. Therefore, f is subadditive.

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Definition 1.2. Let $B : X \times X \to \mathbb{R}_+$, with X again a real linear space. We say that $f : X \to \mathbb{R}$ is **absolutely-B-subadditive**, if the following relation holds true:

$$|f(x+y) - f(x) - f(y)| \le B(x,y) \tag{2}$$

Theorem 1.1. [5] *If $B : X \times X \to \mathbb{R}$ is homogeneous of order zero, and if $f : X \to \mathbb{R}$ is absolutely-B-subadditive, then there exists a single additive function $g : X \to \mathbb{R}$, which "quadratically approximates" f, i.e.*

$$|f(x) - g(x)| \le B(x,x), \quad x \in X \tag{3}$$

Proof. Put $x := 2^{n-1}x$, $y := 2^{n-1}x$ in relation (2). We get

$$\left| \frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}} \right| \le \frac{B(x,x)}{2^n}.$$

By the modulus inequality, one has, on the other hand

$$\left| \frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} \right| \le \left| \frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}} \right| + \left| \frac{f(2^{n-1} x)}{2^{n-1}} - \frac{f(2^{n-2}x)}{2^{n-2}} \right| +$$

$$+ \cdots + \left| \frac{f(2^{m+1} x)}{2^{m+1}} - \frac{f(2^m x)}{2^m} \right| \text{ for } n > m.$$

Thus

$$\left| \frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} \right| \le B(x,x) \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} \right)$$

This inequality easily implies that the sequence of general term $x_n = \dfrac{f(2^n x)}{2^n}$ is fundamental. $\mathbb{R}$ being a complete metric space, (x_n) has a limit; let

$$g(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{4}$$

We now prove that g is additive. Indeed, one has

$$|g(x+y) - g(x) - g(y)| = \lim_{n \to \infty} \left| \frac{f(2^n x + 2^n y)}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right| \le$$

$$\le \lim_{n \to \infty} \frac{B(x,y)}{2^n} = 0.$$

This gives $g(x+y) = g(x) + g(y)$. We now show that g is unique. Let us assume that there exists another additive application h such that

$$|f(x) - h(x)| \le B(x,x).$$

Then

$$|g(x) - h(x)| = |g(x) - f(x) + f(x) - h(x)| \leq 2B(x, x),$$

by assumption. Thus

$$|g(2^n x) - h(2^n x)| \leq 2B(2^n x, 2^n x),$$

implying

$$|g(x) - h(x)| \leq \frac{B(x, x)}{2^{n-1}} \to 0$$

as $n \to \infty$. (Indeed, $g(2^n x) = 2^n g(x)$ and $h(2^n x) = 2^n h(x)$; g and h being additive).

Now, an inductive argument shows that $|f(2^n x) - 2^n f(x)| \leq 2^n B(x, x)$. By dividing with 2^n and letting $n \to \infty$, one has $|f(x) - g(x)| \leq B(x, x)$, i.e. g approximates f in the above defined manner.

Proposition 1.2. *Let $f : (0, +\infty) \to \mathbb{R}$ be such that the application $x \to \frac{f(x)}{x}$ is B-decreasing on $(0, +\infty)$. Then f is B_1-subadditive, where*

$$B_1(x, y) = xB(x + y, x) + yB(x + y, y); \quad x, y \in (0, +\infty).$$

Proof. Since $x, y > 0$; $x + y > x$ implies

$$\frac{f(x + y)}{x + y} \leq \frac{f(x)}{x} + B(x + y, x)$$

and

$$\frac{f(x + y)}{x + y} \leq \frac{f(y)}{y} + B(x + y, x)$$

(here $x + y > y$). Therefore,

$$f(x + y) = \frac{f(x + y)}{x + y}(x + y) \leq \frac{f(x)}{x} \cdot x + xB(x + y, x) + \frac{f(y)}{y} \cdot y + yB(x + y, y) =$$

$$= f(x) + f(y) + B_1(x, y),$$

by the above written two inequalities, and by the definition of B_1.

Definition 1.3. Let Y be a **convex subset** of the real linear space X. Let $A : Y \times Y \times Y \to \mathbb{R}$ be an application of three variables. We say that the function $f : Y \to \mathbb{R}$ is A**-convex** (concave) if the following inequality holds true:

$$f(\lambda u + (1 - \lambda)v) \leq (\geq) \lambda f(u) + (1 - \lambda)f(v)+$$

$$+\lambda(u - v)A(\lambda u + (1 - \lambda)v, u, v) \tag{5}$$

for all $u, v \in Y$, all $\lambda \in [0, 1]$.

Definition 1.4. Let Y be an η-invex set of X (see [8] for definition and related examples or results). We say that $f : Y \to \mathbb{R}$ is an $\eta - A$**-invex** (incave) function, if

$$f(v + \lambda\eta(u, v)) \leq (\geq) \lambda f(u) + (1 - \lambda)f(v) + \lambda(u - v)A(\eta(u, v), u, v) \tag{6}$$

for all $u, v \in Y$, all $\lambda \in [0, 1]$.

Proposition 1.3. *Let* $A : \mathbb{R}^3_+ \to \mathbb{R}$ *and* $f : \mathbb{R}_+ \to \mathbb{R}$ *be an* $A(\cdot, \cdot, 0)$-*concave function. Put* $A_1(\cdot, \cdot) = A(\cdot, \cdot, 0)$ *and assume that* $f(0) = 0$. *Then* f *is a* B_1-*subadditive function, where*

$$B_1(x, y) = -xA_1(x, x + y) - yA_1(y, x + y). \tag{7}$$

Proof. First remark that the A-convexity (concavity) of f is equivalent to the inequality

$$\frac{f(x) - f(z)}{x - z} \leq (\geq) \frac{f(y) - f(z)}{y - z} + A(x, y, z), \quad x < z < y \tag{8}$$

where the application $F_z(x) = \dfrac{f(x) - f(z)}{x - z}$ is an A_z-increasing application for all fixed z, with $A_z(x, y) = A(x, y, z)$. Indeed, let $z < x < y$. Then inequality (8) with $\geq$ can be written also as

$$(y - z)f(x) - (y - z)f(z) \geq (x - z)f(y) - (x - z)f(z) + (x - z)(y - z)A(x, y, z),$$

i.e.

$$(y - z)f(x) \geq (x - z)f(y) + (y - x)f(z) + (x - z)(y - z)A(x, y, z)$$

or

$$f(x) \geq \lambda f(y) + (1 - \lambda)f(z) + (x - z)A(x, y, z),$$

with $\lambda := \dfrac{x - z}{y - z} \in (0, 1)$ and $1 - \lambda = 1 - \dfrac{x - z}{y - z} = \dfrac{y - x}{y - z}$ and $x = \lambda y + (1 - \lambda)z$. Since, by assumption one has $f(0) = 0$ and $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{f(x)}{x}$, from the above remark, the function $\dfrac{f(\cdot)}{(\cdot)}$ is A_1-increasing. Thus, one can write

$$\frac{f(x)}{x} \geq \frac{f(x + y)}{x + y} + A_1(x, x + y), \text{ resp.}$$

$$\frac{f(y)}{y} \geq \frac{f(x+y)}{x+y} + A_1(y, x+y),$$

giving

$$f(x) + f(y) \geq f(x+y)\left(\frac{x}{x+y} + \frac{y}{x+y}\right) + xA_1(x, x+y) + yA_1(y, x+y) =$$

$$= f(x+y) - B_1(x, y).$$

This implies $f(x+y) \leq f(x) + f(y) + B_1(x, y)$, i.e. f is B_1-subadditive, where B_1 is given by (7).

Proposition 1.4. *Let $f : (0, \infty) \to \mathbb{R}$ be a convex function (in the classical sense) and B-subadditive. Then the function g given by $g(x) = \frac{f(x)}{x}$ is a C-increasing function for some $C : (0, \infty) \times (0, \infty) \to \mathbb{R}$.*

Proof. Let $\lambda = \frac{x}{x+h} \in (0, 1)$ with $h > 0$ and $x + h = \lambda x + (1 - \lambda)(2x + h)$. From the B-subadditivity of f one has

$$f(2x + h) \leq f(x) + f(x + h) + B(x, x + h).$$

The convexity of f implies

$$f(x + h) \leq \lambda f(x) + (1 - \lambda)f(2x + h).$$

Therefore,

$$f(x + h) \leq \lambda f(x) + (1 - \lambda)f(x) + (1 - \lambda)f(x + h) + (1 - \lambda)B(x, x + h).$$

This gives

$$\lambda f(x + h) \leq f(x) + (1 - \lambda)B(x, x + h).$$

Here $\lambda = \frac{x}{x+h}$ and $1 - \lambda = \frac{h}{x+h}$, so

$$\frac{x}{x+h}f(x + h) \leq f(x) + \frac{h}{x+h}B(x, x + h),$$

or

$$\frac{f(x + h)}{x + h} \leq \frac{f(x)}{x} + C(x, h),$$

where $C(x, h) = \frac{h}{x} \cdot \frac{B(x, x + h)}{x + h}$, which concludes of the proof of the C-monotonicity of g.

2. **Λ-invex functions ($\Lambda \subseteq [0,1]$, dense)**

Let $\Lambda \subseteq [0,1]$ be a fixed, dense subset of $[0,1]$. As a generalization of the notion of η-cvazi-invexity (see [8]), we shall introduce the notion of **$\eta - \Lambda$-invexity** as follows:

Definition 2.1. ([7]) Let X be a real linear space, $S \subset X$ an η-invex subset of X, where $\eta : X \times X \to X$ (see [8]), and let $f : S \to \mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$. We say that f is an **$\eta - \Lambda$-invex** function, if the following inequality holds true:

$$f(x + \lambda\eta(y,x)) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in S, \text{ all } \lambda \in \Lambda. \tag{9}$$

Remark 2.1. When $\Lambda \equiv [0,1]$, the notion of $\eta - \Lambda$-invexity of f coincides with that of η-cvazi-invexity of f.

Definition 2.2. The set $D(f) = \{x \in S : \ f(x) < +\infty\}$ will be called the **effective domain** of $f : S \to \mathbb{R}_+$.

Definition 2.3. A point $x \in S$ with the property $f(x) = +\infty$ will be called as a **singular point** of f. The **set of all singular points** of f will be denoted by $S(f)$.

In what follows we shall assume that $S = X$, which is a **real normed space**. Let us use the following (standard) notations

$$\underline{f}(x) = \liminf_{y \to x} f(y); \quad \overline{f}(x) = \limsup_{y \to x} f(y).$$

The following result extends theorems due to F. Bernstein and G. Doetsch [1], E. Mohr [4], A. Császár [2].

Theorem 2.1. ([7]) *Let $f : X \to \mathbb{R}_\infty$ be an $\eta - \Lambda$-invex set and let $K \subset D(f)$ be an open, η-invex set. Let us assume that the application $\eta : X \times X \to X$ is continuous in the strong topology and that $\underline{f}(x) > -\infty$ for all $x \in X$. Then the function $\underline{f} : K \to \mathbb{R}$ is η-cvazi-invex.*

Proof. Let $x, y \in K$. There exists $b \in (0,1)$ with $z = x + b\eta(y,x) \in K$. Since we are in the case of normed spaces, we can select sequences (x_k), (y_k) such that $x_k \to x$, $y_k \to y$ $(k \to \infty)$ imply $f(x_k) \to \underline{f}(x)$ and $f(y_k) \to \underline{f}(y)$ $(k \to \infty)$.

Let then $(a_k) \subset \Lambda$ be a sequence such that $a_k \to b$, and put $z_k = x_k + a_k\eta(y_k, x_k)$.

The function η being continuous in the norm topology, one can write $z_k \to x + b\eta(y,x) = z$ and $\underline{f}(x) \leq \liminf_{k\to\infty} f(z_k)$. But from $f(z_k) \leq \max\{f(x_k), f(y_k)\}$, by taking $k \to \infty$ one obtains immediately

$$\underline{f}(z) \leq \liminf_{k\to\infty} f(z_k) \leq \max\left\{\liminf_{k\to\infty} f(x_k), \liminf_{k\to\infty} f(y_k)\right\} =$$

$$= \max\{\underline{f}(x), \underline{f}(y)\},$$

proving the η-cvazi-invexity of the function $\underline{f}$.

Proposition 2.1. *If $f : X \to \mathbb{R}_\infty$ is η-invex (or η-cvazi-invex), then the set $D(f)$ is η-invex set (or η-cvazi-invex set).*

Proof. Let $x, y \in D(f)$. Then $f(x) < +\infty$, $f(y) < +\infty$, so

$$f(x + \lambda\eta(y,x)) \leq \lambda f(y) + (1-\lambda)f(y) < +\infty$$

(in the η-invex case); or

$$f(x + \lambda\eta(y,x)) \leq \max\{f(x), f(y)\} < +\infty$$

(in the η-cvazi-invex case). In any case, one has $x+\lambda\eta(y,x) \in D(f)$ for all $x, y \in D(f)$, all $\lambda \in [0,1]$, proving the η-invexity of the set $D(f)$.

Theorem 2.2. *Let us assume that the real Banach space X and the application η have the following property:*

For $M \subset X$, if $x, x_0 \in intM_0$, then there exists $\lambda \in (0,1)$ and $y \in M$ such that

$$x = x_0 + \lambda\eta(y, x_0). \tag{$*$}$$

Let $f : X \to \mathbb{R}_\infty$ be an $\eta - \Lambda$-invex function and let $x_0 \in intD(f)$ be selected such that $\overline{f}(x_0) < +\infty$. If η is nonexpansive related to the second argument; then $\overline{f}(x) < +\infty$ for all $x \in intD(f)$.

Proof. Let $M := D(f)$ in $(*)$ and let $x, x_0 \in D(f)$, where $\overline{f}(x) = +\infty$, $\overline{f}(x_0) < +\infty$. By condition $(*)$, there exists $\lambda \in \Lambda$ and $y \in D(f)$ such that

$$x = x_0 + \lambda\eta(y, x_0). \tag{10}$$

Select now a sequence (x_k) with $x_k \in D(f) \setminus \{x\}$ such that $x_k \to x$, $f(x_k) \to +\infty$ $(k \to +\infty)$. Thus there exists $k_0 \in \mathbb{N}$ with

$$f(x_k) > f(y) \text{ for all } k \geq k_0. \tag{11}$$

Let z_k be determined by the equation

$$x_k = z_k + \lambda\eta(y, z_k), \quad k \in \mathbb{N}. \tag{12}$$

Equation (10) can be solved for all z_k (k=fixed), since, by letting, with $z_k = z$, the application $g(z) = x - \lambda\eta(y, z)$, $g : X \to X$ becomes a **contraction**. Indeed, one has

$$\|g(z_1) - g(z_2)\| = \lambda\|\eta(y, z_1) - \eta(y, z_2)\| \leq \lambda < 1,$$

η being nonexpansive upon the second argument.

Now Banach's classical contraction principle assures the existence of a unique fix point of the operator g; in other words, equation (10) has a single solution.

We shall prove now that

$$z_k \to x_0. \tag{13}$$

For this aim, remark that

$$\|x_k - x\| = \|z_k - x + \lambda\eta(y, z_k)\| =$$

$$= \|z_k - x_0 + \lambda(\eta(y, x_0) - \eta(y, z_k))\| > \|z_k - x_0\| - \lambda\|\eta(y, x_0) - \eta(y, z_k)\| >$$

$$> \|z_k - x_0\| - \lambda\|z_k - x_0\| = (1 - \lambda)\|z_k - x_0\|.$$

Therefore,

$$\|z_k - x_0\| < \frac{1}{1 - \lambda}\|x_k - x\| \to 0$$

as $k \to \infty$, finishing the proof of relation (14).

Let now z_k be defined uniquely by (10), and let $k \geq k_0$ be given by (11). One can write

$$f(y) < f(x_k) \leq \max\{f(z_k), f(y)\} = f(z_k),$$

so on base of (13), one obtains $\overline{f}(x_0) \geq \lim\limits_{k\to\infty} f(z_k) = +\infty$, which contradicts the assumption $\overline{f}(x_0) = +\infty$.

Remark 2.2. If η has the **nonexpansivity property upon both arguments**, i.e.

$$\|\eta(y, x) - \eta(y_0, x_0)\| \leq \|y - y_0\| + \|x - x_0\|,$$

it is immediately seen that if $M \subseteq X$ is an invex set, then $intM$ will be also invex (for the same η; i.e. η-invex). Thus, for $\Lambda \equiv [0, 1]$, on base of Proposition 2.1, relation $(*)$

holds true for η-cvazi-invex sets. Remark that for $y = y_0$, the nonexpansivity upon the second variable is contained in the above duble nonexpansivity property.

We now prove the main result of this section:

Theorem 2.3. ([6], [7]) *Let us assume that* $f : X \to \mathbb{R}_\infty$ *satisfies the conditions of Theorem 2.2 and that* f *is* **inferior semicontinuous**. *In this case one has the following alternatives: i)* $D(f) = \emptyset$, *ii) If there exists* $x_0 \in intD(f)$ *with* $\overline{f}(x_0) < +\infty$*; then the set* $S(f)$ *of singularities can be written as a numerable intersection of dense sets in* X. *If* $intD(f) \neq \emptyset$, *then* $\overline{f}(x) < +\infty$ *for all* $x \in intD(f)$.

Proof. For $n \in \mathbb{N}$ defined the sets $X_n = \{x \in X : \ f(x) > n\}$, which is an open set. One can write: $S(f) = \cap\{X_n : \ n \in \mathbb{N}\}$. The sets X_n are dense in X, since if not, i.e. if X_{n_0} is not dense ($n_0 \in \mathbb{N}$), then there exists $y_0 \in X$ and a closed ball $B(y_0, r) = B$ such that $B \cap X_{n_0} = \emptyset$. Thus for $x \in B$ we would have $f(x) \leq n_0$. If $intD(f) \neq \emptyset$, by Theorem 2.2 we have $\overline{f}(x) < +\infty$ for all $x \in intD(f)$, which is impossible, by assumption. If $\overline{f}(x_0) = +\infty$ for an $x_0 \in intD(f)$, by Baire's classical lemma one has $S(f) = \cap\{X_n : \ n \in \mathbb{N}\}$ is dense in X. There for $intD(f) = \emptyset$, contradicting $x_0 \in intD(f)$.

Remark 2.3. Theorem 2.3 constitutes a generalization of a theorem by J. Kolumbán [3]. For $\eta(x, y) = x - y$ (i.e. the convex case), we can deduce a generalization of the well known theorem of Banach-Steinhaus on the condensation of singularities.

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BOOK REVIEWS

Matoušek, Jiří, *Lectures on Discrete Geometry*, Springer (Graduate Texts in Mathematics, 212), 2002, 481 pp., Softcover, ISBN 0-387-95334-4.

Discrete geometry is not quite a newcomer on the stage of mathematics. Isolated results belonging to this field can be found already in the works of Descartes, Euler, Dirichlet and, in more modern times, Voronoi, Delaunay, Minkowski, Brunn, Helly, and many others. It deals, mainly, with finite sets of simple geometrical objects such as points, lines, circles or their higher dimensional analogues and it studies things like reciprocal positions of these objects, counts the intersection points or the zones determined by higher dimensional objects (for instance arrangements of hyperplanes, in particular of straight lines) and other problems of the same kind.

In the last few decades the discrete geometry has seen a more rapid development, in connection to some related fields, such as computational geometry or computational geometrical optimization. In spite of the increasing interest in the field, there are still only few reliable textbooks on the market. A notable contribution is, no doubt, the book of Pach and Agarwal (*Combinatorial Geometry*, Wiley, 1995).

The book under review covers, in my opinion, a gap in the pedagogical literature, providing an expository treatment of a wide range of topics in discrete geometry, without assuming too many prerequisites from the reader. We choose just a couple of subjects examined into the book, taken from a list provided by the author himself:

- foundational results from affine and convex geometry, including the Minkowski theorem on lattice points, a couple of words about Voronoi diagrams and Delaunay triangulations a.o.
- combinatorial complexity of geometric configurations (line-point incidences, complexity of arrangements, Davenport-Schinzel sequences, probabilistic methods);
- intersection patterns and transversal of convex sets;
- geometric Ramsey theory, related to the existence, in any sufficiently large configuration, of a subconfiguration which is, in a specific sense, regular;
- polyhedral combinatorics and high-dimensional complexity;
- representation of finite metric spaces by coordinate.

The list is far from being complete. Of course, the field is quite vast, so a lot of subjects had to be left aside. Still, the book is very comprehensive and starts from a low level (only some linear algebra, elementary calculus, probability and combinatorics are assumed), so it will be an ideal to be used both as a textbook and for self-study. The expected audience includes graduate students and researchers in discrete and computational geometry, optimization and computer science. The author is a well-known expert, rather in computational geometry than in discrete geometry and, sometimes, his personal tastes are easily recognized. In facts, in some sense, this book can be used as a "mathematical companion" to a textbook on computational geometry where, usually, the authors focuses on the description and analysis of algorithms rather than on the mathematics which is behind these algorithms.

The book is completed with a lot of examples and exercises, not to mention the impressive number of line diagrams, which cannot miss in such a kind of book.

Paul A. Blaga

I. John Cagnol, Michael P. Polis, Jean-Paul Zolesio (Eds.), *Shape Optimization and Optimal Design,* Lecture Notes in Pure and applied Mathematics, vol. 216, Marcel Dekker, New York-Basel, 2001, ISBN: 0-8247-0556-4.

II. Giuseppe Da Prato, Jean-Paul Zolesio (Eds.), *Partial Differential Equation Methods in Control Analysis,* Lecture Notes in Pure and Applied Mathematics, vol. 188, Marcel Dekker, New York-Basel-Hong Kong, 1997, ISBN: 0-8247-9837-6.

The first volume mentioned comprises papers from the sessions "Distributed Parameter Systems" and "Optimization Methods and Engineering Design" held within the 19th conference System Modeling and Optimization in Cambridge, England.

The second volume presents papers from the Conference on Control and Shape Optimization held at Scuola Normale Superiore di Pisa, Italy. Both the conferences were organized by the International Federation for Information Processing (IFIP).

The papers present the latest developments and major advances in the fields of active and passive control for systems governed by partial differential equations- in particular in shape analysis and optimal shape design.

Traditionally, optimal shape design has been treated as a branch of the calculus of variations, more specifically of optimal control. The subject interfaces with at least four fields: optimization, optimal control, PDEs and their numerical solutions.

The main question that optimal shape design tries to answer is: "What is the best shape for a physical system?".

Many problems that arise in technical and industrial applications can be formulated as the minimization of functionals with respect to a geometrical domain which must belong to an admissible family. Optimal shape design is used in various fields, like those mentioned in the books: fluid mechanics, linear elasticity, thermo-elasticity, soil mechanics, electricity, aircraft industry, material sciences, biodynamics.

The authors of the articles are well known for important results in this field of research.

Some of the aspects treated are:

- shape sensitivity analysis (that is the sensitivity of the solutions with respect to the shape of the domain) for the Navier-Stokes equation, Maxwell's equation, for some problems with singularities (I)
- the study of the material derivative, the shape derivative on a fractured manifold (I), the shape derivative for the Laplace-Beltrami equation (II), the shape hessian for a nondifferentiable variational free boundary problem (II),the shape gradients for mixed finite element formulation (II),the eulerian derivative for non-cylindrical functionals (I)
- numerical aspects (using finite element approximation and other methods, some of them original) for
  - shape problems in linear elasticity (I)
  - parallel solution of contact problems (I)
  - modeling of oxygen sensors (I)
  - control of a periodic flow around a cylinder (I)
  - shape identification problems associated with the stationary heat conduction in 2D(II)
- boundary controllability of thermo-elastic plates (I)
- regularity properties for the weak solutions to certain parabolic equations(II)
- homogenization and continuous dependence for Dirichlet problems, asymptotic analysis on singular perturbations (II), asymptotic analysis of aircraft wing model in subsonic flow (I)
- mapping method in problems governed by hemivariational inequalities (I)
- feedback laws for the optimal control of parabolic variational inequalities

Many more subjects are treated in the 41 papers by 50 authors, which allow the reader to get a good idea about the latest research directions in this very active field of applied mathematics.

Daniela Inoan

Unsöld, Albrecht, Baschek, Bodo, *Der neue Kosmos*, Springer, 2002, 575 pp., Hardcover, ISBN 3-540-44177-7.

The book under review is the 7^{th} edition of Unsöld's textbook "Neue Kosmos", whose first edition was published in 1967. Starting with its third edition the book was updated jointly with Bodo Baschek. He continued to up date and add to contents of the book new after Unsöld pass away. This new edition, that came three years after the sixth edition of the book, contains new results about the Solar System and the Universe as a whole, obtained in these last years.

The book has has four parts: classical astronomy and Solar System, practical astronomy, stellar structure and cosmology and cosmogony. The first part is devoted to the foundations of the astronomy (spherical astronomy, time, celestial mechanics) and a description of the motion of celestial bodies (planets, Sun, Moon). In the last chapters from these part are described the Solar System bodies (planets, satellites, asteroids, comets and meteorites) from the physical point of view. The second part of the book contains a brief introduction in the problems of the practical astronomy. Firstly, the are given the basic notions about radiation and its interaction with the matter and after that there are described the astronomical tools and techniques. The third part of the book is devoted to the physics of the stars. Here are presented the main topics related to the stellar structure and evolution. There are described different types of stars including the Sun. The stellar systems, stellar clusters and galaxies, are described in the last part of the book. Another task of the book is to introduce the problems related to cosmology and cosmogony. Each part begins with a short historical note and at the end of the book there are two appendices devoted to the astronomical units, respectively a list of the constellations. There are also selected problems that could be used during the learning and teaching process.

The book also includes 278 images and line diagrams, including 20 colour plates. The sources of the images are given in an appendix.

The book is highly recommended to students in astronomy and astrophysics, being ideal as a textbook. Let me also mention the impressive graphical qualities of the books, something that, unfortunately, is increasingly rare nowadays.

Cristina Blaga

V. Benci, G. Cerami, M. Degiovanni, D. Fortunato, F. Giannoni, A.M. Micheletti (Eds.), *Variational and Topological Methods in the Study of Nonlinear Phenomena,* Progress in Nonlinear Differential Equations and Their Applications, Vol. 49, Birkhäuser Verlag, Basel-Boston-Berlin 2002, vii + 131 pp., ISBN 3-7643-4278-1 and 0-8176-4278-1.

The articles in this volume are an outgrowth of the international conference *Variational and Topological Methods in the Study of Nonlinear Phenomena*, held in Pisa in January/February 2000, under the framework of the research project *Differential Equations and the Calculus of Variations.* The specific aim of the conference was to celebrate the 60th birthday of Antonio Marino, one of the leaders of the research group, with significant contributions to this area.

The volume contains 9 papers: 1. M. Clapp, *Morse indices and mountain pass orbits of symmetric functionals;* 2. M.J. Esteban and E. Séré, *On some linear and nonlinear eigenvalue problems in relativistic quantum chemistry;* 3. A. Ioffe and E. Schwartzman, *Convexity at infinity and Palais-Smale conditions;* 4. W. Marzantowicz, *Periodic solutions of nonlinear problems with positive oriented periodic coefficients;* 5. M. Mrozek and P. Pilarczyk, *The Conley index and rigorous numerics of attracting periodic orbits;* 6. R. Ortega, *Dynamics of a forced oscillator having an obstacle;* 7. M. del Pinto, P. Felmer and M. Musso, *Spike patterns in the supper-critical Bahri-Coron problem;* 8. P. Sintzoff and M. Willem, *A semilinear elliptic equation on* $\mathbf{R}^{\mathbf{N}}$ *with unbounded coefficients;* 9. R.E.L. Turner, *Traveling waves in natural systems.*

As it can be seen by this enumeration, the contributions highlight recent advances in nonlinear functional analysis, with applications to nonlinear partial or ordinary differential equations, having as unifying theme the use of variational and topological methods. There are worth to mention the applications to biology and chemistry included in the volume.

The volume will be an excellent reference text for researchers and graduate students working in these areas.

S. Cobzaş

Weaver, N., *Mathematical Quantization*, Chapman & Hall / CRC (Studies in Advanced Mathematics), 2001, 278 pp., Hardcover, ISBN 1-58488-001-5.

After almost a century from the creation of quantum mechanics there is still no general agreement on what we should mean by general “quantum theory”, as well as by “quantization”. Besides, there still is a gap between physicists which are, in the end, mainly interested in the phenomenological aspects of quantum theory, and mathematicians, interested in rigor and building more sophisticated theories. Nevertheless, there is a large amount of mathematics (especially functional analysis) that everybody agrees that should be a basic ingredient of any “quantization procedure” and can safely go under the name of “mathematical quantization”.

The book of Weaver intends to expose, in a coherent manner, both the foundational material and some of the contemporary achievements, related especially to noncommutative geometry and quantum groups theory. The basic philosophy of the book is that quantization means replacing sets by Hilbert spaces and the author finds quantum analogues of the main ingredients of classical mathematical physics (topological spaces, distances, measures, a.o.). Of course, the idea itself is not new. After all, even at the initial stage of development of quantum mechanics quantization basically meant just replacing classical observables (functions defined on the configuration space) by operators on Hilbert spaces. However, Weaver is the first to make an extended use of this idea to build the quantum analogues of the classical notions mentioned above, replacing, in particular, the spaces by C^*-algebras and spaces with measure by von Neumann algebras.

The book starts with a brief mathematical review of classical mechanics and continues with Hilbert spaces and linear operators on them. Now come into play the first "quantum" notion: the quantum plane. There follows two chapters on C^*-algebras and von Neumann algebras which are, in the sequel, applied to quantum field theory. The rest of the book is devoted to foundational material in noncommutative geometry (Hilbert modules), Lipschitz algebras and quantum groups.

The intention is to lay the mathematical foundations for physical applications, therefore, no prior knowledge of physics is assumed (although it is, of course, very helpful). In fact, except the chapter devoted to quantum field theory, no applications to physics are discussed, still, someone which is familiar to quantum physics will recognize easily many physical notions and results "in disguise".

The book is a comprehensive exposition of the modern mathematics necessary for quantum theory and the author manages to describe a surprisingly large amount of material in an attractive and clear manner. Of course, it cannot replace the detailed texts in more special topics, anyway, the reader, graduate student or researcher, can get an idea on the state of the art of the theories regarding quantization. Let me also emphasize that, as it is easily understood, only a limited quantity of quantization tools are exposed here. The book can be thought off, in a way, as an introduction into noncommutative geometry (or, rather, into the prerequisites of noncommutative geometry).

To conclude, the book is very well written and provides a lucid and clear exposition of some of the most important tools of quantization theory.

Paul A. Blaga

C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, New Jersey-London-Singapore-Hong Kong 2002, xx+367 pp, ISBN: 981-238-067-1.

This well written book is devoted to convex analysis in infinite dimensional spaces. What makes it different from other existing books on convex analysis and optimization is the fact that the results are presented in their most generality, known at this time, as well as the inclusion of new and recent material. The author is a well known specialist in the field and the book incorporates many of his original results. In order to obtain this generality, the author has included in the first chapter of the book, Ch. 1, *Preliminary results on functional analysis*, a detailed study of convex series (cs) closed, lower cs-closed (lcs-closed), ideally convex, lower ideally convex (li-convex) sets and multivalued mappings, allowing him to prove very general open mapping and closed graph theorems of Ursescu-Robinson type. The chapter contains also a fine study of separation of convex sets and a presentation, with complete proofs, of Ekeland's variational principle and of Borwein-Preis smooth variational principle.

The second chapter of the book, Ch. 2, *Convex analysis in locally convex spaces*, beside classical results, contains also the study of some more general classes of functions, corresponding to the sets studied in the first chapter: cs-closed, cs-complete, lcs-closed, ideally convex, bcs-complete and li-convex functions. The conjugate function, duality formulae, the subdifferential and the ϵ-subdifferential calculus, are also included. The developed machinery is applied to convex programming, perturbed problems, convex optimization with constraint and to minimax theorems.

The last chapter of the book, Ch. 3, *Some results and applications of convex analysis in normed spaces*, contains some deep results in convex analysis that are specific to this framework, which have important applications to optimization and to other areas. We mention the Brønsted-Rockafellar theorem with applications to the proofs of Bishop-Phelps theorem and of Rockafellar's maximal monotonicity of the subdifferential of a convex function. Zagrodny mean value theorem for abstract subdifferentials yields a short proof of the converse of the above result: every cyclically maximal monotone multivalued mappings is subdifferential of a convex functions. The important classes of uniformly convex and uniformly smooth functions are studied, as well as the interplay of their properties and of the differentiability of convex functions, with the geometry of underlying normed space. The last section of this chapter, based on some recent results of S. Simons, is concerned with monotone multivalued mappings.

There are a lot of exercises spread through the book. Some of them contain technical parts of some proofs or examples, while the others are concerned with results which did not fit in the main stream of the exposition.

The book is fairly selfcontained, the prerequisites for the reading being familiarity with basic functional analysis, including topological vector spaces and locally convex spaces.

The book is very well organized, with comprehensive indexes of notation and of notions, and a rich bibliography.

It can be used as a textbook for advanced graduate courses, or as a reference text by specialists.

S. Cobzaş