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# A MOVING FINITE DIFFERENCE METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

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**Abstract**. A moving grid method which has its origin from differential geometry is studied. The method deforms an intial grid according to a vector field calculated by a Poisson equation. The forcing term of the Poisson equation is determined by the time derivative of a positive monitor function. It adapts the grid at each time step by keeping the volume of each cell proportional to the (normalized) time-dependent monitor function. A moving finite difference method is formulated which transforms a time dependent partial differential equation by the grid mapping and then simulates the transformed equation on a fixed orthogonal grid in the computational domain. The method is demonstrated by solving model problems and an incompressible flow problem.

## 1. **Introduction**

In numerical simulation of partial differential equations (PDEs), a well-constructed grid is required to yield satisfactory results. For general grid generation methods we refer to [2],[4],[5], and [6].

Fixed uniform or non-uniform grids have been widely used. In this approach the grid points are distributed in the physical domain prior to a simulation of the PDEs. The same grid point locations are then used throughout the computation. A drawback in using this technique happens when the solution to the PDE exhibits

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large variation due to, for example, shock waves and boundary layers. Because of its static feature the grid is unable to effectively capture such variations.

An approach that can lead to improved accuracy and efficiency is the use of solution-adaptive grids. The idea is to generate the grids according to the salient features of the solution that is being calculated. The objective is to distribute more grid points in regions where the solution possess fine scale structures in order to improve accuracy, and fewer grid points in regions where small changes in the solution occur to improve efficiency.

There are two basic strategies for grid adaptation: local refinement and relocation of nodes. This paper develops a deformation method which moves the nodes according to the numerical solution as it is being computed. A key element of successful moving grid methods is the construction of a transformation $\phi : \Omega \times [0, T] \to \Omega$ which moves the nodes of an initial grid in accordance with the computed solution through a monitor function or an error estimator. To qualify as a transformation, $\phi$ must be one to one (injective) and onto (surjective). Variational methods [1], [3] and elliptic PDE methods [2] have been used to define this transformation.Various aspects of the grid such as orthogonality ("skewness"), smoothness, and cell size are adjusted through a linear combination of individual functionals. The resulting system of PDEs for grid generation are often nonlinear and require intensive computation. Results have also been reported in controlling the cell size through the Jacobian determinants of the transformation [7]. A moving finite element method has been developed by Miller [8] (see [10] for a survey). Recently, methods based on Moving Mesh Partial Differential Equations [11],[12] were developed with remarkable capability to track rapid spatial and temporal transitions for multiple dimensional problems. The special issue of Comput. Methods Appl. Mech. Engerg. edited by Kallinderis [28] contains a collection of papers on adaptive grid methods for compressible CFD. It is an excellent source in the topic covered by this study.

A common weakness of current moving grid methods is that they do not provide mathematical assurance that the "grid transformation $\phi$" is indeed a transformation in three dimensional domains. The method formulated in this paper is based on a deformation method by J. Moser and B. Dacorogna [13],[14] in the study of volume elements. This method was originally developed in the context of Riemannian geometry, and has recently been applied to numerical methodology [15]. It has the advantage of providing direct control over the cell size of the adaptive grid, and the transformation can be easily computed. The method inherently defines a transformation which is necessarily injective, thus ensuring in theory that the grid lines do not cross even in three dimensions. The approach is remarkably robust, in that it can be applied to time dependent differential equations in combination with any commonly used numerical methods (finite element, finite difference, finite volume, etc).

The paper is organized as follows. In Section 2, the mathematical formulation of the deformation is presented. A 3D numerical grid example is shown in Figure 1. In Section 3, a moving finite difference algorithm is formulated. The algorithm is used to solve several model problems. In Section 4, the incompressible Navier-Stokes equation is solved using this method. Finally in Section 5, we give conclusions and indicate further research directions.

## 2. Adaptive Grid Generation by the Deformation Method

2.1. **Mathematical Formulation.** The deformation method generates a time-dependent nodal mapping from a domain $D_1$ to another domain $D_2$. It assures direct volume control through the Jacobian determinant. The way the mapping is constructed assures the existence and uniqueness of the mesh. The vector field in which the nodes move is calculated by a scalar Poisson equation. Thus the mesh lines are quite smooth. Due to the fact that the vector field is irrotational, the cells maintain acceptable shapes during the calculation. The partial differential equations are transformed by the moving mesh mapping and solved on a fixed uniform mesh on a computational domain. For a general description of the method, consider the PDE for $u(\vec{x}, t)$,

$$u_t(\vec{x}, t) = L(u) \tag{1}$$

where $u$ is a scalar or vector variable, $L$ is a differential operator in $\vec{x}$ only, on a physical domain $\Omega_p$ in $\mathrm{R}^d$, $d = 1, 2, 3$, for $t > 0$. Suppose that the solution to (1) has been computed at time step $t = t_{n-1}$, and a preliminary computation has been done at time level $t = t_n$. Assume that we are provided with some positive error estimator (or gradient approximation) $\delta(\vec{x}, t)$. Define the monitor function

$$f(\vec{x}, t) = \frac{C}{\delta(\vec{x}, t)} \tag{2}$$

where $C$ is a positive scaling parameter so that at each time step we have

$$\int_{\Omega} \left( \frac{1}{f(\vec{x}, t_n)} - 1 \right) dA = 0. \tag{3}$$

Note that $f$ is small in regions of large error and becomes larger in regions where the error is small. Let $\Omega_c$ denote the computation domain and $\Omega_p$ denote the physical domain. The deformation method constructs a transformation $\phi : \Omega_c \to \Omega_p$ such that

$$\det \nabla\phi(\vec{\xi}, t) = f(\phi(\vec{\xi}, t), t), \qquad t_{n-1} \le t \le t_n, \tag{4}$$

$$\phi(\vec{\xi}, t_{n-1}) = \phi_{n-1}(\vec{\xi}), \qquad \vec{\xi} \text{ on } \Omega_c,$$

where $\vec{\xi}$ is a node of an initial grid, $\phi_{n-1}(\vec{\xi})$ represents the coordinates of the node at $t = t_{n-1}$. We require that $\phi(\vec{\xi}) \in \partial\Omega$, for all $\vec{\xi} \in \partial\Omega_p$. Note that (4) ensures the size of the transformed cells will conform to the function $f$, i.e. the grid will be appropriately "refined" in regions of large error and "coarsened" in regions of small error.

A steady version of this method has been applied to two-dimensional steady flow problems [19]. The moving grid method has been applied to one-dimensional time-dependent problems [17]. The computation of $\phi$ consists of two steps:

The first step is to find a vector field $\vec{u}(\vec{\xi}, t)$ satisfying

$$\operatorname{div} \vec{u}(\vec{\xi}, t) = -\frac{\partial}{\partial t}\left( \frac{1}{f(\vec{\xi}, t)} \right), \qquad \vec{\xi} \in \Omega_c,\ t_{n-1} \le t \le t_n \tag{5}$$

$$\frac{\partial \vec{v}}{\partial \mathbf{n}} = 0, \quad \vec{\xi} \in \partial\Omega,\ \mathbf{n} = \text{outward normal to } \partial\Omega_c.$$

The vector field $\vec{v}$ can be found by solving for $w$ from the scalar Poisson equation (for a fixed $t$)

$$\triangle w(\vec{\xi}, t) = -\frac{\partial}{\partial t}\left( \frac{1}{f(\vec{\xi}, t)} \right), \qquad \vec{\xi} \in \Omega \tag{6}$$

$$\frac{\partial w}{\partial \mathbf{n}} = 0, \qquad \vec{\xi} \in \partial\Omega_c,$$

then setting $\vec{u} = \nabla w$.

The second step is to solve for the new location $\phi(\vec{\xi}, t)$ at time $t$ of any node $\vec{\xi} \in \overline{\Omega}$ of the grid at $t = 0$ from the ODE system

$$\frac{\partial}{\partial t}\phi(\vec{\xi}, t) = \vec{(v)}(\phi(\vec{\xi}, t), t), \qquad t_{n-1} \le t \le t_n,\ \vec{\xi} \in \Omega, \tag{7}$$

$$\phi(\vec{\xi}, t_{n-1}) = \phi_{n-1}(\vec{\xi}),$$

where the node velocity $\vec{(v)}(\vec{\xi}, t) = f(\vec{\xi}, t)\vec{u}(\vec{\xi}, t)$. The mathematical foundation of the method is provided by the following

**Theorem.** [17] $\det \nabla\phi(\vec{\xi}, t) = f(\phi(\vec{\xi}, t), t)$ for each $\vec{\xi}$ in $D$ each $t > 0$.

The Jacobian determinant of a mapping $\phi(\vec{\xi}, t)$ from $D_1$ to $D_2$ in $\mathrm{R}^n$, $n =$ 1, 2, 3, is

$$J(\phi) = \det \nabla\phi = \frac{|dA'|}{|dA|}, \tag{8}$$

where $dA'$ is the image of a volume (area, in 2D) element $dA$. In our case, $J(\phi)$ $= f(\phi,t) > 0$ since the monitor function $f$ is chosen to be positive. Thus, the theorem assures precise control over the cell size relative to that of the fixed initial grid in both 2D and 3D.

The theorem is proved by showing $\frac{d}{dt}(J(\phi)g(\phi, t)) = 0$ and therefore $Jg = 1$ since $Jg\Big|_{t=0} = 1$. For the convenience of the reader, we include an outline of the proof given in [17].

**Proof.** Let $g(\vec{\xi}, t) = \frac{1}{f(\vec{\xi}, t)}$ $(\Rightarrow g(\phi, t) = \frac{1}{f(\phi, t)})$ and $\vec{u}$ be a vector field satisfying

$$\operatorname{div} \vec{u}(\vec{\xi}, t) = -\frac{\partial}{\partial t} g(\vec{\xi}, t) (\Rightarrow \operatorname{div}_\phi \vec{u}(\phi, t) = -\frac{\partial}{\partial t} g(\phi, t).) \tag{9}$$

Let $\eta = \vec{u}f$ (i.e. $\vec{(v)}(\vec{\xi}, t) = \vec{u}(\vec{\xi}, t)f(\vec{\xi}, t) \Rightarrow \vec{(v)}(\phi, t) = \vec{u}(\phi, t)f(\phi, t)$). Let $J =$ $J(\phi(x, t)) = \det \nabla\phi(\vec{\xi}, t)$, be the Jacobian determinant. First,

$$\frac{d}{dt}\nabla_{\vec{\xi}}\phi(\vec{\xi}, t) = \nabla_{\vec{\xi}}(\frac{d}{dt}\phi(\vec{\xi}, t)) = \nabla_\phi(\vec{(v)}(\phi, t))(\nabla_{\vec{\xi}}\phi). \tag{10}$$

Using the formula: $\frac{d}{dt}M(t) = A(t)M(t) \Rightarrow \frac{d}{dt}(\det\ M(t)) = (\mathrm{Tr}\ A(t))(\det\ M(t))$, we get

$$\frac{dJ}{dt} = J(\mathrm{div}_\phi\,\vec{(v)}(\phi,t)) = J(f\mathrm{div}_\phi\,\vec{u}(\phi,t) + \langle\vec{u},\nabla_\phi f\rangle).$$

by (9),

$$J^{-1}\frac{dJ}{dt} = -f\frac{\partial g}{\partial t} + \langle\vec{u},\nabla_\phi f\rangle. \tag{11}$$

Second,

$$\begin{aligned}\frac{d}{dt}(J(\phi)g(\phi,t)) &= g\frac{dJ}{dt} + J\frac{d}{dt}g(\phi,t)\\ &= g\frac{dJ}{dt} + J(\left\langle\nabla_\phi g,\frac{d\phi}{dt}\right\rangle + \frac{\partial}{\partial t}g(\phi,t))\\ &= g\frac{dJ}{dt} + J(\left\langle\nabla_\phi g, f(\phi,t)\vec{u}(\phi,t)\right\rangle + \frac{\partial}{\partial t}g(\phi,t)).\end{aligned}$$

Third, by (11) and since $fg = 1$ implies $g\nabla f + f\nabla g = 0$

$$\frac{1}{J}\frac{d}{dt}(Jg) = \left[-gf\frac{\partial g}{\partial t} + \left\langle\vec{u},g\nabla_\phi f\right\rangle\right] + \left[\left\langle f\nabla_\phi g,\vec{u}\right\rangle + \frac{\partial}{\partial t}g\right] = 0, \tag{12}$$

implies that

$$\Rightarrow \frac{d}{dt}(Jg) = 0 \Rightarrow Jg = 1 \text{ for each } t > 0, \text{since } Jg\Big|_{t=0} = 1.$$

Thus

$$J(\phi) = f(\phi,t), \quad t > 0.\S$$

Consequently, $\phi$ is indeed injective (non-folding). Also by choosing $f > 0$ to be continuous (or smooth), the Jacobian determinant can be made to change continuously (or smoothly), which is important in obtaining high accuracy in the computation. The deformation method does not have direct control of the orthogonality of the grid lines on the physical domain. The fact that the vector field $\vec{u}$ is irrotational, i.e. $\vec{u} = \nabla w$, and thus curl $\vec{u} = 0$, helps to prevent excessive skewness in the grid. In this paper we transform equation (1) through the grid mapping $\vec{x} = \vec{x}(\vec{\xi},t)$ and solving the transformed equation on an orthogonal grid on the $\vec{\xi}$-domain. This approach enables us to have the benefits of the adaptive grids as well as the advantages of using a fixed orthogonal grid.

The monitor function $f$ is constructed during the solution process. This establishes the dynamic coupling of the mesh movement with the PDE solver through the transformed equation, the Poisson equation, and the deformation ODEs.

Construction of a proper monitor function is a challenging task (see [29]). A common way to construct the monitor function is the equidistribution principle. A posteriori error estimates (if available), residues, and truncation errors, etc. are to be redistributed evenly over the whole domain. In many cases, truncation errors are difficult to compute. Thus, in engineering calculation, gradient of the solution is often used to detect the regions where refined grids are needed. For instance, in the calculation of Euler flows with shock waves [19], we used

$$f = \frac{C_1}{1 + C_2|\nabla p|} \tag{13}$$

where $p$ is the pressure, $C_2$ is a constant for adaptation intensity, $C_1$ is a normalization parameter. For viscous flows one can use the Mach number $M$ in place of the pressure. In general, in addition to the gradient of the unknown variable $z$, terms involving the value of $z$ and the second derivatives of (or curvatures) can also be included. For instance,

$$f = \frac{k}{1 + \alpha|z| + \beta|\nabla z| + \gamma\,|\nabla^2 z|} \tag{14}$$

where $\alpha, \beta,$ and $\gamma$ are parameters controlling the intensity of adaptation.

For interface resolution, $f$ can be constructed using a signed distance function $d$ as follows: Let $f$ be piecewise linear such that

$$f = \begin{cases} 1 & \text{if } |d| > \varepsilon \\ 0.2 & \text{if } d = 0 \end{cases} \tag{15}$$

$f$ is then normalized so that $\int_\Omega(\frac{1}{f} - 1) = 0$, which is required for (4) to be satisfied. We will discuss the issue in more details below.

In [26] a different version of the deformation method is formulated. It is based on the level set evolution equation and the transport formula in fluid dynamics (see, i.e., [30]). It assures the same direct control over the Jacobian determinant.

2.2. **Numerical Examples.** We present numerical examples of the deformation implemented using a finite-difference method. In [18], this method was implemented using a least-squares finite-element method. Two main tasks are required to numerically implement this method:

1. Solve a Poisson equation on $D$

$$w_{\xi\xi} + w_{\eta\eta} = -\frac{\partial}{\partial t}\Big(\frac{1}{f(\xi,\eta,t)}\Big), \qquad (\xi,\eta) \in \Omega_c \tag{16}$$

$$\frac{\partial w}{\partial \vec{n}} = 0 \quad (\xi,\eta) \in \Gamma, \quad \text{where } \vec{n} \text{ is the outward normal,}$$

2. Solve a system of ordinary differential equations (the deformation ODEs)

$$\frac{\partial}{\partial t}\phi(\xi,\eta,t) = f(\phi(\xi,\eta,t),t)\vec{u}(\phi(\xi,\eta,t),t)\,,\ t_{k-1} \le t \le t_k,\ (\xi,\eta) \in \Omega_c \tag{17}$$

$$\phi(\xi,\eta,t_{k-1}) = \phi_{k-1}(\xi,\eta).$$

Let $w_{i,j} = w(i\Delta\xi, j\Delta\eta)$, for $i = 1,\ldots,m$, $j = 1\ldots,n,$ where $\Delta\xi = \frac{1}{m}$, $\Delta\eta = \frac{1}{n}$. For simplicity we assume $\Delta\xi = \Delta\eta = h$ and $m = n$.

A monitor, function $f$, is formed at time $t = t_k$, $k \ge 1$. The monitor function $f$ is normalized to satisfy (3). To accomplish this, let $\overline{f}$ denote the non-normalized monitor function and $f$ denote the normalized monitor function at $t = t_k$. Then

$$f = C \cdot \overline{f} \text{where} \quad C = \int_D \frac{1}{\overline{f}(\xi,\eta,t)}\, dA. \tag{18}$$

Now form $-\frac{\partial}{\partial t}(\frac{1}{f})$, the right-hand side of (16), using the normalized monitor function. The time derivative is approximated on each node in a uniform grid by

$$g_t(\xi,\eta) = -\left(\frac{\frac{1}{f(\xi,\eta,t_k)} - \frac{1}{f(\xi,\eta,t_{k-1})}}{\tau_k}\right), \quad \tau_k = t_k - t_{k-1} \tag{19}$$

The node velocity can then be found by solving (17). In this study (16) is approximated using central difference approximations for both derivatives to obtain

$$\frac{w_{i-1,j} - 2w_{i,j} + w_{i+1,j}}{h^2} + \frac{w_{i,j-1} - 2w_{i,j} + w_{i,j+1}}{h^2} = (\frac{\partial}{\partial t}\frac{1}{f})_{i,j} \tag{20}$$

The resulting system of linear algebraic equations is then solved with the successive over-relaxation (SOR) method implemented in the following two steps

$$\overline{w}_{i,j} = \frac{1}{4}(w^{\text{new}}_{i-1,j} + w^{\text{new}}_{i,j-1} + w^{\text{old}}_{i,j+1} + w^{\text{old}}_{i+1,j} - h^2 \frac{\partial}{\partial t}\frac{1}{f_{i,j}}) - w^{\text{old}}_{i,j} \qquad (21)$$

$$w^{\text{new}}_{i,j} = w^{\text{old}}_{i,j} + \lambda\,\overline{w}_{i,j} \qquad (22)$$

On the boundary, Neumann conditions are implemented with a consistent second-order central-difference scheme. This is done by introducing a "ghost point" outside of the region and approximating the boundary condition with

$$\frac{1}{2h}(w_{i,1} - w_{i,-1}) = 0 \text{ and } \frac{1}{2h}(w_{i,n-1} - w_{i,n+1}) = 0 \qquad (23)$$

$i = 1, \ldots, n-1,$ for the lower and upper boundaries respectively, and

$$\frac{1}{2h}(w_{1,j} - w_{-1,j}) = 0 \text{ and } \frac{1}{2h}(w_{n-1,j} - w_{n+1,j}) = 0 \qquad (24)$$

$j = 1, \ldots, n-1,$ for the left and right boundaries, respectively.

To initialize the SOR iterations at each time step, the approximations at the previous time step can be used. Finally, the vector field $\vec{u}$ and hence the nodal velocities were computed by setting $\vec{u} = \nabla w$ using second-order centered differences. The nodes are then moved by solving the deformation ODEs (17) by a Runge-Kutta method.

**Example 1.** A three dimensional uniform grid in a unit cube of $R^3$ is deformed into a grid concentrated around a pair of spheres and the grid moves appropriately as the spheres merge into each other. The definition of the monitor function is based on the following level set function $d$:

$$d = ((x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2 - r^2)((x-a_2)^2 + (y-b_2)^2 + (z-c_2)^2 - \rho^2)$$

where $(a_1, b_1, c_1)$ is the (moving) center of the first sphere and $(a_2, b_2, c_2)$ is the moving center of the second sphere. The zero set of $d$ consists of the two spheres with varying radii $r$ and $\rho$. In this example, $r = \rho = \textit{constant}$. The spheres initially intersect each other and gradually merge to one sphere. The deformation method deforms an initial uniform grid to a grid adapted to a pair of intersecting circles at $t = 0$ with the monitor function

$$\begin{cases} 1 & d \leq y \leq d - 0.1 \\ 0.3 - 7t(y-d) + (1-t) & d - 0.1 < y \leq d \\ 0.3 + 7t(y-d) + (1-t) & d < y \leq d + 0.1 \\ 1 & d + 0.1 < y \leq 2.0 \end{cases} . \tag{25}$$

The deformation method then continues to deform the adaptive grid for $1 < t \leq 2$ according to the following monitor function

$$\begin{cases} 1 & 0 \leq y \leq d - 0.1 \\ 0.3 - 7t(y-d) & d - 0.1 < y \leq d \\ 0.3 + 7t(y-d) & d < y \leq d + 0.1 \\ 1 & d + 0.1 < y \leq 2.0 \end{cases} . \tag{26}$$

In this example, the time step is $\Delta t = 0.025$. The grids are shown in Figure 1.

## 3. **A Moving Grid Finite Difference Method**

A concern over various moving grid methods is the lack of orthogonality of the grid generated. Indeed, while finite element and finite volume methods can be implemented on non-orthogonal grids, finite difference methods usually are implemented on an orthogonal grid. In this section we describe a common moving grid strategy ([2]) that will be used to implement the deformation method on an orthogonal computational grid.

Let us consider the time dependent equation for a variable $z$ with proper boundary and initial conditions

$$z_t\,(\vec{x}, t) = L(z), \tag{27}$$

where$L$ is a differential operator in $\vec{x}$, only, on a domain $\Omega_p$ in $\mathrm{R}^d$, $d = 1, 2, 3$, for $t > 0$. Suppose a positive monitor function $f(\vec{x}, t)$ has been constructed according to the solution that is being calculated. By the deformation method we construct a transformation $\phi : x^l = \phi^l(\vec{\xi}, t)$, $l = 1, 2, 3$, from a cubic computational domain $\Omega_c$ into $\Omega_p$ at each $t$ such that $J(\phi)(\vec{\xi}, t) = f(\phi(\vec{\xi}, t), t)$. Let $n = 3$ and let $\{\xi_{ijk} \,|\, i, j, k =$

$0,1,2,...,N\}$ be the nodes of a uniform grid on $\Omega_c$. Then the image $x_{ijk}(t)=\phi(\xi_{ijk},t)$ of a node $\xi_{ijk}$ is the new node of the moving grid at time $t$. We now transform (27) to the computational domain $\Omega_c$ with the coordinates $\vec{\xi}=(\xi^1,\ \xi^2,\ \xi^3)$.

Let $Z(\vec{\xi},t)=z(\phi(\vec{\xi},t),t))$. By the Chain Rule, we have,

$$\frac{\partial Z}{\partial t}=\frac{\partial z}{\partial x^i}\frac{\partial \phi^i}{\partial t}+\frac{\partial z}{\partial t}. \tag{28}$$

Thus (27), becomes

$$\frac{\partial Z}{\partial t}=\frac{\partial z}{\partial x^i}\frac{\partial \phi^i}{\partial t}+L(z). \tag{29}$$

Note: The components of the node velocity $\partial\phi^i/\partial t$, $i=1,2,3$ are directly determined by equation (7). In fact one of the advantages of the deformation method is the determination of the node velocity $\partial\phi/\partial t$ by the desired monitor function $f$ through an explicit formula.

To transform the terms in $L(z)$, we use the formulas in [2]. By the deformation method (see Theorem, 2.1) that

$$J(\phi)=f(\phi,t)\ >0.$$

Thus, the transformation formulas are valid.

To demonstrate the method, let us consider the hyperbolic PDE for $d=2$:

$$\frac{\partial z}{\partial t}+a(x,y,t)\frac{\partial z}{\partial x}+b(x,y,t)\frac{\partial z}{\partial y}=0 \tag{30}$$

on the unit square $\Omega=[0,1]\times[0,1]$. We begin with a uniform grid on the computational domain $\Omega_c$, with coordinates $(\xi^1,\xi^2)=(\xi,\eta)$. We are seeking a transformation $\phi:\Omega_c\ \rightarrow\ \Omega_p$ in the form of

$$x=x(\xi,\eta,t),\ y=y(\xi,\eta,t)$$

such that the cell size of the moving grid on $\Omega$ is evenly distributed according to a positive monitor function $f(\xi,\eta,t)$. Let $Z(\xi,\eta,t)=z(x(\xi,\eta,t),y(\xi,\eta,t),t)$. By (28), (30) becomes

$$\frac{\partial Z}{\partial t}-\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}-\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}+a(x,y,t)\frac{\partial z}{\partial x}+b(x,y,t)\frac{\partial z}{\partial y}=0. \tag{31}$$

Differentiating, we have

$$Z_\xi = z_x \frac{\partial x}{\partial \xi} + z_y \frac{\partial y}{\partial \xi}, \tag{32}$$

$$Z_\eta = z_x \frac{\partial x}{\partial \eta} + z_y \frac{\partial y}{\partial \eta}. \tag{33}$$

Since $J = \det \nabla\phi = f > 0$. By Cramer's Rule, we can find $z_x$ and $z_y$ from (32) and (33) and get

$$z_x = \frac{1}{J}\left(Z_\xi \frac{\partial y}{\partial \eta} - Z_\eta \frac{\partial y}{\partial \xi}\right) \tag{34}$$

$$z_y = \frac{1}{J}\left(Z_\eta \frac{\partial x}{\partial \xi} - Z_\xi \frac{\partial x}{\partial \eta}\right) \tag{35}$$

where $J = x_\xi\, y_\eta - y_\xi\, x_\eta$. Let $A(\xi,\eta,t) = a(x((\xi,\eta,t), y(\xi,\eta,t), t)$ and $B(\xi,\eta,t) = b(x(\xi,\eta,t), y(\xi,\eta,t), t)$. Substituting (34) and (35) into (31) and rearranging terms, (31) becomes

$$\frac{\partial Z}{\partial t} + \mathcal{A} Z_\xi + \mathcal{B} Z_\eta = 0 \tag{36}$$

where

$$\mathcal{A} = \frac{1}{J}\left[\left(A - \frac{\partial x}{\partial t}\right)\frac{\partial y}{\partial \eta} - \left(B - \frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial \eta}\right] \tag{37}$$

$$\mathcal{B} = \frac{1}{J}\left[\left(B - \frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial \xi} - \left(A - \frac{\partial x}{\partial t}\right)\frac{\partial y}{\partial \xi}\right] \tag{38}$$

In general, since the Jacobian determinant $J(\phi)$ is positive, the equation is invariant. That is, the transformed equation remains elliptic, parabolic, or hyperbolic depending on the type of the original equation 27.

An algorithm implementing the moving finite-difference method will now be formulated. The algorithm consists of an initialization procedure and a time integration loop. The initialization procedure is an iteration procedure for determining the vector field $\vec{v}$ at $t = 0$. The procedure is needed since the node velocity $\vec{v} = (x_t, y_t)$ at $t = 0$ in the transformed equation can not be determined from the initial value $z_0$ alone. For the subsequent time steps, either an explicit or implicit scheme can be used.

The numerical algorithm in 2D is as follows (extension to 3D is straightforward):

I. Initialization ($t = 0$)

1. Construct a uniform grid in the logical domain $(\xi, \eta)$ in $\Omega_c$.
2. Use initial values $z_0 = z(x, y, 0)$ to construct the monitor function $f$ at $t = 0$.
3. Use the static version of the deformation method to generate an initial adaptive grid on $(x, y)$ in $\Omega_p$ determined by the initial monitor function

$$f(\xi, \eta, 0) = \frac{C_1}{1 + C_2|\nabla z_0|}.$$

Set $g(s) = \frac{C_1}{1+ s\ C_2|\nabla z_0|}$ and deform the uniform grid on $\Omega_p$ according to $g(s)$ in artificial time $s$ from $s = 0$ to $s = 1$.

4. Let $\vec{v}^{(\mathbf{0})}|_{t=0}$ be the vector field $\vec{v}|_{s=1}$ of the static deformation method. Derive the transformed equation at $t = 0$, setting $(x_t,\ y_t) = f|_{t=0}\, \vec{v}^{(\mathbf{0})}|_{t=0}$.
5. Solve the transformed PDE to obtain $U|_{t=\frac{1}{2}\Delta t}$.
6. Use $Z|_{t=0.5\Delta t}$ to construct the monitor function $f|_{t=0.5\Delta t}$.
7. Use values of the monitor function at $t = 0$ and $t = 0.5\Delta t$ to construct the source term of the Poisson equation at $t = 0$:

$$w_{\xi\xi} + w_{\eta\eta} = -\left(\frac{\frac{1}{f|_{t=0.5\Delta t}} - \frac{1}{f|_{t=0}}}{0.5\Delta t}\right)$$

8. Set $\vec{u}^{(\mathbf{1})}|_{t=0} = \nabla w$ at $t = 0$.
9. Compute the transformed equation again at $t = 0$, setting $\vec{v}^{(\mathbf{1})}|_{t=0} = f|_{t=0}\, \vec{u}^{(\mathbf{1})}|_{t=0}$.

If

$$\left|\vec{v}^{(\mathbf{1})}|_{t=0} - \vec{v}^{(\mathbf{0})}_{t=0}\right| < \epsilon,$$

where $\epsilon$ is a preset tolerance, stop; Otherwise, repeat the above procedures until

$$\left|\vec{v}^{(\mathbf{k+1})}|_{t=0} - \vec{v}^{(\mathbf{k})}|_{t=0}\right| < \epsilon.$$

10. Define the transformed equation at $t = 0$ by setting
    $\vec{v} = f|_{t=0}\, \vec{u}^{(\mathbf{k+1})}|_{t=0}$.

II. Time Integration ($t = \Delta t, 2\Delta t,\, 3\Delta t,\, 4\Delta t, ...$)

1. Solve the above PDE and get $Z(\xi, \eta, \Delta t)$.
2. Use $Z(\xi,\, \eta, \Delta t)$ to form the monitor function $f|_{t=\Delta t}$.
3. Use $f|_{t=\Delta t}$ and $f|_{t=0}$ to form the right hand side of the Poisson equation at $t = \Delta t$:

$$w_{\xi\xi} + w_{\eta\eta} = -\left(\frac{\frac{1}{f|_{t=dt}} - \frac{1}{f|_{t=0}}}{\Delta t}\right)$$

4. Set $\vec{u}|_{t=\Delta t} = \nabla w$;
5. Compute the transformed equation at $t = \Delta t$, setting $(x_t, y_t) = f|_{t=\Delta t}\, \vec{u}|_{t=\Delta t}$.
6. Repeat the above procedures for $t = 2\Delta t, 3\Delta t, 4\Delta t, ....$

The moving grid finite-difference scheme is used to solve the following two-dimensional model equations.

**Model Problem I (Weiss Model).** Consider the hyperbolic initial-boundary-value problem

$$z_t \;=\; -\sin 2\pi x\, \cos 2\pi y\, u_x \;+ \sin 2\pi y\, \cos 2\pi x\, u_y \quad x, y \in [0,1]\times[0,1], t > 0 \tag{39}$$

$$z(x, y, 0) =\; 1 - x \quad 0 \le x, y \le 1, \tag{40}$$

$$\begin{cases} z(0, y, t) = 1,\; z(1, y, t) = 0, & y \in [0, 1],\; t > 0 \\ z(x, 1, t) = z(x, 0, t) = 1 - x, & x \in [0, 1],\; t > 0. \end{cases} \tag{41}$$

Since the equation does not have exact solution, we take the numerical solution on a fine uniform grid with $200X200$ nodes as a satisfactory approximation.

The contour plot of the approximate solution is shown in Figure 2. The contour plot exhibits boundary layers at $x = 0, x = 1, y = 0$, and $y = 1$, as well as interior layers at $x = 0.5$. Our task is to generate a moving grid with significantly less nodes which can resolve these layers.

In Figure 3, we showed the moving grid with $100X100$ nodes and in Figure 4 the solution contour plot. The initial grid is uniform. As can be seen by comparison to Figure 2, the layers are resolved well except at the two small regions near $(0.5, 0)$ and $(0.5, 1)$, where boundary and interior layers meet. Clearly, more resolution there is needed.

To provide the needed resolution, an initial grid that is refined near $y = 0$ and $y = 1$ is used. The results are shown in Figure 5 and 6. The contour plot appears to be comparable to that in Figure 2.

This example shows that

1. Grid resolution is a key factor for calculation involving fine structures, as we all agree;

2. Construction of a proper monitor function is a challenging task. It is an active research area (see, i.e., [29]) and some interesting new ideas appear to be very promising. We will explore some of the ideas in subsequent study.

3. Assuming a proper monitor function is formed by the user, the deformation method does generate real-time moving grid with specified cell size distribution as the theory predicts. The term "real-time" means that the grid is updated in one time step as the PDE is being solved. The time $t$ in the grid generation equations is the same as that in the PDE. In particular, $\vec{v}$ is the actual node velocity.

**Model Problem II (Whirlpool Problem).** Consider the hyperbolic problem

$$z_t = -\frac{v_r}{v_{r_{max}}}\frac{y}{r}u_x + \frac{v_r}{v_{r_{max}}}\frac{x}{r}u_y \quad x, y \in [0,1]\times[0,1], t > 0 \tag{42}$$

where

$$r = \sqrt{x^2+y^2+\varepsilon} \quad v_r = \frac{\tanh(r)}{\cosh^2(r)} \text{ and } v_{r_{max}} = 0.385. \tag{43}$$

$$\tag{44}$$

$$\begin{cases} z(x,y,0) = -\tanh(\frac{y}{2}) \quad 0 \le x, y \le 1, \\ \frac{\partial z}{\partial n} = 0 \end{cases} \tag{45}$$

The moving grid and contour plot of the solution are shown in Figure 7, 8. The solution exhibits strong rotation. In contrast to the Lagrange method, our grid nodes do not rotate with the "flow". Instead, they move properly to form cells that are small in the regions where the rotation is strong. The cell shapes remain acceptable due to the fact that the vector $\vec{u}$ is irrotational. In the cavity flow example to be studied below, these characteristics will be seen also.

In the above model problems, the monitor functions are based on the gradient of the variable $z$. Prior to solving the equation for $t > 0$, an initial grid is generated using the static mode of the deformation method. The initial grid is adapted to the initial condition by using the following monitor function:

$$f_{\text{initial}} = \frac{C_1}{1 + s\,C_2|\nabla z_0|} \tag{46}$$

where $s$ is an artificial time that goes from 0 to 1. Ten time steps in the steady mode are used to generate the initial grid. The initial could be taken as an uniform grid as well. If prior information about the solution is known, an adaptive grid could be generated and used as the initial grid.

To solve the transformed equation, the time discretization is accomplished by using a second-order Runge-Kutta scheme given by

$$\tilde{Z}^{n+1}_{i,j} = Z^n_{i,j} + \Delta t\,\mathcal{L}(Z^n_{i,j}) \tag{47}$$

$$Z^{n+1}_{i,j} = \frac{1}{2}Z^n_{i,j} + \frac{1}{2}\tilde{Z}^{n+1}_{i,j} - \frac{1}{2}\Delta t\,\mathcal{L}(\tilde{Z}^{n+1}_{i,j}) \tag{48}$$

Here, $\mathcal{L}$ is the discrete approximation to the differential operator $L$. A second-order essentially non-oscillatory (ENO) method is used to approximate the spatial derivatives of $Z$ as described in [24]. The transformation parameters $x_\xi$, $x_\eta$, $y_\xi$, $y_\eta$ are discretized using central differencing and the time derivatives $x_t$, $y_t$ are interpolated from the computational grid. To approximate the boundary conditions the first-order linear extrapolation scheme given by

$$\begin{cases} Z_{i,0} = 2Z_{i,1} - Z_{i,2} \\ Z_{0,j} = 2Z_{1,j} - Z_{2,j}. \end{cases} \tag{49}$$

is used. The grids are generated using the monitor function $f$ given by

$$f(\xi_i, \eta_j, t) = \frac{C_1}{1 + C_2|\nabla z|}, \tag{50}$$

where $\nabla z$ is transformed and calculated at the uniform nodes $(\xi_i, \eta_j)$ at each time $t$. The constant $C_2$ is the same as in the initial grid stage. The forcing term of the

Poisson equation is approximated using the difference scheme

$$\frac{\partial}{\partial t}\left(\frac{1}{f}\right)^{n}_{i,j} = \frac{3(\frac{1}{f})^{n}_{i,j} - 4(\frac{1}{f})^{n-1}_{i,j} + (\frac{1}{f})^{n-2}_{i,j}}{2\Delta t}. \tag{51}$$

## 4. Navier-Stokes Equation

For an incompressible flow, the governing equations can be written in conservative form as

$$\frac{\partial u_i}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \xi_j}\left[J\left(U_j - V_j\right)u_i\right] = -\frac{1}{J}\frac{\partial}{\partial \xi_j}\left(J\frac{\partial \xi_j}{\partial x_i}p\right) + \frac{1}{J}\frac{1}{Re}\frac{\partial}{\partial \xi_j}\left(Jq_{jk}\frac{\partial u_i}{\partial \xi_k}\right), \tag{52}$$

$$\frac{\partial \xi_k}{\partial x_i}\frac{\partial}{\partial \xi_k}\left[\frac{1}{J}\frac{\partial}{\partial \xi_j}\left(J\frac{\partial \xi_j}{\partial x_i}p\right)\right] = -\frac{1}{J}\frac{\partial D}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i}\frac{\partial}{\partial \xi_j}\left[J\left(U_j - V_j\right)u_i\right]. \tag{53}$$

where in the pressure Poisson equation (PPE) (53), the viscous terms have been removed by using the continuity equation. $U_i$ is the contravariant velocity and $V_i$ is determined by the node velocity $\vec{x}_t$ of the moving grid:

$$U_i = \frac{\partial \xi_i}{\partial x_j}u_j,\; V_i = \frac{\partial \xi_i}{\partial t} = \frac{\partial \xi_i}{\partial t} = \frac{\partial x_j}{\partial t}\frac{\partial \xi_i}{\partial x_j}. \tag{54}$$

The Jacobian and metrics are defined as

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}, q_{jk} = \frac{\partial \xi_j}{\partial x_i}\frac{\partial \xi_k}{\partial x_i}. \tag{55}$$

The deformation method uses a scalar Poisson to get the node velocity vector field and then move the nodes by ODEs. This is in contrast to the elliptic grid generators, which determines node position directly. The flow in a square cavity whose top wall is driven by a lid with uniform velocity has served as a model problem for testing and evaluating numerical techniques. This is a typical complex flow in a simple geometry with a strong vortex near the center and two secondary vortexes. Our task is to generate a moving grid which resolves the vortexes as the computation proceeds. The construction of a suitable monitor function is an important aspect of adaptive algorithms.In this paper, instead of searching for the best possible monitor function, we use a simple and effective monitor function based on the stream function $\psi$ to demonstrate the method. More precisely, in order to resolve the main as well as the secondary vortexes, we use the product of $\psi$ and $\psi - \psi_{\min}$. Also, instead of using the best flow solver, we used a reliable flow solver written by the second author. The present study compares results on moving grids to the benchmark solutions found in [25].

The marker-and-cell (MAC) method is used with collocate grid arrangement in this study. The convective terms are discretized using the QUICK scheme to remove numerical wiggles, while the viscous terms are discretized by central difference scheme. The second-order Adams-Bashforth method is used for time integration to solve the momentum equation (52). Both the pressure Poisson equation (PPE) (53) and the deformation Poisson equation (6) are discretized by central differences and then solved with the ADI method. An explicit 2nd-order Runge-Kutta method is used in solving equation (7). The usual technique: geometric conservation law (GCL) for correcting the errors caused by moving grids is not used in this study. Our experience is that the correction is unnecessary for the viscous flow with low Reynolds number as is the case of the cavity flow. It was necessary for inviscid flows or viscous flows with high Reynolds number. For instance, it was used in [27] for the cylindrical implosion problem.

The monitor function in this study is based on $d = \psi(\psi_{\min} - \psi)$. An initial adapted grid is generated at the artificial time $s = 1$ according to the monitor function $f_{\text{init}} = (1 - s) + sf$ where $f$ is given by

$$f = \begin{cases} 1 & \text{if } |d| > 0.004 \\ 0.344 - 164d & \text{if -0.004} \le d \le 0 \\ 0.344 + 164d & \text{if } 0 \le d \le 0.004 \end{cases} . \tag{56}$$

After the initial time, a moving grid is formed using the same $f$. The moving grid and the stream function contour at the steady state are shown in Figure 9 and 10. The comparisons on stream function and vorticity are given in Table 1. The values presented are nodal values.

TABLE 1. Comparison to Benchmark Solution, Re = 1000

| | Uniform Grid (129 × 129) | Moving Grid (51 × 51) |
|---|---|---|
| Primary Vortex | Benchmark [25] | Moving Grid MAC |
| $\psi_{\min}$ | -0.117929 | -0.117967 |
| $\omega_{\text{v.c}}$ | 2.04968 | 2.19837 |
| Location $(x, y)$ | $(0.5313, 0.5625)$ | $(0.5228, 0.5589)$ |

## 5. Conclusion

We have formulated a moving grid finite difference approach for time dependent PDEs. We have also established rigorously that the method does not lead to tangled grids in three dimensions. Computational experiments indicate that the moving grid method is robust and efficient, and that it can put more nodes in the regions where the need for higher resolution exists. The method is not as fast as on fixed grid (with the same amount of nodes) due to the fact that it requires solving a scalar Poisson equation on a uniform grid at each time step. The method is more efficient when compared to other PDE based grid generators that solve non-linear PDE systems.

A moving finite difference algorithm is presented, which transforms the host partial differential equations via the grid mapping. The transformed equations then are simulated on an orthogonal computational grid. The method is demonstrated by model problems and the Navier-Stokes problem. The calculations showed that the method is capable of significantly enhance resolution where and when it is needed. On the other hand, an additional Poisson equation is solved at each time step, and smaller timestep may be necessary on fine grids. Thus, the method is expected to be used only for solving large, complex PDE systems for which the extra efforts for solving the additional Poisson equation is insignificant and some local resolution enhancement is necessary.

Works are underway to develop methods for adaptive multiple block grids and unstructured meshes for unsteady partial differential equations in 2D and 3D general domains.

## References

[1] Brackbill, J. U. and Saltzman, J. S. *Adaptive Zoning for Singular Problems in Two Dimensions*, J. Comput. Phys., 46(1982).

[2] Thompson, J. F., Warsi, Z. U. A. and Mastin, C. W., *Numerical Grid Generation*, North-Holland, Amsterdam, 1985.

[3] Castillo, J., Steinberg, S. and Roache, P. J., *Mathematical Aspects of Variational Grid Generation*, J. Comput. Appl. Math., 20(1987).

[4] Zegeling, Paul A., *Moving Grid Methods*, Dissertation, The Utrecht University Press, 1992.

[5] Knupp, P. and Steinberg, S., *The Fundamentals of Grid Generation,* CRC Press, Boca Raton, 1993.

[6] Carey, G., *Computational Grid Generation, Adaptation and Solution Strategies*, Taylor and Francis, Washington D.C., 1997.

[7] Anderson, D. A., *Grid Cell Volume Control with an Adaptive Grid Generator*, Appl. Math. and Comput., 35(1990).

[8] Miller, K., *Recent Results on Finite Element Methods with Moving Nodes*, in Accuracy, Estimates and Adaptive Methods in Finite Element Computations, Babuska, Zienkiewicz, Gago, and Oliveira,eds., John Wiley & Sons, 1986.

[9] Arney, D. and Flaherty, J., *An Adaptive Mesh-moving and Local Refinement Method for Time-dependent Partial Differential Equations*, ACM Transactions on Math software, Vol. 16, No. 1, 1900, 48-71.

[10] Hawken, A. et.al., *Review of Some Adaptive Node Movement Techniques in Finite Element and Finite difference Solutions of Partial Difference Equations*, J. Comput. Phys., 95(1991).

[11] Huang, W., Ren, Y. and R. Russell, *Moving Mesh Methods Based on Moving Mesh Partial, Differential Equations*, J. of Comput. Phys., 113(1994).

[12] Huang, W., Ren, Y. and Russell, R., *Moving Mesh Partial Differential Equations (MMPDES) Based on the Equidistribution Principle*, SIAM J. Numer. Anal., 31(1994).

[13] Moser, J., *Volume Elements of a Riemann Manifold*, Trans AMS, 120(1965).

[14] Dacorogna, B. and Moser, J., *On a PDE Involving the Jacobian Determinant*, Ann. Inst. H. Poincare, 7(1990).

[15] Liao, G. and Anderson, D., *A New Approach to Grid Generation*, Appl. Anal., 44(1992).

[16] Liao, G. and Su, J., *A Moving Grid Method for (1+1) Dimension*, Appl. Math. Lett., 8(1995).

[17] Semper, B. and Liao, G., *A Moving Grid Finite Element Method using Grid Deformation*, Numer. Meth. PDEs, 11(1995).

[18] Bochev, P., Liao, G. and dela Pena, G., *Analysis and Computation of Adaptive Moving Grids by Deformation*, Numer. Meth. PDEs, 12(1996).

[19] Liu, F., Ji, S. and Liao, G., *An Adaptive Grid Method with Cell-Volume Control and its Applications to Euler Flow Calculations*, SIAM J. Sci. Comput., 20(1998).

[20] Liao, G. and Su, J., *A Direct Method in Dacorogna-Moser's Approach of Grid Generation Problems*, Appl. Anal., 49(1993).

[21] Liao, G., Pan, T. and Su, J., *A Numerical Grid Generator Based on Moser's Deformation Method*, Numer. Meth. PDEs, 10(1994).

[22] Strikwerda, J. C., *Finite Difference Schemes and PDEs*, Wadswasth & Brooks/Cole, 1989.

[23] Chen, S., Merriman, B., Osher, S. and Smereka, P., *A Simple Level Set Methods for Solving Stefan Problems*, J. of Comput. Phys., 135(1997).

[24] Shu, C. W. and Osher, S., *Efficient Implementation of Essentially Non-oscillatory Shock Capturing Schemes*, J. of Comput. Phys., 77(1988).

[25] Ghia, U., Ghia, K. N. and Shin, C. T., *High-Re Solutions for Incompressible Flow Using the Navier-Stokes Equations and a Multigrid Method*, J. Comput. Phys., 48(1982).

[26] Liao, G., F. Liu, G. dela Pena, D. Peng, and S. Osher, *Level-Set-Based Deformation Methods for Adaptive Grids*, J. Comput. Phys., 159(2000), 103-122.

[27] Liao, G., Lei, Z. and dela Pena, G., *Adaptive grids for resolution enhancement*, Shock Waves, An International Journal on Shock Waves, Detonations and Explosions by Springer, 12(2002), 153-156.

[28] Kallinderis, K. (ed.), *Special Issue: Adaptive Methods for Compressible CFD*, Comput. Methods Appl. Mech. Engerg., 189(2000).

[29] Soni, B., Koomullil, R., Thompson, D. and Thornburg, H., *Solution adaptive strategies based on point redistribution*, Comput. Methods Appl. Mech. Engerg., 189(2000), 1183-1204.

[30] Chorin, A. and Marsden, J., *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag, 1993.

Figure 1.1. (Clockwise from top-left) Grid plots for Example 1 for $t = 1.0$. Cutaway plot, grid slice at $x = 0.5$, grid slice at $z = 0.5$, grid slice at $y = 0.5$.

Figure 1.2. (Clockwise from top-left) Grid plots for example 1 for $t = 5.5$. Cutaway plot, grid slice at $x = 0.5$, grid slice at $y = 0.5$, grid slice at $z = 0.5$.

Figure 2. Weiss Model: Contour Plot. Fixed uniform grid. $200X200$.

Figure 3. Weiss Model: Moving grid with uniform initial grid $100X100$.

Figure 4. Weiss Model: Contour Plot. Moving grid with uniform initial grid. $100X100$.

Figure 5. Weiss Model: Moving grid with adapted initial grid. $100X100$.

Figure 6. Weiss Model: Contour Plot. Moving grid with adapted initial grid. $100X100$.

Figure 7. Wrpool Model: Moving grid with uniform initial grid. $100X100$.

Figure 8. Wrpool Model: Contour Plot. Moving grid with uniform initial grid. $100X100$.

Figure 9. Steady Cavity Flow: Moving grid at steady state. $50X50$.

Figure 10. Steady Cavity Flow: Contour Plot. Moving grid. $50X50$.

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# ON SOME CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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**Abstract**. In Holhoş [1] we defined and studied new classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ of univalent functions with negative coefficients for $0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, $0 < B \le 1$ and

$$\frac{B}{B-A} < \gamma \le \begin{cases} \dfrac{B}{(B-A)\alpha}, & \alpha \neq 0 \\ 1, & \alpha = 0 \end{cases}.$$

In this paper we study the classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ for $0 < \gamma \le \dfrac{B}{B-A}$.

## 1. **Introduction**

Let $\mathbf{U}$ denote the open unit disc: $\mathbf{U} = \{z \; ; \; z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathbf{S}$ denote the class of functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic and univalent in $\mathbf{U}$.

For $f \in \mathbf{S}$ we define the differential operator $\mathbf{D}^n$ ( Sălăgean [5] )

$$\begin{aligned} \mathbf{D}^0 f(z) &= f(z) \\ \mathbf{D}^1 f(z) &= \mathbf{D}f(z) = zf'(z) \end{aligned}$$

and

$$\mathbf{D}^n f(z) = \mathbf{D}(\mathbf{D}^{n-1} f(z)) \quad ; \quad n \in \mathbb{N}^* = \{1, 2, 3, ...\}.$$

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*Key words and phrases.* univalent function, integral operator, Hadarmard product.

We note that if

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

then

$$\mathbf{D}^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \ ; \ z \in \mathbf{U}.$$

Let $T$ denote the subclass of $\mathbf{S}$ which can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \ ; \quad a_k \geq 0, \ \forall k \geq 2. \tag{1}$$

We say that a function $f \in T$ is in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ $\quad 0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $0 < \gamma \leq \dfrac{B}{B-A}$ if

$$\left| \frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1}{(B-A)\gamma \left[ \dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - \alpha \right] - B \left[ \dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1 \right]} \right| < \beta \qquad z \in \mathbf{U}$$

where

$$F_{n,\lambda}(z) = (1-\lambda) D^n f(z) + \lambda D^{n+1} f(z) \ ; \ \lambda \geq 0 \ ; \ f \in T$$

**Remark 1.** *For $n = 0$, $\lambda = 0$, $A = -1$, $B = 1$, $\beta = 1$, the class $T_{0,0}(-1, 1, \alpha, 1, \gamma) = S_0^*(\alpha, \gamma)$ was studied by S.Owa* [2] *and for $n = 0$, $\lambda = 0$, $A = -1$, $B = 1$, the class $T_{0,0}(-1, 1, \alpha, \beta, \gamma) = S_0^*(\alpha, \beta, \gamma)$ was studied by S.Owa in* [3] *and* [4].

## 2. Characterization theorem

**Theorem 2.** *Let $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $0 < \gamma \leq \dfrac{B}{B-A}$. Then a function $f(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$; $\quad a_k \geq 0, \forall k \geq 2$ is in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ if and only if*

$$\begin{aligned} & \sum_{k=2}^{\infty} a_k k^n \left[1 + \lambda(k-1)\right] \left[(k-1) + \beta B(k+1) - \beta\gamma(B-A)(k+\alpha)\right] \leq \\ \leq \quad & \beta\gamma(B-A)(1-\alpha) \end{aligned} \tag{2}$$

*and the result is sharp.*

If we denote

$$\begin{aligned} &\mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) = \\ = \quad &k^n \left[1 + \lambda\left(k-1\right)\right]\left[\left(k-1\right) + \beta B\left(k+1\right) - \beta\gamma\left(B-A\right)\left(k+\alpha\right)\right] \end{aligned} \tag{3}$$

then (2) becomes

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \beta\gamma(B-A)(1-\alpha).$$

*Proof.* Assume that (2) holds and let $|z| = 1$. Then we have

$$\left|zF'_{n,\lambda}\left(z\right) - F_{n,\lambda}\left(z\right)\right| -$$

$$-\beta\left|\left(B-A\right)\gamma\left[zF'_{n,\lambda}\left(z\right) - \alpha F_{n,\lambda}\left(z\right)\right] - B\left[zF'_{n,\lambda}\left(z\right) - F_{n,\lambda}\left(z\right)\right]\right| =$$

$$= \left|zF'_{n,\lambda}\left(z\right) - F_{n,\lambda}\left(z\right)\right| -$$

$$-\beta\left|\left[\left(B-A\right)\gamma - B\right] zF'_{n,\lambda}\left(z\right) + \left[B - (B-A)\gamma\alpha\right] F_{n,\lambda}\left(z\right)\right| =$$

$$= \left|\sum_{k=2}^{\infty} a_k k^n \left(1-k\right)\left[1 + (k-1)\lambda\right] z^k\right| -$$

$$-\beta\left|\left[\left(B-A\right)\gamma - B\right] z - \left[\left(B-A\right)\gamma - B\right]\sum_{k=2}^{\infty} a_k k^{n+1}\left[1 + (k-1)\lambda\right] z^k + \right.$$

$$\left. + \left[B - (B-A)\gamma\alpha\right] z - \left[B - (B-A)\gamma\alpha\right]\sum_{k=2}^{\infty} a_k k^n \left[1 + (k-1)\lambda\right] z^k\right| =$$

$$= \left|\sum_{k=2}^{\infty} a_k k^n \left(1-k\right)\left[1 + (k-1)\lambda\right] z^k\right| -$$

$$-\beta\left|\left(B-A\right)\gamma\left(1-\alpha\right) z - \left[\left(B-A\right)\gamma - B\right]\sum_{k=2}^{\infty} a_k k^{n+1}\left[1 + (k-1)\lambda\right] z^k - \right.$$

$$\left. - \left[B - (B-A)\gamma\alpha\right]\sum_{k=2}^{\infty} a_k k^n \left[1 + (k-1)\lambda\right] z^k\right| \leq$$

$$\leq \sum_{k=2}^{\infty} a_k k^n \left(k-1\right)\left[1 + (k-1)\lambda\right] |z|^k - \beta\left(B-A\right)\gamma\left(1-\alpha\right)|z| +$$

$$+\beta\left|\left(B-A\right)\gamma - B\right|\sum_{k=2}^{\infty} a_k k^{n+1}\left[1 + (k-1)\lambda\right] |z|^k +$$

$$+\beta\left|B-(B-A)\gamma\alpha\right|\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]|z|^k\leq$$

$$\leq\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]\left\{(k-1)+\beta\left[B-(B-A)\,\gamma\right]k+\beta\left[B-(B-A)\,\gamma\alpha\right]\right\}-$$

$$-\beta\gamma(B-A)(1-\alpha)\leq 0$$

Consequently, by the maximum modulus theorem, the functions $f(z)$ is in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.

Conversely, assume that

$$\left|\frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-1}{(B-A)\gamma\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-\alpha\right]-B\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-1\right]}\right|<\beta\Leftrightarrow$$

$$\left|\sum_{k=2}^{\infty}a_k k^n\,(1-k)\left[1+(k-1)\lambda\right]z^k\right|$$

$$\leq\quad\beta\left|(B-A)\,\gamma\,(1-\alpha)\,z-\left[(B-A)\,\gamma-B\right]\sum_{k=2}^{\infty}a_k k^{n+1}\left[1+(k-1)\lambda\right]z^k\;-\right.$$

$$\left.-\left[B-(B-A\gamma\alpha)\right]\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]z^k\right|$$

Since $|\mathrm{Re}\,(z)|\leq|z|$ for any $z$, we have

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=2}^{\infty}a_k k^n\,(k-1)\left[1+(k-1)\lambda\right]z^k}{(B-A)\,\gamma\,(1-\alpha)\,z-\sum\limits_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]\left[B(k+1)-(B-A)\gamma(k+\alpha)\right]z^k}\right\}<\beta$$

Letting $z\to 1$ through real values, upon clearing the denominator in the last inequality we obtain

$$\sum_{k=2}^{\infty}a_k k^n\,(k-1)\left[1+(k-1)\lambda\right]\leq$$

$$\leq\beta\gamma(B-A)(1-\alpha)-\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]\beta\left[B(k+1)-(B-A)\gamma(k+\alpha)\right]$$

and this inequality gives the required condition.

Each function

$$f_k(z) = z - \frac{\beta\gamma(B-A)(1-\alpha)}{k^n\left[1+\lambda\left(k-1\right)\right]\left[k-1+\beta B\left(k+1\right)-\beta\gamma\left(B-A\right)\left(k+\alpha\right)\right]}z^k$$

is an extremal function for the theorem. □

**Remark 3.** *For* $n = 0,\ \lambda = 0,\ A = -1,\ B = 1,\ \beta = 1$ *the result of Theorem 2 was obtained by Owa* [2] *and for* $n = 0,\ \lambda = 0,\ A = -1,\ B = 1,$ *the result of Theorem 2 was obtain by Owa* [3] *.*

## 3. Closure theorems

Let the functions $f_j$ be of the form:

$$f_j\left(z\right) = z - \sum_{k=2}^{\infty} a_{kj} z^k \quad ;\ z \in \mathbf{U} \quad ;\ a_{kj} \geq 0,\ \forall k \geq 2, \quad j = 1, 2, ..., m. \tag{4}$$

We shall prove the following results for the closure of functions in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

**Theorem 4.** *Let the functions* $f_j\left(z\right)$ *defined by* (4) *be in the class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$*. Then the function* $g\left(z\right)$*, defined by*

$$g\left(z\right) = z - \sum_{k=2}^{\infty} b_k z^k \quad ;\ b_k \geq 0,\ \text{with } b_k = \frac{1}{m}\sum_{j=1}^{m} a_{kj}$$

*also belongs to the class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$*.*

*Proof.* As $f_j\left(z\right) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ it follows from Theorem 2 that

$$\sum_{k=2}^{\infty} a_{kj} \mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \beta\gamma(B-A)(1-\alpha) \quad ; \quad j = 1, 2, ..., m.$$

Therefore

$$\begin{aligned} \sum_{k=2}^{\infty} b_k \mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) &= \sum_{k=2}^{\infty} \mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \frac{1}{m}\sum_{j=1}^{m} a_{kj} \\ &\leq \beta\gamma(B-A)(1-\alpha) \end{aligned}$$

hence, by Theorem 2,

$$g\,(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma).$$

□

**Theorem 5.** *Let* $f_j\,(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. *Then the function* $h\,(z)$, *defined by,*

$$h\,(z) = \sum_{j=1}^{m} d_j f_j\,(z)\,;\ \textit{where} \sum_{j=1}^{m} d_j = 1,\ d_j \geq 0, \forall j = 1, 2, ..., m$$

*is also in the class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

*Proof.* By using the definition of $h\,(z)$, we have

$$\begin{aligned} h\,(z) &= \sum_{j=1}^{m} d_j \left[ z - \sum_{k=2}^{\infty} a_{kj} z^k \right] = z \sum_{j=1}^{m} d_j - \sum_{k=2}^{\infty} \sum_{j=1}^{m} d_j a_{kj} z^k = \\ &= z - \sum_{k=2}^{\infty} \left( \sum_{j=1}^{m} d_j a_{kj} z^k \right). \end{aligned}$$

Therefore,

$$\sum_{k=2}^{\infty} \mathcal{D}_n\,(k, A, B, \alpha, \beta, \gamma, \lambda) \left( \sum_{j=1}^{m} d_j a_{kj} \right) =$$

$$= \sum_{k=2}^{\infty} \mathcal{D}_n\,(k, A, B, \alpha, \beta, \gamma, \lambda)\, a_{k1} d_1 + \sum_{k=2}^{\infty} \mathcal{D}_n\,(k, A, B, \alpha, \beta, \gamma, \lambda)\, a_{k2} d_2 + ...$$

$$... + \sum_{k=2}^{\infty} \mathcal{D}_n\,(k, A, B, \alpha, \beta, \gamma, \lambda)\, a_{km} d_m \leq$$

$$\leq d_1 \beta\gamma(B - A)(1 - \alpha) + d_2 \beta\gamma(B - A)(1 - \alpha) + ...$$

$$... + d_m \beta\gamma(B - A)(1 - \alpha) =$$

$$= \beta\gamma(B - A)(1 - \alpha) \sum_{j=1}^{m} d_j = \beta\gamma(B - A)(1 - \alpha)$$

which implies that $h\,(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. □

**Theorem 6.** *Let the functions* $f_1(z) = z - \sum_{k=2}^{\infty} a_{k1} z^k$, $a_{k1} \geq 0, \forall k \geq 2$, *and* $f_2(z) = z - \sum_{k=2}^{\infty} a_{k2} z^k$, $a_{k2} \geq 0, \forall k \geq 2$ *be in the class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, *respectively* $T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$. *Then the function* $p(z)$ *defined by*

$$p(z) = z - \frac{2}{3}\sum_{k=2}^{\infty} (a_{k1} + a_{k2}) z^k \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma).$$

*Proof.* Let $f_1(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ and $f_2(z) \in T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$ ; by using Theorem 2 we get, respectively,

$$\sum_{k=2}^{\infty} a_{k1} \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha)$$

and

$$\sum_{k=2}^{\infty} a_{k2} \mathcal{D}_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha)$$

We have (see (3) )

$$\begin{aligned} 2\sum_{k=2}^{\infty} a_{k2} \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda) &\leq \sum_{k=2}^{\infty} a_{k2} \mathcal{D}_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \\ &\leq \beta\gamma(B - A)(1 - \alpha). \end{aligned}$$

Then

$$\frac{2}{3}\sum_{k=2}^{\infty} a_{k1} \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \frac{2}{3}\beta\gamma(B - A)(1 - \alpha)$$

and

$$\frac{2}{3}\sum_{k=2}^{\infty} a_{k2} \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \frac{1}{3}\beta\gamma(B - A)(1 - \alpha)$$

imply

$$\frac{2}{3}\sum_{k=2}^{\infty} (a_{k1} + a_{k2}) \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha),$$

and from this we deduce that

$$p(z) = z - \frac{2}{3}\sum_{k=2}^{\infty} (a_{k1} + a_{k2}) z^k \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma).$$

□

## 4. **Integral Operators**

**Theorem 7.** *Let the function $f(z)$ defined by (1), be in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, and let $c$ be a real number such that $c>-1$. Then the function $F(z)$ defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \tag{5}$$

*also belongs to the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.*

*Proof.* By using the representation of $F(z)$, it follows that

$$F(z)=z-\sum_{k=2}^{\infty}b_k z^k,\ \textit{where}\ \ b_k=\frac{c+1}{c+k}a_k$$

$$f\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)\Rightarrow\sum_{k=2}^{\infty}a_k\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)\le\beta\gamma(B-A)(1-\alpha)$$

$$\begin{aligned}\sum_{k=2}^{\infty}b_k\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) &= \sum_{k=2}^{\infty}\frac{c+1}{c+k}a_k\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)<\\ &< \sum_{k=2}^{\infty}a_k\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)\\ &\le \beta\gamma(B-A)(1-\alpha)\end{aligned}$$

$\Rightarrow F(z)\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. □

**Theorem 8.** *Let $c$ be a real number such that $c>-1$. If $F(z)\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ then the function $f(z)$ defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt$$

*is univalent in $|z|<R$, where*

$$R=\inf_k\left[\frac{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)}{\beta\gamma(B-A)(1-\alpha)(c+k)k}\right]^{\frac{1}{k-1}},\ k\ge2 \tag{6}$$

*The result is sharp for*

$$f(z)=z-\frac{\beta\gamma(B-A)(1-\alpha)(c+k)z^k}{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)},\qquad k\ge2 \tag{7}$$

*Proof.* Let $F(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$; it follows from (5) that

$$f(z) = \frac{z^{1-c}\left[z^c F(z)\right]'}{c+1} = z - \sum_{k=2}^{\infty} \frac{c+k}{c+1} a_k z^k$$

$$F(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma) \Rightarrow \sum_{k=2}^{\infty} \frac{a_k \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)} \leq 1.$$

If

$$\frac{k(c+k)|z|^{k-1}}{c+1} < \frac{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)}$$

or if

$$|z| < \left[\frac{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)}{\beta\gamma(B-A)(1-\alpha)k(c+k)}\right]^{\frac{1}{k-1}}$$

then

$$\begin{aligned} |f'(z) - 1| &= \left| -\sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k |z|^{k-1} < \\ &< \sum_{k=2}^{\infty} \frac{a_k \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)} \leq 1 \end{aligned}$$

But from $|f'(z) - 1| < 1$, $|z| < R$, we deduce that $f$ is univalent in the disc $|z| < R$.

The result is sharp and the extremal function is given by (7). □

**Theorem 9.** *Let $c \in \mathbb{R}$, $c > -1$. If*

$$F(z) = z - \sum_{k=2}^{\infty} a_k z^k \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$$

*then the function $f(z)$ given by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

*is starlike of order $p$ $(0 \leq p < 1)$ in $|z| < R^*(p,A,B,\alpha,\beta,\gamma)$ where*

$$R^* = \inf_k \left[\frac{(1-p)(c+1)\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{(k-p)(c+k)\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad ; \quad k \geq 2.$$

*The result is sharp.*

*Proof.* It is sufficient to show that $\left|\frac{zf'(z)}{f(z)}-1\right|<(1-p)$, in $|z|<R^*$.

Now

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{-\sum\limits_{k=2}^{\infty}(k-1)\frac{c+k}{c+1}a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}\frac{c+k}{c+1}a_k z^{k-1}}\right|\leq\frac{\sum\limits_{k=2}^{\infty}(k-1)\frac{c+k}{c+1}a_k|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}\frac{c+k}{c+1}a_k|z|^{k-1}}<1-p$$

provided

$$\sum_{k=2}^{\infty}\left(\frac{k-p}{1-p}\right)\left(\frac{c+k}{c+1}\right)a_k|z|^{k-1}<1$$

By using

$$\sum_{k=2}^{\infty}\frac{a_k\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)}\leq 1$$

the inequality

$$\sum_{k=2}^{\infty}\left(\frac{k-p}{1-p}\right)\left(\frac{c+k}{c+1}\right)a_k|z|^{k-1}<1$$

holds if

$$\left(\frac{k-p}{1-p}\right)\left(\frac{c+k}{c+1}\right)|z|^{k-1}<\frac{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)}\quad;\quad k\geq 2$$

or if

$$|z|<\left[\frac{(1-p)(c+1)\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{(k-p)(c+k)\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}}\quad;\quad k\geq 2.$$

Hence, $f(z)\in S^*(p)$ in $|z|<R^*$. The sharpness follows if we take the function $F(z)$, given by

$$F(z)=z-\frac{(B-A)\gamma\beta(1-\alpha)z^k}{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)},\ k\geq 2.$$

□

## 5. The Hadamard products

Let $f,g\in T$ ,

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k;\ a_k\geq 0,\ \forall k\geq 2 \tag{8}$$

and

$$g(z)=z-\sum_{k=2}^{\infty}b_k z^k;\ b_k\geq 0,\ \forall k\geq 2, \tag{9}$$

then we define the Hadamard product of $f$ and $g$ by

$$f * g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k.$$

**Theorem 10.** *If the functions $f$ and $g$ defined by* (8) *and* (9) *belong to the same class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, then the Hadamard product $f * g$ belongs to the class $T_{n,\lambda}(A, B, \alpha, \beta^2, \gamma)$.*

*Proof.* Since $f(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ by using Theorem 2 we have

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B-A)(1-\alpha)$$

and

$$a_k \leq \frac{\beta\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)\left[1+3\beta B - \beta\gamma(B-A)(2+\alpha)\right]} \quad ; \qquad \forall k \geq 2.$$

If $g(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ then

$$\sum_{k=2}^{\infty} b_k \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B-A)(1-\alpha)$$

Since $0 < \beta^2 \leq \beta \leq 1$ we have

$$\sum_{k=2}^{\infty} b_k \mathcal{D}_n\left(k, A, B, \alpha, \beta^2, \gamma, \lambda\right) \leq \sum_{k=2}^{\infty} b_k \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda)$$

and then

$$\begin{aligned} \sum_{k=2}^{\infty} a_k b_k \mathcal{D}_n\left(k, A, B, \alpha, \beta^2, \gamma, \lambda\right) &\leq \sum_{k=2}^{\infty} a_k b_k \mathcal{D}_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \\ &\leq \frac{\beta^2\gamma^2(B-A)^2(1-\alpha)^2}{2^n(1+\lambda)\left[1+3\beta B - \beta\gamma(B-A)(2+\alpha)\right]} \leq \\ &\leq \beta^2\gamma(B-A)(1-\alpha) \end{aligned}$$

because

$$\frac{\beta^2\gamma^2(B-A)^2(1-\alpha)^2}{2^n(1+\lambda)\left[1+3\beta B - \beta\gamma(B-A)(2+\alpha)\right]} \leq \beta^2\gamma(B-A)(1-\alpha) \Leftrightarrow$$

$$\beta^2\gamma(B-A)(1-\alpha)\left\{\gamma(B-A)(1-\alpha) - 2^n(1+\lambda)\left[1+3\beta B - \beta\gamma(B-A)(2+\alpha)\right]\right\} \leq 0$$

$$\Leftrightarrow \gamma(B-A)(1-\alpha) - 2^n(1+\lambda) - 2^n(1+\lambda)\beta B - 2^n(1+\lambda)2\beta B +$$

$$+2^n(1+\lambda)\beta\gamma(B-A)2 + 2^n(1+\lambda)\beta\gamma(B-A)\alpha \leq 0$$

$$\gamma\left(B-A\right)\left(1-\alpha\right)-2^{n}(1+\lambda)+2^{n}(1+\lambda)\beta\left[\gamma(B-A)\alpha-B\right]+$$

$$+2^{n}(1+\lambda)2\beta\left[\gamma(B-A)-B\right]\leq 0$$

According to Theorem 2 we obtain $f*g\in T_{n,\lambda}(A,B,\alpha,\beta^2,\gamma)$. □

**Theorem 11.** *Let* $p>0$ *and* $\frac{p+2-\sqrt{p^2+4p}}{2}\leq\alpha\leq\frac{1}{1+p}$. *If the functions* $f$ *and* $g$ *defined by* (8) *and* (9) *belong to the same class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, *then the Hadamard product* $f*g$ *belongs to the class* $T_{n,\lambda}(A,B,1-p\alpha,\beta^2,\gamma)$.

*Proof.* By using

$$a_k\leq\frac{\beta\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)\left[1+3\beta B-\beta\gamma(B-A)(2+\alpha)\right]}\;;\qquad\forall k\geq 2$$

and

$$\sum_{k=2}^{\infty}b_k\mathcal{D}_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right)\leq\beta\gamma(B-A)(1-\alpha)$$

we obtain

$$\begin{aligned}\sum_{k=2}^{\infty}a_kb_k\mathcal{D}_n\left(k,A,B,1-\alpha,\beta^2,\gamma,\lambda\right) &\leq \sum_{k=2}^{\infty}a_kb_k\mathcal{D}_n\left(k,A,B,\alpha,\beta^2,\gamma,\lambda\right)\leq\\ &\leq \sum_{k=2}^{\infty}a_kb_k\mathcal{D}_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right)\leq\\ &\leq \frac{\beta^2\gamma^2(B-A)^2(1-\alpha)^2}{2^n(1+\lambda)\left[1+3\beta B-\beta\gamma(B-A)(2+\alpha)\right]}\leq\\ &\leq \beta^2\gamma(B-A)(1-\alpha)^2\leq\beta^2\gamma(B-A)\alpha\end{aligned}$$

which implies that $f*g\in T_{n,\lambda}(A,B,1-\alpha,\beta^2,\gamma)$. □

**Corollary 12.** *Let* $\frac{3-\sqrt{5}}{2}\leq\alpha\leq\frac{1}{2}$ *and let the functions* $f$ *and* $g$ *defined by* (8) *and* (9) *be in the same class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. *Then the Hadamard product* $f*g$ *belongs to the class* $T_{n,\lambda}(A,B,1-\alpha,\beta^2,\gamma)$.

**Corollary 13.** *Let* $2-\sqrt{3}\leq\alpha\leq\frac{1}{3}$ *and let the functions* $f$ *and* $g$ *defined by* (8) *and* (9) *be in the same class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. *Then the Hadamard product* $f*g$ *belongs to the class* $T_{n,\lambda}(A,B,1-2\alpha,\beta^2,\gamma)$.

**Remark 14.** *From definition of the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *it is easy to see that if* $0<\beta_1\le\beta_2\le 1$ *then* $T_{n,\lambda}(A,B,\alpha,\beta_1,\gamma)\subset T_{n,\lambda}(A,B,\alpha,\beta_2,\gamma)$.

**Remark 15.** *Since* $0<\beta^2\le\beta\le 1$ *we have* $T_{n,\lambda}(A,B,\alpha,\beta^2,\gamma)\subset T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *and* $T_{n,\lambda}(A,B,1-p\alpha,\beta^2,\gamma)\subset T_{n,\lambda}(A,B,1-p\alpha,\beta,\gamma)$.

## 6. Inclusion properties of the classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$

**Theorem 16.** *Let* $0\le\alpha_2\le\alpha_1<1$; $0<\beta_1\le\beta_2\le 1$ *and* $0<\gamma_1\le\gamma_2\le\dfrac{B}{B-A}$. *Then we have* $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$.

*Proof.* Let $f\in T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)$. Then, by using Theorem 2, we have

$$\sum_{k=2}^{\infty}a_k\mathcal{D}_n\left(k,A,B,\alpha_1,\beta_1,\gamma_1,\lambda\right)\le\beta_1\gamma_1(B-A)(1-\alpha_1). \tag{10}$$

From this we deduce that

$$\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\left[B\left(k+1\right)-\gamma_1(B-A)(k+\alpha_1)\right]\le\gamma_1(B-A)(1-\alpha_1),$$

$$\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\left[-(k+\alpha_1)\right]\le 1-\alpha_1$$

and

$$\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\le 1.$$

Let $\alpha_1=\alpha_2+\delta$; $\beta_1=\beta_2-\varepsilon$; $\gamma_1=\gamma_2-\theta$ where $\delta,\varepsilon,\theta\ge 0$, then from (10) we have

$$\sum_{k=2}^{\infty}a_k\mathcal{D}_n\left(k,A,B,\alpha_2+\delta,\beta_2-\varepsilon,\gamma_2-\theta,\lambda\right)\le\beta_1\gamma_1(B-A)(1-\alpha_1)$$

and because

$$\mathcal{D}_n\left(k,A,B,\alpha_2+\delta,\beta_2-\varepsilon,\gamma_2-\theta,\lambda\right)=\mathcal{D}_n\left(k,A,B,\alpha_2,\beta_2,\gamma_2,\lambda\right)-$$

$$-k^n\left[1+\lambda\left(k-1\right)\right]\varepsilon\left[B\left(k+1\right)-\gamma_1\left(B-A\right)\left(k+\alpha_1\right)\right]-$$

$$-k^n\left[1+\lambda\left(k-1\right)\right]\beta_2\theta\left(B-A\right)\left[-\left(k+\alpha_1\right)\right]-k^n\left[1+\lambda\left(k-1\right)\right]\gamma_2\beta_2\left(B-A\right)\delta$$

we have

$$\sum_{k=2}^{\infty}a_k\mathcal{D}_n\left(k,A,B,\alpha_2,\beta_2,\gamma_2,\lambda\right)\le\beta_1\gamma_1(B-A)(1-\alpha_1)+$$

$$+\varepsilon\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\left[B\left(k+1\right)-\gamma_1\left(B-A\right)\left(k+\alpha_1\right)\right]+$$

$$+\beta_2\theta\left(B-A\right)\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\left[-\left(k+\alpha_1\right)\right]+$$

$$+\gamma_2\beta_2\left(B-A\right)\delta\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\leq$$

$$\leq\beta_1\gamma_1(B-A)(1-\alpha_1)+\varepsilon\gamma_1(B-A)(1-\alpha_1)+\beta_2\theta\left(B-A\right)\left(1-\alpha_1\right)+$$

$$+\beta_2\gamma_2\left(B-A\right)\delta=\beta_2\gamma_2(B-A)(1-\alpha_2)$$

According to Theorem 2 we obtain $f\in T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$ and $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$. □

**Corollary 17.** *Let* $0\leq\alpha_1\leq\alpha_2<1$. *Then we have* $T_{n,\lambda}(A,B,\alpha_1,\beta,\gamma)\supset T_{n,\lambda}(A,B,\alpha_2,\beta,\gamma)$.

**Corollary 18.** *Let* $0<\gamma_1\leq\gamma_2\leq\dfrac{B}{B-A}$. *Then*

$$T_{n,\lambda}(A,B,\alpha,\beta,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha,\beta,\gamma_2).$$

**Theorem 19.** *Let* $0\leq\alpha_2\leq\alpha_1<1$; $0<\beta_1\leq\beta_2\leq1$ *and* $0<\gamma_1\leq\gamma_2\leq\dfrac{B}{B-A}$. *If the functions* $f$ *defined by* (8) *and* $g$ *defined by* (9) *be in the classes* $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)$ *and* $T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$, *respectively, then the Hadamard product* $f*g$ *belongs to the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, *where* $\alpha=\min\left(\alpha_1,\alpha_2\right)$, $\beta=\max\left(\beta_1,\beta_2\right)$ *and* $\gamma=\max\left(\gamma_1,\gamma_2\right)$.

*Proof.* Since

$$\alpha=\min\left(\alpha_1,\alpha_2\right)\Rightarrow\alpha\leq\alpha_1\text{ and }\ \alpha\leq\alpha_2$$

$$\beta=\max\left(\beta_1,\beta_2\right)\Rightarrow\beta\geq\beta_1\text{ and }\beta\geq\beta_2$$

$$\gamma=\max\left(\gamma_1,\gamma_2\right)\Rightarrow\gamma\geq\gamma_1\text{ and }\gamma\geq\gamma_2$$

from Theorem 16 we have $f\in T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)\Rightarrow f\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ and $g\in T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)\Rightarrow g\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. From Theorem 10 and Remark 15 we have $f*g\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. □

**Theorem 20.** *Let* $-1\leq A_2\leq A_1<B_1\leq B_2\leq1$, $0<B_1$. *Then we have* $T_{n,\lambda}(A_1,B_1,\alpha,\beta,\gamma)\subset T_{n,\lambda}(A_2,B_2,\alpha,\beta,\gamma)$.

*Proof.* Let $f \in T_{n,\lambda}(A_1, B_1, \alpha, \beta, \gamma)$ then

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n\left(k, A_1, B_1, \alpha, \beta, \gamma, \lambda\right) \leq \beta\gamma(B_1 - A_1)(1-\alpha).$$

Since $A_1 \geq A_2$, $B_1 \leq B_2 \Rightarrow B_1 - A_1 \leq B_2 - A_2$ and because $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq \dfrac{B}{B-A}$ from (3) we deduce that $\mathcal{D}_n\left(k, A_1, B_1, \alpha, \beta, \gamma, \lambda\right) \geq \mathcal{D}_n\left(k, A_2, B_2, \alpha, \beta, \gamma, \lambda\right)$. We have

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n\left(k, A_2, B_2, \alpha, \beta, \gamma, \lambda\right) \leq \sum_{k=2}^{\infty} a_k \mathcal{D}_n\left(k, A_1, B_1, \alpha, \beta, \gamma, \lambda\right) \leq$$
$$\leq \beta\gamma(B_1 - A_1)(1-\alpha) \leq \beta\gamma(B_2 - A_2)(1-\alpha),$$

and according to Theorem 2 we obtain $f \in T_{n,\lambda}(A_2, B_2, \alpha, \beta, \gamma)$ which imply that $T_{n,\lambda}(A_1, B_1, \alpha, \beta, \gamma) \subset T_{n,\lambda}(A_2, B_2, \alpha, \beta, \gamma)$. □

**Corollary 21.** *Let* $-1 \leq A_2 \leq A_1 < B \leq 1$, $0 < B \leq 1$. *Then we have* $T_{n,\lambda}(A_1, B, \alpha, \beta, \gamma) \subset T_{n,\lambda}(A_2, B, \alpha, \beta, \gamma)$.

**Corollary 22.** *Let* $-1 \leq A < B_1 \leq B_2 \leq 1$, $0 < B_1 \leq B_2 \leq 1$. *Then we have* $T_{n,\lambda}(A, B_1, \alpha, \beta, \gamma) \subset T_{n,\lambda}(A, B_2, \alpha, \beta, \gamma)$.

**Theorem 23.** $T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \supset T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$.

*Proof.* Since $f(z) \in T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$ by using Theorem 2 we have

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_{n+1}\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \beta\gamma(B-A)(1-\alpha)$$

because

$$k^n < k^{n+1};\ \forall k \geq 2 \text{ and } \forall n \geq 0$$

and

$$k^n\left[1 + \lambda\left(k-1\right)\right]\left[\left(k-1\right) + \beta B\left(k+1\right) - \beta\gamma\left(B-A\right)\left(k+\alpha\right)\right] > 0$$

then

$$\mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \mathcal{D}_{n+1}\left(k, A, B, \alpha, \beta, \gamma, \lambda\right)\ ; \qquad \forall n \geq 0$$

and

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \sum_{k=2}^{\infty} a_k \mathcal{D}_{n+1}\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \beta\gamma(B-A)(1-\alpha)$$

According to Theorem 2 we obtain $f \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma) \Rightarrow T_{n,\lambda}(A,B,\alpha,\beta,\gamma) \supset T_{n+1,\lambda}(A,B,\alpha,\beta,\gamma)$. $\square$

**Remark 24.** *From definition of the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ or from Theorem 2 it is easy to see that if $0 \le \lambda_1 \le \lambda_2$ then $T_{n,\lambda_2}(A,B,\alpha,\beta,\gamma) \subset T_{n,\lambda_1}(A,B,\alpha,\beta,\gamma)$.*

**Remark 25.** *From Theorem 23 we have*

$$T_{n,\lambda}(A,B,\alpha,\beta,\gamma) \subset T_{0,\lambda}(A,B,\alpha,\beta,\gamma),$$

*from Remark 24 we have*

$$T_{0,\lambda}(A,B,\alpha,\beta,\gamma) \subset T_{0,0}(A,B,\alpha,\beta,\gamma)$$

*and from Theorem 16 and Theorem 20 we have*

$$T_{0,0}(A,B,\alpha,\beta,\gamma) \subset T_{0,0}(-1,1,0,1,\frac{1}{2})$$

*and*

$$f \in T_{0,0}(-1,1,0,1,\frac{1}{2}) \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 1$$

*the class of starlike functions with negative coefficients. Because these functions are univalent, then all functions in the classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ are univalent.*

## References


[1] Holhoş, A., *Classes of univalent functions with negative coefficients*, Studia Univ. Babeş-Bolyai, Math. 4(2003), 33-42.

[2] Owa, S., *On the starlike functions of order* $\alpha$ and type $\beta$, Math. Japonica 27, No. 6(1982), 723-735.

[3] Owa, S., *On a class of starlike functions,* J. Korean Math. Soc. 19(1982/83) No. 1, 29-38.

[4] Owa, S., *A remark on the Hadamard products of starlike functions II*, Math. Japonica 27, No. 6 (1982), 747-752.

[5] Sălăgean, G., *Subclasses of univalent functions*, Lecture Notes in Math., Springer-Verlag, 1013(1983), 362-372.


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# ON SOME PROPERTIES OF THE STARLIKE SETS AND GENERALIZED CONVEX FUNCTIONS. APPLICATION TO THE MATHEMATICAL PROGRAMMING WITH DISJUNCTIVE CONSTRAINTS

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**Abstract**. In this paper we give an extension for starlike sets of the well known property that any convex set in the n-dimensional Euclidian space is the convex hull of its extremal points. We establish some relationships between two classes of starlike functions and the convex and quasi-convex classes of functions. We consider also the concepts of marginal points and starlike hull of a given set, and we show that a starlike set is the starlike hull of its marginal point set. For the starlike quasi-convex mathematical programming with disjunctive constraints, we show the starlikeness property of its feasible set.

## 1. **Introduction**

The main goal of this paper is to give an extension for starlike sets of the well known property that any convex set in $\mathbb{R}^n$ is the convex hull of its extreme points.

We consider also, in section 2, the classes of starlike convex and starlike quasi-convex functions and we present some relationships between them.

In Section 3, we introduce the concepts of marginal points and starlike hull of a given set, and we show that a starlike set is the starlike hull of its marginal points.

For the starlike quasi-convex mathematical programming with disjunctive constraints, in section 4, we prove the starlikeness property of its feasible set.

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## 2. Preliminaries about starlike sets and functions

We shall present some concepts and preliminaries properties concerning starlike sets and functions useful in order to obtain the main results in this paper.

**Definition 1.** *A set $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$ is called convex, if $\lambda x + (1-\lambda)y \in A$, $\forall x, y \in A$ and $\forall \lambda \in (0,1)$.*

Let $[x, y] = \{\lambda x + (1-\lambda)\, y \in \mathbb{R}^n | \lambda \in [0,1]\}$ be the segment that links the points $x, y \in \mathbb{R}^n$.

The following property holds for any collection of convex sets in $\mathbb{R}^n$ (see, e.g. [12]).

**Proposition 1.** *The intersection of any collection of convex sets in $\mathbb{R}^n$ is a convex set.*

**Definition 2.** *Let $A \subseteq \mathbb{R}^n, A \neq \emptyset$. The convex hull of a set $A$ is the intersection of all convex sets in $\mathbb{R}^n$ containing $A$ and is denoted $convA$.*

**Proposition 2.** *If $B \subseteq A \subseteq \mathbb{R}^n$ and $A$ is a convex set, then $convB \subseteq A$.*

**Definition 3.** *Let $A \subseteq \mathbb{R}^n, A \neq \emptyset$. One point $x \in A$ is called extreme point of $A$, if there exists no two distinct points $x^{'}, x^{''} \in A$ and $\lambda \in (0,1)$ so that $x = \lambda x^{'} + (1-\lambda) x^{''}$. Let denote by $ext(A)$ the set of all extreme points of $A$.*

The following fundamental result is known as Minkowski's Theorem (see, [1], [2], [11]).

**Theorem 3.** *Any convex and compact set in $\mathbb{R}^n$ is the convex hull of its extreme points.*

**Definition 4.** *([15],[4]) Let $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$. The set $A$ is called starlike with respect to the point $x^0 \in A$, if $\lambda x^0 + (1-\lambda)\, y \in A$, $\forall y \in A$ and $\lambda \in (0,1)$. The point $x^0 \in A$ with the above property is said to be starlikeness center of the set $A$. A set $A$ that posses at least one starlikeness center is said to be a starlike set.*

We mention that a characterization of starlike sets in term of their maximal convex subsets was given by Bragard [3].

From Definition 4 it results without difficulty the following two properties:

**Theorem 4.** *If* $A_1, A_2, ..., A_s$ *are starlike sets in* $\mathbb{R}^n$, *having a common starlike center, then* $\bigcup_{k=1}^{s} A_k$ *is a starlike set.*

**Theorem 5.** *If* $A_1, A_2, ..., A_s$ *are starlike sets in* $\mathbb{R}^n$, *having a common starlike center, then* $\bigcap_{k=1}^{s} A_k$ *is a starlike set.*

**Definition 5.** *([5]) Let* $A \subseteq \mathbb{R}^n, A \neq \emptyset$. *The set of all points* $z \in \mathbb{R}^n$, *such that* $[z, x] \subseteq A$, *for any* $x \in A$, *is called starlikeness kernel of the set* $A$. *We denote the starlikeness kernel of the set* $A$ *by* $\ker(A)$.

From Definitions 1, 4, 5 it follows directly:

**Proposition 6.** *(i) The set* $A \subseteq \mathbb{R}^n$ *is a starlike set if and only if* $\ker(A) \neq \emptyset$.
*(ii) For any set* $A \subseteq \mathbb{R}^n$, $\ker(A) \subseteq A$.
*(iii) For any convex set* $A \subseteq \mathbb{R}^n$, $\ker(A) = A$.
*(iv) The starlikeness kernel of the set* $A$, $\ker(A)$ *is a convex set.*

**Definition 6.** *Let* $X \subseteq \mathbb{R}^n$ *be a convex set. A function* $f : X \to \mathbb{R}$ *is called* convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),$$

*for any* $x, y \in X$ *and any* $\lambda \in (0, 1)$. *Let* $cx(X)$ *be the set of all real convex functions defined on the set* $X$.

As a generalization of the convex functions we consider the class of the starlike convex functions.

**Definition 7.** *Let* $X \subseteq \mathbb{R}^n$ *be a starlike set. A function* $f : X \to \mathbb{R}$ *is called* starlike convex *if there exists a point* $x^* \in X$ *such that*

$$f(\lambda x^* + (1-\lambda)y) \leq \lambda f(x^*) + (1-\lambda) f(y) \tag{1}$$

*for any* $y \in X$ *and any* $\lambda \in (0, 1)$ *and*

$$X_r = \{x \in X | f(x) \leq r\} \tag{2}$$

*is a starlike set or an empty set for any* $r \in \mathbb{R}$. *Let* $scx(X)$ *be the set of all real starlike convex functions defined on the set* $X$.

**Definition 8.** *Let $X \subseteq \mathbb{R}^n$ be a convex set. A function $f : X \to \mathbb{R}$ is called* quasi-convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

*for any $x, y \in X$ and any $\lambda \in (0,1)$. Let $qcx(X)$ be the set of all real quasi-convex functions defined on the set $X$.*

**Theorem 7.** *([6, 7]) Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is a convex and nonempty set. The function $f$ is quasi-convex if and only if*

$$X_r = \{x \in X | f(x) \leq r\}$$

*is convex for any $r \in \mathbb{R}$.*

As a generalization of the quasi-convex function class, we consider the family of the starlike quasi-convex functions.

**Definition 9.** *Let $X \subseteq \mathbb{R}^n$ be a starlike set. A function $f : X \to \mathbb{R}$ is called* starlike quasi-convex *if there exists a point $x^* \in X$ such that*

$$f(\lambda x^* + (1-\lambda)y) \leq \max\{f(x^*), f(y)\} \quad (3)$$

*for any $y \in X$ and any $\lambda \in (0,1)$ and*

$$X_r = \{x \in X | f(x) \leq r\}$$

*is a starlike set or an empty set for any $r \in \mathbb{R}$. Let $sqcx(X)$ be the set of all real starlike quasi-convex functions defined on the set $X$.*

In the case of convex functions and more general of quasi-convex functions (see, Theorem 7) any level set $X_r$ is convex.

**Remark 1.** *Concerning the starlike convex functions we note that the condition (1) only do not assure that the level sets $X_r$ defined by (2) are all starlike sets.*

For instance, the function $f : X \to \mathbb{R}$, where

$$X = \{(x_1, 0), (0, x_2) : x_1 \in [-1, 1], x_2 \in [-1, 1]\} \subseteq \mathbb{R}^2$$

and

$$f(x_1,x_2)=\begin{cases}(x_1-0.5)^2, \text{ if } x_1\in[0,1], x_2=0\\(x_1+0.5)^2, \text{ if } x_1\in[-1,0], x_2=0\\(x_2-0.5)^2, \text{ if } x_2\in[0,1], x_1=0\\(x_2+0.5)^2, \text{ if } x_2\in[-1,0], x_1=0\end{cases}$$

satisfies the condition (1) but it do not verify (2), because the level sets $X_r$ for all $r\in[0,0.25)$ are not starlike sets. Therefore, the function $f$ below is not a starlike convex function. But, for instance, the function $f_1 : X \to \mathbb{R}$, where $X$ is the same as in the preceding example and

$$f_1(x,x_2)=\begin{cases}x_1^2, \text{ if } x_1\in[-1,1], x_2=0\\2x_2, \text{ if } x_2\in[-1,1], x=0\end{cases},$$

is a starlike convex function, since it satisfies both condition (1) and (2). The same remark is also true for starlike quasi-convex functions.

The function $f_1$ is a restriction to $X$ of the convex function $f_2 : \mathbb{R}^2 \to \mathbb{R}$, where $f_2(x_1,x_2)=x_1^2+2x_2$.

**Remark 2.** *But not any restriction to a starlike set of a convex function is starlike convex.*

For instance, the function $f_3 : X \to \mathbb{R}$ , where $X$ is the same as in the preceding example and

$$f_3(x_1,x_2)=\begin{cases}(x_1-1)^2, \text{ if } x_1\in[-1,1], x_2=0\\2x_2, \text{ if } x_2\in[-1,1], x_1=0\end{cases}$$

is the restriction to $X$ of the convex function $f_4 : \mathbb{R}^2 \to \mathbb{R}$, where $f_4(x_1,x_2) = (x_1-1)^2+2x_2$.

But the function $f_3$ is not a starlike convex function, because its level set

$$X_0=\{(1,0)\}\bigcup\{(0,x_2) : x_2\in[-1,0]\}$$

is not a starlike set.

**Remark 3.** *There exists also a starlike convex function, which is not a restriction of a convex function.*

Let consider the function $f_5 : X^{'} \to \mathbb{R}$, where

$$X^{'}=X\bigcup\{(x_1,x_2) : x_1=x_2, x_1\in[-1,1], x_2\in[-1,1]\}$$

and

$$f_5(x_1, x_2) = \begin{cases} x_1^2, \text{ if } x_1 \in [-1,1], x_2 = 0 \\ x_2^2, \text{ if } x_2 \in [-1,1], x_1 = 0 \\ 2(x_1 + x_2), \text{ if } x_1 = x_2 \in [-1,1] \end{cases} .$$

The function $f_5$ is a starlike convex function but it is not a restriction to the set $X^{'}$ of a certain convex function, because

$$f_5(\tfrac{1}{2}, \tfrac{1}{2}) = 2 > \tfrac{1}{2} f_5(1,0) + \tfrac{1}{2} f_5(0,1) = 1,$$

while the point $(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(1,0) + \frac{1}{2}(0,1)$ is a convex combination of the points $(1,0)$ and $(0,1)$.

Between the families of convex, starlike convex, quasi-convex and starlike quasi-convex functions there exist the following relationships.

**Theorem 8.** *If $X \subseteq \mathbb{R}^n$ is a convex nonempty set, then the following inclusions hold*
$cx(X) \subseteq scx(X) \subseteq sqcx(X)$,
$cx(X) \subseteq qcx(X) \subseteq sqcx(X)$.

*Proof.* Since a convex set is a starlike set, from Definitions 6 and 7 it follows obviously the inclusion $cx(X) \subseteq scx(X)$. From Definitions 8 and 9, it results obviously the inclusion $qcx(X) \subseteq sqcx(X)$. The inclusion $cx(X) \subseteq qcx(X)$ is also well known, and follows immediately from Definitions 6 and 8. It remain to show only the inclusion $scx(X) \subseteq sqcx(X)$. But this inclusion holds, because the inequality (1) implies (3). ∎

We mention that Tigan [13], [14] was employed the class of starlike quasi-convex functions in order to prove some stability properties for optimization problem with respect to constraint perturbations.

**Theorem 9.** *If the infimum of a starlike quasi-convex function $f$ defined on a starlike set $X \subseteq \mathbb{R}^n$ is finite and the minimum point set is non-empty, then the minimum point set of $f$ is a starlike set.*

*Proof.* Let denote

$$r = \inf \{f(x) : x \in X\}.$$

By theorem hypothesis $r \in \mathbb{R}$. Then since $f$ is a starlike quasi-convex function it follows that the level set

$$X_r = \{x \in X | f(x) \leq r\}$$

is a starlike set. But since $r$ is the infimum of the function $f$ the minimum point set of $f$ is the level set $X_r$. Therefore, the minimum point set of $f$ is a starlike set. ∎

## 3. **Properties concerning starlike sets**

In this section, we will present the concepts of starlike hull and marginal points of a set which extend the notion of convex hull and extremal points of a set and we will give a generalization of the theorem 2, implying these concepts.

**Definition 10.** *(*[8, 9, 10]*) Let $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$. The intersection of all starlike sets in $\mathbb{R}^n$ with the starlikeness center $x^0$, that includes the set $A$ is called the starlike hull with the starlikeness center $x^0$ of the set $A$. This set is denoted by $st(x^0, A)$.*

We can easily show that

$$st\left(x^0, A\right) = \bigcup_{x \in A} [x^0, x]. \tag{4}$$

**Definition 11.** *Let $K \neq \emptyset$, $K \subseteq \mathbb{R}^n$ and $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$. The set*

$$st(K, A) = \bigcup_{y \in K} st(y, A) \tag{5}$$

*is called the starlike hull of the set $A$ with respect to the starlikeness set $K$.*

From Definition 11, it follows immediately the following theorem.

**Theorem 10.** *Let $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$. If $K^{'}$ is a nonempty set and $K^{'} \subseteq K^{''} \subseteq \mathbb{R}^n$, then $st(K^{'}, A) \subseteq st(K^{''}, A)$.*

**Theorem 11.** *If $B \subseteq A \subseteq \mathbb{R}^n$ and $A$ is a starlike set having the starlikeness center $x^0$, then $st(x^0, B) \subseteq A$.*

*Proof.* Let $x \in B$. Since $x \in A$ and the set $A$ is starlike, then $[x^0, x] \subseteq A$. Hence $x$ is an arbitrary point in $B$, by (4), follows that $st(x^0, B) \subseteq A$. ∎

**Theorem 12.** *If $B \subseteq A \subseteq \mathbb{R}^n$, $A$ is a starlike set and $K$ is a nonempty set such that $K \subseteq \ker(A)$, then $st(K, B) \subseteq A$.*

*Proof.* By Theorem 11, $st(y, B) \subseteq A$ for any $y \in K$. Therefore, $\bigcup_{y \in K} st(y, B) \subseteq A$, i.e. $st(K, B) \subseteq A$. ∎

**Definition 12.** *Let $A$ be a subset of $\mathbb{R}^n$ and $x^0 \in A$. A point $x^{'} \in A$ is called a marginal point of $A$ with respect to $x^0$ if there are no $x^{''} \in A$, $x^{''} \neq x^0$ and $\lambda \in (0, 1]$, so that $x^{'} = \lambda x^0 + (1 - \lambda)\, x^{''}$. We denote the set of all marginal points of $A$ with respect to $x^0$ by $mg\left(x^0, A\right)$.*

**Definition 13.** *Let $A$ be a subset of $\mathbb{R}^n$ and $K \subseteq A$. A point $x^{'} \in A$ is called a marginal point of $A$ with respect to $K$ if there are no two distinct points $y \in K, x^{''} \in A$ and $\lambda \in (0, 1]$, such that $x^{'} = \lambda y + (1 - \lambda)\, x^{''}$. We denote the set of all marginal points of $A$ with respect to $K$ by $mg\,(K, A)$.*

From Definitions 12 and 13, it follows immediately the next property.

**Theorem 13.** *Let $A$ be a subset of $\mathbb{R}^n$, $K^{'} \subseteq K^{''} \subseteq A$ and $K \subseteq A$. Then*
*(i) $mg\,(K, A) = \bigcap_{y \in K} mg\,(y, A)\,,$*
*(ii) $mg\,(A, A) = ext(A),$*
*(iii) $mg\left(K^{''}, A\right) \subseteq mg(K^{'}, A).$*

*Proof.* The assertion (i) obviously results from Definitions 12 and 13. The assertion (ii) follows from definitions 13 and 3. The point (iii) of the theorem is a direct consequence of the point (i). ∎

**Definition 14.** *Let $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$ a starlike set. We define the marginal set of $A$ as $mg(A) = mg(\ker(A), A)$.*

The following theorem represents a generalization to the starlike sets of the Minkowski's theorem (see, Theorem 2).

**Theorem 14.** *([8, 9, 10]) Any starlike and compact set $A$ in $\mathbb{R}^n$, having the starlikeness centre $x^0$, is the starlike hull of the centre $x^0$ of $mg\left(x^0, A\right)$, i.e.* $A = st(x^0, mg(x^0, A))$.

*Proof.* From Definition 12 it follows that $mg\left(x^0, A\right) \subseteq A$. Since $A$ is a starlike set, by Theorem 11, it results that

$$st\left(x^0, mg\left(x^0, A\right)\right) \subseteq A \tag{6}$$

Let $x \in A$. If $x \in mg(x^0, A)$, then obviously $x \in st\left(x^0, mg\left(x^0, A\right)\right)$. Let suppose that $x \notin mg\left(x^0, A\right)$. As, by hypothesis $A$ is compact, there exists $x^{'} \in mg\left(x^0, A\right)$, so that $x = \lambda x^0 + (1-\lambda)\, x^{'}$ for a certain $\lambda \in (0, 1]$. Hence, in this case, $x \in st\left(x^0, mg\left(x^0, A\right)\right)$. Therefore, we have

$$A \subseteq st\left(x^0, mg\left(x^0, A\right)\right). \tag{7}$$

From (6) and (7), it follows that $A = st\left(x^0, mg\left(x^0, A\right)\right)$. ∎

**Theorem 15.** *Any starlike and compact set $A$ in $\mathbb{R}^n$ is the starlike hull of the marginal point set $mg\,(A)$ with respect to* $\ker(A)$, *i.e.* $A = st(\ker(A), mg\,(A))$.

*Proof.* By theorem 13, it follows that $mg\,(A) = \bigcap_{y\in\ker(A)} mg\,(y, A)$. Therefore, $mg\,(A) \subseteq mg\,(y, A)$, for any $y \in \ker(A)$. Then, by Theorem 12, we have that $st(\ker(A), mg(A)) \subseteq st(\ker(A), mg(y, A))$ for any $y \in \ker(A)$. But, by Theorem 14 and (5), it results that $st(\ker(A), mg(y, A)) = A$. Therefore, we obtain

$$st(\ker(A), mg(A)) \subseteq A. \tag{8}$$

Let $x \in A$ be an arbitrary element of the set $A$. If $x \in \ker(A)$ or $x \in mg\,(A)$, then we evidently have $x \in st(\ker(A), mg(A))$. Let suppose that $x \notin \ker(A) \cup mg(A)$. Then, there exists $y \in \ker(A)$ and $x^{'} \in mg\,(A)$ such that $x = \lambda y + (1-\lambda)\, x^{'}$ for a certain $\lambda \in (0, 1)$. Therefore, $x \in st(\ker(A), mg(A))$, which implies the inclusion

$$A \subseteq st(\ker(A), mg(A)). \tag{9}$$

Otherwise, in virtue of Definitions 13 and 14, $x \in mg\,(A)$, which contradicts the above assumption. But, from (8) and (9) it results the theorem conclusion. ∎

**Theorem 16.** *If $f$ is a continuous starlike quasi-convex function defined on a starlike compact set $X \subseteq \mathbb{R}^n$, then $f$ has at least a maximum point in the set* $\ker(A) \cup mg(A)$.

*Proof.* Since $f$ is a continuous function on a compact set $X$, $f$ has at least a maximum point $x^* \in X$. If $x^*$ is a marginal point of $X$ then the thorem is true. Suppose that $x^* \notin mg\,(A)$. Then, there exists $y \in \ker(A)$ and $x^{'} \in mg\,(A)$ such that $x^* = \lambda y + (1-\lambda)\, x^{'}$ for a certain $\lambda \in (0,1)$.

Since $f$ is starlike quasi-convex it follows that

$$f(x^*) \leq \max\{f\left(x^{'}\right), f(y)\}.$$

On the other hand, since $x^*$ is a maximum point of $f$ over the set $X$, we have

$$\max\{f\left(x^{'}\right), f(y) \leq f(x^*),$$

from where it follows that $\max\{f\left(x^{'}\right), f(y)\} = f(x^*)$. Therefore, at least one of the points $x^{'}$ or $y$ is a maximum point, which implies the theorem conclusion. ∎

## 4. Application to the starlike quasi-convex programming problem with disjunctive constraints

Let $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{2, ..., m\}$ be starlike quasi-convex functions and let $\Omega = \{b_1, b_2, ..., b_s\} \subseteq \mathbb{R}^m$ be a finite set of vectors in $\mathbb{R}^m$.

By $g = (g_1, g_2, ..., g_m)^T$ we denote the vector of constraint functions for the starlike quasi-convex programming problem with disjunctive constraints (**QS**), and by $b$ we denote $b = \inf \Omega$, where infimum is considered with respect to the usual order relation in $\mathbb{R}^n$.

**QS**. Find

$$\min f(x_1, x_2, ..., x_n)$$

submit to

$$(g(x_1, ..., x_n) \leq b_1) \vee (g(x_1, ..., x_n) \leq b_2) \vee ... \vee (g(x_1, ..., x_n) \leq b_s)\,,$$
$$x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0.$$

Let $S$ be the feasible set for problem **QS** and $S_k$ and $S^{'}$ the sets:

$$S_k = \{x \in \mathbb{R}^n_+ | g(x) \leq b_k\},\ k = 1, 2, ..., s$$
$$S` = \{x \in \mathbb{R}^n_+ | g(x) \leq b\}.$$

Then it results that the following equalities hold

$$S = \cup_{k=1}^{s} S_k \tag{10}$$

$$S^{'} = \cap_{k=1}^{s} S_k \tag{11}$$

where $S_k$, by Theorem 7 and $S^{'}$, by Proposition 1 are both starlike sets, while the set $S$ is generally not convex.

The set $S$ has a property given by following theorem:

**Theorem 17.** *If the following two condition hold: (i) $S^{'} \neq \emptyset$,(ii) $g_i$, $i \in \{1, 2, ..., m\}$ are starlike quasi-convex functions, having all a common starlikeness point, then the feasible set $S$ of problem **QS** is a starlike set.*

*Proof.* Choose an arbitrary point $x^0 \in S^{'}$ and let $x \in S$. By (10) it results that there exists $k \in \{1, 2, ..., s\}$ so that $x \in S_k$. Since $S_k$ is a starlike set and, by (11), $x^0 \in S_k$, we have $[x^0, x] \subseteq S_k \subseteq S$. Therefore, the feasible set $S$ is starlike. ∎

We note that the set $S'$ represents a starlikeness kernel for the set $S$, as resulting from Theorem 17 and Definition 5.

## References


[1] Achmanov S., *Programmation linéaire*, Edition Mir, Moscou, 1984.

[2] Borwein J. M., Lewis A. S., *Convex Analysis and Nonlinear Optimization. Theory and Examples*, Springer Verlag, New York, 2000.

[3] Bragard L., *Propriétés inductive et sous-ensembles maximaux*, Bulletin de la Société Royale des Sciences de Liege, 1-2(1968) 8-13.

[4] Bragard L., *Ensembles étoilés et irradiés dans un espace vectoriel topologiques*, Bulletin de la Société Royale des Sciences de Liege, 1-2(1968), 276-285.

[5] Brun H., *Über Kerneigebiete*, Math. Ann., 73(1913), 436-440.

[6] Bazaraa M. S., Shetty C. M., *Non linear programming theory and algorithms*, John Willey and Sons, New York Chichester Brisbane Toronto, 1979.

[7] Breckner W. W., *Introducere în teoria problemelor de optimizare convexă cu restricţii*, Editura Dacia Cluj, 1974.

[8] Ionac D., *Metode de rezolvare a problemelor de programare matematică*, Teză de doctorat, Universitatea "Babeş-Bolyai" Cluj-Napoca, 1999.

[9] Ionac D., *Some properties of the starlike sets and their relation to mathematical programming by disjunctive constraints*, Analele Universităţii din Oradea, Fascicola Matematică, VII(1999-2000), 90-94.

[10] Ionac D., *Aspecte privind analiza unor probleme de programare matematică*, Editura Treira, Oradea, 2000.

[11] Minkowski H., *Theorie der konvexen Korper, insbesondere Begrundung ihres Oberflachenbegriffs*, In Gesammelte Abhandlungen II, Chelsea, New York, 1967.

[12] Rockafellar R. T., *Convex Analysis*, Princeton University Press, Priceton, New Jersey, 1970.

[13] Tigan S., *Sur quelques propriétés de stabilité concernant les problemes d'optimisation avec contraintes*, Mathematica - Revue d'Analyse Numérique et de Théorie de l'Approximation, 6, 2(1977), 203-225.

[14] Tigan S., *Contribuţii la teoria programării matematice şi aplicaţii ale ei*, Teză de doctorat, Universitatea "Babeş-Bolyai" Cluj-Napoca, 1978.

[15] Valentine F., *Convex sets*, Mc Graw-Hill Book Company, New York, San Francisco, Toronto, London, 1964.

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# THE BETA APPROXIMATING OPERATORS OF FIRST KIND

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**Abstract**. We shall define a general linear transform from which we obtain as particular case the beta first kind transform:

$$\mathcal{B}_{p,q}f = \frac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f(t^a)dt \qquad (*)$$

We consider here only the particular case $a = 1$.
We obtain several positive linear operators as a particular case of this beta first kind transform. We apply the transform $(*)$ to Bernstein's operator $B_n$ and thus we obtain different generalizations of this operator.

## 1. **Introduction**

Many authors introduced and studied positive linear operators, using Euler's beta function of first kind: [1], [2], [4], [6], [7], [8], [11].

Euler's beta function of first kind is defined for $p > 0$, $q > 0$ by the following formula

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \qquad (1.1)$$

The beta transform of the function $f$ is defined by the following formula

$$\mathcal{B}_{p,q}f = \frac{1}{B(p,q)}\int_0^1 t^{p-q}(1-t)^{q-1}p(t)dt$$

We shall define a more general linear transform of a function $f$ from which we obtain as particular case the beta first-kind transform.

For $a, b \in \mathbb{R}$, we define the $(a,b)$-beta transform of a function $f$ (see [6])

$$\mathcal{B}_{p,q}^{(a,b)}f = \frac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f(t^a(1-t)^b)dt \qquad (1.2)$$

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where $B(\cdot,\cdot)$ is the beta function (1.1) and $f$ is any real measurable function defined on $(0,\infty)$ such that $\mathcal{B}_{p,q}^{(a,b)}|f| < \infty$.

If we put in (1.2) $b = 0$ we obtain the first-kind transform of a function $f$

$$\mathcal{B}_{p,q}^{(a)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t^a) dt \tag{1.3}$$

where $B(\cdot,\cdot)$ is the beta function (1.1) and $f$ is any real measurable function defined on $(0,\infty)$ such that $\mathcal{B}_{p,q}^{(a)}|f| < \infty$. Clearly $\mathcal{B}_{p,q}^{(a)}$ is a positive linear functional.

We shall consider here the particular cases $a = 1$ and $a = -1$.

## 2. The beta first kind transform. Case $a = 1$

We shall consider here the particular case $a = 1$

$$\mathcal{B}_{p,q} f = \mathcal{B}_{p,q}^{(1)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t) dt. \tag{2.1}$$

We need to state and prove:

**Lemma 2.1.** *The moment of order $k$ of the functional $\mathcal{B}_{p,q}$ has the following value*

$$\mathcal{B}_{p,q} e_k = \frac{p(p+1)\dots(p+k-1)}{(p+q)\dots(p+q+k-1)} = \frac{(p)_k}{(p+q)_k} \tag{2.2}$$

**Proof.**

$$\mathcal{B}_{p,q} e_k = \frac{1}{B(p,q)} \int_0^1 t^{p+k-1}(1-t)^{q-1} dt = \frac{B(p+k,q)}{B(p,q)} \tag{2.3}$$

By using successively $k$ times the relation

$$B(p+1,q) = \frac{p}{p+q} B(p,q)$$

we find the relation

$$B(p+k,q) = \frac{p(p+1)\dots(p+k-1)}{(p+q)\dots(p+q+k-1)} B(p,q)$$

By replacing it into (2.3) we obtain the desired results (2.2). □

Consequently we obtain

$$\mathcal{B}_{p,q} e_1 = \frac{p}{p+q}, \quad \mathcal{B}_{p,q} e_2 = \frac{p(p+1)}{(p+q)(p+q+1)} \tag{2.4}$$

We impose that $\mathcal{B}_{p,q}e_1 = e_1$, that is $\dfrac{p}{p+q} = x$, or $\dfrac{p}{x} = \dfrac{q}{1-x}$, $x \in (0,1)$, $p > 0$ and we obtain the following linear transform

$$(\mathcal{B}_p f)(x) = \frac{1}{B\left(p, \dfrac{1-x}{x}p\right)} \int_0^1 t^{p-1}(1-t)^{\frac{1-x}{x}p-1} f(t)dt \tag{2.5}$$

**Lemma 2.2.** *One has*

$$\mathcal{B}_p((t-x)^2; x) = \frac{x^2(1-x)}{p+x}.$$

**Proof.** It is obtained from (2.4) for $q = \dfrac{1-x}{x}p$, $p+q = \dfrac{p}{x}$.

$$(\mathcal{B}_p e_2)(x) = \frac{p(p+1)}{\dfrac{p}{x}\left(\dfrac{p}{x}+1\right)} = \frac{p(p+1)x^2}{p(p+x)} = x^2 + \frac{(p+1)x^2}{p+x} - x^2 =$$

$$= x^2 + x^2\frac{p+1-p-x}{p+x} = x^2 + \frac{x^2(1-x)}{p+x}$$

and

$$\mathcal{B}_p((t-x)^2, x) = \frac{x^2(1-x)}{p+x}. \ \square$$

**Particular cases.**

a) Let $\mathcal{B}_\alpha$ be the beta operator defined by

$$(\mathcal{B}_\alpha f)(x) = \frac{1}{B\left(\dfrac{x}{\alpha}, \dfrac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)dt \tag{2.6}$$

$\alpha > 0$, $x \in (0,1)$. If $f$ is defined on $[0,1]$ we set

$$(\mathcal{B}_\alpha f)(0) = f(0), \quad (\mathcal{B}_\alpha f)(1) = f(1).$$

The operator (2.6) has been considered by G. Mülhlbach [7] and it is obtained by (2.5) if we choose in (2.5) $p = \dfrac{x}{\alpha}$.

**Lemma 2.3.** *One has*

$$\mathcal{B}_\alpha((t-x)^2, x) = \frac{\alpha}{1+\alpha}x(1-x).$$

**Proof.** $\mathcal{B}_\alpha e_2 = \dfrac{\dfrac{x}{\alpha}\left(\dfrac{x}{\alpha}+1\right)}{\dfrac{1}{\alpha}\left(\dfrac{1}{\alpha}+1\right)} = \dfrac{x(x+\alpha)}{1+\alpha} = x^2 + \left(\dfrac{x^2+\alpha x}{1+\alpha} - x^2\right) =$

$$= x^2 + \frac{\alpha x - \alpha x^2}{1+\alpha} = x^2 + \frac{\alpha}{1+\alpha}x(1-x) \;\Rightarrow$$

$$\mathcal{B}_\alpha(t-x)^2(x) = \frac{\alpha}{1+\alpha}x(1-x). \;\square$$

For $\alpha = 1/n$ we obtain $\mathcal{B}_{1/n}(t-x)^2(x) = \dfrac{x(1-x)}{n}$.

A slight modification of $\mathcal{B}_\alpha$ is the operator $\mathcal{B}^*_\alpha$ given by

$$(\mathcal{B}^*_\alpha f)(x) = \frac{1}{B\left(\dfrac{x}{\alpha}+1; \dfrac{1-x}{\alpha}+1\right)} \int_0^1 t^{\frac{x}{\alpha}}(1-t)^{\frac{1-\alpha}{\alpha}} f(t)dt, \tag{2.7}$$

$\alpha > 0$, $x \in [0,1]$, which, for $\alpha = 1/n$, $n \in \mathbb{N}$, has been introduced by A. Lupaş [4] and it is obtain by (2.5) if we replace in (2.5) $p = nx+1$.

A significant difference between $\mathcal{B}_\alpha$ and $\mathcal{B}^*_\alpha$ is that $\mathcal{B}_\alpha$ reproduces linear functions whereas $\mathcal{B}^*_\alpha$ does not.

b) Another beta first-kind operator it is obtained by (2.5) for $p = \dfrac{x}{\alpha(1-x)}$.

$$(\overline{\mathcal{B}}_\alpha f)(x) = \frac{1}{B\left(\dfrac{x}{\alpha(1-x)}; \dfrac{1}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha(1-x)}-1}(1-t)^{\frac{1}{\alpha}-1} f(t)dt, \tag{2.8}$$

$\alpha > 0$, $x \in (0,1)$, where $f$ is any real measurable function defined on (0,1) such that $(\overline{\mathcal{B}}_\alpha|f|)(x) < \infty$. The operator (2.7) was introduced by S. Rathore [8] for $\alpha = 1/n$, $n \in \mathbb{N}$.

**Lemma 2.4.** *One has*

$$\overline{\mathcal{B}}_\alpha((t-x)^2;x) = \frac{\alpha x(1-x)^2}{1+\alpha(1-x)}.$$

**Proof.** $\overline{\mathcal{B}}_\alpha e_2 = \dfrac{\dfrac{x}{\alpha(1-x)}\left(\dfrac{x}{\alpha(1-x)}+1\right)}{\dfrac{1}{\alpha(1-x)}\left(\dfrac{1}{\alpha(1-x)}+1\right)} = \dfrac{x(x+\alpha(1-x))}{1+\alpha(1-x)} =$

$$= x^2 + \left(\frac{x^2+\alpha x(1-x)}{1+\alpha(1-x)} - x^2\right) = x^2 + \frac{\alpha x(1-x)^2}{1+\alpha(1-x)} \;\Rightarrow$$

$$\overline{\mathcal{B}}_\alpha(t-x)^2(x) = \frac{\alpha x(1-x)^2}{1+\alpha(1-x)}. \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{\mathcal{B}}_{1/n}(t-x)^2(x) = \frac{x(1-x)^2}{n+1-x}$$

c) Let $\widetilde{\mathcal{B}}_\alpha$ be the operator defined by

$$(\widetilde{\mathcal{B}}_\alpha f)(x) = \frac{1}{B\left(\frac{1}{\alpha}, \frac{1-x}{\alpha x}\right)} \int_0^1 t^{\frac{1}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha x}-1} f(t)dt \tag{2.9}$$

$\alpha > 0$, $x \in (0,1)$. The operator (2.8) is obtained by (2.5) if we choose in (2.5) $p = \dfrac{1}{\alpha}$.

**Lemma 2.5.** *One has*

$$\widetilde{\mathcal{B}}_\alpha((t-x)^2; x) = \frac{\alpha x^2(1-x)}{1+\alpha x}$$

**Proof.** $\widetilde{\mathcal{B}}_\alpha e_2 = \dfrac{\frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)}{\frac{1}{\alpha x}\left(\frac{1}{\alpha x}+1\right)} = \dfrac{\alpha+1}{\alpha^2}\dfrac{\alpha^2 x^2}{1+\alpha x} = \dfrac{\alpha+1}{1+\alpha x}x^2 =$

$$= x^2 + \left(\frac{\alpha+1}{1+\alpha x}x^2 - x^2\right) = x^2 + \frac{\alpha x^2 - \alpha x^3}{1+\alpha x} = x^2 + \frac{\alpha x^2(1-x)}{1+\alpha x} \Rightarrow$$

$$\widetilde{\mathcal{B}}_\alpha(t-x)^2(x) = \frac{\alpha x^2(1-x)}{1+\alpha x}. \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\widetilde{\mathcal{B}}_{1/n}(t-x)^2 = \frac{x^2(1-x)}{n+x}.$$

## 3. The functional $P_n^{(p,q)} f = \mathcal{B}_{p,q}(B_n f)$

Now let us apply the transform (2.1) to the Bernstein operator $B_n$, defined by [3]

$$(B_n f)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right)$$

We may state and prove

**Theorem 3.1.** *The $\mathcal{B}_{p,q}$ transform of $B_nf$ can be expressed under the following form*

$$P_n^{(p,q)}f=\mathcal{B}_{p,q}(B_nf)=\sum_{k=0}^{n}\binom{n}{k}\frac{(p)_k(q)_{n-k}}{(p+q)_n}f\left(\frac{k}{n}\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\frac{p(p+1)\dots(p+k-1)q(q+1)\dots(q+n-k-1)}{(p+q)(p+q+1)\dots(p+q+n-1)}f\left(\frac{k}{n}\right) \tag{3.1}$$

**Proof.**

$$P_n^{(p,q)}f=\mathcal{B}_{p,q}(B_nf)=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{B(p,q)}\int_0^1 t^{p+k-1}(1-t)^{q+n-k+1}dt\cdot f\left(\frac{k}{n}\right)=$$

$$=\sum_{k=0}^{n}\binom{n}{k}\frac{B(p+k,q+n-k)}{B(p,q)}f\left(\frac{k}{n}\right)=$$

$$=\sum_{k=0}^{n}\binom{n}{k}\frac{p(p+1)\dots(p+k-1)q(q+1)\dots(q+n-k-1)}{(p+q)(p+q+1)\dots(p+q+n-1)}f\left(\frac{k}{n}\right).$$

**Theorem 3.2.** *One has*

$$P_n^{(p,q)}e_1=\mathcal{B}_{p,q}(B_ne_1)=\frac{p}{p+q}$$

$$P_n^{(p,q)}e_2=\mathcal{B}_{p,q}(B_ne_2)=\frac{p}{(p+q)(p+q+1)}\left(p+1+\frac{q}{n}\right) \tag{3.2}$$

**Proof.** $P_n^{(p,q)}e_1=\mathcal{B}_{pq}(B_ne_1)=\dfrac{1}{B(p,q)}\displaystyle\int_0^1 t^p(1-t)^{q-1}dt=\dfrac{B(p+1,q)}{B(p,q)}=\dfrac{p}{p+q}$.

$$P_n^{(p,q)}e_2=\mathcal{B}_{p,q}(B_ne_2)=\frac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}\left(t^2+\frac{t(1-t)}{n}\right)dt=$$

$$=\frac{1}{B(p,q)}\left(\int_0^1 t^{p+1}(1-t)^{q-1}dt+\frac{1}{n}\int_0^1 t^p(1-t)^q dt\right)=$$

$$=\frac{B(p+2,q)}{B(p,q)}+\frac{1}{n}\frac{B(p+1,q+1)}{B(p,q)}=\frac{p(p+1)}{(p+q)(p+q+1)}+\frac{1}{n}\frac{pq}{(p+q)(p+q+1)}.\ \square$$

We impose that $P_n^{(p,q)}e_1=\mathcal{B}_{p,q}(B_ne_1)=e_1$, that is $\dfrac{p}{p+q}=x$ or $q=\dfrac{1-x}{x}p$, $x\in(0,1)$, $p>0$. We obtain from Theorem 3.1 and Theorem 3.2 the following results.

**Corollary 3.3.** *One has*

$$P_n^{(p)} f = \mathcal{B}_p(B_n f) = \sum_{k=0}^{n} v_{n,k}(x) f\left(\frac{k}{n}\right) \tag{3.3}$$

*where*

$$v_{n,k}(x) = \binom{n}{k} \frac{p(p+1)\dots(p+k-1)p(1-x)(p(1-x)+x)\dots(p(1-x)+(n-k-1)x)}{p(p+x)\dots(p+(n-1)x)} x^k.$$

**Proof.** If we put in (3.1) $q = p\dfrac{1-x}{x}$, then $p+q = \dfrac{p}{x}$ and we obtain

$$P_n^{(p)} f = \mathcal{B}_p(B_n f) = \sum_{k=0}^{n} v_{n,k}(x) f\left(\frac{k}{n}\right)$$

where

$$v_{n,k}(x) = \binom{n}{k} \frac{(p)_k (q)_{n-k}}{(p+q)_n} = \binom{n}{k} \frac{(p)_k \left(p\dfrac{1-x}{x}\right)_{n-k}}{\left(\dfrac{p}{x}\right)_n} =$$

$$= \binom{n}{k} \frac{p(p+1)\dots(p+k-1)\left(p\dfrac{1-x}{x}\right)\left(p\dfrac{1-x}{x}+1\right)\dots\left(p\dfrac{1-x}{x}+n-k-1\right)}{\dfrac{p}{x}\left(\dfrac{p}{x}+1\right)\dots\left(\dfrac{p}{x}+n-1\right)} =$$

$$= \binom{n}{k} \frac{p(p+1)\dots(p+k-1)p(1-x)(p(1-x)+1)\dots(p(1-x)+(n-k-1)x)}{p(p+x)\dots(p+(n-1)x)} x^k. \square$$

**Corollary 3.4.** *One has* $(P_n^{(p)} e_1)(x) = \mathcal{B}_p(B_n e_1)(x) = x$*;*

$$(P_n^{(p)} e_2)(x) = \mathcal{B}_p(B_n e_2)(x) = x^2 + x(1-x)\frac{nx+p}{n(x+p)};$$

$$P_n^{(p)}(t-x)^2(x) = \mathcal{B}_p(B_n(t-x)^2)(x) = \frac{x(1-x)}{n} \cdot \frac{nx+p}{x+p} \tag{3.4}$$

**Proof.** $(P_n^{(p)} e_1)(x) = \mathcal{B}_p(B_n e_1)(x) = \dfrac{p}{p+q} = \dfrac{px}{p} = x.$

$$(P_n^{(p)} e_2)(x) = \mathcal{B}_p(B_n e_2)(x) = \frac{p}{(p+q)(p+q+1)}\left(p+1+\frac{q}{n}\right) =$$

$$= \frac{p}{\dfrac{p}{x}\left(\dfrac{p}{x}+1\right)}\left(p+1+\frac{p}{n}\frac{1-x}{x}\right) = \frac{x^2}{p+x}\left(p+1+\frac{p}{n}\frac{1-x}{x}\right) =$$

$$= x^2 + \frac{x^2}{p+x}\left(p+1+\frac{p}{n}\frac{1-x}{x}\right) - x^2 = x^2 + x^2\left(\frac{p+1}{p+x} + \frac{p}{n}\frac{1-x}{x(p+x)} - 1\right) =$$

$$= x^2 + x^2\frac{(1-x)(p+nx)}{nx(p+x)} = x^2 + \frac{x(1-x)}{n}\frac{nx+p}{x+p}.$$

$$P_n^{(p)}(t-x)^2(x) = \mathcal{B}_p(B_n(t-x)^2)(x) = \frac{x(1-x)}{n}\frac{nx+p}{x+p}. \square$$

**Particular cases**

a) If we put in (3.3) $p = \dfrac{x}{\alpha}$, $\alpha > 0$, we obtain

$$(P_n^{(\alpha)}f)(x) = \sum_{k=0}^{n} v_{n,k}(x) f\left(\frac{k}{n}\right) \tag{3.5}$$

$$v_{n,k}(x) = \binom{n}{k}\frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1-x)(1-x+\alpha)\dots(1-x+(n-k-1)\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)}$$

This operator has been considered by D. D. Stancu [9], which, for $\alpha = 1/n$, $n \in \mathbb{N}$, has been introduced by L. Lupaş and A. Lupaş [5]:

$$(L_n f)(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(nx)_k(n(1-x))_{n-k}}{(n)_n} f\left(\frac{k}{n}\right) \tag{3.6}$$

**Corollary 3.5.** *One has* $P_n^{(\alpha)}(t-x)^2(x) = \dfrac{1+n\alpha}{n(1+\alpha)}x(1-x)$.

**Remark.** For $\alpha = 1/n$ we obtain

$$P_n(t-x)^2(x) = \frac{2}{n+1}x(1-x).$$

b) Another operator it is obtained by (3.3) for $p = \dfrac{x}{\alpha(1-x)}$, $\alpha > 0$.

$$(\overline{P}^{(\alpha)}f)(x) = \sum_{k=0}^{n} \overline{v}_{n,k}(x) f\left(\frac{k}{n}\right) \tag{3.7}$$

$$\overline{v}_{n,k}(x) = \binom{n}{k}\frac{x(x+\alpha(1-x))\dots(x+(k-1)\alpha(1-x))(1+\alpha)\dots(1+(n-k-1)\alpha)}{(1+\alpha)(1-x)\dots(1+(n-1)\alpha(1-x))}.$$

**Corollary 3.6.** *One has*

$$\overline{P}_n^{(\alpha)}(t-x)^2(x) = \frac{x(1-x)}{n}\cdot\frac{1+n\alpha(1-x)}{1+\alpha(1-x)}$$

**Remark.** For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{P}_n(t-x)^2(x) = \frac{x(1-x)(2-x)}{n+1-x}.$$

c) Let $\widetilde{P}_n^{(\alpha)}$ be the operator defined by

$$(\widetilde{P}_n^{(\alpha)} f)(x) = \sum_{k=0}^{n} \widetilde{v}_{n,k}(x) f\left(\frac{k}{n}\right) \tag{3.8}$$

$$\widetilde{v}_{n,k}(x) = \binom{n}{k} \frac{(1+\alpha)\dots(1+(k-1)\alpha)(1-x)(1-x+\alpha x)\dots(1-x+(n-k-1)\alpha x)}{(1+\alpha x)(1+2\alpha x)\dots(1+(n-1)\alpha x)} x^{n-k}$$

$\alpha > 0$, $x \in (0,1)$. This operator is obtained by (3.3) for $p = 1/\alpha$, $\alpha > 0$.

**Corollary 3.7.** *One has*

$$\widetilde{P}_n^{(\alpha)}(t-x)^2(x) = \frac{x(1-x)}{n} \cdot \frac{1+n\alpha x}{1+\alpha x}$$

**Remark.** For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\widetilde{P}_n(t-x)^2(x) = \frac{x(1-x)(1+x)}{n+x}.$$

From the operators (3.5), (3.7) and (3.8), for $\alpha = 0$ we obtain the operator of S. N. Bernstein.

## 4. The beta first kind transform. Case $a = -1$

We consider now the case $a = -1$. If we put $a = -1$ in (1.3) we obtain

$$\mathbf{B}_{p,q} f = \mathcal{B}_{p,q}^{(-1)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f\left(\frac{1}{t}\right) dt \tag{4.1}$$

**Lemma 4.1.** *The moment of order $k$ $(1 \le k < p)$ of the functional $\mathbf{B}_{p,q}$ has the following value*

$$\mathbf{B}_{p,q} e_k = \frac{(p+q-1)\dots(p+q-k)}{(p-1)\dots(p-k)}, \quad 1 \le k < p \tag{4.2}$$

**Proof.**

$$\mathbf{B}_{p,q} e_k = \frac{1}{B(p,q)} \int_0^1 t^{p-k-1}(1-t)^{q-1} dt = \frac{B(p-k,q)}{B(p,q)} \tag{4.3}$$

By using successively $k$ times the relation

$$B(p-1,q) = \frac{p+q-1}{p-1} B(p,q)$$

we find the relation

$$B(p-k,q)=\frac{(p+q-1)\dots(p+q-k)}{(p-1)\dots(p-k)}B(p,q)$$

By replacing it into (4.3) we obtain the desired results (4.2). $\square$

Consequently we obtain

$$\mathbf{B}_{p,q}e_1=\frac{p+q-1}{p-1},\quad \mathcal{B}_{p,q}e_2=\frac{(p+q-1)(p+q-2)}{(p-1)(p-2)} \tag{4.4}$$

We impose that $B_{p,q}e_1=e_1$, that is $\frac{p+q-1}{p-1}=x$ or $p-1=\frac{q}{x-1}$ and we obtain the following linear transform, defined for $x>1$ and $p>2$:

$$\mathbf{B}_p f=\frac{1}{B(p,(p-1)(x-1))}\int_0^1 t^{p-1}(1-t)^{(p-1)(x-1)-1}f\left(\frac{1}{t}\right)dt \tag{4.5}$$

**Lemma 4.2.** *One has*

$$\mathbf{B}_p((t-x)^2;x)=\frac{x(x-1)}{p-2}$$

**Proof.** It is obtained from Lemma 2.1 for $q=(p-1)(x-1)$, $p+q=1+(p-1)x$ and

$$\mathbf{B}_p e_2(x)=\frac{(p-1)x((p-1)x-1)}{(p-1)(p-2)}=x^2+\frac{(p-1)x^2-x}{p-2}-x^2=$$

$$=x^2+\frac{(p-1)x^2-x-(p-2)x^2}{p-2}=x^2+\frac{x^2-x}{p-2}=x^2+\frac{x(x-1)}{p-2}$$

and

$$\mathbf{B}_p((t-x)^2;x)=\frac{x(x-1)}{p-2}.\ \square$$

**Particular cases**

a) Let $\mathbf{B}_\alpha$ be the beta operator defined by

$$(\mathbf{B}_\alpha f)(x)=\frac{1}{B\left(1+\frac{1}{\alpha},\frac{x-1}{\alpha}\right)}\int_0^1 t^{\frac{1}{\alpha}}(1-t)^{\frac{x-1}{\alpha}-1}f\left(\frac{1}{t}\right)dt \tag{4.6}$$

$\alpha\in(0,1)$, $x\in(1,\infty)$. If $f$ is defined on $[1,\infty)$ we set $(\mathbf{B}_\alpha f)(1)=f(1)$.

This operator is obtained by (4.5) if we choose in (4.5) $p=1+\frac{1}{\alpha}$.

**Lemma 4.3.** *One has*

$$\mathbf{B}_\alpha((t-x)^2;x) = \frac{\alpha}{1-\alpha}x(x-1)$$

**Proof.** $\mathbf{B}_\alpha e_2 = \dfrac{\frac{x}{\alpha}\left(\frac{x}{\alpha}-1\right)}{\frac{1}{\alpha}\left(\frac{1}{\alpha}-1\right)} = \dfrac{x(x-\alpha)}{1-\alpha} = x^2 + \left(\dfrac{x^2-x\alpha}{1-\alpha} - x^2\right) =$

$$= x^2 + \frac{x^2 - x\alpha - x^2 + \alpha x^2}{1-\alpha} = x^2 + \frac{\alpha x(x-1)}{1-\alpha}$$

and

$$\mathbf{B}_\alpha((t-x)^2;x) = \frac{\alpha}{1-\alpha}(x-1). \ \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\mathbf{B}_{\frac{1}{n}}((t-x)^2;x) = \frac{x(x-1)}{n-1}.$$

b) Let $\overline{\mathbf{B}}_\alpha$ be the operator defined by

$$(\overline{\mathbf{B}}_\alpha f)(x) = \frac{1}{B\left(1+\frac{1}{\alpha(x-1)}, \frac{1}{\alpha}\right)} \int_0^1 t^{\frac{1}{\alpha(x-1)}}(1-t)^{\frac{1}{\alpha}-1} f\left(\frac{1}{t}\right) dt \qquad (4.7)$$

$\alpha \in (0,1)$, $x \in \left(1, 1+\dfrac{1}{\alpha}\right)$. This operator is obtained by (4.5) if we choose in (4.5) $p = 1 + \dfrac{1}{\alpha(x-1)}$.

**Lemma 4.4.** *One has*

$$\widetilde{\mathbf{B}}_\alpha((t-x)^2;x) = \frac{\alpha(x-1)^2}{1-\alpha(x-1)}.$$

**Proof.** $\overline{\mathbf{B}}_\alpha e_2 = \dfrac{\frac{x}{\alpha(x-1)}\left(\frac{x}{\alpha(x-1)}-1\right)}{\frac{1}{\alpha(x-1)}\left(\frac{1}{\alpha(x-1)}-1\right)} = \dfrac{x(x-\alpha(x-1))}{1-\alpha(x-1)} =$

$$= x^2 + \frac{x^2-\alpha x(x-1)}{1-\alpha(x-1)} - x^2 = x^2 + \frac{x^2-\alpha x(x-1) - x^2 + \alpha x^2(x-1)}{1-\alpha(x-1)} =$$

$$= x^2 + \frac{\alpha x(x-1)^2}{1-\alpha(x-1)}$$

and

$$\overline{\mathbf{B}}_\alpha((t-x)^2;x) = \frac{\alpha x(x-1)^2}{1-\alpha(x-1)}. \ \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{\mathbf{B}}_{1/n}((t-x)^2;x) = \frac{x(x-1)^2}{n+1-x}.$$

c) Another beta first-kind operator it is obtained by (2.5) for $p = 1 + \dfrac{1}{\alpha x}$.

$$(\widetilde{\mathbf{B}}_\alpha f)(x) = \frac{1}{B\left(1+\dfrac{1}{\alpha x};\dfrac{x-1}{\alpha x}\right)} \int_0^1 t^{\frac{1}{\alpha x}}(1-t)^{\frac{x-1}{\alpha x}-1} f\left(\frac{1}{t}\right) dt \tag{4.8}$$

$\alpha \in (0,1)$, $x \in (1, 1/\alpha)$, where $f$ is any real measurable function defined on $(1, 1/\alpha)$, such that $(\overline{\mathbf{B}}_\alpha|f|)(x) < \infty$.

**Lemma 4.5.** *One has*

$$\widetilde{\mathbf{B}}_\alpha((t-x)^2;x) = \frac{\alpha x^2(x-1)}{1-\alpha x}$$

**Proof.** $$\widetilde{\mathbf{B}}_\alpha e_2 = \frac{\dfrac{1}{\alpha}\left(\dfrac{1}{\alpha}-1\right)}{\dfrac{1}{\alpha x}\left(\dfrac{1}{\alpha x}-1\right)} = \frac{1-\alpha}{1-\alpha x}x^2 = x^2 + \frac{1-\alpha}{1-\alpha x}x^2 - x^2 =$$

$$= x^2 + \frac{x^2-\alpha x^2 - x^2 + \alpha x^3}{1-\alpha x} = x^2 + \frac{\alpha x^2(x-1)}{1-\alpha x}$$

and

$$\widetilde{\mathbf{B}}_\alpha((t-x)^2;x) = \frac{\alpha x^2(x-1)}{1-\alpha x}. \ \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{\mathbf{B}}_{1/n}((t-x)^2;x) = \frac{x^2(x-1)}{n-x}.$$

## References


[1] Adell, J. A., Badia, F. G., De la Cal, J., *Beta-type operators preserve shape properties*, Stochastic Process. Appl., **48**(1993), 1-8.

[2] Adell, J. A., Badia, F. G., De la Cal, J., Plo, L., *On the property of monotonic convergence for Beta operators*, J. of Approx. Theory, **84**(1996), 61-73.

[3] Bernstein, S. N., *Demonstration du théoreme de Weierstrass fondée sur le calcul probabilistiés*, Commun. Kharkow Math. Soc. 13(1912), 1-2.

[4] Lupaş, A., *Die folge der Beta operatoren*, Dissertation, Univ. Stuttgart, Stuttgart, 1972.

[5] Lupaş, L., Lupaş, A., *Polynomials of bynomial type and approximation operators*, Studia Univ. Babeş-Bolyai, Mathematica, XXXII, **4**(1987), 61-69.

[6] Miheşan, V., *Approximation of continuous functions by means of linear positive operators*, Ph. D. Thesis, "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Cluj-Napoca, 1997.

[7] Mühlbach, G., *Verallgemeinerungen der Bernstein und der Lagrangepolynome*, Rev. Roumaine Math. Pures Appl., **15**(1970), 1235-1252.

[8] Rathore, R. K. S., *Linear combinations of linear positive operators and generating relations on special functions*, Ph. D. Thesis, Delhi, 1973.

[9] Stancu, D. D., *Approximation of function by a new class of linear polynomial operators*, Rev. Roum. Math. Pures Appl., **13**(1968), 1173-1194.

[10] Stancu, D. D., *Two classes of positive linear operators*, Analele Univ. Timişoara, **8**(1970), 213-220.

[11] Stancu, D. D., *On the Beta approximating operators of second kind*, Anal. Numer. Theor. Approx., 24, 1-2(1995), 231-239.



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# THE BETA APPROXIMATING OPERATORS OF SECOND KIND

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**Abstract**. We shall define a general linear transform from which we obtain as particular case the beta second kind transform:

$$T_{p,q}f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u^a) du \qquad (*)$$

We consider here only the particular case $a = 1$.

We obtain several positive linear operators as a particular case of this beta second kind transform. We apply the transform $(*)$ to Baskakov's operator and Bleimann, Butzer and Hahn operator respectively and we obtain new generalization of these operators.

## 1. **Introduction**

Many authors introduced and studied positive linear operators, using Euler's beta function of second kind: [1], [2], [5], [6], [7], [9].

Euler's beta function of second kind is defined for $p > 0$, $q > 0$ by the following formula

$$B(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} du \qquad (1.1)$$

The beta transform of the function $f$ is defined by the following formula

$$\mathcal{B}_{p,q}f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u) du$$

We shall define a more general linear transform from which we obtain as particular case the beta second-kind transform.

For $a, b \in \mathbb{R}$ we define the $(a,b)$-beta transform of a function $f$

$$\mathcal{B}_{p,q}^{(a,b)}f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{u^a}{(1+u)^{a+b}}\right) du \qquad (1.2)$$

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where $B(\cdot,\cdot)$ is the beta function (1.1) and $f$ is any real measurable function defined on $(0,\infty)$ such that $\mathcal{B}_{p,q}^{(a,b)}|f|<\infty$.

## 2. The beta second-kind transform. Case $a=1$

Let us denote by $M[0,\infty)$ the linear space of functions defined for $t\geq 0$, bounded and Lebesgue measurable in each interval $[c,d]$, where $0<c<d<\infty$.

If we consider in (1.2) $a+b=0$ we obtain the second-kind transform of function $f\in M[0,\infty)$

$$T_{p,q}^{(a)}f=\mathcal{B}_{p,q}^{(a,-a)}f=\frac{1}{B(p,q)}\int_0^\infty\frac{u^{p-1}}{(1+u)^{p+q}}f(u^a)du \tag{2.1}$$

such that $T_{p,q}^{(a)}|f|<\infty$. Clearly $T_{p,q}^{(a)}$ is a positive linear functional.

We shall consider here only the particular case $a=1$ (see also [9])

$$T_{p,q}f=T_{p,q}^{(1)}f=\frac{1}{B(p,q)}\int_0^\infty\frac{u^{p-1}}{(1+u)^{p+q}}f(u)du \tag{2.2}$$

for $f\in M[0,\infty)$ such that $T_{p,q}|f|<\infty$.

**Remark.** If $a=-1$ we obtain

$$T_{p,q}^{(-1)}f=\frac{1}{B(p,q)}\int_0^\infty\frac{u^{p-1}}{(1+u)^{p+q}}f\left(\frac{1}{u}\right)du$$

Denoting $u=v^{-1}$ we can write

$$T_{p,q}^{(-1)}f=\frac{1}{B(p,q)}\int_0^\infty\frac{\left(\frac{1}{v}\right)^{p-1}}{\left(1+\frac{1}{v}\right)^{p+q}}f(v)\frac{1}{v^2}dv=$$

$$=\frac{1}{B(p,q)}\int_0^\infty\frac{v^{p+q}}{v^{p+1}(1+v)^{p+q}}f(v)dv=$$

$$=\frac{1}{B(p,q)}\int_0^\infty\frac{v^{q-1}}{(v+1)^{p+q}}f(v)dv=T_{p,q}^{(1)}f.$$

That is $T_{p,q}^{(-1)}f=T_{p,q}^{(1)}f=T_{p,q}f$.

**Lemma 2.1.** [9] *The moment of order $k$ $(1\leq k<q)$ of the functional $T_{p,q}$ has the following value*

$$T_{p,q}e_k=\frac{p(p+1)\dots(p+k-1)}{(q-1)\dots(q-k)},\quad 1\leq k<q \tag{2.3}$$

We impose that $T_{p,q}e_1 = e_1$, that is $p = (q-1)x$, $q > 1$ and we obtain

$$(T_q f)(x) = \frac{1}{B((q-1)x, q)} \int_0^\infty \frac{u^{(q-1)x-1}}{(1+u)^{(q-1)(x+1)+1}} f(u)du \tag{2.4}$$

**Lemma 2.2.** *One has*

$$(T_q e_2)(x) = x^2 + \frac{x(x+1)}{q-2}, \quad q > 2$$

$$T_q((t-x)^2; x) = \frac{x(x+1)}{q-2}, \quad q > 2. \tag{2.5}$$

**Proof.** It is obtained from Lemma 2.1 for $p = (q-1)x$. □

**Particular cases**

a) If in (2.4) we choose $q = 1 + \frac{1}{\alpha}$, $\alpha \in (0,1)$, then we give the positive linear operator $L_\alpha$ defined for $\alpha \in (0,1)$ and $x \geq 0$:

$$(L_\alpha f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^\infty \frac{u^{\frac{x}{\alpha}-1}}{(1+u)^{\frac{1+x}{\alpha}+1}} f(u)du \tag{2.6}$$

considered in [9] (see also [1], [2], [7]).

**Lemma 2.3.** *One has*

$$(L_\alpha e_2)(x) = x^2 + \frac{\alpha}{1-\alpha} x(1+x).$$

$$L_\alpha((t-x)^2; x) = \frac{\alpha}{1-\alpha} x(1+x).$$

**Proof.** We take $q = (\alpha+1)/\alpha$ in (2.5). □

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$L_{1/n}((t-x)^2; x) = \frac{x(1+x)}{n-1}.$$

b) If we choose in (2.4) $q = \frac{1}{\alpha(1+x)} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \frac{1}{\alpha} - 1\right)$, we obtain the beta type operator $H_\alpha$, given by

$$(H_\alpha f)(x) = \frac{1}{B\left(\frac{x}{\alpha(1+x)}, \frac{1}{\alpha(1+x)} + 1\right)} \int_0^\infty \frac{u^{\frac{x}{\alpha(1+x)}-1}}{(1+u)^{\frac{1}{\alpha}+1}} f(u)du \tag{2.7}$$

where $f \in M[0,\infty)$ such that $H_\alpha|f| < \infty$, considered by J. Adell [2].

**Lemma 2.4.** *One has*

$$(H_\alpha e_2)(x) = x^2 + \frac{\alpha x(1+x)^2}{1-\alpha(x+1)}$$

$$H_\alpha((t-x)^2;x) = \frac{\alpha x(1+x)^2}{1-\alpha(x+1)}$$

**Proof.** We take $q = \dfrac{1}{\alpha(x+1)} + 1$ in Lemma 2.2. □

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$H_{1/n}((t-x)^2;x) = \frac{x(1+x)^2}{n-1-x}.$$

c) If we put in (2.4) $q = 1 + \dfrac{1}{\alpha x}$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha}\right)$, we obtain the positive linear operator $M_\alpha$, given by

$$(M_\alpha f)(x) = \frac{1}{B\left(\frac{1}{\alpha}, \frac{1}{\alpha x}+1\right)} \int_0^\infty \frac{u^{\frac{1}{\alpha}-1}}{(1+u)^{\frac{1+x}{\alpha x}+1}} f(u)du \tag{2.8}$$

where $f \in M(0,\infty)$ such that $M_\alpha|f| < \infty$.

**Lemma 2.5.** *One has*

$$(M_\alpha e_2)(x) = x^2 + \frac{\alpha x^2(x+1)}{1-\alpha x}$$

$$M_\alpha((t-x)^2;x) = \frac{\alpha x^2(x+1)}{1-\alpha x}$$

**Proof.** The above identities are implied by Lemma 2.2. □

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$M_{1/n}((t-x)^2;x) = \frac{x^2(x+1)}{n-x}.$$

## 3. Generalized Baskakov operator

Let be $\overline{B}_n$ the Baskakov operator [3]

$$(\overline{B}_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right) \tag{3.1}$$

Now let us apply the transform $T_{p,q}$ (2.2) to Baskakov's operator (3.1) and we obtain (see [9])

**Theorem 3.1.** *The $T_{p,q}$ transform of $\overline{B}_n f$ can be expressed by the following form*

$$\overline{T}_n^{(p,q)} f = T_{p,q}(\overline{B}_n f) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(p)_k (q)_n}{(p+q)_{n+k}} f\left(\frac{k}{n}\right) \tag{3.2}$$

*where $(a)_m := a(a+1)\dots(a+m-1)$.*

**Proof.** $\overline{T}_n^{(p,q)} f = T_{p,q}(\overline{B}_n f) =$

$$= \frac{1}{B(p,q)} \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{u^k}{(1+u)^{n+k}} f\left(\frac{k}{n}\right) du =$$

$$= \frac{1}{B(p,q)} \sum_{k=0}^{\infty} \binom{n+k-1}{k} f\left(\frac{k}{n}\right) \int_0^{\infty} \frac{u^{p+k-1}}{(1+u)^{p+q+n+k}} du =$$

$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{B(p+k,q+n)}{B(p,q)} f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(p)_k (q)_n}{(p+q)_{k+n}} f\left(\frac{k}{n}\right). \ \square$$

**Theorem 3.2.** *One has*

$$\overline{T}_n^{(p,q)} e_1 = T_{p,q}(\overline{B}_n e_1) = \frac{p}{q-1} \tag{3.3}$$

$$\overline{T}_n^{(p,q)} e_2 = T_{p,q}(\overline{B}_n e_2) = \frac{p(p+1)}{(q-2)(q-1)} + \frac{1}{n} \frac{p(p+q-1)}{(q-2)(q-1)}.$$

**Proof.** $\overline{T}_n^{(p,q)} e_1 = \dfrac{1}{B(p,q)} \displaystyle\int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} u du$

$$\frac{1}{B(p,q)} \int_0^{\infty} \frac{u^p}{(1+u)^{p+q}} du = \frac{B(p+1,q-1)}{B(p,q)} = \frac{p}{q-1}.$$

$$\overline{T}_n^{(p,q)} e_2 = \frac{1}{B(p,q)} \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} \left(u^2 + \frac{u(u+1)}{n}\right) du =$$

$$= \frac{1}{B(p,q)} \left( \int_0^{\infty} \frac{u^{p+1}}{(1+u)^{p+q}} du + \frac{1}{n} \int_0^{\infty} \frac{u^p}{(1+u)^{p+q-1}} du \right) =$$

$$= \frac{1}{B(p,q)} \left( B(p+2,q-2) + \frac{1}{n} B(p+1,q-2) \right) =$$

$$= \frac{B(p+2,q-2)}{B(p,q)} + \frac{1}{n} \frac{B(p+1,q-2)}{B(p,q)} = \frac{p(p+1)}{(q-2)(q-1)} + \frac{1}{n} \frac{p(p+q-1)}{(q-2)(q-1)}. \square$$

We impose that $\overline{T}_n^{(p,q)} e_1 = e_1$, that is $p = (q-1)x$, $x > 0$, $q > 2$. We obtain from Theorem 3.1 and Theorem 3.2

**Corollary 3.3.** *One has*

$$\overline{T}_n^{(q)} f = T_q(\overline{B}_n f) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{((q-1)x)_k (q)_n}{((q-1)x+q)_{n+k}} f\left(\frac{k}{n}\right) \tag{3.4}$$

**Corollary 3.4.** *One has*

$$(\overline{T}_n^{(q)} e_1)(x) = x, \quad (\overline{T}_n^{(q)} e_2)(x) = x^2 + \frac{x(1+x)}{q-2} \left(1 + \frac{q-1}{n}\right)$$

$$\overline{T}_n^{(q)}((t-x)^2; x) = \frac{x(1+x)}{q-2} \left(1 + \frac{q-1}{n}\right) \tag{3.5}$$

**Proof.** Choosing $p = (q-1)x$ in Theorem 3.2, the conclusion follows. $\square$

**Particular cases**

a) If we put in (3.4) $q = \frac{1}{\alpha} + 1$, $\alpha \in (0,1)$, we obtain the operator considered by D. D. Stancu [8], as a generalization of the Baskakov operator

$$(\overline{L}_n^{(\alpha)} f)(x) = L_\alpha(\overline{B}_n f)(x) = \sum_{k=0}^{\infty} \bar{l}_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) \tag{3.6}$$

where

$$\bar{l}_{n,k}(x,\alpha) = \binom{n+k-1}{k} \frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1+\alpha)(1+2\alpha)\dots(1+n\alpha)}{(1+x+\alpha)(1+x+2\alpha)\dots(1+x+(n+k)\alpha)}$$

**Corollary 3.5.** *One has*

$$\overline{L}_n^{(\alpha)}((t-x)^2; x) = \frac{\alpha x(1+x)}{1-\alpha} \left(1 + \frac{1}{n\alpha}\right)$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{L}_n((t-x)^2; x) = \frac{2x(1+x)}{n-1}.$$

b) For $q = \dfrac{1}{\alpha(x+1)} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha} - 1\right)$, we obtain by (3.4) a new generalization of the Baskakov operator

$$(\overline{H}_n^{(\alpha)} f)(x) = H_\alpha(\overline{B}_n f)(x) = \sum_{k=0}^{\infty} \overline{h}_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) \tag{3.7}$$

where

$$\overline{h}_{n,k}(x,\alpha) =$$

$$= \binom{n+k-1}{k} \frac{x(x+\alpha(1+x))\dots(x+(k-1)\alpha(x+1))(1+\alpha(1+x))\dots(1+n\alpha(1+x))}{(1+x)^{n+k}(1+\alpha)\dots(1+(n+k)\alpha)}$$

**Corollary 3.6.** *One has*

$$\overline{H}_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha x(1+x)^2}{1-\alpha(1+x)}\left(1 + \frac{1}{\alpha n(1+x)}\right)$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{H}_n((t-x)^2;x) = \frac{x(x+1)(x+2)}{n-1-x}.$$

c) If we put in (3.4) $q = 1 + \dfrac{1}{\alpha x}$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha}\right)$, we obtain a new generalization of the Baskakov operator

$$(\overline{M}_n^{(\alpha)} f)(x) = M_\alpha(\overline{B}_n f)(x) = \sum_{k=0}^{\infty} \overline{m}_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) \tag{3.8}$$

where

$$\overline{m}_{n,k}(x,\alpha) = \binom{n+k-1}{k} \frac{(1+\alpha)\dots(1+(k-1)\alpha)(1+\alpha x)\dots(1+n\alpha x)}{(x+1+\alpha x)\dots(x+1+(n+k)\alpha x)} x^k$$

**Corollary 3.7.** *One has*

$$\overline{M}_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha x^2(x+1)}{1-\alpha x}\left(1 + \frac{1}{\alpha n x}\right)$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{M}_n((t-x)^2;x) = \frac{x(x+1)^2}{n-x}.$$

## 4. Generalized Bleimann, Butzer, Hahn operator

Let be $\widetilde{B}_n$ the Bleimann, Butzer, Hahn operator [4]

$$(\widetilde{B}_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} f\left(\frac{k}{n-k+1}\right) \tag{4.1}$$

Now let us apply the transform $T_{p,q}$ (2.2) to Bleimann, Butzer, Hahn's operator (4.1) and we obtain

**Theorem 4.1.** *The $T_{p,q}$ transform of $\widetilde{B}_n f$ can be expressed by the following form*

$$\widetilde{T}_n^{(p,q)} f = T_{p,q}(\widetilde{B}_n f) = \sum_{k=0}^{n} \binom{n}{k} \frac{(p)_k (q)_{n-k}}{(p+q)_n} f\left(\frac{k}{n-k+1}\right) \tag{4.2}$$

**Proof.**

$$\widetilde{T}_n^{(p,q)} f = T_{p,q}(\widetilde{B}_n f) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} \sum_{k=0}^{n} \binom{n}{k} \frac{u^k}{(1+u)^n} f\left(\frac{k}{n-k+1}\right) =$$

$$= \frac{1}{B(p,q)} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n-k+1}\right) \int_0^\infty \frac{u^{p+k-1}}{(1+u)^{p+q+n}} du =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{B(p+k, q+n-k)}{B(p,q)} f\left(\frac{k}{n-k+1}\right) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{(p)_k (q)_{n-k}}{(p+q)_n} f\left(\frac{k}{n-k+1}\right). \ \square$$

**Particular cases**

a) If we put in (4.2), $p = \dfrac{x}{\alpha}$, $q = \dfrac{1}{\alpha} + 1$, $\alpha \in (0,1)$, $x \geq 0$, we obtain the operator introduced by J. Adell [2] as a generalization of the Bleimann, Butzer, Hahn operator

$$(\widetilde{L}_n^{(\alpha)} f)(x) = L_\alpha(\widetilde{B}_n f)(x) = \sum_{k=0}^{n} \widetilde{l}_{n,k}(x,\alpha) f\left(\frac{k}{n-k+1}\right) \tag{4.3}$$

where

$$\widetilde{l}_{n,k}(x,\alpha) = \binom{n}{k} \frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1+\alpha)\dots(1+(n-k)\alpha)}{(x+1+\alpha)\dots(x+1+n\alpha)}$$

b) For $p = \dfrac{x}{\alpha(1+x)}$ and $q = \dfrac{1}{\alpha(1+x)} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha} - 1\right)$, we obtain by (4.2) a new generalization of the Bleimann, Butzer, Hahn operator

$$(\widetilde{H}_n^{(\alpha)} f)(x) == H_\alpha(\widetilde{B}_n f)(x) \sum_{k=0}^{n} \widetilde{h}_{n,k}(x,\alpha) f\left(\frac{k}{n-k+1}\right) \tag{4.4}$$

where

$$\widetilde{h}_{n,k}(x,\alpha) =$$

$$= \binom{n}{k} \frac{x(x+\alpha(1+x))\dots(x+(k-1)\alpha(1+x))(1+\alpha(1+x))\dots(1+(n-k)\alpha(1+x))}{(1+x)^n(1+\alpha)(1+2\alpha)\dots(1+n\alpha)}.$$

c) If we put in (4.2) $p = \dfrac{1}{\alpha}$, $q = \dfrac{1}{\alpha x} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha}\right)$, we obtain a new generalization of the Bleimann, Butzer, Hahn operator

$$(\widetilde{M}_n^{(\alpha)} f)(x) = M_\alpha(\widetilde{B}_n f)(x) = \sum_{k=0}^{n} \widetilde{m}_{n,k}(x,\alpha) f\left(\frac{k}{n-k+1}\right) \tag{4.5}$$

where

$$\widetilde{m}_{n,k}(x,\alpha) = \binom{n}{k} \frac{(1+\alpha)\dots(1+k\alpha)(1+\alpha x)\dots(1+(n-k-1)\alpha x)}{(x+1+\alpha x)\dots(x+1+n\alpha x)} x^k.$$

## References


[1] Adell, J. A., De la Cal, J., *On a Bernstein-type operator associated with the inverse Polya-Eggenberger distribution*, Rend. Circolo Matem. Palermo, Ser. II, Nr. 33(1993), 143-154.

[2] Adell, J. A., Badia, F. G., De la Cal, J., Plo, L., *On the property of monotonic convergence for Beta operators*, J. of Approx. Theory, **84**(1996), 61-73.

[3] Baskakov, V. A., *An example of a sequence of linear positive operators in the space of the continuous functions*, Dokl. Akad. Nauk SSSR, 113(1957), 249-251.

[4] Bleimann, G., Butzer, P. L., Hahn, L., *Bernstein-type operator approximating continuous functions on the semi-axis*, Indag. Math. 42(1980), 255-262.

[5] Miheşan, V., *Approximation of continuous functions by linear positive operators* (in Romanian), Ph. D. Thesis, "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Cluj-Napoca, 1997.

[6] Miheşan, V., *The Beta Approximating Operators of First-Kind*, Studia Univ. Babeş-Bolyai, Mathematica (in press).

[7] Rathore, R. K. S., *Linear combinations of linear positive operators and generating relations on special functions*, Ph. D. Thesis, Delhi, 1973.

[8] Stancu, D. D., *Two classes of positive linear operators*, Analele Univ. Timişoara, **8**(1970), 213-220.

[9] Stancu, D. D., *On the Beta approximating operators of second kind*, Anal. Numer. Theor. Approx., 24, 1-2(1995), 231-239.

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# ORTHOGONAL BASIS IN SOBOLEV SPACE $H_0^1(a,b)$

CRISTINEL MORTICI

**Abstract**. It is the purpose of this work to use the method of double-orthogonal sequences of Bergmann [1] to find an orthogonal basis in the Sobolev space $H_0^1(a,b)$. The elements of the basis are the solutions of some eigenvalue boundary problems.

In practice arise real difficulties in the problem of finding a base in Hilbert spaces. In case of Sobolev spaces a polynomial base is ussualy chosen, but other difficulties appear. Some of them were avoided using the finite element method. We give here a method of elimination of these difficulties using Bergmann's method of double orthogonal sequences [1].

Let $(H,(\cdot,\cdot))$, $(V,<\cdot,\cdot>)$ be real, separable Hilbert spaces and denote by $\|\cdot\|$, $|\cdot|$ the corresponding norms, respectively. In what follows, we use the next result due to Bergmann [1]:

**Theorem 1.** *Assume that $H\subset V$ and the imbedding $H\hookrightarrow V$ is compact,*

$$|x|\le c\,\|x\| \quad , \quad \forall\, x\in H,$$

*for some positive constant c. Then there exist an increasing, unbounded sequence $(\lambda_n)_{n\ge1}$ of positive real numbers and a sequence $(e_n)_{n\ge1}\subset H$ which is orthogonal with respect to both inner products, i.e.*

$$(e_m,e_n)=\lambda_n\delta_{mn} \quad , \quad <e_m,e_n>=\delta_{mn}\,, \tag{1}$$

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*for all positive integers m, n. Moreover,* $(e_n)_{n\geq 1}$ *is complete in H.*

We will give a method to find an orthogonal basis in $H$. In fact, the elements of the basis are the solutions of some optimization problems.

In this sense, denote by $v_1 \in H$ a solution of the problem

$$\sup\{|x| \ ; \ x \in H, \ \|x\| = 1\}.$$

If $v_1, v_2, ..., v_{n-1}$ are already defined, then $v_n \in H$ is chosen as a solution of the problem

$$\sup\{|x| \ ; \ x \in H, \ \|x\| = 1, \ (x, v_i) = 0, \ 1 \leq i \leq n-1\}.$$

Finally,

$$e_n = \frac{1}{|v_n|} \cdot v_n \quad , \quad n \geq 1.$$

For proofs and more details, see [1], [5]. The norms $\|\cdot\|$ and $|\cdot|$ are equivalent on finite dimensional subspaces of $H$.

Indeed, on $H_n = \text{sp}\{e_1, e_2, ..., e_n\}$, $n \geq 1$, we have

$$\frac{1}{c}|x| \leq \|x\| \leq \sqrt{\lambda_n} \cdot |x| \quad , \quad \forall\, x \in H_n.$$

Remark that from (1), we can derive the equalities

$$(e_m, e_n) = \lambda_n < e_m, e_n > \quad , \quad \forall\, m, n \geq 1.$$

Because of completness of the system $(e_n)_{n\geq 1}$, it follows that

$$(e_n, v) = \lambda_n < e_n, v > \quad , \quad \forall\, n \geq 1, \ v \in H. \tag{2}$$

In consequence, the elements of the orthogonal basis $(e_n)_{n\geq 1}$ can be considered as the solutions of the eigenvalue problem (2). In fact, this is an useful method to find a basis in a real separable Hilbert space, as we can see below.

Let $a < b$ be real numbers. We say that $u \in L^2(a, b)$ has generalized derivative (in Sobolev sense) if there exists $g \in L^2(a, b)$ such that

$$\int_a^b u\phi' = -\int_a^b g\phi \quad , \quad \forall\, \phi \in C_0^\infty(a, b).$$

$g$ (unique with this property) is called the generalized derivative of $u$ an denote $g = u'$.

The set of all functions $u \in L^2(a, b)$ with $u(a) = u(b) = 0$, having generalized derivative is denoted by $H_0^1(a, b)$.

$H_0^1(a, b)$ also called Sobolev space is a Hilbert space relative to the scalar product

$$(u, v) = \int_a^b uv + \int_a^b u'v' \quad , \quad u, v \in H_0^1(a, b).$$

Here $u', v'$ deonte the generalized derivatives of $u$, respective $v$. The corresponding norm is

$$\|u\| = \left( \int_a^b u^2 + \int_a^b u'^2 \right)^{1/2} \quad , \quad u \in H_0^1(a, b).$$

Consider also the Hilbert space $L^2(a, b)$ endowed with the usual scalar product

$$< u, v >= \int_a^b uv \quad , \quad u, v \in L^2(a, b)$$

and the usual norm

$$|u| = \left( \int_a^b u^2 \right)^{1/2} \quad , \quad u \in L^2(a, b).$$

The imbedding

$$H_0^1(a, b) \hookrightarrow L^2(a, b)$$

is compact because

$$|u| \le \|u\| \quad , \quad \forall\, u \in H_0^1(a, b).$$

In order to give a method to find an orthogonal basis in $H_0^1(a, b)$, we will use theorem 1. The eigenvalue problem (2) can be written as

$$\int_a^b e_n v + \int_a^b e_n' v' = \lambda_n \int_a^b e_n v \quad , \quad \forall\, v \in H_0^1(a, b),\ n \ge 1. \tag{3}$$

But $v(a) = v(b) = 0$, so

$$\int_a^b e_n' v' = -\int_a^b e_n'' v,$$

if $e_n$ is twice derivable. Hence (3) is equivalent with

$$\int_a^b e_n v - \int_a^b e_n'' v = \lambda_n \int_a^b e_n v,$$

so

$$\int_a^b (e_n'' + (\lambda_n - 1)e_n)v = 0 \quad , \quad \forall\, v \in H_0^1(a,b).$$

We deduce that $(e_n)_{n\geq 1}$ are the eigenfunctions of the following boundary problem

$$\begin{cases} e'' + \lambda e = 0 \\ e(a) = e(b) = 0 \end{cases} , \tag{4}$$

with $\lambda > 0$. The nontrivial solutions of the second order linear equation $e'' + \lambda e = 0$ are

$$e(x) = p \cos \sqrt{\lambda} x + q \sin \sqrt{\lambda} x \quad , \quad x \in (a,b),$$

for reals $p, q$, with $p^2 + q^2 \neq 0$.

The boundary conditions can be written as

$$\begin{cases} p \cos \sqrt{\lambda} a + q \sin \sqrt{\lambda} a = 0 \\ p \cos \sqrt{\lambda} b + q \sin \sqrt{\lambda} b = 0 \end{cases} . \tag{5}$$

If for example $q \neq 0$, we derive

$$-\frac{p}{q} = \tan \sqrt{\lambda} a = \tan \sqrt{\lambda} b,$$

so

$$\sqrt{\lambda} b - \sqrt{\lambda} a = n\pi \Rightarrow \lambda_n = \frac{n^2\pi^2}{(b-a)^2} \ , \ \ n \in \mathbf{N},\, n \geq 1.$$

In conclusion,

$$e_n(x) = -q \tan \frac{n\pi a}{b-a} \cos \frac{n\pi x}{b-a} + q \sin \frac{n\pi x}{b-a} \quad , \quad x \in (a,b)$$

is orthogonal basis in $H_0^1(a,b)$.

## References

[1] Bergmann, St., *The Kernel Function and Conformal Mapping,* New York, AMS, 1950.

[2] Brezis, H., *Annalise Fonctionnelle,* Mason Editeur, 1985.

[3] Deimling, K., *Nonlinear Functional Analysis,* Springer Verlag, 1985.

[4] Nirenberg, L., *Variational and Topological Methods in Nonlinear Problems,* Bull. AMS, 4(81), 267-302.

[5] Sburlan, S., *On a Particular Class of Optimal Problems with Application in the Projection Method,* Operation Research Verfahren XIX(1973), Anton Heim Verlag, 102-108.

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# SIMPLE SUFFICIENT CONDITIONS FOR UNIVALENCE

VIRGIL PESCAR

**Abstract**. We study some integral operators and determine conditions for the univalence of these integral operators.

## 1. Introduction

Let $A$ be the class of the functions $f$ which are analytic in the unit disc $U = \{z \in \mathcal{C}; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$. We denote by S the class of the functions $f \in A$ which are univalent in $U$.

## 2. Preliminary results

We will need the following theorems and lemma.

**Theorem 2.1**[2]. Let $\alpha$ be a complex number, $Re\ \alpha > 0$, and $f \in A$. If

$$\frac{1 - |z|^{2Re\ \alpha}}{Re \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{1}$$

for all $z \in U$, then for any complex number $\beta, Re\ \beta \geq Re\ \alpha$ the function

$$F_\beta(z) = \left[ \beta \int_o^z u^{\beta - 1} f'(u) du \right]^{\frac{1}{\beta}} \tag{2}$$

is in the class $S$.

**Teorem 2.2** [1]. If the function $g$ is regular in $U$ and $|g(z)| < 1$ in $U$, then for all $\xi \in U$ and $z \in U$ the following inequalities hold:

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)} g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z} \xi} \right|, \tag{3}$$

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$$|g'(z)| \leq \frac{1-|g(z)|^2}{1-|z|^2} \tag{4}$$

The equalities hold only in the case $g(z) = \varepsilon \frac{z+u}{1+\overline{u}z}$, where $|\varepsilon| = 1$ and $|u| < 1$.

**The Schwarz Lemma [1].** Let the analytic function $f(z)$ be regular in the unit circle $|\, z \,| < 1$ and let $f(0) = 0$. If, in $|z| < 1, |f(z)| \leq 1$ then

$$|f(z)| \leq |z|, \qquad |z| < 1 \tag{5}$$

where equality can hold only if $f(z) = Kz$ and $|K| = 1$.

## 3. Main results

**Theorem 3.1** Let $\gamma$ be a complex number, $Re\,\gamma \geq 1$ and $g \in A$.

If

$$|g(z)| \leq 1 \tag{6}$$

for all $z \in U$, then the function

$$G_\gamma(z) = \left[\gamma \int_0^z u^{\gamma-1} e^{g(u)} du\right]^{\frac{1}{\gamma}} \tag{7}$$

is in the class $S$.

**Proof.** Let us consider the function

$$f(z) = \int_0^z e^{g(u)}\, du. \tag{8}$$

The function $f$ is regular in U. We have

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| = \left(1-|z|^2\right)|z|\,|g'(z)| \tag{9}$$

From (6) and Theorem 2.2 we obtain

$$|g'(z)| \leq \frac{1}{1-|z|^2} \tag{10}$$

From (9) and (10) we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{11}$$

for all $z \in U$. From (8) we obtain $f'(z) = e^{g(z)}$, then from (11) and Theorem 2.1 for $Re\,\alpha = 1$, it follows that the function $G_\gamma$ is in the class $S$.

**Theorem 3.2.** Let $\gamma$ be a complex number, $Re\gamma = a > 0$, and the function $g \in A$. If

$$|zg'(z)| \leq 1 \tag{12}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}, \tag{13}$$

then the function

$$T_\gamma(z) = \left[\gamma \int_0^z u^{\gamma-1} \left(e^{g(u)}\right)^\gamma du\right]^{\frac{1}{\gamma}} \tag{14}$$

is in the class S.

**Proof.** Let us consider the function

$$f(z) = \int_0^z \left[e^{g(u)}\right]^\gamma du. \tag{15}$$

The function

$$h(z) = \frac{1}{|\gamma|} \frac{zf''(z)}{f'(z)}, \tag{16}$$

where the constant $|\gamma|$ satisfies the inequality (13), is regular in $U$.

From (15) and (16) we obtain

$$h(z) = \frac{\gamma}{|\gamma|} zg'(z), \tag{17}$$

Using (12) and (17) we obtain

$$|h(z)| < 1 \tag{18}$$

for all $z \in U$. From (17) we have $h(0) = 0$ and applying the Schwarz - Lemma we get

$$|h(z)| \leq |z| \tag{19}$$

for all $z \in U$, and hence, we obtain

$$\frac{1 - |z|^{2a}}{a} \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{|\gamma|}{a} \left(1 - |z|^{2a}\right) |z|. \tag{20}$$

Let us consider the function $Q : [0, 1] \to \mathcal{R},\ Q(x) = \left(1 - x^{2a}\right) x,\ \ x = |z|.$ We have

$$Q(x) \leq \frac{2a}{(2a+1)^{\frac{2a+1}{2a}}} \tag{21}$$

for all $x \in [0,1]$. From (21), (20) and (13) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{z\,f''(z)}{f'(z)}\right| \le 1 \tag{22}$$

for all $z \in U$. Then, from (22) and Theorem 2.1 for $Re\alpha = a$ it follows that the function $T_\gamma$ is in the class $S$.

## References


[1] Nehari, Z., *Conformal mapping*, Mc Graw-Hill Book Comp., New York, Toronto, London, 1952.

[2] Pascu, N. N., *An improvement of Becker's univalent criterion,* Proceedings of the Commemorative Session Simion Stoilow, Braşov, 1987, Univ. Braşov, 1987, 43-48.

[3] Pommerenke, C., *Univalent functions*, Göttingen, 1975.



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# THE PROBLEM OF B. V. GNEDENKO FOR PARTIAL SUMMATION SCHEMES ON BANACH SPACE

HO DANG PHUC

**Abstract**. The paper deals with the problem of B. V. Gnedenko for the partial summation scheme of random vectors taking values in a Banach space. A characterization of the limit distribution class of the scheme and some conditions for the limit distribution to be convolutions of semistable distributions are given.

## 1. Introduction and notation

Let $X_1, X_2, ...$ be a sequence of independent random variables and

$$Y_n \quad = \quad a_n \sum_{i=1}^{n} X_i \; + \; x_n \quad , \quad n = 1, 2, ... \tag{1.1}$$

a sequence of normalized sums, which has a proper limit distribution $Q$ for an appropriate choice of normalizing sequence of positive numbers $(a_n)$ tending to 0 and of elements $(x_n)$ from the real line. B. V. Gnedenko posed the problem of characterizing the class of the distributions $\{Q\}$ when among the distributions of the summands $X_i$ there are only $p$ different ones. Let this class be denoted by $G_p$. It is well known that $G_1$ coincides with the class of stable distributions. In [11] Zolotarev and Korolyuk proved that $G_2$ is the class of convolutions of stable distributions pairs. (This theorem is generalized to Banach valued random vectors by Jurek [5]). Further, Zinger shows that in the case of $p > 2$ , the class $G_p$ is broader than the one of stable distribution convolutions [9] and characterized it in [10].

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An extension of the stable distributions class is the one of semistable distributions, i. e. the limit distributions $\{Q\}$ when in (1.1) the index $n$ run not over whole the sequence of natural numbers, but only along some subsequence $(k(n))$ :

$$Y_n \quad = \quad a_n \sum_{i=1}^{k(n)} X_i \ + \ x_n \ , \quad n = 1, 2, ... \quad ,$$

$X_i, i = 1, 2, ...,$ are independent identically distributed and k(n) tends to infinity not too fast:

$$k(n)/k(n+1) \to r \ , \quad 0 < r < 1 \ . \tag{1.2}$$

Under this idea, Chibisova [1,2] generalized the Gnedenko's problem to the partial summation scheme:

$$Y_n \ = \ a_n \left( \sum_{i=1}^{k(1,n)} X_{1,i} \ + ... + \sum_{i=1}^{k(p,n)} X_{p,i} \right) + x_n \ , \ n = 1, 2, ... \ , \tag{1.1$'$}$$

where $X_{j,i}$ , $i = 1, 2, ..., k(i, j);\ j = 1, .., p$ , are independent, $X_{j,i}$ , $i = 1, 2, ..., k(i, j)$, have a common distribution $\mu_j$ for $j = 1, ..., p$ , and

$$1 \ > \ k(j, n)/k(j, n+1) \ \geq \ c \ , \quad j \ = \ 1, ..., p \tag{1.2$'$}$$

for some $c > 0$. The reason for taking (1.2′) instead of (1.2) to restrict the scheme (1.1′) is that for the case when $p > 1$ , if the condition (1.2′) fails, the limit distribution of the scheme (1.1′) would be an arbitrary infinitely divisible distribution without normal component, although (1.2) holds for $k(n) \ = \ k(1, n) + ... + k(p, n)$ (see Theorem 1 [1]).

In this paper, we attempt to study the problem of B. V. Gnedenko for the partial summation scheme of random vectors taking values in a Banach space. A characterization of the limit distributions class of the scheme (Theorem 1) and some conditions for the limit distribution to be convolutions of semistable distributions (Theorems 2, 3 and 4) are given.

In the sequel the following notation will be used: $E$ denotes a separable Banach space, $E'$ its dual space, $< ., . >$ the dual pairing between $E$ and $E'$. Further, $P(E)$ stands for the class of all probability measures on $E$ , $\delta(x)$ the unit mass concentrated at the point $x \in E, *$ the convolution and $\Rightarrow$ the weak convergence of measures in $P(E)$. It is well known that $P(E)$ with the weak convergence topology is a separable metric space (see [6], Theorem II.6.2). Moreover, one can find in this space a *shift-invariant metric* (e. g. the Levy-Prokhorov metric), i. e. a metric $\rho$ such that

$$\rho(\nu * \delta(x)), \mu * \delta(x)) \quad = \quad \rho(\nu, \mu)$$

for all $x \in E,\ \mu,\ \nu \in P(E)$.

Throughout the forthcoming, unless otherwise specified, we shall denote by small italic letters $x, y, z$ elements from $E; a, b, c, r, s, t$ positive numbers, $(n)$ the sequence of all natural numbers and by Greek letters $\gamma, \kappa, \lambda, \mu, \nu$ measures from $P(E)$. Moreover, $(x_n), (y_n), (z_n)$ , also with other subscripts or indexes, will mean sequences of elements from $E$. Similarly, $(a_n), (b_n), (c_n), (r(n)), (s(n)), (t(n))$ mean sequences of positive numbers, $(\gamma_n)$, $(\kappa_n)$, $(\lambda_n), (\mu_n), (\nu_n)$ sequences of measures from $P(E)$ and $(k(n)), (m(n))$, $(n'), (n'')$ subsequences of natural numbers.

A measure $\mu$ is called nondegenerated if it is not concentrated at any point and the power $\mu^{*n}$ is defined recursively by $\mu^{*n} = \mu^{*n-1} * \mu$. Further $\mu$ is said to be *infinitely divisible* if for every $n$ there exists a measure $\mu_n$ such that $\mu = (\mu_n)^{*n}$. By $ID(E)$ we mean the subclass of all infinitely divisible measures from $P(E)$. Then for each $t \geq 0$ and $\mu \in ID(E)$ we can define $\mu^t$ (see [8], for example).

For a measurable map $S$ from $E$ to another Banach space, $S\mu$ stands for the image of $\mu$ by the map $S$. In particular, when $S$ is of the form $a.I$ , where $I$ is the unit operator in $E$ , we write straightly $a.\mu$ instead of $a.I\mu$. We say that $\mu$ belongs to *the domain of semi-attraction*, or more exactly $r-$*semi-attraction*, of $\lambda$ if there exist $(a_n), (k(n))$ and $(x_n)$ such that

$$a_n.\mu^{*k(n)} \; * \; \delta\left(x_n\right) \quad \Rightarrow \quad \lambda$$

and

$$k(n)/k(n+1) \;\rightarrow\; r \quad as \quad n \;\rightarrow\; 0 \quad .$$

It is evident that in this case $\lambda$ is a *semistable measure* (see [3], for example), i. e. $\lambda \in ID(E)$ and $\lambda^r = a.\lambda * \delta(x)$ for some $a$ and $x$.

A sequence $(\lambda_n)$ is said to be *shift-convergent* if there exists a sequence $(x_n)$ such that the sequence $(\lambda_n * \delta(x_n))$ weakly converges and to be compact if every its subsequence contains a convergent subsequence.

## 2. **Main results**

**Theorem 1.** *Let* $c > 0$ , $\lambda$ *be non-degenerated and* $p$ *be a natural number. If there exist sequences* $(k(1,n)), ..., (k(p,n)), (a_n), (x_n)$ *and measures* $\mu_1, ..., \mu_p$ *such that*

$$a_n . \left( \mu_1^{*k(1,n)} \ * \ ... \ * \ \mu_p^{*k(p,n)} \right) \ * \ \delta\left(x_n\right) \quad \Rightarrow \quad \lambda \tag{2.1}$$

*and* $(1.2')$ *holds, then there exist sequences* $(t(1,n)), ..., (t(p,n)), (c_n), (y_n)$*, an element* $y_0$ *and measures* $\lambda_1, ..., \lambda_p \in ID(E)$ *such that* $t(i,n) \geq 1/c, \ n = 1, 2, ...; \ i = 1, ..., p$ *and*

$$\lambda \quad = \quad \lambda_1 \ * \ ... \ * \ \lambda_p \quad * \quad \delta\left(y_0\right) \ , \tag{2.2}$$

$$\lambda \quad = \quad (c_1 ... c_n) \ . \ \left( \lambda_1^{s(1,n)} \ * \ ... \ * \ \lambda_p^{s(p,n)} \right) \ * \ \delta\left(y_n\right) \quad ,$$

*with* $s(i,n) = t(i,1)...t(i,n), n = 1, 2, ...; i = 1, ..., p.$

*Conversely, if (2.2) holds and*

$$s(i,n) \ \rightarrow \ \infty \quad as \quad n \ \rightarrow \ \infty \ , \quad i = 1, ..., p \ , \tag{2.3}$$

*then (2.1) is true.*

The above theorem partially solves the problem of characterizing the limit distributions of partial summation schemes. In the following theorem, the problem is concerned with a special case, when in (2.1) $\mu_1, ..., \mu_p$ belong to domains of semi-attraction of some semistable probability measures:

**Theorem 2.** *Let sequences* $(a_n), (x_n), (k(1,n)), ..., (k(p,n))$ *, a measure* $\lambda$ *and a number* $c > 0$ *be given. Suppose that (2.1) holds and there exist positive numbers* $s(i) < 1$

*and semistable measures* $\nu_i,\ i = 1, ..., p$ *, such that* $\mu_i$ *belongs to the domain of* $s(i)-$*semi-attraction of* $\nu$ *, i. e.*

$$b_i(n)\ .\ \mu_i^{*m(i,n)}\ *\ \delta\,(x_i(n)) \quad \Rightarrow \quad \nu_i \quad as \quad n\ \to\ \infty \tag{2.4}$$

*for some* $(b_i(n))$ , $(x_i(n))$ *and*

$$m(i,n)/m(i,n+1)\ \to\ s(i) \quad as \quad n\ \to\ \infty\ . \tag{2.5}$$

*Then there exist positive numbers* $b_i$ , $t(i)\ \in\ [s(i), 1]$ , $i = 1, ..., p$ *and an element* $x_0$ *such that*

$$\lambda \quad = \quad b_1\ .\ \nu_1^{t(1)}\ *\ ...\ *\ b_p\ .nu_p^{t(p)} \quad * \quad \delta\,(x_0)\ . \tag{2.6}$$

This theorem giving a condition for the limit measure $\lambda$ in (2.1) to be a convolution of semistable measures has been proved on the real line by Chibisova (Theorem 2 [2]) with the additional condition (1.2′). The next theorem is devoted to investigate another condition for this.

**Theorem 3.** *Suppose that (2.1) holds and*

$$a_{n+1}/\ a_n \quad \to \quad a \tag{2.7}$$

$$k(i,n+1)/k(i,n) \to t(i) \quad as \quad n\ \to\ \infty\ ,\ i = 1, ..., p\ , \tag{2.8}$$

*for some positive number* $a$ *and* $t(1), ..., t(p)$ *such that*

$$t(1)\ <\ t(2)\ <\ ...\ <\ t(p)\ . \tag{2.9}$$

*Then there exist semistable measures* $\lambda_i$ , $i = 1, ..., p$ *, and* $y_o$ *such that*

$$\lambda \quad = \quad \lambda_1\ *\ ...\ *\ \lambda_p \quad * \quad \delta\,(y_0)\ . \tag{2.10}$$

*Moreover, there exist sequences* $(z_i(n))$ , $i\ =\ 1,\ ...\ , p$ *such that*

$$a_n{\cdot}\mu_i^{*k(i,n)}\ *\ \delta\,(z_i(n)) \quad \Rightarrow \quad \lambda_i\ , \tag{2.11}$$

*i. e.* $\mu_i$ *belongs to the domain of semi-attraction of* $\lambda_i, i = 1, ..., p$.

**Theorem 4.** *If (2.1) , (2.7) and (2.8) hold then there exists a natural number* $q$ *such that* $\lambda$ *is a convolution of* $q$ *semistable measures.*

## 3. Lemmas and proofs

First we introduce a lemma which will play a crucial role in the following development.

**Lemma 1.** *Let $\mu$ be nondegenerated, $\lambda_n, n = 1, 2, ...,$ and $(x_n)$, $(k(n))$, $(m(n))$ be given. Suppose that*

$$\lambda_n^{k(n)} * \delta\left(x_n\right) \Rightarrow \mu \tag{3.1}$$

*and*

$$m(n)/k(n) \to t \geq 0 \ . \tag{3.2}$$

*Then $\mu \in ID(E)$ and there exists a sequence $(y_n)$ such that*

$$\lambda_n^{*m(n)} * \delta\left(y_n\right) \Rightarrow \mu^t.$$

**Proof.** From (3.1) we can see that if $P$ is any finite-dimensional linear projector of the space $E$ then

$$\left(P\lambda_n\right)^{*k(n)} * \delta\left(Px_n\right) \Rightarrow P\mu.$$

Thus, by the classical argument on finite-dimensional spaces, we infer that $P\mu$ is infinitely divisible, consequently $\mu \in ID(E)$ in view of Corollary 1 [8, p.320]. Now, from (3.2) we can choose a natural number $N$ such that $m(n) < N.k(n)$ for all $n$. Then (3.1) yields

$$\lambda_n^{*Nk(n)} * \delta\left(Nx_n\right) \Rightarrow \mu^{*N}.$$

On the other hand,

$$\lambda_n^{*Nk(n)} = \lambda_n^{*m(n)} * \lambda_n^{*(Nk(n)-m(n))}.$$

Hence, by virtue of Theorem III.2.2 [6], there exists a sequence $(z_n)$ such that the sequence

$$\left(\lambda_n^{*m(n)} * \delta\left(z_n\right)\right) \tag{3.3}$$

is compact. Meanwhile, for every $y' \in E'$ , (3.1) and (3.2) imply

$$\left(y'\lambda_n\right)^{*m(n)} * \delta\left(< y', (m(n)/k(n)).x_n >\right) \Rightarrow \left(y'\mu\right)^t.$$

Then, by the Convergence of Type Theorem, if $\nu$ is any cluster point of the sequence (3.3) then

$$(y'\nu = (y'\nu)^t * \delta(x_{y'})$$

for some real $x_{y'}$. Thus, it follows from Corollary 1 of Lemma 2 [7] the existence of $x(\nu) \in E$ such that $< y', x(\nu) >= x_{y'}$ for all $y' \in E'$ and $\nu = \mu^t * \delta(x(\nu))$. The set

$$\Gamma = \{x(\nu) \ : \ \nu \quad is\ a\ cluster\ point\ of\ (3.3)\}$$

is a compact set provided the compactness of (3.3). Let $\rho$ denote the Levy-Prokhorov metric on $P(E)$. For every $n$ we define $z_n(0)$ by

$$\rho\left(\lambda_n^{*m(n)} * \delta(z_n), \mu^t * \delta(z_n(0))\right) =$$

$$= \min_{x \in \Gamma} \rho\left(\lambda_n^{*m(n)} * \delta(z_n), \mu^t * \delta(x)\right)$$

Then, it is evident that

$$\rho\left(\lambda_n^{*m(n)} * \delta(z_n - z_n(0)), \mu^t\right) \to 0$$

which implies

$$\lambda_n^{*m(n)} * \delta(y_n) \Rightarrow \mu^t$$

with $y_n = z_n - z_n(0)$ , i. e. the conclusion of the lemma is true.

**Proof of Theorem 1.** We invoke Theorem III.5.1 [6] and (2.1) to deduce that there exist sequences $(y_i(n))$, $i = 1, ..., p$ , such that the sequences

$$\left(a_n.\mu_i^{*k(i,n)} * \delta(y_i(n))\right), \quad i = 1, ..., p \tag{3.4}$$

are compact. Then from (1.2′) there are a subsequence $(n')$ , numbers $t(i,1)$ and measures $\lambda_i, \nu_i \in ID(E)$, $i = 1, ..., p$ , such that $t(i,1) \geq 1/c$ and

$$k(i, n'+1)/k(i, n') \to t(i, 1) \tag{3.5}$$

$$a_{n'}.\mu_i^{*k(i,n')} * \delta(y_i(n')) \Rightarrow \lambda_i \tag{3.6}$$

$$a_{n'+1}.\mu_i^{*k(i,n'+1)} * \delta(y_i(n'+1)) \Rightarrow \nu_i \tag{3.7}$$

for $n' \to \infty$ and $i = 1, ..., p$ . Consequently, (2.1) yields

$$\lambda = \lambda_1 * ... * \lambda_p * \delta(y_0) \tag{3.8}$$

$$\lambda = \nu_1 * ... * \nu_p * \delta(z_0).$$

Moreover, in view of Theorem 1 [4] and Lemma 1, the conditions (3.5) through (3.7) imply

$$\nu_i = c_1.\lambda_i^{t(i,1)} * \delta(z_i) \ , \ i = 1, ..., p,$$

for some positive $c_1$ and elements $z_i, \ i = 1, ..., p$. Hence

$$\lambda = c_1. \left(\lambda_1^{t(1,1)} * ... * \lambda_p^{t(p,1)}\right) * \delta(y_1),$$

with $y_1 = z_0 + z_1 + ... + z_p$.

Further, from the compactness of the sequence (3.4), we can pick from $(n')$ another subsequence $(n'')$ such that for some $t(i,2) \geq 1/c$ and $\kappa_i \in ID(E), \ i = 1, ..., p$ , we have

$$k(i, n''+2)/k(i, n''+1) \to t(i,2),$$

$$a_{n''+2}.\mu_i^{*k(i,n''+2)} * \delta(y_i(n''+2)) \Rightarrow \kappa_i$$

as $n'' \to \infty$. Then repeating the above argument we can see that

$$\lambda = (c_1.c_2). \left(\lambda_1^{t(1,1).t(1,2)} * ... * \lambda_p^{t(p,1).t(p,2)}\right) * \delta(y_2),$$

with $c_2 > 0$ and $y_2 \in E$.

The continuation of the above process arrives at the conclusion that for each $n$

$$\lambda = (c_1...c_n). \left(\lambda_1^{s(1,n)} * ... * \lambda_p^{s(p,n)}\right) * \delta(y_n),$$

with $s(i,n) = t(i,1)...t(i,n), t(i,j) \geq 1/c, c_j > 0, j = 1, ..., n; \ i = 1, ..., p$ , which together with (3.8) implies (2.2).

Conversely, let (2.2) and (2.3) hold. Then it follows from Theorem III.5.1 [6] the existence of the sequences $(z_i(n)), \ i = 1, ..., p$ , such that sequences $(\gamma_i(n)), \ i = 1, ..., p$ , are compact, where

$$\gamma_i(n) = b_n.\lambda_i^{s(i,n)} * \delta(z_i(n)), \ n = 1, 2, ...$$

and $b_n = c_1...c_n$.

Let $(n')$ be any subsequence of natural numbers. Since $\lambda$ is nondegenerated and

$$\lambda = \gamma_1(n') * ... * \gamma_p(n') * \delta\left(y_{n'} - z_1(n') - ... - z_p(n')\right),$$

at least one of the subsequences $(\gamma_i(n'))$, $i = 1, ..., p$ , say $(\gamma_1(n'))$ , has a nondegenerated clust point $\gamma_1$ , i. e.

$$b_{n''}.\lambda_1^{s(1,n''} * \delta\left(z_1(n'')\right) \Rightarrow \gamma_1 \quad as \quad n'' \to \infty$$

for some subsequence $(n'')$ of $(n')$. Hence there exists an element $y' \in E'$ such that $y'\gamma_1$ is nondegenerated and

$$b_{n''}.y'\lambda_1^{s(1,n''} * \delta\left(\langle y', z_1(n'')\rangle\right) \ \Rightarrow \ y'\gamma_1 \ .$$

Consequently, since $s(1, n'') \to \infty$ as $n'' \to \infty$ , we see that $b_{n''} \to 0$ as $n'' \to \infty$ . In conclusion, every subsequence $(b_{n'})$ of $(b_{n''})$ contains another subsequence tending to zero, this means $b_n \to 0$ as $n \to \infty$. Then, because for each index $i = 1, ..., p$ the set $\{(\lambda_i)^s : \ 0 \le s \le 1\}$ is compact (see Theorem 5[3]), it is plain that

$$b_n.\lambda_i^{s(i,n)-[s(i,n)]} \Rightarrow \delta(0) \quad as \quad n \to \infty,$$

where $[s]$ denotes the integer part of a number $s$. This implies by virtue of (2.2) that

$$b_n.\left(\lambda_1^{[s(1,n)]} * ... * \lambda_1^{[s(p,n)]}\right) * \delta\left(y_n\right) \ \Rightarrow \ \lambda,$$

which yields (2.1) with $\mu_i = \lambda_i$ , $x_n = y_n$, $k_i(n) = [s(i, n)]$, $i = 1, ..., p;$ $n = 1, 2, ....$ The proof is just complete.

**Proof of Theorem 2.** By the same reason as in Theorem 1, there exist sequences $(y_i(n))$, $i = 1, ..., p$ such that the sequences (3.4) are compact. Then we can find a subsequence $(n')$ of natural numbers and measure $\lambda_i \in ID(E)$, $i = 1, ..., p$ , such that

$$a_{n'}.\mu_i^{*k(i,n')} * \delta\left(y_i(n')\right) \ \Rightarrow \ \lambda_i \ , \ i = 1, ..., p \ . \tag{3.9}$$

Let's fix an index $i$. Then, without loss of generality, we can suppose that

$$m(i, n') \le k(i, n') < m(i, n' + 1).$$

Thus from (2.5) it is clear that the set

$$k(i,n')/m(i,n'+1) : n' \in (n')$$

is compact and we can suppose once more that

$$k(i,n')/m(i,n'+1) \to t(i) \text{ as } n' \to \infty \tag{3.10}$$

for some $t(i) \in [s(i), 1]$. Therefore, taking Theorem 1 [4] and Lemma 1 into account, by virtue of (2.4), (3.9) and (3.10), we infer that $a_{n'}/b_i(n'+1) \to b_i$ and

$$\lambda_i = b_i.\nu_i^{t(i)} * \delta(y_i),$$

for some $b_i > 0,\ y_i \in E$. Hence the theorem is proved in view of (2.1) and (3.9). For the proof of Theorem 3 we need the following lemma:

**Lemma 2.** *Let $\mu \in ID(E)$, $\lambda \in ID(E)$, $(x_n)$, $(t(n))$ and $(a_n)$ be given. Suppose that*

$$a_n.\lambda^{t(n)} * \delta(x_n) \ \Rightarrow\ \mu \tag{3.11}$$

*and*

$$t(n)/t(n+1) \ \to\ t > 0 \ . \tag{3.12}$$

*Then there exist $a > 0$ and $x$ such that*

$$\mu^t = a.\mu * \delta(x) \ , \tag{3.13}$$

*i. e. $m$ is semistable.*

**Proof.** From (3.11) and (3.12), by an argument analogous to that used for the proof of Lemma 1, we can infer that

$$a_n.\lambda^{t(n+1)} * \delta(y_n) \ \Rightarrow\ \mu^{1/t}$$

for some sequence $(y_n)$. Meantime,

$$a_{n+1}.\lambda^{t(n+1)} * \delta(y_{n+1}) \ \Rightarrow\ \mu.$$

Thus, (3.13) follows from Theorem 1 [4] with $a = \lim_{n\to\infty}(a_{n+1}/a_n)$.

**Proof of Theorem 3.** It is evident that (1.2') is true in view of (2.9). Hence, by the same argument as in the proof of Theorem 1, we can see that

$$\lambda = \lambda_1 * ... * \lambda_p * \delta\left(y_0\right) \ , \tag{3.14}$$

$$\lambda = \left(a^n\right).\left(\lambda_1^{t(1)^n} * ... * \lambda_p^{t(p)^n}\right) * \delta\left(y_n\right).$$

Then

$$a^{-n}.\lambda^{t(p)^{-n}} * \delta\left(\left(a.t(p)\right)^{-n}.y_n\right) = \lambda_1^{(t(1)/t(p))^n} * ... * \lambda_p^{(t(p-1)/t(p))^n} * \lambda_p.$$

Meanwhile, since for $i = 1, ..., p-1$ we have $t(i)/t(p) < 1$, Theorem 5 [3] yields

$$\lambda_i^{(t(i)/t(p))^n} \ \Rightarrow \ \delta(0).$$

Hence

$$a^{-n}.\lambda^{t(p)^{-n}} * \delta\left(\left(a.t(p)\right)^{-n}.y_n\right) \ \Rightarrow \ \lambda_p \ . \tag{3.15}$$

Therefore, it follows from Lemma 2 that $\lambda_p$ is semistable. As we have seen in the proof of Theorem 1, there exists a sequence $(y_p(n))$ such that the sequence

$$\left(a_n.\mu_p^{*k(p,n)} * \delta\left(y_p(n)\right)\right) \tag{3.16}$$

is compact. Let $\nu_p$ be any clust point of the sequence (3.16). Then repeating the argument of the proof of Theorem 1 and the above part we can conclude that

$$a^{-n}.\lambda^{t(p)^{-n}} * \delta\left(x_p(n)\right) \ \Rightarrow \ \nu_p$$

for some sequence $(x_p(n))$. This together with (3.15) and Theorem 1 [4] implies the existence of an element $z(\nu_p) \in E$ such that

$$\nu_p = \lambda_p * \delta(z(\nu_p)).$$

Hence, by the same reason used in the proof of Lemma 1 and the compactness of the sequence (3.16) we obtain

$$a_n.\mu_p^{*k(p,n)} * \delta\left(z_p(n)\right) \ \Rightarrow \ \lambda_p \tag{3.17}$$

for some sequence $(z_p(n))$, i. e. (2.11) holds for $i = p$. Now, (2.1) and (3.17) imply the shift convergence of the sequence

$$\left(a_n.\left(\mu_1^{*k(1,n)} * ... * \mu_{p-1}^{*k(p-1,n)}\right)\right)$$

and by the same way as the above we get the semistability of $\lambda_{p-1}$ and (2.11) for $i = p - 1$. The proof is complete after the $p$ times repeated application of the above argument.

**Proof of the Theorem 4.** As in the proof of Theorem 3 we see that (3.14) hold. Then by a renumeration if necessary, we can suppose that

$$t(1) \leq t(2) \leq ... \leq t(p) \ .$$

Let's set

$$r(1) = t(j(1)), \nu_1 = \lambda_1 * ... * \lambda_{j(1)}$$

if $t(1) = ... = t(j(1)) < t(j(1) + 1)$ ,

$$r(2) = t(j(2)), \nu_2 = \lambda_{j(1)+1} * ... * \lambda_{j(2)}$$

if $t(j(1) + 1) = t(j(1) + 2) = ... = t(j(2)) < t(j(2) + 1)$,

. . . . .

$$r(q) = t(p), \nu_q = \lambda_{j(q-1)+1} * ... * \lambda_p$$

Then it follows straightly from (3.14) that

$$q \leq p, \quad r(1) < r(2) < ... < r(q)$$

and

$$\lambda = \nu_1 * ... * \nu_q * \delta(y_0) ,$$

$$\lambda = a^n.\left(\nu_1^{r(1)^n} * ... * \nu_q^{r(q)^n}\right) * \delta(y_n) .$$

Hence, arguing as in the proof of Theorem 3, we get the conclusion of the theorem.

## References

[1] Chibisova, E. D., *A Problem of B. V. Gnedenko in a Partial Summation Scheme* (Russian), Teor. Veroyatnost. i Primenen. **31**, no.**4**(1986), 805-808.

[2] Chibisova, E. D., *On a Partial Summation Scheme* (Russian), Teor. Veroyatnost. i Primenen. **32**, no. **3**(1987), 569-573.

[3] Chung Dong, M., Rajput, B. S., Tortrat, A., *Semistable Laws on Topological Vector Spaces*, Z. Wahrsch. Verw. Gebiete **60**(1982), 209-218.

[4] Csiszar, I., Rajput, B. S., *A Convergence of Types Theorem for Probability Measures on Topological Vector Spaces with Applications to Stable Laws*, Z. Wahrsch. Verw. Gebiete **36**(1976), 1-6.

[5] Jurek, Z. J., *The Problem of B. V. Gnedenko on Banach Spaces*, Bull. Acad. Polon. Sci., Ser. Math., **29**, no. 7-8(1981), 399-407.

[6] Parthasarathy, K. R., *Probability Measures on Metric Spaces*, New York - London Press, 1967.

[7] Tortrat, A., *Structure des Lois Indefiniment Divisibles dans un E.V.T.*, Lecture Notes in Math., Berlin - Heidelberg - New York, 1967, 229-238.

[8] Tortrat, A., *Sur la Structure des Lois Indefiniment Divisibles dans les Espaces Vectoriels*, Z. Wahrsch. Verw. Gebiete **11**(1969), 311-326.

[9] Zinger, A. A., *On a Problem of B. V. Gnedenko* (Russian), Dokl. Akad. Nauk SSSR **6**(162)(1965), 1238-1240.

[10] Zinger, A. A., *On a Class of Limit Distributions for Normalized Sums of Independent Random Variables* (Russian), Teor. Veroyatnost. i Primenen. **10**, no. **4**(1965), 672-692.

[11] Zolotarev, V. M., Korolyuk, V. C., *On a Hypothesis of B. V. Gnedenko* (Russian), Teor. Veroyatnost. i Primenen. **6**, no. **4**(1961), 469-474.

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## BOOK REVIEWS

Alberto Guzman, *Derivatives and Integrals of Multivariable Functions*, Birkhäuser Verlag, Boston-Basel-Berlin 2003, x+319 pp., ISBN: 0-8176-4274-9 and 3-7643-4274-9.

This is a text for a one-semester course in advanced calculus of several variables (differential and integral calculus). The continuity properties of functions as well as some topology questions are treated, within the framework of normed spaces, in a previous book by the same author "*Continuous functions of vector variable*", Birkhäuser 2002. Together they can be used for a one-year advanced course in multivariable calculus. The author indicates in the Preface how it can be used for shorter term courses by skipping some tedious parts (as, e.g., the proof of the implicit function theorem, the treatment of generalized integrals, etc).

Although the present volume contains some references to the previous one, it is fairly self-contained, the prerequisites being familiarity with the topology of the Euclidean space and some linear algebra.

The differentiability is treated in the first three chapters: 1. *Differentiability of multivariable functions,* 2. *Derivatives of scalar functions*, and 3. *Derivatives of vector functions.* They contain the basic definitions and properties, a careful treatment of higher order derivatives and their symmetry, implicit and inverse function theorems, local extrema and conditioned extrema (the rule of Lagrange multipliers), and some geometric applications (curves, surfaces, tangents and normals).

Multivariable integration is treated in chapters 4. *Integrability of multivariable functions* and 5. *Integrals of scalar functions.* Staring with integration on boxes, one defines then Jordan measurable sets (called Archimedean) and integrals over Archimedean domains. Improper integrals are also considered. This chapters contains also a proof of the change of variables formula and a treatment of line and surface integrals.

The last chapter of the book, 6. *Vector integrals and the vector-field theorems*, contains proofs of the fundamental theorems of Green, Stokes and Gauss. Some applications of these theorems to physics are discussed.

The book contains also exercises and their solutions are given at the end of the book –"For emergency use only", as the author says quoting Buck.

The book reflects the teaching experience of the author as well as his taste and scientific ideas. It fits excellently for a course in multivariable calculus for students in mathematics, physics or engineering, preparing them for more advanced courses in real analysis and differential geometry.

S. Cobzaş

Steven G. Krantz, *A Handbook of Real Variables,* Birkhäuser Verlag, Boston-Basel-Berlin 2004, xii+201 pp., ISBN: 0-8176-4329-X and 3-7643-4329-X.

The aim of this handbook is to provide the reader with a quick and accessible treatment of the main ideas of the theory of functions of a real variable (within the limits of Riemann integration), including elements of Fourier analysis and applications to the solving of differential and partial differential equations. It is devoted to scientists who need real analysis, but have no time nor patience to look into the details. So it contains a few proofs, the main idea being to give cute explanations of the notions and to present the basic results along with some examples of application. There are references to textbooks on real analysis for further reading, but these are optional, the book being self-contained.

The topics covered by the book are: basic set theory and real numbers; sequences of numbers and their limits (including Limsup and Liminf); series – convergence tests, operations with series, Riemann's theorem on the rearrangements of conditionally convergent series, the series of the number $e$; the topology of the real line, Cantor set; limits and the continuity of functions, intermediate value property, monotonic functions and their discontinuities; the derivative and the main theorems of differential calculus (mean value theorems), Weierstrass example of a nowhere differentiable function; the Riemann integral and its fundamental properties, the Riemann-Stieltjes integral; sequences and series of functions and Weierstrass approximation theorem; some special functions – the exponential function and the logarithm, the trigonometric functions and their inverses, the Gamma function; Fourier series; the topology of metric spaces and Ascoli-Arzela theorem; Picard's iteration technique for solving differential equations; Fourier analytic methods for solving partial differential equations.

The book will be a very useful tool for physicists, engineers, economists, but also for students in mathematics, computer science, physics or engineering as a helping instrument when working exercises and problems.

S. Cobzaş

Jiri Matousek, *Using the Borsuc-Ulam Theorem – Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer Verlag, Berlin, 2003, ISBN: 3-540-00362-2.

The difficulty with the algebraic topology is the immense amount of preparatory knowledge, the sophisticated technical preludium preceding the proving the significant results of the field. But most part of these results are formulated in terms of the elementary calculus using only the topology and the geometry of the Euclidean space. There is a considerable gap between the geometric intuition and the mathematical machinery permitting its rigorous handling. Hence, last time the algebraic topology was driven out of an usual mathematical curriculum. This circumstance deprives the mathematical student of acquaintance with one of the most brilliant records which the human intelligence ever achieves. The various attempts to make the ideas of the field more accessible produce a sort of popularizing literature renouncing somewhere to rigor. By a difference, Matousek's book is focused not on the whole algebraic topology, but on a part of it, which can be gathered under the title "Borsuc-Ulam theorem, and its relation with the combinatorics". After an introductory chapter concerning the elementary theory of simplicial complexes, the author presents a lot of equivalent versions of the Borsuc-Ulam theorem, proves their interrelations and presents various their proofs. The following sections concerns direct applications, as the well known Ham Sandwich Theorem, Coloring of Necklets, the Lovász-Kneser Theorem, Gales Lemma and Schrijver's Theorem. A next chapter is destined to introduce deeper topological concepts as k-connectedness and cell complexes in order to handle in the last two chapters nonembeddability results as the Van Kampen-Flores Theorem, Sarkaria's Inequality as well as problems concerning multiple points of coincidence with special emphasis on the Topological Tverberg Theorem and the problematic around it.

The book excels in presenting carefully the intuitive background of the material inserting amusing examples and pictures. The idea of presenting first the easier results and interconnections and only them the technically more difficult proofs makes the material more attractive for the beginner, and at the same time an agreeable lecture for the specialist.

A. B. Németh

*Sampling, Wavelets, and Tomography,* John J. Benedetto and Ahmed I. Zayed – Editors, Applied and Numerical Harmonic Analysis, Birkhäuser Verlag, Boston-Basel-Berlin 2004, xxi+344 pp., ISBN: 0-8176-4304-4.

The papers included in the present volume emerged from the materials presented at the biennial Sampling Theory and Applications (SamTA01) Conference, held in Orlando, Florida, in May 2001. The conference celebrated the accomplishments of Claude Elwood Shannon, born on April 30, 1916 and died on February 22, 2001, the creator of modern information theory. This was the third of SamTA conferences, the first one took place in 1997 in Jurmala, Latvia, the second in 1999 in Aveiro, Portugal, and the fourth in 2003 in Strobl, Austria.

Its aim, as it is presented in the first chapter of the book, "*A prelude to sampling, wavelets, and tomography*", by Ahmed I. Zayed, the organizer of the conference and co-editor of the volume, was to emphasize the connections between the three topics mentioned in the title of the volume and. This chapter gives also a general introduction to the remaining chapters, which, written by leading experts in the fields, mathematicians and engineers, deal with mathematical topics as well as with applications. For many years, the research in the sampling has been carried by communication engineers, but, with the advent of new techniques in mathematical analysis, many mathematicians get involved into the matter, leading to new interesting results and interconnections.

A good idea on the contents of the book is given by the headings of its chapters: 2. *Sampling without input constraints: Consistent reconstruction in arbitrary spaces*, by Yoanina C. Eldar; 3. *An introduction to irregular Weyl-Heisenberg frames*, by Peter G. Casazza; 4. *Robustness of regular sampling in Sobolev algebras*, by Hans G. Feichtinger and Tobias Werther; 5. *Sampling theorems for nonbandlimited signals*, by P. P. Vaidyanathan; 6. *Polynomial matrix factorization, multidimensional filter banks, and wavelets*, by N. K. Bose and S. Lertrattanapanich; 7. *Function spaces based on wavelet expansions,* by Stéphane Jaffard; 8. *Generalized frame multiresolution analysis of abstract Hilbert spaces,* by Manos Papadakis; 9. *Sampling theory and parallel-beam tomography*, by Adel Faridani; 10. *Filtered backprojection algorithms for spiral cone beam CT,* by Alexander Katsevich and Guenter Lauritsch; 11. *Adaptive irregular sampling in meshfree flow simulation,* by Armin Iske; 12. *Thin-plate spline interpolation,* by David C. Wilson and Bernard A. Mair.

The book is addressed to mathematicians, scientists, and engineers working on signal and image processing and medical imaging. It is written by experts and for experts, but each chapter has an introductory part written for non-specialists, giving them the possibility to find what the chapter is dealing with.

The book contains important contributions to the areas mentioned in the title: sampling, wavelets and tomography.

S. Cobzaş