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SOLUTIONS TO THE DIOPHANTINE EQUATION $(x+y+z+t)^2 = xyzt$

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Abstract. The main purpose of this paper is to study the Diophantine equation (2). We will indicate nine different infinite families of positive integral solutions to this equation.

1. Introduction

Generally, integral solutions to equations in three or more variables are given in various parametric forms (see [2] or [6]). In the paper [5] it is proved that the Diophantine equation $x+y+z = xyz$ has solutions in the units of the quadratic field $\mathbb{Q}(\sqrt{d})$ if and only if $d = -1$, 2 or 5 and in these cases all solutions are also given. The problem of finding all integral solutions to this equation remains open. In our paper [1] we constructed different families of infinite integral solutions to the equation

$$(x+y+z)^2 = xyz. \tag{1}$$

We have indicated a general method of generating such families of solutions by using the theory of Pell's equations. The problem of finding all solutions to equation (1) is still open.

In this paper we use the theory of general Pell's equations to generate nine infinite families of positive integral solutions to the equation

$$(x+y+z+t)^2 = xyzt. \tag{2}$$

2. The General Pell's Equation $Ax^2 - By^2 = C$

Recall that the equation

$$u^2 - Dv^2 = 1, \tag{3}$$

where D is a positive integer that is not a perfect square is called Pell's equation.

Denoting by $(u_0, v_0) = (1, 0)$ its trivial solution, the main result concerning equation (3) is the following (see [1], pp. 107-110 or [7]): There are infinitely many solutions in positive integers to (3) and all solutions to equation (3) are given by $(u_n, v_n)_{n\geq 0}$, where

$$\begin{cases} u_{n+1} = u_1u_n + Dv_1v_n \\ v_{n+1} = v_1u_n + u_1v_n \end{cases} \tag{4}$$

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Here (u_1, v_1) represents the fundamental solution to (3), that is the minimal solution different from (u_0, v_0).

It is not difficult to see that (4) is equivalent to

$$u_n + v_n\sqrt{D} = (u_1 + v_1\sqrt{D})^n, \quad n \geq 0. \tag{5}$$

Also, relations (5) could be written in the following useful matrix form:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} u_1 & Dv_1 \\ v_1 & u_1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \quad n \geq 0$$

from where

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & Dv_1 \\ v_1 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n \geq 0 \tag{6}$$

From (5) or (6) it follows immediately that

$$\begin{cases} u_n = \dfrac{1}{2}[(u_1 + v_1\sqrt{D})^n + (u_1 - v_1\sqrt{D})^n] \\ \\ v_n = \dfrac{1}{2\sqrt{D}}[(u_1 + v_1\sqrt{D})^n - (u_1 - v_1\sqrt{D})^n],\ n \geq 0 \end{cases} \tag{7}$$

The main method of determining the fundamental solution (u_1, v_1) involves continued fractions. Sometimes this solution is very large, for example if $D = 991$, then

$$\begin{cases} u_1 = 379516400906811930638014896080 \\ v_1 = 12055735790331359447442538767 \end{cases}$$

In what follows we consider the general Pell's equation

$$Ax^2 - By^2 = C, \tag{8}$$

where A, B, C are positive integers with $gcd(A, B) = 1$ and A and B are not perfect squares.

The solvability and unsolvability of equation (8) is discussed in our paper [3]. Concerning this equation we need the following result (see also [4]):

Theorem. *If equation (8) is solvable in positive integers, then it has infinitely many positive integral solutions.*

Proof. We will use the Pell's resolvent associated to equation (8):

$$u^2 - ABv^2 = 1. \tag{9}$$

From the given conditions it follows that AB is not a perfect square so the Pell's equation (9) has infinitely many positive integral solutions. All such solutions are given by (4) or (7), where $D = AB$.

If (x_0, y_0) is a solution to (8) and (u, v) is a solution to (9), then we can construct a new solution to (8) by using the so-called multiplication principle:

$$\begin{cases} x = x_0 u + By_0 v \\ y = y_0 u + Ax_0 v \end{cases} \tag{10}$$

Indeed,

$$Ax^2 - By^2 = A(x_0 u + By_0 v)^2 - B(y_0 u + Ax_0 v)^2 =$$

$$= (Ax_0^2 - By_0^2)(u^2 - ABv^2) = C \cdot 1 = C.$$

Taking into account that the Pell's resolvent has infinitely many solutions, the conclusion follows. □

In the case when equation (8) is solvable, all of its solutions can be expressed in terms of the solutions to the associated general Pell's equation

$$u^2 - ABv^2 = AC. \tag{11}$$

For more details we refer to [3, Theorem 1] or [8].

3. **Infinite Families of Solutions to Equation (2)**

The transformations

$$x = \frac{u+v}{2} + a, \quad y = \frac{u-v}{2} + a, \quad z = b, \quad t = c \tag{12}$$

where a, b, c are positive integers, bring the equation (2) to the form

$$(u + 2a + b + c)^2 = \frac{bc}{4}(u^2 - v^2) + abcu + a^2bc.$$

Setting the conditions $2(2a+b+c) = abc$ and $bc > 4$, we obtain the following general Pell's equation

$$(bc-4)u^2 - bcv^2 = 4[(2a+b+c)^2 - a^2bc]. \tag{13}$$

There are nine triples (a, b, c) up to permutations satisfying the above conditions: (1,6,4), (1,10,3), (2,2,6), (3,4,2), (3,14,1), (5,2,3), (4,1,9), (7,1,6), (12,1,5).

The following table contains the general Pell's equations (13) corresponding to the above triples (a, b, c), their Pell's resolvent, both equations with their fundamental solutions.

(a,b,c)	General Pell's equation (13) and its fundamental solution	Pell's resolvent and its fundamental solution
(1,6,4)	$5u^2 - 6v^2 = 120,\ (12,10)$	$r^2 - 30s^2 = 1,\ (11,2)$
(1,10,3)	$13u^2 - 15v^2 = 390,\ (15,13)$	$r^2 - 195s^2 = 1,\ (14,1)$
(2,2,6)	$2u^2 - 3v^2 = 96,\ (12,8)$	$r^2 - 6s^2 = 1,\ (5,2)$
(3,4,2)	$u^2 - 2v^2 = 72,\ (12,6)$	$r^2 - 2s^2 = 1,\ (3,2)$
(3,14,1)	$5u^2 - 7v^2 = 630,\ (21,15)$	$r^2 - 35s^2 = 1,\ (6,1)$
(4,1,9)	$5u^2 - 9v^2 = 720,\ (42,30)$	$r^2 - 45s^2 = 1,\ (161,24)$
(5,2,3)	$u^2 - 3v^2 = 150,\ (15,5)$	$r^2 - 3s^2 = 1,\ (2,1)$
(7,1,6)	$u^2 - 3v^2 = 294,\ (21,7)$	$r^2 - 3s^2 = 1,\ (2,1)$
(12,1,5)	$u^2 - 5v^2 = 720,\ (30,6)$	$r^2 - 5s^2 = 1,\ (9,4)$

By using the formula (10) we obtain the following sequences of solutions to equations (13):

$$u_m^{(1)} = 12r_m^{(1)} + 60s_m^{(1)}, \quad v_m^{(1)} = 10r_m^{(1)} + 60s_m^{(1)},$$

where $r_m^{(1)} + s_m^{(1)}\sqrt{30} = (11 + 2\sqrt{30})^m,\ m \geq 1$;

$$u_m^{(2)} = 15r_m^{(2)} + 195s_m^{(2)}, \quad v_m^{(2)} = 13r_m^{(2)} + 195s_m^{(2)},$$

where $r_m^{(2)} + s_m^{(2)}\sqrt{195} = (14 + \sqrt{195})^m,\ m \geq 1$;

$$u_m^{(3)} = 12r_m^{(3)} + 24s_m^{(3)}, \quad v_m^{(3)} = 8r_m^{(3)} + 24s_m^{(3)},$$

where $r_m^{(3)} + s_m^{(3)}\sqrt{6} = (5+2\sqrt{6})^m,\ m \geq 1;$

$$u_m^{(4)} = 12r_m^{(4)} + 12s_m^{(4)}, \quad v_m^{(4)} = 6r_m^{(4)} + 12s_m^{(4)},$$

where $r_m^{(4)} + s_m^{(4)}\sqrt{2} = (3+2\sqrt{2})^m,\ m \geq 1;$

$$u_m^{(5)} = 21r_m^{(5)} + 105s_m^{(5)}, \quad v_m^{(5)} = 15r_m^{(5)} + 105s_m^{(5)},$$

where $r_m^{(5)} + s_m^{(5)}\sqrt{35} = (6+\sqrt{35})^m,\ m \geq 1;$

$$u_m^{(6)} = 42r_m^{(6)} + 270s_m^{(6)}, \quad v_m^{(6)} = 30r_m^{(6)} + 210s_m^{(6)},$$

where $r_m^{(6)} + s_m^{(6)}\sqrt{45} = (161+24\sqrt{45})^m,\ m \geq 1;$

$$u_m^{(7)} = 15r_m^{(7)} + 15s_m^{(7)}, \quad v_m^{(7)} = 5r_m^{(7)} + 15s_m^{(7)},$$

where $r_m^{(7)} + s_m^{(7)}\sqrt{3} = (2+\sqrt{3})^m,\ m \geq 1;$

$$u_m^{(8)} = 21r_m^{(8)} + 21s_m^{(8)}, \quad v_m^{(8)} = 7r_m^{(8)} + 21s_m^{(8)},$$

where $r_m^{(8)} + s_m^{(8)}\sqrt{3} = (2+\sqrt{3})^m,\ m \geq 1;$

$$u_m^{(9)} = 30r_m^{(9)} + 30s_m^{(9)}, \quad v_m^{(9)} = 6r_m^{(9)} + 30s_m^{(9)},$$

where $r_m^{(9)} + s_m^{(9)}\sqrt{5} = (9+4\sqrt{5})^m,\ m \geq 1.$

Formulas (12) yield the following nine families of positive integers solutions to the equation (2):

$$x_m^{(1)} = 11r_m^{(1)} + 60s_m^{(1)} + 1, \quad y_m^{(1)} = r_m^{(1)} + 1, \quad z_m^{(1)} = 6, \quad t_m^{(1)} = 4$$

$$x_m^{(2)} = 14r_m^{(2)} + 195s_m^{(2)} + 1, \quad y_m^{(2)} = r_m^{(2)} + 1, \quad z_m^{(2)} = 10, \quad t_m^{(2)} = 3$$

$$x_m^{(3)} = 10r_m^{(3)} + 24s_m^{(3)} + 2, \quad y_m^{(3)} = 2r_m^{(3)} + 2, \quad z_m^{(3)} = 2, \quad t_m^{(3)} = 6$$

$$x_m^{(4)} = 12r_m^{(4)} + 12s_m^{(4)} + 3, \quad y_m^{(4)} = 3r_m^{(4)} + 3, \quad z_m^{(4)} = 4, \quad t_m^{(4)} = 2$$

$$x_m^{(5)} = 18r_m^{(5)} + 105s_m^{(5)} + 3, \quad y_m^{(5)} = r_m^{(5)} + 3, \quad z_m^{(5)} = 14, \quad t_m^{(5)} = 1$$

$$x_m^{(6)} = 36r_m^{(6)} + 240s_m^{(6)} + 4, \quad y_m^{(6)} = 6r_m^{(6)} + 30s_m^{(6)} + 4, \quad z_m^{(6)} = 1, \quad t_m^{(6)} = 9$$

$$x_m^{(7)} = 10r_m^{(7)} + 15s_m^{(7)} + 5, \quad y_m^{(7)} = 5r_m^{(7)} + 5, \quad z_m^{(7)} = 2, \quad t_m^{(7)} = 3$$

$$x_m^{(8)} = 14r_m^{(8)} + 21s_m^{(8)} + 7, \quad y_m^{(8)} = 7r_m^{(8)} + 7, \quad z_m^{(8)} = 1, \quad t_m^{(8)} = 6$$

$$x_m^{(9)} = 18r_m^{(9)} + 30s_m^{(9)} + 12, \quad y_m^{(9)} = 12r_m^{(9)} + 12, \quad z_m^{(9)} = 1, \quad t_m^{(9)} = 5.$$

Remarks. 1) In [9] only solution $(x_m^{(7)}, y_m^{(7)}, z_m^{(7)}, t_m^{(7)})$ is found.

2) Note the atypical form of solution $(x_m^{(6)}, y_m^{(6)}, z_m^{(6)}, t_m^{(6)})$.

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ASYMPTOTICAL VARIANTS OF SOME FIXED POINT THEOREMS IN ORDERED SETS

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Abstract. In this paper we will present some fixed point theorems in ordered sets with condition for operator and its iterates too.

1. **Introduction**

Let $(X, \leq)$ be an ordered set $(X \neq \emptyset)$ and $f : X \to X$ an operator. We denote by

$$F_f = \{x \in X : f(x) = x\}$$

the fixed point set of f.

In this note we need the following results [1-7].

Theorem of Tarski. *Let $(X, \leq)$ be a complete lattice, $f : X \to X$ an increasing operator. Then $F_f \neq \emptyset$ and $(F_f, \leq)$ is a complete lattice.*

Theorem of Birkhoff-Bourbaki. *Let $(X, \leq)$ be right inductive ordered set and let $f : X \to X$ be an expansive operator. Then $F_f \neq \emptyset$.*

Lemma. *Let X be nonempty set and $f, g : X \to X$ two commuting operators. Then:*

(i) $F_g = \emptyset$ *or* $F_g \in I(f)$;

(ii) $F_f = \emptyset$ *or* $F_f \in I(g)$;

2. **The main results**

Theorem 1. *Let $(X, \leq)$ be a an ordered set and $f : X \to X$ an increasing operator. We suppose that there exist $k \in \mathbb{N}^*$ and $Y \subset X$ such that:*

(a) $f^k(X) \subset Y$;

(b) $(Y, \leq)$ *is a complete lattice.*

Then $F_f \neq \emptyset$.

Proof. From (a) and (b) we have that the restriction of iterate f^k has the following properties $f^k|_Y : Y \to Y$ and f^k is an increasing operator.

f is an increasing operator, i.e. for any $x, y \in X$ we have

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

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$$f(x) \le f(y) \Longrightarrow f(f(x)) \le f(f(y))$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f^{k-1}(x) \le f^{k-1}(y) \Longrightarrow f^k(x) \le f^k(y).$$

Since $(Y, \le)$ is a complete lattice, from Theorem of Tarski it follows that $F_{f^k} \neq \emptyset$ and $\left(F_{f^k}, \le\right)$ is a complete lattice. Because f and f^k are commuting operators, from Lemma we have that $f\left(F_{f^k}\right) \subset F_{f^k}$.

We apply the Tarski Theorem to $f : F_{f^k} \to F_{f^k}$ and we conclude that there exists at least a fixed point $\left(\in F_{f^k}\right)$ which means that $F_f \neq \emptyset$.□

Theorem 2. *Let $(X, \le)$ be a an ordered set and $f : X \to X$ be an expansive operator. We suppose that there exist $k \in \mathbb{N}^*$ and $Y \subset X$ such that:*

(a) $f^k(X) \subset Y$;

(b) $(Y, \le)$ *is a right inductive ordered set.*

Then $F_f \neq \emptyset$.

Proof. From (a) we have $f^k|_Y : Y \to Y$. Since f is an expansive operator, i.e.

$$x \le f(x), \quad \forall x \in X,$$

we obtain

$$x \le f(x) \le f(f(x)) = f^2(x) \le \dots \le f^{k-1}(x) \le f^k(x),$$

which means that f^k is an expansive operator. From Theorem of Birkhoff-Bourbaki we have that $F_{f^k} \neq \emptyset$. Let $x^* \in F_{f^k}$, we want to prove that $x^* \in F_f$.

Suppose that x^* is not a fixed point of f: $f(x^*) \neq x^*$. We have two cases: $x^* < f(x^*)$ and $x^* > f(x^*)$.

Case I: $x^* < f(x^*)$

Since f is an expansive operator we deduce

$$x^* < f(x^*) \le f^2(x^*) \le \dots \le f^{k-1}(x^*) \le f^k(x^*) = x^*,$$

which is a contradiction.

Case II: $x^* > f(x^*)$

$$x^* > f(x^*) \ge f^2(x^*) \ge \dots \ge f^{k-1}(x^*) \ge f^k(x^*) = x^*,$$

which is also a contradiction.

Thus we have that $x^* \in F_f$.□

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A MATHEMATICAL MODEL FOR THE STUDY OF GLYCAEMIC HOMEOSTASY

ALEXANDRU BICA

Abstract. A mathematical model for the blood-glucose homeostasis is built in this paper, using the previous models. The results of this paper concern the stability of the equilibrium solutions of a nonlinear differential system which govern the model.

1. **Introduction**

Here, we propose the study of some properties of the mechanisms which are involved in the blood glucose concentration homeostasis. We have in view the models which have been elaborated up to the present and we build a model for the glycaemic homeostasy. In 1965 Ackerman, Gatewood, Rosevear and Molnar [1] have been proposed a model described by a differential linear system in plane where the parameters are the glucose deviation from his constant value (harvested in blood in the morning after fasts overnight) and the similar deviation of a well-balanced average concentration of hormones (insulin, glucagon, growth hormone, epinephrine, cortisone). The destination of the model is to understand the treatment of diabetics in assumption of the administer of some hypoglicaemiant medicine and of glucose. The nonhomogeneous differential system which govern this model is the following:

$$\begin{cases} g' = -m_1 g - m_2 h + J \\ h' = -m_3 h + m_4 g + K \end{cases}$$

where, m_1, m_2, m_3, m_4 are positive constant, $J(t)$ is the rate of infusion from the intestines (or intravenously) of the glucose, $K(t)$ is the intravenous rate of infusion of the insulin, $g = G - G_0, h = H - H_0$. Here, $G = G(t)$ is the blood glucose concentration, $H = H(t)$ is the glucose-regulation hormones concentration in the blood and G_0, H_0 are the constant levels of this concentrations. We can see in the above system that the action of the hormonal concentration, h, is prevalent of the insulin type. In [1], some assumptions are used about the J and K functions and about the constants m_i, $i = \overline{1,4}$ (for instance, $(m_1 + m_3)^2 < 4m_1m_3 + 4m_2m_4$) which permit to solve the system and to obtain the solution in a damping oscillatory form round about the G_0 and H_0 levels.

Afterwards, was been elaborated some models which contain the distinct action of the hyperglicaemiant hormones. A summary presentation of these models can be found in [3]. For instance the Automonov model contain three status parameters

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(insulin and epinephrine concentrations and the glucose concentration in three compartments) and lead to a system with 6 linear differential equations. In [3], starting to the former models, the author build a model (together with the Rhode Island Hospital research workers) considering the blood glucose concentration and the plasmatic levels of hormones (insulin, glucagon, growth hormone, cortisone, thyroxin, epinephrine) and of the free fatty acids and aminoacids. From these result a nonlinear neutral system with 5 differential equations, which also describe various processes included negative feedbacks. Some arguments for nonlinearity are exposed in [3] (we use these arguments and other arguments in this article)

In [4] some algorithms are proposed for the mathematical modelling in glycaemic evolution of diabetics, with applications in treatment schemes. Here, are considered the advanced diabetic cases which present the phenomena of glycosuria, proposing a model with two status parameters : glycaemia and the sugar concentration in urine.

In the construction of the model, in this article, we consider the hypotheses from [1] and [3] and the assertions from the medical monography [2]. Here, we consider the effect of the interaction between the glucose and the hormones concerning on the speed of glycaemic changes and we obtain a nonlinear differential system. Because the glycaemic homeostasy contain negative feedback processes (in accord with [1], [2], [3]), in each equation there is such terms. It is known that the mechanisms of glycaemic homeostasy are so delicacy, and then the effect of the interaction between the glucose and the hormones is attenuated by the great glycaemic values. This effect will be appear in the first equation through the nonlinear term, $\frac{axy}{x+G_0}$.

2. **The construction of the model**

The status parameters are the plasmatic concentrations of the glucose, $G(t)$, of the insulin, $I(t)$, and of the average of hyperglycaemiant hormones, $H(t)$ (glucagon, cortisone, tyroxin, ACTH, growth hormone, epinephrine). Using the reasonings from [1] and [3] we consider that G, I, H are derivable with continuous derivative functions on an interval of $[0,\infty)$. Let G_0, I_0, H_0 be the values of these functions at the initial moment, $t_0 \in [0,\infty)$, which can be known by blood harvesting in the morning after fasts overnight. Our aim is to obtain results using the classification of the singular points in the plane and therefore we consider two dependent variables, $x(t) = G(t) - G_0, y(t) = H(t) - H_0 - (I(t) - I_0)$. Then, the new status parameters are the glycaemic deviation from his equilibrium value and the difference of such deviations for insulin and hyperglycaemiant hormones.

The following hypotheses are used in the construction of the model :

a) Each status variable have influence upon the proper speed of changes into a negative feedback process.

b) An increase of hyperglycaemiant hormones secretion provoke the increase of glycaemia, and the release of insulin secretion lead to a diminution of glycaemia. A glycaemic increase provoke the increase of insulin secretion and the decrease of hyperglycaemiant hormones secretion.

c) The interaction between the glucose and hormones determine a moderate modification of glycaemia. This hypothesis introduce in the first equation of the system a nonlinear term. The intestinal absorption of the alimentary glucose under the action of the intestinal glucagon (a hyperglycaemiant hormone) can be described by this nonlinear term too. This is the reason because the model can be described by an autonomous differential system:

$$(1)\quad \begin{cases} x' = a\dfrac{xy}{x+G_0} - bx + my \\ y' = -cx - dy \end{cases} \quad , a, b, c, d, m > 0$$

with initial conditions:

$$(2): \quad x(0) = 0, y(0) = 0.$$

The terms $-bx$ and $-dy$ represent the negative feedback according to the hypothesis a), the terms my and $-cx$ are introduced by the hypothesis b) and the term $a\dfrac{xy}{x+G_0}$ is the nonlinear term from the hypothesis c). We can see that $x + G_0 = G > 0$, because the glycaemic values are always positive. The constant values a, b, c, d, m and G_0, H_0, I_0 are specific to each person. The constant b, c, d, m have the same signification as in [1] and a is a coefficient of hormonal efficiency. For the most persons we can consider the condition $ac \geq bd + mc$, be fulfilled.

3. **First approximation stability**

We consider the open semiplane, $D = \{(x, y) \in \mathbb{R}^2 : x > -G_0\}$ and the functions

$U, V : D \longrightarrow \mathbb{R}$, given by

$$U(x, y) = a\frac{xy}{x+G_0} - bx + my, V(x, y) = -cx - dy.$$

It can see that $U, V \in C^1(D)$ and so there are locally Lipschitz on D. Then each Cauchy problem, (1)+(2) with initial conditions in D, has a unique maximal solution. About the stability of equilibrium solutions of the system (1) we obtain :

Theorem 3.1. *For each positive values of a, b, c, d, m, G_0 the system (1) has in the set D two echilibrium solutions $P_1(0, 0)$ and $P_2(x_2, y_2)$, with $x_2 < 0, y_2 > 0$, such that $P_1(0, 0)$ is asymptotically stable, and $P_2(x_2, y_2)$ is saddle point. If $(b-d)^2 < 4mc$ then $P_1(0, 0)$ is focus.*

Proof. The equilibrium solutions of the system (1) are the solutions of the algebraic system :

$$\begin{cases} U(x, y) = 0 \\ V(x, y) = 0 \end{cases} \iff \begin{cases} \dfrac{axy}{x+G_0} - bx + my = 0 \\ -cx - dy = 0 \end{cases} ,$$

that is $x_1 = 0, y_1 = 0$ and

$$x_2 = \frac{-G_0(bd+mc)}{ac+bd+mc}, y_2 = \frac{cG_0(bd+mc)}{d(ac+bd+mc)}.$$

For the first approximation stability of the equilibrium solutions $P_1(0, 0)$ and $P_2(x_2, y_2)$ we compute the eigenvalues of the Jacobi matrix for the vectorial field (U, V) in these points. In this sense, for $P_1(0, 0)$:

$$\det(J_{U,V}(0,0) - \lambda I) = \begin{vmatrix} \dfrac{\partial U(0,0)}{\partial x} - \lambda & \dfrac{\partial U(0,0)}{\partial y} \\ \dfrac{\partial V(0,0)}{\partial x} & \dfrac{\partial V(0,0)}{\partial y} - \lambda \end{vmatrix} = \begin{vmatrix} -b-\lambda & m \\ -c & -d-\lambda \end{vmatrix} =$$

$$= 0 \Longleftrightarrow \lambda^2 + (b+d)\lambda + mc + bd = 0.$$

Because $b+d > 0$ and $bd+mc > 0$ we infer that $\text{Re}\lambda_1 < 0, \text{Re}\lambda_2 < 0$. Then $P_1(0,0)$ is asymptotically stable (uniform, because the system is autonomous). If $(b-d)^2 - 4mc \geq 0$ then, this equilibrium point is a node and if $(b-d)^2 - 4mc < 0$, is focus. The condition $(b-d)^2 - 4mc < 0$, is priori asserted in [1], using some experiments, where the values of b and d are greater than m and c, but such that $|b-d| < 2mc$. For the equilibrium point $P_2(x_2, y_2)$,

$$\frac{\partial U(x_2,y_2)}{\partial x} = \frac{aG_0 y_2}{(x_2+G_0)^2} - b = \frac{(bd+mc)^2 + amc^2}{acd}$$

$$\frac{\partial U(x_2,y_2)}{\partial y} = \frac{ax_2}{x_2+G_0} + m = -\frac{bd}{c}$$

$$\frac{\partial V}{\partial x} = -c, \frac{\partial V}{\partial y} = -d.$$

Then, $\det(J_{U,V}(x_2,y_2) - \lambda I) = 0 \Longleftrightarrow$

$$\lambda^2 + \left[d - \frac{(bd+mc)^2 + amc^2}{acd}\right]\lambda - \frac{(bd+mc)^2 + amc^2}{ac} - bd = 0.$$

Because $\lambda_1\lambda_2 = -\dfrac{(bd+mc)^2 + amc^2}{ac} - bd < 0, \quad \forall a,b,c,d,m \in \mathbb{R}_+^*$, we infer that $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 > 0, \lambda_2 < 0$ and then $P_2(x_2,y_2)$ is saddle point, (we can see that $(x_2,y_2) \in D$). The condition $ac \geq bd + mc$ lead to $x_2 \in [-\frac{G_0}{2}, 0)$, (statistical verified). □

Remark 4. *In the phase portrait, the unstable manifold of the saddle point is a curve through this point which arrive in the attractor $P_1(0,0)$, and the stable manifold is the frontier of the attraction basin of the origin. Here is the immediate clinical interpretations: each initial perturbation from the attraction basine of the equilibrium value $(G_0, H_0 - I_0)$ will be attract to this value, prevalent after damping oscillations. For each person there is an glycaemic unstable equilibrium value (x_2), which can be considered a frontier value over there appear hypoglycaemia (sometimes coma). It can see that for the persons with great value for b and d the frontier value is far from the equilibrium value $(G_0, H_0 - I_0)$ and the return to this last value is more fast. This persons are protected by diabetes and hypoglycaemia, having a good glycaemic homeostasy.*

Theorem 4.1. *The system (1) has no periodic solutions.*

Proof. Computing the divergence of the vectorial field (U,V),

$$div(U,V)(x,y) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = \frac{ayG_0}{(x+G_0)^2} - b - d$$

we see that this divergence has constant sign in the inside and in the outside of the parabola :

$$y = \frac{(b+d)}{aG_0}(x+G_0)^2.$$

This parabola is in the first and in the second cadrane.., having the peak $(-G_0, 0)$. So, the origin is in the outside of this parabola and then there is no limit cycle round about the origin, after the Bendixon theorem. Because the second singular point is saddle we infer that there is no limit cycle round about this point. □

5. **Stability after permanent perturbations**

Let be the perturbed system:

$$(3)\qquad \begin{cases} x' = a\dfrac{xy}{x+G_0} - bx + my + R_1(t,x,y) \\ y' = -cx - dy + R_2(t,x,y) \end{cases}$$

with R_1 and R_2 continuous functions on $J \times D$, where $J \subset [0,\infty)$ is interval.

Suppose that the permanent perturbations R_1 and R_2 are bounded in average, that is they have the property: $\forall \varepsilon > 0, \forall T > 0, \exists \eta > 0$ and $\exists \varphi = (\varphi_1, \varphi_2) : J \longrightarrow \mathbb{R}^2$ such that $\int_t^{t+T} \varphi_i(s)ds < \eta, i = \overline{1,2}$ and $|R_i(t,x,y)| < \varphi_i(t), \forall t \in J, \forall (x,y) \in D$ with $\|(x,y)\| < \varepsilon, \quad i = \overline{1,2}$. Then we obtain :

Theorem 5.1. *The zero solution of the system (1) is stable after permanent perturbations bounded in average.*

Proof. Can apply the theorem $1.8'$.(page95) from [5] and use the uniform asymptotic stability of the zero solution (after the first theorem), $\forall a, b, c, d.m > 0$. □

Remark 6. *From the previous theorem follow that the glycaemic value G_0 is resistant to the perturbations of impulse type with great initial values, but bounded in average and rapid estinguished. Such perturbations can be the momentary unloadings of epinephrine (in an emergency). If the permanent perturbations nonbounded in average became frequent, then can be appear some metabolic disorders. Such perturbations lead to the new glycaemic homeostasy configuration, which can be expressed by the autonomous perturbations. We study on the stability after autonomous perturbations.*

Let us consider the system:

$$(4)\qquad \begin{cases} x' = a\dfrac{xy}{x+G_0} - bx + my + f(x,y) \\ y' = -cx - dy + g(x,y) \end{cases}$$

where $f, g \in C^2(D)$ with $f(0,0) = 0, g(0,0) = 0$. We study this system with the first approximation method.

The origin is equilibrium solution of this system having the eigenvalues equation :

$$\lambda^2 + [b + d - \frac{\partial f(0,0)}{\partial x} - \frac{\partial g(0,0)}{\partial y}]\lambda - b\frac{\partial g(0,0)}{\partial y} + \frac{\partial f(0,0)}{\partial x}\cdot\frac{\partial g(0,0)}{\partial y} +$$
$$+ bd - d\frac{\partial f(0,0)}{\partial x} + mc - \frac{\partial f(0,0)}{\partial y}\cdot\frac{\partial g(0,0)}{\partial x} + c\frac{\partial f(0,0)}{\partial y} - m\frac{\partial g(0,0)}{\partial x} = 0.$$

If $\dfrac{\partial f(0,0)}{\partial y} > 0, b > \dfrac{\partial f(0,0)}{\partial x}, d > \dfrac{\partial g(0,0)}{\partial y}$ and $c > \dfrac{\partial g(0,0)}{\partial x}$ then the eigenvalues have negative real part and the solution is uniform asymptotic stable. The clinical interpretation is : if the hyperglycaemiant perturbations not succeed to modify the negative feedback characteristic of the homeostasis mechanism, then the equilibrium value is resistant to such perturbations.

If $\frac{\partial f(0,0)}{\partial x} > b$ and $\frac{\partial g(0,0)}{\partial y} > d$ then the zero solution of the system (4) is unstable. This means that the positive feedback appearance at the both components (glucose and hormones) lead to glycaemic instability.

We study now a particular case of autonomous perturbation, with $g(x,y) = yg(x), g \in C^2(I)$, $I \subset (-G_0, \infty)$, without the condition $g(0) = 0$, which means that the perturbation in the hormonal secretion speed have influence only on the hormonal feedback mechanism..

$$(5) \quad \begin{cases} x' = a\frac{xy}{x+G_0} - bx + my + f(x,y) \\ y' = -cx - dy + yg(x) \end{cases} .$$

Supposing that $0 \in I$, we can write the Taylor formula for the functions f and g :

$$(6) \quad \begin{array}{l} f(x,y) = \frac{\partial f(0,0)}{\partial x} \cdot x + \frac{\partial f(0,0)}{\partial y} \cdot y + \rho_1(x,y) \\ g(x) = g(0) + g'(0)x + \rho_2(x) \end{array}$$

where $\rho_1(x,y)$ and $\rho_2(x)$ contain second order derivatives. For the stability study of the zero solution of this system, after the first approximation method, the eigenvalues equation is :

$$\lambda^2 + [b + d - \frac{\partial f(0,0)}{\partial x} - g(0)]\lambda + bd + mc - d \cdot \frac{\partial f(0,0)}{\partial x} +$$
$$+g(0) \cdot \frac{\partial f(0,0)}{\partial x} + bg(0) + c \cdot \frac{\partial f(0,0)}{\partial y} = 0.$$

Proposition 6.1. *The new feedback components,* $\frac{\partial f(0,0)}{\partial x}$ *and* $g(0)$ *settle on the stability of the zero solution and* $\frac{\partial f(0,0)}{\partial y}$ *establish the shape of the solutions in a neighborhood of origin.*

Proof. If $\frac{\partial f(0,0)}{\partial y} \leq \frac{1}{4c}[b - d - \frac{\partial f(0,0)}{\partial x} + g(0)]^2 - m$ then the origin is a node or a saddle point. Is saddle point if $[b - \frac{\partial f(0,0)}{\partial x}] \cdot [d - g(0)] < c[m + \frac{\partial f(0,0)}{\partial y}]$, and a uniform asymptotic stable node if $[b - \frac{\partial f(0,0)}{\partial x}] \cdot [d - g(0)] > c[m + \frac{\partial f(0,0)}{\partial y}]$ and $d > g(0), b > \frac{\partial f(0,0)}{\partial x}$. The origin is unstable node if $[b - \frac{\partial f(0,0)}{\partial x}] \cdot [d - g(0)] > c[m + \frac{\partial f(0,0)}{\partial y}]$ and $d < g(0), b < \frac{\partial f(0,0)}{\partial x}$. When $\frac{\partial f(0,0)}{\partial y} \leq \frac{1}{4c}[b - d - \frac{\partial f(0,0)}{\partial x} + g(0)]^2 - m$ the eigenvalues are complex conjugated and the shape of the solutions in a neighborhood of the origin is oscillatory. In this case, the sign of $b + d - g(0) - \frac{\partial f(0,0)}{\partial x}$ decide the stability of the zero solution. □

In the case $\frac{\partial f(0,0)}{\partial y} \leq \frac{1}{4c}[b - d - \frac{\partial f(0,0)}{\partial x} + g(0)]^2 - m$ we distinguish the situations :

(i) If $b + d - g(0) - \frac{\partial f(0,0)}{\partial x} > 0$, then $\mathrm{Re}\lambda_{1,2} < 0$ and the origin is asymptotic stable focus.

(ii) If $b + d - g(0) - \dfrac{\partial f(0,0)}{\partial x} < 0$, then $\mathrm{Re}\lambda_{1,2} > 0$ and the origin is unstable focus.

(iii) If $b + d - g(0) - \dfrac{\partial f(0,0)}{\partial x} = 0$, then

$$\lambda_{1,2} = \pm \frac{i}{2}\sqrt{4c[m + \frac{\partial f(0,0)}{\partial y}] - [b - d + g(0) - \frac{\partial f(0,0)}{\partial x}]^2}.$$

Remark 7. *We can realize the clinical interpretations: If the origin is a saddle point, then the negative feedback mechanism of the glucose or of the hormones is overturned and this means a transition from a moderate diabetes to an advanced diabetes.. In the case of asymptotic stable node the negative feedback is preserved and the glycaemic value G_0 is resistant to perturbations. The case of unstable node correspond to positive feedback and advanced diabetes.. If $\dfrac{\partial f(0,0)}{\partial y}$ increase then in the first equation of the system (5) is fortified the insulin action, which means the presence of the insulin therapy. If the origin is an asymptotic stable focus then the insulin dose is best, succeeding to maintain the glycaemy at a nondangerous level. In the case of unstable focus the treatment is inefficient.*

We note $\mu = \dfrac{\partial f(0,0)}{\partial x}$ and consider this value as a parameter and for fixed $g(0)$ let be $\mu_0 = b + d - g(0)$. Suppose that f and g are of C^∞ class and obtain the following result:

Theorem 7.1. *If $f \in C^\infty(D), g \in C^\infty(I)$, $\dfrac{\partial f(0,0)}{\partial y} > 0$ and $\Delta < 0$, then there is a value of the μ parameter for which appear a Hopf bifurcation, corresponding to a periodic solution of the system (5).*

Proof. It can see that $\mathrm{Re}\lambda_{1,2}(\mu_0) = 0$ si $\dfrac{\partial \mathrm{Re}\lambda_{1,2}(\mu_0)}{\partial \mu} = \frac{1}{2} > 0$. Because for each other value of μ the zero solution is a focus, using the theorem of Hopf we infer that there is a periodic solution and the zero solution is a centre. The existence of a periodic solution can be proved in an other way, using the divergence of the vectorial field $(\Phi, \Psi) : D \longrightarrow \mathbb{R}^2$, with

$$\Phi(x,y) = a\frac{xy}{x + G_0} - bx + my + f(x,y)$$
$$\Psi(x,y) = -cx - dy + yg(x)$$

and the curve which limit the regions from D where the divergence have a constant sign, is, $y = \left[\dfrac{b + d - g(x) - \dfrac{\partial f(x,y)}{\partial x}}{aG_0}\right](x + G_0)^2$. If $\mu = \mu_0$, Then the point $(0,0)$ is on this curve, which imply that in each neighborhood of origin the divergence change the sign. Then, according to the theorem of Bendixon, there is a limit cycle round about the origin. □

Remark 8. *If $\mu < \mu_0$ then the origin is an asymptotic stable focus. When $\mu > \mu_0$, the origin is a unstable focus. Therefore, when μ increase crossing through μ_0 appear a Hopf bifurcation and periodic solution with the loss of the stability. We can* *observe that the periodic solutions appear only during the treatment. The bifurcation parameter can be selected also* $g(0)$. *For a person with hyperglycaemia the value* G_0 *is great, differing from the value of a healthy person. Therefore the value* G_0 *from the system (4) or (5) differ by the* G_0 *from the system (1).*

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FITTING OF SOME LINEARISABLE REGRESSION MODELS

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Abstract. In this paper, we obtain fitting conditions for some linearisables regressional models. These conditions are refering to the matrix of sample data.

1. Introduction

The fitting condition for those models, which, by substitution, can be reduced to a linear model, is refering to the matrix of new sample data/variables that results by substitution. In this paper, we consider models such as the polynomial, spline and piecewise linear model and we give for these, fitting conditions in the matrix of initial sample data/variables.

Let be the multiple linear model

$$Y = \alpha_1 X_1 + ... + \alpha_p X_p + \varepsilon \tag{1}$$

and a sample data

$$\mathbf{y}^T = (y_1, y_{2,} ..., y_n) \in \Re^n,$$

$$\mathbf{x} = \left(\mathbf{x}_1 \quad , \mathbf{x}_2, ..., \quad \mathbf{x}_p \right) = \begin{pmatrix} x_{11} & x_{12} & ... & x_{1p} \\ x_{21} & x_{22} & ... & x_{2p} \\ ... & ... & ... & ... \\ x_{n1} & x_{n2} & ... & x_{np} \end{pmatrix} \in \mathbf{M}_{n,p}, n > p.$$

Denoting $\boldsymbol{\alpha}^T = \left(\alpha_1, \quad \alpha_2, \quad ..., \quad \alpha_p \right) \in \Re^p, \boldsymbol{\varepsilon}^T = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in \Re^n$ from (1) we obtain the matriceal form $\mathbf{y} = \mathbf{x}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}$.

The principle of least squares leads to the fitting model:

$$\mathbf{y} = \mathbf{x}\mathbf{a} + \mathbf{e},$$

with

$$\mathbf{a}^T = (a_1, a_2, ..., a_p) \in \Re^p, \mathbf{e}^T = (e_1, e_2, ..., e_n) \in \Re^n \text{and} \sum_{i=1}^{n} e_i^2 = \min.$$

In case of a linear model which contains a constant term we have

$$\mathbf{y} = \mathbf{x}\boldsymbol{\alpha} + \boldsymbol{\varepsilon} = \mathbf{x}_0\boldsymbol{\alpha}_0 + \mathbf{u}\alpha_p + \boldsymbol{\varepsilon} \qquad (2)$$

with $\mathbf{x}_0 = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{p-1}), \boldsymbol{\alpha}_0^T = (\alpha_1, \alpha_2, ..., \alpha_{p-1}), \mathbf{u}^T = (1, 1, ..., 1) \in \Re^n$, $\mathbf{x} = (\mathbf{x_0}, \mathbf{u}), \boldsymbol{\alpha} = \left(\boldsymbol{\alpha}_0^T, \alpha_p\right)$.

The following result is well known in the literature:

Theorem 1.

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i)For model (1) if $\mathbf{x}$ has full column rank (the $\mathbf{x}_j$ are linearly independent) the least squares estimators a_i for $\alpha_i, i = \overline{1,p}$ are uniquely defined by

$$\mathbf{a} = \left(\mathbf{x}^T\mathbf{x}\right)^{-1}\mathbf{x}^T\mathbf{y}, \mathbf{a}^T = (a_1, a_2, ..., a_p) \in \Re^p.$$

ii) For model (2) if $\mathbf{x}$ has full column rank (the $\mathbf{x}_j$ are linearly independent) the least squares estimators a_i for $\alpha_i, i = \overline{1,p}$ are uniquely defined by

$$\mathbf{a}_0 = (a_1, a_2, ..., a_{p-1})^T = \left(\widehat{\mathbf{x}}_0^T\widehat{\mathbf{x}}_0\right)^{-1}\widehat{\mathbf{x}}_0^T\widehat{\mathbf{y}}, \quad a_p = \overline{y} - \sum_{k=1}^{p-1} a_k\overline{x}_k$$

where

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i, \overline{x}_k = \frac{1}{n}\sum_{i=1}^{n} x_{ik}, \widehat{\mathbf{x}}_0 = P\mathbf{x}_0, \widehat{\mathbf{y}} = P\mathbf{y}, P = I - \frac{1}{n}\mathbf{u}\mathbf{u}^T.$$

Remark 2. The Theorem 1 ii) holds for any of the conditions

$$\text{rank}\,(\mathbf{x}) = p, \text{or rank}\,(\mathbf{x}_0) = p - 1$$

because P is a linear transformation and we have

$$\text{rank}\,(\mathbf{x}) = p \Rightarrow \text{rank}\,(\mathbf{x}_0) = p - 1 \Rightarrow \text{rank}\,(P\mathbf{x}_0) = \text{rank}\,(\widehat{\mathbf{x}}_0) = p - 1.$$

2. **Main results**

We consider the polynomial model

$$Y = \alpha_0 + \alpha_1 X + ... + \alpha_r X^r + \varepsilon \qquad (3)$$

with a sample data $(x_i, y_i), i = \overline{1,n}$.

By replacing $X^j = Z_j$, $j = \overline{1,r}$ the model becomes

$$Y = \alpha_0 + \alpha_1 Z_1 + ... + \alpha_r Z_r + \varepsilon.$$

According to Theorem 1, if rank$(z) = r + 1$, then the fitting solution for (3) is given by

$$\mathbf{a} = \left(\widehat{\mathbf{z}}_0^T\widehat{\mathbf{z}}_0\right)^{-1}\widehat{\mathbf{z}}_0^T\widehat{\mathbf{y}}, \quad a_0 = \overline{y} - \sum_{k=1}^{r} a_k\overline{z}_k$$

where

$$\mathbf{z}_0 = \begin{pmatrix} z_{11} & z_{12} & ... & z_{1r} \\ z_{21} & z_{22} & ... & z_{2r} \\ ... & ... & ... & ... \\ z_{n1} & z_{n2} & ... & z_{nr} \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 1 & z_{11} & z_{12} & ... & z_{1r} \\ 1 & z_{21} & z_{22} & ... & z_{2r} \\ ... & ... & ... & ... & ... \\ 1 & z_{n1} & z_{n2} & ... & z_{nr} \end{pmatrix}.$$

In order to give for model (3) a theorem similar to Theorem 1 we search for a relation between the sample data matrix

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$$

and "the substitution matrix"

$$\mathbf{z} = \begin{pmatrix} 1 & x_1 & x_1^2 & ... & x_1^r \\ 1 & x_2 & x_2^2 & ... & x_2^r \\ ... & ... & ... & ... & ... \\ 1 & x_n & x_n^2 & ... & x_n^r \end{pmatrix}.$$

Theorem 3. If there are at least $r+1$ distinct values of the variable X in the sample data matrix, then the least squares fitting solutions of (3) can be written uniquely as

$$\mathbf{a} = \left(\widehat{\mathbf{z}}_0^T \widehat{\mathbf{z}}_0\right)^{-1} \widehat{\mathbf{z}}_0^T \widehat{\mathbf{y}}, \mathbf{a}^T = (a_1, a_2, ..., a_r) \in \Re^r, a_0 = \overline{y} - \sum_{k=1}^{r} a_k \overline{z}_k,$$

where

$$\widehat{\mathbf{z}}_0 = P\mathbf{z}_0, \widehat{\mathbf{y}} = P\mathbf{y}, P = I - \frac{1}{n}\mathbf{u}\mathbf{u}^T$$

and z_0 is the Vandermonde type matrix with n lines, each containing the first r integer powers of the n sample values, without the column which contains the vector $\mathbf{u}^T = (1, 1, ..., 1) \in \Re^n$.

Proof. We assume that the $r+1$ distinct values of X, are the first $r+1$ values, without limiting the generality. Obviously it is necessary that $r+1 \leq n$. If rank$(\mathbf{z}) = r+1$, where $\mathbf{z}$ is the Vandermonde matrix attached to the n values data for X then the theorem holds. Thus it is enough to prove that rank$(\mathbf{z}) = r+1$.

We consider in $\mathbf{z}$ the $r+1$ order minor formed with the rows which contain the $r+1$ distinct values:

$$d = \begin{vmatrix} 1 & x_1 & x_1^2 & ... & x_1^r \\ 1 & x_2 & x_2^2 & ... & x_2^r \\ ... & ... & ... & ... & ... \\ 1 & x_{r+1} & x_{r+1}^2 & ... & x_{r+1}^r \end{vmatrix}.$$

Since the $r+1$ values of the Vandermonde discriminant d are distinct, it follows that $d \neq 0$,and rank$(\mathbf{z}) = r+1$.■

We next consider the model

$$Y = f(X_1, X_2, ..., X_p) + \varepsilon \qquad (4)$$

where

$$f(X_1, X_2, ..., X_p) = \begin{cases} a_1X_1 + a_2X_2 + ... + a_pX_p \ , (X_1, X_2, ..., X_p) \in I \\ b_1X_1 + b_2X_2 + ... + b_pX_p \ , (X_1, X_2, ..., X_p) \in J \end{cases}$$

with I and J, two subsets of $\Re^p$ such as $I \cup J = \Re^p$ and $I \cap J = \emptyset$.

We use the notations:

-$\mathbf{x}_I$ the matrix containing those rows from the sample data matrix which belong to I ,as vectors in $\Re^p$

-$\mathbf{x}_J$ the matrix containing those rows from the sample data matrix which belong to J, as vectors in $\Re^p$

-$\mathbf{y}_I$ the vector containing those components y_i for which $\mathbf{x}_i = (X_1, X_2, ..., X_p)_i \in I$

-$\mathbf{y}_J$ the vector containing those components y_j for which $\mathbf{x}_j = (X_1, X_2, ..., X_p)_j \in J$.

Theorem 4. If $\mathrm{rank}(\mathbf{x}_I) = p$, $\mathrm{rank}(\mathbf{x}_J) = p$ and $2p \leq n$ then the least squares fitting solution of model (4) is uniquely given by

$\mathbf{a} = \left(\mathbf{x}_I^T \mathbf{x}_I\right)^{-1} \mathbf{x}_I^T \mathbf{y}_I$ and $\mathbf{b} = \left(\mathbf{x}_J^T \mathbf{x}_J\right)^{-1} \mathbf{x}_J^T \mathbf{y}_J$, with $\mathbf{a}^T, \mathbf{b}^T \in \Re^p$.

Proof. Using the least squares criteria we have

$$S = \sum_{i=1}^{n} \left[y_i - f\left(x_{i1}, x_{i2}, ..., x_{ip}\right)\right]^2 = \min,$$

$$\frac{\partial S}{\partial a_j} = 0, \frac{\partial S}{\partial b_j} = 0, j = \overline{1, p},$$

$$\frac{\partial S}{\partial a_j} = \frac{\partial}{\partial a_j} \left(\sum_{i=1}^{n} \left[y_i - f\left(x_{i1}, x_{i2}, ..., x_{ip}\right)\right]^2 \right).$$

Denoting $A = \{i \mid (x_{i1}, x_{i2}, ..., x_{ip}) \in I\}$ and $B = \{i \mid (x_{i1}, x_{i2}, ..., x_{ip}) \in J\}$ we obtain

$$\frac{\partial S}{\partial a_j} = \frac{\partial}{\partial a_j} \left(\sum_{i \in A} \left[y_i - f\left(x_{i1}, x_{i2}, ..., x_{ip}\right)\right]^2 \right) + \frac{\partial}{\partial a_j} \left(\sum_{i \in B} \left[y_i - f\left(x_{i1}, x_{i2}, ..., x_{ip}\right)\right]^2 \right) =$$

$$= 2 \sum_{i \in A} \left[y_i - f\left(x_{i1}, x_{i2}, ..., x_{ip}\right)\right] \cdot \left(-\frac{\partial f}{\partial a_j}\right) + 2 \sum_{i \in B} \left[y_i - f\left(x_{i1}, x_{i2}, ..., x_{ip}\right)\right] \cdot \left(-\frac{\partial f}{\partial a_j}\right)$$

From

$$\frac{\partial S}{\partial a_j} = 0$$

we obtain

$$\sum_{i \in A} \left[y_i - (a_1 x_{i1} + ... + a_p x_{ip})\right] \cdot x_{ij} + \sum_{i \in B} \left[y_i - (b_1 x_{i1} + ... + b_p x_{ip})\right] \cdot 0 = 0.$$

Finally, we have

$$\sum_{i \in A} \left[y_i - (a_1 x_{i1} + ... + a_p x_{ip})\right] \cdot x_{ij} = 0.$$

Similarly, from

$$\frac{\partial S}{\partial b_j} = 0$$

we obtain

$$\sum_{i \in B} \left[y_i - (b_1 x_{i1} + ... + b_p x_{ip})\right] \cdot x_{ij} = 0.$$

Then the following holds $\mathbf{x}_I^T \mathbf{x}_I \mathbf{a} = \mathbf{x}_I^T \mathbf{y}_I$, $\mathbf{x}_J^T \mathbf{x}_J \mathbf{b} = \mathbf{x}_J^T \mathbf{y}_J$. From hypothesis we have $\mathrm{rank}(\mathbf{x}_I) = \mathrm{rank}(\mathbf{x}_J) = p$, so follows that

$$\mathbf{a} = \left(\mathbf{x}_I^T \mathbf{x}_I\right)^{-1} \mathbf{x}_I^T \mathbf{y}_I \text{ and } \mathbf{b} = \left(\mathbf{x}_J^T \mathbf{x}_J\right)^{-1} \mathbf{x}_J^T \mathbf{y}_J.$$

We observe that $\mathbf{x}_I \in \mathbf{M}_{n_1,p}$, $\mathbf{x}_J \in \mathbf{M}_{n_2,p}$, $n_1 + n_2 = n$ where

$n_1 = \mathrm{card}\{(x_{i1}, x_{i2}, ..., x_{ip}) \mid (x_{i1}, x_{i2}, ..., x_{ip}) \in I\}$

$n_2 = \mathrm{card}\{(x_{i1}, x_{i2}, ..., x_{ip}) \mid (x_{i1}, x_{i2}, ..., x_{ip}) \in J\}$

where "card" denotes the number of elements of a given set.

Moreover, it is necessary that $p \leq n_1$, $p \leq n_2$ so the condition $2p \leq n$ is required.■

Theorem 5. If in model (4) the function f is given on k subdomains $I_1, I_2, ..., I_k$, then the principle of the least squares leads to

$$\mathbf{a}^S = \left(\mathbf{x}_{I_S}^T \mathbf{x}_{I_S}\right)^{-1} \mathbf{x}_{I_S}^T \mathbf{y}_{I_S},$$

where

$\left(\mathbf{a}^S\right)^T = \left(a_1^S, a_2^S, ..., a_p^S\right) \in \Re^p$, $\forall s \in \overline{1,k}$ and $\mathbf{x}_{I_S}, \mathbf{y}_{I_S}$ are defined as above.

Finally we consider

$$Y = f(X) + \varepsilon \qquad (5)$$

with f a spline function of order $r, r \geq 1$ and with m nodes, $m \geq 1$.

The spline function f has the form

$$f(X) = \alpha_0 + \alpha_1 X + ... + \alpha_r X^r + \sum_{k=1}^{m} \beta_k (X - v_k)_+^r$$

where $v_1 < v_2 < ... < v_m$ are its nodes.

We denote that after substituting $X^j = Z_j, j = \overline{1,r}$, $(X - v_k)_+^r = t_k, k = \overline{1,m}$, the model becomes a linear model with $m + r$ variables and a constant term.

Remark 6.

i) We assume that the nodes $v_k, k = \overline{1,m}$ are given. These can be taken such that in any open interval generated there is at least one value from the n values given for X. In this case $m + 1 \leq n$.

ii) Also we can define a spline function whose nodes are among the sample data of X. If $m < n$ we consider m values of X increasingly ordered as nodes of f such that in any interval $(-\infty, v_1), (v_1, v_2), ..., (v_m, \infty)$ at least one values of X exists. In this case $2m + 1 \leq n$.

In the next theorem we use the notations:

$$V^r(q_1, q_2, ..., q_s) = \begin{pmatrix} q_1 & q_1^2 & ... & q_1^r \\ q_2 & q_2^2 & ... & q_2^r \\ ... & ... & ... & ... \\ q_s & q_s^2 & ... & q_s^r \end{pmatrix},$$

$$V_1^r(q_1, q_2, ..., q_s) = \begin{pmatrix} 1 & q_1 & q_1^2 & ... & q_1^r \\ 1 & q_2 & q_2^2 & ... & q_2^r \\ ... & ... & ... & ... & ... \\ 1 & q_s & q_s^2 & ... & q_s^r \end{pmatrix}$$

$$V'(q_1, q_2, ..., q_s) = \begin{pmatrix} (q_1 - v_1)_+^r & (q_1 - v_2)_+^r & ... & (q_1 - v_m)_+^r \\ (q_2 - v_1)_+^r & (q_2 - v_2)_+^r & ... & (q_2 - v_m)_+^r \\ ... & ... & ... & ... \\ (q_s - v_1)_+^r & (q_s - v_2)_+^r & ... & (q_s - v_m)_+^r \end{pmatrix}.$$

Theorem 7. If $m + r + 1 \leq n$ and among the n values of X there is at least one value situated in each of the $m+1$ open intervals delimited by nodes and there are another r distinct values situated in $(-\infty, v_1)$ then the model is uniquelly fitted by $a_j = c_j, j = \overline{1,r}$, $b_k = c_{r+k}, k = \overline{1,m}$, $\mathbf{c}^T = (c_1, ..., c_{m+r}) \in \Re^{m+r}$, $\mathbf{c} = \left(\widehat{\mathbf{z}}_0^T \widehat{\mathbf{z}}_0\right)^{-1} \widehat{\mathbf{z}}_0^T \widehat{\mathbf{y}}$ where

$$\widehat{\mathbf{z}}_0 = P\mathbf{z}_0, \widehat{\mathbf{y}} = P\mathbf{y}, P = I - \frac{1}{n}\mathbf{u}\mathbf{u}^T, \mathbf{u} = (1, 1, ..., 1) \in \Re^n,$$

$$\mathbf{z}_0 = \left(V^r(x_1, x_2, ..., x_n) : V'(x_1, x_2, ...x_s)\right).$$

Proof. We note that model (5) is a linear model with $m+r$ variables and a constant term. In order for the theorem to remain valid one of the conditions rank($\mathbf{z}$) $= m+r+1$ or rank($\mathbf{z}_0$) $= m+r$ is required. Taking into account that the rank of a matrix is not affected by swaping some rows we consider the values of variable to be ordered as $x_1 \leq x_2 \leq ... \leq x_n$. From hypothesis, there are $r+1$ distinct values in $(-\infty, v_1)$ and in the other intervals there is at least one value. Without loss of generality we take the first $m+r+1$ values such that

$$x_1, x_2, ..., x_{r+1} \in (-\infty, v_1), x_{r+2} \in (v_1, v_2), x_{r+3} \in (v_2, v_3), ..., x_{r+n+1} \in (v_m, \infty) \tag{6}$$

We denote with d the minor formed with the first $m+r+1$ rows of $\mathbf{z}$, so we have $d =$det$M$ with

$$M = \left(V_1^r\left(x_1, x_{2,}..., x_{m+r+1}\right) : V'\left(x_1, x_{2,}..., x_{m+r+1}\right)\right).$$

Since:

$$(x_i - v_k)_+^r = \begin{cases} (x_i - v_k)^r, x_i \geq v_k \\ 0, x_i < v_k \end{cases}$$

we obtain $d =$detM' where

$$M' = \begin{pmatrix} V_1^r\left(x_1, x_{2,}..., x_{r+1}\right) & O_{r+1,m} \\ V_1^r\left(x_{r+2}, x_{r+3,}..., x_{m+r+1}\right) & V \end{pmatrix}$$

and

$$V = \begin{pmatrix} (x_{r+2} - v_1)^r & ... & 0 & 0 \\ ... & ... & ... & ... \\ (x_{m+r} - v_1)^r & ... & (x_{m+r} - v_{m-1})^r & 0 \\ (x_{m+r+1} - v_1)^r & ... & (x_{m+r+1} - v_{m-1})^r & (x_{m+r+1} - v_m)^r \end{pmatrix}$$

Further we obtain

$$d = \left[(x_{m+r+1} - v_m)^r (x_{m+r} - v_{m-1})^r \cdot ... \cdot (x_{r+2} - v_1)^r\right] \cdot$$
$$\cdot \left[(x_{r+1} - x_r)(x_{r+1} - x_{r-1}) \cdot ... \cdot (x_2 - x_1)\right].$$

Since the first $r+1$ values are distinct it follows from (6) that $d \neq 0$ and rank($\mathbf{z}$) $= m+r+1$.

Remark 8. If those m nodes are among the values of X then $2m+r+1 \leq n$.

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UNITARY PRODUCTS AGAIN

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Abstract. Inverses with respect to unitary products are obtained for some functions. These provide further Möbius type of inversion formulas. Lists of powers, products, and summation identities are included.

1. Introduction

The Dirichlet product which is restricted to relatively prime divisors is called the unitary product. It was introduced by Vaidyanathaswamy [14], and it has been further considered by Cohen [1], [2], Davison [3], Gessley [4], Gioia [5], Gioia and Goldsmith [6], Goldsmith [7], Rearick [9], Scheid [10], Sivaramakrishnan [11], Subbarao [12], Subbarao and Gioia [13], and others. We use the notation

$$(\alpha \sqcup \beta)(n) = \sum_{\substack{km=n \\ (k,m)=1}} \alpha(k)\beta(m),$$
$$\alpha^{\sqcup 2} = \alpha \sqcup \alpha, \quad \alpha^{\sqcup -1} = \text{unitary inverse, etc.} \tag{1.1}$$

This product is commutative, associative, and the identity for the Dirichlet product serves as the identity. There are non-trivial divisors of 0 as can be seen by setting $\alpha = \beta = \delta_2$. The unitary product of multiplicative functions is multiplicative. A function α has a $\sqcup$-inverse provided that $\alpha(1) \neq 0$.

In Section 2 we set forth the definitions of the operations and the number-theoretic functions which are to be used. Inverses of a number of functions with respect to the unitary product are given in Section 3. These are used in order to write out some Möbius type of inversion formulas, one of which involves the Möbius function μ. Lists of unitary powers, unitary products, and alternative factorizations are included in the final section. The alternative factorizations provide us with summation identities which involve various number-theoretic functions.

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2. Definitions

The domain of definition of number-theoretic functions is, as usual, taken to be the positive integers. We use the following notations for operations.

$$(\alpha \cdot \beta)(n) = \alpha(n)\beta(n) \quad \text{(pointwise product)}$$

$$(\alpha * \beta)(n) = \sum_{km=n} \alpha(k)\beta(m) \quad \text{(Dirichlet product)}$$

$$(\alpha \Box \beta)(n) = \sum_{[k,m]=n} \alpha(k)\beta(n) \quad \text{(lcm-product)} \tag{2.1}$$

$$\alpha^2 = \alpha \cdot \alpha, \quad \alpha^{*2} = \alpha * \alpha, \quad \alpha^{\Box 2} = \alpha \Box \alpha, \quad \text{etc.}$$

$$\alpha^{*-1} = \text{Dirichlet inverse of } \alpha, \quad \text{etc.}$$

All of these product are associative, commutative, and there are identity elements

$$(\nu_0 \cdot \alpha) = \alpha, \quad \epsilon * \alpha = \alpha, \quad \epsilon \Box \alpha = \alpha, \text{ for all } \alpha. \tag{2.2}$$

No non-trivial divisors of zero exist for the $*$-product, but they do exist for the other products. The set of number theoretic functions which satisfy the condition $\alpha(1) \neq 0$ forms a group under Dirichlet multiplication. In this group $\alpha * \beta$ is a multiplicative function if both α and β are multiplicative; that is, the multiplicative functions form a subgroup; see Niven and Zuckerman [8]. Some properties of all of these multiplications can be found in Scheid [10].

We use ϵ for the identity, $\epsilon(n) = 1$ if $n = 1$, and $= 0$ otherwise. We choose the symbols ν_0 for the multiplicative identity for the pointwise product, $\nu_0 = 1$ for all n; $\nu_k(n) = n^k$; $\kappa(n)$ for the number of square-free divisors of n, $\kappa(n) = 2^{\omega(n)}$; λ for Liouville's function, $\lambda(n) = (-1)^{\Omega(n)}$; μ for the Möbius function, $\mu(n) = 0$ if $p^2|n$, and $= (-1)^{\omega(n)}$ otherwise. Let $\tau_k(n) =$ the number of ways of writing n as a product of k factors, $\tau_2 = \tau$, $\sigma_k(n) =$ the sum of the k^{th} powers of the divisors of n, $\sigma_1 = \sigma$; $\omega(n) =$ the number of prime divisors of n; and $\Omega(n) =$ the total number of divisors of n. In addition we let $\delta_k(n) = 1$ if $n = k$, $= 0$ otherwise; generalized Möbius functions $\mu_k^C = (\mu \cdot \nu_k) = \nu_k^{*-1}$, $\mu_0^C = \mu$; $\mu_k^D = \tau_k^{*-1}$, $\mu_1^D = \mu$; $\mu_k^M = P_k^{*-1}$, $\mu_1^M = \mu$, $J_k = \nu_k * \mu$ Jordan's totient, $J_1 = \phi$; $P_k(n) = 1$ if $n = m^k$, $= 0$ otherwise, $P_2 = P$, the characteristic function for squares; $Q_k(n) = 1$ if n is k^{th}-power-free, $= 0$ otherwise, $Q_2 = Q = \mu^2$; and $S(n) =$ the number of divisors of n^2.

In our work the superscript symbol † is used to denote the unitary analogs of our previously defined number-theoretic functions, instead of the more customary symbol *. This is done in order to avoid possible confusion with $*$-multiplication. We define some of the more important analogous functions which occur naturally in connection with the $\sqcup$-product.

$$\mu^\dagger = (-1)^\omega = \nu_0^{\sqcup -1}. \tag{2.3}$$

$$\mu_k^\dagger = P_k^{\sqcup -1}, \quad \mu_1^\dagger = \mu^\dagger. \tag{2.4}$$

$$\sigma_k^\dagger(n) = \begin{array}{c}\text{sum of the } k^{th} \text{ powers of the} \\ \text{unitary divisors of } n,\end{array} \qquad (2.5)$$

$$\sigma_0^\dagger = \tau^\dagger, \quad \sigma_1^\dagger = \sigma^\dagger.$$

$$\tau_k^\dagger(n) = \nu_0^{\sqcup k}(n) = (\mu^\dagger)^{\sqcup -k} \qquad (2.6)$$

$$\tau_{-1}^\dagger = \mu^\dagger, \quad \tau_0^\dagger = \epsilon, \quad \tau_1^\dagger = \nu_0, \quad \tau_2^\dagger = \tau^\dagger = \kappa.$$

The function $(\mu^\dagger)^{\sqcup k}$ is the unitary analog of μ_k^D; $\mu_k^\dagger$, of μ_k^M. For $k > 0$, the function $\tau_k^\dagger$ counts the number of ways of expressing n as a product of k factors which are relatively prime in pairs.

3. Inverses and inversion formulas

The Möbius inversion formula (for sums over divisors) is given by

$$\alpha = \nu_0 * \beta \;\Leftrightarrow\; \beta = \mu * \alpha. \qquad (3.1)$$

Since the $\sqcup$-inverse of ν_0 is $\mu^\dagger$, an analog of the Möbius inversion formula is

$$\alpha = \nu_0 \sqcup \beta \;\Leftrightarrow\; \beta = \mu^\dagger \sqcup \alpha. \qquad (3.2)$$

A generalization follows from the definitions.

$$\alpha = \tau_k^\dagger \sqcup \beta \;\Leftrightarrow\; \beta = (\mu^\dagger)^{\sqcup k} \sqcup \alpha. \qquad (3.3)$$

Some of the $\sqcup$-inverses have been derived.

$$\kappa^{\sqcup -1} = (\mu^\dagger)^{\sqcup 2} = \tau_{-2}^\dagger. \qquad (3.4)$$

$$P_k^{\sqcup -1} = \mu_k^\dagger. \qquad (3.5)$$

$$Q^{\sqcup -1} = \mu. \qquad (3.6)$$

$$(\kappa \cdot \mu)^{\sqcup -1} = (\kappa \cdot Q). \qquad (3.7)$$

$$(\mu \cdot S)^{\sqcup -1} = (Q \cdot S). \qquad (3.8)$$

$$(\nu_k \cdot Q)^{\sqcup -1} = \mu_k^C. \qquad (3.9)$$

These lead to a number of further inversion formulas. As one example, an alternative analog for the Möbius inversion which retains μ, instead of ν_0, in the formulas reads

$$\alpha = Q \sqcup \beta \;\Leftrightarrow\; \beta = \mu \sqcup \alpha. \qquad (3.10)$$

Some other examples follow.

$$\alpha = P_k \sqcup \beta \;\Leftrightarrow\; \beta = \mu_k^\dagger \sqcup \alpha. \qquad (3.11)$$

$$\alpha = (\kappa \cdot \mu) \sqcup \beta \;\Leftrightarrow\; \beta = (\kappa \cdot Q) \sqcup \alpha. \qquad (3.12)$$

$$\alpha = (\mu \cdot S) \sqcup \beta \;\Leftrightarrow\; \beta = (Q \cdot S) \sqcup \alpha. \qquad (3.13)$$

$$\alpha = (\nu_k \cdot Q) \sqcup \beta \;\Leftrightarrow\; \beta = \mu_k^C \sqcup \alpha. \qquad (3.14)$$

The completely $\sqcup$-multiplicative functions are simply the $*$-multiplicative functions. Hence, if ξ is $*$-multiplicative, we have the three important properties of completely multiplicative functions, see Scheid [10].

$$\xi \cdot (\alpha \sqcup \beta) = (\xi \cdot \alpha) \sqcup (\xi \cdot \beta). \qquad (3.15)$$

$$\xi^{\sqcup -1} = (\xi \cdot \mu^\dagger). \qquad (3.16)$$

$$\xi^{\sqcup r} = (\xi \cdot \tau_k^{\dagger}). \tag{3.17}$$

Further, if η is multiplicative and ξ has a $\sqcup$-inverse, then

$$(\eta \cdot \xi)^{\sqcup -1} = (\eta \cdot \xi^{\sqcup -1)}. \tag{3.18}$$

Hence for any $*$-multiplicative function η we have the general inversion formula

$$\alpha = \eta \sqcup \beta \;\Leftrightarrow\; \beta = (\eta \cdot \mu^{\dagger}) \sqcup \alpha. \tag{3.19}$$

We note that $(\alpha * \beta)(n)$ and $(\alpha \sqcup \beta)(n)$ are equal at squarefree n. Since $Q(n)$ is $*$-multiplicative and equals 0 except at squarefree n, we have

$$(\alpha * \beta) \cdot Q = (\alpha \cdot Q) \sqcup (\beta \cdot Q). \tag{3.20}$$

The operation of pointwise multiplication of functions by Q can be seen to map the $*$-products into the $\sqcup$-products which are evaluated at squarefree numbers. This may seem to be of limited value, but it does give us a way to build up another list from any list of Dirichlet products. From the known result $\lambda * \nu_0 = P$ and evaluation of the pointwise products, we can thus show that $\mu \sqcup Q = \epsilon$, which leads to the Möbius inversion formula (3.10). Formulas (3.16) and (3.20) are a source for various $\sqcup$-inverses.

For completely $*$-multiplicative functions Vaidyanathaswamy [14] had obtained a relation which connects four different products.

$$(\alpha \sqcup \beta) * (\alpha \cdot \beta) = (\alpha \Box \beta). \tag{3.21}$$

Two additional identities of Scheid [10] are of interest, since they also provide connections among various products. The first of these is a corollary of his formula for a product of n-factors; compare with (2.8). For $*$-multiplicative functions

$$\xi * (\alpha \sqcup \beta) = (\xi * \alpha) \sqcup (\xi * \beta) \sqcup (\mu^{\dagger} \cdot \xi). \tag{3.22}$$

$$(\alpha \cdot (\beta * \nu_0)) \sqcup (\beta \cdot (\alpha * \nu_0)) = (\alpha \cdot \beta) * (\alpha \Box \beta). \tag{3.23}$$

4. **Lists of Products**

Since not many explicit unitary products appear in the literature, a number of examples have been obtained. First, a few $\sqcup$-powers are known.

$$\kappa^{\sqcup r} = \tau_{2r}^{\dagger}. \tag{4.1}$$

$$\mu^{\sqcup r} = \mu_r^{D}. \tag{4.2}$$

$$Q^{\sqcup r} = (\tau_r \cdot Q). \tag{4.3}$$

Many special cases of (3.21)-(3.23) are themselves of interest. Several of the special cases resulting from (3.20) have been included in the list.

$$\alpha \sqcup \epsilon = \alpha. \tag{4.4}$$

$$\kappa \sqcup \nu_0 = \tau_3^{\dagger}. \tag{4.5}$$

$$\kappa \sqcup \mu^{\dagger} = \nu_0. \tag{4.6}$$

$$\kappa \sqcup \tau_k^{\dagger} = \tau_{k+2}^{\dagger}. \tag{4.7}$$

$$\kappa \sqcup \phi^{\dagger} = \sigma^{\dagger}. \tag{4.8}$$

$$\lambda^{\sqcup 2} = (\kappa \cdot \lambda). \tag{4.9}$$

$$\mu \sqcup Q_{2k} = \epsilon. \tag{4.10}$$

$$\mu \sqcup (\mu \cdot \tau) = (\mu \cdot S). \tag{4.11}$$

$$\mu \sqcup (\tau \cdot Q) = Q. \tag{4.12}$$

$$\mu_k^C \sqcup (\sigma_k \cdot Q) = Q. \tag{4.13}$$

$$\mu_k^C \sqcup (J_k \cdot Q) = \mu. \tag{4.14}$$

$$\nu_0^{\sqcup 2} = \kappa. \tag{4.15}$$

$$\nu_0^{\sqcup k} = \tau_k^\dagger. \tag{4.16}$$

$$\nu_0 \sqcup \nu_k = \sigma_k^\dagger. \tag{4.17}$$

$$\nu_0 \sqcup \mu^\dagger = \epsilon. \tag{4.18}$$

$$\nu_0 \sqcup \phi^\dagger = \nu_1. \tag{4.19}$$

$$\nu_0 \sqcup J_k^\dagger = \nu_k. \tag{4.20}$$

$$\nu_0 \sqcup \tau_k^\dagger = \tau_{k+1}^\dagger. \tag{4.21}$$

$$\nu_1 \sqcup \mu^\dagger = \phi^\dagger. \tag{4.22}$$

$$\nu_k \sqcup \mu^\dagger = J_k^\dagger. \tag{4.23}$$

$$Q \sqcup (\nu_k \cdot Q) = (\sigma_k \cdot Q). \tag{4.24}$$

$$Q \sqcup (\kappa \cdot \mu) = \mu. \tag{4.25}$$

$$Q \sqcup (\kappa \cdot Q) = (S \cdot Q). \tag{4.26}$$

$$Q \sqcup (\mu \cdot S) = (\mu \cdot \tau). \tag{4.27}$$

$$Q \sqcup (J_k \cdot Q) = (\nu_k \cdot Q). \tag{4.28}$$

$$(\kappa \cdot Q) \sqcup (\mu \cdot S) = \mu. \tag{4.29}$$

$$(\mu \cdot \sigma_k) \sqcup (\nu_k \cdot Q) = \mu. \tag{4.30}$$

$$(\mu \cdot J_k) \sqcup (\nu_k \cdot Q) = Q. \tag{4.31}$$

$$\mu^\dagger \sqcup \sigma_k^\dagger = \nu_k. \tag{4.32}$$

$$\mu^\dagger \sqcup \tau_k^\dagger = \tau_{k-1}^\dagger. \tag{4.33}$$

$$\sigma_k^\dagger \sqcup \phi^\dagger = (\nu_1 \cdot \sigma_{k-1}^\dagger) = (\nu_k \cdot \sigma_{1-k}^\dagger). \tag{4.34}$$

$$\tau^\dagger \sqcup \phi^\dagger = \sigma^\dagger. \tag{4.35}$$

A few examples are presented of mixed products which involve both the Dirichlet and the unitary results. Since $\sigma_k^\dagger$ has a known Dirichlet series generating function, some Dirichlet products which involve it can be obtained.

$$\lambda * (\mu \sqcup (\tau \cdot Q)) = \epsilon. \tag{4.36}$$

$$\nu_1 \sqcup (\nu_1 \Box \mu^\dagger) = \phi^\dagger. \tag{4.37}$$

$$(\nu_k \sqcup \nu_m) * \sigma_{k+m} = (\sigma_k \cdot \sigma_m). \tag{4.38}$$

$$\nu_k * \nu_m * \sigma_{k+m}^\dagger = (\sigma_k \cdot \sigma_m). \tag{4.39}$$

Alternative factorizations can be interpreted as summation identities. Many examples exist; we list a few of them.

$$\lambda^{\Box 2} = \lambda^{\sqcup 2} * \nu_0. \tag{4.40}$$

$$\nu_0 \Box \nu_k = \nu_k * \sigma_k^\dagger. \tag{4.41}$$

$$\nu_k \sqcup \sigma_m^\dagger = \nu_m \sqcup \sigma_k^\dagger. \tag{4.42}$$

$$P * P^{\sqcup 2} = P^{\Box 2}. \tag{4.43}$$

In the spirit of Liouville's summation identities, (4.42) and (4.43) can be rewritten, respectively,

$$\sum_{\substack{rs=n\\(r,s)=1}} \nu_k(r)\sigma_m^\dagger(s) = \sum_{\substack{rs=n\\(r,s)=1}} \nu_m(r)\sigma_k^\dagger(s), \tag{4.44}$$

$$\sum_{hj=n} \sum_{\substack{km=h\\(k,m)=1}} P(j)P(k)P(m) = \sum_{[k,m]=n} P(k)P(m). \tag{4.45}$$

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ROBUST STABILITY OF COMPACT C_0-SEMIGROUPS ON BANACH SPACES

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Abstract. It is a well-known fact that many properties, of a compact semigroup, are preserved under bounded perturbations. In this paper we show that the asymptotic stability is also preserved, provided that the spectral radius of the perturbation is not greater than the modulus of the spectral bound of the semigroup's generator. We achieve our goal by improving Pazy's result concerning the behaviour of the spectrum of the generator.

1. **Preliminaries**

Consider X a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a C_0-semigroup with generator $A : D(A) \subset X \to X$, denoted by $(A, D(A))$.

We use the theoretical notations for $R(\lambda, A)$, $\rho(A)$, $\sigma(A)$, for the resolvent, the resolvent set and, respectively, the spectrum of A. We also use the following notations: the point spectrum $\sigma_p(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not injective}\}$; the spectral bound $s(A) = \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$; the spectral radius $r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$.

Let us also remind that the semigroup $\mathcal{T}$ is *asymptotically stable* if

$$\lim_{t\to\infty} \|T(t)x\| = 0 \text{ for any } x \in X.$$

We also use to say that a certain property of the semigroup is *robust* whenever it is preserved under some bounded perturbations.

2. **About the spectrum and robust asymptotic stability of a compact semigroup**

In the following we shall need an auxiliary result from complex analysis, the proof of which is included for the reader's convenience.

Lemma 1. *Let* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence in* $S_{a,b} = \{\lambda \in C \mid a \leq \operatorname{Re}\lambda \leq b\}$, *where* a, b *are real numbers, such that* $\lim\limits_{n\to-\infty} |\operatorname{Im}\lambda_n| = \infty$. *Then there is* $t > 0$ *such that* $\{e^{t\lambda_n}\}_{n=1}^{\infty}$ *has infinitely many acumulation points.*

Proof. We may assume that $0 \leq \operatorname{Im}\lambda_n$ for all $n \geq 1$. Let $J = [0, 1]$ and $\{q_m\}_{m=1}^{\infty}$ be a dense sequence in $[0, 2\pi]$.

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Let $\{A_n\}_{n=1}^{\infty}$ be an enumeration of the sets $B_{m,k} = \{re^{is} \mid s \in q_m + [0, k^{-1}], e^a \le r \le e^b\}$, for $k, m \in \mathbb{N}$. The claim is that ther is $t > 0$ and a subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ such that $e^{t\lambda_{n_k}} \in A_k$ for every $k \ge 1$. Clearly the assertion follows from this.

To establish the claim, choose $n_1 \in N$ such that $A_1 \subseteq \{re^{is}; s \in (\operatorname{Im}\lambda_{n_1})J, e^a \le r \le e^b\}$. Let $J_1 \subseteq J$ be a closed subinterval with $A_1 = \{re^{is} \mid s \in (\operatorname{Im}\lambda_{n_1})J_1, e^a \le r \le e^b\}$. Inductively, we obtain a subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\lambda_n\}_{n=1}^{\infty}$ and closed intervals $J \supseteq J_1 \subseteq J_2 \supseteq \cdots$ such that $A_k = \{re^{is} \mid s \in (\operatorname{Im}\lambda_{n_k})J_k,\ e^a \le r \le e^b\}$ for $k \ge 1$. Choose any $t \in \bigcap\limits_{k\ge 1} J_k$. Then $e^{t\lambda_{n_k}} \in A_k$ for all $k \ge 1$.

The following theorem improves in the second part one of Pazy's results [3], and using another approach, also gives a more elementary proof, for the first part of the theorem.

Theorem 2. *Let X be a Banach space and $\mathcal{T} = (T(t))_{t\ge 0}$ a compact C_0-semigroup with generator $(A, D(A))$. Then $\sigma(A)$ consists of a sequence of isolated eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, with finite multiplicity, and satisfies $\lim\limits_{n\to\infty} \operatorname{Re}\lambda_n = -\infty$.*

Proof. It is known that $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \ge w\} \subset \rho(A)$ for some $w \in \mathbb{R}$. As $\rho(A) \ne \emptyset$ choose $\eta \in \rho(A)$ and define $R(\eta, A) \in L(X)$. As $(T(t))_{t\ge 0}$ is a compact semigroup it follows that $R(\eta; A)$ is a compact operator [3] which means that $\sigma(R(\eta; A))$ is a sequence of isolated eigenvalues $\{\eta_n\}_{n=1}^{\infty}$ for $R(\eta; A)$ having 0 as the single accumulation point [2]. By the spectral mapping theorem we have $\sigma(R(\eta, A)) = \{0\} \cup \{(\eta - \lambda)^{-1} \mid \lambda \in \sigma(A)\}$. Since eigenvalues of $R(\eta; A)$ correspond to eigenvalues of $(A, D(A))$ having the same finite multiplicity, then, the first part of our claim follows.

Let us denote by $\{\lambda_n\}_{n=1}^{\infty}$ the sequence of eigenvalues of A.

As $\eta_n = \dfrac{1}{\eta - \lambda_n}$, which means $\lambda_n = \eta - \dfrac{1}{\eta_n}$ it follows that for $\eta_n \to 0$, $\lambda_n \to \infty$ and thus $\{\lambda_n\}_{n=1}^{\infty}$ is an unbounded sequence.

Consider now $S_{a,b} = \{\lambda \in \mathbb{C} \mid a \le \operatorname{Re}\lambda \le b\}$ with $a, b \in \mathbb{R}$, $a < b$, and denote by $\{\lambda_{n_k}\}_{k\in N} = S_{a,b} \cap \sigma(A)$.

Suppose that $\{\lambda_{n_k}\}_{k\in N^*}$ is an infinite set. As $\{\lambda_n\}_{n\in N^*}$ is unbounded we deduce that $\lim\limits_{k\to\infty} |\operatorname{Im}\lambda_{n_k}| = \infty$. Then, by Lemma 1, it follows there exists $t_0 > 0$ such that $\{e^{t_0\lambda_{n_k}}\}_{k=1}^{\infty}$ has infinitely many accumulation points. Then the spectral inclusion theorem $e^{t_0\sigma(A)} \subset \sigma(T(t_0))$, and the fact that $\{e^{t_0\lambda_{n_k}}\}_{k\in N^*} \subseteq \{e^{t_0\lambda_n}\}_{n\in N^*}$ imply $\sigma(T(t_0))$ has infinitely many accumulation points. But $T(t_0)$ is a compact operator, therefore it has at most one point of accumulation. That means that $\{\lambda_{n_k}\}_{k\in N^*}$ is always finite for any $a, b \in \mathbb{R}$, $a < b$ and $\lim\limits_{n\to\infty} \operatorname{Re}\lambda_n = -\infty$.

In the following, we shall use a perturbation result from the semigroup theory.

Lemma 3. *Let $(A, D(A))$ be the generator of a C_0–semigroup defined on a Banach space X. If $B \in \mathcal{L}(X)$, then $C = A + B$, where $D(C) = D(A)$, is the generator of a C_0-semigroup $(S(t))_{t\ge 0}$. In addition if $(T(t))_{t\ge 0}$ is compact then $(S(t))_{t\ge 0}$ is also compact.*

Now, let's consider $L(X)$ the space of all linear, bounded operators defined on X.

Theorem 4. *Let X be a Banach space and let $(A, D(A))$ be the infinitesimal generator of an asymptotically stable, compact C_0-semigroup $(T(t))_{t\geq 0}$. If $B \in L(X)$ and B commutes with A, with $r(B) < |s(A)|$, then the semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ generated by $A+B$ is also asymptotically stable.*

Proof. As $(T(t))_{t\geq 0}$ is stable it follows that it is also bounded and therefore $\sigma(A) \subseteq \{\lambda \in C \mid \operatorname{Re}\lambda \leq 0\}$. By Theorem 2.4 [1], a necessary and sufficient condition for the strong stability of $\mathcal{T} = (T(t))_{t\geq 0}$ is $\sigma_p(A) \cap i\mathbb{R} = \emptyset$. Therefore, as $\sigma(A) = \{\lambda_n\}_{n=1}^{\infty}$ and $\operatorname{Re}\lambda_n < 0$ for any $n \in N^*$ and as $\lim\limits_{n\to\infty} \operatorname{Re}\lambda_n = -\infty$ it follows that $s(A) = \max\limits_{n}\{\operatorname{Re}\lambda_n\} < 0$.

By Lemma 3, the semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ generated by $A+B$, is also compact. That means that, by the previous theorem, $\sigma(A+B) = \{\mu_n\}_{n\in N^*}$ with $\lim\limits_{n\to\infty} \operatorname{Re}\mu_n = -\infty$ and so $s(A+B) = \max\limits_{n}\{\operatorname{Re}\mu_n\}$.

If $s(A+B) \leq s(A) < 0$, it means that $s(A+B) < 0$ and thus $\sigma_p(A+B) \cap i\mathbb{R} = \emptyset$ and in this case $\mathcal{S}$ is asymptotically stable. Suppose that $s(A+B) > s(A)$. As B commutes with A, a theorem of Kato [2] assures us, that the Pompeiu–Hausdorff distance between $\sigma(A)$ and $\sigma(A+B)$ does not exceed the $r(B)$, $\operatorname{dist}(\zeta, \sigma(A+B)) \leq r(B)$ if $\zeta \in \sigma(A)$.

As $s(A+B) - s(A) \leq \operatorname*{dist}\limits_{\zeta\in\sigma(A)}(\zeta, \sigma(A+B)) \leq r(B) < |s(A)|$ it follows that $s(A+B) < 0$, which proves the theorem.

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BIVARIATE SHEPARD OPERATORS OF ABEL-GONCIAROV-TYPE

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Let f be a bivariate real-valued function defined on a domain $D \subset \mathbb{R}^2$, $Z = \{z_i|\ z_i = (x_i, y_i),\ i = 1, \ldots, N\} \subset D$ and $\mathcal{I}(f) = \{\lambda_k f|\ k = 1, \ldots, n\}$ a set of informations on f (punctual evaluation of f and of certain of its derivatives).

It is well known the Shepard's interpolation operator S_0, defined by

$$(S_0 f)(x, y) = \sum_{i=1}^{N} A_i(x, y) f(x_i, y_i),$$

where

$$A_i(x, y) = \prod_{\substack{j=1 \\ j \neq i}}^{N} d_j^{\mu}(x, y) \Bigg/ \left(\sum_{k=1}^{N} \prod_{\substack{j=1 \\ j \neq k}}^{N} d_j^{\mu}(x, y) \right)$$

with $d_j(x, y) = ((x - x_j)^2 + (y - y_j)^2)^{1/2}$, $\mu \in \mathbb{R}_+$ and $n = N$.

Remark 1. Here $\lambda_k f = f(x_k)$, $k = 1, \ldots, N$.

A basic characteristic of an approximation operator is its degree of exactness, usually abreviated by "dex".

As $dex(S_0) = 0$, a first problem which appears in regard to the operator S_0 is its low degree of exactness. In order to increase it, the Shepard's operator S_0 was combined with others interpolation operators. In this way were defined Shepard operators of Lagrange-type [6], Taylor-type [1, 3], Hermite-type [5], Birkhoff-type [2], etc.

The most general case, studied in [2], is the Shepard operator of Birkhoff-type.

The information set about f, in this case, is

$$\mathcal{I}_B(f) = \{\lambda_k^{p,q} f|\ \lambda_k^{p,q} f = f^{(p,q)}(x_k, y_k),\ (p, q) \in I_k \subset \mathbb{N}^2,\ k = 1, \ldots, N\}$$

with $|\mathcal{I}_B(f)| = n$.

Now, if $\mathcal{I}_k(f)$ is the information of f at the point z_k ($\mathcal{I}_k(f) = \{\lambda_k^{p,q} f|\ (p, q) \in I_k\}$, $Z_{k,\nu_k} = \{z_{k+j}|\ j = 0, 1, \ldots, \nu_k - 1\}$, $\nu_k \in \mathbb{N}^*$ and $\mathcal{I}_{k,\nu_k}(f) = \{\lambda_{k+j}^{p,q} f|\ (p, q) \in I_{k+j},\ j = 0, 1, \ldots, \nu_k - 1\}$, where $z_{N+i} = z_i$, $i \in \mathbb{N}$, one denotes by $B_k^{r_k}$ the Birkhoff polynomial of the total degree r_k which interpolates the data $\mathcal{I}_{k,\nu_k}(f)$, i.e.

$$\lambda_{k+j}^{p,q} B_k^{r_k} = \lambda_{k+j}^{p,q} f, \quad (p, q) \in I_{k+j}, \quad j = 0, 1, \ldots, \nu_k - 1.$$

The operator S_B defined by

$$S_B f = \sum_{k=1}^{N} A_k B_k^{r_k} f \tag{1}$$

was called combined Shepard operator of Birkhoff-type.

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The degree of exactness of the operator S_B is

$$dex(S_B) = \min\{r_k|\ k = 1, \ldots, N\}.$$

The difficult problem arising in regard to the operator S_B is to select the subsets $\mathcal{I}_{k,\nu_k}(f)$ of the given set $\mathcal{I}_B(f)$ such that the Birkhoff polynomials $B_k^{r_k}$ to exist for all $k = 1, \ldots, N$.

The goal of this paper is to consider some particular cases of Birkhoff-type information for which the existence and the uniqueness of the corresponding Birkhoff interpolation polynomials are assured.

1. **Taylor type information**

As a particular Birkhoff-type information, that was already studied, is Taylor-type information

$$\mathcal{I}_T(f) = \{\lambda_k^{p,q} f|\ \lambda_k^{p,q} f = f^{(p,q)}(z_k),\ p, q \in \mathbb{N},\ p + q \leq n_k,\ k = 1, \ldots, N\}$$

Using this information was constructed so called Shepard operator of Taylor-type S_m [1], [3] defined by

$$(S_m f)(x, y) = \sum_{i=1}^{N} A_i(x, y)(T_i^m f)(x, y)$$

where $T_i^m f$ is the bivariate Taylor operator of the total degree m:

$$(T_i^m f)(x, y) = \sum_{p+q \leq m} \frac{(x - x_i)^p}{p!} \frac{(y - y_i)^q}{q!} f^{(p,q)}(x_i, y_i),$$

respectively $S_{m_1,\ldots,m_N}$ [1]:

$$(S_{m_1,\ldots,m_N} f)(x, y) = \sum_{i=1}^{N} A_i(x, y)(T_i^{m_i} f)(x, y)$$

where $T_i^{m_i}$ is the Taylor's polynomial of the degree m_i and the interpolation nodes x_i.

2. **Bivariate Abel-Gonciarov interpolation**

In the univariate case, the Abel-Gonciarov interpolation problem is based on the information set

$$\mathcal{I}(f) = \{f(x_0), f'(x_1), \ldots, f^{(n)}(x_n)\}$$

The polynomial

$$(P_n f)(x) = \sum_{i=0}^{n} p_{n,i}(x) f^{(i)}(x_i)$$

for which

$$p_{n,i}(x_j) = \delta_{ij}, \quad i, j = 0, 1, \ldots, n$$

is the corresponding Abel-Gonciarov polynomial that interpolates the date $\mathcal{I}(f)$, i.e.

$$(P_n f)^{(i)}(x_i) = f^{(i)}(x_i), \quad i = 0, 1, \ldots, n.$$

Remark 2. For $x_n = x_{n-1} = \cdots = x_1 = x_0$, $P_n f$ becomes the Taylor polynomial $T_n f$:

$$(T_n f)(x) = \sum_{i=0}^{n} \frac{(x-x_0)^i}{i!} f^{(i)}(x_0).$$

For the bivariate case, let us consider the interpolation problem: for a real-valued function defined on D and for a given set of points $Z = \{z_i \in D |\ i = 0, 1, \ldots, n\}$ for which the information

$$\mathcal{I}_n(f) = \{f^{(p,q)}(z_{p+q})|\ p, q \in \mathbb{N},\ p+q = 0, 1, \ldots, n\} \tag{2}$$

exists, find a polynomial $P \in \mathbb{P}^2$ (the set of all bivariate polynomials), with the minimal total degree, that interpolates the data $\mathcal{I}_n(f)$.

Such a problem will be called a bivariate Abel-Gonciarov interpolation problem.

Also, a solution of such a problem is called a bivariate Abel-Gonciarov interpolation polynomial. Next it will be abreviated by $G_n f$, where n is its total degree.

Since $|\mathcal{I}_n(f)| = (n+1)(n+2)/2$, the solution of the above interpolation problem, is a bivariate polynomial of the total degree n:

$$P_n(x,y) = \sum_{i+j \le n} a_{ij} x^i y^j. \tag{3}$$

Lemma 1. *Let be $D \subset \mathbb{R}^2$, $z_i \in D$, $i = 0, 1, \ldots, n$ and $f : D \to \mathbb{R}$ a function for which $\mathcal{I}_n(f)$ exists. Then, for all $n \in \mathbb{N}^*$, there exists an unique polynomial of the total degree n, $G_n f$, that interpolates the data $\mathcal{I}_n(f)$.*

Proof. Applying the interpolation conditions to the polynomial from (3), one obtains

$$\sum_{i+j \le n} i^{[p]} j^{[q]} x_{p+q}^{i-p} y_{p+q}^{j-q} a_{ij} = f^{(p,q)}(x_{p+q}, y_{p+q}), \quad p+q = 0, 1, \ldots, n, \tag{4}$$

which is a $(n+1)(n+2)/2 \times (n+1)(n+2)/2$ linear algebraic system in the unknowns a_{ij}, $i+j \le n$. The matrix of this system, say M, is an upper diagonal matrix with nonzero elements on its diagonal. Hence $\det M \neq 0$ and the proof follows.

Definition 1. The operator G_n is called the bivariate Abel-Gonciarov polynomial interpolation operator.

Remark 3. G_n exists and is unique for all $n \in \mathbb{N}^*$ and $dex(G_n) \ge n$ (lemma 1).

We have

$$(G_n f)(x,y) = \sum_{i+j \le n} g_{ij}(x,y) f^{(i,j)}(x_{i+j}, y_{i+j}), \tag{5}$$

where g_{ij} are the corresponding fundamental interpolation polynomials:

$$g_{ij}^{(p,q)}(x_{i+j}, y_{i+j}) = \begin{cases} 1, & \text{for } (p,q) = (i,j) \\ 0, & \text{otherwise.} \end{cases}$$

3. Combined Shepard operator of Abel-Gonciarov-type

If in (1), instead of $B_k^{r_k}$ is taken Abel-Gonciarov operator $G_k^{r_k}$ that interpolates the information

$$\mathcal{I}_{k,r_k}(f) = \{f^{(p,q)}(z_{k+p+q})|\ p, q \in \mathbb{N},\ p+q = 0, 1, \ldots, r_k\} \tag{6}$$

for $k = 1, \dots, N$, with the specification that $z_{N+k} = z_k$, one obtains the Shepard-type operator $S^G_{r_1,\dots,r_N}$:

$$(S^G_{r_1,\dots,r_N} f)(x, y) = \sum_{k=1}^{N} A_k(x, y)(G^{r_k}_k f)(x, y).$$

Definition 2. The operator $S^G_{r_1,\dots,r_N}$ is called the bivariate Shepard operator of Abel-Gonciarov-type.

From (5), it follows that

$$(G^{r_k}_k f)(x, y) = \sum_{i+j \le r_k} g_{r_k,i,j}(x, y) f^{(i,j)}(x_{k+i+j}, y_{k+i+j})$$

with

$$g^{(p,q)}_{r_k,i,j}(x_{k+i+j}, y_{k+i+j}) = \begin{cases} 1, & \text{for } (p, q) = (i, j) \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3. The existence and uniqueness of the interpolating polynomials $G^{r_k}_k$, $k = 1, \dots, N$ implies the existence and the uniqueness of the operator $S^G_{r_1,\dots,r_N}$.

Theorem 1. *Let be $M = \max\{r_k |\ k = 1, \dots, N\}$. If $\mu > M$ then*

$$(S^G_{r_1,\dots,r_N} f)^{(p,q)}(z_{k+p+q}) = f^{(p,q)}(z_{k+p+q}), \quad p + q = 0, 1, \dots, r_k$$

for all $k = 1, \dots, N$.

Proof. First, we note that

$$\begin{array}{l} A^{(p,q)}_k(x_i, y_i) = 0 \text{ for } (p, q) \in \mathbb{N}^2,\ p + q = 0, 1, \dots, r_i,\ i \ne k \\ A^{(p,q)}_k(x_k, y_k) = 0,\ p + q = 1, \dots, r_k \end{array} \tag{7}$$

It follows that

$$(S^G_{r_1,\dots,r_N} f)^{(p,q)}(z_{i+p+q}) = \sum_{k=1}^{N} (A_k G^{r_k}_k f)^{(p,q)}(z_{i+p+q}) =$$

$$= \sum_{i=1}^{N} A_k(z_{i+p+q})(G^{r_k}_k f)^{(p,q)}(z_{i+p+q})$$

As,

$$A_k(x_i, y_i) = \delta_{ki}$$

$$(G^{r_k}_k f)^{(p,q)}(z_{i+p+q}) = f^{(p,q)}(z_{i+p+q}),$$

one obtains

$$(S^G_{r_1,\dots,r_N} f)^{(p,q)}(z_{i+p+q}) = f^{(p,q)}(z_{i+p+q})$$

for $p + q = 0, 1, \dots, r_i$ and $i = 1, \dots, N$.

Theorem 2. *Let be $r = \min\{r_k |\ k = 1, \dots, N\}$. Then*

$$dex(S^G_{r_1,\dots,r_N}) = r.$$

Proof. We have to check that

$$S^G_{r_1,\dots,r_N} e_{ij} = e_{ij}, \text{ for all } i, j \in \mathbb{N} \text{ with } i + j \le r$$

where $e_{ij}(x,y)x^iy^j$. But, $G_k^{r_k}e_{ij}=e_{ij}$ for $i+j\le r_k$ for all $k=1,\dots,N$ ($dex(G_k^{r_k})=r_k$). Hence $G_k^{r_k}e_{ij}=e_{ij}$ for $i+j\le r$ ($r\le r_k$, $k=1,\dots,N$). It follows that

$$S^G_{r_1,\dots,r_N}e_{ij}=\sum_{k=1}^N A_ke_{ij}=e_{ij}\sum_{k=1}^N A_k=e_{ij}$$

($\sum_{k=1}^N A_k=1$).

Remark 4. For $r_1=\dots=r_N$ the operator $S^G_{r_1,\dots,r_N}$ becomes, say S_r^G, given by

$$(S_r^Gf)(x,y)=\sum_{k=1}^N A_k(x,y)(G_k^rf)(x,y)$$

where G_k^r is the Abel-Gonciarov operator that interpolates the data $\mathcal{I}_{k,r}(f)$. It means that all the polynomials G_k^rf have the same degree r and $dex(S_r^G)=r$.

4. **Particular cases**

1. $\mathcal{I}(f)=\{f^{(p,q)}(x_k,y_k)|\ p+q=0,1;\ k=1,\dots,N\}$

The Abel-Gonciarov polynomials G_k^1f are

$$(G_k^1f)(x,y)=f(x_k,y_k)+(x-x_k)f^{(1,0)}(x_{k+1},y_{k+1})+(y-y_k)g^{(0,1)}(x_{k+1},y_{k+1}),$$

$k=1,\dots,N$, with $x_{N+1}=x$, and $y_{N+1}=y_1$.

The corresponding Shepard operator S_1^G is given by

$$(S_1^Gf)(x,y)=\sum_{k=1}^N A_k(x,y)(G_k^1f)(x,y)$$

and $dex(S_1^G)=1$.

2. As a second example one considers the Shepard operator of Abel-Gonciarov-type which use the information

$$\mathcal{I}(f)=\{f^{(p,q)}(x_i,y_i)|\ p+q=0,1,2;\ i=1,\dots,N\} \tag{8}$$

$$(S_2^Gf)(x,y)=\sum_{k=1}^N A_k(x,y)(G_k^2f)(x,y)$$

where

$$\begin{aligned}(G_k^2f)(x,y)=&f(x_k,y_k)+(x-x_k)f^{(1,0)}(x_{k+1},y_{k+1})+(y-y_k)f^{(0,1)}(x_{k+1},y_{k+1})+\\&+\frac{(x-x_k)(x+x_k-2x_{k+1})}{2}f^{(2,0)}(x_{k+2},y_{k+2})+\\&+[(x-x_{k+1})(y-y_{k+1})-(x_{k+1}-x_k)(y_{k+1}-y_k)]f^{(1,1)}(x_{k+2},y_{k+2})+\\&+\frac{(y-y_k)(y+y_k-2y_{k+1})}{2}f^{(0,2)}(x_{k+2},y_{k+2}).\end{aligned}$$

Is easy to verify that the operator S_2^G interpolates the data (8) and $dex(S_2^G)=2$. Next we give the corresponding graphs for the function $f(x,y)=x\exp(-x^2-y^2)$ and form some Shepard-type operators on the rectangular domain $D=[-2,2]\times[-2,2]$, for $N=10$ and interpolation nodes given bellow:

(-1.9204,1.9204), (-1.9204,-1.6910), (-0.9812,-1.6910), (-0,9812,1.3152), (0.0876,1.5156), (0.2881,-0.0208), (-0.1127,-0.8893), (0.9561,-1.1565), (0.7557,0.8707), (1.5774,-3490)

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SHEPARD METHOD - FROM APPROXIMATION TO INTERPOLATION

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Abstract. In this paper we review the local Shepard method that uses Franke-Little weights, in order to give necessary and sufficient conditions when this method yields an interpolation function. Next we give a practical algorithm to solve this problem, which is based on geometric algorithms. This paper is a refinement of the results presented in [5] and [6].

1. The Local Shepard Method

Let consider the bi-dimensional case of two independent variables x and y for a function $f : z = f(x, y)$, where $(x, y) \in \mathbf{R}^2$ with $(x, y, z) \in \mathbf{R}^3$. Given n interpolation points, we want to find an interpolation function Φ with $z = \Phi(x, y)$ defined for $(x, y) \in D$ in such a way that $F(x_i, y_i) = f(x_i, y_i)$ for all $j = 1, \ldots, n$.

If the nodes (x_j, y_j) $(j = 1, \ldots, n)$ do not form a rectangular grid but are arranged in a completely arbitrary and unordered way, we can use spline surfaces or the Shepard method. The Shepard method has proven well suited for the graphic representation of surfaces. Its approximating function Φ is uniquely determined independently from the ordering of the nodes (x_i, y_i) $(i = 1, \ldots, n)$. The function $f : z = f(x, y)$ for $(x, y)D$, where D is an arbitrary region of the Oxy plane, is approximated for the given nodes (x_i, y_i) by the function

$$\Phi(x, y) = \sum_{i=1}^{n} w_i(x, y) \cdot f_i \tag{1}$$

where the weight functions are defined as

$$w_i(x, y) = \frac{(\rho - r_i(x, y))_+^{\mu}}{\sum_{l=1}^{n} ((\rho - r_l(x, y))_+^{\mu}} \tag{1a}$$

with the notations

$$r_i(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{1b}$$

and

$$s_+^{\mu} = \begin{cases} s^{\mu}, & s \geq 0 \\ 0, & s < 0 \end{cases} \tag{1c}$$

So we can write

$$\Phi(x, y) = \frac{\sum_{i=1}^{n} ((\rho - r_i(x, y))_+^{\mu} \cdot f_i}{\sum_{i=1}^{n} ((\rho - r_i(x, y))_+^{\mu}} \tag{2}$$

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The parameters ρ and μ are determined at the beginning of the interpolation process. The exponent μ can be chosen arbitrarily. If $0 < \mu \leq 1$ the function Φ has peaks at the nodes. If $\mu > 1$ the function is level at the nodes.

The function Φ uses only those nodes (x_j, y_j) within a disc of radius ρ when calculating a new functional value $\Phi(x, y)$, id est, this is a local method. We use a fast local Shepard approximation with Franke-Little weights because of the very reduced complexity order, which is very important for computer graphics applications.

Theorem 1. *Let* $d := \min\{r_j(x_l, y_l) \mid j, l = 1, \ldots, n$ *and* $j \neq l\}$. Φ *(as defined in (2)) is an interpolation function if and only if* $\rho \leq d$.

Proof. The following two conditions are fulfilled for each $(x, y) \in D$:

$$0 \leq w_i(x, y) \leq 1$$
$$\sum_{i=1}^{n} w_i(x, y) = 1$$

The following two conditions are fulfilled if and only if $\rho \leq d$:

$$w_i(x_j, y_j) = 0 \text{ for each } i \neq j \text{ and each } (x, y) \in D;$$
$$w_i(x_i, y_i) = 1 \ (i = 1, \ldots, n). \ \square$$

2. Improving the Local Shepard Method

We need to analyse three problems when using the Shepard method. The first one is how to organise the input data (the set P) in such a way that we can quickly find all the nodes lying inside of a disc having the centre (x, y) and the radius ρ. This problem can be solved in $\boldsymbol{O}(n \log n)$ pre-processing time, and finding all the necessary nodes needs $\boldsymbol{O}(n \log n + k)$ time, where k is the number of the found nodes ([1], [6]).

The second problem is how to determine an acceptable value for the parameter ρ. We impose for this value the following conditions:

- the disc having the radius ρ must cover at least one node from P, wherever we place this disc in the interior of the *ConvHull*(P);
- this disc must not cover too much close nodes: we need to quickly compute the value $\Phi(x, y)$.

This problem can be solved in $\boldsymbol{O}(n \log n)$ time, determining the Delaunay diagram of the set P. This diagram is composed by triangles whose vertices are nodes from P, and the interior of the circumcircle of any triangle does not contain any other node from P. We choose the maximum radius of the circumcircle of any triangle: the value of ρ.

The third problem is how to modify the Shepard method in order to obtain an interpolation function. Precisely, we have to answer the following question. Is the below linear equations system (3) uniquely solvable?

$$\Psi(x_i, y_i) := \frac{\sum_{j=1}^{n} (\rho - r_j(x_i, y_i))_+^{\mu} \cdot z_j}{\sum_{j=1}^{n} (\rho - r_j(x_i, y_i))_+^{\mu}} = f_i \text{ for } i = 1, \ldots, n \tag{3}$$

Theorem 2. *There exists a positive value* μ_0 *in such a way that for any* $\mu > \mu_0$ *the system (3) is uniquely solvable.*

Proof. Suppose, without loss of generality, that $\rho = 1$ (we can apply a simple scaling operation). By elementary transformations (each equation is multiplied by the denominator of $\Psi(x_i, y_i)$), and because $r_i(x_j, y_j) = r_j(x_i, y_i)$ we obtain that the matrix of (3) has the following properties:

it is symmetric, all the values which lie on the main diagonal are 1, and all the

other values lie in the interval [0,1); but is not necessarily diagonal dominant!

Can we determine a value for μ in such a way so that the system matrix of (3) is diagonal dominant?

Let $s_i(\mu) := \sum_{j=1, j\neq 1}^{n} (1 - r_i(x_j, y_j))_+^{\mu}$. We can determine a value μ so that $s_i(\mu) < 1$, because $s_i(\mu)$ is a continuous, decreasing function (all the terms are less than 1), and $\lim_{\mu\to\infty} s_i(\mu) = 0$. □

To calculate the values $\Psi(x, y)$ we need only those points from P, which lie inside the disc having the centre (x, y) and the radius ρ. How can we find an optimal value (minimum) for the parameter ρ so that

(i) Ψ can be defined at least on $ConvHull(P)$;

(ii) $\Psi(x_i, y_i) = f(x_i, y_i)$ for $i = 1, \ldots, n$; id est, Ψ is indeed an interpolation function.

The proposed algorithm is given below.

Step 1. Determine the Delaunay diagram of the set P. Let considers three non-collinear nodes p_i, p_j, p_k that define a triangle of the diagram, and ρ_{ijk} be the circumcircle of the triangle $p_i p_j p_k$. This circle does not contain any other node - this is an important property of the Delaunay diagram ([1], [3]).

Let $d := \min \mathrm{dist}(p_i, p_j) \mid i, j = 1, \ldots, n$ and $i \neq j$, and $\rho = \max \rho_{ijk.}$ The value of d can be quickly determined by scanning all the triangles of the Delaunay diagram. If $\rho \leq d$ then we can easily see that (i) and (ii) are fulfilled.

Step 2. (only if $\rho > d$). In this case only (i) is fulfilled, so we try to solve the system (3). We need to find the values z_i (which are unknown) to fulfil (ii). It is necessary to try different values for the parameter m until the system matrix of (3) is non-singular and well conditioned. At this moment we can solve the system using an iterative method (Jacobi, Gauss-Seidel).

3. Conclusions

The authors of [2] gave a "suggestion" about how to choose the values of the two parameters of the Shepard method:

"To avoid peaks at the nodes, choose $2 \leq \mu \leq 6$. *Our tests indicate that* $0.1 \leq \rho \leq 0.5$ *is the preferred range, where we recommend to choose a small value for* ρ *in case of many available nodes and a larger* ρ *for problems with few nodes.*

For the local method, however, any choice of ρ *near the recommended upper bound of 0.5 leads to unsatisfactory results."*

We don't have any other indication about how to determine these two parameters. The user has to choose manually many couples (ρ, μ) until he gets the expected results.

This is why we gave in this paper an algorithm that automatically determines these two parameters, depending on the topology of the set P. More than that, we improve this method in order to accelerate the computations, and to obtain an interpolation function.

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ON THE NECESSARY AND SUFFICIENT CONDITION FOR THE REGULARITY OF A MULTIDIMENSIONAL INTERPOLATION SCHEME

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Abstract. In this article we will present the necessary and sufficient condition for the regularity of a multidimensional interpolation scheme, in the case when the interpolation indexes are taken from an arbitrary set S from $\mathbb{N}^d$. In particular, if the index set S (of the interpolation space $\mathcal{P}_S$) is inferior with respect to $\mathbb{N}^d$, we obtain the theorem 3.4.2. from [1]. The set Δ_k^d from $\mathbb{N}^d$, given by the relation (1), together with the proposition 1, are the key elements that allow us to approach the theorem in a general context and to give another proof for it, compared of course with the one given in [1].

Let $\mathbb{N}^d = \{\mathbf{i} = (i_1, i_2, \ldots, i_d)/i_k \geq 0, i_k \in \mathbb{N}, k = \overline{1,d}\}$, $|\mathbf{i}| = i_1 + i_2 + \ldots + i_d$, and the definitions:

1. *We say that* $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{N}^d$ *is in the relation "$\leq$" with* $\mathbf{j} = (j_1, j_2, \ldots, j_d) \in \mathbb{N}^d$ *and we write* $\mathbf{i} \leq \mathbf{j}$ *when* $0 \leq i_k \leq j_k$ *for any* $k = \overline{1,d}$.

2. *We say that* $I \subset \mathbb{N}^d$ *is a lower set with respect to* $\mathbb{N}^d$ *if for any* $\mathbf{i} \in I$ *and* $\mathbf{j} \in \mathbb{N}^d$ *so that* $\mathbf{j} \leq \mathbf{i}$, *we have* $\mathbf{j} \in I$. *In the same manner we can define the inferior set with respect to any set* $S \subset \mathbb{N}^d$

If $\Delta_k^d \subset \mathbb{N}^d$,

$$\Delta_k^d = \bigcup_{i_1=0}^{k} \bigcup_{i_2=0}^{k-i_1} \cdots \bigcup_{i_{d-1}=0}^{k-(i_1+\ldots+i_{d-2})} \{i_1, \ldots, i_{d-1}, k - (i_1 + \ldots + i_{d-1})\} \tag{1}$$

we have $T_n^d = \bigcup_{k=0}^{n} \Delta_k^d$, that is

$$T_n^d = \bigcup_{k=0}^{n} \Delta_k^d = \bigcup_{k=0}^{n} \bigcup_{i_1=0}^{k} \bigcup_{i_2=0}^{k-i_1} \cdots \bigcup_{i_{d-1}=0}^{k-(i_1+\ldots+i_{d-2})} \{i_1, \ldots, i_{d-1}, k - (i_1 + \ldots + i_{d-1})\}$$

is a lower set with respect to $\mathbb{N}^d$.

Proposition 1. *Any set* S *from* $\mathbb{N}^d$ *can be written in the form*

$$S = \bigcup_{t=1}^{n} {\Delta'}_{k_t}^{d} \tag{2}$$

where $k_n = \max_{i \in S} |\mathbf{i}|$, $0 \leq k_1 \leq \ldots \leq k_n$, ${\Delta'}_{k_t}^{d} \subset \Delta_{k_t}^d$ *and* $\Delta_{k_t}^d$, $t = \overline{1,n}$, *are given by* (1).

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Proposition 2. *If $P \in \mathcal{P}_S$ and $S \subset \mathbb{N}^d$ given by (2), then for any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in S \subset \mathbb{N}^d$, we have*

$$P(\mathbf{x}) = \sum_{\mathbf{i}\in S} a_{\mathbf{i}} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d} = \sum_{t=1}^{n} P_{k_t}(\mathbf{x}),\ P_{k_t}(\mathbf{x}) = \sum_{\mathbf{i}\in {\Delta'}^d_{k_t}} a_{\mathbf{i}} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d},\ t = \overline{1,n}$$

The $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in S \subset \mathbb{N}^d$ order derivatives of P are:

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_d^{\alpha_d}} P(\mathbf{x}) = \\ = \sum_{\mathbf{i}\in S, \boldsymbol{\alpha}\le \mathbf{i}} a_{\mathbf{i}} \frac{i_1!}{(i_1-\alpha_1)!}\frac{i_2!}{(i_2-\alpha_2)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!} x_1^{i_1-\alpha_1} x_2^{i_2-\alpha_2}\ldots x_d^{i_d-\alpha_d},$$

and those of those of P_{k_t} are:

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_d^{\alpha_d}} P_{k_t}(\mathbf{x}) = \frac{\partial^{k_t}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_{d-1}^{\alpha_{d-1}}\partial x_d^{k_t-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}} P_{k_t}(\mathbf{x}) = \\ = \alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_t-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\, a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_t-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})},$$

for any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

In what follows, whenever we write S we will denote an arbitrary set from $\mathbb{N}^d$. By extending from a set I inferior with respect to $\mathbb{N}^d$ (*see also* [1]) to an arbitrary set S from $\mathbb{N}^d$, we define the following notions.

Definition 1. *A polinomial multidimensional interpolation scheme $(E, \mathcal{P}_S)$ consists of:*

(a) *A set of nodes $Z = \{\mathbf{x}_q\}_{q=1}^m = \{(x_{q,1}, x_{q,2}, \ldots, x_{q,d})\}_{q=1}^m$ from $\mathbb{R}^d$*

(b) *An interpolation space*

$$\mathcal{P}_S = \left\{ P/P(\mathbf{x}) = \sum_{\mathbf{i}\in S} a_{\mathbf{i}} x_1^{i_1} x_2^{i_2}\ldots x_d^{i_d},\ a_{\mathbf{i}} \in \mathbb{R},\ \mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \right\},$$

which is the space of the d variables polinomes with real coefficients where S is an arbitrary subset of $\mathbb{N}^d$, and

(c) *An incidence matrix $E = (e_{q,\boldsymbol{\alpha}})$, $1 \le q \le m$, $\boldsymbol{\alpha} \in S$, where $e_{q,\boldsymbol{\alpha}} = 0$ or 1.*

The interpolation problem associated with $(E, \mathcal{P}_S)$ consists of finding the polinomes $P \in \mathcal{P}_S$, that would satisfy the equations

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_d^{\alpha_d}} P(\mathbf{x}_q) = \mathbf{c}_{q,\boldsymbol{\alpha}} \tag{3}$$

for any $q = \overline{1,m}$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in S$ with $e_{q,\boldsymbol{\alpha}} = 1$ where $c_{q,\boldsymbol{\alpha}}$ are arbitrary real constants

The (3) equations make up a system of linear equations whose unknowns are the real coefficients of the polinome P. The matrix $\mathcal{M}(E, Z)$ of this system is at the same time the matrix of the interpolation scheme $(E, \mathcal{P}_S)$, and it is called *Vandermonde matrix.* If $\mathcal{M}(E, Z)$ is a square matrix, then its determinant, $det\mathcal{M}(E, Z) = \mathcal{D}(E, Z)$, is the determinant of the system (3), and of the interpolation scheme $(E, \mathcal{P}_S)$, and it is called *Vandermonde determinant.*

If in definition 1 we consider that S is a set I inferior with respect to $\mathbb{N}^d$, then, according to [1], $(E, \mathcal{P}_I)$ is a *Birkhoff interpolation scheme*, the E matrix is the *Birkhoff incidence matrix*, and the interpolation polynom P is a *Birkhoff polinome.* Also, according to [1], by various particularisations of the Birkhoff interpolation scheme, we obtain the *Lagrange, Hermite, Taylor* and *Abel interpolation schemes*, with their corresponding incidence matrices and interpolation polinomes.

Definition 2. *Let S be a set from $\mathbb{N}^d$ and $(E, \mathcal{P}_S)$ the corresponding multidimensional interpolation scheme. We say that $E = (e_{q,\boldsymbol{\alpha}})$ is an Abel incidence matrix, if for any $\boldsymbol{\alpha} \in S$, $e_{q,\boldsymbol{\alpha}} = 1$ for exactly one $q \in \overline{1,m}$. The scheme, the polinome and the interpolation problem corresponding to the Abel incidence matrix, are called Abel interpolation scheme, Abel interpolation polinome, respectively Abel interpolation problem.*

Definition 3. *The multidimensional interpolation scheme $(E, \mathcal{P}_S)$ is called normal if $|E| = \dim \mathcal{P}_S$.*

Because in the present article we will work only with normal interpolation schemes, from now on, whenever we discuss an interpolation scheme, we will consider it normal.

Definition 4. *We say that an interpolation scheme $(E, \mathcal{P}_S)$ is*

(a) *singular, if $\mathcal{D}(E, Z) = 0$ for any choice of the set of nodes Z,*
(b) *regular, if $\mathcal{D}(E, Z) \neq 0$ for any choice of the set of nodes Z and*
(c) *almost regular, if $\mathcal{D}(E, Z) \neq 0$ for almost all choices of the set of nodes Z.*

Definition 5. *Two interpolation schemes are equivalent when the systems of their interpolation problems are equivalent.*

Theorem 1. *The interpolation scheme $(E, \mathcal{P}_S)$ is regular if and only if it is equivalent with an Abel interpolation scheme.*

Proof. If E is a Abel matrix and if the order of the coefficients of the interpolation polinome and the order of the derivatives from the interpolation system are those which correspond to the order of the elements of the set S, then the matrix of the interpolation system is superior triangular. If not, by switching lines or (and) columns in $\mathcal{M}(E, Z)$, we can obtain what we have previously shown, and thus it follows that the new determinant is different from the previous one only through its sign. It follows that the determinant of this matrix is:

$$d_S = \pm \prod_{\boldsymbol{\alpha} \in S} \boldsymbol{\alpha}!$$

where $\boldsymbol{\alpha}! = \alpha_1!\alpha_2! \dots \alpha_d!$. It follows that $d_S \neq 0$, and as a result $(E, \mathcal{P}_S)$ is regular.

Conversely, we assume that $(E, \mathcal{P}_S)$ is regular and let P be the solution of the interpolation problem of this interpolation scheme, where:

$$P(\mathbf{x}) = P(x_1, x_2, \dots, x_d) = \sum_{\mathbf{i} \in S} a_{\mathbf{i}} x_1^{i_1} x_2^{i_2} \dots x_d^{i_d}$$

for any $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ and $a_{\mathbf{i}} \in \mathbb{R}^d$ (real constants, since the given scheme is regular).

With the same E and $\mathcal{P}_S$ we show that we have $Q \in \mathcal{P}_S$ (by construction) for which $(E, \mathcal{P}_S)$ is Abel, and the two interpolation systems (of the regular scheme and of the Abel scheme) are equivalent having the same solutions: the $a_{\mathbf{i}} \in \mathbb{R}$. For this we consider the next regrouping $S=\bigcup_{t=1}^{n}{\Delta'}^{d}_{k_t}$ of all indexes of the polinome coefficients P, with $0\le k_1\le k_2\le\ldots\le k_n$, $k_n=\max\limits_{i\in S}|\mathbf{i}|$, ${\Delta'}^{d}_{k_t}\subset\Delta^{d}_{k_t}$ and $\Delta^{d}_{k_t}$, $t=\overline{1,n}$, given by (1) (their existence is ensured by proposition 1). Let be:

$$Q(\mathbf{x})=Q(x_1,x_2,\ldots,x_d)=\sum_{\mathbf{i}\in S}b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=$$

$$=\sum_{t=1}^{n}Q_{k_t}(\mathbf{x}),Q_{k_t}(\mathbf{x})=\sum_{\mathbf{i}\in S}b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}$$

We will determine $b_{\mathbf{i}}$ so that $(E,\mathcal{P}_S)$ is Abel and $b_{\mathbf{i}}=a_{\mathbf{i}}$, for any $\mathbf{i}\in S$. For the beginning let be the system:

$$\sum_{\mathbf{i}\in S,\boldsymbol{\alpha}\le\mathbf{i}}b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}}\tag{4}$$

with $q\in\overline{1,m}$ and $e_{q,\alpha}=1$ for any $\boldsymbol{\alpha}\in S$ and

$$Z=\{\mathbf{x}_q\}_{q=1}^{m}=\{(x_{q,1},x_{q,2}\ldots x_{q,d})\}_{q=1}^{m}\subset\mathbb{R}^d.$$

For $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in{\Delta'}^{d}_{k_n}$ and $\boldsymbol{\alpha}\le\mathbf{i}$, $\mathbf{i}\in{\Delta'}^{d}_{k_n}$. In this case the system (4) is equivalent with

$$\sum_{\mathbf{i}\in{\Delta'}^{d}_{k_n},\boldsymbol{\alpha}\le\mathbf{i}}b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}},$$

that is, according to proposition 2, equivalent with the system

$$\alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\,b_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}=\mathbf{c}_{q,\boldsymbol{\alpha}}.$$

We consider in what follows

$$\mathbf{c}_{q,\boldsymbol{\alpha}}=c_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}\overset{def}{=}$$
$$=\alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\,a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}.$$

It follows that

$$b_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}=a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})},$$

for any $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in{\Delta'}^{d}_{k_n}$, from where we have

$$\dim P_{{\Delta'}^{d}_{k_n}}=$$
$$=\left|\{\boldsymbol{\alpha}\in(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in{\Delta'}^{d}_{k_n}\}\right|=\left|{\Delta'}^{d}_{k_n}\right|$$

(each derivative $\boldsymbol{\alpha}\in{\Delta'}^{d}_{k_n}$ of Q is interpolated only once), and

$$Q_{k_n}(\mathbf{x})=\sum_{\mathbf{i}\in{\Delta'}^{d}_{k_n}}b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=\sum_{\mathbf{i}\in{\Delta'}^{d}_{k_n}}a_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=P_{k_n}(\mathbf{x})$$

If $\boldsymbol{\alpha}\in{\Delta'}^{d}_{k_{n-1}}$, and $\boldsymbol{\alpha}\le\mathbf{i}$, then $\mathbf{i}\in{\Delta'}^{d}_{k_{n-1}}\cup{\Delta'}^{d}_{k_n}$, and the system (4) becomes successively

$$\sum_{\mathbf{i}\in {\Delta'}^d_{k_{n-1}}\cup {\Delta'}^d_{k_n},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}},$$

$$\sum_{\mathbf{i}\in {\Delta'}^d_{k_{n-1}},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}+$$
$$+\sum_{\mathbf{i}\in {\Delta'}^d_{k_n},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}}.$$

We take now

$$\mathbf{c}_{q,\boldsymbol{\alpha}}\overset{def}{=}\sum_{\mathbf{i}\in {\Delta'}^d_{k_n},\boldsymbol{\alpha}\le\mathbf{i}} a_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}+$$
$$+\alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\,a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}.$$

(for $\mathbf{i}\in {\Delta'}^d_{k_n}$, $b_{\mathbf{i}}=a_{\mathbf{i}}$) and considering the proposition 2, we obtain the system

$$\alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\,b_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}=$$
$$=\alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\,a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})},$$

that is

$$b_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}=a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}$$

for any $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in {\Delta'}^d_{k_{n-1}}$.

It follows that

$$\dim P_{{\Delta'}^d_{k_{n-1}}}=$$
$$=\left|\{\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in {\Delta'}^d_{k_{n-1}}\}\right|=\left|{\Delta'}^d_{k_{n-1}}\right|$$

(each derivative $\boldsymbol{\alpha}\in {\Delta'}^d_{k_{n-1}}$ of Q is interpolated only once), respectively

$$Q_{k_{n-1}}(\mathbf{x})=\sum_{\mathbf{i}\in {\Delta'}^d_{k_{n-1}}} b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=\sum_{\mathbf{i}\in {\Delta'}^d_{k_{n-1}}} a_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=P_{k_{n-1}}(\mathbf{x})$$

Continuing in the same manner, we obtain that for $\boldsymbol{\alpha}\in {\Delta'}^d_{k_1}$, the system (4) is successively equivalent with

$$\sum_{\mathbf{i}\in {\Delta'}^d_{k_1}\cup\ldots\cup {\Delta'}^d_{k_n},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}},$$

$$\sum_{\mathbf{i}\in {\Delta'}^d_{k_1},\boldsymbol{\alpha}\leq \mathbf{i}} b_{\mathbf{i}} \frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!} x_{q,1}^{i_1-\alpha_1}\cdots x_{q,d}^{i_d-\alpha_d}+$$

$$+\sum_{\mathbf{i}\in {\Delta'}^d_{k_2}\cup\ldots\cup{\Delta'}^d_{k_n},\boldsymbol{\alpha}\leq \mathbf{i}} b_{\mathbf{i}} \frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!} x_{q,1}^{i_1-\alpha_1}\cdots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}}$$

In this final system we take

$$\mathbf{c}_{q,\boldsymbol{\alpha}} \stackrel{def}{=} \sum_{u=2}^{n}\sum_{\mathbf{i}\in {\Delta'}^d_{k_u},\boldsymbol{\alpha}\leq \mathbf{i}} a_{\mathbf{i}} \frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!} x_{q,1}^{i_1-\alpha_1}\cdots x_{q,d}^{i_d-\alpha_d}+$$

$$+\,\alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\,a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}.$$

(for $\mathbf{i}\in {\Delta'}^d_{k_2}\cup\ldots\cup{\Delta'}^d_{k_n}$, $b_{\mathbf{i}}=a_{\mathbf{i}}$).

It follows that in this case we have also

$$b_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}=a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}$$

for any $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in{\Delta'}^d_{k_1}$, and thus

$$\dim P_{{\Delta'}^d_{k_1}}=$$

$$=\left|\{\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in{\Delta'}^d_{k_1}\}\right|=\left|{\Delta'}^d_{k_1}\right|,$$

(each derivative $\boldsymbol{\alpha}\in{\Delta'}^d_{k_1}$ of Q is interpolated only once), respectively

$$Q_{k_1}(\mathbf{x})=\sum_{\mathbf{i}\in{\Delta'}^d_{k_1}} b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=\sum_{\mathbf{i}\in{\Delta'}^d_{k_1}} a_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=P_{k_1}(\mathbf{x})$$

In the end, it follows that

$$\sum_{t=1}^{n}\dim P_{{\Delta'}^d_{k_t}}=|S|=\dim\mathcal{P}_S,$$

and

$$Q(\mathbf{x})=\sum_{t=1}^{n}Q_{k_t}(\mathbf{x})=\sum_{t=1}^{n}P_{k_t}(\mathbf{x})=P(\mathbf{x})$$

is the solution of the interpolation problem of the Abel scheme $(E,\mathcal{P}_S)$ (each derivative $\boldsymbol{\alpha}\in S$ of Q is interpolated only once). It follows, according to definition 5, that $(E,\mathcal{P}_S)$ is equivalent to an Abel interpolation scheme q.e.d. □

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APPLICATIONS OF DIFFERENTIAL SUBORDINATIONS

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Abstract. In this paper, by using the method of differential subordinations, we obtain a more general condition, from which it could be found the conditions for starlikenes [2], [3], [4].

1. Introduction and definitions

Let $\mathbf{U}$ denote the open unit disk.

Let $H = H(\mathbf{U})$ denote the class of functions analytic in $\mathbf{U}$

For n a positive integer and $a \in \mathbb{C}$, let

$$H\,[a,n] = \left\{\ f \in H\ ;\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \right\}$$

with $H_0 = H\,[0,1]$

Recall the concept of subordination :

Let f and g be in H. The function f is said to be subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function φ analytic in $\mathbf{U}$, with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, such that $f(z) = g(\varphi(z))$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$.

Let $\Psi : \mathbb{C}^3 \times \mathbf{U} \to \mathbb{C}$ and let h be univalent in $\mathbf{U}$. If p is analytic in $\mathbf{U}$ and satisfies the (second-order) differential subordination

$$\Psi(p(z), zp'(z), z^2p''(z), z) \prec h(z) \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordinations, or more simply a dominant, if $p \prec q$ for all p satisfying (1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1).(Note that the best dominant is unique up to a rotation of $\mathbf{U}$).

If we require the more restrictive condition $p \in H[a,n]$, then p will be called an (a,n) - solution, q an (a,n) - dominant and $\widetilde{q}$ the best (a,n) - dominant.

Theorem A. (M. Obradovic, T. Yaguchi, H. Saitoh [4]). Let q be a convex function in $\mathbf{U}$, with $q(0) = 1$ and $\mathrm{Re}[q(z)] > \frac{1}{2}$, $z \in \mathbf{U}$.If $0 \leq \alpha < 1$, p is analytic in $\mathbf{U}$ with $p(0) = 1$ and if

$$(1-\alpha)p^2(z) + (2\alpha - 1)p(z) - \alpha + (1-\alpha)zp'(z) \prec$$
$$(1-\alpha)q^2(z) + (2\alpha - 1)q(z) - \alpha + (1-\alpha)zq'(z) \equiv h(z), \tag{2}$$

then $p \prec q$ and q is the best dominant of (2).

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Theorem B. (Sufficient conditions for starlikeness, P. T. Mocanu, Gh. Oros [2]). Let the function $h(z) = 1+(2\alpha+1)\mu z+\alpha\mu^2 z^2$ where $\alpha > 0$ and $0 < \mu \leq 1+\frac{1}{2\alpha}$.

If $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in $\mathbf{U}$ and satisfies the condition

$$\alpha z p'(z) + \alpha p^2(z) + (1-\alpha)p(z) \prec h(z),$$

then $p(z) \prec 1 + \mu z$ and this result is sharp.

Theorem C. (Sufficient conditions for starlikeness II, P. T. Mocanu, Gh. Oros [3]) Let q be a convex function in $\mathbf{U}$, with $q(0) = 1$, $\operatorname{Re} q(z) > \frac{\alpha-\beta}{2\alpha}$; $\alpha > 0, \alpha+\beta > 0$ and let

$$h(z) = \alpha n z q'(z) + \alpha q^2(z) + (\beta-\alpha)q(z). \tag{3}$$

If the function $p(z) = 1 + p_n z^n + \dots$ satisfies the condition :

$$\alpha z p'(z) + \alpha p^2(z) + (\beta-\alpha)p(z) \prec h(z),$$

where h is given by (3) ,then $p(z) \prec q(z)$ where q is the best dominant.

Theorem D. (S. S. Miller, P. T. Mocanu, [1]) Let q be univalent in $\mathbf{U}$ and let θ in Φ be analytic in a domain D containing $q(\mathbf{U})$, with $\Phi(w) \neq 0$, when $w \in q(\mathbf{U})$. Set $Q(z) = zq(z)\Phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that

i) Q is starlike in $\mathbf{U}$, and

ii) $\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\frac{\theta'[q(z)]}{\Phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right] > 0, \quad z \in \mathbf{U}$.

If p is analytic in $\mathbf{U}$, with $p(0) = q(0)$, $p(\mathbf{U}) \subset D$ and

$$\theta[p(z)] + zp'(z)\Phi[p(z)] \prec \theta[q(z)] + zq'(z)\Phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

2. Main Results

Theorem 1. Let q be convex in $\mathbf{U}$, with $\operatorname{Re}[2aq(z)+b] > 0$, $q(0) = 1$, when $a, b \in \mathbb{C}$, $a \neq 0$ and let

$$h(z) = aq^2(z) + bq(z) + czq'(z); \quad c > 0.$$

If the function $p \in H[1,n]$, i.e. $p(z) = 1 + p_n z^n + \dots$ satisfies the differential subordination :

$$ap^2(z) + bp(z) + czp'(z) \prec h(z),$$

then $p \prec q$ and q is the best $(1,n)$ - dominant.

Proof. (On checking the conditions of Th.D)

Let

$$\theta(w) = aw^2 + bw$$

$$\Phi(w) = c \neq 0 \ , \ \forall w \in q(\mathbf{U})$$

$$Q(z) = zq'(z)\Phi[q(z)] = czq'(z)$$

$$i);\ Q(z) = czq'(z) \text{ is starlike because } q(z) \text{ is convex and } c > 0.$$

$$ii)\ \operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\frac{\theta'[q(z)]}{\Phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right] = \operatorname{Re}\left[\frac{2aq(z)+b}{c} + z\frac{Q'(z)}{Q(z)}\right] =$$

$$= \operatorname{Re}\left[\frac{2aq(z)}{c} + \frac{b}{c} + z\frac{Q'(z)}{Q(z)}\right] > 0,$$

because

$$\operatorname{Re}[2aq(z)+b] > 0.$$

The conditions of Th.D is satisfied and we have: for $p \in H[1,n]$ which satisfies $ap^2(z) + bp(z) + czp'(z) \prec h(z)$ we have $p \prec q$ and q is the best $(1,n)$ - dominant.

Remark 1. For $q(z) = 1+\mu z$ univalent in $\mathbf{U}$, with $\alpha > 0$ and $0 < \mu \leq 1+1/\alpha$, if

$a = \alpha$
$b = 1 - \alpha$
$\Phi(w) = \alpha$

we obtain the result given in Theorem B (P. T. Mocanu, Gh. Oros, [2])

Remark 2. For q convex in $\mathbf{U}$, with $q(0) = 1$, Re $q(z) > \frac{\alpha-\beta}{2\alpha}$, if
$a = \alpha$
$b = \beta - \alpha$
$\Phi(w) = \alpha \cdot u$, when $\alpha > 0$ and $\alpha + \beta > 0$,

we reobtain the Theorem C (P. T. Mocanu, Gh. Oros, [3])

Remark 3. Let q be a convex function in $\mathbf{U}$, with $q(0) = 1$ and Re $q(z) > \frac{1}{2}$; $z \in \mathbf{U}$

If $a = 1 - \alpha$
$b = 2\alpha - 1 \quad ; \alpha \in [0,1)$
$\theta(w) = (1-\alpha)w^2 + (2\alpha - 1)w - \alpha$
$\Phi(w) = 1 - \alpha$,

we obtain the condition for starlikeness of Theorem A. (M. Obradovic, T. Yaguchi, H. Saitoh [4]).

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METHOD ON PARTIAL AVERAGING FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH HUKUHARA'S DERIVATIVE

TERESA JANIAK AND ELŻBIETA ŁUCZAK-KUMOREK

1. Introduction

In classical system of functional-differential equations it is possible to middle both complete and partial equations. Complete averaging was presented by Bogolubov ([1]).

In this paper, we use a partial middling method in the case of functional-differential inclusions with Hukuhara's derivative, i.e. for inclusions of the form

$$D_h X(t) \in F(t, X_t,) \tag{1}$$

where $D_h X$ denotes a Hukuhara's derivative ([2]) of a multivalued mapping X, $X_t : \Theta \to X_t(\Theta) = X(t+\Theta)$ for $\Theta \in [-r, 0]$, $r > 0$, F is a map from $[0, T] \times C_0$ into $CC(R^n)$, and C_0 is a metric space of all continuous mapping $\Phi : [-r, 0] \to \mathrm{Conv}(R^n)$.

The application of this method leads to a reduced form of the initial equations system and is useful in the case when the means of certain functions do not exist.

The results of this paper generalize the results of V. A. Płotnikov ([5]), where the generalized system $\dot{x}(t) \in F(t, x)$ was investigated.

2. Notations and definitions

By $\mathrm{Conv}(R^n)$ we will denote the family of all nonempty compact and convex subsets of the real n-dimensional Euclidean space R^n with the Hausdorff metric H defined by:

$$H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b| \right\}$$

for $A, B \in \mathrm{Conv}(R^n)$, where $|\cdot|$ denotes the Euclidean norm.

It is know that $(\mathrm{Conv}(R^n), H)$ is a complete metric space ([3]). Let $CC(R^n)$ denote the space of all nonempty compact but necessarily convex subsets of $\mathrm{Conv}(R^n)$. By d we will denote the distance between two collections $A, B \in CC(R^n)$ i.e.

$$d(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} H(a, b), \max_{b \in B} \min_{a \in A} H(a, b) \right\} \text{ for } a, b \in \mathrm{Conv}(R^n).$$

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Let us denote by ρ a distance between $A \in CC(R^n)$ and $B \in \mathrm{Conv}(R^n)$ defined by:

$$\rho(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} H(a,b), \sup_{b\in B}\inf_{a\in A} H(a,b)\right\}$$

Let $X : [0,T] \to \mathrm{Conv}(R^n)$ be a given mapping. Using the definition of the difference in $\mathrm{Conv}(R^n)$ the Hukuhara derivative D_hX ([2]) of X may be introduced in the following way:

$$D_hX(t) = \lim_{h\to 0+} 1/h(X(t+h) - X(t)) = \lim_{h\to 0+} 1/h(X(t) - X(t-h)) \tag{2}$$

where X is assumed to belong to the class D of all functions such that both differences in (2) are possible.

The mapping $X : [0,T] \to \mathrm{Conv}(R^n)$ will be called Hukuhara differentiable in $[0,T]$ if D_hX exists for every $t \in [0,T]$.

A function $X : [0,T] \to \mathrm{Conv}(R^n)$ is called absolutely continuous if for every positive number ε there is a positive number δ such that

$$\sum_{i=1}^{k} H(X(\beta_i), X(\alpha_i)) < \varepsilon \ \text{ whenever } \ \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \ldots \le \alpha_k < \beta_k$$

and $\sum_{i=1}^{k}(\beta_i - \alpha_i) < \delta.$

The Aumann-Hukuhara's integral for multifunction $F : [0,T] \to CC(R^n)$ is a collection $G \in CC(R^n)$ defined by:

$$G = \left\{g \in \mathrm{Conv}(R^n) : g = \int_0^t f(t)dt \text{ for } f(t) \in F(t)\right\}$$

where $f : [0,T] \to \mathrm{Conv}(R^n)$ and integral of f on a set $[0,T]$ is the Hukuhara integral defined in the paper ([2]).

Finally, denote by C_α a metric space of all continuous mapping $V : [-r,\alpha] \to \mathrm{Conv}(R^n)$ where $\alpha \ge 0, r > 0$, with metric ρ_α defined by:

$$\rho_\alpha(V_1,V_2) = \sup_{-r\le t\le\alpha} H(V_1(t),V_2(t)) \text{ for } V_1,V_2 \in C_\alpha.$$

We say that X is a solution of (1) with the initial absolutely continuous multifunctions $\Phi : [-r,0] \to \mathrm{Conv}(R^n)$ if X is an absolutely continuous function from $[-r,T]$ into $\mathrm{Conv}(R^n)$ with the properties:

$X(t) = \Phi(t)$ for $t \in [-r,0]$

and X satisfies the inclusions (1) for a.e. $t \in [0,T]$.

3. The theorem on partial middling

Let $F^i : [0,\infty) \times C_0 \to CC(R^n)$ $(i = 1,2)$ satisfy the following conditions:

1° $F^i(\cdot,U) : [0,\infty) \to CC(R^n)$ is measurable for fixed $U \in C_0$

2° there exists a $M > 0$ such that $d(F^i(t,U),\{0\}) \le M$ for $(t,U) \in [0,\infty) \times C_0$

3° $F^i(t,\cdot) : C_0 \to CC(R^n)$ satisfies for fixed $t \in [0,\infty)$ the Lipschitz condition of the form:

$$d\left(F^i(t,U), F^i(t,V)\right) \le K\rho_0(U,V)$$

where $K > 0$, $U, V \in C_0$

4° there exists a limit

$$\lim_{T\to\infty} \frac{1}{T} d\left(\int_0^T F^1(t,U)dt,\ \int_0^T F^2(t,U)dt\right) = 0$$

uniformly with respect to $U \in C_0$.

In this part we shall study differential inclusions of the form

$$D_h X^1(t) \in \varepsilon F^1(t, X_t^1) \quad \text{for a.e.} \quad t \geq 0 \tag{3}$$

and

$$D_h X^2(t) \in \varepsilon F^2(t, X_t^1) \quad \text{for a.e.} \quad t \geq 0 \tag{4}$$

where $\varepsilon > 0$ is a small parameter.

We shall consider (3) and (4) together with the initial conditions

$$X^1(t) = X^2(t) = \Phi(t) \quad \text{for } t \in [-r, 0] \tag{5}$$

where $\Phi : [-r, 0] \to \mathrm{Conv}R^n$ is a given absolutely continuous multifunction.

In paper ([4]) the following theorem has been proved.

Theorem 1. *Let $\delta : [0,T] \to R$ be a non-negative Lebesgue integreable function and let $\Phi \in C_0$ be an absolutely continuous. Suppose that $F : [0,T] \times C_0 \to CC(R^n)$ satisfy the following conditions:*

1) *$F(\cdot, U) : [0,T] \to CC(R^n)$ is measurable for fixed $U \in C_0$*
2) *there exists a $M > 0$ such that $d(F(t,U), \{0\}) \leq M$ for $(t,U) \in [0,T] \times C_0$*
3) *$F(t, \cdot) : C_0 \to CC(R^n)$ satisfies for fixed $t \in [0,T]$ the Lipschitz conditions of the form*

$$d\left(F(t,U), F(t,V)\right) \leq K(t)\rho_0(U,V)$$

where $K : [0,T] \to R^+$ is a Lebesgue integrable function, $U, V \in C_0$.

Furthermore let $Y : [-r,T] \to \mathrm{Conv}(R^n)$ be an absolutely continuous mapping such that

4) $Y(t) = \Phi(t)$ for $t \in [-r, 0]$,
5) $\rho(D_h Y(t), F(t, Y_t) \leq \delta(t)$ for a.e. $t \in [0,T]$.

Then there is a solution X of an initial-value problem:

$$\begin{cases} D_h X(t) \in F(t, X_t) \ \text{for a.e. } t \in [-r, 0], \\ X(t) = \Phi(t) \ \text{for } t \in [-r, 0] \end{cases}$$

such that $H(X(t), Y(t)) \leq \xi(t)$ for $t \in [0,T]$
and $H(D_h X(t),\ D_h Y(t)) \leq \delta(t) + K(t)\xi(t)$ for a.e. $t \in [0,T]$
where $\xi(t) = \int_0^t \delta(s) \exp[m(t) - m(s)]ds$ and $m(t) = \int_0^t K(r)dr$.

Now we can prove the main result of this paper, where in Theorem 2 by $CC(R^n)$ we will denote the spaces of all nonempty compact and convex subsets of $\mathrm{Conv}(R^n)$.

Theorem 2. *Suppose $F^i : [0,\infty) \times C_0 \to CC(R^n)$, $(i = 1, 2,)$ satisfy the conditions 1° – 4°. Then, for each $\eta > 0$ and $T > 0$ there exists a $\varepsilon^0(\eta, T)$ such that for every $\varepsilon \in (0, \varepsilon^0]$ the following conditions are satisfied:*

(i) *for each solution $X^1(\cdot)$ of (3) there exists a solution $X^2(\cdot)$ of (4) such that:*

$$H\left(X^1(t), X^2(t)\right) \le \eta \qquad \text{for } \ t \in \left[-r, \frac{T}{\varepsilon}\right] \tag{6}$$

(ii) *for each solution $X^2(\cdot)$ of (4) there exists a solution $X^1(\cdot)$ of (3) such that (6) holds.*

Proof. Let $X^1(\cdot)$ be a solution of (3) on $[-r, 0]$. In order to prove the theorem we shall consider the solution $X^2(\cdot)$ of the inclusion (4) in such a way that for $t \in [-r, 0]$, $X^1(t) = X^2(t) = \Phi(t)$, hence $H(X^1(t), X^2(t)) = 0 < \eta$. We will prove inequality (6) on the interval $\left[0, \frac{T}{\varepsilon}\right]$. To do this divide the interval $\left[0, \frac{T}{\varepsilon}\right]$ on m-subintervals $[t_i, t_{i+1}]$, where $t_i = \frac{iT}{\varepsilon m}$, $i = 0, 1, \ldots, m-1$ and write a solution $X^1(\cdot)$ in the form:

$$\begin{cases} X^1(t) = \Phi(t) \quad \text{for } \ t \in [-r, 0] \\ X^1(t) = X^1(t_i) + \varepsilon \displaystyle\int_{t_i}^{t} V^1(\tau) d\tau \quad \text{for } \ t \in [t_i, t_{i+1}] \end{cases} \tag{7}$$

where $V^1(t) \in F^1(t, X^1_t)$.

Let us consider a function $Y^1(\cdot)$ defined by

$$\begin{cases} Y^1(t) = \Phi(t) \quad \text{for } \ t \in [-r, 0] \\ Y^1(t) = Y^1(t_i) + \varepsilon \displaystyle\int_{t_i}^{t} U^1_{i+1}(\tau) d\tau \quad \text{for } \quad t \in [t_i, t_{i+1}] \end{cases} \tag{8}$$

where $U^1_{i+1}(\cdot)$, $i = 0, 1, \ldots, m-1$ are measurable functions such that $U^1_{i+1}(t) \in F^1(t, Y^1_{t_i})$ and

$$H\left(V^1(t), U^1_{i+1}(t)\right) = \rho\left(V^1(t), F^1(t, Y^1_{t_i})\right) = \min_{U(t) \in F^1(t, Y^1_{t_i})} H\left(V^1(t), U(t)\right).$$

By virtue of (7) for every $t \in [t_i, t_{i+1}]$ we have

$$\begin{aligned} H\left(X^1(t), Y^1(t_i)\right) &= H\left(X^1(t_i) + \varepsilon \int_{t_i}^{t} V^1(\tau) d\tau, Y^1(t_i)\right) \le \\ &\le H\left(X^1(t_i), Y^1(t_i)\right) + \varepsilon M(t - t_i) \le \delta_i + \varepsilon M(t - t_i) \end{aligned}$$

where $\delta_i = H(X^1(t_i), Y^1(t_i))$, $i = 0, 1, \ldots, m-1$.

Furthermore, for $t \in [t_i, t_{i+1}]$, we have

$$\begin{aligned} &H\left(V^1(t), U^1_{i+1}(t)\right) \le d\left(F^1(t, X^1_t), F^1(t, Y^1(t_i)\right) \le \\ &\le K\rho_0(X^1_t, Y^1_{t_i}) \end{aligned} \tag{9}$$

But

$$\begin{aligned} &\rho_0(X^1_t, Y^1_{t_i}) \le \rho_0(X^1_t, X^1_{t_i}) + \rho_0(X^1_{t_i}, Y^1_{t_i}) = \\ &= \sup\nolimits_{-r \le s \le 0} H\left(X^1(t+s), X^1(t_i+s)\right) + \sup\nolimits_{-r \le s \le 0} H\left(X^1(t_i+s), Y^1(t_i+s)\right) \end{aligned}$$

By the definition of $X^1(\cdot)$ and the properties of multifunction $F^1(t, X^1_t)$ we have:

$$\sup_{-r \le s \le 0} H\left(X^1(t+s), X^1(t_i+s)\right) \le \frac{MT}{m} \quad \text{for} \quad t \in [t_i, t_{i+1}]$$

Furthermore by the definition $X^1(\cdot)$ and $Y^1(\cdot)$ and using of (7) and (8), we have

$$\sup_{-r\le s\le 0} H\left(X^1(t_i+s), Y^1(t_i+s)\right) = \sup_{t_i-r\le\tau\le t_i} \left(H(X^1(\tau), (Y^1(\tau)\right) =$$
$$= \sup_{t_i-r\le\tau\le t_i} \left\{H\left(X^1(t_i)+\varepsilon\int_{t_i}^{\tau} V^1(s)ds, Y^1(t_i)+\varepsilon\int_{t_i}^{\tau} U^1_{i+1}(s)ds\right)\right\} \le$$
$$\le \sup_{t_i-r\le\tau\le t_i} \left\{H\left(X^1(t_i), Y^1(t_i)\right)+\varepsilon H\left(\int_{t_i}^{\tau} V^1(s)ds, \int_{t_i}^{\tau} U^1_{i+1}(s)ds\right)\right\} \le$$
$$\le \delta_i + \sup_{t_i-r\le\tau\le t_i} \varepsilon\int_{t_i}^{\tau} d\left(F^1(s, X^1_s), F^1(s, Y^1_{t_i})\right) ds \le$$
$$\le \delta_i + \sup_{t_i-r\le\tau\le t_i} \varepsilon\left\{\int_{t_i}^{\tau} \left[d\left(F^1(s, X^1_s), \{0\}\right)+d\left(F^1(s, Y^1_{t_i}), \{0\}\right)\right] ds\right\} \le \delta_i + 2\varepsilon Mr.$$

Therefore, inequality (9) for $t\in[ti, ti_{i+1}]$ can be written as follows

$$H\left(V^1(t), U^1_{i+1}(t)\right) \le K\left(\frac{MT}{m}+\delta_i+2\varepsilon Mr\right). \tag{10}$$

By virtue of (7), (8) and (10), it follows

$$\delta_i = H(X^1(t_i), Y^1(t_i)) =$$
$$= H\left(X^1(t_{i-1})+\varepsilon\int_{t_{i-1}}^{t_i} V^1(\tau)d\tau, Y^1(t_{i-1})+\varepsilon\int_{t_{i-1}}^{t_i} U^1_i(\tau)d\tau\right) \le$$
$$\le H(X^1(t_{i-1}), Y^1(t_{i-1}))+\varepsilon H\left(\int_{t_{i-1}}^{t_i} V^1(\tau)d\tau, \int_{t_{i-1}}^{t_i} U^1_i(\tau)d\tau\right) \le$$

$$\le \delta_{i-1}+\varepsilon\int_{t_{i-1}}^{t_i} H\left(V^1(\tau), U^1_i(\tau)\right)d\tau \le \delta_{i-1}+\varepsilon K(t_i-t_{i-1})\left(\frac{MT}{m}+\delta_{i-1}+2\varepsilon Mr\right)$$
$$= \delta_{i-1}+\frac{K\cdot T}{m}\left(\frac{MT}{m}+\delta_{i-1}+2\varepsilon Mr\right) = \delta_{i-1}\left(1+\frac{KT}{m}\right)+\frac{KT}{m}\left(\frac{MT}{m}+2\varepsilon Mr\right)$$
$$= \delta_{i-1}\left(1+\frac{a}{m}\right)+\frac{b}{m},$$

where $a = KT$ and $b = KT\left(\frac{MT}{m}+2\varepsilon Mr\right)$.

Hence,

$$\delta_i \leq \delta_{i-1}\left(1+\frac{a}{m}\right)+\frac{b}{m} \leq \left(1+\frac{a}{m}\right)\left[\delta_{i-2}\left(1+\frac{a}{m}\right)+\frac{b}{m}\right]+\frac{b}{m} \leq$$

$$\leq \left(1+\frac{a}{m}\right)^i \delta_0 + \left(1+\frac{a}{m}\right)^{i-1}\frac{b}{m}+\ldots\frac{b}{m} =$$

$$= \frac{b}{m}\left(1+\left(1+\frac{a}{m}\right)+\ldots+\left(1+\frac{a}{m}\right)^{i-1}\right) = \frac{b}{a}\left(\left(1+\frac{a}{m}\right)^i - 1\right) \leq$$

$$\leq \frac{b}{a}(e^a-1) = \frac{M}{m}(2\varepsilon mr+T)(e^{KT}-1),$$

where $i = 0, 1, \ldots, m-1$.

For $t \in [t_i, t_{i+1}]$ we have

$$H(X^1(t), X^1(t_i)) = H\left(X^1(t_i)+\varepsilon\int_{t_i}^t V^1(\tau)d\tau,\ X^1(t_i)\right) \leq$$

$$\leq \varepsilon H\left(\int_{t_i}^t V^1(\tau)d\tau, \{0\}\right) \leq \varepsilon M \cdot \frac{T}{\varepsilon m} = \frac{MT}{m}$$

and $H(Y^1(t), Y^1(t_i)) \leq \frac{MT}{m}$.

Hence, we obtain

$$\begin{aligned} &H(X^1(t), Y^1(t)) \leq H(X^1(t), X^1(t_i)) + H(X^1(t_i), Y^1(t_i)) \\ &+H(Y^1(t_i), Y^1(t)) \leq \frac{2MT}{m} + \frac{M}{m}(2\varepsilon mr+T)(e^{KT}-1) \end{aligned} \tag{11}$$

Now we shall consider the function

$$\begin{cases} Y^2(t) = \Phi(t) \quad \text{for} \ \ t \in [-r, 0] \\ Y^2(t) = Y^2(t_i) + \varepsilon\displaystyle\int_{t_i}^t U^2_{i+1}(\tau)d\tau \quad \text{for} \ \ t \in [t_i, t_{i+1}] \end{cases} \tag{12}$$

where $U^2_{i+1}(\cdot), i = 0, 1, 2, \ldots, m-1$, are measurable functions such that $U^2_{i+1}(t) \in F^2(t, Y^1_{t_i})$.

Let us notice that by virtue of condition 4^0 for each $\eta_1 > 0$ and $T > 0$ there exists a $\varepsilon^0(\eta_1, T) > 0$ such that for every $\varepsilon \leq \varepsilon^0$ we have the following inequalities:

$$d\left(\frac{\varepsilon m}{iT}\int_0^{\frac{iT}{\varepsilon m}} F^1(t, Y^1_{t_i})dt, \frac{\varepsilon m}{iT}\int_0^{\frac{iT}{\varepsilon m}} F^2(t, Y^1_{t_i})dt\right) \leq \frac{\eta_1}{2i} \tag{13}$$

and

$$d\left(\frac{\varepsilon m}{(i+1)T}\int_0^{\frac{(i+1)T}{\varepsilon m}} F^1(t, Y^1_{t_i})dt, \frac{\varepsilon m}{(i+1)T}\int_0^{\frac{(i+1)T}{\varepsilon m}} F^2(t, Y^1_{t_i})dt\right) \leq \frac{\eta_1}{2(i+1)} \tag{14}$$

where $i = 1, 2, \ldots, m-1$. Let us observe that $\frac{(i+1)T}{\varepsilon m} = t_{i+1}$ and $\frac{iT}{\varepsilon m} = t_i$.

By virtue of (13), (14) and the Hausdorff metric condition we have

$$d\left(\int_{t_i}^{t_{i+1}} F^1\left(t, Y^1_{t_i}\right) dt, \int_{t_i}^{t_{i+1}} F^2\left(t, Y^1_{t_i}\right) dt\right) \leq$$

$$\leq d\left(\int_{0}^{t_{i+1}} F^1\left(t, Y^1_{t_i}\right) dt, \int_{0}^{t_{i+1}} F^2\left(t, Y^1_{t_i}\right) dt\right) +$$

$$+d\left(\int_{0}^{t_i} F^1\left(t, Y^1_{t_i}\right) dt, \int_{0}^{t_i} F^2\left(t, Y^1_{t_i}\right) dt\right) \leq$$

$$\leq \frac{\eta_1}{2i} \cdot \frac{iT}{\varepsilon m} + \frac{\eta_1}{2(i+1)} \cdot \frac{T(i+1)}{\varepsilon m} = \frac{\eta_1 T}{\varepsilon m}.$$

Then

$$H\left(\int_{t_i}^{t_{i+1}} U^1_{i+1}(\tau) d\tau, \int_{t_i}^{t_{i+1}} U^2_{i+1}(\tau) dt\right) \leq \frac{\eta_1 T}{\varepsilon m}$$

and

$$\begin{aligned} &H(Y^1(t_{i+1}), Y^2(t_{i+1})) \leq H(Y^1(t_i), Y^2(t_i)) + \\ &+ \varepsilon H\left(\int_{t_i}^{t_{i+1}} U^1_{i+1}(\tau) d\tau, \int_{ti}^{t_{i+1}} U^2_{i+1}(\tau) dt\right) \leq \\ &\leq H(Y^1(t_i), Y^2(t_i)) + \frac{\eta_1 T}{m} \leq \ldots \leq m \cdot \frac{\eta_1 T}{m} = \eta_1 T, \end{aligned} \tag{15}$$

where $i = 0, 1, \ldots, m-1$.

Using the inequality (15) and the fact that for $t \in [t_i, t_{i+1}]$

$$H(Y^1(t), Y^1(t_i)) \leq \frac{MT}{m} \quad \text{and} \quad H(Y^2(t), Y^2(t_i)) \leq \frac{MT}{m}$$

we have

$$\begin{aligned} &H(Y^1(t), Y^2(t)) \leq H(Y^1(t), Y^1(t_i)) + H(Y^1(t_i), Y^2(t_i)) \\ &+ H(Y^2(t_i), Y^2(t)) \leq \frac{2MT}{m} + \eta_1 T \end{aligned} \tag{16}$$

By assumption 3^0 it follows that

$$d(F^2(t, Y^2_t,) F^2(t, Y^1_{t_i})) \leq K \rho_0(Y^2_t, Y^1_{t_i})$$

Similarly, as in the proof of the inequality (9) and making use of the inequality (16) we obtain

$$\rho_0(Y^2_t, Y^1_{t_i}) \leq \rho_0(Y^2_t, Y^2_{t_i}) + \rho_0(Y^2_{t_i}, Y^1_{t_i})$$

$$\leq \frac{MT}{m} + \frac{2MT}{m} + \eta_1 T = \frac{3MT}{m} + \eta_1 T$$

Hence $d(F^2(t, Y^2_t,) F^2(t, Y^1_{t_i})) \leq K\left(\frac{3MT}{m} + \eta_1 T\right)$.

By virtue of (12) we have:

$$\rho\left(D_h Y^2(t), \varepsilon F^2(t, Y^2_t)\right) = \rho\left(D_h Y^2(t), \varepsilon F^2\left(t, Y^1_{t_i}\right)\right)$$

$$+d\left(\varepsilon F^2\left(t, Y^1_{t_i}\right), \varepsilon F^2\left(t, Y^2_t\right)\right) \leq K\varepsilon\left(\frac{3MT}{m} + \eta_1 T\right).$$

Now, using existence theorem (see Theorem 1) there exists at least a solution $X^2(\cdot)$ of (4) such that for $t \in [0, T/\varepsilon]$

$$H(Y^2(t), X^2(t)) \leq \int_0^t K\varepsilon \left(\frac{3MT}{m} + \eta_1 T\right) \exp[\varepsilon K(t-s)]ds \leq$$

$$\leq \left(\frac{3MT}{m} + \eta_1 T)(e^{KT} - 1\right).$$

Using the inequalities (11) and (16) it follows

$$H\left(X^1(t), X^2(t)\right) \leq H\left(X^1(t), Y^1(t)\right) + H\left(Y^1(t), Y^2(t)\right) + H\left(Y^2(t), X^2(t)\right)$$

$$\leq \frac{4MT}{m} e^{KT} + 2\varepsilon Mre^{KT} + \eta_1 Te^{KT}.$$

Therefore, choosing $m > \frac{12MTe^{KT}}{\eta}, \eta_1 = \frac{\eta}{3Te^{KT}}$ and $\varepsilon < \frac{\eta}{6Mre^{KT}}$ we get the inequality

$$H(X^1(t), X^2(t)) \leq \eta \quad \text{for } t \in [0, T/\varepsilon].$$

Adopting now the procedure presented above we get the condition (ii). In this way the proof is completed for $t \in \left[-r, \frac{T}{\varepsilon}\right]$.

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POLYNOMIAL ORBITS IN DIRECT SUM OF FINITE EXTENSION FIELDS

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Abstract. Let $K_1, \dots, K_n$ be a finite extensions of the field F. We describe the structure of finite orbits and determine its precycle and cycle lengths in the direct sum $K_1 \oplus \dots \oplus K_n$ which are induced by polynomials from F.

Let R be a commutative ring, $k \in \mathbb{N}_0, l \in \mathbb{N}$ and $f \in R[X]$. By a *finite orbit of f in R with precycle length k and cycle length l* we mean a sequence $(x_1, x_2, \dots, x_{k+l})$ of distinct elements of R such that

$$f(x_i) = x_{i+1} \quad \text{for all} \quad i \in \{1, 2, \dots, k+l-1\}, \quad \text{and} \quad f(x_{k+l}) = x_{k+1}.$$

Elements $x_i, i = k+1, \dots, l$ are called *fixpoints of f of order l*. Let $k \in \mathbb{N}_0$. By a *k-iterate of f in R* we mean a polynomial f_k such that

$$f_0(x) = (x), f_1(x) = f(x), f_{k+1}(x) = f(f_k(x))$$

Let K/F be an algebraic field extension. Then Cycl(K/F) is the set of all possible cycle lengths in K of polynomials over F. Consider an algebraic field extension K/F. The following proposition determine the structure of finite orbits in K of polynomials $f \in F[X]$.

Proposition 1. [1] *Let K/F be an algebraic field extension, $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, and let $(x_1, x_2, \dots, x_{k+l})$ be a sequence of distinct elements of K. Then the following assertions are equivalent:*

a): *$(x_1, x_2, \dots, x_{k+l})$ is a finite orbit of a unique polynomial $f \in F[X]$ with precycle length k and cycle length l such that for a certain d*

$$\deg f < \prod_{i=1}^{k+d} \deg_F(x_i).$$

b): *$(x_1, x_2, \dots, x_{k+l})$ is a finite orbit of a polynomial $f \in F[X]$ with precycle length k and cycle length l.*

c): *There holds $F(x_1) \supset F(x_2) \supset \dots \supset F(x_{k+1}) = \dots = F(x_{k+l})$, there exist $d, m \in \mathbb{N}$ and $\tau \in Aut_F\big(F(x_{k+1})\big)$ such that $l = dm$, $\mathrm{ord}(\tau) = \mathrm{m}$, the elements $x_1, \dots, x_{k+d}$ are pairwise not conjugate over F, and*

$$x_{k+\mu d+j} = \tau^{\mu}(x_{k+j}) \quad \textit{for all} \quad j \in \{1, \dots, d\} \quad \textit{and} \quad \mu \in \{1, \dots, m-1\}.$$

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By proposition, let K/F be an algebraic field extension of degree n and N the number of irreducible monic polynomials of degree n over F. Then the set of all possible cycle lengths in K of polynomials over F is given by

$$\mathrm{Cycl(K/F)} = \left\{\mathrm{dm} \;\middle|\; 1 \le \mathrm{d} \le \mathrm{N},\ 1 \le \mathrm{m}|\mathrm{n}\right\}.$$

In the present paper we shall describe the structure of finite polynomial orbits and determine the set of all possible cycle lengths of polynomials over F in the direct sum of finite extension fields $K_1, \dots, K_n$ which is given by

$$\bigcup_{K_i' \subseteq K_i} \mathrm{Cycl}(\mathrm{K}_1' \oplus \ldots \oplus \mathrm{K}_\mathrm{n}'/\mathrm{F}),$$

over all n-tuple $(K_1' \oplus \ldots \oplus K_n')$ with $\mathrm{Cycl}(\mathrm{K}_1' \oplus \ldots \oplus \mathrm{K}_\mathrm{n}'/\mathrm{F})$ are different.

As an application of this general case we can obtain the set of all cycle lengths for special rings which are direct sum of finite extension fields, for example ring of circulant matrices over a finite field which is very important in the coding theory.

First we recall some properties of cycles and polynomials in the following lemmas.

Lemma 1. [1] *Let F be a field, let $f_1, \dots, f_m \in F[X]$, $m \in \mathbb{N}$ be pairwise coprime polynomials, and let $g_1, \dots, g_m \in F[X]$ be any polynomials. Then there exists a unique polynomial $f \in F[X]$ such that*

$$\mathrm{deg(f)} < \prod_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{deg(f_j)} \quad \textit{and} \quad \mathrm{f} \equiv \mathrm{g_j} \mod \mathrm{f_j} \quad \textit{for all} \quad \mathrm{j} \in \{1, \dots, \mathrm{m}\}.$$

Lemma 2. [6] *Let R be a ring. If $a \in R, f_n(a) = a$ and j is the smallest integer satisfying $f_j(a) = a$, then j divides n. Cyclic elements of order n of f coincide with those fixpoints of f_n which are not fixpoints of f_d, where d runs over all proper divisors of n.*

Lemma 3. *All conjugated elements in the finite extension of the field F have the same cycle length of a polynomial $f \in F$.*

Proof. This follows immediately from properties of any automorphism of the algebraic closure of K. □

Theorem 1. *Let K/F be an algebraic field extension of degree n, N the number of irreducible monic polynomials of degree n over F and $s \in \mathbb{N}$. Then the set of all possible cycle lengths of f in the direct sum K^s is given by*
$Cycl(K^s/F) = \{m \cdot \mathrm{lcm(d_1, \dots, d_t)} \mid \mathrm{t} \le \mathrm{s}, \mathrm{d_1, \dots, d_t}$ are distinct, $d_1 + \dots + d_t \le N$ and $m|n\}$.

Proof. Let $(\bar{x}_1, \dots, \bar{x}_l)$ be a cycle of polynomial f in K^s with length l, where $\bar{x}_i = (x_i^{(1)}, \dots, x_i^{(s)})$ and $x_i^{(j)} \in K$.

Then $f_l(\bar{x}_i) = \bar{x}_i$ and $f_l(x_i^{(j)}) = x_i^{(j)}$ for any $i = 1, \dots, l, j = 1, \dots, s$.

For any $j = 1, \dots, s$ consider the least positive integers $l_j \le l$ with $f_{l_j}(x_i^{(j)}) = x_i^{(j)}$. By Lemma 2 we have that l_j divides l and l_j is a cycle length of f in K. Hence by Proposition, l_j can be written in the form $l_j = d_j m_j$, where $d_j = 1, \dots, N$ and $m_j|n$, whence $l = m \cdot \mathrm{lcm(d_1, \dots, d_s)}$, where m is a positive integer which divides n.

From the set $\{d_1, \dots, d_s\}$ choose t elements $d_1, \dots, d_t$, which are different. Assume to the contrary that $d_1 + \cdots + d_t > N$. Then there are positive integers $j_1, j_2 = 1, \dots, s, i_1, i_2 = 1, \dots, l, j_1 \neq j_2, i_1 \neq i_2$ such that elements $x_{i_1}^{j_1}, x_{i_2}^{j_2}$ are conjugated. Lemma 3 implies that cycles $(x_1^{j_1}, \dots, x_{l_1}^{j_1}), (x_1^{j_2}, \dots, x_{l_2}^{j_2})$ must have the same cycle length of the type $d \cdot m$, it means $d_{j_1} = d_{j_2}$. Contradiction.

Let $m, d_1, \dots, d_t$ be positive integers such that $m|n, d_j \leq N,$ and $d_1 + \cdots + d_t \leq N, j = 1, \dots, t \leq s$. Then there is a unique t-tuple of polynomials $f^{(1)}, \dots, f^{(t)}$ over F with cycles $(x_1^{(j)}, \dots, x_{d_j}^{(j)}, \dots, x_{md_j}^{(j)})$, such that $x_1^{(j)}, \dots, x_{d_j}^{(j)}$ are pairwise non conjugated elements. Let $p_i^{(j)}$ be the minimal polynomials of elements x_i^j. Then by Lemma 1 there is a unique polynomial $f \in F[x]$ such that

$$\mathrm{deg(f)} < \prod_{\mathrm{j=1}}^{\mathrm{t}} \prod_{\mathrm{i=1}}^{\mathrm{d_j}} \mathrm{deg(p_i^{(j)})} \quad \text{and} \quad \mathrm{f} \equiv \mathrm{f^{(j)}} \mod \prod_{\mathrm{i=1}}^{\mathrm{d_j}} \mathrm{p_i^{(j)}} \quad \text{for all} \quad \mathrm{j} \in \{1, \dots, \mathrm{t}\}$$

and

$$f_{md_j}(x_i^{(j)}) = x_i^{(j)}.$$

Put $l = m \cdot \mathrm{lcm(d_1, \dots, d_s)} = \mathrm{m} \cdot \mathrm{lcm(d_1, \dots, d_t)}$, then $f_l(\bar{x}_i) = \bar{x}_i$ and so $l \in Cycl(K^s/F)$. □

Theorem 2. *Let $K_1, K_2, \dots, K_r$ be finite extensions of the field F, $s_1, \dots, s_r, r \in \mathbb{N}$. Then*

$$\mathrm{Cycl(K_1^{s_1} \oplus \cdots \oplus K_r^{s_r}/F) = \{lcm(l_i) \mid l_i \in Cycl(K_i^{s_i}/F)\}}.$$

Proof. Let $l \in \mathrm{Cycl(K_1^{s_1} \oplus \cdots \oplus K_r^{s_r}/F)}$. Then there is a polynomial $f \in F[x]$ with the cycle $\big((\bar{x}_1^{(1)}, \dots, \bar{x}_1^{(r)}), \dots, (\bar{x}_l^{(1)}, \dots, \bar{x}_l^{(r)})\big)$, where $\bar{x}_j^{(i)} \in K_i^{s_i}$ for $i = 1, \dots, r, j = 1, \dots, l$. Then

$$(\bar{x}_j^{(1)}, \dots, \bar{x}_j^{(r)}) = f_l\big((\bar{x}_j^{(1)}, \dots, \bar{x}_j^{(r)})\big) = \big(f_l(\bar{x}_j^{(1)}), \dots, f_l(\bar{x}_j^{(r)})\big).$$

Consider the least positive integers $l_i \leq l$ with $f_{l_i}(\bar{x}_j^{(i)}) = \bar{x}_j^{(i)}$. By lemma 2 we have $l_i|l$, therefore $l_i \in \mathrm{Cycl(K_i^{s_i}/F)}$ and $l = \mathrm{lcm(l_i)}$.

Let $l_i \in \mathrm{Cycl(K_i^{s_i}/F)}$ for $i = 1, \dots, r$. Then there are polynomials $f^{(i)}$ over F with cycles $(\bar{x}_1^{(i)}, \dots, \bar{x}_{l_i}^{(i)})$ such that $f_{l_i}^{(i)}(\bar{x}_j^{(i)}) = \bar{x}_j^{(i)}$ for $j = 1, \dots, l_i$. Consider polynomials p_i over F as products of minimal polynomials of non conjugated elements in the cycle $(\bar{x}_1^{(i)}, \dots, \bar{x}_{l_i}^{(i)})$. Now the fact, that different l_i-tuples $(\bar{x}_1^{(i)}, \dots, \bar{x}_{l_i}^{(i)})$ don't consist conjugated elements for $i = 1, \dots, r$, implies that these polynomials are pairwise coprime and by Lemma 1 we have a polynomial $f \in F[x]$ such that

$$f \equiv f^{(i)} \mod p_i \quad \text{for all} \quad j \in \{1, \dots, r\}.$$

Hence $f_{l_i}(\bar{x}_j^{(i)}) = \bar{x}_j^{(i)}$ and if $l = \mathrm{lcm(l_i)}$, then $l \in \mathrm{Cycl(K_1^{s_1} \oplus \cdots \oplus K_r^{s_r}/F)}$. □

Theorem 3. *Let $L_1, \dots, L_n$ are algebraic extensions of a field F, $L_1', \dots, L_n'$ are any subfields such that $F \subseteq L_i' \subseteq L_i$ for $i = 1, \dots, n$.*
Let $\bar{x}_1 = \langle x_1^{(1)}, \dots, x_1^{(n)} \rangle, \dots, \bar{x}_{k+l} = \langle x_{k+l}^{(1)}, \dots, x_{k+l}^{(n)} \rangle$ are different elements of the direct sum $L_1 \oplus \cdots \oplus L_n$.
Let $d(L_i'), t(L_i'), N(L_i')$ are nonnegative integers such that
$d(L_i')$ is the number of non-conjugated elements of L_i' in the set $\{x_{k+1}^{(i)}, \dots, x_{k+l}^{(i)}\}$,
$t(L_i^i)$ is the number of non-conjugated elements of L_i' in the set $\{x_1^{(j)}, \dots, x_k^{(j)}\}-$

$-\{x_{k+1}^{(j*)},\dots,x_{k+l}^{(j*)}\}$ *for* $j=1,\dots,n$ *and some* $j*\in\{1,\dots,n\}$ *such that* $L_{j*}^{'}=L_i^{'}$, $N(L_i^{'})$ *is the number of irreducible polynomials in* $F[X]$ *of degree* $[L_i^{'}:F]$.

Then following assertions are equivalent:

a): *The sequence* $(\bar{x}_1,\dots,\bar{x}_{k+l})$ *is a finite orbit of a polynomial* $f\in F[X]$ *in the direct sum* $L_1\oplus\cdots\oplus L_n$ *with precycle length* k *and cycle of the length* l *in the direct sum* $L_1^{'}\oplus\cdots\oplus L_n^{'}$.

b): *For* $i=1,\dots,n$, *there are sequences of fields*

$$L_i\supseteq F(x_1^{(i)})\supseteq\cdots\supseteq F(x_{k_i}^{(i)})\supseteq F(x_{k_i+1}^{(i)})=\cdots=F(x_{k_i+l_i}^{(i)})=\cdots=F(x_{k+l}^{(i)})=L_i^{'}$$

where $k_i\in\mathbb{N}_0, l_i\in\mathbb{N}, k=\max(\mathrm{k_i})$ *and* $l\in\mathrm{Cycl}(\mathrm{K_1^{s_1}}\oplus\cdots\oplus\mathrm{K_r^{s_r}}/\mathrm{F})=\mathrm{Cycl}(\mathrm{L_1^{'}}\oplus\cdots\oplus\mathrm{L_n^{'}}/\mathrm{F})$ *and for every* $i=1,\dots,n$ *it holds*

$$t(L_i^{'})+\sum_{L_j^{'}=L_i^{'},d(L_j^{'})\ \textit{are distinct}} d(L_j^{'})\le N(L_i^{'}).$$

Proof. Let $(\bar{x}_1,\dots,\bar{x}_{k+l})$ is a finite orbit of a polynomial $f\in F[X]$ in the direct sum of algebraic field extensions $L_1\oplus\cdots\oplus L_n$ with precycle length k and cycle in the direct sum $L_1^{'}\oplus\cdots\oplus L_n^{'}$ of the length l.
Then by definition for $j=1,\dots,k+l-1$ it holds

$$f(\bar{x}_j)=f(\langle x_j^{(1)},\dots,x_j^{(n)}\rangle)=\langle f(x_j^{(1)}),\dots,f(x_j^{(n)})\rangle=\langle x_{j+1}^{(1)},\dots,x_{j+1}^{(n)}\rangle=\bar{x}_{j+1}$$

and for $j=k+1,\dots,k+l$

$$f_l(\bar{x}_j)=f_l(\langle x_j^{(1)},\dots,x_j^{(n)}\rangle)=\langle f_l(x_j^{(1)}),\dots,f_l(x_j^{(n)})\rangle=\langle x_j^{(1)},\dots,x_j^{(n)}\rangle=\bar{x}_j.$$

Then for every $i=1,\dots,n$ there is a finite orbit $(x_1^{(i)},\dots,x_{k+l}^{(i)})$ of the polynomial $f\in F[X]$ in the field L_i.

Consider least positive integers $k_i\in\mathbb{N}_0, l_i\in\mathbb{N}$ such that $f_{l_i}(x_j^{(i)})=x_j^{(i)}$ for every $j>k_i$. Then by definition and lemma2 l_i is the cycle length of i-th finite orbit, k_i is the precycle length of i-th orbit and $k=\max(\mathrm{k_i})$.

By proposition we obtain sequences of fields

$$L_i\supseteq F(x_1^{(i)})\supseteq\cdots\supseteq F(x_{k_i}^{(i)})\supseteq F(x_{k_i+1}^{(i)})=\cdots=F(x_{k+l}^{(i)})=L_i^{'}.$$

Let $K_1,\dots,K_r$ be distinct fields such that $\{K_1,\dots,K_r\}=\{L_1^{'},\dots,L_n^{'}\}$ and suppose that K_i appears s_i times, so

$$L_1^{'}\oplus\cdots\oplus L_n^{'}=K_1^{s_1}\oplus\cdots\oplus K_r^{s_r}.$$

Now assume to the contrary that

$$t(L_i^{'})+\sum_{L_j^{'}=L_i^{'},d(L_j^{'})\ \text{are distinct}} d(L_j^{'})>N(L_i^{'}).$$

Then there is a pair of conjugated elements such that one of them is in some precycle and the second one in some other cycle and it is in contradiction with lemma3. From b) to a) it follows immediately from Lemma1 and Theorem2.

□

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SPLINE APPROXIMATION FOR SOLVING SYSTEM OF FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. In a previous work, [9], the authors introduced a new technique using a spline function to find an approximate solution for first order delay differential equations. In this presented paper, we develop and modify the lemmas in [9] so that the technique can be extended to work for the case of numerical approximation for solving system of first order delay differential equations. Error estimation and convergence are also considered and tested using numerical examples. The stability of the technique is investigated.

1. Introduction

Consider the system of first order delay differential equations of the form:

$$\begin{aligned} y^{'}(x) &= f_1(x, y(x), z(x), y(g(x))),\ a \le x \le b \\ z^{'}(x) &= f_2(x, y(x), z(x), z(g(x))),\ y(x_0) = y_0, z(x_0) = z_0 \\ y(x) &= \phi(x),\ z(x) = \overline{\phi}(x), x \in [a^*, a] \end{aligned} \tag{1}$$

In recent years many studies were devoted to the problems of approximate solutions of system ordinary as well as delay differential equations by spline functions [2-6] and [8-10]. While in [1] A. Ayad investigated the spline approximation for Fredholm integro differential equations. Also G. Micula and H. Akca [7] have studied the numerical solutions of system of differential equations with deviating argument by spline functions. Our introduced method is a one step method o$(h^{m+\alpha})$ in $y^{(i)}(x)$ and $z^{(i)}(x)$ where $i = 0, 1$. The modulus of continuity of $y'(x)$ and $z'(x)$ is o$(h^\alpha), 0 < \alpha \le 1$ and m is an arbitrary positive integer which is equal to the number of iterations used in computing the spline function. Assuming $f_1,\ f_2 \in C([a, b] \times R^3)$ we shall investigate the error estimation and convergence as well as the stability of the method.

2. Description of the spline method

Rewriting the system (1) in the following form:

$$\begin{aligned} y^{'}(x) &= f_1(x, u_1, v_1, u_1^*),\ a \le x \le b \\ z^{'}(x) &= f_2(x, u_1, v_1, v_1^*) \\ y(x_0) &= y_0, z(x_0) = z_0,\ y(x) = \phi(x),\ z(x) = \overline{\phi}(x), x \in [a^*, a] \end{aligned} \tag{2}$$

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The function g is called the delay function and it is assumed to be continuous on the interval $[a,b]$ and satisfies the inequality $a^* \leq g(x) \leq x, x \in [a,b]$ and $\phi, \overline{\phi} \in C[a^*, a]$.

Suppose that $f_1 : [a,b] \times R^3 \to R$ is continuous and satisfies the Lipschitz conditions

$$|f_1(x,u_1,v_1,u_1^*) - f_1(x,u_2,v_2,u_2^*)| \leq L_1\{|u_1-u_2| + |v_1-v_2| + |u_1^*-u_2^*|\} \quad (3)$$

and there exist a constant B_1 so that

$$|u_1^* - u_2^*| \leq B_1 \, |f_1(x,u_1,v_1,u_1^*) - f_1(x,u_2,v_2,u_2^*)| \quad (4)$$

Also Suppose that $f_2 : [a,b] \times R^3 \to R$ is continuous and satisfies the Lipschitz conditions

$$|f_2(x,u_1,v_1,v_1^*) - f_2(x,u_2,v_2,v_2^*)| \leq L_2\{|u_1-u_2| + |v_1-v_2| + |v_1^*-v_2^*|\} \quad (5)$$

and there exist a constant B_2 so that

$$|v_1^* - v_2^*| \leq B_2 \, |f_2(x,u_1,v_1,v_1^*) - f_2(x,u_2,v_2,v_2^*)| \quad (6)$$

$$\forall (x,u_1,v_1,u_1^*), (x,u_2,v_2,u_2^*), (x,u_1,v_1,v_1^*), (x,u_2,v_2,v_2^*) \in ([a,b] \times R^3)$$

These conditions assure the existence of a unique solutions of y and z of system (1).

Let $\triangle$ be a uniform partition of the interval $[a,b]$ defined by the nodes

$\triangle : a = x_0 < x_1 ... < x_k < x_{k+1} ... < x_n = b, x_k = x_0 + kh, h = \frac{b-a}{n} < 1$ and $k = 0(1)n-1$

we define the spline function approximating the solutions y and z by $S(x)$ and $\overline{S}(x)$ where

$$S(x) = \begin{cases} S_\triangle(x), & a \leq x \leq b \\ \phi(x), & a^* \leq x \leq a \end{cases}$$

$$\overline{S}(x) = \begin{cases} \overline{S}_\triangle(x), & a \leq x \leq b \\ \overline{\phi}(x), & a^* \leq x \leq a \end{cases}$$

Choosing the required positive integer m, we define $S_\triangle(x)$ and $\overline{S}_\triangle(x)$ by:

$$S_\triangle(x) = S_k^{[m]}(x) = S_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1(x, S_k^{[m-1]}(x), \overline{S}_k^{[m-1]}(x), S_k^{[m-1]}(g(x)))dx \quad (7)$$

$$\overline{S_\triangle}(x) = \overline{S}_k^{[m]}(x) = \overline{S}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2(x, S_k^{[m-1]}(x), \overline{S}_k^{[m-1]}(x), \overline{S}_k^{[m-1]}(g(x)))dx \quad (8)$$

where $S_{-1}^{[m]}(x_0) = y_0, \overline{S}_{-1}^{[m]}(x_0) = z_0, S_{-1}^{[m]}(g(x_0)) = \phi(g(x_0)), \overline{S}_{-1}^{[m]}(g(x_0)) = \overline{\phi}(g(x_0))$ with $S_{k-1}^{[m]}(x_k)$ and $\overline{S}_{k-1}^{[m]}(x_k)$ are the left hand limit of $S_{k-1}^{[m]}(x)$ and $\overline{S}_{k-1}^{[m]}(x)$ as $x \to x_k$ of the segment $S_\triangle(x)$ and $\overline{S}_\triangle(x)$ defined on $[x_{k-1}, x_k]$. In equation (7), (8) we use the following m iterations for $x \in [x_k\ ,x_{k+1}], k = 0(1)n-1$ and $j = 1(1)m$

$$\begin{aligned} S_k^{[j]}(x) &= S_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1(x, S_k^{[j-1]}(x), \overline{S}_k^{[j-1]}(x), S_k^{[j-1]}(g(x)))dx \qquad (9) \\ \overline{S}_k^{[j]}(x) &= \overline{S}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2(x, S_k^{[j-1]}(x), \overline{S}_k^{[j-1]}(x), \overline{S}_k^{[j-1]}(g(x)))dx \\ S_k^{[0]}(x) &= S_{k-1}^{[m]}(x_k) + M_k\ (x - x_k) \\ \overline{S}_k^{[0]}(x) &= \overline{S}_{k-1}^{[m]}(x_k) + \overline{M}_k\ (x - x_k) \\ \text{where } M_k &= f_1(x_k, S_{k-1}^{[m]}(x_k), \overline{S}_{k-1}^{[m]}(x_k), S_{k-1}^{[m]}(g(x_k))) \text{ and} \\ \overline{M}_k &= f_2(x_k, S_{k-1}^{[m]}(x_k), \overline{S}_{k-1}^{[m]}(x_k), \overline{S}_{k-1}^{[m]}(g(x_k))) \end{aligned}$$

Such $S_\triangle(x)$, $\overline{S}_\triangle(x) \in C[a,b] \times R^3$ are exist and unique.

3. Error estimation and convergence

To estimate the error, we represent the exact solution as described by the following scheme.

$$\begin{aligned} y^{[0]}(x) &= y(x) = y_k + y'(\zeta_k)(x - x_k) \qquad (10) \\ z^{[0]}(x) &= z(x) = z_k + z'(\eta_k)(x - x_k) \end{aligned}$$

where $\zeta_k, \eta_k \in (x_k, x_{k+1})$, $y(x_k) = y_k$, $z(x_k) = z_k$. For $1 \le j \le m$ we write

$$\begin{aligned} y^{[j]}(x) &= y(x) = y_k + \int_{x_k}^{x} f_1(x, y^{[j-1]}(x), z^{[j-1]}(x), y^{[j-1]}(g(x)))dx \qquad (11) \\ z^{[j]}(x) &= z(x) = z_k + \int_{x_k}^{x} f_2(x, y^{[j-1]}(x), z^{[j-1]}(x), z^{[j-1]}(g(x)))dx \end{aligned}$$

Set $\omega(h) = \max\{\omega(y', h), \omega(z', h)\}$ where $\omega(y', h)$ and $\omega(z', h)$ are the modului of continuity for the functions $y'(x)$ and $z'(x)$.

Moreover, we denote to the estimated error of $y(x)$ and $z(x)$ at any point $x \in [a,b]$ by:

$$\begin{aligned} e(x) &= |y(x) - S_\triangle(x)|\,,\ e_k = |y_k - S_\triangle(x_k)| \qquad (12) \\ \overline{e}(x) &= \left|z(x) - \overline{S}_\triangle(x)\right|,\ \overline{e}_k = \left|z_k - \overline{S}_\triangle(x_k)\right| \end{aligned}$$

Lemma 3.1. *[1]. Let α and β be non negative real numbers and $\{A_i\}_{i=1}^m$ be a sequence satisfying $A_1 \ge 0$, $A_i \le \alpha + \beta A_{i+1}$ for $i = 1(1)m-1$ then:*

$$A_1 \le \beta^{m-1} A_m + \alpha \sum_{i=0}^{m-2} \beta^i$$

Lemma 3.2. *[1]. Let α and β be non negative real numbers, $\beta \ne 1$ and $\{A_i\}_{i=0}^k$ be a sequence satisfying*
$A_0 \ge 0$, $A_{i+1} \le \alpha + \beta A_i$ for $i = 0(1)k$ then:

$$A_{k+1} \le \beta^{k+1} A_0 + \alpha \frac{[\beta^{k+1} - 1]}{[\beta - 1]}$$

Definition 3.1. [4] for any $x \in [x_k ,x_{k+1}], k = 0(1)n-1$ and $j = 1(1)m$ we define the operator $T_{kj}(x)$ by:

$$T_{kj}(x) = \left|y^{[m-j]}(x) - S_k^{[m-j]}(x)\right| + \left|z^{[m-j]}(x) - \overline{S}_k^{[m-j]}(x)\right| \tag{13}$$

whose norm is defined by: $\|T_{kj}\| = \max\limits_{x\in[x_k,x_{k+1}]} \{T_{kj}(x)\}$

Lemma 3.3. *For any $x \in [x_k ,x_{k+1}], k = 0(1)n-1$ and $j = 1(1)m$, then*

$$\|T_{km}\| \leq [1 + h(c_0 + \overline{c}_0)] (e_k + \overline{e}_k) + 2h\omega(h) \tag{14}$$

$$\|T_{k1}\| \leq c_1(e_k + \overline{e}_k) + c_2 h^m \omega(h) \tag{15}$$

where $c_0 = \frac{L_1}{1-L_1B_1}, \overline{c}_0 = \frac{L_2}{1-L_2B_2}, c_1 = \sum_{i=0}^{m} (c_0 + \overline{c}_0)^i$ and $c_2 = 2 (c_0 + \overline{c}_0)^{m-1}$ are constants independent of h.

Proof. Using (3), (4), (5), (6), (9), (10), (11) and (12), it is easy to proof the lemma.

Lemma 3.4. *Let $e(x), \overline{e}(x)$ be defined as in (12), then there exist constants $c_3, c_4, \overline{c}_3, \overline{c}_4$ independent of h such that the following inequalities hold:*

$$e(x) \leq (1 + hc_3) e_k + hc_3\overline{e}_k + c_4 h^{m+1}\omega(h) \tag{16}$$

$$\overline{e}(x) \leq h\overline{c}_3 e_k + (1 + h\overline{c}_3) \overline{e}_k + \overline{c}_4 h^{m+1}\omega(h) \tag{17}$$

where $c_3 = c_0c_1$, $c_4 = c_0c_2, \overline{c}_3 = \overline{c}_0c_1$ and $\overline{c}_4 = \overline{c}_0c_2$

Proof. Using (3), (4), (7), (11), (12) and (15) we get:

$$\begin{aligned} e(x) &\leq \left|y(x) - S_k^{[m]}(x)\right| \leq e_k + c_0 \|T_{k1}\| \int_{x_k}^{x} dx \\ &\leq (1 + hc_3) e_k + hc_3\overline{e}_k + c_4 h^{m+1}\omega(h) \end{aligned}$$

Similarly using (5), (6), (8), (11), (12) and (15), we can proof the other part of the lemma where $c_3 = c_0c_1$, $c_4 = c_0c_2, \overline{c}_3 = \overline{c}_0c_1$ and $\overline{c}_4 = \overline{c}_0c_2$ are constants independent of h.

Definition 3.2. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order then we say that $A \leq B$ iff:*

(i) both a_{ij} and b_{ij} are non negative

(ii) $a_{ij} \leq b_{ij} \forall i, j$.

Using matrix notation we let

$$E(x) = [e(x) \quad \overline{e}(x)]^T, E_k = [e_k \quad \overline{e}_k]^T \text{ and } C = [c_4 \quad \overline{c}_4]^T$$

where T stands for the transpose, then from lemma 3.4, we write

$$E(x) \leq (I + hA) E_k + Ch^{m+1}\omega(h) \tag{18}$$

where I is the unit matrix of order 2 and $A = \begin{pmatrix} c_3 & c_3 \\ \overline{c}_3 & \overline{c}_3 \end{pmatrix}$.

Definition 3.3. *Let $T = [T_{i,j}]$ be an $m \times n$ matrix, then we define*

$$\|T\| = \max_i \sum_{j=0}^{n} |t_{i,j}|.$$

Using this definition the inequality (18) yields:

$$\|E(x)\| \leq (1 + h \|A\|) \|E_k\| + \|C\| h^{m+1}\omega(h).$$

This inequality holds for $x \in [a,b]$. Setting $x = x_{k+1}$, we obtain:

$$\|E_{k+1}\| \leq (1 + h\|A\|)\|E_k\| + \|C\| h^{m+1}\omega(h).$$

Using lemma 3.2 and noting that $\|E_0\| = 0$,we get:

$$\begin{aligned}
\|E(x)\| &\leq \|C\| h^{m+1}\omega(h)\frac{\left[(1+h\|A\|)^{k+1}-1\right]}{1+h\|A\|-1} \\
&\leq \frac{\|C\|}{\|A\|}\left[\left(1+\frac{\|A\|(b-a)}{n}\right)^n - 1\right] h^m\omega(h) \\
&\leq \frac{\|C\|}{\|A\|}\left[e^{(\|A\|(b-a))}-1\right] h^m\omega(h) \\
&\leq c_5 h^m\omega(h) = o(h^{m+\alpha})
\end{aligned}$$

where $c_5 = \frac{\|C\|}{\|A\|}\left[e^{(\|A\|(b-a))}-1\right]$ is a constant independent of h. Using definition 3.3, we get:

$$\begin{aligned}
e(x) &\leq c_5 h^m\omega(h) \\
\overline{e}(x) &\leq c_5 h^m\omega(h)
\end{aligned} \tag{19}$$

now we are going to estimate $\left|y^{'}(x) - S^{'}_{\Delta}(x)\right|$. Using (3), (4), (7), (11), (12), (15) and (19), we get:

$$\left|y^{'}(x) - S^{'}_{\Delta}(x)\right| \leq c_6 h^m\omega(h)$$

where $c_6 = c_0[2c_1c_5 + c_2]$ is a constant independent of h. Similarly using (5), (6), (8), (11), (12), (15) and (19), we get:

$$\left|z^{'}(x) - \overline{S}^{'}_{\Delta}(x)\right| \leq c_7 h^m\omega(h)$$

where $c_7 = \overline{c}_0[2c_1c_5 + c_2]$ is a constant independent of h.

Thus from above lemma we have arrived to the following theorem:

Theorem 3.1. *Let $y(x)$, $z(x)$ be the exact solutions of the system (1). If $S_\Delta(x), \overline{S}_\Delta(x)$ given by (7), (8) are the approximate solutions for the problem, $f_1, f_2 \in C\left([a,b]\times R^3\right)$, then the inequalities*

$$\begin{aligned}
\left|y^{(q)}(x) - S^{(q)}_{\Delta}(x)\right| &\leq c_8 h^m\omega(h) \\
\left|z^{(q)}(x) - \overline{S}^{(q)}_{\Delta}(x)\right| &\leq c_9 h^m\omega(h)
\end{aligned}$$

hold for all $x \in [a,b]$ and $q = 0,1$ where c_8 and c_9 are constants independent of h.

4. **Stability of the method**

To study the stability of the method given by (7), (8) we change $S_\Delta(x)$ to $W_\Delta(x)$ and $\overline{S}_\Delta(x)$ to $\overline{W}_\Delta(x)$ where

$$W_\triangle(x) = W_k^{[m]}(x) = W_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1(x, W_k^{[m-1]}(x), \overline{W}_k^{[m-1]}(x), W_k^{[m-1]}(g(x)))dx \quad (20)$$

$$\overline{W}_\triangle(x) = \overline{W}_k^{[m]}(x) = \overline{W}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2(x, W_k^{[m-1]}(x), \overline{W}_k^{[m-1]}(x), \overline{W}_k^{[m-1]}(g(x)))dx \quad (21)$$

$W_{-1}^{[m]}(x_0) = y_0^*, \overline{W}_{-1}^{[m]}(x_0) = z_0^*, W_{-1}^{[m]}(g(x_0)) = \phi(g(x_0)), \overline{W}_{-1}^{[m]}(g(x_0)) = \overline{\phi}(g(x_0))$, with $W_{k-1}^{[m]}(x_k)$ and $\overline{W}_{k-1}^{[m]}(x_k)$ are the left hand limit of $W_{k-1}^{[m]}(x)$ and $\overline{W}_{k-1}^{[m]}(x)$ as $x \to x_k$ of the segment of $W_\triangle(x)$ and $\overline{W}_\triangle(x)$ defined on $[x_{k-1}, x_k]$. In equations (20) and (21), we use the following m iterations. For $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$ and $j = 1(1)m$

$$\begin{aligned} W_k^{[j]}(x) &= W_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_1(x, W_k^{[j-1]}(x), \overline{W}_k^{[j-1]}(x), W_k^{[j-1]}(g(x)))dx \quad (22)\\ \overline{W}_k^{[j]}(x) &= \overline{W}_{k-1}^{[m]}(x_k) + \int_{x_k}^{x} f_2(x, W_k^{[j-1]}(x), \overline{W}_k^{[j-1]}(x), \overline{W}_k^{[j-1]}(g(x)))dx \\ W_k^{[0]}(x) &= W_{k-1}^{[m]}(x_k) + N_k\,(x - x_k) \\ \overline{W}_k^{[0]}(x) &= \overline{W}_{k-1}^{[m]}(x_k) + \overline{N}_k\,(x - x_k) \\ N_k &= f_1(x_k, W_{k-1}^{[m]}(x_k), \overline{W}_{k-1}^{[m]}(x_k), W_{k-1}^{[m]}(g(x_k))) \\ \overline{N}_k &= f_2(x_k, W_{k-1}^{[m]}(x_k), \overline{W}_{k-1}^{[m]}(x_k), \overline{W}_{k-1}^{[m]}(g(x_k))) \end{aligned}$$

Moreover, we use the following notation.

$$e^*(x) = |S_\triangle(x) - W_\triangle(x)|,\ e_k^* = |S_\triangle(x_k) - W_\triangle(x_k)| \quad (23)$$

$$\overline{e}^*(x) = \left|\overline{S}_\triangle(x) - \overline{W}_\triangle(x)\right|,\ \overline{e}_k^* = \left|\overline{S}_\triangle(x_k) - \overline{W}_\triangle(x_k)\right|$$

Definition 4.1. *For any $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$ and $j = 1(1)m$ we define the operator $T_{kj}^*(x)$ by:*

$$T_{kj}^*(x) = \left|S_k^{[m-j]}(x) - W_k^{[m-j]}(x)\right| + \left|\overline{S}_k^{[m-j]}(x) - \overline{W}_k^{[m-j]}(x)\right| \quad (24)$$

whose norm is defined by $\left\|T_{kj}^*\right\| = \max_{x\in[x_k,x_{k+1}]}\{T_{kj}^*(x)\}$.

Lemma 4.1. *For any $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$ and $j = 1(1)m$, then*

$$\|T_{km}^*\| \leq [1 + h(c_0 + \overline{c}_0)]\,(e_k^* + \overline{e}_k^*) \quad (25)$$

$$\|T_{k1}^*\| \leq c_1(e_k^* + \overline{e}_k^*) \quad (26)$$

where $c_0, \overline{c}_0$ and c_1 are constants defined as in lemma 3.3 **Proof**. *Using (3), (4), (5), (6), (9), (22) and (23) it is easy to prove the above lemma*

Lemma 4.2. *Let $e^*(x), \overline{e}^*(x)$ be defined as in (23), then there exist constants $c_3, \overline{c}_3$ independent of h such that the following inequalities hold:*

$$e^*(x) \leq (1 + hc_3)\, e_k^* + hc_3\overline{e}_k^* \tag{27}$$

$$\overline{e}^*(x) \leq h\overline{c}_3 e_k^* + (1 + h\overline{c}_3)\, e_k^* \tag{28}$$

Proof. *Using (3), (4), (5), (6), (7), (8), (20),(21), (23) and (26) the proof is similar to the proof in lemma 3.4.* On the light of definition 3.2 and matrix notation $E^*(x) = [e^*(x)\ \overline{e}^*(x)]^T$ and $E_k^* = [e_k^*\ \overline{e}_k^*]^T$ then from lemma 4.2, we write

$$E^*(x) \leq (I + hA)\, E_k^* \tag{29}$$

where I and A are matrices defined as in (18) using definition 3.3. The inequality (29) yields:
$\|E^*(x)\| \leq (1 + h\,\|A\|)\,\|E_k^*\|$.

This inequality holds for any $x \in [a, b]$. Setting $x = x_{k+1}$,we get:

$$\left\|E_{k+1}^*\right\| \leq (1 + h\,\|A\|)\,\|E_k^*\|$$

Using lemma 3.2, we obtain:

$$\begin{aligned} \|E^*(x)\| &\leq (1 + h\,\|A\|)^{k+1}\,\|E_0^*\| \\ &\leq \left(1 + \frac{\|A\|\,(b-a)}{n}\right)^n \|E_0^*\| \\ &\leq e^{\|A\|(b-a)}\,\|E_0^*\| \\ &\leq c_{10}\,\|E_0^*\| \end{aligned}$$

where $c_{10} = e^{\|A\|(b-a)}$ is a constant independent of h. Now using definition 3.3, we obtain:

$$\begin{aligned} e^*(x) &\leq c_{10}\,\|E_0^*\| \\ \overline{e}^*(x) &\leq c_{10}\,\|E_0^*\| \end{aligned} \tag{30}$$

To estimate $\left|S_\Delta'(x) - W_\Delta'(x)\right|$ we use (3), (4), (7), (20), (23), (26) and (30), we obtain:

$$\left|S_\Delta'(x) - W_\Delta'(x)\right| \leq c_{11}\,\|E_0^*\|$$

where $c_{11} = 2c_0c_1c_{10}$ is a constant independent of h. Similarly using (5), (6), (8), (21), (23), (26) and (30) we get

$$\left|\overline{S}_\Delta'(x) - \overline{W}_\Delta'(x)\right| \leq c_{12}\,\|E_0^*\|$$

where $c_{12} = 2\overline{c}_0c_1c_{10}$ is a constant independent of h. Thus from above lemma we have arrived to the following theorem

Theorem 4.1. *Let $S_\Delta(x), \overline{S}_\Delta(x)$ given by (7), (8) be the approximate solutions of the problem (1) with the initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$ and let $W_\Delta(x), \overline{W}_\Delta(x)$ given by (20), (21) are the approximate solutions for the same problem with the initial conditions $y^*(x_0) = y_0^*$, $z^*(x_0) = z_0^*$ and $f_1, f_2 \in C\left([a, b] \times R^3\right)$ then the inequalities*

$$\begin{aligned} \left|S_\Delta^{(q)}(x) - W_\Delta^{(q)}(x)\right| &\leq c_{13}\,\|E_0^*\| \\ \left|\overline{S}_\Delta^{(q)}(x) - \overline{W}_\Delta^{(q)}(x)\right| &\leq c_{14}\,\|E_0^*\| \end{aligned}$$

hold for all $x \in [a,b]$ *and* $q = 0,1$ $\|E_0^*\| = \max\{|y_0 - y_0^*|, |z_0 - z_0^*|\}$ *where* c_{13}, c_{14} *are constants independent of* h.

5. Numerical example

The method is tested using the following example in the interval $[0,1]$ with step size h=0.1 where $m = 4$ and $m = 5$. To test the stability of the method we do change in the initial condition by adding 0.00001.

Example 5.1. *Consider the system of delay differential equation*

$$\begin{aligned} y'(x) &= y(x) - z(x) + y(x/2) - e^{x/2} + e^{-x},\ 0 \le x \le 1 \\ z'(x) &= -y(x) - z(x) - z(x/2) + e^{-x/2} + e^{x} \\ y(x) &= e^{x},\ z(x) = e^{-x},\ x \le 0,\ y(0) = 1,\ z(0) = 1. \\ \textit{The exact solution is } y &= e^{x},\ z = e^{-x}. \end{aligned}$$

Table I

x	m	First Apr.	Absolute error	Second Apr.	Abs diff. bet. the num. sol.
0.1	4	$y = 1.105170911$	7.6×10^{-9}	1.105182139	1.1×10^{-5}
0.1	5	$y = 1.105170918$	2.9×10^{-11}	1.105182147	1.1×10^{-5}
0.2	4	$y = 1.221402377$	3.8×10^{-7}	1.221415306	1.3×10^{-5}
0.2	5	$y = 1.221402778$	2×10^{-8}	1.221415714	1.3×10^{-5}
0.3	4	$y = 1.349851046$	7.8×10^{-6}	1.349866173	1.5×10^{-5}
0.3	5	$y = 1.349859939$	1.1×10^{-6}	1.349875098	1.5×10^{-5}
0.4	4	$y = 1.491771687$	5.3×10^{-5}	1.491789545	1.8×10^{-5}
0.4	5	$y = 1.491836988$	1.2×10^{-5}	1.491854936	1.8×10^{-5}
0.5	4	$y = 1.648505578$	2.2×10^{-4}	1.648526745	2.1×10^{-5}
0.5	5	$y = 1.64878964$	6.8×10^{-5}	1.648811008	2.1×10^{-5}
0.6	4	$y = 1.821472326$	6.5×10^{-4}	1.821497444	2.5×10^{-5}
0.6	5	$y = 1.822380782$	2.6×10^{-4}	1.822406275	2.5×10^{-5}
0.7	4	$y = 2.012179165$	1.6×10^{-3}	2.012208952	3×10^{-5}
0.7	5	$y = 2.014537772$	7.9×10^{-4}	2.014568184	3×10^{-5}

Table II

x	m	First Abr.	Absolute error	Second Apr. Sol.	Abs. diff. bet. the num. sol.
0.1	4	$z = 0.9048374116$	6.4×10^{-9}	0.9048445718	7.2×10^{-6}
0.1	5	$z = 0.9048374182$	1.8×10^{-10}	0.9048445788	7.2×10^{-6}
0.2	4	$z = 0.8187301857$	5.7×10^{-7}	0.8187347665	4.6×10^{-6}
0.2	5	$z = 0.8187307828$	3×10^{-8}	0.8187353697	4.6×10^{-6}
0.3	4	$z = 0.7408112275$	7×10^{-6}	0.740813402	2.2×10^{-6}
0.3	5	$z = 0.7408189118$	6.9×10^{-7}	0.740821138	2.2×10^{-6}
0.4	4	$z = 0.6702800604$	4×10^{-5}	0.6702799171	1.4×10^{-7}
0.4	5	$z = 0.6703255091$	5.5×10^{-6}	0.6703254446	6.6×10^{-8}
0.5	4	$z = 0.6063734706$	1.6×10^{-4}	0.6063710188	2.5×10^{-6}
0.5	5	$z = 0.606555279$	2.5×10^{-5}	0.6065530007	2.3×10^{-6}
0.6	4	$z = 0.5483243125$	4.9×10^{-4}	0.5483194985	4.8×10^{-6}
0.6	5	$z = 0.5488891836$	7.8×10^{-5}	0.548884684	4.5×10^{-6}
0.7	4	$z = 0.4953086589$	1.3×10^{-3}	0.4953013317	7.3×10^{-6}
0.7	5	$z = 0.4967716293$	1.9×10^{-4}	0.4967648351	6.8×10^{-6}

6. Conclusions

A new technique using spline function approximation to numerically solve the system of first order delay differential equation is presented. The convergence and stability are discussed. Also, error analysis and stability are investigated showed in table I where m the number of iterations. Tables I and II show improvements of error analysis and stability. Also, from the sixth column of the tables one can see that the algorithm is stable.

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ON SOME UNIVALENCE CONDITIONS IN THE UNIT DISK

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Abstract. In this paper we obtain by the method of subordination chains an univalence criterion for analytic functions defined in the unit disk, which generalizes a criterion due to D.Răducanu.

1. **Introduction**

We denote by U_r the disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq 1$ and by $U = U_1$ the unit disk of the complex plane $\mathbb{C}$.

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk U which satisfy the conditions $f(0) = f'(0) - 1 = 0$.

Let f and F be analytic functions in U. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| \leq 1$, and such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

A function $L(z,t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(\cdot,t)$ is analytic and univalent in U, for all $t \geq 0$, and $L(z,s) \prec L(z,t)$, when $0 \leq s \leq t < \infty$.

Theorem 1. [1] *Let* $r \in (0,1]$ *and* $L : U_r \times [0,\infty) \to \mathbb{C}$ *be an analytic function in the disk* U_r*, for all* $t \geq 0$, $L(z,t) = a_1(t) z + \ldots$. *If*

(i) $L(z,\cdot)$ *is locally absolutely continuous in* $[0,\infty)$*, locally uniform with respect to* U_r*,*

(ii) there exists a function $p(z,t)$ *analytic in* U *for all* $t \in [0,\infty)$ *and measurable in* $[0,\infty)$ *for each* $z \in U$*, such that* $\operatorname{Re} p(z,t) > 0$*, for* $z \in U$*,* $t \in [0,\infty)$*, and*

$$\frac{\partial L(z,t)}{\partial t} = z \frac{\partial L(z,t)}{\partial z} p(z,t),$$

for $z \in U_r$*, and for almost all* $t \in [0,\infty)$*,*

(iii) $a_1(t) \neq 0$*, for* $t \geq 0$*,* $\lim_{t\to\infty} |a_1(t)| = \infty$ *and* $\left\{ \frac{L(z,t)}{a_1(t)} \right\}_{t \geq 0}$ *is a normal family in* U_r*,*

then for each $t \geq 0$*,* $L(z,t)$ *has an analytic and univalent extension in* U*.*

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2. **Main Result**

Theorem 2. *Let $f \in \mathcal{A}$ be an analytic function in U of the form $f(z) = z + a_2 z^2 + ...$ for all $z \in U$, $\alpha \in \mathbb{C}$, $a \in \mathbb{R}$ such that* $\left|\frac{2}{a\alpha} - 1\right| \leq 1$ *and* $\operatorname{Re}(a\alpha - 1) > 0$. *If*

$$\left|\left(\frac{2}{a\alpha} - 1\right)\left[1 - (1 - |z|^a)\frac{zf'(z)}{f(z)}\right] + (1 - |z|^a) z \frac{d}{dz}\left[\log \frac{z^{\left(\frac{2}{a}+1\right)}(f'(z))^{\frac{2}{a}}}{(f(z))^{\frac{2}{a}+1}}\right]\right| \leq |z|^a, \tag{1}$$

for all $z \in U$, then f is univalent in U.

Proof. Let $L : U \times [0, \infty) \to \mathbb{C}$ be the function

$$L(z,t) := \left[f\left(e^{-t}z\right)\right]^{1-\alpha}\left[f\left(e^{-t}z\right) + \frac{(e^{at} - 1)e^{-t}zf'(e^{-t}z)}{1 - (e^{at} - 1)\left(\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} - 1\right)}\right]^{\alpha}. \tag{2}$$

Because $f(z) \neq 0$ for all $z \in U \setminus \{0\}$, the function

$$f_1(z,t) := \frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} = 1 + ...$$

is analytic in U. Hence, the function

$$f_2(z,t) := \frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} - 1 = a_2 e^{-t}z + ...$$

is analytic in U.

It follows from

$$f_3(z,t) := 1 + \frac{(e^{at} - 1) f_1(z,t)}{1 - (e^{at} - 1) f_2(z,t)} = e^{at} + ...$$

that there exists an $r \in (0, 1]$ such that f_3 is analytic in U_r and $f_3(z,t) \neq 0$, for all $z \in U_r$, $t \in [0, \infty)$.

We choose an analytic branch in U_r of the function

$$f_4(z,t) := [f_3(z,t)]^{\alpha} = e^{a\alpha t} + ...$$

We have that

$$L(z,t) = \left[f\left(e^{-t}z\right)\right]^{1-\alpha}\left[f\left(e^{-t}z\right) + \frac{(e^{at} - 1)e^{-t}zf'(e^{-t}z)}{1 - (e^{at} - 1)\left(\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} - 1\right)}\right]^{\alpha} \tag{3}$$

$$= f\left(e^{-t}z\right)[f_4(z,t)]^{\alpha} = e^{(a\alpha - 1)t} + ...$$

is an analytic function in U_r.

From (3) we have $L(z,t) = a_1(t) z + ...$, where

$$a_1(t) = e^{(a\alpha - 1)t},$$

$a_1(t) \neq 0$, for all $t \in [0, \infty)$ and $\lim_{t\to\infty} |a_1(t)| = \lim_{t\to\infty} e^{t \operatorname{Re}(a\alpha - 1)} = \infty$.

From (2), by a simple calculation, we obtain

$$\frac{\partial L(z,t)}{\partial t} =$$

$$= e^{-t}zf'\left(e^{-t}z\right)\left[f\left(e^{-t}z\right)\right]^{-\alpha}\left[f\left(e^{-t}z\right)+\frac{\left(e^{at}-1\right)e^{-t}zf'\left(e^{-t}z\right)}{1-\left(e^{at}-1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}-1\right)}\right]^{\alpha}\cdot$$

$$\cdot\left\{-1+\alpha\frac{a+\left(e^{at}-1\right)\left[-1+\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}+\frac{e^{-t}zf''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)}\right]}{1-\left(e^{at}-1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}-1\right)}\right\} \tag{4}$$

$$\frac{\partial L\left(z,t\right)}{\partial z}=$$

$$= e^{-t}zf'\left(e^{-t}z\right)\left[f\left(e^{-t}z\right)\right]^{-\alpha}\left[f\left(e^{-t}z\right)+\frac{\left(e^{at}-1\right)e^{-t}zf'\left(e^{-t}z\right)}{1-\left(e^{at}-1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}-1\right)}\right]^{\alpha}\cdot$$

$$\cdot\left\{1-\alpha\frac{\left(e^{at}-1\right)\left[-1+\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}+\frac{e^{-t}zf''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)}\right]}{1-\left(e^{at}-1\right)\left(\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}-1\right)}\right\} \tag{5}$$

We observe that $\left|\frac{\partial L\left(z,t\right)}{\partial t}\right|$ is bounded on $[0,T]$, for any $T>0$ fixed and $z\in U_r$. Therefore, the function L is locally absolutely continuous in $[0,\infty)$, locally uniform with respect to U_r. We also have $\left|\frac{L\left(z,t\right)}{a_1\left(t\right)}\right|\le k$, for all $z\in U_r$ and $t\in[0,\infty)$. Then, by Montel's Theorem, $\left\{\frac{L\left(z,t\right)}{a_1\left(t\right)}\right\}_{t\in[0,\infty)}$ is a normal family in U_r.

Let $p:U_r\times[0,\infty)\to\mathbb{C}$ be the function defined by

$$p\left(z,t\right)=\frac{\frac{\partial L\left(z,t\right)}{\partial t}}{z\frac{\partial L\left(z,t\right)}{\partial z}}$$

If the function

$$w\left(z,t\right)=\frac{1-p\left(z,t\right)}{1+p\left(z,t\right)}=\frac{z\frac{\partial L\left(z,t\right)}{\partial z}-\frac{\partial L\left(z,t\right)}{\partial t}}{z\frac{\partial L\left(z,t\right)}{\partial z}+\frac{\partial L\left(z,t\right)}{\partial t}}. \tag{6}$$

is analytic in $U\times[0,\infty)$ and $\left|w\left(z,t\right)\right|<1$, for all $z\in U$ and $t\ge0$, then p has an analytic extension with positive real part in U, for all $t\ge0$.

From (4), (5) and (6) we obtain

$$w\left(z,t\right)=\left(\frac{2}{a\alpha}-1\right)\left[e^{at}-\left(e^{at}-1\right)\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}\right]+$$
$$+\left(e^{at}-1\right)\left[\frac{2}{a}+1-\left(\frac{2}{a}+1\right)\frac{e^{-t}zf'\left(e^{-t}z\right)}{f\left(e^{-t}z\right)}+\frac{2}{a}\frac{e^{-t}zf''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)}\right].$$

We have $|w(z,0)| = \left|\frac{2}{a\alpha} - 1\right| \leq 1$ for all $z \in U$, with a and α in the conditions of the theorem. For $t > 0$, $|w(z,t)| < \max_{|z|=1} |w(z,t)| = \left|w\left(e^{i\theta}, t\right)\right|$, where $\theta \in \mathbb{R}$, so we have to proove that $\left|w\left(e^{i\theta}, t\right)\right| \leq 1$.

Consider $u = e^{-t}e^{i\theta}$, then $u \in U$ and $|u| = e^{-t}$. We have

$$\left|w\left(e^{i\theta}, t\right)\right| = \left|\left(\frac{2}{a\alpha} - 1\right)\left[\frac{1}{|u|^a} - \left(\frac{1}{|u|^a} - 1\right)\frac{uf'(u)}{f(u)}\right] + \left(\frac{1}{|u|^a} - 1\right) \cdot \right.$$
$$\left. \cdot \left[\frac{2}{a} + 1 - \left(\frac{2}{a} + 1\right)\frac{uf'(u)}{f(u)} + \frac{2}{a}\frac{uf''(u)}{f'(u)}\right]\right|$$

and from (1) it follows that $\left|w\left(e^{i\theta}, t\right)\right| \leq 1$.

Then, by Theorem 1, the function L is a subordination chain and $L(z,0) = f(z)$ is univalent in U. □

Theorem 3. *Let $f \in \mathcal{A}$ be a locally univalent function in U, $f(z) = z + a_2z^2 + ...$ for all $z \in U$, $a, \alpha \in \mathbb{C}$ such that $\left|\frac{2}{a\alpha} - 1\right| \leq 1$ and $\operatorname{Re}(a\alpha - 1) > 0$. If*

$$\left|\left(\frac{2}{a\alpha} - 1\right)\left[1 - (1 - |z|^a)\frac{zf'(z)}{f(z)}\right] + (1 - |z|^a)\, z\frac{d}{dz}\left[\log \frac{z^{\left(\frac{2}{a}+1\right)}\left(f'(z)\right)^{\frac{2}{a}}}{\left(f(z)\right)^{\frac{2}{a}+1}}\right]\right| \leq |z|^a,$$

for all $z \in U$, where $\frac{z^{\left(\frac{2}{a}+1\right)}\left(f'(z)\right)^{\frac{2}{a}}}{\left(f(z)\right)^{\frac{2}{a}+1}}$ denotes the analytic branch of the function, then f is univalent in U.

Remark 1. *For $a = 2$ we obtain the univalence condition from* [3]

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ON THE DIRECT PRODUCT OF MULTIALGEBRAS

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Abstract. This paper presents some properties of the direct product of a family of multialgebras of the same type.

1. **Introduction**

The multialgebras can be seen as relational systems which generalize the universal algebras. In the same way as in [4] the Cartesian product of a family of structures is organized as a structure it is possible to organize the Cartesian product of the supporting sets of a family of multialgebras as a multialgebra (see [9]). An important tool in the hyperstructure theory is the fundamental relation of a multialgebra (see [5]). The definition of this relation involves the term functions of the universal algebra of the nonempty sets of the given multialgebra and their images for some one element sets. These images of term functions are also used to obtain some identities that furnishes important classes of multialgebras. We will characterize them when our multialgebra is the direct product of a given family of multialgebras and we will prove that such an identity holds for the direct product if it holds for each member of the product.

We will also see that the definition of the multioperations in the direct product is natural in the way that the resulting multialgebra is the product in a category of multialgebras.

2. **Preliminaries**

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence with $n_\gamma \in \mathbb{N} = \{0, 1, \ldots\}$, where $o(\tau)$ is an ordinal and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and let us consider the algebra of the n-ary terms (of type τ) $\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)})$.

Let A be a nonempty set and $P^*(A)$ the family of nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$. One can admit that the support set A of the multialgebra $\mathfrak{A}$ is empty if there are no nullary multioperations among the multioperations $f_\gamma,\ \gamma < o(\tau)$.

Of course, any universal algebra is a multialgebra (we can identify an one element set with its element).

As in [9] we can see the multialgebra $\mathfrak{A}$ as a relational system $(A, (r_\gamma)_{\gamma<o(\tau)})$ if we consider that, for any $\gamma < o(\tau)$, r_γ is the $n_\gamma + 1$-ary relation defined by

$$(a_0, \ldots, a_{n_\gamma-1}, a_{n_\gamma}) \in r_\gamma \ \Leftrightarrow\ a_{n_\gamma} \in f_\gamma(a_0, \ldots, a_{n_\gamma-1}). \tag{1}$$

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Defining for any $\gamma < o(\tau)$ and for any $A_0, \dots, A_{n_\gamma - 1} \in P^*(A)$

$$f_\gamma(A_0, \dots, A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, \dots, a_{n_\gamma-1}) \mid \ a_i \in A_i, \ \forall i \in \{0, \dots, n_\gamma - 1\}\},$$

we obtain a universal algebra on $P^*(A)$ (see [7]). We denote this algebra by $\mathfrak{P}^*(A)$. As in [4], we can construct, for any $n \in \mathbb{N}$, the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(A))$ of the n-ary term functions on $\mathfrak{P}^*(A)$.

A mapping $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type τ is called homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \dots, a_{n_\gamma-1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \dots, h(a_{n_\gamma-1})). \tag{2}$$

A bijective mapping h is a multialgebra isomorphism if both h and h^{-1} are multialgebra homomorphisms. As it results from [7], the multialgebra isomorphisms can be characterized as being those bijective homomorphisms for which (2) holds with equality.

Proposition 1. *For a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \dots, a_{n-1} \in A$ then*

$$h(p(a_0, \dots, a_{n-1})) \subseteq p(h(a_0), \dots, h(a_{n-1})).$$

Proof. We will use the steps of construction of a term.
Step 1. If $\mathbf{p} = \mathbf{x}_i$ $(i \in \{0, \dots, n-1\})$ then

$$\begin{aligned} h(p(a_0, \dots, a_{n-1})) &= h(e_i^n(a_0, \dots, a_{n-1})) = h(a_i) \\ &= e_i^n(h(a_0), \dots, h(a_{n-1})) \\ &= p(h(a_0), \dots, h(a_{n-1})). \end{aligned}$$

Step 2. Suppose that the statement has been proved for $\mathbf{p}_0, \dots, \mathbf{p}_{n_\gamma-1} \in \mathbf{P}^{(n)}(\tau)$ and that $\mathbf{p} = f_\gamma(\mathbf{p}_0, \dots, \mathbf{p}_{n_\gamma-1})$. Then we have

$$\begin{aligned} h(p(a_0, \dots, a_{n-1})) &= h(f_\gamma(p_0, \dots, p_{n_\gamma-1})(a_0, \dots, a_{n-1})) \\ &= h(f_\gamma(p_0(a_0, \dots, a_{n-1}), \dots, p_{n_\gamma-1}(a_0, \dots, a_{n-1}))) \\ &= h(\bigcup\{f_\gamma(b_0, \dots, b_{n_\gamma-1}) \mid b_i \in p_i(a_0, \dots, a_{n-1}), \ i \in \{0, \dots, n_\gamma - 1\}\}) \\ &= \bigcup\{h(f_\gamma(b_0, \dots, b_{n_\gamma-1})) \mid b_i \in p_i(a_0, \dots, a_{n-1}), \ i \in \{0, \dots, n_\gamma - 1\}\} \\ &\subseteq \bigcup\{f_\gamma(h(b_0), \dots, h(b_{n_\gamma-1}))) \mid b_i \in p_i(a_0, \dots, a_{n-1}), \ i \in \{0, \dots, n_\gamma - 1\}\}. \end{aligned}$$

Since for any $i \in \{0, \dots, n_\gamma - 1\}$, $b_i \in p_i(a_0, \dots, a_{n-1})$ it follows

$$h(b_i) \in h(p_i(a_0, \dots, a_{n-1})) \subseteq p_i(h(a_0), \dots, h(a_{n-1})));$$

so we have,

$$\begin{aligned} h(p(a_0, \dots, a_{n-1})) &\subseteq f_\gamma(p_0(h(a_0), \dots, h(a_{n-1})), \dots, p_{n_\gamma-1}(h(a_0), \dots, h(a_{n-1}))) \\ &= f_\gamma(p_0, \dots, p_{n_\gamma-1})(h(a_0), \dots, h(a_{n-1})) \\ &= p(h(a_0), \dots, h(a_{n-1})) \end{aligned}$$

which finishes the proof. □

Remark 1. If for any $\gamma < o(\tau)$ and for all $a_0, \dots, a_{n_\gamma - 1} \in A$ we have equality in (2), then

$$h(p(a_0, \dots, a_{n-1})) = p(h(a_0), \dots, h(a_{n-1})).$$

The proof can be done as before, but it it also results from some properties that can be established for the universal algebra $\mathfrak{P}^*(A)$ (see [1]).

We can easily construct the category of the multialgebras of the same type τ where the morphisms are considered to be the homomorphisms and the composition of two morphisms is the usual mapping composition and we will denote it by $\mathbf{Malg}(\tau)$.

Let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. Using the model offered by [4] and looking at the definitions of the hyperstructures from [2] and also at the generalizations presented in [10], named H_v-structures, we can consider that the n-ary (strong) identity

$$\mathbf{q} = \mathbf{r}$$

is said to be satisfied on a multialgebra $\mathfrak{A}$ if

$$q(a_0, \dots, a_{n-1}) = r(a_0, \dots, a_{n-1})$$

for all $a_0, \dots, a_{n-1} \in A$, where q and r are the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ respectively on $\mathfrak{P}^*(A)$. We can also consider that a weak identity (the notation is intended to be as suggestive as possible)

$$\mathbf{q} \cap \mathbf{r} \neq \emptyset$$

is said to be satisfied on a multialgebra $\mathfrak{A}$ if

$$q(a_0, \dots, a_{n-1}) \cap r(a_0, \dots, a_{n-1}) \neq \emptyset$$

for all $a_0, \dots, a_{n-1} \in A$, where q and r have the same signification as before. Many important particular multialgebras are defined as being those multialgebras which satisfy a given set of identities.

3. Direct products of multialgebras

Given a family of relational systems of the same type $\tau = (n_\gamma + 1)_{\gamma < o(\tau)}$, $(\mathfrak{A}_i = (A_i, (r_\gamma)_{\gamma < o(\tau)}) \mid i \in I)$, in [4] is defined the direct product of this family as being the relational system obtained on the Cartesian product $\prod_{i \in I} A_i$ considering that for $(a_i^0)_{i \in I}, \dots, (a_i^{n_\gamma})_{i \in I} \in \prod_{i \in I} A_i$,

$$((a_i^0)_{i \in I}, \dots, (a_i^{n_\gamma})_{i \in I}) \in r_\gamma \Leftrightarrow (a_i^0, \dots, a_i^{n_\gamma}) \in r_\gamma, \ \forall i \in I.$$

If we consider a family $\{\mathfrak{A}_i\}_{i \in I}$ of multialgebras of type τ and the relational systems defined by (1), the relational system that results on the Cartesian product $\prod_{i \in I} A_i$ from the above considerations is a multialgebra of type τ with the multioperations:

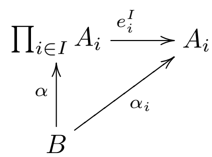
$$f_\gamma((a_i^0)_{i \in I}, \dots, (a_i^{n_\gamma - 1})_{i \in I}) = \prod_{i \in I} f_\gamma(a_i^0, \dots, a_i^{n_\gamma - 1}), \tag{3}$$

for any $\gamma < o(\tau)$. This multialgebra is called the direct product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$. We observe that the canonical projections of the product, e_i^I, $i \in I$, are multialgebra homomorphisms.

Proposition 2. *The multialgebra $\prod_{i \in I} \mathfrak{A}_i$ constructed this way, together with the canonical projections, is the product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$ in the category* $\mathbf{Malg}(\tau)$.

Proof. For any multialgebra $\mathfrak{B}$ and for any family of multialgebra homomorphisms $(\alpha_i : B \to A_i \mid i \in I)$ there is only one homomorphism $\alpha\colon B \to \prod_{i\in I} A_i$ such that $\alpha_i = e_i^I \circ \alpha$ for any $i \in I$.

Indeed, there exists only one mapping α such that the diagram



is commutative. This mapping is defined by $\alpha(b) = (\alpha_i(b))_{i\in I}$. Now, all we have to do is to verify that α is a multialgebra homomorphism. If we consider $\gamma < o(\tau)$ and $b_0, \ldots, b_{n_\gamma-1} \in B$ then

$$\begin{aligned}\alpha(f_\gamma(b_0,\ldots,b_{n_\gamma-1})) &= \{\alpha(b) \mid b \in f_\gamma(b_0,\ldots,b_{n_\gamma-1})\}\\ &= \{(\alpha_i(b))_{i\in I} \mid b \in f_\gamma(b_0,\ldots,b_{n_\gamma-1})\}.\end{aligned}$$

From $b \in f_\gamma(b_0,\ldots,b_{n_\gamma-1})$ it follows that for any $i \in I$,

$$\alpha_i(b) \in \alpha_i(f_\gamma(b_0,\ldots,b_{n_\gamma-1})) \subseteq f_\gamma(\alpha_i(b_0),\ldots,\alpha_i(b_{n_\gamma-1})),$$

so we have

$$\begin{aligned}\alpha(f_\gamma(b_0,\ldots,b_{n_\gamma-1})) &\subseteq \prod_{i\in I} f_\gamma(\alpha_i(b_0),\ldots,\alpha_i(b_{n_\gamma-1}))\\ &= f_\gamma((\alpha_i(b_0))_{i\in I},\ldots,(\alpha_i(b_{n_\gamma-1}))_{i\in I})\\ &= f_\gamma(\alpha(b_0),\ldots,\alpha(b_{n_\gamma-1}))\end{aligned}$$

which finishes the proof. □

Lemma 1. *For every* $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ *and* $(a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I} \in \prod_{i\in I} A_i$, *we have*

$$p((a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I}) = \prod_{i\in I} p(a_i^0,\ldots,a_i^{n-1}). \tag{4}$$

Proof. We will use again the steps of construction of a term.
Step 1. If $\mathbf{p} = \mathbf{x}_j$ $(j \in \{0,\ldots,n-1\})$ then

$$\begin{aligned}&p((a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I}) = e_j^n((a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I}) = (a_i^j)_{i\in I}\\ &=(e_j^n(a_i^0,\ldots,a_i^{n-1}))_{i\in I} = \prod_{i\in I} e_j^n(a_i^0,\ldots,a_i^{n-1}) = \prod_{i\in I} p(a_i^0,\ldots,a_i^{n-1}).\end{aligned}$$

Step 2. Suppose that the statement has been proved for $\mathbf{p}_0,\ldots,\mathbf{p}_{n_\gamma-1} \in \mathbf{P}^{(n)}(\tau)$ and that $\mathbf{p} = f_\gamma(\mathbf{p}_0,\ldots,\mathbf{p}_{n_\gamma-1})$. Then we have

$$\begin{aligned}p((a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I}) &= f_\gamma(p_0,\ldots,p_{n_\gamma-1})((a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I})\\ &= f_\gamma(p_0((a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I}),\ldots,p_{n_\gamma-1}((a_i^0)_{i\in I},\ldots,(a_i^{n-1})_{i\in I}))\\ &= f_\gamma(\prod_{i\in I} p_0(a_i^0,\ldots,a_i^{n-1}),\ldots,\prod_{i\in I} p_{n_\gamma-1}(a_i^0,\ldots,a_i^{n-1}))\end{aligned}$$

But

$$(x_i)_{i\in I} \in f_\gamma(\prod_{i\in I} p_0(a_i^0,\ldots,a_i^{n-1}),\ldots,\prod_{i\in I} p_{n_\gamma-1}(a_i^0,\ldots,a_i^{n-1}))$$

if and only if for each $j \in \{0, \dots, n_\gamma - 1\}$ and $i \in I$, there exists some $b_i^j \in p_j(a_i^0, \dots, a_i^{n-1})$ such that

$$(x_i)_{i\in I} \in f_\gamma((b_i^0)_{i\in I}, \dots, (b_i^{n_\gamma-1})_{i\in I}) = \prod_{i\in I} f_\gamma(b_i^0, \dots, b_i^{n_\gamma-1}),$$

thus

$$\begin{aligned} p((a_i^0)_{i\in I}, \dots, (a_i^{n-1})_{i\in I}) &= \prod_{i\in I} f_\gamma(p_0(a_i^0, \dots, a_i^{n-1}), \dots, p_{n_\gamma-1}(a_i^0, \dots, a_i^{n-1})) \\ &= \prod_{i\in I} f_\gamma(p_0, \dots, p_{n_\gamma-1})(a_i^0, \dots, a_i^{n-1}) \\ &= \prod_{i\in I} p(a_i^0, \dots, a_i^{n-1}) \end{aligned}$$

which finishes the proof of the lemma. □

Proposition 3. *If* $(\mathfrak{A}_i \mid i \in I)$ *is a family of multialgebras such that* $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ *is satisfied on each multialgebra* $\mathfrak{A}_i$ *then* $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ *is also satisfied on the multialgebra* $\prod_{i\in I} \mathfrak{A}_i$.

Proof. Let us consider that $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on each multialgebra $\mathfrak{A}_i$, where $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. This means that for all $i \in I$ and for any $a_i^0, \dots, a_i^{n-1} \in A_i$ we have $q(a_i^0, \dots, a_i^{n-1}) \cap r(a_i^0, \dots, a_i^{n-1}) \neq \emptyset$. Using Lemma 1, it follows that

$$\begin{aligned} q((a_i^0)_{i\in I}, \dots, (a_i^{n-1})_{i\in I}) &\cap r((a_i^0)_{i\in I}, \dots, (a_i^{n-1})_{i\in I}) \\ &= \prod_{i\in I} q(a_i^0, \dots, a_i^{n-1}) \cap \prod_{i\in I} r(a_i^0, \dots, a_i^{n-1}) \\ &= \prod_{i\in I} (q(a_i^0, \dots, a_i^{n-1}) \cap r(a_i^0, \dots, a_i^{n-1})) \neq \emptyset \end{aligned}$$

and the statement is proved. □

Proposition 4. *If* $(\mathfrak{A}_i \mid i \in I)$ *is a family of multialgebras such that* $\mathbf{q} = \mathbf{r}$ *is satisfied on each multialgebra* $\mathfrak{A}_i$ *then* $\mathbf{q} = \mathbf{r}$ *is also satisfied on the multialgebra* $\prod_{i\in I} \mathfrak{A}_i$.

Proof. Consider that $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. For all $i \in I$ and for any $a_i^0, \dots, a_i^{n-1} \in A_i$ we have $q(a_i^0, \dots, a_i^{n-1}) = r(a_i^0, \dots, a_i^{n-1})$. Using Lemma 1, it follows that

$$\begin{aligned} q((a_i^0)_{i\in I}, \dots, (a_i^{n-1})_{i\in I}) &= \prod_{i\in I} q(a_i^0, \dots, a_i^{n-1}) = \prod_{i\in I} r(a_i^0, \dots, a_i^{n-1}) \\ &= r((a_i^0)_{i\in I}, \dots, (a_i^{n-1})_{i\in I}) \end{aligned}$$

and the statement is proved. □

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A FUNCTIONAL CHARACTERIZATION OF THE SYMMETRIC-DIFFERENCE OPERATION

VASILE POP

Abstract. Let M be a set and $\mathcal{P}(M)$ the family of the subsets of M. On $\mathcal{P}(M)$ we consider the set of all binary operations $O(\mathcal{P}(M))$ and on $O(\mathcal{P}(M))$ we define a relation that we call the subordination relation. Then we show that the only group operation on $\mathcal{P}(M)$, subordinate to the union, is the symmetric difference.

1. Introduction

Let M be an arbitrary set and $\mathcal{P}(M) = \{A|\ A \subset M\}$, the family of the subsets of M. On the set of the binary operations on $\mathcal{P}(M)$ we define the following subordination relation:

If $f, g : \mathcal{P}(M)\times\mathcal{P}(M) \to \mathcal{P}(M)$ are binary operation on $\mathcal{P}(M)$, we say that f *is subordinate to* g or that g *subordinates* f, if $f(X,Y) \subset g(X,Y)$ for all $X, Y \in \mathcal{P}(M)$ and we denote $f \le g$.

Our purpose is to determine those operations that confers to $\mathcal{P}(M)$ a group structure and which subordinate the intersection or are subordinated to the union.

2. Main results

For M and $\mathcal{P}(M)$ mentioned above, we denote $O(\mathcal{P}(M))$ the set of all binary operation on the set $\mathcal{P}(M)$:

$$O(\mathcal{P}(M)) = \{f : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathcal{P}(M)|\ f \text{ is a function }\}.$$

Remark 1. a) Among the usual operations, let us mention:
– the operation $\emptyset$: $f(X,Y) = \emptyset$, for all $X, Y \in \mathcal{P}(M)$;
– the operation M: $f(X,Y) = M$, for all $X, Y \in \mathcal{P}(M)$;
– the intersection ($\cap$): $f(X,Y) = X \cap Y$, for all $X, Y \in \mathcal{P}(M)$;
– the union ($\cup$): $f(X,Y) = X \cup Y$, for all $X, Y \in \mathcal{P}(M)$;
– the difference ($\setminus$): $f(X,Y) = X \setminus Y$, for all $X, Y \in \mathcal{P}(M)$;
– the symmetric difference (Δ):

$$f(X,Y) = X\Delta Y = (X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X)$$

b) The following subordination relations hold?

$$\emptyset \le \cap \le \cup \le M.$$

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c) For $f, g \in O(\mathcal{P}(M))$ given operators, the operations $\cap, \cup$ and Δ are defined by:

$$\begin{aligned}(f \cap g)(X,Y) &= f(X,Y) \cap g(X,Y),\\ (f \cup g)(X,Y) &= f(X,Y) \cup g(X,Y),\\ (f \Delta g)(X,Y) &= f(X,Y) \Delta g(X,Y),\end{aligned}$$

for all $X, Y \in \mathcal{P}(M)$.

Proposition 1. *The subordinate relation is an order relation, which determines on $O(\mathcal{P}(M))$ a lattice, where:*

$$\inf\{f,g\} = f \cap g \text{ and } \sup\{f,g\} = f \cup g, \text{ for } f, g \in O(\mathcal{P}(M)).$$

Proof. Let $i, f, g, u \in O(\mathcal{P}(M))$.

If $i \leq f$ and $i \leq g$, then $i(X,Y) \subset f(X,Y)$ and $i(X,Y) \subset g(X,Y)$. So $i(X,Y) \subset (f \cap g)(X,Y)$. The maximal operation i, which verifies this inclusion is $i = f \cap g$.

If $f \leq u$ and $g \leq u$, then $f(X,Y) \subset u(X,Y)$ and $g(X,Y) \subset u(X,Y)$. So $(f \cup g)(X,Y) \subset u(X,Y)$. The minimal operation u, which verifies this inclusion is $u = f \cup g$. □

It is known that the operation Δ determines on $\mathcal{P}(M)$ a group structure and $\Delta \leq U$. We will show that, if M is a finite set, then this property characterizes the symmetric difference, that is Δ is the unique group operation on $\mathcal{P}(M)$, subordinated to the union.

Theorem 1. *If M is a finite set, then the symmetric difference Δ is the unique binary operation on $\mathcal{P}(M)$ which is subordinated to the union and which determines on $\mathcal{P}(M)$ a group structure.*

Proof. a) If we denote by "$*$" an operation which satisfies the requirements of the theorem, from $\emptyset * \emptyset \subset \emptyset$ we have $\emptyset * \emptyset = \emptyset$. So the only element that could be the unit element is $\emptyset$.

b) We show by induction after $|X|$ that $X * X = \emptyset$ for all $X \in \mathcal{P}(M)$.

For $|X| = 0$ we have $x = \emptyset$ and $\emptyset * \emptyset = \emptyset$.

We suppose $X * X = \emptyset$ for all $X \in \mathcal{P}(M)$ with $|X| \leq n$ and let $A \in \mathcal{P}(M)$ with $|A| = n + 1$.

If $X \subset A$, then $X * A \subset X \cup A = A$, so the translation restricted to $\mathcal{P}(M)$ has values in $\mathcal{P}(M)$. Being an injection, it is a surjection, since $\mathcal{P}(A)$ is finite. Thus, there exists the set $B \subset A$ such that $t_A(B) = A * B = \emptyset$. If we suppose that $B \neq A$, then $|B| \leq n$ and from the induction hypothesis we have $B * B = \emptyset$. From $A * B = B * B$ we have $A = B$, which is a contradiction that shows that $A * A = \emptyset$.

c) Using an induction on $|B| = k$ we show that if $A \cap B = \emptyset$, then $A * B = A \cup B$.

For $k = 0$, $A * \emptyset = A \cup \emptyset = A$ is immediately verified since $\emptyset$ is the unit element.

For $k = 1$, $B = \{x\}$, $x \notin A$. If $A * \{x\} = C \subset A \cup \{x\}$ then $C * \{x\} \subset C \cup \{x\}$, that is: $A * (\{x\} * \{x\}) \subset C \cup \{x\}$ or $A * \emptyset \subset C \cup \{x\}$ or $A \subset C \cup \{x\}$. Since $x \notin A$ it follows that $A \subset C$ and $C \subset A \cup \{x\}$. So, either $C = A$ or $C = A \cup \{x\}$. But $C = A * \{x\} \neq A$, so we finally obtain $C = A \cup \{x\}$.

For $k = n + 1$, let $B = B_n \cup \{y\}$ with $|B_n| = n$. $B_n \cap A = \emptyset$ and $y \notin A$, $y \notin B_n$.

We have

$$A * B = A * (B_n \cup \{y\}) = A * (B_n * \{y\}) = (A * B_n) * \{y\} =$$
$$= (A * B_n) \cup \{y\} = (A \cup B_n) \cup \{y\} = A \cup (B_n \cup \{y\}) = A \cup B$$

d) We show that $X * Y = X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. Let $X \cap Y = Z$, $X \setminus Z = U$, $Y \setminus Z = V$ where U, V, Z are disjoint.

We have

$$X * Y = (Z \cup U) * (Z \cup V) \overset{c)}{=} (U * Z) * (Z * V) =$$
$$= U * (Z * Z) * V \overset{b)}{=} U * \emptyset * V \overset{a)}{=} U * V \overset{c)}{=} U \cup V$$
$$= (X \setminus Z) \cup (Y \setminus Z) = (X \setminus Y) \cup (Y \setminus X) = X \Delta Y. \quad \square$$

Theorem 2. *If M is a finite set, then the unique operation on $\mathcal{P}(M)$ which subordinates the intersection and which determines on $\mathcal{P}(M)$ a group structure is the operation $\overline{\Delta}$ defined by:*

$$f(X, Y) = X \overline{\Delta} Y = \overline{X \Delta Y} = M \setminus (X \Delta Y), \quad X, Y \in \mathcal{P}(M).$$

Proof. If we denote by "$\top$" such an operation, then $X \cap Y \subset X \top Y \Leftrightarrow \overline{X \top Y} \subset \overline{X} \cup \overline{Y} \Leftrightarrow \overline{\overline{\overline{X} \top \overline{Y}}} \subset X \cup Y$.

Let us denote $\overline{\overline{X} \top \overline{Y}} = X * Y$ and show that $(\mathcal{P}(M), *)$ is a group.

The function $c : \mathcal{P}(M) \to \mathcal{P}(M)$, $c(X) = \overline{X} = M \setminus X$ is a bijection and the structure induced from the group $(\mathcal{P}(M), \top)$ is $X * Y = c^{-1}(c(X) \top c(Y)) = \overline{\overline{\overline{X} \top \overline{Y}}}$.

Using now the previous theorem and the relation $X * Y \subset X \cup Y$ we deduce that $* = \Delta$, so $\overline{\overline{\overline{X} \top \overline{Y}}} = X \Delta Y$ or, equivalent, $X \top Y = \overline{\overline{X} \Delta \overline{Y}} = \overline{X \Delta Y}$. $\square$

Remark 2. The proofs of the theorems have essentially used the fact that the set M is finite. It is an open problem whether the results take place for infinite sets.

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A NEW SUBCLASS OF CONVEX FUNCTIONS

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Abstract. In this paper we have studied a class of univalent functions defined in the unit disc $U = \{z \in C : |z| < 1\}$.

1. Introduction

Let **A** be the class of the analytic functions in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$, which satisfy the conditions $f(0) = 0$ and $f'(0) = 1$.
We denote by K the class of univalent functions for which we have: $K \subset A$ and for every function $f \in K$ the domain $f(U)$ is a convex set in the complex plane.
It is well known that

$$K = \{f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)} > 0\right) \; for \; all \; z \in U\}.$$

We introduce the notation

$$\mathcal{K}_\lambda = \left\{ f \in A : (\exists)\lambda \in U, \left| \lambda|z|^2 + \left(1 - |z|^2\right) \frac{zf''(z)}{f'(z)} \right| < 1, (\forall) z \in U \right\}.$$

The condition which defines the class $\mathcal{K}_\lambda$ is a univalence criterion,whose proof and generalisation can be found in [4],[5].

2. Preliminaries

Definition 1. *Let f and g be two analytic functions in U. The function f is subordinate to g if there exists an analytic function denoted by Φ with the properties: $|\Phi(z)| < 1$, $z \in U$, $\Phi(0) = 0$ and $f(z) = g(\Phi(z)), z \in U$. The fact that f is subordinate to g will be denoted by $f \prec g$.*

Observation 1. If f and g are two analytic functions in U, g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$ then f is subordinate to g.

To prove our main result we will need the following lemmas.

Lemma A. *If the function f is analytic in U and $z_0 \in U$, then $z_0 f'(z_0)$ is the outward normal to the boundary of the domain $f(U_{r_0})$, where $r_0 = |z_0|$ and $U_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.*

Lemma B. (Miller and Mocanu) [2] *Let q be analytic and univalent in U. $q(0) = a$ and let $p(z) = a + p_n z^n + \ldots$ be analytic in U with $p(z) \neq a, n \geq 1$. If $p \not\prec q$ then there exists points $r_0 e^{i\Theta_0} = z_0 \in U$ and $\zeta_0 \in \partial U$ and $m \geq n$ for which*

(i) $p(U_{r_0}) \subset q(U)$

(ii) $p(z_0) = q(\zeta_0)$

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(iii) $z_0p'(z_0) = m\zeta_0 q'(\zeta_0)$.
If $q(z) = \frac{a+\overline{a}z}{1-z}$ *with* $Re(a) > 0$ *then* $q(U) = \{w \in \mathbb{C} : Re\ w > 0\}$ *and Lemma B becomes:*

Lemma B'. *Let p be analytic in U,* $p(z) = a + p_n z^n + \dots, p \not\equiv a, Re\ a > 0, n \geq 1$.
If $Re\ p(z) \not> 0, z \in U$ *then there exists* $z_0 \in U$, $x, y \in \mathbb{R}$ *for which*
(i) $p(z_0) = ix$
(ii) $z_0 p'(z_0) = y \leq -\frac{1}{2}[x^2+1]$

3. Main result

We observe that if $f_\delta(z) = \dfrac{e^{(1+\delta)z}-1}{1+\delta}$ then $1 + \dfrac{zf''_\delta(z)}{f'_\delta(z)} = 1 + (1+\delta)z$ and so f_δ is not a convex function in U if $\delta > 0$.

Theorem 1. *If* $\lambda \in U$ *then* $\mathcal{K}_\lambda \nsubseteq K$.

Proof. We will prove that for $\lambda \in U$ exists a $\delta > 0$ for which $f_\delta \in \mathcal{K}_\lambda$.
If $\lambda \in U$ then $|\lambda| = 1 - \varepsilon, \epsilon \in (0,1)$ and from the triangle inequality results that:

$$\left|\lambda|z|^2 + \left(1-|z|^2\right)\frac{zf''_\delta(z)}{f'_\delta(z)}\right| \leq (1-\varepsilon)|z|^2 + \left(1-|z|^2\right)(1+\delta)|z|\,,\ z \in U. \qquad (1)$$

Let $r = |z|$ and $g(r) = (1-\varepsilon)r^2 + (1-r^2)(1+\delta)r$. After calculations we get that $g(r) \leq (1+\delta)r(\delta)$ where $r(\delta)$ is the positive root of the equation $g'(r) = 0$. .
To show that there exists $\delta \in (0, +\infty)$ for which

$$\left|\lambda|z|^2 + (1-|z|^2)\frac{zf''_\delta(z)}{f'_\delta(z)}\right| < 1 \text{ for all } z \in U \qquad (2)$$

it is enough to show the existence of δ with the property $(1+\delta)r(\delta) < 1$. The last assertion holds because:

$$\lim_{\delta \to o}(1+\delta)r(\delta) = \frac{|\lambda| + \sqrt{|\lambda|^2+3}}{3} < 1. \qquad (3)$$

This completes the proof of the theorem .

Theorem 2. $\mathcal{K}_{-1} \subset K$.

Proof. 1. We will use Lemma B' to prove our assertion.
If we put $\lambda = -1$ and $p(z) = 1 + \dfrac{zf''(z)}{f'(z)}$ the inequality

$$\left|\lambda|z|^2 + \left(1-|z|^2\right)\frac{zf''(z)}{f'(z)}\right| < 1, z \in U$$

may be rewritten in the following form

$$\left|-1 + \left(1-|z|^2\right)p(z)\right| < 1, z \in U. \qquad (4)$$

If $Re\ p(z) \not> 0, z \in U$ then according to Lemma B' there are $z_0 \in U$ and $x, y \in \mathbb{R}$ so that
(i) $p(z_0) = ix$

(*ii*) $z_0 p'(z_0) = y \le \frac{-1}{2}\left(x^2+1\right)$
and we get that $\left|-1+\left(1-|z_0|^2\right)p(z_0)\right| = \left|-1+\left(1-|z_0|^2\right)ix\right| \ge 1$ which inequality is in contradiction with (4).

Theorem 3. *Let γ be a positive real number.The integral operation I defined by the equality*

$$I(f)(z) = F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \tag{5}$$

satisfies the relation $I(\mathcal{K}_{-1}) \subset \mathcal{K}_{-1}$.

Proof. Let $f \in \mathcal{K}_{-1}$.We must show that the inequality

$$\left|-|z|^2+\left(1-|z|^2\right)\frac{zf''(z)}{f'(z)}\right| < 1, z \in U$$

implies that

$$\left|-|z|^2+\left(1-|z|^2\right)\frac{zF''(z)}{F'(z)}\right| < 1, z \in U.$$

Let $q(z) = \dfrac{zF''(z)}{F'(z)}$. We define the following set

$$B = \left\{r \in [0,+\infty) : \left|-|z|^2+\left(1-|z|^2\right)q(z)\right| < 1, (\forall)z \in \overline{U}_r\right\}$$

where $\overline{U}_r = \{z \in \mathbb{C} : |z| \le r\}$. The set B isn't empty because $0 \in B$. Let $r_0 = \sup B$. For a fixed z the equality $\left|-|z|^2+\left(1-|z|^2\right)(x+iy)\right| = 1$ defines a circle in the $x0y$ system of coordinates. Let's denote this circle by $\mathcal{C}_z$. Because for all $z \in U$ the center $O_1\left(\frac{|z|^2}{1-|z|^2}, 0\right)$ of the circle $\mathcal{C}_z$ is on the real axis $0x$ and the point p(-1,0) is on the circle $\mathcal{C}_z$,we conclude that if $|z_1| < |z_2|$,then every point of the circle $\mathcal{C}_{z_1}$ except p are inside the circle $\mathcal{C}_{z_2}$.The above assertion shows that if $r_0 < 1$, then there exists $z_0 \in U$, $|z_0| = r_0$ so that $\left|-|z_0|^2+\left(1-|z_0|^2\right)q(z_0)\right| = 1$ and the domain $q(U_{r_0})$ is inside the circle $\mathcal{C}_{z_0}$. The border of the domain $q(U_r)$ is tangent to the circle $\mathcal{C}_{z_0}$ in the point $q(z_0)$ which implies that the outward normal $z_0q'(z_0)$ to the border of $q(U_{r_0})$ is outward normal to the circle $\mathcal{C}_{z_0}$. From (5) we get that :

$$q(z) + \frac{zq'(z)}{1+\gamma+q(z)} = \frac{zf''(z)}{f'(z)}, z \in U \tag{6}$$

We will prove that $Re\,\dfrac{1}{1+\gamma+q(z)} > 0, z \in U$.
If $Re\left(1+\gamma+q(z)\right) \not> 0$ for all $z \in U$ then we can apply Lemma B' and we get that there are $z_0 \in U$ and $x, y \in \mathbb{R}$ with the properties

(*a*) $Re\left(1+\gamma+q(z_0)\right) = ix$
(*b*) $z_0q'(z_0) = y \le -\frac{1}{2}(x^2+1)$.

Replacing in (6) results that $Re\left(1+\gamma+q(z_0)+\dfrac{z_0q'(z_0)}{1+\gamma+q(z_0)}\right) = Re\left(ix+\dfrac{y}{ix}\right) = 0$ on the other hand from (6) we get that:

$$Re\left(1+\gamma+q(z_0)+\frac{z_0q'(z_0)}{1+\gamma+q(z_0)}\right) = Re\left(1+\gamma+\frac{z_0f''(z_0)}{f'(z_0)}\right) > 0$$

The contradiction shows that $Re\,(1+\gamma+q(z)) > 0$ for all $z \in U$. Let's return now to the proof of the theorem. The inequality $Re\,\dfrac{1}{1+\gamma+q(z_0)} > 0$ is equivalent to :

$$\left| arg\,\frac{1}{1+\gamma+q(z_0)} \right| < \frac{\pi}{2} \qquad (7)$$

Using (7) and the fact that $z_0q'(z_0)$ is the outward normal to the circle $\mathcal{C}_{z_0}$, we obtain that $q(z_0) + \dfrac{z_0q'(z_0)}{1+\gamma+q(z_0)} \notin Int\,\mathcal{C}_{z_0}$ or equivalently

$$\left| -|z_0|^2 + \left(1-|z_0|^2\right)\left(q(z_0) + \frac{z_0q'(z_0)}{1+\gamma+q(z_0)}\right)\right| \geq 1$$

which implies that $\left| -|z_0|^2 + \left(1-|z_0|^2\right)\dfrac{z_0f''(z_0)}{f'(z_0)}\right| \geq 1, z \in U$ in contradiction with the condition $f \in \mathcal{K}_{-1}$.

Conjecture. If $|\lambda| \leq 1$ and $\mathcal{K}_\lambda \subset K$ then $\lambda = -1$.

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ÉVOLUTION DES COURS GOUVERNÉE PAR UN PROCESSUS DE TYPE ARIMA FRACTIONNAIRE

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Abstract. A suitable model for the evolution of options in a financial market is proposed. It exhibits a long term dependence of options that is not expressed in the usual Black-Scholes model. A fractional processes of ARIMA type is chosen to model the perturbation of the evolution. A solution for a modified model is found.

1. Introduction

Il est bien connu que l'évolution du cours de l'action est habituellement décrite par l'équation de Black et Scholes:

$$dS_t = S_t(\mu dt + \nu dW_t), 0 \leq t \leq T \tag{1.1}$$

où S_t est le prix de l'action à l'instant t, μ et ν sont deux constantes, W_t est un mouvement brownien standard et T est la date d'échéance de l'option à étudier. Pour simplifier, on va se restreindre au cas univarié.

Dans ce modèle (1.1), le rapport relatif $\frac{dS_t}{S_t}$ entre le changement du prix de l'action et lui-même est supposé non seulement proportionnel à la durée du temps de ce changement mais aussi bruit é par le bruit blanc markovien dW_t. Et par conséquent, la solution S_t de (1.1) est un processus de Markov qui ne présente qu'une dépendance très faible et aussi qu'une sorte d'indépendance avec le passé lointain. Mais il est évident que, pour la plupart des processus économiques, l'hypothèse d'absence de mémoire n'est pas tenable. Le prix de l'action S_t à l'instant t peut être influencé par son comportement longtemps avant, ce qui est incompatible avec la propriété de Markov. Et le risque de l'action doit être représenté par un modèle comportant une dépendance. C'est pourquoi nous proposons ici un modèle des cours perturbé par un processus asymptotique à une série temporelle de type ARIMA qui exprime une évolution de longue mémoire.

Considérons d'abord un bruit modélisé par un processus ARIMA Y défini par

$$Y_s = (1 - L)^{-d}\Phi(L)^{-1}\Theta(L)\varepsilon_s, \ s = 0, 1, 2, ..., [T] \tag{1.2}$$

où (ε_s) est un bruit blanc qui est une suite de variables aléatoires de moyennes nulles, non corrélées et de même variance σ, L est l'opérateur de retard, Φ et Θ sont des polynômes de retard ayant leurs racines à l'extérieur du disque unité, d est l'ordre fractionnaire de différentiation.

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On sait que la représentation moyenne mobile du processus peut s'écrire

$$Y = \sum_{k=1}^{s} h_{s-k}^{(d)} \varepsilon_k \tag{1.3}$$

où les coefficients moyennes mobiles peuvent être approchés par

$$h_s^{(d)} \approx \frac{\Theta(1)}{\Phi(1)\Gamma(1)} s^{d-1}$$

avec s grand et où Γ désigne la fonction gamma.

On considère maintenant un processus Z défini par:

$$Z_r = \frac{1}{T^{d-\frac{1}{2}}} Y_{[Tr]}, \ 0 \le r \le 1 \tag{1.4}$$

où $[x]$ est la partie entière de x.

Nous renvoyons les lecteurs aux résultats présentés dans [3] où l'on peut trouver que, par un calcul et par l'application du théorème de Donsker on obtient l'approximation suivante:

$$\begin{aligned} Z_r &= \frac{1}{T^{d-\frac{1}{2}}} \sum_{k=1}^{[Tr]} h_{[Tr]-k}^{(d)} . \varepsilon_k \\ &\approx \frac{\sigma\Theta(1)}{\Phi(1)\Gamma(d)} \sum_{k=1}^{[Tr]} \Big(r - \frac{k}{T}\Big)^{d-1} \Big[W\Big(\frac{k}{T}\Big) - W\Big(\frac{k-1}{T}\Big)\Big] \end{aligned}$$

où W est un mouvement brownien. On peut écrire aussi:

$$Z_r \ \approx \ \frac{\sigma\Theta(1)}{\Phi(1)\Gamma(d)} \int_0^r (r-s)^{d-1} dW_s, \ 0 \le r \le 1. \tag{1.5}$$

L'intégrale stochastique dans (1.5) joue un rôle essentiel dans la description de la présence d'une dépendance à long terme d'un prix d'action dans l'évolution du cours.

On peut revenir au temps t avec $0 \le t \le T$ par un changement de variable $s = \frac{u}{T}$ en remplaçant $\frac{r}{T}$ par t et en notant que le mouvement brownien W_s est un processus auto-similaire, c'est-à-dire, $W_s \equiv W_{\frac{u}{T}} \sim \frac{1}{T} W_u$ (identique en loi). On a alors:

$$Z_r \equiv Z_{tT} = \frac{\sigma\Theta(1)}{\Phi(1)\Gamma(d)} . \frac{1}{T^d} \int_0^t (t-u)^{d-1} dW_u, \ 0 \le t \le T. \tag{1.6}$$

En posant $d - 1 = \alpha \quad (\alpha < \frac{1}{2})$ on considère l'int égrale stochastique dans (1.6):

$$B_t \equiv \int_0^t (t-u)^{\alpha} dW_u, \ 0 \le t \le T, \ \alpha > 0, \tag{1.7}$$

qui sera choisie pour modélisé la perturbation dans notre modèle de long terme de l'évolution du cours.

Nous proposons donc de substituer au modèle (1.1) le modèle suivant

$$dS_t = S_t(\mu dt + \nu dB_t), \quad S_0 \text{ donné}, \tag{1.8}$$

S_t : prix d'option, μ et ν :constantes, B_t défini par (1.7), avec $\alpha = H - \frac{1}{2} > 0$ $(H > \frac{1}{2})$.

On désigne par $\mathcal{F}_t$ le σ-tribu engendré par la variable aléatoire donnée S_0 et par tout $B_s, s \le t : \mathcal{F}_t = \sigma(S_0, B_s, s \le t)$.

Une solution S_t de (1.8) est un processus stochastique $\mathcal{F}_t$-adapté satisfaisant la relation suivante

$$S_t = S_0 + \mu \int_0^t S_s ds + \nu \int_0^t S_s dB_s \tag{1.8$'$}$$

où la dernière intégrale est définie comme suit:

$$\int_0^t S_s dB_s = S_t B_t - \int_0^t B_s dS_s$$

en supposant que S_t soit presque surement borné.

Le modèle (1.8) *est celui de Black et Scholes où on a remplacé le mouvement brownien W_t par le processus fractionnaire B_t afin d'avoir un prix d'option de longue mémoire.*

2. Relation entre la perturbation B_t et le mouvement brownien fractionnaire

Certains auteurs ont aussi considéré le modèle suivant

$$dS_t = \mu S_t dt + \nu S_t dW_t^H \tag{2.1}$$

où W_t^H est un mouvement brownien fractionnaire de paramètre de Hurst H, $0 \le H \le 1$ (voir [2], [5]). On rappelle que W_t^H est un processus gaussien centré avec fonction de covariance donnée par

$$R(s,t) = E(W_s^H W_t^H) = \frac{V_H}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \tag{2.2}$$

où V_H est une constante. Si $H = \frac{1}{2}$ alors $V_H = 1$, $R(s,t) = \frac{1}{2}(s+t-|t-s|) = \min(s,t)$ et on a un mouvement brownien standard ordinaire. Alors (2.1) est le modèle bien connu de Black et Scholes.

L'équation (2.1) ne peut pas être résolue dans le cadre de la théorie de l'intégrale stochastique d'Itô, car W_t^H n'est plus un semi-martingale en général, sauf le cas où $H = \frac{1}{2}$. Des calculs stochastiques nouveaux sont élaborés (voir [2]) pour traiter des telles situations, mais il semble qu'ils sont encore loin des besoins pratiques dans la finance. On sait aussi que le mouvement brownien fractionnaire admet une représentation de la forme

$$W_t^H = \frac{1}{\Gamma(1-\alpha)} \left\{ U_t + \int_0^t (t-s)^\alpha dW_s \right\}, \tag{2.3}$$

où Γ désigne la fonction de gamma, W_t un mouvement brownien standard, $\alpha = H - \frac{1}{2}$, et $U_t = \int_{-\infty}^0 [(t-s)^\alpha - (-s)^\alpha] dW_s$. Parce que U_t est un processus avec des trajectoires absolument continues il suffit de considérer le deuxième terme qui correspond à (1.7).

On a ainsi démontré qu'on a des raisons de choisir B_t défini par (1.7) au lieu de $W_t^H = \frac{1}{\Gamma(1+\alpha)} \{U_t + B_t\}$ et de W_t comme la perturbation du prix de l'action dans un marché financier.

Revenant au modèle (1.8) du paragraphe 1, on va approximer B_t par un semimartigale.

3. Approximation du processus B_t

Pour chercher une solution asymptotique pour le modèle (1.8) on a besoin d'une approximation du processus B_t.

D'abord, pour chaque $\varepsilon > 0$ on définit un processus B_t^ε comme suit

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s, \quad 0 < \alpha < \frac{1}{2} \tag{3.1}$$

où W_t est le mouvement brownien correspondant à B_t dans (1.7).

On voit que B_t^ε est un semimartingale continu:

$$dB_t^\varepsilon = \Big(\int_0^t \alpha(t-s+\varepsilon)^{\alpha-1} dW_s \Big) dt + \varepsilon^\alpha dW_t. \tag{3.2}$$

En effet, d'après le théorème de Fubini, on a:

$$\begin{aligned}\int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u ds &= \int_0^t \Big[\int_u^s (s-u+\varepsilon)^{\alpha-1} ds \Big] dW_u \\ &= \frac{1}{\alpha}\Big[\int_0^t (s-u+\varepsilon)^\alpha dW_u - \varepsilon^\alpha W_t \Big] \\ &= \frac{1}{\alpha}\big[B_t^\varepsilon - \varepsilon^\alpha W_t \big], \text{ d' où:}\end{aligned}$$

$$B_t^\varepsilon = \int_0^t \int_0^s \alpha(s-u+\varepsilon)^{\alpha-1} dW_u ds + \varepsilon^\alpha W_t \text{ et il en result (3.2).}$$

On suppose dans cet article que $\frac{1}{2} < H < 1$, c'est à dire que $0 < \alpha < \frac{1}{2}$. On peut alors établie le résultat suivant

Théorème 1. *B_t^ε converge vers B_t dans $L^2(\Omega)$ lors que ε tend vers 0. Cette convergence est uniforme par rapport à $t \in [0,T]$.*

De la preuve du ce théorème, on obtient aussi l'estimation suivante:

$$\sup_{0 \le t \le T} E|B_t^\varepsilon - B_t|^2 \le K(\alpha)\varepsilon^{\frac{1}{2}+\alpha}, \ \alpha = H - \frac{1}{2}, 0 < \alpha < \frac{1}{2}.$$

Démonstration.

En appliquant le théorème des accroissements finis à la function continument dérivable $u \to u^\alpha$, on a:

$$\begin{aligned}|(t-s+\varepsilon)^\alpha - (t-s)^\alpha| \le & \quad \alpha\varepsilon \sup_{0\le\theta\le1} |(t-s+\theta\varepsilon)^{\alpha-1}| \\ = & \quad \alpha\varepsilon(t-s)^{\alpha-1}, \ \ 0 < \alpha = H - \frac{1}{2} < \frac{1}{2}.\end{aligned} \tag{3.3}$$

D'après l'isométrie de l'intégration d'Itô, on voit que

$$\begin{aligned}E|B_t^\varepsilon - B_t|^2 = & \quad E\Big| \int_0^t \big[(t-s+\varepsilon)^\alpha - (t-s)^\alpha\big] dW_s \Big|^2 \\ = & \quad \int_0^t \big|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\big|^2 ds\end{aligned} \tag{3.4}$$

L'inégalité (3.3) appliquée au membre droite de (3.4) nous donne:

$$\int_0^t \left|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\right|^2 ds \le \alpha^2\varepsilon^2 \int_0^t |t-s|^{2\alpha-2} ds =$$
$$= \alpha^2\varepsilon^2 \int_0^{t-\varepsilon} |t-s|^{2\alpha-2} ds + \alpha^2\varepsilon^2 \int_{t-\varepsilon}^t |t-s|^{2\alpha-2} ds$$
$$\le \alpha^2\varepsilon^2 \frac{\varepsilon^{2\alpha-1}}{1-2\alpha} + \alpha^2\varepsilon^2 \frac{\varepsilon^{2\alpha-1}}{1-2\alpha} = C(\alpha)\varepsilon^{2\alpha+1}, \tag{3.5}$$

où $C(\alpha)$ est une constante ne dépendant que de α : $C(\alpha) = \frac{2\alpha^2}{1-2\alpha}$.

Par conséquent,

$$\sup_{0\le t\le T} \|B_t^\varepsilon - B_t\| \ \le \ K(\alpha)\varepsilon^{\frac{1}{2}+\alpha} \to 0, \tag{3.6}$$

lorsque $\varepsilon \to 0$, où $0 < \alpha < \frac{1}{2}, K(\alpha) = \sqrt{C(\alpha)}$ et $\|\cdot\|$ désigne la norme dans $L^2(\Omega)$.

Donc B_t^ε converge dans $L^2(\Omega)$ vers B_t et la convergence est uniforme par rapport à $t \in [0,T]$. □

Remplacer B_t par le semimartingale B_t^ε permet alors un calcul stochastique usuel sans faire appel à des techniques difficiles comme le calcul de Malliavin.

4. Modèle (1.8) modifié

En se basant sur le Théorème 1 ci-dessus, nous proposons d'étudier ici un modèle modifié qui nous permettera d'utiliser le calcul d'Itô et facilitera les applications pratiques en prenant en compte de conséquences à long terme de chaque prix d'actif.

Pour chaque $\varepsilon > 0$ on associe à (1.8) le modèle asymptotique suivant:

$$dS_t^\varepsilon = \mu S_t^\varepsilon dt + \nu S_t^\varepsilon dB_t^\varepsilon, \quad S_0 = x, \tag{4.1}$$

où B_t^ε est défini comme dans le paragraphe 3 et x est une variable aléatoire positive donnée. Parce que

$$dB_t^\varepsilon = \Big(\int_0^t \alpha(t-s+\varepsilon)^{\alpha-1} dW_s\Big) dt + \varepsilon^\alpha dW_t \tag{4.2}$$

on a

$$dS_t^\varepsilon = S_t^\varepsilon\Big[\mu + \nu\alpha \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s\Big] dt + \varepsilon^\alpha \nu S_t^\varepsilon dW_t. \tag{4.3}$$

En désignant le crochet dans (4.3) par H_t^ε qui est un processus à trajectoires absolument continues, on peut reécrire (4.3) par

$$dS_t^\varepsilon = S_t^\varepsilon H_t^\varepsilon dt + \varepsilon^\alpha \nu S_t^\varepsilon dW_t. \tag{4.4}$$

(4.4) est une équation différentielle stochastique qui peut être résolue par le calcul d'Itô.

$$H_t^\varepsilon = \mu + \nu\alpha \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s. \tag{4.5}$$

Théorème 2. *Pour le modèle modifié*

$$dS_t = S_t(\mu dt + \nu dB_t^\varepsilon),$$

avec la condition initiale $S_0 = x$, *où* x *est une variable aléatoire donnée telle que* $\|x\|^2 = E[x]^2 < \infty$, $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s,\ \alpha > 0$, *nous avons la solution suivante:*

$$S_t^\varepsilon = x \exp\big(\mu t + \frac{1}{2}\nu^2 \varepsilon^{2\alpha} t + \nu B_t^\varepsilon\big). \tag{4.6}$$

Démonstration.

L'équation (4.4) peut s'écrire

$$\frac{dS_t^\varepsilon}{S_t^\varepsilon} = H_t^\varepsilon dt + \varepsilon^\alpha \nu dW_t. \tag{4.7}$$

Appliquons la formule d'Itô à fonction $f(u) = \log u$ avec $u = S_t^\varepsilon > 0$:

$$\log S_t^\varepsilon = \log S_0^\varepsilon + \int_0^t \frac{dS_t^\varepsilon}{S_t^\varepsilon} + \frac{1}{2}\int_0^t -\frac{1}{(S_s^\varepsilon)^2}(\varepsilon^\alpha \nu S_s^\varepsilon)^2 ds.$$

D'où

$$\int_0^t \frac{dS_t^\varepsilon}{S_t^\varepsilon} = \log \frac{S_t^\varepsilon}{S_0^\varepsilon} - \frac{1}{2}(\varepsilon^\alpha \nu)^2 t. \tag{4.8}$$

On déduit de (4.7) et (4.8) que

$$S_t^\varepsilon = S_0^\varepsilon \exp\Big(\frac{1}{2}\nu^2\varepsilon^{2\alpha} t + \nu\varepsilon^a + \int_0^t H_s^\varepsilon ds\Big). \tag{4.9}$$

D'autre part on a

$$\int_0^t H_s^\varepsilon ds = \mu t + \nu\alpha \int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u ds.$$

Comme on a déjà calculé avant l'énoncé du Thérème 1:

$$\begin{aligned}\int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u ds = & \int_0^t \Big[\int_u^s (s-u+\varepsilon)^{\alpha-1} ds\Big] dW_u \\ = & \frac{1}{\alpha}\Big[\int_0^t (s-u+\varepsilon)^\alpha dW_u - \varepsilon^\alpha W_t\Big] \\ = & \frac{1}{\alpha}\big(\varepsilon^\alpha W_t - B_t^\varepsilon\big).\end{aligned}$$

soit

$$\int_0^t H_s^\varepsilon ds = \mu t - \nu\varepsilon^\alpha dW_t + \nu B_t^\varepsilon. \tag{4.10}$$

On suppose que $S_0^\varepsilon = x$ est le cours observé à la date $t = 0$ et est une variable aléatoire indépendante de B_t (c'est -à - dire indépendante de W_t). En remplaçant (4.10) dans (4.9) on obtient enfin:

$$S_t^\varepsilon = x \exp\Big(\frac{1}{2}\nu^2\varepsilon^{2\alpha} t + \mu t + \nu B_t^\varepsilon\Big), \tag{4.11}$$

ce qu'il faudrait démontrer. □

5. **Convergence**

On constate que dans l' expression (4.11), lorsque $\varepsilon \to 0$ le terme $\frac{1}{2}\nu^2\varepsilon^{2\alpha}t$ tend vers 0 tandis que $B_t^\varepsilon \to B_t$ dans $L^2(\Omega)$ uniformement par rapport à $t \in [0, T]$. Alors on considère un processus S_t^* défini par:

$$S_t^* = S_0 \exp(\mu t + \nu B_t). \tag{5.1}$$

Et on a un résultat de convergence comme suivant:
Théorème 3. *Le processus S_t^* défini par le formule (5.1) est la limite dans $L^2(\Omega)$ de S_t^ε lorsque ε tend vers 0. Cette convergence est uniforme par rapport à $t \in [0, T]$.*

Démonstration.
On a

$$\begin{aligned} S_t - S_t^* = &\quad x \exp\big(\mu t - \frac{1}{2}\nu^2\varepsilon^{2\alpha}t + \nu B_t\big) - x\exp\big(\mu t + \nu B_t\big) \\ = &\quad x\exp\big(\mu t + \nu B_t\big)\Big[\exp\big(-\frac{1}{2}\nu^2\varepsilon^{2\alpha}t + \nu(B_t^\varepsilon - B_t)\big) - 1\Big] \end{aligned} \tag{5.2}$$

En désignant par $\|\cdot\|$ le norme dans $L^2(\Omega)$ on voit que

$$\|x\| = Ex^2 < 0 \text{ par hypothèse}, \tag{5.3}$$

$$\|\exp(\mu t + \nu B_t)\| \le e^{\mu t}\exp(\nu\|B_t\|) \le e^{\mu T}\exp\Big(\nu\frac{T^{\frac{1}{2}+\alpha}}{\sqrt{1+2\alpha}}\Big), \tag{5.4}$$

où $\|B_t\|$ est calculé d'après l'isométrie d'Itô:

$$\|B_t\|^2 = E\Big[\int_0^t (t-s)^\alpha dW_s\Big]^2 = E\int_0^t (t-s)^{2\alpha}ds = \frac{t^{1+2\alpha}}{1+2\alpha}.$$

D'autre part, il résulte de la relation $e^A - 1 = A + o(A)$ que

$$\|\exp\Big[-\frac{1}{2}\nu^2\varepsilon^{2\alpha}t + \nu(B_t^\varepsilon - B_t)\Big] - 1\| \le \frac{1}{2}\nu^2\varepsilon^{2\alpha}t + \nu\|B_t^\varepsilon - B_t\| + o\big(\|B_t^\varepsilon - B_t\|\big), \tag{5.5}$$

On a déjà une estimation de $\|B_t^\varepsilon - B_t\|$ par la formule (3.6) du Théorème 1:

$$\|B_t^\varepsilon - B_t\| \le K(\alpha)\varepsilon^{\alpha+\frac{1}{2}}, \tag{5.6}$$

où $K(\alpha)$ est une constante ne dépendant que de α. Par conséquent

$$\|\exp\Big[-\frac{1}{2}\nu^2\varepsilon^{2\alpha}t + \nu(B_t^\varepsilon - B_t)\Big] - 1\| \le \nu^2\varepsilon^{2\alpha}T + 2K(\alpha)\varepsilon^{\alpha+\frac{1}{2}}. \tag{5.7}$$

Il résulte enfin de (5.2), (5.4) et (5.7) que

$$\begin{aligned} \sup_{0\le t\le T}\|S_t^\varepsilon - S_t\| \le &\Big[\exp\big(\mu T + \frac{\nu T^{\frac{1}{2}+\alpha}}{\sqrt{1+2\alpha}}\big)\Big]\Big[\nu^2 T\varepsilon^{2\alpha} + 2K(\alpha)\varepsilon^{\alpha+\frac{1}{2}}\Big] \\ &\to 0. \end{aligned} \tag{5.8}$$

d'où la conclusion du Théorème 3. □

6. Solution du modèle fractionnaire

Nous revenons au modèle fractionnaire proposé au début:

$$\begin{cases} dS_t &= S_t(\mu dt + \nu dB_t) \\ S_0 &= x \text{ donné}, \quad B_t = \int_0^t (t-s)^\alpha dW_s, \quad 0 < \alpha < \frac{1}{2}. \end{cases} \tag{1.8}$$

Définition. *On appelle une solution du modèle fractionaire (1.8) la L^2-limite de la solution du modèle modifié lorsque $\varepsilon \to 0$:*

$$S_t \equiv S_t^* = L^2 - \lim_{\varepsilon \to 0} S_t^\varepsilon.$$

Par cette définition et par Thérème 3 on a maintenant la solution de (1.8): $S_t = x \exp(\mu t + \nu B_t)$.

Existence et Unicité de la solution

L'existence de la solution S_t est assurée par Théorème 3. Par ailleurs, le modèle modifié est donné sous la forme d'une équation différentielle stochastique linéaire gouvernée par un semimartingale avec des coefficients constants et avec la condition $ES_0^2 < \infty$. Alors il existe uniquement une telle solution S_t. L'unicité est au sens de l'espace $L^2(\Omega)$, car si $S_t^{*(1)}$ et $S_t^{*(2)}$ sont deux solutions de S_t^ε dans $L^2(\Omega)$ alors on a

$$\|S_t^{*(1)} - S_t^{*(2)}\| \leq \|S_t^{*(1)} - S_t^\varepsilon\| + \|S_t^\varepsilon - S_t^{*(2)}\| \to 0,$$

lorsque $\varepsilon \to 0$.

7. Sur l'opportunité d'arbitrage

Il est bien connu en mathématiques financières que l'absence d'arbitrage est essentiellement équivalent à l'existence d'une mesure de martingale. Alors une question naturelle se pose: Est-ce-que le principe d'absence d'arbitrage est violé dans notre modèle où le processus gouvernant B_t n'est plus un semimartingale?

La réponse, est que la solution S_t du modèle fractionnaire proposé peut être approximée avec une exactitude arbitraire par une solution S_t^ε du modèle modifié gouverné par un semimartingale où il n'existe aucune opportunité d'arbitrage. C'est là un des avantages de notre approche à calcul stochasticque fractionnaire appliqué à la finance par rapport aux autres approches.

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BOOK REVIEWS

Zdzislaw Denkowski, Stansislaw Migórski and Nikolas S. Papageorgiu, *An Introduction to Nonlinear Analysis,* Vol. I, *Theory*, 689 pp, ISBN 0-306-47392-5, Vol. II, Applications, 823 pp, ISBN 0-306-47456-5, Kluwer Academic Publishers, Boston-Dordrecht-London 2003.

The aim of this two volume treatise is to present some basic results in nonlinear analysis along with some applications. The methods and tools of nonlinear analysis rely heavily on results from other mathematical disciplines, first of all linear functional analysis (Banach space theory), topology and measure theory. In order to make a newcomer acquainted with the needed results from these areas and to prevent him to waste time and energy to browse over various specific books, the authors decided to gather in the first volume the basic results that are often used in nonlinear analysis. In the next we shall pass to a detailed analysis of the volumes.

Volume I: **Theory**

The volume is divided into five chapters corresponding to specific areas included: 1. *Elements of topology* (102 pp); 2. Elements of measure theory (150 pp); 3. *Banach spaces* (149 pp); 4. *Set-valued analysis* (112 pp); 5. *Nonsmooth analysis* (148 pp).

The first chapter contains the basic notions, constructions and results from topology – separation properties, nets and filters, connectedness, compactness, metrizability, continuity and uniform continuity, completeness, topologies on function spaces. Measure theory and integration is treated in the second chapter, and includes basic constructions in measure theory, Radon-Nikodym theorem, measures and measurable functions on topological spaces, Polish and Souslin spaces, Carathéodori functions (Scorza-Dragoni theorem). The third chapter is concerned with the fundamental properties of Banach and Hilbert spaces and of operators acting on them: Hahn-Banach theorem, fundamental principles, weak and weak* topologies, separation of convex sets. A special attention is paid to function spaces (including Sobolev spaces) and their duality, compactness and weak compactness criteria in such spaces.

Although the majority of the results are presented with full proofs, some difficult theorems (as, e.g., Tietze and Urysohn theorems, the paracompactness of metric spaces, Nikodym boundedness theorem, Lyapunov convexity theorem, some extension theorems for measures, James' criterium of weak compactness, Eberlein-Smulian theorem, Bishop-Phelps theorem) are only enounced with exact references to the sources where a proof can be found.

Chapter 4, *Set-valued analysis* (112 pp), presents the basic results of multivalued analysis – various convergence types for sets and multifunctions, continuity, measurability, set-valued measures and integration, measurable and continuous selections. A comprehensive treatment of these topics is given in another two volume treatise published also at Kluwer A. P. by S. Hu and N. S. Papageorgiu, Handbook on multivalued analysis, Vol. I (1997), Vol. II (2000).

The last chapter of this volume is concerned with nonsmooth analysis including differential calculus in Banach spaces, convex functions and their subdifferentials, generalized subdifferentials of locally Lipschitz functions, optimization and minimax theorems, tangent and normal cones.

Volume II, **Applications**

The first chapter of the second volume, *Nonlinear operators and fixed points* (168 pp), discusses nonlinear compact and Fredholm operators, measures of noncompactness and set-contractions, monotone operators, accretive operators and semigroups of nonlinear operators, Ekeland variational principle, fixed points.

After this somewhat transition chapter, making a bridge between the theory treated in the first chapter to the applications from the second one, one passes to more applied topics: 2. *Ordinary differential equations* (144 pp); 3. *Partial differential equations* (228); 4. *Optimal control and calculus of variations* (147 pp); 5. *Mathematical economics* (105 pp). Of course that it is impossible to give in one chapter a comprehensive treatment of the subject, the aim of the authors being rather to emphasize how the techniques developed so far work to give new insights.

For instance, in the second chapter, the approach to differential equations is done via critical point theory and minimax techniques (Mountain Pass, Saddle Point and Linking Theorems). Differential inclusions as well as Hamiltonian systems with emphasis on the existence of periodic trajectories, are also considered.

Partial differential equations, treated in the third chapter, are one of the main domain of applications of nonlinear analysis and, at the same time, a source for many problems and results. Here the main idea is to show that there are some unifying themes, lying underneath the huge amount of apparently unrelated techniques used to solve partial differential equations. The main topics are: eigenvalue problems and maximum principles, nonlinear elliptic problems, evolution equations, Γ-convergence for functions and G-convergence for operators.

Another important field of applications where the method of nonlinear analysis are essential is optimal control, treated in the fourth chapter. Again the treatment is restricted to topics that illustrate the techniques developed in the previous chapters, and they include: existence and relaxation, sensitivity analysis, the maximum principle, Hamilton-Jacobi-Belman equation, viscosity solutions, controllability and observability. The last section of this chapter is devoted to the calculus of variations, a field as old as the calculus itself, but still of great interest.

Finally, the last chapter of the book deals with some problems in mathematical economics, a domain that knew a remarkable progress in the last forty years, and allowed to some mathematicians to win a Nobel prize in economics. From this vast domain the authors selected some topics: Walras equilibria in competitive economies, growth models for both discrete time and continuous time cases, growth models under uncertainty, stochastic games.

Each chapter contains a set of exercises (around 50), followed by solutions, completing the main text. A section of remarks, containing historical comments, references to related results as well as indications for further reading, is also included in each chapter.

This fairly self-contained two volume book is a very good introductory text to a variety of topics in nonlinear analysis and its applications. It, or parts of it, can be used for graduate or post-graduate courses, or as a reference text.

S. Cobzaş

p-Adic Functional Analysis, Lecture Notes in Pure and Applied Mathematics: Vol. 222, A. K. Katsaras, W. H. Schikhof, L. Van Hamme - Editors, M. Dekker, New York 2001, viii+322 pp, ISBN 0-8247-0611-0.

These are the Proceedings of the Sixth International Conference on p-adic Functional Analysis held in 2000 at the University of Ioannina, Greece. Starting with Laredo, Spain 1990, each two years a conference on these topics was held in various countries, most of the proceedings being published by M. Dekker in the same series as the present one.

This conference was attended by about 40 mathematicians from various countries, reputed specialists who, in 30 minutes talks, reported on their latest results in p-adic or non-archimedean (n.a.) analysis. Among the participants were J. Aguayo, H. Ochsenius (Chile), J. Araujo, C. Perez-Garcia (Spain), K. Boussaf, A. Escassut (France), N. De Grande-De Kimpe (Belgium), J. Kakol (Poland), A.K. Katsaras, C.G. Petalas (Greece), A. Khrennikov, K.-O. Lindhal, M. Nilsson (Sweden), A.J. Lemin (Russia), P.N. Natarajan (India), W.H. Schikhof (The Netherlands), B. Dragovich (Yugoslavia), M. Berz (USA), H. Keller (Switzerland), et al.

The volume contains 26 research papers covering a large area of topics in p-adic analysis and its applications as – n.a. locally convex spaces (2 papers) and sequence spaces, n.a. vector measures and integral representations of linear operators, n.a probability measures, compact perturbations of linear operators, spctral radius of derivations, n.a. Banach-Stone theorem, p-adic analytic functions, p-adic differential equations, commutation relations for operators on non-classical Hilbert spaces, dynamical systems (3 papers), embedding n.a. metric spaces in classical L_p-spaces, ultrametric Hopf algebras, Levi-Civita fields, ergodicity of p-adic spheres, and more.

As the preceding ones, this volume will became an indispensable reference for those working in non-archimedean analysis and its applications.

S. Cobzaş