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UNSUPERVISED CLASSIFICATION FOR DESIGNING SPEAKER IDENTIFICATION SYSTEMS

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Abstract. We compare recognition performance of Vector Quantization method (VQ) and Gaussian Mixture Modeling method (GMM) in normal speech conditions. We performed measurements to emphasize the relationship between the size of models –number of clusters or number of components of mixtures– and size of the system. We also conclude that the VQ method is a particular case of the GMM method. The results show that the VQ sometimes overperfoms GMM which has a serious shortcoming, particularly when a mixture distribution consists of several overlapping distributions.

1. **Introduction**

Unsupervised classification is also known as data clustering, which is a generic label for a variety of procedures designed to find natural groupings, or clusters, in multidimensional data based on measured or perceived similarities among the number of clusters, the clusters' shapes and the clusters' sizes. In this article we applied unsupervised classification for designing speaker identification systems and we performed several measurements to show the relationship between number of clusters, used to represent speakers' models and identification system's accuracy.

The goal of a speaker identification system is to automatically determine a speaker's identity using an utterance from the speaker. Such a system may be text-dependent—when the speaker must pronounce a text chosen randomly by the system from a fixed vocabulary—, or may be text-independent, when an arbitrary text is allowed to be uttered. Our system was developed for the text-independent case.

Several methods were studied for text-independent speaker identification systems including Vector Quantization methods (VQ) [1], [2], [3], Gaussian Mixture Model method (GMM) [4] and Hidden Markov Models [6]. These methods belong to the model-based approach. For each speaker a statistical model is created to characterize the speaker's voice. These statistical models do not contain any information about interspeaker variabilities.

In this article we try to show that the Vector Quantization method and the Gaussian Mixture method are both based on unsupervised classification, and the VQ method can be viewed as a particular case of the mixture decomposition method.

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Another observation we make is that for speech data, we use clustering only for reduction of the amount of data. Our objective is to find a reduced set of prototypes that best approximate the original set of features and not to find separable clusters (perhaps no any such cluster exists in speech). So we can conclude that there is no significant difference between K-means clustering algorithms developed by the pattern recognition community and the LBG clustering algorithm described in the speech processing and other communications literature [13].

Clustering can be used not only for separating the data into clusters but also for organising a large amount of data. There are hundreds of clustering algorithms in the literature which can be divided in two main categories: square-error iterative partitional clustering and agglomerative hierarchical clustering. In this article we used only the first approach, so we will describe only this one. This type of clustering algorithms attempt to obtain partitions which minimize the within-cluster scatter or maximize the between-cluster scattering [8].

The partitional clustering algorithm determines a partition of n So for clusteringpatterns in a D-dimensional metric space into M ($M < n$) clusters, such that the patterns in a cluster are more similar to each other than to patterns in different clusters. It is a hard problem to determine the optimal clusters' number (M) even when the type of data is known. In this article we performed some measurements to show how the identification system accuracy is influenced by clustering type and the number of clusters used.

2. **VQ-based Speaker Identification**

In the VQ-based speaker identification system each speaker is represented by a codebook created from some training data uttered by the speaker. Each speaker's model is created in two steps:

- Consider some training data (utterance) from the speaker and extract some type of feature vectors (MFCC [13], LPCC [13])

$$\{x_1, x_2, \ldots, x_n\} \qquad x_i \in \mathbb{R}^D.$$

- Cluster the feature vectors into a fixed number of clusters $\{C_1, C_2, \ldots, C_M\}$, where $M < n$. Take the centroid of each cluster and form a set of M vectors, named also code vectors. This set of code vectors is called codebook and this is the model of a speaker.

This type of speaker identification system is based on square-error clustering. The objective is to obtain a partition that, for a fixed number of clusters minimizes the square-error. The set of n patterns in D dimensions has somehow been partitioned into M clusters $\{C_1, C_2, \ldots, C_M\}$ such that cluster C_k has n_k patterns (feature vectors) and each pattern is in exactly one cluster, so that $\sum_{k=1}^{M} n_k = n$. The mean vector, or center of cluster C_k is defined as the centroid of the cluster:

$$m^{(k)} = \frac{1}{n_k} \sum_{i=1}^{n_k} x_i^{(k)}, \tag{1}$$

where $x_i^{(k)}$ is the ith pattern belonging to cluster C_k [8]. The square-error for cluster C_k, also called within-cluster variation is:

$$e_k^2 = \sum_{i=1}^{n_k} \left(x_i^{(k)} - m^{(k)}\right)^T \left(x_i^{(k)} - m^{(k)}\right). \tag{2}$$

The square-error for the clustering is defined as:

$$E_M^2 = \sum_{i=1}^{M} e_k^2 \tag{3}$$

The objective of this clustering is to find a partition that minimizes (3).

The role of vector quantization (clustering) is to reduce the amount of data and to model the distribution of the feature vectors. The problem of automatically separating training data into groups representing classes is solved by a clustering algorithm. A comparison of clustering algorithms in a VQ-based speaker identification system was made by [5] and the results were that the accuracy of identification of a system generally is not influenced by the clustering algorithm, but is influenced by the number of clusters (codebook size) chosen. So for clustering any efficient and fast algorithm can be used.

The identification procedure can be performed in two ways:

1. comparing the sequence of feature vectors extracted from the unknown speaker utterance $\{x_1, x_2, \ldots, x_T\}$ with all N models (codebooks) in the speaker database [1].
2. forming a codebook from the sequence of these feature vectors and comparing the resulting codebook with the codebooks from the speaker database [3].

For case 1 the identification procedure can be formulated as follows:

Consider a speaker idenification system with N known speakers. We define the codebook for the i^{th} speaker as

$$\lambda_i = \left(m_i^{(1)},\ m_i^{(2)},\ \ldots, m_i^{(M)}\right), \quad i = 1, 2, \ldots, N$$

where $m_i^{(k)}$ is defined by (1).

1. Extract the set of features from the unknown speaker utterance.

$$X = \{x_1, x_2, \ldots, x_T\}, \qquad x_i \in \mathbb{R}^D$$

2. For every model $\lambda_i, \quad i = \overline{1, N}$ compute the distortion

$$d(X,\ \lambda_i) = \frac{1}{T} \sum_{k=1}^{T} \min_{j=1..M} d_E(x_k, m_i^{(j)}),$$

where d_E is the Euclidean metric defined in R^D.

3. Identify the speaker as the one with the smallest distortion:

$$Id = \arg \min_{i=1\ldots N} \{d\,(X,\ \lambda_i)\}$$

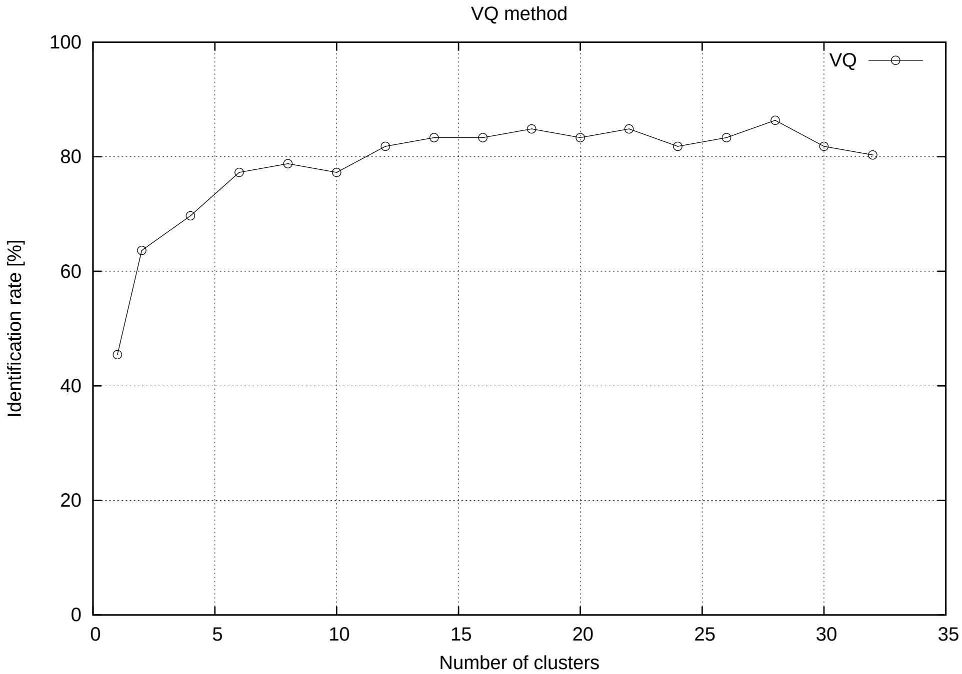


FIGURE 1. Identification rate vs. number of clusters

For case 2 the identification procedure is almost identical to case 1. The only difference is in step (1), where after the feature extraction step is made a codebook from features and this codebook is used in step (2) for calculating the distortions with known speaker models. In this case the algorithm uses a reduced number of distance calculation but performs a clustering to obtain the codebook.

In the application used for measurements we used case 2 for the identification. We trained systems with number of clusters

$$M \in \{1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32\}$$

and measured the identification rates achieved by these systems. The results are represented in figure 1. The parameters of the identification system are presented in the Section 4.

3. GMM-based Speaker Identification

3.1. Preliminaries. Since the primary speaker-dependent information conveyed by the spectrum is about vocal tract shapes, we want to use a speaker model that captures the characteristic vocal tract shapes of a person's voice as manifested in spectral features.

In the statistical speaker model a speaker can be treated as a random source producing the observed feature vectors. The random speaker source is formed by a set of hidden states corresponding to characteristic vocal tract configurations. When the random source is in a particular state, it produces spectral feature vectors from that particular vocal tract configuration. The states are called hidden because we can observe only the spectral feature vectors produced, not the underlying states that produced them. Each state produces spectral feature vectors according to a multidimensional Gaussian probability density function (*pdf*) with a state dependent mean and covariance [4]. The *pdf* for the state i and feature vector x can be expressed as

$$b_i(x) = \frac{1}{(2\pi)^{D/2}|\sum_i|} e^{-\frac{1}{2}(x-\mu_i)^T \sum_i^{-1}(x-\mu_i)} \tag{4}$$

where

- x is a D-dimensional feature vector, $x \in R^D$
- μ_i is the state mean vector, $\mu_i \in R^D$
- Σ_i is the state covariance matrix

The mean vector represents the expected spectral feature vector from the state, and the covariance matrix represents the correlations and variability of spectral features within the state.

The produced feature vector depends on the parameters of the current state $(\mu_i,\ \Sigma_i)$ and the process governing what state the speaker model occupies at any time is modeled as a random process. The following discrete *pdf* associated with the M states describes the probability of being in any state

$$\{p_1, p_2, \ldots, p_M\}, \qquad \text{where } \sum_{i=1}^{M} p_i = 1, \tag{5}$$

and a discrete pdf describes the probability that a transition will occur from one state to any other state,

$$a_{ij} = P(i \to j), \qquad i,j = \overline{1,M} \tag{6}$$

The above definition of the statistical speaker model is known as Hidden Markov Model (HMM) [15]. The HMMs are capable of describing a complex statistical process.

Because our goal is to build speaker's models for text independent speaker recognition we can simplify the statistical speaker model by setting the transition probabilities a_{ij} equal to $1/M$. This means that all state transitions are equally likely.

In following sections we will call each state a component.

3.2. **The Gaussian Mixture Speaker Model.** A Gaussian mixture density of a feature vector x, $\ x \in \mathbb{R}^D$, given the parameter vector λ is a weighted sum of M component densities, and is given by the equality:

$$p(x|\lambda) = \sum_{i=1}^{M} p_i \cdot b_i(x) \tag{7}$$

where

- x is a D-dimensional feature vector
- $b_i(x) \quad i = \overline{1, M}$ are the component densities
- $p_i \quad i = \overline{1, M}$ are the mixture weights.

Each component density is a D-variate Gaussian function defined by the equation (4) with mean vector μ_i and covariance matrix Σ_i and the mixture weights satisfy the constraint

$$\sum_{i=1}^{n} p_i = 1$$

The complete Gaussian mixture density is parameterized by the mean vectors, covariance matrices and mixture weights from all component densities.

These parameters are collectively represented by the symbol:

$$\lambda = (p_i, \mu_i, \Sigma_i), \quad i = \overline{1, M}$$

There are two principal advantages for applying Gaussian mixture densities as a representation of speaker identity. The first is the intuitive notion that the individual component densities of a multi-model density may model some underlying set of acoustic classes. These acoustic classes reflect some general speaker-dependent vocal tract configurations that are useful for characterizing speaker identity. The second advantage of using Gaussian mixture densities for speaker identification is the empirical observation that a linear combination of Gaussian basis functions is capable of representing a large class of sample distributions. One of the powerful attributes of GMM is its ability to form smooth approximations to arbitrarily-shaped densities.

3.3. **Applying the model.** With the GMM as the speaker representation we can then apply this model to speaker identification. The identification system is a maximum likelihood classifier. For a reference group of N speaker models $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, the objective is to find the speaker identity $\hat{s}$ whose mode has the maximum posterior probability for the input feature vector sequence

$$X = \{x_1, x_2, \ldots, x_T\}$$

The minimum-error Bayes' rule for this problem is

$$\hat{s} = \arg \max_{1 \leq s \leq N} P(\lambda_s | X) = \arg \max_{1 \leq s \leq N} \frac{p(X|\lambda_s)}{p(X)} \cdot P(\lambda_s) \tag{8}$$

Assuming equal probabilities of speakers, the $P(\lambda_s)$ and $p(X)$ are constant for all speakers and can be ignored. Equation (8) becomes

$$\hat{s} = \arg \max_{1 \leq s \leq N} p(X|\lambda_s)$$

Assuming independence between observations the decision rule for the speaker identity becomes

$$\hat{s} = \arg \max_{1 \leq s \leq N} \prod_{t=1}^{T} p(x_t|\lambda_s) \tag{9}$$

where

- T is the number of feature vectors
- $p(x_t|\lambda_s)$ is given in equation (7)

Because the logarithm is monotonically increasing, (9) becomes

$$\hat{s} = \arg \max_{1 \leq s \leq N} \sum_{t=1}^{T} \log p(x_t|\lambda_s)$$

3.4. **Estimating GMM parameters.** Given a training speech from a speaker, the goal of speaker model training is to estimate the parameter vector λ for GMM. We will use Maximum Likelihood (ML) estimation technique. The aim of ML estimation is to find the model parameters which maximize the likelihood of the training data.

For a sequence of T training feature vectors

$$X = \{x_1, x_2, \ldots, x_T\}$$

the GMM likelihood can be written as than model with fewert

$$p(X|\lambda) = \prod_{t=1}^{T} p(x_t|\lambda)$$

The ML parameters can be estimated by using a specialized version of the expectation-maximization (EM) algorithm. The basic idea of the EM algorithm is beginning with an initial model λ, to estimate a new model $\overline{\lambda}$, such that $p(X|\overline{\lambda}) \geq p(X|\lambda)$. The new model then becomes the initial model for the next iteration.

On each EM iteration the following estimates are calculated:

Mixture weights::

$$\overline{p_i} = \frac{1}{T} \sum_{t=1}^{T} p(i|x_t, \lambda)$$

Means::

$$\overline{\mu_i} = \frac{\sum_{t=1}^{T} p(i|x_t, \lambda) \cdot x_t}{\sum_{t=1}^{T} p(i|x_t, \lambda)}$$

Covariances::

$$\overline{\Sigma_i} = \frac{\sum_{t=1}^{T} p(i|x_t, \lambda) \cdot x_t x_t^T}{\sum_{t=1}^{T} p(i|x_t, \lambda)} - \bar{\mu}_i \bar{\mu_i}^T$$

If we are using diagonal covariance matrices, we need to update only the diagonal elements in the covariance matrices. For an arbitrary diagonal element σ_i^2 of the covariance matrix of the i^{th} mixture, the variance estimates become:

$$\bar{\sigma_i}^2 = \frac{\sum_{t=1}^{T} p(i|x_t, \lambda) x_t^2}{\sum_{t=1}^{T} p(i|x_t, \lambda)} - \overline{\mu_i}^2$$

The aposteriori probability for component i is given by

$$p(i|x_t, \lambda) = \frac{p_i \cdot b_i(x_t)}{\sum_{k=1}^{M} p_k \cdot b_k(x_t)}$$

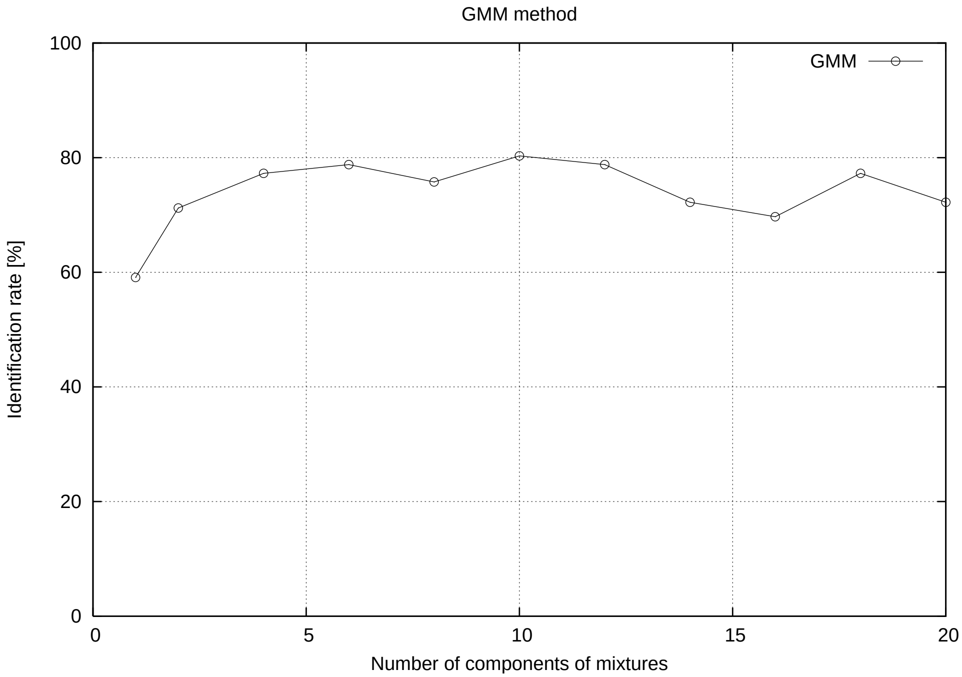


FIGURE 2. Identification rate vs. mixture numbers

Each of the equations involves $p(i|x_t, \lambda)$,which can be interpreted as "fuzzy membership" of x_t to Gaussian i.

Initialization of the GMM models

First, the order M of the model must be large enough to represent the feature distributions. Second, the type of the covariance matrices for the mixture distributions needs to be selected. Diagonal covariance matrices simplify the implementation and are computationally more feasible than models with full covariances. The EM algorithm guarantees to find a *local maximum* likelihood model regardless of the initialization, but different initialization can lead to different local maxima. Usually the means are initialized with centroids of clusters obtained with k-means algorithm and for covariance matrices can be used as initial value the identity matrices.

3.5. **Experimental results.** In the first experiment we tested how the mixtures' components number (model order) influence the identification accuracy. The results are given in figure 2.

In the second experiment we tested if there exists a relationship between the identification system size –speakers known by the system– and the mixture components. We measured the identification rate for systems with mixture's components 1, 2 and 4, increasing the system size (number of speakers) from 1 to 66. As the

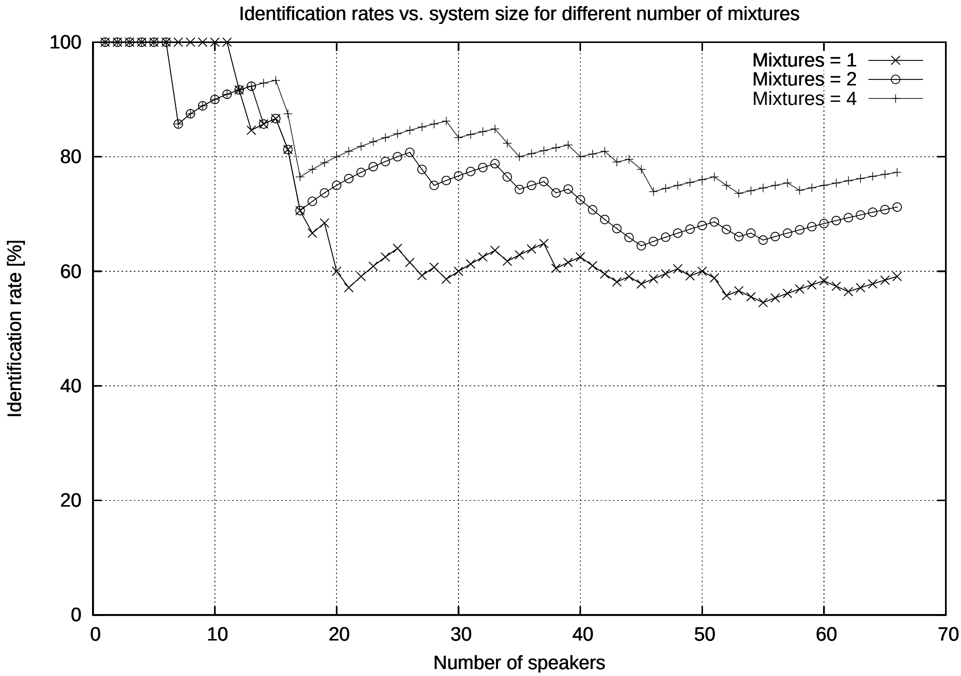


FIGURE 3. Identification rates vs. system size obtained for systems with mixture components 1, 2 and 4

number of speakers increases the model with more mixture components achieves a better performance than the model with fewer components.it was made a The results are shown in figure 3.

4. **Conclusions**

All experiments were done with speech collected from 66 speakers, 29 Hungarian native speakers and 37 Romanian native speakers. 45 of 66 were female speakers and 11 were male speakers. The ages of speakers vary from 14 to 60. The speech was recorded with at least four types of microphones on anonymous soundcards without laboratory conditions. The sampling rate was 16 kHz with 16 bits/sample. Before feature extraction stage a preprocessing was made with direct component (DC) removal and a high emphasis filtering with

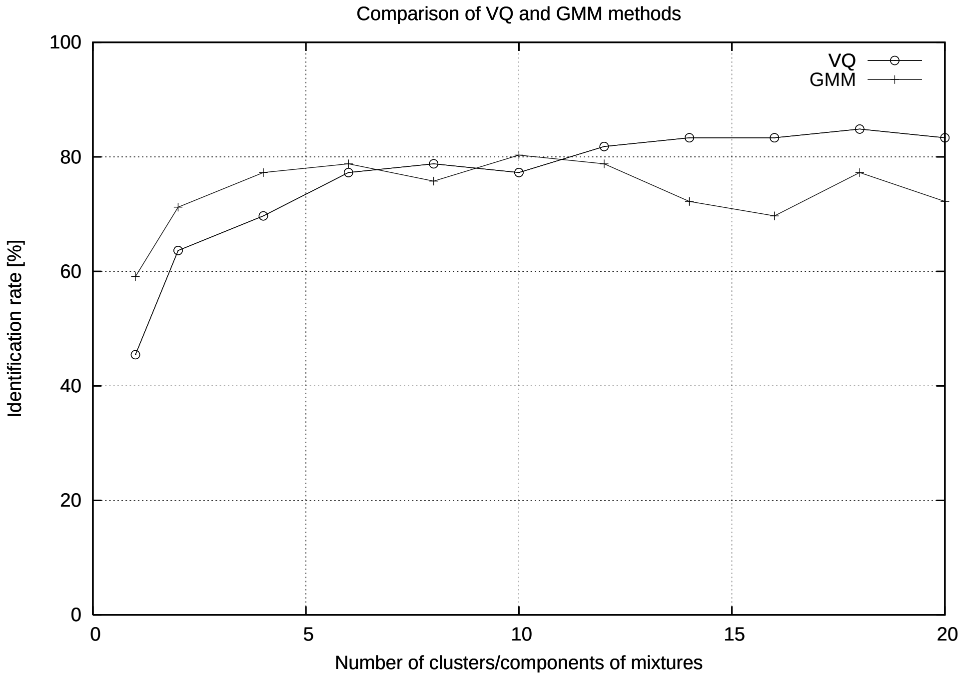
$$H(z) = 1 - 0.95 \cdot z^{-1}$$

and finally, performed a short-term mel-cepstrum analysis with 30 ms Hamming window, with 10 ms shift. The number of mel-cepstral coefficients was 12. For training purposes we used 30s speech and for identification a new set of 1s speech collected from each speaker.

In the stage of estimating the GMM models we initialized the parameters of the model with the following values:

- $p_i = \frac{1}{M}$,
- μ_i were obtained with the clustering algorithm used in the VQ model too
- for Σ_i we used diagonal covariance matrix initialized with the identity matrix, $i = \overline{1, M}$.

Our goal was to compare the VQ method with the GMM method so we used the same features obtained from the same speech database and the same clustering for both methods. Our expectation was that the GMM method will overperform the VQ method, but this is not the case for all values of M. The following figure shows the results obtained for the two systems.



The GMM models' construction is more computer-time consuming and may have a serious shortcoming, particularly when a mixture distribution consists of several overlapping distributions [11].

References

[1] R. K. Soong, A. E. Rosenberg, B. H. Juang, L. R. Rabiner, *A Vector Quantization Approach To Speaker Recognition*, AT&T Technical Journal, 66(1987), 14-26.

[2] J. P. Campbell, *Speaker Recognition: A Tutorial*, Proc. IEEE, vol. 85, no. 9, 1997, 1437-1462.

[3] T. Kinnunen, P. Franti, *Speaker Discriminative Weighting Method for VQ-based Speaker Identification*, Proc. 3^{rd} International Conference on audio- and video-band biometric person authentication, Halmstad, Sweden, 2001, 150-156.
[4] D. A. Reynolds, *Automatic Speaker Recognition Using Gaussian Mixture Speaker Models*, The Lincoln Laboratory Journal, Vol. 8, No. 2, 1995.
[5] T. Kinnunen, Teemu Kilpelainen, Pasi Franti, *Comparison of Clustering Algorithms in Speaker Identification*, Proc. LASTED International Conference, Signal Processing and Communications, Marbella, Spain, 2000, 222-227.
[6] J. M. Naik, L. P. Netsch, G. R. Doddington, *Speaker Verification over Long Distance Telephone Lines*, Proc. ICASSP'89, pp. 524-527, May, 1989.
[7] Bojan Nedic, Herve Bourlard, *Recent Developments in Speaker Verification at IDIAP*, IDIAP-RR 00-26, September 2000.
[8] Anil K. Jain, Robert P. W. Duin, Jiangchang Mao, *Statistical Pattern Recognition: A Review*, IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 22, No.1, 2000.
[9] L. Wang, K. Chen, H. Chi, *Capture Interspeaker Information With a Neural Network for Speaker Identification*, IEEE Transactions on Neural Networks, Vol. 13, No. 2, 2002.
[10] R. O. Duda, P. E. Hart, D. G. Stork, *Pattern Classification*, John Wiley&Sons, 2001.
[11] Keinosuke Fukunaga, *Introduction to Statistical Pattern Recognition*, Second Edition, Morgan Kaufmann, 1990.
[12] F. Jelinek, *Statistical Methods for Speech Recognition*, MIT Press, Third Edition, 2001.
[13] J. R. Deller, Jr. J. H. L. Hansen, J. G. Proakis, *Discrete-Time Processing of Speech Signals*, John Wiley&Sons, 2000.
[14] A. K. Jain, R. C. Dubes, *Algorithms for Clustering Data*, Englewood Cliffs, Prentice Hall, 1988.
[15] L. R. Rabiner, B. H. Juang, *Fundamentals of Speech Recognition*, Prentice-Hall, Englewood Cliffs, 1993.
[16] L. R. Rabiner, *A tutorial on hidden Markov models and selected applications in speech recognition*, Proceedings of IEEE, 77(2), 1989, 257-286.

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DATA DEPENDENCE OF THE FIXED POINTS SET OF WEAKLY PICARD OPERATORS IN GENERALIZED METRIC SPACES

CLAUDIA BACOŢIU

Abstract. In this paper we will extend the results concerning the data dependence of the fixed points set of weakly Picard operators to a generalized metric space (X, d) with $d(x, y) \in \mathbb{R}^n,\ n \in \mathbb{N}^*$.

1. **Introduction**

Definition 1. *Let $x, y \in \mathbb{R}^n,\ x = (x_1, x_2, ..., x_n),\ y = (y_1, y_2, ..., y_n)$. We will consider, by definition:*
- $x \leq y \Leftrightarrow x_i \leq y_i\ \forall i = \overline{1, n}$;
- $\mid x \mid = (\mid x_1 \mid, \mid x_2 \mid, ..., \mid x_n \mid)$;
- $max(x, y) = (max(x_1, y_1), max(x_2, y_2), ..., max(x_n, y_n))$.

Definition 2. *Let X be a nonempty set; an application $d : X \times X \to \mathbb{R}^n_+$ is called generalized metric on X iff:*
(i) $d(x, y) \geq 0\ \forall x, y \in X;\ d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)\ \forall x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in X$.
In this case, (X, d) is said to be a generalized metric space (g.m.s. on short).

The related definitions of completeness, weakly Picard operators, the operator f^∞ in a g.m.s. are the same as in the standard metric spaces.

Definition 3. *If (X, d) is a g.m.s., we will consider the Pompeiu-Hausdorff functional*
$H : P(X) \times P(X) \to (\mathbb{R}_+ \bigcup \{+\infty\})^n, \quad H = (H_1, H_2, ..., H_n)$
$H_i(A, B) := max\{\sup_{a\in A} \inf_{b\in B} d_i(a, b), \sup_{b\in b} \inf_{a\in A} d_i(a, b)\}\ \forall A, B \in P(X)\ \forall i = \overline{1, n}.$

Definition 4. *If (X, d) is a g.m.s., an operator $f : X \to X$ is called C-weakly Picard iff f is weakly Picard and there exists $C \in \mathcal{M}_{n,n}(\mathbb{R})$ such that*
$d(x, f^\infty(x)) \leq Cd(x, f(x))\ \forall x \in X.$

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2. Main results

Theorem 1. *Let (X, d) be a complete g.m.s. and $f, g : X \to X$ two operators. We suppose that:*
(i) there exist $C, D \in \mathcal{M}_{n,n}(\mathbb{R})$ such that f is C-weakly Picard and g is D-weakly Picard;
(ii) there exists $\eta \in \mathbb{R}^n_+$ such that $d(f(x), g(x)) \leq \eta \ \forall x \in X$.
Then $H(F_f, F_g) \leq \max\{C\eta, D\eta\}$.

To prove this theorem we will use the next Lemma:

Lemma 1. *If (X, d) is a g.m.s. and $A,\ B \in P(X)$; $\eta, \zeta \in \mathbb{R}^n_+$ such that:*
$\forall a \in A \ \exists b \in B : \ d(a, b) \leq \eta$;
$\forall b \in B \ \exists a \in A : \ d(a, b) \leq \zeta$.
Then $H(A, B) \leq \max\{\eta, \zeta\}$.

Proof-Theorem 1:
Let $x \in F_g$; then:
$d(x, f^\infty(x)) \leq Cd(x, f(x)) = Cd(g(x), f(x)) \leq C\eta$.
By a similar argument, we have that $d(x, g^\infty(x)) \leq D\eta \ \forall x \in F_f$.
It follows from Lemma 1 that $H(F_f, F_g) \leq \max\{C\eta, D\eta\}$.□
If in Theorem 1 we take f, g A-orbitally contractions, we have:

Theorem 2. *Let (X, d) be a complete g.m.s. and $f, g : X \to X$ two orbitally continuous operators. We suppose that:*
(i) $\exists A \in \mathcal{M}_{n,n}(\mathbb{R})$, $A^k \xrightarrow[k\to\infty]{} 0$ (i.e. the matrix A converges to zero) such that
$d(f^2(x), f(x)) \leq Ad(f(x), x) \ \forall x \in X$ *and*
$d(g^2(x), g(x)) \leq Ad(g(x), x) \ \forall x \in X$;
(ii) there exists $\eta \in \mathbb{R}^n_+$ such that $d(f(x), g(x)) \leq \eta \ \forall x \in X$.
Then:
a) $F_f \neq \emptyset$ and $F_g \neq \emptyset$;
b) $H(F_f, F_g) \leq (I - A)^{-1}\eta$.

3. Applications

We will consider the following systems of integral equations with deviating argument:

$$x(t) = x(a) + \int_a^b K(t, s, x(s))ds \quad \forall t \in [a, b] \tag{1}$$

$$x(t) = x(a) + \int_a^b N(t, s, x(s))ds \quad \forall t \in [a, b] \tag{2}$$

where $K, N \in C([a, b] \times [a, b] \times \mathbb{R}^n, \mathbb{R}^n)$.
By Theorem 2 we have:

Theorem 3. *We suppose that:*
(i) $K(a, s, u) = 0 \in \mathbb{R}^n$ and $N(a, s, u) = 0 \in \mathbb{R}^n \ \forall s \in \mathbb{R} \ \forall u \in \mathbb{R}^n$;

(ii) there exists $\eta \in \mathbb{R}^n_+$ *such that*
$|K(t,s,u) - N(t,s,u)| \leq \eta \ \forall t,s \in [a,b] \ \forall u \in \mathbb{R}^n$;
(iii) there exists $L \in \mathcal{M}_{n,n}(\mathbb{R})$ *such that*
$|K(t,s,u) - K(t,s,v)| \leq L|u-v| \ \forall t,s \in [a,b] \ \forall u,v \in \mathbb{R}^n$ *and*
$|N(t,s,u) - N(t,s,v)| \leq L|u-v| \ \forall t,s \in [a,b] \ \forall u,v \in \mathbb{R}^n$;
(iv) the matrix $(b-a)L$ *converges to zero.*
If S_1 *and* S_2 *are the solutions sets of the systems (1) and (2) in* $C([a,b],\mathbb{R}^n)$ *then:*
a) $S_1 \neq \emptyset$ *and* $S_2 \neq \emptyset$;
b) $H_{\|\cdot\|}(S_1,S_2) \leq [I-(b-a)L](b-a)\eta$;
where we consider the space $C([a,b],\mathbb{R}^n)$ *with the generalized metric induced by the Tchebychev norm* $\|y\| := (\|y_1\|_{C[a,b]}, \|y_2\|_{C[a,b]}, ..., \|y_n\|_{C[a,b]}) \ \forall y \in C([a,b],\mathbb{R}^n)$ *and* $H_{\|\cdot\|}$ *is the related Pompeiu-Hausdorff functional.*

Proof. We consider the operators
$f,g : C([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ defined by

$$f(x)(t) = x(a) + \int_a^b K(t,s,x(s))ds \ \forall t \in [a,b] \ \forall x \in C([a,b];$$

$$g(x)(t) = x(a) + \int_a^b N(t,s,x(s))ds \ \forall t \in [a,b] \ \forall x \in C([a,b]$$

and we will apply the Theorem 2.
We have $f^2(x)(t) = x(a) + \int_a^b K(t,s,f(x(s))ds \ \forall t \in [a,b] \ \forall x \in C([a,b]$,
so $|f^2(x)(t) - f(x)(t)| \leq (b-a)L\|f(x) - x\| \Rightarrow$

$$\|f^2(x) - f(x)\| \leq \underbrace{(b-a)L}_{converges\ to\ 0} \|f(x) - x\|, \text{ so } F_f = S_1 \neq \emptyset.$$

By a similar argument, $\|g^2(x) - g(x)\| \leq (b-a)L\|f(x) - x\|$, so $F_g = S_2 \neq \emptyset$.
We also have $\|f(x) - g(x)\| \leq (b-a)\eta \ \forall x \in C([a,b])$.
We are in the conditions of the Theorem 2 $\Rightarrow H_{\|\cdot\|}(F_f,F_g) \leq [I-(b-a)L](b-a)\eta$, i.e. $H_{\|\cdot\|}(S_1,S_2) \leq [I-(b-a)L](b-a)\eta$.□

References

[1] I. A. Rus, S. Mureşan, *Data Dependence of the Fixed Points Set of Weakly Picard Operators*, Studia Univ. "Babeş-Bolyai", Mathematica, **43**(1998), Nr.1, 79-83.
[2] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, 2001.
[3] A. Petruşel, *Analiza operatorilor multivoci*, Univ. Cluj, 1996.

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ASYMPTOTIC FIXED POINT THEOREMS IN E-METRIC SPACES

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Abstract. In this paper we prove two asymptotical fixed point theorems in E-metric spaces. The first theorem is a variant of Ciric-Reich-Rus theorem in E-metric space, the next theorem is the asymptotic variant of this theorem.

Let E be a real linear space partially ordered by $\leq$ and let $E_+ = \{e \in E : e \geq 0\}$ be the positive cone of E. We consider on E a linear convergence, i.e., a convergence with: ([1])

1.) if $e_n = e$, $\forall n \in \mathbb{N} \Rightarrow \lim e_n = e$.
2.) if $\lim e_n = e$ implies $\lim e_{n'} = e$ for every subsequence $(e_{n'})$ of (e_n).
3.) $\lim e_n = e$ and $\lim f_n = f$ imply $\lim(e_n + f_n) = e + f$.
4.) $\lim e_n = e$ implies $\lim(r \cdot e_n) = r \cdot e$, $\forall r \in \mathbb{R}$.
5.) if $e_n \leq f_n$, $\forall n \in \mathbb{N}$ and $\lim e_n = e$, $\lim f_n = f$ then $e \leq f$.
6.) if $e_n \leq f_n \leq g_n$, $\forall n \in \mathbb{N}$ and $\lim e_n = \lim g_n = e$ then also $\lim f_n = e$.

Let X be a nonempty set and let E be an ordered linear space with a linear convergence. An *E-metric on X* is a mapping $d : X \times X \longrightarrow E_+$ subject to the usual axioms:

1.) $d(x, y) = 0_E$ if and only if $x = y$.
2.) $d(x, y) = d(y, x)$, $\forall x, y \in X$.
3.) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

By *E-metric space* we mean a nonempty set X with an E-metric on X. The ordered space E is briefly called the *metrizing space* for X.

A sequence (x_n) of elements of an E-metric space X is said *convergent toward* $x \in X$ (and we write $x_n \to x$) if $d(x_n, x) \to 0$ as $n \to \infty$.

A sequence (x_n) in X is said to be a *Cauchy sequence* if $d(x_n, x_m) \to 0$, $n, m \to \infty$.

The E-metric space X is said to be *sequentially complete*, if each Cauchy sequence in X converges to a point in X.

A subset Y of an E-metric space X is said to be bounded if the set $\{d(x, y) : x, y \in Y\}$ has an upper bound in E.

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We note $F_A := \{x \in X | A(x) = x\}$ - the fixed point set of A.

In this note we need the following results:

Theorem 1. (Ciric - Reich -Rus, [2], [3]**)** *Let (X, d) be a complet metric space and $A : X \longrightarrow X$ be an operator. Suppose that there exists the numbers a, b, c such that $0 \leq a + b + c < 1$, and let A be a map such that:*

$$d(A(x), A(y)) \leq a \cdot d(x, A(x)) + b \cdot d(y, A(y)) + c \cdot d(x, y), \quad \forall x,\, y \in X.$$

Then A has an unique fixed point.

Lemma 1. ([3]) *Let X be a nonempty set and $f : X \longrightarrow X$ a mapping. If there exists $k \in \mathbb{N}$ such that $F_{f^k} = \{x^*\}$, then $F_f = \{x^*\}$.*

The first result is the generalization of Theorem 1 in E-metric spaces:

Theorem 2. *Let X be a sequentially complete E-metric space. Let $S, T, R : E_+ \longrightarrow E_+$ are increasing operators and let $A : X \longrightarrow X$ be an operator which satisfy the following condition:*

$$d(A(x), A(y)) \leq S\, d(x, A(x)) + T\, d(y, A(y)) + R\, d(x, y), \quad x,\, y \in X.$$

Suppose that

(i) *$1_E - T$ is a bijection*

(ii) *there exists $x_0 \in X$ such that, $\sum_{n \in \mathbb{N}} [(1_E - T)^{-1}(S + R)]^n d(x_0, A(x_0))$ converges.*

Then A has an unique fixed point.

Proof. Let $y = A(x)$. Then

$$d(A(x), A^2(x)) \leq S\, d(x, A(x)) + T\, d(A(x), A^2(x)) + R\, d(x, A(x))$$

and we have that

$$(1_E - T)\, d(A(x), A^2(x)) \leq (S + R)\, d(x, A(x)).$$

Because $(1_E - T)$ is a bijection, we have

$$d(A(x), A^2(x)) \leq (1_E - T)^{-1} \cdot (S + R)\, d(x, A(x))$$

$$\ldots$$

$$d(A^{n+1}(x), A^n(x)) \leq (1_E - T)^{-1}(S + R)\, d(A^n(x), A^{n-1}(x)) \leq \ldots$$
$$\cdots \leq [(1_E - T)^{-1}(S + R)]^n\, d(x, A(x)), \quad \forall n \in \mathbb{N}^*.$$

We want to prove that the $(A^n x_0)_n$ is a Cauchy sequence:

$$d(A^{n+m}(x_0), A^n(x_0)) \leq d(A^{n+m}(x_0), A^{n+m-1}(x_0)) +$$
$$+ d(A^{n+m-1}(x_0), A^{n+m-2}(x_0)) + ... + d(A^{n+1}(x_0), A^n(x_0)) \leq$$
$$\leq [(1_E - T)^{-1}(S + R)]^{n+m-1}\, d(x_0, A(x_0)) +$$
$$+ [(1_E - T)^{-1}(S + R)]^{n+m-2}\, d(x_0, A(x_0)) + \ldots$$
$$\cdots + [(1_E - T)^{-1}(S + R)]^n\, d(x_0, A(x_0)) \to 0. \qquad \text{(ii)}$$

Because the sequence is Cauchy and X is sequencially complete we have that the sequence is convergent and let $x^* = \lim A^n x_0$.

We have

$$d(x^*, A(x^*)) \leq d(x^*, A^n(x_0)) + d(A^n(x_0), A(x^*)) \leq$$
$$\leq d(x^*, A^n(x_0)) + S\, d(A^{n-1}(x_0), A^n(x_0)) + T\, d(x^*, A(x^*)) + R\, d(A^{n-1}(x_0), x^*).$$

Hence

$$(1_E - T)d(x^*, A(x^*)) \leq d(x^*, A^n(x_0)) + S\, d(A^{n-1}(x_0), A^n(x_0)) + $$
$$+ R\, d(A^{n-1}(x_0), x^*)).$$
$$d(x^*, A(x^*)) \leq (1_E - T)^{-1}[d(x^*, A^n(x_0)) + S\, d(A^{n-1}(x_0), A^n(x_0)) +$$
$$+ R\, d(A^{n-1}(x_0), x^*))] \to 0.$$

By letting $n \to \infty$ we have $d(x^*, A(x^*)) = 0$, i.e. $F_A = \{x^*\}$ and $A^n(x_0) \to x^*$. □

The following theorem is the asymptotic variant of the Theorem 1 in E-metric spaces.

Theorem 3. *Let X be a sequentially complete E-metric space. Let $S, T, R : E_+ \longrightarrow E_+$ and let $A : X \longrightarrow X$ be a map for which there exists $k \in \mathbb{N}^*$ such that*

$$d(A^k x, A^k y) \leq S\, d(x, A^k x) + T\, d(y, A^k y) + R\, d(x, y), \quad \forall x,\, y \in X.$$

Suppose that:

(i) *$1_E - T$ is a bijection*

(ii) *there exists $x_0 \in X$ such that, $\sum_{n \in N} [(1_E - T)^{-1}(S + R)]^n d(x_0, A^k(x_0))$ converges.*

Then A has an unique fixed point.

Proof. We apply the Theorem 2 for the iterate A^k and we have that A^k has an unique fixed point. Now apply the lemma and we have that the operator A has an unique fixed point.

Remarks.

1. When $S = 0$ and $T = 0$, we have the asymptotic variant of Banach fixed point theorem in E-metric spaces [1].
2. Let $E = \mathbb{R}^n$ then we have an asymptotic variant of Perov fixed point theorem ([3]).

References

[1] E. de Pascale, G. Marino, P. Pietramala, *The use of the E-metric spaces in the search for fixed points*, Le Matematiche, Vol. **XLVIII** (1993) Fasc. II., 367-376.

[2] B. E. Rhoades, *A collection of contractive definitions*, Mathematics Seminar Notes, Vol. **6**(1978), 229-235.

[3] I. A. Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001.

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APPROXIMATION BY GENERALIZED BRASS OPERATORS

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Abstract. We establish direct and converse theorems for generalized Brass operators and for parameter dependent Brass - type operators, respectively.

1. Introduction

In the paper [8], D. D. Stancu has introduced and investigated a linear positive operator $L_{n,r} : C[0,1] \to C[0,1]$ defined by

$$(L_{n,r}f)(x) \;=\; \sum_{k=0}^{n-r} p_{n-r,k}(x) \;\left[\, (1-x)\; f\left(\frac{k}{n}\right) + x\; f\left(\frac{k+r}{n}\right) \right], \tag{1}$$

where $n > 2r \geq 4$ and $p_{n-r,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-r-k}$, $k = \overline{0, n-r}$. The operator $L_{n,2}$ has been given earlier by H. Brass in [4]. Stancu has established the convergence of the sequence $(L_{n,r})_{n>2r}$, the representation of the remainder in the approximation formula by means of the second - order divided differences and the estimate of the order of approximation using the classical moduli of continuity, respectively.

In what follows we give direct and converse theorems for the operator given above. The converse results will be of Berens - Lorentz type [3] and of strong converse inequality of type B, in the terminology of [7].

Furthermore, let us consider a new, parameter dependent linear positive operator $L^{\alpha}_{n,r} : C[0,1] \to C[0,1]$ defined by
$(L^{\alpha}_{n,r}f)(x) \;=$

$$= \sum_{k=0}^{n-r} w_{n-r,k}(x,\alpha) \cdot \left[\frac{1 - x(n-r-k)\alpha}{1+(n-r)\alpha} \cdot f\left(\frac{k}{n}\right) \;+\; \frac{x+k\alpha}{1+(n-r)\alpha} \cdot f\left(\frac{k+r}{n}\right) \right], \tag{2}$$

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where $n > 2r$ and

$$w_{n-r,k} = \begin{pmatrix} n-r \\ k \end{pmatrix} \cdot \frac{\Pi_{i=0}^{k-1}\ (x+i\alpha)\ \Pi_{j=0}^{n-r-k-1}\ (1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)},$$

where $k = \overline{0, n-r}$ and $\alpha \geq 0$ is a parameter which may depend only on the natural number n. In the case $\alpha = 0$, $L^0_{n,r}$ is the generalized Brass operator defined by (1). Similarly to (1), we shall prove direct and converse theorems for (2).

In the next sections we will use the weighted $K-$ functional for $f \in C[0,1]$ defined by

$$K_{2,\phi}(f,\delta) = \inf \left\{ \|f-g\| + \delta\ \|\phi^2 g''\| \ :\ g \in W^2_\infty(\phi) \right\}, \quad \delta \geq 0.$$

Here $\phi : [0,1] \to \mathbf{R}$ is an admissible step - weight function of the Ditzian - Totik modulus [1, pp. 8 - 9], $\|\cdot\|$ is the supremum norm on $C[0,1]$ and $W^2_\infty(\phi)$ consists of all functions $g \in C[0,1]$ such that g is twice continuously differentiable and $\|\phi^2 g''\|$ is finite. It is well - known that $K_{2,\phi}(f,\delta)$ and $\omega^2_\phi(f,\sqrt{\delta})$ are equivalent [1, p. 11, Theorem 2.1.1], where

$$\omega^2_\phi(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{x\pm h\phi(x)\in[0,1]} |\ f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))\ |$$

is the Ditzian - Totik modulus of smoothness of second order.

2. **Direct and converse theorems**

Our direct result is

Theorem 1. *Let $(L_{n,r})_{n>2r}$ be defined as in (1), $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ and $\phi : [0,1] \to \mathbf{R}$ an admissible step -weight function of the Ditzian - Totik modulus with ϕ^2 concave. Then*

$$|(L_{n,r}f)(x) - f(x)| \leq 4\ K_{2,\phi}\left(f, \frac{n+r(r-1)}{n^2} \cdot \frac{\varphi(x)^2}{\phi(x)^2} \right)$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$.

Proof. By [8, p. 214, Theorem 2.1] we have $L_{n,r}(t-x,x) = 0$ and

$$L_{n,r}((t-x)^2,x) = \frac{n+r(r-1)}{n^2} \cdot \varphi(x)^2$$

On the other hand, the operator $L_{n,r}$ is bounded as follows from

$$\begin{aligned} |(L_{n,r}f)(x)| &\leq \sum_{k=0}^{n-r} p_{n-r,k}(x) \cdot \left[(1-x) \left| f\left(\frac{k}{n}\right) \right| + x \left| f\left(\frac{k+r}{n}\right) \right| \right] \\ &\leq \|f\| \cdot \sum_{k=0}^{n-r} p_{n-r,k}(x) = \|f\| \end{aligned} \tag{3}$$

Now we use [2, p. 398, Theorem 1], obtaining the assertion of the theorem.

Corollary 1. *Let $L_{n,r}$, φ and ϕ be given as in Theorem* 1. *Then*

$$|(L_{n,r}f)(x) - f(x)| \ \leq \ C\ \omega_\phi^2 \left(f, \frac{\sqrt{n+r(r-1)}}{n} \cdot \frac{\varphi(x)}{\phi(x)} \right)$$

for $x \in [0,1]$ and $f \in C[0,1]$, where the constant C depends only on φ and ϕ.

Proof. It is a direct consequence of Theorem 1 and the equivalence between $K_{2,\phi}\left(f, \frac{n+r(r-1)}{n^2} \cdot \frac{\varphi(x)^2}{\phi(x)^2}\right)$ and $\omega_\phi^2\left(f, \frac{\sqrt{n+r(r-1)}}{n} \cdot \frac{\varphi(x)}{\phi(x)}\right)$.

In order to prove the next theorems we need some Bernstein type inequalities.

Lemma 1. *Let $\phi : [0,1] \to \mathbf{R}$ be an admissible step - weight function of the Ditzian - Totik modulus with ϕ^2 concave, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ and $n > 2r \geq 4$. Then for $f \in C[0,1]$*

$$\|\varphi^2 (L_{n,r}f)''\| \ \leq \ 4\ (n-r)\ \|f\| \tag{4}$$

and for smooth functions $g \in C^2[0,1]$

$$\|\varphi^2 (L_{n,r}g)''\| \ \leq \ C_1(r)\ \|\varphi^2 g''\|, \tag{5}$$

$$\|\phi^2 (L_{n,r}g)''\| \ \leq \ C_1(r)\ \|\phi^2 g''\|, \tag{6}$$

where $C_1(r) = 50r^2 + 34r + 17$.

Proof. Let

$$(L^1_{n,r}f)(x) \ = \ \sum_{k=0}^{n-r} p_{n-r,k}(x)\ f\left(\frac{k}{n}\right), \quad x \in [0,1]$$

and

$$(L^2_{n,r}f)(x) \ = \ \sum_{k=0}^{n-r} p_{n-r,k}(x)\ f\left(\frac{k+r}{n}\right), \quad x \in [0,1].$$

Then

$$(L_{n,r}f)(x) \ = \ (1-x) \cdot (L^1_{n,r}f)(x) \ + \ x \cdot (L^2_{n,r}f)(x), \tag{7}$$

$x \in [0,1]$. Furthermore, let $\lambda^i_{n-r,k} : C[0,1] \to \mathbf{R}$ $(i = \overline{1,2})$ be positive linear functionals defined by $\lambda^1_{n-r,k}(f) = f\left(\frac{k}{n}\right)$ and $\lambda^2_{n-r,k}(f) = f\left(\frac{k+r}{n}\right)$, where $k = \overline{0, n-r}$ and $f \in C[0,1]$. Then $\lambda^1_{n-r,k}(1) = \lambda^2_{n-r,k}(1) = 1$. Moreover, if Π_1 denotes the set of all algebraic polynomials of degree at most one then $L^i_{n,r}(\Pi_1) \subset \Pi_1$ for $i = \overline{1,2}$. Therefore, by [2, p. 414, Lemma 3] we obtain

$$\varphi(x)^2\ |(L^i_{n,r}f)''(x)| \ \leq \ 2\ (n-r)\ \|f\| \tag{8}$$

for $x \in [0,1]$, $n > 2r$ and $i = \overline{1,2}$.

On the other hand, in view of (7) we have

$$(L_{n,r}f)''(x) = -2(L^1_{n,r}f)'(x) + 2(L^2_{n,r}f)'(x) + (1-x)(L^1_{n,r}f)''(x) + x(L^2_{n,r}f)''(x). \tag{9}$$

Using [6, p. 305, (2.1)] we obtain

$$(L^1_{n,r}f)'(x) \ = \ (n-r) \ \sum_{k=0}^{n-r-1} \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right] \cdot p_{n-r-1,k}(x)$$

and

$$(L^2_{n,r}f)'(x) \ = \ (n-r) \sum_{k=0}^{n-r-1} \left[f\left(\frac{k+r+1}{n}\right) - f\left(\frac{k+r}{n}\right)\right] \cdot p_{n-r-1,k}(x).$$

Hence

$$\varphi(x)^2 \ |(L^i_{n,r}f)'(x) \ \le \ \frac{1}{2} \ (n-r) \ \|f\|, \tag{10}$$

$i=\overline{1,2}$. Then, by (9), (8) and (10) we obtain

$$\varphi(x)^2 \ |(L_{n,r}f)''(x)| \le (n-r)\|f\| + (n-r)\|f\| + (1-x)\cdot 2(n-r)\|f\|$$
$$+x \cdot 2(n-r)|f\| = 4(n-r)\|f\|,$$

which implies (4).

Furthermore,

$$\lambda^1_{n-r,k}\left(\left(t-\frac{k}{n-r}\right)^2\right) = \left(\frac{k}{n}-\frac{k}{n-r}\right)^2 = r^2\cdot\left(\frac{k}{n(n-r)}\right)^2 \le r^2\cdot\left(\frac{1}{n}\right)^2$$

and

$$\begin{aligned}\lambda^2_{n-r,k}\left(\left(t-\frac{k}{n-r}\right)^2\right) \ &= \ \left(\frac{k+r}{n}-\frac{k}{n-r}\right)^2 = \left[\left(\frac{k}{n}-\frac{k}{n-r}\right)+\left(\frac{r}{n}\right)\right]^2 \\ &\le \ 2\left[\left(\frac{k}{n}-\frac{k}{n-r}\right)^2\left(\frac{r}{n}\right)^2\right] \le (2r)^2\cdot\left(\frac{1}{n}\right)^2\end{aligned}$$

for $n>2r$ and $k=\overline{0,n-r}$. Thus, in view of [2, p. 144, Lemma 3] we have for $g\in C^2[0,1]$:

$$\|\phi^2(L^i_{n,r}g)''\| \ \le \ C'(r) \ \|\phi^2 g''\|, \tag{11}$$

$i=\overline{1,2}$, where $C'(r)=48r^2+32r+8$. By (9), we have
$\phi(x)^2\cdot|(L_{n,r}g)''(x)| \ \le$

$$\begin{aligned} &\le \ \ 2\ \phi(x)^2\cdot|(L^2_{n,r}g)'(x)-(L^1_{n,r}g)'(x)| + \\ &+ \ \ (1-x)\cdot\phi(x)^2 \ |(L^1_{n,r}g)''(x)| + x\cdot\phi(x)^2 \ |(L^2_{n,r}g)''(x)| \end{aligned} \tag{12}$$

Therefore, in view of (11), we have to estimate $\phi(x)^2\cdot|(L^2_{n,r}g)'(x)-(L^1_{n,r}g)'(x)|$. Using Taylor's formulas

$$g\left(\frac{k+1}{n}\right) = g(x)\left(\frac{k+1}{n}-x\right) \ g'(x) + \int_x^{\frac{k+1}{n}} \left(\frac{k+1}{n}-u\right) \ g''(u) \ du$$

and

$$g\left(\frac{k}{n}\right) = g(x) + \left(\frac{k}{n}-x\right) \ g'(x) + \int_x^{\frac{k}{n}} \left(\frac{k}{n}-u\right) \ g''(u) \ du,$$

we obtain

$$
\begin{aligned}
(L^1_{n,r}g)'(x) \ &= \\
&= \ (n-r) \sum_{k=0}^{n-r-1} \left[\left(g\left(\frac{k+1}{n}\right) - g(x)\right) - \left(g\left(\frac{k}{n}\right) - g(x)\right)\right] \cdot p_{n-r-1,k}(x) \\
&= \ (n-r) \left\{ g'(x) \sum_{k=0}^{n-r-1} \left(\frac{k+1}{n} - x\right) p_{n-r-1,k}(x) + \right. \\
&\quad + \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \int_x^{\frac{k+1}{n}} \left(\frac{k+1}{n} - u\right) g''(u) \, du \ - \\
&\quad - g'(x) \sum_{k=0}^{n-r-1} \left(\frac{k}{n} - x\right) p_{n-r-1,k}(x) \ - \\
&\quad \left. - \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - u\right) g''(u) \, du \right\}
\end{aligned}
$$

But, if

$$
(B_{n-r-1}f)(x) \ = \ \sum_{k=o}^{n-r-1} p_{n-r-1,k}(x) \cdot f\left(\frac{k}{n-r-1}\right), \quad f \in C[0,1]
$$

then $B_{n-r-1}(t-x,x) = 0$ and therefore

$$
\sum_{k=0}^{n-r-1} \left(\frac{k+1}{n} - x\right) \cdot p_{n-r-1,k}(x) \ = \ \frac{1}{n} - \frac{r+1}{n} \cdot x
$$

and

$$
\sum_{k=0}^{n-r-1} \left(\frac{k}{n} - x\right) \cdot p_{n-r-1,k}(x) \ = \ -\frac{r+1}{n} \cdot x,
$$

respectively. Thus

$$
\begin{aligned}
(L^1_{n,r}g)'(x) \ &= \\
&= \ (n-r) \cdot \left\{ \frac{1}{n} \cdot g'(x) + \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \int_x^{\frac{k+1}{n}} \left(\frac{k+1}{n} - u\right) g''(u) \, du \ - \right. \\
&- \left. \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - u\right) g''(u) \, du \right\} \qquad (13)
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
(L^2_{n,r}g)'(x) \ &= \\
&= \ (n-r) \cdot \left\{ \frac{1}{n} \cdot g'(x) + \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \int_x^{\frac{k+r+1}{n}} \left(\frac{k+r+1}{n} - u\right) g''(u) \, du \right. \\
&- \left. \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \int_x^{\frac{k+r}{n}} \left(\frac{k+r}{n} - u\right) g''(u) \, du \right\} \qquad (14)
\end{aligned}
$$

Thus (13) and (14) imply
$(L^2_{n,r}g)'(x) - (L^1_{n,r}g)'(x) \ =$

$$= \ (n-r)\cdot\left\{\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+r+1}{n}}\left(\frac{k+r+1}{n}-u\right)g''(u)\,du\ - \right.$$
$$-\ \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+r}{n}}\left(\frac{k+r}{n}-u\right)g''(u)\,du\ -$$
$$-\ \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-u\right)g''(u)\,du\ +$$
$$\left. +\ \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-u\right)g''(u)\,du\right\} \tag{15}$$

So we have to estimate $|\int_x^t (t-u)\ g''(u)\ du|$. Because ϕ^2 is concave, using [2, p. 399, (5)] we obtain

$$\Big|\int_x^t (t-u)\ g''(u)\ du\Big| \ \le\ \Big|\int_x^t |t-u|\cdot|g''(u)|\ du\Big| \le \Big|\int_x^t \frac{|t-u|}{\phi(u)^2}\ du\Big|\cdot\|\phi^2 g''\|$$
$$\le\ \Big|\int_x^t \frac{|t-x|}{\phi(x)^2}\ du\Big|\cdot\|\phi^2 g''\| \ \le\ \frac{(t-x)^2}{\phi(x)^2}\cdot\ \|\phi^2 g''\|$$

Hence

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\ \cdot\ \Big|\int_x^{\frac{k+r+1}{n}}\left(\frac{k+r+1}{n}-u\right)g''(u)\,du\Big| \le$$
$$\le\ \frac{\|\phi^2 g''\|}{\phi(x)^2}\cdot\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\left(\frac{k+r+1}{n}-x\right)^2,$$

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\ \cdot\ \Big|\int_x^{\frac{k+r}{n}}\left(\frac{k+r}{n}-u\right)g''(u)\,du\Big| \le$$
$$\le\ \frac{\|\phi^2 g''\|}{\phi(x)^2}\cdot\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\left(\frac{k+r}{n}-x\right)^2,$$

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\ \cdot\ \Big|\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-u\right)g''(u)\,du\Big| \le$$
$$\le\ \frac{\|\phi^2 g''\|}{\phi(x)^2}\cdot\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\left(\frac{k+1}{n}-x\right)^2$$

and

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \quad \cdot \quad \left| \int_x^{\frac{k}{n}} \left(\frac{k}{n} - u\right) \; g''(u)\; du \right| \le$$

$$\le \quad \frac{\|\phi^2 g''\|}{\phi(x)^2} \cdot \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \; \left(\frac{k}{n} - x\right)^2,$$

respectively. Using again $B_{n-r-1}(t-x,x) = 0$ and $B_{n-r-1}(t^2,x) = x^2 + \frac{x(1-x)}{n-r-1}$ we obtain

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \; \left(\frac{k+r+1}{n} - x\right)^2 \; =$$

$$\begin{aligned}
&= \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \; \left[\left(\frac{k}{n}\right) + 2\frac{k}{n} \cdot \left(\frac{r+1}{n} - x\right) + \left(\frac{r+1}{n} - x\right)^2\right] \\
&= \left(\frac{n-r-1}{n}\right)^2 \cdot \left[x^2 + \frac{x(1-x)}{n-r-1}\right] + 2 \cdot \frac{n-r-1}{n} \cdot \left(\frac{r+1}{n} - x\right) \cdot x \; + \\
&+ \left(\frac{r+1}{n} - x\right)^2 \\
&= \left(\frac{r+1}{n}\right)^2 \cdot x^2 - 2\left(\frac{r+1}{n}\right)^2 \cdot x + \left(\frac{r+1}{n}\right)^2 + \left(\frac{n-r-1}{n}\right)^2 \cdot \frac{x(1-x)}{n-r-1} \\
&\le \left(\frac{r+1}{n}\right)^2 \cdot (1-x)^2 + \frac{1}{4(n-r-1)} \;\le\; \left(\frac{r+1}{n}\right)^2 + \frac{1}{4} \cdot \frac{1}{n-r-1} \\
&= \frac{1}{n} \cdot \left[\frac{(r+1)^2}{n} + \frac{1}{4} \cdot \frac{n}{n-r-1}\right] \;\le\; \frac{1}{n} \cdot \left[\frac{1}{4} \cdot (r+1)^2 + 1\right], \qquad (16)
\end{aligned}$$

because

$$\sup \left\{\frac{n}{n-r-1} \;:\; n > 2r\right\} \;<\; \frac{2r}{2r-r-1} \;\le\; 4,$$

where $n > 2r \ge 4$. With similar arguments we obtain

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \cdot \left(\frac{k+r}{n} - x\right)^2 \; =$$

$$\begin{aligned}
&= \left(\frac{r+1}{n} \cdot x - \frac{r}{n}\right)^2 + \left(\frac{n-r-1}{n}\right)^2 \cdot \frac{x(1-x)}{n-r-1} \\
&\le \left(\frac{r}{n}\right)^2 + \frac{1}{4} \cdot \frac{1}{n-r-1} \;\le\; \frac{1}{n} \cdot \left(\frac{1}{4} \cdot r^2 + 1\right), \qquad (17)
\end{aligned}$$

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \; \left(\frac{k+1}{n} - x\right)^2 \; =$$

$$
\begin{aligned}
&= \left(\frac{r+1}{n}\cdot x-\frac{1}{n}\right)^2+\left(\frac{n-r-1}{n}\right)^2\cdot\frac{x(1-x)}{n-r-1}\\
&\le \left(\frac{r}{n}\right)^2+\frac{1}{4}\cdot\frac{1}{n-r-1} \;\le\; \frac{1}{n}\cdot\left(\frac{1}{4}\cdot r^2+1\right) \qquad (18)
\end{aligned}
$$

and

$$
\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\;\left(\frac{k}{n}-x\right)^2 \;=
$$

$$
\begin{aligned}
&= \left(\frac{r+1}{n}\right)^2\cdot x^2+\left(\frac{n-r-1}{n}\right)^2\cdot\frac{x(1-x)}{n-r-1}\\
&\le \left(\frac{r+1}{n}\right)^2+\frac{1}{4}\cdot\frac{1}{n-r-1} \;\le\; \frac{1}{n}\cdot\left[\frac{1}{4}(r+1)^2+1\right]. \qquad (19)
\end{aligned}
$$

Now, in view of (15), (16), (17), (18) and (19) we obtain
$2\;\phi(x)^2\cdot|(L^2_{n,r}g)'(x)-(L^1_{n,r}g)'(x)|\;\le$

$$
\begin{aligned}
&\le\; 2\;\frac{n-r}{n}\cdot\left\{\frac{1}{2}(r+1)^2+2+\frac{1}{2}r^2+2\right\}\cdot\|\phi^2g''\|\\
&\le\; \big(2r^2+2r+9\big)\cdot\|\phi^2g''\|.
\end{aligned}
$$

Hence, by (12) and (11) we get
$\phi(x)^2\;|(L_{n,r}g)''(x)|\;\le$

$$
\begin{aligned}
&\le\; (2r^2+2r+9)\cdot\|\phi^2g''\|+(1-x)\cdot C'(r)\;\|\phi^2g''\|+x\cdot C'(r)\;\|\phi^2g''\|\\
&=\; (50r^2+34r+17)\cdot\|\phi^2g''\|.
\end{aligned}
$$

This means that $\|\phi^2(L_{n,r}g)''\|\le C_1(r)\cdot\|\phi^2g''\|$, which was to be proved at (6).
If $\phi\equiv\varphi$ then we obtain (5), which completes the proof of lemma.

Remark 1. *If $\phi\equiv\varphi$ then, by Corollary 2, we have*

$$
\|L_{n,r}f-f\|\;\le\;C\;\omega^2_\varphi\left(f,\frac{\sqrt{n+r(r-1)}}{n}\right). \qquad (20)
$$

Thus our first converse theorem will constitute an inverse of (20). More precisely we have

Theorem 2. *If $f\in C[0,1]$ and $k>2r$, $n>2r$, $r\ge 2$ then we have*

$$
K_{2,\varphi}\left(f,\frac{n+r(r-1)}{n^2}\right)\;\le\;\|L_{k,r}f-f\|+C\cdot\frac{k}{n}\cdot K_{2,\varphi}\left(f,\frac{k+r(r-1)}{k^2}\right),
$$

where the constant C depends only on r (it can be chosen as $(r+1)C_1(r)$).

Proof. By Lemma 3 : (4) − (5) we obtain

$$K_{2,\varphi}\left(f,\tfrac{n+r(r-1)}{n^2}\right) \le$$

$$\begin{aligned}
&\le \|f-L_{k,r}f\| + \frac{n+r(r-1)}{n^2}\cdot\|\varphi^2(L_{k,r}f)''\| \\
&\le \|f-L_{k,r}f\| + \frac{n+r(r-1)}{n^2}\cdot\left\{\varphi^2(L_{k,r}(f-g))''\| + \|\varphi^2(L_{k,r}g)''\|\right\} \\
&\le \|f-L_{k,r}f\| + \frac{n+r(r-1)}{n^2}\cdot\left\{4(k-r)\|f-g\| + C_1(r)\cdot\|\varphi^2 g''\|\right\} \\
&= \|f-L_{k,r}f\| + \frac{n+r(r-1)}{n}\cdot\frac{k-r}{n}\cdot\left\{4\|f-g\| + C_1(r)\cdot\frac{1}{k-r}\cdot\|\varphi^2 g''\|\right\} \\
&\le \|f-L_{k,r}f\| + \frac{r+1}{2}\cdot\frac{k}{n}\cdot\left\{4\|f-g\| + C_1(r)\cdot 2\cdot\frac{k+r(r-1)}{k^2}\cdot\|\varphi^2 g''\|\right\} \\
&\le \|L_{k,r}f-f\| + C\cdot\frac{k}{n}\cdot\left\{\|f-g\| + \frac{k+r(r-1)}{k^2}\cdot\|\varphi^2 g''\|\right\}.
\end{aligned}$$

Now taking infimum over all $g \in C^2[0,1]$ we obtain the assertion of our theorem.

Remark 2. *By Corollary 2, the implication*

$$\omega_\phi^2(f,\delta) = O(\delta^\alpha) \Rightarrow |(L_{n,r}f)(x)-f(x)| \le C\left(\frac{\sqrt{n+r(r-1)}}{n}\cdot\frac{\varphi(x)}{\phi(x)}\right)^\alpha$$

holds true for $\alpha \in (0,2)$.

The converse result of Berens - Lorentz type is included in the next theorem

Theorem 3. *Let* $(L_{n,r})_{n>2r}$ *be defined by (1),* $\varphi(x)\sqrt{x(1-x)}$, $x \in [0,1]$ *and* $\phi : [0,1] \to \mathbf{R}$ *an admissible step - weight function of the Ditzian - Totik modulus with* ϕ^2 *and* φ^2/ϕ^2 *concave functions on* $[0,1]$. *Then for* $f \in C[0,1]$ *and* $\alpha \in (0,2)$ *the pointwise approximation*

$$|(L_{n,r}f)(x)-f(x)| \le C\left(\frac{\sqrt{n+r(r-1)}}{n}\cdot\frac{\varphi(x)}{\phi(x)}\right)^\alpha,$$

$x \in [0,1]$ *implies* $\omega_\phi^2(f,\delta) \le C\,\delta^\alpha$, $\delta > 0$.

Proof. We mention that $C > 0$ denotes a constant in this theorem which may depends only on r and it can be different at each occurrence.

The statement of the theorem results from [2, p. 410, Theorem 3] with slight modification using Lemma 3. Indeed, because $n > 2r \ge 4$ we hqve $\frac{n+r(r-1)}{n} < \frac{r+1}{n}$. Thus

$$|(L_{n,r}f)(x)-f(x)| \le C\left(\frac{r+1}{2}\right)^{\alpha/2}\cdot\left(n^{-1/2}\cdot\frac{\varphi(x)}{\phi(x)}\right)^\alpha.$$

By Lemma 3 : (4) we have $|\varphi^2(L_{n,r}f)''\| \le 4n\|f\|$ for $f \in C[0,1]$. Using (6) and step by step the proof of [2, p. 410, Theorem 3] we obtain

$$\omega_\phi^2(f,t) \le C\left(\delta^\alpha + \frac{t^2}{\delta^2}\cdot\omega_\phi^2(f,\delta)\right), \quad 0 < t \le \delta$$

which yields the assertion of the theorem by the well - known Berens - Lorentz lemma [3].

To prove the strong converse inequality of type B for $L_{n,r}$ we need another lemmas.

Lemma 2. *Let* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *and* $n > 2r \geq 4$. *Then for* $f \in C[0,1]$

$$\|\varphi^3 (L_{n,r} f)'''\| \ \leq \ C_2 \ n^{3/2} \|f\| \tag{21}$$

and for smooth functions $g \in C^2[0,1]$

$$\|\varphi^3 (L_{n,r} g)'''\| \ \leq \ C_3(r) n^{1/2} \|\varphi^2 g''\|, \tag{22}$$

where $C_2 = \sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 11$ *and* $C_3(r) = 3C'(r) + 3\sqrt{2} = 144r^2 + 96r + 24 + 3\sqrt{2}$.

Proof. By (9) we have

$$(L_{n,r} f)'''(x) = -3(L^1_{n,r} f)''(x) + 3(L^2_{n,r} f)''(x) + (1-x)(L^1_{n,r} f)'''(x) + x(L^2_{n,r} f)'''(x).$$

Then

$$\begin{aligned} \varphi(x)^3 \cdot |(L_{n,r} f)'''(x)| \ &\leq \ 3\varphi(x)^3 |(L^1_{n,r} f)''(x)| + 3\varphi(x)^3 |(L^2_{n,r} f)''(x)| \\ &+ \ (1-x)\varphi(x)^3 |(L^1_{n,r} f)'''(x)| + x\varphi(x)^3 |(L^2_{n,r} f)'''(x)| \end{aligned} \tag{23}$$

Using (8) we obtain

$$\varphi(x)^3 |(L^i_{n,r} f)''(x)| \leq 2(n-r)\varphi(x) \|f\| \leq (n-r)\|f\| \tag{24}$$

for $x \in [0,1]$, $n > 2r$ and $i = \overline{1,2}$.

Furthermore, by means of the expressions

$$T_{n,s}(x) = \sum_{k=0}^{n} (k - nx)^s \ p_{n,k}(x), \quad n = 1, 2, \ldots, \quad s = 0, 1, 2, \ldots$$

we have the following estimates (see [6, pp. 303 - 304] and [7, p.128, Lemma 9.4.4]]) : $T_{n,2}(x) = n\varphi(x)^2$, $T_{n,4}(x) \leq 11n^2\varphi(x)^4$ and $T_{n,6}(x) \leq 61n^3\varphi(x)^6$, where $x \in [1/n, 1-1/n]$ and $n \geq 2$. In this case $\varphi(x) \geq \frac{1}{\sqrt{2n}}$, $x \in [1/n, 1-1/n]$. Then, for the Bernstein polynomials

$$(B_n f)(x) \ = \ \sum_{k=0}^{n} \ p_{n,k}(x) \ f\left(\frac{k}{n}\right), \quad f \in C[0,1]$$

and for $x \in [1/n, 1-1/n]$ we have

$$\varphi(x)^3 \cdot |(B_n f)'''(x)| \ =$$

$$= \frac{1}{\varphi(x)^3} \cdot \Big| \sum_{k=0}^{n} \ f\left(\frac{k}{n}\right)(k-nx)^3 p_{n,k}(x) - 3(1-2x)\sum_{k=0}^{n} \ f\left(\frac{k}{n}\right)(k-nx)^2 p_{n,k}(x) \ -$$

$$-(3nx(1-x) - 2x(1-x) + 1) \ \sum_{k=0}^{n} \ f\left(\frac{k}{n}\right)(k-nx) p_{n,k}(x) + 2nx(1-x)(1-2x) \ \Big|$$

$$\leq \frac{\|f\|}{\varphi(x)^3} \cdot \Big\{ (T_{n,6}(x))^{1/2} + 3|1-2x| \, (T_{n,4}(x))^{1/2}$$

$$+ |3n\varphi(x)^2 - 2\varphi(x)^2 + 1| \left(T_{n,2}(x)\right)^{1/2} + 2n|1-2x| \cdot \varphi(x)^2 \Big\}$$

$$\leq \frac{\|f\|}{\varphi(x)^3} \cdot \left\{ \sqrt{61} n^{3/2} \varphi(x)^3 + 3\sqrt{11} n \varphi(x)^2 (3n\varphi(x)^2 + 1) n^{1/2} \varphi(x) + 2n\varphi(x)^2 \right\}$$

$$\leq \|f\| \cdot \left\{ \sqrt{61} n^{3/2} + 3\sqrt{22} n^{3/2} + 5n^{3/2} + 2\sqrt{2} n^{3/2} \right\}$$

$$= \left(\sqrt{61} + 3\sqrt{22} + 5 + 2\sqrt{2} \right) n^{3/2} \|f\|. \tag{25}$$

On the other hand, by [1, p. 125, (9.4.3)] we have for $x \in [0, 1/n] \cup [1-1/n, 1]$ and $f \in C[0,1]$:

$$\varphi(x)^3 |(B_n f)'''(x)| \leq n^{-3/2} \cdot \left| n(n-1)(n-2) \sum_{k=0}^{n-3} \left[f\left(\frac{k+3}{n}\right) - 3f\left(\frac{k+2}{n}\right) + \right.\right.$$

$$\left.\left. + \quad 3f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right] p_{n-3,k}(x) \right| \leq 8n^{3/2} \|f\|. \tag{26}$$

Therefore, in view of (25) and (26) we get

$$\varphi(x)^3 |(B_n f)'''(x)| \leq (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|f\| \tag{27}$$

for $f \in C[0,1]$ and $x \in [0,1]$.

Moreover, $(L^1_{n,r} f)(x) = (B_{n-r} g^1_n)(x)$ and $(L^2_{n,r} f)(x) = (B_{n-r} g^2_n)(x)$, where $g^1_n(x) = f\left(\frac{n-r}{n} \cdot x\right)$, $x \in [0,1]$ and $g^2_n(x) = f\left(\frac{n-r}{n} \cdot x + \frac{r}{n}\right)$, $x \in [0,1]$, respectively. Then, by (27) we obtain

$$\begin{aligned} \varphi(x)^3 |(L^1_{n,r} f)'''(x)| &\leq (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|g^1_n\| \\ &\leq (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|f\| \end{aligned}$$

and

$$\begin{aligned} \varphi(x)^3 |(L^1_{n,r} f)'''(x)| &\leq (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|g^2_n\| \\ &\leq (\sqrt{61} + 3\sqrt{22} + 3\sqrt{2} + 5) n^{3/2} \|f\|. \end{aligned}$$

Hence, by (23) and (24) we have

$$\begin{aligned} \varphi(x)^3 |(L_{n,r} f)'''(x)| &\leq 6n\|f\| + (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|f\| \\ &\leq C_2 n^{3/2} \|f\|, \end{aligned}$$

which was to be proved.

For (22) we use [7, p. 87, Lemma 8.4] :

$$\|\varphi^3 (B_n g)'''\| \leq \frac{3}{\sqrt{2}} n^{1/2} \|\varphi^2 g''\|.$$

Hence, by (23), replacing f by g, and (11) with $\phi \equiv \varphi$ we obtain

$$\varphi(x)^3 |(L_{n,r} g)'''(x)| \leq$$

$$\leq 3C'(r) \|\varphi^2 g''\| + (1-x) \cdot \varphi(x)^3 |(B_{n-r} g^1_n)'''(x)| + x \cdot \varphi(x)^3 |(B_{n-r} g^2_n)'''(x)|$$

$$\leq 3C'(r) \|\varphi^2 g''\| + (1-x) \cdot \frac{3}{\sqrt{2}} (n-r)^{1/2} \cdot \|\varphi^2 (g^1_n)''\| + x \cdot \frac{3}{\sqrt{2}} (n-r)^{1/2} \cdot \|\varphi^2 (g^2_n)''\|$$

$$\leq 3C'(r)\|\varphi^2 g''\| + (1-x)\cdot\frac{3}{\sqrt{2}}(n-r)^{1/2}\cdot\left(\frac{n-r}{n}\right)^2\cdot\|\varphi^2 g''\|+$$

$$+x\cdot\frac{3}{\sqrt{2}}(n-r)^{1/2}\cdot\left(\frac{n-r}{n}\right)^2\cdot\|\varphi^2 g''\| \;\leq\; (3C'(r)+3\sqrt{2})n^{1/2}\|\varphi^2 g''\|.$$

Hence $\|\varphi^3(L_{n,r}g)'''\| \;\leq\; C_3(r)n^{1/2}\|\varphi^2 g''\|$, which completes the proof of the lemma.

Lemma 3. *Let* $(L_{n,r})_{n>2r}$ *be defined by (1),* $\varphi(x) = \sqrt{x(1-x)}$, $x\in[0,1]$ *and* $a>0$, $E_{a,n} = \left\{x_0\in[0,1] \mid x_0 \pm an^{-1/2}\varphi(x_0)\in[0,1]\right\}$,

$$g_{M,n,x_0}(t) = \begin{cases} (t-x_0)^2, & \text{if} \quad |t-x_0| \geq Mn^{-1/2}\varphi(x_0) \\ 0, & \text{otherwise.}\end{cases}$$

Then $(L_{n,r}g_{M,n,x_0})(x_0)/(n^{-1}\varphi(x_0)^2) \;\to\; 0$ *as* $M\to\infty$ *uniformly in* n *and* $x_0\in E_{a,n}$.

Proof. Simple computations show, if $x_0\in E_{a,n}$ then $x_0\in\left[\frac{a^2}{n+a^2}, 1-\frac{a^2}{n+a^2}\right]$. This means that

$$\sqrt{n}\varphi(x_0) \;\geq\; \frac{a}{1+a^2}. \tag{28}$$

Therefore, by (7) we obtain

$$\frac{n}{\varphi(x_0)^2}\cdot\left(L_{n,r}g_{m,n,x_0}\right)(x_0) \;=$$

$$= \frac{n}{\varphi(x_0)^2}\cdot\left\{(1-x_0)\sum_{\left|\frac{k}{n}-x_0\right|\geq Mn^{-1/2}\varphi(x_0)} p_{n-r,k}(x_0)\left(\frac{k}{n}-x_0\right)^2 + \right.$$

$$\left. + x_0\sum_{\left|\frac{k+r}{n}-x_0\right|\geq Mn^{-1/2}\varphi(x_0)} p_{n-r,k}(x_0)\left(\frac{k+r}{n}-x_0\right)^2\right\}$$

$$\leq \frac{n}{\varphi(x_0)^2}\cdot\left\{(1-x_0)\sum_{k=0}^{n-r}\frac{1}{M^2}\cdot\frac{n}{\varphi(x_0)^2}\cdot p_{n-r,k}(x_0)\left(\frac{k}{n}-x_0\right)^4 + \right.$$

$$\left. + x_0\sum_{k=0}^{n-r}\frac{1}{M^2}\cdot\frac{n}{\varphi(x_0)^2}\cdot p_{n-r,k}(x_0)\left(\frac{k+r}{n}-x_0\right)^4\right\}$$

$$= \frac{1}{M^2}\cdot\left(\frac{n}{\varphi(x_0)^2}\right)^2\cdot\left\{(1-x_0)\left[\frac{1}{n^4}\cdot T_{n-r,4}(x_0) - 4\cdot\frac{rx_0}{n^4}\cdot T_{n-r,3}(x_0) + \right.\right.$$

$$\left. + 6\cdot\frac{(rx_0)^2}{n^4}\cdot T_{n-r,2}(x_0) + \frac{(rx_0)^4}{n^4}\right] + x_0\left[\frac{1}{n^4}\cdot T_{n-r,4}(x_0) - \right.$$

$$\left.\left. - 4\cdot\frac{r(1-x_0)}{n^4}\cdot T_{n-r,3}(x_0) + 6\cdot\frac{(r(1-x_0))^2}{n^4}\cdot T_{n-r,2}(x_0) + \frac{(r(1-x_0))^4}{n^4}\right]\right\}$$

$$= \frac{1}{M^2}\cdot\left(\frac{n}{\varphi(x_0)^2}\right)^2\cdot\left\{\frac{1}{n^4}\cdot T_{n-r,4}(x_0) - 8\cdot\frac{r}{n^4}\cdot\varphi(x_0)^2\cdot T_{n-r,3}(x_0) + \right.$$

$$+ 6 \cdot \frac{r^2}{n^4} \cdot \varphi(x_0)^2 \cdot T_{n-r,2}(x_0) + \frac{r^4}{n^4} \cdot \varphi(x_0)^2 (1 - 3\varphi(x_0)^2) \Big\} .$$

Hence, by [1, p. 128, Lemma 9.4.4] and (28) we obtain
$\frac{n}{\varphi(x_0)^2} \cdot (L_{n,r} g_{M,n,x_0})(x_0) \ \leq$

$$\begin{aligned}
&\leq \ \frac{1}{M^2} \cdot \left(\frac{n}{\varphi(x_0)^2}\right)^2 \cdot \frac{C}{n^4} \cdot \Big\{ (n-r)^2 \varphi(x_0)^4 + 8r\varphi(x_0)^2 \cdot (T_{n-r,6}(x_0))^{1/2} \ + \\
&+ \ 6r^2 \cdot \varphi(x_0)^2 (n-r) \varphi(x_0)^2 + r^4 \varphi(x_0)^2 (1 + 3\varphi(x_0)^2) \Big\} \\
&\leq \ \frac{C}{M^2} \cdot \frac{1}{n^2 \varphi(x_0)^2} \cdot \Big\{ n^2 \varphi(x_0)^4 + 8r^2 n^{3/2} \varphi(x_0)^5 \ + \\
&+ \ 6r^2 \cdot n\varphi(x_0)^4 + r^4 \varphi(x_0)^2 + 3r^4 \varphi(x_0)^4 \Big\} \ \leq \ \frac{C}{M^2} \ \to \ 0
\end{aligned}$$

as $M \to \infty$. (Here $C > 0$ denotes an absolute constant which can depends only on r and it can be different at each occurrence).

Remark 3. *For $n > 2r$ we have*

$$\frac{1}{\sqrt{n}} \ \leq \ \frac{\sqrt{n + r(r-1)}}{n} \ \leq \ \sqrt{\frac{r+1}{2n}} \cdot \frac{1}{\sqrt{n}}$$

Therefore, by Corollary 2 we have for $\phi \equiv \varphi$ the following direct result:

$$\|L_{n,r} f - f\| \ \leq \ C \ \omega_\phi^2 \left(f, \frac{1}{\sqrt{n}}\right). \tag{29}$$

The constant C may depends only on φ, ϕ and r.

Thus the next theorem will constitute an inverse of type B for (29) :

Theorem 4. *Let $(L_{n,r})_{n>2r}$ be given by (1) and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. Then there exist two constant K and $\tilde{C}$ ($\tilde{C}$ may depends only on φ, ϕ andr) such that for all $f \in C[0,1]$ and m, n with $M \geq Kn$ we have*

$$\omega_\varphi^2 \left(f, \frac{1}{\sqrt{n}}\right) \ \leq \ \tilde{C} \cdot \frac{m}{n} \cdot (\|L_{n,r} f - f\| \ + \ \|L_{m,r} f - f\|) \tag{30}$$

Proof. Using (3), Lemma 3 : (4) - (5), Lemma 6 : (21) - (22) and Lemma 7, we obtain (30) in view of [9, p. 372, Theorem 1].

3. **A new generalized Brass operator**

In this section we establish direct and converse theorems for the operators defined by (2).

Theorem 5. *Let $(L_{n,r}^\alpha)_{n>2r}$ be given by (2) and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. Then there exists an absolute constant $C > 0$ such that for all $f \in C[0,1]$ we have*

$$\|L_{n,r} f - f\| \ \leq \ C \ \omega_\varphi^2 \left(\ f \ , \ \sqrt{\frac{1}{1+\alpha} \cdot \left(\frac{n + r(r-1)}{n^2} + \alpha\right)} \ \right)$$

Proof. By [5, p. 1180, Lemma 3.1] we have for $\alpha > 0$ and $x \in (0,1)$ the following identity

$$w_{n-r,k}(x,\alpha) = \binom{n-r}{k} \cdot \frac{B\left(x\alpha^{-1}+k,(1-x)\alpha^{-1}+n-r-k\right)}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}.$$

Consequently, $L^{\alpha}_{n,r}f$ can be represented by means of the operator (1), as follows

$$(L^{\alpha}_{n,r}f)(x) = \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot (L_{n,r}f)(t)\, dt \tag{31}$$

On the other hand, by (31) and [8, p. 214, Theorem 2.1] we have

$$L^{\alpha}_{n,r}(u-x,x) =$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot L_{n,r}(u-x,t)\, dt$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot \left[L^{\alpha}_{n,r}(u-t,t) + L_{n,r}(t-x,x)\right]\, dt$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, (t-x)\, dt = 0 \tag{32}$$

and

$$L^{\alpha}_{n,r}((u-x)^2,x) =$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot L_{n,r}((u-x)^2,t)\, dt$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot \left[L_{n,r}((u-t)^2,t) + \right.$$

$$\left. + \; 2(t-x)L_{n,r}(u-t,t) + (t-x)^2\right]\, dt$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot \frac{n+r(r-1)}{n^2} \cdot t(1-t)\, dt \; +$$

$$+ \; \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot (t-x)^2\, dt$$

$$= \frac{1}{1+\alpha} \cdot \left(\frac{n+r(r-1)}{n^2} + \alpha\right) \cdot \varphi(x)^2 \tag{33}$$

Furthermore, by (3)

$$|(L^{\alpha}_{n,r}f)(x)| \le \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot |(L_{n,r}f)(t)|\, dt \le \|f\|.$$

So

$$\|L^{\alpha}_{n,r}f\| \le \|f\| \tag{34}$$

for all $f \in C[0,1]$. Now, using (32), (33), (34) and the standard method [1, Chap. 9], we obtain the assertion of the theorem.

In what follows we shall use some lemmas. These are the following:

Lemma 4. *For $(L_{n,r})_{n>2r}$, $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$ and $f\in C[0,1]$ we have*

$$\frac{1}{n}\cdot\|\varphi^2(L_{n,r}f)''\| \;\le\; C_0\;\left(\|L_{n,r}f-f\| \;+\; \|L_{Kn,r}f-f\|\right),$$

where $C_0>0$ is an absolute constant.

Proof. The announced inequality is the estimate (14) for $m=Kn$ given in [9, p. 373], using the estimates (4), (5), (21), (22) and Lemma 7.

Lemma 5. *For $(L_{n,r})_{n>2r}$, $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$ and $f\in C[0,1]$ we have*

$$\|L^{\alpha}_{n,r}f-L_{n,r}f\| \;\le\; \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{n,r}f)''\|.$$

Proof. By (31) and Taylor's formula :

$$(L_{n,r}f)(t)=(L_{n,r}f)(x)+(t-x)(L_{n,r}f)'(x)+\int_x^t (t-u)(L_{n,r}f)''(u)\;du$$

we have

$$\begin{aligned}(L^{\alpha}_{n,r}f)(x)-(L_{n,r}f)(x) &= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\\ &\cdot\;\left[(t-x)(L_{n,r}f)'(x)+\int_x^t (t-u)(L_{n,r}f)''(u)\;du\right]\;dt\\ &= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\\ &\cdot\;\left\{\int_x^t (t-u)(L_{n,r}f)''(u)\;du\right\}\;dt. \qquad (35)\end{aligned}$$

Hence, by [1, p. 140, Lemma 9.6.1] we obtain

$$\begin{aligned}|(L^{\alpha}_{n,r}f)(x)-(L_{n,r}f)(x)| &= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\\ &\cdot\;\left|\int_x^t \frac{|t-u|}{u(1-u)}\cdot u(1-u)\;|(L_{n,r}f)''(u)|\;du\right|\;dt\\ &\le \frac{\|\varphi^2(L_{n,r}f)''\|}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\\ &\cdot\;\frac{(t-x)^2}{x(1-x)}\;dt \;=\; \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{n,r}f)''\;Vert,\end{aligned}$$

which was to be proved.

We have the following result:

Theorem 6. *Let $(L^{\alpha}_{n,r})_{n>2r}$ be given by (2) and $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$. If $\alpha=\alpha(n)$ and $(\alpha/(1+\alpha))\cdot n\,(C_0+C_0\cdot C_1(r)+4K)\;\le\;\tilde{\alpha}\;<\;1$ then*

$$\begin{aligned}(1-\tilde{\alpha})\left(\|L_{n,r}f-f\|+\|L_{Kn,r}f-f\|\right) &\le \|L^{\alpha}_{n,r}f-f\|+\|L^{\alpha}_{Kn,r}f-f\| \;\le\\ &\le (1+\tilde{\alpha})\left(\|L_{n,r}f-f\|+\|L_{Kn,r}f-f\|\right)\end{aligned}$$

for all $f \in C[0,1]$. *Moreover, there exists an absolute constant* $C > 0$ *such that for all* $f \in C[0,1]$ *we have*

$$C^{-1}\ \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)\ \le\ \|L_{n,r}^\alpha f - f\| + \|L_{Kn,r} f - f\|\ \le\ C\ \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right).$$

Proof. We have, in view of Lemma 11 :

$$\begin{aligned}
&\|L_{n,r}^\alpha f - f\| + \|L_{Kn,r}^\alpha f - f\| \ \le \\
&\le\ \|L_{n,r}^\alpha f - L_{n,r} f\| + \|L_{n,r} f - f\| + \|L_{Kn,r}^\alpha f - L_{Kn,r} f - f\| + \|L_{Kn,r} f - f\| \\
&\le\ \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{n,r}f)''\| + \|L_{n,r} f - f\| + \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{Kn,r}f)''\| + \|L_{Kn,r} f - f\|.
\end{aligned}$$

Using Lemma 3 : (4) - (5), we obtain

$$\begin{aligned}
\|\varphi^2(L_{Kn,r}f)''\| &\le \|\varphi^2(L_{Kn,r}(f - L_{n,r}f))''\| + \|\varphi^2(L_{Kn,r}(L_{n,r}f))''\| \\
&\le 4Kn\|f - L_{n,r}f\| + C_1(r)\cdot\|\varphi^2(L_{n,r}f)''\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|L_{n,r}^\alpha f - f\| + \|L_{Kn,r}^\alpha f - f\| \le \frac{\alpha}{1+\alpha}\cdot(1 + C_1(r))\cdot\|\varphi^2(L_{n,r}f)''\| + \\
+\left(\frac{\alpha}{1+\alpha}\cdot 4Kn + 1\right)\cdot\|L_{n,r} f - f\| + \|L_{Kn,r} f - f\|.
\end{aligned}$$

Hence, by Lemma 10 we obtain

$$\begin{aligned}
&\|L_{n,r}^\alpha f - f\| + \|L_{Kn,r} f - f\| \ \le \\
&\le\ \frac{\alpha}{1+\alpha}\cdot nC_0\cdot(1 + C_1(r))\cdot(\ \|L_{n,r} f - f\| + \|L_{Kn,r} f - f\|\ )\ + \\
&+\ \left(\frac{\alpha}{1+\alpha}\cdot 4Kn + 1\right)\cdot\|L_{n,r} f - f\| + \|L_{Kn,r} f - f\| \\
&=\ \left[1 + \frac{\alpha}{1+\alpha}\cdot(nC_0(1 + C_1(r)) + 4K)\right]\cdot\|L_{n,r} f - f\|\ + \\
&+\ \left[1 + \frac{\alpha}{1+\alpha}\cdot nC_0(1 + C_1(r))\right]\cdot\|L_{Kn,r} f - f\| \\
&\le\ (1 + \tilde{\alpha})\cdot(\ \|L_{n,r} f - f\| + \|L_{Kn,r} f - f\|\ ) \qquad (36)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\|L_{n,r} f - f\| + \|L_{Kn,r} f - f\| \ \le \\
&\le\ \|L_{n,r}^\alpha f - L_{n,r} f\| + \|L_{n,r}^\alpha f - f\| + \|L_{Kn,r}^\alpha f - L_{Kn,r} f\| + \|L_{Kn,r}^\alpha f - f\| \\
&\le\ \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{n,r}f)''\| + \|L_{n,r}^\alpha f - f\| + \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{Kn,r}f)''\| + \|L_{Kn,r}^\alpha f - f\|.
\end{aligned}$$

Using Lemma 10 and Lemma 3 : (4) - (5), we obtain

$$\begin{aligned}
\|L_{n,r} f - f\| + \|L_{kn,r} f - f\| \ \le \\
\le \frac{\alpha}{1+\alpha}\cdot nC_0\cdot(\ \|L_{n,r} f - f\| + \|L_{Kn,r} f - f\|\ ) + \|L_{n,r}^\alpha f - f\| + \\
+\frac{\alpha}{1+\alpha}\cdot\left(\ 4Kn\|L_{n,r} f - f\| + C_1(r)\|\varphi^2(L_{n,r}f)''\|\ \right) + \|L_{Kn,r}^\alpha f - f\|
\end{aligned}$$

$$\leq \|L_{n,r}^{\alpha} f - f\| + \|L_{Kn,r}^{\alpha} f - f\| + \frac{\alpha}{1+\alpha} \cdot (nC_0(1 + C_1(r)) + 4Kn) \cdot \|L_{n,r} f - f\| +$$
$$+ \frac{\alpha}{1+\alpha} \cdot nC_0(1 + C_1(r)) \cdot \|L_{Kn,r} f - f\|$$
$$\leq \|L_{n,r}^{\alpha} f - f\| + \|L_{Kn,r}^{\alpha} f - f\| + \tilde{\alpha} \left(\|L_{n,r} f - f\| + \|L_{Kn,r} f - f\| \right).$$

Hence

$$(1 - \tilde{\alpha}) \left(\|L_{n,r} f - f\| + \|L_{Kn,r} f - f\| \right) \leq \|L_{n,r}^{\alpha} f - f\| + \|L_{Kn,r}^{\alpha} f - f\|. \tag{37}$$

In conclusion (36) and (37) imply the assertion of the theorem. Moreover, by (29) and (30), we obtain the second statement of the theorem using the first one.

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References

[1] Z. Ditzian, V. Totik, *Moduli of Smoothness*, Springer - Verlag, New York Berlin Heidelberg London, 1987.
[2] M. Felten, *Local and Global Approximation Theorems for Positive Linear Operators*, J. Approx. Theory, 94(1998), 396-419.
[3] H. Berens, G. G. Lorentz, *Inverse theorems for Bernstein polynomials*, Indiana Univ. Math. J., 21(1972), 693-708.
[4] H. Brass, *Eine Verallgemeinerung der Bernsteinschen Operatoren*, Abh. Math. Sem. Univ. Hamburg, 36(1971), 111-122.
[5] D. D. Stancu, *Approximation of functions by a new class of linear polynomial operators*, Rev. Roumaine Math. Pures Appl., 13(8)(1968), 1173-1194.
[6] R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer - Verlag, Berlin, 1993.
[7] Z. Ditzian, K. G. Ivanov, *Strong converse inequalities*, J. Analyse Math., 61(1993), 61-111.
[8] D. D. Stancu, *Approximation of functions by means of a new generalized Bernstein operator*, Calcolo, 20(2)(1983), 211-229.
[9] V. Totik, *Strong Converse Inequalities*, J. Approx. Theory, 76(1994), 369-375.

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THE FIRST EIGENVALUE AND THE EXISTENCE RESULTS

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Abstract. In this paper we establish some conditions to existence for the solution of the boundary value problem

$$-\frac{1}{q(x)}\left(p(x)\,u'(x)\right)' = f\left(x, u(x), w(p,q)\,u'(x)\right), x \in (0,h)$$
$$u(0) = u(1) = 0$$

The hypotheses from the main result contain assumption on the first eigenvalue of some particular Sturm-Liouville problem. Using the lower boundary for the first eigenvalue, we can give some conditions of existence.

1. Introduction and notation

We consider the equation

$$-\left(p(x)\,u'(x)\right)' + q(x)\,u(x) = \lambda r(x)\,u(x) \tag{1}$$

for $x \in [0,h]$, where $p, p', q, r \in C(0,h)$ and satisfies $p(x) \geq p_0 > 0$, $q(x) \geq 0$, $r(x) \geq r_0 > 0$ for $x \in [0,h]$. The *Sturm-Liouville problem* is to find all *eigenvalues* λ for which the equation (1) has a nontrivial solution which satisfy the boundary condition

$$\alpha u(0) + \beta u'(0) = 0$$
$$\gamma u(h) + \delta u'(h) = 0,$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 \neq 0$ and $\gamma^2 + \delta^2 \neq 0$. The corresponding nontrivial solution is called an *eigenfunction*.

Example 1.1. For the problem

$$-u''(x) = \lambda u(x), x \in [0,\pi]$$
$$u(0) = (\pi) = 0$$

the eigenvalue are $\lambda_k = k^2$, $k \in \mathbb{N}$ and the corresponding eigenfunction is $u_k(x) = A_k \sin kx$, $k \in \mathbb{N}$.

In general, the first eigenvalue λ_1 of the Sturm-Liouville problem is too difficult to determinate. Using the Weinstein's method of intermediate problem we can find a lower boundary for λ_1 see [4]), and by Rayleigh-Ritz method it's possible to determinate an upper boundary for λ_1 .

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Example 1.2. [2] Let be the Sturm-Liouville problem

$$-\left(p\left(x\right)u'\left(x\right)\right)'+q\left(x\right)u\left(x\right)=\lambda u\left(x\right),\ x\in\left[0,h\right]$$

$$u\left(0\right)=u\left(h\right)=0$$

where $p,p',q\in C\left(0,h\right)$, $0<p_0\leq p\left(x\right)\leq p_1$ and $0\leq q\left(x\right)\leq q_1$ on $\left[0,h\right]$. We have the next approximation for the eigenvalues of this problem

$$\frac{p_0\pi^2k^2}{h^2}\leq\lambda_k\leq\frac{p_1\pi^2k^2}{h^2}+q_1, k\in\mathrm{N}$$

In the sequel, we make the following notation:

(N_1) R_β is the set of all measurable functions $q:(0,h)\to[0,\infty)$ such that

$$\int_0^h\left[q\left(x\right)\right]^\beta dx=1$$

where β is a real number, $\beta\neq0$;

(N_2) $m_\beta=\inf\limits_{q\in R_\beta}\lambda_1$ and $M_\beta=\sup\limits_{q\in R_\beta}\lambda_1$;

(N_3) R_α is the set of nonnegative measurable functions p on $(0,h)$ such that

$$\int_0^h\left[p\left(x\right)\right]^\alpha dx=1$$

where α is a real number, $\alpha\neq0$;

(N_4) $m_\alpha=\inf\limits_{q\in R_\alpha}\lambda_1$ and $M_\alpha=\sup\limits_{q\in R_\alpha}\lambda_1$;

(N_5) B is the Euler beta function $B\left(a,b\right)=\int_0^1 x^{a-1}\left(1-x\right)^{b-1}dx$;

(N_6) $C\left(\alpha\right)=\begin{cases}\frac{2\alpha+1}{\alpha}\left(\frac{\alpha+1}{2\alpha+1}\right)^{1-\frac{1}{\alpha}}B^2\left(\frac{1}{2},\frac{1}{2}+\frac{1}{2\alpha}\right), \text{ for } \alpha\in\left(-\infty,-1\right)\cup\left(0,+\infty\right)\\ -4\frac{2\alpha+1}{\alpha}\left(\frac{\alpha+1}{2\alpha+1}\right)^{1-\frac{1}{\alpha}}\left(\int_0^\infty\frac{dt}{\left(1+t^2\right)^{\frac{1}{2}-\frac{1}{2\alpha}}}\right), \text{ for } \alpha\in\left(-\frac{1}{2},0\right)\end{cases}$;

(N_7) The set

$$\Gamma=L_q^2\left(0,1\right)=$$
$$\left\{u:[0,1]\to\mathbb{R}; u \text{ is measurable function and } \int_0^1 q\left(x\right)\left|u\left(x\right)\right|^2dx<\infty\right\}.$$

is endowed with the inner product

$$\left(u,v\right)_\Gamma=\int_0^1 q\left(x\right)u\left(x\right)v\left(x\right)dx \tag{2}$$

and the norm

$$\left\|u\right\|_\Gamma=\left(\int_0^1 q\left(x\right)\left|u\left(x\right)\right|^2dx\right)^{\frac{1}{2}}. \tag{3}$$

(N_8) $L_q^2\left(0,1;\mathbb{R}^2\right)=\left\{u:[0,1]\to\mathbb{R}^2;\int_0^1 q\left(x\right)\left|u\left(x\right)\right|^2dx<\infty\right\}$.

(N_9) The set $H = \left\{u \in L_q^2(0,1); u \text{ is absolute continuous and } u' \in L_p^2(0,1)\right\}$ is endowed with the inner product

$$(u,v)_H = \int_0^1 p(x)\, u'(x)\, v'(x)\, dx \tag{4}$$

and the norm

$$\|u\|_H = \left(\int_0^1 p(x) \left|u'(x)\right|^2 dx\right)^{\frac{1}{2}}. \tag{5}$$

Let us consider the Sturm-Liouville problem

$$u''(x) + \lambda q(x)\, u(x) = 0, x \in (0,h) \tag{6}$$

$$u(0) = u(h) = 0$$

The variational principle implies that the first eigenvalue λ_1 can be founds as

$$\lambda_1 = \inf_{\substack{u \in C_0^\infty(0,h) \\ u \neq 0}} \frac{\int_0^h \left[u'(x)\right]^2 dx}{\int_0^h q(x) \left[u(x)\right]^2 dx} \tag{7}$$

In the following, we remainder a result of Y. Egorov and V. Kondratiev

Lemma 1.1. [1] *If $\beta > 1$, then*

$$m_\beta = \left(\frac{1}{h}\right)^{2-\frac{1}{\beta}} \frac{(\beta-1)^{1+\frac{1}{\beta}}}{\beta(2\beta-1)^{\frac{1}{\beta}}} B^2\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{2\beta}\right) \text{ and } M_\beta = \infty.$$

If $\beta = 1$, then $M_1 = \infty$ and $m_1 = \frac{4}{h}$.
If $0 < \beta < \frac{1}{2}$, then

$$M_\beta = \left(\frac{1}{h}\right)^{2-\frac{1}{\beta}} \frac{(1-\beta)^{1+\frac{1}{\beta}}}{\beta(1-2\beta)^{\frac{1}{\beta}}} B^2\left(\frac{1}{2}, \frac{1}{2\beta}\right) \text{ and } m_\beta = 0.$$

If $\beta < 0$, then

$$M_\beta = \left(\frac{1}{h}\right)^{2-\frac{1}{\beta}} \frac{(1-\beta)^{1+\frac{1}{\beta}}}{\beta(1-2\beta)^{\frac{1}{\beta}}} B^2\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{2\beta}\right) \text{ and } m_\beta = 0.$$

If $\frac{1}{2} \leq \beta < 1$, then $M_\beta = \infty$ and $m_\beta = 0$.

For the Sturm-Liouville problem

$$\left(p(x)\, u'(x)\right)' + \lambda u(x) = 0, \text{ for } x \in (0,1) \tag{8}$$

$$u(0) = u(1) = 0.$$

The first eigenvalue for this problem is given by

$$\lambda_1 = \inf_{\substack{u \in C_0^\infty(0,h) \\ u \neq 0}} \frac{\int_0^1 p(x) \left[u'(x)\right]^2 dx}{\int_0^1 \left[u(x)\right]^2 dx}. \tag{9}$$

We have the following result

Lemma 1.2. [1] *If* $\alpha > -\frac{1}{2}$, $\alpha \neq 0$ *then* $M_\alpha = C(\alpha)$ *and* $m_\alpha = 0$.
If $\alpha < -1$ *then* $m_\alpha = C(\alpha)$ *and* $M_\alpha = \infty$.
If $-1 \leq \alpha \leq -\frac{1}{2}$, *then* $M_\alpha = \infty$ *and* $m_\alpha = 0$.

2. Existence results

In that follows, we assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the *Caratheodory condition*, i.e.

(i) the application $f(x,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous a.e. for $x \in [0,1]$;
(ii) the application $f(\cdot, s) : [0,1] \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}^2$.

Let us consider the nonlinear boundary value problem

$$-\frac{1}{q(x)}\left(p(x)u'(x)\right)' = f\left(x, u(x), w(x)u'(x)\right), \text{ for } x \in (0,1) \tag{10}$$

$$u(0) = u(1) = 0.$$

Consider the operator $A : H \to \Gamma$ defined by

$$A(u)(x) = -\frac{1}{q(x)}\left(p(x)u'(x)\right)'. \tag{11}$$

We have

$$\begin{aligned}(Au,u)_\Gamma &= \int_0^1 q(x)\left[-\frac{\left(p(x)u'(x)\right)'}{q(x)}\right]u(x)\,dx \\ &= -p(x)u'(x)u(x)|_0^1 + \int_0^1 p(x)\left(u'(x)\right)^2 dx = \|u\|_H^2.\end{aligned}$$

Hence,

$$\|u\|_\Gamma^2 \leq \frac{1}{\lambda_1}(Au,u)_\Gamma \leq \frac{1}{\lambda_1}\|Au\|_\Gamma \cdot \|u\|_\Gamma.$$

Therefore,

$$\|u\|_\Gamma \leq \frac{1}{\lambda_1}\|Lu\|_\Gamma. \tag{12}$$

Theorem 2.1. *Suppose that*

(H_1) $w(x) \leq \sqrt{\frac{p(x)}{q(x)}}$ *on* $[0,1]$;
(H_2) *the application* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *satisfies the Caratheodory conditions and*

$$|f(x,s,t)| \leq a|t| + b|s| + g$$

for every $x \in (0,1)$; $t, s \in \mathbb{R}$ *and* $g \in \Gamma$;
(H_3) *there exist* $a, b \in [0,\infty)$ *small enough that*

$$\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1.$$

Then, the problem (10) has at least one solution in H.

Proof. For the beginning, we write problem (10) as a fixed point problem. For this, consider the operator $J : H \to L_q^2\left(0,1;\mathbb{R}^2\right)$ given by

$$J\left(u\right)=\left(u,u'\right)$$

and the Nemitskii operator $N_f : L_q^2\left(0,1;\mathbb{R}^2\right)\to\Gamma$ defined by

$$N_f\left(u\right)\left(x\right)=f\left(x,u_1\left(x\right),w\left(x\right)u_2\left(x\right)\right)$$

where $u=\left(u_1,u_2\right)$. The hypothesis $\left(H_2\right)$ ensures that the Nemitskii operator is well defined and continuous, see [3] for details. We have the diagram

$$H\xrightarrow{J}L_q^2\left(0,1;\mathbb{R}^2\right)\xrightarrow{N_f}\Gamma\xrightarrow{A^{-1}}H$$

Now, we have that the operator $T:H\to H$, $T=A^{-1}N_fJ$ is completely continuous and the problem (10) is equivalent to the equation

$$Tu=u,\ u\in H.$$

We have $\left\|T\right\|_H^2=\left(AT,T\right)_\Gamma\leq\left\|T\right\|_\Gamma\cdot\left\|AT\right\|_\Gamma=\left\|T\right\|_\Gamma\cdot\left\|n_f\right\|_\Gamma$. ¿From (12), we obtain $\left\|T\right\|_\Gamma\leq\frac{1}{\lambda_1}\left\|AT\right\|_\Gamma=\frac{1}{\lambda_1}\left\|N_f\right\|_\Gamma$. So,

$$\left\|T\right\|_H\leq\frac{1}{\sqrt{\lambda_1}}\left\|N_f\right\|_\Gamma. \tag{13}$$

By $\left(H_2\right)$ we have

$$\begin{aligned}\left\|N_f\right\|_\Gamma &= \left(\int_0^1 q\left(x\right)\left|f\left(x,u\left(x\right),w\left(x\right)u'\left(x\right)\right)\right|^2dx\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^1 q\left(x\right)\left\{g\left(x\right)+a\left|u\left(x\right)\right|+b\left|w\left(x\right)u'\left(x\right)\right|\right\}^2dx\right)^{\frac{1}{2}}\\ &\leq \left\|g\right\|_\Gamma+a\left\|u\right\|_\Gamma+b\left(\int_0^1 q\left(x\right)w^2\left(x\right)\left(u'\left(x\right)\right)^2dx\right)^{\frac{1}{2}}\end{aligned}$$

Now, hypothesis $\left(H_1\right)$ implies that

$$\begin{aligned}\left\|N_f\right\|_\Gamma &\leq \left\|g\right\|_\Gamma+a\left\|u\right\|_\Gamma+b\left(\int_0^1 p\left(x\right)\left(u'\left(x\right)\right)^2dx\right)^{\frac{1}{2}}\\ &\leq \left\|g\right\|_\Gamma+a\left\|u\right\|_\Gamma+b\left\|u\right\|_H.\end{aligned}$$

Since $\left\|u\right\|_\Gamma^2\leq\frac{1}{\lambda_1}\left\|u\right\|_H^2$, results $\left\|N_f\right\|_\Gamma\leq\left\|g\right\|_\Gamma+\frac{a}{\sqrt{\lambda_1}}\left\|u\right\|_H+b\left\|u\right\|_H$. Hence, by (13), we obtain

$$\left\|Tu\right\|_H\leq\frac{\left\|g\right\|_\Gamma}{\sqrt{\lambda_1}}+\left(\frac{a}{\lambda_1}+\frac{b}{\sqrt{\lambda_1}}\right)\left\|u\right\|_H.$$

Now, conform to hypothesis $\left(H_3\right)$ we can find a real number $r>0$ such that

$$\left\|Tu\right\|_H<\left\|u\right\|_H \text{ for } u\in H \text{ with } \left\|u\right\|\geq r.$$

By Lerray - Schauder principle, result that equation $Tu = u$ has at least one solution in H. $\square$

In a similar way, we can prove the next result

Theorem 2.2. *Let us consider the boundary value problem*

$$-\frac{1}{q(x)}\left(p(x)\,u'(x)\right)' = f\left(x, u(x), u'(x)\right), \text{ for any } x \in (0,1) \tag{14}$$

$$u(0) = u(1) = 0$$

Suppose that the mapping f satisfies H_2 and

(H_4) *there exist $a, b \in (0, \infty)$ small enough that*

$$\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}}\sqrt{\frac{q_1}{p_0}} < 1$$

Then, the problem (14) *has at least one solution in H.*

An analogous result remains if we consider the interval $[0, h]$.

Example 2.3. For the Sturm - Liouville problem

$$-u'' = \lambda(1 + \sin x)\,u, \quad \text{for} \quad u \in [0, \pi]$$

$$u(0) = u(\pi) = 0$$

it can establish the inequality $0.5394 \le \lambda_1 \le 0.54088$, see [4]. So, the boundary value problem

$$\frac{1}{1 + \sin x} \cdot u''(x) = f(x, u, u'), \text{ for } \ x \in [0, \pi]$$

$$u(0) = u(1) = 0$$

has at least one solution if the mapping $f : [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Caratheodory condition and

$$|f(x, s, t)| \le \frac{|t| + |s|}{4} + g(x)$$

for $x \in (0, \pi)$ and $g \in L^2_q(0, \pi)$.

3. The First Eigenvalue and the Existence Results

Now, in Theorem 2.2 we put the estimation from Lemma 1.2 and obtain the following result

Theorem 3.3. *Consider the nonlinear boundary value problem*

$$-\left(p(x)\,u'(x)\right)' = f\left(x, u(x), u'(x)\right), \text{ for } x \in (0,1) \tag{15}$$

$$u(0) = u(1) = 0$$

Suppose that f satisfies (H_2) and

(H_5) *the nonnegative measurable mapping $p : [0,1] \to \mathbb{R}$ is such that*

$$p_0 = \inf_{x\in(0,1)} p(x) > 0 \text{ and } \int_0^1 p(x)^\alpha dx = 1 \text{ for } \alpha \le -1;$$

(H_6) *there exist the numbers* $a, b \in (0,\infty)$ *small enough that*

$$\frac{a}{m_\alpha} + \frac{b}{\sqrt{p_0 m_\alpha}} < 1,$$

with

$$m_\alpha = \frac{2\alpha+1}{\alpha}\left(\frac{\alpha+1}{2\alpha+1}\right)^{1-\frac{1}{\beta}} B^2\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2\alpha}\right).$$

Then, the problem (15) *has at least one solution in* H.

By Theorem 2.2 and Lemma 1.1, we obtain

Theorem 3.4. *Consider the nonlinear boundary value problem*

$$-\frac{1}{q(x)}u''(x) = f(x, u(x), u'(x)), \text{ for } x \in (0,h) \tag{16}$$

$$u(0) = u(h) = 0$$

Suppose that f *satisfies* (H_2) *and*

(H_7) *the nonnegative measurable mapping* $q : [0,h] \to \mathbb{R}$ *is such that*

$$q_1 = \sup_{x\in(0,h)} q(x) < \infty \text{ and } \int_0^h q(x)^\beta dx = 1 \text{ for } \beta > 1$$

(H_8) *there exists the numbers* $a, b \in (0,\infty)$ *small enough that*

$$\frac{a}{m_\alpha} + \frac{b}{\sqrt{\frac{q_1}{m_\beta}}} < 1,$$

with

$$M_\beta = \left(\frac{1}{h}\right)^{2-\frac{1}{\beta}} \frac{(1-\beta)^{1+\frac{1}{\beta}}}{\beta(1-2\beta)^{\frac{1}{\beta}}} B^2\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{2\beta}\right).$$

Then, the problem (16) *has at least one solution.*

References

[1] Y. Egorov, V. Kondratiev, *On Spectral Theory of Elliptic Operators*, Birkhäuser-Verlag, Berlin, 1996, 153-206.

[2] S. G. Mihlin, *Linear Equation with Partial Derivatives* (in Romanian), Bucharest, 1983.

[3] R. Precup, *Nonlinear Integral Equations*, (in Romanian), Babeş-Bolyai Univ., Cluj-Napoca, Romania, 1993, 77-97 (2001).

[4] Al. Weinstein, *On the Sturm-Lioville Theory and the Eigenvalues of intermediate Problems*, Numerische Mathematik, **5**, 1963, 238-245.

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MODULES OVER TRIANGULATED CATEGORIES AND LOCALIZATION

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Abstract. For a compactly generated triangulated category we gives a new proof for the fact that the category of modules over its subcategory consisting of all compact objects it is not only the colocalization, but also the localization of the category of finitely presented modules over the full triangulated category. We do not only prove the existence of a right adjoint for the restriction functor, but we give it explicitly.

A problem arising in the study of (compactly generated) triangulated categories is to find some abelian categories closely related to a given triangulated one. A such category is the category of finitely presented contravariant functors defined on it with values in the category of abelian groups. We denote it here by Mod-$\mathcal{T}$, where $\mathcal{T}$ is the triangulated category. The Yoneda embedding gives an universal homological functor $\mathbf{h} : \mathcal{T} \to$ Mod-$\mathcal{T}$ [3, 5.1.18]. A result due to Neeman [3, 5.3.9] says, that a triangulated functor between two triangulated categories $\mathcal{T} \to \mathcal{S}$ have a right or a left adjoint if and only if the induced functor Mod-$\mathcal{T} \to$ Mod-$\mathcal{S}$ does. But it is also not easy to deal with the category Mod-$\mathcal{T}$, since it may be not well-powered [3, Appendix C], in the sense that an object may have a proper class (which is not a set) of subobjects (quotients). In the same work of Neeman [3], was observed that a "good" approximation of the category Mod-$\mathcal{T}$ is the category $\mathrm{Ex}((\mathcal{T}^\alpha)^{\mathrm{op}}, \mathcal{A}b)$, whose objects are additive functors $(\mathcal{T}^\alpha)^{\mathrm{op}} \to \mathcal{A}b$ which take coproducts fewer than α objects in products in $\mathcal{A}b$. Here α is a fixed regular cardinal, $\mathcal{T}^\alpha$ is the full subcategory of α-compact objects of $\mathcal{T}$, in the sense of the definition [3, 4.2.7], and it is supposed to be skeletally small. Precisely, the category $\mathrm{Ex}((\mathcal{T}^\alpha)^{\mathrm{op}}, \mathcal{A}b)$ is the colocalization of Mod-$\mathcal{T}$ [3, 6.5.3]. In the case $\mathcal{C} = \mathcal{T}^{\aleph_0}$ that is $\alpha = \aleph_0$ we have $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b) =$ Mod-$\mathcal{C}$ contains all functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$. In this note we find a new proof for the fact that Mod-$\mathcal{C}$ is not only the colocalization, but also the localization of Mod-$\mathcal{T}$, an explicit formula for the right adjoint of the restriction functor being also given.

A few words about terminology and notations: By $\mathcal{A}b$ we shall denote the category of abelian groups. We shall write $\mathcal{A} \to \mathcal{B}$ respectively $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ to emphasize that we deal with a covariant (contravariant) functor between two given categories $\mathcal{A}$ and $\mathcal{B}$. It is well–known that an associative ring R may be regarded as a preadditive

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category with a single object, and then a right R-module means a functor $R^{\rm op} \to \mathcal{A}b$. The additive functors $\mathcal{A}^{\rm op} \to \mathcal{A}b$, defined on an arbitrary preadditive category $\mathcal{A}$ will also call (right) modules over $\mathcal{A}$, or simply $\mathcal{A}$-modules. We denote by $\mathcal{A}(a', a)$ and $\mathrm{Hom}_{\mathcal{A}}(M', M)$ the set of morphism between objects a' and a, in the category $\mathcal{A}$, respectively the class of all natural transformations between $\mathcal{A}$-modules M' and M.

For basic facts about abelian categories, we refer the reader to [4], and for the general theory of triangulated categories to [3]. Even if in the text all references concerning abelian categories are to [4], the personal experience of the author playing a rôle here, this things may be found also in many works, for example in Gabriel's [1].

By a (right) module over a preadditive category $\mathcal{T}$, we understand, as in the case of ordinary modules over a ring, an additive contravariant functor $M : \mathcal{T}^{\rm op} \to \mathcal{A}b$. If $\mathcal{T}$ is skeletally small, then the class of all modules over $\mathcal{T}$ together with the natural transformations between them, form a Grothendieck category, denoted by Mod-$\mathcal{T}$ [4, chapter 4, 4.9], where the limits and the colimits are computed pointwise. Returning to the general case, a module N over the category $\mathcal{T}$ is called *finitely presented*, if there is an exact sequence of functors and natural transformations

$$\mathcal{T}(-, s) \to \mathcal{T}(-, t) \to N \to 0.$$

Denote by Mod-$\mathcal{T}$ the class of all finitely presented modules over $\mathcal{T}$. Note that, even if the class of all modules over $\mathcal{T}$ forms only a illegitimate category, in the sense that the class of the natural transformations between two such modules may be proper, this does not happen with Mod-$\mathcal{T}$. Indeed, by the Yoneda Lemma we infer that the class of all natural transformation between two finitely presented modules is actually a set, Mod-$\mathcal{T}$ together with natural transformations being a good defined category. If, in addition, $\mathcal{T}$ is triangulated, then by [3, 5.1.10], the category Mod-$\mathcal{T}$ is an abelian one.

Let $\mathcal{T}$ be a triangulated category, and $\mathcal{C}$ its full subcategory consisting of all compact objects. Recall that an object $c \in \mathcal{T}$ is called *compact* provided that the covariant functor $\mathcal{T}(c, -) : \mathcal{T} \to \mathcal{A}b$ commutes with direct sums. It is well–known, and also easy to see, that $\mathcal{C}$ is a *thick* subcategory of $\mathcal{T}$, that means, a triangulated subcategory which closed under direct summands. Throughout of this note we assume $\mathcal{T}$ has arbitrary coproducts, $\mathcal{C}$ is a skeletally small category, and it generates $\mathcal{T}$, i.e. $\mathcal{T}(c, x) = 0$ for all $c \in \mathcal{C}$ implies $x = 0$.

The functor $\mathbf{h} : \mathcal{T} \to$ Mod-$\mathcal{T}$, $\mathbf{h}(t) = \mathcal{T}(-, t)$ is a homological embedding, which send any object t of $\mathcal{T}$ into a projective object of Mod-$\mathcal{T}$. Moreover, since $\mathcal{T}$ is idemsplit (that is every idempotent $t \to t$ splits, for all $t \in \mathcal{T}$) [3, 1.6.8], every projective object of Mod-$\mathcal{T}$ is of this form [3, 5.1.11]. Restricting to $\mathcal{C}$ the images of $\mathbf{h}$ on each object $t \in \mathcal{T}$, we obtain a homological functor $\bar{\mathbf{h}} : \mathcal{T} \to$ Mod-$\mathcal{C}$, $\bar{\mathbf{h}}(t) = \mathcal{T}(-, t)|_{\mathcal{C}}$. Clearly $\bar{\mathbf{h}}$ commutes with coproducts, and for any $M \in$ Mod-$\mathcal{C}$, there is an exact sequence

$$\begin{array}{ccccccc} \bigoplus_{j\in J}\bar{\mathbf{h}}(d_j) & \longrightarrow & \bigoplus_{i\in I}\bar{\mathbf{h}}(c_i) & \longrightarrow & M & \longrightarrow & 0\\ \| & & \| & & \| & & \\ \bar{\mathbf{h}}\left(\bigoplus_{j\in J} d_j\right) & \longrightarrow & \bar{\mathbf{h}}\left(\bigoplus_{i\in I} c_i\right) & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

with d_j and c_i belonging to $\mathcal{C}$.

Since $\mathbf{h}$ is an universal homological functor [3, 5.1.18], it results an exact functor $\boldsymbol{\pi}$ making commutative the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\mathbf{h}} & \text{Mod-}\mathcal{T} \\ & \searrow^{\bar{\mathbf{h}}} & \downarrow{\scriptstyle\boldsymbol{\pi}} \\ & & \text{Mod-}\mathcal{C}. \end{array}$$

Because every additive contravariant functor takes finite coproducts to products, we lie in the hypothesis of [3, chapter 6]. It follows that $\boldsymbol{\pi}(M) = M|_{\mathcal{C}}$ [3, 6.5.2], and Mod-$\mathcal{C}$ is the colocalization of Mod-$\mathcal{T}$, what means, the functor $\boldsymbol{\pi}$ has a fully–faithful left adjoint $\mathbf{L} : \text{Mod-}\mathcal{C} \to \text{Mod-}\mathcal{T}$. This adjoint is determined by its right exactness, and by the equality

$$\mathbf{L}\left(\bigoplus_{i\in I}\bar{\mathbf{h}}(c_i)\right) = \mathbf{h}\left(\bigoplus_{i\in I} c_i\right),$$

for all $c_i \in \mathcal{C}$ [3, 6.5.3]. Denote by $v : 1_{\text{Mod-}\mathcal{C}} \to \boldsymbol{\pi}\mathbf{L}$ and $u : \mathbf{L}\boldsymbol{\pi} \to 1_{\text{Mod-}\mathcal{T}}$ the unit, respectively the counit, of this adjunction. It is well–known that the fully–faithfulness of $\mathbf{L}$ is equivalent to the existence of an inverse for v [4, chapter 1, 13.11].

Lemma 1. *Any projective object P of* Mod-$\mathcal{C}$ *is of the form,* $\bar{\mathbf{h}}(c)$ *for an object* $c = \bigoplus_{i\in I} c_i \in \mathcal{T}$*, with* $c_i \in \mathcal{C}$*, and the induced map*

$$\mathcal{T}(c,x) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \bar{\mathbf{h}}(x))$$

is an isomorphism for all $x \in \mathcal{T}$*.*

Proof. A projective object P of Mod-$\mathcal{C}$ is a direct summand of a direct sum $\bigoplus_{j\in J}\mathcal{C}(-,d_j)$, and because $\mathcal{C}$ is idemsplit, it follows $P \cong \bar{\mathbf{h}}(\bigoplus_{i\in I} c_i) = \bar{\mathbf{h}}(c)$.

Using the isomorphism of adjunction, and then the Yoneda isomorphism, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \bar{\mathbf{h}}(x)) &= \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \boldsymbol{\pi}(\mathbf{h}(x))) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}(\bar{\mathbf{h}}(c)), \mathbf{h}(c)) \\ &\cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{h}(c), \mathbf{h}(x)) \cong \mathcal{T}(c,x). \end{aligned}$$

□

We record also an analogous for injectives:

Lemma 2. [2, Lemma 1] *Any injective object Q of* Mod-$\mathcal{C}$ *is of the form* $\bar{\mathbf{h}}(q)$*, for an object $q \in \mathcal{T}$, and the induced map*

$$\mathcal{T}(x, q) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), \bar{\mathbf{h}}(q))$$

is an isomorphism for all $x \in \mathcal{T}$.

Lemma 3. *The assignement $M \mapsto \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M)$ gives a functor*

$$\mathbf{R} : \text{Mod-}\mathcal{C} \to \text{Mod-}\mathcal{T}.$$

Proof. The unique problem which arises is that $\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M) : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$ is actually finitely presented, for all $M \in$ Mod-$\mathcal{C}$.

Choose an injective resolution for M:

$$0 \to M \to Q_1 \to Q_2.$$

Fix an object $x \in \mathcal{T}$. Applying the left exact functor $\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), -)$ to this injective resolution, and using Lemma 2, it follows that there are two objects $q_1, q_2 \in \mathcal{T}$, and a commutative diagram of abelian groups:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), M) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), Q_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), Q_2) \\
& \| & & \| & & \| \\
0 \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), M) & \longrightarrow & \mathcal{T}(x, q_1) & \longrightarrow & \mathcal{T}(x, q_2).
\end{array}$$

Therefore $\mathrm{Hom}(\bar{\mathbf{h}}(-), M)$ is pointwise the kernel of the natural transformation $\mathcal{T}(-, q_1) \to \mathcal{T}(-, q_2)$ between two finitely presented $\mathcal{T}$-modules. Then, by [3, 5.1.10], this functor belongs to Mod-$\mathcal{T}$. □

Now we are ready to give the main result of this note.

Theorem 4. *The functor $\mathbf{R}$: Mod-$\mathcal{C} \to$ Mod-$\mathcal{T}$ is the fully–faithful right adjoint of the functor $\boldsymbol{\pi}$: Mod-$\mathcal{T} \to$ Mod-$\mathcal{C}$, so the category* Mod-$\mathcal{C}$ *is not only the colocalization, but also the localization of the category* Mod-$\mathcal{T}$*.*

Proof. Let $c \in \mathcal{C}$, and $M : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ be a $\mathcal{C}$-module. Then, the Yoneda isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), M) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{T}(-, c)|_{\mathcal{C}}, M) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-, c), M) \cong M(c)$$

shows that $\boldsymbol{\pi}\mathbf{R}(M) = \mathbf{R}(M)|_{\mathcal{C}} = \mathrm{Hom}(\bar{\mathbf{h}}(-), M)|_{\mathcal{C}}$ is naturally isomorphic to M. Denote by $v' : \boldsymbol{\pi}\mathbf{R} \to 1_{\text{Mod-}\mathcal{C}}$ this isomorphism.

Let now $N : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$ be a finitely presented $\mathcal{T}$-module. Then we have

$$\begin{aligned}
\mathbf{R}\boldsymbol{\pi}(N) &= \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), \boldsymbol{\pi}(N)) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}(\bar{\mathbf{h}}(-)), N) \\
&= \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)), N),
\end{aligned}$$

and again an Yoneda isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(\mathbf{h}(-), N) \cong N.$$

The counit $u_{\mathbf{h}(-)} : \mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)) \to \mathbf{h}(-)$ of the adjunction between $\mathbf{L}$ and $\boldsymbol{\pi}$ gives a morphism $\mathrm{Hom}_{\mathcal{T}}(u_{\mathbf{h}(-)}, N)$, which induced by the above isomorphisms an another one

$$u'_N : N \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{h}(-), N) \to \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)), N) \cong \mathbf{R}\boldsymbol{\pi}(N),$$

so a natural transformation $u' : 1_{\text{Mod-}\mathcal{T}} \to \mathbf{R}\boldsymbol{\pi}$.

Fix $c \in \mathcal{C}$, $t \in \mathcal{T}$, $M \in \text{Mod-}\mathcal{C}$ and $N \in \text{Mod-}\mathcal{T}$. The maps $\mathbf{R}(v'_M)$, $v'_{\boldsymbol{\pi}(N)}$ are clearly isomorphisms since v' is so. Moreover, the maps

$$(\boldsymbol{\pi}(u'_N))_c : \boldsymbol{\pi}(N) = N(c) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), N|_{\mathcal{C}})$$

and

$$(u'_{\mathbf{R}(M)})_t : \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(t), M) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(t), \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M)|_{\mathcal{C}})$$

are isomorphisms too, by an analogous argument to the one used for v. Hence, the naturality of these morphisms implies the equalities

$$\mathbf{R}(v'_M)u'_{\mathbf{R}(M)} = 1_{\mathbf{R}(M)} \text{ and } v'_{\boldsymbol{\pi}(N)}\boldsymbol{\pi}(u'_N) = 1_{\boldsymbol{\pi}(N)},$$

which show that $\mathbf{R}$ is the right adjoint of $\boldsymbol{\pi}$, with the unit u' and the counit v'.

Finally the fully–faithfulness of $\mathbf{R}$ is equivalent, by [4, chapter 1, 13.10], to the fact that v' is invertible. □

Remark 5. The subcategory $\mathrm{Ker}\,\boldsymbol{\pi}$ of $\text{Mod-}\mathcal{T}$, consisting of the objects sended by $\boldsymbol{\pi}$ into 0 is both localizing and colocalizing, and the categories $\text{Mod-}\mathcal{T}/\mathrm{Ker}\,\boldsymbol{\pi}$, $\text{Mod-}\mathcal{C}$ and $\mathrm{Ker}\,\boldsymbol{\pi}\backslash\text{Mod-}\mathcal{T}$ are all equivalent.

References

[1] P. Gabriel, *De catégories abeliennes*, Bull. Soc. Math. France, **90**(1962), 323-448.
[2] H. Krause, *Brown representability and flat covers*, J. Pure and Appl. Alg., **157**(2001), 81-86.
[3] A. Neeman, *Triangulated Categories.*
[4] N. Popescu, L. Popescu, *Theory of Categories*, Editura Academiei, Bucureşti, România and Sijthoff & Noordhoff International Publishers, 1979.

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UNIVALENCE CRITERIA CONNECTED WITH ARITHMETIC AND GEOMETRIC MEANS

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Abstract. In this paper we obtain some univalence criteria connected with arithmetic and geometric means of the expressions f/g and f'/g', where f and g are analytic functions in the unit disk.

1. **Introduction**

We let $U_r = \{\ z \in C :\ \ |z| < r\ \}$ denote the disk of z-plane, where $r \in (0,1]$, $U_1 = U$ and $I = [0,\infty)$. Let A be the class of functions f analytic in U such that $f(0) = 0,\ f'(0) = 1$. Our consideration apply the theory of Löwner chains; we first recall here the basic result of this theory, from Pommerenke.

Theorem 1.1. ([4]). *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots,\ a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition $Rep(z,t) > 0$ for all $z \in U,\ t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$ the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

2. **Univalence criteria and arithmetic mean**

In this section we derive several interesting criteria of univalence related to arithmetic mean. The method of prove is based on Theorem 1.1 and on construction of a suitable Löwner chain.

Theorem 2.1. *Let $\alpha,\ \beta,\ \gamma$ be complex numbers such that $\alpha \neq -1$,*

$$|\alpha - \beta| \leq |\beta + 1|, \quad |\gamma - 1| < 1, \quad |\gamma(\alpha+1) - (\beta+1)| \leq |\beta + 1|,$$

and let $f \in A$. If there exists a function $g \in A$ such that the inequalities

$$\left|\ \gamma(\alpha+1)g'(z) - 1 - \beta\frac{f(z)}{g(z)} \cdot \frac{g'(z)}{f'(z)}\ \right| < \left|\ 1 + \beta\frac{f(z)}{g(z)} \cdot \frac{g'(z)}{f'(z)}\ \right| \tag{1}$$

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and

$$\left| \left[\gamma(\alpha+1)g'(z) - 1 - \beta\frac{f(z)g'(z)}{f'(z)g(z)} \right] |z|^2 + (1-|z|^2) \left[(\gamma-1)\left(1+\beta\frac{f(z)g'(z)}{f'(z)g(z)}\right) \right.\right.$$

$$\left.\left. + \frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} + \beta\frac{zg'(z)}{g(z)}\left(1 - \frac{f(z)g'(z)}{f'(z)g(z)}\right) \right] \right| \leq \left| 1 + \beta\frac{f(z)g'(z)}{f'(z)g(z)} \right| \qquad (2)$$

are true for all $z \in U$, then the function

$$F_\gamma(z) = \left(\gamma \int_0^z u^{\gamma-1} f'(u)du \right)^{1/\gamma} \qquad (3)$$

is analytic and univalent in U, where the principal branch is intended.

Proof. Let us prove that there exists a real number $r \in (0,1]$ such that the function $L : U_r \times I \longrightarrow C$, defined formally by

$$L(z,t) = \left(\gamma \int_0^{e^{-t}z} u^{\gamma-1} f'(u)du + \right. \qquad (4)$$

$$\left. + \frac{e^{(2-\gamma)t} - e^{-\gamma t}}{1+\alpha} z^\gamma \left[\frac{f'(e^{-t}z)}{g'(e^{-t}z)} + \beta\frac{f(e^{-t}z)}{g(e^{-t}z)} \right] \right)^{1/\gamma}$$

is analytic in U_r , for all $t \in I$. Let us consider the function

$$h(z,t) = \frac{f'(e^{-t}z)}{g'(e^{-t}z)} + \beta\frac{f(e^{-t}z)}{g(e^{-t}z)} .$$

We have $h(0,t) = 1+\beta$ and we observe that $h(0,t) \neq 0$. Indeed, if $h(0,t) = 0$ then $\beta = -1$ and from the condition $|\alpha-\beta| \leq |\beta+1|$ it follows $\alpha = -1$ which is a contradiction with the hypothesis $\alpha \neq -1$. Therefore there is a disk U_{r_1}, $0 < r_1 \leq 1$, in which $h(z,t) \neq 0$ for all $t \in I$. Denoting

$$h_1(z,t) = \gamma \int_0^{e^{-t}z} u^{\gamma-1} f'(u)du$$

we have $h_1(z,t) = z^\gamma h_2(z,t)$ and is easy to see that h_2 is analytic in U_{r_1} for all $t \in I$, $h_2(0,t) = e^{-\gamma t}$. The function

$$h_3(z,t) = h_2(z,t) + \frac{e^{(2-\gamma)t} - e^{-\gamma t}}{1+\alpha} h(z,t)$$

is also analytic in U_{r_1} and

$$h_3(0,t) = \frac{e^{-\gamma t}}{1+\alpha} \left[\, (\alpha-\beta) + (1+\beta)e^{2t} \, \right] .$$

Let us now prove that $h_3(0,t) \neq 0$ for any $t \in I$. We have $h_3(0,0) = 1$. Assume that there exists $t_0 > 0$ such that $h_3(0,t_0) = 0$. It follows $e^{2t_0} = (\beta-\alpha)/(1+\beta)$ and since $|\alpha-\beta| \leq |\beta+1|$ we get $e^{2t_0} \leq 1$ and this inequality is imposible. Therefore, there is a disk $U_{r_2},\ r_2 \in (0, r_1]$ in which $h_3(z,t) \neq 0$ for all $t \in I$. Then we can choose an uniform branch of $[h_3(z,t)]^{1/\gamma}$ analytic in U_{r_2}, denoted by $h_4(z,t)$, that is equal to

$$a_1(t) = e^{\frac{2-\gamma}{\gamma}t}\left[\frac{(\alpha-\beta)e^{-2t}+(1+\beta)}{1+\alpha}\right]^{1/\gamma}$$

at the origin, and for $a_1(t)$ we fix the principal branch, $a_1(0) = 1$. From these considerations, it follows that the relation (4) may be written as

$$L(z,t) = z \cdot h_4(z,t) = a_1(t)z + a_2(t)z^2 + \dots ,$$

and then the function $L(z,t)$ is analytic in U_{r_2}. Since $|\gamma - 1| < 1$ implies $\mathrm{Re}(2/\gamma) > 1$, it follows that $\lim_{t\to\infty} |a_1(t)| = \infty$. We saw also that $a_1(t) \neq 0$ for all $t \in I$.
From the analyticity of $L(z,t)$ in U_{r_2} it follows that there is a number $r_3,\ 0 < r_3 < r_2$, and a constant $K = K(r_3)$ such that

$$|\ L(z,t)/a_1(t)\ | \ <\ K, \qquad \forall z \in U_{r_3}, \quad t \in I,$$

In consequence, the family $\{\ L(z,t)/a_1(t)\ \}$ is normal in U_{r_3}. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T > 0$ and $r_4,\ 0 < r_4 < r_3$, there exists a constant $K_1 > 0$ (that depends on T and r_4) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| \ <\ K_1, \qquad \forall z \in U_{r_4}, \quad t \in [0,T].$$

It follows that the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_4}. Let us set

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t} \tag{5}$$

and

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} \tag{6}$$

The function $p(z,t)$ is analytic in $U_r,\ 0 < r < r_4$. The function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$, if the function $w(z,t)$ can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$and $t \in I$.
After computation we obtain:

$$w(z,t) = A(z,t)\cdot e^{-2t} + B(z,t)(1-e^{-2t})\ , \tag{7}$$

where

$$A(z,t) = \frac{\gamma(\alpha+1)g'(e^{-t}z) - 1 - \beta\frac{f(e^{-t}z)}{g(e^{-t}z)}\cdot\frac{g'(e^{-t}z)}{f'(e^{-t}z)}}{1+\beta\frac{f(e^{-t}z)}{g(e^{-t}z)}\cdot\frac{g'(e^{-t}z)}{f'(e^{-t}z)}} \tag{8}$$

$$B(z,t) = \gamma - 1 + \tag{9}$$

$$+\frac{\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)} - \frac{e^{-t}zg''(e^{-t}z)}{g'(e^{-t}z)} + \beta\frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)}\left(1 - \frac{f(e^{-t}z)}{g(e^{-t}z)}\cdot\frac{g'(e^{-t}z)}{f'(e^{-t}z)}\right)}{1+\beta\frac{f(e^{-t}z)}{g(e^{-t}z)}\cdot\frac{g'(e^{-t}z)}{f'(e^{-t}z)}}\ .$$

From (1) and (2) we deduce that $w(z,t)$ is analytic in U. In view of (1), from (7) and (8) we have

$$|w(z,0)| = |A(z,0)| = \left| \frac{\gamma(\alpha+1)g'(z) - 1 - \beta\frac{f(z)}{g(z)} \cdot \frac{g'(z)}{f'(z)}}{1 + \beta\frac{f(z)}{g(z)} \cdot \frac{g'(z)}{f'(z)}} \right| < 1 \tag{10}$$

For $z = 0,\ t > 0$, since $|\gamma(\alpha+1) - (1+\beta)| \leq |\beta+1|$ and $|\gamma - 1| < 1$ we get

$$|w(0,t)| = \left| \frac{\gamma(\alpha+1) - (1+\beta)}{1+\beta} e^{-2t} + (\gamma-1)(1-e^{-2t}) \right| < 1 \tag{11}$$

If $t > 0$ is a fixed number and $z \in U$, $z \neq 0$, then the function $w(z,t)$ is analytic in $\bar{U}$ because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{U}$ and it is known that

$$|w(z,t)| = \max_{|\zeta|=1} |w(\zeta,t)| = |w(e^{i\theta},t)|, \quad \theta = \theta(t) \in R \tag{12}$$

Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and because $u \in U$, from (7), (8) and (9) taking into account (2) we get

$$|w(e^{i\theta},t)| \leq 1. \tag{13}$$

From (10), (11), (12) and (13) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$. From Theorem 1.1 it results that $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t \in I$. But $L(z,0) = F_\gamma(z)$ and then the function F_γ defined by (3) is analytic and univalent in U.

Remark. Suitable choices of g yields various type of univalence criteria, so we can take $g(z) \equiv z,\ g(z) \equiv f(z)$ or $g(z) \equiv z \cdot f'(z)$.

If in Theorem 2.1 we take $g(z) \equiv z$ we have the following result.

Corollary 2.1. *Let α, β, γ be complex numbers such that $\alpha \neq -1$,*

$$|\alpha - \beta| \leq |1+\beta|, \quad |\gamma - 1| < 1, \quad |\gamma(\alpha+1) - (\beta+1)| \leq |\beta+1|\,,$$

and let $f \in A$. If the inequalities

$$\left| \gamma(\alpha+1) - 1 - \beta\frac{f(z)}{zf'(z)} \right| < \left| 1 + \beta\frac{f(z)}{zf'(z)} \right|$$

and

$$\left| \left[\gamma(\alpha+1) - 1 - \beta\frac{f(z)}{zf'(z)}\right] \cdot |z|^2 + \right.$$

$$\left. +(1-|z|^2)\left[(\gamma-1)\left(1+\beta\frac{f(z)}{zf'(z)}\right) + \frac{zf''(z)}{f'(z)} + \beta\left(1 - \frac{f(z)}{zf'(z)}\right)\right]\right| \leq \left|1+\beta\frac{f(z)}{zf'(z)}\right|$$

are true for all $z \in U$, then the function F_γ defined by (3) is analytic and univalent in U.

Number of corollaries we can get for particular values of parameters α and β. We shall formulate only two: for $\beta = 0$ and for $\alpha = \beta = 0$.

For $\beta = 0$ we obtain from Corollary 2.1 a generalization of the well-known condition for univalence established by Ahlfors.

Corollary 2.2. *Let α, γ be complex numbers such that $\alpha \neq -1$, $|\alpha| \leq 1$, $|\gamma - 1| < 1$, $|\gamma(\alpha+1) - 1| \leq 1$ and let $f \in A$. If the inequality*

$$\left| [\gamma(\alpha+1)-1]|z|^2 + (1-|z|^2)\left[\frac{zf''(z)}{f'(z)} + \gamma - 1\right]\right| \leq 1$$

is true for all $z \in U$, then the function F_γ defined by (3) is analytic and univalent in U.

In the case $\gamma = 1$ we get $F_1(z) = f(z)$ and we have the univalence criterion found by Ahlfors ([1]).

Corollary 2.3. ([1]). *Let $\alpha \in C$, $|\alpha| \leq 1$, $\alpha \neq -1$ and let $f \in A$. If the inequality*

$$\left| \alpha|z|^2 + (1-|z|^2)\frac{zf''(z)}{f'(z)} \right| \leq 1$$

holds for $z \in U$, then the function f is univalent in U.

For $\alpha = \beta = 0$ we get

Corollary 2.4. *Let $\gamma \in C$, $|\gamma - 1| < 1$. If the inequality*

$$\left| (1-|z|^2)\frac{zf''(z)}{f'(z)} + \gamma - 1 \right| \leq 1$$

is true for all $z \in U$, then the function F_γ defined by (3) is analytic and univalent in U.

We recognize here the expression $(1-|z|^2)zf''(z)/f'(z)$ which appears in the condition for univalence established by Becker. We know that if this value lies in the unit disk U, then the function f is univalent in U and we observe that if this value lies in a disk with the same radius 1, but with the center in the point $1-\gamma$, $|\gamma - 1| < 1$ we obtain the analyticity and the univalence of the function F_γ.

If in Theorem 2.1 we take $f \equiv g$ we have a very simple result given by

Corollary 2.5. *Let α, β, $\gamma \in C$ such that $\alpha \neq -1$, $|\alpha - \beta| \leq |\beta + 1|$, $|\gamma - 1| < 1$, $|\gamma(\alpha+1) - (\beta+1)| \leq |\beta+1|$ and let $f \in A$. If the inequality*

$$\left| f'(z) - \frac{1+\beta}{\gamma(\alpha+1)} \right| < \frac{|1+\beta|}{|\gamma(\alpha+1)|}$$

is true for all $z \in U$, then the function F_γ defined by (3) is analytic and univalent in U.

Example. *Let $\gamma \in C$, $|\gamma - 1| < 1$. Then the function*

$$F(z) = z \cdot \left[1 + \frac{1-|\gamma|}{1+\gamma} \cdot z\right]^{1/\gamma}$$

is analytic and univalent in U.

To prove it consider the function $f \in A$ of the form

$$f(z) = z + \frac{1-|\gamma-1|}{2\gamma} \cdot z^2$$

and we apply corollary 2.5 with $\alpha = \beta$. So we have

$$\left| f'(z) - \frac{1}{\gamma} \right| = \left| \frac{\gamma - 1}{\gamma} + \frac{1 - |\gamma - 1|}{\gamma} \cdot z \right| \leq \frac{|\gamma - 1|}{|\gamma|} + \frac{1 - |\gamma - 1|}{|\gamma|} < \frac{1}{|\gamma|}.$$

Remark. For the case $\gamma = 1$ we have $F_1(z) = f(z)$ and from Theorem 2.1 we find the results obtained by S. Kanas and A. Lecko [3].

3. Univalence criteria and geometric mean

Substituting the arithmetic mean by the geometric one in the construction of the Löwner chain we obtain the following

Theorem 3.1. *Let α, β, γ be complex number such that $|\gamma - 1| < 1$, $Re\gamma > 1/2$ and let $f \in A$, $f'(z)f(z)/z \neq 0$ in U. If there exists a function $g \in A$, $g'(z)g(z)/z \neq 0$ in U, such that the inequalities*

$$\left| f'(z) \left(\frac{g'(z)}{f'(z)} \right)^{\alpha} \cdot \left(\frac{g(z)}{f(z)} \right)^{\beta} - 1 \right| < 1 , \tag{14}$$

$$\left| \left[f'(z) \cdot \left(\frac{g'(z)}{f'(z)} \right)^{\alpha} \left(\frac{g(z)}{f(z)} \right)^{\beta} - 1 \right] \cdot |z|^2 + \right. \tag{15}$$

$$\left. +(1 - |z|^2) \left[\alpha \left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} \right) + \beta \left(\frac{zf'(z)}{f(z)} - \frac{zg'(z)}{g(z)} \right) + \gamma - 1 \right] \right| \leq 1$$

are true for all $z \in U$, then the function

$$F_\gamma(z) = \left(\gamma \int_0^z u^{\gamma - 1} f'(u) du \right)^{1/\gamma} \tag{16}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. The method of the proof is similar to those of Theorem 2.1. Let us define

$$L(z,t) = \left[\int_0^{e^{-t}z} u^{\gamma - 1} f'(u) du + \right. \tag{17}$$

$$\left. + \left(e^{(2-\gamma)t} - e^{-\gamma t} \right) z^{\gamma} \left(\frac{f'(e^{-t}z)}{g'(e^{-t}z)} \right)^{\alpha} \left(\frac{f(e^{-t}z)}{g(e^{-t}z)} \right)^{\beta} \right]^{1/\gamma}$$

It can be shown that $L(z,t)$ is an analytic function in U_r, $r \in (0,1]$ for all $t \in I$, $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$, where

$$a_1(t) = e^{\frac{2-\gamma}{\gamma} t} \left(1 + \frac{1-\gamma}{\gamma} e^{-2t} \right)^{1/\gamma} \tag{18}$$

We fix a determination of $(1/\gamma)^{1/\gamma}$ denoted by δ. For $a_1(t)$ we fix the determination equal to δ for $t = 0$. Since $|\gamma - 1| < 1$ it follows that $\lim_{t\to\infty} |a_1(t)| = \infty$ and from $\mathrm{Re}\gamma > 1/2$ we have $|\gamma - 1| < |\gamma|$ and then $a_1(t) \neq 0$ for all $t \in I$.

Moreover, it can be prove that there is a disk U_{r_0}, $0 < r_0 < r$ such that $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_0} and

$\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_0}. For the functions $p(z,t)$ and $w(z,t)$ defined in (5) and (6), by computation we get

$$w(z,t) = \left[f'(e^{-t}z)\left(\frac{g'(e^{-t}z)}{f'(e^{-t}z)}\right)^{\alpha}\left(\frac{g(e^{-t}z)}{f(e^{-t}z)}\right)^{\beta} - 1 \right]\cdot e^{-2t} +$$

$$+(1-e^{-2t})\left[\alpha\left(\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)} - \frac{e^{-t}zg''(e^{-t}z)}{g'(e^{-t}z)}\right) + \right.$$

$$\left. +\,\beta\left(\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} - \frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)}\right) + \gamma - 1\right].$$

We observe that the function $w(z,t)$ is well-defined and analytic in U for each $t\in I$. The rest of the proof runs exactly as in Theorem 2.1. From Theorem 1.1 it results that the function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t\in I$, in particular $L(z,0)$. But

$$L(z,0) = \left(\int_0^z u^{\gamma-1}f'(u)du\right)^{1/\gamma}$$

and then also the function F_γ defined by (16) is analytic and univalent in U.

For $g(z)\equiv z$ we can deduce the following

Corollary 3.1. *Let $\alpha,\ \beta,\ \gamma\in C$ such that $|\gamma-1|<1$, $Re\gamma>1/2$ and let $f\in A,\ f'(z)f(z)/z\neq 0$ in U. If the inequalities*

$$\left|\left(\frac{z}{f(z)}\right)^{\beta}(\,f'(z)\,)^{1-\alpha} - 1\right| < 1$$

$$\left|\left[\left(\frac{z}{f(z)}\right)^{\beta}(f'(z))^{1-\alpha} - 1\right]|z|^2 + \right.$$

$$\left. +(1-|z|^2)\left[\alpha\frac{zf''(z)}{f'(z)} + \beta\left(\frac{zf'(z)}{f(z)} - 1\right) + \gamma - 1\right]\right| \le 1$$

hold for all $z\in U$, then the function F_γ defined by (16) is analytic and univalent in U.

For $\alpha=0$ and $\beta=1$, from Corollary 3.1 we get

Corollary 3.2. *Let $\gamma\in C$, $|\gamma-1|<1$, $Re\gamma>1/2$ and let $f\in A,\ f'(z)f(z)/z\neq 0$ in U. If the inequalities*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$$

$$\left|\left(\frac{zf'(z)}{f(z)} - 1\right)|z|^2 + (1-|z|^2)\left(\frac{zf'(z)}{f(z)} + \gamma - 2\right)\right| \le 1$$

hold for all $z\in U$, then the function F_γ defined by (16) is analytic and univalent in U.

For $g(z)\equiv f(z)$, from Theorem 3.1 we get the following useful result

Corollary 3.3. *Let $\gamma \in C$, $|\gamma - 1| < 1$, $Re\gamma > 1/2$ and let $f \in A$. If the inequality*

$$| f'(z) - 1 | < 1 \tag{19}$$

hold for all $z \in U$, then the function F_γ defined by (16) is analytic and univalent in U.

Indeed, the inequality (14) becomes (19) and the inequality (15) will be

$$\Big| (f'(z) - 1)|z|^2 + (1 - |z|^2)(\gamma - 1) \Big| \leq 1.$$

This inequality is true under the assumption $|\gamma - 1| < 1$ and in view of (19).

References

[1] L. V. Ahlfors, *Sufficient conditions for quasiconformal extension*, 79(1974), 23-29.

[2] J. Becker, *Löwnersche differentialgleichung und quasikonform fortsetzbare funktionen*, J. Reine Angew. Math., 255(1972), 23-43.

[3] S. Kanas, A. Lecko, *Univalence criteria connected with arithmetic and geometric means, II*, Zeszyty Nauk. Polit. Rzeszowskiej, 154(1996), 49-59.

[4] Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math., 218(1965), 159-173.

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ON THE DIRECT LIMIT OF A DIRECT SYSTEM OF COMPLETE MULTIALGEBRAS

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Abstract. In this paper we will prove that the direct limit of a direct system of complete multialgebras is a complete algebra.

1. Introduction

This paper deals with multialgebras. An important instrument in the study of the multialgebras is fundamental relation of a multialgebra, which can bring us into the class of the universal algebras. In [9] we proved that the fundamental algebra of a multialgebra verifies the identities of the given multialgebra. When trying to obtain multialgebras that verify (even in a weak manner) the identities of their fundamental algebra we obtained a new class of multialgebras. In the particular case of the semihypergroups these multialgebras are the complete semihypergroups that is why we called this multialgebras complete. We will prove that the complete multialgebras form a class of multialgebras closed under the formation of the direct limits of direct systems.

2. Preliminaries

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence with $n_\gamma \in \mathbb{N} = \{0, 1, \ldots\}$, where $o(\tau)$ is an ordinal and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and let us consider the algebra of the n-ary terms (of type τ) $\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)})$.

Let A be a set and $P^*(A)$ the set of the nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$. One can admit that the support set A of the multialgebra $\mathfrak{A}$ is empty if there are no nullary multioperations among the multioperations f_γ, $\gamma < o(\tau)$. Of course, any universal algebra is a multialgebra (we can identify an one element set with its element).

Defining for any $\gamma < o(\tau)$ and for any $A_0, \ldots, A_{n_\gamma-1} \in P^*(A)$

$$f_\gamma(A_0, \ldots, A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \mid\ a_i \in A_i,\ i \in \{0, \ldots, n_\gamma - 1\}\},$$

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we obtain a universal algebra on $P^*(A)$ (see [11]). We denote this algebra by $\mathfrak{P}^*(\mathfrak{A})$. As in [6], we can construct, for any $n \in \mathbb{N}$, the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of the n-ary term functions on $\mathfrak{P}^*(\mathfrak{A})$.

A mapping $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type τ is called homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \dots, a_{n_\gamma - 1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \dots, h(a_{n_\gamma-1})). \tag{1}$$

A bijective mapping h is a multialgebra isomorphism if both h and h^{-1} are multialgebra homomorphisms. The multialgebra isomorphisms can also be characterized as being those bijective homomorphisms for which (1) holds with equality.

Proposition 1. [8, Proposition 1] *For a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \dots, a_{n-1} \in A$ then $h(p(a_0, \dots, a_{n-1})) \subseteq p(h(a_0), \dots, h(a_{n-1}))$.*

The *fundamental relation* of a multialgebra $\mathfrak{A}$ as the transitive closure α^* of the relation α given on A as follows: for $x, y \in A$, $x\alpha y$ if and only if $x, y \in p(a_0, \dots, a_{n-1})$ for some $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, \dots, a_{n-1} \in A$ (see [7] and [9]). The relation α^* is the smallest equivalence relation on A such that the factor multialgebra $\mathfrak{A}/\alpha^*$ is a universal algebra. We denoted the class of $a \in A$ modulo α^* by $\overline{a}$ and A/α^* by $\overline{A}$. We also denoted the algebra $\mathfrak{A}/\alpha^*$ by $\overline{\mathfrak{A}}$ and we called it the *fundamental algebra* of the multialgebra $\mathfrak{A}$.

Proposition 2. [9, Proposition 3] *The following conditions are equivalent for a multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ of type τ :*

(i) *for all $\gamma < o(\tau)$, for all $a_0, \dots, a_{n_\gamma-1} \in A$,*

$$a \in f_\gamma(a_0, \dots, a_{n_\gamma-1}) \;\Rightarrow\; \overline{a} = f_\gamma(a_0, \dots, a_{n_\gamma-1}).$$

(ii) *for all $m \in \mathbb{N}$, for all $\mathbf{q}, \mathbf{r} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, m-1\}\}$, for all $a_0, \dots, a_{m-1}, b_0, \dots, b_{m-1} \in A$,*

$$q(a_0, \dots, a_{m-1}) \cap r(b_0, \dots, b_{m-1}) \neq \emptyset \;\Rightarrow\; q(a_0, \dots, a_{m-1}) = r(b_0, \dots, b_{m-1}).$$

The multialgebras which verify one of the equivalent conditions (i) and (ii) from the previous proposition are generalizations for the complete semihypergroups (see [3, Definition 137]). This fact suggests the following:

Definition 1. A multialgebra which satisfies one of the equivalent conditions from the previous proposition will be called *complete multialgebra*.

Remark 1. As we have seen in [9], a hypergroup $(H, \circ)$ can be identified with a multialgebra $(H, \circ, /, \backslash)$ with three binary multioperations, with $H \neq \emptyset$, $\circ$ associative (i.e. $(a \circ b) \circ c = a \circ (b \circ c)$, for all $a, b, c \in H$) and

$$a/b = \{x \in H \mid a \in x \circ b\},\ b\backslash a = \{x \in H \mid a \in b \circ x\}. \tag{2}$$

Remark 2. In [10] the complete hypergroups are defined as the complete semihypergroups which are hypergroups. For any elements a and b from a complete hypergroup $(H, \circ)$ there exists $b' \in H$ such that $a/b = a \circ b'$ and $b\backslash a = b' \circ a$ (see [10, Theorem 146]). Thus, a hypergroup $(H, \circ)$ is complete if and only if the multialgebra $(H, \circ, /, \backslash)$ from the previous remark is a complete multialgebra.

One can construct the category of the multialgebras of the same type τ where the morphisms are the multialgebra homomorphisms and the product is the usual mapping composition. We will denote this category by $\mathbf{Malg}(\tau)$. The complete multialgebras form a full subcategory $\mathbf{CMalg}(\tau)$ of the category $\mathbf{Malg}(\tau)$.

3. The direct limit of a direct system of complete multialgebras

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of complete multialgebras and let multialgebra $\mathfrak{A}_\infty = (A_\infty, (f_\gamma)_{\gamma < o(\tau)})$ be the direct limit of the direct system $\mathcal{A}$.

Remind that $(I, \leq)$ is a directed preordered set and the mappings φ_{ij} $(i, j \in I,\ i \leq j)$ are such that for any $i, j, k \in I$, with $i \leq j \leq k$, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ and $\varphi_{ii} = 1_{A_i}$, for all $i \in I$. Also remind that the set A_∞ is the direct limit of the direct system of sets $((A_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ and it is obtained as follows: on the disjoint union A of the sets A_i one defines the relation $\equiv$ as follows: for any $x, y \in A$ there exist $i, j \in I$, such that $x \in A_i,\ y \in A_j$, and $x \equiv y$ if and only if $\varphi_{ik}(x) = \varphi_{jk}(y)$ for some $k \in I$ with $i \leq k,\ j \leq k$. This relation on A is an equivalence and A_∞ is the quotient set $A/_\equiv = \{\widehat{x} \mid x \in A\}$ (see [6]).

The multioperations of the direct limit multialgebra are defined as follows: let $\gamma < o(\tau)$ and $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma - 1}} \in A_\infty$ and for any $j \in \{0, \ldots, n_\gamma - 1\}$ let us consider that $x_j \in A_{i_j}$ $(i_j \in I)$. Then

$$\begin{aligned} f_\gamma(\widehat{a_0}, \ldots, \widehat{a_{n_\gamma - 1}}) = \{\widehat{a} \in A_\infty \mid \exists m \in I,\ i_0 \leq m, \ldots,\ i_{n_\gamma - 1} \leq m, \\ a \in f_\gamma(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n_\gamma - 1} m}(a_{n_\gamma - 1}))\}. \end{aligned}$$

Lemma 1. [10, Lemma 2] *Let $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \ldots, a_{n-1} \in A$. If $i_0, \ldots, i_{n-1} \in I$ are such that $a_j \in A_{i_j}$ for all $j \in \{0, \ldots, n-1\}$ then*

$$\begin{aligned} p(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) = \{\widehat{a} \in A_\infty \mid \exists m \in I,\ i_0 \leq m, \ldots,\ i_{n-1} \leq m, \\ a \in p(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n-1} m}(a_{n-1}))\}. \end{aligned}$$

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of multialgebras and let us consider $J \subseteq I$ such that $(J, \leq)$ is also directed. We will denote by $\mathcal{A}_J$ the direct system whose carrier is $(J, \leq)$, consisting of the multialgebras $(\mathfrak{A}_i \mid i \in J)$ and the homomorphisms $(\varphi_{ij} \mid i, j \in J,\ i \leq j)$.

Proposition 3. [10, Proposition 1] *Let $\mathcal{A}$ be a direct system of multialgebras with the carrier $(I, \leq)$ and let us consider $J \subseteq I$ such that $(J, \leq)$ is a directed preordered set cofinal with $(I, \leq)$. Then the multialgebras $\varinjlim \mathcal{A}$ and $\varinjlim \mathcal{A}_J$ are isomorphic.*

Remark 3. This proposition was proved for the case when $(I, \leq)$ is an ordered set. Yet, the antisymmetry of the relation $\leq$ is not involved in the proof.

The main result of this paper is the following:

Theorem 1. *The category $\mathbf{CMalg}(\tau)$ is a subcategory of the category $\mathbf{Malg}(\tau)$ which closed under direct limits of direct systems.*

Proof. Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ a direct system of complete multialgebras, let $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau) \setminus \{\mathbf{x}_j \mid j \in \{0, \ldots, n-1\}\}$ and $\widehat{a_0}, \ldots, \widehat{a_{n-1}}$,

$\widehat{b_0}, \ldots, \widehat{b_{n-1}} \in A_\infty$. We can consider that the representatives $a_0, \ldots, a_{n-1}$, $b_0, \ldots, b_{n-1}$ of the given classes are from the set A_k ($k \in I$). If

$$q(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) \cap r(\widehat{b_0}, \ldots, \widehat{b_{n-1}}) \neq \emptyset$$

then there exists $a \in \bigcup_{i \in I} A_i$ such that

$$\widehat{a} \in q(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) \cap r(\widehat{b_0}, \ldots, \widehat{b_{n-1}}).$$

From $\widehat{a} \in q(\widehat{a_0}, \ldots, \widehat{a_{n-1}})$ it results that there exist $m' \in I$, $m' \geq k$, and $a' \equiv a$ such that

$$a' \in q(\varphi_{km'}(a_0), \ldots, \varphi_{km'}(a_{n-1})) \subseteq A_{m'}.$$

Analogously, from $\widehat{a} \in r(\widehat{b_0}, \ldots, \widehat{b_{n-1}})$ it follows that there exist $m'' \in I$, $m'' \geq k$, and $a'' \equiv a$ such that

$$a'' \in r(\varphi_{km''}(b_0), \ldots, \varphi_{km''}(b_{n-1})) \subseteq A_{m''}.$$

Let $\widehat{x}$ be an arbitrary element from $q(\widehat{a_0}, \ldots, \widehat{a_{n-1}})$. Then there exists $l \in I$ with $k \leq l$ such that

$$x \in q(\varphi_{kl}(a_0), \ldots, \varphi_{kl}(a_{n-1})) \subseteq A_l.$$

From $a' \equiv a \equiv a''$ we deduce the existence of an element $m''' \in I$ with $m' \leq m'''$, $m'' \leq m'''$, such that $\varphi_{m'm'''}(a') = \varphi_{m''m'''}(a'')$. Since $(I, \leq)$ is directed, there exists $m \in I$ with $m''' \leq m$ and $l \leq m$. According to Proposition 1 we have

$$\begin{aligned}\varphi_{lm}(x) &\in \varphi_{lm}(q(\varphi_{kl}(a_0), \ldots, \varphi_{kl}(a_{n-1}))) \\ &\subseteq q(\varphi_{lm}(\varphi_{kl}(a_0)), \ldots, \varphi_{lm}(\varphi_{kl}(a_{n-1}))) \\ &= q(\varphi_{km}(a_0), \ldots, \varphi_{km}(a_{n-1})) \subseteq A_m.\end{aligned}$$

Also,

$$\begin{aligned}\varphi_{m'm}(a') &\in \varphi_{m'm}(q(\varphi_{km'}(a_0), \ldots, \varphi_{km'}(a_{n-1}))) \\ &\subseteq q(\varphi_{m'm}(\varphi_{km'}(a_0)), \ldots, \varphi_{m'm}(\varphi_{km'}(a_{n-1}))) \\ &= q(\varphi_{km}(a_0), \ldots, \varphi_{km}(a_{n-1})) \subseteq A_m\end{aligned}$$

and, analogously,

$$\varphi_{m''m}(a'') \in r(\varphi_{km}(b_0), \ldots, \varphi_{km}(b_{n-1})) \subseteq A_m.$$

But

$$\varphi_{m'm}(a') = \varphi_{m'''m}(\varphi_{m'm'''}(a')) = \varphi_{m'''m}(\varphi_{m''m'''}(a'')) = \varphi_{m''m}(a''),$$

and, since the multialgebra $\mathfrak{A}_m$ is complete it follows that

$$\varphi_{lm}(x) \in q(\varphi_{km}(a_0), \ldots, \varphi_{km}(a_{n-1})) = r(\varphi_{km}(b_0), \ldots, \varphi_{km}(b_{n-1})).$$

Consequently, $\widehat{x} \in r(\widehat{b_0}, \ldots, \widehat{b_{n-1}})$. Thus we have proved that

$$q(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) \subseteq r(\widehat{b_0}, \ldots, \widehat{b_{n-1}}).$$

Similarly, one can show that $q(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) \supseteq r(\widehat{b_0}, \ldots, \widehat{b_{n-1}})$, so, we have

$$q(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) = r(\widehat{b_0}, \ldots, \widehat{b_{n-1}}).$$

Thus the multialgebra $\mathfrak{A}_\infty$ is complete. □

Corollary 1. *Let* $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ *be a direct system of multialgebras. If any* $i,j \in I$ *have an upper bound* $k \in I$ *such that* $\mathfrak{A}_k$ *is a complete multialgebra, then* $\mathfrak{A}_\infty$ *is a complete multialgebra.*

This follows from the previous theorem and Proposition 3 since the set

$$J = \{k \in I \mid \mathfrak{A}_k \text{ is a complete multialgebra}\}$$

(with the restriction of $\leq$ from I) is a directed preordered set cofinal with $(I, \leq)$.

In [12] are proved the following theorems:

Theorem 2. [12, Theorem 3] *Let* $(((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ *be a direct system of semihypergroups. The direct limit* $(H', \circ)$ *of this system is a semihypergroup.*

Theorem 3. [12, Theorem 4] *Let* $(((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ *be a direct system of semihypergroups having the property that for any* $i,j \in I$ *there exists* $k \in I$, $i \leq k$, $j \leq k$ *such that* H_k *is a hypergroup. The direct limit* $(H', \circ)$ *of this system is a hypergroup.*

Remark 4. In [12] is considered that $(I, \leq)$ is partially ordered, but the property holds even if $(I, \leq)$ is only preordered.

Remark 5. If we see each hypergroup $(H_i, \circ_i)$ as a multialgebra $(H_i, \circ_i, /_i, \backslash_i)$ as in Remark 1 we obtain a direct system of multialgebras of type τ. If we consider for this system the direct limit multialgebra $(H_\infty, \circ_\infty, /, \backslash)$ then $H' = H_\infty$, $\circ = \circ_\infty$ and the multioperations $/, \backslash$ are obtained from $\circ$ using (2).

From Remark 2, Theorem 3 and Theorem 1 we have:

Corollary 2. *The direct limit of a direct system of complete (semi)hypergroups is a complete (semi)hypergroup.*

Using, in addition, Corollary 1 we also have:

Corollary 3. *Let* $(((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \leq j))$ *be a direct system of semihypergroups having the property that for any* $i,j \in I$ *there exists* $k \in I$, $i \leq k$, $j \leq k$ *such that* H_k *is a complete (semi)hypergroup. The direct limit* $(H', \circ)$ *of this system is a complete (semi)hypergroup.*

References

[1] S. Breaz, C. Pelea, Multialgebras and term functions over the algebra of their nonvoid subsets, *Mathematica (Cluj)* **43**(66), 2, 2001, 143-149.

[2] S. Burris, H. P. Sankappanavar, A course in universal algebra, *Springer–Verlag, New-York*, 1981.

[3] P. Corsini, Prolegomena of hypergroup theory. Supplement to Riv. Mat. Pura Appl. *Aviani Editore, Tricesimo*, 1993.

[4] P. Corsini, V. Leoreanu, Applications of hyperstructure theory, *Kluwer Academic Publishers, Boston-Dordrecht-London*, 2003.

[5] G. Grätzer, A representation theorem for multi-algebras. *Arch. Math.* **3**, 1962, 452-456.

[6] G. Grätzer, Universal algebra. *D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London* 1968.

[7] C. Pelea, On the fundamental relation of a multialgebra, *Ital. J. Pure Appl. Math.* **10** 2001, 141-146.

[8] C. Pelea, On the direct product of multialgebras, *Studia Univ. Babeş–Bolyai Math.*, **48**, 2, 2003, 93-98.
[9] C. Pelea, Identities and multialgebras, preprint.
[10] C. Pelea, On the direct limit of a direct system of multialgebras, preprint.
[11] H. E. Pickett, Homomorphisms and subalgebras of multialgebras, *Pacific J. Math.* **21** 1967, 327-342.
[12] G. Romeo, Limite diretto di semi-ipergruppi, e ipergruppi d'associatività, *Riv. Mat. Univ. Parma* **8** 1982, 281-288.
[13] T. Vougiouklis, Construction of H_v-structures with desired fundamental structures,*New frontiers in hyperstructures (Molise, 1995)*, 177-188, Ser. New Front. Adv. Math. Ist. Ric. Base, *Hadronic Press, Palm Harbor, FL*, 1996.

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A REMARKABLE STRUCTURE AND CONNECTIONS ON THE TANGENT BUNDLE

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Abstract. The present paper deals with the conformal almost symplectic structure on TM. Starting from the notion of conformal almost symplectic structure in the tangent bundle, we define the notion of general conformal almost symplectic d-linear connection and respective conformal almost symplectic d-linear connection with respect to a conformal almost symplectic structure $\hat{A}$, corresponding to the 1-forms ω and $\tilde{\omega}$ in TM. We determine the set of all general conformal almost symplectic d-linear connections on TM, in the case when the nonlinear connection is arbitrary and we find important particular cases.

1. **Introduction**

The geometry of the tangent bundle (TM, π, M) has been studied by R.Miron and M.Anastasiei in [6], by R.Miron and M.Hashiguchi in [7], by V.Oproiu in [8], by Gh.Atanasiu and I.Ghinea in [1], by R.Bowman in [2], by K.Yano and S.Ishihara in [10],etc.

Concerning the terminology and notations, we use those from [4].

Let M be a real C$^\infty$-differentiable manifold with dimension n, (n=2n') and (TM, π, M) its tangent bundle.

If (x^i) is a local coordinates system on a domain U of a chart on M, the induced system of coordonates on $\pi^{-1}(U)$ is $(x^i, y^i), (i = 1, ..., n)$.

Let N be a nonlinear connection on TM, with the coefficients $N^j{}_i(x, y), (i, j = 1, ..., n)$.

We consider on TM an almost symplectic structure A defined by:

$$A(x,y) = \frac{1}{2}a_{ij}(x,y)dx^i \wedge dx^j + \frac{1}{2}\tilde{a}_{ij}(x,y)\delta y^i \wedge \delta y^j, \tag{1}$$

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where $(dx^i, \delta y^i), (i = 1, ..., n)$ is the dual basis of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, and $(a_{ij}(x,y),\ \tilde{a}_{ij}(x,y))$ is a pair of given d-tensor fields on TM, of the type (0,2), each of them alternate and nondegenerate.

We asociate to the lift A the Obata's operators:

$$\begin{cases} \Phi^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - a_{sj}a^{ir}),\ \Phi^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + a_{sj}a^{ir}), \\ \tilde{\Phi}^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - \tilde{a}_{sj}\tilde{a}^{ir}),\ \tilde{\Phi}^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + \tilde{a}_{sj}\tilde{a}^{ir}). \end{cases} \tag{2}$$

Obata's operators have the same properties as the ones associated with a Finsler space [7].

Let $\mathcal{A}_2(TM)$ be the set of all alternate d-tensor fields, of the type (0,2) on TM. As is easily shown, the relations on $\mathcal{A}_2(TM)$ defined by (3):

$$\begin{cases} (a_{ij} \sim b_{ij}) \Leftrightarrow \left((\exists)\, \lambda(x,y) \in \mathcal{F}(TM), a_{ij}(x,y) = e^{2\lambda(x,y)} b_{ij}(x,y)\right), \\ \left(\tilde{a}_{ij} \sim \tilde{b}_{ij}\right) \Leftrightarrow \left((\exists)\, \mu(x,y) \in \mathcal{F}(TM), \tilde{a}_{ij}(x,y) = e^{2\mu(x,y)} \tilde{b}_{ij}(x,y)\right), \end{cases} \tag{3}$$

is an equivalence relation on $\mathcal{A}_2(TM)$.

Definition 1.1. *The equivalent class: $\hat{A}$ of $\mathcal{A}_2(TM)/_\sim$ to which the almost symplectic tensor field A belongs, is called conformal almost symplectic structure on TM.*

Thus:

$$\hat{A} = \{A' | A'_{ij}(x,y) = e^{2\lambda(x,y)} a_{ij}(x,y) \text{ and } \tilde{A}'_{ij}(x,y) = e^{2\mu(x,y)} \tilde{a}_{ij}(x,y)\}. \tag{4}$$

2. General conformal almost symplectic d-linear connections on TM.

Definition 2.1. *A d-linear connection, D, on TM, with local coefficients $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$, is called general conformal almost symplectic d-linear connection on TM if:*

$$a_{ij|k} = K_{ijk},\ a_{ij}|_k = Q_{ijk},\ \tilde{a}_{ij|k} = \tilde{K}_{ijk},\ \tilde{a}_{ij}|_k = \tilde{Q}_{ijk}, \tag{5}$$

where $K_{ijk}, Q_{ijk}, \tilde{K}_{ijk}, \tilde{Q}_{ijk}$ are arbitrary tensor fields, of the type $(0,3)$ on TM, with the properties:

$$K_{ijk} = -K_{jik},\ Q_{ijk} = -Q_{jik},\ \tilde{K}_{ijk} = -\tilde{K}_{jik},\ \tilde{Q}_{ijk} = -\tilde{Q}_{jik} \tag{6}$$

and |, | denote the h-and respective v-covariant derivatives with respect to D.

Particularly, we have:

Definition 2.2. *A d-linear connection, D, on TM, with local coefficients $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$, for which there exists the 1-forms ω and $\tilde{\omega}$ in TM, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$, $\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}_i \delta y^i$ such that:*

$$\begin{cases} a_{ij|k} = 2\omega_k a_{ij}, \quad a_{ij}|_k = 2\dot{\omega}_k a_{ij}, \\ \tilde{a}_{ij|k} = 2\tilde{\omega}_k \tilde{a}_{ij}, \quad \tilde{a}_{ij}|_k = 2\,\dot{\tilde{\omega}}_k\,\tilde{a}_{ij}, \end{cases} \tag{7}$$

where $|$ *and* $\big|$ *denote the h-and v-covariant derivatives with respect to D,is called conformal almost symplectic d-linear connection on TM, with respect to the conformal almost symplectic structure* $\hat{A}$*, corresponding to the 1-forms* $\omega, \tilde{\omega}$ *and is denoted by:* $D\Gamma(N, \omega, \tilde{\omega})$.

We shall determine the set of all general conformal almost symplectic d-linear connections, with respect to $\hat{A}$.

Let $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}_{jk}, \overset{0}{\tilde{L}^i}_{jk}, \overset{0}{\tilde{C}^i}_{jk}, \overset{0}{C^i}_{jk})$ be the local coefficients of a fixed d-linear connection $\overset{0}{D}$ on TM. Then any d-linear connection, D, on TM, with local coefficients: $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$,can be expresed in the form:

$$\begin{cases} N^i{}_j = \overset{0}{N^i}_j - A^i{}_j, \\ L^i{}_{jk} = \overset{0}{L^i}_{jk} + A^l{}_k \overset{0}{\tilde{C}^i}_{jl} - B^i{}_{jk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}^i}_{jk} + A^l{}_k \overset{0}{C^i}_{jl} - \tilde{B}^i{}_{jk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}^i}_{jk} - \tilde{D}^i{}_{jk}, \\ C^i{}_{jk} = \overset{0}{C^i}_{jk} - D^i{}_{jk}, \\ A^l{}_{j\overset{0}{|}k} = 0, \end{cases} \tag{8}$$

where $(A^i{}_j, B^i{}_{jk}, \tilde{B}^i{}_{jk}, \tilde{D}^i{}_{jk}, D^i{}_{jk})$ are components of the difference tensor fields of $D\Gamma(N)$ from $\overset{0}{D}\ \Gamma(\overset{0}{N})$, [4] and $\overset{0}{|}$, $\overset{0}{\big|}$ denotes the h-and v-covariant derivatives with respect to $\overset{0}{D}$.

Theorem 2.1. *Let* $\overset{0}{D}$ *be a given d-linear connection on TM, with local coefficients* $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}_{jk}, \overset{0}{\tilde{L}^i}_{jk}, \overset{0}{\tilde{C}^i}_{jk}, \overset{0}{C^i}_{jk})$*. The set of all general conformal almost symplectic d-linear connections on TM, with local coefficients* $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ *is given by:*

$$\begin{cases} N^i_{\ j} = \overset{0}{N^i_{\ j}} - X^i_{\ j}, \\ L^i_{\ jk} = \overset{0}{L^i_{\ jk}} + \overset{0}{\tilde{C}^i_{\ jm}} X^m_{\ k} + \frac{1}{2} a^{is} (a_{\underset{sj|k}{0}} + a_{sj} \overset{0}{|}_m X^m_{\ k} - K_{sjk}) + \Phi^{ir}_{hj} X^h_{\ rk}, \\ \tilde{L}^i_{\ jk} = \overset{0}{\tilde{L}^i_{\ jk}} + \overset{0}{C^i_{\ jm}} X^m_{\ k} + \frac{1}{2} \tilde{a}^{is} (\tilde{a}_{\underset{sj|k}{0}} + \tilde{a}_{sj} \overset{0}{|}_m X^m_{\ k} - \tilde{K}_{sjk}) + \tilde{\Phi}^{ir}_{hj} \tilde{X}^h_{\ rk}, \\ \tilde{C}^i_{\ jk} = \overset{0}{\tilde{C}^i_{\ jk}} + \frac{1}{2} a^{is} (a_{sj} \overset{0}{|}_k - Q_{sjk}) + \Phi^{ir}_{hj} \tilde{Y}^h_{\ rk}, \\ C^i_{\ jk} = \overset{0}{C^i_{\ jk}} + \frac{1}{2} \tilde{a}^{is} (\tilde{a}_{sj} \overset{0}{|}_k - \tilde{Q}_{sjk}) + \tilde{\Phi}^{ir}_{hj} Y^h_{\ rk}, \ X^i_{\underset{j|k}{0}} = 0, \end{cases} \tag{9}$$

where $X^i_{\ j},\ X^i_{\ jk},\ \tilde{X}^i_{\ jk},\ \tilde{Y}^i_{\ jk},\ Y^i_{\ jk}$ are arbitrary tensor fields on TM, $\overset{0}{|}, \overset{0}{\big|}$ denote the h-and respective v-covariant derivatives with respect to $\overset{0}{D}$ and $K_{ijk}, Q_{ijk}, \tilde{K}_{ijk}, \tilde{Q}_{ijk}$ are arbitrary tensor fields of the type (0,3) on TM with the properties (6).

Particular cases:

1. If $X^i_{\ j} = X^i_{\ jk} = \tilde{X}^i_{\ jk} = \tilde{Y}^i_{\ jk} = Y^i_{\ jk} = 0$ in Theorem 2.1. we have:

Theorem 2.2. *Let $\overset{0}{D}$ be a given d-linear connection on TM, with local coefficients $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i_{\ jk}}, \overset{0}{\tilde{L}^i_{\ jk}}, \overset{0}{\tilde{C}^i_{\ jk}}, \overset{0}{C^i_{\ jk}})$. Then the following d-linear connection D, with local coefficients $D\Gamma(\overset{0}{N}) = (L^i_{\ jk}, \tilde{L}^i_{\ jk}, \tilde{C}^i_{\ jk}, C^i_{\ jk})$ given by (10) is a general conformal almost symplectic d-linear connection with respect to $\hat{A}$:*

$$\begin{cases} L^i_{\ jk} = \overset{0}{L^i_{\ jk}} + \frac{1}{2} a^{is} (a_{\underset{sj|k}{0}} - K_{sjk}), \\ \tilde{L}^i_{\ jk} = \overset{0}{\tilde{L}^i_{\ jk}} + \frac{1}{2} \tilde{a}^{is} (\tilde{a}_{\underset{sj|k}{0}} - \tilde{K}_{sjk}), \\ \tilde{C}^i_{\ jk} = \overset{0}{\tilde{C}^i_{\ jk}} + \frac{1}{2} a^{is} (a_{sj} \overset{0}{|}_k - Q_{sjk}), \\ C^i_{\ jk} = \overset{0}{C^i_{\ jk}} + \frac{1}{2} \tilde{a}^{is} (\tilde{a}_{sj} \overset{0}{|}_k - \tilde{Q}_{sjk}), \end{cases} \tag{10}$$

where $\overset{0}{|}, \overset{0}{\big|}$ denote the h-and respective v-covariant derivatives with respect to the given d-linear connection $\overset{0}{D}$ and $K_{ijk}, Q_{ijk}, \tilde{K}_{ijk}, \tilde{Q}_{ijk}$ are arbitrary tensor fields of the type (0,3) on TM with the properties (6).

2. If $K_{ijk} = \tilde{K}_{ijk} = \tilde{Q}_{ijk} = Q_{ijk} = 0$ in Theorem 2.1 we have :

Theorem 2.3. *Let $\overset{0}{D}$ be a given d-linear connection on TM, with local coefficients $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i_{\ jk}}, \overset{0}{\tilde{L}^i_{\ jk}}, \overset{0}{\tilde{C}^i_{\ jk}}, \overset{0}{C^i_{\ jk}})$. The set of all almost symplectic d-linear connections on TM, with local coefficients $D\Gamma(N) = (L^i_{\ jk}, \tilde{L}^i_{\ jk}, \tilde{C}^i_{\ jk}, C^i_{\ jk})$ is given by:*

$$\begin{cases} N^i_{\ j} = \overset{0}{N^i_{\ j}} - X^i_{\ j}, \\ L^i_{\ jk} = \overset{0}{L^i_{\ jk}} + \overset{0}{\tilde{C}^i_{\ jm}} X^m_{\ k} + \frac{1}{2}a^{is}(a_{sj\overset{0}{|}k} + a_{sj} \overset{0}{|}_m X^m_{\ k}) + \Phi^{ir}_{hj}X^h_{\ rk}, \\ \tilde{L}^i_{\ jk} = \overset{0}{\tilde{L}^i_{\ jk}} + \overset{0}{C^i_{\ jm}} X^m_{\ k} + \frac{1}{2}\tilde{a}^{is}(\tilde{a}_{sj\overset{0}{|}k} + \tilde{a}_{sj} \overset{0}{|}_m X^m_{\ k}) + \tilde{\Phi}^{ir}_{hj}\tilde{X}^h_{\ rk}, \\ \tilde{C}^i_{\ jk} = \overset{0}{\tilde{C}^i_{\ jk}} + \frac{1}{2}a^{is}a_{sj} \overset{0}{|}_k + \Phi^{ir}_{hj}\tilde{Y}^h_{\ rk}, \\ C^i_{\ jk} = \overset{0}{C^i_{\ jk}} + \frac{1}{2}\tilde{a}^{is}\tilde{a}_{sj} \overset{0}{|}_k + \tilde{\Phi}^{ir}_{hj}Y^h_{\ rk}, \ X^i_{\ j\overset{0}{|}k} = 0, \end{cases} \tag{11}$$

where $X^i_{\ j},\ X^i_{\ jk},\ \tilde{X}^i_{\ jk},\ \tilde{Y}^i_{\ jk},\ Y^i_{\ jk}$ are arbitrary tensor fields on TM and $\overset{0}{|}, \overset{0}{\Big|}$ denote the h-and respective v-covariant derivatives with respect to $\overset{0}{D}$.

3. If $K_{ijk} = 2a_{ij}\omega_k, \tilde{K}_{ijk} = 2\tilde{a}_{ij}\tilde{\omega}_k, \tilde{Q}_{ijk} = 2\tilde{a}_{ij}\dot{\tilde{\omega}}_k, Q_{ijk} = 2a_{ij}\dot{\omega}_k$, such that $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$ and respective $\tilde{\omega} = \tilde{\omega}_i dx_i + \dot{\tilde{\omega}}_i \delta y^i$ are two 1-forms in TM, then from (9) we have the set of all conformal almost symplectic d-linear connections on TM:

Theorem 2.4. *Let $\overset{0}{D}$ be a given d-linear connection on TM, with local coefficients $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i_{\ jk}}, \overset{0}{\tilde{L}^i_{\ jk}}, \overset{0}{\tilde{C}^i_{\ jk}}, \overset{0}{C^i_{\ jk}})$. Then set of all conformal almost symplectic d-linear connections on TM, with respect to $\hat{A}$, corresponding to the 1-forms ω and $\tilde{\omega}$, with local coefficients $D\Gamma(N, \omega, \tilde{\omega}) = (L^i_{\ jk}, \tilde{L}^i_{\ jk}, \tilde{C}^i_{\ jk}, C^i_{\ jk})$ is given by:*

$$\begin{cases} N^i_{\ j} = \overset{0}{N^i_{\ j}} - X^i_{\ j}, \\ L^i_{\ jk} = \overset{0}{L^i_{\ jk}} + \overset{0}{\tilde{C}^i_{\ jm}} X^m_{\ k} + \frac{1}{2}a^{is}(a_{sj\overset{0}{|}k} + a_{sj} \overset{0}{|}_m X^m_{\ k}) - \delta^i_{\ j}\omega_k + \Phi^{ir}_{hj}X^h_{\ rk}, \\ \tilde{L}^i_{\ jk} = \overset{0}{\tilde{L}^i_{\ jk}} + \overset{0}{C^i_{\ jm}} X^m_{\ k} + \frac{1}{2}\tilde{a}^{is}(\tilde{a}_{sj\overset{0}{|}k} + \tilde{a}_{sj} \overset{0}{|}_m X^m_{\ k}) - \delta^i_{\ j}\tilde{\omega}_k + \tilde{\Phi}^{ir}_{hj}\tilde{X}^h_{\ rk}, \\ \tilde{C}^i_{\ jk} = \overset{0}{\tilde{C}^i_{\ jk}} + \frac{1}{2}a^{is}a_{sj} \overset{0}{|}_k - \delta^i_{\ j}\dot{\omega}_k + \Phi^{ir}_{hj}\tilde{Y}^h_{\ rk}, \\ C^i_{\ jk} = \overset{0}{C^i_{\ jk}} + \frac{1}{2}\tilde{a}^{is}\tilde{a}_{sj} \overset{0}{|}_k - \delta^i_{\ j}\dot{\tilde{\omega}}_k + \tilde{\Phi}^{ir}_{hj}Y^h_{\ rk}, \\ \\ X^i_{\ j\overset{0}{|}k} = 0, \end{cases} \tag{12}$$

where $X^i_{\ j},\ X^i_{\ jk},\ \tilde{X}^i_{\ jk},\ \tilde{Y}^i_{\ jk},\ Y^i_{\ jk}$ are arbitrary tensor fields on TM, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$ and respective $\tilde{\omega} = \tilde{\omega}_i dx_i + \dot{\tilde{\omega}}_i \delta y^i$ are arbitrary 1-forms in TM and $\overset{0}{|}, \overset{0}{\Big|}$ denote the h-and respective v-covariant derivatives with respect to $\overset{0}{D}$.

4. If $X^i_{\ j} = X^i_{\ jk} = \tilde{X}^i_{\ jk} = \tilde{Y}^i_{\ jk} = Y^i_{\ jk} = 0$ in Theorem 2.4. we have:

Theorem 2.5. *Let $\overset{0}{D}$ be a given d-linear connection on TM, with local coefficients $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}_{jk}, \overset{0}{\tilde{L}^i}_{jk}, \overset{0}{\tilde{C}^i}_{jk}, \overset{0}{C^i}_{jk})$. Then the following d-linear connection D, with local coefficients $D\Gamma(\overset{0}{N}, \omega, \tilde{\omega}) = (L^i_{\ jk}, \tilde{L}^i_{\ jk}, \tilde{C}^i_{\ jk}, C^i_{\ jk})$ given by (13) is a conformal almost symplectic d-linear connection with respect to $\hat{A}$, corresponding to the 1-forms ω and $\tilde{\omega}$:*

$$\begin{cases} L^i_{\ jk} = \overset{0}{L^i}_{jk} + \frac{1}{2}a^{is}a_{sj\overset{0}{|}k} - \delta^i_j\omega_k, \\ \tilde{L}^i_{\ jk} = \overset{0}{\tilde{L}^i}_{jk} + \frac{1}{2}\tilde{a}^{is}\tilde{a}_{sj\overset{0}{|}k} - \delta^i_j\tilde{\omega}_k, \\ \tilde{C}^i_{\ jk} = \overset{0}{\tilde{C}^i}_{jk} + \frac{1}{2}a^{is}a_{sj}\overset{0}{|}_k - \delta^i_j\dot{\omega}_k, \\ C^i_{\ jk} = \overset{0}{C^i}_{jk} + \frac{1}{2}\tilde{a}^{is}\tilde{a}_{sj}\overset{0}{|}_k - \delta^i_j\dot{\tilde{\omega}}_k, \end{cases} \tag{13}$$

where $\overset{0}{|}$, $\overset{0}{|}$ denote the h-and respective v-covariant derivatives with respect to the given d-linear connection $\overset{0}{D}$ and $\omega = \omega_i dx^i + \dot{\omega}_i\delta y^i$ and respective $\tilde{\omega} = \tilde{\omega}_i dx_i + \dot{\tilde{\omega}}_i\delta y^i$ are two given 1-forms in TM.

5. If we take an almost symplectic d-linear connection as $\overset{0}{D}$ in Theorem 2.5, then (13) becomes:

$$\begin{cases} L^i_{\ jk} = \overset{0}{L^i}_{jk} - \delta^i_j\omega_k, \\ \tilde{L}^i_{\ jk} = \overset{0}{\tilde{L}^i}_{jk} - \delta^i_j\tilde{\omega}_k, \\ \tilde{C}^i_{\ jk} = \overset{0}{\tilde{C}^i}_{jk} - \delta^i_j\dot{\omega}_k, \\ C^i_{\ jk} = \overset{0}{C^i}_{jk} - \delta^i_j\dot{\tilde{\omega}}_k. \end{cases} \tag{14}$$

6. If we take a conformal almost symplectic d-linear connection with respect to $\hat{A}$ as $\overset{0}{D}$ in Theorem 2.4, we have

Theorem 2.6. *Let $\overset{0}{D}$ be a given conformal almost symplectic d-linear connection on TM, with local coefficients: $\overset{0}{D}\Gamma(\overset{0}{N}, \omega, \tilde{\omega}) = (\overset{0}{L^i}_{jk}, \overset{0}{\tilde{L}^i}_{jk}, \overset{0}{\tilde{C}^i}_{jk}, \overset{0}{C^i}_{jk})$. The set of all conformal almost symplectic d-linear connections on TM, with respect to $\hat{A}$, corresponding to the 1-forms ω and $\tilde{\omega}$, with local coefficients $D\Gamma(N, \omega, \tilde{\omega}) = (L^i_{\ jk}, \tilde{L}^i_{\ jk}, \tilde{C}^i_{\ jk}, C^i_{\ jk})$ is given by:*

$$\begin{cases} N^i{}_j = \overset{0}{N}{}^i{}_j - X^i{}_j, \\ L^i{}_{jk} = \overset{0}{L}{}^i{}_{jk} + (\overset{0}{\tilde{C}}{}^i{}_{jm} + \delta^i_j \dot{\omega}_m) X^m{}_k + + \Phi^{ir}_{hj} X^h{}_{rk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}}{}^i{}_{jk} + (\overset{0}{C}{}^i{}_{jm} + \delta^i_j \dot{\tilde{\omega}}_m) X^m{}_k + \tilde{\Phi}^{ir}_{hj} \tilde{X}^h{}_{rk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}}{}^i{}_{jk} + \Phi^{ir}_{hj} \tilde{Y}^h{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \tilde{\Phi}^{ir}_{hj} Y^h{}_{rk}, \\ X^i{}_{\underset{j|k}{0}} = 0, \end{cases} \tag{15}$$

where $X^i_{\ j},\ X^i_{\ jk},\ \tilde{X}^i_{\ jk},\ \tilde{Y}^i_{\ jk},\ Y^i_{\ jk}$ are arbitrary tensor fields on TM, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i$ and respective $\tilde{\omega} = \tilde{\omega}_i dx_i + \dot{\tilde{\omega}}_i \delta y^i$ are two arbitrary 1-forms in TM and $\overset{0}{|}$, $\overset{0}{\Big|}$ denote h-and respective v-covariant derivatives with respect to $\overset{0}{D}$.

7. If we take $X^i_{\ j} = 0$ in Theorem 2.6 we obtain:

Theorem 2.7. *Let $\overset{0}{D}$ be a given conformal almost symplectic d-linear connection on TM, with local coefficients: $\overset{0}{D}\Gamma(\overset{0}{N}, \omega, \tilde{\omega}) = (\overset{0}{L}{}^i{}_{jk}, \overset{0}{\tilde{L}}{}^i{}_{jk}, \overset{0}{\tilde{C}}{}^i{}_{jk}, \overset{0}{C}{}^i{}_{jk})$. The set of all conformal almost symplectic d-linear connections on TM, with respect to $\hat{A}$, which preserve the nonlinear connection $\overset{0}{N}$, corresponding to the 1-forms ω and $\tilde{\omega}$, with local coefficients $D\Gamma(\overset{0}{N}, \omega, \tilde{\omega}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ is given by:*

$$\begin{cases} L^i{}_{jk} = \overset{0}{L}{}^i{}_{jk} + \Phi^{ir}_{hj} X^h{}_{rk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}}{}^i{}_{jk} + \tilde{\Phi}^{ir}_{hj} \tilde{X}^h{}_{rk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}}{}^i{}_{jk} + \Phi^{ir}_{hj} \tilde{Y}^h{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \tilde{\Phi}^{ir}_{hj} Y^h{}_{rk}, \end{cases} \tag{16}$$

where $X^i_{\ j},\ X^i_{\ jk},\ \tilde{X}^i_{\ jk},\ \tilde{Y}^i_{\ jk},\ Y^i_{\ jk}$ are arbitrary tensor fields on TM.

References

[1] Gh. Atanasiu, I. Ghinea, *Connexions Finsleriennes Générales Presque Symplectiques,* An. St. Univ. "Al. I. Cuza", Iaşi, Sect. I a Mat.25 (Supl.), 1979, 11-15.

[2] R. Bowman, R., *Tangent Bundles of Higher Order,* Tensor, N. S., Japonia, 47, 1988, 97-100.

[3] V. Cruceanu, R. Miron, *Sur les connexions compatible à une Structure Mêtrique ou Presque symplectique,* Mathematica (Cluj), 9(32), 1967, 245-252.

[4] M. Matsumoto, *The Theory of Finsler Connections*, Publ. of the Study Group of Geometry 5, Depart. Math., Okayama Univ., 1970, XV+220 pp.
[5] R. Miron, *Asupra Conexiunilor Compatibile cu Structuri Conform Aproape Simplectice sau Conform Metrice,* An. Univ. din Timişoara - seria Şt. Mat. Fiz. V, 1967, 127-133.
[6] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications,* Kluwer Acad. Publ., FTPH, no. 59, 1994.
[7] R. Miron, M. Hashiguchi, *Conformal Finsler Connections*, Rev.Roumaine Math.Pures Appl., 26, 6(1981), 861-878.
[8] V. Oproiu, *On the Differential Geometry of the Tangent Bundle,* Rev. Roum. Math. Pures Appl., 13, 1968, 847-855.
[9] M. Purcaru, *Structuri geometrice remarcabile în geometria Lagrange de ordinul al doilea,* Teză de doctorat, Univ. Babeş-Bolyai Cluj-Napoca, 2002.
[10] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles,* M. Dekker, Inc., New-York, 1973.

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ON CERTAIN CLASSES OF P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS. II

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Abstract. The object of the present paper is to obtain modified Hadamard products (convolutions) of several functions belonging to the classes $T^*(p,\alpha)$ and $C(p,\alpha)$ consisting of analytic and p-valent functions with negative coefficients. We also obtain class preserving integral operator of the form

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\,dt, \quad c > -p$$

for the classes $T^*(p,\alpha)$ and $C(p,\alpha)$. Conversely, when F belongs to $T^*(p,\alpha)$ and $C(p,\alpha)$, radii of p-valence of f defined by the above equation are obtained.

1. Introduction

Let $S(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (p \in \mathbb{N} = \{1, 2, \cdots\})$$

which are analytic and p-valent in the unit disc $\mathrm{U} = \{z : |z| < 1\}$. A function f of $S(p)$ is called p-valent starlike of order α if f satisfies the following conditions

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathrm{U} \tag{1.1}$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi$$

for $0 \le \alpha < p$, $p \in \mathbb{N}$ and $z \in \mathrm{U}$. We denote by $S^*(p,\alpha)$ the class of all p-valent starlike functions of order α. The class $S^*(p,\alpha)$ was studied by Patil and Thakare [3]. Further a function f of $S(p)$ is called p-valent convex of order α if f satisfies the following conditions

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in \mathrm{U} \tag{1.2}$$

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and

$$\int_0^{2\pi} \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi$$

for $0 \le \alpha < p$, $p \in \mathbb{N}$ and $z \in \mathrm{U}$. We denote by $K(p,\alpha)$ the class of all p-valent convex functions of order α.

It follows from (1.1) and (1.2) that

$$f(z) \in K(p,\alpha) \text{ if and only if } zf'(z)/p \in S^*(p,\alpha), \quad 0 \le \alpha < p.$$

Let $T(p)$ denote the subclass of $S(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \ge 0;\ p,\, n \in \mathbb{N}). \tag{1.3}$$

We denote by $T^*(p,\alpha)$ and $C(p,\alpha)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ with $T(p)$, that is $T^*(p,\alpha) = S^*(p,\alpha) \cap T(p)$ and $C(p,\alpha) = K(p,\alpha) \cap T(p)$.

The classes $T^*(p,\alpha)$ and $C(p,\alpha)$ were studied by Owa [2].

In order to prove our results for functions belonging to the classes $T^*(p,\alpha)$ and $C(p,\alpha)$ we shall require the following lemmas given by Owa [2] and Aouf [1].

Lemma 1.1. *Let the function f be defined by (1.3); then $f \in T^*(p,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} (p+n-\alpha) a_{p+n} \le p - \alpha.$$

The result is sharp for the functions

$$f(z) = z^p - \frac{p-\alpha}{p+n-\alpha}\, z^{p+n}, \quad n \in \mathbb{N}. \tag{1.4}$$

Lemma 1.2. *Let the function f be defined by (1.3); then $f \in C(p,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} (p+n)(p+n-\alpha)\, a_{p+n} \le p(p-\alpha).$$

The result is sharp for the functions

$$f(z) = z^p - \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)}\, z^{p+n}, \quad n \in \mathbb{N}.$$

2. Modified Hadamard product

Let the functions f_i be defined, for $i \in \{1, 2, 3\}$, by

$$f_i(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,i}\, z^{p+n} \;\; (a_{p+n,i} \ge 0). \tag{2.1}$$

The modified Hadamard product (convolution) of f_1 and f_2 is defined here by

$$f_1 * f_2(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1}\, a_{p+n,2}\, z^{p+n}.$$

Theorem 2.1. *Let the functions f_i, $i \in \{1, 2\}$, defined by (2.1) be in the class $T^*(p,\alpha)$. Then $f_1 * f_2(z) \in T^*(p, \beta(p,\alpha))$, where*

$$\beta(p,\alpha) = p - \frac{(p-\alpha)^2}{(p+1-\alpha)^2 - (p-\alpha)^2}. \tag{2.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [4], we need to find the largest $\beta = \beta(p,\alpha)$ such that

$$\sum_{n=1}^{\infty} \frac{p+n-\beta}{p-\beta}\, a_{p+n,1}\, a_{p+n,2} \le 1.$$

Since

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,1} \le 1 \text{ and } \sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,2} \le 1,$$

by the Cauchy-Schwarz inequality we have

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha} \sqrt{a_{p+n,1}\, a_{p+n,2}} \le 1.$$

Thus it is sufficient to show that

$$\frac{p+n-\beta}{p-\beta}\, a_{p+n,1}\, a_{p+n,2} \le \frac{p+n-\alpha}{p-\alpha} \sqrt{a_{p+n,1}\, a_{p+n,2}} \quad (n \ge 1),$$

that is

$$\sqrt{a_{p+n,1}\, a_{p+n,2}} \le \frac{(p-\beta)(p+n-\alpha)}{(p-\alpha)(p+n-\beta)}.$$

Note that

$$\sqrt{a_{p+n,1}\, a_{p+n,2}} \le \frac{p-\alpha}{p+n-\alpha} \quad (n \ge 1).$$

Consequently, we need only to prove that

$$\frac{p-\alpha}{p+n-\alpha} \le \frac{(p-\beta)(p+n-\alpha)}{(p-\alpha)(p+n-\beta)} \quad (n \ge 1)$$

or, equivalently, that

$$\beta \le p - \frac{n\,(p-\alpha)^2}{(p+n-\alpha)^2 - (p-\alpha)^2}, \quad (n \ge 1).$$

Since

$$\Psi(n) = p - \frac{n\,(p-\alpha)^2}{(p+n-\alpha)^2 - (p-\alpha)^2}, \quad (n \ge 1), \tag{2.3}$$

is an increasing function of n $(n \ge 1)$, letting $n = 1$ in (2.3) we obtain

$$\beta \le \Psi(1) = p - \frac{(p-\alpha)^2}{(p+1-\alpha)^2 - (p-\alpha)^2},$$

which completes the proof of Theorem 1.

Finally, by taking the functions

$$f_i(z) = z^p - \frac{p-\alpha}{p+1-\alpha}\, z^{p+1}, \quad (i \in \{1,\, 2\}), \tag{2.4}$$

we can see that the result is sharp.

In a similar manner, with the aid of Lemma 1.2, we can prove

Theorem 2.2. *Let the functions f_i, $i \in \{1,\, 2\}$, defined by (2.1) be in the class $C(p,\alpha)$. Then $f_1 * f_2(z) \in C(p, \beta(p,\alpha))$, where*

$$\beta(p,\alpha) = p - \frac{(p-\alpha)^2}{(p+1-\alpha)^2(p+1)/p - (p-\alpha)^2}.$$

The result is sharp for the functions

$$f_i(z) = z^p - \frac{p\,(p-\alpha)}{(p+1)(p+1-\alpha)}\, z^{p+1}, \;\; i \in \{1,\, 2\}. \tag{2.5}$$

Theorem 2.3. *Let the function f_1 defined by (2.1) be in the class $T^*(p,\alpha)$ and let the function f_2 defined by (2.1) be in the class $T^*(p,\gamma)$; then $f_1 * f_2(z) \in T^*(p, \zeta(p,\alpha,\gamma))$, where*

$$\zeta(p,\alpha,\gamma) = p - \frac{(p-\alpha)(p-\gamma)}{(p+1-\alpha)(p+1-\gamma) - (p-\alpha)(p-\gamma)}.$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 2.1, we get

$$\zeta \le \Phi(n) = p - \frac{n\,(p-\alpha)(p-\gamma)}{(p+n-\alpha)(p+n-\gamma) - (p-\alpha)(p-\gamma)}. \tag{2.6}$$

Since the function $\Phi(n)$ is an increasing function of n $(n \ge 1)$, letting $n = 1$ in (2.6) we obtain

$$\zeta \le \Phi(1) = p - \frac{(p-\alpha)(p-\gamma)}{(p+1-\alpha)(p+1-\gamma) - (p-\alpha)(p-\gamma)},$$

which evidently proves Theorem 2.3.

Further, taking

$$f_1(z) = z^p - \frac{p-\alpha}{p+1-\alpha}\, z^{p+1} \;\text{ and }\; f_2(z) = z^p - \frac{p-\gamma}{p+1-\gamma}\, z^{p+1}. \tag{2.7}$$

Theorem 2.4. *Let the function f_1 defined by (2.1) be in the class $C(p,\alpha)$ and the function f_2 defined by (2.1) be in the class $C(p,\gamma)$; then $f_1 * f_2(z) \in C(p, \zeta(p,\alpha,\gamma))$, where*

$$\zeta(p,\alpha,\gamma) = p - \frac{(p-\alpha)(p-\gamma)}{(p+1-\alpha)(p+1-\gamma)(p+1)/p - (p-\alpha)(p-\gamma)}.$$

The result is sharp for the functions

$$f_1(z) = z^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}\, z^{p+1} \;\text{ and }\; f_2(z) = z^p - \frac{p(p-\gamma)}{(p+1-\gamma)(p+1)}\, z^{p+1}.$$

Corollary 2.1. *Let the functions* f_i, $i \in \{1, 2, 3\})$, *defined by (2.1) be in the class* $T^*(p,\alpha)$; *then* $f_1 * f_2 * f_3(z) \in T^*(p, \eta(p,\alpha))$, *where*

$$\eta(p,\alpha) = p - \frac{(p-\alpha)^3}{(p+1-\alpha)^3 - (p-\alpha)^3}.$$

The result is the best possible for the functions

$$f_i(z) = z^p - \frac{p-\alpha}{p+1-\alpha}\, z^{p+1}, \quad i \in \{1, 2, 3\}).$$

Proof. From Theorem 2.1 we have $f_1 * f_2(z) \in T^*(p, \beta(p,\alpha))$, where β is given by (2.2). We use now Theorem 2.3 and we get $f_1 * f_2 * f_3(z) \in T^*(p, \eta(p,\alpha))$, where

$$\eta(p,\alpha) = p - \frac{(p-\alpha)(p-\beta)}{(p+1-\alpha)(p+1-\beta) - (p-\alpha)(p-\beta)} = p - \frac{(p-\alpha)^3}{(p+1-\alpha)^3 - (p-\alpha)^3}.$$

This completes the proof of Corollary 2.1.

Corollary 2.2. *Let the functions* f_i, $i \in \{1, 2, 3\})$, *defined by (2.1) be in the class* $C(p,\alpha)$; *then* $f_1 * f_2 * f_3(z) \in C(p, \eta(p,\alpha))$, *where*

$$\eta(p,\alpha) = p - \frac{(p-\alpha)^3}{(p+1-\alpha)^3(p+1)^2/p^2 - (p-\alpha)^3}.$$

The result is the best possible for the functions

$$f_i(z) = z^p - \frac{p(p-\alpha)}{(p+1-\alpha)(p+1)}\, z^{p+1}, \quad i \in \{1, 2, 3\}).$$

Theorem 2.5. *Let the function* f_1 *defined by (2.1) be in the class* $T^*(p,\alpha)$ *and the function* f_2 *defined by (2.1) be in the class* $T^*(p,\gamma)$; *then* $f_1 * f_2(z) \in C(p, \beta(p,\alpha,\gamma))$, *where*

$$\beta(p,\alpha,\gamma) = p - \frac{(p+1)(p-\alpha)(p-\gamma)}{p(p+1-\alpha)(p+1-\gamma) - (p+1)(p-\alpha)(p-\gamma)}.$$

The result is sharp.

Proof. Since $f_1 \in T^*(p,\alpha)$ and $f_2 \in T^*(p,\gamma)$, we have

$$\sum_{n=1}^{\infty} (p+n-\alpha)\, a_{p+n,1} \le p-\alpha \quad \text{and} \quad \sum_{n=1}^{\infty} (p+n-\gamma)\, a_{p+n,2} \le p-\gamma.$$

It follows that

$$\sum_{n=1}^{\infty} (p+n-\alpha)(p+n-\gamma)\, a_{p+n,1} a_{p+n,2} \le (p-\alpha)(p-\gamma).$$

We want to find the largest $\beta = \beta(p,\alpha,\gamma)$ such that

$$\sum_{n=1}^{\infty} (p+n-\beta)(p+n)\, a_{p+n,1} a_{p+n,2} \le p(p-\beta).$$

This will be certainly satisfied if

$$\frac{(p+n-\beta)(p+n)}{p(p-\beta)} \le \frac{(p+n-\alpha)(p+n-\gamma)}{(p-\alpha)(p-\gamma)} \quad (n \ge 1),$$

or

$$\beta \le p - \frac{n(p+n)(p-\alpha)(p-\gamma)}{p(p+n-\alpha)(p+n-\gamma)-(p+n)(p-\alpha)(p-\gamma)} \quad (n \ge 1).$$

Since

$$K(n) = p - \frac{n(p+n)(p-\alpha)(p-\gamma)}{p(p+n-\alpha)(p+n-\gamma)-(p+n)(p-\alpha)(p-\gamma)} \quad (n \ge 1) \tag{2.8}$$

is an increasing function of n $(n \ge 1)$, letting $n = 1$ in (2.8) we obtain

$$\beta \le K(1) = p - \frac{(p+1)(p-\alpha)(p-\gamma)}{p(p+1-\alpha)(p+1-\gamma)-(p+1)(p-\alpha)(p-\gamma)},$$

and so $\beta(p,\alpha,\gamma) = K(1)$. Finally, the result is sharp for the functions f_1 and f_2 defined by (2.7).

Theorem 2.6. *Let the functions f_i, $i \in \{1, 2\}$, defined by (2.1) be in the class $T^*(p,\alpha)$; then the function*

$$h(z) = z^p - \sum_{n=1}^{\infty} \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) z^{p+n} \tag{2.9}$$

belongs to the class $T^*(p,\delta(p,\alpha))$, where

$$\delta(p,\alpha) = p - \frac{2(p-\alpha)^2}{(p+1-\alpha)^2 - 2(p-\alpha)^2}.$$

The result is sharp.

Proof. By virtue of Lemma 1.1, we obtain

$$\sum_{n=1}^{\infty} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 a_{p+n,1}^2 \le \left\{\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,1}\right\}^2 \le 1 \tag{2.10}$$

and

$$\sum_{n=1}^{\infty} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 a_{p+n,2}^2 \le \left\{\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,2}\right\}^2 \le 1. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) \le 1.$$

Therefore, we need to find the largest δ such that

$$\frac{p+n-\delta}{p-\delta} \le \frac{1}{2} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2,$$

that is

$$\delta \le p - \frac{2n(p-\alpha)^2}{(p+n-\alpha)^2 - 2(p-\alpha)^2} \quad (n \ge 1).$$

Since

$$D(n) = p - \frac{2n(p-\alpha)^2}{(p+n-\alpha)^2 - 2(p-\alpha)^2} \quad (n \geq 1)$$

is an increasing function of n $(n \geq 1)$, we readily have

$$\delta \leq D(1) = p - \frac{2(p-\alpha)^2}{(p+1-\alpha)^2 - 2(p-\alpha)^2}.$$

The result is sharp for the functions f_i, $i \in \{1, 2\}$ given by (2.4).

Theorem 2.7. *Let the functions* f_i, $i \in \{1, 2\}$, *defined by (2.1) be in the class* $C(p, \alpha)$*; then the function* $h(z)$ *defined by (2.9) belongs to the class* $C(p, \delta(p, \alpha))$, *where*

$$\delta(p, \alpha) = p - \frac{2p(p-\alpha)^2}{(p+1)(p+1-\alpha)^2 - 2p(p-\alpha)^2}.$$

The result is sharp for the functions f_i, $i \in \{1, 2\}$ *defined by (2.5).*

3. **Integral operators**

Theorem 3.1. *Let the function* f *defined by (1.3) be in the class* $T^*(p, \alpha)$ *and let* d *be a real number such that* $d > -p$*; then the function* F *defined by*

$$F(z) = \frac{d+p}{z^d} \int_0^z t^{d-1} f(t)\, dt \tag{3.1}$$

also belongs to the class $T^*(p, \alpha)$.

Proof. From the representation of F it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad \text{where} \quad b_{p+n} = \frac{d+p}{d+p+n} a_{p+n}.$$

Therefore

$$\sum_{n=1}^{\infty} (p+n-\alpha) b_{p+n} = \sum_{n=1}^{\infty} (p+n-\alpha) \frac{d+p}{d+p+n} a_{p+n} \leq \sum_{n=1}^{\infty} (p+n-\alpha) a_{p+n} \leq p - \alpha,$$

since $f \in T^*(p, \alpha)$. Hence, by Lemma 1.1, $F \in T^*(p, \alpha)$.

Putting $d = 1 - p$ in Theorem 3.1 we get the following corollary.

Corollary 3.1. *Let the function* f *defined by (1.3) be in the class* $T^*(p, \alpha)$ *and let* F *be defined by*

$$F(z) = \frac{1}{z^{1-p}} \int_0^z \frac{f(t)}{t^p}\, dt;$$

then $F \in T^*(p, \alpha)$.

Theorem 3.2. *Let* d *be a real number such that* $d > -p$. *If* $F \in T^*(p, \alpha)$, *then the function* f *defined by (3.1) is p-valent in* $|z| < R_p^*$, *where*

$$R_p^* = \inf_n \left[\frac{p(p+n-\alpha)(d+p)}{(p+n)(p-\alpha)(d+p+n)} \right]^{1/n}.$$

The result is sharp.

Proof. Let $F(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ $(a_{p+n} \geq 0)$. It follows from (3.1) that

$$f(z) = \frac{z^{1-d}\left(z^d F(z)\right)'}{d+p} = z^p - \sum_{n=1}^{\infty} \frac{d+p+n}{d+p}\, a_{p+n} z^{p+n}.$$

To prove the result it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p \quad \text{for} \quad |z| < R_p^*.$$

Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=1}^{\infty}(p+n)\frac{d+p+n}{d+p}\, a_{p+n} z^n\right| \leq \sum_{n=1}^{\infty}(p+n)\frac{d+p+n}{d+p}\, a_{p+n}\, |z|^n.$$

Thus $|f'(z)/z^{p-1} - p| \leq p$ if

$$\sum_{n=1}^{\infty} \frac{p+n}{p}\frac{d+p+n}{d+p}\, a_{p+n}\, |z|^n \leq 1. \tag{3.2}$$

But Lemma 1.1 confirm that

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n} \leq 1.$$

Thus (3.2) will be satisfied if

$$\frac{(p+n)(d+p+n)}{p(d+p)}|z|^n \leq \frac{p+n-\alpha}{p-\alpha} \quad (n \geq 1)$$

or if

$$|z| \leq \left[\frac{p(p+n-\alpha)(d+p)}{(p+n)(p-\alpha)(d+p+n)}\right]^{1/n} \quad (n \geq 1). \tag{3.3}$$

The required result follows now from (3.3). The result is sharp because the functions

$$f(z) = z^p - \frac{(p-\alpha)(d+p+n)}{(p+n-\alpha)(d+p)}\, z^{p+n} \quad (n \geq 1)$$

are defined by (3.1) when F are given by (1.4).

In a similar manner, with the aid of Lemma 1.2, we can prove the following theorem.

Theorem 3.3. *Let the function f defined by (1.3) be in the class $C(p,\alpha)$ and let d be a real number such that $d > -p$. Then the function F defined by (3.1) also belongs to the class $C(p,\alpha)$.*

Theorem 3.4. *Let d be a real number such that $d > -p$. If $F \in C(p,\alpha)$, then the function f defined by (3.1) is p-valent in $|z| < R_p^{**}$, where*

$$R_p^{**} = \inf_n \left[\frac{(p+n-\alpha)(d+p)}{(p-\alpha)(d+p+n)}\right]^{1/n}.$$

The result is sharp for the functions

$$f(z) = z^p - \frac{p(p-\alpha)(d+p+n)}{(p+n)(p+n-\alpha)(d+p)}\, z^{p+n} \quad (n \geq 1).$$

References

[1] M. K. Aouf, *A generalization of multivalent functions with negative coefficients*, J. Korean Math. Soc. **25** (1988), no. 1, 53-66.
[2] S. Owa, *On certain classes of p-valent functions with negative coefficients*, Simon Stevin **25** (1985), no. 4, 385-402.
[3] D. A. Patil and N. K. Thakare, *On convex hulls and extreme points of p-valent starlike and convex classes with applications*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.) **27** (75) (1983), 145-160.
[4] A. Schild and H. Silverman, *Convolution of univalent functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklodowska Sect. A. **29** (1975), 99-107.

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THE STUDY OF REMAINDER FOR SOME CUBATURE FORMULAS FOR TRIANGULAR DOMAIN

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The purpose of this paper is to give some practical cubature formulas in approximation of the integral

$$I = \int_D f(x,y)\,dxdy \tag{1}$$

where D is a triangular domain, $D = \{(x,y)/x \geq 0, y \geq 0, x+y \leq h\}$ and $f : D \to R$ is an integrable function on D. We would like to construct some practical cubature formulas of the following form:

$$I = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} f(x_i, y_j) + R_{mn}(f)$$

where A_{ij} are the coefficients of the formula and $R_{mn}(f)$ the remainder. We will use the quadrature rules given by Bruno Welfer in [2] for triangular domain and we will study the remainder of these rules and some optimal properties. To obtain the error of the approximation formula we will use a generalization of the Peano Theorem, when the function is a member of the so-called Sard space. We will note with B_{pq} the Sard space where $p, q \in N, p+q = m$. Let $\Omega = [0,h] \times [0,h]$ where $h \in R_+$, then the Sard space $B_{p,q}(0,0)$ is the set of all of the functions $f : \Omega \to R$ with the following properties:

1. $f^{(p,q)} \in C(\Omega)$
2. $f^{(m-j,j)}(\cdot, 0) \in C[0,h]$, $j = 0, ..., q-1$
3. $f^{(i,m-i)}(0, \cdot) \in C[0,h]$, $i = 0, ..., p-1$

Theorem 1. *Let $L : B_{pq}(0,0) \to R$ be a continuous linear functional. If $Ker(L) = P^2_{m-1}$ then*

$$L(f) = \sum_{j<q} \int_0^h K_{m-j,j}(s) f^{(m-j,j)}(s,0)ds + \sum_{i<p} K_{i,m-i}(t) f^{(i,m-i)}(0,t)dt +$$
$$+ \int\int_\omega K_{p,q}(s,t) f^{(p,q)}(s,t)dsdt \tag{2}$$

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where

$$K_{m-j,j}(s) = L^{(x,y)}\left[\frac{(x-s)_+^{m-j-1}}{(m-j-1)!}\frac{y^j}{j!}\right], j<q$$

$$K_{i,m-i}(t) = L^{(x,y)}\left[\frac{x^i}{i!}\frac{(y-t)_+^{m-i-1}}{(m-i-1)!}\right], i<p$$

$$K_{p,q}(s,t) = L^{(x,y)}\left[\frac{(x-s)_+^{p-1}}{(p-1)!}\frac{(y-t)_+^{q-1}}{(q-1)!}\right].$$

First of all we consider the cubature formula with a single knots:

$$\iint\limits_D f(x,y)dxdy = \frac{h^2}{2}f(\frac{h}{3},\frac{h}{3}) + R_1(f). \tag{3}$$

Because the degree of exactness of this formula is equal with 1, we can use the first theorem and follows:

Theorem 2. *If $f^{(2,0)}(\cdot,0)\in C[0,h], f^{(0,2)}(0,\cdot)\in C[0,h]$ and $f^{(1,1)}(x,y)\in C(D)$ then we can give the following delimitation of the error in formula (3):*

$$|R_1(f)| \le \frac{h^4}{72}M_{20}f + \frac{h^4}{72}M_{02}f + \frac{89h^4}{1944}M_{11}f,$$

where

$$M_{20}f = \max_{x\in[0,h]}|f^{(2,0)}(x,0)|, M_{02}f = \max_{y\in[0,h]}|f^{(0,2)}(0,y)|, M_{11}f = \max_D|f^{(1,1)}(x,y)|.$$

Proof. Theorem 1 implies the following error representation:

$$R_1(f) = \int_0^h K_{20}(s)f^{(2,0)}(s,0)ds + \int_0^h K_{02}(s)f^{(0,2)}(0,t)dt$$
$$+\iint\limits_{T_h} K_{11}(s,t)f^{(1,1)}(s,t)dsdt \tag{4}$$

and

$$K_{20}(s) = R^{xy}\left[(x-s)_+\right]$$
$$K_{02}(t) = R^{xy}\left[(y-t)_+\right]$$
$$K_{11}(s,t) = R^{xy}\left[(x-s)_+^0(y-t)_+^0\right].$$

Therefore the so-called Peano-kernels has the representation

$$K_{20}(s) = \begin{cases} \frac{(h-s)^3}{6} - \frac{h^2}{2}(\frac{h}{3}-s), & s\le\frac{h}{3} \\ \frac{(h-s)^3}{6}, & s\ge\frac{h}{3} \end{cases}$$

$$K_{02}(t) = \begin{cases} \frac{(h-t)^3}{6} - \frac{h^2}{2}(\frac{h}{3}-t), & t\le\frac{h}{3} \\ \frac{(h-t)^3}{6}, & t\ge\frac{h}{3} \end{cases}$$

and

$$K_{11}(s,t)=\begin{cases}\frac{(h-t-s)^3}{6}-\frac{h^2}{2}, & 0\le s,t\le\frac{h}{3}\\ \frac{(h-t-s)^3}{6}, & 0\le s\le\frac{h}{3},\frac{h}{3}\le t\le h \ \ or \ \ \frac{h}{3}\le s\le h, 0\le t\le\frac{h}{3}\end{cases}$$

If we study the sign of these functions, we conclude that K_{20} and K_{02} are positive functions at the interval $[0,h]$ and $K_{11}(s,t)\le 0, 0\le s,t\le\frac{h}{3}$ and $K_{11}(s,t)\ge 0 \ \ if \ \ 0\le s\le\frac{h}{3},\frac{h}{3}\le t\le h \ \ or \ \ \frac{h}{3}\le s\le h, 0\le t\le\frac{h}{3}$.

Since

$$\int_0^h K_{20}(s)ds=\frac{h^4}{72},$$

$$\int_0^h K_{02}(t)dt=\frac{h^4}{72},$$

and

$$\int\int_D |K_{11}(s,t)|\,dsdt=\frac{89h^4}{1944},$$

from (4) finally yields the theorem.

Let now consider the following formula:

$$\int_D\int f(x,y)dxdy=\frac{h^2}{6}\left[f(0,\frac{h}{2})+f(\frac{h}{2},0)+f(\frac{h}{2},\frac{h}{2})\right]+R_2(f). \tag{5}$$

The degree of exactness of this formula is 2, therefore we can use the theorem 1 for the representation of the error, an we can give the following delimitation of the approximation error:

Theorem 3. *If* $f^{(3,0)}(\cdot,0)\in C[0,h], f^{(2,1)}(\cdot,0)\in C[0,h], f^{(0,3)}(0,\cdot)\in C[0,h]$ *and* $f^{(1,2)}(s,t)\in C(D)$ *than we have*

$$|R_2(f)|\le M_{30}f\frac{h^5}{720}+M_{21}f\frac{h^5}{364}+M_{03}f\frac{h^5}{720}+M_{12}f\frac{h^5}{24} \tag{6}$$

where

$$M_{30}f=\max_{s\in[0,h]}\left|f^{(3,0)}(s,0)\right|, M_{21}f=\max_{s\in[0,h]}\left|f^{(2,1)}(s,0)\right|,$$

$$M_{03}f=\max_{t\in[0,h]}\left|f^{(0,3)}(0,t)\right| \ \ and\ M_{12}f=\max_{D}\left|f^{(1,2)}(s,t)\right|.$$

Proof. We will use the same method like in the previous theorem, than the error of the formula (5) is

$$R_2(f)=\int_0^h K_{30}(s)f^{(3,0)}(s)ds+\int_0^h K_{21}(s)f^{(2,1)}(s,0)ds+\int_0^h K_{03}(t)f^{(0,3)}(0,t)dt+$$
$$+\int_D\int K_{12}(s,t)f^{(1,2)}(s,t)dsdt$$

where

$$K_{30}(s)=\begin{cases}\frac{(h-s)^4}{24}-\frac{h^3}{6}(\frac{h}{2}-s)^2, & s<h/2\\ \frac{(h-s)^4}{24}, & s\ge h/2\end{cases}$$

$$K_{21}(s)=\begin{cases}\frac{(h-s)^4}{24}-\frac{h^3}{12}(\frac{h}{2}-s), & s<h/2\\ \frac{(h-s)^4}{24}, & s\geq h/2\end{cases}$$

$$K_{12}(s,t)=\begin{cases}\frac{(h-s-t)^3}{6}-\frac{h^2}{6}(\frac{h}{2}-t), & 0<s,t<h/2\\ \frac{(h-s-t)^3}{6}, & otherwise\end{cases}$$

and

$$K_{03}(t)=\begin{cases}\frac{(h-t)^4}{24}-\frac{h^2}{6}(\frac{h}{2}-t)^2, & t<h/2\\ \frac{(h-t)^4}{24}, & t\geq h/2\end{cases}$$

The kernel functions K_{30} and K_{03} are positive on the interval $[0,h]$ and their integral on the same interval is equal with $\frac{h^5}{720}$. Also we have $\max\limits_{s\in[0,h]}|K_{21}(x,y,s)|=\frac{h^4}{384}$ and $\max\limits_{(s,t)\in D}|K_{12}(x,y,s,t)|=\frac{h^3}{12}$, therefore we can give the following delimitation of the absolute error:

$$\begin{aligned}|R_2(f)|\leq&\left|\int_0^h K_{30}(s)f^{(3,0)}(s,0)ds\right|+\left|\int_0^h K_{21}(s)f^{(2,1)}(s,0)ds\right|+\\ &+\left|\int_0^h K_{03}(t)f^{(0,3)}(0,t)dt\right|+\left|\int_D\int K_{12}(s,t)f^{(1,2)}(s,t)dsdt\right|\\ &\leq M_{30}f\int_0^h|K_{30}(s)|\,ds+\frac{h^4}{364}\int_0^h\left|f^{(2,1)}(s,0)\right|ds+\\ &+M_{03}f\int_0^h|K_{03}(t)|\,dt+\frac{h^3}{12}\int_D\int\left|f^{(1,2)}(s,t)\right|dsdt\\ &\leq M_{30}f\frac{h^5}{720}+M_{21}f\frac{h^5}{364}+M_{03}f\frac{h^5}{720}+M_{12}f\frac{h^5}{24}.\end{aligned}$$

Finally we consider the following cubature formula:

$$\begin{aligned}\int_D\int f(x,y)dxdy=&\frac{h^2}{120}\left[3f(0,0)+3f(h,0)+3f(0,h)+8f(\frac{h}{2},0)+\right.\\ &\left.+8f(\frac{h}{2},\frac{h}{2})+8f(0,\frac{h}{2})+27f(\frac{h}{3},\frac{h}{3})\right]+R_3(f).\end{aligned}$$

Because the degree of exactness of this formula is equal with 3, we can give the following theorem for the delimitation of the absolute error:

Theorem 4. *If* $f^{(4,0)}(s,0)\in C[0,h]$, $f^{(3,1)}(s,0)\in C[0,h]$, $f^{(0,4)}(0,t)\in C[0,h]$, $f^{(1,3)}(0,t)\in C[0,h]$ *and* $f^{(2,2)}(s,t)\in C(D)$ *then*

$$|R_3(f)|\leq M_{40}f\frac{h^6}{8640}+M_{31}f\frac{7h^6}{1440}+M_{13}f\frac{7h^6}{1440}+M_{04}f\frac{h^6}{8640}+M_{22}f\frac{h^6}{768},$$

where

$$M_{40}f=\max_{s\in[0,h]}\left|f^{(4,0)}(s,0)\right|,M_{31}f=\max_{s\in[0,h]}\left|f^{(3,1)}(s,0)\right|,M_{13}f=\max_{t\in[0,h]}\left|f^{(1,3)}(0,t)\right|,$$

$$M_{04}f = \max_{t\in[0,h]} \left| f^{(0,4)}(0,t) \right|, M_{22}f = \max_{(s,t)\in D} \left| f^{(2,2)}(s,t) \right|.$$

Proof. We will use again the generalization of the Peano theorem in bidimensional case and we have

$$R_3(f) = \int_0^h K_{40}(s) f^{(4,0)}(s,0)ds + \int_0^h K_{31}(s) f^{(3,1)}(s,0)ds + \int_0^h K_{04}(t) f^{(0,4)}(0,t)dt + \\ + \int_0^h K_{13}(t) f^{(1,3)}(0,t)dt + \int_D \int K_{22}(s,t) f^{(2,2)}(s,t)dsdt$$

where the K_{40}, K_{31}, K_{13} and K_{22} are the so-called kernel functions, and they have the following representation:

$$K_{40}(s) = \begin{cases} \frac{(h-s)^5}{5!} - \frac{h^2}{120}\left[\frac{(h-s)^3}{2} + 8\frac{(\frac{h}{2}-s)^3}{3} + 9\frac{(\frac{h}{3}-s)^3}{2}\right], & s \in \left[0, \frac{h}{3}\right] \\ \frac{(h-s)^5}{5!} - \frac{h^2}{120}\left[\frac{(h-s)^3}{2} + 8\frac{(\frac{h}{2}-s)^3}{3}\right], & s \in \left(\frac{h}{3}, \frac{h}{2}\right) \\ \frac{(h-s)^5}{5!} - \frac{h^2}{120}\frac{(h-s)^3}{2}, & s \in \left[\frac{h}{2}, h\right] \end{cases}$$

$$K_{31}(s) = \begin{cases} -\frac{(h-s)^5}{5!} - \frac{h^2}{120}\left[h\frac{(h-2s)^2}{2} + h\frac{(h-3s)^2}{2}\right], & s \in \left[0, \frac{h}{3}\right] \\ -\frac{(h-s)^5}{5!} - \frac{h^3}{240}(h-2s)^2, & s \in \left(\frac{h}{3}, \frac{h}{2}\right) \\ -\frac{(h-s)^5}{5!} & s \in \left[\frac{h}{2}, h\right] \end{cases}$$

$$K_{13}(t) = \begin{cases} -\frac{(h-t)^5}{5!} - \frac{h^2}{120}\left[h\frac{(h-2t)^2}{2} + h\frac{(h-3t)^2}{2}\right], & t \in \left[0, \frac{h}{3}\right] \\ -\frac{(h-t)^5}{5!} - \frac{h^3}{240}(h-2t)^2, & t \in \left(\frac{h}{3}, \frac{h}{2}\right) \\ -\frac{(h-t)^5}{5!} & t \in \left[\frac{h}{2}, h\right] \end{cases}$$

and

$$K_{22}(s,t) =$$

$$= \begin{cases} \frac{(h-s-t)^4}{4!} - \frac{h^2}{120}\left[8(\frac{h}{2}-s)(\frac{h}{2}-t) + 27(\frac{h}{3}-s)(\frac{h}{3}-t)\right], & 0 \le s,t \le \frac{h}{3} \\ \frac{(h-s-t)^4}{4!} - \frac{h^2}{120}8(\frac{h}{2}-s)(\frac{h}{2}-t) & \frac{h}{3} \le s \le \frac{h}{2}, 0 \le t \le \frac{h}{2}, \\ & 0 \le s \le \frac{h}{3}, \frac{h}{3} \le t \le \frac{h}{2} \\ \frac{(h-s-t)^4}{4!}, & 0 \le s \le \frac{h}{2}, \frac{h}{2} \le t \le h, \\ & \frac{h}{2} \le s \le h, 0 \le t \le \frac{h}{2} \end{cases}$$

The K_{40}, K_{31} and the K_{13} functions do not change their sign at the interval $[0,h]$ and if we calculate the maximum of the function K_{22} we obtain $\max_D |K_{22}(s,t)| =$

$\frac{h^4}{384}$, therefore for the absolute value of the error we have the following delimitation:

$$\begin{aligned}
|R_3(f)| \leq & \left|\int_0^h K_{40}(s)f^{(4,0)}(s,0)ds\right| + \left|\int_0^h K_{31}(s)f^{(3,1)}(s,0)ds\right| + \\
& + \left|\int_0^h K_{04}(t)f^{(0,4)}(0,t)dt\right| + \left|\int_0^h K_{13}(t)f^{(1,3)}(0,t)dt\right| + \\
& + \left|\int_D\int K_{22}(s,t)f^{(2,2)}(s,t)dsdt\right| \\
& \leq M_{40}f\int_0^h |K_{40}(s)|\, ds + M_{31}f\int_0^h |K_{31}(s)|\, ds + M_{04}f\int_0^h |K_{04}(t)|\, dt + \\
& + M_{13}f\int_0^h |K_{13}(t)|\, dt + M_{22}f\frac{h^6}{768} \\
& = M_{40}f\frac{h^6}{8640} + M_{31}f\frac{7h^6}{1440} + M_{04}f\frac{7h^6}{1440} + M_{04}f\frac{h^6}{1440} + M_{22}f\frac{h^6}{768}.
\end{aligned}$$

Remark 1. *The cubature formula (5) has an optimal character, because it satisfy the conditions established by Stroud in* [5] *regarding the minimal number of knots for a cubature formula. If the degree of exactness of a cubature formula is equal with* 2 *then the minimal number of the knots is* $N = n+1$ *where* n *is the dimension number. The cubature formula (5) with the degree of exactness* 2 *and three knots, has a minimal number of knots.*

References

[1] R. E. Barnhill, W. J. Gordon, D. H. Thomas, *The method of successive decomposition for multiple integration*, Research Rep. GMR-1281, General Motors, Warren, Mich., 1972.

[2] Bruno Welfert, *Numerical Analisys*, Lecture Notes University of Arizona.

[3] Gh. Coman, *Analiză numerică*, Ed. Libris, Cluj, 1995.

[4] Gh. Coman, Dimitrie D. Stancu, Petru Blaga, *Analiză numerică şi teoria aproximării*, Presa Univ. Cluj, vol II, 2002.

[5] A. H. Stroud, *Approximate calculation of multiple integrals*, Englewood Cliffs, N.J. Prentice-Hall, Inc. 1971.

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ON A CLASS OF GENERALIZED GAUSS-CHRISTOFFEL QUADRATURE FORMULAE

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Abstract. We consider Gauss-Christoffel-Stancu quadrature rules, over the interval $[-1, 1]$, using m Gaussian nodes and s preassigned multiples nodes, so that the node polynomial of these fixed nodes does not change sign in $(-1, 1)$. The Gaussian nodes x_k of formula (2) are determined so that the degree of exactness of this quadrature formula to be the highest possible. These can be found either by means of the formula (10) or by determining the minimum of the function F of m variables (11). We give explicit formulae for the coefficients and for the remainders. Several illustrative examples are presented for certain preassigned multiple nodes.

1. In a memoir published by E. B. Christoffel in 1858 [1] has been considered a generalization of the classical Gauss quadrature formula, by introducing certain preassigned simple nodes situated outside the integration interval $(-1, 1)$.

This formula has the following form

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{j=1}^{n} B_j f(b_j) + R(f), \tag{1}$$

where b_j are preassigned nodes (the fixed nodes), not situated in the interval $(-1, 1)$, f is an integrable function on this interval and $R(f)$ is the remainder of this quadrature formula. The free nodes x_k are selected so that formula (1) has the highest degree of exactness. We will call x_k the **fundamental** or the **Gaussian nodes**.

2. In 1957 D. D. Stancu [4] has introduced and investigated a quadrature formula using multiple fixed nodes a_i and simple Gaussian nodes x_k.

It has the form

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{i=1}^{s} \sum_{j=0}^{r_i-1} C_{i,j} f^{(j)}(a_i) + R(f). \tag{2}$$

Let us denote by $u(x)$ the node polynomial of the free nodes x_k and by $\omega(x)$ the node polynomial of the fixed nodes, that is

$$u(x) = (x - x_1)(x - x_2) \dots (x - x_m), \tag{3}$$

$$\omega(x) = (x - a_1)^{r_1}(x - a_2)^{r_2} \dots (x - a_s)^{r_s}. \tag{4}$$

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We assume that r_i are natural numbers so that we have $\omega(x) \geqq 0$ on the integration interval $(-1, 1)$.

Given the fixed nodes a_i and their multiplicities r_i, the problem is then to determine the simple nodes x_k and the coefficients A_k and $C_{i,j}$ so that formula (2) has the highest degree of exactness.

In order to find the Gaussian nodes x_k we shall start from the Lagrange-Hermite interpolation formula using the simple nodes x_k, the multiple nodes a_i and other nondetermined simple nodes $t_1, t_2, \dots, t_m$, distinct from the other nodes. It has the form

$$f(x) = (H_{2m+p-1}f)(x) + (R_{2m+p-1}f)(x), \tag{5}$$

where we use as nodes the roots of the polynomial $P(x) = u(x)\omega(x)v(x)$, u and ω being defined at (3) and (4), while

$$v(x) = (x - t_1)(x - t_2)\dots(x - t_m), \quad p = r_1 + r_2 + \dots + r_s.$$

The interpolation polynomial H_{2m+p-1} has the following expression (see [4]):

$$(H_{2m+p-1}f)(x) = \sum_{k=1}^{m} \frac{u_k(x)}{u_k(x_k)} \cdot \frac{v(x)}{v(x_k)} \cdot \frac{\omega(x)}{\omega(x_k)} f(x_k) +$$

$$+ \sum_{h=1}^{m} \frac{u(x)}{u(t_h)} \cdot \frac{v_h(x)}{v_h(t_h)} \cdot \frac{\omega(x)}{\omega(t_h)} f(t_h) +$$

$$+ \sum_{i=1}^{s} \sum_{j=0}^{r_i-1} \sum_{\nu=0}^{r_i-j-1} \frac{(x - a_i)^j}{j!} \left[\frac{(x - a_i)^\nu}{\nu!} \left(\frac{1}{\omega_i(x)} \right)^{(\nu)}_{a_i} \right] \omega_i(x) f^{(j)}(a_i),$$

where

$$u_k(x) = u(x)/(x - x_k), \quad v_h(x) = v(x)/(x - t_h), \quad \omega_i(x) = \omega(x)/(x - a_i)^{r_i}.$$

3. By integrating the preceding interpolation formula we obtain a quadrature formula of the following form

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{h=1}^{m} B_h f(t_h) + \sum_{i=1}^{s} \sum_{j=0}^{r_i-1} C_{i,j} f^{(j)}(a_i) + R(f), \tag{6}$$

where

$$A_k = \int_{-1}^{1} \frac{u_k(x)}{u_k(x_k)} \cdot \frac{v(x)}{v(x_k)} \cdot \frac{\omega(x)}{\omega(x_k)} dx, \tag{7}$$

$$B_h = \int_{-1}^{1} \frac{u(x)}{u(t_h)} \cdot \frac{v_h(x)}{v_h(t_h)} \cdot \frac{\omega(x)}{\omega(t_h)} dx, \tag{8}$$

$$C_{i,j} = \sum_{\nu=0}^{r_i-j-1} \int_{-1}^{1} \frac{(x - a_i)^j}{j!} \left[\frac{(x - a_i)^\nu}{\nu!} \left(\frac{1}{\omega_i(x)} \right)^{(\nu)}_{a_i} \right] \omega_i(x) dx, \tag{9}$$

$$R(f) = \int_{-1}^{1} u(x)v(x)\omega(x) \left[x, \begin{matrix} x_k \\ 1 \end{matrix}, \begin{matrix} t_h \\ 1 \end{matrix}, \begin{matrix} a_i \\ r_i \end{matrix}; f \right] dx.$$

The brackets used in this remainder represent the symbol for divided differences.

4. Now we want to determine the nodes x_k so that we have $B_h = 0$ ($h = 1, 2, \dots, m$) for any values of the parameters $t_1, t_2, \dots, t_m$. It is easy to see that this is equivalent with the condition that the polynomial $u(x)$ is orthogonal on $(-1, 1)$, with respect to the weight function $\omega(x)$, with any polynomial of degree $m-1$, since $t_1, t_2, \dots, t_m$ are arbitrary numbers.

But it is known [4] that we must have

$$U_m(x) = \begin{vmatrix} L_m(x) & L_{m+1}(x) & \dots & L_{m+p}(x) \\ L_m(a_1) & L_{m+1}(a_1) & \dots & L_{m+p}(a_1) \\ L'_m(a_1) & L'_m(a_1) & \dots & L'_{m+p}(a_1) \\ \dots & \dots & \dots & \dots \\ L_m^{(r_1-1)}(a_1) & L_{m+1}^{(r_1-1)}(a_1) & \dots & L_{m+p}^{(r_1-1)}(a_1) \\ L_m(a_2) & L_{m+1}(a_2) & \dots & L_{m+p}(a_2) \\ \dots & \dots & \dots & \dots \\ L_m(a_s) & L_{m+1}(a_s) & \dots & L_{m+p}(a_s) \\ L'_m(a_s) & L'_{m+1}(a_s) & \dots & L'_{m+p}(a_s) \\ \dots & \dots & \dots & \dots \\ L_m^{(r_s-1)}(a_s) & L_{m+1}^{(r_s-1)}(a_s) & \dots & L_{m+p}^{(r_s-1)}(a_s) \end{vmatrix} : \omega(x), \tag{10}$$

where by L_n we denote the Legendre polynomial of degree n:

$$L_n(x) = (2^n \cdot n!)^{-1}[(x^2-1)^n]^{(n)} \quad \text{and} \quad u(x) = \widetilde{U}_m(x).$$

If we take into consideration the formula (8) for the coefficient B_h, we can see that in order to have $B_1 = \dots = B_m = 0$ it is necessary and sufficient that

$$\int_{-1}^{1} \omega(x) u(x) g(x) dx = 0,$$

where $g(x)$ is any polynomial of degree $m-1$.

But it is known [4] that in this case the node polynomial $u(x)$ can be found by means of the formula (10).

We make the remark that the nodes x_k can be found also by determining the minimum of the following function of m variables

$$F(u_1, \dots, u_m) = \int_{-1}^{1} \omega(x)(x-u_1)^2 \dots (x-u_m)^2 dx. \tag{11}$$

5. Because $t_1, t_2, \dots, t_m$ are arbitrary numbers, we can make $t_k \to x_k$ ($k = 1, 2, \dots, m$).

In this case we arrive at the following quadrature formula of Gauss-Christoffel-Stancu type

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{i=1}^{s} \sum_{j=0}^{r_i-1} C_{i,j} f^{(j)}(a_i) + R(f), \tag{12}$$

where

$$A_k = \int_{-1}^{1} \left(\frac{u_k(x)}{u_k(x_k)} \right)^2 \frac{\omega(x)}{\omega(x_k)} dx$$

and

$$R(f) = \int_{-1}^{1} \omega(x)u^2(x) \left[x, \begin{matrix} x_k \\ 2 \end{matrix}, \begin{matrix} a_i \\ r_i \end{matrix}; f\right] dx, \quad k = \overline{1,m};\ i = \overline{1,s}.$$

One observes that all the coefficients A_k are positive.

Assuming that $f \in C^{2m+p}(-1,1)$, by using the mean-value theorem of divided differences we can give the following representation of the remainder

$$R(f) = \frac{f^{(2m+p)}(\xi)}{(2m+p)!} \int_{-1}^{1} \omega(x)u^2(x)dx, \quad \xi \in (-1,1). \tag{13}$$

6. Now we make the remark that if the polynomial of the fixed nodes: $\pm a_1, \pm a_2, \ldots, \pm a_s$ $(2s = r)$ is even, then we can obtain the following equation for determining the Gaussian nodes x_k:

$$\begin{vmatrix} L_m(x) & L_{m+2}(x) & \ldots & L_{m+r}(x) \\ L_m(a_1) & L_{m+2}(a_1) & \ldots & L_{m+r}(a_1) \\ L'_m(a_1) & L'_{m+2}(a_1) & \ldots & L'_{m+r}(a_1) \\ \ldots & \ldots & \ldots & \ldots \\ L_m^{(r_1-1)}(a_1) & L_{m+2}^{(r_1-1)}(a_1) & \ldots & L_{m+r}^{(r_1-1)}(a_1) \\ L_m(a_2) & L_{m+2}(a_2) & \ldots & L_{m+r}(a_2) \\ \ldots & \ldots & \ldots & \ldots \\ L_m(a_s) & L_{m+2}(a_s) & \ldots & L_{m+r}(a_s) \\ L'_m(a_s) & L'_{m+2}(a_s) & \ldots & L'_{m+r}(a_s) \\ \ldots & \ldots & \ldots & \ldots \\ L_m^{(r_s-1)}(a_s) & L_{m+2}^{(r_s-1)}(a_s) & \ldots & L_{m+r}^{(r_s-1)}(a_s) \end{vmatrix} = 0, \tag{14}$$

where by $L_n(x)$ we denote again the Legendre orthogonal polynomial of degree n.

7. If we normalize the orthogonal polynomial given at (10), then we obtain

$$\widehat{U}(x) = \frac{1}{\gamma_m} \sqrt{\frac{(-1)^p \beta_m}{\beta_{m+p} G_m G_{m+1}}} \cdot U_m(x),$$

where γ_m is the coefficient of x^m from the Legendre polynomial $L_m(x)$, that is

$$\gamma_m = \int_{-1}^{1} L_m^2(x)dx = \frac{2}{2m+1}$$

and by G_k we denote the following determinant

$$\begin{vmatrix} L_k(a_1) & L_{k+1}(a_1) & \dots & L_{k+p-1}(a_1) \\ L'_k(a_1) & L'_{k+1}(a_1) & \dots & L'_{k+p-1}(a_1) \\ \dots & \dots & \dots & \dots \\ L_k^{(r_1-1)}(a_1) & L_{k+1}^{(r_1-1)}(a_1) & \dots & L_{k+p-1}^{(r_1-1)}(a_1) \\ L_k(a_2) & L_{k+1}(a_2) & \dots & L_{k+p-1}(a_2) \\ L'_k(a_2) & L'_{k+1}(a_2) & \dots & L'_{k+p-1}(a_s) \\ \dots & \dots & \dots & \dots \\ L_k(a_s) & L_{k+1}(a_s) & \dots & L_{k+p-1}(a_s) \\ \dots & \dots & \dots & \dots \\ L_k^{(r_s-1)}(a_s) & L_{k+1}^{(r_s-1)}(a_s) & \dots & L_{k+p-1}^{(r_s-1)}(a_s) \end{vmatrix}.$$

By using the known Christoffel-Darboux formula from the theory of orthogonal polynomials, we can obtain for the coefficients A_k of the quadrature formula (12) the expressions

$$A_k = \int_{-1}^{1} \frac{\widehat{U}_m(t)\omega(t)dt}{(t-x_k)\widehat{U}'_m(x_k)\omega(x_k)} = \frac{1}{\sqrt{\lambda_m}\,\omega(x_k)\widehat{U}'_m(x_k)\widehat{U}_{m-1}(x_k)}.$$

8. In order to present some illustrations we consider that the fixed nodes are: $a_1 = -1$, $a_2 = 1$, having different orders of multiplicities.

If the polynomial of the fixed nodes is $\omega(x) = (1+x)(1-x)^2$, then the Gaussian nodes can be found by solving the equation

$$\begin{vmatrix} L_m(x) & L_{m+1}(x) & L_{m+2}(x) & L_{m+3}(x) \\ L_m(-1) & L_{m+1}(-1) & L_{m+2}(-1) & L_{m+3}(-1) \\ L_m(1) & L_{m+1}(1) & L_{m+2}(1) & L_{m+3}(1) \\ L'_m(1) & L'_{m+1}(1) & L'_{m+2}(1) & L'_{m+3}(1) \end{vmatrix} = 0.$$

It leads to the solution of the equation

$$(2m+5)[L_m(x) - L_{m+2}(x)] - (2m+3)[L_{m+1}(x) - L_{m+3}(x)] = 0,$$

eliminating the roots of the polynomial $\omega(x)$.

If we take $m = 1$ we find the Gaussian node $x_1 = -\frac{1}{5}$ and the quadrature formula of degree of exactness four

$$\int_{-1}^{1} f(x)dx = \frac{1}{108}\left[27f(-1) + 125f\left(-\frac{1}{5}\right) + 64f(1) - 12f'(1)\right] + \frac{2}{1125}f^{(5)}(\xi),$$

given first in [4].

For $m = 2$ we get the Gaussian nodes

$$x_1 = -\frac{2\sqrt{2}+1}{7}, \quad x_2 = \frac{2\sqrt{2}-1}{7}.$$

By using these nodes and the fixed nodes $a_1 = -1$ (simple) and $a_2 = 1$ (double), we can obtain a quadrature formula of degree of exactness six.

If we now consider that $\omega(x) = (1-x^2)^2$ then the Gaussian nodes and the fixed nodes are the roots of the equation

$$(2m+7)L_m(x) + (2m+3)L_{m+1}(x) - 2(2m+5)L_{m+2}(x) = 0.$$

In the case $m = 3$ we obtain a quadrature formula of degree of exactness nine, namely

$$\int_{-1}^{1} f(x)dx = \frac{1}{105}\left[19f(-1) + f'(-1) + 54f\left(-\frac{1}{\sqrt{3}}\right) + 64f(0) + \right.$$
$$\left. +54f\left(\frac{1}{\sqrt{3}}\right) - f'(1) + 19f(1)\right] + \frac{1}{589396500} f^{(10)}(\xi).$$

Considering also the case $\omega(x) = (1-x^2)^3$, formula (10) leads to the solution of the equation

$$(2m+7)(2m+9)(2m+11)L_m(x) - 3(2m+5)(2m+7)(2m+11)L_{m+2}(x) +$$
$$+3(2m+3)(2m+7)(2m+9)L_{m+4}(x) - (2m+3)(2m+5)(2m+7)L_{m+6}(x) = 0.$$

In the case $m = 2$ we obtain the following quadrature formula of degree of exactness nine

$$\int_{-1}^{1} f(x)dx = \frac{1}{3360}\left[1173f(-1) + 156f'(-1) + 8f''(-1) + 2187f\left(-\frac{1}{3}\right) + \right.$$
$$\left. +2187f\left(\frac{1}{3}\right) + 8f''(1) - 156f'(1) + 1173f(1)\right] - \frac{2}{442047375} f^{(10)}(\xi).$$

For $m = 3$ we get the Gaussian nodes

$$x_1 = -\sqrt{\frac{3}{11}}, \quad x_2 = 0, \quad x_3 = \sqrt{\frac{3}{11}}$$

and a Gauss-Christoffel quadrature formula of degree of exactness eleven.

9. Considering that we have an arbitrary real fixed node a, of multiplicity $2s$, we arrive at a quadrature formula of the form

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{h=0}^{2s-1} B_h f^{(h)}(a) + R(f),$$

where the remainder has the expression

$$R(f) = \frac{f^{(2m+2s)}(\xi)}{(2m+2s)!} \int_{-1}^{1} (x-a)^{2s} \widehat{U}_m^2(x)dx.$$

The Gaussian nodes can be found by solving the equation

$$\begin{vmatrix} L_m(x) & L_{m+1}(x) & \dots & L_{m+2s}(x) \\ L_m(a) & L_{m+1}(a) & \dots & L_{m+2s}(a) \\ L'_m(a) & L'_{m+1}(a) & \dots & L'_{m+2s}(a) \\ \dots & \dots & \dots & \dots \\ L_m^{(2s-1)}(a) & L_{m+1}^{(2s-1)}(a) & \dots & L_{m+2s}^{(2s-1)}(a) \end{vmatrix} = 0,$$

omitting the root a of multiplicity $2s$.

In the case when $\omega(x) = x^2$ and we take $m = 5$ we find the Gaussian nodes

$$-x_1 = x_4 = \sqrt{\frac{21 + 2\sqrt{14}}{33}}, \quad -x_2 = x_3 = \sqrt{\frac{21 - 2\sqrt{14}}{33}}$$

and we are able to obtain a quadrature formula having the degree of exactness eleven, namely

$$\int_{-1}^{1} f(x)dx = \frac{1}{514500}\{440832f(0) + 8960f''(0)+$$
$$+27(5446 - 537\sqrt{14})[f(x_1) + f(x_4)] + 27(5446 + 537\sqrt{14})[f(x_2) + f(x_3)]\}+$$
$$+\frac{1}{476804928600}f^{(12)}(\xi).$$

Considering also the case $\omega(x) = x^4$ and $m = 3$ we get the Gaussian nodes

$$x_1 = -\frac{\sqrt{7}}{3}, \quad x_2 = 0, \quad x_3 = \frac{\sqrt{7}}{3}$$

and the following quadrature formula

$$\int_{-1}^{1} f(x)dx = \frac{1}{36015}\left\{50160f(0) + 3500f''(0) + 10935\left[f\left(-\frac{\sqrt{7}}{3}\right) + f\left(\frac{\sqrt{7}}{3}\right)\right]\right\}+$$
$$+\frac{1}{404157600}f^{(10)}(\xi).$$

Ending this paper we mention that D. D. Stancu and A. H. Stroud have tabulated the values of the nodes, the coefficients and the remainders, with 20 significant digits, in the paper [6].

References

[1] E. B. Christoffel, *Über die Gaussiche Quadratur und eine Veralgemeinerung derselben*, J. Reine Angew. Math. **55**(1858), 61-82.

[2] W. Gautschi, *Numerical Analysis. An Introduction*, Birkhäuser, Boston-Basel-Berlin, 1997.

[3] T. Popoviciu, *Asupra unei generalizări a formulei de integrare numerică a lui Gauss*, Acad. R. P. Rom. Fil. Iaşi, Stud. Cerc. Sti. **6**(1955), 29-57.

[4] D. D. Stancu, *Generalizarea formulei de cuadratură a lui Gauss-Christoffel*, Acad. R. P. Rom. Fil. Iaşi, Stud. Cerc. Sti. **8**(1957), 1-18.

[5] D. D. Stancu, *Sur une classe de polynomes orthogonaux et sur des formules générales de quadrature a nombre minimum de termes*, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine **1**(49), (1957), 479-498.

[6] D. D. Stancu, A. H. Stroud, *Quadrature formulas with simple Gaussian nodes and multiples fixed nodes*, Math. Comp. **17**(1963), 384-394.

[7] D. D. Stancu, Gh. Coman, P. Blaga, *Analiză Numerică şi Teoria Aproximării*, Vol. II, Presa Universitară Clujeană, 2002.

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A NEW CONVEXITY CRITERION

RÓBERT SZÁSZ

Abstract. In this paper we have obtained a simple sufficient condition for the convexity of analytic functions defined in the unit disc $U = \{z \in C : |z| < 1\}$.

1. **Introduction**

Let A be the class of functions which are analytic in the unit disc $U = \{z \in \mathbb{C} \mid |z| < 1\}$ and has the form $f(z) = z + a_2z^2 + a_3z^3 + \dots$.
The analytic function f is said to be in the class $\mathcal{P}$, if and only if $f(0) = 1$ and $Re\, f(z) > 0,\ \forall\ z \in U$. If f and g are analytic in the unit disc U,we say that f is subordinate to g in U if there exist a function Φ analytic in U, so that $\Phi(0) = 0$, $|\Phi(z)| < 1$, and $f(z) = g(\Phi(z))$ for all $z \in U$. The subordination shall be denoted by $f \prec g$. If g is univalent, then f is subordinated to g if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.
We say that the analytic function f is convex in U if it is univalent and $f(U)$ is a convex domain in $\mathbb{C}$. It is well known that a function f is convex if and only if $f'(0) \neq 0$ and $Re\left(1 + \dfrac{z f''(z)}{f'(z)}\right) > 0$, for all $z \in U$. Let K denote the subset of A consisting of convex functions. In order to show our main result,we need the following lemmas.

Lemma 1. *(Herglotz) [1]*
A function f belongs to $\mathcal{P}$ if and only if there is a measure μ on $[0, 2\pi]$ so that

$$f(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, \mathrm{d}\mu(t) \quad and \quad \mu([0, 2\pi]) = 1\ .$$

Lemma 2. *(H.S. Wilf)[4]*
If $Re\left(\dfrac{1}{2} + \sum_{n=1}^{\infty} b_n z^n\right) > 0, \forall\, z \in U$ *and* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *is a convex function, then* $\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z)$.

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2. Main result

Theorem 1. If $f \in A$ and $Re\left(zf''(z) + \frac{z^2}{2}f'''(z)\right) > \frac{-1}{\pi + 4\ln 2}$ for $z \in U$ then $f \in K$.

Proof. If $f \in A$ then it has the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The condition $Re\left(zf''(z) + \frac{z^2}{2}f'''(z)\right) >$
$> \frac{-1}{\pi+4\ln 2} = -c$ is equivalent with $1 + \frac{1}{c}\left(zf''(z) + \frac{z^2}{2}f'''(z)\right) \in \mathcal{P}$ and using the Lemma 1 we obtain the following representation:

$$1 + \frac{1}{2c}\sum_{n=2}^{\infty} n^2(n-1)a_n z^{n-1} = 1 + 2\sum_{n=2}^{\infty} z^{n-1}\int_0^{2\pi} e^{-it(n-1)}\, \mathrm{d}\mu(t)$$

From the last equality we deduce that:

$$a_n = \frac{4c}{n^2(n-1)}\int_0^{2\pi} e^{-it(n-1)}\, \mathrm{d}\mu(t)$$

and

$$f(z) = z + 4c\sum_{n=2}^{\infty}\frac{z^n}{n^2(n-1)}\int_0^{2\pi} e^{-it(n-1)}\, \mathrm{d}\mu(t)$$

After a simple calculation we get that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\frac{1}{4c} + \sum_{n=2}^{\infty}\frac{z^{n-1}}{n-1}\int_0^{2\pi} e^{-it(n-1)}\, \mathrm{d}\mu(t)}{\frac{1}{4c} + \sum_{n=2}^{\infty}\frac{z^{n-1}}{n(n-1)}\int_0^{2\pi} e^{-it(n-1)}\, \mathrm{d}\mu(t)}$$

We introduce the notations $h(z) = \frac{1}{4c} + \sum_{n=2}^{\infty}\frac{z^{n-1}}{n-1}$ and $g(z) = \frac{1}{4c} + \sum_{n=2}^{\infty}\frac{z^{n-1}}{n(n-1)}$. It is easy to observe that $h(z) = \frac{1}{4c} + \log\frac{1}{1-z}$ and h is a convex function.
Because $Re\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{z^n}{n+1}\right) > 0$, $\forall\, z \in U$ and h is convex using Lemma 2 it follows that $g(z) \prec h(z)$.
The convexity of h implies

$$\frac{1}{4c} + \sum_{n=2}^{\infty}\frac{z^{n-1}}{n(n-1)}\int_0^{2\pi} e^{-it(n-1)}\, \mathrm{d}\mu(t) \in h(U), \tag{1}$$

$$\frac{1}{4c}+\sum_{n=2}^{\infty}\frac{z^{n-1}}{n-1}\int_0^{2\pi}e^{-it(n-1)}\,\mathrm{d}\mu(t)\in h(U) \tag{2}$$

for every $z\in U$ and every measure μ for which $\mu\left([0,2\pi]\right)=1$.
If $0<c<\dfrac{1}{4}$ then $Re\ h(z)>0, z\in U$ and we can draw two tangent lines to the curve $\Gamma=\partial h(U)$. Let denote with α the measure of the angle between the two tangent lines which contains $h(U)$. From(1) and (2) follows that

$$|\arg\frac{\frac{1}{4c}+\sum_{n=2}^{\infty}\frac{z^{n-1}}{n-1}\int_0^{2\pi}e^{-it(n-1)}\,\mathrm{d}\mu(t)}{\frac{1}{4c}+\sum_{n=2}^{\infty}\frac{z^{n-1}}{n(n-1)}\int_0^{2\pi}e^{-it(n-1)}\,\mathrm{d}\mu(t)}|<\alpha$$

and so a sufficient condition for $Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ z\in U$ is $\alpha=\dfrac{\pi}{2}$.
The curve Γ has the $h(e^{i\theta})=u(\theta)+iv(\theta),\ \theta\in(0,2\pi)$ parametric representation.
The equality $\alpha=\dfrac{\pi}{2}$ is equivalent with the existence of $\theta_1,\theta_2\in[0,2\pi]$ with the properties

$$\frac{u(\theta_1)}{u'(\theta_1)}=\frac{v(\theta_1)}{v'(\theta_1)}\ ,\ \frac{u(\theta_2)}{u'(\theta_2)}=\frac{v(\theta_2)}{v'(\theta_2)}\ ,\ \frac{v'(\theta_1)}{u'(\theta_1)}\cdot\frac{v'(\theta_2)}{u'(\theta_2)}=-1$$

Because $h(U)$ is symmetric with respect to the real axis,we deduce that $\theta_2=2\pi-\theta_1$ and after calculation we get $c=\frac{1}{\pi+2\ln 2}$. □

Example. *Let λ be a real number so that $0<\lambda<\frac{e-1}{e(\pi+2\ln 2)}$ then the function* $f(z)=z\ +\lambda\displaystyle\int_0^z\!\!\int_0^t\frac{1}{u^2}\int_0^u\frac{s^2}{e^s-1}\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t$ belongs to K.

Proof. After derivation we get that: $zf''(z)+\dfrac{z^2}{2}f'''(z)=\lambda\dfrac{z^2}{e^z-1},\ z\in U$. In [1] had been proved that $q(z)=\dfrac{e^z-1}{z}$ is a convex function in U which implies the inequality:

$$Re\ q(z)>\frac{e-1}{e}\ ,\ z\in U. \tag{2}$$

From (2) follows that $|q(z)|>\dfrac{e-1}{e}, z\in U$ or equivalently

$$\left|\frac{z}{e^z-1}\right|<\frac{e}{e-1}\ ,\ z\in U. \tag{3}$$

Using (3) it is easy to deduce that

$$Re\left(zf''(z)+\frac{z^2}{2}f'''(z)\right)=\lambda\ Re\frac{z^2}{e^z-1}>\lambda\frac{-e}{e-1}=\frac{-1}{\pi+2\ln 2},\ z\in U$$

which is the condition of Theorem 1. □

References

[1] D. J. Hallenbeck, T. H. MacGregor, *Linear problems and convexity techniques in geometric function theory*, Pitman Advanced Publishing Program, 1984.

[2] P. T. Mocanu, *On certain subclasses of starlike functions*, Studia Univ.Babeş-Bolyai, Math. 34, 4(1994) 3-9.

[3] I. Şerb, *The radius of convexity and starlikeness of a particular function*, Mathematica Montisnigri vol. VII(1996), 65-69.

[4] H. S. Wilf, *Subordinating factor sequences for convex maps of the unit circle,* Proc. Amer. Math. Soc. 12(1961), 689-693.

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CORRIGENDUM: ON THE IRRATIONALITY OF SOME ALTERNATING SERIES

J. SÁNDOR AND J. SONDOW

The aim of this note is to point out that Theorem 1 of the first author's paper [1] is incorrect, and to replace it with Theorem A below and give an application.

Theorem 1. *Let (a_n) be a sequence of positive integers such that $a_n(a_1a_2\dots a_{n-1})^2 \to \infty$ as $n \to \infty$. Then the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{a_n(a_1\dots a_{n-1})^2}$ *is irrational.*

The constant sequence $(a_n) = 2, 2, \dots$ is a counterexample. The mistake in the proof lies in assuming that, with $u_i = (a_1 \dots a_{i-1})^{-2}$ and $v_i = a_i$, the sum $\sum_{i=1}^{n}(-1)^{i-1}u_i/v_i$ is a rational number with denominator $v_1 \dots v_n$. In fact, the denominator is $v_n(v_1 \dots v_{n-1})^2$.

The following result is a generalization of Lemma 1 in [1].

Theorem A. *Let $(r_n) = (h_n/k_n)$ be a sequence of rational numbers, with $k_n > 0$, satisfying*

(i) $r_2 < r_4 < r_6 < \dots < r_5 < r_3 < r_1$ and

(ii) $\liminf_{n\to\infty} k_n|r_{n+1} - r_n| = 0$. Then the alternating series

$$r_1 - (r_1 - r_2) + (r_3 - r_2) - (r_3 - r_4) + \dots$$

converges and its sum is irrational.

Proof. It follows from (i) and (ii) that the conditions of Leibniz's alternating series test are satisfied. Thus the series converges and its sum, θ, lies between the partial sums r_n and r_{n+1}, for $n = 1, 2, \dots$ Suppose now that $\theta = a/b$ is rational, $b > 0$. Then (ii) and the inequalities $0 < |\theta - r_n| < |r_{n+1} - r_n|$ imply that $0 < |ak_n - bh_n| < bk_n|r_{n+1} - r_n| < 1$, for some $n \geq 1$. This contradicts the fact that $ak_n - bh_n$ is an integer, completing the proof. □

As an application of Theorem A (or of Lemma 1), we obtain a new proof that *if p_n/q_n is the n-th convergent of an infinite simple continued fraction, $n = 0, 1, 2, \dots$, then the sum of the series $p_0/q_0 + \sum_{n=0}^{\infty}(-1)^n/(q_nq_{n+1})$ is an irrational number,* namely, the value of the continued fraction.

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References

[1] J. Sándor, *On the irrationality of some alternating series*, Studia Univ. Babeş-Bolyai, **33**(1988), 8-12.

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BOOK REVIEWS

Agostino Abbate, Casimer M. DeCusatis, Pankaj K. Das, *Wavelets and Subbands. Fundamentals and Applications*, Birkhäuser, Boston Basel Berlin, 2002, ISBN 0-8176-4136-X.

The volume presents in a typical style some researches and noteworthy directions particularly aimed at providing stimulus and inspiration to workers interested in the broad areas of related to wavelets and their applications. The content of the book is divided into three parts.

In the first part (*Fundamentals*) the authors enlarge on a systematic study of wavelets and subbands concepts. It is written with a concern for simplicity and clarity. The section offers detailed and explanation of some concepts and methods accompanied by carefully selected worked examples. The aim of this part is to familiarize the reader with a lot of basic notions regarding wavelet and subband transforms, such as: Fourier transform as a wave transform, wavelet transform, time-frequency analysis, multiresolution analysis, wavelet frames, connection between wavelets and filters, analysis and synthesis filters, iterated filters for subbands, filter banks for subbands. Also, the link between discrete and continuous wavelets and subbands is explained.

The second part (*Wavelets and Subbands*) includes advanced topics and a more in-depth technical treatment of the subject matter. The information is structured in the following chapters: Time and Frequency Analysis of Signals; Discrete Wavelet Transform: from Frames to Fast Wavelet Transform; Theory of Subband Decomposition; Two-Dimensional Wavelet Transforms and Applications.
Within the scope of this section, the authors investigate the large body of work that has been done in applying wavelet and subband methods to image processing and compression.

The third part (*Applications*) contains some practical applications of wavelets and subbands. Divided in three chapters, these include image processing, image compression, pattern recognition, and signal-to-noise improvement. The communication application concentrates on spread spectrum systems which have applications to wireless communication, digital multitone, code division multiple access and excision.

At the end of the book are inserted four appendices: Fourier Transform, Discrete Fourier Transform, z-Transform and Orthogonal Representation of Signals.

We point out that for additional information, the reader is referred to the many excellent references in the literature which are listed at the end of each part. In the same time, in order to sustain the objectives of the book, a generous bibliography is listed over 20 pages.

In our opinion, the monograph is a valuable text for a broad audience including graduates, researches and professionals in signal processing.

Octavian Agratini

Erik M. Alfsen and Frederic W. Shultz, *Geometry of State Spaces of Operator Algebras,* Mathematics: Theory and Applications, Birkhäuser Verlag, Boston-Basel-Berlin 2003, xiii+467 pp., ISBN 0-8176-4319-2 and 3-7643-4319-2.

The aim of the present book is to give a complete geometric description of the state spaces of operator algebras, meaning to give axiomatic characterizations of those convex sets that are state spaces of C^*-algebras, von Neuman algebras, and of their nonassociative analogs - JB-algebras and JBW-algebras. A previous book by the same authors -*State spaces of operator algebras - basic theory, orientations and C^*-products,* published by Birkhäuser in 2001, contains the necessary prerequisites on C^*-algebras and von Neumann algebras but, for the convenience of the reader, these results are summarized in an appendix at the end of the present book with exact references to previous one for proofs.

The problem of the characterization of state spaces of operator algebras was raised in the early 1950s and was completely solved by the authors of the present book in Acta Mathematica **140** (1978), 155-190, and **144** (1980), 267-305 (the second paper has also H. Hanche-Olsen as co-author). Although the axioms for state spaces are essentially geometric, many of them have physical interpretations. The authors have included a series of remarks concerning these interpretations along with some historical notes.

The book is divided into three parts. Part I (containing Chapters 1 through 6) can serve as an introduction for novices to Jordan algebras and their states. Jordan algebras were originally introduced as mathematical model for quantum mechanics (in 1934 by P. Jordan, J. von Neumann and E. Wigner), starting from the remark that the set of observables is closed under Jordan multiplication, but not necessarily under associative multiplication. Part II (Chapters 7 and 8) develops the spectral theory for affine functions on convex sets. The functional calculus developed in this part reflects a key property of the subalgebra generated by a single element and, physically, it represents the application of a function to the outcome of an experiment. Part III (Chapters 9,10,11) gives the axiomatic characterization of operator algebra state spaces and explain how the algebras can be reconstructed from their state spaces.

This valuable book, together with the previous one on C^*-algebras, presents in a manner accessible to a large audience, the complete solution to a long standing problem, available previously only in research papers, whose understanding requires a solid background from the readers.

It is aimed to specialists in operator algebras, graduate students and mathematicians working in other areas (mathematical physics, foundation of quantum mechanics)

S. Cobzaş

Jan Andres and Lech Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems,* Topological Fixed Point Theory and Its Applications 1, Kluwer Academic Publishers, Dordrecht-Boston-London, 2003, 761 + xvi pp., ISBN 1-4020-1380-9.

The monograph is devoted to the topological fixed point theory for single-valued and multivalued mappings in locally convex spaces and its applications to boundary value problems for ordinary differential equations and inclusions and to multivalued dynamical systems.

Chapter I, Theoretical background (126 pp.) gathers together several topological and analytical notions and results such as: locally convex spaces, absolute retracts (AR-spaces) and absolute neighborhood retracts (ANR-spaces), selections of multivalued mappings, admissible mappings, Lefschetz fixed point theorem, fixed point index in locally convex spaces, Nielson number etc.

In Chapter II, General principles (106 pp.), topological principles necessary for applications are presented, namely: Aronszajn-Browder-Gupta type results on the topological structure of fixed point sets, inverse limit method, topological dimension of fixed point sets, topological essentiality, relative theories of Lefschetz and Nielson, periodic point theorems, fixed point index for condensing maps, approximation methods in the fixed point theory of multivalued mappings, topological degree by means of approximation methods and continuation principles based on fixed point index and coincidence index.

Chapter III, Applications to differential equations and inclusions (366 pp.), is devoted to the applications of the general principles to boundary value problems for ordinary differential equations and inclusions on compact or non-compact intervals and to dynamical systems. The following problems are mainly considered: existence of solutions, topological structure of solution sets, topological dimension of solution sets, multiplicity results, periodic and almost periodic solutions and Wazewski type results.

Three Appendices concerning almost periodic and derivo-periodic functions and multivalued fractals are also included. A large and exhaustive list of References (58 pp.) and a subject Index are added.

The authors are known as experts in their field and most of presented results are their own. The book is self-contained and every chapter concludes by a section of Remarks and Comments giving to the reader historical information and suggestions for further studies.

Authors' intention has been to make deep results of algebraic topology and nonlinear analysis accessible to a wider auditorium and by this, to stimulate the interest of applied mathematicians (mathematical economists, population dynamics experts, theoretical physicists etc.) for such type of methods.

I believe that this monumental monograph will be extremely useful to post-graduate students and researchers in topological fixed point theory, nonlinear analysis, nonlinear differential equations and inclusions, dynamical systems, optimal control and chaos and fractals. This book should stimulate a great deal of interest and research in topological methods in general and in their applications in particular.

Radu Precup

Emmanuele DiBenedetto, *Real Analysis*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Boston-Basel-Berlin 2002, xxiv+485 pp., ISBN 0-8175-4231-5.

The aim of this book is to present at graduate level the basic results in real analysis, needed for researchers in applied analysis - PDEs, calculus of variations, probability, and approximation theory. Assuming only the knowledge of the basic results about the topology of $\mathbb{R}^N$, series, advanced differential calculus and algebra of sets, the author develops the whole machinery of real analysis bringing the reader to the frontier of current research.

The emphasis is on measure and integration in $\mathbb{R}^N$, meaning Lebesgue and Lebesgue-Stieltjes measures, Radon measures, Hausdorff measure and dimension. The topological background, including Tihonov compactness theorem, Tietze and Urysohn theorems, is developed with full proofs. The specific of the book is done by the treatment of some more specialized topics than those usually included in introductory courses of real analysis. Between these topics I do mention a detailed presentation of covering theorems of Vitali and Besicovitch, the Marcinckiewicz integral, the Legendre transform, the Rademacher theorem on the a.e differentiability of Lipschitz functions. Fine topics, as a.e. differentiability of functions with bounded variation and of absolutely continuous functions and the relation with the integral, are worked out.

The spaces L^p are also presented in details in Chapter V - completeness, uniform convexity (via Hanner's inequalities), duality, weak convergence, compactness criteria. The next chapter of the book (Ch. VI) contains a brief introduction to abstract Banach and Hilbert spaces. Distributions, weak differentials and Sobolev spaces are presented in Chapter VII.

The last two chapters of the book, Chapters VIII and IX, contains more specialized topics as maximal functions and Fefferman-Stein theorem, the Calderón-Zygmund decomposition theorem, functions of bounded mean oscillation (BMO), Marcinkiewicz interpolation theorem, embedding theorems for Sobolev spaces, Poincaré inequality, Morrey spaces.

Each chapter is completed by a set of exercises and problems that add new features and shed new light on the results from the main text.

Bringing together, in a relatively small number of pages, important and difficult results in real analysis that are of current use in application to PDEs, Fourier and harmonic analysis and approximation, this valuable book is of great interest to researchers working in these areas, but it can be used for advanced graduate courses in real analysis as well.

Stefan Cobzaş

Stefan Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, New Jersey-London-Singapore-Hong Kong, 2002, ISBN 981-02-4837-7.

In the recent period (especially in the last three decades) functional equations became an important branch of mathematics. The book under review is intended to present a survey of classical results and more recent developments in the theory of functional equations in several variables. Particularly the results of the Polish school of functional equations are emphasized.

The book is divided into three parts. The first one is devoted to the study of additive and convex functions defined on linear spaces endowed with semilinear topologies. The classical results of Bernstein-Doetch, Picard, and Mehdi, concerning the relationship between the continuity and the local boundedness of a convex function, are presented in Chapter 4. Closely related to this problem are the so-called Kuczma-Ger set classes, studied in Chapter 5. The rest of the material included in the first part deals with familiar functional equations like Cauchy, D'Alembert, and quadratic equations.

Part II, entitled *Ulam-Hyers-Rassias Stability of Functional Equations*, is entirely concened with the examination of the stability problem. It has originally been posed by S. M. Ulam in 1940 with regard to the Cauchy functional equation. In 1941 D. H. Hyers gave a significant partial solution to this problem, but a substantial generalization of Hyers' result has been obtained by Th. M. Rassias in 1978. Rassias' paper has rekindled the interest of the mathematicians in the field of stability of functional equations. Since then a great number of articles have appeared in the literature. This second part of Czerwik's book brings together the stability results concerning several functional equations like Cauchy, Pexider, Jensen, D'Alembert, gamma, and quadratic, obtained by many authors.

Particularly valuable, Part III contains a systematic examination of set-valued functional equations, which has been lacking in the mathematical literature. Set-valued versions of the Cauchy, Jensen, Pexider, and quadratic equations are studied in this part. Finally, the author investigates some special kinds of set-valued functions like subadditive, superadditive, subquadratic, K-convex, and K-concave set-valued functions.

Twenty-one of the thirty-seven chapters contain valuable notes at the end, providing useful references to related material. The bibliography counts 216 references.

Written by an expert in domain, the book is an excellent tool for any reader interested to get an idea about the basic results and the latest research directions in the field of functional equations.

Lokenath Debnath, *Wavelet Transforms and Their Applications*, Birkhäuser, Boston Basel Berlin, 2002, ISBN 0-8176-4204-8.

The last two decades have produced tremendous developments in the mathematical theory of wavelets and their great variety of applications. Since wavelet analysis is a relatively new subject, this monograph is intended to be self-contained. The book is designed as a modern and authoritative guide to wavelets, wavelet transform, time-frequency signal analysis and related topics.

It is known that some research workers look wavelets upon as a new basis for representing functions, others consider them as a technique for time-frequency analysis and some others think of them as a new mathematical subject. All these approaches are gathered in this book, which presents an accessible, introductory survey of new wavelet analysis tools and the way they can be applied to fundamental analysis problems. We point out the clear, intuitive style of presentation and the numerous examples demonstrated thorough the book illustrate how methods work in a step by step manner.

This way, the book becomes ideal for a broad audience including advanced undergraduate students, graduate and professionals in signal processing. Also, the book provides the reader with a through mathematical background and the wide variety of applications cover the interdisciplinary collaborative research in applied mathematics. The information is spread over 565 pages and is structured in 9 chapters as follows:

1. Brief Historical Introduction
2. Hilbert Spaces and Orthonormal Systems
3. Fourier Transforms and Their Applications
4. The Gabor Transform and Time-Frequency Signal Analysis
5. The Wigner-Ville Distribution and Time-Frequency Signal Analysis
6. Wavelet Transforms and Basic Properties
7. Multiresolution Analysis and Construction of Wavelets
8. Newland's Harmonic Wavelets
9. Wavelet Transform Analysis of Turbulence.

At the end of the book a key and hints for selected exercises are included.

In order to stimulate further interest in future study and to sustain the present material, a generous bibliography is listed.

Octavian Agratini

Andrzej Granas and James Dugundji, *Fixed point theory,* Springer-Verlag, New York-Berlin-Heidelberg, 2003, 13 figs. xv + 690 pages, ISBN 0-387-00173-5.

Fixed point theory represents one of the most powerful tools for various problems from pure, applied and computational mathematics. The abstract theory, the computation of fixed points and various applications, mainly for proving the existence of solutions to several classes of nonlinear operator equations, occupies a central place in today's mathematics. Over 150 monographs and proceedings, as well as more then 10, 000 papers deal with this topic. Two very new journals are entirely dedicated to fixed point theory and its applications.

The new edition of Granas and Dugundji's book is, in my opinion, the most important and complete survey in the last years on fixed point theory and its applications. The book goes through almost all the basic results in fixed point theory, from elementary theorems to advanced topics, from ordered, metric or topological structures to algebraic topology. The main text is self-contained, the necessary background material being collected in an appendix at the end of the book. Each chapter ends with "Miscellaneous Results and Examples" and some very important "Notes and Comments". Several nice photographs of famous mathematicians in the field pigment the text.

The book is organized in six important parts, each of them containing several chapters, twenty on the whole.

Part I ("Elementary Fixed Point Theorems", 74 pages) includes basic results and applications in ordered and metric structures. The main topics of this part are: the Banach contraction principle, the continuation method for contractive maps, the Knaster-Tarski and Tarski-Kantorovitch theorems, the Bishop-Phelps result, Caristi's fixed point theorem, Nadler's extension of Banach contraction principle to set-valued operators, the KKM operator theory and the fixed point theory for nonexpansive operators.

Part II (Theorem of Borsuk and Topological Transversality, 112 pages) presents several fundamental results in the topological fixed point theory: the antipodal theorem of Borsuk (and as consequence, the Brouwer fixed point theorem), Schauder's fixed point principles, the infinite-dimensional version of Borsuk theorem, the theory on topological transversality based on the notion of essential map, the Leray-Schauder principle and the nonlinear alternative. As applications, the Fan coincidence theorem, the mini-max inequality and the Kakutani and Ryll-Nardzewski theorems are also presented.

Part III (Homology and Fixed Points, 50 pages) is dedicated to the Lefschetz-Hopf theorem for polyhedra.

Part IV (Leray-Schauder Degree and Fixed Point Index, 120 pages) presents the notions of topological degree and fixed point index. This part starts with the presentation of Brouwer's degree, defined for maps on the Euclidian spaces, and then the concept is extended for compact maps in normed linear spaces. Further on, the case of an arbitrary metric absolute neighborhood retracts is also considered. Bifurcation results in absolute neighborhood retracts and existence theorems for boundary value problems related to partial differential equations are nice applications of this theory.

Part V (The Lefschetz-Hopf Theory, 122 pages) is deals with the Lefschetz fixed point theorem and the Hopf index theorem. Several extensions of the Lefschetz theory to wider classes of maps and spaces are also included.

Part VI (Selected Topics, 97 pages) contains advanced topics of algebraic topology: Finite-Codimensional Čech Cohomology, Vietoris Fractions ans Coincidence Theory.

The Bibliography is organized as follows:

► General Reference Texts (Monographs, Lecture Notes, Surveys, Articles) with more than 700 titles

► Additional References with more than 400 titles.

An Appendix, a List of Symbols, an Index of Names and an Index of Terms are also included.

From the above considerations, it is more than obviously that this new edition of Granas-Dugundji's monograph is, in fact, a new book. New and interesting results and applications can be found all over the book. The style is alert and pleasant. The technical presentation of the book is exceptional.

The book **Fixed Point Theory, by Andrzej Granas and James Dugundji**, which appeared in the series **Springer Monograph in Mathematics**, is an inspired publication of Springer-Verlag Publishing House and I am sure that it will be a very useful work for anyone (postgraduate students, Ph.D. students, researchers, etc.) who is involved in fixed point theory in particular and nonlinear analysis in general.

Adrian Petruşel

Srdjan Stojanovic, *Computational financial mathematics using Mathematica: optimal trading stocks and options*, Birkhäuser Verlag, Boston - Basel - Berlin, 2003, XI+481 pages.

The book consists in 481 pages i.e. 8 chapters, a bibliography and an index and includes CD-ROM. Srdjan Stojanovic taught the course on Financial Mathematics at the University of Cincinnati since 1998 and at Purdue University during the academic year 2001-2002. This book is an expanded version of those courses, built with the help of the students during the time when Srdjan Stojanovic taught them computational financial mathematics and MATHEMATICA^R programming.

A very interesting and very actual book, because now, the computer make an integrand part of our life. The author, himself, underlines in the Introduction, that the book is addressed to students and professors of academic programs in financial mathematics (like computational finance and financial engineering). Anyway, the mathematical background would be Calculus, Differential Equations and Probability, but varies according to the objectives of the reader. The book is, as recommends the author, divided in some parts according to the required mathematical level as follows: the basics (for the Chapters 1-4), intermediate level (the Chapters 5 and 7), advanced level (for the Chapters 6 and 8).

In the Chapter 1, **Cash Account Evolution**, ordinary differential equations are solving with Mathematica^R, and symbolic and numerical solutions of ODEs are presented.

The Chapter 2, **Stock Price Evolution**, explains to the reader what are stocks and then presents the stock price modeling, i.e. some stochastic differential equations. An other aim of this chapter is to be acquainted with Itô calculus and with multivariable and symbolic Itô calculus. Also, some relationship between SDEs and PDEs are presented.

In the Chapter 3, **European Style Stock Options**, the first paragraph deals with the notion of stock option. Then, the Black and Scholes PDE and hedging are presented and the Black and Scholes PDE are symbolically solved. Also, the generalized Black and Scholes formulas are presented.

In the Chapter 4, **Stock Market Statistics**, the stock market data import and manipulation are presented. Then, the chapter deals with the volatility estimates, i.e. scalar case, and also deals with the appreciation rate estimates (the scalar case) and the statistical experiments (Bayesian and non-Bayesian). In the same chapter, the vector basic price model statistics and the dynamic statistics, like the filtering of conditional Gaussian processes, are treated.

In the Chapter 5, **Implied Volatility for European Options**, the option market data is presented. After that, the Black and Scholes theory is made obvious vs. market data (the implied volatility) and then, the numerical PDEs, the optimal control and the implied volatility are studied.

The Chapter 6, **American Style Stock Options**, deals with the american options, the obstacle problems and presents the general implied volatility for american options.

Very important, the Chapter 7, **Optimal Portfolio Rules,** presents the utility of wealth, the Merton's optimal portfolio rule derived and implemented, the portfolio rules under appreciation rate uncertainty, the portfolio optimization under equality constraints, the portfolio optimization under inequality constraints.

In the Chapter 8, **Advanced Trading Strategies,** the reduced Monge-Ampère PDEs of advanced portfolio hedging and the hypoelliptic obstacle problems in optimal momentum trading are presented.

As we have already said, the book is accompanied by a CD-ROM, but the book is not a software product. Informations about further developments might be available at the web site CFMLab.com. The reader may direct comments to the same address.

Diana Andrada Filip

Advances in Gabor Analysis, Hans G. Feichtinger and Thomas Strohmer - Editors, Applied and Numerical Harmonic Analysis, Birkhäuser Verlag, Boston-Basel-Berlin 2003, xviii+356 pp., ISBN 0-8176-4239-0 and 3-7643-4239-0.

In 1946 Dennis Gábor (Nobel prize for physics in 1971) had the idea to use linear combinations of a set of regularly spaced, discrete time and frequency translates of a single Gaussian function to expand arbitrary square-integrable functions. The idea turned out to be a very fruitful and far-reaching one, with spectacular applications to quantum mechanics and electrical engineering. The Heisenberg uncertainty principle, discussed at large in one of the included chapters, is the core of the time-frequency analysis and of Gabor analysis. Gabor analysis attracted many first rate mathematicians due to the highly non-trivial mathematics lying behind it. A strong impulse came from the development of frames in Hilbert space, leading to important problems of practical computation - rate of convergence, stability, density. In the last time, M.A. Rieffel, R.E. Howe and T.J. Steger found some unexpected connections with operator algebras.

The present book can be considered as a continuation of two previous ones: *Gabor Analysis and Algorithms: Theory and Applications,* H. G. Feichtinger and T. Strohmer - Editors, Birkhäuser 1998, and the book by K. Gröchenig, *Foundations of Time-frequency Analysis,* Birkhäuser 2001. It contains survey chapters, but new results that have been not published previously are also included. The introductory chapter of the book, written by H.G. Feichtinger and T. Strohmer, contains a clear outline of the contents as well as some comments on the future developments in Gabor analysis.

Beside this introductory chapter, the book contains other eleven chapters, written by different authors, and dealing with various questions in Gabor analysis and its applications: uncertainty principles, Zak transforms, Weil-Heisenberg frames, Gabor multipliers, Gabor analysis and operator algebras, approximation methods, localization properties, optimal stochastic encoding, applications to digital signal processing and to wireless communication.

Written by leading experts in the field, the volume appeals, by its interdisciplinary character, to a large audience, both novices and experts, theoretically inclined researchers and practitioners as well. It brilliantly illustrates how application areas and pure and applied mathematics can work together with profit for all.

S. Cobzaş

Enrico Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, London-Singapore-Hong Kong, 2003, vii+403 pp., ISBN 981 238 043 4.

Let Ω be a domain in $\mathbb{R}^n$ and $F(x,u,z)$ a function from $\Omega\times\mathbb{R}^N\times\mathbb{R}^{n\times N}$ to $\mathbb{R}$. One denotes $x=(x_i)_{1\le i\le n}$, $u=(u^\alpha)_{1\le\alpha\le N}$, and $z=(z_i^\alpha)$, $1\le i\le n$, $1\le\alpha\le N$. The fundamental problem of the calculus of variations consists in finding a function $u:\Omega\to\mathbb{R}^N$ which minimizes the integral functional

$$\mathcal{F}(u,\Omega)=\int_\Omega F(x,u(x),Du(x))dx, \tag{1}$$

provided u satisfies some suitable conditions, the most frequent being a boundary condition, $u=U$ on $\partial\Omega$. Supposing F of class C^1, replacing u by $u+t\varphi$, where $\varphi=U$ on $\partial\Omega$, it follows that $g(t)=\mathcal{F}(u+t\varphi,\Omega)$ has a minimum at $t=O$, implying $g'(0)=0$. This condition leads to Euler (called sometimes Euler-Lagrange) equations

$$\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial z_i^\alpha}(x,u(x),Du(x))\right)-\frac{\partial F}{\partial u^\alpha}(x,u(x),Du(x))=0, \tag{2}$$

for $\alpha=1,...,N$, that give a necessary condition of minimum. This approach is useful when the Euler equations can be explicitly integrated, particularly for $n=N=1$, leading to an explicit solution of the minimum problem, but with growing difficulties in higher dimensions.

The *direct method* in the calculus of variations, initiated by Riemann, consists in proving the existence of the minimum of $\mathcal{F}$ and discovering its properties, mainly regularity, without appealing to Euler equations. The usual assumptions, under which such an approach works, are lower semicontinuity of $\mathcal{F}$ and some convexity conditions (convexity, quasi-convexity, polyconvexity, rank-one convexity) on the function F, combined with some compactness hypotheses on the domain of $\mathcal{F}$, requiring compactness criteria in appropriate function spaces.

The present book follows this approach to the study of minima of the functional (1), an outline of its contents, along with some historical remarks, being given in the Introduction to the book. The book is divided into ten chapters: 1. *Semi-classical theory,* 2. *Measurable functions,* 3. *Sobolev spaces,* 4. *Convexity and semicontinuity,* 5. *Quasi-convex functionals,* 6. *Quasi-minima,* 7. *Hölder continuity,* 8. *First derivatives,* 9. *Partial regularity,* 10. *Higher derivatives.*

The prerequisites for the reading are basic properties of the Lebesgue integral and L^p spaces, and some elements of functional analysis, the more special topics being presented with full proofs in the second and the third chapters of the book. Compactness in L^p, Morrey-Campanato spaces, John-Nirenberg theorem on BMO functions, the interpolation theorems of Marcinkiewicz and Stampachia, and elements of Hausdorf measure in the second chapter, and a short introduction to Sobolev spaces (including embedding and trace theorems, Poincaré and Sobolev-Poincaré inequalities) in the third one.

The various aspects of the direct methods in the calculus of variations a treated in the rest of the book - semicontinuity (Chapters 4 and 5) and regularity (Chapters 6 to 10). A special emphasis is put on quasi-minima.

The field owes much to the Italian school of mathematics, starting with L. Tonelli, and continuing with the substantial contributions and ideas of E. De Giorgi, C. Miranda, M. Giaquinta, G. Modica, G. Anzelotti, L. Ambrosio and the author. The book reflects very well these contributions, along with those of other well known mathematicians as J. Moser, J. Nash, B. Dacorogna, S. Benstein, O. Ladyzenskaia, A. Ioffe, K. Uhlenbeck, J. Maly, H. Federer, L. Evans, R. Gariepy, et al.

It can be recommended for graduate courses or post-graduate courses in the calculus of variations, or as reference text.

J. Kolumbán

Israel Gohberg, Seymour Goldberg and Marinus A. Kaashoek, *Basic Classes of Linear Operators,* Birkhäuser Verlag, Basel-Boston-Berlin 2003, xvii+423 pp., ISBN 3-7643-6930-2.

The book provides an introduction to Hilbert and Banach spaces, with emphasis on operator theory, its aim being to stimulate the students to expand their knowledge of operator theory. It is designed for senior undergraduate and graduate students, the prerequisites being familiarity with linear algebra and Lebesgue integration (an appendix contains some results in this area with references). At the same time, the book is written in such a way that it can serve as an introduction to the two volume treatise by the same authors, *Classes of Linear Operators*, published by Birkhäuser, 1990 (Vol I), and 1993 (Vol. II).

The book is based on a previous one of the authors, *Basic Operator Theory,* Birkhäuser 1981, but the present one differs substantially from the previous one. The changes reflect the experience gained by the authors by using the old text in various courses, as well as the recent developments in operator theory. This affected the choice of the topics, proofs and exercises. They included more examples of concrete classes of linear operators as, for instance, Laurent, Toeplitz and singular integral operators, the theory of traces and determinants in an infinite dimensional setting and Fredholm theory. The theory of unbounded operators is expanded.

The material is presented in a way to make a natural transition from linear algebra and analysis to operator theory, keeping it at an elementary level. The main part of the book (Chapters I to X) deals with the Hilbert case. It starts with a chapter on the geometry of Hilbert spaces, and continues with the study of operators acting on them. This study comprises bounded linear operators, Laurent and Toeplitz operators on Hilbert space, unbounded operators, and spectral theory (including the operational calculus). As applications one considers the oscillations of an elastic string and iterative methods for solving linear equations in Hilbert space (relying on spectral theory).

The Banach space setting is treated in Chapters XI to XVI. These contain the basic principles of Banach spaces, linear operators, compact operators, Poincaré operators and their determinants and traces, Fredholm operators, Toeplitz and singular integral operators. The last chapter of the book, Chapter XVII, is concerned with some fix point theorems for non linear operators.

Each chapter ends with a set of exercises, chosen to expand reader's comprehension of the material or to add new results.

By the careful choice of the topics and by the numerous examples included, the book provides the reader with a firm foundation in operator theory, and demonstrates the power of the theory in applications. A list for further reading is presented at the end of the book.

S. Cobzaş

Nonlinear Analysis and its Applications to Differential Equations, M. R. Grossinho, M. Ramos, C. Rebelo and L. Sanchez, Editors, Progress in Nonlinear Differential Equations and Their Applications; Vol. 43, Birkhäuser, Boston-Basel-Berlin, 2001, 380 pp., ISBN 0-8176-4188-2.

This volume presents a significant part of the material given in the autumn school on "Nonlinear Analysis and Differential Equations" held at the CMAF (Centro de Matemática e Aplicações Fundamentais), University of Lisbon, in September-October 1998.

Part 1: Short courses (143 pp.), includes key articles offering a systematic approach to some classes of problems in ordinary differential equations and partial differential equations: C. De Coster and P. Habets, An overview of the method of lower and upper solutions for ODEs; E. Feireisl, On the long-time behaviour of solutions to the Navier-Stokes equations of compressible flow; J. Mawhin, Periodic solutions of systems with p-Laplacian-like operators; W.M. Oliva, Mechanics on Riemannian manifolds; R. Ortega, Twist mappings, invariant curves and periodic differential equations; and K. Schmitt, Variational inequalities, bifurcation and applications.

Part 2: Seminar papers, includes short articles representative of the recent research of participants: F. Alessio, M. Calanchi and E. Serra, Complex dynamics in a class of reversible equations; L. Almeida and Y. Ge, Symmetry and monotonicity results for solutions of certain elliptic PDEs on manifolds; J. Andres, Nielsen number and multiplicity results for multivalued boundary value problems; D. Arcoya and J.L. Gámez, Bifurcation theory and application to semilinear problems near the resonance parameter; P. Benevieri, Orientation and degree for Fredholm maps of index zero between Banach spaces; A. Cabada, E. Liz and R.L. Pouso, On the method of upper and lower solutions for first order BVPs; A. Cañada, J.L. Gámez and J.A. Montero, Nonlinear optimal control problems for diffusive elliptic equations of logistic type; A. Capietto, On the use of time-maps in nonlinear boundary value problems; P. Drábek, Some aspects of nonlinear spectral theory; C. Fabry and A. Fonda, Asymmetric nonlinear oscillators; T. Faria, Hopf bifurcation for a delayed predator-prey model and the effect of diffusion; M. Fečkan, Galerkin-averaging method in infinite-dimensional spaces for weakly nonlinear problems; D. Franco and J.J. Nieto, PBVPs for ordinary impulsive differential equations; M.R. Grossinho, F. Minhós and S. Tersian, Homoclinic and periodic solutions for some classes of second order differential equations; J. Jacobsen, Global bifurcation for Monge-Ampère operators; M. Kunze, Remarks on boundedness of semilinear oscillators; D. Lupo and K.R. Payne, The dual variational method in nonlocal semilinear Tricomi problems; F. Pacella, Symmetry properties of positive solutions of nonlinear differential equations involving the p-Laplace operator; A.M. Robles-Pérez, A maximum principle with applications to the forced Sine-Gordon equation; A.V. Sarychev and D.F.M. Torres, Lipschitzian regularity conditions for the minimizing trajectories of optimal control problems; and I. Schindler and K. Tintarev, Abstract concentration compactness and elliptic equations on unbounded domains.

We recommend this book to those mathematicians working in nonlinear analysis, ordinary differential equations, partial differential equations and related fields.

Radu Precup

Steven G. Krantz and Harold R. Parks, *The Implicit Function Theorem - History, Theory and Applications,* Birkhäuser, Boston-Basel-Berlin, 20002, ISBN 0-8176-4285-4 and 3-7643-4285-4.

The Implicit Function Theorem (IFT) and its closest relative - the Inverse Function Theorem - are two fundamental results of mathematical analysis with deep and far reaching applications to various domains of mathematics, as partial differential equations, differential geometry, geometric analysis, optimization. The aim of the present book is to present some fundamental implicit function theorems along with some nontrivial applications.

Some historical facts concerning the evolution of the ideas of function and implicit function, are presented in the second chapter of the book, *History.* It turns out that the origins of the notion of implicit function can be traced back to I. Newton (in 1669), G.W. Leibniz, who used implicit differentiation as early as 1684, and J.-L. Lagrange, who applied in 1670 the inverse function theorem for real analytic functions to some problems in celestial mechanics. The first explicit formulation of the implicit function theorem for holomorphic functions was done by A. Cauchy, and the first real variable formulation belongs to U. Dini in the academic year 1876/77 at the University of Pisa. In the third chapter, *Basic ideas,* the authors present two proofs of the IFT (the finite dimensional case) - one by induction and one via the inverse function theorem and Banach contraction principle.

Ch. 4, *Applications,* deals with existence results for differential equations (Picard's theorem), numerical homotopy methods and smoothness of the distance function to a smooth manifold.

Ch. 5, *Variations and generalizations,* is concerned with IFT for non-smooth functions and for function with degenerate Jacobian.

The highlight of the book is Ch. 6, *Advanced implicit function theorems,* presenting Hadamard's global inverse function theorem and the famous Nash-Moser implicit function theorem.

A Glossary of notions and a bibliography complete the book.

Collecting together disparate ideas in an important area of mathematical analysis and presenting them in an accessible but rigorous way, the book is of great interest to mathematicians, graduate or advanced undergraduate students, who want to learn or to apply the powerful tools supplied by implicit function theorems.

Tiberiu Trif

Sergiu Kleinerman and Francesco Nicolò, *The Evolution Problem in General Relativity,* Progress in Mathematical Physics, Vol. 25, Birkhäuser Verlag, Boston - Basel - Berlin 2003, xxii+385 pp., ISBN 3-7643-4254-4 and 0-8176-4254-4.

From the Preface: "The aim of the present book is to give a new self-contained proof of the global stability of the Minkowski space, given in D. Christodoulou and S. Kleinerman, *The global nonlinear stability of the Minkowski space,* Princeton Mathematical Series, Vol. 41. Princeton 1993 (Ch-Kl). We provide a new self-contained proof of the main part of that result, which concerns the full solution of the radiation problem in vacuum, for arbitrary asymptotically flat initial data sets. This can be also interpreted as a proof of the global stability of the external region of Scwarzschild spacetime.

The proof, which is a significant modification of the argument in Ch-Kl, is based on a *double null foliation* of spacetime instead of the *mixed null-maximal foliation* used in Ch-Kl. this approach is more naturally adapted to the radiation features of the Einstein equations and leads to important technical simplifications."

The book is fairly self-contained, the basic notions from differential geometry being reviewed in the first chapter. This chapter contains also a review of known results on Einstein equations and initial data value problems in general relativity, and the formulation of the main result.

The rest of the book is devoted to technical preparations for the proof, and to the proof of the main result. These chapters are headed as follows: 2. *Analytic methods in the study of the initial value problems,* 3. *Definitions and results,* 4. *Estimates for the connection coefficients,* 5. *Estimates for the Riemann curvature tensor,* 6. *The error estimates,* 7.*The initial hypersurface and the last slice,* and 8. *Conclusions.* This last chapter contains a rigorous derivation of the Bondi mass as well as of the connection between the Bondi mass and the ADM mass.

This important monograph, presenting the detailed proof of an important result in general relativity, is of great interest to researchers and graduate students in mathematics, mathematical physics, and physics in the area of general relativity.

Paul A. Blaga

JuliánLópez-Gómez, *Spectral theory and nonlinear functional analysis,* Research Notes in Mathematics, Vol. 426, Chapman & Hall/CRC, New York Washington 2001, xii+265 pp., ISBN 1-58488-249-2.

The general abstract problem this monograph deals with is the following one: For U and V real Banach spaces consider the operator

$$\mathfrak{F} : \mathbb{R} \times U \to V$$

of the form

$$\mathfrak{F}(\lambda, u) = \mathfrak{L}(\lambda)u + \mathfrak{N}(\lambda, u)$$

and the associated equation

$$(*) \qquad \mathfrak{F}(\lambda, u) = 0$$

where the following conditions are assumed to be satisfied:

The construction of the spectral theory is based on appropriate definitions of the notions of bifurcation point, nonlinear eigenvalue and algebraic eigenvalue. One of the principal goals of the monograph is characterizing the nonlinear eigenvalues of $\mathfrak{L}$ by means of a so called *generalized algebraic multiplicity* of $\mathfrak{L}$ at λ_0. As the author says "Our generalized algebraic multiplicity, subsequently denoted by $\chi[\mathfrak{L}(\lambda); \lambda_0]$ is a natural number that provides a finite order algorithm to calculate the change of the Leray-Schauder degree as λ crosses λ_0, thereby ascertaining and establishing the deep relationship between algebraic/analytic and topological invariants arising in nonlinear functional analysis."

The algebraic multiplicity can be defined if and only if λ_0 is an algebraic eigenvalue of $\mathfrak{L}$. The most crucial property of the algebraic multiplicity is established by Theorem 1.2.1: λ_0 is a nonlinear eigenvalue of $\mathfrak{L}$ if and only if $\chi[\mathfrak{L}(\lambda); \lambda_0]$ is odd.

If $V = U$ and $\mathfrak{L}(\lambda) = T - \lambda I_U$, where T is a continuous linear operator acting in U, and I_U stands for the identity operator of U, if λ_0 is an isolated eigenvalue of T and $T - \lambda_0 I_U$ is Fredholm of index zero, then the order ν of λ_0 is an algebraic eigenvalue of $\mathfrak{L}$. In this case $\chi[\mathfrak{L}(\lambda); \lambda_0] = \dim N[(T - \lambda_0 I_U)^\nu]$. Hence $\chi[\mathfrak{L}(\lambda); \lambda_0]$ equals the classical algebraic multiplicity of λ_0 as an eigenvalue of T.

If $U = V = \mathcal{R}^d, d \geq 1,$ and λ_0 is an algebraic eigenvalue of $\mathfrak{L}$, then $\chi[\mathfrak{L}(\lambda); \lambda_0]$ is odd if and only if the determinant of $\mathfrak{L}(\lambda)$ in any basis changes sign as λ crosses λ_0.

We have selected some basic properties of the algebraic multiplicity as they are listed in the Introduction of the monograph. This Introduction is in fact a well written detailed abstract of the book, with historical comments and, whenever possible, finite dimensional examples and counterexamples involved. As a general aspect of the approach, we remark the intention of the author to present the results in the most simple, but significant context. Hence, more sophisticated topological notions, variational aspects and monotonicity techniques appear only in the last two sections 6 and 7.

The book summarizes the authors new results in the nonlinear bifurcation theory some of which were subjects of the various lectures presented by him in the last decade. They extend and complete classical contributions in the field due to Ize, Fitzpatrick and Pejsachowicz, Rabinowitz, Rabier, Magnus, Ramm and others.

The book is addressed to researchers in nonlinear functional analysis and operator equations. It can be used also for advanced graduate or postgraduate courses.

A. B. Németh

Piotr Mikusiński and Michael D. Taylor, *An Introduction to Multivariable Analysis - From Vector to Manifold*, Birkhäuser Verlag, Basel-Boston-Berlin 2002, x+295 pp., ISBN 0-8176-4234-X and 3-7643-4234-X.

The aim of the present book is to provide a quick and smooth introduction to multivariable calculus, including differential calculus and Lebesgue integration in $\mathbb{R}^N$, and culminating with calculus on manifolds. The more geometric and intuitive approach, based on K-vectors and wedge product, allows the authors to overcome some of the difficulties encountered in the study of differential forms and, at a same time, to give full and rigorous coverage of the fundamental theorems, including Stokes generalized theorem.

The first two chapters of the book, Ch. 1, *Vectors and volumes*, and Ch. 2, *Metric spaces,* contain the algebraic and topological background needed for the development of multivariable analysis. As more specialized topics included in the first chapter we mention the Binet-Cauchy formula for determinants with applications to K-dimensional volumes of parallelipipeds in $\mathbb{R}^N$.

Differential calculus for mappings from open subsets of $\mathbb{R}^N$ to $\mathbb{R}^M$, including Taylor's formula and inverse and implicit function theorems, is developed in Ch. 3, *Differentiation.* As application one proves the Lagrange multiplier rule.

Lebesgue integration is developed in Ch. 4, *The Lebesgue integral,* following the approach proposed in the book by Jan and Piotr Mikusiński *Introduction to Analysis - from Number to Integral*, J. Wiley 1993, in the case of functions of one variable. The building starts from "bricks", which are intervals $[a, b) \subset \mathbb{R}^N$, and the integrals of step functions (= linear combinations of characteristic functions of bricks), and defining then the integrable functions f and their integrals by the conditions

$\int f = \sum_{n=1}^{\infty} \int f_n$, where (f_k) is a sequence of step functions with $\sum_{n=1}^{\infty} \int |f_n| < \infty$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ whenever $\sum_{n=1}^{\infty} |f_n(x)| < \infty$. Measurable sets are defined later as sets whose characteristic functions are integrable.

Ch. 5, *Integral on manifolds*, starts with the proof of the change of variables formula for the integrals, the most complex proof in the book, and continues with the study of C^r manifolds embedded in $\mathbb{R}^N$ and the integrals of real-valued functions defined on manifolds.

Ch. 6, *$K-$vectors and wedge products*, develops the fundamental properties of $K-$ vectors in $\mathbb{R}^N$, the wedge and dot products, and the Hodge star operator. These are essential tools for the next and the last chapter of the book, Ch. 7, *Vector analysis on manifolds,* the highlight of the book. This chapter is dealing with integration of differential forms on oriented manifolds and the proofs of fundamental theorems of the calculus on manifolds: Stokes theorem and Poincaré lemma. The particular case of Green formula is emphasized.

The authors strongly motivate the abstract notions by a lot of intuitive examples and pictures. The exercises at the end at each section range from computational to theoretical.

The book is highly recommended for basic undergraduate or graduate courses in multivariable analysis for students in mathematics, physics, engineering or economics.

Ştefan Cobzaş

Laurenţiu Modan, *Calcul Diferenţial Real*, Editura CISON, Bucureşti, 2002.

This book is firstly destined to the students of Computer Science Faculties, but it is also recommended to the students of all other universities, where the Real Analysis is teaching.

Having as subject *Real Differential Calculus* the book looks for giving correct reasonings to the students, so that to permit them a high mathematical education becoming good specialists, endowed by the logic which insure them finding the best decision in the domain they will work.

The author chosed a walk between theoretical and practical knowledges supported by many excellent examples and exercises.

The book develops its all fundamental notions in 4 chapters: *Elements of topology*, *Elements of numerical and function sequences in* $\mathbb{R}$ *and* $\mathbb{R}^n$, *Numerical and function series, including Taylor and MacLaurin power series*, and finally *Functions of several variables.*

Of the end of the book in *Appendix* the author presents four sets of special problems of the great didactical interest.

This book is warmly recommended to the users, not only for its content and presentation, but also for its mathematical beauty.

Gh. Micula

Jürgen Moser, *Selected Chapters in the Calculus of Variations*, (Lectures notes by Oliver Knill), Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag, Basel-Boston-Berlin 2003, xvi+132, ISBN 3-7643-2185-7.

This book is based on the lectures presented by J. Moser in the spring of 1998 at the Eidgenössische Technische Hochschule (ETH) Zürich. The course was attended by students in the 6th and 8th semesters, by some graduate students and visitors from ETH. The German version of the notes was typed in the summer of of 19988 and J. Moser carefully corrected it the same year in September. A translation was done in 2002 and figures were included, but the original text remained essentially unchanged.

The lectures are concerned with a new development in the calculus of variations - the so called Aubry-Mather Theory. It has its origins in the research of the theoretical physicist S. Aubry on the motion of electrons in two dimensional crystal, and in that of J. Mather on monotone twist maps, appearing as Poincaré maps in mechanics. They were studied by G. Birkhoff in 1920s, but it was J. Mather in 1982 who succeeded to make substantial progress proving the existence of a class of closed invariant subsets, called now Mather sets. The unifying topic of both Aubry and Mather approaches is that of some variational principles, a point that the book makes very clear.

The material is grouped in three chapters: 1. *One-dimensional variational problems;* 2. *Extremal fields and global minimals;* 3. *Discrete systems, Applications.*

The first chapter collects the basic results from the classical theory, the notion of extremal fields being a central one. In the second chapter the variational problems on the 2-dimensional torus are investigated, leading to the notion of Mather set. In the last chapter the connection with monotone twist maps is made, as a starting point of Mather's theory, and the discrete variational problems lying at the basis of Aubry's theory are presented.

The aim of the book is not to present the things in their greatest generality, but rather to emphasize the relations of the newer developments with classical notions.

The progress made in the area since 1998 is shortly presented in an Appendix along with some additional literature.

The book is ideal for advanced courses in the calculus of variations and its applications.

S. Cobzaş

Vladimir Müller, *Spectral Theory of Linear Operators (and spectral Systems in Banach Algebras),* Operator Theory: Advances and Applications, Vol. 139, Birkhäuser Verlag, Basel-Boston-Berlin 2003, x+381, ISBN 3-7643-6912-4.

The book is devoted to the basic results in spectral theory in Banach algebras for both single elements and for n-tuples of commuting elements, with emphasis on the spectral theory of operators on Banach and Hilbert spaces. The unifying idea, allowing to present in an axiomatic and elementary way various types of spectra - the approximate point spectrum, Taylor spectrum, local spectrum, essential spectrum, etc - is that of regularity in a Banach algebra. A regularity is a subset R of a unital Banach algebra $\mathcal{A}$ having some nice properties as : $ab \in R \iff a, b \in R$; $e \in R$; $\mathrm{Inv}(\mathcal{A}) \subset R$. The spectrum of an element $a \in \mathcal{A}$ with respect to a regularity R is defined by $\sigma_R(a) = \{\lambda \in \mathbb{C} : a - \lambda e \notin R\}$. For $R = \mathrm{Inv}(\mathcal{A})$ one obtains the usual spectrum $\sigma(a)$ of the element a. This notion, introduced and studied by the author and V. Kordula (Studia Math. **113** (1995), 127-139), is sufficiently general to cover many interesting cases of spectra but, at the same time, sufficiently strong to have non-trivial consequences as, e.g., the spectral mapping theorem.

The first chapter of the book, Ch. I, *Banach algebras*, presents the basic results on spectral theory in Banach algebras, including the axiomatic theory of spectrum via regularities. A special attention is paid to approximate point spectrum and its connection with removable and non-removable ideals. In the second chapter, Ch. II, *Operators,* these notions and results are specified to the very important case of operators on Banach and Hilbert spaces. Chapter III, *Essential spectrum,* is concerned with spectra in the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$, and with Fredholm and Browder operators. Although having a rather involved definition, the Taylor spectrum for commuting finite systems of operators seems to be the natural extension of ordinary spectrum for single operators, due mainly to the existence of functional calculus for functions analytic in a neighborhood of it. The presentation of Taylor functional calculus is done in the fourth chapter in an elementary way, without the use of sheaf theory and cohomological methods, following the ideas from a paper by the author of the book (Studia Math. **150** (2002), 79-97).

The last chapter of the book, Ch. IV, *Orbits and capacity*, is concerned with the study of orbits, meaning sequences $\{T^n x : n = 0, 1, ...\}$ in Banach or Hilbert spaces, a notion closely related to those of local spectral radius and capacity of an operator. Some Baire category results of the author on the boundedness of the orbit are included.

The book is clearly written and contains a lot of material, some of it appearing for the first time in book form. At the same time, the author tried successfully to keep the presentation at an elementary level, the prerequisites being basic functional analysis, topology and complex function theory (some needed results are collected in an Appendix at the end of the book).

The book, or parts of it, can be used for graduate or postgraduate courses, or as a reference text.

S. Cobzaş

Manfred Reimer, *Multivariate Polynomial Approximation*, International Series of Numerical Mathematics, Vol. 144, Birkhäuser Verlag, Basel-Boston-Berlin, 2003, pp. 358. ISBN 3-7643-1638-1.

This monograph brings a new breath over an old field of research - the approximation of functions by using multivariate polynomials. Besides surveying both classical and recent results in this field, the book also contains a certain amount of new material. The theory is characterized both by a large variety of polynomials which can be used and by a great richness of geometric situations which occur. Among these approached families of polynomials, we recall: Gegenbauer polynomials, the polynomial systems of Appell and Kampé de Feriét, the space $\mathbb{P}^r(S^{r-1})$, $r \in \mathbb{N} \setminus \{1\}$, of polynomial restriction, onto the unit sphere S^{r-1} and its subspaces, the most important of them being rotation-invariant subspaces.

The author investigates polynomial approximation to multivariate functions which are defined by linear operators. The reader will meet Bernstein polynomials, the Weierstrass theorem, the concept of best approximation and interpolatory projections in the space of the continuous real functions defined on a compact subset of $\mathbb{R}^r$, $r \in \mathbb{N}$, as well.

Distinct sections are devoted to quadratures. For example, the following are presented: Gauss quadratures, quadrature on the sphere, the geometry of nodes and weights in a positive quadrature, quadrature on the ball.

Hyperinterpolation represents another important concept treated by Manfred Reimer. It is a generalization of interpolation which shares with it the advantage of an easy evaluation but achieves simultaneously the growth order of the minimal projections. This new positive discrete polynomial approximation method is established on the sphere and then it is carried over to the balls of lower dimension.

By using summation methods such as Cesàro method or a method based on the Newman-Shapiro kernels, positive linear approximation operators are generated. A special consideration is given to the approximation on the unit ball B^r, $r \geq 2$. More precisely, orthogonal projections, Appell series and summation methods, interpolation on the ball are studied.

Among the book's outstanding features is the inclusion of some applications and a large variety of problems. As regards the applications, the author studies a recovery problem for real functions F belonging to a given space X and which are to be reconstructed from the values λF, where λ runs in a family of linear functionals on X. This way are presented both Radon transform, k-plane transform and reconstruction by approximation. As regards the problems, these are attached to help the reader to become familiar with the multivariate theory. All exercises are solved in a separate appendix.

Multivariate Polynomial Approximation includes the author's own research results developed over the last ten years, some of which build upon the results of others and some that introduce new research opportunities. His approach and proofs are straightforward constructive making the book accessible to graduate students in pure and applied mathematics and to researchers as well.

Octavian Agratini

Luigi Ambrosio and Paolo Tilli, *Selected Topics on "Analysis in Metric Spaces"*, APPUNTI, Scuola Normale Superiore, Pisa 2000, 133 pp.

The aim of these notes, based on a course taught by the first author in the academic year 1988-89 at the Scuola Normale Superiore di Pisa, is to present the main mathematical prerequisites needed to study or to do research in the field of Analysis in Metric Spaces. This relatively new and rapidly expanding area of investigation has as target to transpose to the case of metric spaces as many as possible results from classical analysis. In order to obtain consistent results, one supposes the metric space (X, d) endowed with a Borel measure μ that is finite on bounded sets and doubling, meaning that it values on balls in X satisfy the inequality $\mu(B_{2r}(x) \leq C\mu(B_r(x))$. Important contributions to the subject have been done by the authors of this book, by their coworkers from SNS di Pisa, and by M. Gromov, J. Cheeger, P. Hajlasz, P. Koskela, J. Heinonen.

The book contains six chapters: 1. *Some preliminaries in measure theory*; 2. *Hausdorff measures and covering theorems in metric spaces*; 3. *Lipschitz functions in metric spaces*; 4. *Geodesic problem and Gromov-Hausdorff convergence*; 5. *Sobolev spaces in a metric framework*; 6. *A quick overview on the theory of integration.*

Although in some places the proofs are only sketched with the specification of a source, the book covers a lot of topics. The last chapter of the book presents De Giorgi's approach to the theory of integration based on Cavalieri's formula.

The bibliography at the end of the book contains the basic references in the field.

Written in a clear and pleasant style, the book is a good introductory text to this promising area of investigation - the Analysis on Metric Spaces.

S. Cobzaş

Lectures Notes on Analysis in Metric Spaces, a cura di Luigi Ambrosio and Francesco Serra Cassano, Scuola Normale Superiore, Pisa 2000, 121 pp.

The book contains the notes of an international Summer School on Analysis in Metric Spaces, organized by L. Ambrosio, N. Garofalo, P. Serapioni, and F. Serra Cassano in May of 1999 at the Scuola Normale Superiore di Pisa.

There are included five papers, representing the edited and a little expanded versions of lectures delivered at the school: 1. Thierry Coulhon, *Random walks and geometry on infinite graphs*, pp. 5-36; 2. Guy David *Uniform rectifiability and quasi-minimal sets,* pp. 37-54; 3. Pekka Koskela, *Upper gradients and Poincaré inequalities,* pp. 55-69; 4. Stephen Semmes, *Derivatives and difference quotients for Lipschitz or Sobolev functions on various spaces,* pp. 71-103; 5. Richard L. Wheeden, *Some weighted Poincaré estimates in spaces of homogeneous type,* pp. 105-121.

The main concern of Analysis in Metric Spaces is to see to what extent results from classical analysis extend to the more general framework of metric spaces. Among these results I do mention the introduction of Sobolev spaces via the methods of upper gradients and Poincaré inequalities, treated in several papers in the volume. The first paper discusses the discrete case of analysis on graphs, with special emphasis on Cayley graphs.

Surveying new results, some of them belonging to the authors of the contributions, in this rapidly growing field of investigation situated at the border between analysis, topology and geometry, the book is of great interest for researchers working in this area, as well as for people desiring to become acquainted with its powerful methods.

S. Cobzaş

Steven G. Krantz and Harold R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Boston-Basel-Berlin 2002, xii+205 pp., ISBN 0-8175-4264-1.

Complex analytic functions of one or several complex variables are presented in a lot of books, at introductory level and at advanced as well.

Their older and poorer relatives - the real analytic functions - having totally different features, found their first book treatment in the first edition of the present book, published by Birkhäuser in 1992. Real analytic functions are an essential tool in the study of embedding problem for real analytic manifolds. They have also applications in PDEs and in other areas of analysis.

With respect to the first edition, beside the revision of the presentation, some new material on topologies on spaces of real analytic functions and on the Weierstrass preparation theorem, has been added.

The basic results on real analytic functions are presented in the first two chapters: Ch. *Elementary properties,* and Ch. 2 *Multivariable calculus of real analytic functions*, including implicit and inverse function theorems, Cauchy-Kowalewski theorem.

Chapters 3, *Classical topics* and 4 ,*Some questions in hard analysis*, contain more advanced topics as Besicovitch's theorem, Whitney's extension and approximation theorems, quasi-analytic classes and Gevrey classes, Puiseux series.

Ch. 5, *Results motivated by PDEs*, is concerned with topics as division of distributions, the FBI transform (FBI comes here from the name of mathematical physicists Fourier, Bros and Iagnolitzer), and Paley-Wiener theorem.

The last chapter, Ch. 6, *Topics in geometry*, contains a discussion of some deep and difficult results as embedding of real analytic manifolds, sub- and semi-analytic sets, the structure theorem for real analytic varieties.

Bringing together results scattered in various journals or books and presenting them in a clear and systematic manner, the book is of interest first of all for analysts, but also for applied mathematicians and for researcher in real algebraic geometry.

Stefan Cobzaş

Steven G. Krantz, *Handbook of Logic and Proof Techniques for Computer Science*, Birkhäuser Boston, Inc., Boston, MA; Springer-Verlag, New York, 2002. xx+245pp., ISBN 0-8176-4220-X.

Logic plays a key role in modern mathematics and computer science. However, the vast number of topics, the unusual and sometimes difficult formalism and terminology, made most of the modern logic inaccessible to all but the experts.

The present book is a comprehensive overview of the most important topics in modern logic emphasizing ideas essential in Computer Science as axiomatics, completeness, consistency, decidability, independence, recursive functions, model theory, P/NP completeness. Some of these topics are: first-order logic, semantics and syntax, axiomatics and formalism in mathematics, the axioms of set theory, elementary set theory, recursive functions, the number systems, methods of mathematical proof, the axiom of choice, proof theory, category theory, complexity theory, Boolean algebra, the word problem.

The book was written to be accessible for non experts. It contains definitions, plenty of concrete examples and a clear presentation of the main ideas, on the other hand avoids complicated proofs, difficult notations, difficult formalisms. Self-contained, the book is designed for those mathematicians, engineers and especially computer scientists who need a quick understanding of some key ideas from logic.

The vast bibliography also makes this book an excellent modern logic resource for the working mathematician.

Csaba Szántó

Bhimsen K. Shivamoggi, *Perturbations Methods for Differential Equations,* Birkhäuser Verlag, Basel-Boston-Berlin 2003, xiv+354, ISBN 3-7643-4189-0 and 0-8176-4189-0.

The mathematical problems associated with nonlinear equations, generally, are very complex. So that, one practical approach is to seek the solutions of these nonlinear equations as the perturbations of known solutions of a linear equation. A perturbative solution of a nonlinear problem becomes viable if it is close to the solution of another problem we already know how to solve.

After a chapter containing the asymptotic series and expansions, this book presents the regular perturbation methods for differential and partial differential equations. Other methods, such as the strained coordinates method, the averaging method, the matched asymptotic expansion method, the multiple scales method, are also very detailed presented. Very important is the fact that each chapter contains certain important applications, especially to fluid dynamics, but also to solid mechanics and plasma physics. Moreover, each chapter contains a section of specific exercises, and an appendix with basic mathematical tools.

Many methods and procedures are very well described without technical proofs. It is obvious the intention of the author to convince the reader to understand the phenomena and to learn how to apply correctly the suitable presented method.

"Perturbation Methods for Differential Equations" can serve as a textbook for undergraduate students in applied mathematics, physics and engineering. Researchers in these areas will also find the book an excellent reference. A comprehensive bibliography and an index complete the book.

Gh. Micula

Alain Escassut, *Ultrametric Banach Algebras,*, World Scientific, London - Singapore - Hong Kong 2003, xiii+275 pp., ISBN 981-238-194-5.

The book is concerned with the spectral theory of ultrametric Banach algebras over an algebraically closed complete ultrametric field. As it is well known, in the classical case due to Gelfand's representation theory every commutative complex Banach algebra can be viewed as an algebra of functions on a compact space, whose points characterize all maximal ideals, which are all of codimension 1. Any such algebra admits a holomorphic functional calculus.

In the ultrametric case the situation is much more complicated, because an ultrametric Banach algebra can have maximal ideals of infinite codimension. It turns out that a key role in constructing an ultrametric spectral theory is played by the family of multiplicative semi-norms on an ultrametric Banach algebra, allowing to define a kind of Gelfand transform. There exists also a spectral semi-norm defined by $\|a\|_{si} = \lim_n \|a^n\|^{1/n}$, which is also equal to the supremum of all multiplicative semi-norms. It is also possible to construct a holomorphic functional calculus. The basic idea is to associate methods based on affinoid algebras (called also "Tate algebras") with methods based on holomorphic functional calculus involving very thin properties of analytic functions of one variable. The present book is the first that treats together both of these subjects. Concerning holomorphic mappings in the ultrametric case, references are given to another book by the same author, *Analytic Elements in p-adic Analysis*, World Scientific, Singapore 1995.

The author is well known specialist in the field and the book is largely based on his original results.

The book will be of interest to researchers in non-archimedean analysis (or ultrametric analysis), a field having its origins in the work of the Dutch mathematicians A. F. Monna and T. A. Springer, and which still is in the focus of attention of several research centers. Recently there have been found some applications of non-archimedean analysis to mathematical physics, see V. S. Vladimirov, I. V. Volovich and E. J. Zelenov, *p-Adic Analysis and Mathematical Physics,* World SCientific, Singapore 1994.

S. Cobzaş

Mathematics and War, Editors: Bernhelm Booß-Bavnek and Jens Høyrup, Birkhäuser Verlag, Basel-Boston-Berlin 2003, viii+416 pp., ISBN 3-7643-1634-9.

The volume contains some of the contributions delivered at the International Meeting on Mathematics and War, held in Karlskrona, Sweden, from 29 to 31 august, 2002, together with some invited papers. The idea was to bring together mathematicians, historians, philosophers and military, to discuss some of the interconnections between warfare and mathematics. As it is well known after the World War II, there has been a strong mathematization of warfare and of the concepts of modern war, which in its turn deeply influenced the development of some areas of mathematics.

The papers included in the volume deal with topics ranging from historical, philosophical, ethical aspects of the problem, to more technical aspects like the functioning of weapons, the actual planning of war and information warfare. The perspectives the authors approaches the treated theme also differ from one to other - some papers are written from a pacifist point of view (more or less explicitly), while others are not. Some of the papers are dealing with history, but focused on the last sixty years, e.g., WW II and the Kosovo war.

The volume is organized in four parts: I. *Perspectives from mathematics,* II. *Perspectives from the military,* III. *Ethical issues,* IV. *Enlightenment perspectives.*

The first part contains studies on military work in mathematics 1914-1945 (R. Siegmund-Schultz), on the Enigma code breaking (E. Rakus-Anderson), on the defence work of A. N. Kolmogorov (A. N. Shiryaev), on the discovery of maximum pricilple by Lev Pontryagin (R. V. Gamkrelidze), and on the mathematics and war in Japan (S. Fukutomi).

The second part is written by military and deals with topics as information warfare (U. Bernhard and I. Ruhmann), the exposure of civilians under the modern "safe" warfare (E. Schmägling), duels of systems and forces (H. Löfstedt).

The third part is concerned with N. Bohr's and A. Turing's involving in military research (I. Aaserud and A. Hodges, respectively), and K. Ogura and the "Great Asia War" (T. Makino).

The last part contains two studies - one on mathematical thinking and international law (I. M. Jarvard), and one on modeling the conflict and cooperation (J. Scheffran).

The aim of the volume is to draw the attention of scientists, military and philosophers on the dramatic consequences that the use of science, particularly of mathematics, for military purposes can have on the development of humanity, and to trace some possible way of preventing this disaster.

S. Cobzaş