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INTERPOLATING ON SOME NODES OF A GIVEN TRIANGLE

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Abstract. We consider the interpolation problem for some data on several nodes of a given triangle. We show that an interpolant may be found by dividing the initial problem into two subproblems, each one with some fewer nodes. The main result is given in Theorem 5.

1. **Introduction**

We are interested in interpolation on certain nodes on a given triangle, using the generalized Newton algorithm [1], [2]. This algorithm enables us to divide the interpolation problem into two smaller subproblems.

We shall recall first some known results. Denote by $\Pi_k(\mathbb{R}^s)$ the space of polynomials in s variables and of degree at most k and by $\#(A)$ the cardinal of a set A.

Theorem 1. *(Gasca-Maeztu) see* [2]*. Let N be a set of $\frac{1}{2}(k+1)(k+2)$ nodes in $\mathbb{R}^s$, where $s \geq 2$. Suppose that there exist the hyperplanes H_0, $H_1, ..., H_k$ in $\mathbb{R}^s$ such that*

a) $N \subset H_0 \cup H \cup ... \cup H_k$;

b) $\#(N \cap H_i) = i + 1, \quad 0 \leq i \leq k$.

Then arbitrary data on N can be interpolated by elements of $\Pi_k(\mathbb{R}^s)$.

The previous result generalizes the following theorem of Micchelli:

Theorem 2. *see* [1]*. Interpolation of arbitrary data by an element of* $\Pi_m(\mathbb{R}^2)$ *is uniquely possible on a set N of $\frac{1}{2}(m+1)(m+2)$ nodes if there exist $m+1$ lines $L_0, L_1, ..., L_m$ whose union contains N and that have the property that each L_i contains exactly $i+1$ nodes, $i = 0, ..., m$.*

Next we present the Newton algorithm and its generalization (see [1] and [2]).

Algorithm 3. *(The Newton algorithm for univariate polynomial interpolation). Let g be a polynomial that interpolates a function f at the distinct nodes $x_1, ..., x_n$ and let $h = \prod_{i=1}^{n}(x - x_i)$. Then for suitable c, $g + ch$ will interpolate f at $x_1, ..., x_n, x_{n+1}$, where x_{n+1} is a new node. The algorithm is applied repeatedly, starting with $n = 1$. The polynomial g can be of degree $n-1$, but this is not necessarily.*

Algorithm 4. *(The generalized Newton algorithm). Let X be an arbitrary linear space. Let g be a function (not necessarily a polynomial) that interpolates the given function $f : X \to \mathbb{R}$ at the distinct nodes $x_1, ..., x_n$. Let x_{n+1} be a new node. We require a function h that takes the value 0 at $x_1, ..., x_n$, but has a nonzero value at x_{n+1}. For an appropriate value of c, $g + ch$ will interpolate f at $x_1, ..., x_n, x_{n+1}$.*

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At the next level of generalization, we replace $\{x_1, ..., x_n\}$ by any set of nodes N, which needs not be finite. We assume that g interpolates f on N (in symbols, $g|N = f|N$). Let y be a new node, $y \notin N$. We require a functional h such that $h|N = 0$ and $h(y) \neq 0$. We may assume $h(y) = 1$. Then $g + f(y)h$ interpolates f on $N \cup \{y\}$.

In a further level of generalization we use $g+rh$ as an interpolant, but permit r to be a function more general than simply a constant. We use the notation

$$Z = \{x \in X : h(x) = 0\}.$$

Consider a set N of nodes, let g interpolate f on $N \cap Z(h)$ and r interpolate $(f-g)/h$ on $N \backslash Z(h)$. Then $g + rh$ interpolates f on N.

As pointed out in [1] and [2], this last generalization of the Newton algorithm is successfully applied in Theorem 2.

An immediate conclusion is that this abstract version of the Newton algorithm enables the dividing of an interpolation problem into two smaller subproblems, where smallness refers to the number of interpolation conditions.

2. **Interpolating on some nodes of a given triangle**

Let $f : T_h \to \mathbb{R}$ be a function defined on the triangle

$$T_h = \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq h,\, 0 \leq y \leq h,\, x + y \leq h\right\}, \quad h \in \mathbb{N}^*. \tag{1}$$

Let $N = \{X_1,\, X_2,\, X_3,\, X_4,\, X_5,\, X_6\}$ be a set of six nodes situated on the edges of the triangle T_h, where $X_1(0, \frac{h}{3})$, $X_2(0, \frac{2h}{3})$, $X_3(\frac{h}{4}, 0)$, $X_4(\frac{h}{2}, 0)$, $X_5(\frac{3h}{4}, 0)$, $X_6(\frac{h}{2}, \frac{h}{2})$. We consider the Lagrange interpolation functionals

$$\Lambda_L = \{\lambda_i(f) \,|\, \lambda_i(f) = f(x_i),\ 1 \leq i \leq 6\}.$$

The problem we deal with here is to

$$\text{find } r \in \Pi_2(\mathbb{R}^2) \text{ such that } r \text{ interpolates } f \text{ with regard to } \Lambda_L. \tag{2}$$

We find an answer in the following way. Let L_0, L_1, L_2 denote the lines of the triangle, such that $\{X_6\} \subset L_0$, $\{X_1,\, X_2\} \subset L_1$, $\{X_3,\, X_4,\, X_5\} \subset L_2$. The problem here satisfies the hypothesis of the Micchelli's theorem. Therefore, this result assures that there exists an interpolant in $\Pi_2(\mathbb{R}^2)$ for f with regard to Λ_L and this interpolant is unique.

Next our purpose is to find this interpolant. For doing this we use the generalized Newton algorithm.

Let l_2 denote an element of $\Pi_1(\mathbb{R}^2)$ whose zero set is the line L_2,

$$Z(l_2) = \{(x, y) : l_2(x, y) = 0\} = L_2.$$

We have $N \cap L_2 = \{X_3,\, X_4,\, X_5\}$, $l_2(x, y) = y$.

Let $p_2 \in \Pi_2(\mathbb{R}^2)$ interpolate f on $N \cap L_2 = \{X_3, X_4, X_5\}$. Therefore, p_2 has the form

$$p_2(x, y) = a_0 x^2 + b_0 x + c_0,$$

where a_0, b_0 and c_0 can be determined from the interpolation conditions:

$$\begin{cases} p_2(\frac{h}{4}, 0) = f(\frac{h}{4}, 0) \\ p_2(\frac{h}{2}, 0) = f(\frac{h}{2}, 0) \\ p_2(\frac{3h}{4}, 0) = f(\frac{3h}{4}, 0). \end{cases} \tag{3}$$

Its expression is

$$\begin{aligned} p_2(x,y) &= (\tfrac{8}{h^2}x^2 - \tfrac{6}{h}x + 1)f(\tfrac{3h}{4},0) + (\tfrac{8}{h^2}x^2 - \tfrac{10}{h}x + 3)f(\tfrac{h}{4},0) \\ &\quad + (-\tfrac{16}{h^2}x^2 + \tfrac{16}{h}x - 3)f(\tfrac{h}{2},0). \end{aligned}$$

Let $q_1 \in \Pi_1(\mathbb{R}^2)$ interpolate $(f - p_2)/l_2$ on $N \backslash Z(l_2) = \{X_1, X_2, X_6\}$. Therefore, q_1 has the form

$$q_1(x,y) = a_1 x + b_1 y + c_1,$$

where a_1, b_1 and c_1 can be determined from the interpolation conditions:

$$\begin{cases} q_1(0,\frac{h}{3}) = \frac{f(0,\frac{h}{3}) - p_2(0,\frac{h}{3})}{\frac{h}{3}} \\ q_1(0,\frac{2h}{3}) = \frac{f(0,\frac{2h}{3}) - p_2(0,\frac{2h}{3})}{\frac{2h}{3}} \\ q_1(\frac{h}{2},\frac{h}{2}) = \frac{f(\frac{h}{2},\frac{h}{2}) - p_2(\frac{h}{2},\frac{h}{2})}{\frac{h}{2}}. \end{cases} \tag{4}$$

According to the generalized Newton algorithm we have that $r = p_2 + l_2 q_1$ interpolates f on N. So the interpolation problem is divided into two smaller subproblems, with fewer interpolation conditions. The subproblems involve the determination of p_2 and q_1, each regarding three interpolation conditions.

Since r obeys the interpolation conditions we obtain that r solves the interpolation problem on N.

The problem becomes easier to solve if we apply twice the generalized Newton algorithm. We have to find an interpolant for q_1 on the set $M := \{X_1,\ X_2,\ X_6\}$. Let l_1 denote an element of $\Pi_1(\mathbb{R}^2)$ whose zero set is the line L_1,

$$Z(l_1) = \{(x,y) : l_1(x,y) = 0\} = L_1.$$

We have $M \cap L_1 = \{X_1,\ X_2\}$, $l_1(x,y) = x$.

Let $p_1 \in \Pi_1(\mathbb{R}^2)$ interpolate q_1 on $M \cap L_1 = \{X_1,\ X_2\}$. Therefore, p_1 has the form

$$p_1(x,y) = a_2 y + b_2,$$

where a_2 and b_2 can be determined from the interpolation conditions:

$$\begin{cases} p_1(0,\frac{h}{3}) = q_1(0,\frac{h}{3}) \\ p_1(0,\frac{2h}{3}) = q_1(0,\frac{2h}{3}). \end{cases} \tag{5}$$

By (4), (5) becomes

$$\begin{cases} p_1(0,\frac{h}{3}) = \frac{f(0,\frac{h}{3}) - p_2(0,\frac{h}{3})}{\frac{h}{3}} \\ p_1(0,\frac{2h}{3}) = \frac{f(0,\frac{2h}{3}) - p_2(0,\frac{2h}{3})}{\frac{2h}{3}}. \end{cases} \tag{6}$$

Its expression is

$$\begin{aligned} p_1(x,y) &= \tfrac{3}{2h}(\tfrac{3}{h}y - 1)f(0,\tfrac{2h}{3}) + \tfrac{3}{h}(-\tfrac{3}{h}y + 2)f(0,\tfrac{h}{3}) \\ &\quad + \tfrac{3}{2h}(-\tfrac{3}{h}y + 1)p_2(0,\tfrac{2h}{3}) + \tfrac{3}{h}(\tfrac{3}{h}y - 2)p_2(0,\tfrac{h}{3}). \end{aligned}$$

Let $q_0 \in \Pi_0(\mathbb{R}^2)$ interpolate $(q_1 - p_1)/l_1$ on $M \backslash Z(l_1) = \{X_6\}$. Therefore, q_0 is constant:

$$q_0 = \frac{q_1(\frac{h}{2},\frac{h}{2}) - p_1(\frac{h}{2},\frac{h}{2})}{\frac{h}{2}}.$$

According to the generalized Newton algorithm we have that $p_1 + l_1 q_0$ interpolates q_1 on $M = \{X_1, X_2, X_6\}$. So the interpolation problem here involves the determination of p_1, regarding two interpolation conditions, the initial interpolation problem becoming much easier to solve. The polynomial $p_1 + l_1 q_0$ verifies the interpolation conditions so it interpolates $q_1$on M. We conclude with the following result.

Theorem 5. *The initial interpolation problem (2) on N is solved by*

$$r = p_2 + l_2 q_1 = p_2 + l_2(p_1 + l_1 q_0).$$

We shall illustrate the above theory with two practical examples. Consider $h = 10$ in (1), $f_1 : T_{10} \to \mathbb{R}$, $f_1(x,y) = x^2 + y^2$ and $f_2 : T_{10} \to \mathbb{R}$, $f_2(x,y) = \sqrt{x^2+y^2}$. Consider r_i the interpolant of f_i, $i = 1, 2$. Figures 1 and 2 display the error $|f_i(x,y) - r_i(x,y)|$, plotted in Matlab.

3. The generalized Newton algorithm for linear functionals

As pointed out in [2], the generalized Newton algorithm can be applied not only for point-evaluation functionals, but for arbitrary linear functionals. Consider a linear space E and $\Phi_1, \Phi_2, ...$ some linear functionals defined on E. Let f be an element of E to be interpolated. We assume that an element g is available in E such that $\Phi_i(g) = \Phi_i(f)$ for $1 \le i \le n$. Next, select h in E so that $\Phi_i(h) = 0$ for $1 \le i \le n$ and $\Phi_{n+1}(h) = 1$. The new interpolant will be of the form $g + ch$, where $c = \Phi_{n+1}(f)$.

We illustrate this by solving a problem proposed in [2].

Problem 6. *Find $p \in \Pi_3(\mathbb{R})$ such that $p(0) = 3$, $p'(1) = 4$, $\int_0^1 p(x)dx = 5$ and $\int_0^1 x^2 p(x)dx = 6$.*

Proof. We use the generalized Newton algorithm for linear functionals. We consider the linear functionals defined by: $\Phi_1(f) = f(0)$, $\Phi_2(f) = \int_0^1 f(x)dx$, $\Phi_3(f) = \int_0^1 x^2 f(x)dx$, $\Phi_4(f) = f'(1)$, for some given f.

We assume that there exists g such that

$$\begin{aligned} &\Phi_i(g) = \Phi_i(p), \qquad i = 1,2,3 \\ &\Phi_4(g) = 0. \end{aligned} \tag{7}$$

Select now h such that

$$\begin{aligned} &\Phi_i(h) = 0, \quad i = 1,2,3, \\ &\Phi_4(h) = 1. \end{aligned} \tag{8}$$

The interpolant of p is $g + ch$, where $c = \Phi_4(f) = 4$. Therefore we have to find the interpolant of p, $r := g + 4h$ from $\Pi_3(\mathbb{R})$. We do this taking into account (7) and (8). We have $g \in \Pi_3(\mathbb{R})$ and $h \in \Pi_3(\mathbb{R})$ so g and h have the following expressions

$$\begin{aligned} g(x) &= a_1 x^3 + b_1 x^2 + c_1 x + d_1, \\ h(x) &= a_2 x^3 + b_2 x^2 + c_2 x + d_2, \end{aligned}$$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{R}$, and moreover $a_1^2 + a_2^2 \neq 0$.

Solving the systems (7) and (8) we obtain the polynomials

$$\begin{aligned} g(x) &= -\tfrac{984}{5}x^3 + 366x^2 - \tfrac{708}{5}x + 3, \\ h(x) &= \frac{6}{5}x^3 - \frac{3}{2}x^2 + \frac{2}{5}x. \end{aligned}$$

Therefore, the interpolant of p is

$$r(x) = -192x^3 + 360x^2 - 140x + 3 \in \Pi_3(\mathbb{R}).$$

□

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MINIMUM VALUE OF A MATRIX NORM WITH APPLICATIONS TO MAXIMUM PRINCIPLES FOR SECOND ORDER ELLIPTIC SYSTEMS

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Abstract. The purpose of this paper is to use an estimation of minimum value of a matrix norm to improve some results given by I.A.Rus in 1969, 1973, and A.S. Muresan in 1975.

1. Introduction

Let us consider the following operator:

$$Lu := \sum_{i,j=1}^{m} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} A_i \frac{\partial u}{\partial x_i} + A_0 u,$$

where $A_{ij}, A_i, A_0 \in C(\overline{\Omega}, M_n(\mathbb{R}))$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain.

Let us also consider the following systems:

$$Lu = 0, \tag{1}$$

$$Lu = f, \tag{2}$$

where $f \in C(\overline{\Omega}, \mathbb{R}^n)$.

There are some maximum principles for the solutions of (1) (see for example [2], [5] and [8]).

In [5] the following principle is given:

Theorem 1. *Suppose that:*

1. *the system (1) is strongly elliptic,*
2. $e^* Le < 0$, *for each* $e \in C^2(\overline{\Omega}, \mathbb{R}^n)$, *with* $\|e\| := \left(\sum\limits_{i=1}^{n} e_i^2\right)^{1/2} = 1.$

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ *is a solution of (1), then* $\|u\| := \left(\sum\limits_{i=1}^{n} u_i^2\right)^{1/2}$ *attains his maximum value on* $\partial\Omega$.

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The aim of this paper is to find conditions which imply condition 2 of Theorem 1. This will be done in section 2 of this paper. In section 3 we shall try to improve some estimations for the norm of solution of system (2), estimations given in [4] and [6].

Let $A \in M_n(\mathbb{R}), J$ the Jordan normal form of A. We know that there exist a nonsingular matrix T such that $A = TJT^{-1}$.
We will denote:

$$\begin{aligned} \widetilde{\alpha} &= \begin{cases} \frac{1}{n}\sum\limits_{k=1}^{s} n_k\lambda_k, \lambda_k \in \mathbb{R} \\ \frac{1}{n}\sum\limits_{k=1}^{s} n_k \ Re\lambda_k, \lambda_k \in \mathbb{C}\backslash\mathbb{R} \end{cases} \\ \gamma_F &= \|T\|_F \cdot \left\|T^{-1}\right\|_F \\ m_F &= \|J - \widetilde{\alpha} I\|_F \end{aligned}$$

where λ_k are the eigenvalues of A, n_k is the number of λ_k which appears in Jordan blocks (generated by λ_k) and $\|\cdot\|_F$ is the euclidean norm of a matrix (see [1]).

We shall use the following result given in [1]:

Theorem 2. *Let $\varphi_{\|\cdot\|} : \mathbb{R} \to \mathbb{R}, \varphi_{\|\cdot\|}(\alpha) = \|A - \alpha I_n\|$, $\|\cdot\|$ being one of the following norms: $\|\cdot\|_F$, $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$. In these conditions:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) \le \sqrt{n}\gamma_F m_F.$$

Remark 1. *In case of euclidean norm $\|\cdot\|_F$ and spectral norm $\|\cdot\|_2$ we have that $\varphi_{\|\cdot\|}(\widetilde{\alpha}) \le \gamma_F m_F$ (see [1]). Because $n \ge 2$, if $m_F \ne 0$, then:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) < \sqrt{n}\gamma_F m_F.$$

2. Main result for the solution of system (1)

In this section we shall give conditions under which condition 2 of Theorem 1 holds in case $A_{ij} = a_{ij}I_n$, $a_{ij} \in C(\overline{\Omega})$. Suppose that there exist $\delta > 0$ such that:

$$\sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j \ge \delta^2 \|\xi\|^2, \xi \in \mathbb{R}^n. \tag{3}$$

Theorem 3. *Suppose that (3) holds and:*

$$\xi^* A_0(x)\xi \le -\frac{1}{4\delta^2} n \|\xi\|^2 \sum_{i=1}^{m} (\gamma_F^i m_F^i)^2, \forall \xi \in \mathbb{R}^n, \forall x \in \Omega. \tag{4}$$

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n), u \ne 0$, is a solution of (1), then $\|u\| := \left(\sum\limits_{k=1}^{n} u_k^2\right)^{1/2}$ attains his maximum value on $\partial\Omega$.

Proof. Our result is based on the following remark which appears in [5]:

If, for each $x \in \Omega$, there exist $\widetilde{\alpha_i}(x) \in \mathbb{R}, i = \overline{1,m}$, such that:

$$\xi^* \begin{pmatrix} -a_{11}I_n & -a_{12}I_n & \dots & -a_{1m}I_n & 0 \\ -a_{21}I_n & -a_{22}I_n & \dots & -a_{2m}I_n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_{m1}I_n & -a_{m2}I_n & \dots & -a_{mm}I_n & 0 \\ A_1(x) - \widetilde{\alpha_1}(x)I_n & A_2(x) - \widetilde{\alpha_2}(x)I_n & \dots & A_m(x) - \widetilde{\alpha_m}(x)I_n & A_0(x) \end{pmatrix} \xi < 0, \tag{5}$$

for all $\xi \in \mathbb{R}^{(m+1)n}, \xi \neq 0, \forall x \in \Omega$ then condition 2 of Theorem 1 holds.

So, it is enough to show that (4) implies (5).

Now it's easy to see that if, for each $x \in \Omega$, there exist $\varepsilon_i(x) > 0$ and $\widetilde{\alpha_i}(x) \in \mathbb{R}$, such that

$$\|A_i(x) - \widetilde{\alpha_i}(x)I_n\| < 2\varepsilon_i(x), i = \overline{1,m}, \tag{6}$$

$$\xi^* A_0(x)\xi \leq -\frac{1}{\delta^2}\|\xi\|^2 \sum_{i=1}^{m} \varepsilon_i^2(x), \forall \xi \in \mathbb{R}^n, \tag{7}$$

then (5) holds.

For simplicity we shall prove this in case $m = n = 2$.
We have:

$$\xi^* \begin{pmatrix} -a_{11}I_2 & -a_{12}I_2 & 0 \\ -a_{21}I_2 & -a_{22}I_2 & 0 \\ A_1(x) - \widetilde{\alpha_1}(x)I_2 & A_2(x) - \widetilde{\alpha_2}(x)I_2 & A_0(x) \end{pmatrix} \xi \leq -\delta^2(\xi_1^2 + \xi_3^2) -$$

$$-\delta^2(\xi_2^2 + \xi_4^2) + \delta^2(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) + \frac{1}{4\delta^2}\|\xi'\|^2 \|A_1(x) - \widetilde{\alpha_1}(x)I_2\|^2 +$$

$$+\frac{1}{4\delta^2}\|\xi'\|^2 \|A_2(x) - \widetilde{\alpha_2}(x)I_2\|^2 + \xi'^* A_0(x)\xi' <$$

$$< \frac{\varepsilon_1^2(x) + \varepsilon_2^2(x)}{\delta^2}\|\xi'\|^2 - \frac{\varepsilon_1^2(x) + \varepsilon_2^2(x)}{\delta^2}\|\xi'\|^2 = 0,$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{R}^6, \xi \neq 0, \xi' = (\xi_5, \xi_6) \in \mathbb{R}^2, \xi' \neq 0$.

Now, according to Theorem 2 and Remark 1 we have that if $m_F^i \neq 0, i = \overline{1,m}$, then for each $x \in \Omega$, there exist $\widetilde{\alpha_i}(x) \in \mathbb{R}$ such that $\|A_i(x) - \widetilde{\alpha_i}(x)I_n\| < \sqrt{n}\gamma_F^i m_F^i$. So choosing $\varepsilon_i(x) = \frac{1}{2}\sqrt{n}\gamma_F^i m_F^i$, the proof is done.

Remark 2. *If $m_F^i = 0, i = \overline{1,m}$, then the conclusion of Theorem 3 holds if*

$$\xi^* A_0(x)\xi < 0, \forall \xi \in \mathbb{R}^n, \xi \neq 0, x \in \Omega$$

Example 1. *Let us consider the system (1) in case $m = n = 2$ with $A_1 = A_2 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}$. We suppose that $a_2, a_3 > 0$. In this case we shall have: $\widetilde{\alpha}_1 = \widetilde{\alpha}_2 = a_1$, $\gamma_F^{A_1} = \gamma_F^{A_2} = \frac{a_2+a_3}{\sqrt{a_2a_3}}, m_F^{A_1} = m_F^{A_2} = \sqrt{2a_2a_3}, A_1 - a_1I_2 = A_2 - a_1I_2 = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}, \varepsilon_1 = \varepsilon_2 = a_2 + a_3$.*
The condition (4) becomes:

$$\xi^* A_0(x)\xi \leq -\frac{2}{\delta^2}(a_2 + a_3)^2 \|\xi\|^2, \xi \in \mathbb{R}^2, x \in \Omega. \tag{8}$$

If (3) and (8) holds, then we have:

$$\xi^* \begin{pmatrix} -a_{11}I_2 & -a_{12}I_2 & 0 \\ -a_{21}I_2 & -a_{22}I_2 & 0 \\ A_1 - a_1 I_2 & A_2 - a_2 I_2 & A_0(x) \end{pmatrix} \xi \leq \frac{1}{4\delta^2} \left\|\xi'\right\|^2 (a_2^2 + a_3^2) +$$

$$+\frac{1}{4\delta^2} \left\|\xi'\right\|^2 (a_2^2 + a_3^2) - \frac{2}{\delta^2}(a_2 + a_3)^2 \left\|\xi'\right\|^2 = \frac{1}{4\delta^2} \left[a_2^2 + a_3^2 - 4(a_2 + a_3)^2\right] \left\|\xi'\right\|^2 < 0,$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{R}^6, \xi \neq 0, \xi' = (\xi_5, \xi_6) \in \mathbb{R}^2, \xi' \neq 0.$

So, if (3) and (8) holds then, if $u \in C^2(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega}, \mathbb{R}^2), u \neq 0,$ *is a solution of (1) in case* $m = n = 2$*, with* $A_1 = A_2 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}, a_2, a_3 > 0,$ *then* $\|u\|$ *attains his maximum value on* $\partial\Omega$*.*

3. **Estimations for the solution of system (2)**

In this section we shall try to improve some estimation for the norm of the solution of system (2), estimations given in [4] and [6]. For other estimations see [3] and [8].

Theorem 4. *([4],[6]): Suppose that:*

1. *the system (2) is strongly elliptic,*
2. $e^* L e \leq -p^2$*, for each* $e \in C^2(\overline{\Omega}, \mathbb{R}^n)$*, with* $\|e\| = 1, p \in \mathbb{R}^*$*.*

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ *is a solution of (2), then:*

$$|u(x)| \leq \max\left\{\max_{x \in \partial\Omega} |u(x)|, \frac{1}{p^2} \max_{x \in \overline{\Omega}} |f(x)|\right\}, x \in \overline{\Omega}.$$

As in section 2, we shall try to find conditions under which condition 2 of Theorem 4 holds. In this way we shall be able to find a value of p.

In case m=1, system (2) becomes:

$$Ly := \frac{d^2 y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = f(x), \tag{9}$$

where $B, C \in C([a, b], M_n(\mathbb{R})), f \in C([a, b], \mathbb{R}^n)$.

If $m_F \neq 0$, then we have the following result:

Theorem 5. *Suppose that:*

$$e^* C(x) e \leq -\frac{1}{4} n(\gamma_F m_F)^2, \tag{10}$$

$\forall e \in C^2([a,b],\mathbb{R}^n), \|e\| = 1, \forall x \in]a, b[$.

If $y \in C^2([a, b], \mathbb{R}^n), y \neq 0,$ *is a solution of (9), then:*

$$|y(x)| \leq \max\left\{|y(a)|, |y(b)|, \frac{4}{n\gamma_F^2 m_F^2 - \left\|B(x) - \widetilde{\beta}(x) I_n\right\|^2} \max_{x \in [a,b]} |f(x)|\right\}, x \in [a, b].$$

Proof. According to Theorem 2 and Remark 1 we have that, for each $x \in]a,b[$, there exist $\widetilde{\beta}(x) \in \mathbb{R}$ such that $\left\| B(x) - \widetilde{\beta}(x) I_n \right\| < \sqrt{n}\gamma_F m_F$.

We have:
$e^* L e = -\|e'\|^2 + e^* B(x) e' + e^* C(x) e = -\|e'\|^2 + e^* \left(B(x) - \widetilde{\beta}(x) I_n \right) e' + e^* C(x) e \leq -\|e'\|^2 + \left\| B(x) - \widetilde{\beta}(x) I_n \right\| \|e'\| + e^* C(x) e \leq \frac{1}{4} \left\| B(x) - \widetilde{\beta}(x) I_n \right\|^2 + e^* C(x) e \leq \frac{1}{4} \left\| B(x) - \widetilde{\beta}(x) I_n \right\|^2 - \frac{1}{4} n (\gamma_F m_F)^2 = -p^2(x) < 0.$

So $e^* L e \leq -p^2$ and hence and from Theorem 4, Theorem 5 is proved.

Remark 3. *In case that $m_F = 0$, if there exist $p \neq 0$ such that $e^* C(x) e \leq -p^2, \forall x \in]a,b[$, then the conclusion becomes:*

$$|y(x)| \leq \max \left\{ |y(a)|, |y(b)|, \frac{1}{p^2} \max_{x \in [a,b]} |f(x)| \right\}, x \in [a,b].$$

Example 2. *Let us consider the system (9) with* $B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}$ *and* $a_2, a_3 > 0$. *In this case we shall have:*
$\widetilde{\beta} = a_1$, $\gamma_F = \frac{a_2 + a_3}{\sqrt{a_2 a_3}}$, $m_F = \sqrt{2 a_2 a_3}$, *and* $B - \widetilde{\beta} I_2 = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}$. *The relation* (10), *becomes:*

$$e^* C(x) e \leq -(a_2 + a_3)^2, x \in]a,b[. \tag{11}$$

If (11) holds and $y \in C^2([a,b], \mathbb{R}^2)$ is a solution of (9), then:

$$|y(x)| \leq \max \left\{ |y(a)|, |y(b)|, \frac{4}{3a_2^2 + 8 a_2 a_3 + 3 a_3^2} \max_{x \in [a,b]} |f(x)| \right\}, x \in [a,b].$$

In case $m = 2$, $A_{ij} = I_n$, we shall consider the system:

$$Lu := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + A(x,y) \frac{\partial u}{\partial x} + B(x,y) \frac{\partial u}{\partial y} + C(x,y) u = f(x,y), \tag{12}$$

where $A, B, C \in C(\overline{\Omega}, M_n(\mathbb{R}))$, $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $\Omega \subseteq \mathbb{R}^2$ is a bounded domain.

If $m_F^A \neq 0$, $m_F^B \neq 0$, then we have the following result:

Theorem 6. *Suppose that:*

$$e^* C(x,y) e \leq -\frac{1}{4} n \left[\left(\gamma_F^A m_F^A \right)^2 + \left(\gamma_F^B m_F^B \right)^2 \right], \tag{13}$$

$\forall e \in C^2(\overline{\Omega}, \mathbb{R}^n), \|e\| = 1, \forall (x,y) \in \Omega$.

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n), u \neq 0$, is a solution of (12), then:

$$|u(x,y)| \leq \max \left\{ \max_{(x,y) \in \partial\Omega} |u(x,y)|, \frac{4}{p^2(x,y)} \max_{(x,y) \in \overline{\Omega}} |f(x,y)| \right\}, (x,y) \in \overline{\Omega},$$

where

$$p^2(x,y) = n \left(\gamma_F^A m_F^A \right)^2 + n \left(\gamma_F^B m_F^B \right)^2 - \|A(x,y) - \widetilde{\alpha}(x,y) I_n\|^2 - \left\| B(x,y) - \widetilde{\beta}(x,y) I_n \right\|^2.$$

Proof. According to Theorem 2 and Remark 1, if $m_F^A \neq 0, m_F^B \neq 0$, for each $(x,y) \in \Omega$, there exist $\widetilde{\alpha}(x,y), \widetilde{\beta}(x,y) \in \mathbb{R}$ such that:

$$\|A(x,y) - \widetilde{\alpha}(x,y)I_n\| < \sqrt{n}\gamma_F^A m_F^A$$

$$\left\|B(x,y) - \widetilde{\beta}(x,y)I_n\right\| < \sqrt{n}\gamma_F^B m_F^B$$

We have:

$$e^*Le = -\|e'_x\|^2 - \left\|e'_y\right\|^2 + e^*(A(x,y) - \widetilde{\alpha}(x,y)I_n)e'_x + e^*(B(x,y) - \widetilde{\beta}(x,y)I_n)e'_y +$$

$$+e^*C(x,y)e \leq \frac{1}{4}\|A(x,y) - \widetilde{\alpha}(x,y)I_n\|^2 + \frac{1}{4}\left\|B(x,y) - \widetilde{\beta}(x,y)I_n\right\|^2 + e^*C(x,y)e \leq$$

$$\leq \frac{1}{4}\left[\|A(x,y) - \widetilde{\alpha}(x,y)I_n\|^2 + \left\|B(x,y) - \widetilde{\beta}(x,y)I_n\right\|^2 - n\left(\gamma_F^A m_F^A\right)^2 - n\left(\gamma_F^B m_F^B\right)^2\right] =$$

$$= -p^2(x,y) < 0.$$

So $e^*Le \leq -p^2$ and hence and from Theorem 4, Theorem 6 is proved.

Remark 4. *In case that* $m_F^A = 0, m_F^B \neq 0$, *if* $e^*C(x,y)e \leq -\frac{1}{4}n\left(\gamma_F^B m_F^B\right)^2$, *then the conclusion holds with* $p^2(x,y) = n\left(\gamma_F^B m_F^B\right)^2 - \left\|B(x,y) - \widetilde{\beta}(x,y)I_n\right\|^2$.

Example 3. *Let us consider the system (12) with* $A = B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}$. *We suppose that* $a_2, a_3 > 0$. *In this case we shall have:* $\widetilde{\alpha} = \widetilde{\beta} = a_1$, $\gamma_F^A = \gamma_F^B = \frac{a_2+a_3}{\sqrt{a_2 a_3}}$, $m_F^A = m_F^B = \sqrt{2a_2a_3}$, $A - \widetilde{\alpha}I_2 = B - \widetilde{\beta}I_2 = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}$.

$$e^*C(x,y)e \leq -2(a_2 + a_3)^2, \tag{14}$$

$$e^*Le \leq \frac{a_2^2 + a_3^2}{2} - 2(a_2 + a_3)^2 < 0.$$

If (14) holds and $u \in C^2(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega}, \mathbb{R}^2), u \neq 0$, *is a solution of (12), we have:*

$$|u(x,y)| \leq \max\left\{\max_{(x,y)\in\partial\Omega}|u(x,y)|, \frac{2}{3a_2^2 + 8a_2a_3 + 3a_3^2}\max_{(x,y)\in\overline{\Omega}}|f(x,y)|\right\}, (x,y) \in \overline{\Omega}.$$

Let us consider now $A_{ij} = a_{ij}I_n$, $a_{ij} \in C(\overline{\Omega})$. System (2) becomes:

$$Lu := \sum_{i,j=1}^{m} a_{ij}(x)I_n\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} A_i(x)\frac{\partial u}{\partial x_i} + A_0(x)u = f(x), \tag{15}$$

where $A_i, A_0 \in C(\overline{\Omega}, M_n(\mathbb{R})), f \in C(\overline{\Omega}, \mathbb{R}^n)$.

If $m_F^i \neq 0$, then we have the following result:

Theorem 7. *Suppose (3) holds and:*

$$e^*A_0(x)e \leq -\frac{1}{4\delta^2}n\sum_{i=1}^{m}(\gamma_F^i m_F^i)^2, \forall e \in C^2(\Omega, \mathbb{R}^n), \|e\| = 1, \forall x \in \Omega. \tag{16}$$

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n), u \neq 0$, *is a solution of (15), then:*

$$|u(x)| \leq \max\left\{\max_{x\in\partial\Omega}|u(x)|, \frac{4\delta^2}{n\sum_{i=1}^{m}(\gamma_F^i m_F^i)^2 - \sum_{i=1}^{m}\|A_i(x) - \widetilde{\alpha}_i(x)I_n\|^2}\max_{x\in\overline{\Omega}}|f(x)|\right\},\ x\in\overline{\Omega}.$$

Remark 5. *In case that* $m_F^i = 0$, *if there exist* $p \neq 0$ *such that* $e^*A_0(x)e \leq -p^2$, *then:*

$$|u(x)| \leq \max\left\{\max_{x\in\partial\Omega}|u(x)|, \frac{1}{p^2}\max_{x\in\overline{\Omega}}|f(x)|\right\}, x \in \overline{\Omega}.$$

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BIVARIATE SPLINE-POLYNOMIAL INTERPOLATION

GH. COMAN AND MARIUS BIROU

Let $\Delta \subseteq \mathbb{R}^2$ be an arbitrary domain, f a real-valued function defined on Δ, $Z = \{z_i |\ z_i = (x_i, y_i),\ i = \overline{1,N}\} \subset \Delta$ and $I(f) = \{\lambda_k f |\ k = 1, \ldots, N\}$ a set of informations about f (evaluations of f and of certain of its derivatives at $z_1, \ldots, z_N$).

A general interpolation problem is: for a given function f find a function g that interpolates the data $I(f)$ i.e.

$$\lambda_k f = \lambda_k g, \quad k = \overline{1,N}.$$

A solution of such a problem can be obtain by the generalization of the bivariate Lagrange formula for the rectangular grid $\Pi = \{x_0, \ldots, x_m\} \times \{y_0, \ldots, y_n\}$:

$$f(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v(y)}{(y-y_j)v'(y_j)} f(x_i, y_j) + (R_{mn}f)(x,y) \qquad (1)$$

where

$$(R_{mn}f)(x,y) = u(x)[x, x_0, \ldots, x_m; f(\cdot, y)] + \sum_{i=0}^{m} \frac{u(x)v(y)}{(x-x_i)u'(x_i)} [y, y_0, \ldots, y_n; f(x_i, \cdot)]$$

with $u(x) = (x-x_0)\ldots(x-x_m)$ and $v(y) = (y-y_0)\ldots(y-y_n)$.

A first generalization of the formula (1) was given by J.F. Steffensen [4]:

$$f(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v_i(y)}{(y-y_j)v_i'(y_j)} + (R_{m,n_i}f)(x,y) \qquad (2)$$

where

$$(R_{m,n_i}f)(x,y) = u(x)[x, x_0, \ldots, x_m; f(\cdot, y)] + \sum_{i=0}^{m} \frac{u(x)v_i(y)}{(x-x_i)u'(x_i)} [y, y_0, \ldots, y_{in_i}; f(x_i, \cdot)]$$

with

$$v_i(y) = (y-y_0)\ldots(y-y_{n_i}).$$

The interpolation grid here is $\Pi_1 = \{(x_i, y_{ij}) |\ i = \overline{0,m},\ j = \overline{0,n_i}\}$.

A second generalization of the Lagrange interpolation formula (1), that is also an extension of the Steffensen formula (2) was given by D.D. Stancu [2]:

$$f(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f(x_i, y_{ij}) +$$
$$+ (R_{m,n_i}f)(x,y) \qquad (3)$$

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where

$$(R_{m,n_i}f)(x,y) = u(x)[x,x_0,\dots,x_m;f(\cdot,y)] + \sum_{i=0}^{m} \frac{u(x)v_i(y)}{(x-x_i)u'(x_i)}[y,y_{i0},\dots,y_{in_i};f(x,\cdot)]$$

with $v_i(y) = (y-y_{i0})\dots(y-y_{in_i})$ and the interpolation set

$$\Pi_2 = \{(x_i,y_{ij})|\ i=\overline{0,m},\ j=\overline{0,n_i}\}.$$

Remark 1. The Steffensen formula (3) does not solve the general interpolation problem, Π_1 is only a particular case of the interpolatory set $\{z_1,\dots,z_N\}$.

Remark 2. Formula (3) is really a solution of the considered general problem. Indeed, let $Z_k \subset Z$ be the set of nodes (x_i,y_i), $i=\overline{1,N}$ with the same abscises x_k, i.e. $Z_k = \{(x_k,y_{kj})|\ j=\overline{0,n_k}\}$ for all $k=0,1,\dots,m$. We have $Z_i \neq Z_j$ for $i \neq j$ and $Z = Z_0 \cup \dots \cup Z_m$.

If L_m^x is the Lagrange's operator for the interpolates nodes $x_0,\dots,x_m$ and $L_{n_i}^y$, $i=\overline{0,m}$ are the Lagrange's operators for the nodes $y_{i0},\dots,y_{in_i}$ respectively, then we have

$$f = L_m^x f + R_m^x f \tag{4}$$

with

$$(L_m^x f)(x,y) = \sum_{i=0}^{m} \frac{u(x)}{(x-x_i)u'(x_i)} f(x_i,y)$$

and

$$f(x_i,\cdot) = (L_{n_i}^y f)(x_i,\cdot) + (R_{n_i}^y f)(x_i,\cdot), \quad i=\overline{0,m} \tag{5}$$

with

$$(L_{n_i}^y f)(x_i,y) = \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f(x_i,y_{ij}).$$

If the remainder terms are written with the divided differences, from (4) and (5) follows formula (3).

Remark 3. Usually the degree m of the operator L_m^x is more greater than the largest degree of $L_{n_i}^y$ i.e. $m \gg \max\{n_0,\dots,n_m\}$, which imply a large computational complexity of the polynomial interpolation from (3), say

$$(Pf)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v_i(y)}{(y-y_{ij})v'(y_{ij})} f(x_i,y_{ij}).$$

From this reason and the another ones, instead of Lagrange polynomial operator L_m^x we can use a spline interpolation function of Lagrange, Hermite or Birkhoff type.

1. Spline polynomial interpolation of Lagrange type

Let $S_{L,2n-1}^x$ be the spline interpolation operator of the degree $2n-1$, that interpolates the function f with regard to the variable x at the nodes (x_k,y), $k=\overline{0,m}$ i.e.

$$(S_{L,2n-1}^x f)(x,y) = \sum_{i=0}^{n-1} a_i x^i + \sum_{j=0}^{m} b_j (x-x_j)_+^{2n-1} \tag{6}$$

for which

$$\begin{cases} (S^x_{L,2n-1}f)(x_k,y) = f(x_k,y), & k=\overline{0,m} \\ (S^x_{L,2n-1}f)^{(p,0)}(\alpha,y) = 0, & p=\overline{n,2n-1},\ \alpha > x_m \end{cases} \tag{7}$$

The spline function of Lagrange type can also be written in the form

$$(S^x_{L,2n-1}f)(x,y) = \sum_{k=0}^{m} s_k(x) f(x_k,y)$$

where s_k are the corresponding cardinal splines i.e., they are of the same form (6), but with the interpolatory conditions

$$s_k(x_j) = \delta_{kj}, \quad k,j=\overline{0,m}.$$

This way, formula (3) becomes

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} s_i(x)\frac{v_i(y)}{(y-y_{ij})v'(y_{ij})} f(x_i,y_{ij}) + (Rf)(x,y) \tag{8}$$

where $(Rf)(x,y)$ is the remainder term.

Taking into account that for $f(\cdot,y) \in C^n[x_0,x_m]$

$$(R^x_{L,2n-1}f)(x,y) = \int_{x_0}^{x_m} \varphi_n(x,s) f^{(n,0)}(s,y)ds$$

with

$$\varphi_n(x,s) = R^x\left[\frac{(x-s)_+^{n-1}}{(n-1)!}\right]$$

it follows

Theorem 1. If $f \in C^{n,0}(\Delta)$ then

$$(Rf)(x,y) = \int_{x_0}^{x_m} \varphi_n(x,s) f^{(n,0)}(s,y)ds + \tag{9}$$

$$+\sum_{i=0}^{m} s_i(x)v_i(y)[y,y_{i0},\dots,y_{in_i};f(x_i,\cdot)]$$

and if $f \in C^{n,p+1}(\Delta)$ with $p=\max\{n_0,\dots,n_m\}$ we have

$$(Rf)(x,y) = \int_{x_0}^{x_m} \varphi_n(x,s) f^{(n,0)}(s,y)ds + \tag{10}$$

$$+\sum_{i=0}^{m} s_i(x) \int_{y_{i0}}^{y_{in_i}} \psi_{n_i}(y,t) f^{(0,n_i+1)}(x_i,t)$$

with

$$\psi_{n_i}(y,t) = \frac{(y-t)_+^{n_i}}{n_i!} - \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})}\frac{(y_{ij}-t)_+^{n_i}}{n_i!}.$$

2. Spline polynomial interpolation of Hermite type

Let $S^x_{H,2n-1}$ be the spline interpolation operator of the degree $2n-1$, that interpolates the function f and certain of its derivatives with regard to the variable x at the nodes (x_k, y), $k=\overline{0,m}$, i.e.

$$(S^x_{H,2n-1}f)(x,y) = \sum_{i=0}^{n-1} a_i x^i + \sum_{k=0}^{m}\sum_{j=0}^{q_k} b_{kj}(x-x_k)_+^{2n-j-1} \tag{11}$$

for which

$$\begin{cases} (S^x_{H,2n-1}f)^{(j,0)}(x_k,y) = f^{(j,0)}(x_k,y), & k=\overline{0,m},\ j=\overline{0,q_k} \\ (S^x_{H,2n-1}f)^{(p,0)}(\alpha,y) = 0, & p=\overline{n,2n-1},\ \alpha > x_m \end{cases} \tag{12}$$

The spline function of Hermite type can also be written in the form

$$(S^x_{H,2n-1}f)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{q_k} s_{kj}(x) f^{(j,0)}(x_k,y)$$

where s_{kj} are the corresponding cardinal splines i.e., they are of the same form (11), but with the interpolatory conditions

$$\begin{cases} s_{kj}^{(q)}(x_\nu) = 0, & k=\overline{0,m},\ \nu \neq k,\ q=\overline{0,q_\nu} \\ s_{kj}^{(q)}(x_k) = \delta_{jq}, & q=\overline{0,q_k} \\ s_{kj}^{(p)}(\alpha) = 0, & p=\overline{n,2n-1},\ \alpha > x_m \end{cases}$$

This way, formula (3) becomes

$$f(x,y) = \sum_{i=0}^{m}\sum_{l=0}^{q_i} s_{il}(x) \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f^{(l,0)}(x_i,y_{ij}) + (Rf)(x,y) \tag{13}$$

where

$$v_i(y) = (y-y_{i0})\dots(y-y_{in_i})$$

and $(Rf)(x,y)$ is the remainder term.

In this case the set of information about f is

$$I(f) = \{f^{(l,0)}(x_i,y_{ij}) |\ i=\overline{0,m},\ j=\overline{0,n_i},\ l=\overline{0,q_i}\} \tag{14}$$

Taking into account that for $f(\cdot,y) \in C^n[x_0,x_m]$

$$(R^x_{H,2n-1}f)(x,y) = \int_{x_0}^{x_m} \varphi_H(x,s) f^{(n,0)}(s,y)ds$$

with

$$\varphi_H(x,s) = \frac{(x-s)_+^{n-1}}{(n-1)!} - \sum_{i=0}^{m}\sum_{l=0}^{q_i} s_{il}(x)\frac{(x_i-s)_+^{n-l-1}}{(n-l-1)!}$$

and that

$$(Rf)(x,y) = (R^x_{H,2n-1}f)(x,y) + \sum_{i=0}^{m}\sum_{l=0}^{q_i} s_{il}(x)(R^y_{L,n_i} f^{(l,0)})(x_i,y)$$

it follows

Theorem 2. If $f \in C^{n,0}(\Delta)$ then

$$(Rf)(x,y) = \int_{x_0}^{x_m} \varphi_H(x,s) f^{(n,0)}(s,y)ds + \tag{15}$$

$$+ \sum_{i=0}^{m} \sum_{l=0}^{q_i} s_{il}(x) v_i(y) [y, y_{i0}, \ldots, y_{in_i}; f^{(l,0)}(x_i, \cdot)]$$

and if $f \in C^{n,p+1}(\Delta)$ with $p = \max\{n_0, \ldots, n_m\}$ then

$$(Rf)(x,y) = \int_{x_0}^{x_m} \varphi_H(x,s) f^{(n,0)}(s,y)ds + \tag{16}$$

$$+ \sum_{i=0}^{m} \sum_{l=0}^{q_i} s_{il}(x) \int_{y_{i0}}^{y_{i,n_i}} \psi_{n_i}(y,t) f^{(l,n_i+1)}(x_i,t)dt$$

with

$$\psi_{n_i}(y,t) = \frac{(y-t)_+^{n_i}}{n_i!} - \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} \frac{(y_{ij}-t)_+^{n_i}}{n_i!}$$

3. Spline polynomial interpolation of Birkhoff type

Let $S^x_{B,2n-1}$ be the spline interpolation operator of the degree $2n-1$, that interpolates the function f and certain of its partial derivatives to the variable x at the nodes (x_k, y), $k = \overline{0,m}$, i.e.

$$(S^x_{B,2n-1} f)(x,y) = \sum_{i=0}^{n-1} a_i x^i + \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj} (x - x_k)_+^{2n-j-1} \tag{17}$$

for which

$$\begin{cases} (S^x_{B,2n-1} f)^{(j,0)}(x_k, y) = f^{(j,0)}(x_k, y), & k = \overline{0,m},\ j \in I_k \\ (S^x_{B,2n-1} f)^{(p,0)}(\alpha, 0) = 0, & p = \overline{n, 2n-1},\ \alpha > x_m \end{cases} \tag{18}$$

The spline function of Birkhoff type can also be written in the form

$$(S^x_{B,2n-1} f)(x,y) = \sum_{k=0}^{m} \sum_{j \in I_k} s_{kj}(x) f^{(j,0)}(x_k, y)$$

where s_{kj} are the corresponding cardinal splines i.e., they are of the same form (17), but with the interpolatory conditions:

$$\begin{cases} s_{kj}^{(q)}(x_\nu) = 0, & k = \overline{0,m},\ \nu \neq k,\ q \in I_\nu \\ s_{kj}^{(q)}(x_k) = \delta_{jq}, & q \in I_k \\ s_{kj}^{(p)}(\alpha) = 0, & p = \overline{n, 2n-1},\ \alpha > x_m \end{cases}$$

If the set of informations of f is

$$I(f) = \{ f^{(l,0)}(x_i, y_{ij}) |\ i = \overline{0,m},\ j = \overline{0,n_i},\ l \in I_i \} \tag{19}$$

we can use the interpolation formula of Lagrange

$$f^{(l,0)}(x_i, y) = (L^y_{n_i} f^{(l,0)})(x_i, y) + (R^y_{L,n_i} f^{(l,0)})(x_i, y) \tag{20}$$

This way, formula (3) becomes

$$f(x,y)=\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)\sum_{j=0}^{n_i}\frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})}f^{(l,0)}(x_i,y_{ij})+(Rf)(x,y) \tag{21}$$

where $(Rf)(x,y)$ is the remainder term.

Taking into account that for $f(\cdot,y)\in C^n[x_0,x_m]$

$$(R^x_{B,2n-1}f)(x,y)=\int_{x_0}^{x_m}\varphi_B(x,s)f^{(n,0)}(s,y)ds$$

with

$$\varphi_B(x,s)=\frac{(x-s)_+^{n-1}}{(n-1)!}-\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)\frac{(x_i-s)_+^{n-l-1}}{(n-l-1)!}$$

and that

$$(Rf)(x,y)=(R^x_{B,2n-1}f)(x,y)+\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)(R^y_{L,n_i}f^{(l,0)})(x_i,y)$$

it follows

Theorem 3. If $f\in C^{n,0}(\Delta)$ then

$$(Rf)(x,y)=\int_{x_0}^{x_m}\varphi_B(x,s)f^{(n,0)}(s,y)+ \tag{22}$$

$$+\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)v_i(y)[y,y_{i0},\dots,y_{in_i};f^{(l,0)}(x_i,\cdot)]$$

with $v_i(y)=(y-y_{i0})\dots(y-y_{in_i})$ and if $f\in C^{n,p+1}(\Delta)$ with $p=\max\{n_0,\dots,n_m\}$ then

$$(Rf)(x,y)=\int_{x_0}^{x_m}\varphi_B(x,s)f^{(n,0)}(s,y)ds+ \tag{23}$$

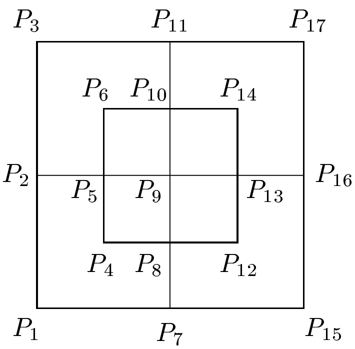
$$+\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)v_i(y)\int_{y_{i0}}^{y_{in_i}}\psi_{n_i}(y,t)f^{(l,n_i+1)}(x_i,t)dt$$

where

$$\psi_{n_i}(y,t)=\frac{(y-t)_+^{n_i}}{n_i!}-\sum_{j=0}^{n_i}\frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})}\frac{(y_{ij}-t)_+^{n_i}}{n_i!}$$

4. **Example**

One considers the function $f(x,y)=\exp(-x^2-y^2)$ on the rectangular domains $\Delta=[-1,1]\times[-1,1]$ and the interpolation nodes P_1-P_{17}



$P_1 = (-1,-1)$
$P_2 = (-1,0)$
$P_3 = (-1,1)$
$P_4 = (-0.5,-0.5)$
$P_5 = (-0.5,0)$
$P_6 = (-0.5,0.5)$
$P_7 = (0,-1)$
$P_8 = (0,-0.5)$
$P_9 = (0,0)$
$P_{10} = (0,0.5)$
$P_{11} = (0,1)$
$P_{12} = (0.5,-0.5)$
$P_{13} = (0.5,0)$
$P_{14} = (0.5,0.5)$
$P_{15} = (1,-1)$
$P_{16} = (1,0)$
$P_{17} = (1,1)$

We will use the formulas (8), (13) and (21) for $n = 2$ (cubic spline with regard the variable x).

In fig. 1 is given the graph for the function f.

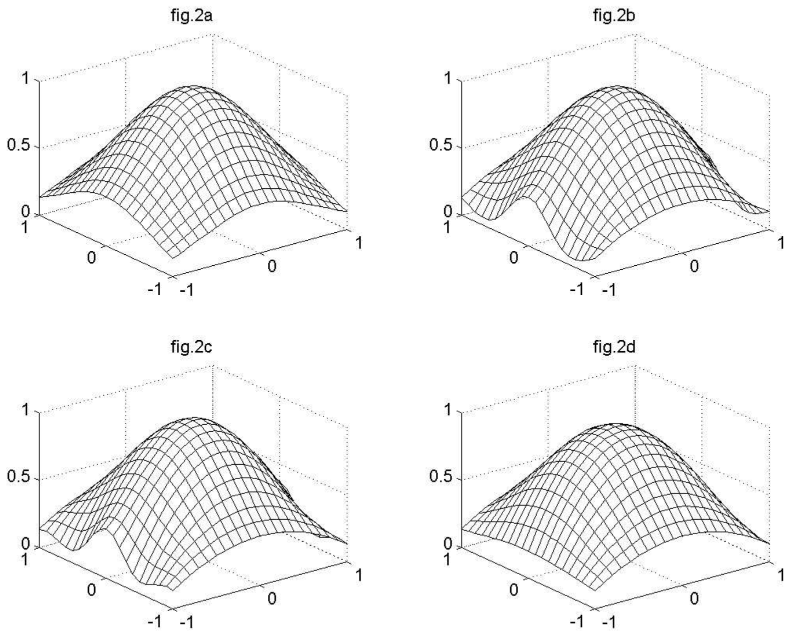
In fig. 2 is used the information of Lagrange type.

In fig. 3 is used information of Hermite type, i.e.

$$\{f^{(j,0)}(P_i): \ i = \overline{1,17}, \ j = 0,1\}$$

In fig. 4 is used a set of information of Birkhoff type:

$$\{f(P_i): \ i = 1,2,3,15,16,17\} \cup \{f^{(1,0)}(P_i): \ i = \overline{4,14}\}.$$



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FREE CONVECTION BOUNDARY LAYER OVER A VERTICAL PERMEABLE CONE EMBEDDED IN A FLUID SATURATED POROUS MEDIUM WITH INTERNAL HEAT GENERATION

TEODOR GROŞAN AND ŞERBAN RAREŞ POP

1. Introduction

Heat transfer through porous media has important practical applications such as oil extraction, thermal insulation, geophysical flows, water waste disposal, etc. Recent monographs by Ingham and Pop (1998, 2002), Nield and Bejan (1999), Vafai (2000) and Pop and Ingham (2001) give excellent summary of the work on the subject.

The phenomenon of internal heat generation is present in many situations, especially in the field of nuclear energy and composite superconductors (see Horvat et al., 2001; Malinowski, 1993). Studies in natural convection driven by internal heat generation has been done by Roberts (1967), Jahn and Reinke (1974), Hardee and Nilson (1977), Stewart and Dona (1988), Crepeau and Clarksean (1997), etc.

The present paper studies the free convection from a vertical permeable cone embedded in a fluid saturated porous medium the effects of internal heat generation being present. The case of a variable temperature at the cone surface is considered, see Fig. 1.

2. Basic equation

Under Boussinesq and boundary layer approximation the governing equations can be written as:

$$\frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial y} = 0 \tag{1}$$

$$u = \frac{g \cos\gamma K \beta}{\nu}(T - T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \frac{q^{'''}}{\rho C_p} \tag{3}$$

where $r = x\cos\gamma$ is the cone radius, ν is the kinematic viscosity, K the permeability,α_m is the thermal diffusivity, $q^{'''}$ is the internal heat, ρ the density and C_p is the specific heat at constant pressure. Indexes w and ∞ refer to the cone surface and ambient conditions.

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Further we introduce the stream function ψ given by

$$ru = \frac{\partial \psi}{\partial x}, \ \ rv = \frac{\partial \psi}{\partial y} \tag{4}$$

so that the equations (1)-(3) become:

$$\frac{1}{r}\frac{\partial \psi}{\partial y} = \frac{g\cos\gamma K\beta}{\nu}(T - T_\infty) \tag{5}$$

$$\frac{1}{r}\left(\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y}\right) = \alpha_m \frac{\partial^2 T}{\partial y^2} + \frac{q^{'''}}{\rho C_p} \tag{6}$$

subject to the boundary conditions:

$$-\frac{\partial \psi}{\partial x} = ax^n, \ T_w = T_\infty + Ax^\lambda \ for \ y = 0 \tag{7}$$

$$\frac{\partial \psi}{\partial y} \to 0, \ T \to T_\infty \ for \ y \to 0 \tag{8}$$

where A is a positive constant and a, n, λ are constants with $a > 0$ for injection and $a < 0$ for suction.

In order to solve equations (5)-(6)we introduce the following similarity variables:

$$\psi = \alpha_m r Ra_x^{1/2} f(\eta), \ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \ \eta = Ra_x^{1/2}(y/x) \tag{9}$$

where Ra is the local Rayleigh number defined as by:

$$Ra_x = \frac{g\beta K\cos\gamma(T - T_\infty)x}{\nu\alpha_m} = \frac{g\beta K\cos\gamma A x^{\lambda+1}}{\nu\alpha_m} \tag{10}$$

In order that the similarity solution of equations (5)-(6) exist, we assume following Crepeau and Clarksean (1997), that the internal heat generation q''' is given by:

$$q^{'''} = \frac{k_m(T_w - T_\infty)}{x^2} Ra_x e^{-\eta} \tag{11}$$

On using (9) and (11) in (5) and (6) we obtain the following ordinary differential equations of the motion:

$$f' = \theta \tag{12}$$

$$\theta^{''} + \frac{\lambda+3}{2} f\theta' - f\lambda f'\theta + e^{-\eta} = 0 \tag{13}$$

Combining (12) and (13) we get

$$f^{'''} + \frac{\lambda+3}{2} f\theta' - \lambda f'^2 + e^{-\eta} = 0 \tag{14}$$

along with the boundary conditions

$$f(0) = -f_w, \ f'(0) = 1, \ f \to 0 \ for \ \eta \to \infty \tag{15}$$

where f_w is the mass flux parameter given by

$$f_w = -\frac{2a}{\lambda+3}\left(\frac{\alpha_m \nu}{g\beta K\cos\gamma A}\right)^{1/2} \tag{16}$$

For the above similar equations we considered that

$$n = \frac{\lambda - 1}{2} \tag{17}$$

as it was found in Postelnicu et al. (2000), for the corresponding flat plate case.

In this case, the local Nusselt number is given by:

$$Nu_x/Ra_x^{1/2} = -f''(\lambda, 0) \tag{18}$$

3. **Results and discussions**

Equation (14) with the boundary condition (17) has been solved numerically using a shooting method (see, Chakraborty 1998) for $\lambda = 0, 1/3$ and $1/2$ and $f_w = -2, -1, -0.5, 0, 0.5, 1$ and 2. Results obtained for the Nusselt number were compared in Table 1 with the results previously obtained by Cheng et al.(1985), and we can see that the results are in a very good agreement.

This table shows also that the presence of internal heat generation leads to the decrease of the local heat transfer. Figures 2-4 present the non-dimensional temperature profiles in the absence of internal heat generation and figures 5-7 the same profiles when internal heat generation is present. It can be seen from these figures that the thickness of boundary layer increases with the increase of the mass flux parameter, f_w. This phenomenon is more significant for law values of λ.

Figure 8 shows the variation of Nusselt number, $f''(\lambda, 0)$, with the mass flux parameter f_w. It is noticed that in the both cases with and without internal heat generation, the heat transfer is more significant for higher values of the mass flux parameter.

Table 1. Values of the local Nusselt number, $-f''(\lambda, 0)$

λ	Without internal heat generation		With internal heat generation
	Cheng et al. (1985)	present results	present results
0	0.769	0.7687	0.1963
1/3	0.921	0.9210	0.3937
1/2	0.992	0.9900	0.4799

Figure 1. Physical model

Figure 2. Velocity profiles for $\lambda = 0$ and some values of the mass flux parameter f_w when the effect of internal heat generation is not present

Figure 3. Velocity profiles for $\lambda = 1/3$ and some values of the mass flux parameter f_w when the effect of internal heat generation is not present

Figure 4. Velocity profiles for $\lambda = 1/2$ and some values of the mass flux parameter f_w when the effect of internal heat generation is not present

Figure 5. Velocity profiles for $\lambda = 0$ and some values of the mass flux parameter f_w when the effect of internal heat generation is present

Figure 6. Velocity profiles for $\lambda = 1/3$ and some values of the mass flux parameter f_w when the effect of internal heat generation is present

Figure 7. Velocity profiles for $\lambda = 1/2$ and some values of the mass flux parameter f_w when the effect of internal heat generation is present

Figure 8. Variation of the local Nusselt number, $-f''(0)$, with the mass flux parameter f_w when the effect of internal heat generation is not present (........) and when the effect of internal heat generation is present (____)

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CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

AMELIA ANCA HOLHOŞ

Abstract. In this paper we define and study new classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ of univalent functions with negative coefficients.

1. Introduction

Let $\mathbf{U}$ denote the open unit disc: $\mathbf{U} = \{z\ ;\ z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathbf{S}$ denote the class of functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic and univalent in $\mathbf{U}$.

For $f \in \mathbf{S}$ we define the differential operator $\mathbf{D}^n$ (Sălăgean [1])

$$\begin{aligned} \mathbf{D}^0 f(z) &= f(z) \\ \mathbf{D}^1 f(z) &= \mathbf{D}f(z) = zf'(z) \end{aligned}$$

and

$$\mathbf{D}^n f(z) = \mathbf{D}(\mathbf{D}^{n-1} f(z)) \quad ; \quad n \in \mathbb{N}^* = \{1, 2, 3, ...\}.$$

We note that if

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

then

$$\mathbf{D}^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j\ ;\ z \in \mathbf{U}.$$

Let T denote the subclass of $\mathbf{S}$ which can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} |a_k|\, z^k \tag{1}$$

We say that a function $f \in T$ is in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ $\quad 0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, $0 < B \le 1$ if

$$\left| \frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1}{(B-A)\gamma \left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - \alpha \right] - B \left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1 \right]} \right| < \beta \quad , \quad z \in \mathbf{U} \tag{2}$$

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$$\frac{B}{B-A} < \gamma \leq \begin{cases} \dfrac{B}{(B-A)\alpha} & ; \quad \alpha \neq 0 \\ 1 & ; \quad \alpha = 0 \end{cases} \tag{3}$$

where

$$F_{n,\lambda}(z) = (1-\lambda)D^n f(z) + \lambda D^{n+1} f(z) \ \ ; \ \lambda \geq 0 \ \ ; \ f \in T \tag{4}$$

Remark 1.

$$\begin{aligned} F_{0,\lambda}(z) &= z - \sum_{k=2}^{\infty} [1+(k-1)\lambda] \, |a_k| \, z^k \\ F_{1,\lambda}(z) &= z - \sum_{k=2}^{\infty} k\,[1+(k-1)\lambda] \, |a_k| \, z^k \\ &\ldots\ldots\ldots\ldots\ldots \\ F_{n,\lambda}(z) &= z - \sum_{k=2}^{\infty} k^n\,[1+(k-1)\lambda] \, |a_k| \, z^k \end{aligned} \tag{5}$$

For $n = 0$, $T_{0,\lambda}(A,B,\alpha,\beta,\gamma) = T^*_\lambda(A,B,\alpha,\beta,\gamma)$ *and for* $n = 1$, $T_{1,\lambda}(A,B,\alpha,\beta,\gamma) = C^*_\lambda(A,B,\alpha,\beta,\gamma)$.

The class $T^*_\lambda(A,B,\alpha,\beta,\gamma)$ *and* $C^*_\lambda(A,B,\alpha,\beta,\gamma)$ *was studied by S.B.Joshi and H.M.Srivastava* [3] *and S.B.Joshi* [2].

2. **Characterization theorem**

Theorem 2. *Let* $f \in T$, $f(z) = z - \sum\limits_{k=2}^{\infty} |a_k| \, z^k$. *Then* $f(z)$ *is in the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *if and only if*

$$\begin{aligned} &\sum_{k=2}^{\infty} |a_k| \, k^n \, [1+\lambda(k-1)] \, \{(k-1) + \beta \, [(B-A)\gamma(k-\alpha) - B(k-1)]\} \leq \\ \leq \ \ &\beta\gamma(B-A)(1-\alpha) \end{aligned} \tag{6}$$

and the result is sharp.

If we denote

$$\begin{aligned} &D_n\,(k,A,B,\alpha,\beta,\gamma,\lambda) = \\ = \ \ &k^n\,[1+\lambda\,(k-1)]\,\{(k-1) + \beta\,[(B-A)\,\gamma\,(k-\alpha) - B\,(k-1)]\} \end{aligned} \tag{7}$$

then (6) becomes

$$\sum_{k=2}^{\infty} |a_k| \, D_n\,(k,A,B,\alpha,\beta,\gamma,\lambda) \leq \beta\gamma(B-A)(1-\alpha) \tag{8}$$

Proof. Assume that

$$\begin{aligned} &\sum_{k=2}^{\infty} |a_k| \, k^n \, [1+\lambda(k-1)] \, \{(k-1) + \beta \, [(B-A)\gamma(k-\alpha) - B(k-1)]\} \leq \\ \leq \ \ &\beta\gamma(B-A)(1-\alpha) \end{aligned}$$

and let $|z| = 1$. Then we have

$$\begin{aligned}
&\left|zF'_{n,\lambda}(z) - F_{n,\lambda}(z)\right| - \\
&-\beta\left|(B-A)\gamma\left[zF'_{n,\lambda}(z) - \alpha F_{n,\lambda}(z)\right] - B\left[zF'_{n,\lambda}(z) - F_{n,\lambda}(z)\right]\right| \\
= \; &\left|zF'_{n,\lambda}(z) - F_{n,\lambda}(z)\right| - \\
&-\beta\left|\left[(B-A)\gamma - B\right] zF'_{n,\lambda}(z) + \left[B - (B-A)\gamma\alpha\right] F_{n,\lambda}(z)\right| \\
= \; &\left|\sum_{k=2}^{\infty} k^n (1-k)\left[1+(k-1)\lambda\right]|a_k|\, z^k\right| - \\
&-\beta\left|\left[(B-A)\gamma - B\right] z - \left[(B-A)\gamma - B\right]\sum_{k=2}^{\infty} k^{n+1}\left[1+(k-1)\lambda\right]|a_k|\, z^k + \right. \\
&\left. +\left[B-(B-A)\gamma\alpha\right] z - \left[B-(B-A)\gamma\alpha\right]\sum_{k=2}^{\infty} k^{n}\left[1+(k-1)\lambda\right]|a_k|\, z^k\right| \\
= \; &\left|\sum_{k=2}^{\infty} k^n (1-k)\left[1+(k-1)\lambda\right]|a_k|\, z^k\right| - \\
&-\beta\left|(B-A)\gamma(1-\alpha) z - \left[(B-A)\gamma - B\right]\sum_{k=2}^{\infty} k^{n+1}\left[1+(k-1)\lambda\right]|a_k|\, z^k - \right. \\
&\left. -\left[B-(B-A)\gamma\alpha\right]\sum_{k=2}^{\infty} k^{n}\left[1+(k-1)\lambda\right]|a_k|\, z^k\right| \\
\le \; &\sum_{k=2}^{\infty} k^n (k-1)\left[1+(k-1)\lambda\right]|a_k|\,|z|^k - \beta(B-A)\gamma(1-\alpha)|z| + \\
&+\beta\left[(B-A)\gamma - B\right]\sum_{k=2}^{\infty} k^{n+1}\left[1+(k-1)\lambda\right]|a_k|\,|z|^k \\
&+\beta\left[B-(B-A)\gamma\alpha\right]\sum_{k=2}^{\infty} k^{n}\left[1+(k-1)\lambda\right]|a_k|\,|z|^k \\
\le \; &\sum_{k=2}^{\infty} k^n\left[1+(k-1)\lambda\right]|a_k|\left\{(k-1) + \beta\left[(B-A)\gamma(k-\alpha) - B(k-1)\right]\right\} - \\
&\quad -\beta\gamma(B-A)(1-\alpha) \le 0
\end{aligned}$$

Consequently, by the maximum modulus theorem , the functions $f(z)$ is in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Conversely, assume that

$$\left|\frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1}{(B-A)\gamma\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - \alpha\right] - B\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1\right]}\right| < \beta \Longleftrightarrow$$

$$\left|\sum_{k=2}^{\infty} k^n (1-k)[1+(k-1)\lambda]|a_k| z^k\right|$$

$$\leq \beta \left|(B-A)\gamma(1-\alpha)z - [(B-A)\gamma - B]\sum_{k=2}^{\infty} k^{n+1}[1+(k-1)\lambda]|a_k| z^k - \right.$$

$$\left. - [B-(B-A\gamma\alpha)]\sum_{k=2}^{\infty} k^n [1+(k-1)\lambda]|a_k| z^k\right|$$

Since $|\mathrm{Re}(z)| \leq |z|$ for any z, we have

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=2}^{\infty} k^n (k-1)[1+(k-1)\lambda]|a_k| z^k}{\beta(B-A)\gamma(1-\alpha)z - \sum\limits_{k=2}^{\infty} k^n [1+(k-1)\lambda]|a_k|[(B-A)\gamma(k-\alpha) - B(k-1)] z^k}\right\} < \beta$$

Letting $z \to 1$ through real values, upon clearing the denominator in the last inequality we obtain

$$\sum_{k=2}^{\infty} k^n (k-1)[1+(k-1)\lambda]|a_k| \leq$$

$$\beta\gamma(B-A)(1-\alpha) - \sum_{k=2}^{\infty} k^n [1+(k-1)\lambda]|a_k|\beta[(B-A)\gamma(k-\alpha) - B(k-1)]$$

and this inequality gives the required condition.

The function

$$f(z) = z - \frac{\beta\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)\{1+\beta[(B-A)\gamma(2-\alpha)-B]\}} z^2$$

is an extremal function for the theorem. □

Remark 3. *For $n=0$ and $n=1$ the result of Theorem 1 was obtain by Joshi and Srivastava* [3].

3. **Closure Theorems**

Let the functions f_j be of the form:

$$f_j(z) = z - \sum_{k=2}^{\infty} |a_{kj}| z^k \quad ; \; z \in \mathbf{U} \quad ; \; j = 1, 2, ..., m \tag{9}$$

we shall prove the following results for the closure of functions in the classe $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Theorem 4. *Let the functions $f_j(z)$ defined by* (3.1) *be in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. Then the function $h(z)$, defined by*

$$h(z) = z - \sum_{k=2}^{\infty} |b_k| z^k \quad ; \quad \text{with } b_k = \frac{1}{m}\sum_{j=1}^{m} |a_{kj}| \tag{10}$$

also belongs to the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Proof. As $f_j(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ it follows from Theorem 1. that

$$\sum_{k=2}^{\infty} |a_{kj}| D_n(k,A,B,\alpha,\beta,\gamma,\lambda) \le \beta\gamma(B-A)(1-\alpha) \quad ; \quad j=1,2,...,m \tag{11}$$

Therefore

$$\begin{aligned} \sum_{k=2}^{\infty} |b_k| D_n(k,A,B,\alpha,\beta,\gamma,\lambda) &= \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) \frac{1}{m} \sum_{j=1}^{m} |a_{kj}| \\ &\le \beta\gamma(B-A)(1-\alpha) \end{aligned}$$

hence, by Theorem 1 ,

$$h(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma).$$

□

Remark 5. *For $n=0$ we obtain Theorem 1 as Joshi*[2]. *For $n=1$ we obtain Theorem 2 as Joshi*[2].

Theorem 6. *Let $f_j(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. Then the function $h(z)$, defined by,*

$$h(z) = \sum_{j=1}^{m} |d_j| f_j(z); \text{ where } \sum_{j=1}^{m} |d_j| = 1 \tag{12}$$

is also in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.

Proof. By using definition of $h(z)$, we have

$$\begin{aligned} h(z) &= \sum_{j=1}^{m} |d_j| \left[z - \sum_{k=2}^{\infty} |a_{kj}| z^k \right] = z \sum_{j=1}^{m} |d_j| - \sum_{k=2}^{\infty} \sum_{j=1}^{m} |d_j| |a_{kj}| z^k \\ &= z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} |d_j| |a_{kj}| z^k \right) \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) \left(\sum_{j=1}^{m} |d_j| |a_{kj}| \right) \\ &= \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) |a_{k1}| |d_1| + \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) |a_{k2}| |d_2| + \\ &\quad ... + \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) |a_{km}| |d_m| \\ &\le |d_1| \beta\gamma(B-A)(1-\alpha) + |d_2| \beta\gamma(B-A)(1-\alpha) + \\ &\quad ... + |d_m| \beta\gamma(B-A)(1-\alpha) \\ &= \beta\gamma(B-A)(1-\alpha) \sum_{j=1}^{m} |d_j| = \beta\gamma(B-A)(1-\alpha) \end{aligned}$$

which implies that $h(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. □

Remark 7. *For $n=0$ obtain Theorem 3 as Joshi*[2]. *For $n=1$ we obtain Theorem 4 as Joshi*[2].

Theorem 8. *Let the functions*

$$f_1(z) = z - \sum_{k=2}^{\infty} |a_{k1}| z^k \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$$

and

$$f_2(z) = z - \sum_{k=2}^{\infty} |a_{k2}| z^k \in T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma).$$

Then the function $p(z)$ *defined by*

$$p(z) = z - \frac{2}{3}\sum_{k=2}^{\infty} (|a_{k1} + a_{k2}|) z^k \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma).$$

Proof. Let $f_1(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ and $f_2(z) \in T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$; by using Theorem 1. we get , respectively ,

$$\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k1}| \leq \beta\gamma(B - A)(1 - \alpha) \tag{13}$$

and

$$\sum_{k=2}^{\infty} D_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| \leq \beta\gamma(B - A)(1 - \alpha) \tag{14}$$

We have (see (7))

$$\begin{aligned} 2\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| &\leq \sum_{k=2}^{\infty} D_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| \leq \\ &\leq \beta\gamma(B - A)(1 - \alpha) \end{aligned}$$

$$\frac{2}{3}\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k1}| \leq \frac{2}{3}\beta\gamma(B - A)(1 - \alpha)$$

$$\frac{2}{3}\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| \leq \frac{1}{3}\beta\gamma(B - A)(1 - \alpha) \Rightarrow$$

$$\frac{2}{3}\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) [|a_{k1}| + |a_{k2}|] \leq \beta\gamma(B - A)(1 - \alpha) \Rightarrow$$

$$p(z) = z - \frac{2}{3}\sum_{k=2}^{\infty} |a_{k1} + a_{k2}| z^k \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma).$$

□

4. **Integral Operators**

Theorem 9. *Let the functions* $f(z)$ *defined by* (1)*, be in the class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$*, and let* c *be a real number such that* $c > -1$.

Then the function $F(z)$ *defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\, dt \tag{15}$$

also belongs to the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Proof. By using the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} |b_k| z^k, where \quad |b_k| = \frac{c+1}{c+k} |a_k| \tag{16}$$

$$f \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma) \Rightarrow \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) |a_k| \leq \beta\gamma(B-A)(1-\alpha)$$

$$\begin{aligned} \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) |b_k| &= \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) \frac{c+1}{c+k} |a_k| < \\ &< \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) |a_k| \\ &\leq \beta\gamma(B-A)(1-\alpha) \end{aligned}$$

$\Rightarrow F(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. □

Theorem 10. *Let c be a real number such that $c > -1$. If $F(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ then the function $f(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

is univalent in $|z| < R$, where

$$R = \inf_k \left[\frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)}{\beta\gamma(B-A)(1-\alpha)(c+k)k} \right]^{\frac{1}{k-1}}, \ k \geq 2 \tag{17}$$

The result is sharp for

$$f(z) = z - \frac{\beta\gamma(B-A)(1-\alpha)(c+k) z^k}{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)}, \qquad k \geq 2 \tag{18}$$

Proof. Let $F(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$, it folows from (15) that

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{c+1} = z - \sum_{k=2}^{\infty} \frac{c+k}{c+1} |a_k| z^k \tag{19}$$

$$F(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma) \Rightarrow \sum_{k=2}^{\infty} D_n(k,A,B,\alpha,\beta,\gamma,\lambda) |a_k| \leq \beta\gamma(B-A)(1-\alpha) \Rightarrow$$

$$\sum_{k=2}^{\infty} \frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda) |a_k|}{\beta\gamma(B-A)(1-\alpha)} \leq 1 \tag{20}$$

If

$$\frac{k(c+k)|z|^{k-1}}{c+1} < \frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)}$$

or if

$$|z| < \left[\frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)}{\beta\gamma(B-A)(1-\alpha)k(c+k)} \right]^{\frac{1}{k-1}} \tag{21}$$

then

$$\begin{aligned} |f'(z)-1| &= \left| -\sum_{k=2}^{\infty} k \frac{c+k}{c+1} |a_k| z^{k-1} \right| \leq \sum_{k=2}^{\infty} k \frac{c+k}{c+1} |a_k| |z|^{k-1} < \\ &< \sum_{k=2}^{\infty} \frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)|a_k|}{\beta\gamma(B-A)(1-\alpha)} \leq 1 \end{aligned}$$

But from $|f'(z)-1|<1, |z|<R$, we deduce that f is univalent in the disc $|z|<R$.

The result is sharp and the extremal function is given by (18). □

Theorem 11. *Let* $c \in \mathbb{R}$, $c > -1$. *If*

$$F(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$$

then the function $f(z)$ *given by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,dt$$

is starlike of order p $(0 \leq p < 1)$ *in* $|z| < R^*(p,A,B,\alpha,\beta,\gamma)$ *where*

$$R^* = \inf_k \left[\frac{(1-p)(c+1) D_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{(k-p)(c+k)\beta\gamma(B-A)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad ; \quad k \geq 2.$$

The result is sharp.

Proof. Is sufficient to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| < (1-p)$, in $|z| < R^*$.

Now

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{-\sum_{k=2}^{\infty} (k-1) \frac{c+k}{c+1} |a_k| z^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{c+k}{c+1} |a_k| z^{k-1}} \right| \leq \frac{\sum_{k=2}^{\infty} (k-1) \frac{c+k}{c+1} |a_k| |z|^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{c+k}{c+1} |a_k| |z|^{k-1}} < 1-p$$

provided

$$\sum_{k=2}^{\infty} \left(\frac{k-p}{1-p} \right) \left(\frac{c+k}{c+1} \right) |a_k| |z|^{k-1} < 1$$

By using

$$\sum_{k=2}^{\infty} \frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)|a_k|}{\beta\gamma(B-A)(1-\alpha)} \leq 1$$

the inequality

$$\sum_{k=2}^{\infty} \left(\frac{k-p}{1-p} \right) \left(\frac{c+k}{c+1} \right) |a_k| |z|^{k-1} < 1$$

holds if

$$\frac{k-p}{1-p} \frac{c+k}{c+1} |z|^{k-1} < \frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)} \quad ; \quad k \geq 2$$

or if

$$|z| < \left[\frac{(1-p)(c+1) D_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{(k-p)(c+k)\beta\gamma(B-A)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad ; \quad k \geq 2.$$

Hence, $f(z) \in S^*(p)$ in $|z| < R^*$. The sharpness follows if we take the function $F(z)$, given by

$$F(z) = z - \frac{(B-A)\gamma\beta(1-\alpha)z^k}{D_n(k, A, B, \alpha, \beta, \gamma, \lambda)}, \ k \geq 2.$$

□

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ON LACUNARY INVARIANT SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

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Abstract. The purpose of this paper is to introduce and study some sequence spaces which are defined by combining the concepts of lacunary convergence, invariant mean and the sequence of modulus functions We also examine some topological properties of these spaces.

1. **Introduction**

Let ℓ_∞ and $c$denote the Banach spaces of real bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup\limits_k |x_k|$, respectively.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or σ-mean if and only if

i. $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

ii. $\phi(e) \geq 0$, where *e= (1,1,1,..)* and,

iii. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

Let V_σ denote the set of bounded sequence all of whose invariant means are equal. In particular, if σ is the translation $n \to n+1$, then a σ-mean reduce to a Banach limit (see, Banach [1]) and set V_σ reduce to $\hat{c}$, the spaces of all almost convergent sequences (see, Lorentz [7]).

If $x = (x_n)$, write $Tx = Tx_n = (x_{\sigma(n)})$. It can be shown (Schaefer [16]) that $V_\sigma = \left\{ x \in \ell_\infty : \lim\limits_k t_{kn}(x) = \ell, \text{ uniformly in n,} \right\}$ $\ell = \sigma - \lim x$, where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + ... + x_{\sigma^k(n)}}{k+1}.$$

Here $\sigma^k(n)$ denote the k^{th} iterate of the mapping σ at n. The mapping σ is one to one and such that $\sigma^k(n) \neq n$ for all positive integers n and k. Thus a σ- mean ϕ extends the limit functional on c, the spaces of convergent sequence, in the sense that $\phi(x) = \lim x$ for all $x \in c$. (see, Mursaleen [11]).

We call V_σ as the space of σ-convergent sequences.

A sequence $x = (x_k)$ is said to be strongly σ-convergent (Mursallen [12]) if there exists a number ℓ such that $\lim\limits_k \frac{1}{k} \sum\limits_{j=1}^{k} \left| x_{\sigma^j(n)} - \ell \right| = 0$ uniformly in n.

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We denote $[V_\sigma]$ as the set of all strongly σ-convergent sequences. In case $\sigma(n) = n+1$, $[V_\sigma]$ reduce to $[\hat{c}]$, the space of all strong almost convergent sequence (Maddox [8]).

Also the strongly almost convergent sequences was studied by Freedman et all [4], independently.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, \mathrm{k_r}]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al [4] as:

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - \ell| = 0, \text{for some}\, \ell \right\}$$

Recently, the concept of lacunary strong σ-convergence was introduced by Savas [14] which is a generalization of the idea of lacunary strong almost convergence due to Das and Mishra [2].

A modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

i. $f(x) = 0$ if and only if $x = 0$
ii. $f(x+y) \le f(x) + f(y)$, for all $x, y > 0$
iii. f is increasing,
iv. f is continuous from the right at zero.

Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from conditions (ii) and (iv) that f is continuous everywhere on [0,∞).

A modulus function may be bounded or unbounded. For example, $f(t) = \frac{t}{t+1}$ is bounded but $f(t) = t^p$ $(0 < p \le 1)$ is unbounded.

Ruckle [13] and Maddox [9], Savas [15] and other authors used modulus function to construct new sequence spaces.

Recently, Kolk ([6], [7]) gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ i.e.,

$$X(f_k) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$$

In this paper by combining lacunary sequence , invariant mean and a sequence of modulus functions, we define the following new sequence spaces:

$$\left[w_\sigma^0, F\right]_\theta = \left\{ x : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k\left(|t_{kn}(x)|\right) = 0, \text{uniformly in n} \right\}$$

$$[w_\sigma, F]_\theta = \left\{ x : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k\left(|t_{kn}(x-l)|\right) = 0, \text{uniformly in n, for some l} \right\}$$

$$[w_\sigma^\infty, F]_\theta = \left\{ x : \sup_{r,n} \frac{1}{h_r} \sum k \in I_r f_k\left(|t_{kn}(x)|\right) < \infty \right\}$$

$$[w_\sigma, F] = \left\{ x : \lim_r \frac{1}{m} \sum_{k=1}^{m} f_k\left(|t_{kn}(x-l)|\right) = 0, \text{uniformly in n, for some l} \right\}$$

Some sequence spaces are obtained by specializing F, θ, σ. For example, if

$\theta = (2^r)$, $\sigma(n) = n+1$and $f_k(x) = x$ for all k, then $[w_\sigma, F]_\theta = \hat{w}$ (see, Das and Sahoo [3]). If $\sigma(n) = n+1$ and $f_k(x) = f$ for all k, then $[w_\sigma, F]_\theta = [\hat{w}(f)]_\theta$ and $[w_\sigma, F] = [\hat{w}(f)]$ (see, Mursaleen and Chishti [12]).

When $\sigma(n) = n+1$, the spaces $[w_\sigma^0, F]_\theta$, $[w_\sigma, F]_\theta$ and $[w_\sigma^\infty, F]_\theta$ reduce to the spaces $[\hat{w}_0, F]_\theta$, $[\hat{w}, F]_\theta$ and $[\hat{w}_\infty, F]_\theta$ respectively, where

$$[\hat{w}, F]_\theta = \Big\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k(|d_{kn}(x - l)|) = 0,$$

$$\text{uniformly in n, for some l} \Big\}$$

and

$$d_{nk}(x) = \frac{x_n + x_{n+1} + \ldots + x_{n+k}}{k+1}$$

If $\theta = (2^r)$, then $[w_\sigma^0, F]_\theta = [w_\sigma^0, F]$, $[w_\sigma, F]_\theta = [w_\sigma, F]$ and $[w_\sigma^\infty, F]_\theta = [w_\sigma^\infty, F]$.

2. **Main Results**

We have

Theorem 2.1. *For any a sequence of modulus functions $F = (f_k)$, $[w_\sigma^0, F]_\theta$, $[w_\sigma, F]_\theta$, $[w_\sigma^\infty, F]_\theta$ and $[w_\sigma, F]$ are linear spaces over the set of complex numbers.*

Proof. We shall prove the result only for $[w_\sigma^0, F]_\theta$. The others can be treated similarly. Let $x, y \in [w_\sigma^0, F]_\theta$and $\alpha, \beta \in$ C. Then there exist integers H_α and $K_\beta$such that $|\alpha| < H_\alpha$ and $|\beta| < K_\beta$. We have

$$\tfrac{1}{h\,r} \sum_{k \in I_r} f_k(|t_{kn}(\alpha x - \beta y)|) \le H_\alpha \tfrac{1}{h_r} \sum_{k \in I_r} f_k(|t_{kn}(x)|)$$

$$+ K_\beta \frac{1}{h_r} \sum_{k \in I_r} f_k(|t_{kn}(y)|)$$

This implies $\alpha x + \beta y \in [w_\sigma^0, F]_\theta$ □

We will now give a lemma.

Lemma 2.2. *Let f be a modulus and let $0 < \delta < 1$. Then for each $|t_{kn}(x)| > \delta$ for all k and n we have*

$$f(|t_{kn}(x)|) \le 2f(1)\,\delta^{-1}|t_{kn}(x)|$$

Proof.

$$f(|t_{kn}(x)|) \le f\left(1 + \left[\frac{|t_{kn}(x)|}{\delta}\right]\right) \le f(1) + f\left(\left[\frac{|t_{kn}(x)|}{\delta}\right]\right)$$

$$\le f(1)\left(1 + \frac{|t_{kn}(x)|}{\delta}\right) \le 2f(1)\,\delta^{-1}|t_{kn}(x)|$$

□

Theorem 2.3. *For a sequence of modulus functions $F = (f_k)$ and any lacunary sequence $\theta = (k_r)$,*

$$[w_\sigma, F]_\theta \subset [w_\sigma^\infty, F]_\theta\,.$$

Proof. Let $F = (f_k)$ be a sequence of modulus functions and $x \in [w_\sigma, F]_\theta$. Put $\sup_k f_k(1) = M$. We can write

$$\frac{1}{h_r}\sum_{k\in I_r} f_k(|t_{kn}(x)|) \le \frac{1}{h_r}\sum_{k\in I_r} f_k(|t_{kn}(x-l)|) + \frac{1}{h_r}\sum_{k\in I_r} f_k(|l|)$$

$$\le \frac{1}{h_r}\sum_{k\in I_r} f_k(|t_{kn}(x-l)|) + T_l M$$

where T_l is integer number such that $|l| < T_l$. Hence $x \in [w_\sigma^\infty, F]_\theta$.

Now for any lacunary sequence $\theta = (k_r)$, we give connection between $[w_\sigma, F]_\theta$ and $[w_\sigma, F]$. $\square$

Theorem 2.4. *Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_r q_r > 1$. Then for sequence of modulus functions $F = (f_k)$,*

$$[w_\sigma, F] \subset [w_\sigma; F]_\theta$$

Proof. Suppose that $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r > 1 + \delta$ for all r. Then for $x \in [w_\sigma, F]$, we write

$$\frac{1}{k_r}\sum_{k=1}^{k_r} f_k(|t_{kn}(x-l)|) \ge \frac{1}{k_r}\sum_{k=1}^{k_r} f_k(|t_{kn}(x-l)|) + \frac{1}{k_r}\sum_{k=1}^{k_{r-1}} f_k(|t_{kn}(x-l)|)$$

$$= \frac{1}{k_r}\sum_{k\in I_r} f_k(|t_{kn}(x-l)|)$$

$$\ge \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{k\in I_r} f_k(|t_{kn}(x-l)|)$$

By taking limit as $r \to \infty$ uniformly in, hence we obtain $x \in [w_\sigma, F]_\theta$. This completes the proof. $\square$

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup q_r < \infty$. Then for any sequence of modulus functions $F = (f_k)$,*

$$[w_\sigma, F]_\theta \subset [w_\sigma, F]$$

Proof. If $\limsup q_r < \infty$, there exists $H > 0$ such that $q_r < H$ for all $r \ge 1$. Let $x \in [w_\sigma, F]_\theta$ and $\varepsilon > 0$. There exists $R > 0$ such that for every $j \ge R$ and all n

$$A_j = \frac{1}{h_j}\sum_{k\in I_j} f_k(|t_{kn}(x-l)|) < \varepsilon.$$

We can also find $M > 0$ such that $A_j < K$ for all $j = 1, 2, ...$ Now let m be any integer with $k_{r-1} < m \le k_r$, where $r > R$. We have

$$\frac{1}{m}\sum_{k=1}^{m} f_k(|t_{kn}(x-l)|) \le \frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f_k(|t_{kn}(x-l)|)$$

$$= \frac{1}{k_{r-1}}\sum_{j=1}^{r}\sum_{k\in I_j} f_k(|t_{kn}(x-l)|)$$

$$= \frac{1}{k_{r-1}} \sum_{j=1}^{R} \sum_{k \in I_j} f_k\left(\left|t_{kn}(x-l)\right|\right) + \frac{1}{k_{r-1}} \sum_{j=R+1}^{r} \sum_{k \in I_j} f_k\left(\left|t_{kn}(x-l)\right|\right)$$

$$\leq \frac{1}{k_{r-1}} \left(\sup_{j \leq R} A_j\right) k_R + \frac{1}{k_{r-1}} \varepsilon \left(\sum_{j=R+1}^{r} h_j\right)$$

$$\leq \frac{1}{k_{r-1}} M k_R + \frac{1}{k_{r-1}} \varepsilon \left(h_{R+1} + h_{R+2} + \ldots + h_r\right)$$

$$\leq \frac{1}{k_{r-1}} M k_R + \varepsilon H$$

Since $k_{r-1} \to \infty$ as $r \to \infty$, it follows that

$$\frac{1}{m} \sum_{k=1}^{m} f_k\left(\left|t_{kn}(x-l)\right|\right) \to 0$$

uniformly in n and consequently $x \in [w_\sigma, F]$. Hence the proof completes. □

Theorem 2.6. *Let $\theta = (k_r)$ be a lacunary sequence 1< $\liminf_r q_r \leq \limsup q_r < \infty$. Then for any sequence of modulus functions $F = (f_k)$,*

$$[w_\sigma, F]_\theta = [w_\sigma, F]$$

Proof. Theorem 2.6 follows the theorems 2.5 and 2.4. □

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ON A GENERAL CLASS OF GAMMA APPROXIMATING OPERATORS

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Abstract. Many authors introduced and studied positive linear operators, using Euler's gamma function Γ_p, $p > 0$. We shall define a more general linear transform $\Gamma_p^{(a,b)}$, $a, b \in \mathbb{R}$, from which we obtain as particular cases the gamma first-kind transform and the gamma second-kind transform. For different values of a and b we obtain several gamma type operators studied in the literature.

1. **Introduction**

Many authors introduced and studied positive linear operators, using Euler's gamma function: [3], [4], [7], [8], [9].

We shall define a more general linear transform from which we obtain as particular cases the gamma first-kind transform and the gamma second-kind transform.

Euler's gamma function is defined for $p > 0$ by the following formula

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt \tag{1.1}$$

which can be written as

$$\Gamma(p) = \int_0^1 \ln^{p-1}\left(\frac{1}{u}\right) du \tag{1.2}$$

For $a, b \in \mathbb{R}$ we define the (a, b)-gamma transform of a function f by the functional (see also [5])

$$(\Gamma_p^{(a,b)} f)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f(xe^{-bt}(t/p)^a) dt \tag{1.3}$$

where Γ is defined by (1.1) (or (1.2)) and $f \in L_{1,loc}(0, \infty)$ such that $\Gamma_p^{(a,b)}|f| < \infty$. The above relation is equivalent with

$$(\Gamma_p^{(a,b)} f)(x) = \frac{1}{\Gamma(p)} \int_0^1 \ln^{p-1}\left(\frac{1}{u}\right) f\left(xu^b\left(\frac{1}{p}\ln\frac{1}{u}\right)^a\right) du \tag{1.4}$$

For different values of a and b we obtain several gamma type operators studied by many authors.

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2. The Gamma first-kind transform

If we put in (1.3) $b = 0$ we obtain the gamma first-kind transform of function f

$$(\Gamma_p^{(a)} f)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f\left(x\left(\frac{t}{p}\right)^a\right) dt \tag{2.1}$$

where $f \in L_{1,loc}[0,\infty)$ such that $\Gamma_p^{(a)}|f| < \infty$.

One observes that $\Gamma_p^{(a)}$ is a positive linear functional.

We state and prove:

Lemma 2.1. *The moment of order k of the functional $\Gamma_p^{(a)}$ has the following value*

$$(\Gamma_p^{(a)} e_k)(x) = \frac{\Gamma(p+ka)}{p^{ka}\Gamma(p)} x^k, \quad x > 0. \tag{2.2}$$

Proof. By using (1.1) we easily obtain

$$(\Gamma_p^{(a)} e_k)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} x^k \left(\frac{t}{p}\right)^{ka} dt$$

$$= \frac{x^k}{\Gamma(p)p^{ka}} \int_0^\infty e^{-t} t^{p+ka-1} dt = \frac{\Gamma(p+ka)}{p^{ka}\Gamma(p)} x^k. \ \square$$

Consequently we obtain

$$\Gamma_p^{(a)} e_1(x) = \frac{\Gamma(a+p)}{p^a \Gamma(p)} x, \quad \Gamma_p^{(a)} e_2(x) = \frac{\Gamma(p+2a)}{p^{2a}\Gamma(p)} x^2 \tag{2.3}$$

Particular cases

Case 1. If we consider $a = 1$ in (2.1) we obtain

$$(\Gamma_p f)(x) = (\Gamma_p^{(1)} f)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f\left(\frac{xt}{p}\right) dt \tag{2.4}$$

For $a = 1$, Lemma 2.1 leads us to the following

Corollary 2.2. *The moment of order k, $k \in \mathbb{N}$, of the functional Γ_p has the following values*

$$(\Gamma_p e_k)(x) = \frac{\Gamma(p+k)}{\Gamma(p)p^k} x^k, \quad x > 0.$$

We deduce

$$(\Gamma_p e_1)(x) = x, \quad (\Gamma_p e_2)(x) = \frac{p+1}{p} x^2, \quad \Gamma_p((t-x)^2; x) = \frac{x^2}{p}.$$

If we choose $p = n$, $n \in \mathbb{N}$ in (2.4) then we obtain Post-Wider's positive linear operator, defined for $f \in L_{1,loc}(0,\infty)$ by

$$(P_n f)(x) = \frac{1}{\Gamma(n)} \int_0^\infty e^{-t} t^{n-1} f\left(\frac{xt}{n}\right) dt. \tag{2.5}$$

If we replace p by nx, for $n \in \mathbb{N}$ and $x \geq 0$, in (2.4), we reobtain Rathore's positive linear operator [8], defined for $f \in L_{1,loc}(0,\infty)$ by

$$(R_n f)(x) = \frac{1}{\Gamma(nx)} \int_0^\infty e^{-t} t^{nx-1} f\left(\frac{t}{n}\right) dt. \tag{2.6}$$

Corollary 2.3. *One has*

$$P_n((t-x)^2;x) = \frac{x^2}{n}, \quad R_n((t-x)^2;x) = \frac{x}{n}.$$

Proof. It is obtained from Corollary 2.2.

Szasz's operator is defined by the following formula

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \geq 0. \tag{2.7}$$

If we apply gamma transform (2.4) to Szasz's operator we obtain the following positive linear operator.

Theorem 2.4. *The following identity*

$$\Gamma_p(S_n f)(x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{p^p (p)_k}{(nx+p)^{p+k}} f\left(\frac{k}{n}\right), \quad x > 0 \tag{2.8}$$

holds true. Here $(p)_0 = 1$ *and* $(p)_k = p(p+1)\dots(p+k-1)$, $k \geq 1$.

Proof.

$$\begin{aligned}
\Gamma_p(S_n f)(x) &= \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-t} t^{p-1} e^{-\frac{ntx}{p}} \sum_{k=0}^{\infty} \left(\frac{ntx}{p}\right)^k \frac{1}{k!} f\left(\frac{k}{n}\right) \\
&= \frac{1}{\Gamma(p)} \sum_{k=0}^{\infty} \frac{(nx)^k}{p^k k!} f\left(\frac{k}{n}\right) \int_0^{\infty} e^{-t\left(\frac{nx}{p}+1\right)} t^{p+k-1} dt \\
&= \frac{1}{\Gamma(p)} \sum_{k=0}^{\infty} \frac{(nx)^k}{p^k k!} f\left(\frac{k}{n}\right) \Gamma(p+k) \left(\frac{p}{nx+p}\right)^{p+k} \\
&= \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{p^p (p)_k}{(nx+p)^{p+k}} f\left(\frac{k}{n}\right).
\end{aligned}$$

In [2] A. Lupaş considered the operator L_n, defined for $f \in C[0,\infty)$ by

$$(L_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right) \tag{2.9}$$

which reproduces linear functions.

This operator is similar with Szasz's operator. In [2] the author asks to find properties of operator L_n. Some approximation properties were given in [1]. In the following theorem we shall prove that this operator can be obtained by the composite of Rathore's operator with Szasz's operator.

Theorem 2.5. *a) If* P_n *is the Post-Wider's operator (2.5) then* $B_n^* f = P_n(S_n f)$, *where* B_n^* *is the Baskakov's operator*

$$(B_n^* f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \geq 0.$$

b) If R_n *is the Rathore's operator (2.6) then*

$$L_n f = R_n(S_n f).$$

Proof. The proof is obtained from Theorem 2.4 for $p = n$ in the first case and for $p = nx$ in the second case.

Corollary 2.6. *The operator L_n can be written in the following manner*

$$(L_n f)(x) = \sum_{k=0}^{\infty} \frac{(nx)_k}{n^k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right].$$

Proof. We apply Theorem 2.4(b), using for the Szasz's operator the following formula

$$(S_n f)(x) = \sum_{k=0}^{\infty} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] x^k.$$

Case 2. If we replace $a = -1$ in (2.1) we obtain the following gamma transform

$$(\widetilde{\Gamma}_p f)(x) = (\Gamma_p^{(-1)} f)(x) = \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-t} t^{p-1} f\left(\frac{px}{t}\right) dt \tag{2.10}$$

where Γ is the gamma function (1.1), $p > 0$, and $f \in L_{1,loc}(0, \infty)$ such that $\widetilde{\Gamma}_p |f| < \infty$.

One observes that $\widetilde{\Gamma}_p$ is a positive linear functional.

Lemma 2.7. *The moment of order k, $k \in \mathbb{N}$, $k < p$, of the functional $\widetilde{\Gamma}_p$ has the following value*

$$\widetilde{\Gamma}_p e_k(x) = \frac{\Gamma(p-k)}{\Gamma(p)} (px)^k, \quad x > 0.$$

Proof. It is obtained from Lemma 2.1, for $a = -1$.

We deduce

$$(\widetilde{\Gamma}_p e_2)(x) = x^2 + \frac{x^2}{p-1}; \quad \widetilde{\Gamma}_p((t-x)^2; x) = \frac{x^2}{p-1}.$$

If we put $p = n + 1$ in (2.9) we obtain the gamma operator introduced and studied by A. Lupaş and M. Müller [4]

$$(G_n f)(x) = \frac{1}{n!} \int_0^{\infty} e^{-t} t^n f\left(\frac{(n+1)x}{t}\right) dt \tag{2.11}$$

Corollary 2.8.

$$G_n((t-x)^2; x) = \frac{x^2}{n}.$$

Proof. It is obtained from Lemma 2.7 for $p = n + 1$.

Several papers have dealt with these operators: [3], [4], [9].

3. The Gamma second-kind transform

If we choose in (1.3) $a = 0$ then we obtain the gamma second-kind transform of a function f

$$(\Gamma_p^{(b)} f)(x) = \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-t} t^{p-1} f(xe^{-bt}) dt \tag{3.1}$$

where Γ is the gamma function (1.1), $p > 0$, and $f \in L_{1,loc}[0, \infty)$ such that $\Gamma_p^{(b)} |f| < \infty$.

We consider here only the case $b = 1$.

$$(\Gamma_p^* f)(x) = (\Gamma_p^{(1)} f)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f(xe^{-t}) dt \tag{3.2}$$

Formula (3.2) is equivalent with (see (1.4))

$$(\Gamma_p^* f)(x) = \frac{1}{\Gamma(p)} \int_0^1 \ln^{p-1} \frac{1}{u} f(ux) du$$

Clearly, Γ_p^* is a positive linear functional.

Lemma 3.1. *The moment of order k, $k \in \mathbb{N}$, of the functional Γ_p^* has the following value*

$$(\Gamma_p^* e_k)(x) = \frac{x^k}{(k+1)^p}$$

Proof. We can write successively

$$(\Gamma_p^* e_k)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} (xe^{-t})^k dt = \frac{x^k}{\Gamma(p)} \int_0^\infty t^{p-1} e^{-t(k+1)} dt =$$
$$= \frac{x^k}{\Gamma(p)} \frac{\Gamma(p)}{(k+1)^p} = \frac{x^k}{(k+1)^p}.$$

By using (3.1), for $p = \alpha$, $\alpha > 0$ we obtain the positive linear operator

$$(A_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} f(xe^{-t}) dt, \tag{3.3}$$

or equivalent (see (3.2))

$$(A_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 \ln^{\alpha-1} \frac{1}{t} f(tx) dt. \tag{3.4}$$

This operator was introduced by the author in [5] and it is strongly related with Cesaro means of order α (see [5]). This operator is an approximating operator for $\alpha \to 0$, for example, $\alpha = 1/n$, $n \in \mathbb{N}$.

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MINIMAL CURVES IN ALMOST MINKOWSKI MANIFOLD

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Abstract. In a Lorentz manifold $[M, g]$ with a global timelike vector field Z which respects $g(Z, Z) = -1$ and its distribution is involutive, we consider a topological norm and this corresponding length of curves. We find the local equations of minimal curve of this length functional.

1. **Introduction**

Let M be a $(n+1)$ dimensional connected paracompact without boundary manifold and g a nondegenerate bilinear form with diagonal form +,+,...,+,- to each tangent space.

Given a global vector field Z so that $g(Z, Z) = -1$ on M, we say that the structure (M, g, Z) is a *time-normalized space-time manifold.*

Definition 1.1. A time-normalized space-time manifold (M, g, Z) is an *almost Minkowski manifold* if the distribution:

$$x \in M \longmapsto \{X \in T_xM \mid g(X, Z) = 0\} \text{ is involutive.}$$

Definition 1.2. A time-normalized space-time manifold (M, g, Z) for which there is $f : M \to \mathbf{R}$ so that $Z = \nabla f$ is called a *functional normalized space-time manifold* and it is noted (M, g, f).

Remark 1. Obviously any functional normalized space-time manifold is an almost Minkowski manifold.

In [5] the necessary and sufficient conditions for a time-normalized space-time manifold are given to be an almost Minkowski manifold.

Proposition 1.1. Let (M, g, Z) be a time-normalized space-time manifold with $H^1(M) = \{0\}$. The necessary and sufficient conditions for the existence of an atlas $\overline{A}$ of M, so that the local coordinates of g respects:

$$\partial_{n+1} = Z \text{ and } \frac{\partial g_{an+1}}{\partial x^b} = \frac{\partial g_{bn+1}}{\partial x^a}, \ \forall a, b = \overline{1, n+1} \tag{1}$$

is (M, g, Z) to be a functional normalized space-time manifold.

Proof. We define the $1-$form $\omega = g_{an+1}dx^a$. Then:

$$d\omega = \left(\frac{\partial g_{an+1}}{\partial x^b} - \frac{\partial g_{bn+1}}{\partial x^a} \right) dx^a \wedge dx^b$$

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and the condition (1) implies $d\omega = 0$ and by hypothesis ω is an exact 1-form and there exist $f : M \to \mathbf{R}$ so that:

$$df = \omega \Leftrightarrow \frac{\partial f}{\partial x^a} dx^a = g_{an+1} dx^a \text{ or}$$

$$g^{ab} \frac{\partial f}{\partial x^a} = \delta^b_{n+1} \Rightarrow g^{ab} \frac{\partial f}{\partial x^a} \partial_b = \partial_{n+1} \Rightarrow \nabla f = Z$$

Reciprocally if $\nabla f = Z$ then $g_{an+1} = \dfrac{\partial f}{\partial x^a}$ and $\dfrac{\partial g_{an+1}}{\partial x^b} = \dfrac{\partial g_{bn+1}}{\partial x^a}$

Remark 2. The restrictive condition of almost Minkowski manifold does not imply the chronologicity. For example if $M = S^1 \times \mathbf{R}$ with $g = -d\theta^2 + dt^2$ admits the curve $\gamma(s) = (s, t_0)$ which is a closed timelike curve, and $\left(M, g; \frac{\partial}{\partial \theta}\right)$ is a almost Minkowski manifold.

If $Lor(M)$ denotes the set of Lorentz metrics. with partial ordering relation:

$$g_{1 \leq} g_2 \Leftrightarrow \forall p \in M,\ \forall X \in T_pM,\ g_1(X,X) \leq 0 \Rightarrow g_2(X,X) \leq 0$$

then by [1, Prop. 6.4.9] the functional-normalized space-time manifold can be characterize by the following statement:

Proposition 1.2. A Lorentz manifold (M, g) can be become a functional normalization Lorentz manifold if and only if a causal metric g_1 exists, so that $g \leq g_1$.

2. Minimal curve in almost Minkowski manifold

Definition 2.1. We call $Z-$*norm* on a almost Minkowski manifold the application: $|\ |_Z : TM \to \mathbf{R}$, defined by:

$$|X|_Z = |g(X,Z)| + \sqrt{g(X,Z)^2 + g(X,X)}$$

Remark 1. a) It is proved in [5] that:

$$|X|_Z = \min\{\lambda \geq 0,\ -\lambda Z_x \leq X \leq \lambda Z_x\}$$

where the ordering relation on T_xM is defined by:

$$X \leq Y \Leftrightarrow (X = Y) \vee (g(Y - X, Y - X) < 0 \wedge g(Z_x, Y - X) < 0)$$

b) For a almost Minkowski manifold (M, g, Z), the expression of the norm in the preferential atlas $\overline{A}$ (which exists [5]) with:

$$\begin{cases} \partial_{n+1} = Z \\ g_{an+1} = -\delta^a_{n+1}, \forall a = \overline{1, n+1} \end{cases}$$

is:

$$|X|_Z = \left|X^{n+1}\right| + \sqrt{g_{ij} X^i X^j}, \forall i, j = \overline{1,n} \text{ where } X = X^a \partial_a, a = \overline{1, n+1}$$

If $p_1,\ p_2 \in M$, we note the $\Omega_{p_1 p_2}$ the set of $\mathcal{C}^\infty$ curves from p_1 to p_2 and its subsets:

$$\Omega^+_{p_1p_2} = \{\gamma : [\alpha, \beta] \to M,\ g\left(\gamma'(t), Z_{\gamma(t)}\right) > 0,\ \forall t \in [\alpha, \beta]\}$$
$$\Omega^-_{p_1p_2} = \{\gamma : [\alpha, \beta] \to M,\ g\left(\gamma'(t), Z_{\gamma(t)}\right) < 0,\ \forall t \in [\alpha, \beta]\}$$
$$\Omega^0_{p_1p_2} = \{\gamma : [\alpha, \beta] \to M,\ g\left(\gamma'(t), Z_{\gamma(t)}\right) = 0,\ \forall t \in [\alpha, \beta]\}$$

Definition 2.2. For $\gamma \in \Omega_{p_1p_2}$ we define the $Z-$length of γ by:

$$L_Z(\gamma) = \int_\alpha^\beta |\gamma'(t)|_Z \, dt$$

Theorem 2.1. If $\gamma_0 \in \Omega^+_{p_1p_2}$ exist so that:

$$L_Z(\gamma_0) \leq L_Z(\gamma)\,,\ \forall \gamma \in \Omega^+_{p_1p_2}$$

then there is a parametrization of γ_0 which in local preferential coordinates verifies:

$$h_{ab}\frac{d^2x_0^b}{ds^2} + \frac{1}{2}\left[\frac{\partial h_{ab}}{\partial x^c} + \frac{\partial h_{ac}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^a}\right]\frac{dx_0^b}{ds}\frac{dx_0^c}{ds} = 0,\ \forall a = \overline{1, n+1}$$

where:

$$h_{ab} = \begin{cases} g_{ab} \text{ if } (a,b) \neq (n+1, n+1) \\ 0 \text{ if } (a,b) = (n+1, n+1) \end{cases}$$

and the local equations of γ_0 are:

$$\begin{cases} x^a = x_0^a(s) \\ a = \overline{1, n+1},\ s \in [\alpha', \beta'] \end{cases} \text{ with } \left(g\left(\frac{d\gamma}{ds}, Z\right)^2 + g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) = 1 \right)$$

Proof. Let be $\gamma \in \Omega^+_{p_1p_2}$ with the local preferential atlas
$\overline{A} : \begin{cases} x^a = x^a(t) \\ a = \overline{1, n+1} \end{cases}$, $t \in [\alpha, \beta]$ Then $g\left(\frac{d\gamma}{dt}, Z_{\gamma(t)}\right) > 0$ implies
$\frac{dx^{n+1}}{dt} < 0$. and the Z-length functional is:

$$L_Z(\gamma) = \int_\alpha^\beta \left\{ -\frac{dx^{n+1}}{dt} + \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} \right\} dt \text{ where } i, j \in \overline{1, n}$$

We denote:

$$G\left(t, (x^a), \left(x^{a'}\right)\right) = -\frac{dx^{n+1}}{dt} + \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}$$

and we calculate:

$$\frac{\partial G}{\partial x^m} = \frac{\frac{\partial g_{ij}}{\partial x^m}\frac{dx^i}{dt}\frac{dx^j}{dt}}{2\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}}\ ;\ \frac{\partial G}{\partial x^{m'}} = +\delta_m^{n+1} + \frac{g_{im}\frac{dx^i}{dt}\left(1 - \delta_m^{n+1}\right)}{\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}}$$

Because $(g_{ij})_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}}$ is a matrix of a positive definite bilinear form, it is possible to find a parametrization of γ so that $g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1$ or $g\left(\frac{d\gamma}{ds}, Z\right)^2 + g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) = 1$. Then the Euler-Lagrange system of the Z-length functional for the curves of $\Omega^+_{p_1p_2}$ becomes:

$$\frac{\partial G}{\partial x^m} - \frac{d}{ds}\left(\frac{\partial G}{\partial x^{m'}}\right) = 0$$

or

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^m}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{d}{ds}\left[-\delta_m^{n+1} + g_{im}\frac{dx^i}{ds}\left(1 - \delta_m^{n+1}\right)\right] = 0.$$

For $m \neq n+1$ we have:

$$g_{im}\frac{d^2x^i}{ds^2} + \frac{1}{2}\left[\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right]\frac{dx^i}{ds}\frac{dx^j}{ds} + \frac{\partial g_{im}}{\partial x^{n+1}}\frac{dx^i}{ds}\frac{dx^{n+1}}{ds} = 0$$

or

$$g_{im}\frac{d^2x^i}{ds^2} + \frac{1}{2}\left[\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right]\frac{dx^i}{ds}\frac{dx^j}{ds} +$$
$$+\frac{1}{2}\left[\frac{\partial g_{im}}{\partial x^{n+1}} + \frac{\partial g_{n+1m}}{\partial x^i} - \frac{\partial g_{in+1}}{\partial x^m}\right]\frac{dx^i}{ds}\frac{dx^j}{ds} +$$

$$+\frac{1}{2}\left[\frac{\partial g_{jm}}{\partial x^{n+1}}+\frac{\partial g_{n+1m}}{\partial x^{j}}-\frac{\partial g_{n+1j}}{\partial x^{m}}\right]\frac{dx^{m+1}}{ds}\frac{dx^{j}}{ds}+$$
$$+\frac{1}{2}\left[\frac{\partial g_{n+1m}}{\partial x^{n+1}}+\frac{\partial g_{n+1m}}{\partial x^{n+1}}-\frac{\partial g_{n+1n+1}}{\partial x^{m}}\right]\frac{dx^{n+1}}{ds}\frac{dx^{n+1}}{ds}=0$$

that is:

$$g_{am}\frac{d^2x^a}{ds^2}+\frac{1}{2}\left[\frac{\partial g_{bm}}{\partial x^{a}}+\frac{\partial g_{am}}{\partial x^{b}}-\frac{\partial g_{ab}}{\partial x^{m}}\right]\frac{dx^{a}}{ds}\frac{dx^{b}}{ds}=0 \tag{2}$$

where $a,b\in\overline{1,n+1}$

For $m=n+1$ the Euler-Lagrange equation become:

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^{m}}\frac{dx^{i}}{ds}\frac{dx^{j}}{ds}=0$$

or

$$\left(g_{n+1a}-g_{n+1n+1}\right)\frac{d^2x^a}{ds^2}+\frac{1}{2}\left[\frac{\partial g_{bn+1}}{\partial x^{a}}+\frac{\partial g_{an+1}}{\partial x^{b}}-\frac{\partial g_{ab}}{\partial x^{n+1}}\right]\frac{dx^{a}}{ds}\frac{dx^{b}}{ds}=0 \tag{3}$$

Therefore the relations 2 and 3 implies:

$$h_{ab}\frac{d^2x_0^b}{ds^2}+\frac{1}{2}\left[\frac{\partial h_{ab}}{\partial x^{c}}+\frac{\partial h_{ac}}{\partial x^{b}}-\frac{\partial h_{bc}}{\partial x^{a}}\right]\frac{dx_0^{b}}{ds}\frac{dx_0^{c}}{ds}=0,\ \forall a=\overline{1,n+1}$$

where:

$$h_{ab}=\begin{cases} g_{ab}\text{ if }(a,b)\neq(n+1,n+1)\\ 0\text{ if }(a,b)=(n+1,n+1)\end{cases}$$

Remark 2. The analogous statement for the case $\Omega^{-}_{p_1p_2}\neq\phi$. For $\Omega^{0}_{p_1p_2}\neq\phi$ the equation $g\left(\frac{d\gamma}{dt},Z\right)=0$ becomes $-x^{n+1'}=0$ hence $x^{n+1}=\mathbf{k}$, where $\mathbf{k}$ is a constant. If exist $\gamma_0\in\Omega^{0}_{p_1p_2}$ so that

$$L_Z(\gamma_0)\leq L_Z(\gamma),\ \forall\gamma\in\Omega^{0}_{p_1p_2}$$

than we can find a parametrization of γ_0 so that its local preferential coordinates verify:

$$\begin{cases}\widetilde{h}_{ik}\frac{d^2x^i}{ds^2}+\frac{1}{2}\left[\frac{\partial\widetilde{h}_{ik}}{\partial x^{j}}+\frac{\partial\widetilde{h}_{jk}}{\partial x^{i}}-\frac{\partial\widetilde{h}_{ij}}{\partial x^{k}}\right]\frac{dx^{i}}{ds}\frac{dx^{j}}{ds}=0\\ \frac{dx^{n+1}}{ds}=0,\ \forall i,j,k=\overline{1,n}\end{cases}$$

where $\widetilde{h}_{ij}=g_{ij}\left(x^1,x^2,..,x^n,\mathbf{k}\right)$.

If p_1,p_2 are on the same pure timelike curve, meaning that:

$\exists\gamma_0:[\alpha,\beta]\to M,\ \gamma(\alpha)=p_1,\ \gamma(\beta)=p_2,\ \gamma'(t)=\lambda(t)\,Z_{\gamma(t)}$, where $\lambda(t)>0$ or $\lambda(t)<0$, we can find a parametrization of γ_0 in the preferential coordinates, so that:

$$\frac{dx_0^a}{dt}=\pm\delta^a_{n+1},\ \forall a=\overline{1,n+1}\text{ and:}$$

$$\begin{aligned} L_Z(\gamma_0) &= \left|x_0^{n+1}(\beta)-x_0^{n+1}(\alpha)\right|\leq\\ &\leq \int_\alpha^\beta\left\{\left|\frac{dx^{n+1}(t)}{dt}\right|+\sqrt{g_{ij}\frac{dx^i(t)}{dt}\frac{dx^j(t)}{dt}}\right\}dt=L_Z(\gamma)\end{aligned}$$

for every $\gamma\in\Omega^{+}_{p_1p_2}$ if $\lambda>0$ and $\gamma\in\Omega^{-}_{p_1p_2}$ if $\lambda<0$.

Corollary 2.1. Let (M,g,f) be a almost Minkowski manifold. For any $p_1\in M$,there is a neighborhood V_1, so that for any $p_2\in V_1$ with $\Omega^{+}_{p_1p_2}\neq\Phi$ there is at least a curve $\gamma_0\in\Omega^{+}_{p_1p_2}$ which is minimal for the $Z-$length functional

Remark 3. We can state the same results for $\Omega^{-}_{p_1 p_2}$ and $\Omega^{0}_{p_1 p_2}$.

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ON A PARTICULAR FIRST ORDER NONLINEAR DIFFERENTIAL SUBORDINATION II

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Abstract. We find conditions on the complex-valued functions B, C, D in unit disc U and the positive constants M and N such that

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

implies $|p(z)| < N$, where p is analytic in U, with $p(0) = 0$.

1. Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

We let Q denote the class of functions q that are holomorphic and injective in $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial U : \ \lim_{z \to \zeta} q(z) = \infty\right\}$$

and furthermore $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$, where $E(q)$ is called exception set.

In order to prove the new results we shall use the following:

Lemma A. [1] (Lemma 2.2.d p. 24) *Let $q \in Q$, with $q(0) = a$, and let*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$, $r_0 < 1$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, and

(iii) $\mathrm{Re}\dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\mathrm{Re}\left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$

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In [1] chapter IV, the authors have analyzed a first-order linear differential subordination

$$B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{1}$$

where B, C, D and h are complex-valued functions in the unit disc U. A more general version of (1) is given by

$$B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega, \tag{2}$$

where $\Omega \subset \mathbb{C}$.

In [2] we found conditions on the complex-valued functions B, C, D, E in the unit disc U and the positive constants M and N such that

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)| < M$$

implies $|p(z)| < N$, where $p \in \mathcal{H}[0, n]$.

In this paper we shall consider a particular first-order nonlinear differential subordination given by the inequality

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M \tag{3}$$

We find conditions on the complex-valued functions B, C, D such that (3) implies $|p(z)| < N$ where $p \in \mathcal{H}[0, n]$.

In some cases, given the functions B, C, D and the constant M we will determine an appropriate N such that (3) implies $|p(z)| < N$.

2. Main results

The results in [2] can certainly be used in the special case when $E(z) \equiv 0$. However, in this case we can improve those results by the following theorem:

Theorem. *Let $M > 0$, $N > 0$, and let n be a positive integer. Suppose that the functions $B, C, D : U \to \mathbb{C}$ satisfy $B(z) \neq 0$,*

$$\begin{cases} (i)\ \mathrm{Re}\dfrac{D(z)}{B(z)} \geq -n \\ (ii)\ |nB(z) + D(z)| \geq \dfrac{1}{N}[M + N^2|C(z)|]. \end{cases} \tag{4}$$

If $p \in \mathcal{H}[0, n]$ and

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M \tag{5}$$

then

$$|p(z)| < N.$$

Proof. If we let

$$W(z) = B(z)zp'(z) + C(z)p^2(z) + D(z)p(z), \tag{6}$$

then from (6) we obtain

$$|W(z)| = |B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)|. \tag{7}$$

From (7) and (5) we have

$$|W(z)| < M, \quad z \in U. \tag{8}$$

Assume that $|p(z)| \not< N$, which is equivalent with $p(z) \not\prec Nz = q(z)$.

According to Lemma A, with $q(z) = Nz$, there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $r_0 < 1$, $\theta_0 \in [0, 2\pi)$, $\zeta \in \partial U$, $|\zeta| = 1$ and $m \geq n$, such that $p(z_0) = N\zeta$ and $z_0 p'(z_0) = mN\zeta$.

Using these conditions in (7) we obtain for $z = z_0$

$$|W(z_0| = |B(z_0)mN\zeta + C(z_0)N^2\zeta + D(z_0)N\zeta| = \tag{9}$$
$$= |N[B(z_0)m + D(z_0)] + C(z_0)N^2\zeta| \geq$$
$$\geq N|B(z_0)m + D(z_0)| - N^2|C(z_0)|.$$

Since $m \geq n$ and $B(z) \neq 0$, from condition (i) we have

$$\left| m + \frac{D(z)}{B(z)} \right| \geq \left| n + \frac{D(z)}{B(z)} \right|,$$

and

$$|mB(z) + D(z)| \geq |nB(z) + D(z)|.$$

For $z = z_0$, we have

$$|mB(z_0) + D(z_0)| \geq |nB(z_0) + D(z_0)|.$$

Using this last result and condition (ii) together with (9) we deduce that

$$|W(z_0)| \geq N[nB(z_0) + D(z_0)| - N^2|C(z_0)| \geq M.$$

Since this contradicts (8) we obtain the desired result $|p(z)| < N$. □

Instead of prescribing the constant N in Theorem, in some cases we can use (ii) to determine an appropriate $N = N(M, n, B, C, D)$ so that (5) implies $|p(z)| < N$. This can be accomplished by solving (ii) for N and by taking the supremum of the resulting function over U. The conditions (ii) is equivalent to

$$|C(z)|N^2 - N|nB(z) + D(z)| + M \leq 0. \tag{10}$$

The inequality (10) holds if:

$$|nB(z) + D(z)|^2 \geq 4|C(z)|. \tag{11}$$

In this case we let

$$N = \sup_{|z|<1} \frac{|nB(z) + D(z)| - \sqrt{|nB(z) + D(z)|^2 - 4M|C(z)|}}{2|C(z)|} =$$
$$= \sup_{|z|<1} \frac{2M}{|nB(z) + D(z)| + \sqrt{|nB(z) + D(z)|^2 - 4M|C(z)|}}$$

If this supremum is finite, we have the following version of the Theorem:

Corollary. *Let $M > 0$ and let n be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$, and the functions $B, C, D : U \to \mathbb{C}$, with $B(z) \neq 0$, $C(z) \neq 0$, satisfy:*

$$\operatorname{Re}\left[\frac{D(z)}{B(z)}\right] \geq -n, \quad |nB(z) + D(z)| \geq 4|C(z)|$$

and let

$$N = \sup_{|z|<1} \frac{2M}{|nB(z) + D(z)| + \sqrt{|nB(z) + D(z)|^2 - 4M|C(z)|}} < \infty$$

Then

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

implies

$$|p(z)| < N.$$

If $n = 1$, $B(z) = 3 + z$, $C(z) = 1$, $D(z) = 1 - z$, $M = 1$, $N = 2 - \sqrt{3}$.

In this case from Corollary, we deduce

Example 1. If $p \in \mathcal{H}[0, 1]$, then

$$|(3 + z)zp'(z) + p^2(z) + (1 - z)p(z)| < 1$$

implies

$$|p(z)| < 2 - \sqrt{3}.$$

If $n = 3$, $B(z) = 1 + z$, $C(z) = 2$, $D(z) = 4 - 3z$, $M = 2$, $N = \dfrac{7 - \sqrt{33}}{4}$. In this case from Corollary, we deduce:

Example 2. If $p \in \mathcal{H}[0, 3]$, then

$$|(1 + z)zp'(z) + 2zp^2(z) + (4 - 3z)p(z)| < 2$$

implies

$$|p(z)| < \frac{7 - \sqrt{33}}{4}.$$

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SOME SUFFICIENT CONDITIONS FOR UNIVALENCE

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Abstract. In this paper we prove the analyticity and the univalence of the functions which are defined by means of integral operators. In particular cases we find some known results.

1. Introduction

We denote by $U_r = \{z \in C : |z| < r\}$ the disk of z-plane, where $r \in (0, 1]$, $U_1 = U$ and $I = [0, \infty)$.

Let A be the class of functions f analytic in U such that $f(0) = 0,\ f'(0) = 1$. Let S denote the class of function $f \in A,\ f$ univalent in U. The usual subclasses of S consisting of starlike functions and α-convex functions will be denoted by S^* respectively M_α.

Definition 1.1. *([2])Let $f \in A,\ f(z)f'(z) \neq 0$ for $0 < |z| < 1$ and let $\alpha \geq 0$. We denote by*

$$M(\alpha, f) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(\frac{zf''(z)}{f'(z)} + 1) \tag{1}$$

If $ReM(\alpha, f) > 0$ in U, then f is said to be an α- convex function ($f \in M_\alpha$).

Theorem 1.1. *([2]). The function $f \in M_\alpha$ if and only if there exists a function $g \in S^*$ such that*

$$f(z) = \left(\frac{1}{\alpha}\int_0^z \frac{g^{\frac{1}{\alpha}}(u)}{u}du\right)^\alpha \tag{2}$$

Definition 1.2. *([5]) Let $f \in A$. We said that $f \in S^*(a, b)$ if*

$$\left|\frac{zf'(z)}{f(z)} - a\right| < b, \qquad |z| < 1, \tag{3}$$

where

$$a \in C,\ Rea \geq b,\ |a - 1| < b. \tag{4}$$

Theorem 1.2. *([1]) Let $f \in A$. If for all $z \in U$*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{5}$$

then the function f is univalent in U.

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2. Preliminaries

Theorem 2.1. *([4]) Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots,\ a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r, \tag{6}$$

where $p(z,t)$ is analytic in U and satisfies the condition $Rep(z,t) > 0$ for all $z \in U,\ t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$ the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

3. Main results

Theorem 3.1. *Let $f,\ g \in A$ and let α be a complex number, $|\alpha - 1| < 1$. If*

$$(1-|z|^2)\left|(\alpha-1)\frac{zg'(z)}{g(z)} + \frac{zf''(z)}{f'(z)}\right| \leq 1, \quad \forall z \in U, \tag{7}$$

then the function

$$H(z) = \left(\alpha\int_0^z g^{\alpha-1}(u)f'(u)du\right)^{1/\alpha} \tag{8}$$

is analytic and univalent in U.

Proof. Let us prove that there exists a real number $r \in (0,1]$ such that the function $L : U_r \times \to C$ defined formally by

$$L(z,t) = \left[\int_0^{e^{-t}z} g^{\alpha-1}(u)f'(u)du + (e^t - e^{-t})zg^{\alpha-1}(e^{-t}z)f'(e^{-t}z)\right]^{1/\alpha} \tag{9}$$

is analytic in U_r for all $t \in I$.
Since $g \in A$, the function $h(z) = \frac{g(z)}{z}$ is analytic in U and $h(0) = 1$. Then there is a disk U_{r_1}, $0 < r_1 \leq 1$, in which $h(z) \neq 0$ for any $z \in U_{r_1}$ and we choose the uniform branch of $(h(z))^{\alpha-1}$ equal to 1 at the origin, denoted by h_1.
For the function

$$h_2(t) = \int_0^{e^{-t}z} u^{\alpha-1}h_1(u)f'(u)du$$

we have $h_2(z,t) = z^\alpha h_3(z,t)$ and is easy to see that h_3 is also analytic in U_{r_1}. The function

$$h_4(z,t) = h_3(z,t) + (e^t - e^{-t})e^{-t(\alpha-1)}h_1(e^{-t}z)f'(e^{-t}z)$$

is analytic in U_{r_1} and we have $h_4(0,t) = e^{(2-\alpha)t}[1 + (1/\alpha - 1)e^{-2t}] \neq 0$ for any $t \in I$. It results that there exist $r_2 \in (0, r_1]$ such that $h_4(z,t) \neq 0$ in U_{r_2}. Then we can choose an uniform branch of $[h_4(z,t)]^{1/\alpha}$ analytic in U_{r_2} denoted by $h_5(z,t)$, which is equal to $a_1(t) = e^{(2-\alpha)t/\alpha}[1 + (1/\alpha - 1)e^{-2t}]^{1/\alpha}$ at the origin and for $a_1(t)$ we fix a determination.
From this considerations it results that the relation (9) may be written as

$$L(z,t) = zh_5(z,t) = a_1(t)z + a_2(t)z^2 + \dots$$

and then the function $L(z,t)$ is analytic in U_{r_2}.
Since $|\alpha - 1| < 1$ it results that $\lim_{n\to\infty}|a_1(t)| = \infty$.

It is easy to prove that $L(z,t)$ is locally absolutely continous in I, locally uniformly with respect to U_{r_3} and that $\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_3}, $r_3 \in (0, r_2]$. It follows that the function $p(z,t)$ defined by (6) is analytic in U_r, $r \in (0, r_3]$, for all $t \geq 0$.
In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$, it is sufficient to prove that the function $w(z,t)$ defined in U_r by

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}$$

can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.
After computation we obtain

$$w(z,t) = (1-e^{-2t})\left[(\alpha-1)\frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)} + \frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}\right] \tag{10}$$

From (7) we deduce that the function $w(z,t)$ is analytic in the unit disk U. We have $w(z,0) = 0$ and also $|w(0,t)| = |(1-e^{-2t})(\alpha-1)| < |\alpha-1| < 1$.
Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and taking into account the relation (7) we have

$$|w(e^{i\theta},t)| = (1-|u|^2)\left|(\alpha-1)\frac{ug'(u)}{g(u)} + \frac{uf''(u)}{f'(u)}\right| \leq 1$$

Using the maximum principle for all $z \in U\backslash\{0\}$ and $t > 0$ we conclude that $|w(z,t)| < 1$ and finally we have $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.
From Theorem 2.1 it results that the function

$$L(z,0) = \left(\int_0^z g^{\alpha-1}(u)f'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U and then the function H defined by (8) is analytic and univalent in U.
For particular choices of f and g we get the following

Corollary 3.1. *Let $f \in A$ and let $\alpha \in C$, $|\alpha-1| < 1$. If*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-|\alpha-1|(1-|z|^2), \quad \forall z \in U, \tag{11}$$

then the function

$$F(z) = \left(\alpha\int_0^z u^{\alpha-1}f'(u)du\right)^{1/\alpha} \tag{12}$$

is analytic and univalent in U.

Proof. For the function $g(z) = z$, from (7) we have

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)} + \alpha - 1\right| \leq 1$$

We observe that if the condition (5) of Theorem 1.2 will be replaced by the strong condition (11), then we have not only the univalence of the function f $(\alpha = 1)$, but we obtain also the univalence of the function F defined by (12).

Corollary 3.2. *Let $g \in A$ and let $\alpha \in C$, $|\alpha - 1| < 1$. If*

$$(1 - |z|^2)\left|(\alpha - 1)\frac{zg'(z)}{g(z)}\right| \leq 1, \quad \forall z \in U, \tag{13}$$

then the function

$$G(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)du\right)^{1/\alpha} \tag{14}$$

is analytic and univalent in U.

Proof. If we take $f(z) = z$, from (7) we obtain the relation (13). So we find a result from paper [3].

For the function $f \in A$, $f'(z) = \frac{g(z)}{z}$, from theorem 3.1 we get the following

Theorem 3.2. *Let $g \in A$ and let $\alpha \in C$, $|\alpha - 1| < 1$. If*

$$(1 - |z|^2)\left|\alpha\frac{zg'(z)}{g(z)} - 1\right| \leq 1, \quad \forall z \in U, \tag{15}$$

then the function

$$G(z) = \left(\alpha \int_0^z \frac{g^{\alpha}(u)}{u}du\right)^{1/\alpha} \tag{16}$$

is analytic and univalent in U.

The operator (16) is just the integral operator introduced by Prof. P. T. Mocanu in the integral representation of α-convex functions.

Corollary 3.3. *Let $g \in A$, $\alpha \in C$, $|\alpha - 1| < 1$. If*

$$\left|\frac{zg'(z)}{g(z)} - \frac{1}{\alpha}\right| \leq \frac{1}{|\alpha|}, \qquad \forall z \in U,$$

then the function G defined by (16) is analytic and univalent in U.

Remark . Let $\alpha \in (0,\ 2)$ and let $g \in S^*(\frac{1}{\alpha}, \frac{1}{\alpha})$. Then the function G defined by (16) is analytic and univalent in U.

Indeed, if we consider $a = 1/\alpha$ and $b = 1/|\alpha|$, the conditions (4) are satisfied for $\alpha \in (0,\ 2)$. But here we obtain only the univalence of G.

If in theorem 3.1 we take $f \equiv g$,we have

Theorem 3.3. *Let $f \in A$ and let $\alpha \in C$, $|\alpha - 1| < 1$. If*

$$(1 - |z|^2)\left|(\alpha - 1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right| \leq 1, \qquad \forall z \in U \tag{17}$$

then the function f is univalent in U.

Corollary 3.4. *Let $f \in A$, $\beta \in C$, $Re\beta > \frac{1}{2}$. If*

$$|\ M(\ \beta,\ f\ ) - \beta\ | \leq |\beta| \tag{18}$$

for all $z \in U$, then the function f is univalent in U.

Proof. For $\beta = 1/\alpha$, from $|\alpha - 1| < 1$ we get $\mathrm{Re}\beta > \frac{1}{2}$ and

$$\left(\frac{1}{\beta} - 1\right)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} = \frac{1}{\beta}\,[\ M(\beta,\ f) - \beta\ ]$$

Remark. In the case $\beta > \frac{1}{2}$, the condition (18) implies $\mathrm{Re}M(\beta,\ f) > 0$ and from Theorem 1.1 we get that f is a β-convex function.

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BOOLE LATTICES OF IDEMPOTENTS IN A RING

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Abstract. In this paper, we will show that in a ring R, there exist maximal subsets of commuting idempotents. On these maximal subsets, one can define Boole lattice structures which induce Boole rings which usually are not subrings of R. If R is a Boole ring, we obtain the Stone's Theorem.

Let V be a linear space over the skewfield K and $EndV$ the set of endomorphisms of V.

If $f, g \in EndV$ then the functions $f + g : V \to V$ and $g \circ f : V \to V$ defined by:

$$(f+g)(x) = f(x) + g(x) \text{ and } (f \circ g)(x) = f(g(x))$$

are endomorphisms of V, that is $f + g, f \circ g \in EndV$.

The set $EndV$ is a ring with respect to the operations defined above. This ring is not commutative if $dimV \geq 2$. An endomorphism f of V is called projector of V if $f^2 = f$.

Starting from the papers of W.J. Gordon [3] and W.J. Gordon and E.W.Cheney [4], F.J. Delvos and W. Schempp are presenting in their book [1] the construction of lattices of projectors from $EndV$ which are commutative. They use these lattices in the approximation and interpolation theory.

In this paper, we will associate Boole lattices to a ring (associative) with the unit R. An element $a \in R$ with the property $a^2 = a$ is called idempotent. Thus, the projectors of V correspond to the idempotent elements of the ring $EndV$.

If R is a Boole ring, i.e. every element of $a \in R$ is idempotent, then these lattices coincide with the Boole lattice associated to R, according to Stone's Theorem which establishes a bijection between Boole lattices and Boole rings. Note that every Boole ring R is commutative and $2a = 0$ for $\forall a \in R$.

Let $I(R) = \left\{a \in R | a^2 = a\right\}$, $\mathcal{P}(I(R))$ the set of subsets of $I(R)$ and

$$\mathcal{P}' = \{X \in \mathcal{P}(I(R)) \,|\forall a, b \in X; ab = ba\}.$$

Remarks. a) We have $I(R) = R$ if and only if R is a Boole ring. In this case, $\mathcal{P}' = \mathcal{P}(R)$ and R is the only maximal element of $\mathcal{P}'$.

b) We have $\{0, 1\} \in \mathcal{P}'$.

Theorem 1. 1) For every $X \in \mathcal{P}'$ there exists a maximal element Y in $(\mathcal{P}', \subseteq)$ such that $X \subseteq Y$.

2) If Y is a maximal element of $\mathcal{P}'$ then $0, 1 \in Y$.

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Proof. 1) If $\mathcal{P}''$ is a non empty chain from $\mathcal{C} = \{X' \in \mathcal{P}' | X \subseteq X'\}$ then

$$\bigcup_{X' \in \mathcal{P}''} X' \in \mathcal{C}$$

Thus, according to Zorn's lemma there exist maximal elements in $\mathcal{C}$.

2) The elements 0 and 1 are idempotent and they commute with every $y \in Y$. So $Y \cup \{0, 1\} \in \mathcal{P}'$ and using the maximality of Y in $\mathcal{P}'$ it results that $0, 1 \in Y$.

Theorem 2. If Y is a maximal element in $(\mathcal{P}', \subseteq)$ then:

i) Y is stable with respect to the multiplication in R, i.e.

$$x, y \in Y \Rightarrow xy \in Y$$

ii) The relation " $\leq$ " defined on Y by

$$x \leq y \Leftrightarrow xy = x$$

is an ordering relation and 0 respectively 1 is the least respectively the greatest element in $(Y, \leq)$.

iii) The ordered set $(Y, \leq)$ is a Boole lattice. In this lattice we have

$$x \wedge y = xy,\ x \vee y = x + y - xy \text{ and } x' = 1 - x \tag{1}$$

where $x \wedge y = inf\,(x, y)$, $x \vee y = sup\,(x, y)$ and x' is the complement of x.

Proof. i) From $x, y \in Y$ it results that x, y are idempotents which commute, which implies that

$$(xy)^2 = xyxy = x^2y^2 = xy$$

so xy is idempotent. Since the elements of Y are commuting, it results that for $\forall z \in Y$ we have

$$(xy)\,z = x\,(yz) = x\,(zy) = (xz)\,y = (zx)\,y = z\,(xy)\,,$$

so xy commutes with every element of Y.

This means that $Y \cup \{xy\} \in \mathcal{P}'$, which together with the maximality of Y in $(\mathcal{P}', \subseteq)$, implies that $xy \in Y$.

ii) Since the elements of Y are idempotents, it results that for $\forall x \in X$ we have

$$x^2 = x,$$

so $x \leq x$. Thus the " $\leq$ " relation is reflexive. If $x, y, z \in Y$ then

$$x \leq y \text{ and } y \leq z \Rightarrow xy = x \text{ and } yz = y$$

Using the fact that y is idempotent we deduce that

$$(xy)\,(yz) = xy \Rightarrow xyz = xy \Rightarrow xz = x \Rightarrow x \leq z$$

So the " $\leq$ " relation is also transitive.

If $x, y \in Y$ then using the fact that the elements of Y are commuting, we have:

$$x \leq y \text{ and } y \leq x \Rightarrow xy = x \text{ and } yx = y \Rightarrow x = y$$

which shows that the relation " $\leq$ " is antisymmetric.

We have proved that " $\leq$ " is an order relation on Y. For $\forall x \in Y$, from

$$0x = 0 \text{ and } x1 = x$$

it results that $0 \leq x$ and $x \leq 1$.

So, 0 and 1 are the least respectively the greatest element in $(Y, \leq)$.

iii) For $\forall x, y \in Y$ we have

$$(xy)\,x = x^2 y = xy, (xy)\,y = xy^2 = xy$$

and

$$(x + y - xy)\,x = x^2 + yx - xyx = x + yx - yx^2 = x + yx - yx = x,$$
$$(x + y - xy)\,y = xy + y^2 - xy^2 = xy + y - xy = y$$

which shows that $xy \leq x$, $xy \leq y$ and $x \leq x + y - xy$ and $y \leq x + y - xy$, so xy is a lower bound and $x + y - xy$ is a upper bound of x and y.

If $z \in Y$ then

$$z \leq x, z \leq y \Rightarrow zx = z, zy = z \Rightarrow z^2\,(xy) = z^2 \Rightarrow z\,(xy) = z \Rightarrow z \leq xy$$

and

$$x \leq z, y \leq z \Rightarrow xz = x, yz = y$$

from which it results

$$z\,(x + y - xy) = zx + zy - zxy = x + y - xy$$

hence $z \leq x + y - xy$.

So xy is the greatest lower bound and $x + y - xy$ is the least upper bound of x and y.

So

$$x \wedge y = xy \text{ and } x \vee y = x + y - xy$$

Thus we have proved that $(Y, \leq)$ is a lattice.

Now we will show that this lattice is also distributive. If $x, y, z \in Y$ then

$$x \wedge (y \vee z) = x\,(y + z - yz) = xy + xz - xyz,$$

$$(x \wedge y) \vee (x \wedge z) = (xy) \vee (xz) = xy + xz - xyxz = xy + xz - xyz$$

so

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{2}$$

Here we notice that the identity (2) is true in a lattice if and only if the following identity is also true:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Thus $(Y, \leq)$ is a distributive lattice having 0 the least element and 1 the greatest element.

For $\forall x \in Y$ we have

$$x \wedge (1 - x) = x - x^2 = x - x = 0,$$

$$x \vee (1 - x) = x + 1 - x - x\,(1 - x) = 1 - x + x = 1$$

which shows that $x' = 1 - x$ is the complement of x.

So we proved that $(Y, \leq)$ is a Boole lattice.

Corollary. If $X \in \mathcal{P}'$ then the relation " $\leq$ " defined on X

$$x \leq y \Leftrightarrow xy = x$$

is an order relation.

Theorem 3. If Y is a maximal element in $(\mathcal{P}', \subseteq)$ then $\oplus$ defined by

$$x \oplus y = x + y - 2xy$$

is an operation on Y and Y is a Boolean ring with respect to $\oplus$ and the multiplication induced by the multiplication in R.
Proof. Applying Stone's Theorem to the Boole lattice $(Y, \vee, \wedge)$ it follows that the equalities

$$\begin{aligned} x * y &= (x \wedge y') \vee (x' \wedge y) \\ xy &= x \wedge y \end{aligned} \tag{3}$$

are defining operations in Y and $(Y, *, \cdot)$ is a Boole ring.

From (1) and (3) it results :

$$\begin{aligned} x * y &= [x(1-y)] \vee [(1-x)y] = x(1-y) + (1-x)y - x(1-y)(1-x)y = \\ &= x - xy + y - xy - x(1 - x - y + yx)y = \\ &= x + y - xy - xy - xy + x^2y + xy^2 - xyxy = \\ &= x + y - xy - xy - xy + xy + xy - xy = x + y - 2xy = x \oplus y \end{aligned}$$

Corollary. a) If Y is a maximal element in $\mathcal{P}'$ then the ring $(Y, \oplus, \cdot)$ is a subring of R if and only if

$$2x = 0 \text{ for } \forall x \in Y \tag{4}$$

We know that the ring $(Y, \oplus, \cdot)$ is a subring of R if and only if

$$x \oplus y = x + y \text{ for } \forall x, y \in Y \tag{5}$$

and

$$(5) \Leftrightarrow 2xy = 0; \forall x, y \in Y \Leftrightarrow 2x = 0, \forall x \in Y$$

The last equivalence takes place because $1 \in Y$.

b) If Y is a maximal element in $\mathcal{P}'$ and $R \neq \{0\}$, then the ring $(Y, \oplus, \cdot)$ is a subring of R if and only if the characteristic of R is 2.

The condition (4) is verified if and only if $2x = 0$ for $x = 1 \in R$ which implies that R has the characteristic 2.

c) Let K be a field of characteristic greater than 2 (in particular, K could be $\mathbb{R}$ or $\mathbb{C}$) and $R = EndV$. In this case, if Y is a maximal element in $\mathcal{P}'$, then the ring $(Y, \oplus, \cdot)$ is not a subring of R.

We know that if $\alpha \in K$ then the function

$$t_\alpha : V \to V,\ t_\alpha(x) = \alpha x$$

is an endomorphism of V, i.e. $t_\alpha \in EndV = R$, and $\varphi : K \to R$, $\varphi(\alpha) = t_\alpha$ is a unitary and injective homomorphism of rings.

So the characteristic of R concides to the characteristic of K, so R has a characteristic different from 2.

d) **Stone's Theorem.** If $(R, +, \cdot)$ is a Boole ring then R is a Boole lattice with respect to the operations " $\vee$ " and " $\wedge$ " defined by

$$x \vee y = x + y - xy \text{ and } x \wedge y = xy$$

The correspondence $(R, +, \cdot) \longmapsto (R, \vee, \wedge)$ introduces a bijection between the class of Boole rings and the class of Boole lattices.

On the other side, if $(R, \vee, \wedge)$ is a Boole lattice then R is a Boole ring with respect to the operations " $+$ " and " $\cdot$ " defined by

$$x + y = (x \wedge y') \vee (x' \wedge y) \text{ and } xy = x \wedge y$$

If we denote the above bijection by φ, we have $\varphi^{-1}(R, \vee, \wedge) = (R, +, \cdot)$

This theorem results from Theorems 2 and 3 and from Corollary a), taking into account that in this case $\mathcal{P}' = \mathcal{P}(R)$ and $Y = R$.

Theorem 4. If R is a ring with unit and $\phi \neq X \in \mathcal{P}'$, i.e. X is a non empty subset of R composed by idempotents which commute, then the elements of R such as

$$x_{11}x_{12}\dots x_{1n_1} \vee x_{21}x_{22}\dots x_{2n_2} \vee \dots \vee x_{k1}x_{k2}\dots x_{kn_k} \tag{6}$$

where $x_{ij} \in X$ $(i = 1,\dots,k; j = 1,\dots,n_k; k, n_k \in \mathbb{N}^*)$ are idempotents and commute between them.

The set $\overline{X}$ of elements such as (6) is a distributive lattice with respect to the relation " $\leq$ ". This lattice is generated by X.

Proof. From Theorem 1 it results that there exists a maximal element $Y \in \mathcal{P}'$ such that $X \subseteq Y$ and from Theorem 2 we know that $(Y, \leq)$ is a distributive lattice.

From $X \subseteq Y$ and the fact that $(Y, \leq)$ is a lattice, it results that $\overline{X} \subseteq Y$.

So elements such as (6) are idempotents and commute between them. If y, z are elements such as (6), that is $y, z \in \overline{X}$, then it is obvious that $y \vee z \in \overline{X}$ and from the distributivity of the operation " $\wedge$ " (which coincides with " $\cdot$ ") with respect to " $\vee$ ", it results that yz is also such as (6).

This means that $\overline{X}$ is a sublattice of $(Y, \leq)$.

From (6) it results that $X \subseteq \overline{X}$. If Z is a sublattice of Y and $X \subseteq Z$, then from (6) it follows that $\overline{X} \subseteq Z$.

Thus we have proved that $\overline{X}$ is the least sublattice of Y which includes X.

So $\left(\overline{X}, \leq\right)$ is a distributive lattice generated by X.

Remarks. Considering $R = EndV$ in Theorem 4, where V is a linear space over $\mathbb{R}$ or $\mathbb{C}$, we obtain Propositions 6 and 7, section 1.2 from [1].

Theorem 5. If R is a ring with unit and $\phi \neq X \in \mathcal{P}'$ and $X^c = \{1 - x | x \in X\}$ then the elements such as (6) where $x_{ij} \in X \cup X^c$, $(i = 1,\dots,k;\ j = 1,\dots,n_k;\ k, n_k \in \mathbb{N}^*)$ are idempotents which commute and the set $\overline{\overline{X}}$ of this elements is a Boole lattice generated by X.

Proof. If $x, y \in X \cup X^c$ then we can easily prove that x and y are idempotents and $xy = yx$. Using Theorem 1 it follows that there exists a maximal element $Y \in \mathcal{P}'$ such that $X \cup X^c \subseteq Y$.

From Theorem 2 it results that $(Y, \leq)$ is a Boole lattice and from the proof of Theorem 4 it follows that $\overline{\overline{X}}$ is the sublattice of Y generated by $X \cup X^c$.

So $\overline{\overline{X}}$ is a distributive lattice. If $x \in X$ then

$$\begin{aligned} x \wedge (1 - x) &= x - x^2 = x - x = 0, \\ x \vee (1 - x) &= x + 1 - x - x(1 - x) = 1 - x + x^2 = 1 - x + x = 1 \end{aligned}$$

which shows that $0, 1 \in \overline{\overline{X}}$.

Using de Morgan's formulas in the Boole lattice Y

$$(y \vee z)' = y' \wedge z' \text{ and } (y \wedge z)' = y' \vee z'$$

for $\forall y, z \in Y$, and using the distributiveness, it follows that the complement of an element of $\overline{\overline{X}}$ is also in $\overline{\overline{X}}$.

So $\overline{\overline{X}}$ is a Boole sublattice of the lattice Y.

We have $X \subseteq \overline{\overline{X}}$ and if Z is a Boole sublattice of Y which includes X then $\overline{\overline{X}} \subseteq Z$.

This means that $\overline{\overline{X}}$ is the smallest Boole lattice including X that is $\overline{\overline{X}}$ is generated by X.

Remarks. Considering, in Theorem 5, $R = EndV$ where V is a linear space over $\mathbb{R}$ or $\mathbb{C}$ it results the construction of the Boole algebra of projectors, given in section 1.4. of [1].

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A REPRESENTATION OF p-CONVEX SET-VALUED MAPS WITH VALUES IN $\mathbb{R}$

DORIAN POPA

Abstract. For a p-convex set-valued map with compact values in $\mathbb{R}$ is given a representation theorem as a sum of an additive function and a compact interval.

1. Introduction

Let X be a real vector space. We denote by $\mathcal{P}_0(X)$ the set of all nonempty subsets of X. A subset D of X is said to be *p-convex*, where p is a real number in the interval $(0,1)$, if for every $x,y \in D$ we have:

$$(1-p)x + py \in D.$$

It is known (see [4]) that every p-convex and closed subset of a real topological vector space is a convex set. A $\frac{1}{2}$-convex set is called *midconvex* set.

Let D be a p-convex and nonempty subset of X. A set-valued map $F : D \to \mathcal{P}_0(\mathbb{R})$ is said to be *p-convex* if for every $x,y \in D$ we have:

$$(1-p)F(x) + pF(y) \subseteq F((1-p)x + py).$$

A function $f : D \to \mathbb{R}$ is said to be *p-convex* (*concave*) if for every $x,y \in D$ we have:

$$f((1-p)x + py) \leq (\geq)(1-p)f(x) + pf(y).$$

The following assertions, which are true for midconvex set-valued maps and functions [3], holds for p-convex set-valued maps and functions.

A set valued map $F : D \to \mathcal{P}_0(\mathbb{R})$ is p-convex if and only if the *graph* of F, defined by

$$Graph\, F = \{(x,y) \in X \times \mathbb{R} : \ y \in F(x)\},$$

is a p-convex subset of the vector space $X \times \mathbb{R}$.

A function $f : D \to \mathbb{R}$ is p-convex if and only if the *epigraph* of f, defined by

$$Epi\, f = \{(x,t) \in X \times \mathbb{R} : \ f(x) \leq t\},$$

is a p-convex subset of the vector space $X \times \mathbb{R}$.

Example 1.1. Let $f,g : D \to \mathbb{R}$ be two functions such that $f(x) \leq g(x)$ for every $x \in D$. Then the set valued map $F : D \to \mathcal{P}_0(\mathbb{R})$ given by the relation

$$F(x) = [f(x), g(x)]$$

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for every $x \in D$ is p-convex if and only if f is p-convex and g is p-concave.
Proof. Let $x, y \in D$. We have

$$(1-p)F(x) + pF(y) = [(1-p)f(x) + pf(y), (1-p)g(x) + pg(y)]$$

and

$$F((1-p)x + py) = [f((1-p)x + py), g((1-p)x + py)].$$

The relation

$$(1-p)F(x) + pF(y) \subseteq F((1-p)x + py)$$

holds if and only if we have

$$f((1-p)x + py) \leq (1-p)f(x) + pf(y)$$

and

$$(1-p)g(x) + pg(y) \leq g((1-p)x + py),$$

hence f is p-convex and g is p-concave.
Remark 1.1. If $F : D \to \mathcal{P}_0(\mathbb{R})$ is a p-convex set-valued map with closed values, then it is convex valued.
Proof. Let $x \in D$. We have

$$(1-p)F(x) + pF(x) \subseteq F((1-p)x + px) = F(x),$$

hence $F(x)$ is a p-convex subset of $\mathbb{R}$ and being closed it is a convex subset of $\mathbb{R}$. □

The goal of this paper is to give a representation of p-convex set-valued maps with compact values in $\mathbb{R}$. For additive set-valued function this problem was studied by H. Rädstrom [8]. Later K. Nikodem [5], gave a characterization of midconvex set-valued maps with compact values in $\mathbb{R}$. A representation of the solutions of a generalization of Jensen equation for set-valued maps is given by the author in [7]. K. Nikodem, F. Papalini and S. Vercillo [6], established conditions under which every midconvex set-valued function can be represented as a sum of an additive function and a convex set-valued function. We prove that an analogous result holds for p-convex set-valued maps with compact values in $\mathbb{R}$.

2. Main results

For the characterization of p-convex set-valued maps with compact values in $\mathcal{P}_0(\mathbb{R})$ we need some lemmas.
Lemma 2.1. ([2]) *Let $p \in (0,1)$. Denote by $(P_n)_{n\geq 1}$ the sequence of sets defined as follows: $P_1 = \{0, p, 1\}$; if $P_n = \{0, p_n^{(1)}, \ldots, p_n^{(2^n-1)}\}$, where*

$$0 < p_n^{(1)} < \cdots < p_n^{(2^n-1)} < 1,$$

is defined, put

$$P_{n+1} = P_n \cup \{(1-p)p_n^{(k-1)} + pp_n^{(k)} : \ 1 \leq k \leq 2^n\}$$

where $p_0^{(0)} = 0$ and $p_n^{(2n)} = 1$. Then the set

$$P = \bigcup_{n\geq 1} P_n$$

is dense in the interval $[0,1]$.

Lemma 2.2. ([2]) *Let X be a real linear space and D a p-convex and nonempty subset of X. Then D is q-convex for each $q \in P$, where P is the set defined in Lemma 2.1.*

Lemma 2.3. *Let X be a real linear space, D a p-convex and nonempty subset of X. If a set-valued map $F : D \to \mathcal{P}_0(Y)$ is p-convex then it is q-convex for every $q \in P$, where P is the set defined in Lemma 2.1.*

Proof. From the p-convexity of F it results that $Graph\, F$ is a p-convex subset of $X \times \mathbb{R}$, and using Lemma 2.2 we obtain that $Graph\, F$ is q-convex for every $q \in P$. Then F is q-convex for every $q \in P$. □

Theorem 2.1. *Let D be a linear subspace of the real linear space X and $F : D \to \mathcal{P}_0(\mathbb{R})$ be a p-convex set-valued map with bounded values. Then there exists an additive function $a : D \to \mathbb{R}$ and two real numbers s, t, $s \leq t$, such that for every $x \in D$*

$$a(x) + s \leq F(x) \leq a(x) + t.$$

Proof. Following the method used in [5], for any $x \in D$ put $f(x) = \inf F(x)$ and $g(x) = \sup F(x)$. Then $f : D \to \mathbb{R}$ is p-convex and $g : D \to \mathbb{R}$ is p-concave. Indeed, for every $x, y \in X$ we have:

$$\begin{aligned} f((1-p)x + py) &= \inf F((1-p)x + py) \\ &\leq \inf((1-p)F(x) + pF(y)) \\ &= \inf((1-p)F(x) + \inf(pF(y)) \\ &= (1-p)f(x) + pf(y), \end{aligned}$$

hence f is a p-convex function and analogously g is a p-concave function. We have also

$$f(x) \leq F(x) \leq g(x)$$

for every $x \in D$.

Let $h : D \to \mathbb{R}$, $h(x) = g(x) - f(x)$ for every $x \in D$. Obviously h is p-concave and $h(x) \geq 0$ for every $x \in D$. We prove that h is a constant function.

The function $-h$ is p-convex, hence the set $Epi(-h)$ is p-convex and it follows from Lemma 2.2 that $Epi(-h)$ is q-convex for every $q \in P$. It follows that $-h$ is a q-convex function for $q \in P$, hence h is q-concave for $q \in P$.

Suppose that h is nonconstant. Then there exist $x, y \in X$, $x \neq y$, such that $h(x) < h(y)$. Using the density of P in $[0,1]$ it follows that there exists $t > 1$, $\frac{1}{t} \in P$, such that:

$$t(h(x) - h(y)) + h(y) < 0.$$

From the q-concavity of h with $q \in P$ we get:

$$\begin{aligned} h(x) &= h\left(\frac{1}{t}(tx + (1-t)y) + \left(1 - \frac{1}{t}\right)y\right) \\ &\geq \frac{1}{t}h(tx + (1-t)y) + \left(1 - \frac{1}{t}\right)h(y) \end{aligned}$$

and foreward it follows

$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y) = t(h(x) - h(y)) + h(y) < 0,$$

contradiction with nonnegativity of the values of h.

Hence there exists $c \in \mathbb{R}$ such that $h(x) = c$, for every $x \in X$.

The function $f = g - c$ is p-concave and being p-convex satisfies the relation

$$f((1-p)x + py) = (1-p)f(x) + pf(y). \tag{1}$$

We prove that there exists an additive function $a : D \to \mathbb{R}$ and $k \in \mathbb{R}$ such that $f(x) = a(x) + k$ for every $x \in D$.

For $x = 0$ and $y \in D$ in (1) we have

$$f(py) = pf(y) + (1-p)f(0). \tag{2}$$

For $y = 0$ and $x \in D$ in (1) we have

$$f((1-p)x) = (1-p)f(x) + pf(0). \tag{3}$$

Let $u, v \in D$. From (1), (2), (3) we have

$$\begin{aligned} f(u+v) &= f\left((1-p)\frac{u}{1-p} + p\frac{v}{p}\right) \\ &= (1-p)f\left(\frac{u}{1-p}\right) + pf\left(\frac{v}{p}\right) \\ &= (1-p)f\left(\frac{u}{1-p}\right) + pf(0) + pf\left(\frac{v}{p}\right) \\ &+ (1-p)f(0) - (1-p)f(0) - pf(0) \\ &= f(u) + f(v) - f(0). \end{aligned}$$

The function $a : D \to \mathbb{R}$, $a(x) = f(x) - f(0)$, $x \in D$, is additive. Indeed for any $x, y \in X$ we have:

$$a(x+y) = f(x+y) - f(0) = f(x) + f(y) - f(0) - f(0) = a(x) + a(y).$$

Denoting $s = f(0)$ we obtain $f(x) = a(x) + s$ and $g(x) = a(x) + t$ for every $x \in D$, where $t = s + c$. □

Corollary 2.1. *Let D be a linear subspace of a real linear space X and $F : D \to \mathcal{P}_0(\mathbb{R})$ be a p-convex set-valued map with compact values.*

Then there exists an additive function $a : D \to \mathbb{R}$ and a compact interval I in $\mathbb{R}$ such that

$$F(x) = a(x) + I$$

for every $x \in D$.

Proof. In view of Theorem 2.1, there exist an additive function $a : D \to \mathbb{R}$ and $s, t \in \mathbb{R}$, $s \leq t$, such that

$$a(x) + s \leq F(x) \leq a(x) + t$$

for every $x \in D$. Taking account of the Remark 1.1, $F(x)$ is a convex subset of $\mathbb{R}$, hence

$$F(x) = [a(x) + s, a(x) + t] = a(x) + I$$

for every $x \in D$, where $I = [s, t]$. □

Remark 2.1. If p is a rational number in the interval $(0, 1)$ then the converse of Corollary 2.1 is true.

Proof. Let $a : D \to \mathbb{R}$ be an additive function, I a compact interval in $\mathbb{R}$ and $F(x) = a(x)+I$ for every $x \in D$. Taking into account that a is rationally homogeneous [1] it follows that

$$\begin{aligned} F((1-p)x+py) &= a((1-p)x+py) \\ &= (1-p)a(x)+pa(y)+(1-p)I+pI \\ &= (1-p)F(x)+pF(y) \end{aligned}$$

for every $x, y \in D$. □

The results proved in Theorem 2.1 and Corollary 2.1 are extensions of the results obtained in [5] for midconvex-valued maps.

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ON A CLASS OF PARAMETRIC PARTIAL LINEAR COMPLEX VECTOR FUNCTIONAL EQUATIONS

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Abstract. In this paper one class of parametric complex vector partial linear functional equations is solved.

0. **Introduction**

First we introduce the following notations. Let $\mathcal{V}$, $\mathcal{V}'$ be finite dimensional complex vector spaces and $\mathbf{Z}_i$, $i \in \mathbf{N}$, be vectors in $\mathcal{V}$. We may assume that $\mathbf{Z}_i = (z_{i1}(t), \ldots, z_{in}(t))^T$, where $z_{ij}(t)$ $(1 \leq j \leq n)$ are complex functions and $\mathbf{O} = (0, \ldots, 0)^T$ is the zero-vector in $\mathcal{V}$ or $\mathcal{V}'$. We also denote by $\mathcal{V}^0$ the subspace of all real vectors in $\mathcal{V}$ (thus $\mathcal{V} = \mathcal{V}^0 + i\mathcal{V}^0$), and by $\mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ the space of linear mappings $\mathcal{V}^0 \to \mathcal{V}'$. Let (m, n) be the greatest common divisor of m and n.

In the present paper our object of investigation will be the following functional equation

$$\sum_{i=1}^{m+n} f_i\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j}, \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{i+m+j}\Big) = \mathbf{O} \tag{1}$$

$$(\mathbf{Z}_{m+n+i} \equiv \mathbf{Z}_i, \quad a \in \mathbf{C}),$$

where $\mathbf{C}$ is the field of complex numbers and $f_i : \mathcal{V}^2 \to \mathcal{V}'$ $(1 \leq i \leq m+n)$ are unknown complex vector functions.

The above equation for $a = 1$ was solved in [1] under the assumption that the functions and variables are real. But the argument given there is valid only if the greatest common divisor of m and n is 1. Also, one special general case is solved in [2]. The theorems of [2] concerning the cases $m \neq n$ should be modified to give the general continuous solutions.

1. **Main Results**

Now we will give the following results.

Theorem 1. *If $a = 1$, $(m, n) = 1$ and $m + n > 2$, then the general continuous solution of the functional equation (1) is*

$$f_i(\mathbf{U}, \mathbf{V}) = F_1(\mathbf{U} + \mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U} + \mathbf{V})\mathrm{Im}\,\mathbf{U} + G_i(\mathbf{U} + \mathbf{V}) \tag{2}$$

$$(1 \leq i \leq m+n),$$

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so that

$$\sum_{i=1}^{n+m} G_i(\mathbf{U}) = -m[F_1(\mathbf{U})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U})\mathrm{Im}\,\mathbf{U}],$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ and $G_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n-1)$ *are arbitrary continuous complex vector functions.*

Proof. We accept the convention to reduce the indices $\mathrm{mod}(m+n)$. If we set

$$\mathbf{S} = \sum_{i=1}^{m+n} \mathbf{Z}_i,$$

$$\mathbf{T}_i = \mathbf{Z}_i + \mathbf{Z}_{i+1} + \cdots + \mathbf{Z}_{i+m-1} - \frac{m\mathbf{S}}{m+n} \qquad (1 \le i \le m+n-1), \tag{3}$$

the vectors $\mathbf{T}_i$ $(1 \le i \le m+n-1)$ and $\mathbf{S}$ are independent since $(m,n) = 1$. The equation (1) becomes

$$\sum_{i=1}^{m+n-1} f_i\Big(\mathbf{T}_i + \frac{m\mathbf{S}}{m+n}, \ \frac{n\mathbf{S}}{m+n} - \mathbf{T}_i\Big) \tag{4}$$

$$+f_{m+n}\Big(-\mathbf{T}_1 - \mathbf{T}_2 - \cdots - \mathbf{T}_{m+n-1} + \frac{m\mathbf{S}}{m+n}, \ \frac{n\mathbf{S}}{m+n} + \mathbf{T}_1 + \mathbf{T}_2 + \cdots + \mathbf{T}_{m+n-1}\Big) = \mathbf{O}.$$

We introduce the new notations

$$f_i\Big(\mathbf{U} + \frac{m\mathbf{S}}{m+n}, \ \frac{n\mathbf{S}}{m+n} - \mathbf{U}\Big) = g_i(\mathbf{U}, \mathbf{S}) \quad (1 \le i \le m+n),$$

i.e.,

$$f_i(\mathbf{U}, \mathbf{V}) = g_i\Big(\frac{n\mathbf{U} - m\mathbf{V}}{m+n}, \ \mathbf{U} + \mathbf{V}\Big) \quad (1 \le i \le m+n). \tag{5}$$

The equation (4) is transformed into

$$\sum_{i=1}^{m+n-1} g_i(\mathbf{T}_i, \mathbf{S}) + g_{m+n}(-\mathbf{T}_1 - \mathbf{T}_2 - \cdots - \mathbf{T}_{m+n-1}, \ \mathbf{S}) = \mathbf{O}. \tag{6}$$

By the substitution $\mathbf{T}_1 = \mathbf{T}_2 = \cdots = \mathbf{T}_{r-1} = \mathbf{T}_{r+1} = \cdots = \mathbf{T}_{m+n-1} = \mathbf{O}$, we obtain

$$g_r(\mathbf{T}_r, \mathbf{S}) = -g_{m+n}(-\mathbf{T}_r, \mathbf{S}) - H_r(\mathbf{S}) \quad (1 \le r \le m+n-1). \tag{7}$$

Putting (7) into (6), we get

$$g_{m+n}(-\mathbf{T}_1 - \mathbf{T}_2 - \cdots - \mathbf{T}_{m+n-1}, \ \mathbf{S}) = \sum_{i=1}^{m+n-1} g_{m+n}(-\mathbf{T}_i, \mathbf{S}) + \sum_{i=1}^{m+n-1} H_i(\mathbf{S}). \tag{8}$$

We conclude that the function

$$K(\mathbf{U}, \mathbf{S}) = g_{m+n}(\mathbf{U}, \mathbf{S}) + \frac{1}{m+n-2} \sum_{i=1}^{m+n-1} H_i(\mathbf{S}) \tag{9}$$

satisfies the functional equation

$$K(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_{m+n-1}, \ \mathbf{S}) = \sum_{i=1}^{m+n-1} K(\mathbf{Z}_i, \mathbf{S}). \tag{10}$$

Using the continuity of K, from (10) we deduce that for fixed $\mathbf{S}$

$$K(\mathbf{U},\mathbf{S}) = c_1 \mathrm{Re}\,\mathbf{U} + c_2 \mathrm{Im}\,\mathbf{U},$$

where $\mathrm{Re}\,\mathbf{U}$ resp. $\mathrm{Im}\,\mathbf{U}$ denotes the real resp. imaginary part of $\mathbf{U}$. The mappings $c_1, c_2 \in \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ may depend upon $\mathbf{S}$. Hence,

$$K(\mathbf{U},\mathbf{V}) = F_1(\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{V})\mathrm{Im}\,\mathbf{U}, \tag{11}$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ are continuous functions.

From (9), (11) and (7) we obtain

$$\begin{aligned} g_{m+n}(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{V})\mathrm{Re}\,\mathbf{U} + f_2(\mathbf{V})\mathrm{Im}\,\mathbf{U} - \frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{V}), \\ g_r(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{V})\mathrm{Im}\,\mathbf{U} - H_r(\mathbf{V}) \\ &+ \frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{V}) \quad (1 \le r \le m+n-1). \end{aligned} \tag{12}$$

From (5) and (12) we deduce that

$$\begin{aligned} f_r(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\left(\frac{n\mathbf{U}-m\mathbf{V}}{m+n}\right) + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\left(\frac{n\mathbf{U}-m\mathbf{V}}{m+n}\right) \\ &-H_r(\mathbf{U}+\mathbf{V}) + \frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{U}+\mathbf{V}) \quad (1 \le r \le m+n-1), \\ f_{m+n}(\mathbf{U}+\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\left(\frac{n\mathbf{U}-m\mathbf{V}}{m+n}\right) + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\left(\frac{n\mathbf{U}-m\mathbf{V}}{m+n}\right) \\ &-\frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{U}+\mathbf{V}). \end{aligned} \tag{13}$$

By denoting

$$\begin{aligned} &-F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\left[\frac{m(\mathbf{U}+\mathbf{V})}{m+n}\right] - F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\left[\frac{m(\mathbf{U}+\mathbf{V})}{m+n}\right] \\ &+\frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{U}+\mathbf{V}) - H_r(\mathbf{U}+\mathbf{V}) = G_r(\mathbf{U}+\mathbf{V}) \quad (1 \le r \le m+n-1), \\ &-F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\left[\frac{m(\mathbf{U}+\mathbf{V})}{m+n}\right] - F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\left[\frac{m(\mathbf{U}+\mathbf{V})}{m+n}\right] \\ &-\frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{U}+\mathbf{V}) = G_{m+n}(\mathbf{U}+\mathbf{V}), \end{aligned}$$

from (13) we get (2).

The converse can be established by a straightforward verification. □

Example 1. The general continuous solution of the functional equation

$$f_1(\mathbf{Z}_1+\mathbf{Z}_2,\ \mathbf{Z}_3) + f_2(\mathbf{Z}_2+\mathbf{Z}_3,\ \mathbf{Z}_1) + f_3(\mathbf{Z}_3+\mathbf{Z}_1,\ \mathbf{Z}_2) = \mathbf{O}$$

is given by

$$\begin{aligned} f_1(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_1(\mathbf{U}+\mathbf{V}), \\ f_2(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_2(\mathbf{U}+\mathbf{V}), \end{aligned}$$

$f_3(\mathbf{U},\mathbf{V}) = -F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\,(\mathbf{U}+2\mathbf{V}) - F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\,(\mathbf{U}+2\mathbf{V}) - G_1(\mathbf{U}+\mathbf{V}) - G_2(\mathbf{U}+\mathbf{V})$, where $F_1, F_2 : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ and $G_1, G_2 : \mathcal{V} \to \mathcal{V}'$ are arbitrary continuous complex vector functions.

Corollary. *The general continuous solution of the vector functional equation*

$$\sum_{i=1}^{m+n} g_i(\mathbf{Z}_i + \cdots + \mathbf{Z}_{i+m-1},\ \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_{m+n}) = \mathbf{O}$$

if $(m,n) = 1$ *and* $m + n > 2$ *is given by*

$$g_i(\mathbf{U},\mathbf{V}) = F_1(\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_i(\mathbf{V}) \quad (1 \le i \le m+n),$$

$$\sum_{i=1}^{m+n} G_i(\mathbf{V}) = -m[F_1(\mathbf{V})\mathrm{Re}\,\mathbf{V} + F_2(\mathbf{V})\mathrm{Im}\,\mathbf{V}],$$

where $F_1, F_2 : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$, $G_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n-1)$ *are arbitrary continuous complex vector functions.*

Proof. Put $f_i(\mathbf{U},\mathbf{V}) = g_i(\mathbf{U},\mathbf{U}+\mathbf{V})$ in Theorem 1. □

Theorem 2. *The general continuous solution of the complex vector functional equation (1) if* $a = 1$, $(m,n) = d > 1$, $m/d = p$, $n/d = q$ *and* $p + q > 2$ *is given by*

$$f_{id+j}(\mathbf{U},\mathbf{V}) = F_{1j}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{2j}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{ij}(\mathbf{U}+\mathbf{V})$$
$$(0 \le i \le p+q-1, \quad 1 \le j \le d),$$

$$\sum_{i=0}^{p+q-1} G_{ij}(\mathbf{U}) = H_j(\mathbf{U}) - p[F_{1j}(\mathbf{U})\mathrm{Re}\,\mathbf{U} + F_{2j}(\mathbf{U})\mathrm{Im}\,\mathbf{U}] \quad (1 \le j \le d), \tag{14}$$

$$\sum_{j=1}^{d} H_j(\mathbf{U}) = \mathbf{O},$$

where

$$F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1,2;\ 1 \le j \le d),$$
$$H_j : \mathcal{V} \to \mathcal{V}' \quad (1 \le j \le d-1),$$
$$G_{ij} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2;\ 1 \le j \le d)$$

are arbitrary continuous complex vector functions.

Proof. We set

$$f_i(\mathbf{U},\mathbf{V}) = g_i(\mathbf{U},\mathbf{U}+\mathbf{V}) \quad (1 \le i \le m+n) \tag{15}$$

and we obtain

$$\sum_{i=1}^{m+n} g_i(\mathbf{Z}_i + \mathbf{Z}_{i+1} + \cdots + \mathbf{Z}_{i+m-1},\ \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_{m+n}) = \mathbf{O}. \tag{16}$$

Let us introduce the new vectors

$$\mathbf{V}_i = \mathbf{Z}_i + \mathbf{Z}_{i+1} + \cdots + \mathbf{Z}_{i+d-1} \quad (1 \le i \le m+n) \quad \text{so that} \quad \mathbf{V}_{i+m+n} = \mathbf{V}_i \tag{17}$$

and

$$\mathbf{W} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_{m+n}. \tag{18}$$

They are not independent because

$$\sum_{i=0}^{p+q-1} \mathbf{V}_{id+j} = \mathbf{W} \quad (1 \le j \le d). \tag{19}$$

The vectors $\mathbf{V}_i$ $(1 \le i \le m+n-d)$ and $\mathbf{W}$ are independent because the rank of the matrix of linear forms determining them is $m+n-d+1$, which is easy to verify. In the sequel we will use all vectors (17) and (18) but we must have always in mind that (19) holds. The equation (16) becomes

$$\sum_{i=1}^{m+n} g_i(\mathbf{V}_i + \mathbf{V}_{i+d} + \cdots + \mathbf{V}_{i+(p-1)d},\ \mathbf{W}) = \mathbf{O}.$$

It can be written in the following form

$$\sum_{j=1}^{d} \sum_{i=0}^{p+q-1} g_{id+j}(\mathbf{V}_{id+j} + \mathbf{V}_{(i+1)d+j} + \cdots + \mathbf{V}_{(i+p-1)d+j},\ \mathbf{W}) = \mathbf{O}.$$

If we set here

$$\mathbf{V}_{id+j} = \mathbf{O} \quad (0 \le i \le p+q-2;\ j = 1,2,\dots,r-1,r+1,\dots,d),$$
$$\mathbf{V}_{(p+q-1)d+j} = \mathbf{W} \quad (j = 1,2,\dots,r-1,r+1,\dots,d),$$

we get

$$\sum_{i=0}^{p+q-1} g_{id+r}(\mathbf{V}_{id+r} + \mathbf{V}_{(i+1)d+r} + \cdots + \mathbf{V}_{(i+p-1)d+r},\ \mathbf{W}) - \frac{H_r(\mathbf{W})}{p+q} = \mathbf{O} \quad (1 \le r \le d)$$

and

$$\sum_{r=1}^{d} H_r(\mathbf{W}) = \mathbf{O}.$$

By using the corollary of Theorem 1 we get

$$g_{id+r}(\mathbf{U},\mathbf{V}) = F_{1r}(\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{2r}(\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{ir}(\mathbf{V}) \quad (0 \le i \le p+q-1;\ 1 \le r \le d),$$
$$\sum_{i=0}^{p+q-1} G_{ir}(\mathbf{V}) = H_r(\mathbf{V}) - p[F_{1r}(\mathbf{V})\mathrm{Re}\,\mathbf{V} + F_{2r}(\mathbf{V})\mathrm{Im}\,\mathbf{V}] \quad (1 \le r \le d),$$

where

$$F_{ir} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1,2;\ 1 \le r \le d),$$
$$G_{ir} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2;\ 1 \le r \le d),$$
$$H_r : \mathcal{V} \to \mathcal{V}' \quad (1 \le r \le d-1)$$

are arbitrary continuous complex vector functions. By application of (15) these formulas give (14).

It is easy to prove that the functions $f_i : \mathcal{V}^2 \to \mathcal{V}'$ $(1 \le i \le m+n)$ defined by (15) satisfy the complex vector functional equations (1). □

Example 2. The general continuous solution of the functional equation

$$f_1(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4,\ \mathbf{Z}_5 + \mathbf{Z}_6) + f_2(\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_5,\ \mathbf{Z}_6 + \mathbf{Z}_1)$$
$$+ f_3(\mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_5 + \mathbf{Z}_6,\ \mathbf{Z}_1 + \mathbf{Z}_2) + f_4(\mathbf{Z}_4 + \mathbf{Z}_5 + \mathbf{Z}_6 + \mathbf{Z}_1,\ \mathbf{Z}_2 + \mathbf{Z}_3)$$
$$+ f_5(\mathbf{Z}_5 + \mathbf{Z}_6 + \mathbf{Z}_1 + \mathbf{Z}_2,\ \mathbf{Z}_3 + \mathbf{Z}_4) + f_6(\mathbf{Z}_6 + \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3,\ \mathbf{Z}_4 + \mathbf{Z}_5) = \mathbf{O}$$

is given by

$$f_1(\mathbf{U},\mathbf{V}) = F_{11}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{21}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{01}(\mathbf{U}+\mathbf{V}),$$
$$f_2(\mathbf{U},\mathbf{V}) = F_{12}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{22}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{02}(\mathbf{U}+\mathbf{V}),$$
$$f_3(\mathbf{U},\mathbf{V}) = F_{11}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{21}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{11}(\mathbf{U}+\mathbf{V}),$$

$$f_4(\mathbf{U},\mathbf{V}) = F_{12}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{22}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{12}(\mathbf{U}+\mathbf{V}),$$
$$f_5(\mathbf{U},\mathbf{V}) = -F_{11}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,(\mathbf{U}+2\mathbf{V}) - F_{21}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,(\mathbf{U}+2\mathbf{V})$$
$$+H_1(\mathbf{U}+\mathbf{V}) - G_{01}(\mathbf{U}+\mathbf{V}) - G_{11}(\mathbf{U}+\mathbf{V}),$$
$$f_6(\mathbf{U},\mathbf{V}) = -F_{12}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,(\mathbf{U}+2\mathbf{V}) - F_{22}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,(\mathbf{U}+2\mathbf{V})$$
$$-H_1(\mathbf{U}+\mathbf{V}) - G_{01}(\mathbf{U}+\mathbf{V}) - G_{12}(\mathbf{U}+\mathbf{V}),$$

where

$$F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1, 2),$$
$$G_{ij} : \mathcal{V} \to \mathcal{V}' \quad (i = 0, 1;\ j = 1, 2),$$
$$H_1 : \mathcal{V} \to \mathcal{V}'$$

are arbitrary continuous complex vector functions.

Theorem 3. *The most general solution of (1) if $a = 1$ and $m = n$ is*

$$f_i(\mathbf{U},\mathbf{V}) \quad (1 \le i \le m) \quad \textit{are} \quad \textit{arbitrary},$$
$$f_{m+i}(\mathbf{U},\mathbf{V}) = H_i(\mathbf{U}+\mathbf{V}) - f_i(\mathbf{V},\mathbf{U}) \quad (1 \le i \le m), \tag{20}$$
$$\sum_{i=1}^{m} H_i(\mathbf{U}) = \mathbf{O},$$

where $H_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m-1)$ are arbitrary functions.

Proof. Put $f_i(\mathbf{U},\mathbf{V}) = G_i(\mathbf{U},\mathbf{U}+\mathbf{V})$. □

Example 3. The most general solution of the equation

$$f_1(\mathbf{Z}_1+\mathbf{Z}_2,\ \mathbf{Z}_3+\mathbf{Z}_4) + f_2(\mathbf{Z}_2+\mathbf{Z}_3,\ \mathbf{Z}_4+\mathbf{Z}_1)$$
$$+f_3(\mathbf{Z}_3+\mathbf{Z}_4,\ \mathbf{Z}_1+\mathbf{Z}_2) + f_4(\mathbf{Z}_4+\mathbf{Z}_1,\ \mathbf{Z}_2+\mathbf{Z}_3) = \mathbf{O}$$

is

$$f_1(\mathbf{U},\mathbf{V}),\ f_2(\mathbf{U},\mathbf{V}) \quad \textit{are} \quad \textit{arbitrary},$$
$$f_3(\mathbf{U},\mathbf{V}) = H_1(\mathbf{U}+\mathbf{V}) - f_1(\mathbf{V},\mathbf{U}),$$
$$f_4(\mathbf{U},\mathbf{V}) = -H_1(\mathbf{U}+\mathbf{V}) - f_2(\mathbf{V},\mathbf{U}),$$

where $H_1 : \mathcal{V} \to \mathcal{V}'$ is an arbitrary function.

Theorem 4. *If $a^{m+n} \neq 1$ and $m \neq n$, the general solution of the functional equation (1) is given by*

$$f_i(\mathbf{U},\mathbf{V}) = F_i(\mathbf{U}+a^m\mathbf{V}) - F_{i+n}(a^n\mathbf{U}+\mathbf{V}) + A_i \quad (1 \le i \le m+n), \tag{21}$$

where $F_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n)$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum\limits_{i=1}^{m+n} A_i = \mathbf{O}$.

Proof. If we introduce new functions g_i by the equation

$$f_i(\mathbf{U},\mathbf{V}) = g_i(\mathbf{U}+a^m\mathbf{V},\ a^n\mathbf{U}+\mathbf{V}) \quad (1 \le i \le m+n), \tag{22}$$

then equation (1) becomes

$$\sum_{i=1}^{m+n} g_i\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{m+n-1-j}\mathbf{Z}_{m+i+j},$$
$$\sum_{j=0}^{m-1} a^{m+n-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{m+i+j}\Big) = \mathbf{O},$$

i.e.,

$$\sum_{i=1}^{m+n} g_i\Big(\sum_{j=0}^{m+n-1} a^j \mathbf{Z}_{m+i-1-j},\ \sum_{j=0}^{m+n-1} a^j \mathbf{Z}_{i-1-j}\Big) = \mathbf{O}. \tag{23}$$

Since $a^{m+n} \neq 1$, this transformation is possible. Also we may introduce new vectors $\mathbf{V}_i$ by

$$\mathbf{V}_i = \sum_{j=0}^{m+n-1} a^j \mathbf{Z}_{m+i-1-j} \quad (1 \leq i \leq m+n)$$

but the equation (23) takes the form

$$\sum_{i=0}^{m+n} g_i(\mathbf{V}_i, \mathbf{V}_{i+n}) = \mathbf{O}. \tag{24}$$

By putting $\mathbf{V}_j = \mathbf{O}$ $(j = 1, 2, \ldots, i-1, i+1, \ldots, i+n-1, i+n+1, \ldots, m+n)$ we obtain

$$g_i(\mathbf{V}_i, \mathbf{V}_{i+n}) = F_i(\mathbf{V}_i) + G_i(\mathbf{V}_{i+n}) \quad (1 \leq i \leq m+n). \tag{25}$$

On the basis of the expression (25), the equation (24) becomes

$$\sum_{i=1}^{m+n} [F_i(\mathbf{V}_i) + G_i(\mathbf{V}_{i+n})] = \mathbf{O},$$

or

$$\sum_{i=1}^{m+n} [F_i(\mathbf{V}_i) + G_{m+i}(\mathbf{V}_i)] = \mathbf{O}. \tag{26}$$

From (26) it follows that

$$G_{i+m}(\mathbf{V}_i) = -F_i(\mathbf{V}_i) + A_i \quad (1 \leq i \leq m+n), \tag{27}$$

where A_i are arbitrary constant complex vectors with the property

$$\sum_{i=1}^{m+n} A_i = \mathbf{O}.$$

On the basis of the expression (27), the equality (25) has the form

$$g_i(\mathbf{U}, \mathbf{V}) = F_i(\mathbf{U}) + F_{i+n}(\mathbf{V}) + A_i \quad (1 \leq i \leq m+n), \tag{28}$$

where $\sum\limits_{i=1}^{m+n} A_i = \mathbf{O}$.

On the basis of the equalities (28) and (22), we obtain (21). □

Example 4. If $a^3 \neq 1$, the general solution of the functional equation

$$f_1(a^2\mathbf{Z}_1 + a\mathbf{Z}_2 + \mathbf{Z}_3,\ \mathbf{Z}_4) + f_2(a^2\mathbf{Z}_2 + a\mathbf{Z}_3 + \mathbf{Z}_4,\ \mathbf{Z}_1)$$
$$+ f_3(a^2\mathbf{Z}_3 + a\mathbf{Z}_4 + \mathbf{Z}_1,\ \mathbf{Z}_2) + f_4(a^2\mathbf{Z}_4 + a\mathbf{Z}_1 + \mathbf{Z}_2,\ \mathbf{Z}_3) = \mathbf{O}$$

is given by

$$f_1(\mathbf{U}, \mathbf{V}) = F_1(\mathbf{U} + a^3\mathbf{V}) - F_2(a\mathbf{U} + \mathbf{V}) + A_1,$$
$$f_2(\mathbf{U}, \mathbf{V}) = F_2(\mathbf{U} + a^3\mathbf{V}) - F_3(a\mathbf{U} + \mathbf{V}) + A_2,$$
$$f_3(\mathbf{U}, \mathbf{V}) = F_3(\mathbf{U} + a^3\mathbf{V}) - F_4(a\mathbf{U} + \mathbf{V}) + A_3,$$
$$f_4(\mathbf{U}, \mathbf{V}) = F_4(\mathbf{U} + a^3\mathbf{V}) - F_1(a\mathbf{U} + \mathbf{V}) - A_1 - A_2 - A_3,$$

where $F_i : \mathcal{V} \to \mathcal{V}'$ $(i = 1, 2, 3, 4)$ are arbitrary complex vector functions, and A_i $(i = 1, 2, 3)$ are arbitrary constant complex vectors.

Theorem 5. *If $a^{m+n} \neq 1$ and $m = n$, the most general solution of the functional equation (1) is*

$$f_{i+m}(\mathbf{U}, \mathbf{V}) = -f_i(\mathbf{V}, \mathbf{U}) + A_i \quad (1 \leq i \leq m), \tag{29}$$

where $f_i : \mathcal{V}^2 \to \mathcal{V}'$ $(1 \leq i \leq m)$ and A_i $(1 \leq i \leq m)$ are arbitrary complex constant vectors such that $\sum_{i=1}^{m} A_i = \mathbf{O}$.

Proof. By the transformations which were exhibited in the proof of the previous theorem we may bring the equation (1) to the form (24).

For $\mathbf{V}_j = \mathbf{O}$ $(j = 1, 2, \dots, i-1, i+1, \dots, i+m-1, i+m+1, \dots, 2m)$ the equation (24) becomes

$$g_i(\mathbf{V}_i, \mathbf{V}_{i+m}) + g_{i+m}(\mathbf{V}_{i+m}, \mathbf{V}_i) = A_i \quad (1 \leq i \leq m), \tag{30}$$

where A_i $(1 \leq i \leq m)$ are arbitrary complex constant vectors. By substituting (30) into (1), we obtain that it must hold

$$\sum_{i=1}^{m} A_i = \mathbf{O}.$$

On the basis of this equality and (30), we obtain (29). □

Example 5. If $a^4 \neq 1$, the most general solution of the functional equation

$$f_1(a\mathbf{Z}_1 + \mathbf{Z}_2,\ a\mathbf{Z}_3 + \mathbf{Z}_4) + f_2(a\mathbf{Z}_2 + \mathbf{Z}_3,\ a\mathbf{Z}_4 + \mathbf{Z}_1)$$
$$+ f_3(a\mathbf{Z}_3 + \mathbf{Z}_4,\ a\mathbf{Z}_1 + \mathbf{Z}_2) + f_4(a\mathbf{Z}_4 + \mathbf{Z}_1,\ a\mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{O}$$

is given by

$$f_i(\mathbf{U}, \mathbf{V}) \quad (i = 1, 2) \quad \textit{are} \quad \textit{arbitrary},$$
$$f_3(\mathbf{U}, \mathbf{V}) = -f_1(\mathbf{U}, \mathbf{V}) + A,$$
$$f_4(\mathbf{U}, \mathbf{V}) = -f_1(\mathbf{U}, \mathbf{V}) - A,$$

where A is an arbitrary complex constant vector.

If $a^{m+n} = 1$, then the functional equation (1) may be transformed in the following way.

We introduce new vectors by the equality

$$\mathbf{V}_i = a^{1-i}\mathbf{Z}_i, \quad \text{i.e.,} \quad \mathbf{Z}_i = a^{i-1}\mathbf{V}_i \quad (1 \leq i \leq m+n).$$

Then the equation (1) becomes

$$\sum_{i=1}^{m+n} f_i\Big(a^{m-2+i}\sum_{j=0}^{m-1}\mathbf{V}_{i+j},\ a^{m+n-2+i}\sum_{j=0}^{n-1}\mathbf{V}_{m+i+j}\Big) = \mathbf{O}. \tag{31}$$

Now, if we put

$$g_i(\mathbf{U}, \mathbf{V}) = f_i(a^{m-2+i}\mathbf{U},\ a^{m+n-2+i}\mathbf{V}) \quad (1 \leq i \leq m+n),$$

i.e.,

$$f_i(\mathbf{U}, \mathbf{V}) = g_i(a^{n+2-i}\mathbf{U},\ a^{m+n+2-i}\mathbf{V}) \quad (1 \leq i \leq m+n), \tag{32}$$

the functional equation (31) takes the form

$$\sum_{i=1}^{m+n} g_i\Big(\sum_{j=0}^{m-1}\mathbf{V}_{i+j},\ \sum_{j=0}^{n-1}\mathbf{V}_{m+i+j}\Big) = \mathbf{O}. \tag{33}$$

The equation (33) is just the equation (1) for $a = 1$.

Theorem 6. *If $a^{m+n} = 1$, $(m, n) = 1$ and $m + n > 2$, then the general continuous solution of the functional equation (1) is given by*

$$f_i(\mathbf{U}, \mathbf{V}) = F_1(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Re}\,(a^{n+2-i}\mathbf{U}) \tag{34}$$

$$+F_2(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Im}\,(a^{n+2-i}\mathbf{U}) + G_i(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})$$

$(1 \le i \le m+n)$, so that

$$\sum_{i=1}^{m+n} G_i(\mathbf{U}) = -m[F_1(\mathbf{U})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U})\mathrm{Im}\,\mathbf{U}], \tag{35}$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ and $G_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n-1)$ are arbitrary continuous complex vector functions.

Proof. The proof immediately follows from (33), (32) and Theorem 1. □

Theorem 7. *If $a^{m+n} = 1$, $(m, n) = d > 1$, $m/d = p$, $n/d = q$ and $p + q > 2$, then the general continuous solution of the functional equation (1) is*

$$f_{id+j}(\mathbf{U}, \mathbf{V}) = F_{1j}(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Re}\,(a^{n+2-i}\mathbf{U}) \tag{36}$$

$$+F_{2j}(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Im}\,(a^{n+2-i}\mathbf{U}) + G_{ij}(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})$$

$$(0 \le i \le p+q-1; \quad 1 \le j \le d)$$

so that

$$\sum_{i=0}^{p+q-1} G_{ij}(\mathbf{U}) = H_j(\mathbf{U}) - p[F_{1j}(\mathbf{U})\mathrm{Re}\,\mathbf{U} + F_{2j}(\mathbf{U})\mathrm{Im}\,\mathbf{U}] \quad (1 \le j \le d), \tag{37}$$

$$\sum_{j=1}^{d} H_j(\mathbf{U}) = \mathbf{O}, \tag{38}$$

where

$$F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1, 2; \quad 1 \le j \le d),$$

$$G_{ij} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2; \quad 1 \le j \le d),$$

$$H_j : \mathcal{V} \to \mathcal{V}' \quad (1 \le j \le d-1)$$

are arbitrary continuous complex vector functions.

Proof. On the basis of the expressions (33), (32) and Theorem 2 we derive the proof of the theorem. □

Theorem 8. *If $a^{m+n} = 1$ and $m = n$, then the most general solution of the functional equation (1) is given by*

$$\begin{aligned} f_i(\mathbf{U}, \mathbf{V}) & \qquad (1 \le i \le m) \quad \textit{are} \quad \textit{arbitrary}, \\ f_{m+i}(\mathbf{U}, \mathbf{V}) &= H_i(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V}) \\ &- f_i(a^{n+2-i}\mathbf{U},\ a^{m+n+2-i}\mathbf{V}) \quad (1 \le i \le m), \end{aligned} \tag{39}$$

where $H_i : \mathcal{V} \to \mathcal{V}'$ are arbitrary complex vector functions such that $\sum_{i=1}^{m} H_i(\mathbf{U}) = \mathbf{O}$.

Proof. The proof immediately follows from (33), (32) and Theorem 3. □

2. A Special Functional Equation

Now, we will solve the following functional equation

$$\sum_{i=1}^{m+n} f\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j},\ \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{i+m+j}\Big) = \mathbf{O}, \tag{40}$$

which is obtained as a special case of the equation (1) for $f_i = f$ $(1 \leq i \leq m+n)$.

Theorem 9. *If $a^{m+n} \neq 1$, then the most general solution of the complex vector functional equation (40) is given by*

$$f(\mathbf{U},\mathbf{V}) = \begin{cases} F(\mathbf{U}+a^m\mathbf{V}) - F(a^n\mathbf{U}+\mathbf{V}) & (m \neq n), \\ G(\mathbf{U}+a^m\mathbf{V},\ a^m\mathbf{U}+\mathbf{V}) - G(a^m\mathbf{U}+\mathbf{V},\ \mathbf{U}+a^m\mathbf{V}) & (m = n), \end{cases} \tag{41}$$

where $F : \mathcal{V} \to \mathcal{V}'$, $G : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions.

Proof. We set

$$f(\mathbf{U},\mathbf{V}) = g(\mathbf{U}+a^m\mathbf{V},\ a^n\mathbf{U}+\mathbf{V}) \tag{42}$$

into (40) and deduce that

$$\sum_{i=1}^{m+n} g\Big(\sum_{j=0}^{m+1} a^{m-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{m+n-1}\mathbf{Z}_{i+m+j},$$

$$\sum_{j=0}^{m-1} a^{m+n-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{i+m+j}\Big) = \mathbf{O},$$

i.e.,

$$\sum_{i=1}^{m+n} g\Big(\sum_{j=0}^{m+n-1} a^{j}\mathbf{Z}_{i+m-1-j},\ \sum_{j=0}^{m+n-1} a^{j}\mathbf{Z}_{i-1-j}\Big) = \mathbf{O}. \tag{43}$$

This transformation of the equation (40) is possible since $a^{m+n} \neq 1$.

Now we introduce new vectors

$$\mathbf{V}_i = \sum_{j=0}^{m+n+1} a^j\mathbf{Z}_{i-1-j} \quad (1 \leq i \leq m+n). \tag{44}$$

The linear forms (44) are independent since their determinant is $(a^{m+n}-1)^{m+n-1}$.

Making use of these notations, the equation (43) becomes

$$\sum_{i=1}^{m+n} g(\mathbf{V}_i, \mathbf{V}_{i+n}) = \mathbf{O}. \tag{45}$$

If $m \neq n$, we set $\mathbf{V}_1 = \mathbf{V}_2 = \cdots = \mathbf{V}_{m-1} = \mathbf{V}_{m+1} = \mathbf{V}_{m+2} = \cdots = \mathbf{V}_{m+n-1} = \mathbf{O}$ and we get

$$g(\mathbf{U},\mathbf{V}) = F(\mathbf{U}) + F_1(\mathbf{V}). \tag{46}$$

We substitute g from (46) into (45) and obtain

$$\sum_{i=1}^{m+n} [F(\mathbf{V}_i) + F_1(\mathbf{V}_i)] = \mathbf{O},$$

which implies that $F_1(\mathbf{V}_i) = -F(\mathbf{V}_i)$. Hence,

$$g(\mathbf{U},\mathbf{V}) = F(\mathbf{U}) - F(\mathbf{V}). \tag{47}$$

If $m = n$, the equation (43) yields

$$g(\mathbf{U},\mathbf{V}) + g(\mathbf{V},\mathbf{U}) = \mathbf{O},$$

i.e.,

$$g(\mathbf{U},\mathbf{V}) = G(\mathbf{U},\mathbf{V}) - G(\mathbf{V},\mathbf{U}). \tag{48}$$

From (42), (47) and (48) we conclude that (41) holds. It is easy to verify that (41) satisfies (40). □

Example 6. If $a^3 \neq 1$, then the most general solution of the functional equation

$$f(a\mathbf{Z}_1 + \mathbf{Z}_2,\ \mathbf{Z}_3) + f(a\mathbf{Z}_2 + \mathbf{Z}_3,\ \mathbf{Z}_1) + f(a\mathbf{Z}_3 + \mathbf{Z}_1,\ \mathbf{Z}_2) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = F(\mathbf{U} + a^2\mathbf{V}) - F(a\mathbf{U} + \mathbf{V}),$$

where $F : \mathcal{V} \to \mathcal{V}'$ is an arbitrary complex vector function.

Example 7. If $a^4 \neq 1$, the most general solution of the functional equation

$$f(a\mathbf{Z}_1 + \mathbf{Z}_2,\ a\mathbf{Z}_3 + \mathbf{Z}_4) + f(a\mathbf{Z}_2 + \mathbf{Z}_3,\ a\mathbf{Z}_4 + \mathbf{Z}_1)$$
$$+ f(a\mathbf{Z}_3 + \mathbf{Z}_4,\ a\mathbf{Z}_1 + \mathbf{Z}_2) + f(a\mathbf{Z}_4 + \mathbf{Z}_1,\ a\mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = G(\mathbf{U} + a^2\mathbf{V},\ a^2\mathbf{U} + \mathbf{V}) - G(a^2\mathbf{U} + \mathbf{V},\ \mathbf{U} + a^2\mathbf{V}),$$

where $G : \mathcal{V}^2 \to \mathcal{V}'$ is an arbitrary complex vector function.

Theorem 10. *If $a^{m+n} = 1$, $(m,n) = 1$ and $m + n > 2$, then the general continuous solution of the functional equation (40) is given by*

$$f(\mathbf{U},\mathbf{V}) = \sum_{i=1}^{m+n} [F_1(a^i\mathbf{U} + a^{i+m}\mathbf{V})\mathrm{Re}\,(a^i\mathbf{U}) + F_2(a^i\mathbf{U} + a^{i+m}\mathbf{V})\mathrm{Im}\,(a^i\mathbf{U})] \tag{49}$$
$$+ \sum_{i=1}^{m+n-1} [G_i(a^i\mathbf{U} + a^{i+m}\mathbf{V}) - G_i(a^i\mathbf{U} + a^m\mathbf{V})]$$
$$-m[F_1(\mathbf{U} + a^m\mathbf{V})\mathrm{Re}\,(\mathbf{U} + a^m\mathbf{V}) + F_2(\mathbf{U} + a^m\mathbf{V})\mathrm{Im}\,(\mathbf{U} + a^m\mathbf{V})],$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1,2)$ and $G_i : \mathcal{V} \to \mathcal{V}'$ $(1 \leq i \leq m+n-1)$ are arbitrary complex vector functions.

Proof. Let us put $\mathbf{Z}_i = a^{i-1}\mathbf{T}_i$ $(1 \leq i \leq m+n)$. The equation (40) becomes

$$\sum_{i=1}^{m+n} f\Big(a^{m+i-2}\sum_{j=0}^{m-1}\mathbf{T}_{i+j},\ a^{m+n-2+i}\sum_{j=0}^{n-1}\mathbf{T}_{m+i-j}\Big) = \mathbf{O}. \tag{50}$$

Now we make the substitutions

$$f(a^{m+i-2}\mathbf{U},\ a^{m+n-2+i}\mathbf{V}) = f_i(\mathbf{U},\mathbf{V}) \quad (1 \leq i \leq m+n),$$

i.e.,

$$f(\mathbf{U},\mathbf{V}) = f_i(a^{n-i+2}\mathbf{U},\ a^{m+n+2-i}\mathbf{V}) \quad (1 \leq i \leq m+n), \tag{51}$$

and we obtain

$$\sum_{i=1}^{m+n} f_i\Big(\sum_{j=0}^{m-1}\mathbf{T}_{i+j},\ \sum_{j=0}^{n-1}\mathbf{T}_{m+i+j}\Big) = \mathbf{O}. \tag{52}$$

The equation (52) is just the equation (1) for $a = 1$, and its solution is determined by Theorem 1.

By an application of Theorem 1, and by (51) we get

$$f_i(\mathbf{U},\mathbf{V}) = P_1(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Re}\,(a^{n+2-i}\mathbf{U}) \quad (53)$$

$$+P_2(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Im}\,(a^{n+2-i}\mathbf{U}) + Q_i(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})$$

$(1 \le i \le m+n)$, so that

$$\sum_{i=1}^{m+n} Q_i(\mathbf{U}) = -m[P_1(\mathbf{U})\mathrm{Re}\,\mathbf{U} + P_2(\mathbf{U})\mathrm{Im}\,\mathbf{U}],$$

where $P_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ and $Q_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n)$ are continuous complex vector functions. By addition of all equations (53) and putting

$$P_1(\mathbf{U}) = (m+n)F_1(\mathbf{U}), \quad P_2(\mathbf{U}) = (m+n)F_2(\mathbf{U}),$$

$$Q_i(\mathbf{U}) = (m+n)G_{n+2-i}(\mathbf{U}) \quad (i = 1, 2, \ldots, m+n)$$

we obtain (49). □

Example 8. If $a^3 = 1$, the general continuous solution of the functional equation

$$f(a\mathbf{Z}_1 + \mathbf{Z}_2,\ \mathbf{Z}_3) + f(a\mathbf{Z}_2 + \mathbf{Z}_3,\ \mathbf{Z}_1) + f(a\mathbf{Z}_3 + \mathbf{Z}_1,\ \mathbf{Z}_2) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = F_1(a\mathbf{U}+\mathbf{V})\mathrm{Re}\,(a\mathbf{U}) + F_2(a\mathbf{U}+\mathbf{V})\mathrm{Im}\,(a\mathbf{U})$$
$$+F_1(a^2\mathbf{U}+\mathbf{V})\mathrm{Re}\,(a^2\mathbf{U}) + F_2(a^2\mathbf{U}+\mathbf{V})\mathrm{Im}\,(a^2\mathbf{U})$$
$$-F_1(\mathbf{U}+a^2\mathbf{V})\mathrm{Re}\,(\mathbf{U}+2a^2\mathbf{V}) - F_2(\mathbf{U}+a^2\mathbf{V})\mathrm{Im}\,(\mathbf{U}+2a^2\mathbf{V})$$
$$+G_1(a\mathbf{U}+\mathbf{V}) - G_1(\mathbf{U}+a^2\mathbf{V}) + G_2(a^2\mathbf{U}+a\mathbf{V}) - G_2(\mathbf{U}+a^2\mathbf{V}),$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ and $G_i : \mathcal{V} \to \mathcal{V}'$ $(i = 1, 2)$ are arbitrary complex vector functions.

Theorem 11. *If $a^{m+n} = 1,\ (m, n) = d > 1,\ m/d = p,\ n/d = q$ and $p + q > 2$, then the general continuous solution of the functional equation (40) is given by*

$$f(\mathbf{U},\mathbf{V}) = \sum_{j=-1}^{d-2}\sum_{i=0}^{p+q-1} [F_{1,j+2}(a^{n-id-j}\mathbf{U} + a^{-id-j}\mathbf{V})\mathrm{Re}\,(a^{n-id-j}\mathbf{U}) \quad (54)$$

$$+F_{2,j+2}(a^{n-id-j}\mathbf{U} + a^{-id-j}\mathbf{V})\mathrm{Im}\,(a^{n-id-j}\mathbf{U}) + G_{i,j+2}(a^{n-id-j}\mathbf{U} + a^{-id-j}\mathbf{V})],$$

so that

$$\sum_{i=0}^{p+q-1} G_{ij}(\mathbf{U}) = H_j(\mathbf{U}) - p[F_{1j}(\mathbf{U})\mathrm{Re}\,(\mathbf{U}) + F_{2j}(\mathbf{U})\mathrm{Im}\,(\mathbf{U})] \quad (1 \le j \le d)$$

and

$$\sum_{j=1}^{d} H_j(\mathbf{U}) = \mathbf{O},$$

where

$$F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1, 2; \quad 1 \le j \le d),$$
$$G_{ij} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2; \quad 1 \le j \le d),$$
$$H_j : \mathcal{V} \to \mathcal{V}' \quad (1 \le j \le d-1)$$

are arbitrary continuous complex vector functions.

Proof. We can start from equation (50). From (49) and (50) on the basis of Theorem 2 we get

$$\begin{aligned}f(\mathbf{U},\mathbf{V}) = P_{1j}(a^{n-id-j+2}\mathbf{U} + a^{m+n+2-id-j}\mathbf{V})\mathrm{Re}\,(a^{n-id-j+2}\mathbf{U}) \\ +P_{2j}(a^{n-id-j+2}\mathbf{U} + a^{m+n+2-id-j}\mathbf{V})\mathrm{Im}\,(a^{n-id-j+2}\mathbf{U}) \\ +Q_{ij}(a^{n-id-j+2}\mathbf{U} + a^{n+m+2-id-j}\mathbf{V}) \quad (0 \le i \le p+q-1;\ 1 \le j \le d),\end{aligned} \tag{55}$$

$$\sum_{i=0}^{p+q-1} Q_{ij}(\mathbf{U}) = K_j(\mathbf{U}) - p[P_{1j}(\mathbf{U})\mathrm{Re}\,(\mathbf{U}) + P_{2j}(\mathbf{U})\mathrm{Im}\,(\mathbf{U})] \quad (1 \le j \le d), \tag{56}$$

$$\sum_{j=1}^{d} K_j(\mathbf{U}) = \mathbf{O}, \tag{57}$$

where

$$\begin{gathered}P_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1, 2; \quad 1 \le j \le d), \\ Q_{ij} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2; \quad 1 \le j \le d), \\ K_j : \mathcal{V} \to \mathcal{V}' \quad (1 \le j \le d-1)\end{gathered}$$

are continuous functions.

We take into account (56) and (57) and we add together all equations (55). In this way we obtain (55) with

$$\begin{gathered}P_{1j}(\mathbf{U}) = (m+n)F_{1j}(\mathbf{U}), \quad P_{2j}(\mathbf{U}) = (m+n)F_{2j}(\mathbf{U}), \\ Q_{ij}(\mathbf{U}) = (m+n)G_{ij}(\mathbf{U}), \quad K_j(\mathbf{U}) = (m+n)H_j(\mathbf{U}) \\ (0 \le i \le p+q-2; \quad 1 \le j \le d). \ \square\end{gathered}$$

Example 9. If $a^6 = 1$, then the general continuous solution of the functional equation

$$\begin{gathered}f(a^3\mathbf{Z}_1 + a^2\mathbf{Z}_2 + a\mathbf{Z}_3 + \mathbf{Z}_4,\ a\mathbf{Z}_5 + \mathbf{Z}_6) + f(a^3\mathbf{Z}_2 + a^2\mathbf{Z}_3 + a\mathbf{Z}_4 + \mathbf{Z}_5,\ a\mathbf{Z}_6 + \mathbf{Z}_1) \\ +f(a^3\mathbf{Z}_3 + a^2\mathbf{Z}_4 + a\mathbf{Z}_5 + \mathbf{Z}_6,\ a\mathbf{Z}_1 + \mathbf{Z}_2) + f(a^3\mathbf{Z}_4 + a^2\mathbf{Z}_5 + a\mathbf{Z}_6 + \mathbf{Z}_1,\ a\mathbf{Z}_2 + \mathbf{Z}_3) \\ +f(a^3\mathbf{Z}_5 + a^2\mathbf{Z}_6 + a\mathbf{Z}_1 + \mathbf{Z}_2,\ a\mathbf{Z}_3 + \mathbf{Z}_4) + f(a^3\mathbf{Z}_6 + a^2\mathbf{Z}_1 + a\mathbf{Z}_2 + \mathbf{Z}_3,\ a\mathbf{Z}_4 + \mathbf{Z}_5) = \mathbf{O}\end{gathered}$$

is given by

$$\begin{gathered}f(\mathbf{U},\mathbf{V}) = F_{11}(a\mathbf{U} + a^5\mathbf{V})\mathrm{Re}\,(a\mathbf{U}) + F_{21}(a\mathbf{U} + a^5\mathbf{V})\mathrm{Im}\,(a\mathbf{U}) \\ +F_{11}(a^3\mathbf{U} + a\mathbf{V})\mathrm{Re}\,(a^3\mathbf{U}) + F_{21}(a^3\mathbf{U} + a\mathbf{V})\mathrm{Im}\,(a^3\mathbf{U}) \\ -F_{11}(a^5\mathbf{U} + a^3\mathbf{V})\mathrm{Re}\,(a^5\mathbf{U} + 2a^3\mathbf{V}) - F_{21}(a^5\mathbf{U} + a^3\mathbf{V})\mathrm{Im}\,(a^5\mathbf{U} + 2a^3\mathbf{V}) \\ +F_{12}(\mathbf{U} + a^4\mathbf{V})\mathrm{Re}\,(\mathbf{U}) + F_{22}(\mathbf{U} + a^4\mathbf{V})\mathrm{Im}\,(\mathbf{U}) \\ +F_{12}(a^2\mathbf{U} + \mathbf{V})\mathrm{Re}\,(a^2\mathbf{U}) + F_{22}(a^2\mathbf{U} + \mathbf{V})\mathrm{Im}\,(a^2\mathbf{U}) \\ -F_{12}(a^4\mathbf{U} + a^2\mathbf{V})\mathrm{Re}\,(a^4\mathbf{U} + 2a^2\mathbf{V}) - F_{22}(a^4\mathbf{U} + a^2\mathbf{V})\mathrm{Im}\,(a^4\mathbf{U} + 2a^2\mathbf{V}) \\ +G_{01}(a\mathbf{U} + a^5\mathbf{V}) - G_{01}(a^5\mathbf{U} + a^3\mathbf{V}) + G_{02}(a\mathbf{U} + a^4\mathbf{V}) - G_{02}(a^4\mathbf{U} + a^2\mathbf{V}) \\ +G_{11}(a^3\mathbf{U} + a\mathbf{V}) - G_{11}(a^5\mathbf{U} + a^3\mathbf{V}) + G_{12}(a^2\mathbf{U} + \mathbf{V}) - G_{12}(a^4\mathbf{U} + a^2\mathbf{V}) \\ +H_1(a^5\mathbf{U} + a^3\mathbf{V}) - H_1(a^4\mathbf{U} + a^2\mathbf{V}),\end{gathered}$$

where $F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i, j = 1, 2)$; $G_{ij} : \mathcal{V} \to \mathcal{V}'$ $(i = 0, 1;\ j = 1, 2)$ and $H_1 : \mathcal{V} \to \mathcal{V}'$ are arbitrary continuous complex vector functions.

Theorem 12. *If $a^{m+n} = 1$ and $m = n$, the most general solution of the functional equation (40) is given by*

$$f(\mathbf{U},\mathbf{V}) = \sum_{i=1}^{m}[F_1(a^i\mathbf{U},\ a^{n+i}\mathbf{V}) - F_i(a^i\mathbf{U},\ a^{n+i}\mathbf{V}) + H_i(a^{n+i}\mathbf{V} + a^i\mathbf{U})],$$
$$\sum_{i=1}^{m} H_i(\mathbf{U}) = \mathbf{O}, \tag{58}$$

where $F_i : \mathcal{V}^2 \to \mathcal{V}'$ $(1 \le i \le m)$ and $H_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m-1)$ are arbitrary complex vector functions.

Proof. We start again from the equation (50). According to Theorem 3 and (49) we have

$$f(\mathbf{U},\mathbf{V}) = P_i(a^{m-i+2}\mathbf{U},\ a^{m+n+2-i}\mathbf{V}) \quad (1 \le i \le m),$$

$$f(\mathbf{U},\mathbf{V}) = Q_i(a^{m-i+2}\mathbf{U} + a^{m+n+2-i}\mathbf{V}) - P_i(a^{m+n+2-i}\mathbf{V},\ a^{m+2-i}\mathbf{U}) \quad (1 \le i \le m),$$

$$\sum_{i=1}^{m} Q_i(\mathbf{U}) = \mathbf{O}. \tag{59}$$

By addition we get (58) with

$$P_i(\mathbf{U},\mathbf{V}) = 2mF_{m-i+2}(\mathbf{U},\mathbf{V}), \quad Q_i(\mathbf{U}) = 2mH_{m-i+2}(\mathbf{U}). \ \square$$

Example 10. If $a^4 = 1$, the most general solution of the functional equation

$$f(a\mathbf{Z}_1 + \mathbf{Z}_2,\ a\mathbf{Z}_3 + \mathbf{Z}_4) + f(a\mathbf{Z}_2 + \mathbf{Z}_3,\ a\mathbf{Z}_4 + \mathbf{Z}_1)$$

$$+f(a\mathbf{Z}_3 + \mathbf{Z}_4,\ a\mathbf{Z}_1 + \mathbf{Z}_2) + f(a\mathbf{Z}_4 + \mathbf{Z}_1,\ a\mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = F_1(a\mathbf{U},\ a^3\mathbf{V}) - F_1(a\mathbf{V},\ a^3\mathbf{U}) + F_2(a^2\mathbf{U},\ \mathbf{V})$$

$$-F_2(a^2\mathbf{V},\ \mathbf{U}) + H_1(a^3\mathbf{U} + a\mathbf{V}) - H_1(\mathbf{U} + a^2\mathbf{V}),$$

where $F_i : \mathcal{V}^2 \to \mathcal{V}'$ $(i = 1, 2)$ and $H_1 : \mathcal{V} \to \mathcal{V}'$ are arbitrary complex vector functions.

Now, as special cases we obtain the results given in [3,4,5].

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ALGORITHMS FOR APPROXIMATION WITH LOCALLY SUPPORTED RATIONAL SPLINE PREWAVELETS ON THE SPHERE

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Abstract. In [5], some locally supported rational spline prewavelets on the sphere were constructed. We present here another two properties of them and some algorithms for decomposition, reconstruction and approximation, together with some numerical tests. A comparison with the spherical harmonics approach shows the advantage of the small support of our prewavelets.

1. **Introduction**

Consider the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$, $\mathbb{S}^2 = \left\{ \mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1 \right\}$. In [5] the following construction were made.

We considered the polyhedron Π having the bound Ω, the vertices situated on $\mathbb{S}^2$ and triangular faces such that no face contains the origin O and O is situated inside the polyhedron. The set of its faces was denoted $\mathcal{T}^0 = \left\{ T_1^0, T_2^0, \ldots, T_n^0 \right\}$. Then we projected each triangle of $\mathcal{T}^0$ onto $\mathbb{S}^2$, getting a triangulation of the sphere, denoted $\mathcal{U}^0 = \left\{ U_1^0, U_2^0, \ldots, U_n^0 \right\}$, where $U_i^0 = p\left(T_i^0\right)$ and $p : \Omega \longrightarrow \mathbb{S}^2$, is the radial projection

$$p(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right), \text{ for all } (x,y,z) \in \Omega.$$

We divided each triangle T_k^0 into four triangles, taking the mid-points of the edges. Thus, we obtained a refined triangulation of Ω, denoted $\mathcal{T}^1 = \left\{ T_1^1, T_2^1, \ldots, T_{4n}^1 \right\}$. Continuing the refinement process we built the triangulations $\mathcal{T}^j$ for arbitrary level $j \in \mathbb{N}$. The projection $\mathcal{U}^j = p\left(\mathcal{T}^j\right)$ is a triangulation of $\mathbb{S}^2$. We denoted by V^j the set of all vertices of plane triangles in $\mathcal{T}^j$.

Let $M_1 M_i M_k$ be a triangle of $\mathcal{T}^j$, with the vertices of coordinates (x_1, y_1, z_1), (x_i, y_i, z_i), (x_k, y_k, z_k) respectively. Let $M_1' M_i' M_k'$ be its radial projection onto $\mathbb{S}^2$. Then we defined the functions $\varphi_{M_1}^j$, associated to the vertex M_1, as

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$$\varphi_{M_1}^j(\eta_1,\eta_2,\eta_3) =$$

$$= \begin{cases} \begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ x_i & y_i & z_i \\ x_k & y_k & z_k \end{vmatrix} \cdot \begin{vmatrix} \eta_1 & \eta_2 & \eta_3 & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_i & y_i & z_i & 1 \\ x_k & y_k & z_k & 1 \end{vmatrix}^{-1}, \text{ on each triangle } M_1'M_i'M_k' \text{ of } \mathcal{U}^j, \\ 0, \text{ on the triangles of } \mathcal{U}^j \text{ that do not contain } M_1'. \end{cases}$$

It is immediately that the function $\varphi_{M_1}^j$ is continuous on $\mathbb{S}^2$ and the set $\{\varphi_v^j,\ v \in V^j\}$ is a basis of the space $\mathcal{V}^j = \operatorname{span}\{\varphi_v^j,\ v \in V^j\}$. We denoted by V_v^j the set of the j-level neighbors of the vertex v.

Due to the refinement relation

$$\varphi_v^j = \varphi_v^{j+1} + \frac{1}{2}\sum_{w \in V_v^{j+1}} \varphi_w^{j+1},\ v \in V^j,\ j \in \mathbb{N},$$

we deduce that $\mathcal{V}^j \subseteq \mathcal{V}^{j+1}$. Then we defined an inner product on $\mathbb{S}^2$ based on the coarsest triangulation $\mathcal{T}^0$:

$$\begin{aligned} \langle F, G\rangle_* &= \langle F\circ p, G\circ p\rangle_\Omega \\ &= \sum_{T\in\mathcal{T}^0}\int_{p(T)} F(\boldsymbol{\eta})\, G(\boldsymbol{\eta})\, \frac{2d_T^2}{|a_T\eta_1 + b_T\eta_2 + c_T\eta_3|^3} d\omega(\boldsymbol{\eta}), \end{aligned}$$

where $\boldsymbol{\eta} = (\eta_1,\eta_2,\eta_3)$, the numbers a_T, b_T, c_T, d_T are the coefficients of x, y, z and 1 of the polynomial function

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

and the triangle T has the vertices $M_i(x_i, y_i, z_i)$, $i = 1, 2, 3$. The inner product $\langle\cdot,\cdot\rangle_*$ may be interpreted as a "multi-weighted" inner product, with the weights

$$w_T(\eta_1,\eta_2,\eta_3) = \frac{2d_T^2}{|a_T\eta_1 + b_T\eta_2 + c_T\eta_3|^3}. \tag{1}$$

Afterwards and we considered the space $\mathcal{W}^j$ as the orthogonal complement of $\mathcal{V}^j$ into $\mathcal{V}^{j+1}$:

$$\mathcal{V}^{j+1} = \mathcal{V}^j \bigoplus \mathcal{W}^j. \tag{2}$$

The spaces $\mathcal{W}^j$ were called the *wavelet spaces*. We determined a basis in each $\mathcal{W}^j$, consisting of prewavelets of small supports. This basis consists in the following functions:

$$\psi_u^j(\boldsymbol{\eta}) = \sigma_{a_1,u}^j(\boldsymbol{\eta}) + \sigma_{a_2,u}^j(\boldsymbol{\eta}), \tag{3}$$

with

$$\begin{aligned} \sigma_{a_1,u}^j(\boldsymbol{\eta}) &= s_{a_1}\varphi_{a_1}^{j+1}(\eta) + \sum_{u\in V_{a_1}^{j+1}} s_w\varphi_w^{j+1}(\boldsymbol{\eta}), \\ \sigma_{a_2,u}^j(\boldsymbol{\eta}) &= s_{a_2}\varphi_{a_2}^{j+1}(\boldsymbol{\eta}) + \sum_{u\in V_{a_2}^{j+1}} t_w\varphi_w^{j+1}(\boldsymbol{\eta}), \end{aligned}$$

where u is a "new" vertex, mid-point of the edge $[a_1a_2]$, $s_{a_1} = -\frac{3}{2s_1}$, $s_{a_2} = -\frac{3}{2s_2}$, $s_{b_i} = \frac{3}{28s_1} + \theta(i, s_1)$, $t_{c_i} = \frac{3}{28s_2} + \theta(i, s_2)$. Here s_1 and s_2 are the number of neighbors of the vertices a_1 resp. a_2, $\theta(i, s) = \frac{\lambda^i + \lambda^{s-i}}{\sqrt{21}(1-\lambda^s)}$, $\lambda = \frac{-5+\sqrt{21}}{2}$. By $b_0, b_1, \ldots, b_{s_1-1}$ we denoted the ordered neighbors of a_1, starting with $b_0 = u$ and by $c_0, c_1, \ldots, c_{s_2-1}$ we denoted the ordered neighbors of a_2, starting with $c_0 = u$.

The set $\{\psi_u^j,\ u \in V^{j+1} \backslash V^j\}$ was proved to be a stable basis of $L^2(\mathbb{S}^2)$ (see [5], Section 3).

In the next section we present the algorithms of decomposition and reconstruction.

2. **Decomposition and reconstruction**

Consider $\{\varphi_v^j\}_{v \in V^j}$ basis of $\mathcal{V}^j$ and $\{\psi_u^j\}_{u \in V^{j+1} \backslash V^j}$ basis of $\mathcal{W}^j$. With a fixed ordering of the vertices in V^j and in $V^{j+1} \backslash V^j$, we can regard these bases as row vectors:

$$\Phi^j = \left(\varphi_v^j\right)_{v \in V^j} \quad \text{and } \Psi^j = \left(\psi_u^j\right)_{u \in V^{j+1} \backslash V^j}.$$

Then any elements $f^j = \sum_{v \in V^j} f_v^j \varphi_v^j$ and $g^j = \sum_{u \in V^{j+1} \backslash V^j} g_u^j \psi_u^j$ in $\mathcal{V}^j$ resp. $\mathcal{W}^j$ can be written as

$$f^j = \Phi^j \mathbf{f}^j \quad \text{resp.} \quad g^j = \Psi^j \mathbf{g}^j, \tag{4}$$

where $\mathbf{f}^j$ is the column vector $\left(f_v^j\right)_{v \in V^j}$ and $\mathbf{g}^j$ is the column vector $\left(g_u^j\right)_{u \in V^{j+1} \backslash V^j}$.

Since $\mathcal{V}^{j-1}$and $\mathcal{W}^{j-1}$are subspaces of $\mathcal{V}^j$, there exist two unique matrices P^j and Q^j such that

$$\Phi^{j-1} = \Phi^j P^j \quad \text{and} \quad \Psi^{j-1} = \Phi^j Q^j. \tag{5}$$

Take now $f^j \in \mathcal{V}^j$. Equation (2) implies that there exist unique $f^{j-1} \in \mathcal{V}^{j-1}$ and $g^{j-1} \in \mathcal{W}^{j-1}$ such that

$$f^j = f^{j-1} + g^{j-1}. \tag{6}$$

Substituting (4) into (6) yields the following equation:

$$\Phi^j \mathbf{f}^j = \Phi^{j-1} \mathbf{f}^{j-1} + \Psi^{j-1} \mathbf{g}^{j-1}$$

and then, using (5) and the fact that Φ^j is a basis for $\mathcal{V}^j$, we find

$$\left(P^j\ Q^j\right) \begin{pmatrix} \mathbf{f}^{j-1} \\ \mathbf{g}^{j-1} \end{pmatrix} = \mathbf{f}^j. \tag{7}$$

The block matrix $\left(P^j\ Q^j\right)$ is called the *two-scaled matrix.* It is nonsingular and it must be inverted in order to compute the coefficient vectors $\mathbf{f}^{j-1}$ and $\mathbf{g}^{j-1}$ from a given coefficient vector $\mathbf{f}^j$. Repeating the above calculations for the levels $j = m, m-1, \ldots, 1$, we obtain the decomposition algorithm.

Algorithm D

Input : $m \in \mathbb{N}$ highest level
$\mathbf{f}^m = (f_v^m)_{v \in V^m}$ the values of a given function $f^m \in \mathcal{V}^m$
at the nodes $v \in V^m$.

(i) For each level $j = m, m-1, \ldots, 1,$ solve the linear system (7)
and get $\mathbf{f}^{j-1}$ and $\mathbf{g}^{j-1}$.

Output : $\mathbf{g}^j$ $(j = 0, 1, \ldots, m-1)$ wavelet coefficients,
$\mathbf{f}^0$ coefficient of approximation.

Thus, the function $f^m \in \mathcal{V}^m$ was decomposed into

$$f^m = f^0 + g^0 + g^1 + \ldots + g^{m-1},$$

meaning an approximation $f^0 \in \mathcal{V}^0$ and a sum of details (wavelets) $g^i \in \mathcal{W}^i$, $i = 0, 1, \ldots, m-1$.

Let us come back to the system (7) .The entries of P^j and Q^j are evaluations of the bases Φ^j resp. Ψ^j. Their expressions are

$$p_{wv}^j = \varphi_v^{j-1}(w) = \begin{cases} 1 & \text{if } w = v, \\ \frac{1}{2} & \text{if } w \in V_v^j, \\ 0 & \text{otherwise} \end{cases}, \text{ resp. } q_{wu}^j = \psi_u^{j-1}(w).$$

The system (7) can be written

$$\begin{pmatrix} I & Q_1^j \\ P_2^j & Q_2^j \end{pmatrix} \begin{pmatrix} \mathbf{f}^{j-1} \\ \mathbf{g}^{j-1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1^j \\ \mathbf{f}_2^j \end{pmatrix} \tag{8}$$

Using the Schur complement matrix $\tilde{Q}_2^j = Q_2^j - P_2^j Q_1^j$, we reduce the system (8) to

$$\begin{pmatrix} I & Q_1^j \\ 0 & \tilde{Q}_2^j \end{pmatrix} \begin{pmatrix} \mathbf{f}^{j-1} \\ \mathbf{g}^{j-1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1^j \\ \mathbf{f}_2^j - P_2^j \mathbf{f}_1^j \end{pmatrix}.$$

This means that we have to solve the system

$$\tilde{Q}_2^j \mathbf{g}^{j-1} = \mathbf{f}_2^j - P_2^j \mathbf{f}_1^j$$

for computing $\mathbf{g}^{j-1}$ and then calculate $\mathbf{f}^{j-1}$ from the substitution

$$\mathbf{f}^{j-1} = \mathbf{f}_1^j - Q_1^j \mathbf{g}^{j-1}.$$

Besides the lower dimension, the system (8) has also the advantage that is better conditioned than the system (7).

The next step is to write the reconstruction algorithm.

Algorithm R

Input : $m \in \mathbb{N}$ highest level

$\mathbf{g}^j, j = 0, 1, \ldots, m-1$ coefficient vectors of a given function $g^j \in \mathcal{W}^j$

$\mathbf{f}^0$ coefficient vector of a given function $f^0 \in \mathcal{V}^0$

For each level $j = 1, 2, \ldots, m$

(**i**) (**a**) compute $\mathbf{f}_1^j$ from $\mathbf{f}_1^j = \mathbf{f}^{j-1} + Q_1^j \mathbf{g}^{j-1}$,

(**b**) compute $\mathbf{f}_2^j$ from $\mathbf{f}_2^j = P_2^j \mathbf{f}_1^j + \widetilde{Q}_2^j \mathbf{g}^{j-1}$

Output : $\mathbf{f}^m$

The locality of the supports of our bases has the advantage that the matrices P_2^j and $\widetilde{Q}_2^j$ are sparse. In P_2^j, on each column we have two nonzero entries and in $\widetilde{Q}_2^j$, on each column and row we have $n = \max\{11, \{2t(v) - 1, v \in V^0\}\}$ nonzero entries. Here $t(v)$ denotes the number of neighbors of the vertex v.

3. Thresholding

A typically application of wavelets is data compression using thresholding. Numerical examples will be given in Section 5.

A given function $f^m \in \mathcal{V}^m$ is first decomposed into its components $f^0, g^0, g^1, \ldots, g^{m-1}$, using the algorithm **D,** with Schur complement. The wavelet components $g^j \in \mathcal{W}^j$ are replaced by the functions $\widehat{g}^j \in \mathcal{W}^j$, by modifying their coefficients according to a particular strategy (for more details see [7]). Here we use the strategy called *hard thresholding,* which means that for a threshold $thr > 0$, we set, for $u \in V^{j+1} \backslash V^j$,

$$\widehat{g}_u^j = \begin{cases} g_u^j, & \text{if } \left|g_u^j\right| \geq thr, \\ 0, & \text{otherwise.} \end{cases}$$

The ratio of number of subsequent nonzero coefficients to the total number

$$\frac{\sum_{j=0}^{m-1} \left|\left\{u \in V^{j+1} \backslash V^j : \ \widehat{g}_u^j \neq 0\right\}\right|}{\sum_{j=0}^{m-1} \left|V^{j+1} \backslash V^j\right|}$$

is called the *compression rate.*

Reconstruction with the algorithm **R,** applied to the modified functions $\widehat{g}^j$, yields an approximant $\widehat{f}^m \in \mathcal{V}^m$ of the original function f^m, given by

$$\widehat{f}^m = f^0 + \widehat{g}^0 + \widehat{g}^1 + \ldots + \widehat{g}^{m-1}.$$

The resulting approximation error is

$$e^m = f^m - \widehat{f}^m = \sum_{j=0}^{m-1} \left(g^j - \widehat{g}^j\right).$$

4. Other properties of our prewavelets

We prove here two properties which were not mentioned in [5].

Proposition 1. *The function* $\mathbf{1}_{\mathbb{S}^2} : \mathbb{S}^2 \longrightarrow \mathbb{R}$, $\mathbf{1}_{\mathbb{S}^2}(\eta) = 1$ *for all* $\eta \in \mathbb{S}^2$, *belongs to the space* $\mathcal{V}^0$ *and therefore to all the spaces* $\mathcal{V}^j$. *As a consequence, the prewavelets have a vanishing moment of order zero.*

Proof. First we show that on each triangle $U = A_1A_2A_3$ of $\mathcal{U}^0$, $A_i\left(x_i, y_i, z_i\right)$, $i = 1, 2, 3$ we have

$$\varphi^0_{A_1} + \varphi^0_{A_2} + \varphi^0_{A_3} = 1, \tag{9}$$

which is equivalent to

$$\frac{\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}{\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}} + \frac{\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \end{vmatrix}}{\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 & 0 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_1 & y_1 & z_1 & 1 \end{vmatrix}} + \frac{\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}}{\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 & 0 \\ x_3 & y_3 & z_3 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix}} = 1$$

for all $(\eta_1, \eta_2, \eta_3) \in U$. This is immediately if we split the determinant from the denominator after the last column.

Now let us take an arbitrary point (η_1, η_2, η_3) of the sphere. It will be situated on a spherical triangle $\widetilde{U} \in \mathcal{U}^0$ having the vertices M_1', M_2', M_3', which are the projections of the points M_1, M_2, M_3, situated on the polyhedron. Then we can write

$$1 = \mathbf{1}_{\mathbb{S}^2}\left(\eta_1, \eta_2, \eta_3\right) = \varphi^0_{M_1}\left(\eta_1, \eta_2, \eta_3\right) + \varphi^0_{M_2}\left(\eta_1, \eta_2, \eta_3\right) + \varphi^0_{M_3}\left(\eta_1, \eta_2, \eta_3\right).$$

Since at (η_1, η_2, η_3) all other pyramidal functions φ^0_v, $v \in V^0$, take the value zero, we may write

$$\mathbf{1}_{\mathbb{S}^2} = \sum_{v \in V^0} \varphi^0_v.$$

As a consequence, we can state that for every element g^{j-1} of the wavelet space $\mathcal{W}^{j-1}$,

$$\left\langle \mathbf{1}_{\mathbb{S}^2}, g^{j-1} \right\rangle_* = 0 \text{ for all } j \in \mathbb{N}^*.$$

This means that our wavelets have a vanishing moment of order zero with respect to the scalar product $\langle \cdot, \cdot \rangle_*$:

$$0 = \sum_{T \in \mathcal{T}^0} \int_{p(T)} g^{j-1}\left(\boldsymbol{\eta}\right) w_T\left(\eta_1, \eta_2, \eta_3\right) d\omega\left(\boldsymbol{\eta}\right),$$

with w_T the weight-functions given by (1).

Since

$$\begin{aligned}
\left\langle \mathbf{1}_{\mathbb{S}^2}, g^{j-1} \right\rangle_* &= \left\langle \mathbf{1}_{\mathbb{S}^2} \circ p, g^{j-1} \circ p \right\rangle_\Omega = \left\langle \mathbf{1}_\Omega, g^{j-1} \circ p \right\rangle_\Omega \\
&= \sum_{T \in \mathcal{T}^0} \frac{1}{a\left(T\right)} \int_T \left(g^{j-1} \circ p\right)\left(\mathbf{x}\right) d\Omega\left(\mathbf{x}\right) \\
&= \frac{1}{3} \sum_{[w_1 w_2 w_3] \in \mathcal{T}^j} \left(g^{j-1} \circ p\right)\left(w_1\right) + \left(g^{j-1} \circ p\right)\left(w_2\right) + \left(g^{j-1} \circ p\right)\left(w_3\right) \\
&= \frac{1}{3} \sum_{w \in V^j} t\left(w\right) g^{j-1}\left(p\left(w\right)\right),
\end{aligned}$$

we finally obtain

$$\sum_{w \in V^j} t\left(w\right) g^{j-1}\left(p\left(w\right)\right) = 0.$$

□

Next we apply this result to obtain another identity which show the fact that a sum of prewavelets ψ_u^{j-1} is constant over coarse and fine vertices.

Proposition 2. *Let*

$$\Sigma^{j-1}(\eta) = \sum_{u \in V^j \setminus V^{j-1}} t(u)\, \psi_u^{j-1}(\eta), \;\; \eta \in \mathbb{S}^2.$$

Then we have

$$\Sigma^{j-1}(p(w)) = \begin{cases} 3 & \text{if } w \in V^j \setminus V^{j-1}, \\ -9 & \text{if } w \in V^{j-1}. \end{cases}$$

Proof. For $u \in V^j \setminus V^{j-1}$, the number of its neighbors is $t(u) = 6$. Therefore we can write the weighted sum as

$$\Sigma^{j-1}(\eta) = 6 \sum_{u \in V^j \setminus V^{j-1}} \psi_u^{j-1}(\eta).$$

First let w be a fine vertex, i.e. $w \in V^j \setminus V^{j-1}$. From the previous proposition we have

$$0 = \sum_{u \in V^j} t(u)\, \psi_w^{j-1}(p(u)) = \sum_{u \in V^j \setminus V^{j-1}} t(u)\, \psi_w^{j-1}(p(u)) + \sum_{v \in V^{j-1}} t(v)\, \psi_w^{j-1}(p(v)). \tag{10}$$

With w being the mid-point of an edge $[a_1 a_2]$, $a_1, a_2 \in V^{j-1}$, we obtain from (3) that

$$\begin{aligned} \sum_{v \in V^{j-1}} t(v)\, \psi_w^{j-1}(p(v)) &= \sum_{v \in V^{j-1}} t(v)\, \sigma_{a_1,w}^{j-1}(p(v)) + t(v)\, \sigma_{a_2,w}^{j-1}(p(v)) \\ &= t(a_1)\, \sigma_{a_1,w}^{j-1}(p(a_1)) + t(a_2)\, \sigma_{a_2,w}^{j-1}(p(a_2)) \\ &= -\frac{3}{2} - \frac{3}{2} = -3. \end{aligned} \tag{11}$$

The symmetry property $\psi_w^{j-1}(p(u)) = \psi_u^{j-1}(p(w))$ yields

$$\Sigma^{j-1}(p(w)) = \sum_{u \in V^j \setminus V^{j-1}} t(u)\, \psi_u^{j-1}(p(w)) = \sum_{u \in V^j \setminus V^{j-1}} t(u)\, \psi_w^{j-1}(p(u)) = 3,$$

taking into account (10) and (11).

Finally, let $v \in V^{j-1}$ be a coarse vertex.Then

$$\begin{aligned} \Sigma^{j-1}(p(v)) &= \sum_{u \in V_v^j} t(u)\, \sigma_{v,u}^{j-1}(p(v)) = -\frac{3}{2t(v)} \sum_{u \in V_v^j} t(u) \\ &= -\frac{3}{2t(v)} 6t(v) = -9. \end{aligned}$$

□

5. Some numerical tests

To illustrate the efficiency of our prewavelets, we took as the initial polyhedron the regular octahedron and we performed five levels of decomposition. The total number of vertices at the level five is 4098. We considered a data set *jump* consisting of 36×72 measurements on the sphere at the points $P_{ij}(\theta_i, \varphi_j)$, given by

TABLE 1. Reconstruction errors for some compression rates

comp. rate	nr. of zero coeff.	$\left\|\mathbf{e}^5\right\|_\infty$	$\left\|\mathbf{e}^5\right\|_2$	mean $\left(\mathbf{e}^5\right)$
0.05	3888	59.2191	992.7856	12.5437
0.1	3683	17.6250	220.3438	2.5425
0.25	3070	1.4023	12.9496	0.1287
0.5	2046	0.0527	0.4462	0.0039
0.75	1024	0.0005	0.0034	$1.97 \cdot 10^{-5}$

their spherical coordinates (θ, φ), where $(\theta_i)_{1\le i\le 36}$ are equidistant nodes of the interval $[-\pi, \pi]$ and $(\varphi_j)_{1\le j\le 36}$ are equidistant nodes of the interval $[-\pi/2, \pi/2]$. This dataset is constant over the sphere, except to a small portion, where it has a very big jump (see Figure 1). Such functions appear in crystallography (see [6]).

First we approximated this data with the function $f^5 \in \mathcal{V}^5$ (figure 2). The measured approximation errors were

$$
\begin{aligned}
e_1 &= \frac{1}{36\cdot 72}\sum_{i=1}^{36}\sum_{j=1}^{72}\left|f^5(i,j) - jump(i,j)\right| = 1.0984,\\
e_2 &= \left(\frac{1}{36\cdot 72}\sum_{i=1}^{36}\sum_{j=1}^{72}\left|f^5(i,j) - jump(i,j)\right|^2\right)^{1/2} = 0.4424.
\end{aligned}
$$

Then we performed the decomposition, thresholding and reconstruction using the algorithms described in Section 2 and Section 3. We denoted by $\mathbf{e}^5$ the vector $\mathbf{f}^5 - \widehat{\mathbf{f}}^5 = \left(f^5_v - \widehat{f}^5_v\right)_{v\in V^5}$ and we measured the errors

$$
\begin{aligned}
\left\|\mathbf{e}^5\right\|_\infty &= \max_{\eta\in\mathbb{S}^2}\left|\mathbf{e}^5(\eta)\right| = \max_{v\in V^5}\left|\mathbf{e}^5(v)\right|,\\
\left\|\mathbf{e}^5\right\|_2 &= \left(\sum_{v\in\mathcal{V}^5}\left|f^5_v - \widehat{f}^5_v\right|^2\right)^{1/2},\\
\operatorname{mean}\left(\mathbf{e}^5\right) &= \frac{1}{|V^5|}\sum_{v\in V^5}\left|\mathbf{e}^5(v)\right|.
\end{aligned}
$$

The errors are tabulated in Table 1.

To compare our approach, whose strength is the locality of the prewavelets support, we took the case of spherical harmonic polynomials. For more details about spherical harmonics, see [4]. The basis functions are the polynomial kernels. Their supports are localized, but not local. An example of a polynomial kernel is given in Figure 5. Here we can see that its support covers the whole sphere. The wavelet decomposition was described in [1], Chapter 3. We performed 6 levels of decomposition. At the level $j = 6$, the total number of vertices was $2^{2j+1} = 8192$. Figure 6 show the approximation at the level 6. The oscillations around the jump, which occur because of the global support, are avoided in our approach.

Finally, let us mention that, to our knowledge, no construction of locally supported continuous prewavelets was made so far.

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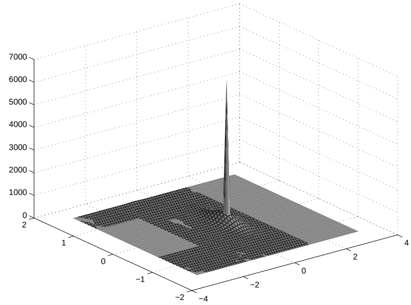

FIGURE 1. The initial dataset *jump*, represented in spherical coordinates.

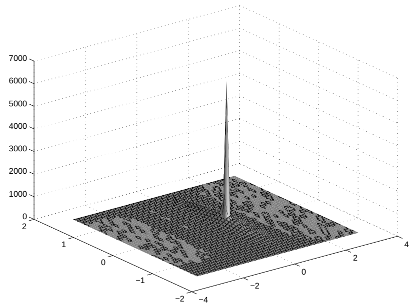

FIGURE 2. The approximation f^5 at the level 5.

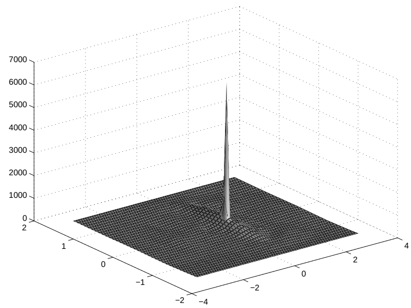

FIGURE 3. Approximation with the compression rate 0.05.

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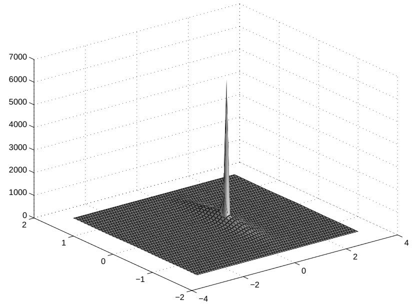

FIGURE 4. Approximation with the compression rate 0.1.

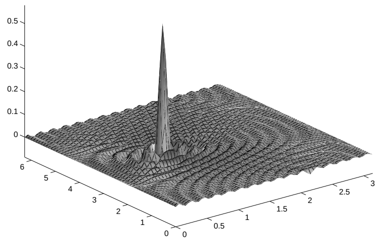

FIGURE 5. An example of kernel of spherical harmonics: localized but supported on the whole sphere.

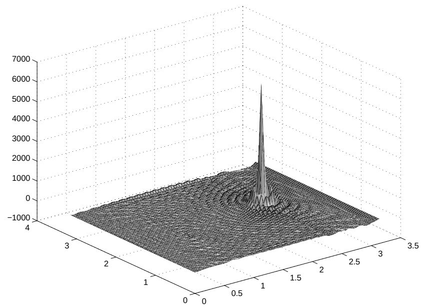

FIGURE 6. Approximation at the level 6, using the kernels of spherical harmonics.

BOOK REVIEWS

H.M. Antia, *Numerical Methods for Scientists and Engineers*, Birkhäuser Verlag, Boston–Basel–Berlin, 2002, 864 pages, Hardcover, ISBN: 3-7643-6715-6.

This book is a comprehensive exposition (almost 900 pages) of numerical methods usable in science and engineering. It is intended to fulfill the difficult task to cover all elementary topics in numerical computations and to discuss them such that to enable their practical implementation.

The first two chapters are on errors in numerical computation.

The third chapter is dedicated to methods for solving linear algebraic systems (direct methods and iterative stationary methods). Chapter 4 deals with interpolation. An advanced topic of this chapter is interpolation of more dimensions. The next chapter is on (numerical) differentiation. Numerical integration (including multivariate) is the subject of chapter 5. Nonlinear equations in $\mathbb{R}$ and $\mathbb{R}^n$ are treated in the seventh chapter. Chapter 8 is on optimization. Chapter 9, "Functional approximation", presents various least square approximations, FFT and Laplace transform, Cebyshev, Padé and L_1-approximation. Algebraic eigenvalue problem is the subject of chapter 10. Chapter 11 treats ordinary differential equations – numerical methods for initial value problems and two-points boundary value problems. In chapter 12 one finds an advanced topics – integral equations – not too present in older classical books. The last chapter, 13, is dedicated to the numerical methods for partial differential equations.

Bibliographies accompany each chapter.

The book contains also over 500 exercises and problems, whose answer and hints are in appendix A. Over 100 well-chosen worked out examples, which illustrate the usability of the methods and their pitfalls are also included. The accompanying CD contains good quality Fortran and C programs and tests (appendix B and C, only on CD).

Intended audience: students, computer scientists, researchers in science and engineering, practicing engineers.

Radu T. Trîmbiţaş

Roger Godement, *Analyse Mathématique,* Springer, Berlin - Heidelberg - New York.
I. *Convergence, fonctions elementaires,* (1998), 2éme édition corrigé 2001, xx+458 pp, ISBN 3 540 42057-6 ;
II. *Calcul différentiel et intégral, séries de Fourier, fonctions holomorphes,* (1998), 2éme édition corrigé 2003, viii+490 pp, ISBN 3 540 00655-9;
III. *Fonctions analytiques, différentielles et variétés, surfaces de Riemann,* 2002, ix+338 pp, ISBN 3 540 66142-5;
IV. *Intégration et théorie spéctrale, analyse harmonique, le jardin des délices*

modulaires, 2003, xii+599 pp, ISBN 3 540 43841-6.

This four volume book is an unusual treatise on mathematical analysis, in the sense that the main target of the author is not to present the results in their strict logical connections with shortest proofs possible (called *Blitzbeweise* by the author), but rather to emphasize the historical evolution of mathematical ideas. Due to this nonlinear character of the exposure, there are some repetitions, but each new approach to a subject sheds a new light on it, revealing new faces and opening new perspectives. These repetitions lead to the increase of the size of the book but, as a former member of Bourbaki group, the author adopted one of their basic principles : "don't spare paper". By the numerous comments and footnotes spread through the four volumes, the author put in evidence the sinuous way the notions and results travelled before reaching the clarity and logical rigor of the 20th century. The volumes contain a lot of examples of miscalculations and wrong reasonings (*derapages* or *acrobatie sans filet*) of great mathematicians as Newton, the Bernoullis, Euler, Cauchy, Fourier, a.o., leading to correct or to false results. This shows that many notions and results were often seized by the intuition of the great creators of mathematical analysis (*les Fondateurs*), initially in an obscure and confusing manner. For us many of these thinks look very simple and clear, but this took sometimes fifty or even hundred of years years of evolution, discussions, or arguments. Beside these comments of mathematical character there are a lot of political and social considerations concerning pure and applied mathematics (mainly its military and social applications), and on the responsibility of scientists and governs for the use of the research for military purposes. In the last 25 years the author was deeply involved in such questions, and some of his ideas and conclusions are collected in a *Postface* at the end of the second volume, with special references to armament race and the construction of A and H bombs by the USA and SSSR.

The first two volumes of the book (Chapters I through VII) are concerned with the differential and integral calculus of the functions of one variable (real or complex), including elements of Fourier analysis and holomorphic functions. Some results on differential calculus in $\mathbb{R}^n$ are treated (including the implicit function theorem in $\mathbb{R}^2$). An appendix to Chapter III contains some results on metric, normed and inner product spaces. A specific feature of the book is the early treatment of some topics considered as advanced - summable families (*convergence en vrac*), analytic functions, Radon measures, Schwartz distributions, Weierstrass theory of elliptic functions. A more advanced treatment of some of these results can be found in the third volume of the book.

The third volume contains three chapters: VIII. *La théorie de Cauchy,* IX. *Différentielles et intégrales à plusieurs variables,* and X. *La surface de Riemann d'une fonction algébrique.* In Chapter VIII one continues the study of holomorphic functions, started in the second volume, with the Cauchy integral formula and its applications to the calculus of residues, to complex Fourier transform (including the Paley-Wiener theorem) and to Mellin transform. Chapter IX contains a discussion on tensors, differential varieties, differential forms and their integration, culminating with Stokes theorem. The last chapter of this volume contains a brief introduction to

Riemann surfaces, a subject that was yet touched at the end of the first volume when dealing with the functions Arg z and Log z.

The last volume of the treatise contains two chapters: XI. *Intégration et transformation de Fourier*, and XII. *Le jardin des délices modulaires ou, l'opium des mathématiciens.* The chapter on integration, based on the famous course taught for a long period by the author at the University Paris VII, develops the integration theory following Daniell's approach, like Bourbaki. One constructs the spaces L^p, including completeness and duality results, and one proves Lebesgue-Fubini and Lebesgue-Nikodym theorems. The author insists on the notion of Polish space, a term suggested by him to Bourbaki when he was a member of the group, a that was adopted immediately by Bourbaki and by the mathematical community as well. The construction of Haar invariant measure on a locally compact group G, with applications to Fourier transform on $L^1(G)$ and $L^2(G)$, is included. This chapter contains also an introduction to operators on Hilbert space, and to unitary representations of locally compact topological groups.

The last chapter of the treatise is devoted to more specialized topics related to modular functions - theta and L series, elliptic functions and integrals, the Lie algebra SL(2). It can be used as an introduction to this area of research with very reach possibilities of generalization.

Reflecting author's encyclopaedic knowledge of mathematics and written in a live and attractive style (a perfect illustration of the famous "French spirit"), the book will be a valuable help for those teaching mathematical analysis or desiring to be acquainted with the evolution of the mathematical ideas. The historical, social and ethical comments accompanying the main text, reflects the complex personality of the author and his broad interests.

S. Cobzaş

Robert L. Ellis and Israel Gohberg, *Orthogonal Systems and Convolution Operators*, Operator Theory: Advances and Applications, Vol. 140, Birkhäuser Verlag, Basel-Boston-Berlin 2003, xvi+236, ISBN: 3-7643-6929-9.

The Szegö polynomials are polynomials that are obtained by the Gramm-Schmidt orthogonalization process from $1, z, z^2, ...$ in the space $L^2(\mathbb{T})$, $\mathbb{T}$ the unit circle, that are orthogonal with respect to the inner product

$$< f, g >_\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{g(e^{it})} \omega(e^{it}) f(e^{it}) dt. \tag{1}$$

Here ω is a positive integrable weight function. G. Szegö proved that all the zeros of the Szegö polynomials lie inside the unit circle. M. G. Krein extended Szegö's theorem to the case of a not necessarily positive weight function ω, when the inner product (1) need not to be positive definite, and the corresponding space is called a space with indefinite inner product. In this case the distribution of the zeros of orthogonal polynomials is much more complicated than in the definite case, and is connected with the numbers of positive and negative eigenvalues of some Toeplitz matrices associated with the orthogonal polynomials. Krein investigated also the continuous analogues of orthogonal polynomials, by replacing the Toeplitz matrix by a convolution operator on $L^2(0, a)$. Together with H. Langer, he proved an analogue of Krein's theorem.

Based on the research done by the authors and their colleagues for nearly fifteen years, the book is devoted to a unified and thorough presentation of these results, along with many extensions and generalizations. These extensions are concerned with matrix- and operator-valued polynomials, functions and operators, considered both in the discrete and continuous case. The unifying theme is that of the orthogonalization with invertible squares in modules over C^*-algebras. One of the main features of the book is the interplay between polynomials and operator theory – the theory of Toeplitz matrices, Wiener-Hopf operators, Fredholm operators and their indices.

The book book is of interest for analysts but, as the developed theory has many applications to some engineering problems (mainly in signal processing and prediction theory), the engineers and physicists will find a lot of interesting things in it too.

P. T. Mocanu

Writing the History of Mathematics: Its Historical Development, Editors: Joseph W. Dauben and Christoph J. Scriba, Historical Studies-Science Networks, Birkhäuser Verlag, Basel-Boston-Berlin 2002, xxxvii+689 pp, ISBN: 3-7643-6167-0.

"History of mathematics' is concerned with the development in time of the unfolding of mathematics, whereas "historiography of mathematics" deals with scholarly research, reconstruction and description of the past development of the history of mathematics. The aim of the present book is to provide a perspective on how and why history of mathematics has developed in various countries and at different times.

The idea to initiate a study of the history of history of mathematics was settled down at a meeting in Oberwolfach, Germany, in the early 1990s. Subsequently J. Dauben and Ch. Scriba accepted the editorial responsibility of the project, and the task was accomplished with the help of more than forty collaborators, and with support from several institutions and organizations.

The first part of the book, *Countries*, contains 20 chapters, the first 19, written by different authors, deals with the history of history of mathematics in different countries. The last chapter, *Postscriptum*, discusses some connections of the history of mathematics with teaching and society, the impact of electronic media, and the humanism of mathematics.

The second part of the book contains three hundred short biographies of prominent contributors to the history of mathematics, including some great names of contemporary mathematics – J. Dieudonné, A. N. Kolmogorov, D. J. D Struik, B. L. van der Warden, A. Weil. Some portraits are also included.

The third part contains a list of abbreviations, the bibliography and an index.

All the included material was drafted and circulated, being revised and rewritten several times, and finally reconsidered in the context of the entire project by the team of scholars overseeing the project.

The result is this monumental book, that is of great interest not only to mathematicians but also to people interested in the history of science in general.

Paul A. Blaga

Philippe Blanchard and Erwin Brüning, *Mathematical Methods in Physics–Distributions, Hilbert Space Operators and Variational Methods,* Progress in Mathematical Physics, Vol. 26, Birkhäuser Verlag, Boston-Basel-Berlin 2002, pp. xxii+463, ISBN: 3-7643-4228-5 and 0-8176-4228-5.

The book contains a detailed exposition of the basic mathematical facts and tools needed in quantum mechanics and in classical and quantum field theory. The book is divided into three parts: I, it Distributions, II, *Hilbert space operators* and III, *Variational methods*.

The first part contains a fairly complete presentation of Schwarz theory of distributions, including the necessary elements of locally convex spaces. The emphasis is on the analytical aspects of the theory as: Fourier transform, convolution and approximation of distributions by regularizing sequences, holomorphic functions and the relations of distributions with boundary values of analytic functions. Many carefully chosen and very interesting examples, as the distribution δ of Dirac, the principal value distribution, the Sokhotski-Plemelji formula, give a strong motivation for the developed topics and make the lecture more attractive. Applications of the theory of distributions to ODEs and PDEs are given. This part ends with a discussion of other classes of generalized functions – Gelfand-Shilov generalized functions and Komatsu hyperfunctions.

In the second part, after some introductory results on the geometry of Hilbert spaces and orthonormal bases, one passes to the study of linear operators on Hilbert space and their spectral theory. The focus is on properties needed for the study of Schrödinger operator in quantum mechanics. The authors consider the principal classes of bounded and unbounded operators on Hilbert space: self-adjoints, symmetric, closable, unitary, trace operators, Hilbert-Schmidt operators. One insists on the C^*-algebra properties of the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on the Hilbert space $\mathcal{H}$. As application one considers the interpretation of the spectrum of self-adjoint Hamiltonian.

The last part of the book is concerned with direct methods in the calculus of variations and constrained minimization, with applications to boundary and eigenvalue problems. A short presentation of the Hohenberg-Kohn variational principle is included. The authors have written another book *Variational methods in mathematical physics. A unified approach*, Springer Verlag, Berlin 1992, devoted to variational methods.

The book is fairly self-contained, the prerequisites being advanced differential calculus (including Lebesgue integration) and familiarity with basic results in ODEs and PDEs Four appendices contain some supplementary material from topology, functional analysis and algebra.

By presenting in a rigorous way and with many historical comments the basic results needed for quantum mechanics, the book will be of great interest to physicists and engineers using the mathematical apparatus in their research. For mathematicians interested in an accurate presentation of non-trivial applications of relatively abstract areas of mathematics, the book is a valuable source of examples.

S. Cobzaş

Yves Nievergelt, *Foundations of Logic and Mathematics*, Birkhäuser Verlag, Boston, 2002, xvi + 416 pp., Hardcover, ISBN 0-8176-4249-8.

This book is a modern introduction to the foundations of Logics and Mathematics, written with a permanent care for the possible applications of some rather classical topics in modern fields of science and especially in Computer Science.

The present volume is structured into two main parts, namely *A. Theory*, containing Chapters 0-4, and *B. Applications*, containing Chapters 5-7.

Chapter 0 sets the fundamentals concerning Boolean algebraic logic, discussing logical formulae, logical truth and connectives, tautologies and contradictions, methods of proof and Karnaugh tables. Chapter 1 refers to logic and deductive reasoning, having as main topics propositional and classical implicational calculus, proofs by contraposition, proofs with connectives or quantifiers and predicate calculus. Chapter 2 contains the basic material of Set Theory, including operations for sets, relations, functions, equivalence and ordering relations. Chapter 3 deals with mathematical induction, definition and (arithmetic) properties of natural numbers, integers and rational numbers, also referring to finite and infinite cardinality and ending with some arithmetic in finance. Chapter 4 discusses decidability and completeness, the selected topics being on logics for scientific reasoning, incompleteness, automated theorem proving, transfinite methods, transitive sets and ordinals and regularity of well-formed sets.

Chapter 5 presents the relationship between Number Theory and Code Theory, containing the classical Euclidean Algorithm, digital expansion of integers, properties of primes and modular arithmetic as well as some very practical information on modular codes, such as the International Standard Book Number (ISBN) code, the Universal Product Code (UPC) and the Bank Identification Code, and Rivest-Shamir-Adleman (RSA) codes in public key cryptography. Chapter 6 deals with (cyclic) permutations, arrangements and combinations, elements of probabilities, the most of these with the finality of describing the ENIGMA machines. Chapter 7, which is mainly an introduction to Graph Theory, discusses several types of graphs (directed, undirected, path-connected, weighted or bipartite), Euler and Hamiltonian circuits, trees, but also some of their applications in science, concerning the shape of molecules and hydrocarbons or sequences of radioactive decays.

The book is well written, concise and organized and contains an impressive quantity of information on rather different topics. I should emphasize the numerous examples (more than 1000) and exercises (again more than 1000) throughout the text as well as the several projects at the end of each chapter, that propose some more difficult problems, sometimes suggesting further bibliographic sources.

I warmly recommend the volume to students in Mathematics and Computer Science, but also to those interested in the foundations of these sciences.

Septimiu Crivei

M. M. Rao, Z. D. Ren, *Applications of Orlicz Spaces*, Marcel Dekker, Inc., New York-Basel, 2002.

This book is written by well-known specialists in the theory of Orlicz spaces. Their book "Theory of Orlicz spaces", Marcel Dekker, New York 146, 1991 and the work of S.T. Chen "Geometry of Orlicz spaces", Dissertationes Math., 356 (1996), 1-204, together with the present volume cover a great part of the modern theory of Orlicz spaces and its applications.

In order to obtain complete solutions for some problems, the authors prefer to work in Orlicz spaces $L^{\phi}(S, \Sigma, \mu)$, where ϕ is an N-function (instead of a general Young function) and where the measure space (S, Σ, μ) is either purely atomic or diffuse and finite (σ-finite). On the other hand they consider both the cases when $L^{\phi}(S, \Sigma, \mu)$ is endowed with the Orlicz or Luxemburg norm. Exact values for several geometric constants of Orlicz spaces are computed in the case $\phi := \phi_s, s \in (0, 1)$, where ϕ_s is an intermediate function between $\phi_0(u) = u^2$ and a given N-function.

In Chapter II one obtains lower and upper bounds for James constant and for von Neumann-Jordan constant of Orlicz spaces endowed equally with Orlicz and Luxemburg norms. Exact values for these constants are obtained for $L^{\phi_s}([0, 1]), L^{\phi_s}(\mathbb{R})$ and ℓ^{ϕ_s} endowed with both norms. In chapters III-V similar estimates are given for other geometric constants as: the normal structure coefficient, weak convergent sequence coefficient, Jung constant, Kottman constant and for the packing constant of Orlicz spaces. All such estimates are expressed in terms of quantitative indices of N-functions. In chapter VI the authors consider some problems of Fourier Analysis in Orlicz and generalized Orlicz spaces. So, they present conditions implying the almost everywhere convergence of Fourier or conjugate Fourier series of all $f \in L^{\phi}([0, 1])$, or that the Haar system of functions forms an unconditional basis in $L^{\phi}([0, 1])$. In the next chapter applications to prediction theory are presented – for instance a necessary and sufficient condition for a prediction operator (with respect to a Chebyshev subspace of $L^{\phi}(\mu)$) to be linear. Other applications in the field of stochastic analysis and of partial differential equations with solutions in Orlicz-Sobolev spaces are also presented.

The book is well-written, self-contained, with many bibliographical comments, suggestive examples and a rich list of references (from old ones to very recent titles). Many of the results in the book were not yet known thirty years ago and some were even not known ten years ago. The book is recommended to graduate students and research workers in the field of Banach space theory, probability, partial differential equations, approximation theory etc.

Ioan Şerb

Hrushikesh N. Mhaskar and Devidas V. Pai, *fundamentals of Approximation Theory,* Alpha Science International Ltd., 2000, xv+541 pp., Hardcover, ISBN 1-84265-016-5.

Understood in a broad sense, approximation is one of the major themes of mathematics – approximate mathematical objects with simpler ones, easier to handle. The development of the computers made it even more important – the numerical algorithms are based on discretization techniques that are, in fact, approximation procedures.

The book under review is dedicated to a comprehensive presentation of basic tools and results in approximation theory, understood as a mathematical discipline.

Its characteristic features are the clarity of the exposure, a careful choice of the included topics and the permanent interplay between classical and abstract (meaning functional analytic) tools.

The best idea on its content can be given by a short presentation of the chapters. Ch. I, *Density theorems*, deals with Weierstrass type theorems for both trigonometric polynomials and algebraic (of Fejér's and Bernstein), Korovkin's theorems, Stone-Weierstrass theorem.

In Ch. II, *Linear Chebyshev approximation*, after presenting some results on the existence and uniqueness of best approximation in abstract normed spaces, the authors pass to the concrete case of uniform approximation by polynomials, including existence, Chebyshev alternation theorem, Haar spaces and uniqueness, strong uniqueness and continuity of the metric projection operator. A special attention is paid to discretization and algorithms for computing the best approximation polynomials (Remes algorithms).

Ch. III, *The degree of approximation*, is concerned with quantitative aspects of approximation theory, emphasizing the the connections between the smoothness properties (expressed in terms of some moduli of continuity and smoothness) and the degree of approximation. The chapter contains both direct and converse deep theorems, belonging to Jackson, Favard, Markov, Bernstein. Bernstein's theorem on the approximation by analytic functions is included too.

Ch. IV, *Interpolation*, is an introduction to various interpolation procedures – Lagrange, Taylor, Abel-Gonchearov, Hermite. Evaluations of the errors are included.

Ch. V, *Fourier series,* contains a brief introduction to the subject, with emphasis on convergence and summability.

Ch. VI, *Spline functions*, aims to give a short but thorough introduction to spline functions, viewed as a new tool of approximation, and showing how the ideas developed in the first four chapters look like in this case.

Ch. VII, *Orthogonal polynomials*, introduce the reader to this very important area of mathematical analysis.

The last chapter of the book, Ch. VIII, *Best approximation in normed linear spaces*, is concerned with best approximation in abstract setting. The authors put in evidence the deep relations between the geometry of the normed space and its approximation properties - existence and uniqueness of best approximation, continuity of the metric projection, convexity of Chebyshev sets. The last section is concerned with optimal recovery problems.

Each chapter ends with a section of historical notes and a set of exercises. Some of these are routine, completing the main text, but there also challenging exercises, taken from current research papers. For these ones detailed hints are included.

A comprehensive bibliography of 302 items is included.

The authors are well known specialists in the domain and the book incorporates a lot of their original results.

Based on an over that 10 years teaching experience, the book can be used for special graduate or post-graduate courses. The chapters are relatively independent, so that parts of the book can be used for different courses. The prerequisites are advanced calculus and basic topology, measure theory and functional analysis.

Covering, in a clear and comprehensive manner, the basic results in approximation theory, both classical and abstract as well, I think the book will become a standard reference in the field.

S. Cobzaş

A. Brown and Ken R. Goodearl, *Lectures on Algebraic Quantum Groups.* Advanced courses in mathematics - CRM Barcelona, Birkhäuser Verlag, Basel-Boston-Berlin, 2002, ix+349 pp., Softcover, ISBN 3-7643-6714-8.

The term 'quantum groups' refers to a rapidly growing field of mathematics and mathematical physics which appeared in the 1980's theoretical physics and statistical mechanics. The volume under review is an expanded version of the lectures given by the authors in September 2000 at the Centre de Ricerca Mathemàtica in Barcelona, and it focuses on two types of algebras. First, there are the so called 'quantum coordinate rings' which are deformations of the classical coordinate rings of algebraic groups or related algebraic varieties. The second type consists of 'quantized enveloping algebras', which are deformations of universal enveloping algebras of semisimple Lie algebras or of affine Kac-Moody Lie algebras.

The book is divided into three parts. Part I contains the fundamental background material. The second part deals with generic quantized coordinate rings, while the third part focuses on quantized algebras at roots of unity. The presentation begins at a point accessible to a graduate student. Later, the style becomes more informal; only sketches of proofs are given, and some topics are presented in a survey manner. There are also many exercises aimed at the non expert reader. Some topics such as the nature of the prime spectrum of a generic quantized algebra, and the relationship between the Hopf algebra structure of the algebra and the Poisson algebra structure of the centre are covered for the first time in book form.

The authors are important contributors to the subject, and their book is a very useful addition to the literature. I warmly recommend it to anyone interested in quantum groups.

Andrei Marcus

Miklós Laczkovich, *Conjecture and Proof*, The Mathematical Association of America, Washington, DC, 2001, x+118 pp., Softcover, ISBN 0-88385-722-7.

The book under review is an extended version of the lectures given by the author at an one-semester course based on creative problem solving of the Budapest Semesters in Mathematics. This is a program for American and Canadian students initiated by Paul Erdős, László Lovász, Vera T. Sós, and László Babai. The book is divided into two parts (Proofs of Impossibility–Proofs of Nonexistence, Constructions–Proofs of Existence)and discusses questions from various fields of mathematics: number theory, algebra and geometry. It contains important and interesting results like the transcendence of e, the Banach-Tarski paradox, the existence of Borel sets of arbitrary finite class, while the necessary prerequisites are kept at the level of an introductory calculus course. All these features will make this volume into a valuable source of inspiration for students and teachers of mathematics.

Andrei Marcus

Nicolai N. Vorobiev, *Fibonacci Numbers*, Birkhäuser Verlag, Basel–Boston–Berlin 2002, x+176 pp, ISBN 3-7643-6135-2.

The book under review is translated from the Russian 6th edition by Mircea Martin and it presents the bearing of Fibonacci numbers on mathematics.

In Chapter 1 (The Simplest Properties of Fibonacci numbers) the basic properties of Fibonacci numbers are given, as *Binet's formula* and applications of this result. In Chapter 2 (Number-Theoretic Properties of Fibonacci Numbers) the main aim is the study of the divisibility of Fibonacci numbers. We mention here Theorem 11 saying that *any to consecutive Fibonacci numbers are relatively prime*, and the more general result Theorem 12, which says that $\gcd(u_m, u_n) = u_{\gcd(m,n)}$. In Chapter 3, entitled Fibonacci Numbers and Continued Fractions, the continued fractions are described using Fibonacci numbers. Legendre's Theorem, Vahlen's Theorem, Borel's Theorem, and Hurwitz's Theorem about continuous fractions are presented. In Chapter 4, "Fibonacci Numbers and Geometry", the author presents connection between Fibonacci numbers and results of classical geometry and graph theory. In Chapter 5 (Fibonacci Numbers and Search Theory) specific variants of minimum problems are discussed: "estimate the minimizing point $\overline{x}$ together with the minimum value $f(\overline{x})$ taken by f at this point" (Problem A) and "approximate the minimizing point $\overline{x}$".

It is well know the fact that Fibonacci numbers have had an important impact on areas as art, architecture, political economy, and other domains, hence many specialists in other domains than mathematics should be interested by them. The book under review is very well written, the prerequisites for reading it are minimal hence it is easy to read. Also, the book will be useful for any student, teacher, and researcher.

Simion Breaz

Toma Albu, *Cogalois Theory*, Marcel Dekker, New York–Basel 2003, xii+341 pp, ISBN 0-8247-0949-7.

The classical Galois theory says that E/F is a finite Galois extension, then the lattice Intermediate(E/F) of all intermediate fields is anti-isomorphic to the lattice of subgroups of Gal(E/F). There are however field extensions which are not necessarily Galois, but have a dual property, that is, there is a lattice isomorphism between Intermediate(E/F) and the lattice of subgroups of a group Δ canonically associated to E/F. Such extensions are called extensions with Δ-Cogalois correspondence.

The book under review is the first which offers a systematic investigation of this concept. One should note that the term Cogalois appeared in literature in 1980 in a paper of C. Greither and D.K. Harrison, while the term extension with Cogalois correspondence was introduced by the author and F. Nicolae.

The volume is divided into two parts. The first part deals with finite extensions, and consists of 10 chapters. These chapters contain the necessary preliminaries and investigate the following aspects of the theory: G-radical extensions, Cogalois extensions, Cogalois connections associated to G-radical extensions, strongly

G-Kummer extensions (which are extensions with G/F^*-Cogalois correspondence), almost G-Cogalois extensions, finite Galois extensions which are Cogalois, radical, Kneser or G-Cogalois, Kummer extensions. Applications to Algebraic Number Theory and connections with graded algebras and Hopf algebras are also discussed.

The second part considers infinite extensions and has 5 chapters. The first problem here is to find suitable generalizations of the above concepts. The author discusses infinite G-Kneser extensions, infinite G-Cogalois extensions, infinite Kummer extensions, and infinite Galois G-Cogalois extensions, which involve profinite groups.

The author is an important contributor to the subject, and the volume contains many of his results. The book is carefully written and it is accessible to graduate students. Familiarity with basic abstract algebra, Galois theory and some Galois cohomology is assumed. Over 250 exercises, an up-to-date bibliography and an extensive index add to the value of the book.

This volume is especially recommended to students and researchers in Algebraic Number Theory, but any algebraist will find here interesting ideas and information.

Andrei Marcus

M. W. Wong, *Wavelet Transforms and Localization Operators*, Operator Theory: Advances and Applications, Vol. 136, Birkhäuser Verlag, 2002, pp. 156. ISBN: 3-7643-6789-X.

Wavelet analysis is an emerging mathematical discipline, that has begun to play a serious role in a broad range of applications, including signal processing, data and image compression, solution of partial differential equations, modeling multiscale phenomena, and statistics. In the present book, the author studies wavelet transforms and localization operators in the context of infinite-dimensional and square-integrable representations of locally compact and Hausdorff groups. At the same time, fruitful approaches have been developed as regards Daubechies operators on the Weyl-Heisenberg group, localization operators on the affine group, wavelet multipliers on the Euclidean space, the book providing the reader with the spectral theory of wavelet transforms and localization operators in the form of Schatten - von Neumann norm inequalities. The information is structured in 26 chapters as follows:

Introduction / Schatten - von Neumann Classes / Topological Groups / Haar Measures and Modular Functions / Unitary Representations / Square-Integrable Representations / Wavelet Transforms / A Sampling Theorem / Wavelet Constants / Adjoints / Compact Groups / Localization Operators / S_p Norm Inequalities / Trace Class Norm Inequalities / Hilbert-Schmidt Localization Operators / Two-Wavelet Theory / The Weyl-Heisenberg Group / The Affine Group / Wavelet Multipliers / The Landau-Pollak-Slepian Operator / Products of Wavelet Multipliers / Products of Daubechies Operators / Gaussians / Group Actions and Homogeneous Spaces / A Unification / The Affine Group Action on $\mathbb{R}$.

In order to sustain the above material, a good bibliography containing 108 titles is listed. The author offers clear explanations of every concept and method making the book accessible and valuable to researchers and graduate students alike.

Octavian Agratini